STELLAR DYNAMICS

BY THE SAME AUTHOR

Spherical Astronomy
The Sun, the Stars, and the Universe
Astrophysics
Astronomy

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Position Line Tables

STELLAR DYNAMICS

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PREFACE

The birth of Stellar Dynamics may quite fairly be associated with Kapteyn's discovery of star-streaming in 1904; within the short space of a third of a century the subject has grown to such dimensions as to constitute one of the most important departments of theoretical and observational astronomy. Hard on the heels of Kapteyn's discovery came the mathematical researches of Eddington on the two-drifts theory of stellar motions and of Schwarzschild on the alternative ellipsoidal hypothesis; in each case the theory was submitted to as stringent a test as the rather limited observational material available at the time permitted. With the recent rapid increase in the number of accurate proper motions and radial velocities, this kinematical part of the subject has steadily expanded and, so far as one can predict, it will form the basis of many extensive observational programmes in the years to come.

The region of dynamics is definitely entered with the researches of Jeans and Eddington on stellar systems, almost a quarter of a century ago. Further development only became possible with the increase of accurate observations and in 1926 the subject received a fresh impetus when Oort published his first investigation on galactic rotation.

Such, in brief, are the main stages in the history of Stellar Dynamics.

This book is an attempt to present the subject in its full mathematical and observational development. After an introductory chapter, in which the correction of observational statistics is given a place, the mathematical theory of a single star-drift is described in Chapter II with considerable detail. The following chapter discusses the problem of the solar motion in many of its aspects, historical, theoretical and practical. Chapter IV is devoted to the theory of the two star-streams, with its mathematical foundations resting on the theorems of the second chapter. Chapter V deals in detail with Schwarzschild's ellipsoidal hypothesis; in both Chapters IV and V special emphasis has been laid on the practical applications of the several theoretical methods described.

Chapter VI is devoted to a discussion of the various methods of deriving statistical parallaxes of the stars from a knowledge of their proper motions, with due consideration of their distribution in the sky. The information supplied in this way is of great value; in particular, the scale of the distances of the globular clusters depends largely at present on the application of the statistical method to the Cepheids. In the succeeding chapter Dyson's formula for the distribution of the stars is made the basis of an investigation dealing with proper motions; results involving the two-streams theory and the ellipsoidal theory are given in some detail. General theorems on stellar statistics are brought together in Chapter VIII. Although much of the theoretical work described was published a quarter of a century ago, practical applications have been comparatively few owing, in the main, to the general paucity of observational material. Chapter IX deals with moving, open and globular clusters with sections on the galactic absorbing cloud.

Chapter X contains the fundamental work of Jeans and Eddington on the Dynamics of Stellar Systems, and Chapter XI deals with Galactic Rotation, to a great extent, from the observational standpoint. The final chapter on the Dynamics of the Galaxy incorporates several recent researches on different aspects of galactic phenomena.

Although I have consulted a very considerable number of memoirs, it has been impossible to treat, within the compass of a single volume of reasonable dimensions, all the various view-points of the subject. Another author would, almost certainly, have proceeded on somewhat different lines, but I hope that the systematic account of Stellar Dynamics given in the book will form the foundation on which the reader can build a more complete structure.

There remains the pleasurable duty of thanking several astronomers for valuable assistance. To Sir Arthur Eddington, O.M., I am specially indebted, first, for my introduction to the study of stellar motions while I served under him at Cambridge and, second, for helpful and illuminating discussions on practically every department of the subject. Dr Alan Fletcher has read the manuscript with the utmost care and patience; in detecting errors and removing obscurities and ambiguities his assistance has been invaluable and I am very grateful to him for all the time and labour he has so ungrudgingly devoted to a long and tedious task. I am also deeply obliged to Dr S. Chandrasekhar for his careful reading of the proofs. Finally, it is a pleasure to express my thanks again to the Officials and Staff of the University Press for their attention and care while the book was assuming its final shape.

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W. M. S.

CHAPTER I

INTRODUCTION

1·1. *General description of the galactic system.*

The astronomical objects visible in a telescope—with the exception of those recognised as members of the solar system—may be broadly classified as follows: stars, open star clusters, diffuse nebulae, planetary nebulae, globular clusters, spiral and other extra-galactic nebulae. The assembly of stars is generally referred to as the *galactic system*, and within its bounds are to be found the objects mentioned above with the exception of the globular clusters and the spiral and extra-galactic nebulae. The spiral nebulae are now believed to be independent galaxies, comparable in size with the galactic system and separated from each other by distances many times greater than their diameters. The globular clusters, of which about one hundred are known, are believed to form a spherical system concentric with the galactic system.

Modern research has established that the galactic system has the spatial characteristics of an oblate spheroid. The median plane defines the *galactic equator* (the coordinates of its north pole are: R.A. 190°, declination + 28°) and in its immediate neighbourhood are to be found the great star-clouds forming the Milky Way. The star-density—that is, the number of stars per unit volume of space—is greatest in the galactic equator and decreases rapidly towards the galactic poles. The galactic equatorial plane is thus a plane of maximum stellar concentration and a plane of maximum extension. The sun's position in the galaxy is not known with any great accuracy; however, it is believed to be a little on the north side of the galactic equator and distant about ten thousand parsecs from the centre of the system. The excentric position of the sun may be qualitatively inferred from the observed positions of the globular clusters in the sky, if it be assumed that these have a more or less symmetrical distribution with respect to the galaxy as a whole; nearly all the globular clusters are to be found in one hemisphere of the sky, with almost equal numbers on either side of the galactic equator. The direction of the centre of the system of globular clusters is in the constellation of Sagittarius* and it is significant that here the Milky Way clouds are densest, from which the inference is drawn that the centre of the galactic system is also in the direction of

* The position given by Shapley, *Star Clusters*, 22, 1930, is: R.A. $17^h\ 28^m$, dec. $-29°$, very close to the point common to Sagittarius, Ophiuchus and Scorpio.

Sagittarius. Investigations based on counts of stars also fortify this conclusion, but perhaps the most striking confirmation is derived from considerations based on the idea that the galactic system is in a state of rotation. In several instances the rotation of extra-galactic nebulae has actually been observed spectroscopically and it is not unreasonable to assume that our own galactic system has similar dynamical characteristics. The hypothesis has been put to the test within the last decade and various investigations have confirmed the rotation about a distant centre whose direction is almost exactly that foreshadowed by the distribution of globular clusters. The use of the expression "galactic rotation" is somewhat misleading—the underlying idea is that of the orbital motion of the individual stars about a distant centre of attraction analogous to the orbital revolution of the planets around the sun. As we shall see later, it is also possible from the appropriate observational material to make an estimate of the sun's distance from the galactic centre and of the gravitational mass responsible for the orbital motions of the stars.

There is some evidence that the stars in the neighbourhood of the sun form a loose cluster—known as the *local cluster*—with characteristics of distribution somewhat different from those of the galactic system as a whole. For example, the bright stars of spectral class B have a plane of concentration inclined at about 12° to the galactic equator; on the other hand, the faint B-type stars, which are presumably at much greater distances, are situated symmetrically with respect to the Milky Way and thus conform to the general galactic distribution.

The greater part of the succeeding pages will be devoted to studying the motions and spatial distribution of the galactic stars; we shall omit detailed consideration of the recessional velocities of the spiral nebulae.

1·21. *Spectral types of the stars.*

From considerations of the characteristics of their spectra, the stars are arranged by astrophysicists in the following main spectral classes:

O, B, A, F, G, K, M,

with the further decimal subdivision of each class, e.g. A0, A1, ..., A9. The sequence is a continuous one; in particular it is a sequence of decreasing effective temperature. In this work we shall not be concerned with the physical foundations of spectral classification; it is sufficient for our purpose to accept this system of classification so that stars of very similar physical characteristics may be considered together, as a group, in relation to particular problems of stellar distribution and of stellar motions in the galactic system. We add for reference* Table 1, in which are given the colours and the approximate effective temperatures of stars of different spectral types.

Table 1. *Colours and effective temperatures according to spectral class*

Spectral class	Colour	Effective temperature	
		Giants	Main series
O	Blue		> 30,000°
B0	Blue		23,000
A0	Bluish-white		11,200
F0	White		7,400
G0	Yellow	5500°	6,000
K0	Orange	4100	5,100
M0	Red	3050	3,400

1·22. *Stellar magnitudes.*

Hipparchus, more than two thousand years ago, was the first to classify the stars, visible to the naked eye, according to apparent brightness. The twenty brightest stars were designated stars of the first magnitude, and stars just visible to the naked eye of the sixth magnitude, stars of intermediate brightness being assigned to intermediate magnitude classes. The accurate classification, according to brightness, of the myriads of stars visible even in a telescope of moderate aperture evidently requires to be based on precise principles, and magnitude has now come to mean a number, on a certain scale, associated with the brightness of a star. If m_1 and m_2 denote the magnitudes of two stars on this scale and l_1 and l_2 their apparent brightness or luminosity, the difference of magnitude $m_2 - m_1$ is defined by the formula

$$\frac{l_1}{l_2} = 10^{-0\cdot4(m_1-m_2)}. \qquad \ldots\ldots(1)$$

A difference of five magnitudes thus corresponds to a ratio of 100:1 in brightness and a difference of one magnitude to a ratio of 2·512:1. The zero of the magnitude scale is chosen arbitrarily. On the visual magnitude scale adopted in practice the magnitude 1·0 corresponds very closely to the mean brightness of the two nearly equally bright stars Altair and Aldebaran. The various magnitude systems will be briefly noticed.

(*a*) *Visual magnitudes.*

These are determined from observations made by the eye directly. The instrument used, called a photometer, is generally one of two types. In one type the brightness of a particular star is compared with that of a

* Russell, Dugan and Stewart, *Astronomy*, 734, 1927. More recent and detailed information is given by G. P. Kuiper, *Ap. J.* **86**, 180, 1937.

standard star such as Polaris, while in the other type an artificial source of light is the standard by which comparisons are effected.

(*b*) *Photographic magnitudes.*

If a photograph is taken of a field of stars, the brightest stars will, as a rule, give rise to the largest images or, if the plate is placed outside the focus of the objective, to the images of greatest density. Thus measures of the diameters of the images or, in the second case, of the density or degree of blackening of the extra-focal images, furnish a method of comparison of brightness so far as actinic effect is concerned. As the ordinary photographic plate is more sensitive to blue light than to yellow or red light, a blue star will form a larger image on the plate than a yellow star of the same visual magnitude and will consequently have the smaller magnitude on the photographic scale. The zero of the photographic magnitude scale is adjusted in such a way that, for a star of spectral type A0, the photographic magnitude is defined to be the same as the visual magnitude.

(*c*) *Photovisual magnitudes.*

These are essentially equivalent to visual magnitudes but are determined photographically by allowing only the light to which the eye is most sensitive to fall on a special kind of photographic plate which in this instance is sensitive to the same range of radiations as the eye. This is achieved by placing a yellow filter in front of the photographic plate.

(*d*) *Photo-electric magnitudes.*

These are measured by means of a photometric apparatus embodying a photo-electric cell. The relation between photo-electric magnitudes and, say, photographic magnitudes is dependent on the particular kind of cell in use. Very great accuracy can be attained by a photometer of this type and the instrument is employed mainly in the detection and measurement of the light changes in variable stars.

(*e*) *Bolometric magnitudes.*

Visual, photographic and photo-electric magnitudes are concerned with different sections of a star's total radiation; in the case of visual magnitudes, for example, it is the section of the spectrum to which the eye is sensitive. Magnitudes based on the total radiation of the stars are called bolometric magnitudes. So far, very few direct determinations have been made, but it is possible to calculate bolometric magnitudes fairly accurately from the visual magnitudes when the effective temperatures are known.

The difference between the photographic magnitude m_p and the visual or photovisual magnitude m_v of a star, in the sense $m_p - m_v$, is called the *colour index*. The values* of the colour index (C.I.) for different spectral classes are shown in Table 2.

Table 2. *Colour indices*

Spectral class	C.I.	Spectral class	C.I.	
			Giants	Dwarfs
B0	$-0^m{\cdot}33$	G0	$+0^m{\cdot}67$	$+0^m{\cdot}57$
B5	$-0{\cdot}18$	G5	$+0{\cdot}92$	$+0{\cdot}65$
A0	$0{\cdot}00$	K0	$+1{\cdot}12$	$+0{\cdot}78$
A5	$+0{\cdot}20$	K5	$+1{\cdot}57$	$+0{\cdot}98$
F0	$+0{\cdot}33$	M0	$+1{\cdot}73$	$+1{\cdot}45$
F5	$+0{\cdot}47$			

From type G0 onwards in the table the stars are divided into "giants" and "dwarfs", that is, stars of high intrinsic luminosity and stars of low intrinsic luminosity. In investigations involving large numbers of faint stars for which a rough separation into spectral classes is regarded as important, the colour indices can be readily determined photographically and the corresponding spectral classes inferred within fairly narrow limits.

1·23. *Stellar parallaxes.*

To Bessel, in 1838, belongs the distinction of the first positive determination of the distance of a star (61 Cygni) and within a few months Henderson and Struve announced successful parallax measurements of α Centauri and Vega respectively. As the earth moves in its orbit around the sun, the direction of a near star, as viewed against the background of the very faint and, presumably, very distant stars, alters by a minute amount which depends, amongst other things, on the distance d of the star and on the radius a of the earth's orbit (which in this connection may be assumed circular). The angle of parallax, p, is defined by $\sin p = a/d$ or, expressing p in seconds of arc,

$$p = \frac{a}{d \sin 1''}, \qquad \text{......(1)}$$

since the angle of parallax is at most an extremely minute quantity—for the nearest star it is $0''{\cdot}76$. The angular displacement of the parallax star, due to a change in the earth's position in its orbit, is a function of p; hence if the displacement can be measured, the value of p can be obtained. A parallax determined according to these principles is called a *trigonometrical parallax*. At the present time, such parallaxes are obtained photographically; generally, at least a score of plates with two or three exposures per plate are necessary for a reasonably accurate determination of the

* Russell, Dugan and Stewart, *Astronomy*, 734, 1927. See also a paper by Miss Payne in *Harvard Annals*, 89, No. 6, 1935.

parallax of a single star and even then the probable error of the result is usually estimated to be about $\pm 0''{\cdot}010$. The normal practice is to measure the parallax displacement with reference to stars of the tenth to the twelfth magnitude. The resulting parallax is, accordingly, a relative parallax, for the faint comparison stars are also displaced by amounts depending on their parallaxes which are generally of the order of $0''{\cdot}003$. The determination of the absolute parallax of the star thus involves the determination of the parallaxes of faint stars which are too distant for the application of the trigonometrical method. The necessary information relating to the faint stars is obtained by statistical methods depending on principles to be considered in detail later.

The unit of stellar distance in general use is the *parsec* which is defined to be the distance corresponding to an angle of parallax equal to $1''$. Thus, from (1),

$$1 \text{ parsec} = a \operatorname{cosec} 1'' = 206{,}265 \text{ astronomical units}$$

or, since

$$1 \text{ astronomical unit} = 149{\cdot}5 \times 10^6 \text{ km.},$$

$$1 \text{ parsec} = 30{\cdot}84 \times 10^{12} \text{ km. or } 19{\cdot}16 \times 10^{12} \text{ miles.}$$

The distance of a star of parallax p is evidently $1/p$ parsecs, where p is expressed in seconds of arc.

The *light-year* is another unit of distance generally encountered in popular writings; it is the distance traversed by light in the course of a year. As the velocity of light *in vacuo* is 299,800 km./sec.,

$$1 \text{ light-year} = 9{\cdot}46 \times 10^{12} \text{ km. or } 5{\cdot}88 \times 10^{12} \text{ miles.}$$

We have also the relation, easily derived from the previous data,

$$1 \text{ parsec} = 3{\cdot}26 \text{ light-years.}$$

1·24. *Absolute magnitudes.*

Assuming that there is no absorption of light in interstellar space, the apparent brightness of a star as viewed in the sky depends on the intrinsic luminosity of the star and on its distance from us. If all the stars were at the same distance, a knowledge of their apparent magnitudes would, by formula (1) of section 1·22, enable us to compare their relative luminosities. When the distances of several stars are known, we can calculate the magnitudes they would be observed to have if they were situated at the same distance from the sun. The apparent brightness, l, of a star at a distance d varies as $1/d^2$. Hence, if L denotes its brightness if it were at a distance D,

$$\frac{L}{l} = \frac{d^2}{D^2} \qquad \text{......(1)}$$

or, if p and P are the parallaxes corresponding to d and D,

$$\frac{L}{l} = \frac{P^2}{p^2}. \qquad \text{......(2)}$$

Denote by m and M the apparent magnitudes, say on the visual scale, corresponding to the brightnesses l and L. Then we have

$$\frac{P^2}{p^2} = 10^{0\cdot4(m-M)}$$

or, on taking logarithms,*

$$M - m = 5\,\mathrm{Log}\,p - 5\,\mathrm{Log}\,P.$$

For the comparison of the intrinsic luminosities of the stars, the value of P adopted in practice is $0''\cdot1$ (corresponding to a distance of 10 parsecs) and the magnitude M is then called the *absolute magnitude*. It is given, by the preceding formula, in terms of the apparent magnitude m and the parallax p by

$$M = m + 5 + 5\,\mathrm{Log}\,p. \qquad \text{......(3)}$$

It is sometimes found convenient to express the intrinsic luminosity of a star in terms of the sun's luminosity as the unit. It has been calculated that on the visual scale the sun's apparent magnitude† is $-26^{\mathrm{m}}\cdot72$; this of course corresponds to its geocentric distance of 1 astronomical unit or sin 1″ parsecs. Putting $p = 206{,}265$ in (3), we find that the sun's absolute magnitude is $+4^{\mathrm{m}}\cdot85$. If L is now taken to mean the luminosity of a star of absolute magnitude M, in terms of the sun's luminosity as the unit, we have

$$\mathrm{Log}\,L = 0\cdot4(4\cdot85 - M), \qquad \text{......(4)}$$

from which the luminosity of the star can be easily calculated provided its absolute magnitude is known.

It is found that the stars vary greatly in absolute magnitude; at one end of the scale are stars of absolute magnitude -5 and at the other end are stars of absolute magnitude $+15$, the corresponding luminosities being of the order of 10^4 and 10^{-4} respectively times the sun's luminosity. The very luminous stars are the giants and the feebly luminous stars are the dwarfs. There is no precise line of demarcation between giants and dwarfs, but as a rough working rule it may be assumed that stars with absolute magnitudes algebraically less than $+2$ are giants and that stars with absolute magnitudes greater than $+2$ are dwarfs. The sun is, accordingly, a dwarf star, its spectral type being G0.

Within recent years, various small differences in the spectra of giant and dwarf stars of the same spectral types have been detected, notably the differences in the relative intensities of several absorption lines. For stars

* The logarithm of a quantity x to base 10 will be denoted by Log x and to base e by log x.

† H. Spencer Jones, *General Astronomy*, 305, 1934.

of known distance, these differences exhibit a definite correlation with absolute magnitude. Thus, for a star sufficiently bright for spectroscopic observation, the absolute magnitude may be inferred and, further, its parallax can then be calculated by means of (3). Parallaxes determined in this way are called *spectroscopic parallaxes*.

1·31. *Proper motions.*

In 1718 it was noticed by Halley that the positions of three bright stars —Sirius, Arcturus and Aldebaran—were appreciably different from the positions as recorded in the catalogue of Hipparchus, compiled more than eighteen centuries before, due allowance being made for the effects of precession on the coordinates of the stars during the interval. This could only be interpreted as due to the individual motion of the three stars at right angles to the line of sight against the otherwise apparently unchanging background of the stars. Since Halley's time precise observations have shown that every star examined is in motion and the conception of a fixed stellar background must, in theory at least, be abandoned. However, we can retain the idea of a fixed stellar background in practice if we have in mind the stars, say, of the twentieth magnitude which are, in the main, so distant that linear cross-velocities of the ordinary stellar size would be insufficient to change the directions of the stars by an observable amount even after the lapse of a century. The rate of change of direction which we are considering here is called *proper motion* and is usually measured in seconds of arc per annum.

Proper motions of the bright stars are derived from meridian-circle observations covering an interval of at least 50 years in general. If α_1, δ_1 are the coordinates of a star observed at time t_1 and referred to the mean equator and equinox for the beginning of the year of observation, and if α_2, δ_2 are the coordinates at time t_2 and referred to the same mean equator and equinox as in the first observation, the coordinates of the star have altered in (t_2-t_1) years at the yearly rates of $(\alpha_2-\alpha_1)/(t_2-t_1)$ and $(\delta_2-\delta_1)/(t_2-t_1)$ in right ascension and declination respectively. These quantities are the proper motions of the star in right ascension and declination respectively and they are generally denoted by μ_α and μ_δ respectively. It is to be understood that all the corrections customary in meridian-circle work have been applied. In particular, the coordinates α_2 and δ_2 are obtained after the removal of the effects of precession for the appropriate number of years; consequently, an error in the constants of precession is reflected in the deduced proper motions. The possibility that such an error exists is taken account of in certain statistical investigations involving proper motions (for example, see section 3·32).

The proper motions of the faint stars are determined most easily by photography and considerable progress in extending our knowledge of the motions of stars as faint as the fifteenth magnitude has been made in recent years in representative areas of the sky. In 1906 Kapteyn put forward his *Plan of Selected Areas* and, as one part of the Plan, several observatories have measured, or are measuring, the proper motions of stars in the areas accessible to their telescopes. The selected areas, 206 in number, are centred on the parallels of declination $0°$, $\pm 15°$, $\pm 30°$, $\pm 45°$, $\pm 60°$, $\pm 75°$ together with two polar areas. There are, in addition, 46 "special areas" to deal with representative or special features of the Milky Way. Parallax plates can also be used for the purpose of obtaining proper motions; in securing them the utmost precautions are taken as regards both mechanical and observational conditions, and the proper motions derived from the comparison of such plates with others taken after a suitable interval (usually 10 to 20 years) are generally of a high order of accuracy. The method now widely adopted is to photograph the parallax field at the second epoch through the glass; that is to say, the plate is placed in the telescope with its film-side away from the incident beam which, accordingly, has to pass through the plate glass before affecting the emulsion. This procedure enables the two plates—the parallax plate and the reversed plate—to be placed film to film so that corresponding images can be made to overlap approximately. Actually the plates are given a small relative displacement with the result that each star is represented by a pair of images resembling a double star. The displacement of one image from its companion is measured in directions parallel to the equator and to the meridian corresponding to the equatorial coordinates of the centre of the plate. A number of stars distributed over the region are used as comparison stars and the reduction of the measures for all the stars leads to proper motions relative to the mean motion of the comparison stars. If the proper motion of one star has been obtained from meridian-circle observations, the correction to be applied to the relative proper motions to convert them into absolute proper motions (that is, according to the system of meridian proper motions) is at once obtained. Actually, owing to the errors inherent in the observations of both meridian and relative proper motions, it would be necessary in practice to have several stars with well-determined meridian proper motions in order to provide the necessary accuracy for this correction. It is the exception rather than the rule to have a sufficient number of such stars in any photographic region and accordingly an indirect method of ascertaining the correction has to be employed; this is based on the parallactic motions of stars of definite magnitude groups, for which mean parallaxes are known. The method will be further considered in Chapter VI.

If S and S' are the positions of a star on the celestial sphere at the beginning and end of a year and referred to the same mean equator and equinox, the arc SS' is the *annual total proper motion*, denoted by μ. If P is the north pole of the equator concerned, the angle PSS' is the *position angle** of S' with respect to S. Denoting it by θ, we have the formulae

$$\mu_\alpha \cos\delta = \mu \sin\theta; \quad \mu_\delta = \mu\cos\theta. \qquad \text{......(1)}$$

In practical applications we shall always assume that μ, μ_α and μ_δ are expressed in seconds of arc.

The linear velocity corresponding to the total proper motion μ can be found if the star's parallax is known. This linear velocity will be referred to as the cross-velocity or the tangential or transverse velocity. If d is the distance of the star in kilometres, and T is the transverse velocity expressed in kilometres per second, we have

$$\mu \sin 1'' = \frac{Tn}{d},$$

where n is the number of seconds in a year; hence, by (1) of section 1·23,

$$T = \frac{a}{n}\cdot\frac{\mu}{p}.$$

Inserting the values $149{\cdot}5 \times 10^6$ and $31{\cdot}56 \times 10^6$ for a and n respectively, we obtain

$$T = 4{\cdot}74\,\frac{\mu}{p}$$

or, writing $\kappa \equiv 4{\cdot}74$,

$$T = \frac{\kappa\mu}{p}. \qquad \text{......(2)}$$

1·32. *Radial velocities.*

The rate at which a star (or any other heavenly body) is approaching the earth or receding from it can be measured directly by the spectroscope. The velocity so obtained is the component, in the line of sight, of the star's spatial velocity; it is called the line-of-sight velocity or the *radial velocity*. The star's spectrum, obtained by means of a slit-spectroscope, is photographed and alongside the stellar spectrum a comparison spectrum produced by a terrestrial source of light, such as an iron arc, is also photographed. As the wave-lengths of the iron lines are known with high accuracy, the wave-length of any line in the stellar spectrum can be obtained. The difference between this wave-length and the normal laboratory wave-length of the element or compound concerned gives the displacement of the stellar line. According to the Doppler-Fizeau principle, a line of normal wave-length λ is displaced towards the red end of the spectrum—that is, in the direction of increasing wave-length—by $u\lambda/c$, where u is the velocity of

* The position angle at the point S is measured *eastwards* from the meridional arc SP.

recession of the star with reference to the observer and c is the velocity of light. If the star is approaching the observer, the displacement of the stellar line is towards the violet end of the spectrum and is given numerically by the formula just mentioned. The convention as to the *sign* of the radial velocity is that a velocity of recession is positive and a velocity of approach is negative.

The important point about such spectroscopic observations is that the radial velocity of a star is determined directly in kilometres per second.

The velocities so measured are relative to the observer. They are affected by two variable factors, one the component in the line of sight of the observer's linear velocity due to the earth's diurnal motion, and the other a similar component due to the earth's orbital motion around the sun. These components are easily calculated and when they are removed from the star's observed radial velocity, the radial velocity *relative to the sun* is obtained. It is in this latter sense that the term "radial velocity" will be employed in succeeding pages.

A few stars have radial velocities of two, three or four hundred km./sec., but for the great majority of the stars the radial velocities lie within the comparatively small range of -40 to $+40$ km./sec.

1·33. *The equatorial linear components of a stellar velocity.*

Fig. 1 represents the celestial sphere centred at the sun, O; OZ is parallel to the earth's axis and the great circle XYN, of which Z is the pole, is the equator. We take OX, OY and OZ as the equatorial system of rectangular coordinate axes, OX being directed towards the vernal equinox, and OY towards the point on the equator with right ascension 90°.

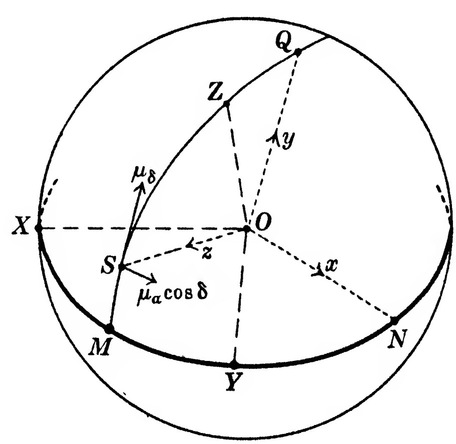

Fig. 1

Consider a star, whose equatorial coordinates are (α, δ), at S. The R.A. component μ_α of its proper motion gives rise to an arcual displacement at S of amount $\mu_\alpha \cos\delta$, parallel to the equatorial plane and perpendicular to the meridional plane through S. If x denotes the corresponding *linear* velocity, x is given—according to (2) of section 1·31—by

$$x = \kappa \frac{\mu_\alpha}{p} \cos\delta \qquad \ldots\ldots(1)$$

and the direction of this velocity is parallel to ON, where N is the point $(\alpha+90^\circ, 0)$.

The component μ_δ of the proper motion is parallel to OQ, where Q is the point, distant 90° from S, on the prolongation of the meridian SZ. The corresponding linear velocity y is given by

$$y = \kappa\frac{\mu_\delta}{p}. \qquad \ldots\ldots(2)$$

The radial velocity R—which we shall temporarily denote here by z for purposes of symmetry—is directed along OS.

The axes ON, OQ and OS clearly form a rectangular system with reference to which the components of the star's linear velocity are (x, y, z).

Let (u, v, w) denote the components of the star's linear velocity with reference to the equatorial system OX, OY, OZ. Resolving along ON, OQ and OS in order, we obtain (remembering that N is the pole of the great circle SZQ),

$$x = u\cos XN + v\cos YN,$$

$$y = u\cos XQ + v\cos YQ + w\cos ZQ,$$

$$z = u\cos XS + v\cos YS + w\cos ZS.$$

The coefficients of u, v and w in these formulae are easily expressed in terms of α and δ; for example,

$$\cos XQ = \cos XM \cos MQ = \cos\alpha\cos(90^\circ+\delta)$$
$$= -\cos\alpha\sin\delta.$$

Replacing x and y by the expressions in (1) and (2) and writing R for z, we have the formulae

$$-u\sin\alpha + v\cos\alpha = \kappa\frac{\mu_\alpha}{p}\cos\delta, \qquad \ldots\ldots(3)$$

$$-u\cos\alpha\sin\delta - v\sin\alpha\sin\delta + w\cos\delta = \kappa\frac{\mu_\delta}{p}, \qquad \ldots\ldots(4)$$

$$u\cos\alpha\cos\delta + v\sin\alpha\cos\delta + w\sin\delta = R. \qquad \ldots\ldots(5)$$

This system of equations will be found useful later.

The expression of each of the components u, v and w in terms of μ_α, μ_δ and R can be obtained either by solving the equations (3), (4) and (5) or directly as follows.

Resolving along OX, OY, OZ in order we obtain

$$u = x\cos XN + y\cos XQ + z\cos XS,$$

$$v = x\cos YN + y\cos YQ + z\cos YS,$$

$$w = x\cos ZN + y\cos ZQ + z\cos ZS,$$

from which the desired formulae are easily found to be

$$u = -\kappa \frac{\mu_\alpha}{p} \sin\alpha \cos\delta - \kappa \frac{\mu_\delta}{p} \cos\alpha \sin\delta + R\cos\alpha\cos\delta, \qquad \ldots\ldots(6)$$

$$v = \kappa \frac{\mu_\alpha}{p} \cos\alpha \cos\delta - \kappa \frac{\mu_\delta}{p} \sin\alpha \sin\delta + R\sin\alpha\cos\delta, \qquad \ldots\ldots(7)$$

$$w = \kappa \frac{\mu_\delta}{p} \cos\delta + R\sin\delta. \qquad \ldots\ldots(8)$$

1·41. *Galactic coordinates.*

In many problems, it is a matter of great importance to investigate the positions and motions of the stars with reference to the galactic equator—the median plane of the galactic system. The position of the galactic equator or, more particularly, the position of the galactic pole with respect to the usual equatorial system of coordinates, can be estimated in various ways: (*a*) the Milky Way clouds provide one source of information; (*b*) from counts of stars it is possible to define a plane of maximum stellar distribution, regions of obscuration by dark nebulae being of course avoided in the compilation of the statistics; (*c*) several classes of celestial objects, for example, Cepheid variables and stars spectroscopically designated with the *c*-characteristic show a strong concentration towards the Milky Way and the plane of symmetry can be estimated. The several methods agree in placing the north galactic pole* near R.A. 190°, declination +28° (mean equator and equinox for 1900·0).

The conversion of equatorial coordinates into galactic coordinates is easily effected. In Fig. 2, P is the pole of the mean equator for 1900·0, $W(\alpha_0, \delta_0)$ is the pole of the galactic equator UTV, S is the position of a star (α, δ) and ♈ is the vernal equinox. The position of S is specified with reference to the galactic equator by means of its longitude G (the arc UT—measured from the ascending node U in the direction $\overrightarrow{UT}$) and its latitude g (the arc TS). Galactic latitudes are reckoned positive if the objects concerned are in the hemisphere containing the north celestial pole. From the figure the

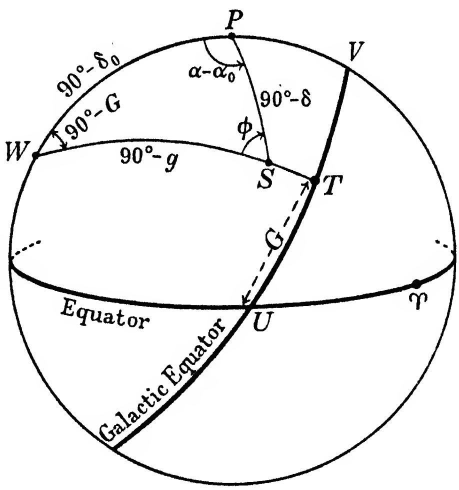

Fig. 2

* A. Kohlschütter adopts +27° as the declination of the galactic pole in calculations of certain galactic quantities relating to stars of Boss's *Preliminary General Catalogue* (*P.G.C.*): v. *Veröff. der Univ. Sternwarte zu Bonn*, No. 22, 1930.

following formulae are easily found by means of which the galactic longitude and latitude of any star can be calculated, the values of α_0 and δ_0 being supposed known:

$$\sin(\alpha-\alpha_0)\tan G = \cos\delta_0\tan\delta - \sin\delta_0\cos(\alpha-\alpha_0), \qquad \ldots\ldots(1)$$

$$\sin g = \sin\delta_0\sin\delta + \cos\delta_0\cos\delta\cos(\alpha-\alpha_0). \qquad \ldots\ldots(2)$$

Some astronomers measure galactic longitudes, not from the ascending node U, but from other points on the galactic equator suggested by particular lines of investigation. The direction of the centre of the galactic system, for example, would give a suitable point of departure for measuring longitudes, but as this is not known with sufficient accuracy it would be unwise—at any rate, at present—to adopt this system in the construction of tables.

It is to be noted that, in our definition, U is the ascending node of the galactic equator on the celestial mean equator for a specified epoch, namely 1900·0; accordingly, the mean coordinates of the stars for 1900·0 must be used. Extensive tables giving the galactic longitudes and latitudes have been compiled at the Lund Observatory* at intervals of one degree in right ascension and in declination.

In Fig. 2, the angle WSP, denoted by ϕ, is called the *galactic parallactic angle*. It is conventionally measured from the galactic meridian SW to the meridian SP, in the direction of the arrow, from 0° to 360°. The *spherical angle* WSP of the spherical triangle, as shown in Fig. 2, can be calculated from the formula

$$\sin(\alpha-\alpha_0)\cot WSP = \cos\delta\tan\delta_0 - \sin\delta\cos(\alpha-\alpha_0). \qquad \ldots\ldots(3)$$

In this instance (Fig. 2) there is no doubt as to the appropriate quadrant in which ϕ lies; in other cases, the rule to be observed is that ϕ lies between 0° and 180° when α lies between α_0 and $\alpha_0+180°$ (that is, between 190° and 10°), and ϕ lies between 180° and 360° when α lies between 10° and 190°. The Lund tables (*loc. cit.*) contain also the values of ϕ.

We can now obtain the components μ_G and μ_g of proper motion in galactic coordinates in terms of μ_α and μ_δ. From Fig. 3 it is easily seen that

$$\mu_G\cos g = \mu_\alpha\cos\delta\cos\phi + \mu_\delta\sin\phi \qquad \ldots\ldots(4)$$

and

$$\mu_g = -\mu_\alpha\cos\delta\sin\phi + \mu_\delta\cos\phi. \qquad \ldots\ldots(5)$$

1·42. *The galactic linear components of a stellar velocity.*

In Fig. 4, let P and W be respectively the poles of the celestial equator and of the galactic equator. As in Fig. 1, for a star at S the linear components of velocity x, y and z—where x and y are given by (1) and (2) of

* *Lund Annals*, No. 3, 1932. Other tables are to be found in *Harvard Annals*, **56**, 2–7, 1912 and *Pubbl. Specola Vaticana*, No. 14, 1929—the latter calculated by P. Emanuelli. For rough purposes, O. R. Walkey's table (*M.N.* **74**, 201, 1914) may be found useful.

section 1·33, and $z \equiv R$, the radial velocity—are parallel to the radii ON, OQ and OS, N being the pole of the meridian PS and the arc PQ being equal to the star's declination δ.

We require to find the components ξ, η and ζ of the star's velocity, parallel to the galactic rectangular axes OU, OV and OW, the galactic longitude of V being 90°.

Let l_3, m_3, n_3 be the direction-cosines of OS with respect to OU, OV and OW. Then

$$l_3 = \cos SU, \quad m_3 = \cos SV, \quad n_3 = \cos SW. \qquad \ldots\ldots(1)$$

Consider any point Z with equatorial coordinates (A, D). Then

$$\cos SZ = \sin\delta \sin D + \cos\delta \cos D \cos(A - \alpha). \qquad \ldots\ldots(2)$$

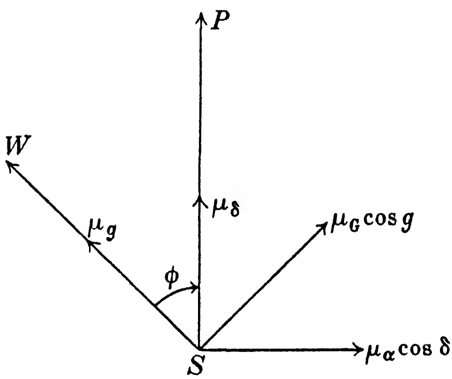

Fig. 3

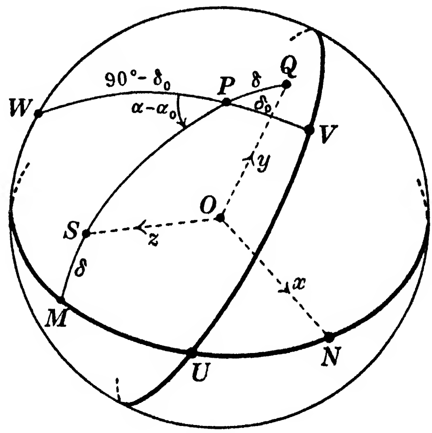

Fig. 4

Now the equatorial coordinates of U, V and W are as follows:

$$U, (\alpha_0 + 90°, 0); \quad V, (\alpha_0 + 180°, 90° - \delta_0); \quad W, (\alpha_0, \delta_0).$$

Hence, from (2),

$$\left.\begin{aligned} l_3 &\equiv \cos SU = \cos\delta \sin(\alpha - \alpha_0) \\ m_3 &\equiv \cos SV = \sin\delta \cos\delta_0 - \cos\delta \sin\delta_0 \cos(\alpha - \alpha_0) \\ n_3 &\equiv \cos SW = \sin\delta \sin\delta_0 + \cos\delta \cos\delta_0 \cos(\alpha - \alpha_0) \end{aligned}\right\}. \qquad \ldots\ldots(3)$$

Let l_1, m_1, n_1 be the direction-cosines of ON with respect to OU, OV, OW. Then

$$l_1 = \cos NU, \quad m_1 = \cos NV, \quad n_1 = \cos NW.$$

Now the equatorial coordinates of N are $(\alpha + 90°, 0)$; hence, by putting $(\alpha + 90°)$ for α and 0 for δ in (3), we can write down the values of l_1, m_1 and n_1. The results are

$$\left.\begin{aligned} l_1 &\equiv \cos NU = \cos(\alpha - \alpha_0) \\ m_1 &\equiv \cos NV = \sin\delta_0 \sin(\alpha - \alpha_0) \\ n_1 &\equiv \cos NW = -\cos\delta_0 \sin(\alpha - \alpha_0) \end{aligned}\right\}. \qquad \ldots\ldots(4)$$

Again, let l_2, m_2, n_2 denote the direction-cosines of OQ with respect to OU, OV, OW. Then

$$l_2 = \cos QU, \quad m_2 = \cos QV, \quad n_2 = \cos QW.$$

The equatorial coordinates of Q are $(180°+\alpha, 90°-\delta)$; hence, putting $(180°+\alpha)$ for α and $(90°-\delta)$ for δ in (3), we obtain

$$\left.\begin{aligned} l_2 &\equiv \cos QU = -\sin\delta \sin(\alpha-\alpha_0) \\ m_2 &\equiv \cos QV = \cos\delta\cos\delta_0 + \sin\delta\sin\delta_0\cos(\alpha-\alpha_0) \\ n_2 &\equiv \cos QW = \cos\delta\sin\delta_0 - \sin\delta\cos\delta_0\cos(\alpha-\alpha_0) \end{aligned}\right\}. \quad \text{......(5)}$$

We can now express ξ, η and ζ in terms of x, y and z; the formulae are

$$\left.\begin{aligned} \xi &= l_1 x + l_2 y + l_3 z \\ \eta &= m_1 x + m_2 y + m_3 z \\ \zeta &= n_1 x + n_2 y + n_3 z \end{aligned}\right\}, \quad \text{......(6)}$$

in which $$x = \kappa\frac{\mu_\alpha}{p}\cos\delta, \quad y = \kappa\frac{\mu_\delta}{p}, \quad z = R$$

and the values of $l_1, l_2, \ldots n_3$ are given by (3), (4) and (5). The values of $l_1, \ldots n_3$ have been computed by Kohlschütter* for stars in Boss's *P.G.C.* for which the complete data required in (6) are known.

When tables for the conversion of equatorial coordinates into galactic coordinates are available—such as the Lund tables—the calculation of the components ξ, η, ζ can be effected more easily than by the method summarised in the formulae (6). It is advisable first to compute the values of μ_G and μ_g by means of the formulae (4) and (5) in section 1·41. We have then a system in galactic coordinates analogous in every way to the system in equatorial coordinates treated in section 1·33; the components ξ, η, ζ are accordingly given by the analogues of (6), (7) and (8) of that section. The formulae are:

$$\xi = -\kappa\frac{\mu_G}{p}\sin G\cos g - \kappa\frac{\mu_g}{p}\cos G\sin g + R\cos G\cos g, \quad \text{......(7)}$$

$$\eta = \kappa\frac{\mu_G}{p}\cos G\cos g - \kappa\frac{\mu_g}{p}\sin G\sin g + R\sin G\cos g, \quad \text{......(8)}$$

$$\zeta = \kappa\frac{\mu_g}{p}\cos g + R\sin g. \quad \text{......(9)}$$

It may be verified that (7), (8) and (9) are the same as the three formulae of (6). For example, in the first formula of (6) the coefficient of x is l_1 or $\cos(\alpha-\alpha_0)$. Now apply the polar analogue of the fundamental formula to the triangle PWS (Fig. 2); we have

$$\cos(\alpha-\alpha_0) = -\sin G\cos\phi + \cos G\sin\phi\sin g$$

and the right-hand side of this formula is the coefficient of x in (7) when the formulae for μ_G and μ_g are substituted.

* *Veröff. Bonn*, No. 22, 1930.

1·5. *The solar motion.*

The discovery of stellar motions inevitably led to the inference that the sun—itself a star—is also in motion. But at this stage we require to distinguish between the observed motion of a star and the motion of the sun. As regards the star, its movement is inferred from its proper motion and radial velocity and these are obtained relative to the sun; actually, the measures of proper motion are essentially geocentric but, owing to the star's great distance compared with the sun's distance from the earth, the observed proper motion may be regarded as identical with the proper motion relative to the sun; also, as we have previously noted, we mean by the term "radial velocity" the line-of-sight motion relative to the sun. Thus the sun is the point of reference for the specification of the observed components of stellar velocities. As regards the solar motion, however, there is no such simple point of reference and the best we can do, having regard to the comparatively small section of the stellar system which can be adequately surveyed by our instruments, is to define the motion of the sun with respect to the group of stars in the immediate neighbourhood of the sun, say within a sphere of radius one thousand parsecs. The centre of mass of this group of stars suggests itself as the most suitable point of reference for the solar motion; but, as stellar masses are known only in a comparatively few instances, this procedure is at present impracticable. As, however, a star's motion is sensibly rectilinear over very long intervals of time, it is sufficient to take as the theoretical reference point the centre of position of the group of stars. It is in this sense that the solar motion is defined. A detailed discussion will be deferred to Chapter III but, meanwhile, it may be stated that with reference to the naked-eye stars the study of proper motions and of radial velocities separately places the direction in which the sun is moving near the point of the celestial sphere at R.A. 270° and declination +30°. From the radial velocities the solar speed is found to be approximately 19½ km./sec. The point of the heavens towards which the solar motion is directed is called the *solar apex*; the antipodal point is the solar *antapex*.

We can now divide the observed velocity of a star (that is, relative to the sun) into two parts, one part relative to the point of reference connected with the group of stars concerned, the other part depending on the solar motion. The former is the star's *motus peculiaris*; the latter is called the *parallactic motion*. It was for long assumed or taken for granted that the peculiar motions of the stars were entirely haphazard in character. However, in 1904 Kapteyn's discovery of star-streaming introduced a new feature into the laws governing the distribution of stellar velocities, namely, the recognition that there is a certain direction, associated with the galactic system, parallel to which the stars show an unmistakable preference for moving. If the galactic system is in, or near, a steady state, the argument of symmetry—and according to observation the galactic system may be described as symmetrical as regards its large-scale features—leads to the expectation that the direction of preferential motion should be parallel to the galactic equator. Most investigations dealing with the preferential motions of the stars confirm this point satisfactorily.

1·6. *Stellar masses.*

The masses of the stars can be determined directly only from observations of binary systems. A binary is a double star, each component of which revolves around the common centre of mass in an elliptic orbit under the force of gravitational attraction. Sir William Herschel was the first to demonstrate the existence of such systems in the heavens. When each component is visible in the telescope, the system is known as a *visual binary*.

If m_1 and m_2 are the masses of the components of a visual binary, T the orbital period and a (in linear measure) the semi-major axis of the orbit of one star relative to the other, Kepler's third law gives

$$\frac{4\pi^2 a^3}{T^2} = G(m_1+m_2), \qquad \text{......(1)}$$

where G is the constant of gravitation in terms of the units employed for T, a, m_1 and m_2. For the earth's orbit around the sun, we have similarly

$$\frac{4\pi^2 a_0^3}{T_0^2} = Gm_0, \qquad \text{......(2)}$$

in which m_0 denotes the sun's mass, the earth's mass being neglected in comparison with m_0. For units we take the solar mass m_0 to be unity, a_0 to be one astronomical unit of distance and T_0 to be one year. In terms of these units, we have from (2)

$$G = 4\pi^2, \qquad \text{......(3)}$$

and hence (1) becomes

$$\frac{a^3}{T^2} = m_1+m_2. \qquad \text{......(4)}$$

The observed orbit is the projection of the true orbit on the plane at right angles to the line of sight, and the apparent separation of the components is measured in seconds of arc. The study of the observed orbit leads, in particular, to the period T, the inclination of the plane of the true orbit to the plane perpendicular to the line of sight, and the angular measure α corresponding to the true semi-major axis a. If p is the parallax, $a = \alpha/p$ in astronomical units, α and p being expressed in seconds of arc. Hence from (4)

$$m_1+m_2 = \frac{\alpha^3}{p^3 T^2}. \qquad \text{......(5)}$$

If the parallax is known, the sum of the masses can be easily calculated by means of this formula. It is found that the masses of visual binary systems are approximately, on the average, twice the mass of the sun.

This last result leads to an indirect method of estimating the parallax of a binary when the orbit, or a portion of the orbit, has been sufficiently well observed so as to give satisfactory values of α and T. If we equate (m_1+m_2) to 2 in (5), we obtain p in terms of the quantities α and T derived from the observations. Parallaxes obtained in this way are usually called *dynamical parallaxes*. It is to be noted that if the true value of the total mass of the system is, say, twice the hypothetical value, the dynamical parallax is $2^{1/3}$, or approximately, $1\frac{1}{4}$ times the true parallax. Even in this probably exceptional instance, an error of 25 % in a parallax determination is not unsatisfactory and in the case of a parallax of $0''\cdot 01$ the accuracy is superior to that attainable by the direct trigonometrical method.

The individual masses m_1 and m_2 can only be determined if the orbit of one component about the common centre of mass of the system can be observed. In some instances, such as the binary system of Sirius, meridian-circle observations can be used to give the necessary information; in other instances photographic methods are employed.

In 1889, E. C. Pickering discovered the first of another class of binaries by means of the spectroscope—these are known as *spectroscopic binaries*. In such a system the components are very much closer together than in the case of visual binaries and almost invariably beyond the resolving powers of the ordinary telescope; moreover, the relative orbital motion, in linear measure, is also very much greater and is of a magnitude easily measurable by the spectroscope. If each component of a spectroscopic binary is bright enough to register its own characteristic lines in the spectrum, the line-of-sight component of the velocity of each star about the common centre of mass can be deduced. The study of the velocity curves enables the values of $m_1 \sin^3 i$ and $m_2 \sin^3 i$—and, consequently, of the ratio $m_1 : m_2$—to be derived, where i is the inclination of the orbital plane to the plane perpendicular to the line of sight. Unless the value of i can be determined, as in the case of a very few systems that are also close visual binaries, the information provided by the spectroscopic binaries can only be utilised statistically to yield average values of the individual masses.

The following table,* quoted by Spencer Jones, gives the relation of mass (or rather, the mass multiplied by $\sin^3 i$) and of mass-ratio to spectral type, with the number of spectroscopic binaries for which the requisite information was then available.

The main apparent difference, other than differences due to spectral characteristics, between visual and spectroscopic binaries lies in the relative dimensions of their orbits or, owing to the comparatively small range in stellar masses, in the orbital periods. The periods of visual binaries range

* H. Spencer Jones, *General Astronomy*, 335, 1934.

Table 3. *Masses of spectroscopic binaries*

Spectral class	$m_1 \sin^3 i$	$m_2 \sin^3 i$	$m_1 : m_2$	Number
O6–B4	13·18	10·50	1·25	21
B5–B9	5·05	3·40	1·49	9
A0–A4	1·71	1·01	1·69	21
A5–F4	1·80	1·24	1·45	18
F5–G4	1·01	0·89	1·13	15
G5–K4	0·87	0·68	1·28	3

from a few years to several centuries; the periods of spectroscopic binaries range from a third of a day to several thousand days. This separation into two classes is due to the limitations of the respective observational methods and not to intrinsic differences.

A third class of binary from which valuable information can be derived is that of the *eclipsing variable* which is essentially of the same character as, and is also frequently observed as, a spectroscopic binary. If the line of sight is in or near the orbital plane, eclipses of one component by the other will clearly occur, thus leading to a diminution of the light reaching the observer. The study of the light-curve of a typical eclipsing binary yields the radii of the component stars as fractions of their linear separation, the inclination of the orbital plane to the line of sight, the relative brightness of the components and, with an assumption as to the ratio of the masses, the mean densities of the two stars. If the system can also be observed as a spectroscopic binary, the dimensions of the system and the masses of the stars can be deduced; both spectra must be observable if the assumption of the mass ratio is to be avoided. In many of the best-observed systems the problem of deriving the quantities just mentioned is complicated by several other factors—the ellipsoidal forms of the stars (if they are very near together), darkening at the limb (due to the diminution of light, proceeding to the observer, from the centre of the disc towards the limb) and the reflection effect (the brightening of one star by means of the incident radiation emitted by the other).

From all the information garnered mainly within the last few years, the masses of the stars are found to range from about one-sixth of the sun's mass to about twenty times the sun's mass. A few stars exceed the latter figure; the most massive star so far investigated is B.D. + 6° 1309, a spectroscopic binary with components at least 85 and 70 times respectively more massive than the sun.

An indirect method of finding the mass of a star is based on Eddington's *mass-luminosity relationship*.* From purely theoretical considerations, Eddington established a formula connecting the mass and absolute magnitude

* A. S. Eddington, *M.N.* **84**, 308, 1924.

(bolometric) of a giant star of which the material, owing to its extremely low density, is in the state of a "perfect gas". Unexpectedly, it was found that dwarf stars of well-determined masses, with densities comparable with that of water, all satisfied the relationship; the conclusion naturally followed that dwarf stars, despite their high density, are also in the condition of a perfect gas, the explanation being based on the extreme smallness of the stellar ions as compared with terrestrial atoms. The relationship—shown diagrammatically in Fig. 5—is thus applicable to giant and dwarf stars alike, the only exceptions being the "white dwarfs" in which the mean

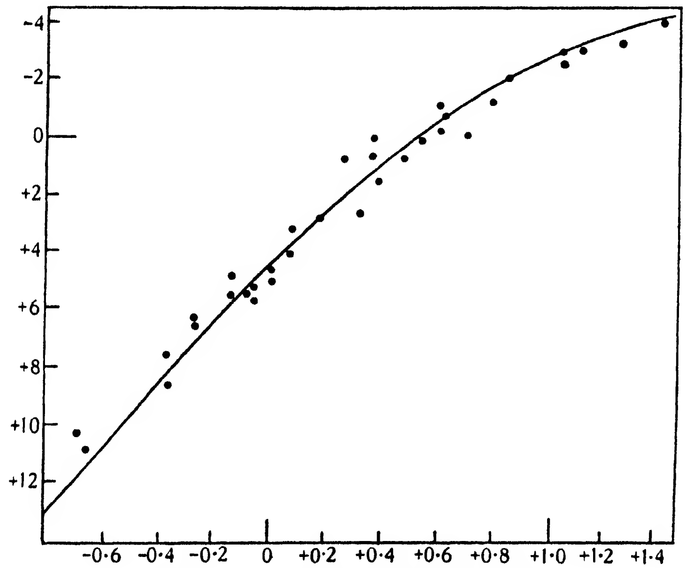

Fig. 5. *The Mass-Luminosity Relationship*

The abscissae are the logarithms of the mass; the ordinates are the absolute bolometric magnitudes. The full-line curve represents the theoretical relationship between mass and luminosity. Each dot is plotted from observational data obtained from individual stars.

density of the stellar material is of the order of 10^5 times the density of water. Thus the diagram can be used to estimate the mass of a star if its absolute bolometric magnitude is known.

1·7. *Stellar evolution and the time-scale.*

The recognition of the great diversity in the absolute luminosities of the stars dates from the beginning of the century; in particular, the great diversity in the absolute luminosities of stars of the *same* spectral type (G, K, M) was pointed out by E. Hertzsprung,* who coined the terms "giants" and "dwarfs" to express the distinction between stars of high

* *A.N.* **179**, 373, 1909; *Potsdam Publ.* **63**, 1911.

intrinsic brightness and low intrinsic brightness. In 1913, H. N. Russell* provided additional material to support this view; he had at his disposal the measured parallaxes of several hundreds of stars with their spectral classification, and from these he calculated the absolute magnitudes. The relation between spectral type and absolute magnitude is shown schematically in Fig. 6—known as the Hertzsprung-Russell diagram. In particular, it will be noticed that stars of type M fall into two sharply divided groups; in one group near X, the absolute magnitude is about -2 and in the second group near Z, the absolute magnitude is about $+12$. Thus stars of the first group are of the order 10^5 or 10^6 times more luminous than stars of the

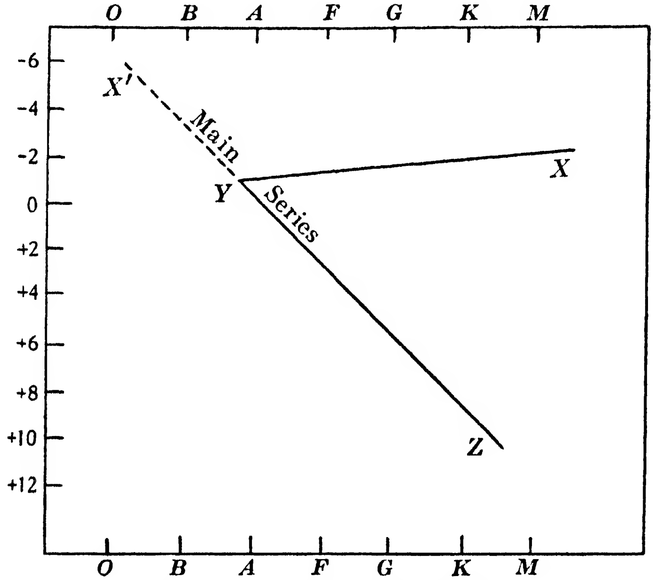

Fig. 6. *The Hertzsprung-Russell diagram*

The absolute magnitudes (ordinates) are plotted against spectral type

second group; these are respectively the giants and the dwarfs of spectral class M. As the spectral types are the same for the two groups, thus indicating similarity in the effective temperatures, the difference in luminosity is to be ascribed to the great diversity in surface area and consequently in the radii of the stars. Later determinations of stellar parallaxes have added weight to the relationship between intrinsic luminosity and spectral type as exhibited by the diagram. One of the most conclusive pieces of evidence, relating to the branch $X'YZ$ (*the main series*) in Fig. 6, has been provided by Hertzsprung† in his study of over 1000 stars in the Pleiades cluster. With proper motion as the criterion, the stars belonging to the cluster can

* *Observatory*, **36**, 324, 1913. † *M.N.* **89**, 660, 1929.

be separated from the non-cluster stars; as the parallax of the cluster is known, the absolute magnitudes of the cluster stars can be easily obtained from the apparent magnitudes. As most of the stars are too faint to permit of their spectral types being determined in the usual way, Hertzsprung measured their colour indices which are related to spectral types as shown in Table 2, p. 5. The result of the investigation was to show that the stars of the Pleiades reproduced the main series in the diagram.

An additional observational feature of the Hertzsprung-Russell diagram is that the sequence XYZ is one of increasing density; the stars at X are extremely diffuse and the stars at Z have densities several times the density of water. This suggested to Russell that the course of evolution of a star was in the sense $X \rightarrow Y \rightarrow Z$ in accordance with the theoretical investigations of Lane and Ritter. A star in a highly diffused state, such as a giant M star, contracts under gravitational attraction and its temperature rises. Accordingly, it is supposed to pass from X to Y where the density reaches such a value that the gain of heat energy by contraction is balanced by the loss due to radiation. Thereafter, that is, along YZ the star gradually cools as the density increases and finally reaches the state of a dwarf star of class M. The turning-point at Y was originally believed to mark the point where the star ceased to be in the condition of a perfect gas.

There was general acceptance of this theory until 1924 when Eddington established his mass-luminosity relationship illustrated in Fig. 5. In Russell's theory, stellar mass was not one of the physical factors directly involved although, even with the small amount of information available at the time, it was noticed that the stars of smallest mass were found near Z in Fig. 6. But, since mass is observed to be correlated with luminosity, the Hertzsprung-Russell diagram acquires a new feature, namely, the diminution of stellar masses in the direction $X \rightarrow Y \rightarrow Z$. If Russell's theory of stellar evolution is to be maintained, it follows that if a star starts its luminous career as a giant M star and passes through the sequence of changes indicated by XYZ, it must lose approximately 99 % of its mass in the process. (If the mass is finally one-fifth of the sun's mass when the star is a dwarf of type M, its mass originally as a giant M star may be postulated to be of order 20 times the solar mass—a not extravagant value.) According to the theory of relativity, mass and energy are interrelated entities so that, by the simple process of radiating light and heat, a star is automatically losing mass at a rate that can be calculated. For example, due to this cause, the sun's mass is decreasing at the rate of about four million tons per second. But an important physical question arises as to the mechanism whereby mass is eventually converted into radiant energy. It may be noted in passing that gravitational contraction can provide a certain amount of energy which is converted into radiation, but it is wholly inadequate to account for more than a small fraction of the energy radiated by the sun during the period of its existence as a luminous star. There is a consistent body of evidence giving 2.10^9 years as the age of the earth and presumably the sun must be at least as old. But gravitational contraction of a diffuse globe to the present dimensions of the sun can produce energy sufficient only for the comparatively short interval of twenty million years. Thus contraction must be ruled out as anything more than a minor contributory agent with respect to the supply of stellar energy. Two suggestions as to the source of stellar energy, or to the mechanism at work within a star, have been offered. In one it is supposed that matter is actually destroyed through the annihilation of protons and electrons, the energy of mass being converted into radiant energy; in the other, the source of energy is found in the synthesis of the atoms of the various chemical elements from hydrogen atoms.

If the first hypothesis is true, it is possible for a star to begin its evolutionary career as a giant of type M and by the process of self-annihilation to pass along the sequence of states represented in the Hertzsprung-Russell diagram. Moreover, it can be calculated that about 7.10^{12} years would be necessary for the sun to arrive at its present stage if it started as a massive star. When we pay regard to the stars with masses smaller than that of the sun, we can conclude that the time-scale is of the order of 10^{14} years. If, on the other hand, stellar energy is derived from the synthesis of the elements from hydrogen, it would appear that the loss of mass could amount only to rather less than 1 % and, accordingly, the Hertzsprung-Russell diagram, although representing facts, cannot represent the course of events in the life-history of a star.

The theory of stellar evolution is thus at a deadlock, and all that we can say definitely is that, according to present knowledge and present ideas, Russell's original theory can only be saved by the hypothesis of the annihilation of matter within the star. But this implies a time-scale of the order of 10^{13} or 10^{14} years which is now seriously challenged by the relativistic theory of the expansion of the universe as evidenced in the recessional motions of the extra-galactic nebulae; in this latter theory the time-scale is not greater than 10^{10} years. Various suggestions have been put forward to attempt a reconciliation between the two time-scales but, for the present, no observational evidence of a wholly conclusive character has been produced to settle what is, perhaps, the most baffling and the most important problem of astronomy to-day.*

* For a discussion of the arguments in favour of each of the two time-scales, see *Observatory*, **58**, 108, 1935.

1·81. *The correction of an observed frequency curve.*

Many important investigations depend on the compilation of statistics exhibiting the number of stars with an observed value of a particular characteristic x, which may be parallax, proper motion, magnitude, etc. The practical procedure is to determine the number, y, of stars with values of the characteristic lying between $x-\frac{1}{2}\Delta x$ and $x+\frac{1}{2}\Delta x$, where Δx denotes a small step in the characteristic x; in parallax statistics, for example, Δx may be taken to be $0''\cdot005$, and in magnitude statistics $0^{m}\cdot2$ or $0^{m}\cdot5$. Plotting y against x we obtain a series of points and it is generally possible to draw a smooth curve $y = v(x)$ to give a good representation of the observed frequency. The problem is to deduce the true frequency curve $y = u(x)$, when information as to the precision of the observations is available.

We have to recognise that each observation is liable to an error which we shall denote by ϵ, and, if these errors follow the Gaussian law, the proportion of errors falling between ϵ and $\epsilon+d\epsilon$ is given by $Ce^{-h^2\epsilon^2}d\epsilon$, in which h is known as the modulus of precision. The constant C is found from the consideration that for all errors in the range $-\infty<\epsilon<\infty$

$$C\int_{-\infty}^{\infty} e^{-h^2\epsilon^2}\,d\epsilon = 1,$$

so that

$$C = h/\sqrt{\pi}.$$

The probable error, r, is defined to be such that the proportion of errors for which $-r\leqslant\epsilon\leqslant r$ is 0·5; that is to say, the expectation of the error falling within this range is equal to the chance that it is outside the range. From the definition,

$$\frac{1}{2} = \frac{h}{\sqrt{\pi}}\int_{-r}^{r} e^{-h^2\epsilon^2}\,d\epsilon = \frac{2}{\sqrt{\pi}}\int_{0}^{hr} e^{-x^2}\,dx.$$

From numerical tables of the last integral, it is found that

$$hr = 0\cdot4769. \qquad \text{......(1)}$$

When the probable error, r, of a deduced result is known—such as the measure of the parallax of a star—h can be obtained from (1); for a group of parallax observations made at a particular observatory, the probable error varies very little from one determination to another and, accordingly, h may be regarded as a constant associated with this group of observations.

We have

$u(x)\,dx$ = *true* number of stars with the value of the characteristic between x and $x+dx$;

$v(x)\,dx$ = *observed* number of stars with the value of the characteristic between x and $x+dx$.

If ϵ is the error* of observation when the *observed* characteristic is x, the

* In the sense of being applied algebraically to the true value to give the observed value.

true value is $x-\epsilon$, and the proportion of observations with errors lying between ϵ and $\epsilon+d\epsilon$ is

$$\frac{h}{\sqrt{\pi}} e^{-h^2\epsilon^2} d\epsilon.$$

Let t denote the true value of the characteristic corresponding to the observed value x; then

$$t = x-\epsilon.$$

The number of stars with true values of the characteristic between t and $t+dt$ is $u(t)\,dt$ and, of these, a number

$$u(t) \cdot \frac{h}{\sqrt{\pi}} e^{-h^2\epsilon^2} dt\, d\epsilon$$

will be observed with errors lying between ϵ and $\epsilon+d\epsilon$.

Change the variables t, ϵ to x, ϵ. Then, since

$$dx\, d\epsilon \equiv \frac{\partial(t+\epsilon, \epsilon)}{\partial(t, \epsilon)} dt\, d\epsilon = dt\, d\epsilon,$$

the number of stars with the observed characteristic between x and $x+dx$, the errors lying between ϵ and $\epsilon+d\epsilon$, is

$$\frac{h}{\sqrt{\pi}} u(x-\epsilon)\, e^{-h^2\epsilon^2} dx\, d\epsilon.$$

The total number, $v(x)\,dx$, of stars with observed characteristics between x and $x+dx$ will be obtained by summing the previous expression for all values of ϵ between $-\infty$ and $+\infty$. Thus we obtain

$$v(x) = \frac{h}{\sqrt{\pi}} \int_{-\infty}^{\infty} u(x-\epsilon)\, e^{-h^2\epsilon^2} d\epsilon. \qquad \ldots\ldots(2)$$

By means of this formula the true frequency function $u(x)$ is to be found from the given observed frequency function $v(x)$.

In actual practice the function $v(x)$ is identified with a curve of which the ordinates at points x_i $(i=1, 2, \ldots n)$ are determined in the first instance from the observed numbers of stars with characteristics lying within small intervals $x_i-\alpha$ to $x_i+\alpha$. The curve is thus based on n points corresponding to n distinct values $x_1, x_2, \ldots x_n$ of x; and when the observations are smoothed we may still regard the smooth curve as being determined from n points. Thus $v(x)$ may be supposed to be a polynomial of degree $(n-1)$.

It is clear from (2) that, if $u(x)$ is a polynomial of degree $(n-1)$, $v(x)$ is also a polynomial of degree $(n-1)$; conversely, if $v(x)$ is regarded as a polynomial, as derived from the observations, the function $u(x)$ is a polynomial of the same degree.

Expanding $u(x-\epsilon)$ by Taylor's theorem and denoting $\frac{d^n}{dx^n}u(x)$ by $u_n(x)$, we find that (2) can be written

$$v(x) = \frac{h}{\sqrt{\pi}}\int_{-\infty}^{\infty}\left\{u(x)-\epsilon u_1(x)+\frac{\epsilon^2}{2!}u_2(x)-\frac{\epsilon^3}{3!}u_3(x)+\ldots\right\}e^{-h^2\epsilon^2}\,d\epsilon,$$

in which the series terminates, the last term being $\frac{(-1)^{n-1}}{(n-1)!}\epsilon^{n-1}u_{n-1}(x)$.

Also
$$\int_{-\infty}^{\infty}\epsilon^{2p+1}e^{-h^2\epsilon^2}\,d\epsilon = 0$$

for integral values of p. Hence

$$v(x) = \frac{h}{\sqrt{\pi}}\int_{-\infty}^{\infty}\left\{u(x)+\frac{\epsilon^2}{2!}u_2(x)+\frac{\epsilon^4}{4!}u_4(x)+\ldots\right\}e^{-h^2\epsilon^2}\,d\epsilon. \quad \ldots\ldots(3)$$

Now
$$\xi \equiv \int_{-\infty}^{\infty}e^{-h^2\epsilon^2}\,d\epsilon = \frac{\sqrt{\pi}}{h}$$

and, as the integral is uniformly convergent,

$$\frac{d\xi}{dh} \equiv -2h\int_{-\infty}^{\infty}\epsilon^2 e^{-h^2\epsilon^2}\,d\epsilon = -\frac{\sqrt{\pi}}{h^2}.$$

Hence
$$\int_{-\infty}^{\infty}\epsilon^2 e^{-h^2\epsilon^2}\,d\epsilon = \frac{\sqrt{\pi}}{2h^3}.$$

Similarly, we can derive the general formula, for positive integral values of p,

$$\int_{-\infty}^{\infty}\epsilon^{2p}e^{-h^2\epsilon^2}\,d\epsilon = \frac{\sqrt{\pi}}{h}\cdot\frac{(2p)!}{p!(4h^2)^p}. \quad \ldots\ldots(4)$$

Hence (3) becomes

$$v(x) = u(x)+\frac{1}{4h^2}u_2(x)+\frac{1}{32h^4}u_4(x)+\ldots+\frac{1}{p!(4h^2)^p}u_{2p}+\ldots.$$

Regarding the probable error r—and consequently $1/h$ by (1)—as a small quantity (otherwise the statistics would be of comparatively little value), we derive $u(x)$ in terms of $v(x)$ and its derivatives by the process of successive approximations. Thus,

(i) $u(x) = v(x)$,

(ii) $u(x) = v(x)-\frac{1}{4h^2}v_2(x)$,

(iii) $u(x) = v(x)-\frac{1}{4h^2}\frac{d^2}{dx^2}\left\{v(x)-\frac{1}{4h^2}v_2(x)\right\}-\frac{1}{32h^4}v_4(x)$

$$= v(x)-\frac{1}{4h^2}v_2(x)+\frac{1}{32h^4}v_4(x).$$

It is easily seen that the general formula is

$$u(x) = v(x) - \frac{1}{4h^2} v_2(x) + \frac{1}{2!(4h^2)^2} v_4(x) - \frac{1}{3!(4h^2)^3} v_6(x) + \ldots, \qquad \ldots\ldots(5)$$

a formula first given by Eddington.*

Eddington's proof depends on the use of symbolic operators. Let D denote the operational symbol d/dx. Then

$$u(x-\epsilon) = e^{-\epsilon D} . u(x)$$

by the symbolic form of Taylor's theorem. Hence (2) becomes

$$v(x) = \frac{h}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-h^2\epsilon^2 - \epsilon D} . u(x)\, d\epsilon.$$

Now
$$\int_{-\infty}^{\infty} e^{-h^2\epsilon^2 - a\epsilon}\, d\epsilon = \frac{\sqrt{\pi}}{h} e^{a^2/4h^2}.$$

Hence
$$v(x) = e^{D^2/4h^2} . u(x),$$

so that
$$u(x) = e^{-D^2/4h^2} . v(x)$$

or
$$u(x) = v(x) - \frac{1}{4h^2} v_2(x) + \frac{1}{2!(4h^2)^2} v_4(x) - \ldots,$$

which is formula (5).

In this proof, questions relating to the convergency of the function $u(x)$ arise in the general case but, as explained previously, we are concerned in practice only with n different numbers corresponding to the values $x_1, x_2, \ldots x_n$ of the characteristic, and in the present case both the functions $u(x)$ and $v(x)$ can be described as polynomials.

There remains the evaluation of the functions $v_2(x)$, $v_4(x)$, ... We shall suppose that $v(x)$ is obtained from the smooth curve of the observations for each of the series of values of the abscissa

$$\ldots x-3\alpha, \quad x-2\alpha, \quad x-\alpha, \quad x, \quad x+\alpha, \quad x+2\alpha, \ldots.$$

Further, we shall suppose that the interval α is small and we shall neglect powers of α higher than the fourth. Assume that the tabular differences corresponding to a value x have been found, and let b and d denote the second and fourth differences respectively. Then

$$b = v(x+\alpha) + v(x-\alpha) - 2v(x),$$

whence
$$b = \alpha^2 v_2(x) + \frac{\alpha^4}{12} v_4(x).$$

Also
$$\begin{aligned} d &= v(x+2\alpha) - 4v(x+\alpha) + 6v(x) - 4v(x-\alpha) + v(x-2\alpha) \\ &= \alpha^4 v_4(x). \end{aligned}$$

* *M.N.* **73**, **359**, **1913**. Some criticisms of a theoretical nature are given by H. Jeffreys, *M.N.* **98**, **190**, **1938**.

From these results we obtain

$$v_4(x) = \frac{d}{\alpha^4}, \qquad \text{......(6)}$$

$$v_2(x) = \frac{12b-d}{12\alpha^2}. \qquad \text{......(7)}$$

Also the value of h is presumed to have been calculated by (1) from the given probable error of the observations. From (6) and (7), the formula (5) becomes—keeping only the first three terms—

$$u(x) = v(x) - \frac{12b-d}{48h^2\alpha^2} + \frac{d}{32h^4\alpha^4}. \qquad \text{......(8)}$$

Thus the value of the true frequency function corresponding to any given value of x can be determined and in this way the true frequency curve $y = u(x)$ can be obtained.

1·82. *The correction of observed mean values.*

As before, the number of stars with the observed characteristic between x and $x+dx$, with errors lying between ϵ and $\epsilon+d\epsilon$, is

$$\frac{h}{\sqrt{\pi}} u(x-\epsilon)\, e^{-h^2\epsilon^2}\, dx\, d\epsilon.$$

Let $\bar{\epsilon}$ denote the average error for the stars with the observed characteristic between x and $x+dx$; then $\bar{\epsilon}$ is given by

$$\bar{\epsilon}\int_{-\infty}^{\infty} u(x-\epsilon)\, e^{-h^2\epsilon^2}\, d\epsilon = \int_{-\infty}^{\infty} \epsilon u(x-\epsilon)\, e^{-h^2\epsilon^2}\, d\epsilon$$

or, with the help of (2) of the previous section,

$$\bar{\epsilon}\frac{\sqrt{\pi}}{h} v(x) = \int_{-\infty}^{\infty} \epsilon u(x-\epsilon)\, e^{-h^2\epsilon^2}\, d\epsilon. \qquad \text{......(1)}$$

When integrated by parts, the right-hand side of (1) becomes

$$-\frac{1}{2h^2}\Big[u(x-\epsilon)\, e^{-h^2\epsilon^2}\Big]_{-\infty}^{\infty} + \frac{1}{2h^2}\int_{-\infty}^{\infty} e^{-h^2\epsilon^2} \frac{d}{d\epsilon} u(x-\epsilon)\,.\,d\epsilon,$$

of which the integrated part vanishes at both limits. Also,

$$\frac{d}{d\epsilon}\,.\,u(x-\epsilon) = -\frac{d}{dx}\,.\,u(x-\epsilon).$$

Hence
$$\bar{\epsilon}\frac{\sqrt{\pi}}{h} v(x) = -\frac{1}{2h^2}\frac{d}{dx}\int_{-\infty}^{\infty} u(x-\epsilon)\, e^{-h^2\epsilon^2}\, d\epsilon,$$

from which
$$\bar{\epsilon} = -\frac{1}{2h^2}\cdot\frac{v'(x)}{v(x)}, \qquad \text{......(2)}$$

a result* depending only on the observed distribution.

* This result, due to A. S. Eddington, was first given by F. W. Dyson, *M.N.* **86**, 686, 1926.

It is to be remembered that ϵ has been used in the sense of being applied algebraically to the true value of the characteristic to give the observed value; consequently, the quantity, as derived from (2), to be applied to the observed statistics will be given by $+\frac{1}{2h^2}\cdot\frac{v'(x)}{v(x)}$.

Formula (2) has been used extensively by Dyson* and Nassau† for the correction of parallaxes with a given mean probable error. The following table relating to parallaxes measured at the Leander McCormick Observatory is given by Nassau:‡

Table 4. *Statistics of parallaxes*

Limits of p (unit 0″·001)	Observed number of stars	Correction to p (unit 0″·001)	Corrected number of stars
−30 to −25	3	+29	0
−25 „ −20	7	26	0
−20 „ −15	7	22	0
−15 „ −10	17	18	0
−10 „ − 5	27	13	0
− 5 „ 0	39	8	2
0 „ 5	46	4	13
5 „ 10	51	+ 2	150
10 „ 15	56	− 1	109
15 „ 20	50	− 2	65
20 „ 25	43	− 3	57
25 „ 30	45	− 4	42
30 „ 35	43	− 5	38
35 „ 40	32	− 5	42
40 „ 45	33	− 6	17
45 „ 50	29	− 7	20
50 „ 55	21	− 6	18
55 „ 60	12	− 6	13
60 „ 65	17	− 6	16
65 „ 70	13	− 6	14
70 „ 75	15	− 6	13
75 „ 80	12	− 7	7
80 „ 85	6	− 8	7
85 „ 90	6	− 7	3
90 „ 95	5	− 7	2
95 „ 100	5	− 6	3

In the second column are the numbers of stars observed to have parallaxes, p, within the limits indicated in the first column. These numbers give the broken curve in Fig. 7; the full-line curve is the smoothed curve $y = v(x)$. Formula (2) may be written approximately

$$\bar{\epsilon} = -\frac{1}{2h^2}\frac{1}{y}\cdot\frac{\Delta y}{\Delta x}. \qquad \text{......(3)}$$

Corresponding to the interval x to $x+\Delta x$ (or p to $p+\Delta p$) in the abscissae the value Δy can be taken from the smoothed curve. For $\Delta x \equiv \Delta p = 0''{\cdot}005$

* *Loc. cit.* † J. J. Nassau, *M.N.* 88, 441 and 583, 1928. ‡ *M.N.* 88, 584, 1928.

and the probable error $\pm 0''\cdot 0096$ (from which $1/h = 0\cdot 0096/0\cdot 4769$ numerically), as given by the observers, formula (3) becomes

$$\bar{\epsilon} = -0''\cdot 040\frac{\Delta y}{y}, \qquad \ldots\ldots(4)$$

so that the correction to be applied to the observed statistics is $+0''\cdot 040\Delta y/y$. This correction is given in the third column of the table. Each value of the observed parallax is corrected and the distribution of the corrected parallaxes is shown in the last column. The result of applying the corrections is to remove practically all of the negative parallaxes from the distribution.

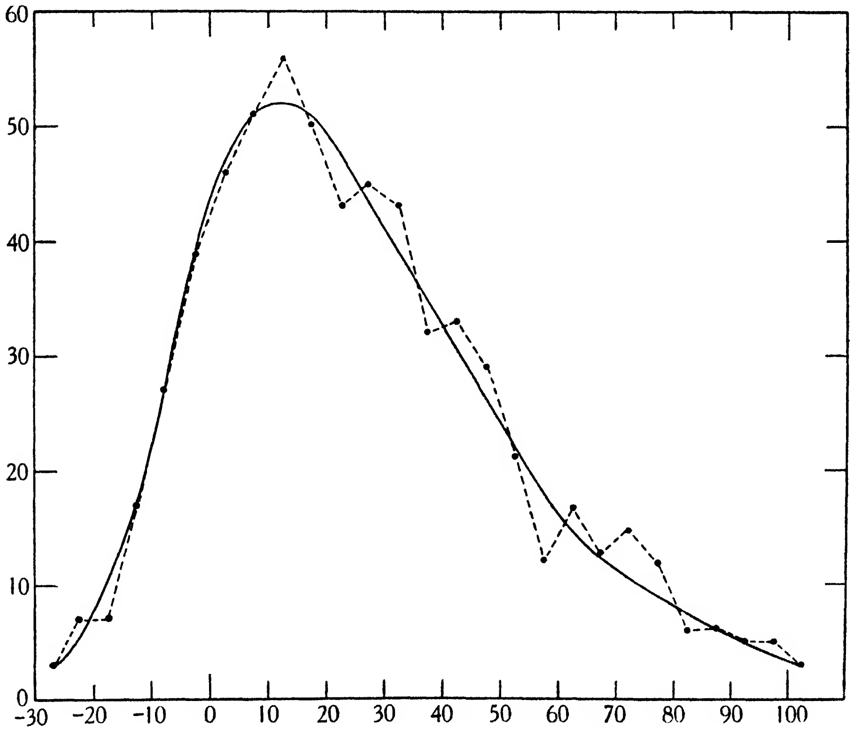

Fig. 7. *Parallax Statistics* (*Nassau*). The abscissae are the values of the parallax, p, the unit being 0".001; the ordinates are the numbers of stars with parallaxes within intervals of 0".005.

Applying the above method, Dyson (*loc. cit.*) has corrected the parallaxes measured at the Royal Observatory, Greenwich, and computed the absolute magnitudes of the stars concerned and their transverse linear motions, using for the latter the observed proper motions. An extension of the method in connection with the combination of trigonometrical and spectroscopic parallaxes has been given by T. Nicolini.*

* *Catania, Contributi astrofisici*, No. 37, 1937.

CHAPTER II

A SINGLE STAR-DRIFT

2·1. In 1906 Eddington* introduced the term *drift* to denote an isolated assembly of stars whose linear velocities relative to a system of coordinate axes are entirely haphazard. The axes are fixed in direction and the origin is chosen so that

$$\Sigma u = \Sigma v = \Sigma w = 0, \qquad \ldots\ldots(1)$$

where (u, v, w) are the components of motion of a star parallel to these axes. There is the further consideration implied in the term "haphazard", namely that, corresponding to a given numerical value, u_0, of a velocity component, say u, the number of stars with positive values of u between u_0 and u_0+du_0 is equal to the number of stars with negative values of u between $-u_0$ and $-(u_0+du_0)$. The formulae (1) show that we can regard the centre of position of the stars forming the drift as at rest (this is identical with the centre of mass if the stars are all assumed to have equal masses) and we can take this point as the origin of coordinates. With this origin and system of axes the motion of the sun with reference to the assembly of stars can be defined without ambiguity. In the same way we define the *velocity of the drift* to be the motion of this origin relative to parallel axes through the sun. Actually, the motion of an individual star at any instant will be determined by the gravitational potential of the system and, over long periods of time, we should have to take into account the accelerations produced. No linear acceleration and no curvature in the path of a single star have hitherto been detected which can be ascribed to gravitational causes—we exclude, of course, the members of binary or multiple systems—and, accordingly, the theoretical concept of a drift can be related to the practical study of stellar motions.

Before 1904 the investigators of the solar motion assumed that the stellar system, as then explored, formed a single drift, but Kapteyn's discovery of star-streaming showed that this hypothesis was not in accord with the observed facts; in other words, that the individual motions of the stars were not distributed at random. Eddington's development of the theory involved, mainly, the division of the stars into two drifts and agreement between theory and observation followed. In this chapter we shall be concerned with a single drift of stars and the results will be used later in discussion of the two-drift theory.

* *M.N.* **67**, 34, 1906.

Let (u, v, w) denote the components of the linear velocity W of a star with respect to the coordinate axes associated with a drift. Then

$$W^2 = u^2 + v^2 + w^2. \qquad ......(2)$$

The mathematical expression corresponding to a haphazard distribution of velocities must satisfy several conditions. In the first place, we have to rule out infinite velocities and thus the functional expression must vanish for infinite values of u or v or w; secondly, the expression must be independent of the orientation of the axes; thirdly, it must be a function of u^2 and v^2 and w^2, since the distribution is the same for negative as for positive velocities. These conditions imply that the function is of the form $F(W)$, where W is given by (2), it being assumed that $F \to 0$ as u or v or $w \to \infty$. We therefore have that the proportion of drift-stars with linear components between (u, v, w) and $(u+du, v+dv, w+dw)$ is

$$F(W)\, du\, dv\, dw. \qquad ......(3)$$

This distribution of velocities is called a *spherical velocity distribution*.

In the practical applications to be considered later, the form of the function will be taken to be

$$F(W) \equiv Ce^{-h^2W^2} \equiv Ce^{-h^2(u^2+v^2+w^2)}, \qquad ......(4)$$

which is the Maxwellian frequency law, C and h being certain constants. As Eddington remarked,* "we are not at the moment concerned with what law stellar motions are likely to follow; that is a dynamical problem. We are rather choosing a standard of comparison with which to compare the actual distribution of motions and that standard ought to be the simplest possible. Further, there is a special propriety in taking Maxwell's Law as it is the nearest possible approach to an absolutely chaotic state of motion." Also, it is to be noted, that with Maxwell's Law there is no correlation between the u, v and w components of velocity.

We now develop the consequence of the general distribution of velocities, as given by (3), on formal lines.

2·21. *The mean random radial speed for a drift.*

Consider a small area of the celestial sphere, with the sun as centre, at S, in which there are N stars per unit area with the given spherical velocity-distribution. We here assume that our point of observation—the sun—is at rest with respect to the drift; consequently, all velocities concerned will be random velocities. This procedure is equivalent to observing the stars of the drift in any direction from any point fixed with respect to the coordinate axes associated with the drift, and we shall assume that the stars in any sample volume of space obey the velocity-distribution law (3).

* *Stellar Movements*, 128, 1914.

Let the w-axis be the radius passing through S. Then w is the random or peculiar radial velocity which we shall generally denote by R. It is to be noted that, theoretically, the values of R range from $-\infty$ to $+\infty$.

If dN is the number of stars, per unit area at S, with velocity components between (u, v, w) and $(u+du, v+dv, w+dw)$, we can write

$$dN = BF(W)\,du\,dv\,dw,$$

where, on summing for all theoretically possible values of u, v and w, B is defined by

$$N = B\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(W)\,du\,dv\,dw,$$

or, since $F(W)$ is an even function of u, v and w,

$$N = 8B\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} F(W)\,du\,dv\,dw. \qquad \text{......(1)}$$

In this formula the integration is taken through the octant of the sphere

$$r^2 = u^2+v^2+w^2$$

of which the radius, r, tends to an infinite value and for which u, v and w play the part of current coordinates.

We set

$$u = W\cos\phi\sin\theta, \quad v = W\sin\phi\sin\theta, \quad w = W\cos\theta, \qquad \text{......(2)}$$

so that the element of volume of the sphere is $W^2\sin\theta\,dW\,d\theta\,d\phi$. Hence

$$N = 8B\int_{0}^{\infty} W^2F(W)\,dW\int_{0}^{\pi/2}\sin\theta\,d\theta\int_{0}^{\pi/2}d\phi$$

or

$$N = 4\pi BQ_1, \qquad \text{......(3)}$$

where

$$Q_1 = \int_{0}^{\infty} W^2F(W)\,dW. \qquad \text{......(4)}$$

Let $\bar{R}$ denote the mean arithmetical value of the random radial velocities, R—or the *mean random radial speed*—of the stars concerned. Then the number of stars for which R lies between w and $w+dw$, for all possible values of u and v, is

$$B\,dw\int_{-\infty}^{\infty}\int_{-}^{\infty} F(W)\,du\,dv$$

or

$$4B\,dw\int_{0}^{\infty}\int_{0}^{\infty} F(W)\,du\,dv.$$

There is an equal number of stars for which R lies between $-w$ and $-(w+dw)$. Considering only the *arithmetical* values of R, we then obtain for $\bar{R}$ the formula

$$N\bar{R} = 8B\int_{0}^{\infty} w\,dw\int_{0}^{\infty}\int_{0}^{\infty} F(W)\,du\,dv.$$

Using the transformations in (2), we find that

$$N\bar{R} = 8B\int_0^\infty W^3F(W)\,dW\int_0^{\pi/2}\sin\theta\,\cos\theta\,d\theta\int_0^{\pi/2}d\phi,$$

whence $$N\bar{R} = 2\pi BQ_2, \qquad \text{......(5)}$$

where $$Q_2 = \int_0^\infty W^3F(W)\,dW. \qquad \text{......(6)}$$

From (3) and (5) we obtain $$\bar{R} = \frac{Q_2}{2Q_1}. \qquad \text{......(7)}$$

It is to be noticed that this value of the mean radial speed depends only on the form of the frequency function F and not on the numerical distribution of the stars. It is thus constant for all areas of the celestial sphere.

The results just obtained will be used in the sections immediately following and in Chapter X.

2·22. *The mean random transverse speed.*

With the same convention as in section 2·21 with regard to the w-axis, the transverse linear component of the motion of a star at S is $(u^2+v^2)^{1/2}$, which we denote by T. We denote by $\bar{T}$ the mean of the arithmetical values of T for all the stars concerned. The number of stars with transverse velocity components between (u, v) and $(u+du, v+dv)$ is, for all possible values of w,

$$B\,du\,dv\int_{-\infty}^{\infty}F(W)\,dw.$$

Accordingly, $\bar{T}$ is given by

$$N\bar{T} = 4B\int_{-\infty}^{\infty}dw\int_0^\infty\int_0^\infty (u^2+v^2)^{1/2}F(W)\,du\,dv,$$

the coefficient 4 arising since the number of stars with negative u-components is equal to the number with positive components, with a similar argument as to the v-components. The transformation (2) of the previous section leads to

$$N\bar{T} = 8B\int_0^\infty W^3F(W)\,dW\int_0^{\pi/2}\sin^2\theta\,d\theta\int_0^{\pi/2}d\phi,$$

whence, in terms of the function Q_2, previously defined,

$$N\bar{T} = \pi^2 BQ_2$$

or, by (3) of the previous section,

$$\bar{T} = \frac{\pi Q_2}{4Q_1}. \qquad \text{......(1)}$$

As in the case of $\bar{R}$, this value of the mean random transverse speed depends only on the form of the frequency function F and is independent of the distribution of the stars.

From (7) of section 2·21 and (1) above, we obtain

$$\bar{T} = \frac{\pi}{2}\bar{R} \qquad \text{......(2)}$$

and *this result is independent of the frequency function F.*

2·23. *The mean linear speed.*

Let $\bar{W}$ denote the mean linear speed and we have at once

$$N\bar{W} = 8B\int_0^\infty \int_0^\infty \int_0^\infty WF(W)\,du\,dv\,dw,$$

which, with the usual transformation, becomes

$$N\bar{W} = 8B\int_0^\infty W^3F(W)\,dW \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{\pi/2} d\phi.$$

Hence, we derive in the same way as before,

$$N\bar{W} = 4\pi B Q_2$$

and, using (3) of section 2·21, $\bar{W} = \frac{Q_2}{Q_1}$. ......(1)

We then have the result $\bar{R} : \bar{T} : \bar{W} = 2 : \pi : 4$, ...(2)

which is independent of the form of the function F.

2·24. *Formulae for $\bar{R}$, $\bar{T}$ and $\bar{W}$ according to the Maxwellian law.*

When $F(W) = Ce^{-h^2W^2}$ (formula (4) of section 2·1), we have

$$Q_1 = C\int_0^\infty W^2 e^{-h^2W^2}\,dW,$$

so that, by 1·81 (4), $Q_1 = \frac{C\sqrt{\pi}}{4h^3}$.

Now $y \equiv \int_0^\infty We^{-h^2W^2}\,dW = \frac{1}{2h^2}$

and, as the integral is uniformly convergent,

$$\frac{dy}{dh} \equiv -2h\int_0^\infty W^3e^{-h^2W^2}\,dW = -\frac{1}{h^3}.$$

Hence we have $Q_2 = \frac{C}{2h^4}$.

From the results for $\bar{R}$, $\bar{T}$ and $\bar{W}$ in terms of Q_1 and Q_2 we finally obtain

$$\bar{R} = \frac{1}{h\sqrt{\pi}}, \qquad \text{......(1)}$$

$$\bar{T} = \frac{\sqrt{\pi}}{2h}, \qquad \text{......(2)}$$

$$\bar{W} = \frac{2}{h\sqrt{\pi}}. \qquad \text{......(3)}$$

We regard the modulus h as a constant defining the dispersion of velocities in the drift and it can be evaluated by means of (1), (2) or (3) if any one of the quantities $\bar{R}$, $\bar{T}$ and $\bar{W}$ is known.

Another criterion that may be adopted is the value, W_1, of the median speed such that the proportion of stars for which $0 \leqslant W \leqslant W_1$ is equal to the proportion for which $W > W_1$. In this case,

$$\int_0^{W_1} e^{-h^2W^2}\,dW = \tfrac{1}{2}\int_0^{\infty} e^{-h^2W^2}\,dW$$

or

$$\int_0^{hW_1} e^{-x^2}\,dx = \tfrac{1}{2}\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{4}.$$

Tables of the integral give

$$hW_1 = 0{\cdot}4769,$$

which may be used as a definition of h.

2·3. *The drift-curve.*

We shall now find the distribution of the transverse velocities for a small area of the sky at S (Fig. 8), the sun being the centre of the celestial sphere, taking into account the drift-velocity relative to the sun. If the solar motion, U, is directed towards A_0 (the apex), the drift-velocity relative to the sun will be U in the direction of the antipodal point A (the antapex). We take the area at S to be defined by the two small circles at angular distances λ and $\lambda + d\lambda$ from A and by the two "meridians" ϕ and $\phi + d\phi$. The area at S is thus $\sin\lambda\, d\lambda\, d\phi$.

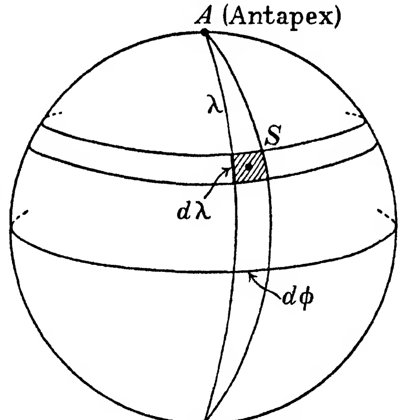

Fig. 8

As before, we shall take the w-axis of the haphazard motions to be the radius through S. The projection of the drift-velocity, U, on the tangential plane at S is $U\sin\lambda$ and we shall take the corresponding direction on the tangent plane to be the u-axis. We write

$$V = U\sin\lambda. \qquad \ldots\ldots(1)$$

The transverse velocity, as *observed* from the sun, for any star is compounded of the haphazard transverse velocity OQ (Fig. 9) with components (u, v) and the constant velocity V (the parallactic velocity) represented by

OP in the direction of the u-axis. The *observed transverse velocity* is thus OR—which we denote by T—making an angle θ with the direction of the parallactic motion.

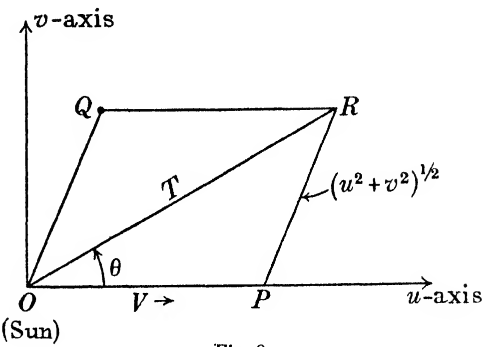

Fig. 9

Let n denote the number of stars in the small region at S (Fig. 8). We shall assume that the Maxwellian law holds for this sample of stars, so that, if dn is the number of stars with random transverse velocity components between (u, v) and $(u+du, v+dv)$,

$$dn = Ae^{-h^2(u^2+v^2)}\,du\,dv. \qquad \text{......(2)}$$

Hence
$$n = A\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-h^2(u^2+v^2)}\,du\,dv$$

and therefore
$$A = \frac{nh^2}{\pi}. \qquad \text{......(3)}$$

It is required to find the frequency of the observed transverse velocities in the small sector, defined by θ and $\theta+d\theta$, of the tangent plane.

From Fig. 9, we have

$$u^2+v^2 = T^2-2TV\cos\theta+V^2.$$

Also,
$$u = T\cos\theta - V, \quad v = T\sin\theta,$$

from which
$$du\,dv = \frac{\partial(u,v)}{\partial(T,\theta)}\,dT\,d\theta = T\,dT\,d\theta.$$

Using these results in (2), we obtain

$$dn = \frac{nh^2}{\pi}e^{-h^2(T^2-2TV\cos\theta+V^2)}\,T\,dT\,d\theta; \qquad \text{......(4)}$$

dn is thus the number of stars moving in the sector θ, $\theta+d\theta$ and with observed velocities between T and $T+dT$. The total number of stars moving in the small sector is obtained by summing dn for all values of T between 0 and ∞; consequently, if we denote this number by $n(\theta)\,d\theta$ or $\rho\,d\theta$, we obtain

$$\rho \equiv n(\theta) = \frac{nh^2}{\pi}\int_0^{\infty} Te^{-h^2(T^2-2TV\cos\theta+V^2)}\,dT. \qquad \text{......(5)}$$

We write
$$x = h(T-V\cos\theta), \qquad \text{......(6)}$$
$$\tau = hV\cos\theta, \qquad \text{......(7)}$$

so that
$$hT = x+\tau.$$

Then (5) becomes

$$\rho \equiv n(\theta) = \frac{n}{\pi} e^{-h^2V^2} \,.\, e^{\tau^2} \int_{-\tau}^{\infty} (x+\tau)\, e^{-x^2}\, dx.$$

But
$$\int_{-\tau}^{\infty} (x+\tau)\, e^{-x^2}\, dx = \tfrac{1}{2} e^{-\tau^2} + \tau \int_{-\tau}^{\infty} e^{-x^2}\, dx. \qquad \text{......(8)}$$

Hence we have
$$\rho \equiv n(\theta) = \frac{n}{\pi} e^{-h^2V^2} \left\{ \tfrac{1}{2} + \tau e^{\tau^2} \int_{-\tau}^{\infty} e^{-x^2}\, dx \right\}. \qquad \text{......(9)}$$

Eddington,* to whom this formula is due, defines a function $f(\tau)$ by

$$f(\tau) = \frac{2}{\sqrt{\pi}} \left\{ \tfrac{1}{2} + \tau e^{\tau^2} \int_{-\tau}^{\infty} e^{-x^2}\, dx \right\}. \qquad \text{......(10)}$$

We thus write
$$\rho \equiv n(\theta) = \frac{n}{2\sqrt{\pi}} e^{-h^2V^2} f(\tau). \qquad \text{......(11)}$$

The values of $f(\tau)$ can be derived from the data† of Table 5.

Table 5. *Values of* $\log f(\tau)$

τ	$\log f(\tau)$	τ	$\log f(\tau)$	τ	$\log f(\tau)$
−1·2	9·0411	−0·1	9·6763	1·0	0·7461
−1·1	9·0874	0·0	9·7514	1·1	0·8751
−1·0	9·1355	0·1	9·8303	1·2	1·0103
−0·9	9·1856	0·2	9·9131	1·3	1·1520
−0·8	9·2378	0·3	0·0001	1·4	1·3003
−0·7	9·2923	0·4	0·0916	1·5	1·4555
−0·6	9·3493	0·5	0·1876	1·6	1·6177
−0·5	9·4088	0·6	0·2886	1·7	1·7871
−0·4	9·4711	0·7	0·3947	1·8	1·9637
−0·3	9·5363	0·8	0·5061	1·9	2·1478
−0·2	9·6046	0·9	0·6232	2·0	2·3393
−0·1	9·6763	1·0	0·7461		

We can write the function $f(\tau)$ in the alternative form

$$f(\tau) = \frac{2}{\sqrt{\pi}} \left[\tfrac{1}{2} + \tau e^{\tau^2} \left\{ \frac{\sqrt{\pi}}{2} + K(\tau) \right\} \right], \qquad \text{......(12)}$$

where
$$K(\tau) = \int_0^{\tau} e^{-x^2}\, dx. \qquad \text{......(13)}$$

The values of $K(\tau)$ are easily obtained from tables such as that in Brunt's *The Combination of Observations* (2nd edition), 234, 1931.

Now ρ is a function of θ—it is given by (11)—and the curve obtained by calculating ρ for different values of θ between 0° and 360° is called a *drift-curve*. It is clear that a drift-curve is symmetrical about the radius vector corresponding to $\theta = 0$.

* *M.N.* 67, 34, 1906.

† A. S. Eddington, *Stellar Movements*, 129, 1914.

Fig. 10 shows four drift-curves* drawn for the values 0·3, 0·6, 1·0 and 1·5 of hV. In each case O is the origin from which the radii vectores, ρ, are drawn, and the u-axis, or the direction of the parallactic motion, is given by $\theta = 0$.

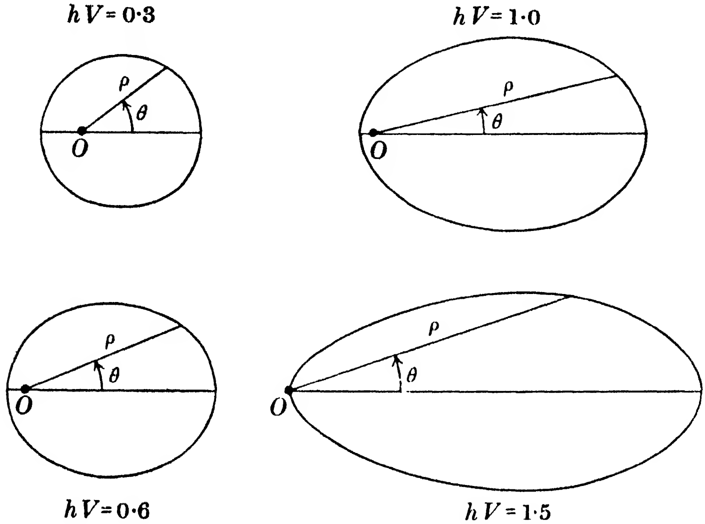

Fig. 10

2·41. *The mean value of the observed linear transverse motions in a given direction, for a small area of the sky.*

The number, dn, of stars with transverse velocities between T and $T+dT$ and lying in the sector between θ and $\theta+d\theta$ is given by (4) of section 2·3. As $n(\theta)\,d\theta$ is the total number of stars with velocities of all magnitudes in this sector, the mean value of T for the sector—we denote it by T_1—is given by

$$T_1\,.\,n(\theta)\,d\theta = \frac{nh^2}{\pi}d\theta \int_0^\infty e^{-h^2(T^2-2TV\cos\theta+V^2)}\,T^2\,dT$$

or, using (6) and (7) of section 2·3,

$$hT_1\,.\,n(\theta) = \frac{n}{\pi}e^{-h^2V^2+\tau^2}\int_{-\tau}^\infty (x+\tau)^2\,e^{-x^2}\,dx.$$

Now
$$\int_{-\tau}^\infty x^2 e^{-x^2}\,dx = -\tfrac{1}{2}\tau e^{-\tau^2} + \tfrac{1}{2}\int_{-\tau}^\infty e^{-x^2}\,dx$$

and
$$\int_{-\tau}^\infty 2xe^{-x^2}\,dx = e^{-\tau^2}.$$

Hence
$$hT_1\,.\,n(\theta) = \frac{n}{\pi}e^{-h^2V^2+\tau^2}\left\{\frac{\tau}{2}e^{-\tau^2} + (\tau^2+\tfrac{1}{2})\int_{-\tau}^\infty e^{-x^2}\,dx\right\}$$

* These have been taken from Eddington's *Stellar Movements*, 88, 1914.

or, from (11) of section 2·3,

$$hT_1 . f(\tau) = \frac{2}{\sqrt{\pi}}\left[\tau\left\{\tfrac{1}{2}+\tau e^{\tau^2}\int_{-\tau}^{\infty} e^{-x^2}\,dx\right\}+\frac{e^{\tau^2}}{2}\int_{-\tau}^{\infty} e^{-x^2}\,dx\right]$$

$$= \tau f(\tau)+\frac{1}{\sqrt{\pi}}e^{\tau^2}\int_{-\tau}^{\infty} e^{-x^2}\,dx,$$

whence

$$hT_1 = \tau+\frac{e^{\tau^2}\int_{-\tau}^{\infty} e^{-x^2}\,dx}{\sqrt{\pi}f(\tau)}, \qquad \ldots\ldots(1)$$

which may be written $hT_1 = g(\tau)$, ...(2)

where $g(\tau)$ is the right-hand side of (1) and may be expressed in the alternative form

$$g(\tau) = \tau+\frac{f(\tau)-\frac{1}{\sqrt{\pi}}}{2\tau f(\tau)}. \qquad \ldots\ldots(3)$$

The values of $g(\tau)$ have been tabulated by Eddington;* they are given in Table 6.

Table 6. *Values of* $g(\tau)$

τ	$g(\tau)$	τ	$g(\tau)$	τ	$g(\tau)$
−1·0	0·565	−0·1	0·845	0·8	1·315
−0·9	0·589	0·0	0·886	0·9	1·381
−0·8	0·614	0·1	0·930	1·0	1·449
−0·7	0·641	0·2	0·977	1·1	1·520
−0·6	0·670	0·3	1·027	1·2	1·594
−0·5	0·701	0·4	1·079	1·3	1·669
−0·4	0·734	0·5	1·134	1·4	1·747
−0·3	0·768	0·6	1·191	1·5	1·827
−0·2	0·805	0·7	1·252	1·6	1·908
−0·1	0·845	0·8	1·315	1·7	1·991

2·42. *The mean value of the observed linear transverse motions in all directions, for a small area of the sky.*

We write the number, $n(\theta)\,d\theta$, of stars moving in the sector θ, $\theta+d\theta$ as

$$n(\theta)\,d\theta = Bf(\tau)\,d\theta,$$

where

$$B = \frac{n}{2\sqrt{\pi}}e^{-h^2V^2}.$$

For this number the mean value of T is T_1, as deduced in the previous section. Let T_2 denote the mean value of the observed transverse linear motions for all the stars, n in number, between the directions $\theta = 0$ and $\theta = 2\pi$. We then have

$$nT_2 = \int_0^{2\pi} T_1 n(\theta)\,d\theta$$

* *Stellar Movements*, 141, 1914.

or, from symmetry about $\theta = 0$,

$$nT_2 = 2B\int_0^{\pi} T_1 f(\tau)\,d\theta.$$

But, using (12) and (13) of section 2·3 and (1) of section 2·41,

$$hT_1 f(\tau) = \tau f(\tau) + \tfrac{1}{2}e^{\tau^2} + \frac{1}{\sqrt{\pi}}e^{\tau^2}K(\tau)$$

$$= (\tau^2+\tfrac{1}{2})\,e^{\tau^2} + \frac{1}{\sqrt{\pi}}\{\tau + (2\tau^2+1)\,e^{\tau^2}K(\tau)\}.$$

Hence $$\frac{nhT_2}{2B} = \int_0^{\pi}(\tau^2+\tfrac{1}{2})\,e^{\tau^2}d\theta + \frac{1}{\sqrt{\pi}}\int_0^{\pi}\{\tau+(2\tau^2+1)e^{\tau^2}K(\tau)\}\,d\theta. \qquad\ldots\ldots(1)$$

Since $\tau = hV\cos\theta$, we have $\int_0^{\pi}\tau\,d\theta = 0,$

and we can write

$$\int_0^{\pi}(2\tau^2+1)\,e^{\tau^2}K(\tau)\,d\theta = \int_0^{\pi/2}(2\tau^2+1)\,e^{\tau^2}K(\tau)\,d\theta + \int_{\pi/2}^{\pi}(2\tau^2+1)\,e^{\tau^2}K(\tau)\,d\theta. \qquad\ldots\ldots(2)$$

Putting $(\pi-\theta)$ for θ in the second integral on the right of (2), we find that it becomes

$$\int_0^{\pi/2}(2\tau^2+1)\,e^{\tau^2}K(-\tau)\,d\theta.$$

Also $K(-\tau) = -K(\tau)$; hence the right-hand side of (2) vanishes. The formula (1) then becomes

$$\frac{nhT_2}{2B} = \int_0^{\pi}(\tau^2+\tfrac{1}{2})\,e^{\tau^2}d\theta$$

or, inserting the expression for B,

$$T_2 = \frac{1}{2h\sqrt{\pi}}\,e^{-h^2V^2}\int_0^{\pi}(2\tau^2+1)\,e^{\tau^2}d\theta.$$

We put $$(hV)^2 = 2b, \qquad\ldots\ldots(3)$$

so that $$\tau = \sqrt{2b}\cos\theta. \qquad\ldots\ldots(4)$$

Hence $$T_2 = \frac{1}{2h\sqrt{\pi}}\int_0^{\pi}e^{-2b\sin^2\theta}(1+4b\cos^2\theta)\,d\theta \qquad\ldots\ldots(5)$$

or, in the alternative form,

$$T_2 = \frac{1}{2h\sqrt{\pi}}\,e^{-b}\int_0^{\pi}e^{b\cos 2\theta}(1+2b+2b\cos 2\theta)\,d\theta. \qquad\ldots\ldots(6)$$

The integral on the right can be expressed in terms of Bessel functions of imaginary argument, since

$$\int_0^{\pi}e^{b\cos 2\theta}\,d\theta = \pi J_0(ib) \qquad\ldots\ldots(7)$$

and $$\int_0^\pi \cos 2\theta\, e^{b\cos 2\theta}\, d\theta = \pi \frac{dJ_0(ib)}{db} = -i\pi J_1(ib). \qquad \ldots\ldots(8)$$

Hence $$T_2 = \frac{\sqrt{\pi}}{2h} e^{-b}\{(1+2b)\, J_0(ib) - 2ibJ_1(ib)\}.$$

If n is an odd integer, positive or negative, $J_n(ib)$ is a purely imaginary quantity; if n is even or zero, $J_n(ib)$ is real. Defining real functions $I_n(b)$ by

$$I_n(b) = e^{-\frac{1}{2}in\pi} J_n(ib), \qquad \ldots\ldots(9)$$

we obtain $$T_2 = \frac{\sqrt{\pi}}{2h} e^{-b}\{(1+2b)\, I_0(b) + 2bI_1(b)\}.$$

Now, by (2) of 2·24, the mean *random* transverse linear speed, $\bar{T}$, is given by

$$\bar{T} = \frac{\sqrt{\pi}}{2h}.$$

Hence we have* $$T_2 = \bar{T}\psi(b), \qquad \ldots\ldots(10)$$

where $$\psi(b) = e^{-b}\{(1+2b)\, I_0(b) + 2bI_1(b)\}. \qquad \ldots\ldots(11)$$

The values of $e^{-x}I_0(x)$ and $e^{-x}I_1(x)$ are tabulated in G. N. Watson's *Theory of Bessel Functions*, 698–713, 1922, and the values of the function $\psi(b)$ are thus easily found. They are given in Table 7.

Table 7. *Values of* $\psi(b)$

hV	b	$\psi(b)$	hV	b	$\psi(b)$
0·0	0·0	1·000	1·0	0·50	1·446
0·1	0·005	1·005	1·1	0·605	1·529
0·2	0·02	1·020	1·2	0·72	1·616
0·3	0·045	1·045	1·3	0·845	1·706
0·4	0·08	1·078	1·4	0·98	1·800
0·5	0·125	1·121	1·5	1·125	1·896
0·6	0·18	1·172	1·6	1·28	1·994
0·7	0·245	1·231	1·7	1·445	2·094
0·8	0·32	1·297	1·8	1·62	2·196
0·9	0·405	1·369	1·9	1·805	2·299
1·0	0·50	1·446	2·0	2·00	2·404

We add for reference the following formulae for the I-functions (the modified Bessel functions):

$$\int_0^\pi e^{b\cos 2\theta}\, d\theta = \pi I_0(b), \qquad \ldots\ldots(12)$$

$$\int_0^\pi \cos 2\theta\, e^{b\cos 2\theta}\, d\theta = \pi \frac{dI_0(b)}{db}, \qquad \ldots\ldots(13)$$

$$\frac{dI_0(b)}{db} = I_1(b). \qquad \ldots\ldots(14)$$

* W. M. Smart, *M.N.* **95**, 127, 1934. The results given in sections 2·43 to 2·46 following are also given in this paper.

Also,
$$\int_0^{\pi} e^{b\cos\phi}\,d\phi = \tfrac{1}{2}\int_0^{2\pi} e^{b\cos\phi}\,d\phi$$
$$= \int_0^{\pi} e^{b\cos 2\theta}\,d\theta,$$
on writing $\phi = 2\theta$. We thus have the alternative formula
$$\int_0^{\pi} e^{b\cos\theta}\,d\theta = \pi I_0(b) = \pi J_0(ib). \qquad \text{......(15)}$$

The modified Bessel function of order m is given by
$$\pi I_m(b) = \int_0^{\pi} e^{b\cos\theta}\cos m\theta\,d\theta. \qquad \text{......(16)}$$

2·43. *The mean value of the observed linear transverse speeds for the whole sky.*

We shall assume that the stars are distributed uniformly over the celestial sphere. The procedure in any statistical investigation based on the result of this section must consequently be modified by considering only the *means* of the observed quantities over each small region (of standard area) considered, irrespective of the number of stars utilised in such areas.

Let N denote the number of stars per unit area of the sphere. The number in the zone between λ and $\lambda + d\lambda$ (Fig. 8) is accordingly $2\pi N \sin\lambda\,d\lambda$ and the mean observed transverse linear speed of these stars is T_2, as given by (10) in the previous section, where
$$2b = (hV)^2.$$
Now, since V is the projection of the drift-velocity, U, on the tangent plane at any point of the zone λ, $\lambda + d\lambda$, we have $V = U\sin\lambda$, so that
$$2b = (hU)^2\sin^2\lambda. \qquad \text{......(1)}$$
Set
$$2c = (hU)^2 = \eta^2. \qquad \text{......(2)}$$
Let T_3 denote the mean value of the observed linear transverse speeds for the whole sky. The total number of stars is $4\pi N$. We have, in consequence, for the whole sphere,
$$4\pi N T_3 = 2\pi N\int_0^{\pi} T_2 \sin\lambda\,d\lambda$$
or
$$2T_3 = \overline{T}\int_0^{\pi}\sin\lambda\,\psi(b)\,d\lambda,$$
which can be written, since $b = c\sin^2\lambda$,
$$T_3 = \overline{T}\int_0^{\pi/2}\sin\lambda\,\psi(b)\,d\lambda. \qquad \text{......(3)}$$
But
$$db = 2c\sin\lambda\,\cos\lambda\,d\lambda.$$
Hence
$$T_3 = \frac{\overline{T}}{2\sqrt{c}}\int_0^{c}\frac{\psi(b)}{\sqrt{c-b}}\,db.$$

Consider the integral on the right of this last formula; it is, by 2·42 (11),

$$\int_0^c \frac{db}{\sqrt{c-b}} e^{-b}\{(1+2b)I_0(b)+2bI_1(b)\}.$$

Integrating by parts, we find that it becomes, using 2·42 (14),

$$-2[\sqrt{c-b}\,e^{-b}\{(1+2b)I_0(b)+2bI_1(b)\}]_0^c$$
$$+2\int_0^c \sqrt{c-b}\,e^{-b}\left[I_0(b)+I_1(b)+2b\left\{\frac{d^2I_0}{db^2}+\frac{1}{b}\frac{dI_0(b)}{db}-I_0(b)\right\}\right]db.$$

The first line reduces to $2\sqrt{c}$. Also, Bessel's differential equation for $J_0(z)$ is

$$\frac{d^2J_0(z)}{dz^2}+\frac{1}{z}\frac{dJ_0(z)}{dz}+J_0(z) = 0, \qquad \text{......(4)}$$

which becomes, on writing $z = ib$,

$$\frac{d^2J_0(ib)}{db^2}+\frac{1}{b}\frac{dJ_0(ib)}{db}-J_0(ib) = 0$$

or, since $I_0(b) = J_0(ib)$, by 2·42 (9),

$$\frac{d^2I_0(b)}{db^2}+\frac{1}{b}\frac{dI_0(b)}{db}-I_0(b) = 0. \qquad \text{......(5)}$$

The integral concerned thus reduces to

$$2\sqrt{c}+2\int_0^c \sqrt{c-b}\,e^{-b}\{I_0(b)+I_1(b)\}\,db.$$

We thus obtain

$$\frac{T_3}{\overline{T}} = 1+\frac{1}{\sqrt{c}}\int_0^c \sqrt{c-b}\,e^{-b}\{I_0(b)+I_1(b)\}\,db$$

or, since the expression on the right of the preceding equation is a function of η—we have, from (2), $2c = \eta^2$—we can write it as

$$T_3 = \overline{T}\chi(\eta), \qquad \text{......(6)}$$

where $$\chi(\eta) = 1+\frac{1}{\sqrt{c}}\int_0^c \sqrt{c-b}\,e^{-b}\{I_0(b)+I_1(b)\}\,db. \qquad \text{......(7)}$$

We shall later prove that $\chi(\eta)$ is identical with a function $\phi(\eta)$, given by

$$\phi(\eta) = \tfrac{1}{2}e^{-\eta^2}+\frac{1}{2\eta}(2\eta^2+1)K(\eta), \qquad \text{......(8)}$$

where $K(\eta)$ is defined by (13) of section 2·30. The values of $\phi(\eta)$ are given in Table 9, p. 51.

Meanwhile we shall assume that T_3 is given by

$$T_3 = \overline{T}\phi(\eta), \qquad \text{......(9)}$$

where $$\eta = hU \qquad \text{......(10)}$$

and U is the drift-velocity relative to the sun.

2·44. *The mean value of the observed radial speeds for a small area of the sky.*

We now consider the *arithmetical* values of the radial velocities—or the radial speeds—of n stars in the small area $\sin\lambda\, d\lambda\, d\phi$ at S (Fig. 8). Denote by dn the number of stars with haphazard radial velocities between R and $R+dR$. As before the w-axis (or the R-axis) is radial at S. Using the general frequency function $F(W)$ of section 2·1, we have

$$dn = n\,dR \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(W)\,du\,dv$$

in which the sum is taken for all possible values of u and of v between $-\infty$ and $+\infty$. The integral on the right will be a function of R (we have $W^2 = u^2+v^2+R^2$) and hence the above formula can be written

$$dn = CG(R)\,dR, \qquad \text{......(1)}$$

where C is a constant and it is assumed that $G(R)$ is an even function of R. Later we shall use the Maxwellian form, namely $e^{-h^2R^2}$, for $G(R)$, but meanwhile it is convenient to retain the functional form. It follows from (1) that

$$n = C\int_{-\infty}^{\infty} G(R)\,dR = 2C\int_{0}^{\infty} G(R)\,dR, \qquad \text{......(2)}$$

which is to be regarded as an equation to determine C when n and $G(R)$ are supposed known.

First consider the hemisphere in Fig. 8 for which $0 \leqslant \lambda \leqslant 90°$. The projection of the parallactic motion, U, on the w-axis (or R-axis) at S is $U\cos\lambda$. We denote it by ρ, so that

$$\rho = U\cos\lambda. \qquad \text{......(3)}$$

The observed radial velocity, R_0, of any star at S will thus be the sum of the parallactic component, ρ, and the haphazard radial velocity, R, and accordingly

$$R_0 = R+\rho. \qquad \text{......(4)}$$

In (4), ρ is a positive quantity for $0 \leqslant \lambda \leqslant 90°$, and R, being the haphazard radial velocity, can be positive or negative.

Let $2n_1$ denote the number of stars with random radial velocities such that $0 \leqslant |R| \leqslant \rho$; this is, of course, the number with radial speeds between 0 and ρ. Let $2n_2$ denote the number of stars with random radial velocities such that $|R| > \rho$. Then

$$n = 2(n_1+n_2). \qquad \text{......(5)}$$

From (1),

$$n_1 = C\int_{0}^{\rho} G(R)\,dR \qquad \text{......(6)}$$

and

$$n_2 = C\int_{\rho}^{\infty} G(R)\,dR. \qquad \text{......(7)}$$

We now form the sum $\Sigma \mid R_0 \mid$ for the n stars at S. The n stars are divided into three groups:

(i) (n_1+n_2) stars, for each of which R is positive.
R_0 is accordingly positive and

$$| R_0 | = | R | + \rho,$$

so that, summing for the (n_1+n_2) stars,

$$\sum_{n_1+n_2} | R_0 | = \sum_{R=0}^{\infty} | R | + (n_1+n_2)\rho. \qquad \text{......(8)}$$

(ii) n_1 stars, for each of which R is negative and $0 \leqslant | R | \leqslant \rho$.
R_0 is accordingly positive and therefore

$$| R_0 | = -| R | + \rho,$$

so that, summing for the n_1 stars,

$$\sum_{n_1} | R_0 | = -\sum_{R=0}^{\rho} | R | + n_1\rho. \qquad \text{......(9)}$$

(iii) n_2 stars, for each of which R is negative and $| R | > \rho$.
Thus R_0 is negative and consequently

$$-| R_0 | = -| R | + \rho$$

or

$$| R_0 | = | R | - \rho,$$

so that, summing for the n_2 stars,

$$\sum_{n_2} | R_0 | = \sum_{R=\rho}^{\infty} | R | - n_2\rho. \qquad \text{......(10)}$$

Hence, adding the results given by (8), (9) and (10), we have for the n stars at S

$$\sum_{n} | R_0 | = 2\sum_{R=\rho}^{\infty} | R | + 2n_1\rho. \qquad \text{......(11)}$$

Let R_2 denote the mean value of the observed radial speeds of the n stars; then

$$\sum_{n} | R_0 | = nR_2.$$

Also, since by (1) the number, dn, of stars with random radial velocities between R and $R+dR$ is $CG(R)\,dR$, we obtain the result

$$\sum_{R=\rho}^{\infty} | R | = C\int_{\rho}^{\infty} RG(R)\,dR.$$

Hence, using (6), the formula (11) becomes

$$nR_2 = 2C\int_{\rho}^{\infty} RG(R)\,dR + 2\rho C\int_{0}^{\rho} G(R)\,dR$$

or

$$nR_2 = 2C\int_{0}^{\infty} RG(R)\,dR + 2C\int_{0}^{\rho} (\rho - R)\,G(R)\,dR. \qquad \text{......(12)}$$

The mean random radial speed, $\bar{R}$, is given by

$$n\bar{R} = 2C\int_0^\infty RG(R)\,dR.$$

Also, if we set

$$Q = \int_0^\infty G(R)\,dR, \qquad \text{......(13)}$$

we have from (2)

$$n = 2CQ.$$

Hence (12) becomes

$$R_2 = \bar{R} + \frac{1}{Q}\int_0^\rho (\rho - R)\,G(R)\,dR. \qquad \text{......(14)}$$

It is evident that this formula also holds for a small region at S', antipodal to S. The mean haphazard radial speed, $\bar{R}$, is that investigated in section 2·21.

We shall now find the expression for R_2 in (14) when the random linear velocities are distributed according to the Maxwellian function

$$F(W) = e^{-h^2(u^2+v^2+w^2)}.$$

Then $G(R) = e^{-h^2R^2}$ and, by (1) of section 2·24,

$$\bar{R} = \frac{1}{h\sqrt{\pi}}.$$

Also, (13) gives

$$Q = \frac{\sqrt{\pi}}{2h} = \frac{1}{2h^2\bar{R}}.$$

Hence (14) becomes

$$\frac{R_2}{\bar{R}} = 1 + 2h^2\int_0^\rho (\rho - R)\,e^{-h^2R^2}\,dR.$$

Write

$$x = hR,$$

$$\xi = h\rho \equiv hU\cos\lambda. \qquad \text{......(15)}$$

Then we have

$$\frac{R_2}{\bar{R}} = 1 + 2\int_0^\xi (\xi - x)\,e^{-x^2}\,dx,$$

which we write in the form

$$R = \bar{R}F(\xi), \qquad \text{......(16)}$$

where it is easily seen that

$$F(\xi) = e^{-\xi^2} + 2\xi K(\xi). \qquad \text{......(17)}$$

It is to be noted that

$$\xi = \eta\cos\lambda, \qquad \text{......(18)}$$

where $\eta = hU$, according to (2) of section 2·43. Table 8 gives the values of $F(\xi)$.

Table 8. *Values of* $F(\xi)$

ξ	$F(\xi)$	ξ	$F(\xi)$	ξ	$F(\xi)$
0·00	1·000	0·70	1·453	1·40	2·503
0·05	1·003	0·75	1·515	1·45	2·588
0·10	1·010	0·80	1·579	1·50	2·673
0·15	1·022	0·85	1·646	1·55	2·759
0·20	1·040	0·90	1·716	1·60	2·845
0·25	1·062	0·95	1·788	1·65	2·932
0·30	1·089	1·00	1·861	1·70	3·019
0·35	1·120	1·05	1·936	1·75	3·106
0·40	1·156	1·10	2·014	1·80	3·194
0·45	1·196	1·15	2·093	1·85	3·282
0·50	1·240	1·20	2·173	1·90	3·370
0·55	1·288	1·25	2·254	1·95	3·458
0·60	1·340	1·30	2·336	2·00	3·546
0·65	1·395	1·35	2·419		
0·70	1·453	1·40	2·503		

2·45. *The mean value of the observed radial speeds for the whole sky.*

We shall again assume that the stars are distributed uniformly over the sphere so that, in practical applications of the formulae, the *means* of the observed quantities over each small region (of standard area) are to be taken, irrespective of the numbers of stars in these areas. As the results for antipodal areas S' are the same as for the areas S, we need consider only one of the hemispheres, of which the antapex, A, is the pole.

Let N denote the number of stars per unit area of the sphere. The number in the zone between λ and $\lambda+d\lambda$ (Fig. 8) is $2\pi N\sin\lambda\, d\lambda$, and the mean observed radial speed of these stars is R_2, given by (14) of section 2·44. The total number of stars in the hemisphere is $2\pi N$.

Let R_3 denote the mean observed radial velocity for the hemisphere. Then

$$2\pi N R_3 = 2\pi N \int_0^{\pi/2} R_2 \sin\lambda\, d\lambda.$$

Hence
$$R_3 = \bar{R} + \frac{1}{Q}\int_0^{\pi/2}\int_0^{\rho} (\rho - R)\, G(R) \sin\lambda\, d\lambda\, dR.$$

But $\rho = U\cos\lambda$; hence
$$\sin\lambda\, d\lambda = -\frac{1}{U}\, d\rho$$

and
$$R_3 = \bar{R} + \frac{1}{QU}\int_0^{U} d\rho \int_0^{\rho} (\rho - R)\, G(R)\, dR. \qquad \text{......(1)}$$

Now $d\rho \int_0^{\rho} (\rho - R)\, G(R)\, dR$ is the summation of the function $(\rho - R)\, G(R)$ over the strip AB (Fig. 11) of width $d\rho$, OQ bisecting the angle between the ρ-axis and the R-axis, and the double integral is the summation of the function over the triangle QOP in which $OP = PQ = U$. Changing the order of integration we sum first over the strip CD of width dR, that is between OE and U, thus obtaining

$$dR\int_{OE}^{U}(\rho-R)\,G(R)\,d\rho$$

or $$G(R)\,dR\int_{EC}^{U}(\rho-R)\,d\rho.$$

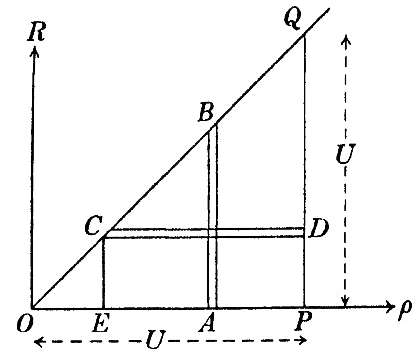

Fig. 11

We then sum over the triangle QOP, that is from $R=0$ to $R=U$. Thus the double integral is

$$\int_0^U G(R)\,dR\int_R^U(\rho-R)\,d\rho,$$

which is equivalent to

$$\tfrac{1}{2}\int_0^U(U-R)^2\,G(R)\,dR.$$

Hence $$R_3=\bar{R}+\frac{1}{2QU}\int_0^U(U-R)^2\,G(R)\,dR. \qquad \text{......(2)}$$

In the case of a Maxwellian distribution of random velocities, we have, as before,

$$G(R)=e^{-h^2R^2};\quad \bar{R}=\frac{1}{h\sqrt{\pi}};\quad Q=\frac{1}{2h^2\bar{R}}.$$

Also, writing as before $hU=\eta,\quad hR=x,$

(2) becomes $$R_3=\bar{R}\phi(\eta), \qquad \text{......(3)}$$

where $$\phi(\eta)=1+\frac{1}{\eta}\int_0^\eta(\eta-x)^2\,e^{-x^2}\,dx \qquad \text{......(4)}$$

or, in terms of the integral $K(\eta)\equiv\int_0^\eta e^{-x^2}\,dx,$

$$\phi(\eta)=\tfrac{1}{2}e^{-\eta^2}+\left(\eta+\frac{1}{2\eta}\right)K(\eta). \qquad \text{......(5)}$$

This result may be obtained directly from (1) as follows. We have

$$\frac{R_3}{\bar{R}}=1+\frac{1}{QU\bar{R}}\int_0^U d\rho\int_0^\rho(\rho-R)\,e^{-h^2R^2}\,dR.$$

Set $h\rho=\xi$ and, as before, $hR=x$, $hU=\eta$. Then

$$\frac{R_3}{\bar{R}}=1+\frac{2}{\eta}\int_0^\eta d\xi\int_0^\xi(\xi-x)\,e^{-x^2}\,dx$$

$$=1+\frac{2}{\eta}\int_0^\eta\{\xi K(\xi)+\tfrac{1}{2}e^{-\xi^2}-\tfrac{1}{2}\}\,d\xi$$

$$=\frac{1}{\eta}K(\eta)+\frac{2}{\eta}\int_0^\eta\xi K(\xi)\,d\xi.$$

But
$$2\int_0^\eta \xi K(\xi)\,d\xi = \Big[\xi^2 K(\xi)\Big]_0^\eta - \int_0^\eta \xi^2 \frac{dK}{d\xi}\,d\xi$$
$$= \eta^2 K(\eta) - \int_0^\eta \xi^2 e^{-\xi^2}\,d\xi$$
$$= \eta^2 K(\eta) + \tfrac{1}{2}\eta\, e^{-\eta^2} - \tfrac{1}{2}K(\eta),$$
and the result, (5), follows immediately.

The values of $\phi(\eta)$ are given in Table 9.

Table 9. *Values of* $\phi(\eta)$

η	$\phi(\eta)$	η	$\phi(\eta)$	η	$\phi(\eta)$
0·0	1·000	0·7	1·156	1·4	1·553
0·1	1·003	0·8	1·201	1·5	1·622
0·2	1·013	0·9	1·250	1·6	1·693
0·3	1·029	1·0	1·304	1·7	1·766
0·4	1·052	1·1	1·362	1·8	1·841
0·5	1·081	1·2	1·423	1·9	1·917
0·6	1·116	1·3	1·487	2·0	1·994
0·7	1·156	1·4	1·553		

2·46. *Proof of the identity* $\phi(\eta) \equiv \chi(\eta)$.

By (4) of section 2·45, we have
$$\phi(\eta) = 1 + \frac{1}{\eta}\int_0^\eta (\eta - x)^2 e^{-x^2}\,dx$$
$$= 1 + \frac{1}{\eta}\int_0^\eta (\eta - x)^2 \left\{\sum_0^\infty (-1)^r \frac{x^{2r}}{r!}\right\} dx$$
$$= 1 + \frac{1}{\eta}\int_0^\eta \sum_0^\infty \frac{(-1)^r}{r!}\{\eta^2 x^{2r} - 2\eta x^{2r+1} + x^{2r+2}\}\,dx.$$
The integrand is a uniformly convergent series. Hence
$$\phi(\eta) = 1 + \sum_0^\infty \frac{(-1)^r \eta^{2r+2}}{r!}\left\{\frac{1}{2r+1} - \frac{2}{2r+2} + \frac{1}{2r+3}\right\}$$
$$= 1 + 2\sum_0^\infty \frac{(-1)^r}{r!} \cdot \frac{\eta^{2r+2}}{(2r+1)(2r+2)(2r+3)}. \qquad \text{......(1)}$$
Now consider $\chi(\eta)$. From (3) and (6) of section 2·43, we have
$$\chi(\eta) = \int_0^{\pi/2} \sin\lambda\, \psi(b)\,d\lambda, \qquad \text{......(2)}$$
where
$$b = c\sin^2\lambda \quad \text{and} \quad c = \tfrac{1}{2}(hU)^2 = \tfrac{1}{2}\eta^2. \qquad \text{......(3)}$$
We shall first express $\psi(b)$ as a power series in b.

Since $\bar{T} = \dfrac{\sqrt{\pi}}{2h}$, we can write (5) of section 2·42 as
$$T_2 = \frac{\bar{T}}{\pi}\int_0^\pi e^{-2b\sin^2\theta}(1 + 4b\cos^2\theta)\,d\theta. \qquad \text{......(4)}$$

Also, by (10) of the same section,

$$T_2 = \bar{T}\psi(b). \qquad \ldots\ldots(5)$$

Hence we define $\psi(b)$ by means of (4) and (5) as

$$\psi(b) = \frac{2}{\pi}\int_0^{\pi/2} e^{-2b\sin^2\theta}\{(1+4b)-4b\sin^2\theta\}\,d\theta.$$

The integrand can be expanded into a uniformly convergent series and we obtain

$$\frac{\pi}{2}\psi(b) = \int_0^{\pi/2}\left[1+\sum_1^{\infty}(-1)^r(2b)^r\left\{\frac{\sin^{2r}\theta}{r!}-\frac{2\sin^{2r-2}\theta}{(r-1)!}+\frac{2\sin^{2r}\theta}{(r-1)!}\right\}\right]d\theta.$$

But
$$\int_0^{\pi/2}\sin^{2r}\theta\,d\theta = \frac{2r-1\,.\,2r-3\,.\ldots.\,1}{2r\,.\,2r-2\,.\ldots.\,2}\cdot\frac{\pi}{2}$$

$$= \frac{(2r)!}{2^{2r}(r!)^2}\cdot\frac{\pi}{2}.$$

Hence
$$\psi(b) = 1+\sum_1^{\infty}(-1)^r(2b)^r\left\{\frac{(1+2r)(2r)!}{2^{2r}(r!)^3}-\frac{2(2r-2)!}{2^{2r-2}\{(r-1)!\}^3}\right\}.$$

The right-hand side is seen to reduce to

$$1+2\sum_1^{\infty}(-1)^{r-1}\left(\frac{b}{2}\right)^r\frac{(2r-2)!}{(r-1)!\,(r!)^2}$$

and accordingly we can write

$$\psi(b) = 1+2\sum_0^{\infty}(-1)^r\left(\frac{b}{2}\right)^{r+1}\frac{(2r)!}{r!\{(r+1)!\}^2}. \qquad \ldots\ldots(6)$$

Insert now the series given by (6) in (2) and we have, putting $b = c\sin^2\lambda$,

$$\chi(\eta) = 1+2\int_0^{\pi/2}\sum_0^{\infty}(-1)^r\left(\frac{c}{2}\right)^{r+1}\frac{(2r)!}{r!\{(r+1)!\}^2}\sin^{2r+3}\lambda\,d\lambda.$$

Now
$$\int_0^{\pi/2}\sin^{2r+3}\lambda\,d\lambda = \frac{2r+2\,.\,2r\,.\,2r-2\,.\ldots.\,2}{2r+3\,.\,2r+1\,.\ldots.\,1} = \frac{2^{2r+2}\{(r+1)!\}^2}{(2r+3)!}.$$

Hence
$$\chi(\eta) = 1+2\sum_0^{\infty}(-1)^r\frac{(2c)^{r+1}}{(2r+3)(2r+2)(2r+1)\,.\,r!}$$

or, since $2c = \eta^2$ by (2) of section 2·43,

$$\chi(\eta) = 1+2\sum_0^{\infty}(-1)^r\frac{\eta^{2r+2}}{(2r+3)(2r+2)(2r+1)\,.\,r!} \qquad \ldots\ldots(7)$$

and the expression on the right is the same as that in (1); hence

$$\chi(\eta) = \phi(\eta)$$

and the identity is established.

It follows from the preceding sections that

$$R_3 : T_3 = \bar{R} : \bar{T}$$

and consequently that $$R_3 : T_3 = 2 : \pi. \qquad \ldots\ldots(8)$$

Thus the ratio of the mean observed radial speed to the mean observed transverse speed for the whole sky is the same as the ratio of the mean random radial velocity to the mean random transverse velocity in the drift. The result (8) seems to have been first stated explicitly by W. J. Luyten.*

2·51. *Proof of the relations,* $R_3 : T_3 : W_3 = 2 : \pi : 4$.

As in the previous sections, R_3 and T_3 denote respectively the mean observed radial and transverse speeds, taken over the whole sky, for a single drift of stars. We denote by W_3, in a similar way, the mean value of the observed total speeds, W_0, for the whole sky. The following proof of the formulae was given by F. J. W. Whipple.†

Consider the stars, n per unit area of the sky, with the *same* velocity W_0 relative to the sun and in the *same* direction. Let OZ (Fig. 12) be the given direction of W_0, the sun being at the centre, O, of the celestial sphere. Assuming uniform distribution of the stars over the sphere, we have that the number of stars with the given value of W_0 in the zone θ to $\theta + d\theta$ is $2\pi n \sin\theta \, d\theta$. But the radial velocity of these stars, observed with reference to the sun, is $W_0 \cos\theta$ and their transverse velocity is $W_0 \sin\theta$. Hence, if R_W and T_W denote the mean observed radial speed and the mean observed transverse speed respectively for all the stars of velocity W_0 in directions parallel to OZ, and taken over the hemisphere $0 \leqslant \theta \leqslant \pi/2$, we have—since the total number of such stars over the hemisphere is $2\pi n$—

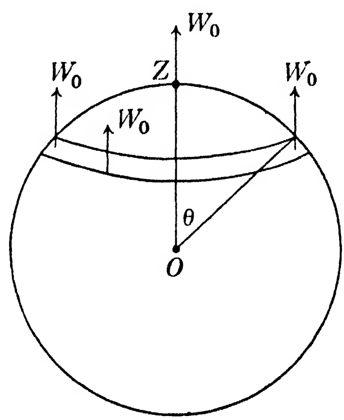

Fig. 12

$$2\pi n R_W = 2\pi n W_0 \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta$$

and $$2\pi n T_W = 2\pi n W_0 \int_0^{\pi/2} \sin^2\theta \, d\theta.$$

Hence $$R_W = \tfrac{1}{2} W_0 \qquad \ldots\ldots(1)$$

and $$T_W = \frac{\pi}{4} W_0. \qquad \ldots\ldots(2)$$

The hemisphere, $\pi/2 \leqslant \theta \leqslant \pi$, provides identical results and so the formulae (1) and (2) hold for the whole sphere.

* *Proc. Nat. Acad. of Sciences*, **11**, 192, 1925. † *M.N.* **95**, 442, 1935.

Averaging for all possible values of W_0 and for all directions we obtain from (1) and (2)

$$R_3 = \tfrac{1}{2} W_3 \qquad \text{......(3)}$$

and

$$T_3 = \frac{\pi}{4} W_3, \qquad \text{......(4)}$$

whence

$$R_3 : T_3 : W_3 = 2 : \pi : 4. \qquad \text{......(5)}$$

It is thus seen that the relations

$$\bar{R} : \bar{T} : \bar{W} = 2 : \pi : 4,$$

given by (2) of section 2·23, are a particular result of the more general formula (5).

2·52. *Whipple's proof of the formulae* $R_3 = \bar{R}\phi(\eta)$, $T_3 = \bar{T}\phi(\eta)$.

In previous sections the average observed transverse and radial speeds have been obtained for any small area of the sky (these results will be used in a later chapter) and the mean observed transverse and radial speeds for the whole sky follow by integrating over the celestial sphere. To obtain the results for the whole sky, Whipple* proceeds by integrating in a different order and by making use of the formulae of the preceding section.

Let W denote the haphazard linear velocity of a star, with components u, v and w. Then, with a Maxwellian distribution of velocities, the number of stars with velocity components between (u, v, w) and $(u+du, v+dv, w+dw)$ is

$$Ce^{-h^2(u^2+v^2+w^2)}\,du\,dv\,dw,$$

where C is a constant related to the total number, N, of the stars concerned.

Let the direction of W make an angle θ with the direction of the solar motion, which will be taken as the w-axis (Fig. 13), and write

$$u = W\cos\phi\sin\theta,$$
$$v = W\sin\phi\sin\theta,$$
$$w = W\cos\theta.$$

Then the number of stars with haphazard velocities between W and $W+dW$, in directions between θ and $\theta+d\theta$, ϕ and $\phi+d\phi$ is

$$Ce^{-h^2W^2}\frac{\partial(u, v, w)}{\partial(W, \theta, \phi)}\,dW\,d\theta\,d\phi$$

or $$Ce^{-h^2W^2}W^2\sin\theta\,dW\,d\theta\,d\phi.$$

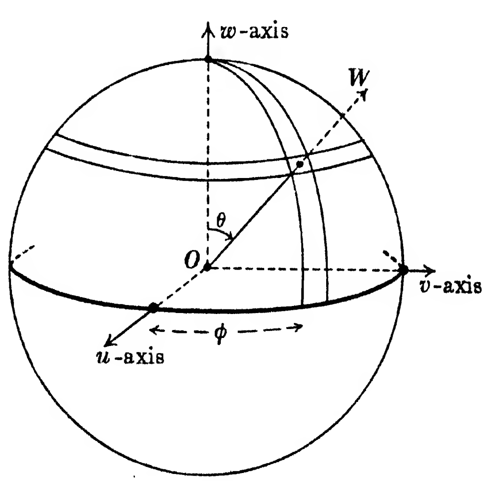

Fig. 13

* *Loc. cit.*

Hence, if dN denotes the number of stars with haphazard velocities between W and $W+dW$ and in directions lying between the cones θ and $\theta+d\theta$,

$$dN = 2\pi C e^{-h^2W^2}\, W^2 \sin\theta\, dW\, d\theta.$$

We thus obtain
$$N = 2\pi C \int_0^\infty W^2 e^{-h^2W^2}\, dW \int_0^\pi \sin\theta\, d\theta,$$

from which
$$C = \frac{Nh^3}{\pi^{3/2}},$$

so that
$$dN = \frac{2Nh^3}{\sqrt{\pi}} \cdot e^{-h^2W^2}\, W^2\, dW \sin\theta\, d\theta.$$

If W_0 denotes the velocity, relative to the sun, of a star with haphazard velocity W, we have

$$W_0^2 = W^2 - 2WU\cos\theta + U^2,$$

where U is the solar motion.

Denoting, as before, by W_3 the mean speed, relative to the sun, for all stars in the drift, we obtain

$$W_3 = \frac{2h^3}{\sqrt{\pi}} \int_0^\infty \int_0^\pi (W^2 - 2WU\cos\theta + U^2)^{1/2} e^{-h^2W^2}\, W^2\, dW \sin\theta\, d\theta,$$

in which it is to be understood that the square root has the positive sign.

Integrating first with respect to θ, we have

$$W_3 = \frac{2h^3}{3U\sqrt{\pi}} \int_0^\infty \{(W+U)^3 - (W \sim U)^3\}\, e^{-h^2W^2}\, W\, dW.$$

But
$$\begin{aligned}
&\int_0^\infty \{(W+U)^3 - (W \sim U)^3\}\, e^{-h^2W^2}\, W\, dW \\
&= \int_0^U \{(W+U)^3 - (U-W)^3\}\, e^{-h^2W^2}\, W\, dW \\
&\quad + \int_U^\infty \{(W+U)^3 - (W-U)^3\}\, e^{-h^2W^2}\, W\, dW \\
&= 2\int_0^U (W^3 + 3WU^2)\, e^{-h^2W^2}\, W\, dW \\
&\quad + 2\int_U^\infty (3W^2U + U^3)\, e^{-h^2W^2}\, W\, dW \\
&= 2\int_0^\infty (3W^2U + U^3)\, e^{-h^2W^2}\, W\, dW \\
&\quad - 2\int_0^U (U-W)^3 e^{-h^2W^2}\, W\, dW.
\end{aligned}$$

Hence

$$W_3 = \frac{4h^3}{3\sqrt{\pi}} \left\{ \int_0^\infty (3W^3 + U^2W)\, e^{-h^2W^2}\, dW - \frac{1}{U}\int_0^U (U-W)^3 e^{-h^2W^2}\, W\, dW \right\}.$$

Write $$\frac{1}{h\sqrt{\pi}} = \bar{R}, \quad hU = \eta \text{ and } hW = x.$$

Then $$W_3 = \tfrac{4}{3}\bar{R}\left\{\int_0^\infty (3x^3+\eta^2 x)\, e^{-x^2} dx - \frac{1}{\eta}\int_0^\eta (\eta-x)^3 e^{-x^2} x\, dx\right\}.$$

But $$\int_0^\infty x^3 e^{-x^2}\, dx = \int_0^\infty x e^{-x^2}\, dx = \tfrac{1}{2}$$

and $$\int_0^\eta (\eta-x)^3\, e^{-x^2} x\, dx = -\left[\frac{1}{2}(\eta-x)^3 e^{-x^2}\right]_0^\eta - \frac{3}{2}\int_0^\eta (\eta-x)^2 e^{-x^2} dx$$
$$= \frac{1}{2}\eta^3 - \frac{3}{2}\int_0^\eta (\eta-x)^2 e^{-x^2} dx.$$

Hence we obtain $$W_3 = 2\bar{R}\left\{1 + \frac{1}{\eta}\int_0^\eta (\eta-x)^2 e^{-x^2} dx\right\}.$$

But the expression within the parentheses is $\phi(\eta)$ by (4) of section 2·45; consequently $$W_3 = 2\bar{R}\phi(\eta).$$

But from (5) of section 2·51, $$R_3 = \tfrac{1}{2}W_3.$$

Hence $$R_3 = \bar{R}\phi(\eta).$$

Similarly $$T_3 = \frac{\pi}{2}\bar{R}\phi(\eta).$$

But $$\bar{T} = \frac{\pi}{2}\bar{R}.$$

Hence $$T_3 = \bar{T}\phi(\eta).$$

But by (6) of section 2·43, $$T_3 = \bar{T}\chi(\eta)$$

and we accordingly have the result that $\chi(\eta) = \phi(\eta)$. Whipple's procedure thus contains implicitly the proof of the identity of the functions $\phi(\eta)$ and $\chi(\eta)$, as given by the expressions 2·43 (7) and 2·45 (4), which we proved directly in section 2·46.

2·61. *The frequency function of the observed transverse velocities for the whole sky.*

The principal formula of this section was given by W. J. Luyten* in a paper on "The mathematical expression of the law of tangential velocities". The following is a modification of his work.

From (4) of section 2·3, if there are n stars in a small area of the sky at an angular distance λ from the solar antapex (Fig. 8), the number dn of stars

* *Proc. Nat. Acad. of Sciences*, **11**, 87, 1925.

with transverse linear components between T and $T+dT$, relative to the sun, and in the sector $\theta, \theta+d\theta$ is given by

$$dn = \frac{nh^2}{\pi} e^{-h^2(T^2-2TV\cos\theta+V^2)}\, T\,dT\,d\theta,$$

where $$V = U\sin\lambda, \qquad \text{......(1)}$$

U being the solar motion. Hence, if dn_1 denotes the number of observed transverse velocities between T and $T+dT$ for all values of θ,

$$dn_1 = \frac{2nh^2}{\pi} e^{-h^2(T^2+V^2)}\, T\,dT \int_0^\pi e^{2h^2TV\cos\theta}\, d\theta$$

or, in terms of the modified Bessel function of zero order, by 2·42 (15),

$$dn_1 = 2nh^2 e^{-h^2(T^2+V^2)} I_0(2h^2TV)\, T\,dT. \qquad \text{......(2)}$$

This formula holds for all small areas in the zone between λ and $\lambda+d\lambda$ and it will accordingly hold for the whole zone between λ and $\lambda+d\lambda$, whose area is $2\pi\sin\lambda\, d\lambda$.

Let N be the number of stars per unit area of the sky, so that $4\pi N$ is the total number over the sky. If n, dn and dn_1 now refer to the area of the zone between λ and $\lambda+d\lambda$, we have

$$n = 2\pi N \sin\lambda\, d\lambda.$$

Hence (2) becomes, using (1),

$$dn_1 = 4\pi N h^2 \sin\lambda\, d\lambda\, e^{-h^2(T^2+U^2\sin^2\lambda)} I_0(2h^2TU\sin\lambda)\, T\,dT.$$

If $\Psi(T)$ denotes the frequency function for the observed transverse velocities over the whole sky, so that $\Psi(T)\,dT$ is the proportion of stars with transverse velocities between T and $T+dT$, we have

$$4\pi N\Psi(T)\,dT = 4\pi N h^2\, T\,dT \int_0^\pi \sin\lambda\, e^{-h^2(T^2+U^2\sin^2\lambda)} I_0(2h^2TU\sin\lambda)\, d\lambda.$$

We write $$hT = z, \quad hU = \eta$$

and $$2h^2TU\sin\lambda \equiv 2\eta z\sin\lambda = \xi.$$

Then
$$\begin{aligned} I_0(2h^2TU\sin\lambda) &\equiv J_0(i\xi) \\ &= 1 + \frac{(\xi/2)^2}{1^2} + \frac{(\xi/2)^4}{1^2 . 2^2} + \dots \\ &= \sum_{j=0}^{\infty} \frac{\eta^{2j} z^{2j} \sin^{2j}\lambda}{(j\,!)^2}. \end{aligned}$$

Hence $$\Psi(T) = h^2 T e^{-z^2} \int_0^\pi e^{-\eta^2\sin^2\lambda} \left\{ \sum_0^\infty \frac{\eta^{2j} z^{2j} \sin^{2j+1}\lambda}{(j\,!)^2} \right\} d\lambda,$$

which we shall write in the form

$$\Psi(T) = 2h^2 T e^{-z^2} \sum_0^\infty B_j z^{2j}, \qquad \text{......(3)}$$

where $$B_j = \frac{\eta^{2j}}{(j\,!)^2} \int_0^{\pi/2} e^{-\eta^2\sin^2\lambda} \sin^{2j+1}\lambda\, d\lambda.$$

Expanding the exponential function in the previous formula, we obtain

$$B_j = \frac{\eta^{2j}}{(j\,!)^2}\int_0^{\pi/2}\left\{\sin^{2j+1}\lambda - \frac{\eta^2}{1\,!}\sin^{2j+3}\lambda + \frac{\eta^4}{2\,!}\sin^{2j+5}\lambda - \ldots\right\}d\lambda$$

$$= \frac{\eta^{2j}}{(j\,!)^2}\left[\frac{2j\,.\,2j-2\,.\ldots 2}{2j+1\,.\,2j-1\,.\ldots 1} - \frac{2j+2\,.\,2j\,.\ldots 2}{2j+3\,.\,2j+1\,.\ldots 1}\,.\,\frac{\eta^2}{1\,!}\right.$$

$$\left. + \frac{2j+4\,.\,2j+2\,.\ldots 2}{2j+5\,.\,2j+3\,.\ldots 1}\,.\,\frac{\eta^4}{2\,!} - \ldots\right]$$

and, finally,

$$B_j = \frac{(2\eta)^{2j}}{(2j+1)\,!}\left\{1 - \frac{2(j+1)}{2j+3}\,.\,\frac{\eta^2}{1\,!} + \frac{2^2(j+1)(j+2)}{(2j+3)(2j+5)}\,.\,\frac{\eta^4}{2\,!} - \ldots\right\}. \qquad \ldots\ldots(4)$$

The formulae (3) and (4) define the frequency function.

Since the observed transverse velocities, T, are signless—(2) has been obtained, effectively, by integration with respect to θ from $\theta = 0$ to $\theta = 2\pi$ —the definition of $\Psi(T)$ gives

$$\int_0^\infty \Psi(T)\,dT = 1$$

and hence, using (3),

$$2\int_0^\infty z\,e^{-z^2}\left(\sum_0^\infty B_j z^{2j}\right)dz = 1. \qquad \ldots\ldots(5)$$

Since

$$\int_0^\infty z^{2j+1}e^{-z^2}\,dz = \frac{j\,!}{2},$$

(5) becomes

$$\sum_0^\infty j\,!\,B_j = 1, \qquad \ldots\ldots(6)$$

which is the relation connecting the coefficients, B_j.

2·62. *The mean value, T_3, of the observed transverse speeds for the whole sky.*

The results of the previous section can be employed to derive the formula for T_3, the mean value of the observed transverse speeds for all the stars of the drift scattered uniformly over the sky. We have

$$4\pi N T_3 = 4\pi N\int_0^\infty T\Psi(T)\,dT,$$

which, by (3) of section 2·61, becomes

$$T_3 = \frac{2}{h}\int_0^\infty e^{-z^2}\left(\sum_0^\infty B_j z^{2j+2}\right)dz. \qquad \ldots\ldots(1)$$

Now

$$\int_0^\infty z^{2j+2}e^{-z^2}\,dz = \frac{\sqrt{\pi}}{2}\,.\,\frac{1\,.\,3\,.\,5\,.\ldots 2j+1}{2^{j+1}}$$

$$= \frac{\sqrt{\pi}}{2}\,.\,\frac{(2j+2)\,!}{2^{2j+2}(j+1)\,!}$$

and, using the formula $$\bar{T} = \frac{\sqrt{\pi}}{2h},$$

we obtain $$T_3 = 2\bar{T}\sum_0^\infty \frac{(2j+2)!}{2^{2j+2}(j+1)!}B_j.$$

Inserting the value of B_j given by (4) of the previous section, we have

$$T_3 = \bar{T}\sum_0^\infty \frac{\eta^{2j}}{j!}\left\{1 - \frac{2j+2}{2j+3}\cdot\frac{\eta^2}{1!} + \frac{2j+2\,.\,2j+4}{2j+3\,.\,2j+5}\cdot\frac{\eta^4}{2!} - \dots\right\}. \qquad \dots\dots(2)$$

It is easily found that, up to the term in η^6,

$$T_3 = \bar{T}\left\{1 + \frac{\eta^2}{1\,.\,3} - \frac{\eta^4}{3\,.\,5\,.\,2!} + \frac{\eta^6}{5\,.\,7\,.\,3!}\right\}. \qquad \dots\dots(3)$$

This result was given by Luyten* and it is readily verified that the series on the right of (3) is identical with the corresponding terms of the expression of $\chi(\eta)$ in series form in (7) of section 2·46.

2·71. *The representation of the formula for a drift-curve by a Fourier Series.*

By (11) of section 2·3 the radius vector, ρ, of a drift-curve, which is inclined at an angle θ to the axis of symmetry, is given by

$$\rho = \frac{n}{2\sqrt{\pi}}e^{-h^2V^2}f(\tau)$$

or, using the form of $f(\tau)$ in (12) of section 2·3,

$$\rho = \frac{n}{\pi}e^{-h^2V^2}\left[\tfrac{1}{2} + \tau e^{\tau^2}\left\{\frac{\sqrt{\pi}}{2} + K(\tau)\right\}\right], \qquad \dots\dots(1)$$

where $$\tau = hV\cos\theta, \qquad \dots\dots(2)$$

$$K(\tau) = \int_0^\tau e^{-x^2}\,dx, \qquad \dots\dots(3)$$

and n is the number of the stars forming the drift. Following Eddington,† to whom the following analysis is due, we write, by Fourier's theorem, since ρ is a function of θ,

$$\rho = \frac{n}{\pi}B + \frac{n}{\pi}C\cos\theta + \frac{n}{\pi}D\cos 2\theta + \frac{n}{\pi}E\cos 3\theta + \frac{n}{\pi}F\cos 4\theta + \dots$$
$$+ B_1\sin\theta + B_2\sin 2\theta + \dots,$$

where $$\frac{n}{\pi}B = \frac{1}{2\pi}\int_0^{2\pi}\rho\,d\theta \quad\text{or}\quad nB = \tfrac{1}{2}\int_0^{2\pi}\rho\,d\theta, \qquad \dots\dots(4)$$

$$nC = \int_0^{2\pi}\rho\cos\theta\,d\theta, \qquad \dots\dots(5)$$

* *Proc. Nat. Acad. of Sciences*, **11**, 90, 1925.
† *M.N.* **68**, 588, 1908.

$$nD = \int_0^{2\pi} \rho \cos 2\theta d\theta, \qquad \ldots\ldots(6)$$

$$nE = \int_0^{2\pi} \rho \cos 3\theta d\theta, \qquad \ldots\ldots(7)$$

$$nF = \int_0^{2\pi} \rho \cos 4\theta d\theta \qquad \ldots\ldots(8)$$

and
$$B_j = \frac{1}{\pi} \int_0^{2\pi} \rho \sin j\theta d\theta.$$

The coefficients B_j all vanish owing to the symmetry of the drift-curve. We then have

$$\rho = \frac{n}{\pi}\{B + C\cos\theta + D\cos 2\theta + E\cos 3\theta + F\cos 4\theta + \ldots\} \qquad \ldots\ldots(9)$$

and we require to evaluate the coefficients $B, C, \ldots F$.

(i) *Evaluation of B.*

We have
$$\int_0^{2\pi} \rho d\theta = n.$$

Hence, by (4),
$$B = \tfrac{1}{2}.$$

(ii) *Evaluation of C.*

From (5), using (1), we have

$$nC = \frac{n}{\pi} e^{-h^2V^2} \int_0^{2\pi} \cos\theta \left[\tfrac{1}{2} + \tau e^{\tau^2}\left\{\frac{\sqrt{\pi}}{2} + K(\tau)\right\}\right] d\theta,$$

where τ is given by (2). We shall write, as in previous sections,

$$h^2V^2 = 2b. \qquad \ldots\ldots(10)$$

Then
$$C = \frac{hV}{\pi} e^{-2b} \int_0^{2\pi} \cos^2\theta \, e^{2b\cos^2\theta} \left\{\frac{\sqrt{\pi}}{2} + K(\tau)\right\} d\theta.$$

Now
$$L \equiv \int_0^{2\pi} \cos^2\theta \, e^{2b\cos^2\theta} K(\tau) \, d\theta$$

$$= 2\int_0^{\pi/2} \cos^2\theta \, e^{2b\cos^2\theta} K(\tau) \, d\theta + 2\int_{\pi/2}^{\pi} \cos^2\theta \, e^{2b\cos^2\theta} K(\tau) \, d\theta$$

and, writing $(\pi - \theta)$ for θ in the last integral, it becomes

$$2\int_0^{\pi/2} \cos^2\theta \, e^{2b\cos^2\theta} K(-\tau) \, d\theta.$$

Also $K(-\tau) = -K(\tau)$; hence $L = 0$.

We then have

$$C = \frac{hV}{2\sqrt{\pi}} e^{-2b} \int_0^{2\pi} \cos^2\theta \, e^{2b\cos^2\theta} \, d\theta \qquad \ldots\ldots(11)$$

or
$$C = \frac{hV}{\sqrt{\pi}} e^{-b} \int_0^{\pi} \cos^2\theta \, e^{b\cos 2\theta} \, d\theta. \qquad \ldots\ldots(12)$$

Now, by 2·42 (12),
$$\int_0^\pi e^{b\cos 2\theta}\,d\theta = \pi I_0(b) \qquad \ldots\ldots(13)$$
and consequently
$$\int_0^\pi e^{2b\cos^2\theta}\,d\theta = \pi e^b I_0(b). \qquad \ldots\ldots(14)$$
From (13), by differentiation with respect to b, we derive
$$2\int_0^\pi \cos^2\theta\, e^{2b\cos^2\theta}\,d\theta = \pi e^b\left\{I_0(b)+\frac{dI_0(b)}{db}\right\}$$
$$= \pi e^b\{I_0(b)+I_1(b)\}$$
on using 2·42 (14).

Hence we obtain from (11) and the last result
$$C = \frac{\sqrt{\pi}}{2}hVe^{-b}\{I_0(b)+I_1(b)\}, \qquad \ldots\ldots(15)$$
in which $b = \frac{1}{2}h^2V^2$. We can now readily calculate C from (15).

(iii) *Evaluation of D.*

We have, from (6),
$$nD = \int_0^{2\pi} \rho\cos 2\theta\,d\theta$$
$$= 2\int_0^{2\pi}\rho\cos^2\theta\,d\theta - \int_0^{2\pi}\rho\,d\theta$$
$$= 4\int_0^{\pi}\rho\cos^2\theta\,d\theta - n. \qquad \ldots\ldots(16)$$
Referring to 2·3 (5) we see that ρ is defined by
$$\rho = \frac{nh^2}{\pi}\int_0^\infty e^{-h^2(T^2+V^2-2TV\cos\theta)}\,T\,dT, \qquad \ldots\ldots(17)$$
whence
$$\int_0^\pi \rho\cos^2\theta\,d\theta = \frac{nh^2}{\pi}\int_0^\infty\int_0^\pi e^{-h^2(T^2+V^2-2TV\cos\theta)}\cos^2\theta\,.\,T\,dT\,d\theta$$
$$= \frac{nh^2}{\pi}\int_0^\infty e^{-h^2(T^2+V^2)}\left\{\int_0^\pi \cos^2\theta\, e^{2h^2TV\cos\theta}\,d\theta\right\}T\,dT. \qquad \ldots\ldots(18)$$
Write $2h^2TV = z$. Then the integral with regard to θ in (18) is
$$\int_0^\pi \cos^2\theta\, e^{z\cos\theta}\,d\theta. \qquad \ldots\ldots(19)$$
Now by 2·42 (15),
$$\int_0^\pi e^{z\cos\theta}\,d\theta = \pi I_0(z),$$
from which
$$\int_0^\pi \cos^2\theta\, e^{z\cos\theta}\,d\theta = \pi\frac{d^2I_0(z)}{dz}.$$

Hence $$\int_0^\pi \rho\cos^2\theta\,d\theta = nh^2 e^{-2b}\int_0^\infty e^{-h^2T^2}\frac{d^2I_0(z)}{dz^2}\,T\,dT.$$

But by 2·43 (5), $$\frac{d^2I_0(z)}{dz^2} = I_0(z)-\frac{1}{z}\frac{dI_0(z)}{dz}.$$

Hence

$$\int_0^\pi \rho\cos^2\theta\,d\theta = nh^2 e^{-2b}\int_0^\infty e^{-h^2T^2}\left\{TI_0(z)-\frac{1}{4h^4V^2}\frac{d}{dT}I_0(z)\right\}dT. \qquad \ldots(20)$$

Also, from (17),

$$n = 2\int_0^\pi \rho\,d\theta = \frac{2nh^2}{\pi}e^{-2b}\int_0^\infty e^{-h^2T^2}\left\{\int_0^\pi e^{z\cos\theta}\,d\theta\right\}T\,dT$$

or $$n = 2nh^2 e^{-2b}\int_0^\infty e^{-h^2T^2}I_0(z)\,T\,dT. \qquad \ldots\ldots(21)$$

Hence, from (20) and (21),

$$2\int_0^\pi \rho\cos^2\theta\,d\theta = n-\frac{n}{2h^2V^2}e^{-2b}\int_0^\infty e^{-h^2T^2}\frac{d}{dT}I_0(z)\,dT. \qquad \ldots\ldots(22)$$

Now,

$$\int_0^\infty e^{-h^2T^2}\frac{d}{dT}I_0(z)\,dT = \Big[e^{-h^2T^2}I_0(z)\Big]_0^\infty + 2h^2\int_0^\infty e^{-h^2T^2}I_0(z)\,T\,dT. \qquad \ldots\ldots(23)$$

The value of the integrated part on the right of (23) is -1 and the integral is given by (21). We thus obtain from (22)

$$2\int_0^\pi \rho\cos^2\theta\,d\theta = n+\frac{ne^{-2b}}{2h^2V^2}-\frac{n}{2h^2V^2}$$

and finally, from (16), $$D = 1-\frac{1}{h^2V^2}(1-e^{-h^2V^2}). \qquad \ldots\ldots(24)$$

(iv) *Evaluation of E and F.*

By following the previous methods the expressions for E and F can be easily obtained. The results are

$$E = \frac{\sqrt{\pi}}{2}hVe^{-b}\left\{I_0(b)+\left(1-\frac{2}{b}\right)I_1(b)\right\} \qquad \ldots\ldots(25)$$

or, perhaps more simply, $$E = C-\frac{2\sqrt{\pi}}{hV}e^{-b}I_1(b) \qquad \ldots\ldots(26)$$

and $$F = 1-\frac{2}{h^4V^4}\{(3+h^2V^2)e^{-h^2V^2}+2h^2V^2-3\} \qquad \ldots\ldots(27)$$

or $$F = \frac{1}{b}(b+1-e^{-2b}-3D). \qquad \ldots\ldots(28)$$

F is easily expressed as a series in hV; thus, writing $(hV)^2 = x$, we obtain

$$F = \frac{2x^2}{4!}-\frac{4x^3}{5!}+\frac{6x^4}{6!}-\frac{8x^5}{7!}+\ldots, \qquad \ldots\ldots(29)$$

which converges rapidly for values of x less than unity; (29) is a convenient form for calculating F for values of hV between 0 and 1. For larger values, it is better to obtain the values of F from (28), the values of D having previously been calculated.

The values of C, D, E and F are given in Table 16, p. 127, for values of hV between 0 and 2·0—the values of C, D and E have been taken from Eddington's table.*

The principal formulae of this section will be employed in Chapter IV for the purpose of deriving analytically the constants of the two star-streams.

2·72. *General method of deriving the Fourier constants of a drift-curve.*

The following analysis has been given by A. Fletcher.†

We write the Fourier series for ρ in the form

$$\rho = \frac{n}{\pi}(\tfrac{1}{2}A_0 + A_1\cos\theta + A_2\cos 2\theta + \ldots). \qquad \ldots\ldots(1)$$

The identification of the constants in (1) with those in the previous section is

$$\tfrac{1}{2}A_0 = B = \tfrac{1}{2} \text{ (or } A_0 = 1), \quad A_1 = C, \quad A_2 = D, \quad A_3 = E, \quad A_4 = F.$$

Then, from (1),

$$\frac{n}{\pi}A_m = \frac{1}{\pi}\int_0^{2\pi}\rho\cos m\theta\,d\theta = \frac{2}{\pi}\int_0^{\pi}\rho\cos m\theta\,d\theta.$$

By 2·71 (17),
$$\rho = \frac{nh^2}{\pi}\int_0^{\infty} e^{-h^2(T^2+V^2-2TV\cos\theta)}\,T\,dT, \qquad \ldots\ldots(2)$$

so that
$$A_m = \frac{2h^2}{\pi}\int_0^{\pi}\cos m\theta\left\{\int_0^{\infty} e^{-h^2(T^2+V^2-2TV\cos\theta)}\,T\,dT\right\}d\theta$$

$$= \frac{2h^2}{\pi}e^{-h^2V^2}\int_0^{\infty} Te^{-h^2T^2}\left\{\int_0^{\pi} e^{2h^2VT\cos\theta}\cos m\theta\,d\theta\right\}dT. \qquad \ldots\ldots(3)$$

Now the modified Bessel function $I_m(x)$ is defined by

$$\pi I_m(x) = \int_0^{\pi} e^{x\cos\theta}\cos m\theta\,d\theta.$$

Hence, writing $t \equiv hT$ in (3), we have

$$A_m = 2e^{-h^2V^2}\int_0^{\infty} te^{-t^2}I_m(2hVt)\,dt.$$

As before, we write
$$b = \tfrac{1}{2}h^2V^2.$$

Then
$$A_m = 2e^{-2b}\int_0^{\infty} te^{-t^2}I_m(\sqrt{8b}\,t)\,dt. \qquad \ldots\ldots(4)$$

* *M.N.* 68, 592, 1908.
† *M.N.* 96, 877, 1936.

This integral is convergent at its lower limit provided $m > -2$. Also, we have the formulae*

$$I_{-\frac{1}{2}}(b) = \sqrt{\frac{2}{\pi b}} \cosh b,$$

$$I_{\frac{1}{2}}(b) = \sqrt{\frac{2}{\pi b}} \sinh b,$$

$$I_{\frac{3}{2}}(b) = \sqrt{\frac{2}{\pi b}} \left(\cosh b - \frac{1}{b} \sinh b\right),$$

$$I_2(b) = I_0(b) - \frac{2}{b} I_1(b).$$

Using these in conjunction with the expressions for A_0, A_1, A_2 and A_3 ($A_1 \equiv C$ is given in section 2·71 by (15), $A_2 \equiv D$ by (24) and $A_3 \equiv E$ by (25)), we see that

$$A_0 = \sqrt{\tfrac{1}{2}\pi b}\, e^{-b}(I_{-\frac{1}{2}} + I_{\frac{1}{2}}) = 1,$$

$$A_1 = \sqrt{\tfrac{1}{2}\pi b}\, e^{-b}(I_0 + I_1),$$

$$A_2 = \sqrt{\tfrac{1}{2}\pi b}\, e^{-b}(I_{\frac{1}{2}} + I_{\frac{3}{2}}),$$

$$A_3 = \sqrt{\tfrac{1}{2}\pi b}\, e^{-b}(I_1 + I_2),$$

the argument of the I-functions being b.

The formulae just given suggest that the A's follow the general expression

$$A_m = \sqrt{\tfrac{1}{2}\pi b}\, e^{-b}\{I_{\frac{1}{2}(m-1)}(b) + I_{\frac{1}{2}(m+1)}(b)\}. \qquad \ldots\ldots(5)$$

Thus, referring to (4), we have to prove that

$$\int_0^\infty t e^{-t^2} I_m(\sqrt{8b}\,t)\, dt = \sqrt{\frac{\pi b}{8}}\, e^b \{I_{\frac{1}{2}(m-1)}(b) + I_{\frac{1}{2}(m+1)}(b)\}. \qquad \ldots\ldots(6)$$

Consider the integral ($\equiv L$) on the left-hand side of this equation, and write x for t^2. Then

$$L = \tfrac{1}{2}\int_0^\infty e^{-x} I_m(\sqrt{8bx})\, dx$$

$$= \tfrac{1}{2}\int_0^\infty e^{-x} \sum_{r=0}^\infty \frac{(2bx)^{r+\frac{1}{2}m}}{r!\,\Gamma(r+m+1)}\, dx$$

$$= \tfrac{1}{2}\sum_{r=0}^\infty \frac{(2b)^{r+\frac{1}{2}m}}{r!\,\Gamma(r+m+1)} \int_0^\infty e^{-x} x^{r+\frac{1}{2}m}\, dx$$

$$= \tfrac{1}{2}\sum_{r=0}^\infty \frac{(2b)^{r+\frac{1}{2}m}\,\Gamma(r+\frac{1}{2}m+1)}{r!\,\Gamma(r+m+1)}. \qquad \ldots\ldots(7)$$

* G. N. Watson, *Theory of Bessel Functions*, 53–55, 1922.

Consider now the right-hand side of (6). We use the following relation due to Kummer,*

$$e^{b}I_{\nu}(b)=\frac{1}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{(2b)^{r+\nu}\,\Gamma(r+\nu+\frac{1}{2})}{r!\,\Gamma(r+2\nu+1)}. \qquad \text{......(8)}$$

We thus have

$$\sqrt{2\pi b}\,e^{b}I_{\frac{1}{2}(m-1)}(b)=\sum_{r=0}^{\infty}\frac{(2b)^{r+\frac{1}{2}m}\,\Gamma(r+\frac{1}{2}m)}{r!\,\Gamma(r+m)} \qquad \text{......(9)}$$

and

$$\sqrt{2\pi b}\,e^{b}I_{\frac{1}{2}(m+1)}(b)=\sum_{r=0}^{\infty}\frac{r(2b)^{r+\frac{1}{2}m}\,\Gamma(r+\frac{1}{2}m)}{r!\,\Gamma(r+m+1)}. \qquad \text{......(10)}$$

Adding (9) and (10), we obtain

$$\begin{aligned}\sqrt{2\pi b}\,e^{b}\{I_{\frac{1}{2}(m-1)}(b)+I_{\frac{1}{2}(m+1)}(b)\}&=\sum_{r=0}^{\infty}\frac{(2b)^{r+\frac{1}{2}m}\,\Gamma(r+\frac{1}{2}m)}{r!\,\Gamma(r+m+1)}\{(r+m)+r\}\\&=2\sum_{r=0}^{\infty}\frac{(2b)^{r+\frac{1}{2}m}\,\Gamma(r+\frac{1}{2}m+1)}{r!\,\Gamma(r+m+1)}.\end{aligned} \qquad \text{......(11)}$$

We thus see from (7) and (11) that the general relation (5) has been established.

The Fourier coefficient A_m can be readily expressed in the form of a series by means of the companion formula of (8), namely,

$$e^{-b}I_{\nu}(b)=\frac{1}{\sqrt{\pi}}\sum_{r=0}^{\infty}(-1)^{r}\frac{(2b)^{r+\nu}\,\Gamma(r+\nu+\frac{1}{2})}{r!\,\Gamma(r+2\nu+1)}.$$

We then obtain, by a procedure similar to that in deriving (11),

$$\begin{aligned}\sqrt{2\pi b}\,e^{-b}\{I_{\frac{1}{2}(m-1)}(b)+I_{\frac{1}{2}(m+1)}(b)\}&=\sum_{r=0}^{\infty}(-1)^{r}\frac{(2b)^{r+\frac{1}{2}m}\,\Gamma(r+\frac{1}{2}m)}{r!\,\Gamma(r+m+1)}\{(r+m)-r\},\end{aligned}$$

whence, by (5),

$$A_m=\frac{m}{2}\sum_{r=0}^{\infty}(-1)^{r}\frac{(hV)^{2r+m}\,\Gamma(r+\frac{1}{2}m)}{r!\,\Gamma(r+m+1)}. \qquad \text{......(12)}$$

This series is convenient when m is even. Thus, for $m = 4$, we readily find that

$$A_4=\frac{2}{4!}(hV)^4-\frac{4}{5!}(hV)^6+\frac{6}{6!}(hV)^8-\dots.$$

* Watson, *Theory of Bessel Functions*, 191, 1922. The formula (8) may be derived for $\nu > -\frac{1}{2}$, which is all that matters in the present connection, from the formula

$$I_{\nu}=\frac{(\frac{1}{2}b)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^{\pi}e^{b\cos\theta}\sin^{2\nu}\theta\,d\theta.$$

We then have

$$\begin{aligned}e^{b}I_{\nu}(b)&=\frac{2(\frac{1}{2}b)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^{\pi}e^{2b\cos^2(\theta/2)}2^{2\nu}\sin^{2\nu}\frac{\theta}{2}\cos^{2\nu}\frac{\theta}{2}\frac{d\theta}{2}\\&=\frac{2^{\nu+1}(\frac{1}{2}b)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^{\pi/2}e^{2b\cos^2\phi}\sin^{2\nu}\phi\cos^{2\nu}\phi\,d\phi.\end{aligned}$$

Expand the exponential in the integral and integrate term by term; the result (8) is then obtained. Similarly, the series for $e^{-b}I_{\nu}$, used in establishing (12), can be obtained.

CHAPTER III

THE SOLAR MOTION

3·11. *Definition of the solar motion.*

We shall consider a group of stars—in the ideal case, scattered over the sky—and we shall suppose that for each star the proper motion has been observed or that, alternatively, the radial velocity of each star is known. The group consists, consequently, of stars which depend for their choice on the capabilities of the instruments used for the measurement of either proper motion or radial velocity. For example, the proper motions of the naked-eye stars of Boss's *Preliminary General Catalogue* as faint as the sixth magnitude have been determined with great accuracy from meridian-circle observations spread over several scores of years; a large proportion of these stars have also been observed spectroscopically for the determination of radial velocity. If the solar motion is to be derived from the proper motions of Boss's stars, the magnitude and direction of the solar motion are to be defined with reference to this particular group of stars and to no other; in the same way, if we employ the radial velocity measures of stars of a particular catalogue the solar motion is to be defined strictly with reference to such stars.

Consider a group of N stars and let the coordinates of a star be (x, y, z) with reference to rectangular axes through a particular point as origin. We can clearly choose the origin so that

$$\Sigma x = \Sigma y = \Sigma z = 0, \qquad \text{......(1)}$$

and we define this point as the *geometrical centre* of the group. The geometrical centre is evidently the same as the centre of mass, or centroid, of the group if all the stars are all of the same mass. Frequently, the geometrical centre is referred to as the centroid but this latter term must not be identified, in this connection, with the centre of mass.

Let (U, V, W) denote the rectangular components of motion of any star with reference to axes fixed in direction and passing through the geometrical centre. Then, by (1),

$$\Sigma U = \Sigma V = \Sigma W = 0. \qquad \text{......(2)}$$

Let (ξ, η, ζ) be the components of the sun's motion with respect to the axes considered. Further, let (u, v, w) denote the components of the star's motion relative to the sun and with respect to axes parallel to those of the first system. We then have

$$U = u+\xi, \quad V = v+\eta, \quad W = w+\zeta, \qquad \text{......(3)}$$

and summing for the N stars of the group and using (2) we obtain

$$\Sigma u + N\xi = 0, \quad \Sigma v + N\eta = 0, \quad \Sigma w + N\zeta = 0. \qquad \ldots\ldots(4)$$

In these equations, the components (u, v, w) of a stellar velocity, relative to the sun, can only be found—see section 1·33, where the components are defined for the usual equatorial system of axes—if the components, μ_α and μ_δ, of the proper motion, the star's radial velocity and the parallax, p, are all known.

It is to be remarked that the assembly of stars under consideration need not form a "drift" in the technical sense in which this term has been used in Chapter II; in other words, we are not concerned with a particular law governing the distribution of the velocity vectors (U, V, W).

As remarked previously, there is no observational evidence that the motion of any given single star is other than uniform and rectilinear and so the geometrical centre of the group will have, at least for several centuries, a uniform and rectilinear motion with reference, for example, to the centre of the whole galactic system; thus the geometrical centre forms a dynamically convenient point of reference to which the motion of the sun can be related.

As we shall see later, the equations (4) are readily adapted, with the addition of certain assumptions, to the practical determination of the solar motion either from the proper motions alone or from the radial velocities alone.

3·12. *Herschel's investigation of the solar motion.*

Sir William Herschel* was the first to investigate the *direction* of the sun's motion. As judged by modern standards, he had at his disposal a very meagre amount of observational information—the proper motions of but thirteen stars, in all, were available when he made his first attack on the problem. Let us assume for the moment that each of these stars is at rest with reference to fixed axes as defined in the previous section and that the sun, S, alone is in motion, towards the solar apex (Fig. 14). A star, X, will consequently appear to have an equal linear motion, relative to the sun, in the opposite direction XA, that is to say, in the direction of the solar antapex. This apparent motion of the star will have a component along XB, perpendicular to the line of sight, and this transverse component will

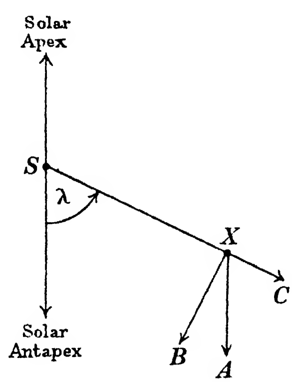

Fig. 14

* *Phil. Trans.* **73**, 247, 1783; *Collected Scientific Papers*, I, 108, 1912.

be observed as a proper motion along the great circle arc joining X to the position of the antapex on the celestial sphere. It is easy to see, in this case, that the amount of the proper motion varies inversely as the distance of the star from the sun and directly as $\sin\lambda$, where λ is the angular distance of the star from the antapex. On the hypothesis stated, the proper motions of all the stars in the group would be directed towards a definite point in the sky (the solar antapex) which could be simply ascertained from the observational data. However, the hypothesis is very far from the truth and, accordingly, the observed proper motion of a star is actually the combination of the effect produced by the reversed solar motion* and by the star's individual motion (or peculiar motion) with respect to the fixed axes. Nevertheless, the directions of the observed total proper motions should be expected to indicate a general convergence towards a particular point in the sky. This was the argument advanced by Herschel, and with the data at his disposal he placed the apex of the solar motion near the star λ Herculis.

Let the observed proper motion of a star, X, be along the great circle XY (Fig. 15) and let θ denote the position angle PXY; let λ be the angular distance of X from the antapex A, and χ the position angle of A with reference to X. Also, let AB ($\equiv d$) be the great circle arc drawn perpendicular from A to XY. Then

$$\sin d = \sin\lambda \sin(\theta-\chi). \qquad \ldots\ldots(1)$$

We have similar formulae for the other stars. Knowing θ for each star, we have to determine a point A such that the distribution of the values of d will indicate the maximum degree of convergency. If we regard the angles $(\theta-\chi)$, calculated for an assumed position of A, as of the nature of accidental errors, we choose that position of A for which Σd^2 is a minimum, applying the usual procedure in the theory of errors. In Herschel's time the theory of errors had not been developed and the criterion he applied in effect, although not expressed in mathematical language, concerned the choice of A for which Σd was a minimum, the length d of the great circle arc AB being reckoned positive in each instance.

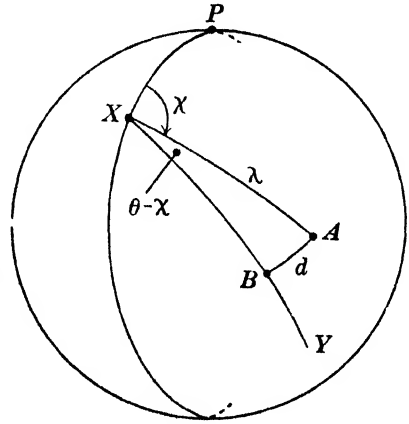

Fig. 15

Any method involving the use of formula (1) directly is inconvenient in practice, as it involves a vast amount of computation; we shall subsequently

* The component along XB (Fig. 14) of the reversed solar motion gives rise to the *parallactic proper motion*; the component along XC is the *parallactic radial velocity*.

derive other methods depending on the same basic idea and comparatively easy to apply, however abundant the observational material may be.

3·13. *A simple method of determining an approximate position of the solar apex.*

The method to be described appears to have been first given by Russell, Dugan and Stewart.* If we consider a large number of stars scattered over the sky, it is clear that for stars on the same meridian as the solar apex the general tendency of motion, so far as the proper motions are concerned, will be in declination; consequently, for this right ascension we should expect the number of positive values of μ_α to be the same as the number of negative values. For stars in right ascension, say, 2^h greater than that of the apex, the number of positive values of μ_α should exceed the number of negative values, since the position angle of the antapex, towards which the general tendency of motion takes place, for stars in this meridian is between 0° and 180°. Similarly for stars with right ascensions, say, 2^h less than that of the apex, the number of negative values of μ_α should be expected to exceed the number of positive values. Similar arguments apply to stars on meridians in the neighbourhood of that of the antapex.

Let now N_1 and N_2 denote, respectively, the number of positive and of negative values of μ_α for stars lying between two meridians, say, 1^h apart. Let P denote, algebraically, the relative preponderance of positive values over negative values, so that

$$P = \frac{N_1 - N_2}{N_1 + N_2}.$$

We can then find very easily, from the observed data, the values of P corresponding to different mean values, α, of the right ascension. Drawing a graph with the values of α as abscissae and the values of P as ordinates, we readily find the two values of the right ascension for which P vanishes. That value near which P changes from negative to positive is evidently the right ascension of the apex, and the other is the right ascension of the antapex. The two values should, theoretically, differ by 12^h, although in any application of the method this difference is not likely to be obtained exactly. A similar procedure with the values of μ_δ leads to an approximate determination of the declination of the apex.

Fig. 16 is adapted from the results of an investigation by F. K. Edmondson,† based on the proper motions of 7602 stars in Schlesinger's *Catalogue of Bright Stars*; the ordinates are the values of $100P$, corresponding to intervals of 1^h in right ascension. It will be seen that the right ascensions of the apex and of the antapex are close to 18^h and 6^h respectively. To fix the

* *Astronomy*, **2**, 659, 1927.

† *A.J.* **41**, 143, 1931.

right ascensions more accurately, Edmondson found the values of P, for intervals of 5 minutes in the right ascension, for some distance on either side of 6^h and of 18^h; and by drawing a straight line through the corresponding points of the graph so as to fit the observations as closely as possible, the right ascensions of the apex and antapex were found to be $18^h\,13^m$ and $6^h\,3^m$ respectively. If we apply the condition that the difference in these right

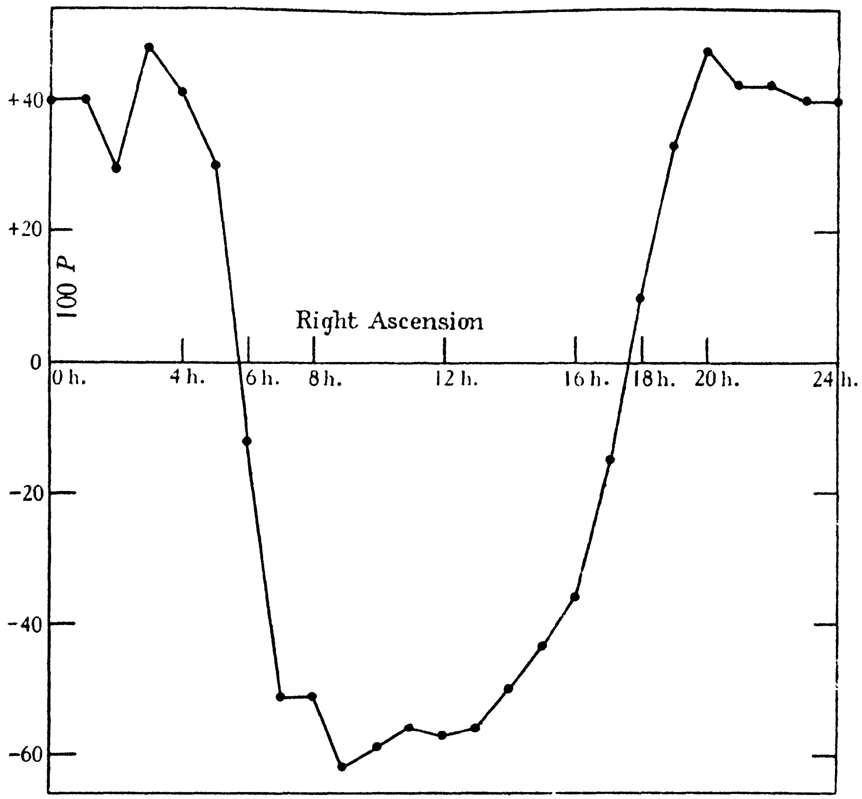

Fig. 16

ascensions should be 12^h, we may take the right ascension of the apex to be $18^h\,8^m$ or $272°$. The declination of the apex was found in a similar manner to be $+33°{\cdot}6$. It is to be noticed that the final determination of the right ascension of the apex depends only on stars within 30^m or so of the right ascension of the apex and of the antapex; a similar argument applies to the declinations. We add that this position of the apex is in very good agreement with that found by more general methods which take account of the proper motions of all the stars, irrespective of their positions on the celestial sphere.

3·21. *The method of Bravais.*

Bravais* first considered the solar motion as defined with reference to the centre of mass of the system of stars forming a selected group; but as stellar masses were entirely unknown in his time he was later compelled, in the application of his method, to suppose that the masses of the stars were all the same. This procedure, as we have seen, is equivalent to determining the solar motion with respect to the geometrical centre of the group—the conventional definition of the sun's motion.

We shall take, as the fundamental equations, the formulae (4) of section 3·11, namely:

$$\Sigma u + N\xi = 0, \quad \Sigma v + N\eta = 0, \quad \Sigma w + N\zeta = 0, \qquad \ldots\ldots(1)$$

in which, it may be recalled, (u, v, w) are the components of a star's linear velocity relative to the sun and, as we shall now assume, with respect to the usual equatorial system of axes; also (ξ, η, ζ) are the components of the solar motion, referred to parallel axes through the geometrical centre, and N is the number of stars in the group.

Let r denote the heliocentric distance of a star and (l, m, n) the direction-cosines of the heliocentric radius vector to the star referred to a parallel system of axes moving with the sun. The heliocentric coordinates of the star are then given by

$$x = lr, \quad y = mr, \quad z = nr.$$

Also
$$l = \cos\alpha\cos\delta, \quad m = \sin\alpha\cos\delta, \quad n = \sin\delta. \qquad \ldots\ldots(2)$$

Then, relative to the sun, we obtain

$$\dot{x} = \dot{l}r + l\dot{r}, \text{ etc.}$$

or
$$u = \dot{l}r + l\rho, \quad v = \dot{m}r + m\rho, \quad w = \dot{n}r + n\rho, \qquad \ldots\ldots(3)$$

where ρ denotes the star's radial velocity relative to the sun. Also

$$\left.\begin{aligned} \dot{l} &= -\mu_\alpha \sin\alpha\cos\delta - \mu_\delta\cos\alpha\sin\delta \\ \dot{m} &= \mu_\alpha\cos\alpha\cos\delta - \mu_\delta\sin\alpha\sin\delta \\ \dot{n} &= \mu_\delta\cos\delta \end{aligned}\right\}. \qquad \ldots\ldots(4)$$

In these expressions we shall assume that the unit of time is a year; consequently, μ_α and μ_δ are, for the present, the components of the observed proper motion expressed in circular measure. The formulae (1) and (3) then give

$$\left.\begin{aligned} N\xi &= -\Sigma\dot{l}r - \Sigma l\rho \\ N\eta &= -\Sigma\dot{m}r - \Sigma m\rho \\ N\zeta &= -\Sigma\dot{n}r - \Sigma n\rho \end{aligned}\right\}. \qquad \ldots\ldots(5)$$

* *Liouville's Journal*, **8**, 435. 1843.

or, from (4),

$$\left.\begin{aligned} N\xi &= \quad \Sigma r\mu_\alpha \sin\alpha\ \cos\delta + \Sigma r\mu_\delta \cos\alpha\ \sin\delta - \Sigma l\rho \\ N\eta &= -\Sigma r\mu_\alpha \cos\alpha\ \cos\delta + \Sigma r\mu_\delta \sin\alpha\ \sin\delta - \Sigma m\rho \\ N\zeta &= \qquad\qquad\qquad\qquad -\Sigma r\mu_\delta \cos\delta \qquad -\Sigma n\rho \end{aligned}\right\}. \qquad \ldots\ldots(6)$$

These equations, (6), are the equations of Bravais. They involve the three observable quantities μ_α, μ_δ and ρ and if, in addition, the heliocentric distances of the stars are known the equations are sufficient for determining the components of the solar motion and the direction in the sky in which the sun is travelling.

3·22. *Modification of the equations.*

In the time of Bravais, the spectroscopic method of measuring radial velocities had not even been foreshadowed; consequently, the equations (6) cannot be used with anything more than a knowledge of the observed proper motions—we leave over for the present the question raised by the appearance of the stellar distances, r, in the equations. An assumption as to the distribution of the linear velocities of the stars is evidently required. We proceed as follows.

Multiply the three equations in (3) of section 3·21 by l, m and n; adding, we obtain

$$\rho = lu + mv + nw, \qquad \ldots\ldots(1)$$

since

$$l^2 + m^2 + n^2 = 1$$

and

$$l\dot{l} + m\dot{m} + n\dot{n} \equiv \frac{1}{2}\frac{d}{dt}(l^2 + m^2 + n^2) = 0.$$

Also, if R denotes the peculiar linear velocity of a star, in the direction defined by the direction-cosines (l, m, n), with respect to fixed axes, we have similarly

$$R = lU + mV + nW, \qquad \ldots\ldots(2)$$

where (U, V, W) are the components of the star's linear velocity relative to the fixed axes. Hence, since $U = u + \xi$, etc., we obtain, from (1) and (2),

$$\rho = R - (l\xi + m\eta + n\zeta). \qquad \ldots\ldots(3)$$

Substitute this expression for ρ in (5) of section 3·21. Then

$$\left.\begin{aligned} \xi\,\Sigma(1-l^2) - \eta\,\Sigma lm - \zeta\,\Sigma ln &= -\Sigma\dot{l}r - \Sigma lR \\ -\xi\,\Sigma lm + \eta\,\Sigma(1-m^2) - \zeta\,\Sigma mn &= -\Sigma\dot{m}r - \Sigma mR \\ \xi\,\Sigma ln - \eta\,\Sigma mn + \zeta\,\Sigma(1-n^2) &= -\Sigma\dot{n}r - \Sigma nR \end{aligned}\right\}. \qquad \ldots\ldots(4)$$

These are accurate formulae.

We now introduce the assumption made by Bravais for utilising the equations in association with the observed proper motions of the stars. Assume that

$$\Sigma lR = \Sigma mR = \Sigma nR = 0, \qquad \ldots\ldots(5)$$

the summation extending over the group of N stars. Now R is the individual or peculiar radial velocity of a star with reference to the system of fixed axes and the equations (5) may be stated in the form that the sum of the projections of the peculiar radial velocities in any given direction is zero. The assumptions summarised in (5) form a natural corollary if the space-motions of the general assembly of stars are distributed at random in accordance with the single-drift hypothesis and if the group with which we are concerned is a representative sample of the totality of stars. For, if we consider a small area of the sky with N_1 stars, the contribution to the value of $\sum_N lR$ provided by these N_1 stars is, taking l constant, $l\sum_{N_1} R$ and, R being now a random velocity, this sum tends to vanish; accordingly $\sum_N lR$ may be considered to be zero.

But the single-drift hypothesis does not actually represent the distribution of stellar velocities. As is well known, stellar motions are represented almost equally well on the two-streams theory and on the ellipsoidal hypothesis of Schwarzschild. In the first, the totality of stars is supposed to be formed from two intermingled aggregations. If the number of stars in each drift is the same, the geometrical centre of one aggregation will move in a particular direction with velocity V relative to the geometrical centre of the totality of stars, while the geometrical centre of the other will move with velocity V in the opposite direction. This direction defines the axis of preferential motion.

Consider now a number of stars in a small region of the sky and suppose that they form a representative sample of the stars in general. There will be a number of stars whose radial velocities relative to fixed axes will consist, first, of a common part—namely, the projection of V in the direction of the region—and second, of the peculiar radial velocities relative to the geometrical centre of the drift or stream concerned. There will also be an equal number of stars belonging to the second stream and their radial velocities relative to the fixed axes will consist of a common part, namely, the projection of $-V$ in the direction of the region, together with the peculiar radial velocities associated with the stream. The numbers being equal, the contribution of all the stars of the region to the sum ΣlR may be expected to vanish. Thus, the assumptions represented by (5) are in conformity with the two-streams theory, with equal numbers of stars in the streams. If the numbers of stars in the streams are not the same, say n_1 and n_2, the speeds V_1 and V_2, of the streams relative to the geometrical centre of the totality of stars are still in opposite directions and, since this centre is taken as the "standard of rest", we have

$$n_1 V_1 = n_2 V_2. \qquad \text{......(6)}$$

Suppose now that the group of stars in a given direction of the sky contains numbers belonging to the two streams in the proportion of n_1 to n_2; the contribution of the two aggregates to the sum ΣlR will again vanish by virtue of (6) and of the random character of the radial velocities of each stream relative to its own geometrical centre. We conclude then that the assumptions (5) are in accordance with the two-streams theory in general. The ellipsoidal theory gives a similar result as, in this theory, speeds of a given amount are equally probable in each of two opposite directions. The conclusion at which we arrive is that the assumption of Bravais, as represented by equations (5), is in accordance with the known distribution of stellar velocities.

Formulae (4) then become

$$\left.\begin{aligned} \xi\,\Sigma(1-l^2)-\eta\,\Sigma lm-\zeta\,\Sigma ln &= -\Sigma\dot{l}r \\ -\xi\,\Sigma lm+\eta\,\Sigma(1-m^2)-\zeta\,\Sigma mn &= -\Sigma\dot{m}r \\ -\xi\,\Sigma ln-\eta\,\Sigma mn+\zeta\,\Sigma(1-n^2) &= -\Sigma\dot{n}r \end{aligned}\right\}. \qquad \ldots\ldots(7)$$

We shall now write $\xi=-X,\quad \eta=-Y,\quad \zeta=-Z,$ ......(8)

so that the components of the solar motion are $(-X,\ -Y,\ -Z)$; thus the motion of a star relative to the sun, ignoring its individual or peculiar motion, has components (X, Y, Z).

Using (8) and inserting the values of $\dot{l}$, $\dot{m}$, $\dot{n}$ given in (4) of section 3·21, we write (7) as follows:

$$\left.\begin{aligned} -AX+cY+bZ &= \Sigma r\mu_\alpha \sin\alpha\cos\delta+\Sigma r\mu_\delta\cos\alpha\sin\delta \\ cX-BY+aZ &= -\Sigma r\mu_\alpha\cos\alpha\cos\delta+\Sigma r\mu_\delta\sin\alpha\sin\delta \\ bX+aY-CZ &= -\Sigma r\mu_\delta\cos\delta \end{aligned}\right\}, \qquad \ldots\ldots(9)$$

in which

$$\left.\begin{aligned} A &\equiv \Sigma(1-l^2) = \Sigma(1-\cos^2\alpha\cos^2\delta) \\ B &\equiv \Sigma(1-m^2) = \Sigma(1-\sin^2\alpha\cos^2\delta) \\ C &\equiv \Sigma(1-n^2) = \Sigma\cos^2\delta \\ a &\equiv \Sigma mn = \Sigma\sin\alpha\sin\delta\cos\delta \\ b &\equiv \Sigma nl = \Sigma\cos\alpha\sin\delta\cos\delta \\ c &\equiv \Sigma lm = \Sigma\sin\alpha\cos\alpha\cos^2\delta \end{aligned}\right\}. \qquad \ldots\ldots(10)$$

The formulae (9) are those to be used in determining the solar motion from the observed proper motions of the stars. It is to be noticed that the distances, r, enter into the right-hand sides of the three formulae (9); we shall consider this feature more fully later.

If the stars with which we are dealing in this problem of the solar motion are numerous and uniformly distributed over the sky, it is easily seen that

$$a=b=c=0 \qquad \ldots\ldots(11)$$

and that A, B and C each tend to $\frac{2}{3}N$, where N is the total number of stars considered. For example, we can find the value of C as follows. The number of stars per unit area of the sphere is $N/4\pi$ and therefore the number in the zone between the parallels δ and $\delta + d\delta$ of declination is

$$\frac{N}{4\pi} 2\pi \cos\delta \,.\, d\delta.$$

Hence $$C \equiv \Sigma \cos^2\delta \to 2\int_0^{\pi/2} \frac{N}{2} \cos^3\delta \,.\, d\delta = \tfrac{2}{3}N.$$

The values of A and B are easily obtained in a similar manner.

These results, however, are most simply derived as follows. From the first three formulae of (10),

$$A + B + C = 2N,$$

and, for uniform distribution of stars over the sphere, $A = B = C$ since the choice of axes has no special significance. Hence

$$A = B = C = \tfrac{2}{3}N.$$

3·31. *Airy's method.*

In deriving the formulae for the solar motion, with the components of the proper motions as the observational data, by Airy's method* we shall first assume that each member of the group of stars, to which the solar motion is to be referred, is at rest relative to the geometrical centre. Let (x, y, z) denote the coordinates of a star, measured from the sun at a given epoch τ, with respect to the usual equatorial system of axes, the equinox and equator being specified for this epoch τ; these axes are accordingly considered to be fixed. Let r be the corresponding heliocentric distance of the star; then

$$x = lr, \quad y = mr, \quad z = nr,$$

where (l, m, n) are the direction-cosines of the heliocentric radius vector to the star at the epoch τ; l, m and n are given by (2) of section 3·21. Taking, as in the previous section, $(-X, -Y, -Z)$ to be the components of the solar motion relative to the given fixed axes (we take the unit of time to be one year), we see that the coordinates of the star, relative to the sun, at the end of a year—that is, at time $(\tau + 1)$—are

$$X + lr, \quad Y + mr, \quad Z + nr.$$

If r_1 is now the corresponding heliocentric distance of the star and (l_1, m_1, n_1) are the direction-cosines, we have

$$X + lr = l_1 r_1, \quad Y + mr = m_1 r_1, \quad Z + nr = n_1 r_1, \qquad \text{......(1)}$$

where $$l_1 = \cos\alpha_1 \cos\delta_1, \quad m_1 = \sin\alpha_1 \cos\delta_1, \quad n_1 = \sin\delta_1, \qquad \text{......(2)}$$

α_1 and δ_1 being the right ascension and declination of the star at time $(\tau + 1)$ with reference to the equatorial system of axes at time τ. The differences,

* *Memoirs, R.A.S.* 28, 143, 1859.

$(\alpha_1-\alpha)$ and $(\delta_1-\delta)$, are the components of proper motion resulting from the solar motion; we write

$$\alpha_1-\alpha = P_\alpha, \quad \delta_1-\delta = P_\delta. \qquad \text{......(3)}$$

From (2) and (1), $\qquad \tan\alpha_1 = \dfrac{m_1}{l_1} = \dfrac{Y+mr}{X+lr},$

which may be written, by virtue of (2) of section 3·21,

$$\tan\alpha_1 = \tan\alpha \,.\, \frac{1+Y/mr}{1+X/lr}. \qquad \text{......(4)}$$

Now X/r and Y/r are small quantities—for the nearest star r is approximately $4 \,.\, 10^{13}$ km. and, as the solar motion is about 20 km. per second, X and Y are not greater than $6 \,.\, 10^8$ km. per annum—hence, neglecting all quantities of order smaller than X/r or Y/r, we obtain from (3) and (4)

$$\tan\alpha + \sec^2\alpha \,.\, P_\alpha = \tan\alpha\left(1-\frac{X}{lr}+\frac{Y}{mr}\right),$$

from which $\qquad -\dfrac{X}{r}\sin\alpha + \dfrac{Y}{r}\cos\alpha = P_\alpha \cos\delta. \qquad \text{......(5)}$

Again, from (2) and (1),

$$\cot^2\delta_1 = \frac{l_1^2+m_1^2}{n_1^2} = \frac{(X+lr)^2+(Y+mr)^2}{(Z+nr)^2},$$

from which, on keeping small quantities of the first order only,

$$\cot^2\delta - 2\cot\delta\,\operatorname{cosec}^2\delta \,.\, P_\delta = \frac{l^2+m^2+2lX/r+2mY/r}{n^2+2nZ/r}$$

$$= \cot^2\delta\left\{1+\frac{2X\cos\alpha}{r\cos\delta}+\frac{2Y\sin\alpha}{r\cos\delta}\right\}\left\{1-\frac{2Z}{r\sin\delta}\right\},$$

whence $\qquad -\dfrac{X}{r}\cos\alpha\sin\delta - \dfrac{Y}{r}\sin\alpha\sin\delta + \dfrac{Z}{r}\cos\delta = P_\delta. \qquad \text{......(6)}$

The formulae (5) and (6) have been derived by the procedure adopted by Airy; they are essentially equivalent to the formulae (3) and (4) of section 1·33. We have to remember that in (5) and (6), if we express r in kilometres, the unit for X, Y and Z is the velocity of 1 km. per annum and that P_α and P_δ are the components of the annual parallactic motion expressed in circular measure.

We now consider the general problem in which the observed proper motion of a star is compounded of the parallactic motion and the angular motion resulting from the star's individual linear motion with respect to fixed axes, which we shall suppose, as before, to be defined with reference to the geometrical centre of the group of stars under consideration. Let (u, v, w)

denote the linear components of the individual velocity of a star at X (Fig. 17), the axes being chosen so that the u-component is parallel to the equator and perpendicular to the meridian, the v-component is tangential to the meridian at X and the w-component is radial. The last component has no effect on the proper motion of the star. The unit of time being a year, the u-component gives rise to an annual angular motion, u/r, along the parallel of declination at X. Hence, if μ_α and μ_δ denote the observed annual proper motion in right ascension and declination (expressed in circular measure), we have

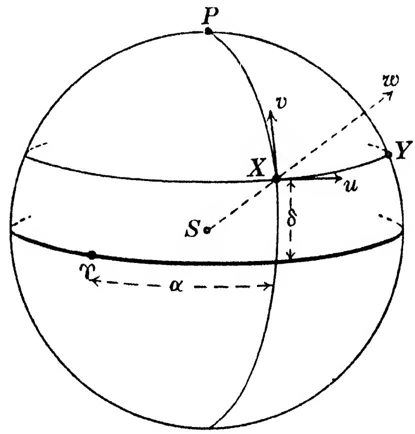

Fig. 17

$$\mu_\alpha \cos\delta = P_\alpha \cos\delta + \frac{u}{r}.$$

Similarly,
$$\mu_\delta = P_\delta + \frac{v}{r}.$$

Equations (5) and (6) now become

$$-\frac{X}{r}\sin\alpha + \frac{Y}{r}\cos\alpha + \frac{u}{r} = \mu_\alpha\cos\delta, \qquad \text{......(7)}$$

$$-\frac{X}{r}\cos\alpha\sin\delta - \frac{Y}{r}\sin\alpha\sin\delta + \frac{Z}{r}\cos\delta + \frac{v}{r} = \mu_\delta. \qquad \text{......(8)}$$

3·32. *The corrections to the proper motions due to errors in the precessional and other constants.*

The accuracy of the observed proper motions derived from meridian-circle observations depends, *inter alia*, upon the accuracy of the precessional constants. Due to precession the right ascension of a star increases at the annual rate, f, given by*

$$f = \chi\cos\epsilon - \lambda + \chi\sin\epsilon\sin\alpha\tan\delta \equiv m' - \lambda + n\sin\alpha\tan\delta, \qquad \text{......(1)}$$

where χ is the luni-solar precession, λ is the planetary precession and ϵ is the obliquity of the ecliptic. In reductions of star places, Newcomb's values of these constants are employed and, although these are believed to be of a high order of accuracy, small errors can have a considerable effect on the values of the proper motions derived in this way.

Let (α, δ) denote the equatorial coordinates of a star referred to the mean

* See, for example, the author's *Spherical Astronomy* (2nd Edn.), p. 238, 1936.

equinox and equator for epoch t_0, and (α_1, δ_1) the coordinates referred to the mean equinox and equator for epoch (t_0+t). It is to be supposed that α and α_1 are obtained from meridian-circle observations near t_0 and t_0+t, the actual interval between the observations being t_1. We have, due to precessional and proper motion effects alone,

$$\alpha_1 = \alpha + ft + \mu_\alpha t_1,$$

from which

$$\mu_\alpha = \frac{\alpha_1 - \alpha}{t_1} - \frac{ft}{t_1}. \qquad \text{......(2)}$$

The observed value of μ_α is thus found from (2) using Newcomb's constants in the expression for f in (1). If, however, $m' + \Delta m'$, $\lambda + \Delta\lambda$ and $n + \Delta n$ are the true values of m', λ and n respectively, the true value of the right ascension component of the proper motion, which we denote by (μ_α), is given by

$$(\mu_\alpha) = \frac{\alpha_1 - \alpha}{t_1} - \frac{ft}{t_1} - \Delta f \frac{t}{t_1}, \qquad \text{......(3)}$$

where $\Delta f = \Delta m' - \Delta\lambda + \Delta n \sin\alpha \tan\delta$. Hence, from (2) and (3),

$$(\mu_\alpha) = \mu_\alpha - \Delta f \frac{t}{t_1}. \qquad \text{......(4)}$$

As the observations have been supposed to be made near t_0 and t_0+t, t and t_1 are nearly equal and, as in (4) Δf must be regarded as an extremely small quantity, it will be sufficient to write $t = t_1$, so that

$$(\mu_\alpha) = \mu_\alpha - \Delta f. \qquad \text{......(5)}$$

Also, the true value will be in error due to an erroneous value of the motion of the equinox; if the error of the latter is Δe, we have finally

$$(\mu_\alpha) = \mu_\alpha - \Delta f + \Delta e, \qquad \text{......(6)}$$

which we write in the form

$$(\mu_\alpha) = \mu_\alpha - \Delta k - \Delta n \sin\alpha \tan\delta, \qquad \text{......(7)}$$

where

$$\Delta k = \Delta m' - \Delta\lambda - \Delta e. \qquad \text{......(8)}$$

Thus on the right of (7) in section 3·31 we have to write in place of $\mu_\alpha \cos\delta$

$$(\mu_\alpha - \Delta k - \Delta n \sin\alpha \tan\delta)\cos\delta.$$

Again,

$$\delta_1 = \delta + gt + \mu_\delta t_1,$$

where

$$g = n\cos\alpha,$$

and we obtain in a similar way, denoting the true value of the proper motion component in declination by (μ_δ),

$$(\mu_\delta) = \mu_\delta - \Delta n \cos\alpha. \qquad \text{......(9)}$$

We have then to replace μ_δ on the right of (8), section 3·31, by $\mu_\delta - \Delta n \cos\alpha$.

The equations for determining the solar motion are now, adding ϵ_α and ϵ_δ—the accidental errors of observation in parallel and in declination—

$$-\frac{X}{r}\sin\alpha+\frac{Y}{r}\cos\alpha+\Delta k\cos\delta+\Delta n\sin\alpha\sin\delta+\frac{u}{r}+\epsilon_\alpha=\mu_\alpha\cos\delta, \quad \text{......(10)}$$

$$-\frac{X}{r}\cos\alpha\sin\delta-\frac{Y}{r}\sin\alpha\sin\delta+\frac{Z}{r}\cos\delta+\Delta n\cos\alpha+\frac{v}{r}+\epsilon_\delta=\mu_\delta. \quad \text{......(11)}$$

3·33. *Application of Airy's method; first hypothesis.*

The complete solution of equations (10) and (11) of the previous section involves the determination of the quantities X, Y, Z, Δk and Δn from the observed values of the components of proper motion; we shall suppose for the moment that the distances, r, are known. We require also definite information concerning the distribution of the individual linear velocity components, u and v. We can make two hypotheses: first, that the values of u/r and v/r, which are angular motions expressed in the same way as $\mu_\alpha\cos\delta$ and μ_δ, are of the nature of accidental errors; second, that the distribution of u and v is associated in a definite way with the random motions of the stars forming the group under consideration.

In the first hypothesis, we can then suppose the accidental errors u/r and ϵ_α combined; this is equivalent to omitting u/r and v/r from the equations and regarding the errors ϵ_α, ϵ_δ now as the combined accidental errors with probable errors depending on those of u/r and $\mu_\alpha\cos\delta$, and of v/r and μ_δ. The equations can then be solved by the method of least squares. For example, (10) of section 3·32 gives rise to the four normal equations:

$$X\Sigma\frac{1}{r^2}\sin^2\alpha-Y\Sigma\frac{1}{r^2}\sin\alpha\cos\alpha-\Delta k\Sigma\frac{1}{r}\sin\alpha\cos\delta$$
$$-\Delta n\Sigma\frac{1}{r}\sin^2\alpha\sin\delta=-\Sigma\frac{1}{r}\mu_\alpha\sin\alpha\cos\delta,$$

$$-X\Sigma\frac{1}{r^2}\sin\alpha\cos\alpha+Y\Sigma\frac{1}{r^2}\cos^2\alpha+\Delta k\Sigma\frac{1}{r}\cos\alpha\cos\delta$$
$$+\Delta n\Sigma\frac{1}{r}\sin\alpha\cos\alpha\sin\delta=\Sigma\frac{1}{r}\mu_\alpha\cos\alpha\cos\delta,$$

$$-X\Sigma\frac{1}{r}\sin\alpha\cos\delta+Y\Sigma\frac{1}{r}\cos\alpha\cos\delta+\Delta k\Sigma\cos^2\delta$$
$$+\Delta n\Sigma\sin\alpha\sin\delta\cos\delta=\Sigma\mu_\alpha\cos^2\delta,$$

$$-X\Sigma\frac{1}{r}\sin^2\alpha\sin\delta+Y\Sigma\frac{1}{r}\sin\alpha\cos\alpha\sin\delta+\Delta k\Sigma\sin\alpha\sin\delta\cos\delta$$
$$+\Delta n\Sigma\sin^2\alpha\sin^2\delta=\Sigma\mu_\alpha\sin\alpha\sin\delta\cos\delta.$$

In a similar way the equation (11) of section 3·32 gives rise to four normal equations involving X, Y, Z and Δn. The solutions can then be effected in the usual way.

However, the hypothesis is hardly likely to be valid for stars at varying distances as it implies greater linear peculiar velocities at greater distances from the sun. Of course, if the various members of our group are at nearly the same distance from the sun, the hypothesis is not open to the same objection. But in statistical investigations of the solar motion we have, generally, insufficient information about the distances of the stars and, actually, in dealing with faint stars, whose proper motions are derived photographically, direct information concerning distances is almost wholly lacking; in this case, the distances may be distributed between comparatively wide limits, so that the objection to the hypothesis remains.

3·34. *Application of Airy's method: second hypothesis.*

We consider now the hypothesis that the irregularities in the proper motion components are entirely due to the random linear motions of the stars. Omitting ϵ_α and ϵ_δ, we write the equations (10) and (11) of section 3·32 as

$$-X\sin\alpha + Y\cos\alpha + r\Delta k\cos\delta + r\Delta n\sin\alpha\sin\delta + u = r\mu_\alpha\cos\delta,$$

$$-X\cos\alpha\sin\delta - Y\sin\alpha\sin\delta + Z\cos\delta + r\Delta n\cos\alpha + v = r\mu_\delta.$$

Suppose that X, Y, Z, u and v are expressed in kilometres per second, that μ_α, μ_δ, Δk, Δn are expressed in seconds of arc per annum and that, instead of the distance, r, we use the parallax, p (in seconds of arc); then the equations become

$$-X\sin\alpha + Y\cos\alpha + \Delta k\,.\frac{\kappa}{p}\cos\delta + \Delta n\,.\frac{\kappa}{p}\sin\alpha\sin\delta + u = \frac{\kappa}{p}\mu_\alpha\cos\delta, \qquad \ldots\ldots(1)$$

$$-X\cos\alpha\sin\delta - Y\sin\alpha\sin\delta + Z\cos\delta + \Delta n\,.\frac{\kappa}{p}\cos\alpha + v = \frac{\kappa}{p}\mu_\delta, \qquad \ldots\ldots(2)$$

where $\kappa \equiv 4{\cdot}74$.

We now form the normal equations in the usual manner (we take $\kappa\Delta k$ and $\kappa\Delta n$ as the unknowns instead of Δk and Δn). Remembering that u and v are random in character, the four normal equations derived from (1) are:

$$X\Sigma\sin^2\alpha - Y\Sigma\sin\alpha\cos\alpha - \kappa\Delta k\,\Sigma\frac{1}{p}\sin\alpha\cos\delta - \kappa\Delta n\,\Sigma\frac{1}{p}\sin^2\alpha\sin\delta = -\kappa\Sigma\frac{1}{p}\mu_\alpha\sin\alpha\cos\delta, \qquad \ldots\ldots(3)$$

$$-X\Sigma\sin\alpha\cos\alpha+Y\Sigma\cos^2\alpha+\kappa\Delta k\Sigma\frac{1}{p}\cos\alpha\cos\delta$$
$$+\kappa\Delta n\Sigma\frac{1}{p}\sin\alpha\cos\alpha\sin\delta=\kappa\Sigma\frac{1}{p}\mu_\alpha\cos\alpha\cos\delta, \qquad \ldots\ldots(4)$$

$$-X\Sigma\frac{1}{p}\sin\alpha\cos\delta+Y\Sigma\frac{1}{p}\cos\alpha\cos\delta+\kappa\Delta k\Sigma\frac{1}{p^2}\cos^2\delta$$
$$+\kappa\Delta n\Sigma\frac{1}{p^2}\sin\alpha\sin\delta\cos\delta=\kappa\Sigma\frac{1}{p^2}\mu_\alpha\cos^2\delta, \qquad \ldots\ldots(5)$$

$$-X\Sigma\frac{1}{p}\sin^2\alpha\sin\delta+Y\Sigma\frac{1}{p}\sin\alpha\cos\alpha\sin\delta+\kappa\Delta k\Sigma\frac{1}{p^2}\sin\alpha\sin\delta\cos\delta$$
$$+\kappa\Delta n\Sigma\frac{1}{p^2}\sin^2\alpha\sin^2\delta=\kappa\Sigma\frac{1}{p^2}\mu_\alpha\sin\alpha\sin\delta\cos\delta. \qquad \ldots\ldots(6)$$

The four normal equations derived from (2) are, similarly,

$$X\Sigma\cos^2\alpha\sin^2\delta+Y\Sigma\sin\alpha\cos\alpha\sin^2\delta-Z\Sigma\cos\alpha\sin\delta\cos\delta$$
$$-\kappa\Delta n\Sigma\frac{1}{p}\cos^2\alpha\sin\delta=-\kappa\Sigma\frac{1}{p}\mu_\delta\cos\alpha\sin\delta, \qquad \ldots\ldots(7)$$

$$X\Sigma\sin\alpha\cos\alpha\sin^2\delta+Y\Sigma\sin^2\alpha\sin^2\delta-Z\Sigma\sin\alpha\sin\delta\cos\delta$$
$$-\kappa\Delta n\Sigma\frac{1}{p}\sin\alpha\cos\alpha\sin\delta=-\kappa\Sigma\frac{1}{p}\mu_\delta\sin\alpha\sin\delta, \qquad \ldots\ldots(8)$$

$$-X\Sigma\cos\alpha\sin\delta\cos\delta-Y\Sigma\sin\alpha\sin\delta\cos\delta+Z\Sigma\cos^2\delta$$
$$+\kappa\Delta n\Sigma\frac{1}{p}\cos\alpha\cos\delta=\kappa\Sigma\frac{1}{p}\mu_\delta\cos\delta, \qquad \ldots\ldots(9)$$

$$-X\Sigma\frac{1}{p}\cos^2\alpha\sin\delta-Y\Sigma\frac{1}{p}\sin\alpha\cos\alpha\sin\delta+Z\Sigma\frac{1}{p}\cos\alpha\cos\delta$$
$$+\kappa\Delta n\Sigma\frac{1}{p^2}\cos^2\alpha=\kappa\Sigma\frac{1}{p^2}\mu_\delta\cos\alpha. \qquad \ldots\ldots(10)$$

Combining (3) and (7), we have

$$X\Sigma(1-\cos^2\alpha\cos^2\delta)-Y\Sigma\sin\alpha\cos\alpha\cos^2\delta-Z\Sigma\cos\alpha\sin\delta\cos\delta$$
$$-\kappa\Delta k\Sigma\frac{1}{p}\sin\alpha\cos\delta-\kappa\Delta n\Sigma\frac{1}{p}\sin\delta$$
$$=-\kappa\Sigma\frac{1}{p}\mu_\alpha\sin\alpha\cos\delta-\kappa\Sigma\frac{1}{p}\mu_\delta\cos\alpha\sin\delta. \qquad \ldots\ldots(11)$$

Combining (4) and (8), we have

$$-X\Sigma\sin\alpha\cos\alpha\cos^2\delta+Y\Sigma(1-\sin^2\alpha\cos^2\delta)-Z\Sigma\sin\alpha\sin\delta\cos\delta$$
$$+\kappa\Delta k\Sigma\frac{1}{p}\cos\alpha\cos\delta=\kappa\Sigma\frac{1}{p}\mu_\alpha\cos\alpha\cos\delta-\kappa\Sigma\frac{1}{p}\mu_\delta\sin\alpha\sin\delta.$$
$$\ldots\ldots(12)$$

Combining (6) and (10), we have

$$-X\Sigma\frac{1}{p}\sin\delta+Z\Sigma\frac{1}{p}\cos\alpha\cos\delta+\kappa\Delta k\,\Sigma\frac{1}{p^2}\sin\alpha\sin\delta\cos\delta$$
$$+\kappa\Delta n\,\Sigma\frac{1}{p^2}(1-\sin^2\alpha\cos^2\delta)=\kappa\Sigma\frac{1}{p^2}\mu_\alpha\sin\alpha\sin\delta\cos\delta+\kappa\Sigma\frac{1}{p^2}\mu_\delta\cos\alpha. \quad \text{......(13)}$$

The equations (11), (12) and (13) together with (5) and (9) are the five combined normal equations from which X, Y, Z, $\kappa\Delta k$ and $\kappa\Delta n$ can be found.

Comparing these equations with (9) and (10) of section 3·22, we see that we have reproduced the equations of Bravais, with the addition, of course, of the terms in $\kappa\Delta k$ and $\kappa\Delta n$.

The group of equations (11), (12), (13), (5) and (9) cannot be solved unless the various values of the parallax, p, are known. In default of this information, it is customary to restrict the choice of stars to be used in the equations by considering only stars of a limited range of magnitude and by omitting stars with very large proper motions. Other things being equal, a large proper motion suggests that the star is comparatively near. It is then assumed that the remaining stars have the same parallax p_0, and writing X_1 for $\frac{p_0}{\kappa}X$, Y_1 for $\frac{p_0}{\kappa}Y$ and Z_1 for $\frac{p_0}{\kappa}Z$, the equations of condition, (1) and (2), are (omitting the random components u and v):

$$-X_1\sin\alpha+Y_1\cos\alpha+\Delta k\cos\delta+\Delta n\sin\alpha\sin\delta=\mu_\alpha\cos\delta, \quad \text{......(14)}$$
$$-X_1\cos\alpha\sin\delta-Y_1\sin\alpha\sin\delta+Z_1\cos\delta+\Delta n\cos\alpha=\mu_\delta, \quad \text{......(15)}$$

from which the normal equations are formed in the usual way, the trigonometrical factors being the same as in (11), (12), (13), (5) and (9).

These equations, (14) and (15), we shall call Airy's equations; with or without the terms in Δk, Δn they are the equations generally employed for determining the solar motion when the proper motions furnish the observational material.

3·35. *The solution of L. Boss.*

The equations (14) and (15) of the previous section were used by L. Boss* for the stars of the *Preliminary General Catalogue*. Stars with annual proper motions greater than 0″·2 were omitted. The mean magnitude was $5^{\mathrm{m}}\cdot7$. Boss's solution must be regarded as the best that has hitherto been obtained by Airy's method owing to the high accuracy of the proper motions of the *P.G.C.* stars, and the results are likely to be accepted as the standard values for some time to come. The results† are in our notation (Boss denotes the

* *A.J.* **26**, 95, 111, 187, 1910. † *Loc. cit.* p. 112.

components of the solar motion by (X, Y, Z) whereas we have denoted them by $(-X, -Y, -Z)$, and he also takes the century to be the unit of time):

$$X_1 = -0''{\cdot}0003, \quad Y_1 = +0''{\cdot}0318, \quad Z_1 = -0''{\cdot}0217. \quad \ldots\ldots(1)$$

If

$$M = (X_1^2 + Y_1^2 + Z_1^2)^{\frac{1}{2}}, \quad \ldots\ldots(2)$$

the coordinates of the solar apex (α_0, δ_0) are found from

$$M \cos\alpha_0 \cos\delta_0 = +0''{\cdot}0003,$$
$$M \sin\alpha_0 \cos\delta_0 = -0''{\cdot}0318,$$
$$M \sin\delta_0 = +0''{\cdot}0217,$$

whence

$$\alpha_0 = 270^\circ{\cdot}5, \quad \delta_0 = +34^\circ{\cdot}3. \quad \ldots\ldots(3)$$

Also $M = 0''{\cdot}0385$, which is the annual parallactic motion for the stars at an angular distance of 90° from the solar apex, the parallax of the stars being p_0. From the results obtained, we can easily derive the value of p_0 used implicitly in the equations. We have

$$X_1 = \frac{p_0}{\kappa} X, \text{ etc.}$$

and consequently the solar speed, V_0, is given by

$$V_0 \equiv (X^2 + Y^2 + Z^2)^{\frac{1}{2}} = \frac{\kappa}{p_0} M. \quad \ldots\ldots(4)$$

Since X, Y and Z are expressed in km./sec., V_0 is expressed in the same way. If we assume that $V_0 = 19{\cdot}5$ km./sec. as obtained from a study of the radial velocities of the stars, it is then found that

$$p_0 = 0''{\cdot}0094. \quad \ldots\ldots(5)$$

The numerical values of Δk and Δn, as obtained by Boss, may also be noted; they are

$$\Delta k = -0''{\cdot}0037, \quad \Delta n = +0''{\cdot}0034,$$

the year as before being taken to be the unit of time. From various considerations, outside the scope of this book, concerning the values of $\Delta\lambda$ and Δe (see section 3·32), the definitive values of Δk and Δn were taken by Boss to be

$$\Delta k = -0''{\cdot}0032, \quad \Delta n = +0''{\cdot}0023. \quad \ldots\ldots(6)$$

Hence the corrections to the annual proper motions of the *P.G.C.* stars are, from (6):

for μ_α; $\quad +0^s{\cdot}00021 - 0^s{\cdot}00015 \sin\alpha \tan\delta, \quad \ldots\ldots(7)$

for μ_δ; $\quad -0''{\cdot}0023 \cos\alpha. \quad \ldots\ldots(8)$

Further corrections relating to the system of Boss stars have been subsequently obtained by Raymond;* these have been based on more recent meridian observations.

* *A.J.* **36**, 129, 1926; **37**, 88, 1927.

3·41. *The solar motion from radial velocities.*

Denoting, as before, the components of the solar motion by $(-X, -Y, -Z)$ in km./sec., we see that, relative to the sun, the star will be displaced radially away from the sun with the velocity

$$P_\rho \equiv lX + mY + nZ,$$

where (l, m, n) are the direction-cosines of the line joining the sun to the star and P_ρ is the radial component of the parallactic motion. The observed radial velocity will also include the peculiar radial velocity R referred to fixed axes through the geometrical centre of the group of stars considered. In addition, a constant term K, representing any systematic peculiarity in the radial velocities—such as would result from incorrect wave-lengths of the lines in the comparison source—is generally added to the equation of condition which then takes the form, on inserting the values of l, m and n as given in (2) of section 3·21,

$$X \cos\alpha \cos\delta + Y \sin\alpha \cos\delta + Z \sin\delta + K + R = \rho, \qquad \text{......(1)}$$

in which ρ is the observed radial velocity (relative to the sun) in km./sec.

Assuming that we may regard the peculiar radial velocities, R, as having the characteristics of accidental errors, the formula (1) leads to the four normal equations:

$$X\Sigma\cos^2\alpha\cos^2\delta + Y\Sigma\sin\alpha\cos\alpha\cos^2\delta + Z\Sigma\cos\alpha\sin\delta\cos\delta + K\Sigma\cos\alpha\cos\delta = \Sigma\rho\cos\alpha\cos\delta, \qquad \text{......(2)}$$

$$X\Sigma\sin\alpha\cos\alpha\cos^2\delta + Y\Sigma\sin^2\alpha\cos^2\delta + Z\Sigma\sin\alpha\sin\delta\cos\delta + K\Sigma\sin\alpha\cos\delta = \Sigma\rho\sin\alpha\cos\delta, \qquad \text{......(3)}$$

$$X\Sigma\cos\alpha\sin\delta\cos\delta + Y\Sigma\sin\alpha\sin\delta\cos\delta + Z\Sigma\sin^2\delta + K\Sigma\sin\delta = \Sigma\rho\sin\delta, \qquad \text{......(4)}$$

$$X\Sigma\cos\alpha\cos\delta + Y\Sigma\sin\alpha\cos\delta + Z\Sigma\sin\delta + NK = \Sigma\rho, \qquad \text{......(5)}$$

where, in (5), N is the total number of stars under consideration.

Denoting the solar velocity by V_0 and the coordinates of the solar apex by (α_0, δ_0), we have

$$\left.\begin{aligned} V_0\cos\alpha_0\cos\delta_0 &= -X \\ V_0\sin\alpha_0\cos\delta_0 &= -Y \\ V_0\sin\delta_0 &= -Z \end{aligned}\right\}, \qquad \text{......(6)}$$

from which V_0, α_0 and δ_0 can be calculated when the values of X, Y and Z have been derived from the normal equations (2)...(5). It is to be remarked that the solar velocity, V_0, is obtained in km./sec.

3·42. *The solar motion determined from the radial velocities according to the method of Bravais.*

We begin with the formulae (4) of section 3·11, writing the components of the solar motion as $(-X, -Y, -Z)$ relative to fixed axes whose origin we may assume to be the geometrical centre of the group of N stars; the equations are:

$$\sum_N u - NX = 0, \quad \sum_N v - NY = 0, \quad \sum_N w - NZ = 0, \qquad \text{......(1)}$$

u, v and w being the components of a star's linear velocity with respect to the sun and the usual system of equatorial axes OX, OY, OZ (Fig. 18). But we can describe the linear velocity of a star at S by the radial velocity ρ and rectangular components P and Q in the tangent plane at S, the direction of P being perpendicular to the meridian and Q tangential to it. The components ρ, P and Q are relative to the sun.

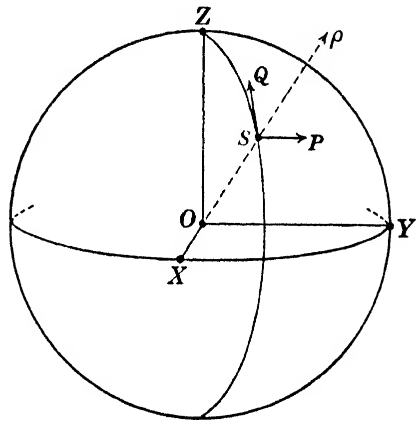

Fig. 18

Now P is composed of a parallactic component, p, and a component, p', due to the star's individual velocity relative to the geometrical centre, so that

$$P = p + p'. \qquad \text{......(2)}$$

Similarly

$$Q = q + q', \qquad \text{......(3)}$$

where q is the parallactic component along the tangent to the meridian at S and q' is the component in this direction of the star's individual velocity. The direction-cosines of P (or of p) are

$$-\sin\alpha, \quad +\cos\alpha, \quad 0$$

and the direction-cosines of Q (or of q) are

$$-\cos\alpha\sin\delta, \quad -\sin\alpha\sin\delta, \quad +\cos\delta.$$

Also the direction-cosines of ρ are

$$+\cos\alpha\cos\delta, \quad +\sin\alpha\cos\delta, \quad +\sin\delta.$$

Hence

$$p = -X\sin\alpha + Y\cos\alpha, \qquad \text{......(4)}$$

$$q = -X\cos\alpha\sin\delta - Y\sin\alpha\sin\delta + Z\cos\delta. \qquad \text{......(5)}$$

These equations are essentially the same as (7) and (8) of section 3·31.

Now u is the sum of the projections, along OX, of P, Q and ρ, and therefore

$$u = -P\sin\alpha - Q\cos\alpha\sin\delta + \rho\cos\alpha\cos\delta.$$

Hence from (2), (3), (4) and (5) we obtain

$$\begin{aligned} u = -p'\sin\alpha - \sin\alpha(-X\sin\alpha + Y\cos\alpha) - q'\cos\alpha\sin\delta \\ -\cos\alpha\sin\delta(-X\cos\alpha\sin\delta - Y\sin\alpha\sin\delta + Z\cos\delta) + \rho\cos\alpha\cos\delta. \end{aligned} \quad \text{......(6)}$$

The first equation of (1) can be written as

$$\sum_N (u - X) = 0.$$

Hence inserting the expression for u, given by (6), in this formula, we obtain, after some simplification,

$$\begin{aligned} X\Sigma\cos^2\alpha\cos^2\delta + Y\Sigma\sin\alpha\cos\alpha\cos^2\delta + Z\Sigma\cos\alpha\sin\delta\cos\delta \\ + \Sigma p'\sin\alpha + \Sigma q'\cos\alpha\sin\delta = \Sigma\rho\cos\alpha\cos\delta. \end{aligned}$$

We assume now that the sums of the random tangential velocity components in any small area of the sky vanish and, accordingly, we are left with the equation

$$\begin{aligned} X\Sigma\cos^2\alpha\cos^2\delta + Y\Sigma\sin\alpha\cos\alpha\cos^2\delta \\ + Z\Sigma\cos\alpha\sin\delta\cos\delta = \Sigma\rho\cos\alpha\cos\delta. \end{aligned} \quad \text{......(7)}$$

Let us examine more closely the assumption which we have just made. Consider a small region of the sky at S in which there are n stars; we can write $\Sigma p'\sin\alpha$ as $\sin\alpha\,\Sigma p'$. If the n stars form a representative sample of the stars in general, we should expect the sum $\sum_n p'$ to vanish, or tend to vanish. This is evident if the components of the individual motions are of a haphazard character as in a single drift and it is also true for star-streaming. In the latter case we shall assume for simplicity that the velocities V_1 and V_2 of the two drifts, *relative to the geometrical centre* of all the stars belonging to the two drifts, are equal and opposite; this implies that there are equal numbers of stars in the two drifts and we shall also assume that this holds for the representative sample of n stars in the small region at S under consideration. Taking the $\frac{1}{2}n$ stars belonging to drift I, the component of a linear velocity perpendicular to the meridian at S consists of the projections of the drift velocity V_1 in this direction together with the random component relative to the geometrical centre of the drift. Similarly, taking the $\frac{1}{2}n$ stars belonging to drift II, the component of a linear velocity perpendicular to the meridian at S consists of the projection of the drift-velocity V_2 in this direction together with the corresponding random component. But the direction of V_2 is opposite to that of V_1; consequently, since $V_1 = V_2$, the systematic parts of $\sum p'$ disappear and the remaining random parts ensure, under ideal conditions, that $\sum_n p' = 0$. The argument is similar if the drift-velocities V_1 and V_2 are different (relative to the geometrical centre of the whole assembly of stars), for in this case the numbers of stars in the two drifts are inversely in the ratio of the drift-velocities.

It will be noticed that, with the exception of the K term, the equation (7) is the same as equation (2) of section 3·41. The latter, being a normal equation, was derived on the assumption that the peculiar velocities had the characteristics of accidental errors.

By considering the equations

$$\Sigma v - NY = 0 \quad \text{and} \quad \Sigma w - NZ = 0,$$

we obtain the equations (3) and (4) of section 3·41—with the exception, of course, of the K term.

3·43. *Observational results from the radial velocities.*

A fairly recent determination of the solar motion from radial velocities, based on a homogeneous set of observations, is that of W. W. Campbell and J. H. Moore.* The measures were made at the Lick Observatory and at the Lick southern station at Santiago, Chile. After excluding stars belonging to moving clusters and also 37 "high velocity stars", 2148 stars were available for use in the general solution. In this investigation, the criterion adopted for a "high velocity" star is as follows; assuming that the solar motion is 20 km./sec. and that the apex is at R.A. 270°, declination +30°, the parallactic component P_ρ is found for each star; if $\rho - P_\rho$ exceeds 60 km./sec., where ρ is the observed radial velocity, the star is classed as a high velocity star and is accordingly excluded from the equations. It has been found† that the high velocity stars have systematic motions towards one hemisphere of the sky. We shall discuss this group of stars in a subsequent chapter.

The 2148 stars were divided into 94 groups depending on their position on the celestial sphere and the mean was taken for each region. Thus there were 94 equations of condition of the type of (1), section 3·41, and these were weighted according to the number of stars in each region. The results are:

$$\alpha_0 = 270°\cdot 6, \quad \delta_0 = +29°\cdot 2, \quad V_0 = 19\cdot 7\,\text{km./sec.}, \quad K = +1\cdot 3\,\text{km./sec.}$$

This position of the solar apex thus differs by about 5° from the position derived from the proper motions; the difference, it should be noted, is almost entirely in declination.

* *Lick Publications*, **16**, 1928.

† Adams and Joy, *Ap. J.* **49**, 179, 1919; Strömberg, *Ap. J.* **56**, 265, 1922.

The results obtained by Smart and Green* from the radial velocities of 3683 stars taken from Schlesinger's *Catalogue of Bright Stars*, 1930, are

$$\alpha_0 = 267°\cdot 0, \quad \delta_0 = +32°\cdot 0, \quad V_0 = 19\cdot 5\,\text{km./sec.}, \quad K = +0\cdot 8\,\text{km./sec.}$$

There is a significant increase of about 3° in the declination of the apex.

The fact that the K term is non-vanishing—a zero value is to be expected on the hypothesis of random motions or of either kind of preferential motion—implies, from the strictly kinematical point of view, that the group of stars with which we are concerned is expanding at a mean rate of about 1 km./sec. But when the O and B type stars are alone considered, the value of the K term from Campbell and Moore's results is about $+5$ km./sec. A similar result is found in the investigation of Smart and Green.* Also the contribution to the K term made by the other spectral classes is comparatively trifling and well within the limits of accidental error. Thus the incidence of a non-zero value for the K term is to be attributed entirely to the stars of types O and B.

The kinematical explanation for the existence of the K term was always received with caution as it was realised that small systematic errors in the adopted laboratory wave-lengths used in the measurement of stellar spectra could adequately account for the apparent phenomenon. Part of the K term for the O and B stars (about $+1$ to $+2$ km./sec.) can be attributed to the gravitational displacement of the spectral lines towards the red end of the spectrum as predicted by the relativity theory and this displacement is of appreciable amount only in the case of the O and B type stars.† It has been stated by the authors of the paper just quoted that the gravitational displacement can account for the whole of the K term—its value they reduce to about $+2$ km./sec.—but their arguments are erroneous.‡ Making allowance for the gravitational displacement, we find that there is a residual K term amounting to $+3$ or $+4$ km./sec. Whatever the final physical or kinematical explanation of the K term may be, it is important to preserve it in the equations of condition as it represents a systematic tendency, perhaps real or perhaps spurious, of the observed radial velocities to be larger algebraically than they should be on any of the usual hypotheses as to the distribution of stellar velocities.

3·44. *Solar motion and spectral type.*

The numerical results quoted in the previous section for the solar motion are based upon the radial velocities of all spectral types (Smart and Green omit the O type stars from their statistics), and the solar motion, it must be

* W. M. Smart and H. E. Green, *M.N.* **96**, 471, 1936.

† See the calculations of J. S. Plaskett and J. A. Pearce, *M.N.* **94**, 679, 1934.

‡ W. M. Smart, *M.N.* **96**, 568, 1936.

remembered, is defined with reference to the particular group of stars concerned. When the solutions are made for stars of a single spectral class (e.g. for stars of type B) alone, the numerical results show certain characteristic differences. We take the investigation (*loc. cit.*) of Campbell and Moore as typical in this connection. Later researches may, and probably will, produce small modifications in the numerical results, but it is generally accepted that the principal features are clearly established and represent a definite correlation between motion and spectral type. As regards the position of the apex derived from groups of stars of different spectral type the variations from the position (270°, +30°) are probably of an accidental nature, due in some measure to the comparatively small number of stars in each spectral group. Assuming that no special significance need be attached to such variations and taking the position of the solar apex to be (270°, +30°) for each spectral group, the equation of condition becomes

$$V_0 \cos\lambda + K = \rho,$$

where λ is the angular distance of the star from the antapex (we omit the peculiar velocity R which, as before, is assumed to have no effect in the normal equations). The following table due to Campbell and Moore (*loc. cit.*) exhibits the results for the main spectral subdivisions.

Table 10. *Solar motion* (*Campbell and Moore*)

Spectral class	Number of stars	V_0 (km./sec.)	K (km./sec.)	Average residual velocity (km./sec.)
B (Oe5–B5)	284	22·7	+4·9	8·7
A (B8–A3)	500	18·6	+1·7	9·9
F (A5–F4)	199	19·7	+0·3	12·5
G (F5–G4)	244	18·6	−0·2	14·8
K (G5–K4)	687	18·0	+0·3	15·3
M (K5–Mb)	234	22·1	+0·7	16·1
B to M	2148	19·7	+1·3	—

The last line gives the results for all stars of spectral types from Oe5 to Mb and corresponds to the general solution.

The principal features of the table are, first, the much greater value of the solar motion with reference to the group of B type stars and to the group of M type stars than for the remaining groups A, F, G and K; second, the large value of the K term for the B type stars as compared with the almost insignificant values derived for the other spectral classes; and third, the unmistakable progression of the average residual velocities from type B towards type M. The average residual velocity is obtained by removing from each observed measure the parallactic component $V_0 \cos\lambda$ and the value of K and forming the mean from the residuals without regard to sign. This procedure is greatly facilitated by the use of special methods among which may be mentioned those of Dufton* and of Pearce and Hill.†

In the following table (Table 11), the results obtained by Smart and Green‡ are shown. The solutions were made according to galactic zones and in the table G_0 and g_0 denote the galactic longitude and latitude respectively of the solar antapex found for each spectral class (Type B means B1 to B9 and so on).

Table 11. *Solar motion (Smart and Green)*

Spectral class	Number of stars	V_0 (km./sec.)	K (km./sec.)	G_0 (degrees)	g_0 (degrees)
B	645	22·4	+4·7	209·9	−22·5
A	742	17·1	+0·0	193·5	−28·5
F	523	18·1	−0·6	194·8	−32·4
G	433	17·2	−1·0	205·2	−17·8
K	1118	19·7	−0·2	205·4	−21·8
M	222	19·8	+0·0	220·2	−24·1
A to M	3038	18·2	−0·2	202·2	−24·8
B to M	3683	19·5	+0·8	204·6	−24·8

It will be noticed that in Table 11 the solar motion with respect to the M type stars is not so conspicuously large as in Campbell and Moore's results. The penultimate line shows that the K term is practically zero for stars of spectral classes A to M and that it is a phenomenon associated with the B type stars alone.

3·45. *The relationship between absolute magnitude and linear velocity.*

Let us first consider the space velocity of a star relative to the sun. Its components (u, v, w) with respect to the usual equatorial system of axes are (see section 1·33):

$$u = \rho \cos\alpha \cos\delta - \frac{\kappa}{p}\{\mu_\alpha \sin\alpha \cos\delta + \mu_\delta \cos\alpha \sin\delta\},$$

$$v = \rho \sin\alpha \cos\delta + \frac{\kappa}{p}\{\mu_\alpha \cos\alpha \cos\delta - \mu_\delta \sin\alpha \sin\delta\},$$

$$w = \rho \sin\delta + \frac{\kappa}{p}\mu_\delta \cos\delta,$$

where ρ is the observed radial velocity, p is the parallax and $\kappa = 4{\cdot}74$.

If $(-X, -Y, -Z)$ denote, as usual, the components of the solar motion, V_0, we have

$$-X = V_0 \cos\alpha_0 \cos\delta_0, \quad -Y = V_0 \sin\alpha_0 \cos\delta_0, \quad -Z = V_0 \sin\delta_0.$$

* *M.N.* **92**, 688, 1932.

† *Publ. of the Dominion Astr. Obs., Victoria*, **6**, No. 4, 1931.

‡ *M.N.* **96**, 471, 1936.

The components of a star's space-velocity relative to the geometrical centre are then $u-X$, $v-Y$, $w-Z$, from which the space-velocity U, given by

$$U = \{(u-X)^2+(v-Y)^2+(w-Z)^2\}^{\frac{1}{2}},$$

can be readily found from the observational material and the assumed values for the solar motion, V_0, and the solar apex (α_0, δ_0). We shall refer to U as the absolute velocity.

Owing to the great increase in the number of measured parallaxes, especially by the spectroscopic method, it is now possible to obtain a sufficient amount of material for such statistical investigations as that on the correlation of absolute velocities with absolute magnitudes. Among such investigations may be mentioned that of Adams, Strömberg and Joy,* one definite result being that the average absolute velocity increases by about 3 km./sec. for an increase of one magnitude on the absolute magnitude scale. It is to be noted, however, that as the observed parallax is used for determining both the absolute magnitude and the absolute velocity, these two latter quantities are not wholly independent; consequently, it is preferable to use a method which does not involve the parallax in determining the absolute velocities. This can only be done by means of the radial velocities alone. Removing the parallactic component from the radial velocities we obtain the absolute radial velocities referred to fixed axes and it is these radial velocities which we wish to correlate with absolute magnitude and also with spectral type. The following results, shown in Table 12, have recently been obtained by B. Boss,† using only stars found in Schlesinger's *Catalogue of Bright Stars*, 1930: the parallaxes were taken from this catalogue and the radial velocities from Moore's *General Catalogue of Radial Velocities*.‡

Table 12. *Mean absolute radial velocity (km./sec.)*

Absolute magni-tude, M	Spectral type					
	B	A	F	G	K	M
−1·5	7·7	9·4	12·1 (+1·7)	12·0 (−0·4)	14·2 (−0·3)	16·9 (+0·4)
−0·5	8·9	11·5	13·7 (+1·7)	14·3 (+0·2)	16·1 (0·0)	20·8 (+2·6)
+0·5	7·0	12·0	15·2 (+1·5)	14·4 (−1·3)	17·0 (−0·8)	15·9 (−3·9)
+1·5	—	11·1	12·7 (−2·6)	17·4 (0·0)	20·3 (+0·9)	18·4 (−3·1)
+2·5	—	8·7	15·4 (−1·6)	20·1 (+1·1)	20·4 (−0·7)	—
+3·5	—	—	14·3 (−4·3)	23·8 (+3·1)	29·7 (+7·0)	—
+4·5	—	—	20·6 (+0·3)	22·3 (0·0)	21·3 (−3·1)	—
+5·5	—	—	—	25·4 (+1·4)	—	—

The table shows that the absolute radial velocity, R, increases on the whole with absolute magnitude, M, and spectral class between F and M. An empirical formula is suggested, to represent the facts presented by the table, namely,

$$R = A + xC + MD,$$

* *Ap. J.* **54**, 9, 1921. † *A.J.* **44**, 182, 1935. ‡ *Lick Publ.* **18**, 1932.

where A, C and D are constants and x a number attached to a spectral type on the assumption that the spectral series is linear; thus for types B, A, F, G, K, M, the corresponding values of x are taken to be $-2{\cdot}5$, $-1{\cdot}5$, $-0{\cdot}5$, $+0{\cdot}5$, $+1{\cdot}5$, $+2{\cdot}5$. A least squares solution of the equation yields

$$A = +13{\cdot}88, \quad C = +2{\cdot}05, \quad D = +1{\cdot}65.$$

The quantities in parentheses in Table 12 are the differences between the observed quantities shown in the table and the corresponding values of R calculated by means of the above formula. The value of D shows that for any spectral class or for all the classes combined the absolute radial velocity increases by 1·65 km./sec. for each unit increase in the absolute magnitude. According to the mass-luminosity relationship, the mass of a star is a function of its absolute magnitude and, accordingly, the previous result may be expressed in the form that the more massive a star is, the less is its absolute speed. Although no great emphasis need be placed on the empirical formula, it would appear that, qualitatively, the conclusions we have mentioned are reliable.

3·5. *Kapteyn's equations.*

In the previous sections the analysis has been developed for the usual equatorial system of axes, the observed quantities being the components, μ_α and μ_δ, of proper motion and the radial velocity, ρ. In certain researches it is convenient to take one of the axes of coordinates to be defined by the direction opposite to that of the solar motion. In Fig. 19, A is the antapex of the solar motion and we take OA to be the z-axis; OJ and OK are the x- and y-axes; we can specify J, if we wish, as either of the points of intersection of the equator with the great circle of which A is the pole. The observed annual proper motion of a star at X is resolved into two components: (i) υ, towards the antapex, and (ii) τ, perpendicular to the great circle AXB. Expressing all linear velocities in km./sec., the linear velocity corresponding to υ is $\kappa\upsilon/p$, where p is the parallax of the star and $\kappa = 4{\cdot}74$; similarly the linear velocity corresponding to τ is $\kappa\tau/p$.

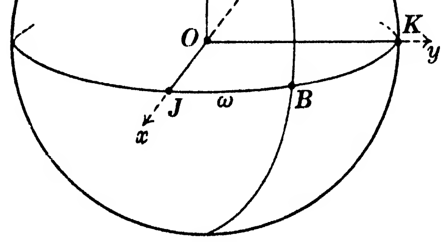

Fig. 19

Let λ denote the angular distance of X from the antapex and ω the arc

JB. The direction-cosines of the vectors τ, υ and ρ, with respect to the x-, y-, z-axes, are then as follows:

For τ:	$-\sin\omega$,	$\cos\omega$,	0.
For υ:	$-\cos\lambda\cos\omega$,	$-\cos\lambda\sin\omega$,	$\sin\lambda$.
For ρ:	$\sin\lambda\cos\omega$,	$\sin\lambda\sin\omega$,	$\cos\lambda$.

If the components of the linear velocity of the star at X, parallel to the three axes, are x, y and z respectively, we have

$$-\frac{\kappa\tau}{p}\sin\omega-\frac{\kappa\upsilon}{p}\cos\lambda\cos\omega+\rho\sin\lambda\cos\omega=x, \qquad \ldots\ldots(1)$$

$$\frac{\kappa\tau}{p}\cos\omega-\frac{\kappa\upsilon}{p}\cos\lambda\sin\omega+\rho\sin\lambda\sin\omega=y, \qquad \ldots\ldots(2)$$

$$\frac{\kappa\upsilon}{p}\sin\lambda \qquad +\rho\cos\lambda \qquad =z. \qquad \ldots\ldots(3)$$

The velocities x and y are components of the peculiar linear velocity of the star and the component z is made up of the parallactic velocity V_0 and the component of the peculiar velocity in the direction of the z-axis. Summing (1), (2) and (3) for N stars scattered over the sky, we expect that Σx and Σy will vanish or tend to vanish and that Σz will tend to the value NV_0. In the ideal case, we then have the equations

$$-\kappa\Sigma\frac{\tau}{p}\sin\omega-\kappa\Sigma\frac{\upsilon}{p}\cos\lambda\cos\omega+\Sigma\rho\sin\lambda\cos\omega=0, \qquad \ldots\ldots(4)$$

$$\kappa\Sigma\frac{\tau}{p}\cos\omega-\kappa\Sigma\frac{\upsilon}{p}\cos\lambda\sin\omega+\Sigma\rho\sin\lambda\sin\omega=0, \qquad \ldots\ldots(5)$$

$$\kappa\Sigma\frac{\upsilon}{p}\sin\lambda \qquad +\Sigma\rho\cos\lambda \qquad =NV_0, \qquad \ldots\ldots(6)$$

which are substantially the same formulae as given by Kapteyn* and his collaborators.

If we know the parallaxes, p, of the stars in addition to the other observed quantities μ_α, μ_δ and ρ, we can try various positions of the antapex and the correct position will be determined by the consideration that the left-hand sides of (4) and (5) will both be zero. The z-axis is accordingly found and the formula (6) then gives the solar motion. This method, originally suggested by Kapteyn, involves a vast amount of computing and from this point of view is hardly to be recommended.

Alternatively, we may suppose that the position of the antapex is known with sufficient accuracy and we can employ (6) in a form to be found below for calculating the solar motion; actually, the determinations of the solar apex for stars as faint as the eighth magnitude are remarkably accordant

* *Groningen Publ.* **29**, **6**, **1918**.

and it would appear that no serious error would result, so far as (6) is concerned, by assuming the coordinates of the apex to be (270°, +34°), the position found by L. Boss, as stated in section 3·35.

Eliminating τ between (1) and (2), we obtain

$$-\frac{\kappa v}{p}\cos\lambda + \rho\sin\lambda = x\cos\omega + y\sin\omega. \qquad \text{......(7)}$$

Writing $z = V_0 + z_1$ in (3), where z_1 is the linear component of the star's peculiar motion in the direction of the z-axis, we have

$$\frac{\kappa v}{p}\sin\lambda + \rho\cos\lambda = V_0 + z_1. \qquad \text{......(8)}$$

Eliminating v between (7) and (8), we obtain

$$\rho = V_0\cos\lambda + (x\cos\omega + y\sin\omega)\sin\lambda + z_1\cos\lambda$$

or

$$\rho = V_0\cos\lambda + \epsilon, \qquad \text{......(9)}$$

where ϵ is evidently the radial component of the star's peculiar motion. It is to be remembered that ρ is the observed radial velocity, that is, measured relative to the sun. We have an equation of the form (9) for each of the N stars and the best we can do in determining the value of V_0 is to suppose that ϵ has the characteristics of an accidental error and then to apply the method of least squares. We thus obtain

$$V_0 = \frac{\Sigma\rho\cos\lambda}{\Sigma\cos^2\lambda}, \qquad \text{......(10)}$$

the summations extending over the N stars.

If we are dealing with B type stars, a K term must be added to (9), and the normal equations formed in the usual way.

This formula, (10), is the most convenient one for finding the value of the solar motion when the position of the solar apex is assumed, for λ can be found readily for any given star by means of diagrams specially constructed for this purpose.*

We remark here—the subject will be more fully treated in Chapter VI—that the equations (1) to (3) can be used to find the mean parallax of the group of stars concerned, if V_0 has been determined by means of (10), or otherwise. Eliminating ρ between (7) and (8), we find that

$$V_0 p\sin\lambda = \kappa v + p\{(x\cos\omega + y\sin\omega)\cos\lambda - z_1\sin\lambda\}$$

or

$$V_0 p\sin\lambda = \kappa v + \epsilon_1, \qquad \text{......(11)}$$

where ϵ_1, it is assumed, has the characteristics of an accidental error, at any rate if the dispersion in the parallaxes is small.

* J. M. Baldwin, *M.N.* **89**, 453, 1929; J. A. Pearce and S. N. Hill, *Victoria Publ.* **4**, 49, 1927; W. M. Smart, *M.N.* **83**, 465, 1923.

In this case we may apply the method of least squares to the N equations of type (11), and we obtain, denoting the mean parallax by $\bar{p}$,

$$V_0 \bar{p} = \frac{\kappa \Sigma v \sin\lambda}{\Sigma \sin^2\lambda}. \qquad \text{......(12)}$$

3·61. *The practical determination of the solar motion.*

In investigations of the solar motion which are based either on proper motions or on radial velocities according to the theoretical principles outlined in previous sections, it is the practice to divide the sky into areas of several hundred square degrees and to form the equations of condition for the centres of the various areas. For example, if we are dealing with radial velocities and if $(-X, -Y, -Z)$ denote, as usual, the linear components of the solar motion with respect to the assembly of stars for which we have the required observational material, the equation of condition for a particular star in a given area is

$$lX + mY + nZ + K = \rho, \qquad \text{......(1)}$$

where (l, m, n) are the direction-cosines for the star with reference to one of the usual systems of coordinate axes—equatorial or galactic—and ρ is the observed radial velocity. This equation is equivalent to (1) of section 3·41, the peculiar radial component R being omitted. In a strict calculation we should require to form an equation of the form of (1) for each star; but for the sake of economy in calculation when the data refer to several thousands of stars, the usual procedure is to form equations of condition of the type

$$l_0 X + m_0 Y + n_0 Z + K = \bar{\rho}, \qquad \text{......(2)}$$

where (l_0, m_0, n_0) are the direction-cosines for the centre of the region and $\bar{\rho}$ is the mean algebraic radial velocity of the stars in the area, and to attach the weight N to the equation, N being the number of stars in the area. This procedure is tantamount to making the assumption that all the N stars are situated at the centre of the region and that their radial velocities, if the stars are so situated, are the same as the observed velocities. A similar argument applies to the determination of the solar motion from the observed proper motions. It will be shown that the general result is to introduce a systematic error into one or more of the quantities to be found, this error depending on the extent of the areas into which the sky is divided. For areas of four or five hundred square degrees, a systematic error of the order of half a kilometre per second can result in one component of the solar motion; relatively, this is about one-quarter* of the magnitude of the precessional

* The results of section 3·35 show that Δk and Δn are each numerically about one-tenth of the annual parallactic motion, M, at an angular distance of 90° from the antapex, compared with which the correction of ½ km./sec. is about 1/40 of the total solar motion.

constants Δk and Δn discussed in section 3·32. With somewhat larger areas, the correction to the solar motion would be, relatively, of the same order of magnitude as the quantities Δk and Δn. It is, therefore, important to investigate the effects of the corrections due to the grouping of the stars.

Clearly, the ideal way to deal with the problem is to subdivide the sky into areas not exceeding a hundred square degrees, for then the systematic corrections are comparatively innocuous. But this entails a vast expenditure of labour and time which might justifiably be regarded as not commensurate with the subsequent gain in precision. The alternative is to continue to deal with large areas, thereby keeping the equations of condition comparatively small in number, and to correct the observed quantities—or rather the means over the areas—in accordance with the systematic effects which can be easily calculated under normal circumstances. We now investigate the theoretical expressions for these systematic corrections.

Let (x, y, z) denote the components of the linear motion of a star, corresponding to the annual proper motion components $\mu_\alpha \cos\delta$, μ_δ and to the observed radial velocity ρ. Then

$$x = \frac{\kappa}{p}\mu_\alpha \cos\delta, \quad y = \frac{\kappa}{p}\mu_\delta, \quad z = \rho, \qquad \ldots\ldots(3)$$

where $\kappa = 4{\cdot}74$ and p is the star's parallax. If (u, v, w) are the components of the star's motion, relative to the sun, with respect to the usual system of equatorial axes, we have the following equations connecting (x, y, z) and (u, v, w):

$$x = -u\sin\alpha + v\cos\alpha, \qquad \ldots\ldots(4)$$

$$y = -u\cos\alpha\,\sin\delta - v\sin\alpha\,\sin\delta + w\cos\delta, \qquad \ldots\ldots(5)$$

$$z = u\cos\alpha\,\cos\delta + v\sin\alpha\,\cos\delta + w\sin\delta. \qquad \ldots\ldots(6)$$

The formulae (4) and (5) are analogous to (4) and (5) of section 3·42, while (6) is analogous to (1) of section 3·41. These equations can be written in the alternative forms:

$$u = -x\sin\alpha - y\cos\alpha\sin\delta + z\cos\alpha\,\cos\delta, \qquad \ldots\ldots(7)$$

$$v = x\cos\alpha - y\sin\alpha\,\sin\delta + z\sin\alpha\,\cos\delta, \qquad \ldots\ldots(8)$$

$$w = y\cos\delta + z\sin\delta. \qquad \ldots\ldots(9)$$

Let (A, D) be the coordinates of the centre, R, of the region. Let (x', y', z') denote the values of (x, y, z) if the star were observed at R with the components (u, v, w) unaltered. Then

$$x' = -u\sin A + v\cos A, \qquad \ldots\ldots(10)$$

$$y' = -u\cos A\,\sin D - v\sin A\,\sin D + w\cos D, \qquad \ldots\ldots(11)$$

$$z' = u\cos A\,\cos D + v\sin A\,\cos D + w\sin D. \qquad \ldots\ldots(12)$$

From (10) we obtain

$$x'-x = -x-u\sin A+v\cos A,$$

whence, using (7) and (8),

$$x'-x = -x\{1-\cos(\alpha-A)\}-y\sin(\alpha-A)\sin\delta+z\sin(\alpha-A)\cos\delta \quad\ldots\ldots(13)$$

Similarly, we derive

$$y'-y = x\sin(\alpha-A)\sin D+y\{\cos D\cos\delta+\cos(\alpha-A)\sin D\sin\delta-1\}$$
$$+z\{\cos D\sin\delta-\cos(\alpha-A)\sin D\cos\delta\}, \quad\ldots\ldots(14)$$

$$z'-z = -x\sin(\alpha-A)\cos D+y\{\sin D\cos\delta-\cos(\alpha-A)\cos D\sin\delta\}$$
$$+z\{\sin D\sin\delta+\cos(\alpha-A)\cos D\cos\delta-1\}. \quad\ldots\ldots(15)$$

The three formulae (13), (14) and (15) are accurate formulae, and the values of $x'-x$, $y'-y$, $z'-z$ can be found if all the observational data are known for each star.

The formulae are also accurate if we are dealing with parallactic motion alone. In this case, writing X, Y, and Z for u, v and w in (4), (5) and (6), we see that (4) and (5) become simply Airy's equations (section 3·34) for determining the solar motion from the proper motions and (6) the usual equation for the radial velocities (we omit here the consideration of the Δk, Δn and the K terms). Also, the values of x, y and z on the right of (13), (14) and (15) become the corresponding components of the parallactic motion.

We shall suppose that a given region is defined by the meridians $A-\phi$ and $A+\phi$ and by parallels of declination $D-\theta$ and $D+\theta$. For parallactic motion it will be seen that, if the stars are uniformly distributed over the region, the values of $\Sigma(x'-x)$, $\Sigma(y'-y)$ and $\Sigma(z'-z)$ are of order θ^2 and ϕ^2, the angles θ and ϕ being expressed in circular measure and the summations being taken over the whole area. It follows that, in the general case, when using the formulae for $\Sigma(x'-x)$, etc. it will be adequate to replace x, y and z on the right of (13), (14) and (15) by the corresponding parallactic values.

N being the number of stars in the region, we write

$$\Sigma x'-\Sigma x = NC_x, \quad\ldots\ldots(16)$$

$$\Sigma y'-\Sigma y = NC_y, \quad\ldots\ldots(17)$$

$$\Sigma z'-\Sigma z = NC_z, \quad\ldots\ldots(18)$$

in which C_x, C_y and C_z can be found from the expressions on the right of (13), (14) and (15) with x, y, z referring to parallactic motion.

For example, if we introduce the terms Δk and Δn in the first of Airy's formulae (compare (1) of section 3·34), the correct equation of condition for the region as a whole is

$$-X\sin A+Y\cos A+\frac{\kappa}{p}\Delta k\cos D+\frac{\kappa}{p}\Delta n\sin A\sin D = \frac{1}{N}\Sigma x+C_x, \quad\ldots\ldots(19)$$

its weight being N.

As already stated, the value of C_x for large areas is comparable in magnitude with the values of Δk and Δn.

We now consider the evaluation of the corrections, C, it being assumed that the stars are uniformly distributed over the particular region concerned. It may be noted here that the equations so far derived hold for galactic coordinates on the understanding that α, A and δ, D refer in this case to galactic longitude and latitude respectively.

3·62. (i) *Evaluation of C_x.*

Let R in Fig. 20 denote the centre (A, D) of the region and X the position of a star (α, δ) in the region. Let P be the north pole and M the position of the antapex (α_0, δ_0); the solar motion is V_0. We denote by λ and χ the angular distance of X from M and the position angle of M with reference to X, respectively; λ' and χ' refer similarly to R. Also write

$$\alpha_0 - A = B; \quad \alpha - A = \psi.$$

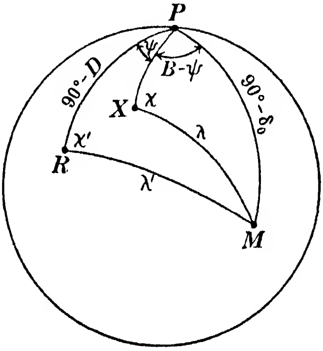

Fig. 20

We consider the region to be defined by the meridians $A-\phi$ and $A+\phi$ and by the parallels of declination δ_1 and δ_2, where

$$\delta_1 = D - \theta; \quad \delta_2 = D + \theta. \qquad \ldots\ldots(1)$$

An element of area of the region is $\cos\delta\, d\delta\, d\psi$, and the total area, Δ, of the region is given by

$$\Delta = \int_{\delta_1}^{\delta_2} \cos\delta\, d\delta \int_{-\phi}^{\phi} d\psi$$

$$= 2\phi(\sin\delta_2 - \sin\delta_1),$$

whence, by (1),
$$\Delta = 4\phi \sin\theta \cos D. \qquad \ldots\ldots(2)$$

Equating $(x' - x)$ to the difference of the parallactic components at R and X, we have

$$x' - x = V_0 \sin\lambda' \sin\chi' - V_0 \sin\lambda \sin\chi$$

and hence
$$x' - x = V_0 \cos\delta_0 \{\sin B - \sin(B - \psi)\}.$$

Let n denote the number of stars per unit area so that the number in the element of area is $n \cos\delta\, d\delta\, d\psi$. Also

$$N = n\Delta. \qquad \ldots\ldots(3)$$

Hence we obtain

$$\frac{\Sigma x' - \Sigma x}{V_0 \cos\delta_0} = n\Delta \sin B - n \int_{\delta_1}^{\delta_2} \int_{-\phi}^{\phi} \cos\delta \sin(B - \psi)\, d\delta\, d\psi,$$

so that, by (16) of section 3·61 and by (3),

$$\frac{C_x}{V_0 \cos \delta_0} = \sin B - \frac{1}{\Delta}\{\sin \delta_2 - \sin \delta_1\}\{\cos(B-\phi) - \cos(B+\phi)\},$$

and, by (1) and (2),

$$C_x = V_0 \cos \delta_0 \sin(\alpha_0 - A)\left\{1 - \frac{\sin \phi}{\phi}\right\}. \qquad \text{......(4)}$$

With the assumption as to the uniformity of distribution of the stars over the region, this is an accurate formula. If ϕ is no greater than 20°, say, it will be sufficient to write (4) in the form

$$C_x = \tfrac{1}{6}\phi^2 V_0 \cos \delta_0 \sin(\alpha_0 - A), \qquad \text{......(5)}$$

where ϕ is in circular measure.

(ii) *Evaluation of* C_y.

At X the parallactic component in declination is $V_0 \sin \lambda \cos \chi$. We then have

$$\begin{aligned} y' - y &= V_0 \sin \lambda' \cos \chi' - V_0 \sin \lambda \cos \chi \\ &= V_0\{\sin \delta_0 \cos D - \cos \delta_0 \sin D \cos B\} \\ &\qquad - V_0\{\sin \delta_0 \cos \delta - \cos \delta_0 \sin \delta \cos(B - \psi)\}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\Sigma y' - \Sigma y}{V_0} &= \sin \delta_0 \left\{N \cos D - n \int_{\delta_1}^{\delta_2} \cos^2 \delta \, d\delta \int_{-\phi}^{\phi} d\psi\right\} \\ &\quad - \cos \delta_0 \left\{N \sin D \cos B - n \int_{\delta_1}^{\delta_2} \sin \delta \cos \delta \, d\delta \int_{-\phi}^{\phi} \cos(B - \psi)\, d\psi\right\}, \end{aligned}$$

from which

$$\begin{aligned} C_y &= V_0 \sin \delta_0 \left\{\cos D - \frac{\phi}{\Delta}(\sin 2\theta \cos 2D + 2\theta)\right\} \\ &\quad - V_0 \cos \delta_0 \left\{\sin D \cos B - \frac{1}{\Delta} \sin \phi \cos B \sin 2\theta \sin 2D\right\} \end{aligned}$$

or

$$\begin{aligned} C_y &= V_0 \sin \delta_0 \cos D \left\{1 - \frac{1}{2\cos^2 D}\left(\frac{\theta}{\sin \theta} + \cos \theta \cos 2D\right)\right\} \\ &\quad - V_0 \cos \delta_0 \sin D \cos(\alpha_0 - A)\left\{1 - \frac{\sin \phi \cos \theta}{\phi}\right\}. \qquad \text{......(6)} \end{aligned}$$

This is an accurate formula. Its approximate form, in which θ^4 and ϕ^4 are neglected, is

$$\begin{aligned} C_y &= \tfrac{1}{12} V_0 \theta^2 \sin \delta_0 \sec D (3 \cos 2D - 1) \\ &\quad - \tfrac{1}{6} V_0 (3\theta^2 + \phi^2) \cos \delta_0 \sin D \cos(\alpha_0 - A). \qquad \text{......(7)} \end{aligned}$$

(iii) *Evaluation of* C_z.

At X the parallactic radial component is $V_0 \cos\lambda$. Hence

$$z' - z = V_0 \cos\lambda' - V_0 \cos\lambda,$$

from which

$$\frac{z'-z}{V_0} = \sin\delta_0 \sin D + \cos\delta_0 \cos D \cos B$$
$$- \{\sin\delta_0 \sin\delta + \cos\delta_0 \cos\delta \cos(B-\psi)\}.$$

We then have

$$\frac{\Sigma z' - \Sigma z}{V_0} = \sin\delta_0 \left\{N \sin D - n\int_{\delta_1}^{\delta_2} \sin\delta\cos\delta\, d\delta \int_{-\phi}^{\phi} d\psi\right\}$$
$$+ \cos\delta_0 \left\{N\cos D\cos B - n\int_{\delta_1}^{\delta_2} \cos^2\delta\, d\delta \int_{-\phi}^{\phi} \cos(B-\psi)\, d\psi\right\},$$

from which we readily obtain

$$C_z = V_0 \sin\delta_0 \sin D\,(1-\cos\theta)$$
$$+ V_0 \cos\delta_0 \cos D \cos(\alpha_0 - A)\left\{1 - \frac{\sin\phi}{\phi}\left(\frac{2\theta + \sin 2\theta \cos 2D}{4\sin\theta\cos^2 D}\right)\right\}, \quad \ldots\ldots(8)$$

which is an accurate formula. In the approximate form, it becomes

$$C_z = \tfrac{1}{2} V_0 \theta^2 \sin\delta_0 \sin D$$
$$+ \tfrac{1}{6} V_0 \cos\delta_0 \cos D \cos(\alpha_0 - A)\{\phi^2 + \theta^2(3 - 2\sec^2 D)\}. \quad \ldots\ldots(9)$$

3·63. *Practical application.*

It transpires* that the whole effect of the C_x, C_y and C_z corrections is thrown into the components X, Y and Z of the parallactic motion. Thus the systematic errors introduced in the practical method of determining the solar motion and related constants leave Δk, Δn and K unaffected and produce errors only in the solar motion constants. These errors are not negligible when the areas of the regions are about four or five hundred square degrees as in the investigations of L. Boss and of Campbell and Moore, alluded to in previous sections. With the increase of observational material it might be thought desirable, or even essential, to divide the sphere into much smaller areas—say of a hundred square degrees—for which the systematic errors would be of negligible amount. But, since the whole sky contains about 40,000 square degrees, this would mean the treatment of about 400 equations of condition, and with four, five or six unknowns in the equations the amount of labour required to achieve the solutions would be prodigious.

But the greater part of this labour is unnecessary if we utilise the values of C_x, C_y and C_z in our equations; in other words, we correct our observed

* For a full discussion *v.* W. M. Smart, *M.N.* **96**, 461, 1936.

quantities according to the dimensions of the regions into which we subdivide the sky. It is thus possible to work with very much larger areas than have been used previously without sacrificing accuracy in the solutions. It is only necessary that the stars in any given region should be fairly uniformly distributed over the area. With this condition holding, the equation of condition formed for the centre of the region, with attached weight equal to the number of the stars in the region, is essentially the same, when the appropriate value of C_x or C_y or C_z is added, as if the equation of condition were formed for each star and the sum taken.

When the regions are arranged in zones of declination or of galactic latitude, the values of C_x, C_y and C_z can be easily and rapidly calculated.

In Smart and Green's determination* of the solar motion, areas as large as 900 to 1200 square degrees are employed; the number of equations of condition is reduced to thirty-four and the labour of forming the normal equations is thereby lessened to a very great extent.

* *M.N.* **96**, 471, 1936.

CHAPTER IV

THE TWO STAR-STREAMS

4·1. *Kapteyn's discovery of the star-streams.*

We have seen in section 2·3 that for an assembly of stars forming a single drift the distribution of proper motions, according to position angle, for any small area of the sky can be represented by a single drift-curve, symmetrical about a line giving the projection of the solar motion on the tangent plane at the point of the sky considered. Until the end of last century, it was generally assumed—in the absence of sufficient observational data and as a convenient working hypothesis—that the motions of the stars were entirely haphazard, and on this basis the characteristics of the solar motion were investigated in numerous researches. Kobold* was the first to recognise that this hypothesis of the haphazard distribution of the motions of the stars came into conflict with the observational data. Actually, in two papers† in 1895 and in 1897, Kapteyn had noticed what he considered to be anomalies in the distribution of the directions of proper motions—that is to say, distinct deviations from symmetrical drift-curves—but he later concluded‡ that the cause of such anomalies was to be looked for, not in any systematic effect connected with the real proper motions for different regions of the sky, but for the most part in a constant or systematic error in the declination components of the proper motions. However, Kapteyn was forced, a few years later, to abandon this attempted explanation, and in 1904, at an international scientific congress held in St Louis, he announced the discovery of the two star-streams.§ In particular he showed, in the language of drift-motions, that for a given region of the sky the distribution of proper motions in position angle could be adequately represented by the combination of two drift-curves, each with its own characteristic shape, dimensions and direction of the axis of symmetry. The combination of such information from different parts of the sky led to the conception of two streams of stars moving in opposite directions in space.||

In 1906 Eddington published¶ his convenient method of analysing the proper motions of the stars in any region of the sky and of deriving the characteristics of the streams. We shall now describe this method.

* *A.N.* **144**, 33, 1897; **150**, 257, 1899.

† *Publ. Acad. of Sciences, Amsterdam.*

‡ *Groningen Publ.* **5**, 3, 1900.

§ See also *British Association Report*, 1905, p. 257.

|| Kapteyn's mathematical analysis is to be found in *M.N.* **72**, 743, 1912.

¶ *M.N.* **67**, 34, 1906.

4·21. *Eddington's method of deriving the constants of the drift-curves from proper motions.*

It is assumed that, in a given region of the sky at S, the observed distribution of the proper motions in position angle is the result of superposing two drifts—called drift I and drift II—in which N_1 and N_2 are the numbers of stars, θ_1 and θ_2 are the position angles in which the drifts are pointing, and hV_1 and hV_2 are the projections on the tangent plane at S of the space-velocities of drift I and drift II with respect to the sun, the velocities being measured in terms of a theoretical unit $1/h$ proportional to the mean peculiar motion in either drift (see section 2·24). For a single drift of n stars, with linear velocities distributed according to the Maxwellian law, we have from section 2·3 for the number $n(\theta)\,d\theta \equiv \rho\,d\theta$ of stars moving between angles θ and $\theta + d\theta$ measured with respect to the axis of symmetry of the drift,

$$n(\theta)\,d\theta \equiv \rho\,d\theta = \frac{n\,d\theta}{2\sqrt{\pi}}\,e^{-h^2V^2}f(\tau), \qquad \ldots\ldots(1)$$

where hV is the projection of the drift-velocity (that is, the velocity of the assembly of stars as a whole) with respect to the sun, on the tangent plane at S. Also

$$\tau = hV\cos\theta. \qquad \ldots\ldots(2)$$

We now use θ in the sense of position angle so that, for a given value of θ, the angles between the corresponding direction and the directions of the axes of symmetry of drift I and drift II are $\theta - \theta_1$ and $\theta - \theta_2$, where θ_1 and θ_2 are the position angles of the axes of symmetry of the drifts. For drift I we accordingly have—denoting the radius vector of the drift-curve by ρ_1—

$$\rho_1\,d\theta = \frac{N_1}{2\sqrt{\pi}}\,d\theta\,e^{-h^2V_1^2}f(\tau_1), \qquad \ldots\ldots(3)$$

$$\tau_1 = hV_1\cos(\theta - \theta_1). \qquad \ldots\ldots(4)$$

Similarly, we have for drift II,

$$\rho_2\,d\theta = \frac{N_2}{2\sqrt{\pi}}\,d\theta\,e^{-h^2V_2^2}f(\tau_2), \qquad \ldots\ldots(5)$$

$$\tau_2 = hV_2\cos(\theta - \theta_2). \qquad \ldots\ldots(6)$$

Hence, for the distribution of proper motions resulting from the two drifts, the number, $\rho\,d\theta$, of stars moving between position angles θ and $\theta + d\theta$ is given by

$$\rho\,d\theta = \rho_1\,d\theta + \rho_2\,d\theta. \qquad \ldots\ldots(7)$$

Eddington's "trial and error" method consists in fitting two drift-curves so as to give as close a representation as possible of the observed distribution of the proper motions. In general, there are five constants to be found for the formulae (3) and (5) so that $\rho\,d\theta$, given by (7), may be in as good accordance as possible, for all position angles, with the observed values; the constants are: θ_1, hV_1 and N_1 for drift I, and θ_2, hV_2 and N_2 for drift II, with the condition that $N_1+N_2 = N$, where N is the total number of stars in the region considered.

Owing to the incidence of errors in the proper motions and to the comparatively small number of stars in a region for which proper motion data

Table 13. *Values of* $\phi(hV, \theta-\theta_0)$

$\theta-\theta_0$ \ hV		0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
0°	360°	0·67	0·79	0·91	1·05	1·20	1·36	1·52	1·69	1·87
5	355	0·67	0·78	0·91	1·05	1·19	1·35	1·50	1·67	1·84
10	350	0·67	0·78	0·90	1·04	1·18	1·34	1·48	1·64	1·80
15	345	0·67	0·77	0·90	1·02	1·16	1·30	1·44	1·58	1·71
20	340	0·66	0·76	0·88	1·00	1·13	1·26	1·37	1·50	1·61
25	335	0·65	0·75	0·87	0·97	1·08	1·20	1·30	1·40	1·48
30	330	0·65	0·74	0·85	0·94	1·03	1·13	1·21	1·30	1·36
35	325	0·64	0·73	0·83	0·90	0·98	1·06	1·12	1·18	1·22
40	320	0·64	0·72	0·80	0·86	0·93	0·98	1·02	1·06	1·07
45	315	0·63	0·70	0·77	0·82	0·87	0·91	0·94	0·95	0·95
50	310	0·63	0·69	0·74	0·78	0·81	0·83	0·85	0·84	0·82
55	305	0·62	0·67	0·71	0·74	0·76	0·77	0·76	0·75	0·71
60	300	0·61	0·65	0·68	0·70	0·70	0·70	0·68	0·65	0·62
65	295	0·60	0·63	0·65	0·66	0·65	0·64	0·61	0·57	0·53
70	290	0·59	0·62	0·62	0·62	0·60	0·58	0·54	0·51	0·45
75	285	0·58	0·60	0·59	0·58	0·55	0·52	0·48	0·44	0·39
80	280	0·57	0·58	0·56	0·54	0·51	0·48	0·43	0·38	0·33
85	275	0·57	0·56	0·54	0·51	0·48	0·43	0·39	0·34	0·29
90	270	0·56	0·54	0·51	0·48	0·44	0·39	0·35	0·30	0·25
95	265	0·55	0·52	0·49	0·45	0·41	0·36	0·31	0·27	0·22
100	260	0·54	0·51	0·47	0·43	0·38	0·33	0·28	0·23	0·19
105	255	0·53	0·50	0·45	0·40	0·35	0·30	0·25	0·21	0·17
110	250	0·52	0·48	0·43	0·38	0·33	0·28	0·23	0·19	0·15
115	245	0·52	0·47	0·42	0·36	0·31	0·26	0·21	0·17	0·14
120	240	0·51	0·46	0·40	0·34	0·29	0·24	0·19	0·16	0·12
125	235	0·50	0·44	0·38	0·33	0·27	0·22	0·18	0·14	0·11
130	230	0·50	0·43	0·37	0·31	0·26	0·21	0·17	0·13	0·10
135	225	0·49	0·43	0·36	0·30	0·25	0·20	0·16	0·13	0·09
140	220	0·49	0·42	0·35	0·29	0·24	0·19	0·15	0·12	0·09
145	215	0·48	0·41	0·34	0·28	0·23	0·18	0·14	0·11	0·08
150	210	0·48	0·41	0·33	0·27	0·22	0·17	0·13	0·10	0·08
155	205	0·48	0·40	0·33	0·27	0·21	0·17	0·13	0·10	0·08
160	200	0·47	0·39	0·32	0·26	0·21	0·16	0·12	0·10	0·07
165	195	0·47	0·39	0·32	0·26	0·20	0·16	0·12	0·09	0·07
170	190	0·47	0·39	0·32	0·25	0·20	0·16	0·12	0·09	0·07
175	185	0·47	0·39	0·31	0·25	0·20	0·16	0·12	0·09	0·07
180	180	0·47	0·39	0·31	0·25	0·20	0·16	0·12	0·09	0·07

can usually be obtained, it is not possible in practice to use too small a sector, $d\theta$, for which to compare the theoretical number of proper motions with the observed number. In investigations of this type where the total number of proper motions is a few hundreds, the sector is usually taken to be 10°, so that $d\theta = \pi/18$.

Let r_1 denote the theoretical number of stars, say in drift I, moving in the sector $\theta-5°$ to $\theta+5°$. Then

$$r_1 = \int_{\theta-5°}^{\theta+5°} \rho\, d\theta = \rho_1 \int_{\theta-5°}^{\theta+5°} d\theta = \frac{\pi}{18}\rho_1,$$

Table 13. *Values of* $\phi(hV,\ \theta-\theta_0)$

$\theta-\theta_0$ \ hV		1·0	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8
0°	360°	2·05	2·24	2·43	2·62	2·81	3·01	3·21	3·40	3·60
5	355	2·02	2·21	2·38	2·57	2·76	2·94	3·15	3·32	3·51
10	350	1·97	2·14	2·28	2·44	2·62	2·77	2·94	3·10	3·27
15	345	1·86	2·05	2·12	2·25	2·38	2·50	2·63	2·72	2·85
20	340	1·73	1·87	1·93	2·02	2·10	2·17	2·25	2·28	2·32
25	335	1·57	1·65	1·70	1·76	1·80	1·82	1·87	1·86	1·82
30	330	1·41	1·46	1·48	1·50	1·50	1·44	1·46	1·42	1·40
35	325	1·24	1·25	1·27	1·25	1·22	1·17	1·07	1·08	1·05
40	320	1·08	1·07	1·04	1·03	0·98	0·93	0·86	0·80	0·74
45	315	0·94	0·90	0·87	0·82	0·76	0·70	0·64	0·57	0·51
50	310	0·80	0·76	0·70	0·65	0·59	0·53	0·47	0·41	0·35
55	305	0·68	0·63	0·58	0·52	0·46	0·40	0·34	0·28	0·24
60	300	0·57	0·51	0·46	0·40	0·35	0·30	0·25	0·20	0·16
65	295	0·48	0·43	0·37	0·31	0·27	0·22	0·18	0·14	0·11
70	290	0·40	0·35	0·30	0·25	0·21	0·17	0·13	0·10	0·08
75	285	0·34	0·29	0·24	0·20	0·16	0·13	0·10	0·07	0·06
80	280	0·29	0·24	0·20	0·16	0·13	0·10	0·08	0·05	0·04
85	275	0·24	0·20	0·16	0·13	0·10	0·08	0·06	0·04	0·03
90	270	0·21	0·17	0·13	0·10	0·08	0·06	0·04	0·03	0·02
95	265	0·18	0·14	0·11	0·08	0·06	0·05	0·04	0·02	0·02
100	260	0·15	0·12	0·10	0·07	0·05	0·05	0·04	0·02	0·01
105	255	0·14	0·11	0·08	0·06	0·05	0·04	0·03	0·02	0·01
110	250	0·12	0·09	0·07	0·05	0·04	0·04	0·03	0·02	0·01
115	245	0·11	0·08	0·06	0·04	0·04	0·03	0·03	0·02	0·01
120	240	0·09	0·07	0·05	0·04	0·03	0·03	0·03	0·01	0·01
125	235	0·09	0·06	0·05	0·04	0·03	0·02	0·02	0·01	0·01
130	230	0·08	0·06	0·04	0·03	0·02	0·02	0·02	0·01	0·01
135	225	0·07	0·05	0·04	0·03	0·02	0·02	0·02	0·01	0·01
140	220	0·07	0·05	0·04	0·03	0·02	0·02	0·02	0·01	0·01
145	215	0·06	0·05	0·04	0·03	0·02	0·02	0·02	0·01	0·01
150	210	0·06	0·05	0·03	0·03	0·02	0·02	0·02	0·01	0·01
155	205	0·06	0·04	0·03	0·03	0·02	0·02	0·02	0·01	0·01
160	200	0·05	0·04	0·03	0·03	0·02	0·02	0·01	0·01	0·01
165	195	0·05	0·04	0·03	0·03	0·02	0·02	0·01	0·01	0·01
170	190	0·05	0·04	0·03	0·03	0·02	0·02	0·01	0·01	0·01
175	185	0·05	0·04	0·03	0·03	0·02	0·02	0·01	0·01	0·01
180	180	0·05	0·04	0·03	0·03	0·02	0·02	0·01	0·01	0·01

where ρ_1 is the mean value of ρ in the sector. Since the sector is small, we take the value of ρ_1 to be equivalent to the value of ρ corresponding to θ. Thus we obtain

$$r_1 = 0{\cdot}04923 N_1 e^{-h^2 V_1^2} f(\tau_1). \qquad \ldots\ldots(8)$$

The values of $\log f(\tau)$, for values of τ between $-1{\cdot}3$ and $+2{\cdot}0$, have been given by Eddington.* It is however more convenient in the calculations to use numerical values of a function $\phi(hV, \theta-\theta_1)$, given by

$$\phi(hV, \theta-\theta_1) = e^{-h^2V^2} f(\tau), \qquad \ldots\ldots(9)$$

so that
$$r_1 = 0{\cdot}04923 N_1\, \phi(hV_1, \theta-\theta_1). \qquad \ldots\ldots(10)$$

Table 13 gives the values† of the function ϕ for values of $\theta-\theta_0$ at 5° intervals and for values of hV from 0·1 to 1·8 at intervals of 0·1; ($\theta_0 \equiv \theta_1$ or θ_2).

In the same way, if r_2 denotes the number of stars belonging to drift II and moving in the sector $\theta-5°$ to $\theta+5°$,

$$r_2 = 0{\cdot}04923 N_2\, \phi(hV_2, \theta-\theta_2). \qquad \ldots\ldots(11)$$

In most regions of the sky it is found that one drift (drift I) is usually more prominent than the other and it is usually easy to obtain approximate values of θ_1, hV_1 and N_1 by little more than inspection. On the other hand, drift II—owing to the much smaller values of hV_2—gives a less distinctive distribution from which it is somewhat difficult to ascertain reliable values of θ_2 and hV_2 at a first attempt.

An example will be worked out in the next section to illustrate the procedure. It may, however, be conveniently stated here that the observed statistics of the proper motions—derived as the number of stars moving in each of the 10° sectors $\theta-5°$ to $\theta+5°$ for the following thirty-six values of θ: 5°, 15°, ... 355°—are smoothed by taking the means of three adjacent sectors of 10°. This procedure tends to eliminate accidental irregularities in the observed distribution. For the sake of consistency, the calculated distribution (based on the figures of Table 13) is smoothed in a similar way.

4·22. *Example of the analysis of proper motions.*

Table 14 gives the distribution in position angle of the proper motions of 684 stars, formed by combining the results of two adjacent regions (XXVI and XXVII) measured by G. H. ten Bruggencate‡ at Groningen. The combined region, denoted by M (centre at R.A. $14^{h}\,5^{m}$; dec. $+34°{\cdot}2$), was treated by the author§ according to the method of the previous section. In Table 14, r_0 denotes the observed number of stars moving between position angles $\theta-5°$ and $\theta+5°$ for $\theta = 5°, 15°, 25°, \ldots 355°$ (fractional numbers occur as the counts have been smoothed by taking, as the effective number moving in a sector $\theta-5°$ to $\theta+5°$, the mean of the observed numbers in the three sectors $\theta-15°$ to $\theta-5°$, $\theta-5°$ to $\theta+5°$, and $\theta+5°$ to $\theta+15°$).

The data of Table 14 are plotted in Fig. 21 (full-line curve), the position angle θ being taken as the abscissa and the number, r_0, of stars as the

* *M.N.* **67**, 37, 1906.
† W. M. Smart, *M.N.* **87**, 128, 1926.
‡ *B.A.N.* **3**, 35, 1925.
§ *M.N.* **88**, 144, 1927.

Table 14. *Distribution of proper motions in position angle*

Position angle, θ	Number of stars, r_0	Position angle, θ	Number of stars, r_0	Position angle, θ	Number of stars, r_0
5°	9·7	125°	21·3	245°	22·7
15	8·3	135	22·7	255	28·0
25	7·7	145	23·3	265	32·3
35	8·3	155	22·7	275	35·3
45	7·7	165	22·0	285	35·3
55	8·3	175	21·0	295	34·3
65	8·3	185	19·7	305	31·3
75	8·3	195	17·3	315	28·7
85	8·3	205	16·0	325	27·0
95	9·0	215	16·0	335	22·0
105	11·3	225	16·7	345	15·5
115	19·0	235	18·7	355	12·0

ordinate. The curve shows two distinct maxima near $\theta = 150°$ and 290°. Assuming that 290° refers to drift I, we show the results of three solutions by trying various values of hV_1, hV_2, N_1 and N_2 as follows:

	hV_1	θ_1	N_1	hV_2	θ_2	N_2
Solution (i)	1·2	290°	324	0·7	150°	360
(ii)	1·0	290	360	0·7	150	324
(iii)	1·0	290	360	0·6	150	324

The corresponding curves are shown or indicated in Fig. 21. The solution (iii), which seems to represent the observed distribution of proper motions most successfully, is taken as the definitive solution—it is represented in the figure by a broken line.

The details of solution (iii) are given in Table 15. The second column contains the values (denoted by ϕ_1) of the function $\phi(hV, \theta-\theta_1)$ for $hV = 1·0$ and $\theta_1 = 290°$. These values are simply extracted from Table 13. In the fourth column we have the smoothed values of ϕ_1 obtained as explained above. The third and fifth columns give similar information with respect to theoretical values for drift II ($hV_2 = 0·6, \theta_2 = 150°$).

The numbers under the heading r_1 are obtained by multiplying the smoothed values of ϕ_1 (column 4) by $0·04923N_1 \equiv 17·7$ ($N_1 = 360$). The next column, r_2, is obtained in a similar way by multiplying the smoothed values of ϕ_2 by $0·04923N_2 \equiv 15·9$ ($N_2 = 324$). The numbers under the heading $r \equiv r_1 + r_2$ give the theoretical distribution. The penultimate column gives the observed distribution, r_0, and the last column shows the differences, to the nearest integer, between the observed and theoretical distributions.

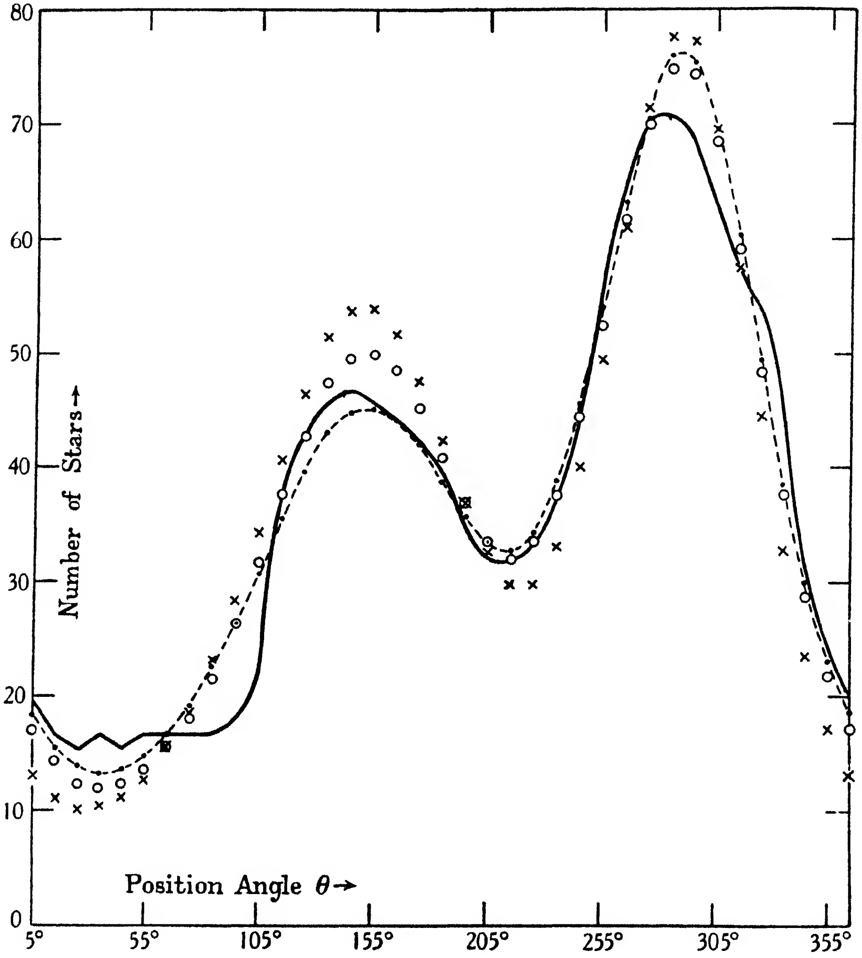

Fig. 21

The full-line curve is the observed distribution of proper motions.

× × × ... denotes solution (i).
o o o ... " (ii).
—·—·—· " (iii).

Table 15. *Solution (iii) (Region M)*

Position angle θ	ϕ_1	ϕ_2	ϕ_1 (smoothed)	ϕ_2 (smoothed)	r_1	r_2	$r \equiv r_1 + r_2$	r_0 (observed)	Observed minus Calculated
5°	0·34	0·18	0·35	0·18	6·2	2·9	9·1	9·7	+1
15	0·24	0·20	0·25	0·20	4·4	3·2	7·6	8·3	+1
25	0·18	0·22	0·19	0·23	3·4	3·7	7·1	7·7	+1
35	0·14	0·26	0·14	0·26	2·5	4·1	6·6	8·3	+2
45	0·11	0·30	0·11	0·31	1·9	4·9	6·8	7·7	+1
55	0·09	0·36	0·09	0·36	1·6	5·7	7·3	8·3	+1
65	0·07	0·43	0·07	0·44	1·2	7·0	8·2	8·3	0
75	0·06	0·52	0·06	0·53	1·1	8·4	9·5	8·3	−1
85	0·06	0·64	0·06	0·64	1·1	10·2	11·3	8·3	−3
95	0·05	0·77	0·05	0·77	0·9	12·2	13·1	9·0	−4
105	0·05	0·91	0·05	0·91	0·9	14·5	15·4	11·3	−4
115	0·05	1·06	0·05	1·06	0·9	16·9	17·8	19·0	+1
125	0·05	1·20	0·05	1·19	0·9	18·9	19·8	21·3	+1
135	0·06	1·30	0·06	1·28	1·1	20·4	21·5	22·7	+1
145	0·06	1·35	0·06	1·33	1·1	21·1	22·2	23·3	+1
155	0·07	1·35	0·07	1·33	1·2	21·1	22·3	22·7	0
165	0·09	1·30	0·09	1·28	1·6	20·4	22·0	22·0	0
175	0·11	1·20	0·11	1·19	1·9	18·9	20·8	21·0	0
185	0·14	1·06	0·14	1·06	2·5	16·9	19·4	19·7	0
195	0·18	0·91	0·19	0·91	3·4	14·5	17·9	17·3	−1
205	0·24	0·77	0·25	0·77	4·4	12·2	16·6	16·0	−1
215	0·34	0·64	0·35	0·64	6·2	10·2	16·4	16·0	0
225	0·48	0·52	0·50	0·53	8·8	8·4	17·2	16·7	−1
235	0·68	0·43	0·70	0·44	12·4	7·0	19·4	18·7	−1
245	0·94	0·36	0·95	0·36	16·8	5·7	22·5	22·7	0
255	1·24	0·30	1·25	0·31	22·1	4·9	27·0	28·0	+1
265	1·57	0·26	1·56	0·26	27·6	4·1	31·7	32·3	+1
275	1·86	0·22	1·82	0·23	32·2	3·7	35·9	35·3	−1
285	2·02	0·20	1·97	0·20	34·9	3·2	38·1	35·3	−3
295	2·02	0·18	1·97	0·18	34·9	2·9	37·8	34·3	−4
305	1·86	0·17	1·82	0·17	32·2	2·7	34·9	31·3	−4
315	1·57	0·16	1·56	0·16	27·6	2·5	30·1	28·7	−1
325	1·24	0·16	1·25	0·16	22·1	2·5	24·6	27·0	+2
335	0·94	0·16	0·95	0·16	16·8	2·5	19·3	22·0	+3
345	0·68	0·16	0·70	0·16	12·4	2·5	14·9	15·5	+1
355	0·48	0·17	0·50	0·17	8·8	2·7	11·5	12·0	+1

4·23. *Dyson's graphical method of analysis.*

In this method* the principal object is the representation of the observed distribution of the proper motions in position angle by two symmetrical curves, the position angles of the maxima being identified with the directions of the axes of symmetry of the two constituent drifts. The procedure may be best explained by an example. Fig. 22 shows the observed distribution of proper motions in Dyson's region *B* (*loc. cit.*) with centre at R.A. 0°, dec. 0°. Inspection shows that (i) there is a pronounced maximum between position angles 90° and 100° (this refers to drift I), (ii) there is a second maximum between 180° and 210° (this refers to drift II), and (iii) the effect of drift II dies away at 280° approximately. If now the maximum of drift II is at

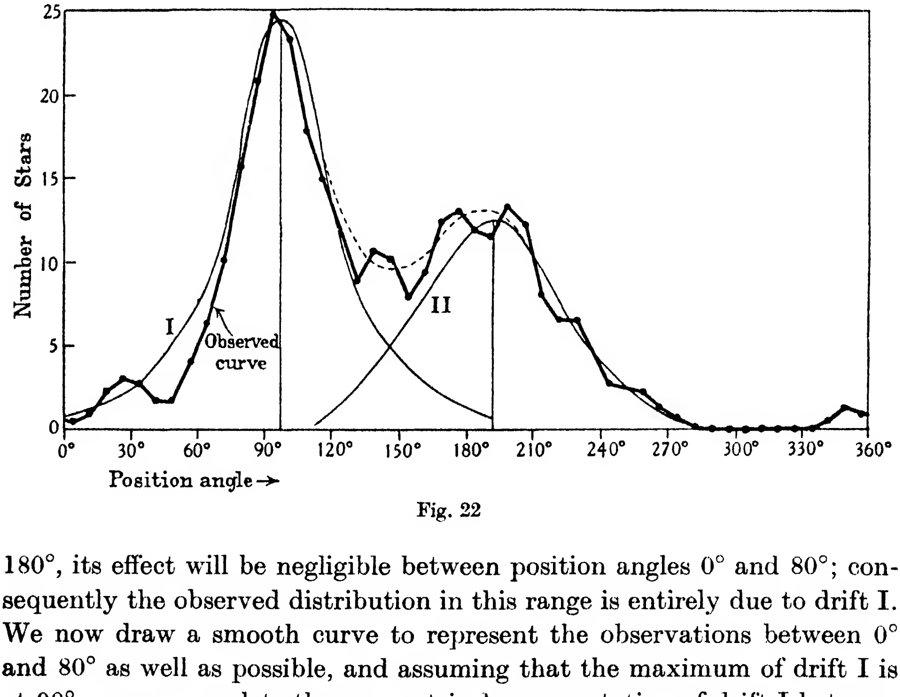

Fig. 22

180°, its effect will be negligible between position angles 0° and 80°; consequently the observed distribution in this range is entirely due to drift I. We now draw a smooth curve to represent the observations between 0° and 80° as well as possible, and assuming that the maximum of drift I is at 90° we can complete the symmetrical representation of drift I between 100° and say 200°, where its effect becomes negligible. Subtracting the ordinates, at each position angle between 100° and 200°, of this curve from the ordinates of the observed curve, we are left with the distribution of proper motions due to drift II alone. The maximum of this latter distribution is found to be near 190°, and a symmetrical curve is drawn to fit the distribution as well as possible. We now subtract the ordinates of this symmetrical curve from the ordinates of the observed curve between say 80°

* F. W. Dyson, *Proc. Roy. Soc. Edin.* **28**, pt. iii, 231, 1908.

and 200° and we are left with substantially the distribution of proper motions due to drift I. A symmetrical curve can now be drawn to represent the contribution of drift I to the observed distribution; its maximum is now found to be near 95°. The contribution of drift II can now be derived in a similar way and the maximum inferred as before; it is found to be about 190° or 195°.

The combination of the two symmetrical curves (drift I and drift II) should then give a good representation of the observed distribution. This combined curve is shown by a dotted line.

It is to be remarked that the symmetrical curves drawn in the above processes are not necessarily true drift-curves, although, in the great majority of cases where the effect of star-streaming is well defined, they are unlikely to deviate appreciably from true drift-curves. The procedure determines θ_1 and θ_2—the position angles in which the drifts are pointing—but the values of the drift-velocities are not determined. This method of analysis has been extensively used, notably by Dyson and ten Bruggencate.*

4·31. *The apices and space-velocities of the two drifts.*

When proper motion data are analysed for several different regions of the sky, it is usually found that one drift has associated with it a drift-velocity, relative to the sun, much greater than that of the second drift; the former is accordingly shown up more distinctively—unless N_1 is very much less than N_2—in diagrams giving the distribution of proper motions, of which Fig. 22 is an illustration. As the value of a drift-velocity, hV_1 for example, deduced from the analysis of a given region at S, is the projection on the tangent plane at S of the *space-velocity* of the drift relative to the sun, it follows that the space-velocity of one drift relative to the sun is greater than the space-velocity of the other drift; these drifts are conventionally designated drift I and drift II respectively.

Consider the assembly of stars forming drift I and let hU_1 denote the space-velocity of this drift with respect to the sun. This space velocity will have a certain direction and on the celestial sphere centred at the sun this direction will be defined by a certain point, called the *apex* of drift I. The apex of drift II is defined in a similar way. The two drifts are thus characterised by their apices and their space-velocities hU_1, hU_2 relative to the sun.

4·32. *Determination of the apices and the space-velocities of the drifts from proper motions (first method).*

In Fig. 23 consider a region at S with equatorial coordinates (α, δ) and let (α_1, δ_1) denote the coordinates of the apex A of a drift. Let hU denote

* *B.A.N.* 3, 35, 1925.

the space-velocity of the drift; then if hV is the projection of hU on the tangent plane at S, we have

$$hV = hU \sin SA, \qquad \text{......(1)}$$

since the direction of hV is tangential at S to the great circle SA. Thus, P being the pole of the equator, the position angle, θ, of the axis of symmetry of the drift-curve at S is the angle PSA. As we have shown, the analysis of the proper motions at S gives the values of hV and θ for the particular drift concerned and the value of θ defines the great circle through S on which the apex, A, must lie. Similarly we obtain the values of hV for other regions at $S_1, S_2, \ldots$ and also the great circles through $S_1, S_2, \ldots$ defined by the corresponding values of θ. Ideally, all such great circles should pass through the apex, A.

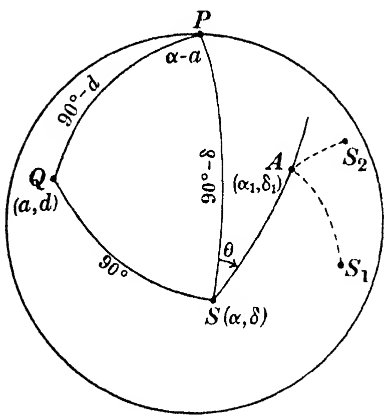

Fig. 23

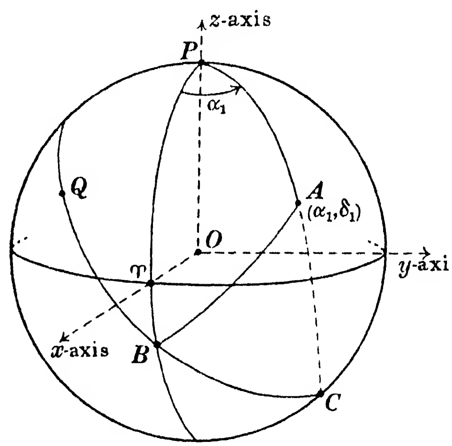

Fig. 24

Let $Q\ (a, d)$ denote the coordinates of the pole, Q, of the great circle SA. Then, since the great circles SA, S_1A, S_2A, ... are theoretically concurrent, the corresponding poles Q, Q_1, Q_2, ... lie on a great circle and one pole of this great circle is the apex, A.

From the triangle PQS we have

$$\sin d = \cos\delta \sin\theta, \qquad \text{......(2)}$$

$$\cot(a-\alpha) = \sin\delta \tan\theta. \qquad \text{......(3)}$$

These equations give the position of Q.

Let

$$lx + my = z \qquad \text{......(4)}$$

denote the equation of the plane, on which the poles Q, Q_1, Q_2, ... lie, with respect to the usual system of equatorial axes and let the corresponding great circle cut the meridians $\alpha = 0$, $\alpha = 90°$ in B and C respectively (Fig. 24). Denote PB and PC by ψ and ϕ respectively. Then, taking the radius of the sphere to be unity, the coordinates of B are $(\sin\psi, 0, \cos\psi)$. Hence, from (4),

$$l = \cot\psi. \qquad \text{......(5)}$$

Similarly,

$$m = \cot\phi. \qquad \text{......(6)}$$

From the triangle PAB, in which A is the apex (α_1, δ_1) and $AB = 90°$ (since A is a pole of the great circle BC), we have

$$\cos AB = \cos\psi \sin\delta_1 + \sin\psi \cos\delta_1 \cos\alpha_1,$$

whence

$$\cos\alpha_1 = -\cot\psi \tan\delta_1$$

or, from (5),

$$\cos\alpha_1 = -l\tan\delta_1. \qquad \text{......(7)}$$

Similarly, we obtain from the triangle PAC,

$$\sin\alpha_1 = -m\tan\delta_1. \qquad \text{......(8)}$$

Now the equatorial coordinates (a, d) of Q can be calculated by (2) and (3), and as the rectangular coordinates of Q are

$$\cos a \cos d, \quad \sin a \cos d, \quad \sin d,$$

we have, from (4), $\quad l\cos a \cos d + m \sin a \cos d = \sin d, \qquad$ ......(9)

in which the coefficients of l and m are now supposed known. For n regions of the sky we have n equations of the form (9) which can be solved by least squares to give the values of l and m. The coordinates (α_1, δ_1) of the apex are then easily found by means of (7) and (8).

The angular distance, SA, of a region S from the apex A can now be calculated and we obtain, in this way, n equations of the form

$$hV = hU \sin SA \qquad \text{......(10)}$$

from which to calculate the space-velocity, hU, of the drift, hV being supposed known from the analysis of each region. The solution for hU from (10) is effected by the method of least squares; it is expressed by

$$hU = \frac{\Sigma hV \sin SA}{\Sigma \sin^2 SA}.$$

The above method,* which is analogous to Bessel's method of finding the solar apex, suffers from the disadvantage that only part of the observed quantities (namely, the position angles, θ, of the axes of symmetry of the drift-curves) is used in the determination of the coordinates of the apex of the drift. The method, however, is important when the regions of the sky are analysed by Dyson's method.

* For a numerical application, see *M.N.* **87**, 134, 1926. See also Eddington's *Stellar Movements*, **83**, 119, 1914.

4·33. *Determination of the apices and the space-velocities of the drifts (second method).*

Consider drift I in a region at $S(\alpha, \delta)$, the drift-velocity and the position angle of the axis of symmetry of the drift-curve being hV_1 and θ_1 respectively. Let (X_1, Y_1, Z_1) be the components of the space-velocity, hU_1, of the drift relative to the sun and referred to the usual system of equatorial axes. Then, the observed projected drift-velocity, hV_1, has components $hV_1 \sin\theta_1$ and $hV_1 \cos\theta_1$ perpendicular to the meridian through S and tangential to the meridian, respectively.

Using the formulae of section 1·33, we have

$$-X_1 \sin\alpha + Y_1 \cos\alpha = hV_1 \sin\theta_1, \qquad \text{......(1)}$$

$$-X_1 \cos\alpha \sin\delta - Y_1 \sin\alpha \sin\delta + Z_1 \cos\delta = hV_1 \cos\theta_1. \qquad \text{......(2)}$$

These are the equations of condition and when similar equations are formed for all the regions concerned, a solution by least squares yields the values of X_1, Y_1 and Z_1. The method is thus analogous to Airy's method for determining the solar motion.

The normal equations formed from (1) are:

$$X_1 \Sigma \sin^2\alpha - Y_1 \Sigma \sin\alpha \cos\alpha = -\Sigma hV_1 \sin\theta_1 \sin\alpha, \qquad \text{......(3)}$$

$$-X_1 \Sigma \sin\alpha \cos\alpha + Y_1 \Sigma \cos^2\alpha = \Sigma hV_1 \sin\theta_1 \cos\alpha. \qquad \text{......(4)}$$

The normal equations formed from (2) are:

$$X_1 \Sigma \cos^2\alpha \sin^2\delta + Y_1 \Sigma \sin\alpha \cos\alpha \sin^2\delta - Z_1 \Sigma \cos\alpha \sin\delta \cos\delta$$
$$= -\Sigma hV_1 \cos\theta_1 \cos\alpha \sin\delta, \qquad \text{......(5)}$$

$$X_1 \Sigma \sin\alpha \cos\alpha \sin^2\delta + Y_1 \Sigma \sin^2\alpha \sin^2\delta - Z_1 \Sigma \sin\alpha \sin\delta \cos\delta$$
$$= -\Sigma hV_1 \cos\theta_1 \sin\alpha \sin\delta, \qquad \text{......(6)}$$

$$-X_1 \Sigma \cos\alpha \sin\delta \cos\delta - Y_1 \Sigma \sin\alpha \sin\delta \cos\delta + Z_1 \Sigma \cos^2\delta$$
$$= \Sigma hV_1 \cos\theta_1 \cos\delta. \qquad \text{......(7)}$$

Combining (3) with (5), and (4) with (6), and rewriting (7), we have the group of equations to determine X_1, Y_1 and Z_1:

$$X_1 \Sigma (\sin^2\alpha + \cos^2\alpha \sin^2\delta) - Y_1 \Sigma \sin\alpha \cos\alpha \cos^2\delta - Z_1 \Sigma \cos\alpha \sin\delta \cos\delta$$
$$= -\Sigma hV_1 (\sin\theta_1 \sin\alpha + \cos\theta_1 \cos\alpha \sin\delta), \qquad \text{......(8)}$$

$$-X_1 \Sigma \sin\alpha \cos\alpha \cos^2\delta + Y_1 \Sigma (\cos^2\alpha + \sin^2\alpha \sin^2\delta) - Z_1 \Sigma (\sin\alpha \sin\delta \cos\delta)$$
$$= \Sigma hV_1 (\sin\theta_1 \cos\alpha - \cos\theta_1 \sin\alpha \sin\delta), \qquad \text{......(9)}$$

$$-X_1 \Sigma \cos\alpha \sin\delta \cos\delta - Y_1 \Sigma \sin\alpha \sin\delta \cos\delta + Z_1 \Sigma \cos^2\delta$$
$$= \Sigma hV_1 \cos\theta_1 \cos\delta. \qquad \text{......(10)}$$

These last three equations are analogous to Airy's equations derived in section 3·34.

The coordinates (α_1, δ_1) of the apex of the drift are given by

$$\tan\alpha_1 = Y_1/X_1,$$
$$\tan\delta_1 = Z_1/(X_1^2+Y_1^2)^{1/2},$$

and the space-velocity, hU_1, of the drift by

$$hU_1 = (X_1^2+Y_1^2+Z_1^2)^{1/2}.$$

A similar procedure gives the corresponding quantities for drift II.

4·34. *The vertex of star-streaming.*

The result of many investigations into the systematic motions of the stars obtained from the distribution of proper motions shows that the phenomena can be explained satisfactorily on the two-streams theory. In particular, as we have seen, we can derive the components of the space velocities of each drift, *relative to the sun*, in terms of a theoretical unit $1/h$ and with reference to the usual equatorial system of axes. If we are to consider the solar motion in relation to the stars observed for proper motion, we have to remember that the totality of stars consists of two assemblies of stars, each *ex hypothesi* with a Maxwellian distribution of velocities. From the definition of solar motion the totality of stars defines a standard of rest and consequently the space-motions of the two drifts with reference to this standard of rest must lie in opposite directions. We can conveniently consider the centroid, or geometrical centre, of the totality of stars as the standard position and so, relative to this centroid, drift I will appear to be moving in a particular direction and drift II in the opposite direction. This direction is of fundamental importance in the distribution of stellar motions and it defines two antipodal points in the sky, called the *vertices* of star-streaming. The totality of stars being viewed from the centroid, the axis joining the vertices has the characteristic property that the general tendency of motion is parallel to this axis, which is an axis of symmetry. In the theory with which we are at present dealing, the emphasis is laid on the division of the totality of stars into two groups or streams which, as we shall see later, are intermingled in space. In the next chapter, the *tendency* of the stars to move *parallel to the axis of symmetry* is the starting-point of an alternative theory (the "ellipsoidal theory" of Schwarzschild) to explain or coordinate the peculiarities in the distribution of stellar motions.

We assume that the components X_1, Y_1, Z_1 of the space-velocities, relative to the sun, of drift I have been obtained by the method of the previous section, and also the components X_2, Y_2, Z_2 of drift II. The components x, y, z of the velocity of drift I relative to drift II are given by

$$x = X_1 - X_2, \quad y = Y_1 - Y_2, \quad z = Z_1 - Z_2,$$

the components of the solar motion disappearing from the differences $X_1 - X_2$, etc.

Let (A, D) denote the coordinates of a vertex. Then the values of A, D are clearly found from the relations

$$x = k\cos A\cos D, \quad y = k\sin A\cos D, \quad z = k\sin D,$$

where
$$k = (x^2+y^2+z^2)^{1/2}.$$

Thus
$$\tan A = y/x,$$
$$\tan D = z/(x^2+y^2)^{1/2}.$$

When the galactic coordinates of the vertex are obtained from the values of A and D, it is found that the vertex lies on the galactic equator. Thus the axis of star-streaming is related in a significant way to the plane of symmetry of the galactic system.

4·35. *The solar motion.*

We assume that the coordinates (α_1, δ_1) of the apex of drift I and the coordinates (α_2, δ_2) of the apex of drift II have been calculated. For example, the results* of Eddington's analysis of the proper motions of the stars of Boss's *P.G.C.* are:

Coordinates of A_1 $(90°{\cdot}8, -14°{\cdot}6)$; of A_2 $(287°{\cdot}8, -64°{\cdot}1)$. The angular distance A_1A_2 (which we denote by ϵ) is calculated by means of the formula

$$\cos\epsilon = \sin\delta_1\sin\delta_2 + \cos\delta_1\cos\delta_2\cos(\alpha_1-\alpha_2). \qquad \ldots\ldots(1)$$

Consider now two vectors SA_1', SA_2' (Fig. 25), including the angle ϵ, of magnitudes hU_1 and hU_2 respectively; they define the space-velocities of drift I and drift II respectively, relative to the sun S. Let n_1 and n_2 denote the total number of stars belonging to drift I and to drift II, respectively, for all the regions of the sky considered. The solar motion is defined with reference to the totality, (n_1+n_2), of stars. Clearly, the solar motion will be represented by a vector, lying in the plane $SA_1'A_2'$, which we shall provisionally designate by BS cutting $A_1'A_2'$ in A_0. This point A_0 will accordingly define the direction of the antapex of the solar motion with reference to the known directions SA_1' and SA_2'.

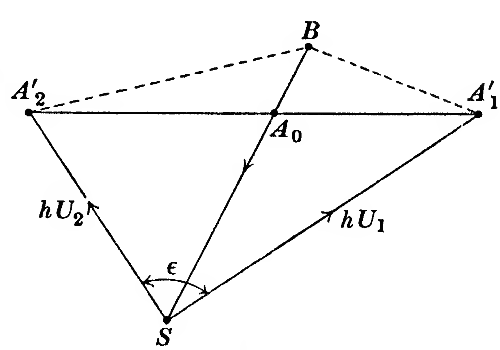

Fig. 25

Considering drift I, we have:

The space-velocity of drift I relative to the centroid of the (n_1+n_2) stars = the space-velocity of drift I relative to the sun + the velocity of the sun

* *M.N.* 71, 35, 1910.

relative to the centroid of the (n_1+n_2) stars; or, in terms of the vectors in Fig. 25, the space-velocity of drift I relative to the centroid of the (n_1+n_2) stars is obtained by compounding the vectors BS and SA_1' and is thus represented by the vector BA_1'. Similarly, the space-velocity of drift II relative to the centroid of the (n_1+n_2) stars is represented by the vector BA_2'. But, in order that the centroid of the (n_1+n_2) stars may be taken as the standard of rest, the vectors BA_1' and BA_2' must be in opposite directions; this can only be so if B lies on the straight line $A_1'A_2'$. Hence the solar motion is defined by the vector A_0S, where A_0 is at present a point between A_1' and A_2'. Consequently, the space-velocities of drift I and of drift II relative to the centroid of the (n_1+n_2) stars are represented by the vectors A_0A_1' and A_0A_2' respectively. Further, the position of A_0 is defined explicitly by the relation

$$n_1a_1 = n_2a_2, \qquad \ldots\ldots(2)$$

where a_1 and a_2 are the lengths of A_0A_1' and A_0A_2', this relation expressing simply the consideration that the centroid of the (n_1+n_2) stars is taken as the centre of rest. From the known values of hU_1 and hU_2 (i.e. of SA_1' and SA_2'), of ϵ and of n_1 and n_2, the position of A_0 can be easily calculated by elementary methods and hence the characteristics of the solar motion can be found. We proceed as follows.

Let hU_0 denote the solar motion reversed. Then $(n_1+n_2).hU_0$ is the resultant of $n_1.hU_1$ and $n_2.hU_2$ (Fig. 26).

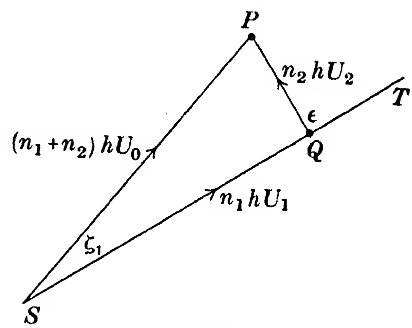

Fig. 26

Let ζ_1 denote the angle PSQ (or A_0SA_1' in Fig. 25). Then we have

$$(n_1+n_2)^2(hU_0)^2 = n_1^2(hU_1)^2+n_2^2(hU_2)^2+2n_1n_2(hU_1).(hU_2)\cos\epsilon \qquad \ldots(3)$$

and

$$\tan\zeta_1 = \frac{n_2(hU_2)\sin\epsilon}{n_1(hU_1)+n_2(hU_2)\cos\epsilon}. \qquad \ldots\ldots(4)$$

Since ϵ is supposed to have been found by means of (1), these formulae, (3) and (4), enable us to calculate hU_0 and ζ_1.

Consider now in Fig. 27 the celestial sphere, with the sun as centre, in which A_1 and A_2 are the apices of drift I and drift II respectively. Since the direction of the antapex of the solar motion lies in the plane defined by the vectors SA_1' and SA_2' (Fig. 25), on the celestial sphere the antapex—which we denote by A—lies on the great circle arc A_1A_2. Furthermore, as we have found the angle, ζ_1, between the direction of the apex of drift I and the direction of the antapex, the great circle arc A_1A is ζ_1.

We can now obtain the coordinates (α_0, δ_0) of the solar antapex, A, as follows. From the triangle PA_1A_2 (Fig. 27), in which $PA_1 = 90° - \delta_1$, $PA_2 = 90° - \delta_2$, $A_1PA_2 = \alpha_1 - \alpha_2$, $A_1A_2 = \epsilon$, all these quantities being supposed known and δ_1, δ_2 being used in their algebraic significance, we have

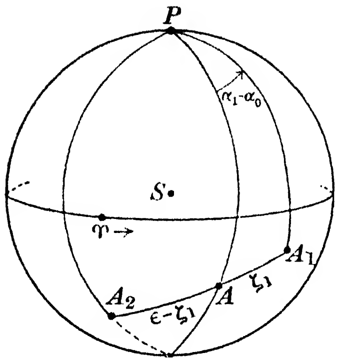

Fig. 27

$$\sin PA_1A_2$$
$$= \sin(\alpha_1 - \alpha_2)\cos\delta_2 \operatorname{cosec}\epsilon, \qquad \ldots\ldots(5)$$

from which the angle PA_1A_2 is calculated—for the usual positions found for A_1 and A_2 it is easily seen that PA_1A_2 lies between 90° and 180°. The declination, δ_0, of the antapex A is then calculated from the formula

$$\sin\delta_0 = \sin\delta_1 \cos\zeta_1 + \cos\delta_1 \sin\zeta_1 \cos PA_1A_2 \qquad \ldots\ldots(6)$$

and the right ascension, α_0, of A from

$$\sin(\alpha_1 - \alpha_0) = \sin\zeta_1 \sin PA_1A_2 \sec\delta_0. \qquad \ldots\ldots(7)$$

An alternative, and simpler, method of calculating hU_0, α_0 and δ_0 is as follows.

Projecting the vectors represented in Fig. 26 on the equatorial axes, we have at once

$$(n_1+n_2) \,.\, hU_0 \,.\, \cos\alpha_0 \cos\delta_0 = n_1 \,.\, hU_1 \,.\, \cos\alpha_1 \cos\delta_1 + n_2 \,.\, hU_2 \,.\, \cos\alpha_2 \cos\delta_2, \qquad \ldots\ldots(8)$$

$$(n_1+n_2) \,.\, hU_0 \,.\, \sin\alpha_0 \cos\delta_0 = n_1 \,.\, hU_1 \,.\, \sin\alpha_1 \cos\delta_1 + n_2 \,.\, hU_2 \,.\, \sin\alpha_2 \cos\delta_2, \qquad \ldots\ldots(9)$$

$$(n_1+n_2) \,.\, hU_0 \,.\, \sin\delta_0 = n_1 \,.\, hU_1 \,.\, \sin\delta_1 + n_2 \,.\, hU_2 \,.\, \sin\delta_2, \qquad \ldots\ldots(10)$$

from which hU_0, α_0 and δ_0 are readily derived.

4·36. *Numerical results for the stream constants.*

During the last thirty years a large number of investigations into the systematic motions of the stars have been made, using proper motion data. The results—whether for the brighter or for the fainter stars—may be said to be in fairly good accordance, and it will be sufficient to state, for reference, Eddington's conclusions since his research* dealt with the stars of Boss's *P.G.C.*, the proper motions of which have a high degree of accuracy; moreover, the stars are well distributed over the whole sky.

* *M.N.* 71, 4, 1910.

(*a*) *Drift constants.*

	Drift I	Drift II
R.A. of apex	91°	288°
Dec. of apex	− 15°	− 64°
Space velocity relative to sun (unit $1/h$)	1·516	0·855
Percentage of stars	60	40

(*b*) *Vertex.*

(i) Equatorial coordinates of vertex: R.A. 274°; Dec. − 12°
(ii) Galactic coordinates of vertex: Long. 347°; Lat. − 0°·5
(iii) Relative speed of drifts (unit $1/h$): 1·868

(*c*) *Solar motion.*

R.A. of apex: 267°
Dec. of apex: + 36°
Speed of solar motion (unit $1/h$): 0·908

As we have seen in Chapter III, the position of the solar apex found by L. Boss,* using the proper motions of the *P.G.C.*, was (270°·5, + 34°·3).

It may be recalled that in Airy's method the magnitudes of the proper motions appear in the equations of condition whereas in the application of the two-streams theory it is only the distribution of proper motions in position angle that is utilised. The agreement between the two sets of results, obtained from the same data by wholly different methods, is noteworthy.

As shown in Chapter III, the determination of the solar motion by means of the observed radial velocities of the stars leads to the evaluation of the solar speed in km./sec. and the result may be taken to be 19·5 km./sec. If V_0 denotes the solar speed in km./sec.,

$$V_0 = (hV_0) \cdot \frac{1}{h},$$

so that, on inserting the numerical values of V_0 ($\equiv 19{\cdot}5$) and hV_0 ($\equiv 0{\cdot}908$) from (*c*) above, it is easily found that the theoretical unit $1/h$ is equivalent to 21·5 km./sec.

4·4. *The mean parallax of the stars of the two drifts.*

We have seen in section 2·41 that for a given region of the sky the mean linear speed, $\overline{T}$ (there denoted by T_1), of stars moving in a direction making an angle θ with the axis of symmetry of a single drift-curve is given by

$$\overline{T} = \frac{1}{h} g(\tau), \qquad \ldots\ldots(1)$$

where $\tau = hV\cos\theta$ and $g(\tau)$ is the function tabulated on p. 41. $\overline{T}$ is determined by this equation in terms of the theoretical unit, $1/h$, and it can

* *A.J.* **26**, 111, 1910.

afterwards be expressed in km./sec. by substituting in (1) the value of $1/h$ in km./sec. if the value of $1/h$ km./sec. is substituted in (1).

Let now θ denote position angle and let θ_1, θ_2 be the position angles of the axes of symmetry of drift I and drift II. Hence, if $\overline{T}_1$ and $\overline{T}_2$ denote the mean linear speeds of stars belonging to drift I and drift II respectively and moving in position angle θ, we have

$$\overline{T}_1 = \frac{1}{h}g(\tau_1); \quad \tau_1 = hV_1 \cos(\theta - \theta_1), \qquad \text{......(2)}$$

$$\overline{T}_2 = \frac{1}{h}g(\tau_2); \quad \tau_2 = hV_2 \cos(\theta - \theta_2). \qquad \text{......(3)}$$

If μ is the annual proper motion corresponding to the linear transverse speed, T, of a star whose parallax is p, we have

$$\kappa\mu = pT,$$

where $\kappa = 4{\cdot}74$; μ and p are expressed in seconds of arc and T in km./sec. Let $\overline{\mu}_1$ denote the mean proper motion of stars of drift I in position angle θ. We define the mean parallax, p_1, of these stars by

$$\kappa\overline{\mu}_1 = p_1\overline{T}_1.$$

Hence $$\kappa\overline{\mu}_1 = p_1 \cdot \frac{1}{h} g(\tau_1). \qquad \text{......(4)}$$

Similarly, for drift II, $$\kappa\overline{\mu}_2 = p_2 \cdot \frac{1}{h} g(\tau_2). \qquad \text{......(5)}$$

Let n_1 and n_2 be the number of stars, moving in a small sector with position angle θ, belonging to the two drifts; these numbers are supposed to be known from the analysis of the region (for example, in Table 15, p. 109; the columns r_1 and r_2 give the theoretical numbers concerned). We then have, letting $\overline{\mu}$ denote the mean of the observed proper motions of all the stars moving in position angle θ,

$$(n_1 + n_2)\,\overline{\mu} = n_1\overline{\mu}_1 + n_2\overline{\mu}_2$$

and hence $$\kappa(n_1 + n_2)\,\overline{\mu} = n_1 p_1 \cdot \frac{1}{h} g(\tau_1) + n_2 p_2 \cdot \frac{1}{h} g(\tau_2). \qquad \text{......(6)}$$

This is an equation of condition involving two unknowns, p_1 and p_2, all the other quantities being assumed known from the analysis of the region; in practice, the procedure of considering sectors of 10°, say, is followed for determining the quantities concerned.

For a given region, we have as many equations of condition of the form of (6) as sectors in which the data are regarded as numerically adequate. A least-squares solution leads to the appropriate values of p_1 and p_2 for the region considered.

Eddington's analysis* of the Groombridge stars—about 4500 in number, mainly between magnitudes 6 and 9, and within 52° of the north pole—yields, on weighting equally each of the seven regions investigated,

$$\frac{1}{h}\frac{p_1}{\kappa} = 0''{\cdot}0304, \quad \frac{1}{h}\frac{p_2}{\kappa} = 0''{\cdot}0331,$$

and, putting $1/h = 21{\cdot}5$, $\kappa = 4{\cdot}74$, the mean parallaxes are

$$p_1 = 0''{\cdot}0067, \quad p_2 = 0''{\cdot}0073.$$

The conclusion to be drawn from these results is that, so far as the Groombridge stars are concerned, the two drifts are intermingled systems with practically identical mean parallaxes.

A similar conclusion† is reached with reference to the Boss stars from a discussion of a large composite region containing 1122 stars. The analysis gives

$$\frac{1}{h}.\frac{p_1}{\kappa} = 0''{\cdot}0694, \quad \frac{1}{h}.\frac{p_2}{\kappa} = 0''{\cdot}0738,$$

from which

$$p_1 = 0''{\cdot}0153, \quad p_2 = 0''{\cdot}0163.$$

Again, the mean parallaxes of the two drifts (in this instance, consisting of the naked-eye stars) are practically identical.

4·5. *The effect of accidental errors in the proper motions.*

A general investigation requires a knowledge of the distribution of the stars in space; we shall here consider only the simplified problem in which we suppose the stars of a drift to have the same parallax, p. An error $\Delta\mu$ in the proper motion (taking one component) leads to an error ΔT, given by

$$\Delta T = \frac{\kappa}{p}\Delta\mu, \qquad \text{......(1)}$$

in the corresponding linear velocity.

It is easy to see in a general way that the effect of accidental errors in the observed proper motions will be a tendency to conceal the characteristic drift-motion (as exemplified in a drift-curve) especially for stars with small proper motions; the observed drift-curve will be somewhat less "elliptical" in form and accordingly the deduced drift-velocity will be smaller than the true drift-velocity. We can express this otherwise by saying that the true theoretical unit, $1/h$, of speed will be less than the corresponding observed unit, $1/h_0$.

We shall suppose that the errors, ϵ, in the components, u, of linear velocity follow the Gaussian error-law, so that the proportion of velocities with errors between ϵ and $\epsilon + d\epsilon$ is

$$\frac{g}{\sqrt{\pi}} e^{-g^2\epsilon^2} d\epsilon. \qquad \text{......(2)}$$

* *Stellar Movements*, 113, 1914. † *Stellar Movements*, 116, 1914.

Since the stars are supposed to be at the same distance, the errors of the proper motions obey a similar law.

Let the linear velocity, as affected by an error ϵ, be denoted by v; we shall call v the "observed velocity". The corresponding true velocity is $v-\epsilon$. It is to be remarked that as we suppose u to be expressed in terms of the theoretical unit $1/h$, we regard ϵ as expressed in terms of this unit. For an assembly of N stars, the number with haphazard linear velocities between u and $u+du$ is

$$\frac{Nh}{\sqrt{\pi}} e^{-h^2u^2} du.$$

Hence the number with true velocities between $v-\epsilon$ and $v+dv-\epsilon$ is

$$\frac{Nh}{\sqrt{\pi}} e^{-h^2(v-\epsilon)^2} dv$$

and of these the number which have an observed velocity between v and $v+dv$ is

$$\frac{Nh}{\sqrt{\pi}} e^{-h^2(v-\epsilon)^2} dv \frac{g}{\sqrt{\pi}} e^{-g^2\epsilon^2} d\epsilon.$$

Summing throughout the range of ϵ, we find that the total number of stars with an observed velocity between v and $v+dv$ is

$$\frac{Ngh}{\pi} dv \int_{-\infty}^{\infty} e^{-h^2(v-\epsilon)^2-g^2\epsilon^2} d\epsilon,$$

which may be written

$$\frac{Ngh}{\pi} e^{-\frac{g^2h^2}{g^2+h^2}v^2} dv \int_{-\infty}^{\infty} e^{-(g^2+h^2)\left(\epsilon-\frac{h^2v}{g^2+h^2}\right)^2} d\epsilon.$$

Hence the number is $$\frac{Ngh}{\sqrt{\pi}(g^2+h^2)^{\frac{1}{2}}} e^{-\frac{g^2h^2}{g^2+h^2}v^2} dv$$

or $$\frac{Nh_0}{\sqrt{\pi}} e^{-h_0^2v^2} dv, \qquad \ldots\ldots(3)$$

where $$h_0^2 = \frac{g^2h^2}{g^2+h^2}$$

or $$\frac{1}{h_0^2} = \frac{1}{g^2} + \frac{1}{h^2}. \qquad \ldots\ldots(4)$$

Thus the frequency function of the observed linear velocities is of the same form as the frequency function of the true velocities, the theoretical unit, $1/h_0$, however being greater than the true unit $1/h$.

Since the distribution of the errors of the linear velocities is given by (2), the probable error is $0{\cdot}477/g$ in terms of the theoretical unit $1/h$. If we denote the probable error by x_0 when expressed in km./sec., we have

$$x_0 = \frac{0{\cdot}477}{g} \cdot \frac{21{\cdot}5}{1/h}. \qquad \ldots\ldots(5)$$

But, by (1), the probable error, μ_0, of the proper motions is related to x_0 by

$$x_0 = \frac{\kappa}{p}\mu_0, \qquad \ldots\ldots(6)$$

where $\kappa = 4{\cdot}74$, p is expressed in seconds of arc and μ_0 in seconds of arc per annum. Hence from (5) and (6)

$$\frac{0{\cdot}477}{g}\,.\,21{\cdot}5h = \frac{4{\cdot}74}{p}\mu_0$$

or

$$\frac{1}{g} = 0{\cdot}462\frac{\mu_0}{p}\cdot\frac{1}{h}. \qquad \ldots\ldots(7)$$

From (4) and (7) we then have

$$\frac{1}{h_0^2} = \frac{1}{h^2}\left\{1 + (0{\cdot}462)^2\frac{\mu_0^2}{p^2}\right\}. \qquad \ldots\ldots(8)$$

Taking the mean parallax of the Boss stars to be $0''{\cdot}0158$ and the probable error, μ_0, to be $\pm 0''{\cdot}0055$ per annum, we find that

$$\frac{1}{g} = 0{\cdot}161\left(\frac{1}{h}\right),$$

$$\frac{1}{h_0} = 1{\cdot}013\left(\frac{1}{h}\right).$$

The difference between $1/h_0$ and $1/h$ is almost negligible, due to the comparatively small probable error of the proper motions and to the considerable value of the parallax.

Considering the Groombridge stars, we have the estimated probable error of the proper motions to be $\pm 0''{\cdot}007$ per annum and, taking the mean parallax to be $0''{\cdot}0070$, we find that

$$\frac{1}{g} = 0{\cdot}462\left(\frac{1}{h}\right),$$

$$\frac{1}{h_0} = 1{\cdot}10\left(\frac{1}{h}\right).$$

In this instance, owing to accidental errors, the drift-velocities obtained from the analysis of a given region require to be increased by about 10 %.

4·61. *Eddington's analytical method of deriving the drift constants.*

In Chapter II, section 2·71, we expressed the radius vector, ρ, of a single drift-curve in the form of a Fourier series thus:

$$\rho = \frac{n}{\pi}B + \frac{n}{\pi}C\cos\theta + \frac{n}{\pi}D\cos 2\theta + \frac{n}{\pi}E\cos 3\theta + \frac{n}{\pi}F\cos 4\theta + \ldots,$$

where

$$nB = \tfrac{1}{2}\int_0^{2\pi}\rho\, d\theta, \quad nC = \int_0^{2\pi}\rho\cos\theta\, d\theta, \text{ etc.}$$

In these formulae θ is the angle between the radius vector for the drift-curve and the axis of symmetry. Regarding θ now as the position angle and taking θ_1 to be the position angle of the axis of symmetry of drift I, we rewrite the formulae as

$$\left.\begin{aligned} \int_0^{2\pi} \rho\cos(\theta-\theta_1)\,d\theta &= n_1 C_1 \\ \int_0^{2\pi} \rho\cos 2(\theta-\theta_1)\,d\theta &= n_1 D_1 \\ \int_0^{2\pi} \rho\cos 3(\theta-\theta_1)\,d\theta &= n_1 E_1 \\ \int_0^{2\pi} \rho\cos 4(\theta-\theta_1)\,d\theta &= n_1 F_1 \end{aligned}\right\}. \qquad \ldots\ldots(1)$$

There is a similar set for drift II. In these formulae, C_1, D_1, ... are functions of the velocity, hV_1, of drift I and the corresponding quantities C_2, D_2, ... are functions of the velocity, hV_2, of drift II; their values are shown in Table 16, p. 127. The following procedure is due to Eddington.†

Considering drift I, we have

$$\int_0^{2\pi} \rho\, e^{i\theta}\, d\theta = e^{i\theta_1}\int_0^{2\pi} \rho\, e^{i(\theta-\theta_1)}\, d\theta$$

$$= e^{i\theta_1}\int_0^{2\pi} \rho\cos(\theta-\theta_1)\,d\theta + i\,e^{i\theta_1}\int_0^{2\pi}\rho\sin(\theta-\theta_1)\,d\theta \qquad \ldots\ldots(2)$$

$$= n_1 C_1 e^{i\theta_1}, \qquad \ldots\ldots(3)$$

since, owing to the symmetry of the drift-curve, the second integral on the right of (2) vanishes.

Let C_1^* denote the complex quantity $C_1 e^{i\theta_1}$. We then have, from (3),

$$\int_0^{2\pi} \rho\, e^{i\theta} d\theta = n_1 C_1^*. \qquad \ldots\ldots(4)$$

Similarly, we obtain, from (1),

$$\int_0^{2\pi} \rho\, e^{2i\theta} d\theta = n_1 D_1^*, \qquad \ldots\ldots(5)$$

$$\int_0^{2\pi} \rho\, e^{3i\theta} d\theta = n_1 E_1^*, \qquad \ldots\ldots(6)$$

$$\int_0^{2\pi} \rho\, e^{4i\theta} d\theta = n_1 F_1^*, \qquad \ldots\ldots(7)$$

where $\quad C_1^* = C_1 e^{i\theta_1}, \quad D_1^* = D_1 e^{2i\theta_1}, \quad E_1^* = E_1 e^{3i\theta_1}, \quad F_1^* = F_1 e^{4i\theta_1}.$ $\qquad \ldots\ldots(8)$

† *M.N.* **68**, 588, 1908.

We now consider the theoretical formulae in relation to the observations.

Since $$\int_0^{2\pi} \rho \cos\theta d\theta \equiv \int_0^{2\pi} \cos\theta (\rho d\theta)$$

and since $\rho d\theta$ is the theoretical number of stars of a drift moving between position angles θ and $\theta + d\theta$, the integral may be interpreted as the sum of the values of $\cos\theta$ contributed by the stars concerned. We have a similar interpretation for $\int_0^{2\pi} \rho \sin\theta d\theta$.

Now for all the stars of the region belonging to drifts I and II we can form from the observational material the sums $\Sigma \cos\theta$ and $\Sigma \sin\theta$. Let

$$L = \frac{1}{n}(\Sigma \cos\theta + i \Sigma \sin\theta), \qquad \text{......(9)}$$

where n ($\equiv n_1 + n_2$) is the total number of stars in the two drifts. L is thus a complex quantity derived from the observations. Then

$$nL \equiv \Sigma(\cos\theta + i \sin\theta)$$

is to be equated to the sum of the two integrals of the form $\int_0^{2\pi} \rho e^{i\theta} d\theta$ for drift I and drift II; thus the equation of condition is

$$nL = n_1 C_1^* + n_2 C_2^*. \qquad \text{......(10)}$$

Let $$n_1 = \frac{n}{2}(1+\alpha) \qquad \text{......(11)}$$

and, since $n = n_1 + n_2$, $$n_2 = \frac{n}{2}(1-\alpha). \qquad \text{......(12)}$$

Hence (10) becomes $$2L = (1+\alpha) C_1^* + (1-\alpha) C_2^*. \qquad \text{......(13)}$$

Similarly, we derive from the observations complex quantities M, N, O given by

$$\left.\begin{aligned} M &= \frac{1}{n}(\Sigma \cos 2\theta + i \Sigma \sin 2\theta) \\ N &= \frac{1}{n}(\Sigma \cos 3\theta + i \Sigma \sin 3\theta) \\ O &= \frac{1}{n}(\Sigma \cos 4\theta + i \Sigma \sin 4\theta) \end{aligned}\right\}, \qquad \text{......(14)}$$

giving rise to the equations of condition

$$\left.\begin{aligned} 2M &= (1+\alpha) D_1^* + (1-\alpha) D_2^* \\ 2N &= (1+\alpha) E_1^* + (1-\alpha) E_2^* \\ 2O &= (1+\alpha) F_1^* + (1-\alpha) F_2^* \end{aligned}\right\}. \qquad \text{......(15)}$$

The equations (13) and (15) thus constitute four complex equations of condition and since the real and imaginary parts must be separately satisfied, the system of equations is equivalent to eight real equations of condition from which the five drift constants—hV_1, hV_2, θ_1, θ_2 and α—are to be determined. Now the system (13) and (15) is one of diminishing weight, since errors in the observed position angle θ will in general give rise to larger errors, for example, in the sums $\Sigma \cos 2\theta$ and $\Sigma \sin 2\theta$, than in the sums $\Sigma \cos\theta$ and $\Sigma \sin\theta$.

Eddington, accordingly, employs only the equation (13) and the first two of (15). He writes

$$C_1^* = \zeta_1 P_1, \quad D_1^* = \zeta_1 P_1^2, \quad E_1^* = \frac{\zeta_1}{\gamma_1} P_1^3 \qquad \text{......(16)}$$

and

$$C_2^* = \zeta_2 P_2, \quad D_2^* = \zeta_2 P_2^2, \quad E_2^* = \frac{\zeta_2}{\gamma_2} P_2^3. \qquad \text{......(17)}$$

From (16), $\quad P_1 C_1^* = D_1^*, \quad$ or $\quad P_1 C_1 e^{i\theta_1} = D_1 e^{2i\theta_1}.$

Thus

$$P_1 = \frac{D_1}{C_1} e^{i\theta_1}$$

and, accordingly, P_1 is a complex quantity with argument θ_1. It follows from the relation $C_1^* = \zeta_1 P_1$ that ζ_1 is a real quantity; it is also a function of hV_1. The values of $|P| \equiv D/C$ are tabulated in the second column of Table 16 for different values of hV, and in the third column are the corresponding values of $\zeta \equiv C/|P|$. The values of C, D, E and F in the remaining columns of the table are calculated by means of the formulae given in Chapter II, section 2·71.

Also

$$\frac{E_1^*}{D_1^*} = \frac{P_1}{\gamma_1}, \quad \text{or} \quad \frac{\gamma_1 E_1 e^{3i\theta_1}}{D_1 e^{2i\theta_1}} = |P_1| e^{i\theta_1}, \qquad \text{......(18)}$$

from which it is seen that γ_1 (and, similarly, γ_2) is real.

Moreover, it is found that the values of γ_1 (and of γ_2), as calculated from (18), that is, from

$$\gamma = \frac{D}{E}|P| = \frac{D^2}{CE},$$

are sensibly constant for the range of hV from 0 to 1·8, and Eddington writes simply

$$\gamma_1 = \gamma_2 \equiv \gamma = 1{\cdot}163. \qquad \text{......(19)}$$

If E_0 denotes the value calculated from $E_0 \equiv \dfrac{D\,.\,|P|}{1{\cdot}163}$, the values of $E - E_0$ range from $-0{\cdot}008$ to $+0{\cdot}009$ for values of hV in the range 0 to 1·8. For $hV = 1{\cdot}9$ and 2·0, the respective values of $E - E_0$ are $+0{\cdot}014$ and $+0{\cdot}019$. The approximation (19) is thus of satisfactory accuracy.

Making the substitutions (16) and (17) in (13) and (15), we obtain

$$2L = (1+\alpha)\zeta_1 P_1 + (1-\alpha)\zeta_2 P_2, \qquad \text{......(20)}$$

$$2M = (1+\alpha)\zeta_1 P_1^2 + (1-\alpha)\zeta_2 P_2^2, \qquad \text{......(21)}$$

$$2\gamma N = (1+\alpha)\zeta_1 P_1^3 + (1-\alpha)\zeta_2 P_2^3, \qquad \text{......(22)}$$

where, in the last equation, γ has the value 1·163.

Let $$(1+\alpha)\zeta_1 = (1+\beta)k, \qquad \text{......(23)}$$
$$(1-\alpha)\zeta_2 = (1-\beta)k, \qquad \text{......(24)}$$
from which $$k = \tfrac{1}{2}(1+\alpha)\zeta_1 + \tfrac{1}{2}(1-\alpha)\zeta_2. \qquad \text{......(25)}$$

Since α, ζ_1 and ζ_2 are real, it follows that k and β are real. We now have the system of equations

$$\frac{2L}{k} = (1+\beta)P_1 + (1-\beta)P_2, \qquad \text{......(26)}$$

$$\frac{2M}{k} = (1+\beta)P_1^2 + (1-\beta)P_2^2, \qquad \text{......(27)}$$

$$\frac{2\gamma N}{k} = (1+\beta)P_1^3 + (1-\beta)P_2^3. \qquad \text{......(28)}$$

Table 16. *Values of the Fourier constants*

hV	$\lvert P\rvert$	ζ	C	D	E	F
0·0	0·0000	1·585	0·0000	0·0000	0·0000	0·0000
0·1	·0564	1·573	·0884	·0050	·0002	·0000
0·2	·1125	1·561	·1755	·0197	·0017	·0001
0·3	·1680	1·548	·2600	·0437	·0057	·0007
0·4	·2227	1·531	·3409	·0759	·0134	·0020
0·5	·2762	1·510	·4171	·1152	·0253	·0047
0·6	·3284	1·486	·4879	·1602	·0420	·0094
0·7	·3789	1·459	·5528	·2094	·0637	·0165
0·8	·4275	1·431	·6115	·2614	·0901	·0266
0·9	·4739	1·401	·6640	·3147	·1211	·0401
1·0	·5179	1·371	·7103	·3679	·1558	·0570
1·1	·5595	1·342	·7507	·4200	·1937	·0774
1·2	·5984	1·313	·7856	·4701	·2337	·1011
1·3	·6345	1·285	·8156	·5175	·2752	·1279
1·4	·6678	1·259	·8410	·5617	·3172	·1573
1·5	·6984	1·235	·8626	·6024	·3590	·1888
1·6	·7262	1·213	·8807	·6396	·4000	·2219
1·7	·7514	1·192	·8959	·6732	·4396	·2559
1·8	·7742	1·174	·9087	·7035	·4774	·2904
1·9	·7946	1·157	·9194	·7305	·5133	·3249
2·0	·8128	1·142	·9284	·7546	·5468	·3590

Now, since ζ_1 is a function of hV_1 and ζ_2 is a function of hV_2, k is by (25) a function of α, hV_1 and hV_2; similarly, from (23), β is a function of α, hV_1 and hV_2 and, consequently, α is a function of hV_1, hV_2 and β. For purposes of solution, we can thus regard the five quantities to be determined as hV_1, hV_2, θ_1, θ_2 and β.

From (25) it is seen that k is a weighted mean between ζ_1 and ζ_2, the weights being proportional to the numbers of stars in the two drifts. Further, Table 16 shows that ζ varies between fairly narrow limits, and for the usual values of hV for the two drifts in different parts of the sky and for values of α round about zero (corresponding to equal numbers of stars in the drifts), the value of k may be taken to lie between 1·35 and 1·45; for a first approximation it may be assumed that

$$k = 1{\cdot}40. \qquad \text{......(29)}$$

The left-hand sides of (26), (27) and (28) are thus known complex numbers. We have to find from these equations a real number, β, and two complex numbers, P_1 and P_2, with arguments θ_1 and θ_2 respectively.

From (26), (27) and (28) we obtain easily

$$\frac{4}{k^2}(N\gamma k - LM) = (1-\beta^2)(P_1-P_2)^2(P_1+P_2) \qquad \text{......(30)}$$

and

$$\frac{4}{k^2}(Mk - L^2) = (1-\beta^2)(P_1-P_2)^2, \qquad \text{......(31)}$$

from which

$$\frac{N\gamma k - LM}{Mk - L^2} = P_1 + P_2 \equiv 2K. \qquad \text{......(32)}$$

Thus K is a complex quantity whose numerical value is easily found.

From (26) and (32) we obtain

$$P_1 = K + \frac{L - Kk}{k\beta}, \qquad \text{......(33)}$$

$$P_2 = K - \frac{L - Kk}{k\beta}, \qquad \text{......(34)}$$

from which, on substituting in (27),

$$\beta = \frac{L - Kk}{\{(L-Kk)^2 + (Mk - L^2)\}^{\frac{1}{2}}}. \qquad \text{......(35)}$$

From (26) and (31) we express P_1 and P_2 in terms of β, k, L and M; the results are

$$kP_1 = L + \left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2}}(Mk - L^2)^{\frac{1}{2}}, \qquad \text{......(36)}$$

$$kP_2 = L - \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}(Mk - L^2)^{\frac{1}{2}}. \qquad \text{......(37)}$$

If the numerical value of β is found from (35), the formula (36) enables us to calculate P_1, from which we obtain $|P_1|$ and the argument θ_1; from Table 16, the value of hV_1 corresponding to $|P_1|$ is easily obtained. Similarly, the values of hV_2 and θ_2 are derived. The values of ζ_1 and ζ_2 are then found from the table for the appropriate values of hV_1 and hV_2. Finally, α is obtained from the formula

$$\frac{1-\alpha}{1+\alpha} = \frac{1-\beta}{1+\beta}\cdot\frac{\zeta_1}{\zeta_2} \qquad \text{......(38)}$$

derived by dividing (24) by (23).

However, this procedure will break down in practice inasmuch as the three complex equations (26), (27) and (28) in five unknowns are fundamentally equivalent to six real equations which are hardly likely to be simultaneously satisfied when the observational values of $2L/k$, $2M/k$ and

$2\gamma N/k$ are substituted in these equations. The result is that the numerical value of β derived from (35) will generally be a complex quantity, although *by definition* β is a *real* quantity. From a previous remark, the weight of (28) may be expected to be less than that of (26) and (27). Taking the real parts of (26) and (27) and their imaginary parts, we have four equations uniting five unknowns and these are sufficient to determine hV_1, hV_2, θ_1 and θ_2 in terms of α; in this way, we shall have P_1, P_2 and β determined as functions of α. We can now determine the right-hand side of (28) as a function of α; denote it by $2\gamma N_\alpha/k$. We thus obtain

$$\delta N \equiv N - N_\alpha$$

as a function of α.

For different values of α, we shall have corresponding values of the residuals δN (which are, in general, complex quantities), and the value, α_0, to be chosen is that which makes the sum of the squares of the residuals (obtained from the real and imaginary parts) a minimum, subject to the condition that β is a real quantity. It follows that $|\delta N|$ is to be a minimum.

With given numerical values of L and M, it is seen from (32) that N and K are connected by a linear relation, from which it follows that δN and δK are connected by a linear relation. Hence, in our solution, the condition is that $|\delta K|$ is to be a minimum, corresponding to the value, α_0, of α, subject to the condition that β is real. Let K_0 be the value of K corresponding to α_0; then $|K - K_0|$ is the minimum value of δK.

From (35),

$$\frac{1}{\beta^2} = 1 + \left\{\frac{(Mk - L^2)^{\frac{1}{2}}}{L - Kk}\right\}^2, \qquad \text{......(39)}$$

and since β is to be real we have to choose $L - Kk$ so that the argument of the resulting quantity (call it for the moment $L - K_1 k$) is the same as, or differs by 180° from, the argument of $(Mk - L^2)^{\frac{1}{2}}$. On the Argand diagram (Fig. 28) let A denote the complex quantity $L - Kk$ and B the complex quantity $(Mk - L^2)^{\frac{1}{2}}$. The sign of the root is chosen so that OB is nearer the direction of motion of drift I relative to drift II than the opposite direction BO. Since $L - K_1 k$ has the same argument as $(Mk - L^2)^{\frac{1}{2}}$, it will be represented by a point C_1 lying in OB.

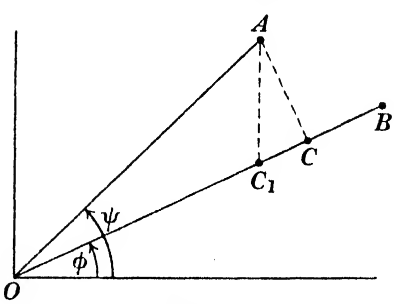

Fig. 28

Now, the vector $\overrightarrow{C_1 A}$ = vector $\overrightarrow{OA}$ − vector $\overrightarrow{OC_1}$,

that is,

$$\overrightarrow{C_1 A} = \text{vector}\{(L - Kk) - (L - K_1 k)\} = \text{vector}\{-k(K - K_1)\}.$$

Thus, for $(K-K_1)$ to be as small as possible, $\overrightarrow{C_1A}$ must be as small as possible. Thus the condition is satisfied for the foot, C, of the perpendicular from A to OB; consequently, C represents the complex number $L-K_0k$.

Let ψ and ϕ denote respectively the arguments of $L-Kk$ and $(Mk-L^2)^{\frac{1}{2}}$.

Then $$L-K_0k = |L-Kk|\cos(\psi-\phi)\,e^{i\phi}. \qquad \ldots\ldots(40)$$

Also $$\frac{1}{\beta^2} = 1+\frac{|(Mk-L^2)|}{|(L-K_0k)|^2}. \qquad \ldots\ldots(41)$$

Further, the sign of β is positive or negative according as $(L-K_0k)$ is of the same sign as, or of opposite sign to, that of $(Mk-L^2)^{\frac{1}{2}}$.

With the numerical value of β derived from (41), P_1 is calculated from (33). The value of hV_1 is then deduced from the entries in Table 16 corresponding to the value of $|P_1|$ and the argument of P_1 is the position angle of the drift. The values of hV_2 and θ_2 are obtained in a similar way. With the numerical values of hV_1 and hV_2 we find ζ_1 and ζ_2 from the table and then α is obtained from (38).

We can finally calculate k from (25). If this value is not in substantial agreement with the assumed value (generally taken to be 1·40 for a first approximation), the entire solution should be repeated with this new value of k.

4·62. *Example of the determination of the drift constants by the analytical method.*

The proper motion data are taken from a memoir* by V. Nechvile, "Recherches sur les mouvements propres de 3802 étoiles". The regions 19 to 22 are grouped together to form a composite region F containing 954 stars.†

In Table 17, the second column gives the effective number, r_0, of stars observed to move in the various sectors $\theta-5°$ to $\theta+5°$; the occurrence of fractional numbers in this column is due to the fact that the counts have been smoothed by taking as the effective number of stars moving in the sector $\theta-5°$ to $\theta+5°$ the mean of the observed numbers in this sector and its two flanking sectors. The next column contains the sum of the values of $\cos\theta$, that is $\Sigma\cos\theta$, for all the stars moving in the various sectors; it is sufficient to assume for the present purpose that $\Sigma\cos\theta = r_0\cos\theta$. The entries in the third column are then easily calculated. A similar procedure is adopted for the other columns.

* *Publ. de l'Obs. national de Prague*, No. 4, 1927.

† The distribution of the proper motions in position angle is given by W. M. Smart and H. E. Green, *M.N.* 89, 149, 1928.

From the totals at the foot of the table we obtain from (9) and (14)*

$$L = -0{\cdot}032+0{\cdot}167i \quad \equiv 0{\cdot}170E(100^\circ{\cdot}8), \qquad \ldots\ldots(a)$$

$$M = +0{\cdot}150+0{\cdot}397i \quad \equiv 0{\cdot}424E(69^\circ{\cdot}3), \qquad \ldots\ldots(b)$$

$$N = -0{\cdot}136-0{\cdot}013i,$$

and, with $\gamma = 1{\cdot}163$,

$$N\gamma = -0{\cdot}158-0{\cdot}015i \quad \equiv 0{\cdot}159E(185^\circ{\cdot}4),$$

where $E(\theta)$ denotes $e^{i\theta}$.

Table 17. *The calculation of L, M and N*

Position angle θ	r_0	$\Sigma\cos\theta \equiv r_0\cos\theta$	$\Sigma\sin\theta \equiv r_0\sin\theta$	$\Sigma\cos 2\theta \equiv r_0\cos 2\theta$	$\Sigma\sin 2\theta \equiv r_0\sin 2\theta$	$\Sigma\cos 3\theta \equiv r_0\cos 3\theta$	$\Sigma\sin 3\theta \equiv r_0\sin 3\theta$
5°	27·7	+27·6	+ 2·4	+ 27·3	+ 4·8	+ 26·7	+ 7·2
15	37·0	35·7	9·6	32·0	18·5	26·2	26·2
25	50·0	45·3	21·1	32·1	38·3	+ 13·0	48·3
35	61·0	50·0	35·0	20·9	57·3	− 15·8	58·9
45	60·7	42·9	42·9	+ 0·0	60·7	42·9	42·9
55	59·0	33·9	48·3	− 20·2	55·5	57·0	+ 15·3
65	49·0	20·7	44·4	31·5	37·5	47·3	− 12·7
75	36·7	9·5	35·5	31·8	18·3	25·9	25·9
85	26·7	+ 2·3	26·6	26·3	+ 4·6	− 6·9	25·8
95	21·7	− 1·9	21·6	21·4	− 3·8	+ 5·6	21·0
105	17·3	4·5	16·7	15·0	8·6	12·2	12·2
115	12·0	5·0	10·9	7·7	9·2	11·6	− 3·1
125	11·3	6·5	9·3	3·9	10·6	10·9	+ 2·9
135	9·7	6·9	6·9	− 0·0	9·7	6·9	6·9
145	13·7	11·2	7·8	+ 4·7	12·8	+ 3·5	13·1
155	16·7	15·1	7·1	10·7	12·8	− 4·3	16·1
165	22·0	21·3	5·7	19·1	11·0	15·7	15·7
175	40·0	39·8	+ 3·5	39·4	− 7·0	39·6	+ 10·4
185	48·0	47·9	− 4·2	47·3	+ 8·3	46·4	− 12·4
195	51·3	49·6	13·3	44·4	25·6	36·3	36·3
205	49·0	44·4	20·7	31·5	37·6	− 12·7	47·4
215	47·0	38·4	26·9	16·0	44·1	+ 12·2	45·4
225	44·3	31·3	31·3	+ 0·0	44·3	31·3	31·3
235	29·3	16·8	23·9	− 10·0	27·5	28·3	− 7·6
245	19·3	8·2	17·5	12·4	14·8	18·6	+ 5·0
255	10·0	2·6	9·7	8·7	5·0	7·1	7·1
265	9·0	− 0·8	9·0	8·9	+ 1·6	+ 2·3	8·7
275	8·0	+ 0·7	8·0	7·8	− 1·4	− 2·1	7·8
285	6·7	1·8	6·5	5·9	3·3	4·8	4·8
295	6·3	2·7	5·8	4·1	4·9	6·2	+ 1·7
305	5·7	3·2	4·6	− 1·9	5·3	5·4	− 1·5
315	5·7	3·9	3·9	+ 0·0	5·7	3·9	3·9
325	5·7	4·6	3·2	1·9	5·2	− 1·4	5·3
335	7·3	6·6	3·1	4·7	5·5	+ 1·9	7·0
345	9·7	9·3	2·5	8·3	4·8	6·8	6·8
355	20·7	+ 20·6	− 1·8	+ 20·3	− 3·6	+ 20·0	− 5·4
	Totals	− 30·9	+ 159·4	+ 143·1	+ 379·1	− 129·5	− 12·0

* The formulae, in numerals, quoted in this section refer to section 4·61.

If we perform the solution with $k \equiv 1{\cdot}40$, as in (29), we arrive at values of 1·3 approximately for hV_1 and hV_2 and about 0·05 for α. The result is that k, calculated from formula (25), is somewhat less than 1·40. In the subsequent work, we put $k = 1{\cdot}30$. We then obtain

$$N\gamma k = -0{\cdot}205 - 0{\cdot}020i \equiv 0{\cdot}206E(185^\circ{\cdot}6). \qquad \ldots\ldots(c)$$

From (a), (b) and (c), we find

$$LM = 0{\cdot}072E(170^\circ{\cdot}1) \equiv -0{\cdot}071 + 0{\cdot}012i,$$
$$N\gamma k - LM = -0{\cdot}134 - 0{\cdot}032i \equiv 0{\cdot}138E(193^\circ{\cdot}4),$$
$$L^2 = 0{\cdot}029E(201^\circ{\cdot}6) \equiv -0{\cdot}027 - 0{\cdot}011i.$$

Also,
$$MK = 0{\cdot}195 + 0{\cdot}516i,$$
$$Mk - L^2 = 0{\cdot}222 + 0{\cdot}527i \equiv 0{\cdot}572E(67^\circ{\cdot}2)$$

and
$$(Mk - L^2)^{\frac{1}{2}} \equiv 0{\cdot}756E(33^\circ{\cdot}6). \qquad \ldots\ldots(d)$$

Now by (32),

$$2K \equiv \frac{N\gamma k - LM}{Mk - L^2} = \frac{0{\cdot}138E(193^\circ{\cdot}4)}{0{\cdot}572E(67^\circ{\cdot}2)} = 0{\cdot}241E(126^\circ{\cdot}2).$$

Hence
$$K = -0{\cdot}071 + 0{\cdot}097i$$

and
$$Kk = -0{\cdot}092 + 0{\cdot}126i.$$

We then have
$$L - Kk = 0{\cdot}060 + 0{\cdot}041i \equiv 0{\cdot}073E(34^\circ{\cdot}3) \qquad \ldots\ldots(e)$$

and
$$(L - Kk)^2 = 0{\cdot}005E(68^\circ{\cdot}6) \equiv 0{\cdot}002 + 0{\cdot}005i.$$

Writing
$$X \equiv (L - Kk)^2 + Mk - L^2,$$

we derive
$$X = 0{\cdot}224 + 0{\cdot}532i \equiv 0{\cdot}577E(67^\circ{\cdot}2)$$

and
$$X^{\frac{1}{2}} = 0{\cdot}760E(33^\circ{\cdot}6).$$

If β is calculated by means of (35), we have

$$\beta \equiv \frac{L - Kk}{X^{\frac{1}{2}}} = \frac{0{\cdot}073E(34^\circ{\cdot}3)}{0{\cdot}760E(33^\circ{\cdot}6)} = 0{\cdot}096E(0^\circ{\cdot}7),$$

from which
$$\beta = 0{\cdot}096 + 0{\cdot}001i.$$

Thus β comes out to be nearly real. However, to illustrate the method, we shall keep β in its complex form and we have to modify the procedure in accordance with formulae (40) and (41).

From (e) and (d), $\psi = 34^\circ{\cdot}3$ and $\phi = 33^\circ{\cdot}6$. Hence, by (40),

$$L - K_0k = 0{\cdot}073 \cos\{34^\circ{\cdot}3 - 33^\circ{\cdot}6\}\, E(33^\circ{\cdot}6)$$
$$= 0{\cdot}073E(33^\circ{\cdot}6).$$

Consequently,
$$\frac{(Mk - L^2)^{\frac{1}{2}}}{L - K_0k} = \frac{0{\cdot}756}{0{\cdot}073} = 10{\cdot}36,$$

and, from (41), $$\frac{1}{\beta^2} = 1 + (10{\cdot}36)^2 = 108{\cdot}3,$$

from which $$\beta = +0{\cdot}096,$$

the sign of β being determined in accordance with the rule previously mentioned.

With this value of β, we find

$$\left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2}}(Mk - L^2)^{\frac{1}{2}} = 0{\cdot}908 \times 0{\cdot}756E(33^\circ{\cdot}6)$$
$$= 0{\cdot}572 + 0{\cdot}380i.$$

Adding the value of $L \equiv -0{\cdot}032 + 0{\cdot}167i$, we obtain from (36)

$$kP_1 = 0{\cdot}540 + 0{\cdot}547i \equiv 0{\cdot}769E(45^\circ{\cdot}4),$$

from which, with $k = 1{\cdot}30$,

$$P_1 = 0{\cdot}592E(45^\circ{\cdot}4).$$

Similarly, $$P_2 = 0{\cdot}602E(202^\circ{\cdot}1).$$

We then obtain, using the values of $|P|$ in Table 16,

$$hV_1 = 1{\cdot}18, \quad \theta_1 = 45^\circ{\cdot}4,$$
$$hV_2 = 1{\cdot}21, \quad \theta_2 = 202^\circ{\cdot}1.$$

With these values of hV_1 and hV_2, we find from Table 16

$$\zeta_1 = 1{\cdot}318, \quad \zeta_2 = 1{\cdot}310.$$

Formula (38) now enables the value of α to be found; it is

$$\alpha = +0{\cdot}093.$$

Thus the numbers of stars in the drifts are 521 and 433 respectively. We now *calculate* k from (25); the result is

$$k = 1{\cdot}314,$$

in good agreement with the assumed value 1·30. A re-calculation with this new value of k is unnecessary.

The results* of the analysis by the "trial and error" method of section 4·21 are (the corresponding drift-velocities are denoted by hV_1' and hV_2' in the paper referred to):

$$hV_1 = 1{\cdot}3, \quad \theta_1 = 45^\circ,$$
$$hV_2 = 1{\cdot}2, \quad \theta_2 = 200^\circ,$$

and the numbers of stars in the two drifts are 509 and 445 respectively. The two methods of solution give, in this instance, very accordant results.

When the stars are distributed between the two drifts in nearly equal proportions, as in the example just worked out, the agreement between the results of the two methods is generally satisfactory; otherwise, the analytical

* W. M. Smart and H. E. Green, *M.N.* **89**, 151, 1928.

method of the previous section is not well adapted for determining α and the "trial and error" method is to be preferred.

4·63. *Modification of the analytical method.*

Although earlier investigations, mainly of the brighter stars, appeared to show that the stars of drift I were more numerous than the stars of drift II (for example, in Eddington's analysis* of the Boss stars, the ratio was found to be 3 : 2), recent work on the faint stars has in the main suggested an approximate equality in the numbers. If we assume α to be zero, our unknowns are reduced to four, viz. hV_1, hV_2, θ_1 and θ_2, and the two equations, (20) and (21) of section 4·61, suffice to determine the drift-velocities and the position angles θ_1, θ_2. We have, then, with the same definition of the symbols,

$$2L = \zeta_1 P_1 + \zeta_2 P_2,$$

$$2M = \zeta_1 P_1^2 + \zeta_2 P_2^2$$

and

$$k = \tfrac{1}{2}(\zeta_1 + \zeta_2), \qquad \text{......(1)}$$

$$\frac{1-\beta}{1+\beta} = \frac{\zeta_2}{\zeta_1} \equiv \epsilon^2. \qquad \text{......(2)}$$

The solutions are then contained in the following equations (equivalent to (36) and (37) of section 4·61):

$$kP_1 = L + \epsilon(Mk - L^2)^{\frac{1}{2}}, \qquad \text{......(3)}$$

$$kP_2 = L - \frac{1}{\epsilon}(Mk - L^2)^{\frac{1}{2}}. \qquad \text{......(4)}$$

The numerical work is evidently greatly reduced. Eddington has applied† this method in analysing the Cambridge proper motions.

In a preliminary solution of the equations (3) and (4) we can put ϵ equal to unity and take a suitable value of k. We then obtain hV_1 and hV_2 and with these values we obtain from Table 16 the corresponding values of ζ_1 and ζ_2, which lead to new values of k and ϵ given by (1) and (2). With these values of k and ϵ, the work may be repeated, if considered necessary.

We illustrate the method by means of the data of section 4·62. Putting $k = 1{\cdot}28$ and $\epsilon = 1$, we have

$$(Mk - L^2)^{\frac{1}{2}} = 0{\cdot}751E(33^\circ{\cdot}4) = 0{\cdot}627 + 0{\cdot}413i,$$

$$L = \qquad - 0{\cdot}032 + 0{\cdot}167i.$$

Hence, from (3) and (4),

$$kP_1 = \quad 0{\cdot}595 + 0{\cdot}580i = 0{\cdot}831E(44^\circ{\cdot}3),$$

$$kP_2 = -0{\cdot}659 - 0{\cdot}246i = 0{\cdot}703E(200^\circ{\cdot}5),$$

from which $|P_1| = 0{\cdot}647$ and $|P_2| = 0{\cdot}549$.

* *M.N.* 71, 38, 1910; see also section 4·36. † *M.N.* 87, 138, 1926.

The solutions then are:

$$hV_1 = 1{\cdot}34,\ \theta_1 = 44^\circ{\cdot}3;\quad hV_2 = 1{\cdot}08,\ \theta_2 = 200^\circ{\cdot}5.$$

With these values of hV_1 and hV_2, we find

$$\zeta_1 = 1{\cdot}271 \quad \text{and} \quad \zeta_2 = 1{\cdot}348,$$

from which

$$k = 1{\cdot}31 \quad \text{and} \quad \epsilon = 1{\cdot}03.$$

Repeating the solution with these values of k and ϵ, we obtain

$$hV_1 = 1{\cdot}29,\ \theta_1 = 44^\circ{\cdot}0;\quad hV_2 = 1{\cdot}04,\ \theta_2 = 200^\circ{\cdot}5.$$

4·7. *The three-drift hypothesis.*

When the proper motions in a given region of the sky are analysed by the methods of the preceding pages, it is generally found that the direction of the apex of drift I is fairly close to the direction of the solar antapex. This may be easily seen from a diagram. In Fig. 29, A_1 denotes the position, on the celestial sphere, of the apex of drift I and A the solar antapex (we assume that the right ascensions of A_1 and A are 6^h and that the declinations are -14° and -34°—in accordance with the results of observation). For a region at R the directions of A and A_1 from R have a comparatively small separation. It is only for regions such as S (and the antipodal regions), with right ascensions approximately between 5^h and 7^h and declinations between -14° and -34°, that the separation between these directions is considerable. Now if, in addition to the two assemblies of stars forming drift I and drift II, there is a third at rest in space with reference to the totality of the stars, this third group will give rise to a drift—when the solar motion is taken into account—with its drift-velocity in the direction of the solar antapex. In addition to drift II, we should then have two drifts with their drift-velocities, in general, not much separated in direction and the combined effect would be to give a distribution of proper motions resembling a single drift. This is illustrated in Fig. 30, taken from a paper by Halm.* The full-line curve gives the distribution resulting from the combination of two drifts, one moving in position angle 90° with velocity $hV = 1{\cdot}05$, the other in position angle 135° with velocity 1·23, the numbers of stars in the two drifts being assumed equal. The curve is clearly unsymmetrical, but

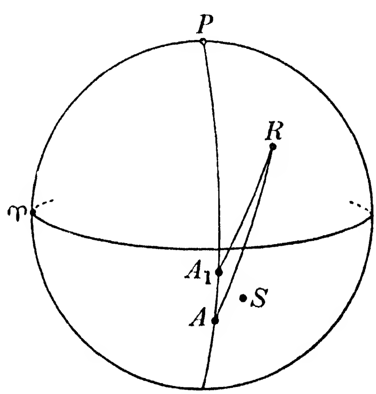

Fig. 29

* *M.N.* **71**, 620, 1911.

the deviations are so inconsiderable that a single drift-curve can be readily found to give a good representation of the original curve. This single drift-curve is shown by the broken-line and corresponds to position angle 116° and velocity 0·90.

Investigations of the motions of the bright "Orion" stars (that is, of spectral type B) of Boss's *P.G.C.* suggest that there is little or no evidence of preferential motions as with stars of the other spectral types, for which star-streaming is unmistakable. The B type stars, in fact, form a single drift with the drift-velocity directed towards the solar antapex. In Eddington's analysis of the Boss stars, the B type stars were omitted. Halm, however, has shown (*loc. cit.*) in a rediscussion of the material that, in regions (such as *S* in Fig. 29) where the directions of the apex of drift I and of the solar ant-

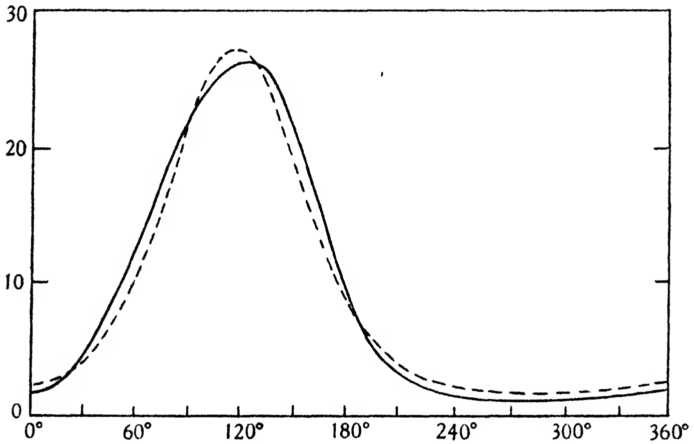

Fig. 30

apex are widely separated, there is evidence of a residual antapical drift of stars (other than of type B) to which, with the Orion stars, he gave the designation "drift O". In the other regions where there is little divergence between the direction of the apex of drift I and the direction of the solar antapex, drift O is supposed to be concealed in the analysis on the two-drift hypothesis, as it is mainly combined with drift I in the manner illustrated in Fig. 30. Eddington* had remarked on the presence of a third drift in two of the Boss regions which he analysed, but was inclined to regard it as of minor importance in comparison with the two great star-drifts.

So far as can be ascertained, the emergence of drift O is not so evident in the analysis of the proper motions of faint stars and it is generally ignored in statistical discussions. Unless drift O is of an importance comparable with that of drift I and drift II, this procedure is in accordance with the method

* *M.N.* 71, 40, 1910.

of solution by successive approximations, the two-drift hypothesis providing the first approximation. It may be said that this first approximation represents, on the whole, the observed distribution of proper motions very satisfactorily and a more refined investigation of the proper motion data on the basis of a three-drift hypothesis must await more accurate and more plentiful observational material.

4·81. *The radial velocities and the two-drift hypothesis.*

Hitherto, it has not been found practicable to use the radial velocities as a means of deriving the drift constants but, nevertheless, it is extremely important to ascertain whether the observed radial velocities are statistically in accordance with the two-streams theory. One method of testing the theory is as follows. From the observed radial velocities of stars scattered over the sky, the solar motion is found by the method described in Chapter III. If λ is the angular distance of a star from the solar antapex, the observed radial velocity of the star is made up of (*a*) the projection of the reversed solar speed in the direction of the star—in other words, the parallactic component—of magnitude $V_0 \cos\lambda$, where V_0 is the solar speed, and (*b*) a part, R, due to the star's motion in the drift. Knowing V_0 and λ we can remove the parallactic component from the observed radial velocity and so obtain the component, R.

Consider now N stars in a small region of the sky at S, and suppose that, of these N stars, N_1 belong to drift I and N_2 to drift II, where $N = N_1 + N_2$. Let the velocity of drift I relative to the centre of rest of the totality of stars be V_1, directed towards the vertex B, and let the direction of S make an angle ϕ with the direction of B. Thus, the component of drift I in the direction of S is $V_1 \cos\phi$. Consequently, the radial velocity of any star at S (freed from the solar motion) is made up of a constant part $V_1 \cos\phi$ and a haphazard component v. The following analysis is analogous to that in section 2·44.

Of the N_1 stars, n_1 will have positive* radial velocities given by $V_1 \cos\phi + v$, if $-V_1 \cos\phi < v < \infty$, and the remainder, n_2, $(N_1 \equiv n_1 + n_2)$, will have negative radial velocities if $v < -V_1 \cos\phi$. Now the number of stars with random velocities between v and $v + dv$ is

$$\frac{hN_1}{\sqrt{\pi}} e^{-h^2v^2} dv$$

and consequently, if ρ_1 denotes the mean radial speed of the n_1 stars,

$$n_1\rho_1 = \frac{hN_1}{\sqrt{\pi}} \int_{-V_1 \cos\phi}^{\infty} (v + V_1 \cos\phi)\, e^{-h^2v^2} dv$$

* We consider $V_1 \cos\phi$ here to be positive. As we shall be concerned with *speeds*, it is easily seen that the formulae (4) and (5) are independent of the sign of $V_1 \cos\phi$.

or, writing $$hv = x, \quad hV_1 \cos\phi = \tau_1, \qquad \text{......(1)}$$

$$n_1\rho_1 = \frac{N_1}{h\sqrt{\pi}}\int_{-\tau_1}^{\infty}(x+\tau_1)\,e^{-x^2}\,dx. \qquad \text{......(2)}$$

Similarly, if ρ_2 denotes the mean linear speed of the n_2 stars,

$$n_2\rho_2 = \frac{hN_1}{\sqrt{\pi}}\int_{V_1\cos\phi}^{\infty}(v-V_1\cos\phi)\,e^{-h^2v^2}\,dv$$

$$= \frac{N_1}{h\sqrt{\pi}}\int_{\tau_1}^{\infty}(x-\tau_1)\,e^{-h^2v^2}\,dv. \qquad \text{......(3)}$$

Let $\bar{R}_1$ denote the mean radial speed (freed from solar motion) of the N_1 stars; then $$N_1\bar{R}_1 = n_1\rho_1 + n_2\rho_2,$$

from which, by means of (2) and (3),

$$h\sqrt{\pi}\,\bar{R}_1 = \int_{-\tau_1}^{\infty}(x+\tau_1)\,e^{-x^2}\,dx + \int_{\tau_1}^{\infty}(x-\tau_1)\,e^{-x^2}\,dx$$

$$= 2\int_{-\tau_1}^{\infty}xe^{-x^2}\,dx + 2\tau_1\int_0^{\tau_1}e^{-x^2}\,dx,$$

or, in terms of the integral

$$K(\tau) \equiv \int_0^{\tau}e^{-x^2}\,dx,$$

$$h\sqrt{\pi}\,\bar{R}_1 = e^{-\tau_1^2} + 2\tau_1 K(\tau_1). \qquad \text{......(4)}$$

The function on the right-hand side of (4) is the same as the function $F(\tau_1)$ defined in (17) of section 2·44; the numerical values are given in Table 8. Accordingly, we write

$$h\sqrt{\pi}\,\bar{R}_1 = F(\tau_1). \qquad \text{......(5)}$$

Similarly, the mean radial speed, $\bar{R}_2$ (freed from the solar motion), of the N_2 stars of drift II is given by

$$h\sqrt{\pi}\,\bar{R}_2 = F(\tau_2), \qquad \text{......(6)}$$

where $$\tau_2 = hV_2\cos\phi. \qquad \text{......(7)}$$

Let $\bar{R}_0$ denote the mean radial speed (freed from the solar motion) of the $N(\equiv N_1+N_2)$ stars. Then

$$(N_1+N_2)\,\bar{R}_0 = N_1\bar{R}_1 + N_2\bar{R}_2,$$

or, writing $N_1 = \alpha(N_1+N_2)$, and using (5) and (6),

$$h\sqrt{\pi}\,\bar{R}_0 = \alpha F(\tau_1) + (1-\alpha)\,F(\tau_2). \qquad \text{......(8)}$$

It may be remarked that α and the stream-velocities, hV_1 and hV_2, are related by the equation $$\alpha V_1 = (1-\alpha)V_2.$$

The formula (8) has been used by Eddington* in a particular case for testing the two-streams hypothesis. Denoting by R' the value of $\bar{R}_0$ for a region at the vertex, where $\phi = 0$, and by R'' the value of $\bar{R}_0$ for a region 90° from the vertex, we have, since $R'' = \dfrac{1}{h\sqrt{\pi}}$,

$$\frac{R'}{R''} = \alpha F(hV_1) + (1-\alpha) F(hV_2). \qquad \ldots\ldots(9)$$

The theoretical value of R'/R'' as derived from (9) with the known values of α, hV_1 and hV_2 was found to be in fairly good agreement with the value obtained from the observed radial velocities.

The formula (8) is a general one and it can be applied to any region of the sky. The disadvantage of the method, however, is that the solar motion has to be removed from the observed radial velocities of the stars in each area. In the next section we demonstrate a method that is easy to apply in practice.

4·82. Consider a region of the sky at angular distances λ_1 and λ_2 from the apices of drift I and drift II respectively. By formula (16) of section 2·44, the mean observed speed (that is, relative to the sun) for stars in drift I is $\bar{R}F(\tau_1)$, where

$$\tau_1 = hU_1 \cos\lambda_1$$

and $F(\tau)$ is the function tabulated on p. 49, hU_1 is the space-velocity of the drift relative to the sun, and $\bar{R}$ is the mean random radial speed for the stars in the drift. There is a similar expression for the mean observed speed of the stars belonging to drift II. With the meaning of α as defined in the previous section, it is seen that, if ρ_0 denotes the mean observed speed of all the stars in the region,

$$\bar{\rho}_0 = \bar{R}\{\alpha F(\tau_1) + (1-\alpha) F(\tau_2)\}, \qquad \ldots\ldots(1)$$

where

$$\tau_2 = hU_2 \cos\lambda_2,$$

the space-velocity of drift II relative to the sun being hU_2.

We suppose that the drift-velocities hU_1 and hU_2, the positions of the apices of the drifts and the value of α have all been determined from the analysis of the proper motions. Also, this analysis gives the solar speed expressed in terms of the theoretical unit $1/h$; for example, Eddington's value for the solar speed (section 4·36) is 0·908 in terms of the theoretical unit, and identifying this with the usual value of 19·5 km./sec. obtained from the radial velocities as in Chapter III, section 3·43, we have

$$\frac{1}{h} = \frac{19{\cdot}5}{0{\cdot}908}\text{ km./sec.}$$

Also, by (1) of section 2·24, $\bar{R} = \dfrac{1}{h\sqrt{\pi}}$.

Hence

$$\bar{R} = 12{\cdot}1\text{ km./sec.} \qquad \ldots\ldots(2)$$

* *Stellar Movements*, 144, 1914.

The right-hand side of (1) can now be calculated in km./sec. and the result compared with the value of $\bar{\rho}_0$ obtained from the observed radial speeds in the region. A recent investigation* showed that for the radial velocities of Schlesinger's *Catalogue of Bright Stars*, 1930 and with Eddington's values of the drift constants obtained from the analysis of Boss's proper motions, theory and observation were in close accordance. It should be remarked that practically all the radial velocity stars are also Boss stars.

We can proceed (*loc. cit.*) in a somewhat different way by regarding (1) as an observational equation in which $\bar{R}$ and α are to be determined, the value of $\bar{\rho}_0$ for each region being given by the observations. As before, we suppose the space-velocities of the drifts relative to the sun and the drift apices to be known. We then have as many equations of the form (1) as regions, and by a least squares solution we can derive $\bar{R}$ and α. For convenience, write

$$\bar{R} = x; \quad \alpha = 0{\cdot}5+\beta; \quad \bar{R}\beta = y$$

and set $$\tfrac{1}{2}F(\tau_1)+\tfrac{1}{2}F(\tau_2) = a, \quad F(\tau_1)-F(\tau_2) = b.$$

Then (1) becomes $$ax+by = \bar{\rho}_0, \qquad \ldots\ldots(3)$$

in which a, b and $\bar{\rho}_0$ are known for each region; (3) is then an equation of condition for the unknowns x and y.

The radial velocities of 3679 stars were taken from Schlesinger's catalogue. Stars omitted from consideration were as follows: (*a*) stars of spectral type other than B to M, (*b*) stars belonging to open clusters, and (*c*) stars in Oort's category of "high velocity stars".† Two solutions were made, (i) for all spectral types B to M, and (ii) for all spectral types A to M. The second solution corresponds to Eddington's analysis of the proper motions of the Boss stars, as in this analysis the B type stars were omitted. The results are:

$$\text{(i)} \quad \bar{R} = 11{\cdot}8 \text{ km./sec.}; \quad \alpha = 0{\cdot}52,$$

$$\text{(ii)} \quad \bar{R} = 12{\cdot}2 \text{ km./sec.}; \quad \alpha = 0{\cdot}49.$$

The solution (ii) is the appropriate one with which to compare the value of $\bar{R}$ as given in (2); the two values of $\bar{R}$ are almost identical and we conclude that the radial velocities are confirmatory of—certainly not antagonistic to—the two-streams theory. The solution (ii) also indicates that the stars utilised in this analysis are divided almost equally between the two streams.

* W. M. Smart, *M.N.* **96**, 165, 1936. † *Groningen Publ.* **40**, 30, 1926.

CHAPTER V

THE ELLIPSOIDAL THEORY

5·11. *Schwarzschild's hypothesis.*

We have seen in the previous chapter that the observed distribution of stellar motions can be explained satisfactorily by the assumption of two assemblies of stars which are, so far as can be ascertained, intermingled in space. The motion of one drift relative to the other defines a well-determined axis lying in the galactic equator and fixing the vertices of star-streaming. If we imagine that our observations are made from a position at rest with reference to the totality of stars, the distribution of velocities will be such that one assembly will appear to have, as a whole, a motion parallel to the axis, the individual motions being compounded of this common motion and the haphazard motions, while the other assembly will appear to have, as a whole, a motion parallel to the axis but in the opposite direction, the individual velocities again being compounded of this common motion and the random motions. To our imaginary observer the phenomenon may be described as a greater mobility parallel to the axis joining the vertices than in directions perpendicular to the axis. It was this aspect of the distribution of stellar motions that led Schwarzschild* to postulate a velocity function of the form

$$e^{-K^2U^2-H^2(V^2+W^2)},$$

in which (U, V, W) are the components of the linear velocity of a star, the U-axis corresponding to the axis of star-streaming, and symmetry about this axis being assumed. The exponent in this formula is related to the equation of an ellipsoid (with two equal axes), namely

$$K^2U^2+H^2(V^2+W^2) = 1,$$

in which the velocity components play the part of coordinates. If K is less than H, the mean speed component parallel to the U-axis is, as we shall show, greater than the mean speed component perpendicular to this axis. The U-axis, therefore, gives the direction of greatest mobility.

Let N be the number of stars with the given velocity function. If dN is the number of stars with velocity components between (U, V, W) and $(U+dU, V+dV, W+dW)$, Schwarzschild's hypothesis gives

$$dN = Ae^{-K^2U^2-H^2(V^2+W^2)}dU\,dV\,dW. \qquad \text{......(1)}$$

* *Göttingen Nach.* 1907, p. 614.

The coefficient A is obtained by summing for all possible values of U, V and W. Thus

$$N = A\int_{-\infty}^{\infty} e^{-K^2U^2}dU \int_{-\infty}^{\infty} e^{-H^2V^2}dV \int_{-\infty}^{\infty} e^{-H^2W^2}dW,$$

so that

$$A = \frac{NKH^2}{\pi^{\frac{3}{2}}}. \qquad \ldots\ldots(2)$$

Let $\bar{U}$ denote the mean velocity component parallel to the U-axis, taken without regard to sign. The total number, δN, of stars with components between U and $U+dU$ is obtained by summing (1) for all possible values of V and W. Hence, using (2),

$$\delta N = \frac{NKH^2}{\pi^{\frac{3}{2}}} e^{-K^2U^2}dU \int_{-\infty}^{\infty} e^{-H^2V^2}dV \int_{-\infty}^{\infty} e^{-H^2W^2}dW,$$

from which

$$\delta N = \frac{NK}{\pi^{\frac{1}{2}}} e^{-K^2U^2}dU. \qquad \ldots\ldots(3)$$

Since we are concerned only with the arithmetical values of U in finding the value of $\bar{U}$, we have, from (3),

$$N\bar{U} = \frac{2NK}{\pi^{\frac{1}{2}}}\int_0^{\infty} Ue^{-K^2U^2}dU,$$

from which

$$\bar{U} = \frac{1}{K\pi^{\frac{1}{2}}}. \qquad \ldots\ldots(4)$$

Similarly,

$$\bar{V} = \bar{W} = \frac{1}{H\pi^{\frac{1}{2}}}. \qquad \ldots\ldots(5)$$

Hence, if $K < H$, the mean component in the U-direction is greater than the mean component in a perpendicular direction; in other words, the U-axis is the axis of greatest mobility.

Referring to (1), we see that $\bar{U}$ and $\bar{V}$ are proportional to the semi-axes of the velocity ellipsoid.

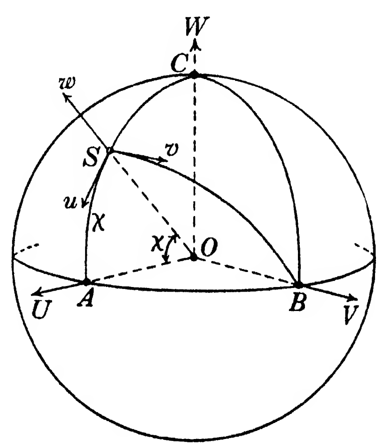

Fig. 31

5·12. *The velocity ellipse.*

Consider a small area of the sky at S (Fig. 31) in which there are n stars with the given ellipsoidal distribution. The centre, O, is at rest with respect to the assembly of stars and we are to find the distribution of velocities in the plane at S perpendicular to the line of sight. Let OA represent the U-axis and let BC be the great circle of which A is the pole. Since the velocity distribution is symmetrical about OA, we can choose the

V and W axes according to convenience. Let the W axis lie in the plane OAS. The V-axis defines the direction of a pole of the great circle ASC.

We now choose new rectangular axes, the w-axis being radial at S and the u and v axes in the tangent plane at S; the u-axis is taken to be tangential to the great circle SA at S; the v-axis is tangential to the great circle SB.

Let χ denote the angle AOS, that is, the angular distance of S from the vertex A. From Fig. 31, it is evident that

$$\left.\begin{aligned} u &= U\sin\chi - W\cos\chi \\ v &= V \\ w &= U\cos\chi + W\sin\chi \end{aligned}\right\} \qquad \ldots\ldots(1)$$

or

$$\left.\begin{aligned} U &= u\sin\chi + w\cos\chi \\ V &= v \\ W &= -u\cos\chi + w\sin\chi \end{aligned}\right\}. \qquad \ldots\ldots(2)$$

Hence

$$K^2U^2 + H^2(V^2+W^2) = u^2(K^2\sin^2\chi + H^2\cos^2\chi) + H^2v^2$$
$$+ w^2(K^2\cos^2\chi + H^2\sin^2\chi) + 2uw(K^2-H^2)\sin\chi\cos\chi. \qquad \ldots\ldots(3)$$

Set, in (3),

$$\alpha = K^2\cos^2\chi + H^2\sin^2\chi \quad \text{and} \quad \beta = (K^2-H^2)\sin\chi\cos\chi.$$

The number of stars with velocity components between (U, V, W) and $(U+dU,\ V+dV,\ W+dW)$ is given by formula (1) of the previous section; hence the number with velocity components between (u, v, w) and $(u+du,\ v+dv,\ w+dw)$ is

$$A\,du\,dv\,dw\,e^{-u^2(K^2\sin^2\chi + H^2\cos^2\chi) - H^2v^2 - \alpha w^2 - 2\beta uw}.$$

Let dn now denote the number of stars with transverse velocities between (u, v) and $(u+du, v+dv)$. This number will be obtained by summing the previous expression for all values of w between $-\infty$ and $+\infty$. Hence

$$dn = A\,du\,dv\,e^{-u^2(K^2\sin^2\chi + H^2\cos^2\chi) - H^2v^2}\int_{-\infty}^{\infty} e^{-\alpha w^2 - 2\beta uw}\,dw. \qquad \ldots\ldots(4)$$

Now

$$\int_{-\infty}^{\infty} e^{-\alpha w^2 - 2\beta uw}\,dw = e^{\frac{\beta^2}{\alpha}u^2}\int_{-\infty}^{\infty} e^{-\alpha\left(w+\frac{\beta u}{\alpha}\right)^2}\,dw$$
$$= \sqrt{\frac{\pi}{\alpha}}\,e^{\frac{\beta^2}{\alpha}u^2}. \qquad \ldots\ldots(5)$$

Hence, from (4) and (5),

$$dn = A\sqrt{\frac{\pi}{\alpha}}\,du\,dv\,e^{-u^2\left\{K^2\sin^2\chi + H^2\cos^2\chi - \frac{\beta^2}{\alpha}\right\} - H^2v^2},$$

which may be written $\qquad dn = C\,e^{-k^2u^2 - h^2v^2}\,du\,dv, \qquad \ldots\ldots(6)$

where $$k^2 = \frac{K^2 H^2}{K^2 \cos^2 \chi + H^2 \sin^2 \chi} \qquad \ldots\ldots(7)$$

and $$h = H. \qquad \ldots\ldots(8)$$

The ellipse given by the equation

$$k^2 u^2 + h^2 v^2 = 1 \qquad \ldots\ldots(9)$$

is called the *velocity ellipse.*

From (7) and (8), we obtain

$$\frac{K^2 \cos^2 \chi + H^2 \sin^2 \chi}{K^2} = \frac{h^2}{k^2},$$

from which $$\left(\frac{H^2}{K^2} - 1\right) \sin^2 \chi = \frac{h^2}{k^2} - 1. \qquad \ldots\ldots(10)$$

This is an important formula. Since $K < H$, it is evident that $k < h$, so that the u-axis is the major axis of the velocity ellipse and is consequently the axis of greatest mobility in the tangent plane. Also, since $h = H$, the minor axis is the same for all parts of the sky.

The formula (10) gives the relation between the ratio of the axes of the velocity ellipse and the ratio of the axes of the velocity ellipsoid. As will be shown later, the value of h/k can be obtained, for a particular region, from the observed proper motions and the combination of results from different regions gives the value of H/K and the direction of the U-axis, that is, the direction of the vertices of preferential motion.

5·21. *The distribution of the observed transverse motions in a given region of the sky.*

Consider now the effect of the parallactic motion on the distribution of the transverse linear velocities in the region of the sky at S. As in the previous section we define the u-axis to be the tangent at S to the great circle joining S to the vertex A. Also, the number, dn, of stars with components between (u, v) and $(u+du, v+dv)$ is, rewriting (6) of the previous section,

$$dn = Ce^{-k^2u^2 - h^2v^2}\, du\, dv, \qquad \ldots\ldots(1)$$

from which it is easily found that

$$C = \frac{nhk}{\pi}, \qquad \ldots\ldots(2)$$

where n is the total number of stars in the region at S.

Let U_0, V_0 denote the linear components of the parallactic motion for the region referred to the u and v axes. We require a convention as to the positive direction of the u-axis; we shall take the positive direction to be such that the component U_0 of the parallactic motion is positive. The positive direction of the v-axis is chosen in a similar way. Let OA, in Fig. 32, represent the vector (U_0, V_0) and let AB represent the peculiar velocity of a star, with components u and v. The star will then be observed to have a linear velocity r (represented by OB) in a direction making an angle ϕ with the u-axis.

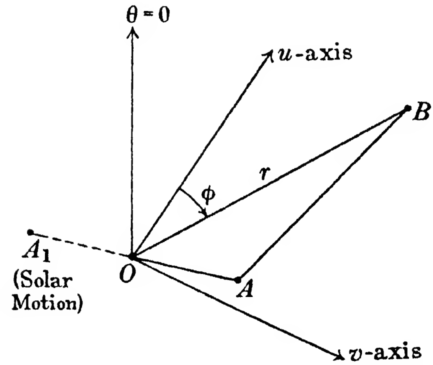

Fig. 32

If (x, y) are the components of the observed velocity with respect to the u and v axes, we have

$$x = r\cos\phi, \quad y = r\sin\phi. \qquad \ldots\ldots(3)$$

Also $\quad x = u + U_0, \quad y = v + V_0.$

Hence the number, dn, of linear velocities with components between (x, y) and $(x+dx, y+dy)$ is given from (1) and (2) by

$$dn = \frac{nhk}{\pi} dx\,dy\, e^{-k^2(x-U_0)^2 - h^2(y-V_0)^2}. \qquad \ldots\ldots(4)$$

From (3), $\qquad dx\,dy = r\,dr\,d\phi.$

Consequently, the number of stars with linear velocities between r and $r+dr$ and moving in the sector ϕ, $\phi+d\phi$ is given by

$$dn = \frac{nhk}{\pi} d\phi\,.\,r\,dr\, e^{-k^2(r\cos\phi - U_0)^2 - h^2(r\sin\phi - V_0)^2}. \qquad \ldots\ldots(5)$$

Let $\rho\,d\phi \equiv n(\phi)\,d\phi$ denote the total number of stars moving in the sector ϕ, $\phi+d\phi$. Then, from (5), by summing for all possible values of r between 0 and ∞,

$$\rho\,d\phi \equiv n(\phi)\,d\phi = \frac{nhk}{\pi} d\phi\, e^{-k^2U_0^2 - h^2V_0^2}$$

$$\times \int_0^\infty r\,dr\, e^{-r^2(k^2\cos^2\phi + h^2\sin^2\phi) + 2r(k^2U_0\cos\phi + h^2V_0\sin\phi)}. \qquad \ldots\ldots(6)$$

Let $\qquad p = +(k^2\cos^2\phi + h^2\sin^2\phi)^{\frac{1}{2}}, \qquad \ldots\ldots(7)$

$$\xi = \frac{1}{p}(k^2U_0\cos\phi + h^2V_0\sin\phi) \qquad \ldots\ldots(8)$$

and $\qquad x = pr - \xi. \qquad \ldots\ldots(9)$

It is to be noted that, with the conventions stated above, ξ is positive for the positive directions of the u and v axes (corresponding to $\phi = 0$ and $\phi = \pi/2$).

The integral in (6) becomes, by means of (7), (8) and (9),

$$\frac{1}{p^2} e^{\xi^2} \int_{-\xi}^{\infty} (x+\xi)\, e^{-x^2}\, dx$$

or

$$\frac{1}{p^2}\left\{\tfrac{1}{2} + \xi e^{\xi^2} \int_{-\xi}^{\infty} e^{-x^2}\, dx\right\}$$

or $\frac{\sqrt{\pi}}{2p^2} f(\xi)$, where $f(\xi)$ is Eddington's function introduced in Chapter II, section 2·3. We thus have, from (6),

$$\rho \equiv n(\phi) = \frac{B^2}{p^2} f(\xi), \qquad \ldots\ldots(10)$$

where

$$B^2 = \frac{nhk}{2\sqrt{\pi}} e^{-k^2 U_0^2 - h^2 V_0^2}. \qquad \ldots\ldots(11)$$

The polar curve derived from (10) gives the distribution of the transverse linear velocities relative to the sun.

5·22. *Characteristics of the polar curve.*

From the formula for $f(\xi)$, it is seen that $f(\xi)$ increases as ξ increases and therefore $f(\xi)$ will have its maximum value when ξ has its maximum value. Writing $t \equiv \tan\phi$, we have, from (7) and (8) of the previous section,

$$\xi = \frac{k^2 U_0 + h^2 V_0 t}{(k^2 + h^2 t^2)^{\frac{1}{2}}},$$

from which

$$\frac{d\xi}{dt} = \frac{h^2 k^2 (V_0 - U_0 t)}{(k^2 + h^2 t^2)^{\frac{3}{2}}}$$

and hence

$$\frac{d\xi}{d\phi} = \frac{h^2 k^2 (V_0 \cos\phi - U_0 \sin\phi)}{(k^2 \cos^2\phi + h^2 \sin^2\phi)^{\frac{3}{2}}}.$$

Thus ξ is a maximum or a minimum when $\tan\phi = V_0/U_0$, that is, in the direction of the parallactic motion or in the opposite direction. Further, it is easily seen from the expression for $d^2\xi/d\phi^2$ that ξ, and consequently $f(\xi)$, is a maximum in the direction of the parallactic motion and a minimum in the opposite direction.

In this respect the function $f(\xi)$ resembles the function $f(\tau)$ associated with a single drift-curve which also has its maximum in the direction of the parallactic motion and its minimum in the opposite direction. The function $f(\tau)$ is also symmetrical about the direction of parallactic motion, but this is not the case with the function $f(\xi)$, as may be easily seen as follows. Let $\phi = \psi + \alpha$, where $\tan\psi = V_0/U_0$. Then

$$\cos\phi = \cos\psi\left(\cos\alpha - \frac{V_0}{U_0}\sin\alpha\right),$$

$$\sin\phi = \cos\psi\left(\sin\alpha + \frac{V_0}{U_0}\cos\alpha\right),$$

from which

$$p^2 = k^2 \cos^2\psi \left(\cos\alpha - \frac{V_0}{U_0}\sin\alpha\right)^2 + h^2\cos^2\psi\left(\sin\alpha + \frac{V_0}{U_0}\cos\alpha\right)^2,$$

which is of the form

$$p^2 = A\cos 2\alpha + B\sin 2\alpha + C.$$

Also, $\xi \equiv \xi(\alpha) = \dfrac{\cos\psi}{pU_0}\{(h^2-k^2)\,U_0V_0\sin\alpha + (h^2V_0^2 + k^2U_0^2)\cos\alpha\}.$

It is therefore clear, in the first place, that $\xi(2\pi-\alpha) \neq \xi(\alpha)$ and, accordingly, $f(\xi)$ is not symmetrical about the line $\alpha = 0$, that is, about the axis of the parallactic motion.

An example (approximate) of the curve $\rho_1 = f(\xi)$ is shown by the broken-line curve of Fig. 33.

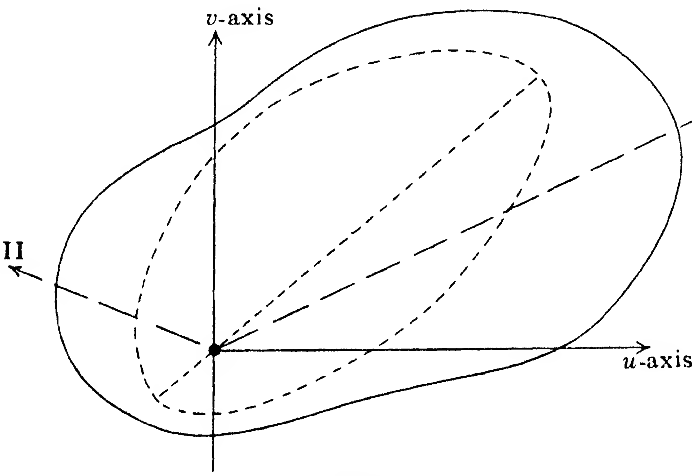

Fig. 33

Consider now the function $f(\xi)/p^2$. We have

$$\frac{d(p^2)}{d\phi} = (h^2-k^2)\sin 2\phi,$$

$$\frac{d^2(p^2)}{d\phi^2} = 2(h^2-k^2)\cos 2\phi.$$

Since $k<h$, p^2 is a minimum for $\phi = 0°$ and $180°$ and a maximum for $\phi = 90°$ and $270°$. Thus the effect of the factor, $1/p^2$, applied to the radii vectores of the curve $\rho_1 = f(\xi)$ is to extend the values of ρ_1 to a maximum extent for $\phi = 0°$ and $180°$ and to a minimum extent for $\phi = 90°$ and $180°$. The result is, in general, a bilobed curve of the form shown by the full-line curve in Fig. 33, with features resembling the curve obtained by the combination of two drift-curves.

5·31. *Determination of the polar curve constants from the proper motions (first method).*

In the two-streams theory, there are five constants to be determined from the data relating to the transverse motions for a particular area of the sky; they are: N_1, hV_1 and θ_1 for drift I and $(N-N_1)$, hV_2 and θ_2 for drift II, it being assumed that all the stars, N in number, belong to one or other of the two drifts. In the ellipsoidal theory, the constants to be determined are four in number; they are: U_0, V_0, k/h and θ_0, where θ_0 is the position angle of the axis of greatest mobility and consequently defines the orientation of the u-axis in the tangent plane.

The following analysis assumes that $\theta'-\theta_0$ lies between 0° and 90°, θ' being the position angle of the parallactic motion. Thus θ is to increase in a sense which, strictly, is known only after some progress has been made with the solution. In practice, we quickly gain the requisite information and, if necessary, reverse the sense of increasing θ temporarily while performing the solution. In combining results from different regions, as in section 5·4, the values of θ_0 and θ' must naturally conform to the usual convention. The point of these remarks occurs in connection with the final term in equation (3) below.

Consider the number, $n_1 d\theta$, of stars moving in the sector defined by the position angles θ, $\theta+d\theta$. Then, by (10) of section 5·21,

$$n_1 = \frac{B^2}{p^2} f(\xi), \qquad \text{......(1)}$$

where p and ξ are now given by

$$p^2 = k^2 \cos^2(\theta-\theta_0) + h^2 \sin^2(\theta-\theta_0), \qquad \text{......(2)}$$

$$\xi = \frac{1}{p}\{k^2 U_0 \cos(\theta-\theta_0) + h^2 V_0 \sin(\theta-\theta_0)\}. \qquad \text{......(3)}$$

Let $n_2 d\theta$ denote the number of stars moving in the opposite sector defined by the position angles $180°+\theta$, $180°+\theta+d\theta$. Then p remains the same for both sectors but ξ changes sign. Hence

$$n_2 = \frac{B^2}{p^2} f(-\xi). \qquad \text{......(4)}$$

From (1) and (4), we obtain

$$\frac{n_1}{n_2} = \frac{f(\xi)}{f(-\xi)} \equiv \psi(\xi). \qquad \text{......(5)}$$

The values of $\log \psi(\xi)$ are given in Table 18.* It will be noticed that $\log \psi(\xi)$ differs very little from the function $0{\cdot}155\xi$.

* The function $\psi(\xi)$ is not to be confused with the function $\psi(b)$ of section 2·42.

Table 18. *Values of* $\log \psi(\xi)$

ξ	$\log \psi(\xi)$	ξ	$\log \psi(\xi)$
0·0	0·000	0·4	0·620
0·1	0·154	0·5	0·779
0·2	0·309	0·6	0·939
0·3	0·464	0·7	1·102
0·4	0·620	0·8	1·268

The observed proper motions give the ratio of the numbers n_1 and n_2 for opposite sectors or, if the distribution of the proper motions is exhibited in the form of a polar curve, the ratio of the radii vectores in position angles θ and $\theta + 180°$. Hence, by means of Table 18, the value of ξ is found. In this way, ξ is found for each value of the position angle θ between $\theta = 0°$ and $\theta = 180°$—for the remainder of the range 180° to 360°, the value of ξ for position angle $\theta + 180°$ is numerically equal to, but of opposite sign to, the value of ξ for position angle θ.

Again, for each value of ξ the corresponding value of $f(\xi)$ can be found by means of Table 5 (p. 39), and n_1 being known formula (1) enables us to calculate the corresponding value of B/p. It is to be remembered that $\xi, f(\xi)$, $\psi(\xi)$ and B/p are all functions of θ (B is a constant for the particular region concerned).

Suppose now that radii vectores, r_1, are drawn for different values of the position angle θ, their lengths being given by the appropriate values of B/p. The theoretical locus traced out is given, from (2), by

$$r_1^2\{k^2\cos^2(\theta-\theta_0)+h^2\sin^2(\theta-\theta_0)\} = B^2, \qquad \ldots\ldots(6)$$

which is an equation of an ellipse (the auxiliary ellipse). Referred to its principal axes its Cartesian equation is

$$k^2x^2+h^2y^2 = B^2. \qquad \ldots\ldots(7)$$

An alternative method, suitable for a least-squares solution, is as follows. We write (6) as

$$x+y\cos 2\theta+z\sin 2\theta = \frac{2}{r_1^2}, \qquad \ldots\ldots(8)$$

where

$$x = \frac{1}{B^2}(h^2+k^2),$$

$$y = -\frac{1}{B^2}(h^2-k^2)\cos 2\theta_0,$$

$$z = -\frac{1}{B^2}(h^2-k^2)\sin 2\theta_0,$$

and the value of $2/r_1^2$ is known for each position angle θ. The linear equation in x, y and z is solved by least squares; the values of θ_0, k/B, h/B are then easily derived.

Since, by convention, the positive direction of the u-axis is such that U_0 is positive, we see from (3) that ξ is positive for $\theta = \theta_0$ (p is the positive square root of the right-hand side of (2)). Hence the data indicate which direction of the major axis of the auxiliary ellipse corresponds to $\theta = \theta_0$. In the same way, the positive direction of the v-axis $\theta_0 + \pi/2$ is found. This defines the sense in which θ must increase. If ξ, determined from the data, happens to be negative at $\theta_0 + 90°$, the assumed sense of increasing θ must be reversed (or the analysis modified).

From the values of B/p and ξ, we now calculate the quantity $B/p\xi$ for each value of the position angle. If we plot the points whose radii vectores, r_2, are given by $B/p\xi$, corresponding to the various values of θ, we obtain by means of (3) the theoretical locus

$$r_2\{k^2U_0\cos(\theta-\theta_0)+h^2V_0\sin(\theta-\theta_0)\} = B, \qquad \text{......(9)}$$

which is the equation of a straight line. Referred to the principal axes of the auxiliary ellipse, its Cartesian equation is

$$k^2U_0x+h^2V_0y = B. \qquad \text{......(10)}$$

Denoting the intercepts of this straight line on the principal axes of the auxiliary ellipse by c and d, we have

$$k^2U_0 = \frac{B}{c}, \quad h^2V_0 = \frac{B}{d}. \qquad \text{......(11)}$$

A simple procedure is to plot the points whose polar coordinates are $(B/p\xi, \theta)$, as obtained from the data, and to draw a straight line to satisfy as accurately as possible the points so plotted. As we now know the value of θ_0, we have the positions of the axes $\theta = \theta_0$ and $\theta = \theta_0 + 90°$ and the intercepts, c and d, on these axes can be obtained by measurement.

The equation (9) can also be used as the basis of a least-squares solution, if desired.

From (8) and (11), $$kU_0 = \frac{a}{c}, \quad hV_0 = \frac{b}{d}. \qquad \text{......(12)}$$

We can write these in the form, using (8),

$$U_0 = \frac{a^2}{bc}\cdot\frac{1}{h}, \quad V_0 = \frac{b}{d}\cdot\frac{1}{h}. \qquad \text{......(13)}$$

These last formulae determine U_0 and V_0 in terms of a theoretical unit, $1/h$.

Let θ' denote the position angle of the parallactic motion, corresponding to the vector OA in Fig. 32. Then $\tan(\theta'-\theta_0) = V_0/U_0$, or, from (13),

$$\tan(\theta'-\theta_0) = \frac{b^2c}{a^2d}. \qquad \text{......(14)}$$

If a check on the calculations is required, it is to be noted that

$$B^2 = \frac{nhk}{2\sqrt{\pi}} e^{-k^2U_0^2 - h^2V_0^2},$$

from which
$$n \equiv 2\sqrt{\pi}\, ab\, e^{\frac{a^2}{c^2}+\frac{b^2}{d^2}}. \qquad \text{......(15)}$$

The quantities a, b, c and d on the right of this formula are measured quantities and the identity may be used for purposes of verification.

However, it may happen that one of the four quantities a, b, c, d can be obtained from the graphical work with rather less certainty than the other three; under these circumstances the formula (15) should be used, with the known value of n, to calculate this particular quantity.

5·32. *Example of the calculation of the polar curve constants.*

We shall consider the region centred at ($0^h\,16^m$, $+50°$) for which we have data* of the proper motions of 545 stars, measured photographically at Cambridge. An analysis on the two-streams theory yielded the results:

$$hV_1 = 1{\cdot}5,\ \theta_1 = 105°; \quad hV_2 = 0{\cdot}8,\ \theta_2 = 190°,$$

where hV_1, hV_2 are the drift velocities and θ_1, θ_2 are the position angles of the drift apices; also the numbers of stars in the two drifts are practically equal. With these values of the drift constants the theoretical distribution has been calculated and the second column of Table 19 shows the theoretical number, r, of stars moving in the 30° sector $\theta - 15°$ to $\theta + 15°$ for values 0°, 10°, 20°, ... 350° of θ. The theoretical distribution and the observed distribution are also shown diagrammatically in Fig. 34.

Instead of using the *observed* data, we shall use the theoretical distribution on the two-drift theory for calculating the corresponding ellipsoidal constants; we shall thus illustrate the method of the previous section and, in addition, exhibit the relation between the theoretical two-drift curve and the theoretical ellipsoidal distribution.

The third column of Table 19 contains the values of $f(\xi)/f(-\xi)$ which are found by dividing the value of r in the second column, corresponding to a particular value of θ, by the value of r for $\theta + 180°$—in accordance with (5) of section 5·31. It is unnecessary to make the calculations for more than eighteen consecutive values of θ, as the remainder of the values are simply the reciprocals of those found. The fourth column contains the logarithms of $f(\xi)/f(-\xi)$, and the fifth column the corresponding values of ξ deduced from Table 18. The entries of this last column are readily completed, since the value of ξ for a position angle $\theta + 180°$ is equal numerically but of opposite sign to the value for θ. The sixth column contains the values of $f(\xi)$ deduced

* *M.N.* 87, 123, 1926.

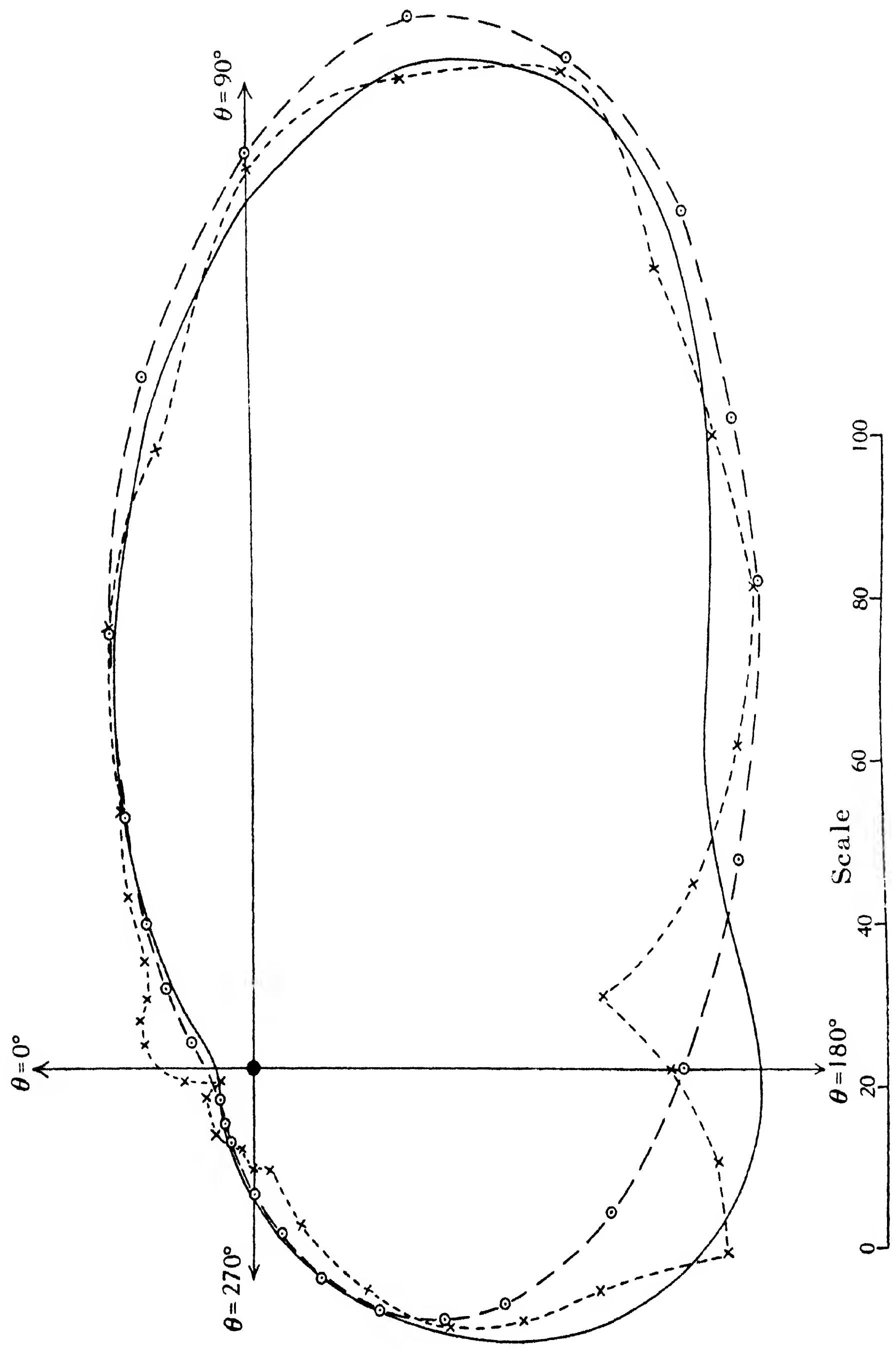

Fig. 34

Full-line curve, theoretical two-drift curve; × × ×, *observed distribution of proper motions;* ⊙ ⊙, *theoretical ellipsoidal distribution.*

from Table 5, p. 39. The entries in the seventh column are formed by dividing the value of r in the second column by the corresponding value of $f(\xi)$; by (1) of section 3·51, this quotient is B^2/p^2. The eighth column contains the values of B/p, which is the radius vector, r_1, of the auxiliary ellipse. The entries in the last column are the values of $B/p\xi$, or r_2, the radius vector associated with the straight line of formula (9).

Table 19. *Calculation of the polar curve constants*

θ	r	$\dfrac{f(\xi)}{f(-\xi)}$	$\log\dfrac{f(\xi)}{f(-\xi)}$	ξ	$f(\xi)$	$\dfrac{B^2}{p^2}$	$\dfrac{B}{p}\equiv r_1$	$\dfrac{B}{p\xi}\equiv r_2$
0°	5·4	—	—	−0·702	0·19	27·7	5·26	− 7·5
10	5·9	—	—	−0·685	0·20	29·3	5·41	− 7·9
20	7·3	—	—	−0·615	0·23	32·0	5·65	− 9·2
30	9·8	—	—	−0·510	0·25	38·8	6·21	−12·2
40	14·4	—	—	−0·367	0·31	46·7	6·85	−18·5
50	22·5	—	—	−0·192	0·41	54·9	7·41	−38·5
60	35·9	—	—	0·010	0·57	62·9	7·94	—
70	55·9	—	—	0·200	0·82	68·5	8·26	+41·7
80	81·4	3·80	0·580	0·375	1·18	69·0	8·33	22·2
90	106·9	6·36	0·803	0·503	1·56	68·5	8·26	16·4
100	124·6	9·51	0·978	0·623	2·06	60·6	7·81	12·4
110	128·3	12·34	1·091	0·692	2·44	52·4	7·25	10·5
120	117·9	13·71	1·137	0·720	2·62	45·0	6·71	9·3
130	100·3	13·74	1·138	0·721	2·62	38·3	6·17	8·5
140	83·7	13·29	1·124	0·714	2·56	32·7	5·71	8·0
150	73·4	12·88	1·110	0·704	2·49	29·5	5·43	7·7
160	69·6	12·65	1·102	0·698	2·46	28·3	5·32	7·6
170	69·6	13·13	1·118	0·702	2·48	28·1	5·29	7·5
180	70·8	13·12	1·118	0·702	2·48	28·5	5·35	7·6
190	70·5	11·95	1·077	0·685	2·38	29·6	5·43	7·9
200	67·3	9·22	0·965	0·615	2·03	33·1	5·75	9·3
210	61·2	6·24	0·795	0·510	1·58	38·8	6·21	12·2
220	53·2	3·69	0·567	0·367	1·17	45·4	6·76	18·5
230	44·1	1·96	0·292	0·192	0·80	55·2	7·41	+38·5
240	35·2	0·98	$\bar{1}$·992	−0·010	0·55	64·1	8·00	—
250	27·5	0·49	$\bar{1}$·692	−0·200	0·40	69·0	8·33	−41·7
260	21·4	0·26	$\bar{1}$·420	−0·375	0·31	69·0	8·33	−22·2
270	16·8	—	—	−0·503	0·26	65·8	8·13	−16·4
280	13·1	—	—	−0·623	0·22	61·0	7·81	−12·5
290	10·4	—	—	−0·692	0·20	51·8	7·19	−10·4
300	8·6	—	—	−0·720	0·19	45·4	6·76	− 9·3
310	7·3	—	—	−0·721	0·19	38·5	6·21	− 8·6
320	6·3	—	—	−0·714	0·19	33·3	5·78	− 8·1
330	5·7	—	—	−0·704	0·19	29·2	5·41	− 7·7
340	5·5	—	—	−0·698	0·19	28·2	5·32	− 7·6
350	5·3	—	—	−0·702	0·19	27·2	5·21	− 7·4

In Fig. 35 the radii vectores, r_1, are drawn in their respective position angles. The full-line curve evidently approximates closely to an ellipse with semi-axes

$$a = 8{\cdot}4 \quad \text{and} \quad b = 5{\cdot}3;$$

also the position angle of the axis is either 77° or 257°. But since ξ is positive for position angles 70° and 80° (see column 5 of Table 19), the positive

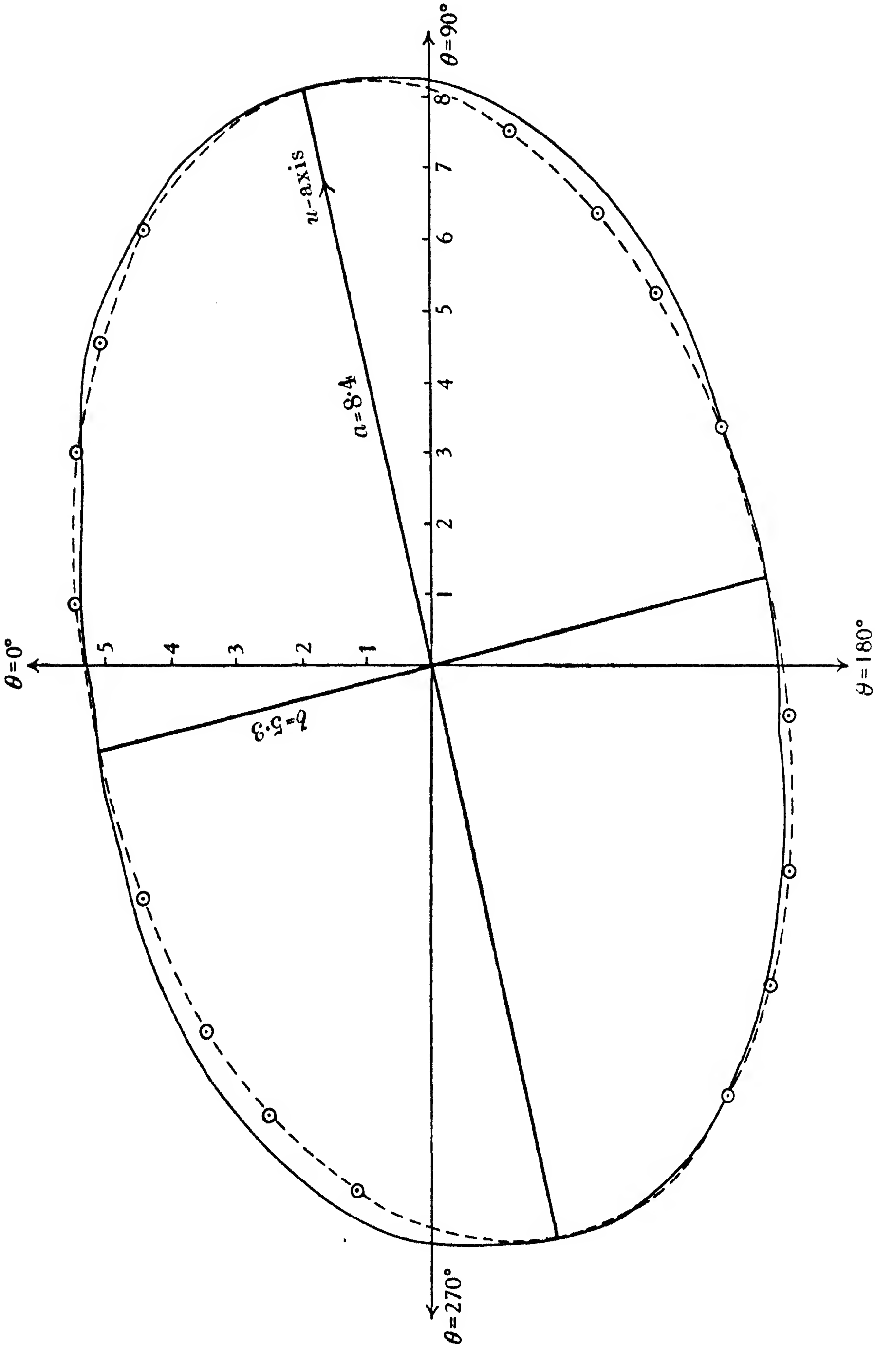

Fig. 35

direction of the u-axis is in position angle 77°. The broken-line curve in Fig. 35 is the ellipse

$$\frac{x^2}{(8\cdot4)^2}+\frac{y^2}{(5\cdot3)^2}=1,$$

which differs very little from the curve $r_1 = B/p$.

Also $\xi > 0$ for $\theta = 167°$, so that the usual convention for θ increasing is also the right one for the present purpose.

In Fig. 36, the radii vectores, r_2 ($\equiv B/p\xi$), are plotted in their respective

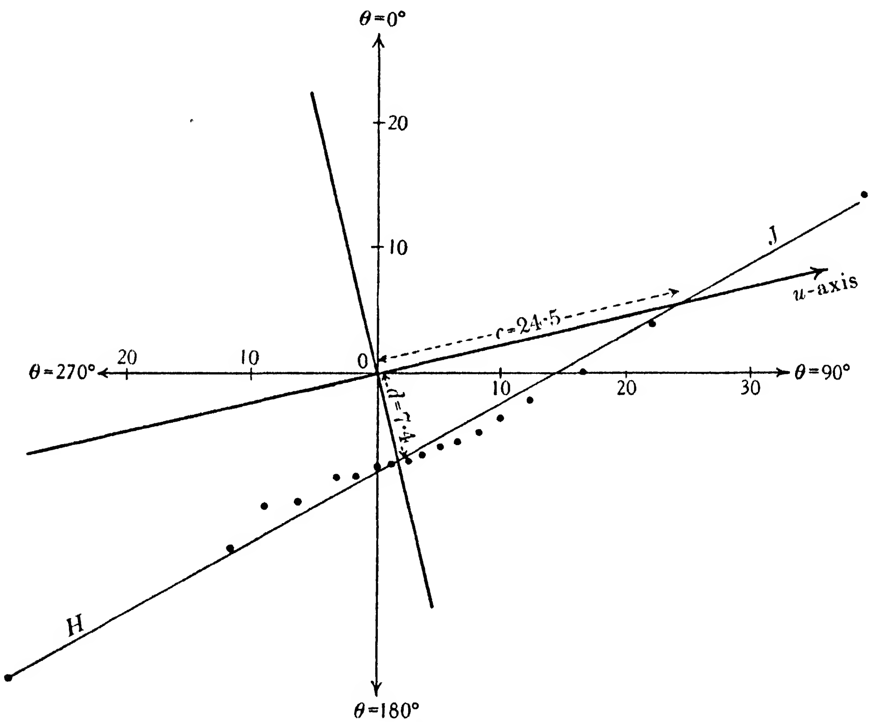

Fig. 36

position angles. The straight line HJ is drawn to satisfy the points as well as possible. The measured intercepts on the u and v axes are

$$c = 24\cdot5 \quad \text{and} \quad d = 7\cdot4.$$

In this instance the straight line HJ is only a fair representation of the points plotted.

Summarising the results from Figs. 35 and 36, we have

$$\theta_0 = 77°; \quad a = 8\cdot4, \; b = 5\cdot3; \; c = 24\cdot5, \; d = 7\cdot4.$$

From these values we find, by means of the formulae of the preceding section,

$$\frac{k}{h} = 0{\cdot}631,$$

$$kU_0 = 0{\cdot}343, \quad hV_0 = 0{\cdot}716,$$

$$\tan(\theta' - \theta_0) = 1{\cdot}318, \quad \theta' - \theta_0 = 53^\circ \text{ or } 233^\circ.$$

Since U_0, V_0 are by definition positive, $\theta' - \theta_0$ is by definition between 0° and 90°; accordingly we have

$$\theta' = 130^\circ.$$

We now proceed to apply the check given by (15) of section 5·31, noticing however that an adjustment of scale is first of all necessary. Actually, the curves in Fig. 34 are drawn through the extremities of radii vectores in position angles θ equal in magnitude to the numbers of stars in the corresponding sectors $\theta - 15^\circ$ to $\theta + 15^\circ$. The theoretical radius vector, ρ, is thus related to the radius vector, r, by

$$\int_{\theta-15^\circ}^{\theta+15^\circ} \rho \, d\theta = r,$$

or, if $\bar{\rho}$ is the mean value of ρ in the sector,

$$\bar{\rho} \cdot \frac{\pi}{6} = r.$$

The curves drawn in Fig. 34 consequently correspond to a number $\pi n/6$ of stars and the check formula becomes

$$\frac{\pi n}{6} = 2\sqrt{\pi}\, ab\, e^{\frac{a^2}{c^2} + \frac{b^2}{d^2}}.$$

Inserting the values of a, b, c and d, we find that

$$n = 567,$$

whereas the actual number of stars in the region is 545.

We have seen that the determination of the straight line in Fig. 36 is somewhat approximate. Assuming that $c = 24{\cdot}5$ and taking the values of a and b already found from the auxiliary ellipse, we use the check equation above for calculating d. The result is $d = 7{\cdot}7$.

The following values are then found:

$$\frac{k}{h} = 0{\cdot}631, \quad \theta_0 = 77^\circ,$$

$$kU_0 = 0{\cdot}343, \quad hV_0 = 0{\cdot}688,$$

$$\tan(\theta' - \theta_0) = 1{\cdot}266, \quad \theta' = 129^\circ.$$

The parallactic motion is given by $(U_0^2 + V_0^2)^{\frac{1}{2}}$ or, in terms of the theoretical unit $1/h$ (which is constant for all parts of the sky, since $h = H$), by

$$h(U_0^2 + V_0^2)^{\frac{1}{2}} \equiv \left\{\frac{h^2}{k^2}(kU_0)^2 + (hV_0)^2\right\}^{\frac{1}{2}} = 0{\cdot}877.$$

Table 20. *Theoretical ellipsoidal distribution of proper motions*

θ	$\theta-\theta_0$	$0{\cdot}0107 \times \cos 2(\theta-\theta_0)$	$\frac{p^2}{B^2}$	$\frac{p}{B}$	$0{\cdot}0408 \times \cos(\theta-\theta_0)$	$0{\cdot}130 \times \sin(\theta-\theta_0)$	$\frac{\xi p}{B}$	ξ	$f(\xi)$	$\frac{B^2}{p^2}f(\xi)$
0°	− 77°	−0·0096	0·0345	0·186	+0·0092	−0·1267	−0·1175	−0·63	0·21	6·1
20	− 57	− 43	292	171	+ 222	− 1090	− 868	−0·51	0·25	8·6
40	− 37	+ 29	219	148	+ 326	− 782	− 446	−0·30	0·34	15·5
50	− 27	+ 63	186	136	+ 364	− 590	− 226	−0·17	0·42	22·9
60	− 17	+ 89	160	126	+ 390	− 380	+ 10	+0·01	0·57	35·6
70	− 7	+ 104	145	120	+ 405	− 158	+ 247	+0·21	0·83	57·2
80	+ 3	+ 106	142	119	+ 407	+ 68	+ 475	+0·40	1·23	86·7
90	+ 13	+ 96	153	124	+ 398	+ 292	+ 690	+0·56	1·72	112·6
100	+ 23	+ 74	175	132	+ 376	+ 508	+ 884	+0·67	2·30	131·8
110	+ 33	+ 43	205	143	+ 342	+ 708	+ 1050	+0·73	2·71	131·9
120	+ 43	+ 8	241	155	+ 298	+ 887	+ 1185	+0·76	2·94	121·7
140	+ 63	− 63	312	177	+ 185	+ 1158	+ 1343	+0·76	2·90	93·1
160	+ 83	− 104	353	188	+ 50	+ 1290	+ 1340	+0·71	2·58	73·0
180	+103	− 96	345	186	− 92	+ 1267	+ 1175	+0·63	2·08	60·3
200	+123	− 43	292	171	− 222	+ 1090	+ 868	+0·51	1·56	53·4
220	+143	+ 29	219	148	− 326	+ 782	+ 446	+0·30	1·00	45·8
240	+163	+ 89	160	126	− 390	+ 380	− 10	−0·01	0·55	34·6
260	+183	+ 106	142	119	− 407	− 68	− 475	−0·40	0·30	20·7
280	+203	+ 74	175	132	− 376	− 508	− 884	−0·67	0·20	11·5
300	+223	+ 8	241	155	− 298	− 887	− 1185	−0·76	0·18	7·5
320	+243	− 63	312	177	− 185	− 1158	− 1343	−0·76	0·19	6·0
340	+263	− 104	353	188	− 50	− 1290	− 1340	−0·71	0·19	5·4

The two-drift theory* gives (273°·2, +43°·6) for the coordinates of the solar apex, as deduced from this and other regions, and the position angle, θ', for the parallactic motion in this region is easily calculated to be 122°, in good agreement with the value deduced above from the ellipsoidal theory.

We can now construct the theoretical ellipsoidal distribution corresponding to the values of a, b, c and d ($\equiv 7{\cdot}7$) already found. The appropriate formulae are

$$\frac{p^2}{B^2} = \frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right) - \frac{1}{2}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\cos 2(\theta-\theta_0),$$

$$\frac{\xi p}{B} = \frac{1}{c}\cos(\theta-\theta_0) + \frac{1}{d}\sin(\theta-\theta_0).$$

With the values of a, b, c and d derived above, we obtain

$$\frac{p^2}{B^2} = 0{\cdot}0249 - 0{\cdot}0107\cos 2(\theta-\theta_0),$$

$$\frac{\xi p}{B} = 0{\cdot}0408\cos(\theta-\theta_0) + 0{\cdot}130\sin(\theta-\theta_0).$$

The separate steps in the calculation are shown in Table 20, omitting the less important position angles. In the actual computations the various quantities were generally found to an additional decimal place (not shown in the table). The final column gives the theoretical ellipsoidal distribution of the proper motions and the corresponding curve is shown in Fig. 34. It is seen that the ellipsoidal curve and the two-drift curve represent the observations equally well and are very much alike except between position angles 180° to 220°, where it is rather difficult to decide whether one theoretical distribution is in better accord with the observed curve than the other.

5·4. *Combination of results from different regions of the sky.*

As we have seen, the analysis of a single region according to the ellipsoidal hypothesis gives

(i) k/h, the ratio of the semi-axes of the velocity ellipse,

(ii) θ_0, the position angle of the u-axis, or in other words, the position angle of the vertex,

(iii) $h(U_0^2+V_0^2)^{\frac{1}{2}}$, the parallactic velocity in terms of the theoretical unit $1/h$ or $1/H$.

(iv) θ', the position angle of the parallactic motion.

Denoting, as before, the angular distance of a region from the vertex by χ, we have, from (10) of section 5·12,

$$\left(\frac{H^2}{K^2}-1\right)\sin^2\chi = \frac{h^2}{k^2}-1 \qquad \text{......(1)}$$

or

$$\zeta\sin\chi = \zeta_0, \qquad \text{......(2)}$$

where

$$\zeta = \left(\frac{H^2}{K^2}-1\right)^{\frac{1}{2}} \text{ and } \zeta_0 = \left(\frac{h^2}{k^2}-1\right)^{\frac{1}{2}}. \qquad \text{......(3)}$$

* *M.N.* 87, 137, 1926.

In Fig. 37 let R be the centre of a region, the coordinates of R being (α, δ). Let V be the vertex with coordinates (A, D). The angle PRV is the observed position angle θ_0. We have, from the triangle PRV,

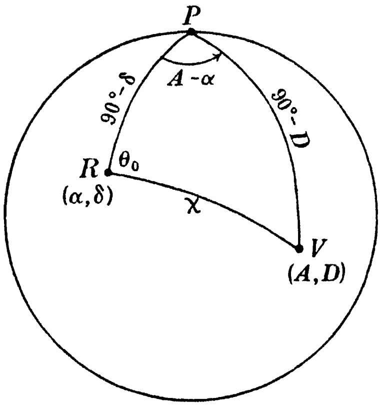

Fig. 37

$$\sin\chi \cos\theta_0 = \sin D \cos\delta - \cos D \sin\delta \cos(A-\alpha),$$

$$\sin\chi \sin\theta_0 = \cos D \sin(A-\alpha),$$

from which, by means of (2),

$$\zeta_0 \cos\theta_0 = \zeta \sin D \cos\delta - \zeta \cos D \sin\delta \cos(A-\alpha), \quad \text{......(4)}$$

$$\zeta_0 \sin\theta_0 = \zeta \cos D \sin(A-\alpha). \quad \text{......(5)}$$

Let

$$\left.\begin{aligned} X &= \zeta \cos D \cos A \\ Y &= \zeta \cos D \sin A \\ Z &= \zeta \sin D \end{aligned}\right\}. \quad \text{......(6)}$$

We then have, from (4), (5) and (6),

$$-X\cos\alpha \sin\delta - Y\sin\alpha \sin\delta + Z\cos\delta = \zeta_0 \cos\theta_0, \quad \text{......(7)}$$

$$-X\sin\alpha \quad + Y\cos\alpha \quad = \zeta_0 \sin\theta_0. \quad \text{......(8)}$$

As h/k is known, from the analysis of a region, the numerical values of $\zeta_0 \cos\theta_0$ and $\zeta_0 \sin\theta_0$ are known for each region. A least-squares solution of the equations (7) and (8) yields the values of X, Y and Z, and from (6) we obtain

$$\tan A = Y/X,$$

$$\tan D = Z/(X^2 + Y^2)^{\frac{1}{2}},$$

$$\zeta = (X^2 + Y^2 + Z^2)^{\frac{1}{2}}.$$

Thus the coordinates of the vertex can be found and the last formula, with the help of (3), enables us to calculate the ratio K/H of the axes of the velocity ellipsoid.

In a similar way we deduce the coordinates of the antapex of the solar motion and the value of the space velocity of the sun relative to the assembly of stars. Let λ be the angular distance of R from the antapex and let $-hU$ be the space-velocity of the solar motion in terms of the theoretical unit, $1/h$ or $1/H$. Then the projection of the parallactic velocity on the tangent plane at R is $hU \sin\lambda$. Hence

$$hU \sin\lambda = h(U_0^2 + V_0^2)^{\frac{1}{2}},$$

which we write in the form $\quad \eta \sin\lambda = \eta',$

where $\quad \eta = hU \quad \text{and} \quad \eta' = h(U_0^2 + V_0^2)^{\frac{1}{2}}.$

η' is a numerical quantity derived from the analysis of the proper motions in the region. If (A', D') are the coordinates of the antapex, A', we have, from the analysis, the position angle, θ', for each region. As before,

$$\sin\lambda\cos\theta' = \sin D'\cos\delta - \cos D'\sin\delta\cos(A'-\alpha),$$

$$\sin\lambda\sin\theta' = \cos D'\sin(A'-\alpha),$$

from which

$$\eta'\cos\theta' = \eta\sin D'\cos\delta - \eta\cos D'\sin\delta\cos(A'-\alpha),$$

$$\eta'\sin\theta' = \eta\cos D'\sin(A'-\alpha).$$

Write
$$X_1 = \eta\cos D'\cos A',$$
$$Y_1 = \eta\cos D'\sin A',$$
$$Z_1 = \eta\sin D'.$$

Then
$$-X_1\cos\alpha\sin\delta - Y_1\sin\alpha\sin\delta + Z_1\cos\delta = \eta'\cos\theta',$$
$$-X_1\sin\alpha \quad + Y_1\cos\alpha \quad = \eta'\sin\theta'.$$

The quantities on the right-hand sides of these equations are known for each region. A least-squares solution yields the values of X_1, Y_1 and Z_1 and from these values the coordinates of the antapex and the value of the solar speed are derived from

$$\tan A' = Y_1/X_1,$$
$$\tan D' = Z_1/(X_1^2+Y_1^2)^{\frac{1}{2}},$$
$$hU = (X_1^2+Y_1^2+Z_1^2)^{\frac{1}{2}}.$$

Schwarzschild's investigation* of the proper motions of the Groombridge stars, treated earlier by Eddington, leads to the following results:

Coordinates of vertex: R.A. 93°, Dec. + 6°.

Coordinates of solar apex: R.A. 266°, Dec. + 33°.

Solar motion, hU: 0·70.

K/H: 0·63.

Eddington's results† for the vertex from the same data are: R.A. 95°, Dec. + 3°, almost identical with Schwarzschild's results.

Schwarzschild's position of the solar apex is within a degree or two of the position derived in several more recent investigations based on the proper motions of the brighter stars.

* *Göttingen Nach.* 1907, p. 614.
† *M.N.* 67, 34, 1906.

5·51. *Determination of the polar curve constants from the proper motions (second method).*

As we have seen in section 5·21, formula (4), the number, dn, of stars with linear velocity components between (x, y) and $(x+dx, y+dy)$ is given by

$$dn = F(x, y)\, dx dy, \qquad \ldots\ldots(1)$$

where
$$F(x, y) = \frac{nhk}{\pi} e^{-k^2(x-U_0)^2 - h^2(y-V_0)^2} \qquad \ldots\ldots(2)$$

and
$$x = u + U_0, \quad y = v + V_0, \qquad \ldots\ldots(3)$$

U_0, V_0 being the components of the parallactic motion and u, v the components of the peculiar velocity of a star in the tangent plane.

Following Schwarzschild,* we shall denote the projection of the *solar motion* on the tangent plane by S, making an angle ϕ_0 with the u-axis; ϕ_0 will thus define the direction of the *solar apex* for the region concerned. Since U_0 and V_0 are positive, ϕ_0 is in the third quadrant. We have

$$U_0 = -S\cos\phi_0, \quad V_0 = -S\sin\phi_0, \qquad \ldots\ldots(4)$$

so that
$$F(x, y) = \frac{nhk}{\pi} e^{-k^2(x+S\cos\phi_0)^2 - h^2(y+S\sin\phi_0)^2}. \qquad \ldots\ldots(5)$$

We shall find an expression for the total number of stars moving in the sector ϕ to ϕ' when the corresponding radii vectores are parallel to two conjugate diameters of the velocity ellipse

$$k^2u^2 + h^2v^2 = 1.$$

In this ellipse, let two radii vectores make angles ϕ and ϕ' with the u-axis and let σ and σ' be the corresponding eccentric angles. If ϕ, ϕ' correspond to conjugate diameters,

$$\tan\phi \tan\phi' = -k^2/h^2. \qquad \ldots\ldots(6)$$

Also
$$\tan\phi = \frac{k}{h}\tan\sigma \quad \text{and} \quad \tan\phi' = \frac{k}{h}\tan\sigma', \qquad \ldots\ldots(7)$$

with the relation
$$\sigma' = \sigma + 90^\circ. \qquad \ldots\ldots(8)$$

In Fig. 38 let OX and OY be the x and y axes (parallel to the u and v axes). Let OP and OQ be parallel to two conjugate diameters of the velocity ellipse. In the figure Op and Oq are parallel to the corresponding radii of the auxiliary circle of the ellipse.

Let L_1 denote the number of stars moving in the sector POQ. Then, from (1),

$$L_1 = \iint F(x, y)\, dx dy, \qquad \ldots\ldots(9)$$

where the integration is taken over the infinite sector bounded by OP and OQ.

* *Göttingen Nach.* 1908, p. 191.

We now choose new axes through O such that

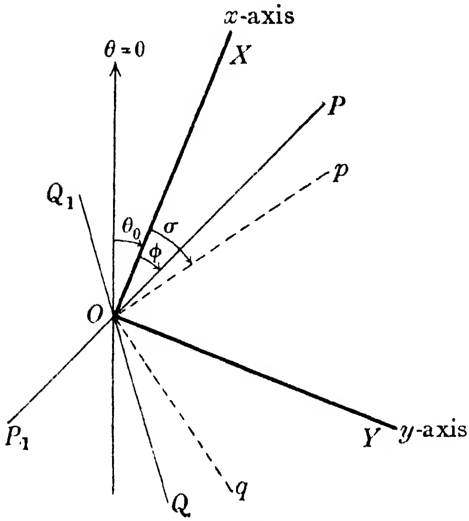

Fig. 38

$$\xi = kx\cos\sigma + hy\sin\sigma, \quad \text{......(10)}$$

$$\eta = -kx\sin\sigma + hy\cos\sigma, \quad \text{...(11)}$$

which are equivalent to

$$kx = \xi\cos\sigma - \eta\sin\sigma, \quad \text{......(12)}$$

$$hy = \xi\sin\sigma + \eta\cos\sigma. \quad \text{......(13)}$$

From (11), the ξ-axis, for which $\eta = 0$, is given by

$$kx\sin\sigma - hy\cos\sigma = 0$$

or
$$y = x\,.\frac{k}{h}\tan\sigma.$$

But, by (7), the last equation is

$$y = x\tan\phi.$$

But this is the equation of OP with respect to OX and OY. Hence OP is the ξ-axis.

From (10), in a similar way, we see that OQ is the η-axis.

We now express $F(x,y)$ as a function of ξ and η. We have, from (5), (12) and (13),

$$F(x,y) \equiv F_1(\xi,\eta) = \frac{nhk}{\pi} e^{-(\xi\cos\sigma - \eta\sin\sigma + kS\cos\phi_0)^2 - (\xi\sin\sigma + \eta\cos\sigma + hS\sin\phi_0)^2}.$$

Let
$$\mu = S(k\cos\phi_0\cos\sigma + h\sin\phi_0\sin\sigma), \quad \text{......(14)}$$

$$\nu = S(-k\cos\phi_0\sin\sigma + h\sin\phi_0\cos\sigma), \quad \text{......(15)}$$

and we obtain
$$F_1(\xi,\eta) = \frac{nhk}{\pi} e^{-(\xi+\mu)^2-(\eta+\nu)^2}. \quad \text{......(16)}$$

Also
$$dx\,dy = \frac{\partial(x,y)}{\partial(\xi,\eta)} d\xi\,d\eta.$$

Hence, using (12) and (13), we have

$$dx\,dy = \frac{1}{hk} d\xi\,d\eta.$$

Consequently, from (9),

$$L_1 = \frac{1}{hk}\iint F_1(\xi,\eta)\,d\xi\,d\eta,$$

where the integration is over the infinite sector bounded by OP and OQ. The limits of ξ are 0 and ∞ and the limits of η are 0 and ∞. Hence, by (16),

$$L_1 = \frac{n}{\pi}\int_0^\infty\int_0^\infty e^{-(\xi+\mu)^2-(\eta+\nu)^2} d\xi\,d\eta.$$

Let
$$\Theta(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}\,dt. \qquad \ldots\ldots(17)$$

Then
$$\int_0^\infty e^{-(\xi+\mu)^2}\,d\xi = \int_\mu^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}\{1-\Theta(\mu)\}.$$

Similarly
$$\int_0^\infty e^{-(\eta+\nu)^2}\,d\eta = \frac{\sqrt{\pi}}{2}\{1-\Theta(\nu)\}.$$

Hence
$$L_1 = \frac{n}{4}\{1-\Theta(\mu)\}\{1-\Theta(\nu)\}. \qquad \ldots\ldots(18)$$

The function $\Theta(t)$ is the "probability integral", values of which are tabulated for example in Brunt's *The Combination of Observations* (2nd edition), 234, 1931.

It is to be remembered that L_1 gives the number of stars moving in a sector bounded by two radii vectores which are parallel to two conjugate semi-diameters of the velocity ellipse.

Consider now the sector QOP_1 in Fig. 38. The eccentric angle corresponding to OQ is σ' and if μ_1, ν_1 are the values of μ, ν as defined by (14) and (15), we have

$$\mu_1 = S(k\cos\phi_0\cos\sigma' + h\sin\phi_0\sin\sigma'),$$
$$\nu_1 = S(-k\cos\phi_0\sin\sigma' + h\sin\phi_0\cos\sigma').$$

These become, since $\sigma' = \sigma + 90°$,
$$\mu_1 = +\nu \quad \text{and} \quad \nu_1 = -\mu.$$

Hence, the number, L_2, of proper motions for the sector QOP_1 is, using (18), given by
$$L_2 = \frac{n}{4}\{1-\Theta(\nu)\}\{1-\Theta(-\mu)\}.$$

But, from (17),
$$\Theta(-t) = -\Theta(t).$$

Consequently,
$$L_2 = \frac{n}{4}\{1-\Theta(\nu)\}\{1+\Theta(\mu)\}.$$

In the same way we obtain L_3 and L_4 for the sectors P_1OQ_1 and Q_1OP.

The results are summarised as follows:

$$\left.\begin{array}{ll}\text{Sector } \phi \text{ to } \phi': & L_1 = \frac{n}{4}\{1-\Theta(\mu)\}\{1-\Theta(\nu)\}\\ \phi' \text{ to } 180°+\phi: & L_2 = \frac{n}{4}\{1+\Theta(\mu)\}\{1-\Theta(\nu)\}\\ 180°+\phi \text{ to } 180°+\phi': & L_3 = \frac{n}{4}\{1+\Theta(\mu)\}\{1+\Theta(\nu)\}\\ 180°+\phi' \text{ to } \phi: & L_4 = \frac{n}{4}\{1-\Theta(\mu)\}\{1+\Theta(\nu)\}\end{array}\right\}, \qquad \ldots\ldots(19)$$

where μ, ν are defined by (14) and (15).

From (19) we easily deduce the following:

$$L_1 L_3 = L_2 L_4, \qquad \ldots\ldots(20)$$

$$\Theta(\mu) = \frac{L_2 - L_1}{L_2 + L_1} = \frac{L_3 - L_4}{L_3 + L_4} = \frac{L_2 + L_3 - L_1 - L_4}{L_1 + L_2 + L_3 + L_4}, \qquad \ldots\ldots(21)$$

$$\Theta(\nu) = \frac{L_4 - L_1}{L_4 + L_1} = \frac{L_3 - L_2}{L_3 + L_2} = \frac{L_3 + L_4 - L_1 - L_2}{L_1 + L_2 + L_3 + L_4}, \qquad \ldots\ldots(22)$$

and $$n = L_1 + L_2 + L_3 + L_4. \qquad \ldots\ldots(23)$$

At this stage, we shall make a simple transformation of the analytical expressions for μ and ν in (14) and (15).

Let $$k\cos\phi_0 = m\cos\sigma_0, \quad h\sin\phi_0 = m\sin\sigma_0. \qquad \ldots\ldots(24)$$

Then $$\mu = mS\cos(\sigma_0 - \sigma), \qquad \ldots\ldots(25)$$

$$\nu = mS\sin(\sigma_0 - \sigma). \qquad \ldots\ldots(26)$$

From (24), $\tan\phi_0 = (k/h)\tan\sigma_0$; hence (7) shows that σ_0 is the eccentric angle corresponding to ϕ_0.

As before, let θ_0 be the position angle of the u-axis and let θ_1 be the position angle of the solar apex. Then

$$\phi = \theta - \theta_0, \quad \phi' = \theta' - \theta_0, \quad \phi_0 = \theta_1 - \theta_0,$$

where, as previously, ϕ and ϕ' refer to conjugate diameters.

From the statistics of the proper motions we can find, corresponding to a particular value of θ, the numbers l_1, l_2, l_3 and l_4 of stars moving in the quadrants θ to $\theta + \pi/2$, $\theta + \pi/2$ to $\theta + \pi$, $\theta + \pi$ to $\theta + 3\pi/2$ and $\theta + 3\pi/2$ to θ. As the data are usually arranged so as to exhibit the number of stars with proper motions, for example, in 10° sectors, the numbers l_1, l_2, l_3 and l_4 are easily found for each value of θ.

We consider two particular cases.

5·52. *The direction of the solar motion.*

We take $\phi = \phi_0$, so that OP in Fig. 38 corresponds to the direction of the solar motion. Then $\sigma = \sigma_0$ and it follows from (26), (25) and (19) that

$$\nu = 0, \quad \mu = mS, \quad L_1 = L_4 \quad \text{and} \quad L_2 = L_3.$$

Consequently, $$L_1 + L_2 = L_3 + L_4. \qquad \ldots\ldots(1)$$

But $L_1 + L_2$ is the theoretical number of proper motions between ϕ_0 and $\phi_0 + 180°$, that is, between position angles θ_1 and $\theta_1 + 180°$, and this number must be equal to the observed number $l_1 + l_2$ between θ_1 and $\theta_1 + 180°$. Hence

$$l_1 + l_2 = L_1 + L_2.$$

Similarly, $$l_3 + l_4 = L_3 + L_4.$$

Hence, from (1), $$l_1 + l_2 - l_3 - l_4 = 0. \qquad \ldots\ldots(2)$$

This is the condition by means of which the direction of the solar motion is to be obtained. The procedure is as follows. Obtain the value of the quantity $(l_1+l_2-l_3-l_4)$ for each value of the position angle (0°, 10°, ...). The statistics will then show the position angle for which this quantity vanishes. As l_3 for $\theta=\theta_1$ is the same as l_1 for $\theta_1+180°$ and so on for l_2, l_3 and l_4, there will be two position angles, differing by 180°, satisfying the relation (2), say ψ and $\psi+180°$. A rule is required to determine which of these values corresponds to θ_1, the position angle of the direction of the solar apex. If l_1, l_2, l_3 and l_4 are formed for ψ, then $\theta_1=\psi$ if $l_3>l_1$ and $l_4>l_2$. Otherwise, $\psi+180°$ is to be identified with θ_1. Actually, it is usually easy in practice to discriminate between the two values, as the value corresponding to $\theta_1+180°$ indicates roughly the direction in which most of the stars are moving.

The inequalities above may be formally proved as follows.

By definition, the number of stars moving in the quadrant ϕ_0 to $\phi_0+\pi/2$ is l_1 which, on referring to formula (10) of section 5·21, is seen to be given by

$$l_1 = B^2\int_{\phi_0}^{\phi_0+\pi/2}\frac{1}{p^2}f(\xi)\,d\phi,$$

where

$$\xi = \frac{1}{p}(k^2U_0\cos\phi+h^2V_0\sin\phi)$$

or, writing as before

$$U_0=-S\cos\phi_0,\quad V_0=-S\sin\phi_0,$$

$$\xi = -\frac{S}{p}(k^2\cos\phi\,\cos\phi_0+h^2\sin\phi\,\sin\phi_0). \qquad \text{......(3)}$$

Setting $\phi=\phi_0+\alpha$, the expression for ξ becomes

$$\xi = -\frac{S}{p}\{(k^2\cos^2\phi_0+h^2\sin^2\phi_0)\cos\alpha+(h^2-k^2)\sin\phi_0\cos\phi_0\sin\alpha\}.$$

Also,

$$p = +\{k^2\cos^2(\phi_0+\alpha)+h^2\sin^2(\phi_0+\alpha)\}^{\frac{1}{2}}. \qquad \text{......(4)}$$

In the quadrant under consideration $0\leqslant\alpha\leqslant\pi/2$, and as $k<h$ the value of ξ is negative and, say, equal to $-\tau$, where τ is a positive quantity. It is to be noted that from the definition of the positive directions of the u and v axes, $180°\leqslant\phi_0\leqslant 270°$.

We now have

$$l_1 = B^2\int_0^{\pi/2}\frac{1}{p^2}f(-\tau)\,d\alpha. \qquad \text{......(5)}$$

Consider now the opposite quadrant, defined by $\phi_0+\pi$ and $\phi_0+3\pi/2$, in which the number of stars is l_3.

Setting, in (3), $\phi=\phi_0+\pi+\alpha$, where $0\leqslant\alpha\leqslant\pi/2$, we have for this sector that ξ is positive and equal to τ. Also p is given by (4). Hence

$$l_3 = B^2\int_0^{\pi/2}\frac{1}{p^2}f(\tau)\,d\alpha. \qquad \text{......(6)}$$

It is noted that, in (5) and (6), p and τ are functions of α.

Now $f(-\tau) < f(\tau)$ for τ positive; hence $l_3 > l_1$. Similarly $l_4 > l_2$.

These inequalities enable us to decide which of the two possible values of the position angle, for which $(l_1+l_2-l_3-l_4)$ vanishes, corresponds to the direction of the solar apex.

5·53. *The direction of the u-axis and the value of k/h.*

The direction of the u-axis is given by $\phi = 0$ and the corresponding direction of the ξ-axis is parallel to the major axis of the velocity ellipse; consequently, the conjugate diameter is given by $\phi' = 90°$. It follows that, in this case, $L_1 = l_1$ and similarly $L_2 = l_2$, $L_3 = l_3$ and $L_4 = l_4$. Hence, by (20) of section 5·51,

$$l_1 l_3 - l_2 l_4 = 0. \qquad \ldots\ldots(1)$$

To determine the position angle, θ_0, of the u-axis, the quantity $(l_1 l_3 - l_2 l_4)$ is formed from the data for each value of θ and by inspection, or interpolation, the value of the position angle corresponding to the vanishing of $(l_1 l_3 - l_2 l_4)$ is found. It is clear that there will be four such values, corresponding to $\phi = 0°$, $90°$, $180°$ and $270°$, since for each of these values the ξ and η axes are perpendicular. It is only necessary to distinguish between $\phi = 0$ and $\phi = 90°$, and the manner of doing this will be explained later.

Suppose that when $l_1 l_3 - l_2 l_4 = 0$ the associated value of θ corresponds to the direction of the u-axis, $\phi = 0$. Let μ_0 and ν_0 denote the values of μ and ν for this direction. Then, by (21), (22) and (23) of section 5·51, we have —since $L_1 = l_1$, etc.—

$$\Theta(\mu_0) = \frac{1}{n}(l_2+l_3-l_1-l_4), \qquad \ldots\ldots(2)$$

$$\Theta(\nu_0) = \frac{1}{n}(l_3+l_4-l_1-l_2). \qquad \ldots\ldots(3)$$

It is to be remembered that l_1, l_2, l_3 and l_4 now refer to the particular value, θ_0, of the position angle, and as this latter is supposed known as the result of the application of the condition (1), the values of l_1, l_2, l_3 and l_4 in (2) and (3) are obtained from the statistics. Thus the numerical values of the functions $\Theta(\mu_0)$ and $\Theta(\nu_0)$ are readily found from (2) and (3), and from tables of the function the values of μ_0 and ν_0 are obtained.

But when $\phi = 0$, we have $\sigma = 0$, so that, by (14) and (15) of section 5·51,

$$\mu_0 = Sk\cos\phi_0,$$

$$\nu_0 = Sh\sin\phi_0,$$

or, since $\phi_0 = \theta_1 - \theta_0$,

$$Sk = \mu_0 \sec(\theta_1-\theta_0), \qquad \ldots\ldots(4)$$

$$Sh = \nu_0 \operatorname{cosec}(\theta_1-\theta_0). \qquad \ldots\ldots(5)$$

Now the value of θ_1 is known from the analysis of the previous section and we have also the value of θ_0. Hence we can find the numerical value of k/h, from (4) and (5), given by

$$\frac{k}{h} = \frac{\mu_0}{\nu_0}\tan(\theta_1 - \theta_0), \qquad \ldots\ldots(6)$$

all the quantities on the right being known. The necessary condition is that k/h should be less than unity. If this condition is satisfied for the value of θ_0, we have obtained the positive or negative direction of the u-axis and therefore the direction of the vertices for the given region.

If, however, k/h is greater than unity as a consequence of the calculation according to (6), it is evident that this result is associated with the minor axis of the velocity ellipse.

The positive directions of the two axes are easily determined, since they are such that the direction of the parallactic motion lies between them.

It is clear from (4), (5) and (6) that, if S is small—that is, when the region under consideration is near the solar apex or antapex—or if $\theta_1 - \theta_0$ is close to a multiple of 90°, the resulting value of k/h may not be well determined. In these circumstances the following procedure is preferable to that just outlined.

We have, from (20) of section 5·51,

$$L_1 L_3 = L_2 L_4.$$

Hence

$$L_1(L_3 + L_4) = L_4(L_1 + L_2),$$

so that

$$\frac{L_1}{L_4} = \frac{L_1 + L_2}{L_3 + L_4} = \frac{l_1 + l_2}{l_3 + l_4}.$$

Consider any position angle θ and let θ' be the position angle corresponding to the conjugate semi-diameter next in order of position angle. Then L_1 is the number of proper motions between θ and θ' and L_4 is the number between $180° + \theta'$ and θ. Also $l_1 + l_2$ is the number between θ and $\theta + 180°$ and $l_3 + l_4$ is the number between $\theta + 180°$ and θ. The ratio $(l_1 + l_2) : (l_3 + l_4)$ can be readily obtained from the data for each value of θ. Let this ratio be denoted by q; then

$$\frac{L_1}{L_4} = q. \qquad \ldots\ldots(7)$$

Also, the relation between θ and θ'—obtained from (6) of 5·51—is given by

$$\frac{k^2}{h^2} = -\tan(\theta - \theta_0)\tan(\theta' - \theta_0). \qquad \ldots\ldots(8)$$

If θ' can be found for a given value of θ, k/h is easily found from (8).

Obtain from the data the numbers, L_1', of proper motions between θ and $\theta + 100°$, θ and $\theta + 110°$, θ and $\theta + 120°$, ... and in the same way the numbers

L_4', between $\theta+280°$ and θ, $\theta+290°$ and θ, ...; calculate the corresponding values of L_1'/L_4', and from these values find by inspection or interpolation the position angle $\theta+\alpha$ for which L_1'/L_4' is equal to the number q in (7). Thus $\theta+\alpha = \theta'$, the position angle corresponding to the conjugate semi-diameter associated with θ. The value of k/h is then calculated by means of (8).

5·54. *Example of the method.*

We shall illustrate the method of the previous three sections by means of the proper motions of 219 stars between magnitudes 9·0 and 14·0 in Region 14 of the *Radcliffe Catalogue of Proper Motions in the Selected Areas, 1 to 115* (1934). The counts in 10° sectors $\theta-5°$ to $\theta+5°$ are given on p. xxvi of the Introduction. The statistics are smoothed by adding the numbers in the three sectors $\theta-15°$ to $\theta-5°$, $\theta-5°$ to $\theta+5°$ and $\theta+5°$ to $\theta+15°$ for the values 5°, 15°, 25°, ... of θ. We shall regard the numbers so obtained as referring to 3×219 or 657 stars for the 10° sectors $\theta-5°$ to $\theta+5°$, ... for the values 5°, 15°, 25°, ... of θ. The data are shown in Table 21.

In Table 22, the numbers corresponding to l_1, l_2, l_3 and l_4 are given. Thus for a particular position angle θ, l_1 is the number of proper motions in the quadrant θ to $\theta+90°$, l_2 is the number in the quadrant $\theta+90°$ to $\theta+180°$, l_3 is the number in the quadrant $\theta+180°$ to $\theta+270°$ and l_4 is the number in the quadrant $\theta+270°$ to θ.

The quantities $(l_1+l_2-l_3-l_4)$ and $(l_1l_3-l_2l_4)$ are formed and their values are given in the last two columns of the table for position angles between 0° and 170°. For the values of θ between 180° and 350°, the numbers $(l_1+l_2-l_3-l_4)$ repeat themselves with change of sign.

By interpolation, $(l_1+l_2-l_3-l_4)$ vanishes for $\theta = 73°$ and also for 253°. Hence the position angle of the solar apex is 73° or 253°. But we have the condition $l_3 > l_1$ and this inequality is observed for $\theta = 73°$ and not for 253°. Hence, the position angle, θ_1, of the solar apex is given by

$$\theta_1 = 73°.$$

Again $(l_1l_3-l_2l_4)$ vanishes for $\theta = 48°$ or 138°, so that the position angle θ_0, of the vertex (or its antipodal point) is either 48° or 138°.

For the value $\theta = 138°$ we find the corresponding values of l_1, l_2, l_3 and l_4 by interpolation in Table 22; the results are

$$l_1 = 191, \quad l_2 = 237, \quad l_3 = 127, \quad l_4 = 102.$$

By formulae (2) and (3) of section 5·53 we find, for $\theta = 138°$,

$$\Theta(\mu_0) \equiv \frac{l_2+l_3-l_1-l_4}{n} = \frac{71}{657} = 0{\cdot}108,$$

$$\Theta(\nu_0) \equiv \frac{l_3+l_4-l_1-l_2}{n} = -\frac{199}{657} = -0{\cdot}303,$$

and from the tables of the function $\Theta(t)$, we obtain

$$\mu_0 = 0{\cdot}096, \quad \nu_0 = -0{\cdot}275.$$

Now $$\theta_1 - \theta_0 = 73° - 138° = 295°.$$

Hence, by (4) and (5) of section 5·53, we have

$$Sk = 0{\cdot}096 \sec 295°, \quad Sh = -0{\cdot}275 \operatorname{cosec} 295°,$$

from which $$Sk = 0{\cdot}227,$$

$$Sh = 0{\cdot}303,$$

and $$k/h = 0{\cdot}75.$$

Thus $k < h$ and the value for θ_0 is 138° and not 48°.

As Sk and Sh have rather small values, the ratio k/h is not very well determined. We proceed to calculate k/h by way of formula (8) of section 5·53.

Table 21. *Data for the Radcliffe region* 14

θ	Number in sector $\theta - 5°$ to $\theta + 5°$	θ	Number	θ	Number	θ	Number
5°	13	95°	6	185°	24	275°	33
15	11	105	12	195	11	285	33
25	10	115	19	205	13	295	32
35	9	125	21	215	13	305	25
45	5	135	23	225	19	315	22
55	5	145	23	235	16	325	20
65	6	155	27	245	22	335	20
75	7	165	31	255	26	345	19
85	6	175	29	265	29	355	17

Table 22. *Values of* l_1, l_2, l_3 *and* l_4

θ	l_1	l_2	l_3	l_4	$l_1+l_2-l_3-l_4$	$l_1 l_3 - l_2 l_4$
0°	72	191	173	221	−131	−298 × 10²
10	65	209	182	201	−109	−302
20	66	208	204	179	−109	−238
30	75	202	223	157	−103	−150
40	87	194	235	141	− 95	− 69
50	105	190	238	124	− 67	+ 14
60	123	183	242	109	− 45	+ 98
70	144	178	240	95	− 13	+176
80	168	173	233	83	+ 25	+248
90	191	173	221	72	+ 71	+298
100	209	182	201	65	+125	+302
110	208	204	179	66	+167	+238
120	202	223	157	75	+193	+150
130	194	235	141	87	+201	+ 69
140	190	238	124	105	+199	− 14
150	183	242	109	123	+193	− 98
160	178	240	95	144	+179	−176
170	173	233	83	168	+155	−248

Consider the position angle $\theta = 170°$. Then

$$\frac{l_1+l_2}{l_3+l_4} = \frac{406}{251} = 1{\cdot}62.$$

Find, by means of Table 21 and Table 22, the number L_1' for the sectors $170°+\alpha$, where $\alpha = 90°, 100°, 110°, \ldots$, and the numbers L_4' for the corresponding angles. We have the following results, the last column giving the quotient L_1'/L_4'.

Table 23

θ	$\theta+\alpha$	L_1'	L_4'	L_1'/L_4'
170°	260°	173	168	1·03
	270	202	162	1·25
	280	235	156	1·51
	290	268	144	1·86

By interpolation, it is seen that $L_1'/L_4' = 1{\cdot}62$ when $\theta+\alpha = 283°$. Hence

$$\theta' \equiv \theta+\alpha = 283°$$

and by (8) of section 5·53,

$$\frac{k^2}{h^2} = -\tan(170°-138°)\tan(283°-138°)$$

$$= 0{\cdot}438,$$

from which

$$\frac{k}{h} = 0{\cdot}66.$$

The original choice of 170° for θ in this computation is dictated by the consideration that greater accuracy is likely to result in the calculation of k/h if θ is chosen so that $\tan(\theta-\theta_0)$ and $\tan(\theta'-\theta_0)$ are as nearly equal as possible numerically, in which event $\tan(\theta-\theta_0)$ is approximately equal to k/h. With a rough idea of the value of k/h, θ can be found to the nearest 10° and the calculation outlined above is based on this value of θ. Actually, with the value of k/h just derived we find that $\theta = 170°$, assuming that $\tan(\theta-\theta_0) = k/h$ numerically. Thus the most satisfactory value (that is, 170°) has been chosen.

With $k/h = 0{\cdot}66$ and $Sh = 0{\cdot}303$, it follows that $Sk = 0{\cdot}201$, which we may consider a somewhat more reliable value than that previously determined. The collected results for this region are (accepting the second value of k/h):

$\theta_0 \equiv$ position angle of vertex: 138°.

$\theta_1 \equiv$ position angle of solar apex: 73°.

k/h: 0·66.

$Sk = 0{\cdot}20$ and $Sh = 0{\cdot}30$.

The projection, S, of the solar velocity on the tangential plane of the region is 0·30 in terms of the theoretical unit, $1/h$, or $1/H$.

5·6. *The general equation of the velocity ellipsoid.*

Referred to its principal axes, the equation of the velocity ellipsoid can be written

$$f \equiv \frac{U^2}{s_1} + \frac{V^2}{s_2} + \frac{W^2}{s_3} = 1, \qquad \text{......(1)}$$

in which s_2 and s_3 are no longer necessarily equal.

The number of stars with velocities between (U, V, W) and $(U+dU, V+dV, W+dW)$ is

$$Pe^{-f}\,dU\,dV\,dW, \qquad \text{......(2)}$$

where P is a constant given in terms of the total number, N, of stars under consideration by

$$N = P\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-f}\,dU\,dV\,dW. \qquad \text{......(3)}$$

Consider now a small region of the sky with N stars whose *motus peculiares* obey the general ellipsoidal law. Let a velocity have components (u, v, w) with respect to rectangular axes chosen so that the w-axis is the line of sight, and u and v axes lying in the tangent plane. If (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) denote the direction-cosines of the u, v, w axes respectively with respect to the U, V, W system, we have

$$\left.\begin{aligned} U &= l_1u + l_2v + l_3w \\ V &= m_1u + m_2v + m_3w \\ W &= n_1u + n_2v + n_3w \end{aligned}\right\}, \qquad \text{......(4)}$$

so that the equation of the ellipsoid referred to the u, v, w axes is, from (1),

$$F(u, v, w) \equiv au^2 + bv^2 + cw^2 + 2fvw + 2gwu + 2huv = 1, \qquad \text{......(5)}$$

where

$$\left.\begin{aligned} a &= \frac{l_1^2}{s_1} + \frac{m_1^2}{s_2} + \frac{n_1^2}{s_3} \\ b &= \frac{l_2^2}{s_1} + \frac{m_2^2}{s_2} + \frac{n_2^2}{s_3} \\ c &= \frac{l_3^2}{s_1} + \frac{m_3^2}{s_2} + \frac{n_3^2}{s_3} \\ f &= \frac{l_2l_3}{s_1} + \frac{m_2m_3}{s_2} + \frac{n_2n_3}{s_3} \\ g &= \frac{l_3l_1}{s_1} + \frac{m_3m_1}{s_2} + \frac{n_3n_1}{s_3} \\ h &= \frac{l_1l_2}{s_1} + \frac{m_1m_2}{s_2} + \frac{n_1n_2}{s_3} \end{aligned}\right\}. \qquad \text{......(6)}$$

The number, δN, of stars with linear components between (u, v) and $(u+du, v+dv)$ is found by summing for all possible values of w, so that

$$\delta N = P\,du\,dv \int_{-\infty}^{\infty} e^{-F} dw.$$

Now
$$F \equiv au^2 + bv^2 + 2huv - \frac{(fv+gu)^2}{c} + c\left(w + \frac{fv+gu}{c}\right)^2$$
$$\equiv \frac{1}{c}(b_0 u^2 - 2h_0 uv + a_0 v^2) + c\left(w + \frac{fv+gu}{c}\right)^2,$$

where $a_0, b_0, \ldots h_0$ are the minors of $a, b, \ldots h$ in the determinant

$$\Delta = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}.$$

Hence
$$\delta N = P\,du\,dv\, e^{-\frac{1}{c}(b_0u^2 - 2h_0uv + a_0v^2)} \int_{-\infty}^{\infty} e^{-c\left(w + \frac{fv+gu}{c}\right)^2} dw.$$

The value of the integral is $\sqrt{\pi/c}$. Hence

$$\delta N = P\sqrt{\frac{\pi}{c}}\, e^{-\frac{1}{c}(b_0u^2 - 2h_0uv + a_0v^2)}\, du\,dv. \qquad \ldots\ldots(7)$$

Thus the motions, projected on the tangent plane of the region considered, correspond to the velocity ellipse whose equation is

$$\frac{1}{c}(b_0 u^2 - 2h_0 uv + a_0 v^2) = 1. \qquad \ldots\ldots(8)$$

It is to be remarked that (8) is the equation of the cross-section of the enveloping cylinder of the velocity ellipsoid whose generators are parallel to the direction given by the region concerned; the velocity ellipse is thus the *outline* of the velocity ellipsoid seen from an infinite distance in the direction concerned; this property enables us to visualise the variation of the velocity ellipse in different parts of the sky.

Again, let $\delta_1 N$ denote the number of stars with velocity components between u and $u+du$. Then

$$\delta_1 N = P\sqrt{\frac{\pi}{c}}\, du\, e^{-\frac{b_0}{c}u^2} \int_{-\infty}^{\infty} e^{-\frac{1}{c}(a_0v^2 - 2h_0uv)} dv$$
$$= P\sqrt{\frac{\pi}{c}}\, du\, e^{-\frac{1}{c}\left(\frac{a_0b_0 - h_0^2}{a_0}\right)u^2} \int_{-\infty}^{\infty} e^{-\frac{a_0}{c}\left(v - \frac{h_0}{a_0}u\right)^2} dv$$
$$= P\frac{\pi}{\sqrt{a_0}}\, du\, e^{-\frac{1}{c}\left(\frac{a_0b_0 - h_0^2}{a_0}\right)u^2}.$$

But
$$a_0 b_0 - h_0^2 = c\Delta.$$

Hence
$$\delta_1 N = \frac{P\pi}{\sqrt{a_0}} e^{-\frac{\Delta}{a_0}u^2} du, \qquad \text{......(9)}$$

from which, by summing for all possible values of u, we obtain

$$N = \frac{P\pi}{\sqrt{a_0}} \sqrt{\frac{\pi a_0}{\Delta}}.$$

Thus, P is given in terms of N by the formula

$$P = N \sqrt{\frac{\Delta}{\pi^3}}. \qquad \text{......(10)}$$

The distribution of transverse linear velocities is given, from (7) and (10), by

$$\delta N = \frac{N}{\pi} \sqrt{\frac{\Delta}{c}} e^{-\frac{1}{c}(b_0 u^2 - 2h_0 uv + a_0 v^2)} du\, dv \qquad \text{......(11)}$$

and $\delta_1 N$ is given by
$$\delta_1 N = N \sqrt{\frac{\Delta}{\pi a_0}} e^{-\frac{\Delta}{a_0}u^2} du. \qquad \text{......(12)}$$

Also, if dN is the number of stars with components of velocity between (u, v, w) and $(u+du, v+dv, w+dw)$, we have

$$dN = N \sqrt{\frac{\Delta}{\pi^3}} e^{-F(u,v,w)} du\, dv\, dw. \qquad \text{......(13)}$$

5·71. *The treatment of radial velocities; a theorem concerning the mean peculiar radial speed for a small region of the sky.*

Let δN now denote the number of stars in a given region with radial velocity components between w and $w+dw$. By analogy with formula (12) of the previous section, we have

$$\delta N = N \sqrt{\frac{\Delta}{\pi c_0}} e^{-\frac{\Delta}{c_0}w^2} dw. \qquad \text{......(1)}$$

This is the frequency function for the peculiar radial velocities. Now, by (6) of section 5·6,

$$\begin{aligned} c_0 &\equiv ab - h^2 \\ &= \left(\frac{l_1^2}{s_1} + \frac{m_1^2}{s_2} + \frac{n_1^2}{s_3}\right)\left(\frac{l_2^2}{s_1} + \frac{m_2^2}{s_2} + \frac{n_2^2}{s_3}\right) - \left(\frac{l_1 l_2}{s_1} + \frac{m_1 m_2}{s_2} + \frac{n_1 n_2}{s_3}\right)^2 \\ &= \Sigma \frac{1}{s_1 s_2}(l_1 m_2 - l_2 m_1)^2. \end{aligned}$$

But $l_1 m_2 - l_2 m_1 = n_3$, etc.; hence

$$c_0 = \frac{1}{s_1 s_2 s_3}(l_3^2 s_1 + m_3^2 s_2 + n_3^2 s_3). \qquad \text{......(2)}$$

Also, the determinant, Δ, is an invariant for change of axes. Consequently, Δ is equal to the corresponding determinant when the equation of the ellipsoid is referred to its principal axes, as in (1) of section 5·6. Thus

$$\Delta = \frac{1}{s_1 s_2 s_3}. \qquad \text{......(3)}$$

Accordingly, by means of (2) and (3), the expression for δN in (1) becomes

$$\delta N = Qe^{-\frac{w^2}{l_3^2 s_1 + m_3^2 s_2 + n_3^2 s_3}} dw, \qquad \text{......(4)}$$

where

$$Q = \frac{N}{\{\pi(l_3^2 s_1 + m_3^2 s_2 + n_3^2 s_3)\}^{\frac{1}{2}}}. \qquad \text{......(5)}$$

The expression, $l_3^2 s_1 + m_3^2 s_2 + n_3^2 s_3$, has a geometrical significance. Let p denote the length of the perpendicular from the origin to that tangent plane of the velocity ellipsoid whose normal is parallel to the line of sight. Then p is given by

$$p^2 = l_3^2 s_1 + m_3^2 s_2 + n_3^2 s_3, \qquad \text{......(6)}$$

so that

$$\delta N = Qe^{-w^2/p^2} dw \qquad \text{......(7)}$$

and

$$Q = \frac{N}{p\sqrt{\pi}}. \qquad \text{......(8)}$$

Let R_0 denote the mean radial *speed* for the region considered. Then

$$NR_0 = 2Q \int_0^\infty we^{-w^2/p^2} dw$$

$$= Qp^2;$$

from this result and (8),

$$R_0 = \frac{p}{\sqrt{\pi}} \qquad \text{......(9)}$$

or

$$\pi R_0^2 = l_3^2 s_1 + m_3^2 s_2 + n_3^2 s_3. \qquad \text{......(10)}$$

Formula (9) embodies the theorem that the mean peculiar radial speed for a region is $1/\sqrt{\pi}$ times the length of the perpendicular from the origin to that tangent plane of the velocity ellipsoid whose normal is parallel to the line of sight; and formula (10) is an expression of this theorem when the equation of the ellipsoid is referred to its principal axes.

5·72. Consider now the equation of the velocity ellipsoid referred to the usual system of equatorial (or galactic) axes; we write it

$$F(U, V, W) \equiv AU^2 + BV^2 + CW^2 + 2FVW + 2GWU + 2HUV = 1. \qquad \text{......(1)}$$

The mean radial speed, R_0, in the direction (l, m, n) is given by

$$\pi R_0^2 = p^2, \qquad \text{......(2)}$$

where p is the length of the perpendicular from the origin to the tangent

plane of (1) whose normal has direction-cosines (l, m, n). The equation of the plane is accordingly
$$lU+mV+nW = p. \qquad \text{......(3)}$$
The condition that (3) should be a tangent plane to (1) is
$$\begin{vmatrix} A, & H, & G, & l \\ H, & B, & F, & m \\ G, & F, & C, & n \\ l, & m, & n, & p^2 \end{vmatrix} = 0.$$
Expanding the determinant we obtain the relation between p and the coefficients A, B, ... H in the form
$$A_0 l^2 + B_0 m^2 + C_0 n^2 + 2F_0 mn + 2G_0 nl + 2H_0 lm = p^2 \Delta',$$
where A_0, B_0, ... H_0 are the minors of A, B, ... H in the determinant
$$\Delta' = \begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix}. \qquad \text{......(4)}$$
Consequently,
$$\pi R_0^2 = a_1 l^2 + b_1 m^2 + c_1 n^2 + 2f_1 mn + 2g_1 nl + 2h_1 lm, \qquad \text{......(5)}$$
where
$$a_1 = \frac{A_0}{\Delta'}, \quad b_1 = \frac{B_0}{\Delta'}, \quad \ldots \quad h_1 = \frac{H_0}{\Delta'}. \qquad \text{......(6)}$$

Each region of the sky provides an equation of the form (5), in which R_0, l, m and n are all supposed known. A least-squares solution then gives the values of a_1, b_1, ... h_1.

It is to be remembered that R_0, in our notation, is the mean *motus peculiaris*, without regard to sign, in the line of sight for the region considered. As each observed radial velocity contains the parallactic component in the line of sight, the observed velocity must first be corrected for the effects of the solar motion so as to yield the appropriate value of R_0. In general, this procedure requires that all the radial velocity material must be first analysed to give the direction and amount of the solar motion.

5·73. *Determination of the lengths and directions of the principal axes of the velocity ellipsoid.*

We assume that the coefficients a_1, b_1, ... h_1 of 5·72 (5) have been determined. The ellipsoid
$$a_1 U^2 + b_1 V^2 + c_1 W^2 + 2f_1 VW + 2g_1 WU + 2h_1 UV = 1 \qquad \text{......(1)}$$
is the reciprocal ellipsoid of the velocity ellipsoid with respect to a concentric sphere of unit radius, and since a_1, b_1, ... h_1 have been found the reciprocal ellipsoid is known. Also, the directions of the principal axes of

(1) are the same as those of the velocity ellipsoid and the lengths of corresponding semi-axes are reciprocal; these theorems are at once evident if the equation of the velocity ellipsoid is referred to its principal axes.

The direction-cosines (L, M, N) of the principal axes of (1) are easily found as follows. The diametral plane of the straight line

$$\frac{U}{L} = \frac{V}{M} = \frac{W}{N} \qquad \text{......(2)}$$

is $$U(a_1 L + h_1 M + g_1 N) + V(h_1 L + b_1 M + f_1 N) + W(g_1 L + f_1 M + c_1 N) = 0. \qquad \text{......(3)}$$

Since (2) is a principal axis, it is perpendicular to (3); hence

$$\frac{a_1 L + h_1 M + g_1 N}{L} = \frac{h_1 L + b_1 M + f_1 N}{M} = \frac{g_1 L + f_1 M + c_1 N}{N}.$$

Setting each of these ratios equal to λ, we have

$$\left.\begin{aligned} (a_1 - \lambda) L + h_1 M + g_1 N = 0 \\ h_1 L + (b_1 - \lambda) M + f_1 N = 0 \\ g_1 L + f_1 M + (c_1 - \lambda) N = 0 \end{aligned}\right\}. \qquad \text{......(4)}$$

Eliminating L, M and N, we see that λ is a root of

$$\begin{vmatrix} a_1 - \lambda, & h_1, & g_1 \\ h_1, & b_1 - \lambda, & f_1 \\ g_1, & f_1, & c_1 - \lambda \end{vmatrix} = 0, \qquad \text{......(5)}$$

that is, λ is a root of

$$\lambda^3 - \lambda^2(a_1 + b_1 + c_1) + \lambda(A_1 + B_1 + C_1) - \Delta_1 = 0, \qquad \text{......(6)}$$

where $$\Delta_1 = \begin{vmatrix} a_1, & h_1, & g_1 \\ h_1, & b_1, & f_1 \\ g_1, & f_1, & c_1 \end{vmatrix}$$

and A_1, B_1, C_1 are the minors of a_1, b_1, c_1 in Δ_1.

Substituting each of the three roots of (6) in (4), we obtain the three sets of values for (L, M, N).

Now consider the cone

$$a_1 U^2 + b_1 V^2 + c_1 W^2 + 2f_1 VW + 2g_1 WU + 2h_1 UV = \frac{U^2 + V^2 + W^2}{r^2} \qquad \text{......(7)}$$

passing through the extremities of all semi-diameters, of length r, of the reciprocal ellipsoid, (1). When r is equal to the length of a principal semi-axis, the cone (7) degenerates into a pair of planes; the appropriate condition is

$$\begin{vmatrix} a_1 - 1/r^2, & h_1, & g_1 \\ h_1, & b_1 - 1/r^2, & f_1 \\ g_1, & f_1, & c_1 - 1/r^2 \end{vmatrix} = 0, \qquad \text{......(8)}$$

where now r denotes the length of a principal semi-axis.

Comparing this equation with (5), we see that the lengths of the principal semi-axes of (1) are given by $1/\sqrt{\lambda_1}$, $1/\sqrt{\lambda_2}$, $1/\sqrt{\lambda_3}$, where $\lambda_1, \lambda_2, \lambda_3$ are the roots of (6). It follows that the lengths of the principal semi-axes of the velocity ellipsoid are $\sqrt{\lambda_1}$, $\sqrt{\lambda_2}$, and $\sqrt{\lambda_3}$.

We have thus shown how to find the lengths and directions of the axes of the velocity ellipsoid in terms of the quantities $a_1, b_1, \ldots h_1$.

5·81. *The general treatment of proper motions.*

As before, we shall write the general equation of the velocity ellipsoid, referred to equatorial (or galactic) axes, as

$$F(U, V, W) \equiv AU^2 + BV^2 + CW^2 + 2FVW + 2GWU + 2HUV = 1, \quad \ldots(1)$$

so that the number, dn, of stars with peculiar velocities between (U, V, W) and $(U+dU, V+dV, W+dW)$ is given, according to 5·16 (13), by

$$dn = n\sqrt{\frac{\Delta'}{\pi^3}}\, e^{-F(U,V,W)}\, dU\, dV\, dW, \ldots(2)$$

where Δ' is defined by 5·72 (4) and n is the total number of stars concerned.

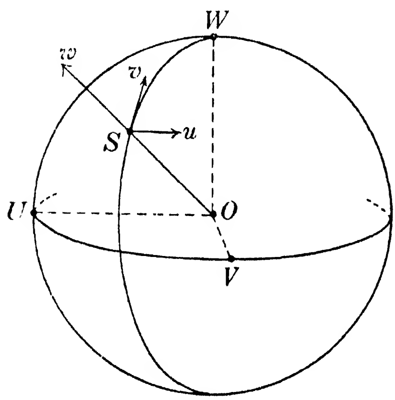

Fig. 39

Consider a small region at S (Fig. 39) with n stars obeying the frequency law (2) and transform to the axes as shown in the figure, the w-axis being radial. Let (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) be the direction-cosines of the u, v and w axes with respect to the U, V and W axes. If (α, δ) are the coordinates of S, we have

$$\left.\begin{aligned} l_1 &= -\sin\alpha, & m_1 &= \cos\alpha, & n_1 &= 0 \\ l_2 &= -\cos\alpha\sin\delta, & m_2 &= -\sin\alpha\sin\delta, & n_2 &= \cos\delta \\ l_3 &= \cos\alpha\cos\delta, & m_3 &= \sin\alpha\cos\delta, & n_3 &= \sin\delta \end{aligned}\right\}. \quad \ldots\ldots(3)$$

Also

$$\left.\begin{aligned} U &= l_1 u + l_2 v + l_3 w \\ V &= m_1 u + m_2 v + m_3 w \\ W &= n_1 u + n_2 v + n_3 w \end{aligned}\right\}. \quad \ldots\ldots(4)$$

By means of (4), $F(U, V, W)$ is transformed into

$$f(u, v, w) \equiv au^2 + bv^2 + cw^2 + 2fvw + 2gwu + 2huv = 1, \quad \ldots\ldots(5)$$

where, for example,

$$a = Al_1^2 + Bm_1^2 + Cn_1^2 + 2Fm_1n_1 + 2Gn_1l_1 + 2Hl_1m_1, \quad \ldots\ldots(6)$$

$$f = Al_2l_3 + Bm_2m_3 + Cn_2n_3 + F(m_2n_3 + m_3n_2) + G(n_2l_3 + n_3l_2) + H(l_2m_3 + l_3m_2). \quad \ldots\ldots(7)$$

From 5·6 (11), the distribution of the peculiar transverse components (u, v) is given by

$$\delta n = \frac{n}{\pi}\sqrt{\frac{\Delta}{c}}\, e^{-\frac{1}{c}(b_0u^2-2h_0uv+a_0v^2)}\,du\,dv, \qquad \text{......(8)}$$

where a_0, h_0, b_0 are the minors of a, h, b in the determinant

$$\Delta = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}. \qquad \text{......(9)}$$

Following Fleming,* to whom this investigation is due, we write (8) in the form

$$\delta n = \frac{n}{\pi}\sqrt{\frac{\Delta}{c}}\, e^{-pu^2-2suv-qv^2}\,du\,dv, \qquad \text{......(10)}$$

where $$p = b_0/c, \quad q = a_0/c, \quad s = -h_0/c. \qquad \text{......(11)}$$

It is to be noticed that since (5) is obtained from (1) by change of axes, the determinants Δ and Δ' are equal, by the theorem of invariancy. Thus we can write (9) in the alternative form

$$\Delta = \begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix}. \qquad \text{......(12)}$$

5·82. We consider now the *observed* transverse motions. Let $(-U_0, -V_0)$ denote the components of the solar motion, projected on the tangent plane at S, parallel respectively to the u and v axes. If x and y are the components of the observed linear transverse velocities, we have

$$x = u + U_0, \quad y = v + V_0. \qquad \text{......(1)}$$

The formula 5·81 (10) becomes

$$\delta n = \frac{n}{\pi}\sqrt{\frac{\Delta}{c}}\, e^{-p(x-U_0)^2-q(y-V_0)^2-2s(x-U_0)(y-V_0)}\,dx\,dy, \qquad \text{......(2)}$$

which gives the number of stars with observed components between (x, y) and $(x+dx, y+dy)$.

Put $$x = r\sin\theta, \quad y = r\cos\theta. \qquad \text{......(3)}$$

If the general equation, 5·81 (1), of the velocity ellipsoid is referred to equatorial axes, it will be seen by referring to Fig. 39 that θ is the position angle of an observed proper motion.

From (3), $$dx\,dy = r\,dr\,d\theta.$$

* *M.N.* **97**, 173, 1937.

Hence the number of stars moving in the sector θ to $\theta+d\theta$ with total transverse motions between r and $r+dr$ is

$$\frac{n}{\pi}\sqrt{\frac{\Delta}{c}}\,r\,dr\,d\theta\,e^{-p(r\sin\theta-U_0)^2-q(r\cos\theta-V_0)^2-2s(r\sin\theta-U_0)(r\cos\theta-V_0)}.$$

Let $n(\theta)\,d\theta$ denote the total number of stars moving in the sector $d\theta$; this number is obtained by integrating the previous expression with respect to r from 0 to ∞. We thus obtain

$$n(\theta) = \frac{n}{\pi}\sqrt{\frac{\Delta}{c}}\,e^{-pU_0^2-qV_0^2-2sU_0V_0}\int_0^\infty re^{-\rho^2r^2+2\rho\xi r}\,dr, \qquad \text{......(4)}$$

where

$$\rho^2 = p\sin^2\theta+q\cos^2\theta+2s\sin\theta\cos\theta, \qquad \text{......(5)}$$

$$\xi = \frac{1}{\rho}\{(pU_0+sV_0)\sin\theta+(sU_0+qV_0)\cos\theta. \qquad \text{......(6)}$$

Write

$$\rho r-\xi = \tau$$

in the integral in (4). This integral becomes

$$\frac{e^{\xi^2}}{\rho^2}\int_{-\xi}^\infty(\tau+\xi)\,e^{-\tau^2}\,d\tau$$

or, in terms of Eddington's function $f(\xi)$,

$$\frac{\sqrt{\pi}}{2\rho^2}f(\xi).$$

Hence (4) becomes

$$n(\theta) = \frac{n}{2\rho^2}\sqrt{\frac{\Delta}{\pi c}}\,e^{-pU_0^2-qV_0^2-2sU_0V_0}f(\xi). \qquad \text{......(7)}$$

Write

$$L\equiv\frac{n}{2}\sqrt{\frac{\Delta}{\pi c}}\,e^{-pU_0^2-qV_0^2-2sU_0V_0} \qquad \text{......(8)}$$

so that

$$n(\theta) = \frac{L}{\rho^2}f(\xi). \qquad \text{......(9)}$$

The number of stars moving in the sector θ to $\theta+d\theta$ is then given by

$$n(\theta)\,d\theta = \frac{L}{\rho^2}f(\xi)\,d\theta. \qquad \text{......(10)}$$

Now let $n'(\theta)\,d\theta$ denote the number of stars moving in the sector $\theta+\pi$ to $\theta+d\theta+\pi$. The addition of π to θ in (5) leaves ρ unaltered but in (6) ξ changes sign. Hence

$$n'(\theta)\,d\theta = \frac{L}{\rho^2}f(-\xi)\,d\theta. \qquad \text{......(11)}$$

Subtracting (11) from (10), we have

$$\{n(\theta)-n'(\theta)\}\,d\theta = \frac{L}{\rho^2}d\theta\{f(\xi)-f(-\xi)\}. \qquad \text{......(12)}$$

By the definition of $f(\xi)$—see 2·3 (10)—

$$f(\xi)-f(-\xi) = \frac{2}{\sqrt{\pi}}\left\{\tfrac{1}{2}+\xi e^{\xi^2}\int_{-\xi}^{\infty} e^{-\tau^2}d\tau\right\} - \frac{2}{\sqrt{\pi}}\left\{\tfrac{1}{2}-\xi e^{\xi^2}\int_{\xi}^{\infty} e^{-\tau^2}d\tau\right\}$$
$$= 2\xi e^{\xi^2}.$$

Hence
$$\{n(\theta)-n'(\theta)\}\,d\theta = \frac{2L\xi}{\rho^2}e^{\xi^2}d\theta. \qquad \text{......(13)}$$

5·83. Write 5·82 (6) for convenience in the form

$$\xi = \frac{A\sin\theta + B\cos\theta}{\rho}.$$

We find on differentiating with respect to θ that

$$\rho^3\frac{d\xi}{d\theta} = \rho^2(A\cos\theta - B\sin\theta) - \rho\frac{d\rho}{d\theta}(A\sin\theta + B\cos\theta).$$

Also, from 5·82 (5),

$$\rho\frac{d\rho}{d\theta} = (p-q)\sin\theta\cos\theta + 2s\cos^2\theta - s.$$

After some reduction we obtain

$$\rho^3\frac{d\xi}{d\theta} = (pq-s^2)(U_0\cos\theta - V_0\sin\theta).$$

Let
$$\eta = \frac{(pq-s^2)^{\frac{1}{2}}(U_0\cos\theta - V_0\sin\theta)}{\rho}. \qquad \text{......(1)}$$

Then
$$\rho^2\frac{d\xi}{d\theta} = (pq-s^2)^{\frac{1}{2}}\eta. \qquad \text{......(2)}$$

It is easily shown that

$$\rho^2(\xi^2+\eta^2) = (p\sin^2\theta + q\cos^2\theta + 2s\sin\theta\cos\theta)(pU_0^2 + qV_0^2 + 2sU_0V_0),$$

from which
$$\xi^2+\eta^2 = \epsilon^2, \qquad \text{......(3)}$$

where
$$\epsilon^2 = pU_0^2 + qV_0^2 + 2sU_0V_0. \qquad \text{......(4)}$$

It is to be remarked that ϵ is independent of θ; hence

$$\xi\frac{d\xi}{d\theta} = -\eta\frac{d\eta}{d\theta}.$$

We thus obtain from (2)

$$\frac{\xi}{\rho^2}d\theta = -\frac{d\eta}{(pq-s^2)^{\frac{1}{2}}}. \qquad \text{......(5)}$$

Also from (4) and 5·82 (8)

$$2L = n\sqrt{\frac{\Delta}{\pi c}}\,e^{-\epsilon^2}. \qquad \text{......(6)}$$

Using (3), (5) and (6), we find that 5·82 (13) becomes

$$\{n(\theta)-n'(\theta)\}\,d\theta = -n\sqrt{\frac{\Delta}{\pi c(pq-s^2)}}\,e^{-\eta^2}d\eta. \qquad \text{......(7)}$$

By 5·81 (11), $$pq-s^2 = \frac{1}{c^2}(a_0b_0-h_0^2).$$

But $$a_0b_0-h_0^2 = c\Delta,$$

so that $$\Delta = c(pq-s^2). \qquad \text{......(8)}$$

Hence (7) becomes $$\{n(\theta)-n'(\theta)\}\,d\theta = -\frac{n}{\sqrt{\pi}}e^{-\eta^2}\,d\eta. \qquad \text{......(9)}$$

5·84. *Counts of proper motions in semi-circular sectors.*

Let $N(\theta)$ denote the total number of proper motions in the sector θ to $\theta+\pi$ and $N'(\theta)$ the number in the sector $\theta+\pi$ to $\theta+2\pi$. Then

$$N(\theta)-N'(\theta) = \int_\theta^{\theta+\pi}\{n(\theta)-n'(\theta)\}\,d\theta.$$

Hence, by 5·83 (9), $$N(\theta)-N'(\theta) = -\frac{n}{\pi}\int_\eta^{\eta_1}e^{-\eta^2}\,d\eta, \qquad \text{......(1)}$$

where η and η_1 correspond to θ and $\theta+\pi$ respectively.

Now, by 5·83 (1), η changes sign when $\theta+\pi$ is substituted for θ. Hence $\eta_1 = -\eta$; and (1) becomes

$$N(\theta)-N'(\theta) = -\frac{n}{\sqrt{\pi}}\int_\eta^{-\eta}e^{-\eta^2}\,d\eta.$$

Introducing the error integral

$$\Theta(\eta) = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2}\,d\eta \qquad \text{......(2)}$$

and writing $$M(\theta) \equiv N(\theta)-N'(\theta), \qquad \text{......(3)}$$

we finally obtain $$M(\theta) = n\Theta(\eta). \qquad \text{......(4)}$$

5·851. *Determination of the solar apex.*

In practical investigations it is customary, as previously described, to work with counts in 10° sectors. If $n_0(\theta_1)$ is the number of proper motions in the sector $\theta_1-5°$ to $\theta_1+5°$,

$$n_0(\theta_1) = \int_{\theta_1-5°}^{\theta_1+5°}n(\theta)\,d\theta,$$

or, taking $n(\theta_1)$ to be the mean value of $n(\theta)$ in the sector,

$$n_0(\theta_1) = \frac{\pi}{18}n(\theta_1).$$

Similarly, $$n_0'(\theta_1) = \frac{\pi}{18}n'(\theta_1).$$

Then corresponding to $\theta \equiv \theta_1-5°$, we have

$$M(\theta) = \Sigma\{n_0(\theta_1)-n_0'(\theta_1)\},$$

where the summation extends for values of θ_1 between $\theta+5°$ and $\theta+175°$.

The values of $M(\theta)$, derived from a given region, can then be plotted against θ.

By 5·84 (4), $M(\theta)$ vanishes when η vanishes. But, by 5·83 (1), $\eta = 0$ for a position angle θ_0 given by

$$\tan\theta_0 = U_0/V_0. \qquad \text{......(1)}$$

This value θ_0 is obtained at once from the graph of $M(\theta)$. θ_0 is the position angle of the projected solar motion on the tangent plane of the region. Denote $\tan\theta_0$ by λ, which is now supposed known; then

$$U_0 - \lambda V_0 = 0. \qquad \text{......(2)}$$

But, if $(-U_1, -V_1, -W_1)$ are the components of the solar motion referred to the (U, V, W) axes of Fig. 39, we have, by 5·81 (4),

$$U_1 = l_1 U_0 + l_2 V_0 + l_3 W_0,$$
$$V_1 = m_1 U_0 + m_2 V_0 + m_3 W_0,$$
$$W_1 = n_1 U_0 + n_2 V_0 + n_3 W_0,$$

where $-W_0$ is the radial component of the solar motion. From these we have

$$U_0 = l_1 U_1 + m_1 V_1 + n_1 W_1,$$
$$V_0 = l_2 U_1 + m_2 V_1 + n_2 W_1.$$

Hence the condition (2) becomes

$$U_1(l_1 - \lambda l_2) + V_1(m_1 - \lambda m_2) + W_1(n_1 - \lambda n_2) = 0. \qquad \text{......(3)}$$

Each region furnishes an equation of condition of this type, in which the coefficients of U_1, V_1 and W_1 are known; by a least-squares solution the ratios

$$U_1 : V_1 : W_1$$

are obtained. The coordinates of the solar apex are then found in the usual way from the formulae

$$\tan A = V_1/U_1,$$
$$\tan D = W_1/(U_1^2 + V_1^2)^{\frac{1}{2}}.$$

5·852. *Example.*

As an illustration of the method we give Table 24 containing the statistics* resulting from the analysis of a region† by Fleming's method; the second and third columns are taken directly from the data of the second paper mentioned.

The values of $M(\theta)$ in the last column are given only for values of θ between 5° and 175°; between 185° and 355° they are reproduced as in the last column with a change of sign throughout.

It will be noticed that $M(\theta)$ vanishes when $\theta \equiv \theta_0 = 126°$. The value of θ_0 calculated from the coordinates of the solar apex (273°·6, +43°·6), as derived in the second paper, is 121°, with which Fleming's value is in good agreement.

* J. Fleming, *M.N.* **97**, 181, 1937. † W. M. Smart, *M.N.* **87**, 122, 1927.

Fleming's analysis of the eight Cambridge groups of regions by the method of the previous section yielded the following coordinates of the solar apex:

$$A = 272^\circ{\cdot}4, \quad D = +43^\circ{\cdot}1,$$

in good agreement with the position already quoted and obtained by an entirely different procedure.

Table 24. *Values of $M(\theta)$ for the Cambridge Group I*

θ	n_0	n_0'	$N(\theta)$	$N'(\theta)$	$M(\theta)$
5°	7	29	393	152	241
15	3	23	415	130	285
25	7	19	435	110	325
35	7	14	447	98	349
45	6	16	454	91	363
55	14	12	464	81	383
65	16	4	462	83	379
75	27	5	450	95	355
85	34	4	428	117	311
95	50	3	398	147	251
105	40	3	351	194	157
115	40	3	314	231	83
125	33	3	277	268	9
135	28	3	247	298	− 51
145	31	1	222	323	−101
155	20	3	192	353	−161
165	15	2	175	370	−195
175	15	5	162	383	−221

5·86. *Determination of the constants of the velocity ellipsoid.*

From 5·83 (1), we have

$$\rho^2 = \frac{(pq-s^2)(U_0\cos\theta - V_0\sin\theta)^2}{\eta^2}.$$

Let T_0 denote the projection of the solar motion (which we denote here by X) on the tangent plane of the region under investigation. If λ is the angular distance of the region from the solar apex,

$$T_0 = X\sin\lambda.$$

Also, $$-U_0 = T_0\sin\theta_0, \quad -V_0 = T_0\cos\theta_0,$$

where θ_0 is the position angle of the projection of the solar motion on the tangent plane. We suppose that θ_0 has been found by the method of section 5·851.

We then have $$\rho^2 = \frac{(pq-s^2)X^2\sin^2\lambda\,\sin^2(\theta-\theta_0)}{\eta^2}$$

or $$\rho^2 = K\frac{\sin^2(\theta-\theta_0)}{\eta^2}, \qquad \text{......(1)}$$

where $$K = (pq-s^2)X^2\sin^2\lambda. \qquad \text{......(2)}$$

Now, by 5·84 (4), $$\Theta(\eta) = \frac{1}{n}M(\theta).$$

As n and $M(\theta)$ are known from the statistics for each value of θ used, the corresponding value of $\Theta(\eta)$ is easily found; from tables of the function $\Theta(\eta)$, the value of η corresponding to a given θ can then be obtained. Thus the value of

$$\zeta \equiv \frac{\sin^2(\theta-\theta_0)}{\eta^2} \qquad \text{......(3)}$$

can be calculated for each value of θ.

We thus have, from (1) and (3),

$$\rho^2 = K\zeta.$$

Now ρ^2 is given by 5·82 (5); accordingly

$$p\sin^2\theta + q\cos^2\theta + 2s\sin\theta\cos\theta = K\zeta,$$

or, on writing $\quad p = PK, \quad q = QK, \quad s = SK, \qquad$ ......(4)

we obtain $\quad P\sin^2\theta + Q\cos^2\theta + 2S\sin\theta\cos\theta = \zeta. \qquad$ ......(5)

It will be assumed that the counts of proper motions are made in 10° sectors for the values 0°, 10°, 20°, ... of θ. It follows that for each region we shall have eighteen equations of condition of the form of (5), in which the separate values of ζ are known. A solution by least squares will give the values of P, Q and S for the region concerned.

The normal equations are:

$$P\Sigma\sin^4\frac{r\pi}{18} + Q\Sigma\sin^2\frac{r\pi}{18}\cos^2\frac{r\pi}{18} + 2S\Sigma\sin^3\frac{r\pi}{18}\cos\frac{r\pi}{18} = \Sigma\zeta\sin^2\frac{r\pi}{18},$$

$$P\Sigma\sin^2\frac{r\pi}{18}\cos^2\frac{r\pi}{18} + Q\Sigma\cos^4\frac{r\pi}{18} + 2S\Sigma\sin\frac{r\pi}{18}\cos^3\frac{r\pi}{18} = \Sigma\zeta\cos^2\frac{r\pi}{18},$$

$$2P\Sigma\sin^3\frac{r\pi}{18}\cos\frac{r\pi}{18} + 2Q\Sigma\sin\frac{r\pi}{18}\cos^3\frac{r\pi}{18} + 4S\Sigma\sin^2\frac{r\pi}{18}\cos^2\frac{r\pi}{18} = \Sigma\zeta\sin\frac{r\pi}{9},$$

where the summations are taken from $r = 0$ to $r = 17$.

Now $\quad \cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

and

$$\Sigma\cos^4\frac{r\pi}{18} = \frac{1}{8}\left\{\Sigma\cos\frac{2r\pi}{9} + 4\Sigma\cos\frac{r\pi}{9} + 54\right\}$$
$$= \frac{1}{8}\{0+0+54\}$$
$$= \frac{27}{4}.$$

Similarly

$$\Sigma\sin^4\frac{r\pi}{18} = \frac{27}{4},$$

$$\Sigma\sin^2\frac{r\pi}{18}\cos^2\frac{r\pi}{18} = \frac{9}{4},$$

$$\Sigma\sin\frac{r\pi}{18}\cos^3\frac{r\pi}{18} = \Sigma\sin^3\frac{r\pi}{18}\cos\frac{r\pi}{18} = 0.$$

The normal equations are thus

$$\left.\begin{aligned} 3P+Q &= \tfrac{4}{9}\Sigma\zeta\sin^2\frac{r\pi}{18} \\ P+3Q &= \tfrac{4}{9}\Sigma\zeta\cos^2\frac{r\pi}{18} \\ S &= \tfrac{1}{9}\Sigma\zeta\sin\frac{r\pi}{9} \end{aligned}\right\}. \qquad \text{......(6)}$$

These equations enable us to calculate the values of P, Q and S for the region.

From (2) and (4) we have

$$\frac{1}{K} = (PQ-S^2)X^2\sin^2\lambda. \qquad \text{......(7)}$$

Also, by 5·83 (8),

$$pq-s^2 = \frac{\Delta}{c},$$

so that (2) can be written

$$K = \frac{\Delta}{c}X^2\sin^2\lambda. \qquad \text{......(8)}$$

Multiplying (7) and (8), we obtain

$$c = \Delta X^4(PQ-S^2)\sin^4\lambda$$

or

$$c = \gamma(PQ-S^2)\sin^4\lambda, \qquad \text{......(9)}$$

where

$$\gamma \equiv \Delta X^4. \qquad \text{......(10)}$$

It is to be noticed that Δ and X are constants, the former being the invariant given by 5·81 (12); hence γ is independent of the region concerned. Also $PQ-S^2$ and λ are to be determined for each region.

Now c is given by the analogue of 5·81 (6), so that (9) can now be written

$$Al_3^2+Bm_3^2+Cn_3^2+2Fm_3n_3+2Gn_3l_3+2Hl_3m_3 = \gamma(PQ-S^2)\sin^4\lambda. \qquad \text{......(11)}$$

This is the equation of condition for a given region. A least-squares solution of (11) gives the values of $A, B, \ldots H$ in terms of γ. The directions and relative lengths of the principal semi-axes of the velocity ellipsoid are then found by the method described in section 5·73.

5·91. *The treatment of parallax stars.*

By parallax stars we mean those stars whose parallaxes are known, as well as the components of proper motion and the radial velocities. We thus can determine the linear components of motion relative to the sun by the formulae of sections 1·33 or 1·42 with respect to equatorial or galactic axes. We assume that the solar motion is completely known; hence, using the linear components $(-X, -Y, -Z)$ of the solar motion with respect to these axes, we obtain the linear components of the *motus peculiares* of the stars.

Consider now the velocity ellipsoid referred to its principal axes. The linear components of the *motus peculiares* are now (U, V, W) and the number of stars with velocity components between (U, V, W) and $(U+dU, V+dV, W+dW)$ is

$$\frac{N}{\pi^{\frac{3}{2}}(s_1 s_2 s_3)^{\frac{1}{2}}} e^{-\frac{U^2}{s_1}-\frac{V^2}{s_2}-\frac{W^2}{s_3}} dU\,dV\,dW,$$

where N is the total number of stars under consideration.

Let
$$U = r\sin\theta\cos\phi \equiv lr,$$
$$V = r\sin\theta\sin\phi \equiv mr,$$
$$W = r\cos\theta \equiv nr.$$

Then
$$dU\,dV\,dW = r^2 \sin\theta\, d\theta\, d\phi\, dr \equiv r^2 dr\, dS,$$

where $dS = \sin\theta\, d\theta\, d\phi$. The number of stars with linear velocities between r and $r+dr$ within a cone of solid angle dS is

$$\frac{Nr^2 dr\, dS}{\pi^{\frac{3}{2}}(s_1 s_2 s_3)^{\frac{1}{2}}} e^{-r^2\left(\frac{l^2}{s_1}+\frac{m^2}{s_2}+\frac{n^2}{s_3}\right)}. \qquad \ldots\ldots(1)$$

Now, the radius vector, ρ, of the velocity ellipsoid in the direction (l, m, n) is given by

$$\frac{1}{\rho^2} = \frac{l^2}{s_1} + \frac{m^2}{s_2} + \frac{n^2}{s_3}. \qquad \ldots\ldots(2)$$

Hence (1) can be written
$$Ke^{-\frac{r^2}{\rho^2}} r^2 dr.$$

The total number of stars within the cone is

$$K\int_0^\infty e^{-\frac{r^2}{\rho^2}} r^2 dr = K\rho^3 \int_0^\infty x^2 e^{-x^2} dx$$

or
$$K\frac{\sqrt{\pi}}{4}\rho^3.$$

This result shows that the number of stars moving in a given direction is proportional to the cube of the radius vector drawn to the velocity ellipsoid in this direction.

Let W_0 denote the mean linear velocity in the given direction. Then

$$W_0 = \frac{K\int_0^\infty e^{-\frac{r^2}{\rho^2}} r^3 dr}{K\frac{\sqrt{\pi}}{4}\rho^3}$$

$$= \frac{4\rho}{\sqrt{\pi}}\int_0^\infty x^3 e^{-x^2} dx,$$

whence
$$W_0 = \frac{2}{\sqrt{\pi}}\rho. \qquad \ldots\ldots(3)$$

Consider now the velocity ellipsoid

$$AU^2+BV^2+CW^2+2FVW+2GWU+2HUV = 1$$

referred to the usual system of axes. For linear peculiar motions in a given direction (l, m, n) with respect to these axes, we have

$$W_0 = \frac{2}{\sqrt{\pi}}\rho,$$

where ρ is now given by

$$\frac{1}{\rho^2} = Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm.$$

Hence the equation of condition is

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm = \frac{4}{\pi}\cdot\frac{1}{W_0^2}. \qquad \ldots\ldots(4)$$

If we use equatorial coordinates, we can define l, m, n by

$$l = \cos\alpha_1 \cos\delta_1, \quad m = \sin\alpha_1 \cos\delta_1, \quad n = \sin\delta_1$$

and then

$$U = r\cos\alpha_1 \cos\delta_1, \quad V = r\sin\alpha_1 \cos\delta_1, \quad W = r\sin\delta_1. \quad \ldots\ldots(5)$$

It is to be remarked that α_1 and δ_1 must not be confused with the equatorial coordinates of the stars. Since we are supposed to know the values of U, V and W for each star, we derive the corresponding values of r, α_1 and δ_1 from (5).

For a small region* in the neighbourhood of (l, m, n) we obtain the value of W_0 which, with the values of l, m, n, can be substituted in (4).

The solution of (4) by the method of least-squares yields the values of A, B, ... H. We thus obtain the equation of the velocity ellipsoid referred to the equatorial axes.

The application of the method described in 5·73 leads to the determination of the lengths of the semi-axes of the velocity ellipsoid and their directions.

5·92. *Numerical results.*

There have been numerous investigations on the derivation of the characteristics of the velocity ellipsoid, many of them undertaken at Lund Observatory by C. V. L. Charlier, K. G. Malmquist and W. Gyllenberg. It cannot be pretended that the various results are in completely satisfactory agreement; but this is hardly surprising when one considers the somewhat inadequate observational material available in most of these investigations.

Charlier's researches may be summed up in the following table,† which

* In "velocity-space", and not in the sky.

† *The motion and distribution of the stars: Memoirs of the University of California*, **7**, 74, 1926.

gives his results obtained by three different methods: (1) from radial velocities; (2) from parallax stars; (3) from proper motions. In the last method—which we have not described—it is assumed that the mean parallaxes of the stars of different groups are known. Charlier expresses the ellipsoidal velocity function in terms of the standard deviations σ_1, σ_2 and σ_3, which are related to s_1, s_2 and s_3 in our notation by

$$s_1 = 2\sigma_1^2, \quad s_2 = 2\sigma_2^2, \quad s_3 = 2\sigma_3^2. \qquad \text{......(1)}$$

In the table, we give Charlier's values of σ_1, σ_2 and σ_3.

(G_1, g_1), (G_2, g_2), (G_3, g_3) are the galactic coordinates defining the directions of the principal axes.

Table 25

	Radial velocities	Parallax stars	Proper motions
Number of stars:	1986	646	4182
$\sigma_1 \equiv \sqrt{s_1/2}$	19·9 km./sec.	27·9 km./sec.	23·4 km./sec.
G_1	341°2	341°3	339°0
g_1	−5°7	+2°8	−3°9
$\sigma_2 \equiv \sqrt{s_2/2}$	13·4 km./sec.	19·4 km./sec.	15·1 km./sec.
G_2	69°6	71°4	70°0
g_2	+16°2	+7°6	−13°4
$\sigma_3 \equiv \sqrt{s_3/2}$	15·6 km./sec.	16·1 km./sec.	12·1 km./sec.
G_3	270°0	233°1	52°9
g_3	+72°8	+83°1	+75°8

The lengths $\sqrt{s_1}$, $\sqrt{s_2}$, $\sqrt{s_3}$ of the semi-axes of the velocity ellipsoid can be easily found by means of (1) from the data of the table.

The axis corresponding to σ_1 is the longest axis of the ellipsoid and it lies nearly in the galactic equator. The three values of G_1 are in good agreement with the direction of the vertex as obtained from the theory of the two star streams. The third axis is directed approximately towards the galactic pole. The lengths of the second and third semi-axes are, on the average, not very dissimilar.

CHAPTER VI

STATISTICAL PARALLAXES DERIVED FROM STELLAR MOTIONS

6·1. In this chapter we investigate the methods adopted for obtaining mean parallaxes for particular groups of stars, usually selected according to apparent magnitude or spectral type, when the proper motions of the individual stars are known. A parallax determination of a single star involves a large amount of work—about a score of plates, with two or three exposures on each plate, have to be secured and measured—and for stars at distances greater than a hundred parsecs the probable errors of the trigonometrical parallaxes generally exceed in magnitude the quantities to be derived. An annual output of fifty parallaxes from an observatory, or a department of an observatory, devoted to this class of work represents a considerable achievement. Despite the progress that has been made in recent years, the number of stars with accurately determined parallaxes (of which the probable error is, say, a third or a quarter of the parallax itself) is comparatively small and such stars are generally our nearest stellar neighbours. Within recent years the trigonometrical method has been supplemented by the spectroscopic method which, however, is based on a knowledge of representative parallaxes as determined by the trigonometrical method. In the latter method, the faintness of the object whose distance is to be measured is no practical disadvantage, say, to the tenth or eleventh magnitude, since the necessary exposures can be made with modern refractors in a minute or two. The spectroscopic method, however, suffers from the disadvantage that the star must be comparatively bright, say, brighter than the sixth or seventh magnitude, for otherwise very long exposures must be given. It is hardly surprising that up to the present time the measures of spectroscopic parallaxes scarcely go beyond the naked-eye stars. Thus, in practice, each method has its definite limitations.

Moreover, in the trigonometrical method the parallax of the star under investigation is relative to the mean parallax of the comparison stars used in the reductions. These are generally stars of the tenth or eleventh magnitude and are presumably at considerably greater distances from us. To obtain the absolute parallax of the star we require to know the mean parallax of the comparison stars and this can only be ascertained by statistical methods based on the proper motions of representative stars of these magnitudes. Proper motions are comparatively easy to measure by the photographic method, and it is probably true to affirm that the proper motions of a hundred stars can be as easily measured with, at least, the same relative accuracy as the parallax of a single star. The proper motions of large numbers of stars as faint as the fifteenth or sixteenth magnitude have been measured up to the present and these proper motions furnish the material for the statistical investigations of mean parallax.

6·21. *The υ-components of proper motion.*

We shall assume that the magnitude and direction of the solar motion are known with respect to the magnitude class of stars with which we are concerned. Actually, the solar speed V_0, measured in km./sec., has only been determined from observations of the brightest stars owing to the practical limitations experienced in deriving the radial velocities from the spectra of stars fainter than the sixth or seventh magnitudes. The position of the solar apex, on the other hand, can be derived by the methods of Chapters III, IV or V from the proper motions alone, whatever the magnitudes of the stars observed may be. We shall consider first a single drift of stars and later investigate the modifications, if any, of the various methods resulting from the preferential motions of the stars, using as our basis either the two-streams theory or the ellipsoidal theory.

As in section 3·5 we resolve the annual proper motion of a star at X (Fig. 40) into two components, the first, υ, directed towards the antapex A and the second, τ, perpendicular to the great circle XA. The component υ thus consists of the parallactic component in the direction XA and a part due to the star's peculiar motion. Denote the solar speed by V_0, in km./sec., the angular distance of X from A by λ and the parallax of the star by p; the parallactic component of proper motion is then given by $\frac{pV_0}{\kappa}\sin\lambda$ and we can write

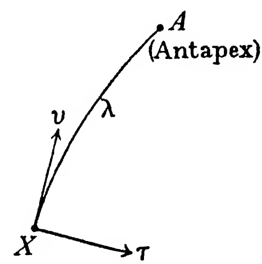

Fig. 40

$$\upsilon = \frac{pV_0}{\kappa}\sin\lambda + \upsilon', \qquad \ldots\ldots(1)$$

where υ' is the part of the observed proper motion (in the direction XA) due to the star's peculiar motion and $\kappa = 4{\cdot}74$.

Let us consider a number of stars, in various parts of the sky, with their peculiar linear velocities distributed according to the Maxwellian law. Assuming first that the parallaxes are the same, the quantities υ' will be distributed according to the law of errors. Consequently, the value of p will be found from (1) by applying the principles of the method of least-squares. In this case, we shall have

$$p = \frac{\kappa}{V_0}\frac{\Sigma\,\upsilon\sin\lambda}{\Sigma\sin^2\lambda}, \qquad \ldots\ldots(2)$$

the summations extending over all the stars. This formula is equivalent to (12) of section 3·5. With the assumptions made, (2) may be expected to give a reliable value of the common parallax, p.

In the more general case, we now assume that the stars, N in number, with which we are dealing can be divided into m groups, in each of which the parallax may be regarded as constant. This is a theoretical subdivision and it is not suggested that we can place a particular star in any one of the groups.

For each star in the first group, containing n_1 stars, and of parallax p_1, we have

$$v = \frac{p_1 V_0}{\kappa} \sin\lambda + v';$$

the value of p_1 as determined from the n_1 stars will be obtained from

$$\frac{p_1 V_0}{\kappa} = \frac{\sum_{n_1} v \sin\lambda}{\sum_{n_1} \sin^2\lambda}. \qquad \text{......(3)}$$

We shall have similar equations for the other groups.

Let $\bar{p}$ denote the mean parallax of the N stars, where $N = n_1 + n_2 + \ldots + n_m$. Assuming that the different groups have the same distribution over the parts of the sky for which data are available, we may take the mean value of $\Sigma \sin^2\lambda$ to be the same for each group and, denoting it by Σ_0, we write (3) as

$$\frac{p_1 V_0}{\kappa} = \frac{\sum_{n_1} v \sin\lambda}{n_1 \Sigma_0}. \qquad \text{......(4)}$$

Now

$$\bar{p} = \frac{n_1 p_1 + n_2 p_2 + \ldots}{N}.$$

Hence, by (4),

$$\frac{\bar{p} V_0}{\kappa} = \frac{\sum_{n_1} v \sin\lambda + \sum_{n_2} v \sin\lambda + \ldots}{N \Sigma_0}$$

or, since $N\Sigma_0 = \sum_N \sin^2\lambda$, in which $\sin^2\lambda$ is summed for all the N stars,

$$\bar{p} = \frac{\kappa}{V_0} \frac{\sum_N v \sin\lambda}{\sum_N \sin^2\lambda}. \qquad \text{......(5)}$$

Formula (5) gives the statistical mean parallax on the assumption that the stars form a single drift. The procedure shows that $\bar{p}$, as derived from (5), is an arithmetical mean.

In many practical applications, the data will come from several small regions (n, say, in number) in different parts of the sky and λ may be taken to be the same for each star in any particular region. If $\bar{v}$ denotes the mean value of v for the stars in a region, the appropriate formula for $\bar{p}$ is

$$\bar{p} = \frac{\kappa}{V_0} \frac{\sum\limits_n \bar{v} \sin \lambda}{\sum\limits_n \sin^2 \lambda}, \qquad \text{......(6)}$$

where the summations are taken over the n regions.

We have now to examine the applicability of (5) when the stars are divided into two drifts, as actually observed.

Consider the space velocities, V_1 and V_2, of drift I and drift II relative to the sun, S. As in section 4·35 they are represented by vectors SA_1, SA_2 (Fig. 41), the lengths of SA_1, SA_2 being proportional to V_1, V_2. If $A_1 A_2$ is divided at O so that

$$OA_1 : OA_2 = N_2 : N_1, \qquad \text{......(7)}$$

where N_1 and N_2 are the numbers of stars in the two drifts, the solar motion, V_0, is given by the vector OS, since O corresponds to the motion of the geometrical centre or centroid of the totality of stars. Also, SO gives the direction of the antapex of the solar motion. Again, the space-velocities W_1 and W_2 of the drifts with respect to O are given by the vectors OA_1 and OA_2, so that, by (7),

$$N_1 W_1 = N_2 W_2. \qquad \text{......(8)}$$

Leaving the random motions of the stars, relative to their appropriate drifts, out of consideration for a moment, we see that the systematic motion along SO of a star belonging to drift I and measured relative to the sun consists of a systematic velocity V_0 along SO and a systematic velocity

$$W_1 \cos A_1 OA.$$

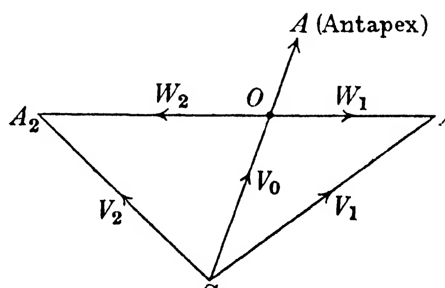

Fig. 41

The sum of such systematic motions of the N_1 stars of drift I is, accordingly,

$$N_1 V_0 + N_1 W_1 \cos A_1 OA.$$

Similarly, the sum of the systematic motions of the N_2 stars of drift II relative to the sun and in the direction SO is

$$N_2 V_0 + N_2 W_2 \cos A_2 OA.$$

Hence, from (8), the sum of the systematic motions, relative to the sun, of the $(N_1 + N_2)$ stars along SO is

$$(N_1 + N_2) V_0.$$

If we consider a small area of the sky, at an angular distance, λ, from the antapex, in which the stars are distributed between the two drifts in the ratio $N_1 : N_2$, it follows that when all the stars belonging to the two drifts are taken together the systematic motion of each star may be taken simply as $V_0 \sin \lambda$.

In addition, we have the random motion of each star in the drifts in the direction of the antapex giving rise to the proper motion component υ' as in (1). In all regions in which the ratio $N_1 : N_2$ holds the probable magnitude of υ' is the same for each region. Where this ratio is not strictly observed the values of υ' for a given region will contain a residuum of systematic motion which may be positive or negative and, consequently, in combining all such regions this effect may be regarded as an accidental error and its effect in deriving (5) from (1) as negligible. We conclude, then, that (5) is valid when streaming is taken into account.

This method, as summarised in (5) or (6), has been extensively used to determine mean parallaxes. It is to be remembered that, in the absence of observational evidence, the solar motion with respect to faint stars, say of magnitudes 9 to 15, is assumed to be the same (namely 19·5 km./sec.) as that found for the naked-eye stars.

6·22. *The τ-components of proper motion.*

Consider, first, a single drift of stars. We shall suppose as before that the magnitude and direction of the solar motion are known. The τ-component of the proper motion of a star will then correspond to its random linear velocity in one coordinate. From the observed radial velocity of a star and the radial component of the parallactic velocity we obtain at once the random radial velocity of the star, and from a large number of stars we find the mean random speed, $\bar{R}$. Since $\bar{R}$ can only be obtained at present from the brighter stars, we have to assume that the observed value of $\bar{R}$ is appropriate to the investigations based on faint stars. Accordingly, for a number of stars in a given region of the sky we take the mean linear speed corresponding to the mean of the τ-components of proper motion to be this value of $\bar{R}$. If the stars are all at the same distance, we have at once

$$p = \kappa\bar{\tau}/\bar{R}, \qquad \ldots\ldots(1)$$

where $\bar{\tau}$ is the mean *arithmetical* value of the τ-components.

Consider now different groups of stars with parallaxes $p_1, p_2, \ldots$. In the ideal case, the value of $\bar{R}$ will be the same for each group on the single drift hypothesis; we shall have, accordingly,

$$p_1 = \kappa\bar{\tau}_1/\bar{R}, \quad p_2 = \kappa\bar{\tau}_2/\bar{R}, \ldots.$$

The mean parallax, $\bar{p}$, will then be given by

$$\bar{p} = \frac{\kappa\bar{\tau}}{\bar{R}}, \qquad \text{......(2)}$$

where $\bar{\tau}$ now denotes the mean of the τ-components, regardless of sign, for all the stars of the various groups.

In a later section (6·41) we investigate the validity (or otherwise) of (2) when account is taken of the preferential motions of the stars.

6·23. *The υ'-residuals.*

A third method of investigating mean parallaxes is based on the values of υ'. For a single drift of stars all having the same parallax p the distribution of the values of υ' must be expected to be the same as that of the τ-components. Consequently, the parallax is given by

$$\bar{p} = \frac{\kappa\overline{\upsilon'}}{\bar{R}}, \qquad \text{......(1)}$$

where $\overline{\upsilon'}$ denotes the mean of the arithmetical values of υ'. It is to be understood that the residuals υ' are found after the parallactic motion has been deduced by any of the known methods. If the stars are at different distances, it is only the mean parallactic motion that can be found and the resulting values of the residuals υ' are, in general, different from the true residuals as deduced from the true parallactic motion for each star. The use of (1) is thus restricted to stars at the same distance or to stars known to be of the same absolute magnitude (such as the stars belonging to a subdivision of spectral class B) or to stars whose absolute magnitudes are known except for an undetermined constant (such as the Cepheids and cluster variables). In these last two classes, the apparent magnitudes enable us to calculate the relative distances of the stars and so to determine what their observed proper motions would be if they were all at some standard distance. Proper motions so adjusted are called *reduced proper motions*.

The effect of preferential motion in the derivation of mean parallaxes by (1) will be investigated subsequently (section 6·44).

6·31. *The mean peculiar radial speed for a small area of the sky.*

In the next few sections we take the ellipsoidal hypothesis as the more convenient mathematical expression for preferential motion and we assume that the individual linear motions are distributed according to the law

$$e^{-K^2U^2-H^2(V^2+W^2)}\,dU\,dV\,dW. \qquad \text{......(1)}$$

The U-axis is directed towards the vertex and $K < H$.

From (4) and (5) of section 5·11, the mean speeds, $\overline{U}$ and $\overline{V}$, parallel to the U and V axes are given by

$$\overline{U} = \frac{1}{K\sqrt{\pi}} \quad \text{and} \quad \overline{V} = \frac{1}{H\sqrt{\pi}} \qquad \ldots\ldots(2)$$

or, writing c and a for $\overline{U}$ and $\overline{V}$ respectively,

$$c = \frac{1}{K\sqrt{\pi}}, \quad a = \frac{1}{H\sqrt{\pi}}, \qquad \ldots\ldots(3)$$

and (1) can be expressed as

$$e^{-\frac{U^2}{\pi c^2}-\frac{V^2+W^2}{\pi a^2}}\, dU\, dV\, dW. \qquad \ldots\ldots(4)$$

It is to be noted that $c > a$.

The intensity of streaming may be defined as c/a and we write

$$\frac{c}{a} = \cosh b. \qquad \ldots\ldots(5)$$

In the absence of preferential motion, we have $c = a$ and consequently $b = 0$. We are concerned only with the ratio of c to a.

Consider a small region of the sky, at an angular distance χ from the vertex A, containing N stars (Fig. 42). For a star at X, the U-component of the preferential motion will be along XY which is parallel to OA; we take the V-axis in the plane of the great circle AX. Thus the W-component does not contribute to the radial preferential motion, R.

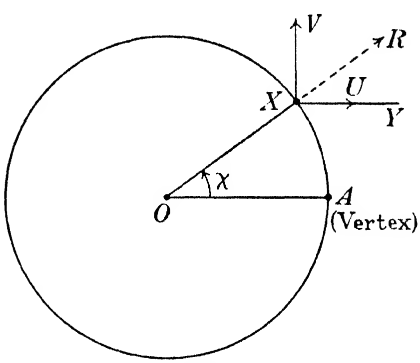

Fig. 42

The mean radial speed, which we now denote by $\overline{R}$, at X is given by formula (10) of section 5·71. With our choice of axes in Fig. 42 the direction-cosines of OX are $\cos\chi$, $\sin\chi$, 0. Hence, from (4),

$$\overline{R}^2 = c^2\cos^2\chi + a^2\sin^2\chi. \qquad \ldots\ldots(6)$$

We can derive this formula directly as follows.

The number, dN, of stars with linear velocities, resolved in the plane of the great circle AX, with components between (U, V) and $(U+dU, V+dV)$ is given by

$$dN = Ce^{-K^2U^2-H^2V^2}\, dU\, dV, \qquad \ldots\ldots(7)$$

where

$$C = \frac{NHK}{\pi}. \qquad \ldots\ldots(8)$$

The radial velocity of any one of these stars is $U\cos\chi + V\sin\chi$ and we require to form the mean value of this quantity, regardless of sign, for all the stars in the region. As we are concerned with speeds, we consider only positive values of U; the values of V, however, can be of either sign. We are thus dealing effectively with $\frac{1}{2}N$ stars. For a given value of U (a positive quantity), the radial velocity will be positive if $V > -U\cot\chi$; hence, the radial speeds, R, can be divided into two groups in the first of which, the radial velocity being positive,

$$R = \sin\chi\,(U\cot\chi + V),$$

where the range of V is given by

$$-U\cot\chi < V < \infty,$$

and, in the second group, the speed R (the corresponding radial velocity is negative) is given by

$$R = \sin\chi\,(V - U\cot\chi),$$

where the range of V is now such that

$$V > U\cot\chi.$$

Forming the sum for all possible values of U between 0 and ∞, we obtain the mean radial speed $\bar{R}$ for the $\frac{1}{2}N$ stars from

$$\tfrac{1}{2}N\bar{R} = C\sin\chi\int_0^\infty dU e^{-K^2U^2}\left\{\int_{-U\cot\chi}^{\infty}(U\cot\chi + V)\,e^{-H^2V^2}\,dV + \int_{U\cot\chi}^{\infty}(V - U\cot\chi)\,e^{-H^2V^2}\,dV\right\}.$$

Let

$$\left.\begin{aligned}\xi &= U\cot\chi\\ \eta &= K\tan\chi\end{aligned}\right\}. \qquad \text{......(9)}$$

Then

$$N\bar{R} = 2C\sin\chi\tan\chi\int_0^\infty d\xi\,e^{-\eta^2\xi^2}\left\{\int_{-\xi}^{\infty}(\xi + V)\,e^{-H^2V^2}\,dV + \int_{\xi}^{\infty}(V - \xi)\,e^{-H^2V^2}\,dV\right\},$$

from which it is found that

$$N\bar{R} = 2C\sin\chi\tan\chi\int_0^\infty d\xi\,e^{-\eta^2\xi^2}\left\{\frac{1}{H^2}e^{-H^2\xi^2} + 2\xi\int_0^{\xi}e^{-H^2V^2}\,dV\right\}$$

$$= 2C\sin\chi\tan\chi\left\{\frac{1}{H^2}\int_0^\infty e^{-(H^2+\eta^2)\xi^2}\,d\xi + 2\int_0^\infty \xi e^{-\eta^2\xi^2}\,d\xi\int_0^{\xi}e^{-H^2V^2}\,dV\right\} \qquad \text{...(10)}$$

$$\equiv \frac{\sqrt{\pi}C\sin\chi\tan\chi}{H^2(H^2+\eta^2)^{\frac{1}{2}}} + 4C\sin\chi\tan\chi I,$$

where I denotes the double integral in (10).

In Fig. 43 let OB bisect the angle between the ξ and V axes (OA and OD). The integral denotes the summation of the function of ξ and V over the infinite area between OA and OB, first summing with respect to V over the strip PQ, where $PQ = OP = \xi$, and then summing all such strips between $\xi = 0$ and $\xi = \infty$. Changing the order of integration, we first sum over the strip RS for ξ between OT ($\equiv V$) and ∞. Hence

$$I = \int_0^\infty e^{-H^2V^2}\,dV \int_V^\infty \xi e^{-\eta^2\xi^2}\,d\xi$$

$$= \frac{1}{2\eta^2}\int_0^\infty e^{-H^2V^2-\eta^2V^2}\,dV$$

$$= \frac{\sqrt{\pi}}{4\eta^2}\frac{1}{(\eta^2+H^2)^{\frac{1}{2}}}.$$

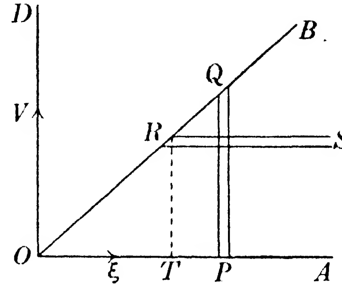

Fig. 43

Hence $N\bar{R} = \sqrt{\pi}C\sin\chi\tan\chi\dfrac{(\eta^2+H^2)^{\frac{1}{2}}}{\eta^2H^2}$,

or, using (3), (8) and (9),

$$\bar{R} = (c^2\cos^2\chi + a^2\sin^2\chi)^{\frac{1}{2}},$$

which is the same as (6).

It is to be noted that the mean peculiar radial speed, $\bar{R}$, is a function of χ.

In the absence of streaming we have $c = a$, so that the mean radial speed is a. Thus the mean peculiar radial speed when there is streaming is greater than the mean peculiar radial speed in a single drift by a factor f, defined by

$$f = \frac{1}{a}(c^2\cos^2\chi + a^2\sin^2\chi)^{\frac{1}{2}}, \qquad \text{......(11)}$$

or, in terms of b in (5), by

$$f = (1+\sinh^2 b\cos^2\chi)^{\frac{1}{2}}. \qquad \text{......(12)}$$

6·32. *The mean peculiar radial speed for uniform distribution over the whole sky.*

Let $\bar{R}_u$ denote the mean peculiar radial speed for stars uniformly distributed over the celestial sphere. Then

$$4\pi\bar{R}_u = \int_0^\pi \bar{R}\,.\,2\pi\sin\chi\,d\chi.$$

We shall write

$$\bar{R}_u = af_u. \qquad \text{......(1)}$$

Then

$$f_u = \int_0^{\pi/2}(1+\sinh^2 b\cos^2\chi)^{\frac{1}{2}}\sin\chi\,d\chi$$

$$\equiv \int_0^1 (1+x^2\sinh^2 b)^{\frac{1}{2}}\,dx$$

or

$$f_u = \tfrac{1}{2}\cosh b + \frac{b}{2\sinh b}. \qquad \text{......(2)}$$

This result was given by Eddington.*

* *Stellar Movements*, 157, 1914.

6·33. *The mean peculiar radial speed for uniform distribution over the galactic equator.*

The various determinations of the vertex of the preferential motions place it in or near the galactic equator; we shall accordingly assume that the galactic latitude of the vertex is zero.

Let $\bar{R}_0$ denote the mean peculiar radial speed in the case under consideration, and let

$$\bar{R}_0 = af_0. \qquad \ldots\ldots(1)$$

For stars on the galactic equator we can regard χ as the galactic longitude measured from the vertex. Consequently

$$2\pi f_0 = \int_0^{2\pi} (1+\sinh^2 b \cos^2\chi)^{\frac{1}{2}}\, d\chi,$$

whence

$$f_0 = \frac{2c}{a\pi}\int_0^{\pi/2} (1-\tanh^2 b \sin^2\chi)^{\frac{1}{2}}\, d\chi. \qquad \ldots\ldots(2)$$

Setting

$$k = \tanh b \equiv \frac{1}{c}(c^2-a^2)^{\frac{1}{2}}, \qquad \ldots\ldots(3)$$

we write (2) as

$$f_0 = \frac{2c}{a\pi}E_2(k), \qquad \ldots\ldots(4)$$

where $E_2(k)$ is the elliptic integral of the second kind with modulus k.

When the intensity (c/a) of streaming is known, f_0 can be obtained from tables of the elliptic integrals. This result was first given by A. Fletcher.*

6·34. *The mean peculiar radial speed for uniform distribution over a parallel of galactic latitude.*

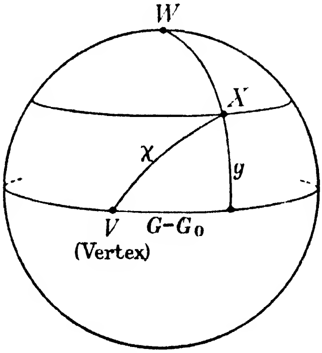

Fig. 44

Consider a small area at X (Fig. 44), whose galactic coordinates are (G, g), at an angular distance χ from the vertex. The mean value, $\bar{R}$, of the radial speeds at X is given by af, where f is defined by (12) of section 6·31.

Also, $\cos\chi = \cos g \cos(G-G_0)$,

where G_0 is the galactic longitude of the vertex.

Writing the small area at X as $\cos g\, dg\, dG$, we have, if $\bar{R}_g$ denotes the mean peculiar radial speed for uniform distribution of the stars between the parallels, g and $g+dg$, of galactic latitude,

$$\bar{R}_g \,.\, 2\pi \cos g\, dg = a \cos g\, dg \int_0^{2\pi} \{1+\sinh^2 b \cos^2 g \cos^2(G-G_0)\}^{\frac{1}{2}}\, dG,$$

* *M.N.* **92**, 780, 1932.

or, setting $$\bar{R}_g = af_g, \qquad \ldots\ldots(1)$$

$$f_g = \frac{2}{\pi}(1+\sinh^2 b\cos^2 g)^{\frac{1}{2}}\,E_2(k_1), \qquad \ldots\ldots(2)$$

where $$k_1^2 = \frac{\sinh^2 b\cos^2 g}{1+\sinh^2 b\cos^2 g}. \qquad \ldots\ldots(3)$$

These equations* may be written more concisely in the form

$$f_g = \frac{2}{\pi}\sec\theta\,E_2(\sin\theta), \qquad \ldots\ldots(4)$$

in which θ is defined by

$$\tan\theta = \sinh b\cos g. \qquad \ldots\ldots(5)$$

6·35. *The "factor of exaggeration" for the mean peculiar radial speeds.*

The results of the three previous sections show that the mean peculiar radial speed, $\bar{R}$, is a function of position on the celestial sphere. It follows that the determination of mean parallaxes from the formulae in sections 6·22 and 6·23 depends on the appropriate value of $\bar{R}$ to be used. For example, B type stars and Cepheids are strongly concentrated towards the plane of the Milky Way and for them the value of $\bar{R}$—assuming that they partake of the usual preferential motion—is considerably greater than the value to be used if streaming were absent. In the formulae of sections 6·22 and 6·23 it is assumed that the mean peculiar linear speed corresponding to the mean of the τ-components and of the υ'-residuals is the same as the mean peculiar radial speed which can be obtained from the observed radial velocities. With star-streaming as a fundamental feature of stellar motions this is no longer true, and, accordingly, we must adjust $\bar{\tau}$ and $\bar{R}$, for example, to the same standard of mean motions; in particular, we take this standard to be that corresponding to a Maxwellian distribution of velocities (that is, a single drift), for in this case the mean linear speeds for different directions are the same. Thus, the mean peculiar radial speed, considered in the previous three sections, is f times greater than it would be if star-streaming were absent.

Fletcher (*loc. cit.*) defines the percentage "exaggeration" of the mean peculiar radial speeds by $100(f-1)$ and writes it as

$$e = 100(f-1). \qquad \ldots\ldots(1)$$

The percentage exaggerations for uniform distribution over the sphere, over the galactic equator and over the parallel of galactic latitude are denoted respectively by e_u, e_0 and e_g. The values of these quantities, calculated from the previous formulae for f, are given in the following two tables.

* A. Fletcher, *M.N.* **92**, 782, 1932.

Table 26. *Values of e_u and e_0*

c/a	e_u	e_0
1·0	0·0	0·0
1·2	6·9	10·2
1·4	14·2	20·8
1·6	21·9	31·7
1·8	29·9	42·9
2·0	38·0	54·2
2·2	46·4	65·7

Table 27. *Values of e_g for different galactic latitudes*

c/a	0°	15°	30°	45°	60°	75°	90°
1·0	0	0	0	0	0	0	0
1·2	10	10	8	5	3	1	0
1·4	21	20	16	11	6	2	0
1·6	32	30	25	17	9	3	0
1·8	43	41	34	24	13	4	0
2·0	54	51	43	31	17	5	0
2·2	66	62	52	38	21	6	0

6·41. *The effect of preferential motion on the mean τ-component for a small area of the sky.*

Consider the stars in a small region of the sky at S; we shall assume at first that their parallaxes are all identical. Let A be the solar antapex and V the vertex of preferential motion (Fig. 45) and let SA and SV be λ and χ respectively. The velocity ellipse of the transverse motions at S has its major axis directed along the great circle SV. We denote the mean arithmetical values of the peculiar *angular* motions of the stars by ξ and η, the former along SV and the latter in a perpendicular direction. Since the stars are all assumed to be at the same distance, ξ and η are proportional to the major and minor axes of the velocity ellipse. The mean of the arithmetical values of the τ-components of the peculiar angular motions is then given by analogy with (6) of section 6·31 by

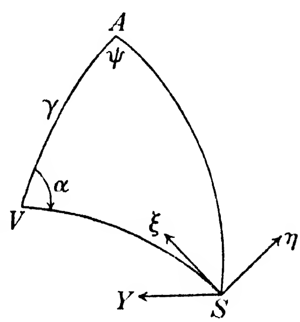

Fig. 45

$$\bar{\tau}^2 = \xi^2 \sin^2 S + \eta^2 \cos^2 S, \qquad \text{......(1)}$$

in which S denotes the angle ASV. Also, by 5·12 (8), η is the same for all parts of the sky; in the absence of streaming the corresponding value of $\bar{\tau}$, which we denote by $\bar{\tau}_0$, is simply η, by (1). Thus, owing to preferential motion the mean τ-component is exaggerated by the factor f', given by

$$f' \equiv \frac{\bar{\tau}}{\eta} = \left\{\frac{\xi^2}{\eta^2} \sin^2 S + \cos^2 S\right\}^{1/2}. \qquad \text{......(2)}$$

Now, from 5·12 (10) we have the relation

$$\left(\frac{H^2}{K^2}-1\right)\sin^2\chi = \frac{h^2}{k^2}-1 \qquad \text{......(3)}$$

between the constants of the velocity ellipsoid and of the velocity ellipse. Since K and H are inversely proportional to the mean speeds, c and a, in the directions of the major and minor axis respectively, we have

$$\frac{H}{K}=\frac{c}{a},$$

and similarly

$$\frac{h}{k}=\frac{\xi}{\eta}.$$

Hence, from (3),

$$\frac{\xi^2}{\eta^2} = 1+\left(\frac{c^2}{a^2}-1\right)\sin^2\chi = 1+\sinh^2 b\,\sin^2\chi, \qquad \text{......(4)}$$

and (2) becomes $\qquad f' = \{1+\sinh^2 b\,\sin^2\chi\,\sin^2 S\}^{1/2}.$

But $\qquad \sin\chi\,\sin S = \sin\gamma\,\sin\psi,$

where ψ is the angle VAS, and γ is the angular distance of the solar antapex from the vertex and may be supposed known; its value is in the neighbourhood of 45°. Thus

$$f' = \{1+\sinh^2 b\,\sin^2\gamma\,\sin^2\psi\}^{1/2}. \qquad \text{......(5)}$$

It is seen that f' is a function of a single coordinate ψ, which Fletcher,* to whom the above result is due, designates the apical longitude of the region at S. Thus f' is the same for all points on the meridian of apical longitude ψ.

Consider another group of equidistant stars at S with a different value of the parallax. Then the factor f' will be the same for this new group as for the previous group, since the expression on the right of (5) is independent of the parallax. It follows that, for an assembly of stars at S, the value of $\bar{\tau}$ obtained from all the stars will be greater by the factor f' than the corresponding value, $\bar{\tau}_0$, which would have been obtained if the assembly formed a single drift.

6·42. *The effect of preferential motion on the mean τ-component for stars distributed uniformly over the sky.*

The corresponding factor, which we denote by f'_u, is simply obtained by integrating 6·41 (5) over the sphere, that is, between the values 0 and 2π of ψ. Thus

$$f'_u = \frac{1}{2\pi}\int_0^{2\pi}\{1+\sinh^2 b\,\sin^2\gamma\,\sin^2\psi\}^{1/2}\,d\psi, \qquad \text{......(1)}$$

or, as in sections 6·33 and 6·34,

$$f'_u = \frac{2}{\pi}\sec\theta' E_2(\sin\theta'), \qquad \text{......(2)}$$

where θ' is given by $\qquad \tan\theta' = \sinh b\,\sin\gamma. \qquad \text{......(3)}$

* *Loc. cit.* p. 782.

The values of the mean percentage exaggeration, e'_u, which is defined by

$$e'_u = 100(f'_u - 1),$$

are given in Table 28 for the three values 35°, 45° and 55° of γ.

Table 28. *Values of* e'_u

c/a	$\gamma = 35°$	$\gamma = 45°$	$\gamma = 55°$
1·0	0	0	0
1·2	4	5	7
1·4	7	11	15
1·6	12	17	22
1·8	16	24	31
2·0	21	31	39
2·2	26	38	48

6·43. *The effect of preferential motion on the mean τ-components for stars distributed uniformly over the galactic equator.*

In Fig. 45 let S now be a region on the galactic equator which will thus be defined by VS. Denote the angle AVS by α, which may be supposed known from the positions of A and V derived from observations.

If f'_0 denotes the mean value of f round the galactic equator,

$$f'_0 = \frac{1}{2\pi}\int_0^{2\pi} \{1 + \sinh^2 b \sin^2\gamma \sin^2\psi\}^{1/2} d\chi, \qquad \text{......(1)}$$

in which ψ and χ are related by the formula, derived from the triangle ASV,

$$\cos\gamma \cos\alpha = \sin\gamma \cot\chi - \sin\alpha \cot\psi. \qquad \text{......(2)}$$

Writing
$$\cot\chi = l \cot\psi + m, \qquad \text{......(3)}$$

where
$$l = \sin\alpha \operatorname{cosec}\gamma, \quad m = \cos\alpha \cot\gamma, \qquad \text{......(4)}$$

and expressing $d\chi$ in terms of ψ by (3), we obtain

$$f'_0 = \frac{1}{\pi}\int_0^{\pi} \frac{p\cos^4\psi + q\cos^2\psi + r}{f\cos^4\psi + g\cos^2\psi + h} \cdot \frac{l\, d\psi}{(1+\kappa^2\sin^2\psi)^{1/2}}, \qquad \text{......(5)}$$

where
$$\kappa^2 = \sinh^2 b \sin^2\gamma,$$
$$p = \kappa^2(1 - l^2 + m^2),$$
$$q = l^2(1+\kappa^2) - (1+m^2)(1+2\kappa^2),$$
$$r = (1+m^2)(1+\kappa^2),$$
$$f = 1 - 2l^2 + 2m^2 + (l^2+m^2)^2,$$
$$g = 2\{l^2 - l^2m^2 - (1+m^2)^2\},$$
$$h = (1+m^2)^2.$$

Further, setting

$$k^2 = \kappa^2/(1+\kappa^2), \quad k_1^2 = (1-k^2)^{1/2}, \quad \phi = \frac{\pi}{2} - \psi,$$

we obtain from (5)

$$f_0' = \frac{2lk_1}{\pi} \int_0^{\pi/2} \frac{p \sin^4 \phi + q \sin^2 \phi + r}{f \sin^4 \phi + g \sin^2 \phi + h} \cdot \frac{d\phi}{(1-k^2 \sin^2 \phi)^{1/2}}. \qquad \ldots\ldots(6)$$

By an approximate method Fletcher was able to obtain numerical values of f_0' from (1) corresponding to $\gamma = 45°$ and $\alpha = 35°$, values given by the most reliable positions of the antapex and vertex. Subsequently, Fletcher and A. Mulligan,* using the formula (6), calculated the exact values of f_0' for $c/a = 1·6$ and $2·0$. As these agreed sufficiently well with the values obtained by the approximate method, they concluded that it was unnecessary to repeat the long calculations involved in the formula (6). Write, in accordance with previous procedure,

$$e_0' = 100(f_0' - 1).$$

The values of e_0', as found by Fletcher, are given in Table 29.

Table 29. *Values of* e_0'

($\gamma = 45°$, $\alpha = 35°$)

c/a	e_0'	c/a	e_0'
1·0	0	1·6	13
1·2	4	1·8	18
1·4	8	2·0	23
1·6	13	2·2	29

6·44. *The v′-residuals.*

Assuming that the stars in a region of the sky at S have the same parallax we find, as in section 6·41, that the mean of the arithmetical values of v' is given by

$$\overline{v'}^2 = \xi^2 \cos^2 S + \eta^2 \sin^2 S,$$

and denoting by f'' the factor of exaggeration, we have as before

$$f'' = \overline{v'}/\eta,$$

so that

$$f'' = \{1 + \sinh^2 b \sin^2 \chi \cos^2 S\}^{1/2}.$$

Define e_u'' and e_0'' by means of

$$e_u'' = 100(f_u'' - 1), \quad e_0'' = 100(f_0'' - 1),$$

where f_u'' and f_0'' are respectively the mean values of f over the sphere and over the galactic equator, for uniform distribution in each case. The following table computed by Fletcher by a method of approximation gives the values of e_u'' and e_0'' for $\gamma = 35°$ and $\alpha = 35°$:

* *M.N.* **95**, 737, 1935.

Table 30. *Values of* e''_u *and* e''_0

c/a	e_u''	e_0''
1·0	0	0
1·2	9	7
1·4	18	13
1·6	27	21
1·8	37	28
2·0	47	36
2·2	57	44

6·45. *Application to the calculation of mean parallaxes.*

In deriving the mean arithmetical values of the υ and τ-components in a small area of the sky for insertion in the formulae (6) of section 6·21 and (2) of section 6·22, we have to remember that the components of proper motion are subject to accidental errors; consequently, it is essential, if the highest accuracy is aimed at, to correct the statistics for the effect of accidental errors. Generally, in the process of determining proper motions, it is possible to estimate the values, ρ, of the probable error for the $\mu_\alpha \cos\delta$ and μ_δ components, and usually these two values are equal or approximately equal. If we take them to be the same, the probable errors of the υ-components and of the τ-components are each ρ numerically and the observed distribution of the υ-components, for example, can be corrected by the method of 1·81 or 1·82. We then obtain the true distribution of the υ-components and from it the value of $\bar{\upsilon}$ to be used in (6) of section 6·21. We obtain similarly the value of $\bar{\tau}$ to be used in (2) of section 6·22.

We shall denote by $\bar{\tau}_0$ and $\bar{R}_0$ the means of the arithmetical values of the τ-components and of the radial speeds (freed from the solar motion) as observed and corrected for accidental errors. If we insert these values in (2) of section 6·22, we obtain the "calculated" value of the mean parallax, which we denote by $\bar{p}_c$. Thus

$$\bar{p}_c = \kappa\bar{\tau}_0/\bar{R}_0. \qquad \text{......(1)}$$

But the use of this formula implies, as we have seen, that the mean linear speed corresponding to the τ-components is the same as the mean radial speed; thus, on the hypothesis of preferential motion, the application of (1) gives an erroneous value of the mean parallax. If we denote by $\bar{p}_t$ the true value of the mean parallax, we have

$$\bar{p}_t = \kappa\bar{\tau}/\bar{R},$$

where $\bar{\tau}$ and $\bar{R}$ refer to the absence of preferential motion. Now, by the previous sections,

$$\bar{R}_0 = f\bar{R} \quad \text{and} \quad \bar{\tau}_0 = f'\bar{\tau},$$

where f and f' refer to any one of the distributions of the stars in the sky. Accordingly,

$$\bar{p}_t = \frac{f}{f'} \cdot \bar{p}_c. \qquad \ldots\ldots(2)$$

Thus, the true value of the mean parallax is obtained by applying the appropriate factor f/f' to the value as computed by (2) of section 6·22; for example, if the stars are distributed uniformly over the galactic equator the appropriate formula is

$$\bar{p}_t = \frac{f_0}{f'_0} \cdot \bar{p}_c.$$

In the case of the υ'-residuals we obtain in a similar way

$$\bar{p}_t = \frac{f}{f''} \cdot \bar{p}_c, \qquad \ldots\ldots(3)$$

where f, f'' refer to one of the distribution of stars considered.

It is to be emphasised that the f factors only apply to the formulae for $\bar{p}$ derived from *peculiar* motions; the formulae (5) and (6) of section 6·21 for $\bar{p}$ derived from *parallactic* motions remain valid when there is streaming.

6·46. *The corrections to absolute magnitude.*

It is sometimes convenient, in the application of the previous formulae, to deal with absolute magnitudes. Let $\bar{m}$ denote the mean apparent magnitude of the stars concerned and $\bar{M}_c$ the mean absolute magnitude corresponding to the value $\bar{p}_c$ as obtained from (1) of section 6·45. Then

$$\bar{M}_c = \bar{m} + 5 + 5\,\mathrm{Log}\,\bar{p}_c.$$

Similarly, the true absolute magnitude, $\bar{M}_t$, is given by

$$\bar{M}_t = \bar{m} + 5 + 5\,\mathrm{Log}\,\bar{p}_t.$$

Hence, writing $\Delta M' = \bar{M}_t - \bar{M}_c,$

we have from the τ-components

$$\Delta M' = 5\,\mathrm{Log}\,(f/f').$$

In a similar way, if $\Delta M''$ denotes the corresponding correction for absolute magnitude when the υ'-residuals are employed, we obtain

$$\Delta M'' = 5\,\mathrm{Log}\,(f/f'').$$

We can thus calculate from the data of Tables 26–30 the values of $\Delta M'_u$, $\Delta M'_0$ and $\Delta M''_u$, $\Delta M''_0$ corresponding to uniform distribution over the sky and over the galactic equator. The results* are summarised in the following tables (for various values of γ and $\alpha = 35°$):

* Fletcher and Mulligan, *M.N.* **95**, 741, 1935.

Table 31. *Corrections to absolute magnitude* (τ*-components*)

c/a	$\Delta M_u'$			$\Delta M_0'$ ($\gamma = 45°$, $\alpha = 35°$)
	$\gamma = 35°$	$\gamma = 45°$	$\gamma = 55°$	
	M	M	M	M
1·0	0·00	0·00	0·00	0·00
1·2	+0·07	+0·03	0·00	+0·13
1·4	+0·13	+0·06	−0·01	+0·24
1·6	+0·19	+0·08	−0·01	+0·33
1·8	+0·24	+0·10	−0·01	+0·41
2·0	+0·28	+0·12	−0·02	+0·48
2·2	+0·32	+0·13	−0·02	+0·55

Table 32. *Corrections to absolute magnitude* (υ'*-residuals*)

c/a	$\Delta M_u''$			$\Delta M_0''$ ($\gamma = 45°$, $\alpha = 35°$)
	$\gamma = 35°$	$\gamma = 45°$	$\gamma = 55°$	
	M	M	M	M
1·0	0·00	0·00	0·00	0·00
1·2	−0·07	−0·03	0·00	+0·07
1·4	−0·12	−0·06	0·00	+0·14
1·6	−0·17	−0·09	0·00	+0·19
1·8	−0·21	−0·11	−0·01	+0·24
2·0	−0·24	−0·13	−0·01	+0·27
2·2	−0·27	−0·15	−0·01	+0·31

For normal streaming the ratio of the minor axis of the velocity spheroid to the major axis is found to be about 0·6, so that c/a is approximately 1·7. Taking the usually accepted value of γ (the angular distance between the vertex and the antapex) to be 45°, the tables show that for stars uniformly distributed over the sky the correction to the absolute magnitude is fairly small and of opposite sign for the τ and υ' formulae, so that the average of the mean parallaxes derived from these formulae requires practically no correction. But when the stars are concentrated on or near the galactic equator, the correction is substantial.

The most important application concerns the mean parallaxes of the Cepheids or of the cluster variables on which are based the estimates of the distances of the globular clusters and extragalactic nebulae in which these objects are found. The galactic Cepheids are, almost without exception, too remote for the successful determination of parallax by the trigonometrical method, and it is by means of their proper motions that estimates of their mean distances can be made. From observations of these variables in a particular star cluster it is found that the periods of light-variation are related in a definite way to the mean apparent magnitudes and, as we can assume for practical purposes that all the stars in the cluster are at the same distance from us, it follows that the relation can be expressed as being between the periods and the relative luminosities. This relation is known as the *period-luminosity law* and it is assumed to hold for all Cepheids, whether in the galactic system or elsewhere. If we can find accurately the parallax of a single Cepheid, relative luminosities can be converted into absolute magnitudes and the law can then be expressed as a relationship between period and absolute magnitude. In this way, the distance of a remote cluster or nebula, containing one or more of these variables, can be estimated, for the observed periods fix the absolute magnitudes, M, and if the apparent magnitudes, m, are measured the parallax is deduced from the formula

$$M = m + 5 + 5 \operatorname{Log} p.$$

More strictly, the period-luminosity relationship involves the period P, the bolometric absolute magnitude M and the effective temperature T_e, and as given by Jeans* its mathematical expression is

$$\operatorname{Log} P + 0{\cdot}23M + 3 \operatorname{Log} T_e = C, \qquad \ldots\ldots(1)$$

where C is a constant whose value has been determined to be 11·35. The evaluation of C depends, of course, on the evaluation of the parallax of at least one star, so that the corresponding value of M can be obtained.

Consider a group of galactic Cepheids at varying distances and, for simplicity, suppose that the values of T_e are the same for all these stars. By measuring the periods P_1 and P_2 of any two stars, we have

$$0{\cdot}23\,(M_1 - M_2) = \operatorname{Log}(P_2/P_1), \qquad \ldots\ldots(2)$$

from which the difference of their absolute magnitudes can be calculated. If we suppose that the second star is moved to the same distance from us as the first star, the apparent magnitude, m_2', of the second star would then be given by

$$M_2 = m_2' + 5 + 5 \operatorname{Log} p_1,$$

and, since

$$M_1 = m_1 + 5 + 5 \operatorname{Log} p_1,$$

we obtain from these and (2),

$$m_2' = m_1 - \frac{1}{0{\cdot}23} \operatorname{Log}(P_2/P_1). \qquad \ldots\ldots(3)$$

Thus, regarding the parallax p_1 to correspond to a standard distance d_1, we can find from (3) the apparent magnitude of the second star if it were situated at this standard distance. But its apparent magnitude is actually m_2 and the relation between its actual distance d_2 and the standard distance d_1 is given by

$$m_2 - m_2' = 5 \operatorname{Log}(d_2/d_1). \qquad \ldots\ldots(4)$$

The observed proper motion of the second star corresponds of course to its actual distance, but by multiplying the proper motion by the factor

* *Astronomy and Cosmogony* (2nd ed.), 386, 1929.

d_2/d_1 as calculated from (4) we obtain the proper motion which the second star would have if it were at the distance d_1. Such a proper motion is called a *reduced proper motion*. In this way we can deal with a group of stars which may now be all supposed to be at the same distance, and the distribution of the reduced proper motions is then the distribution of the transverse linear velocities. The application of the formula (3) of section 6·45 in the case of the v'-residuals is then valid. It is to be noted however that the introduction of the factor d_2/d_1 may have serious effects on the probable errors of the reduced proper motions unless the observed proper motions have been determined with almost complete accuracy.

It is by such processes as have been described in the previous sections of this chapter that the Cepheids can be arranged on the usual absolute magnitude scale.

6·5. *Formula for the mean parallax derived from the total proper motions.*

We denote by μ the total proper motion of a star so that, with the usual notation,

$$\mu^2 = \mu_\alpha^2 \cos^2 \delta + \mu_\delta^2.$$

In the case of photographic proper motions the values of $\mu_\alpha \cos \delta$ are found directly from the measures. The values of μ can be rapidly obtained by plotting $\mu_\alpha \cos \delta$ and μ_δ on squared paper.

In section 2·42 we derived the formula for the mean transverse linear speed, T_2, for a small region of the sky when the stars concerned form a single drift; it is

$$T_2 = \overline{T}\psi(b), \qquad \text{......(1)}$$

where $\overline{T}$ is the mean random transverse speed, $b = \frac{1}{2}h^2V^2$, in which hV is the projection of the space-velocity of the drift, relative to the sun, on the tangent plane for the small region and ψ is the function

$$\psi(b) = e^{-b}\{(1+2b)\,I_0(b) + 2bI_1(b)\}, \qquad \text{......(2)}$$

whose values are given in Table 7 (p. 43).

Basing our procedure on the two-streams theory we shall have, for the given region, the proportion, α, of the total number of stars belonging to drift I and the proportion $(1-\alpha)$ belonging to drift II. If hV_1 and hV_2 are the projections of the space-velocities of the two drifts on the tangent plane, the mean transverse linear speed, T, of all the stars concerned will be given, by means of (1), by

$$T = \overline{T}\{\alpha\psi(b_1) + (1-\alpha)\,\psi(b_2)\}, \qquad \text{......(3)}$$

where

$$b_1 = \tfrac{1}{2}h^2V_1^2, \quad b_2 = \tfrac{1}{2}h^2V_2^2. \qquad \text{......(4)}$$

Suppose at first that all the stars have the same parallax, p. Then, if $\bar{\mu}$ is the mean of the total proper motions, T is given in km./sec. by

$$T = \frac{\kappa\bar{\mu}}{p}, \qquad \ldots\ldots(5)$$

where $\kappa = 4{\cdot}74$.

Hence, from (3) and (5),

$$p = \frac{\kappa\bar{\mu}}{\bar{T}\{\alpha\psi(b_1)+(1-\alpha)\,\psi(b_2)\}}. \qquad \ldots\ldots(6)$$

Also, from section 2·24, $$\bar{T} = \frac{\sqrt{\pi}}{2h} \qquad \ldots\ldots(7)$$

in terms of the theoretical unit $1/h$.

We shall suppose that from an analysis of several regions of the sky the various drift constants have been derived, as described in Chapter IV. To particularise, we assume that the following are known for each region:

(i) α, (ii) hV_1 and hV_2, (iii) the value of the solar motion in terms of the theoretical unit $1/h$. From (ii) we obtain the values of b_1 and b_2 to be used in (6), and from (iii) and the value of the solar motion in km./sec., as found from radial velocity measures, we can express the theoretical unit in km./sec. and so obtain $\bar{T}$ in km./sec. by means of (7).

Thus with $\bar{\mu}$ given by the observations we can calculate p by means of (6).

In the general case with N stars in the region divided up into m groups, containing n_1, n_2, ... n_m stars, the stars in each group having the same parallax, the mean parallax, $\bar{p}$, of all the N stars is given by

$$N\bar{p} = n_1p_1+n_2p_2+\ldots+n_mp_m.$$

If each group is supposed to be a representative sample of the two drifts, we see from (6), inasmuch as the denominator of (6) is constant for each group, that

$$\bar{p} = \frac{\kappa\bar{\mu}}{\bar{T}\{\alpha\psi(b_1)+(1-\alpha)\,\psi(b_2)\}}, \qquad \ldots\ldots(8)$$

where $\bar{\mu}$ now denotes the mean total proper motion of the N stars. In the application of this formula we can, if we like, restrict the stars to those lying within prescribed limits of magnitude.

It is to be remembered that the value of $\bar{\mu}$ will be affected by the accidental errors inherent in the measurement of the proper motion components; accordingly, it will be necessary to use the corrected value of $\bar{\mu}$ in (8). We now investigate the methods of allowing for accidental errors in the total proper motions.

6·61. *The correction of the total proper motions (first method).*

We consider here the statistics of a quantity which is given as the resultant of two rectangular components, the measurement of each of which is subject to the Gaussian law of errors. In particular, we take the quantity to be the total proper motion μ given by

$$\mu^2 = \xi^2 + \eta^2, \qquad \ldots\ldots(1)$$

where $\xi = \mu_\alpha \cos\delta$ and $\eta = \mu_\delta$, so that an error in μ depends on the error in ξ and the error in η. We shall assume that the quantities ξ and η are measured with the same probable error r; accordingly, the modulus, h, in the Gaussian error law is given by

$$h = \frac{0{\cdot}4769}{r}. \qquad \ldots\ldots(2)$$

Following Kapteyn and van Rhijn,* to whom the method to be described is due, we denote by $Nf(\mu)\,d\mu$ the number of stars with values of the *true* total proper motion between μ and $\mu + d\mu$, N being the total number of stars concerned.

Let $N\phi(\mu_0)\,d\mu_0$ denote the number of stars with values of the *observed* total proper motion between μ_0 and $\mu_0 + d\mu_0$. The observational statistics will furnish the form of the frequency function $\phi(\mu_0)$ and we require to deduce the true frequency function $f(\mu)$.

In Fig. 46 let OS define the true total proper motion μ of a star, the coordinates of S being (ξ, η) with respect to the axes OA, OB. As a result of errors in ξ and η the observed total proper motion, μ_0, is represented by a radius vector such as OT. Let (x, y) denote the coordinates of T with respect to the axes OX, OY as shown in the figure, OX being drawn through S. Then

$$\mu_0^2 = x^2 + y^2.$$

Thus the error in μ_0 is due to error components $(x-\mu)$ along OX and y along OY. It follows from the equality of the probable errors for ξ and η that the probable errors for measures with respect to the axes OX and OY are also equal and that the modulus in each coordinate is h, as given by (2). Hence, the probability that, in the direction of the OX axis, the observational error will lie between $(x-\mu)$ and $(x-\mu+dx)$ is

$$\frac{h}{\sqrt{\pi}} e^{-h^2(x-\mu)^2}\,dx.$$

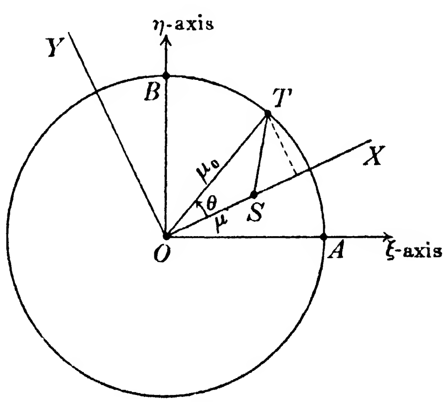

Fig. 46

* *Groningen Publ.* No. 30, 44, p. 1920.

Similarly, the probability that, in the direction of the OY axis, the observational error will be between y and $y+dy$ is

$$\frac{h}{\sqrt{\pi}}e^{-h^2y^2}\,dy.$$

Hence the probability that the observed total proper motion will have component errors lying between $(x-\mu)$ and $(x-\mu+dx)$ and between y and $y+dy$ is

$$\frac{h^2}{\pi}e^{-h^2\{(x-\mu)^2+y^2\}}\,dx\,dy.$$

Transforming to polar coordinates (μ_0, θ), so that

$$(x-\mu)^2+y^2 = \mu_0^2-2\mu\mu_0\cos\theta+\mu^2$$

and
$$dx\,dy = \mu_0 d\mu_0 d\theta,$$

we find the probability that the observed total proper motion will have values between μ_0 and $\mu_0+d\mu_0$ lying in directions between θ and $\theta+d\theta$ to be

$$\frac{h^2}{\pi}\mu_0\,e^{-h^2(\mu_0{}^2+\mu^2-2\mu\mu_0\cos\theta)}\,d\mu_0\,d\theta,$$

and consequently the probability that the total proper motion lies between μ_0 and $\mu_0+d\mu_0$ for all possible values of θ from 0 to 2π is $\zeta\,d\mu_0$, where

$$\zeta\,d\mu_0 = \frac{2h^2}{\pi}\mu_0\,e^{-h^2(\mu_0{}^2+\mu^2)}\,d\mu_0\int_0^{\pi}e^{2h^2\mu_0\mu\cos\theta}\,d\theta. \qquad \ldots\ldots(3)$$

The integral on the right is expressible in terms of the modified Bessel function of zero order and of imaginary argument as

$$\pi I_0(2h^2\mu_0\mu).$$

We can then write (3) as

$$\zeta\,d\mu_0 = 2h^2\mu_0\,e^{-h^2(\mu_0{}^2+\mu^2)}\,I_0(2h^2\mu_0\mu)\,d\mu_0. \qquad \ldots\ldots(4)$$

Consider now, for a given value of μ, the number $Nf(\mu)\,d\mu$ of stars with values of the true proper motion between μ and $\mu+d\mu$. Then the corresponding number with observed values between μ_0 and $\mu_0+d\mu_0$ is

$$Nf(\mu)\,d\mu\,.\,\zeta\,d\mu_0,$$

where $\zeta\,d\mu_0$ is given by (4). Summing for all possible values of μ between 0 and ∞, we obtain the total number of stars with observed values of the total proper motions between μ_0 and $\mu_0+d\mu_0$; this number is, by definition, $N\phi(\mu_0)\,d\mu_0$. Hence

$$N\phi(\mu_0)\,d\mu_0 = N\,d\mu_0\int_0^{\infty}\zeta f(\mu)\,d\mu;$$

from this result and (4)

$$\phi(\mu_0) = 2h^2\mu_0\,e^{-h^2\mu_0{}^2}\int_0^{\infty}I_0(2h^2\mu_0\mu)f(\mu)\,e^{-h^2\mu^2}\,d\mu. \qquad \ldots\ldots(5)$$

Theoretically, given the form of $\phi(\mu_0)$, we can determine the form of $f(\mu)$ from this integral equation. But, in the general case, it is evident that the difficulties in the way of solving for $f(\mu)$ would be at least formidable and possibly unsurmountable. Instead, we can find the form of the function $\phi(\mu_0)$ for assumed functional forms of $f(\mu)$. In particular, the integral on the right of (5) can be evaluated in simple form if $f(\mu)$ is given by

$$f(\mu) = A\mu e^{-\beta^2\mu^2}, \qquad \ldots\ldots(6)$$

the constant A being determined from the relation

$$\int_0^\infty Nf(\mu)\,d\mu = N,$$

which leads to

$$A\int_0^\infty \mu e^{-\beta^2\mu^2}\,d\mu = 1,$$

whence

$$A = 2\beta^2 \qquad \ldots\ldots(7)$$

and

$$f(\mu) = 2\beta^2\mu e^{-\beta^2\mu^2}. \qquad \ldots\ldots(8)$$

By invoking the principles of section 4·5, it is clear that $\phi(\mu_0)$ will be given by a function similar to that in (8), with a modulus k defined by

$$\frac{1}{k^2} = \frac{1}{\beta^2} + \frac{1}{h^2}.$$

We verify this result by direct integration of (5).

We then have

$$\phi(\mu_0) = 4h^2\beta^2\mu_0 e^{-h^2\mu_0^2}\int_0^\infty \mu e^{-(h^2+\beta^2)\mu^2} I_0(2h^2\mu\mu_0)\,d\mu. \qquad \ldots\ldots(9)$$

Write

$$H^2 = h^2+\beta^2, \quad y = 2h^2\mu_0. \qquad \ldots\ldots(10)$$

If K denotes the integral in (9), we have

$$K = \int_0^\infty \mu e^{-H^2\mu^2} I_0(\mu y)\,d\mu. \qquad \ldots\ldots(11)$$

Now

$$I_0(\xi) \equiv J_0(i\xi) = \sum_0^\infty \frac{\xi^{2n}}{2^{2n}(n\,!)^2}. \qquad \ldots\ldots(12)$$

Hence

$$K = \int_0^\infty \sum_0^\infty \frac{y^{2n}}{2^{2n}(n\,!)^2}\mu^{2n+1} e^{-H^2\mu^2}\,d\mu.$$

We have the formula

$$\int_0^\infty x^{2n+1} e^{-H^2x^2}\,dx = \frac{1}{2}\cdot\frac{n\,!}{H^{2n+2}} \qquad \ldots\ldots(13)$$

(which may be derived by differentiating n times the uniformly convergent integral

$$\int_0^\infty xe^{-ax^2}\,dx = \frac{1}{2a} \text{ with respect to } a)$$

and, consequently,
$$K = \frac{1}{2H^2}\sum_0^\infty \frac{1}{n!}\left(\frac{y^2}{4H^2}\right)^n$$
$$= \frac{1}{2H^2}e^{y^2/4H^2}$$

or
$$K = \frac{1}{2H^2}e^{\frac{h^4\mu_0^2}{h^2+\beta^2}}. \qquad \ldots\ldots(14)$$

Hence
$$\phi(\mu_0) = \frac{2h^2\beta^2\mu_0}{h^2+\beta^2}e^{-\frac{h^2\beta^2}{h^2+\beta^2}\mu_0^2}$$

or, defining k by
$$k^2 = \frac{h^2\beta^2}{h^2+\beta^2}, \qquad \ldots\ldots(15)$$
$$\phi(\mu_0) = 2k^2\mu_0 e^{-k^2\mu_0^2}. \qquad \ldots\ldots(16)$$

Thus $\phi(\mu_0)$ is of the same analytical form as $f(\mu)$.

We conclude that if the observed frequency function of the total proper motions is given by (16), in which k may now be supposed known, the true frequency function $f(\mu)$ is given by
$$f(\mu) = 2\beta^2\mu e^{-\beta^2\mu^2},$$

where
$$\beta^2 = \frac{h^2k^2}{h^2-k^2}. \qquad \ldots\ldots(17)$$

6·62. *Derivation of the theoretical correction.*

It is found in practice that the observed distribution of the total proper motions can be represented satisfactorily by one or more functions of the form given by (16) in the previous section. In the general case, we can consequently assume that
$$\phi(\mu_0) = 2A_1k_1^2\mu_0 e^{-k_1^2\mu_0^2} + 2A_2k_2^2\mu_0 e^{-k_2^2\mu_0^2} + \ldots, \qquad \ldots\ldots(1)$$
where the constants $A_1, A_2, \ldots, k_1, k_2, \ldots$ are determined from the statistics of the observed total proper motions. In consequence, the distribution of the true total proper motions is given by
$$f(\mu) = 2A_1\beta_1^2\mu e^{-\beta_1^2\mu^2} + 2A_2\beta_2^2\mu e^{-\beta_2^2\mu^2} + \ldots, \qquad \ldots\ldots(2)$$

in which
$$\beta_1^2 = \frac{h^2k_1^2}{h^2-k_1^2}, \quad \beta_2^2 = \frac{h^2k_2^2}{h^2-k_2^2}, \ldots. \qquad \ldots\ldots(3)$$

Actually, it is more satisfactory to determine the constants $A_1, A_2, \ldots, k_1, k_2, \ldots$ from a function $\Phi(\mu_0)$ which gives the proportion of the observed total proper motions greater than a given value μ_0. Thus
$$\Phi(\mu_0) = \int_{\mu_0}^\infty \phi(\mu_0)\,d\mu_0,$$

which leads to
$$\Phi(\mu_0) = A_1e^{-k_1^2\mu_0^2} + A_2e^{-k_2^2\mu_0^2} + \ldots \qquad \ldots\ldots(4)$$

with the condition, since $\Phi(0) = 1$,
$$A_1 + A_2 + \ldots = 1.$$

I have found it more convenient to work with centennial proper motions, and in the tabular matter and numerical work that follow, the century will be taken as the unit of time.

Table 33 gives the values of the function $e^{-k^2\mu_0^2}$ for different values of k and for values of the observed centennial proper motions between $\mu_0 = 0''\cdot0$ and $\mu_0 = 8''\cdot0$.

Table 33. *Values of* $e^{-k^2\mu_0^2}$

μ_0 \ k	0·1	0·2	0·3	0·4	0·5	0·6
0″0	1·000	1·000	1·000	1·000	1·000	1·000
0·5	0·998	0·990	0·978	0·961	0·939	0·914
1·0	0·990	0·961	0·914	0·852	0·779	0·698
1·5	0·978	0·914	0·817	0·698	0·570	0·445
2·0	0·961	0·852	0·698	0·527	0·368	0·237
2·5	0·939	0·779	0·570	0·368	0·210	0·105
3·0	0·914	0·698	0·445	0·237	0·105	0·039
3·5	0·885	0·613	0·332	0·141	0·047	0·012
4·0	0·852	0·527	0·237	0·077	0·018	0·003
5·0	0·779	0·368	0·105	0·018	0·002	
6·0	0·698	0·237	0·039	0·003		
7·0	0·613	0·141	0·012	0·000		
8·0	0·527	0·077	0·003			

In the following table the values of β as a function of k and of the probable error, r, of the centennial proper motion components, $\mu_\alpha \cos\delta$ and μ_δ, are given for numerical values of r between $0''\cdot2$ and $1''\cdot0$ per century; the value of h corresponding to a particular value of r is calculated first from (2) of 6·61 and with this value of h, β is obtained from (17) of section 6·61 for each value of k between 0·1 and 0·7.

Table 34. *Values of* β

k \ r	0″2	0″3	0″4	0″5	0″6	0″7	0″8	0″9	1″0
0·1	0·100	0·100	0·100	0·100	0·101	0·101	0·102	0·102	0·102
0·2	0·201	0·201	0·203	0·205	0·207	0·209	0·213	0·216	0·220
0·3	0·303	0·306	0·311	0·316	0·324	0·334	0·347	0·364	0·386
0·4	0·405	0·413	0·426	0·441	0·463	0·494	0·539	0·609	0·731
0·5	0·512	0·527	0·552	0·593	0·643	0·737	0·920	1·506	
0·6	0·620	0·649	0·694	0·771	0·914				
0·7	0·732	0·784	0·865	1·030					

When the constants $A_1, A_2, \ldots, k_1, k_2, \ldots$ have been obtained, the *theoretical* correction—which we denote by c—to be applied to the mean of the observed total proper motions to give the mean of the true total proper motions is easily derived.

We have
$$\bar{\mu} = \int_0^\infty \mu f(\mu)\,d\mu$$
$$= \frac{\sqrt{\pi}}{2}\left\{\frac{A_1}{\beta_1}+\frac{A_2}{\beta_2}+\ldots\right\}, \qquad \ldots\ldots(5)$$

and, similarly,
$$\bar{\mu}_0 = \int_0^\infty \mu_0 \phi(\mu_0)\,d\mu_0$$
$$= \frac{\sqrt{\pi}}{2}\left\{\frac{A_1}{k_1}+\frac{A_2}{k_2}+\ldots\right\}, \qquad \ldots\ldots(6)$$

and, writing
$$c = \bar{\mu}-\bar{\mu}_0, \qquad \ldots\ldots(7)$$

we have
$$c = -\frac{\sqrt{\pi}}{2}\Sigma A_n\left(\frac{1}{k_n}-\frac{1}{\beta_n}\right). \qquad \ldots\ldots(8)$$

Reference to 6·61 (17) shows that β is greater than the corresponding value of k and consequently the correction c is always negative. Thus the effect of correcting for accidental error is to diminish the mean of the observed total proper motions.

Generally, the observed distribution shows a somewhat greater proportion of the larger total proper motions than is allowed for by the theoretical distribution given by the function $\phi(\mu_0)$ or the function $\Phi(\mu_0)$. By omitting several of these larger proper motions from the statistics (the number is readily found by "trial and error"), the remaining distribution—which we call the *adjusted distribution*—gives, as a rule, a sufficiently satisfactory representation between $\mu_0 = 0''\cdot 0$ and $\mu_0 = 4''\cdot 0$, say, of the function $\Phi(\mu_0)$ composed of one or more of the exponential functions. Let N be the total number of stars and N' the number in the adjusted distribution. The theoretical correction then applies only to the N' stars and we take as the mean, $\bar{\mu}$, of the true proper motions the expression given by

$$\bar{\mu} = \bar{\mu}_0 + \frac{N'}{N}c. \qquad \ldots\ldots(9)$$

In this way we correct the great majority of the total proper motions for accidental error and leave uncorrected a few of the larger ones, together with the residual numbers between the theoretical and observed distributions of the N' stars. The result is a slight under-correction of $\bar{\mu}_0$ but this may be balanced empirically by taking a slightly larger value of the estimated probable error, r.

6·63. *Example of the application of the method.*

Table 35 gives the relevant details* concerning the total proper motions, in a particular region, of 115 stars between magnitudes 9·0 and 9·4 on Argelander's scale or of mean magnitude 9·5 on the Harvard scale (visual).

* W. M. Smart, *M.N.* 96, 141, 1935.

Table 35

μ_0	No. (observed)	$\frac{1}{N'}N'(\mu_0)$	Φ ($k=0{\cdot}48$)
$0''{\cdot}0$		1·00	1·00
	5		
0·5		0·95	0·94
	12		
1·0		0·83	0·79
	26		
1·5		0·57	0·59
	16		
2·0		0·41	0·40
	18		
2·5		0·23	0·24
	8½		
3·0		0·14	0·13
	7½		
3·5		0·07	0·06
	2		
4·0		0·05	0·03
>4·0	5		

In the second column are the numbers of stars with observed centennial total proper motions between the limits $0''{\cdot}0$–$0''{\cdot}5$, $0''{\cdot}5$–$1''{\cdot}0$, etc. (a star with $\mu_0 = 3''{\cdot}0$, for example, contributes ½ to the interval $2''{\cdot}5$–$3''{\cdot}0$ and ½ to the interval $3''{\cdot}0$–$3''{\cdot}5$). The number of total proper motions greater than $4''{\cdot}0$ is assumed to be 5 (actually the number of proper motions greater than $4''{\cdot}0$ is 20), so that in the adjusted distributed $N' = 100$; thus 15 of the total proper motions exceeding $4''{\cdot}0$ per century are unaccounted for in the adjusted distribution.

Denote by $N'(\mu_0)$ the number of stars in the adjusted distribution with total proper motions exceeding μ_0; thus, for $\mu_0 = 4''$, $N'(\mu_0) = 5$: for $\mu_0 = 3''{\cdot}5$, $N'(\mu_0) = 7$: for $\mu_0 = 3''{\cdot}0$, $N'(\mu_0) = 14\frac{1}{2}$ and so on. The third column contains the values of $\frac{1}{N'}N'(\mu_0)$, and this set of values represents the observed adjusted frequency distribution. In the last column are the values of the function $\Phi(\mu_0) \equiv e^{-k^2\mu_0^2}$ for the value $k = 0{\cdot}48$; the value of k is most readily obtained from graphs of the function Φ for various values of k, so as to give as accurate a representation of the observed adjusted distribution as possible. In this particular example it is found that a good representation can be obtained when Φ is represented by one exponential function. Inspection of the last two columns shows that there is satisfactory agreement between the observed adjusted distribution and the theoretical distribution given by $k = 0{\cdot}48$.

The probable error, r, is estimated to be $0''{\cdot}4$ numerically; hence with this value of r and $k = 0{\cdot}48$, we find from Table 33 that $\beta = 0{\cdot}525$. Formula (8) of section 6·62 then gives $c = -0''{\cdot}158$.

The statistics of the proper motions give $297''{\cdot}3$ as the sum of the total proper motions for the 115 stars; hence $\bar{\mu}_0 = 2''{\cdot}585$. Formula (9) of section 6·62 then gives

$$\bar{\mu} = 2''{\cdot}585 - \frac{100}{115} \times 0''{\cdot}158,$$

from which

$$\bar{\mu} = 2''{\cdot}448.$$

This is the value to be used in (8) of section 6·5.

6·64. *The correction of the total proper motions (second method).*

In this method we derive the mean value of the true total próper motion corresponding to a given value of the observed proper motion.

The formula (4) of section 6·61 gives the probability $\zeta d\mu_0$ that for a given μ the observed total proper motion lies between μ_0 and $\mu_0 + d\mu_0$. Also $f(\mu)\, d\mu$ is the proportion of stars with true total proper motions between μ and $\mu + d\mu$. Hence the proportion of stars with observed total proper motions between μ_0 and $\mu_0 + d\mu_0$ and with true proper motions between μ and $\mu + d\mu$ is

$$\zeta f(\mu)\, d\mu\, d\mu_0. \qquad \ldots\ldots(1)$$

We may equally well describe this result as giving the proportion of stars with true total proper motions between μ and $\mu + d\mu$ for observed values between μ_0 and $\mu_0 + d\mu_0$.

Denote by $\bar{\mu}_1$ the mean of the true total proper motions for a given value μ_0 of the observed total proper motions. Then

$$\bar{\mu}_1 = \frac{\int_0^\infty \mu \zeta f(\mu)\, d\mu}{\int_0^\infty \zeta f(\mu)\, d\mu}, \qquad \ldots\ldots(2)$$

whence, on inserting the expression for ζ given by (4) of section 6·61,

$$\bar{\mu}_1 = \frac{\int_0^\infty \mu e^{-h^2\mu^2} f(\mu)\, I_0(2h^2\mu\mu_0)\, d\mu}{\int_0^\infty e^{-h^2\mu^2} f(\mu)\, I_0(2h^2\mu\mu_0)\, d\mu}. \qquad \ldots\ldots(3)$$

Assuming, in accordance with (2) of section 6·62, that $f(\mu)$ is given by

$$f(\mu) = 2\Sigma A_j \beta_j^2 \mu e^{-\beta_j^2\mu^2},$$

we then have

$$\bar{\mu}_1 = \frac{\Sigma A\beta^2 \int_0^\infty \mu^2 e^{-\mu^2(h^2+\beta^2)} I_0(2h^2\mu\mu_0)\, d\mu}{\Sigma A\beta^2 \int_0^\infty \mu e^{-\mu^2(h^2+\beta^2)} I_0(2h^2\mu\mu_0)\, d\mu}, \qquad \ldots\ldots(4)$$

in which we drop the suffix j for the sake of simplicity. Writing as before

$$H^2 = h^2 + \beta^2, \quad y = 2h^2\mu_0, \qquad \text{......(5)}$$

(4) becomes

$$\bar{\mu}_1 = \frac{\Sigma A\beta^2 \int_0^\infty \mu^2 e^{-H^2\mu^2} I_0(\mu y)\, d\mu}{\Sigma A\beta^2 \int_0^\infty \mu e^{-H^2\mu^2} I_0(\mu y)\, d\mu}. \qquad \text{......(6)}$$

The integrals in the denominator are of the form in (11) of section 6·61; hence the denominator of (6) is

$$\Sigma \frac{A\beta^2}{2(h^2+\beta^2)} e^{\frac{h^4\mu_0^2}{h^2+\beta^2}}. \qquad \text{......(7)}$$

Consider a typical integral

$$L \equiv \int_0^\infty \mu^2 e^{-H^2\mu^2} I_0(\mu y)\, d\mu$$

in the numerator of (6). Using the series for $I_0(\xi)$ given by (12) of section 6·61, we have

$$L = \int_0^\infty \sum_0^\infty \left(\frac{y^2}{4}\right)^n \frac{1}{(n\,!)^2} \mu^{2n+2} e^{-H^2\mu^2} d\mu.$$

But

$$\int_0^\infty x^{2n+2} e^{-H^2x^2} dx = \frac{\sqrt{\pi}}{2} \cdot \frac{1.3.5\ldots.2n+1}{2^{n+1}} \cdot \frac{1}{H^{2n+3}}.$$

Hence

$$L = \frac{\sqrt{\pi}}{4H^3} \sum_0^\infty \left(\frac{y^2}{4H^2}\right)^n \frac{2n+1}{n\,!} \cdot \frac{1.3.5\ldots.2n-1}{2.4.6\ldots.2n}. \qquad \text{......(8)}$$

We write

$$q = y^2/4H^2,$$

so that, by (5),

$$q = \frac{h^4\mu_0^2}{h^2+\beta^2}. \qquad \text{......(9)}$$

Also, we have

$$\frac{1.3.5\ldots.2n-1}{2.4.6\ldots.2n} = \frac{1}{\pi}\int_0^\pi \cos^{2n}\theta\, d\theta.$$

Hence (8) becomes

$$L = \frac{1}{4\sqrt{\pi}H^3} \int_0^\pi \left\{\sum_0^\infty (1+2n) \frac{(q\cos^2\theta)^n}{n\,!}\right\} d\theta$$

$$= \frac{1}{4\sqrt{\pi}H^3} \int_0^\pi (1+2q\cos^2\theta)\, e^{q\cos^2\theta}\, d\theta,$$

or, on writing

$$q = 2b,$$

$$L = \frac{e^{2b}}{4\sqrt{\pi}H^3} \int_0^\pi (1+4b\cos^2\theta)\, e^{-2b\sin^2\theta}\, d\theta. \qquad \text{......(10)}$$

But, by referring to section 2·42, it is seen that the integral in (10) is $\pi\psi(b)$, where, in the notation of that section,

$$\psi(b) = e^{-b}\{(1+2b)\, I_0 + 2bI_1(b)\}.$$

Thus
$$L = \frac{\sqrt{\pi}\, e^{2b}}{4(h^2+\beta^2)^{\frac{3}{2}}}\, \psi(b), \qquad \text{......(11)}$$

or, in terms of q,
$$L = \frac{e^q G(q)}{2(h^2+\beta^2)^{\frac{3}{2}}}, \qquad \text{......(12)}$$

where $G(q)$ is defined to be
$$\frac{\sqrt{\pi}}{2}\, \psi\left(\frac{q}{2}\right). \qquad \text{......(13)}$$

From (6), (7) and (12) we then obtain

$$\bar{\mu}_1 = \frac{\Sigma \dfrac{A\beta^2}{(h^2+\beta^2)^{\frac{3}{2}}} e^q G(q)}{\Sigma \dfrac{A\beta^2}{h^2+\beta^2} e^q}, \qquad \text{......(14)}$$

where q is defined by (9).

If the function $f(\mu)$ contains one exponential only, we have the result in this case

$$\bar{\mu}_1 = \frac{G(q)}{(h^2+\beta^2)^{\frac{1}{2}}}. \qquad \text{......(15)}$$

Kapteyn and van Rhijn expressed the function $G(q)$ as an infinite series which is rapidly convergent for $q < 1$ and in this way they evaluated the function for such restricted values of q. For $q > 1$, they effected the evaluation of the function by numerical integration. The calculation of the values of $G(q)$ by this procedure is extremely laborious. The preceding demonstration* shows that $G(q)$ can be expressed in terms of the function $\psi(q/2)$, the evaluation of which can be very simply effected by means of the tables for $e^{-b}I_0(b)$ and $e^{-b}I_1(b)$ given in Watson's *Bessel Functions*.

The following table gives the values of $G(q)$ as calculated by Kapteyn and van Rhijn.†

Table 36. *Values of* $G(q)$

q	$G(q)$	q	$G(q)$	q	$G(q)$
0·0	0·89	7·0	2·73	40·0	6·38
1·0	1·28	8·0	2·91	50·0	7·13
2·0	1·61	9·0	3·08	60·0	7·80
3·0	1·89	10·0	3·24	70·0	8·41
4·0	2·13	15·0	3·93	80·0	8·98
5·0	2·34	20·0	4·52	90·0	9·51
6·0	2·54	30·0	5·55	100·0	10·02
7·0	2·73	40·0	6·38	150·0	12·31

* W. M. Smart, *M.N.* 96, 136, 1935.

† *Groningen Publ.* No. 30, p. 63, 1920.

6·65. *Example of finding the values,* $\bar{\mu}_1$.

Taking the statistics of the adjusted distribution in section 6·63, we have $\beta = 0{\cdot}525$ and h, calculated from $h = 0{\cdot}477/r$, is found to be 1·19 for $r = 0''{\cdot}4$. Thus we obtain

$$q = 1{\cdot}19\mu_0^2.$$

Also

$$(h^2+\beta^2)^{\frac{1}{2}} = 1{\cdot}303.$$

For the values $1''{\cdot}0, \ldots\ 4''{\cdot}0$ of μ_0 we find the corresponding values of $\bar{\mu}_1$, using the entries of Table 36, as follows:

μ_0	$\bar{\mu}_1$
1″0	1″03
2·0	1·75
3·0	2·57
4·0	3·39

6·7. *Determination of the mean parallax derived from the total proper motions.*

The formula for the mean parallax is, from (8) of section 6·5,

$$\bar{p} = \frac{\kappa\bar{\mu}}{\bar{T}\{\alpha\psi(b_1)+(1-\alpha)\,\psi(b_2)\}}, \qquad \ldots\ldots(1)$$

in which $\bar{\mu}$ is now the corrected mean total proper motion.

We have to express $\bar{T}$ in km./sec. This can be done when the solar motion is known both in terms of the theoretical unit, $1/h$, and in km./sec. Taking as an example the analysis* of the Cambridge photographic proper motions, it is found that the theoretical value of the solar motion is 0·881; identifying this with the generally adopted value of 19·5 km./sec., we find that

$$\bar{T} = \frac{\sqrt{\pi}}{2}\cdot\frac{19{\cdot}5}{0{\cdot}881} = 19{\cdot}6\,\text{km./sec.}$$

Hence, on inserting the value of $\kappa \equiv 4{\cdot}74$ in (1), we have

$$\bar{p} = 0{\cdot}242\frac{\bar{\mu}}{D},$$

where

$$D = \alpha\psi(b_1)+(1-\alpha)\,\psi(b_2). \qquad \ldots\ldots(2)$$

With the values of α and the drift velocities hV_1 and hV_2 for the region concerned, D is readily calculated and hence $\bar{p}$ is obtained.

For example, for the region to which the data of section 6·63 refer, D is found to be 1·582 and, as the corrected value of $\bar{\mu}$ is $2''{\cdot}448$, the mean parallax† of the stars, in the region, of mean magnitude 9·5 on the Harvard-scale is found to be $0''{\cdot}0037$.

* *M.N.* **87**, 137, 1926.

† For further details and results, the reader is referred to *M.N.* **96**, 132, 1935.

CHAPTER VII

THE SPACE DISTRIBUTION OF THE STARS DERIVED FROM THEIR PROPER MOTIONS

7·1. *Density laws.*

In this chapter we consider one of several suggested laws of stellar distribution and investigate its theoretical implications concerning the distribution of proper motions. Observation and theory can then be brought together for comparison and it will then be possible to decide as to the degree of justification for the assumed law of stellar density.

Let $D(r, g)$ denote the number of stars per unit volume of space at a distance r from the sun and in galactic latitude g. As the sun is believed to be very near the galactic plane, the form of $D(r, g)$ must be such that this function decreases, for r constant, from the galactic equator towards the galactic poles in accordance with the well-authenticated thinning-out of the stars towards the poles. There is a certain amount of evidence in favour of a "local cluster" with the sun occupying a fairly central position; for example, Charlier's investigation* of the B type stars is in accordance with this suggestion, and van Rhijn's research† on the absorption of light in interstellar space provides a certain amount of additional confirmation. In consequence we postulate, for a given galactic latitude, a function $D(r)$ decreasing with the heliocentric distance r in the expectation that it will be of the nature of a first approximation, at least, to the true distribution.

We shall see in a subsequent chapter that there is a general method of determining the form of $D(r)$ from the observed distribution of stellar motions, but its practical application has met with difficulties depending on the uncertainty and insufficiency of observational data. Just as we can regard the Maxwellian law concerning the distribution of linear velocities as a standard of comparison between theoretical considerations and observational facts, so in a similar way we can treat an empirical law of density. There is a further practical consideration of a different nature. Statistics of proper motions, for example, are frequently incomplete so far as the smallest values are concerned, and a general discussion of systematic motions of stars with proper motions exceeding an assigned minimum value is invalid unless the space distribution of such stars is, in some way, taken into account. With an assumed law of stellar density, even if it is but a first approximation, the effect of omitting the smallest proper motions can be found. For example, the omission of annual proper motions less than

* *Lund Medd.* Ser. II, No. 14, 1916.

† *Groningen Publ.* No. 47, 1936.

$0''\cdot02$ for any part of the sky, generally leads, on the two-streams theory, to values of the drift velocities conspicuously greater than the values obtained from the analysis of proper motions of all magnitudes. Such high values are not representative of the star-drifts and must be regarded as spurious. But with the introduction of a density law, they can be correlated with the true velocities of the drifts and can serve to give reliable information concerning the systematic motions of the stars.

Other than $D(r) \equiv \text{constant}$, the simplest empirical density law is that of Seeliger, whose name is associated with the formula

$$D(r) = D_0 r^{-s}, \qquad \text{......(1)}$$

in which s is positive, and D_0 a constant.

In 1912, Schwarzschild proposed the formula

$$D(r) = D_0 e^{-a(y-b)^2}, \qquad \text{......(2)}$$

in which $y = \log r$ and D_0, a and b are constants.

In 1913, Dyson suggested the formula

$$D(r) = \frac{D_0}{r} e^{-k^2r^2}, \qquad \text{......(3)}$$

in which D_0 and k are constants.

The results of his investigation into the motions of the B type stars suggested to Charlier the adoption of the formula

$$D(r) = D_0 e^{-k^2r^2}, \qquad \text{......(4)}$$

where, again, D_0 and k are constants.

In this chapter we shall be concerned only with formula (3) and its implications.

7·21. *Investigation of the peculiar proper motions.*

In the absence of systematic motion, we shall assume that the proportion of stars with linear motions in one coordinate between v and $v + dv$ is given by

$$\frac{h}{\sqrt{\pi}} e^{-h^2v^2} dv. \qquad \text{......(1)}$$

We suppose that, in a particular region of the sky, we can resolve the observed proper motions in such a direction that (1) holds. For example, on the ellipsoidal theory, if we select a region on the great circle passing through the vertex of star-streaming and the solar apex, the motions perpendicular to this great circle will be haphazard and in accordance with (1).

If the region subtends a small solid angle S, the element of volume of the cone between distances r and $r + dr$ is $Sr^2 dr$ and the number of stars in this volume is $Sr^2 D(r)\, dr$.

Writing the number of stars, within the cone and at distances between r and $r+dr$, as $f(r)\,dr$, we define $f(r)$ as the *linear density*. Then

$$f(r) = Sr^2 D(r),$$

or, by (3) of the previous section,

$$f(r) = Are^{-k^2r^2}, \qquad \ldots\ldots(2)$$

where A and k are constants.

As a generalisation we shall take, following Dyson,*

$$f(r) = Ar^{\lambda} e^{-k^2r^2}, \qquad \ldots\ldots(3)$$

where $\lambda > 0$. This formula contains two disposable constants λ and k, A being determinable later in terms of λ and k and the total number, N, of stars under consideration.

Since the total number of stars is N, we have from (3)

$$N = A\int_0^\infty r^{\lambda}e^{-k^2r^2}\,dr,$$

or, on putting $k^2r^2 = y$,

$$2N = \frac{A}{k^{\lambda+1}}\int_0^\infty y^{\frac{\lambda-1}{2}}\, e^{-y}\,dy.$$

Hence

$$A = \frac{2Nk^{\lambda+1}}{\Gamma\left(\frac{\lambda+1}{2}\right)}. \qquad \ldots\ldots(4)$$

This formula determines A in terms of N, λ and k.

The number, dN, of stars at distances between r and $r+dr$ and with linear velocity components, in the assigned direction, between v and $v+dv$ is given by

$$dN = \frac{Ah}{\sqrt{\pi}} r^{\lambda}e^{-k^2r^2}\,e^{-h^2v^2}\,dr\,dv. \qquad \ldots\ldots(5)$$

If τ denotes the proper motion (in circular measure) corresponding to the linear velocity v, the units of time for v and τ being the same,

$$v = r\tau,$$

and for the stars at distances between r and $r+dr$,

$$dv = r\,d\tau.$$

Hence the number, dN, of stars with proper motions between τ and $\tau+d\tau$ and at distances between r and $r+dr$ is given by

$$dN = \frac{Ah}{\sqrt{\pi}} r^{\lambda+1}\, e^{-r^2(k^2+h^2\tau^2)}\,dr\,d\tau.$$

* *M.N.* 73, 334, 1913.

Summing now for stars with proper motions between the limits τ and $\tau+d\tau$ for all distances between 0 and ∞, and denoting the number so obtained by $N(\tau)\,d\tau$, we have

$$N(\tau)\,d\tau = \frac{Ah}{\sqrt{\pi}}d\tau\int_0^\infty r^{\lambda+1}\,e^{-r^2(k^2+h^2\tau^2)}\,dr$$

$$= \frac{Ah}{2\sqrt{\pi}}\frac{\Gamma\left(\frac{\lambda+2}{2}\right)d\tau}{(k^2+h^2\tau^2)^{\frac{\lambda+2}{2}}}.$$

Set $$k = ah. \qquad \text{......(6)}$$

Then we can write

$$N(\tau)\,d\tau = NC\frac{d\tau}{a}\left(1+\frac{\tau^2}{a^2}\right)^{-\left(\frac{\lambda+2}{2}\right)}, \qquad \text{......(7)}$$

where C is found, with the help of (4), to be

$$C = \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{\lambda+2}{2}\right)}{\Gamma\left(\frac{\lambda+1}{2}\right)}. \qquad \text{......(8)}$$

It is to be noted that a is expressed in terms of the same units as τ.

The number of proper motions with numerical values greater than τ (τ being positive) is, from (7),

$$2NC\int_\tau^\infty\left(1+\frac{\tau^2}{a^2}\right)^{-\frac{\lambda+2}{2}}\frac{d\tau}{a}, \qquad \text{......(9)}$$

and if we denote the number when $\tau = \tau_1$ by N_1 and the number when $\tau = \tau_2$ by N_2, we can find from (9) the ratio of N_1 to N_2. In particular when τ is large compared with a, we have, as an approximation,

$$N_1 : N_2 = \int_{\tau_1}^\infty\left(\frac{\tau}{a}\right)^{-(\lambda+2)}d\tau : \int_{\tau_2}^\infty\left(\frac{\tau}{a}\right)^{-(\lambda+2)}d\tau$$

$$= \tau_2^{\lambda+1} : \tau_1^{\lambda+1}. \qquad \text{......(10)}$$

This last relation, (10), can be used, in conjunction with the statistics of τ, for estimating the value of λ and for testing the legitimacy of the assumption regarding the form of the density function.

7·22. *Application to the τ-components of the Carrington stars.*

The analysis of the previous section is due to Dyson (*loc. cit.*). To test the formulae he used the proper motions of 3708 stars in Carrington's *Circumpolar Catalogue*, which includes all stars between declinations $+81°$ and $+90°$ down to the tenth magnitude. For the comparatively small area of the sky represented by this catalogue, the parallactic motion may be taken to be towards the point (6^h, 0°), and this is also approximately the direction of the velocity of drift I and opposite to the direction of drift II on the two-streams theory. By finding the components, τ, of the observed proper motions in the direction perpendicular to that of the parallactic motion, we obtain angular motions which are almost completely free from the effects of the preferential motions and are consequently of the type to which the foregoing analysis can be applied.

At the outset, the statistics of the τ-components were corrected for accidental error, partly by an empirical process and partly according to the method of section 1·81.

Dyson compared the distribution of the τ-components with formula (7) of section 7·21, taking $d\tau$ to be 0″·4 (centennial) and the three values 0·8, 1·0 and 1·2 of λ and different values for a. He concluded that the observations were best satisfied when λ = 1·0 and a = 1″·47.

The following table gives the comparison of the observed distribution with the theoretical distribution for λ = 1·0 and a = 1″·47.

Table 37

Limits of τ	Number of stars		
	Observed	Calculated	Difference
0″·0–0″·4	634	604	+30
0·4–0·8	555	575	−20
0·8–1·2	503	517	−14
1·2–1·6	440	443	− 3
1·6–2·0	372	366	+ 6
2·0–2·4	281	288	− 7
2·4–2·8	215	224	− 9
2·8–3·2	162	163	− 1
3·2–3·6	121	120	+ 1
3·6–4·0	94	88	+ 6
4·0–4·4	64	64	0
4·4–4·8	51	47	+ 4
4·8–6·0	76	83	− 7
6·0–8·0	66	59	+ 7
8·0–10·0	33	26	+ 7
10·0–15·0	25	24	+ 1
> 15·0	16	18	− 2

7·23. *Evaluation of k.*

Taking the function $f(r)$ to be given by (2) and (4) of section 7·21 ($\lambda = 1$), we have

$$f(r) = Are^{-k^2r^2} \qquad \ldots\ldots(1)$$

and

$$A = 2k^2N. \qquad \ldots\ldots(2)$$

With the parsec as the unit of distance, the parallax, p, in seconds of arc is

$$p = \frac{1}{r},$$

and the mean parallax, $\bar{p}$, of the stars is given by

$$\bar{p} = 2k^2 \int_0^\infty \frac{1}{r} r e^{-k^2r^2} dr = k\sqrt{\pi} \qquad \ldots\ldots(3)$$

or

$$\bar{p} = 1{\cdot}772k. \qquad \ldots\ldots(4)$$

The mean parallax can be determined numerically by means of the υ-components, as explained in section 6·21. Dyson assumed that the antapex of the solar motion is at (6^h, $-35°$), so that, for the small polar area under consideration, the component of the parallactic motion due to the solar speed, V_0, is along the great circle joining the pole to the point (6^h, $0°$). For a star with parallax, p, the parallactic centennial motion υ in this direction is given by

$$V_0 \cos 35° = \frac{4{\cdot}74\upsilon}{100p},$$

and inserting $V_0 = 19{\cdot}5$ km./sec., we obtain

$$\bar{p} = \frac{\bar{\upsilon}}{412 \cos 35°}.$$

The observed value of $\bar{\upsilon}$ was found to be $1''{\cdot}41$; consequently

$$\bar{p} = 0''{\cdot}00418. \qquad \ldots\ldots(5)$$

From (4) and (5) it is found that

$$k = 0{\cdot}00236. \qquad \ldots\ldots(6)$$

From these results the percentage of the Carrington stars at different distances or between different limits of parallax are readily calculated; the details are shown in the following table given by Dyson.*

Table 38

Distance (parsecs)	Parallax	Percentage of stars
0–40	> 0″·025	0·9
40–100	0″·025 – 0″·010	5·0
100–200	0·010 – 0·005	15·1
200–400	0·005 – 0·0025	40·1
400–667	0·0025–0·0015	31·5
667–1000	0·0015–0·0010	7·1
> 1000	< 0″·0010	0·3

7·31. *Investigation of the systematic components.*

Consider again the small polar area treated by Dyson. The coordinates of the vertex of preferential motions are (94°, + 12°) and consequently the great circle joining the pole to the solar antapex passes very close to the vertex. Consider the components, υ, of proper motion of the stars at the pole resolved

* *M.N.* 73, 342, 1913.

along this great circle; they will correspond to the linear peculiar velocities given by the major axis of the velocity ellipse for the polar region combined with the parallactic linear velocity in the same direction.

We write the velocity ellipse at the pole in the form

$$g^2u^2+h^2v^2 = 1, \quad (g<h).$$

The peculiar linear motions in the direction concerned are, by the ellipsoidal theory, distributed according to the frequency law

$$\frac{g}{\sqrt{\pi}}e^{-g^2u^2}du, \qquad \ldots\ldots(1)$$

where g is inversely proportional to the major axis of the velocity ellipse for the polar region.

If w is the total linear velocity in the direction we are considering and V is the component of the parallactic motion in this direction,

$$u = w-V,$$

and hence the total linear velocities are distributed according to the frequency law

$$\frac{g}{\sqrt{\pi}}e^{-g^2(w-V)^2}dw. \qquad \ldots\ldots(2)$$

Since v is the component of proper motion corresponding to w,

$$w = vr. \qquad \ldots\ldots(3)$$

With the linear density law expressed by (1) and (2) of the previous section, the number of stars between distances r and $r+dr$ and with linear velocities between w and $w+dw$ is

$$\frac{2Ngk^2}{\sqrt{\pi}}re^{-k^2r^2}e^{-g^2(w-V)^2}dr\,dw. \qquad \ldots\ldots(4)$$

If dN denotes the number of stars at distances between r and $r+dr$ and with proper motion components between v and $v+dv$, (3) and (4) give

$$dN = \frac{2Ngk^2}{\sqrt{\pi}}r^2e^{-k^2r^2-g^2(rv-V)^2}dr\,dv. \qquad \ldots\ldots(5)$$

Hence the total number, $N(v)\,dv$, of stars at all distances with proper motion components between v and $v+dv$ is given by

$$N(v)\,dv = \frac{2Ngk^2}{\sqrt{\pi}}dv\int_0^\infty r^2e^{-r^2(k^2+g^2v^2)+2g^2Vvr-g^2V^2}dr. \qquad \ldots\ldots(6)$$

Set $$r(k^2+g^2v^2)^{\frac{1}{2}}-\frac{g^2Vv}{(k^2+g^2v^2)^{\frac{1}{2}}} = x \qquad \ldots\ldots(7)$$

and $$gv = k\tan\theta. \qquad \ldots\ldots(8)$$

Then $$kr = \cos\theta(x+gV\sin\theta), \qquad \ldots\ldots(9)$$

$$g\,dv = k\sec^2\theta\,d\theta, \qquad \ldots\ldots(10)$$

and (6) becomes

$$N(v)\,dv = \frac{2N}{\sqrt{\pi}}\cos\theta\,d\theta\,e^{-g^2V^2\cos^2\theta}\int_{-gV\sin\theta}^{\infty}(x+gV\sin\theta)^2e^{-x^2}dx,$$

or, on writing

$$\tau = gV\sin\theta, \qquad \text{......(11)}$$

$$N(v)\,dv = \frac{2N}{\sqrt{\pi}}\cos\theta\,d\theta\,e^{-g^2V^2}\left\{e^{\tau^2}\int_{-\tau}^{\infty}(x+\tau)^2e^{-x^2}dx\right\}.$$

But

$$e^{\tau^2}\int_{-\tau}^{\infty}(x+\tau)^2e^{-x^2}dx = \tfrac{1}{2}(1+2\tau^2)\,e^{\tau^2}\int_{-\tau}^{\infty}e^{-x^2}dx+\tfrac{1}{2}\tau$$

$$= \frac{1}{2}\frac{d}{d\tau}\left\{\tau e^{\tau^2}\int_{-\tau}^{\infty}e^{-x^2}dx\right\}.$$

Hence, using (11),

$$N(v)\,dv = \frac{N}{\sqrt{\pi}}d\theta\frac{d}{d\theta}\left\{\sin\theta\,e^{-g^2V^2\cos^2\theta}\int_{-gV\sin\theta}^{\infty}e^{-x^2}dx\right\}. \qquad \text{......(12)}$$

If $N(0, +v)$ denotes the number of stars with values of v between 0 and $+v$, we obtain, from (12),

$$N(0, +v) = \frac{N}{\sqrt{\pi}}\sin\theta\,e^{-g^2V^2\cos^2\theta}\int_{-gV\sin\theta}^{\infty}e^{-x^2}dx, \qquad \text{......(13)}$$

since $\theta = 0$ when $v = 0$.

Similarly, the number, $N(0, -v)$, of stars with values of v between 0 and $-v$ is given by

$$N(0, -v) = \frac{N}{\sqrt{\pi}}\sin\theta\,e^{-g^2V^2\cos^2\theta}\int_{gV\sin\theta}^{\infty}e^{-x^2}dx. \qquad \text{......(14)}$$

From (13) and (14),

$$N(0,v)+N(0, -v) = N\sin\theta\,e^{-g^2V^2\cos^2\theta} \qquad \text{......(15)}$$

and

$$N(0,v)-N(0, -v) = \frac{2N}{\sqrt{\pi}}\sin\theta\,e^{-g^2V^2\cos^2\theta}\int_{0}^{gV\sin\theta}e^{-x^2}dx. \qquad \text{......(16)}$$

Similarly, denoting by $N(v,\infty)$ and $N(-v, -\infty)$ the numbers of stars with proper motion components between v and ∞, and between $-v$ and $-\infty$ respectively,

$$N(v,\infty)+N(-v, -\infty) = N-\{N(0,v)+N(0, -v)\}. \qquad \text{......(17)}$$

This last formula gives the number of stars with proper motion components greater numerically than v.

When v is large, θ is near to $\pi/2$ and, writing

$$\alpha = \pi/2-\theta, \qquad \text{......(18)}$$

the number, N_1, of large proper motions (taken regardless of sign) is given by

$$N_1 = N\{1-\cos\alpha\,e^{-g^2V^2\sin^2\alpha}\}.$$

Hence, as α is now supposed small, we have—up to the second order in α—

$$N_1 = \frac{N}{2}(1+2g^2V^2)\alpha^2. \qquad \text{......(19)}$$

But, from (8) and (18), $\tan\alpha = \dfrac{k}{gv}$,

or, with sufficient accuracy, $\alpha = \dfrac{k}{gv}$. ......(20)

Hence, (19) becomes $N_1 = \dfrac{Nk^2}{2g^2v^2}(1+2g^2V^2)$. ......(21)

Similarly, if N_2 denotes the difference between the number of stars with large positive proper motions and the number with large negative proper motions (the limits being numerically the same),

$$\begin{aligned} N_2 &= N(v,\infty) - N(-v, -\infty) \\ &= N(0,\infty) - N(0, -\infty) - \{N(0,v) - N(0, -v)\}. \qquad \text{......(22)} \end{aligned}$$

But from (13) and (14) and from the consideration that $\theta = \pi/2$ when $v = \infty$ as deduced from (8),

$$N(0,\infty) = \frac{N}{\sqrt{\pi}}\left\{\frac{\sqrt{\pi}}{2} + \int_0^{gV} e^{-x^2}\,dx\right\} \qquad \text{......(23)}$$

and

$$N(0,-\infty) = \frac{N}{\sqrt{\pi}}\left\{\frac{\sqrt{\pi}}{2} - \int_0^{gV} e^{-x^2}\,dx\right\}. \qquad \text{......(24)}$$

Hence from (22), using (23), (24) and (16),

$$N_2 = \frac{2N}{\sqrt{\pi}}\int_0^{gV} e^{-x^2}\,dx - \frac{2N}{\sqrt{\pi}}\sin\theta\, e^{-g^2V^2\cos^2\theta}\int_0^{gV\sin\theta} e^{-x^2}\,dx,$$

or, with the previous approximation with regard to α,

$$N_2 = \frac{2N}{\sqrt{\pi}}\int_0^{gV} e^{-x^2}\,dx - \frac{2N}{\sqrt{\pi}}\{1 - \tfrac{1}{2}\alpha^2 - g^2V^2\alpha^2\}\left[\int_0^{gV} e^{-x^2}\,dx - \int_{gV\cos\alpha}^{gV} e^{-x^2}\,dx\right].$$

Also, $\displaystyle\int_{gV\cos\alpha}^{gV} e^{-x^2}dx$ is approximately, since α is small, equal to

$$e^{-g^2V^2}gV(1-\cos\alpha) \quad \text{or} \quad \tfrac{1}{2}\alpha^2 gVe^{-g^2V^2}.$$

Hence, inserting the value of α given by (20), we obtain

$$N_2 = \frac{N}{\sqrt{\pi}}\frac{k^2}{g^2v^2}\left\{(1+2g^2V^2)\int_0^{gV} e^{-x^2}\,dx + gVe^{-g^2V^2}\right\}. \qquad \text{......(25)}$$

The formulae (21) and (25), which were derived by Dyson,* can be used in conjunction with the observational data.

* *M.N.* **73, 402, 1913.**

7·32. *Application to the Carrington stars.*

Of the 3735 Carrington stars used by Dyson in this particular investigation 87 had values of υ greater than 12″ per century and 18 had negative values exceeding 12″ per century numerically, so that $N_1 = 105$. Hence, from (21) of the previous section,

$$\frac{1}{2}\frac{k^2}{g^2} + k^2V^2 = \frac{105}{3735}(12)^2. \qquad \text{......(1)}$$

From (3) of 7·23, $$kV = \frac{V\bar{p}}{\sqrt{\pi}},$$

where $\bar{p}$ is the mean parallax.

Also $V\bar{p}$ is the parallactic proper motion which was found to be 1″·44 (a later value than that used in section 7·23). Hence

$$kV = \frac{1''\cdot 44}{\sqrt{\pi}}. \qquad \text{......(2)}$$

Substituting (2) in (1), we obtain

$$\frac{k}{g} = 2''\cdot 61. \qquad \text{......(3)}$$

But we had from section 7·22 $$\frac{k}{h} = 1''\cdot 47. \qquad \text{......(4)}$$

Hence, from (3) and (4), $$\frac{g}{h} = 0\cdot 563. \qquad \text{......(5)}$$

But g and h are respectively inversely proportional to the axes of the velocity ellipse at the pole. The ratio of the axes of the velocity ellipsoid is given by, from formula (10) of section 5·12,

$$\left(\frac{H^2}{K^2} - 1\right)\sin^2\chi = \frac{h^2}{g^2} - 1.$$

With the usual position of the vertex, $\chi = 78°$, so that, with the help of (5),

$$\frac{H}{K} = 0\cdot 55,$$

a result which may be regarded as in fair agreement with the results based on the methods of Chapter V.

From (2) and (3) $$gV = \frac{1\cdot 44}{2\cdot 61}\frac{1}{\sqrt{\pi}} = 0\cdot 311, \qquad \text{......(6)}$$

and with this value of gV, we obtain from (13) and (23) of section 7·31

$$N(\upsilon, \infty) = \frac{N}{\upsilon^2}\cdot 3\cdot 40\{0\cdot 60 + 0\cdot 37\},$$

when υ is taken to be a large proper motion.

Similarly, $$N(-\upsilon, -\infty) = \frac{N}{\upsilon^2} . 3\cdot40\,\{0\cdot60 - 0\cdot37\}.$$

With $N = 3735$ and $\upsilon = 12''$ per century,

$$N(\upsilon, \infty) = 85, \quad N(-\upsilon, -\infty) = 20,$$

as compared with the observed numbers 87 and 18 respectively; the agreement between theory and observation is thus satisfactory.

Again, formula (13) of section 7·31 enables us to calculate the number of stars with values of υ between υ_1 and υ_2, and formula (14) of section 7·31 is available in the same way for the corresponding negative values of υ. Before comparing the theoretical and observed distributions, it is first necessary to take account of the accidental errors. Instead of correcting the observed distribution, Dyson put the effect of the errors into the theoretical distribution. Accordingly, in Table 39, "theoretical number" implies the number as calculated from the formula, together with the effect of accidental error; it is to be compared with the "observed number" in the adjoining column. The details for the Carrington stars, as given by Dyson (*loc. cit.*), are in Table 39.

Table 39

Centennial proper motion (υ)	Theoretical number	Observed number	Centennial proper motion (υ)	Theoretical number	Observed number
$< -12''$	16	18	$+3''$ to $+4''$	304	257
$-12''$ to $-8''$	19	13	$+4$,, $+5$	201	182
-8 ,, -6	28	23	$+5$,, $+6$	125	126
-6 ,, -5	33	30	$+6$,, $+7$	81	86
-5 ,, -4	57	50	$+7$,, $+8$	53	48
-4 ,, -3	104	91	$+8$,, $+9$	36	34
-3 ,, -2	211	225	$+9$,, $+10$	27	36
-2 ,, -1	347	359	$+10$,, $+12$	34	39
-1 ,, 0	480	517	$+12$,, $+15$	27	27
0 ,, $+1$	553	574	$+15$,, $+20$	20	30
$+1$,, $+2$	528	530	$> +20''$	27	30
$+2$,, $+3$	425	410			

The agreement between the theoretical and observed results is, as Dyson remarked, very satisfactory.

A further test can be applied as follows. The total number, $N(0, \infty)$, of positive proper motions is given, from 7·31 (23), by

$$N(0, \infty) = \frac{N}{2}\left\{1 + \frac{2}{\sqrt{\pi}}\int_0^{gV} e^{-x^2}\,dx\right\} = \frac{N}{2}\{1 + \Theta(gV)\},$$

$\Theta(t)$ being the probability integral.* With the value of gV given in (6), we find that

$$N(0, \infty) = 0\cdot67N.$$

* For numerical values of $\Theta(t)$, see Brunt's *The Combination of Observations* (2nd ed.), 234, 1931.

Thus 67 % of the stars have positive values of v according to the theory. Table 39 shows that the observed number is 2409, or 64½ %, in good agreement with the theoretical value.

7·33. *Relation to the two-streams theory.*

The preceding tests have been made with reference to the ellipsoidal hypothesis. On the two-streams theory the polar region is such that the directions of the speeds of both drift I and drift II are given approximately by the great circle joining the pole to the point (6^h, $0°$). In this case, we can use formula (21) of section 7·31 for each drift, with the proviso that the modulus, g, in this formula is to be replaced by h, the modulus in the perpendicular direction. Assuming that the stars are equally divided between the drifts and that the projections of the space velocities of the drifts on the tangent plane at the pole are V_1 and V_2, the number, N_1', of stars belonging to drift I with large proper motions (taken regardless of sign) and the corresponding number, N_2', for drift II are given by

$$N_1' = \left(\frac{N}{2}\right)\frac{k^2}{2h^2v^2}(1+2h^2V_1^2),$$

$$N_2' = \left(\frac{N}{2}\right)\frac{k^2}{2h^2v^2}(1+2h^2V_2^2).$$

Hence, of the N stars forming the two drifts, the number, N_1, ($\equiv N_1'+N_2'$), will have proper motions greater numerically than v, where

$$N_1 = \frac{Nk^2}{2h^2v^2}\{1+h^2(V_1^2+V_2^2)\}.$$

Thus with the numerical data corresponding to 7·32 (1), we have

$$\frac{k^2}{h^2}\{1+h^2(V_1^2+V_2^2)\} = 2\,.\frac{105}{3735}\,.\,(12)^2.$$

But, as we have seen, $k/h = 1''\!.47$. Hence

$$h^2V_1^2+h^2V_2^2 = 2{\cdot}75. \qquad \text{......(1)}$$

Also, with the stars divided equally between the two drifts, the mean linear parallactic motion is $\frac{1}{2}(hV_1+hV_2)$, which by 7·32 (2) we see is equal to $h\dfrac{1{\cdot}44}{k\sqrt{\pi}}$.

Hence $$hV_1+hV_2 = \frac{2{\cdot}88}{\sqrt{\pi}}\cdot\frac{1}{1{\cdot}47} = 1{\cdot}10. \qquad \text{......(2)}$$

From (1) and (2), we find that

$$hV_1 = 1{\cdot}58, \quad hV_2 = -0{\cdot}48. \qquad \text{......(3)}$$

The negative sign associated with hV_2 in (3) indicates that the drift velocities for the polar region are in opposite directions, as can easily be deduced from the coordinates of the apices of the drifts.

Eddington's results from the analysis* of the Carrington stars brighter than magnitude 8·9 give the numerical values of hV_1 and hV_2 to be 1·40 and 0·35, in fair agreement with the values in (3). It should be added however that as the direction of the velocity of drift II is approximately 20° different from the direction of the velocity of drift I for the polar area, exact agreement could hardly be anticipated. In consequence, the τ-components treated in section 7·21 include the effect of the velocity of drift II in this direction, and the theoretical formulae, pertaining to the two-streams theory, which we have used cannot be applied with complete strictness to a comparison with the observations.

7·41. *General theory.*

We investigate now a general theory† applicable to any region of the sky; as we have seen, Dyson's researches deal with a special region of the sky in which the proper motions can be resolved in two directions, one approximately free from systematic motions and the other containing the greater part of the systematic motions.

We consider first a single drift. Suppose that there are N stars in a small region subtending a solid angle S. If V is the projection of the space velocity of the drift on the tangent plane at the centre of the region and θ is the angle between the direction of the projected drift velocity and any other direction in the tangent plane, the number of stars with transverse linear velocities between w and $w+dw$ and in directions between θ and $\theta+d\theta$ is, by 2·3 (4),

$$\frac{Nh^2}{\pi}d\theta\, w\, dw\, e^{-h^2(w^2+V^2-2Vw\cos\theta)},$$

in which h is the modulus of the drift.

Denote the star-density by $D(r)$ per unit volume. The element of volume of the cone of solid angle S between the distances r and $r+dr$ is Sr^2dr and the number of stars in this volume is $Sr^2D(r)\,dr$. Of these stars the number with transverse linear velocities between w and $w+dw$ and in directions between θ and $\theta+d\theta$ is

$$\frac{Sh^2}{\pi}r^2D(r)\,w\,e^{-h^2(w^2+V^2-2Vw\cos\theta)}\,dr\,d\theta\,dw.$$

If μ is the proper motion corresponding to w,

$$w = r\mu.$$

Hence the number of stars in the element of volume, at a distance r, with proper motions between μ and $\mu+d\mu$ and in directions between θ and $\theta+d\theta$ is

$$\frac{Sh^2}{\pi}r^4D(r)\,\mu\,e^{-h^2(r^2\mu^2+V^2-2Vr\mu\cos\theta)}\,d\mu\,d\theta\,dr.$$

* *M.N.* **67**, 53, 1906.

† W. M. Smart, *M.N.* **88**, 567, 1928; **89**, 93, 1928.

(i) *The distribution of proper motions in any direction* θ.

Let $M(\mu,\theta)\,d\mu\,d\theta$ denote the number of stars in the whole cone with proper motions between μ and $\mu+d\mu$ and in directions between θ and $\theta+d\theta$. Then

$$M(\mu,\theta) = \frac{Sh^2\mu}{\pi}\int_0^\infty r^4\,D(r)\,e^{-h^2(r^2\mu^2+V^2-2Vr\mu\cos\theta)}\,dr. \qquad \ldots\ldots(1)$$

We assume that $D(r)$ is given by (3) of section 7·1, but for convenience we shall write it in the form

$$D(r) = \frac{A}{r}\,e^{-h^2k^2r^2}. \qquad \ldots\ldots(2)$$

The total number of stars in the cone is given by

$$N = \int_0^\infty SD(r)\,r^2dr = AS\int_0^\infty re^{-h^2k^2r^2}\,dr,$$

from which

$$AS = 2Nh^2k^2. \qquad \ldots\ldots(3)$$

From (1) and (2),

$$M(\mu,\theta) = \frac{ASh^2\mu}{\pi}\int_0^\infty r^3e^{-h^2\{r^2(\mu^2+k^2)+V^2-2Vr\mu\cos\theta\}}\,dr. \qquad \ldots\ldots(4)$$

Let

$$h^2(\mu^2+k^2) = a^2, \qquad \ldots\ldots(5)$$

$$a\left(r-\frac{V\mu\cos\theta}{\mu^2+k^2}\right) = x, \qquad \ldots\ldots(6)$$

$$\frac{hV\mu\cos\theta}{(\mu^2+k^2)^{\frac{1}{2}}} = y. \qquad \ldots\ldots(7)$$

Then

$$M(\mu,\theta) = \frac{ASh^2\mu}{\pi a^4}\,e^{-h^2V^2}\,e^{y^2}\int_{-y}^\infty (x+y)^3\,e^{-x^2}\,dx. \qquad \ldots\ldots(8)$$

Let

$$G(y) = \int_{-y}^\infty e^{-x^2}\,dx. \qquad \ldots\ldots(9)$$

Expand $(x+y)^3$ under the integral in (8) and make use of the following results:

$$\int_{-y}^\infty xe^{-x^2}\,dx = \tfrac{1}{2}e^{-y^2}, \qquad \ldots\ldots(10)$$

$$\int_{-y}^\infty x^2e^{-x^2}\,dx = \tfrac{1}{2}G(y)-\tfrac{1}{2}ye^{-y^2}, \qquad \ldots\ldots(11)$$

$$\int_{-y}^\infty x^3e^{-x^2}\,dx = \tfrac{1}{2}y^2e^{-y^2}+\tfrac{1}{2}e^{-y^2}. \qquad \ldots\ldots(12)$$

Then, using (3),

$$M(\mu,\theta) = \frac{Nh^4k^2}{\pi a^4}\,\mu e^{-h^2V^2}\,[(2y^3+3y)\,e^{y^2}\,G(y)+y^2+1]. \qquad \ldots\ldots(13)$$

This formula, combined with (3), gives the distribution of proper motions in any direction θ.

(ii) *The number of proper motions greater than μ in any direction θ.*

Let $N(\mu,\theta)\,d\theta$ denote the number of stars with proper motions between μ and ∞ in directions between θ and $\theta+d\theta$. Then

$$N(\mu,\theta) = \int_{\mu}^{\infty} M(\mu,\theta)\,d\mu. \qquad \text{......(14)}$$

From (7), $$y\,dy = h^2V^2\cos^2\theta\,\frac{k^2}{(\mu^2+k^2)^2}\,\mu\,d\mu;$$

from this equation and (5) we obtain

$$\frac{\mu\,d\mu}{a^4} = \frac{y\,dy}{h^6k^2V^2\cos^2\theta}. \qquad \text{......(15)}$$

Write $$hV\cos\theta = \tau. \qquad \text{......(16)}$$

Then from (7) we see that $y = \tau$ when $\mu = \infty$.

From (13), (14), (15) and (16) we obtain

$$N(\mu,\theta) = \frac{Ne^{-h^2V^2}}{\pi\tau^2}\int_{y}^{\tau}\{(2y^4+3y^2)\,e^{y^2}\,G(y)+y^3+y\}\,dy. \qquad \text{......(17)}$$

But, since $$\frac{dG(y)}{dy} = e^{-y^2},$$

we can write the integral in (17) as

$$\int_{y}^{\tau}\left\{G(y)\frac{d}{dy}(y^3e^{y^2})+y^3e^{y^2}\frac{dG(y)}{dy}+y\right\}dy.$$

Thus $$N(\mu,\theta) = \frac{Ne^{-h^2V^2}}{\pi\tau^2}\,[\tau^2\{\tau e^{\tau^2}\,G(\tau)+\tfrac{1}{2}\}-y^2\{ye^{y^2}\,G(y)+\tfrac{1}{2}\}].$$

Now Eddington's function $f(x)$ is given in 2·3 (10) by

$$\frac{\sqrt{\pi}}{2}f(x) = \tfrac{1}{2}+x\,e^{x^2}\,G(x).$$

Hence $$N(\mu,\theta) = \frac{Ne^{-h^2V^2}}{2\sqrt{\pi}}\left\{f(\tau)-\frac{y^2}{\tau^2}f(y)\right\}. \qquad \text{......(18)}$$

Describing proper motions which exceed a certain value of μ as *restricted proper motions*, we see that (18) gives rise to the frequency curve of restricted proper motions and is analogous to the formula for the drift curve derived in section 2·3. It is clear that $\theta = 0$ is an axis of symmetry, as in the drift curve.

Let us suppose now that the drift curve and the frequency curve of the restricted proper motions have an axis of symmetry in position angle θ_1. Then, from (16), $$\tau = hV\cos(\theta-\theta_1), \qquad \text{......(19)}$$

in which θ signifies position angle.

Let
$$\eta = \frac{\mu}{(\mu^2+k^2)^{\frac{1}{2}}}, \qquad \text{......(20)}$$

so that, by (7),
$$y = \eta\tau. \qquad \text{......(21)}$$

Formula (20) shows that η cannot exceed unity. Defining θ' by

$$\cos(\theta'-\theta_1) = \eta\cos(\theta-\theta_1), \qquad \text{......(22)}$$

we have
$$y = hV\cos(\theta'-\theta_1). \qquad \text{......(23)}$$

Introduce the function

$$\phi(hV, \theta-\theta_1) \equiv e^{-h^2V^2}f(\tau) \qquad \text{......(24)}$$

whose values are given in Table 13, pp. 104, 105. Then we have

$$N(\mu,\theta) = \frac{N}{2\sqrt{\pi}}\{\phi(hV,\theta-\theta_1)-\eta^2\phi(hV,\theta'-\theta_1)\}, \qquad \text{......(25)}$$

or, in terms of Eddington's function, $f(\tau)$,

$$N(\mu,\theta) = \frac{N}{2\sqrt{\pi}}e^{-h^2V^2}\{f(\tau)-\eta^2 f(\eta\tau)\}. \qquad \text{......(26)}$$

In any region of the sky we have an equation of the form, (25), for drift I and a similar equation for drift II. If the drift constants ($N_1, N_2, \theta_1, \theta_2$, hV_1 and hV_2) are known, we can readily calculate the theoretical distribution of the restricted proper motions belonging to the two drifts, by means of the values of $\phi(hV, \theta-\theta_1)$ given in Table 13 (pp. 104, 105), for any assumed value of the parameter, k, occurring in the density function $D(r)$. Thus we can compare the theoretical distribution (corresponding to a definite value of k) of the restricted proper motions with the observed distribution and, if the stellar density follows the assumed law, we are then enabled to deduce the value of k for the region concerned. It may be anticipated that k is likely to vary with galactic latitude.

7·42. *Approximate formula for $N(\mu,\theta)$ when k/μ is small.*

By 7·41 (20) it is seen that η approaches unity as k/μ tends to zero. Let

$$\eta = 1-\alpha. \qquad \text{......(1)}$$

Then, up to the second order in k/μ,

$$\alpha = \frac{k^2}{2\mu^2}. \qquad \text{......(2)}$$

Also, it is easily found that up to the first order in α

$$f(\tau)-\eta^2 f(\eta\tau) = \alpha\left\{(3+2\tau^2)f(\tau)-\frac{1}{\sqrt{\pi}}\right\},$$

where $f(\tau)$ is Eddington's function; this result can be obtained at once from the equations following (5) of section 7·5. We thus derive from 7·41 (26),

$$N(\mu,\theta) = \frac{N}{4\sqrt{\pi}}\frac{k^2}{\mu^2}e^{-h^2V^2}\left\{(3+2\tau^2)f(\tau)-\frac{1}{\sqrt{\pi}}\right\}, \qquad \ldots\ldots(3)$$

or, in terms of the ϕ function,

$$N(\mu,\theta) = \frac{N}{4\sqrt{\pi}}\frac{k^2}{\mu^2}\left\{\phi(hV,\theta-\theta_1)-\frac{1}{\sqrt{\pi}}e^{-h^2V^2}\right\}. \qquad \ldots\ldots(4)$$

7·43. *A special case of the density function.*

The analysis of section 7·41 breaks down when the value of k is zero. The density function is then given by

$$D(r) = \frac{A}{r}. \qquad \ldots\ldots(1)$$

Let N be the number of stars within a cone of solid angle S and extending to a distance R. Then

$$N = AS\int_0^R r\,dr,$$

so that

$$AS = \frac{2N}{R^2}. \qquad \ldots\ldots(2)$$

The formulae (5), (6), (7) and (19) of section 7·41 become

$$h\mu = a, \quad h\mu r-\tau = x, \quad y = \tau = hV\cos(\theta-\theta_1) \qquad \ldots\ldots(3)$$

and $M(\mu,\theta)$ is now given by

$$M(\mu,\theta) = \frac{ASe^{-h^2V^2}}{\pi h^2\mu^3}e^{\tau^2}\int_{-\tau}^{h\mu R-\tau}(x+\tau)^3e^{-x^2}dx.$$

If R is large we can write the upper limit of the integral as ∞, without introducing any appreciable error. It is found that

$$M(\mu,\theta) = \frac{CF(\tau)}{\mu^3}, \qquad \ldots\ldots(4)$$

where C is independent of V, μ and τ and

$$F(\tau) = (\tau^2+\tfrac{3}{2})\phi(hV,\theta-\theta_1)-\frac{1}{2\sqrt{\pi}}e^{-h^2V^2}. \qquad \ldots\ldots(5)$$

The number, $N(\mu,\theta)\,d\theta$, of stars with proper motions greater than μ in position angles between θ and $\theta+d\theta$ is then given by

$$N(\mu,\theta) = \frac{CF(\tau)}{2\mu^2}. \qquad \ldots\ldots(6)$$

7·44. *The number of stars in a drift with total proper motions greater than μ.*

Let $R(\mu)$ denote the number of stars in a region of the sky, belonging to a drift, with total proper motions exceeding a given value μ. Then

$$R(\mu) = \int_0^{2\pi} N(\mu, \theta)\, d\theta$$

or, from (26) of section 7·41,

$$R(\mu) = \frac{N}{2\sqrt{\pi}} e^{-h^2V^2} \int_0^{2\pi} \{f(\tau) - \eta^2 f(\eta\tau)\}\, d\theta. \qquad \ldots\ldots(1)$$

If $\rho\, d\theta$ is the number of stars moving between position angles θ and $\theta + d\theta$, no restriction being made on the magnitude of the proper motions, we have, from 2·3 (11),

$$\rho\, d\theta = \frac{N}{2\sqrt{\pi}} e^{-h^2V^2} f(\tau)\, d\theta,$$

in which τ has the signification of the previous sections. Now

$$\int_0^{2\pi} \rho\, d\theta = N,$$

where N is the total number of stars in the drift. Hence

$$\int_0^{2\pi} f(\tau)\, d\theta = 2\sqrt{\pi}\, e^{h^2V^2}. \qquad \ldots\ldots(2)$$

Similarly,

$$\int_0^{2\pi} f(\eta\tau)\, d\theta \equiv \int_0^{2\pi} f(\eta h V \cos\theta)\, d\theta$$

$$= 2\sqrt{\pi}\, e^{\eta^2 h^2 V^2}. \qquad \ldots\ldots(3)$$

Consequently,

$$R(\mu) = N\{1 - \eta^2 e^{-h^2V^2(1-\eta^2)}\}. \qquad \ldots\ldots(4)$$

7·5. *The pseudo-drift curve of restricted proper motions.*

It is found that the polar curve

$$\rho = N(\mu, \theta), \qquad \ldots\ldots(1)$$

representing the distribution of the restricted proper motions in position angle, resembles very closely a drift curve of which the velocity is greater than that in the true drift curve which would have been obtained for the same region if no restriction had been placed on the magnitude of the total proper motions. The curve given by (1) is called, in consequence, a *pseudo-drift curve* and we denote the pseudo-velocity by hV_1. As θ_1 is the position angle for the axis of symmetry of the true drift curve, it will also be the position angle of the axis of symmetry of the pseudo-drift curve, and we shall write

$$\tau_1 = hV_1 \cos(\theta - \theta_1). \qquad \ldots\ldots(2)$$

Identifying the curve, given by (1), with a drift curve relating to a total number, $R(\mu)$, of stars, we write

$$N(\mu,\theta) = \frac{R(\mu)}{2\sqrt{\pi}} e^{-h^2V_1^2} f(\tau_1); \qquad \ldots\ldots(3)$$

whence, using (26) of section 7·41 and (4) of section 7·44,

$$\frac{e^{-h^2V^2}\{f(\tau) - \eta^2 f(\eta\tau)\}}{1 - \eta^2 e^{-h^2V^2(1-\eta)^2}} = e^{-h^2V_1^2} f(\tau_1). \qquad \ldots\ldots(4)$$

It is to be remembered that (4) is not a true equation, as the sign of equality only implies that the function on the left can be represented satisfactorily by the drift function on the right in terms of a pseudo-drift velocity, hV_1.

The close approximation of the left-hand side of (4) to a drift curve is illustrated by the entries* in Table 40.

Table 40

$\theta-\theta_1$	A	B	$\theta-\theta_1$	A	B
0°	4·03	4·04	90°	1·20	1·20
10	3·96	3·96	100	0·99	1·00
20	3·75	3·73	110	0·81	0·85
30	3·42	3·37	120	0·69	0·73
40	3·01	2·94	130	0·60	0·64
50	2·57	2·51	140	0·53	0·57
60	2·13	2·12	150	0·48	0·52
70	1·76	1·76	160	0·42	0·49
80	1·44	1·45	170	0·41	0·48

The entries in the columns A are derived from the left-hand side of (4) for a true drift velocity, hV, of 0·5 and for $\eta = 0{\cdot}7$ (corresponding to $k = 2''{\cdot}0$ centennially and a minimum centennial proper motion of 2″). In the columns B are the values derived from the right-hand side of (4) with $hV_1 = 0{\cdot}6$. In each case the sums for three adjacent 10° sectors are given—this conforms to the usual practice of smoothing the counts in the 10° sectors.

A curve based on the entries, A, would evidently be indistinguishable from a curve based on the entries B; in other words, the frequency curve of the restricted proper motions is practically identical with a true drift curve. On analysing the statistics of the restricted proper motions (for which the true drift velocity is hV) we associate the curve with a drift curve in which the velocity is hV_1. Thus we can regard hV_1 as given by the observations of the restricted proper motions and it is then required to derive the true drift velocity hV pertaining to an assembly of stars in which there is no restriction as to the magnitude of the proper motions.

Table 41 gives the values of hV_1, corresponding to several values of hV and of k, when the limiting proper motion is 2″ per century.

* W. M. Smart, *M.N.* **89**, 96, 1928.

Table 41. *Values of hV_1 in terms of hV and k*

k (per century) \ hV	0·0	0·5	1·0	1·5
2·0	0·0	0·60	1·10	1·58
1·0	0·0	0·66	1·22	1·65
0·5	0·0	0·68	1·25	1·71
0·0	0·0	0·70	1·26	1·73

Except when η is close to unity (corresponding to values of k near zero—or to large values of the limiting proper motion, μ), the entries in this table have been derived by means of formula (4). When η is near unity, this formula is unsuitable and we proceed as follows.

Let
$$\eta = 1-\alpha,$$
where α is to be regarded as a small quantity. Up to terms* in α^2, we have

$$f(\tau)-\eta^2 f(\eta\tau) = \alpha\left\{2f(\tau)+\tau\frac{df}{d\tau}\right\}-\alpha^2\left\{f(\tau)+2\tau\frac{df}{d\tau}+\tfrac{1}{2}\tau^2\frac{d^2f}{d\tau^2}\right\}. \quad \ldots\ldots(5)$$

But
$$\frac{\sqrt{\pi}}{2}f(\tau) = \tfrac{1}{2}+\tau e^{\tau^2}\int_{-\tau}^{\infty} e^{-x^2}\,dx,$$

from which
$$\tau\frac{df}{d\tau} = (1+2\tau^2)f(\tau)-\frac{1}{\sqrt{\pi}},$$
$$\frac{d^2f}{d\tau^2} = (6+4\tau^2)f(\tau)-\frac{2}{\sqrt{\pi}}.$$

Inserting these in (5), we obtain
$$e^{-h^2V^2}\{f(\tau)-\eta^2 f(\eta\tau)\} = A\alpha-B\alpha^2,$$

where
$$A = e^{-h^2V^2}\left\{(3+2\tau^2)f(\tau)-\frac{1}{\sqrt{\pi}}\right\},$$
$$B = e^{-h^2V^2}\left\{(3+7\tau^2+2\tau^4)f(\tau)-\frac{1}{\sqrt{\pi}}(2+\tau^2)\right\},$$

or, in a form more suitable for computation,
$$A = (3+2\tau^2)\,\phi(hV,\theta-\theta_1)-\frac{1}{\sqrt{\pi}}e^{-h^2V^2},$$
$$B = (3+7\tau^2+2\tau^4)\,\phi(hV,\theta-\theta_1)-\frac{1}{\sqrt{\pi}}(2+\tau^2)\,e^{-h^2V^2}.$$

In a similar manner we obtain
$$1-\eta^2 e^{-h^2V^2(1-\eta^2)} = F\alpha-G\alpha^2,$$

where
$$F = 2(1+h^2V^2), \quad \ldots\ldots(6)$$
$$G = 1+5h^2V^2+2h^4V^4.$$

* For the expansions up to terms in α^3, *v.* W. M. Smart, *M.N.* 89, 97, 1928.

Thus (4) becomes

$$F(hV,\eta,\tau)\equiv\frac{A-B\alpha}{F-G\alpha}=\phi(hV_1,\theta-\theta_1). \qquad ......(7)$$

The values of A, B, F and G and the values of the function $F(hV,\eta,\tau)$ were found at intervals of 10° or 20° in $(\theta-\theta_1)$. These latter values are to be associated with the corresponding values for a true drift curve with velocity hV_1, so chosen as to give as satisfactory a representation as possible of the function $F(\eta,\tau)$.

In the limit when $\eta = 1$, (7) reduces to

$$\frac{(3+2\tau^2)\,\phi(hV,\theta-\theta_1)-\frac{1}{\sqrt{\pi}}e^{-h^2V^2}}{2(1+h^2V^2)}=\phi(hV_1,\theta-\theta_1). \qquad ......(8)$$

This is the form corresponding to $k=0$, the density function $D(r)$ being given by

$$D(r)=\frac{A}{r}.$$

Fig. 47* illustrates the results for $hV=0{\cdot}5$. The curve A is the true drift curve in polar coordinates, with $\theta_1=0$. Curve B is drawn from the entries A of Table 40 and is the frequency curve of the restricted proper motions (with $\mu=2''$ as the limiting centennial value) for $k=2{\cdot}0$. It is indistinguishable from a true drift curve with velocity 0·6. Curve C, derived from (8), corresponds to $k=0{\cdot}0$; the differences between this curve and a true drift curve with velocity 0·70 are so small that in practical applications C would be identified with the latter.

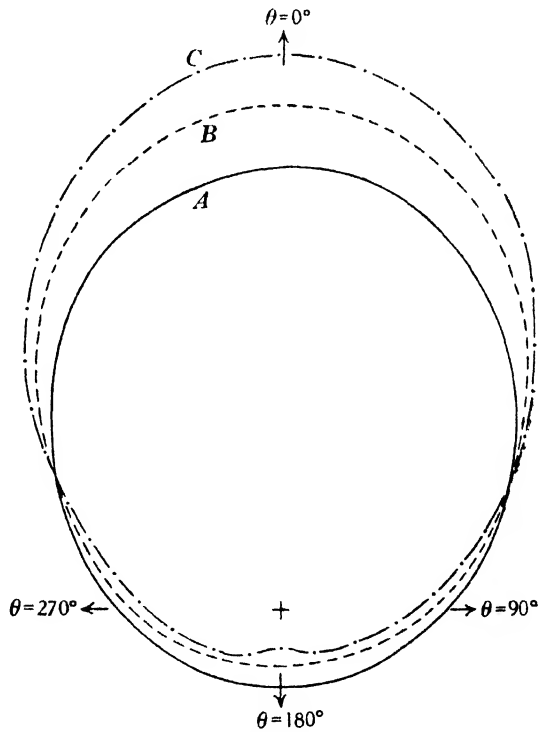

Fig. 47

* *M.N.* 89, 99, 1928.

7·6. *Tests of the density formula.*

There are several ways in which the formulae of the previous sections may be tested. In each case we assume that, for a given region of the sky, the analysis of all the proper motions, whatever their magnitudes may be, into two constituent drifts has been made. We thus know the constants of the drifts for the region concerned.

Method (*a*).

If the stars are sufficiently numerous we can find the observed numbers of stars moving in an assigned sector, say $\theta - 5°$ to $\theta + 5°$, with centennial proper motions* greater than $2''$, $3''$, $4''$ and so on. From 7·41 (13) we can calculate the corresponding values of $M(\mu, \theta)$ for two or three values of k, say 0·0, 1·0 and 2·0. If the density law is satisfied, the observed quantities should satisfy the theoretical curve $y = M(\mu, \theta)$, in which θ is constant, and consequently it should be possible to estimate the value of k. The process can be repeated for different sectors and it will be seen whether the deduced values of k are all consistent. This method, however, is hardly practicable at present, as, for a satisfactory application, the observational material is not yet sufficiently abundant.

Method (*b*).

The frequency curves of the restricted proper motions, for a given minimum value of μ, can be analysed into two pseudo-drift curves. We obtain, then, the two pseudo-drift velocities, hV_1, for the two drifts. The corresponding true drift velocities, hV, being known, the data of Table 41 will enable us to find the appropriate value of k for each drift. Repeating the process for a new minimum value of μ, we ought to find again the same values of k. The method suffers from the same defect as (*a*), namely, the present inadequacy of the observational material.

Method (*c*).

In this method, which is at present the most practicable of the three, we find the total number of stars in the region with proper motions exceeding a given value of μ. If $R(\mu)$ denotes this number, we have

$$R(\mu) = R_1(\mu) + R_2(\mu),$$

in which R_1 and R_2 are each given by 7·44 (4) in terms of the total numbers N_1 and N_2 of stars in the true drifts and of the corresponding true drift velocities. For a given value of k, we can then draw the curve

$$y = R(\mu)$$

for values of μ, say, $2''$, $3''$, $4''$, ...

* These should first be corrected for accidental errors.

Fig. 48 shows two such curves* calculated for $k = 0$ and $k = 1{\cdot}6$, the velocities of the true drifts being 1·5 and 0·7. In each curve the number of stars corresponding to $\mu = 2''$ has been taken arbitrarily to be 50. The curves are well separated and, if the observational data are in accordance with the theoretical considerations on which the method is based, it ought to be possible to estimate k for a given region to within 0·2 or so.

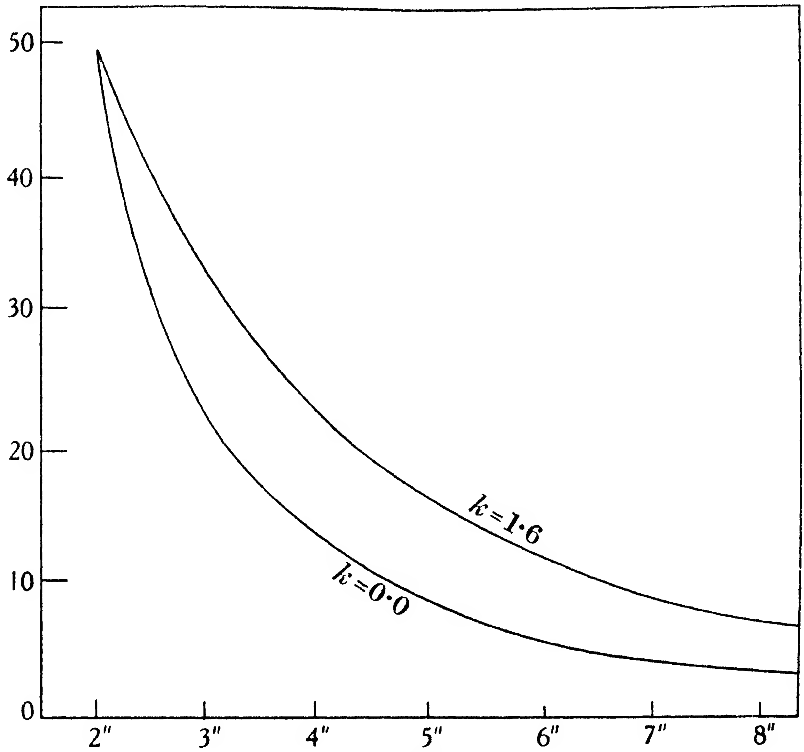

Fig. 48

The method was applied to the Cambridge proper motions (*loc. cit.*) with $2''$ per century as the minimum proper motion. A positive result of the investigation—the observational material was comparatively meagre—was that k increased with galactic latitude. Since, by a modification of 7·23 (3) in accordance with the change of notation in section 7·41, the mean parallax $\bar{p}$ is $\sqrt{\pi}\, hk$, this result is in accordance with the observed fact that the mean parallaxes of the stars within a given range of magnitude increase with galactic latitude. It was also estimated that, for galactic latitudes 0°, 30° and 60°, k had the values of 0·3, 1·0 and 1·6 approximately. According to

* *M.N.* **88**, 581, 1928.

these results the mean parallax increases at a greater rate than that indicated by investigations based on the methods of Chapter VI.

When the limiting value of μ is large, the value of η to be used in the formula for $R(\mu)$ is close to unity and the relation between the true drift velocity and the pseudo-drift velocity is the same as if k were close to zero in the entries of Table 41. From this point of view, Dyson's proper motions,* exceeding 20″ per century, have been re-analysed.† From the values of the pseudo-drift velocities, the true drift velocities were deduced in the manner already described. As the coordinates of the drift apices may be assumed known, the space velocities of the drifts can be readily calculated and these were found to be in good agreement with the values derived by the methods of Chapter IV.

7·71. *The density law and the ellipsoidal hypothesis.*

We now undertake a parallel investigation on‡ the basis of the ellipsoidal hypothesis. Let the equation of the velocity ellipse for a given region with N stars be

$$g^2u^2+h^2v^2 = 1.$$

Then the number of stars with transverse linear velocities between (u, v) and $(u+du, v+dv)$ is

$$\frac{Ngh}{\pi}e^{-g^2u^2-h^2v^2}\,du\,dv$$

and, consequently, the number within these limits of velocity in the volume-element Sr^2dr of the cone, defined by the region, is

$$\frac{Sgh}{\pi}r^2D(r)\,e^{-g^2u^2-h^2v^2}\,du\,dv\,dr.$$

Let U, V be the components of the parallactic motion and w the resultant transverse linear velocity making an angle with the u-axis. Then

$$u = w\cos\theta-U, \quad v = w\sin\theta-V.$$

Also

$$du\,dv = w\,dw\,d\theta.$$

Let $M(\mu,\theta)\,d\mu\,d\theta$ denote, as before, the number of total proper motions between μ and $\mu+d\mu$ in the sector θ to $\theta+d\theta$. Since $w = r\mu$, we obtain, on writing $\frac{A}{r}e^{-h^2k^2r^2}$ for $D(r)$,

$$M(\mu,\theta) = \frac{ASgh}{\pi}\mu\int_0^\infty r^3e^{-h^2k^2r^2-g^2(r\mu\cos\theta-U)^2-h^2(r\mu\sin\theta-V)^2}\,dr$$

$$\equiv\frac{ASgh}{\pi}\mu\int_0^\infty r^3e^{-(a^2r^2-2br+c^2)}\,dr,$$

* *Proc. Roy. Soc. Edin.* **28**, pt. iii, 231, 1908.

† *M.N.* **89**, 101, 1928. For further applications of the method, *v. M.N.* **89**, 147, 1928; **90**, 112, 1929.

‡ W. M. Smart, *M.N.* **89**, 105, 1928.

where
$$a^2 = h^2k^2+\mu^2(g^2\cos^2\theta+h^2\sin^2\theta), \qquad \ldots\ldots(1)$$
$$b = \mu(g^2U\cos\theta+h^2V\sin\theta), \qquad \ldots\ldots(2)$$
$$c^2 = g^2U^2+h^2V^2. \qquad \ldots\ldots(3)$$

Write
$$\frac{b}{a} = y, \quad ar = x+y. \qquad \ldots\ldots(4)$$

Then
$$M(\mu,\theta) = \frac{ASgh\mu}{\pi a^4}e^{y^2-c^2}\int_{-y}^{\infty}(x+y)^3e^{-x^2}dx.$$

Using the results of section 7·41, we have
$$AS = 2Nh^2k^2$$

and
$$M(\mu,\theta) = \frac{Ngh^3k^2\mu}{\pi a^4}e^{-c^2}[(2y^3+3y)e^{y^2}G(y)+y^2+1]. \qquad \ldots\ldots(5)$$

This equation determines the function $M(\mu,\theta)$.

Let $N(\mu,\theta)\,d\theta$ denote the number of proper motions exceeding μ in the sector θ to $\theta+d\theta$. Then
$$N(\mu,\theta) = \int_{\mu}^{\infty}M(\mu,\theta)\,d\mu. \qquad \ldots\ldots(6)$$

Now
$$y^2 = \frac{b^2}{a^2} = \frac{\mu^2(g^2U\cos\theta+h^2V\sin\theta)^2}{h^2k^2+\mu^2(g^2\cos^2\theta+h^2\sin^2\theta)}.$$

Put
$$\alpha^2 = g^2\cos^2\theta+h^2\sin^2\theta, \qquad \ldots\ldots(7)$$
$$\beta = g^2U\cos\theta+h^2V\sin\theta. \qquad \ldots\ldots(8)$$

Then
$$y^2 = \frac{\beta^2\mu^2}{h^2k^2+\alpha^2\mu^2}, \qquad \ldots\ldots(9)$$

from which
$$y\,dy = \frac{\beta^2h^2k^2}{a^4}\mu\,d\mu. \qquad \ldots\ldots(10)$$

From (5), (6) and (10), we obtain
$$N(\mu,\theta) = \frac{Ngh}{\pi\beta^2}e^{-c^2}\int_{y}^{y_1}[(2y^4+3y^2)e^{y^2}G(y)+y^3+y]\,dy,$$
where $y_1 = \beta/\alpha$, corresponding to $\mu = \infty$.

The integral on the right has been evaluated in section 7·41. We then obtain, in terms of Eddington's function,
$$N(\mu,\theta) = \frac{Ngh}{2\alpha^2\sqrt{\pi}}e^{-c^2}\left[f\left(\frac{\beta}{\alpha}\right)-\frac{\alpha^2\mu^2}{h^2k^2+\alpha^2\mu^2}f(y)\right]. \qquad \ldots\ldots(11)$$

This last equation gives the distribution of the restricted proper motions in position angle. It can be readily verified that (11) reduces to 7·41 (26) when g is put equal to h—the distribution is then that of a single drift.

7·72. *Limiting case for $k/\mu \to 0$.*

Regard k/μ as a small quantity. From (9) of the previous section,

$$y = \frac{\beta}{\alpha} - \frac{\beta h^2 k^2}{2\mu^2 \alpha^3}.$$

Hence
$$f(y) = f\left(\frac{\beta}{\alpha}\right) - \frac{\beta h^2 k^2}{2\mu^2\alpha^3}\left(\frac{\partial f}{\partial y}\right)_0,$$

where $\left(\frac{\partial f}{\partial y}\right)_0$ denotes the value of $\frac{\partial f}{\partial y}$ when $y = \frac{\beta}{\alpha}$. Consequently,

$$f\left(\frac{\beta}{\alpha}\right) - \frac{\alpha^2\mu^2}{h^2k^2 + \alpha^2\mu^2} f(y) = \frac{h^2k^2}{\mu^2\alpha^2}\left\{f\left(\frac{\beta}{\alpha}\right) + \frac{1}{2}\frac{\beta}{\alpha}\left(\frac{\partial f}{\partial y}\right)_0\right\}. \quad \text{......(1)}$$

Also,
$$y\frac{\partial f}{\partial y} = (2y^2 + 1) f(y) - \frac{1}{\sqrt{\pi}}.$$

After some reduction, we obtain, from (11) of section 7·71,

$$N(\mu, \theta) = \frac{Ngh^3k^2 e^{-c^2}}{4\mu^2\alpha^4\sqrt{\pi}}\left[\left\{3 + 2\left(\frac{\beta}{\alpha}\right)^2\right\} f\left(\frac{\beta}{\alpha}\right) - \frac{1}{\sqrt{\pi}}\right], \quad \text{......(2)}$$

which may be written
$$N(\mu, \theta) = \frac{Ck^2 F(\theta)}{\mu^2}, \quad \text{......(3)}$$

where C is a constant for the region.

This is a similar result to that in 7·42 (6).

7·73. *The number of stars in the region with total proper motions exceeding μ.*

As before, let $R(\mu)$ denote the number of stars in the region with total proper motions greater than μ in all position angles. Then

$$R(\mu) = \int_0^{2\pi} N(\mu, \theta)\, d\theta$$
$$= \frac{Ngh}{2\sqrt{\pi}} e^{-c^2}\left[\int_0^{2\pi} \frac{1}{\alpha^2} f\left(\frac{\beta}{\alpha}\right) d\theta - \int_0^{2\pi} \frac{\mu^2}{h^2k^2 + \alpha^2\mu^2} f(y)\, d\theta\right]. \quad \text{......(1)}$$

Let $\rho\, d\theta$ be the number of stars in the region with total proper motions between 0 and ∞ in the sector θ to $\theta + d\theta$. Then

$$\rho = N(0, \infty),$$

and from (11) of section 7·71 we have, putting $\mu = 0$ in this formula,

$$\rho = \frac{Ngh}{2\sqrt{\pi}} e^{-c^2} \frac{1}{\alpha^2} f\left(\frac{\beta}{\alpha}\right).$$

Now
$$\int_0^{2\pi} \rho\, d\theta = N.$$

Hence $$\int_0^{2\pi} \frac{1}{\alpha^2} f\left(\frac{\beta}{\alpha}\right) d\theta = \frac{2\sqrt{\pi}}{gh} e^{c^2}, \qquad \text{......(2)}$$

where $$c^2 = g^2U^2 + h^2V^2. \qquad \text{......(3)}$$

Consider now the second integral on the right of (1) and denote it by I. Writing A^2 for $\frac{h^2k^2 + \alpha^2\mu^2}{\mu^2}$, we have, since

$$\alpha^2 = g^2 \cos^2\theta + h^2 \sin^2\theta, \qquad \text{......(4)}$$

$$A^2 = \frac{h^2k^2}{\mu^2} + g^2 + (h^2 - g^2)\sin^2\theta$$

$$\equiv G^2 + (H^2 - G^2)\sin^2\theta,$$

or $$A^2 = G^2\cos^2\theta + H^2\sin^2\theta, \qquad \text{......(5)}$$

where $$G^2 = g^2 + \frac{h^2k^2}{\mu^2}, \quad H^2 = h^2 + \frac{h^2k^2}{\mu^2}. \qquad \text{......(6)}$$

Comparing (4) and (5), we see that A^2 and α^2 are analogous functions of θ.

Also, from 7·71 (9),

$$y^2 = \frac{\beta^2}{A^2}, \qquad \text{......(7)}$$

in which $$\beta = g^2U\cos\theta + h^2V\sin\theta$$

$$\equiv G^2U_1\cos\theta + H^2V_1\sin\theta \equiv B, \qquad \text{......(8)}$$

where $$G^2U_1 = g^2U, \quad H^2V_1 = h^2V. \qquad \text{......(9)}$$

From (7) and (8), we have

$$I = \int_0^{2\pi} \frac{1}{A^2} f\left(\frac{B}{A}\right) d\theta,$$

in which B and A are functions of θ analogous to β and α.

We can now write down the value of I, by means of (2); the result is

$$I = \frac{2\sqrt{\pi}}{GH} e^{C^2}, \qquad \text{......(10)}$$

where C is analogous to c and is given by

$$C^2 = G^2U_1^2 + H^2V_1^2. \qquad \text{......(11)}$$

Using (6) and (9), we find that

$$C^2 = \frac{g^2U^2}{1 + \frac{h^2k^2}{g^2\mu^2}} + \frac{h^2V^2}{1 + \frac{k^2}{\mu^2}}. \qquad \text{......(12)}$$

We have now from (1), with the help of (2) and (10),

$$R(\mu) = N\left\{1 - \frac{gh}{GH} e^{C^2 - c^2}\right\}, \qquad \text{......(13)}$$

or, writing
$$\epsilon = \frac{g}{h}, \qquad \ldots\ldots(14)$$

$$R(\mu) = N\left[1 - \frac{\epsilon\mu^2}{(k^2+\epsilon^2\mu^2)^{\frac{1}{2}}(k^2+\mu^2)^{\frac{1}{2}}} e^{-k^2\left\{\frac{g^2U^2}{k^2+\epsilon^2\mu^2}+\frac{h^2V^2}{k^2+\mu^2}\right\}}\right]. \qquad \ldots\ldots(15)$$

The value of $R(\mu)$ can be obtained from (15) when ϵ (the ratio of the axes in the velocity ellipse) and gU, hV are known from the analysis of all the stars in the region, without restriction as to the magnitude of the proper motions, following the method of Chapter v

7·81. *Comparison of $R(\mu)$ as obtained from the ellipsoidal and two-streams theories.*

The following table* shows the values of $R(\mu)$, as calculated on the two theories, for a typical region; E. denotes "ellipsoidal theory" and T.S. "two-streams theory"; μ and k are expressed in centennial measure; it is assumed that $R(\mu) = 100$ for $\mu = 2''$ in each instance.

Table 42. *Values of $R(\mu)$*

μ	$k = 2''\!.0$		$k = 1''\!.0$		$k = 0''\!.5$		$k = 0''\!.0$	
	E.	T.S.	E.	T.S.	E.	T.S.	E.	T.S.
2″	100	100	100	100	100	100	100	100
4	54·4	53·6	34·7	34·3	27·7	27·3	25·0	25·0
6	30·2	29·2	17·0	16·3	12·6	12·6	11·1	11·1
8	19·2	18·4	9·8	9·4	7·1	7·1	6·2	6·2

Within the range of μ considered in the table, the two formulae for $R(\mu)$ lead to very much the same numerical results, and it is hardly likely that the values of $R(\mu)$ obtained from actual counts of observed proper motions would be able to differentiate between the two theories. If we continue our calculations for limiting values of $\mu > 8''$, the agreement between the pairs of columns would become more exact, since, in the limit as $k/\mu \to 0$, the two expressions for $R(\mu)$ are of the same form, namely C/μ^2.

It may be concluded that the frequency distribution of restricted proper motions is practically identical in the two theories.

7·82. *An approximate relation between the ellipsoidal and the two-streams constants.*

In the preceding section we have seen that if k/μ is small the two expressions for $R(\mu)$ derived on the ellipsoidal and two-streams theories may be taken to be identical.

* *M.N.* 89, 112, 1928.

Taking the ellipsoidal formula for $R(\mu)$, as in 7·73 (15), and writing it in the form

$$R(\mu) = N(1 - Pe^{-Q}),$$

we find, up to the second order in k/μ (which we regard as a small quantity),

$$P = 1 - \frac{k^2}{2\mu^2}\left(1 + \frac{1}{\epsilon^2}\right),$$

$$Q = \frac{k^2h^2}{\mu^2}(U^2 + V^2),$$

so that

$$R(\mu) = \frac{Nk^2}{2\mu^2}\left\{2h^2(U^2 + V^2) + 1 + \frac{1}{\epsilon^2}\right\}. \qquad \ldots\ldots(1)$$

On the two-streams theory,

$$R(\mu) = R_1(\mu) + R_2(\mu),$$

where $R_1(\mu)$ refers to drift I with constants N_1, hV_1, and $R_2(\mu)$ to drift II with constants N_2, hV_2. Now

$$R_1(\mu) = N_1\{1 - \eta^2 e^{-h^2V_1^2(1-\eta^2)}\}.$$

Write $\alpha = 1 - \eta$ (since k/μ is small, η is a little less than unity); then

$$\alpha = 1 - \frac{\mu}{(\mu^2 + k^2)^{\frac{1}{2}}} = \frac{k^2}{2\mu^2}$$

and, with the same approximation as before and using 7·5 (6),

$$R_1(\mu) = \frac{N_1k^2}{\mu^2}(1 + h^2V_1^2).$$

Assuming that the two drifts contain equal numbers of stars, we have

$$R(\mu) = \frac{Nk^2}{2\mu^2}(2 + h^2V_1^2 + h^2V_2^2). \qquad \ldots\ldots(2)$$

Equating (1) and (2), we obtain

$$\frac{1}{\epsilon^2} = 1 + h^2V_1^2 + h^2V_2^2 - 2h^2(U^2 + V^2). \qquad \ldots\ldots(3)$$

Now $h(U^2 + V^2)^{\frac{1}{2}}$ is the transverse parallactic motion, for the region concerned, in terms of the theoretical unit $1/h$, so that if hV_0 denotes the solar motion and λ the angular distance of the region from the solar antapex,

$$h(U^2 + V^2)^{\frac{1}{2}} = hV_0 \sin\lambda.$$

Accordingly, (3) becomes

$$\frac{1}{\epsilon^2} = 1 + h^2V_1^2 + h^2V_2^2 - 2h^2V_0^2 \sin^2\lambda. \qquad \ldots\ldots(4)$$

This equation enables us to calculate the ratio of the axes of the velocity ellipse when the drift velocities and the solar motion are known.

Considering the region discussed in section 5·32, we have

$$hV_1 = 1{\cdot}5, \quad hV_2 = 0{\cdot}8.$$

From the complete analysis* of eight regions of which the preceding region is one, $hV_0 = 0{\cdot}88$, and from the given position of the solar apex it is found that $\lambda = 120°$ for the region under consideration. Inserting these numerical values in (4), we derive

$$\epsilon \equiv \frac{g}{h} = 0{\cdot}61.$$

The analysis, in section 5·32, of the proper motions in the same region gave

$$\epsilon = 0{\cdot}63.$$

The two results, obtained by widely different methods, are in good agreement.

* W. M. Smart, *M.N.* 87, 137, 1926.

CHAPTER VIII

GENERAL THEOREMS OF STELLAR STATISTICS

8·11. *Introduction.*

In previous chapters we have dealt with several types of investigations by making in each an initial assumption concerning the law governing the distribution of a particular characteristic amongst the stars; by following out the implications of such an assumption we have been able to compare theoretical conclusions with observed facts. If there is reasonable agreement between theory and observation, we are entitled to affirm that the original assumption is likely to be of the character of a fundamental law or, at least, a good approximation to an actual law. For example, we have studied the implications of the ellipsoidal distribution of peculiar velocities and have shown that the observed features of stellar motions support the original assumption. A statement of the same character may be made equally well as regards the two-streams theory. But the true frequency function of peculiar velocities cannot be represented exactly by both theoretical distributions and all we conclude is that the true distribution can be imitated very successfully by either of the two theories.

In this chapter we invert, to a large extent, the procedure just outlined and we show how it is possible to deduce from observation certain frequency functions* associated with various characteristics of the stars.

We have to distinguish between an observed or *apparent* characteristic of a star and the real or *absolute* characteristic. The apparent brightness of a star, for example, depends on the star's intrinsic luminosity and on its distance from us. We assume at first that interstellar space is perfectly transparent; in later sections we consider the effects of a galactic absorbing medium. We can obtain from observation the distribution of apparent luminosities, but this does not represent something of fundamental importance. What is of importance to an understanding of the stellar system is the manner in which the stars are arranged according to intrinsic luminosity. This is but one example.

8·12. *Two fundamental theorems.*

Consider an absolute characteristic X; this may be absolute magnitude or intrinsic luminosity or peculiar linear velocity (the latter being supposed here to be independent of position in the galaxy). We denote by x the corre-

* Most of these in sections 8·1–8·3 are due to Schwarzschild (*A.N.* **190**, 361, 1912).

sponding apparent characteristic (apparent magnitude or apparent luminosity or proper motion). If r denotes the distance of a star from the sun (we shall generally measure r in parsecs), we have that x is a function of X and r, that is,

$$x = f(X, r),$$

and inversely,

$$X = F(x, r). \qquad \ldots\ldots(1)$$

We shall suppose that $\phi(X)$ is the frequency function of X, so that the proportion of stars with the characteristic X between X and $X+dX$ is $\phi(X)\,dX$; consequently, if X_1 and X_2 are the limiting values of X for the assembly of stars concerned,

$$\int_{X_1}^{X_2} \phi(X)\,dX = 1. \qquad \ldots\ldots(2)$$

Let $D(r)$ be the density function. If a small region of the sky subtends a solid angle S, the element of volume of the cone with generators passing through the periphery of the region is Sr^2dr and the number of stars in this element of volume is

$$Sr^2 D(r)\,dr.$$

Of these stars, the number with the absolute characteristic between X and $X+dX$ is

$$Sr^2 D(r)\,\phi(X)\,dr\,dX,$$

and these will be observed to have the apparent characteristic between x and $x+dx$.

Also, for the element of volume considered, r may be regarded as constant; and

$$dr\,dX \equiv \frac{\partial(r, X)}{\partial(r, x)}\,dr\,dx = \frac{\partial X}{\partial x}\,dr\,dx.$$

Consequently, the number of stars in the cone at distances between r and $r+dr$, and with apparent characteristics between x and $x+dx$, is

$$dN\,dx \equiv Sr^2 D(r)\,\phi(X)\frac{\partial X}{\partial x}\,dr\,dx, \qquad \ldots\ldots(3)$$

in which X is expressed as a function of r and x by means of (1).

Let now $b(x)\,dx$ denote the total number of stars within the cone with the apparent characteristic between x and $x+dx$. Then the function $b(x)$ is given by summing the expression on the right of (3) for all values of r between 0 and ∞; hence

$$b(x) = S\int_0^\infty r^2 D(r)\,\phi(X)\frac{\partial X}{\partial x}\,dr. \qquad \ldots\ldots(4)$$

This is the first theorem.

Let $p(x)$ denote the mean parallax of stars with the apparent characteristic x. Defining the parallax, p, of a single star by $1/r$ so that, as r is measured in parsecs, p will be given in seconds of arc, we have from (3)

$$p(x) = \int_{r=0}^{\infty} \frac{1}{r} dN \div \int_{r=0}^{\infty} dN,$$

whence

$$p(x) = \frac{\int_0^\infty r\, D(r)\, \phi(X) \frac{\partial X}{\partial x}\, dr}{\int_0^\infty r^2 D(r)\, \phi(X) \frac{\partial X}{\partial x}\, dr} \qquad \ldots\ldots(5)$$

or

$$b(x)\,.\,p(x) = S \int_0^\infty r\, D(r)\, \phi(X) \frac{\partial X}{\partial x}\, dr. \qquad \ldots\ldots(6)$$

This is the second theorem.

If the functions $b(x)$ and $p(x)$ can be determined from the observations, (4) and (6) are two integral equations from which the functions $D(r)$ and $\phi(X)$ can theoretically be determined.

We can extend the previous procedure to the case when ϕ is a function of two (or more) absolute characteristics X_1, X_2. Then if $b(x_1, x_2)\, dx_1 dx_2$ denotes the number of stars with apparent characteristics between x_1 and $x_1 + dx_1$, and between x_2 and $x_2 + dx_2$, it is clear that $b(x_1, x_2)$ will be given by

$$b(x_1, x_2) = S \int_0^\infty r^2 D(r)\, \phi(X_1, X_2) \frac{\partial X_1}{\partial x_1} \cdot \frac{\partial X_2}{\partial x_2}\, dr, \qquad \ldots\ldots(7)$$

in which X_1 is supposed to be expressed in terms of r and x_1, and X_2 in terms of r and x_2. There is an equation, similar to (5), giving the mean parallax function $p(x_1, x_2)$.

8·13. *Apparent and absolute luminosities.*

Let L and l denote the absolute and apparent luminosities of a star. We define the former as the luminosity the star would appear to have if it were at unit distance ($r = 1$). Then, since the apparent brightness of a star varies inversely as the square of the distance,

$$L = lr^2. \qquad \ldots\ldots(1)$$

In (4) and (6) of section 8·12, X and x are to be replaced by L and l respectively. Also

$$\frac{\partial L}{\partial l} = r^2.$$

Hence

$$b(l) = S \int_0^\infty r^4 D(r)\, \phi(lr^2)\, dr \qquad \ldots\ldots(2)$$

and

$$a(l) \equiv b(l)\,.\,p(l) = S \int_0^\infty r^3 D(r)\, \phi(lr^2)\, dr. \qquad \ldots\ldots(3)$$

In these formulae we have explicitly assumed that a function, $\phi(L)$, of the absolute luminosities, independent of r, exists for the stars concerned. In practice we associate a function ϕ with each of the various spectral divisions (or subdivisions) in the Hertzsprung-Russell diagram, distinguishing between giants and the stars of the main series.

8·14. *The total apparent brightness in any region of the sky.*

Considering stars of a single spectral division, the mean absolute luminosity, $\bar{L}$, is given by

$$\bar{L} = \int_0^\infty L\,\phi(L)\,dL. \qquad \text{......(1)}$$

If we assume that the mean absolute luminosity of the stars in the element of volume, $Sr^2 dr$, of the cone is given by (1), these stars will have an apparent luminosity $\bar{L}/r^2$ on the average. As the number of stars in the volume-element is $SD(r)\,r^2 dr$, the total apparent luminosity arising from these stars is $\bar{L}S\,D(r)\,dr$.

We then have for the total apparent luminosity, λ, for the region

$$\lambda = \bar{L}S\int_0^\infty D(r)\,dr,$$

equivalent to the brightness of a single star of apparent magnitude $-2{\cdot}5\,\text{Log}\,\lambda$, assuming that zero apparent magnitude corresponds to $l = 1$.

The stars of the other spectral divisions furnish similar results.

8·15. *Apparent and absolute magnitudes.*

The absolute magnitude, M, of a star is given in section 1·24 in terms of the apparent magnitude m and parallax p by

$$M = m + 5 + 5\,\text{Log}\,p. \qquad \text{......(1)}$$

M is thus defined in terms of the standard distance of 10 parsecs.

We write, for convenience,

$$M_1 = M - 5, \qquad \text{......(2)}$$

so that M_1 is defined in terms of the standard distance of 1 parsec, and

$$M_1 = m + 5\,\text{Log}\,p. \qquad \text{......(3)}$$

In this formula the base of the logarithm is 10.

We shall refer to M_1 in this connection as the *modified absolute magnitude*.

Also, with r measured in parsecs, we have $p = 1/r$, and so

$$M_1 = m - 5\,\text{Log}\,r. \qquad \text{......(4)}$$

Let $\phi(M_1)$ denote the frequency function of the modified absolute magnitudes. From (4),

$$\frac{\partial M_1}{\partial m} = 1.$$

Hence, $$b(m) = S\int_0^\infty r^2 D(r)\,\phi(m-5\,\mathrm{Log}\,r)\,dr \qquad \ldots\ldots(5)$$

and $$b(m)\,.\,p(m) = S\int_0^\infty r\,D(r)\,\phi(m-5\,\mathrm{Log}\,r)\,dr. \qquad \ldots\ldots(6)$$

In these formulae, $b(m)$ is the number and $p(m)$ is the mean parallax of stars of apparent magnitude m.

We write $$\rho = -5\,\mathrm{Log}\,r,$$

so that $$r = e^{-c\rho}, \qquad \ldots\ldots(7)$$

where $$c = \tfrac{1}{5}\log_e 10 = 0{\cdot}4605. \qquad \ldots\ldots(8)$$

Then $$b(m) = cS\int_{-\infty}^{\infty} e^{-3c\rho}\,D(e^{-c\rho})\,\phi(m+\rho)\,d\rho, \qquad \ldots\ldots(9)$$

$$b(m)\,.\,p(m) = cS\int_{-\infty}^{\infty} e^{-2c\rho}\,D(e^{-c\rho})\,\phi(m+\rho)\,d\rho, \qquad \ldots\ldots(10)$$

or, on setting $$\Delta(\rho) \equiv cSe^{-3c\rho}\,D(e^{-c\rho}), \qquad \ldots\ldots(11)$$

$$b(m) = \int_{-\infty}^{\infty} \Delta(\rho)\,\phi(m+\rho)\,d\rho, \qquad \ldots\ldots(12)$$

$$a(m) \equiv b(m)\,.\,p(m) = \int_{-\infty}^{\infty} e^{c\rho}\,\Delta(\rho)\,\phi(m+\rho)\,d\rho. \qquad \ldots\ldots(13)$$

If the functions $b(m)$ and $a(m)$ can be derived from observations, the functions $\Delta(\rho)$ and $\phi(m+\rho)$ can theoretically be found by means of (12) and (13).

8·16. *Linear velocity and proper motion.*

Let T denote the transverse linear velocity of a star in a given region and μ the corresponding total annual proper motion. Then

$$T = \mu r. \qquad \ldots\ldots(1)$$

With r measured in parsecs and μ in seconds of arc, T will be measured in terms of the unit κ or 4·74 km./sec. Here T and μ correspond to the characteristics X and x.

Let $\psi(T)$ be the frequency function of the linear velocities. Then by (4) and (6) of section 8·12,

$$b(\mu) = S\int_0^\infty r^3 D(r)\,\psi(\mu r)\,dr, \qquad \ldots\ldots(2)$$

$$a(\mu) \equiv b(\mu)\,.\,p(\mu) = S\int_0^\infty r^2 D(r)\,\psi(\mu r)\,dr, \qquad \ldots\ldots(3)$$

in which $b(\mu)$ and $p(\mu)$ are respectively the number and mean parallax of stars with proper motion μ.

Set $$r = e^{\rho} \quad \text{and} \quad \mu = e^{\alpha}. \qquad \ldots\ldots(4)$$

Then (2) becomes

$$b(\mu) \equiv b_1(\alpha) = S \int_{-\infty}^{\infty} e^{4\rho} D(e^{\rho}) \psi(e^{\rho+\alpha}) \, d\rho, \qquad \ldots\ldots(5)$$

and setting $$Se^{4\rho} D(e^{\rho}) = \Delta_1(\rho), \qquad \ldots\ldots(6)$$

$$\psi(e^{\rho+\alpha}) = \psi_1(\rho+\alpha), \qquad \ldots\ldots(7)$$

the formula (5) becomes

$$b(\mu) \equiv b_1(\alpha) = \int_{-\infty}^{\infty} \Delta_1(\rho) \psi_1(\rho+\alpha) \, d\rho. \qquad \ldots\ldots(8)$$

Similarly,

$$a(\mu) \equiv a_1(\alpha) = b(\mu) . p(\mu) = \int_{-\infty}^{\infty} e^{-\rho} \Delta_1(\rho) \psi_1(\rho+\alpha) \, d\rho. \quad \ldots\ldots(9)$$

The formulae (8) and (9) are integral equations from which the functions $\Delta_1(\rho)$ and $\psi_1(\rho+\alpha)$ can be determined when the functions $b(\mu)$ and $a(\mu)$ have been obtained from observations.

8·17. *The number and mean parallax of stars of magnitude m and proper motion μ.*

Denote, as before, the frequency function of the modified absolute magnitudes by $\phi(M_1)$ and the frequency function of the linear velocities by $\psi(T)$.

Let $b(m, \mu) \, dm \, d\mu$ be the number of stars with apparent magnitudes between m and $m+dm$ and proper motions between μ and $\mu+d\mu$. Then, assuming that there is no correlation between M_1 and T, we have

$$b(m, \mu) \, dm \, d\mu = S \int_0^{\infty} r^2 D(r) \phi(M_1) \frac{\partial M_1}{\partial m} dm \, \psi(T) \frac{\partial T}{\partial \mu} d\mu \, dr,$$

whence $$b(m, \mu) = S \int_0^{\infty} r^3 D(r) \phi(m - 5 \operatorname{Log} r) \psi(\mu r) \, dr. \qquad \ldots\ldots(1)$$

Similarly, the mean parallax $p(m, \mu)$ is given by

$$a(m, \mu) \equiv b(m, \mu) . p(m, \mu) = S \int_0^{\infty} r^2 D(r) \phi(m - 5 \operatorname{Log} r) \psi(\mu r) \, dr. \quad \ldots\ldots(2)$$

Setting $$r = e^{-c\rho},$$

where $c = 0{\cdot}4605$ as in 8·15 (8),

and $$cSe^{-4c\rho} D(e^{-c\rho}) \psi(\mu e^{-c\rho}) = \Delta_1(\mu, \rho), \qquad \ldots\ldots(3)$$

we have $$b(m, \mu) = \int_{-\infty}^{\infty} \Delta_1(\mu, \rho) \phi(m+\rho) \, d\rho \qquad \ldots\ldots(4)$$

and $$b(m, \mu) . p(m, \mu) = \int_{-\infty}^{\infty} e^{c\rho} \Delta_1(\mu, \rho) \phi(m+\rho) \, d\rho. \qquad \ldots\ldots(5)$$

8·18. *The mean proper motion of stars of apparent magnitude m.*

From the preceding section, the number of stars with a given assigned apparent magnitude m and with proper motions between μ and $\mu+d\mu$ is proportional to $b(m,\mu)\,d\mu$ and, if $\mu(m)$ denotes the mean proper motion of all stars of apparent magnitude m, we have

$$\mu(m) = \frac{\int_0^\infty \mu b(m,\mu)\,d\mu}{\int_0^\infty b(m,\mu)\,d\mu}.$$

From 8·17 (1), this formula becomes

$$\mu(m) = \frac{\int_0^\infty r^3 D(r)\,\phi(m-5\,\mathrm{Log}\,r)\left\{\int_0^\infty \psi(\mu r)\,\mu\,d\mu\right\}dr}{\int_0^\infty r^3 D(r)\,\phi(m-5\,\mathrm{Log}\,r)\left\{\int_0^\infty \psi(\mu r)\,d\mu\right\}dr},$$

or, on the assumption that the frequency function of the linear velocities T is independent of r,

$$\mu(m) = \frac{\int_0^\infty r\,D(r)\,\phi(m-5\,\mathrm{Log}\,r)\,dr\int_0^\infty T\,\psi(T)\,dT}{\int_0^\infty r^2 D(r)\,\phi(m-5\,\mathrm{Log}\,r)\,dr\int_0^\infty \psi(T)\,dT}.$$

But
$$\int_0^\infty T\,\psi(T)\,dT = \overline{T}\int_0^\infty \psi(T)\,dT,$$

where $\overline{T}$ is the mean transverse linear velocity. From (5) and (6) of section 8·15 we obtain

$$\mu(m) = p(m)\,.\,\overline{T}, \qquad \ldots\ldots(1)$$

where $p(m)$ is the mean parallax of stars of apparent magnitude m.

As (1) holds for each small range of magnitude, we obtain

$$\overline{\mu} = \overline{p}\,.\,\overline{T}, \qquad \ldots\ldots(2)$$

where $\overline{\mu}$ and $\overline{p}$ denote the mean proper motion and parallax of stars within any given magnitude range.

From (1), $\quad b(m)\,.\,\mu(m) = b(m)\,.\,p(m)\,.\,\overline{T}.$

Hence, from (13) and (12) of section 8·15,

$$b(m)\,.\,\mu(m) = \overline{T}\int_{-\infty}^\infty e^{cp}\,\Delta(\rho)\,\phi(m+\rho)\,d\rho \qquad \ldots\ldots(3)$$

and
$$b(m) = \int_{-\infty}^\infty \Delta(\rho)\,\phi(m+\rho)\,d\rho. \qquad \ldots\ldots(4)$$

If the functions $b(m)$ and $\mu(m)$ are obtained from observations and if we suppose that $\overline{T}$ can be found for the given region, say from a representative number of stars of known parallax, the formulae (3) and (4) are two integral equations from which the functions $\Delta(\rho)$ and $\phi(m+\rho)$ can theoretically be obtained.

8·21. *The Fourier integrals.*

We begin with the well-known double integral

$$f(x) = \frac{1}{\pi}\int_0^\infty d\beta \int_{-\infty}^\infty f(\alpha)\cos\beta(x-\alpha)\,d\alpha, \qquad \text{......(1)}$$

where the function $f(x)$ satisfies Dirichlet's conditions* and

$$\int_{-\infty}^\infty f(x)\,dx$$

is absolutely convergent.

We write (1) as

$$f(x) = \frac{1}{\pi}\int_0^\infty \cos\beta x\left\{\int_{-\infty}^\infty f(\alpha)\cos\beta\alpha\,d\alpha\right\}d\beta + \frac{1}{\pi}\int_0^\infty \sin\beta x\left\{\int_{-\infty}^\infty f(\alpha)\sin\beta\alpha\,d\alpha\right\}d\beta$$

or

$$f(x) = \int_0^\infty P(\beta)\cos\beta x\,d\beta + \int_0^\infty Q(\beta)\sin\beta x\,d\beta, \qquad \text{......(2)}$$

where

$$\left.\begin{aligned} P(\beta) &= \frac{1}{\pi}\int_{-\infty}^\infty f(\alpha)\cos\beta\alpha\,d\alpha \\ Q(\beta) &= \frac{1}{\pi}\int_{-\infty}^\infty f(\alpha)\sin\beta\alpha\,d\alpha \end{aligned}\right\}. \qquad \text{......(3)}$$

From (2),

$$f(x) = \frac{1}{2}\int_0^\infty \{P(\beta)-iQ(\beta)\}\,e^{i\beta x}\,d\beta + \frac{1}{2}\int_0^\infty \{P(\beta)+iQ(\beta)\}\,e^{-i\beta x}\,d\beta. \qquad \text{......(4)}$$

Let

$$2F(\beta) = P(\beta)+iQ(\beta). \qquad \text{......(5)}$$

Then, by (3) and (5),

$$F(\beta) = \frac{1}{2\pi}\int_{-\infty}^\infty f(\alpha)\,e^{i\beta\alpha}\,d\alpha. \qquad \text{......(6)}$$

Also, from (3), $P(-\beta) = P(\beta)$ and $Q(-\beta) = -Q(\beta)$.

Hence

$$2F(-\beta) = P(\beta)-iQ(\beta).$$

But, from (4),

$$f(x) = \int_0^\infty F(-\beta)\,e^{i\beta x}\,d\beta + \int_0^\infty F(\beta)\,e^{-i\beta x}\,d\beta,$$

from which, on writing $-\beta$ for β in the first integral,

$$f(x) = \int_{-\infty}^0 F(\beta)\,e^{-i\beta x}\,d\beta + \int_0^\infty F(\beta)\,e^{-i\beta x}\,d\beta$$

or

$$f(x) = \int_{-\infty}^\infty F(\beta)\,e^{-i\beta x}\,d\beta. \qquad \text{......(7)}$$

* Whittaker and Watson, *Modern Analysis* (4th Edn.), pp. 163–167, 1927.

The integrals $$F(\beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\alpha)\, e^{i\beta\alpha}\, d\alpha \qquad \text{......(8)}$$

and $$f(x) = \int_{-\infty}^{\infty} F(\beta)\, e^{-i\beta x}\, d\beta \qquad \text{......(9)}$$

are the Fourier integrals and we describe f and F as conjugate functions.

8·22. *The integral equations.*

The principal formulae of sections 8·15, 8·16, 8·17 and 8·18 can all be summarised in the following two forms:

$$b(x) = \int_{-\infty}^{\infty} \Delta(\rho)\, \phi(x+\rho)\, d\rho, \qquad \text{......(1)}$$

$$a(x) = \int_{-\infty}^{\infty} e^{k\rho}\, \Delta(\rho)\, \phi(x+\rho)\, d\rho. \qquad \text{......(2)}$$

In (2),
$k = c$ for 8·15 (13),
$k = -1$ for 8·16 (9),
$k = c$ for 8·17 (5),
$k = c$ for 8·18 (3).

We shall indicate by an asterisk the Fourier function which is conjugate to a given function. In this notation, the formulae (8) and (9) of the previous section are

$$f^*(\beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\alpha)\, e^{i\beta\alpha}\, d\alpha, \qquad \text{......(3)}$$

$$f(x) = \int_{-\infty}^{\infty} f^*(\beta)\, e^{-i\beta x}\, d\beta. \qquad \text{......(4)}$$

In general, the formulae (1) and (2) are such that the functions $b(x)$ and $a(x)$ may be supposed given by observation. The problem is to solve these equations so as to derive the functions Δ and ϕ. The solutions were first effected by Schwarzschild† by means of the Fourier integrals.

We first consider the case when $b(x)$ and $\phi(x)$ are known and it is required to find $\Delta(x)$.

8·23. *Solution of the integral equations, the functions $b(x)$, or $a(x)$, and $\phi(x)$ being known.*

It is clear that the integral equation involving $a(x)$ is of the same type, in this case, as the equation involving $b(x)$; we consider only the latter.

Multiply (1) of section 8·22 by $e^{i\omega x}\, dx$ and integrate between $-\infty$ and $+\infty$. Then

$$\int_{-\infty}^{\infty} b(x)\, e^{i\omega x}\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Delta(\rho)\, \phi(x+\rho)\, e^{i\omega x}\, dx\, d\rho$$

$$= \int_{-\infty}^{\infty} \Delta(\rho)\, e^{-i\omega\rho}\, d\rho \int_{-\infty}^{\infty} \phi(x+\rho)\, e^{i\omega(x+\rho)}\, dx. \qquad \text{......(1)}$$

† *A.N.* **185**, 81, 1910. See also Charlier, *Lund Medd.* Ser. II, No. 8, 1912.

In the second integral on the right-hand side, let $x+\rho=\alpha$, so that for a given value of ρ it can be written

$$\int_{-\infty}^{\infty}\phi(\alpha)\,e^{i\omega\alpha}\,d\alpha,$$

which is equivalent, by 8·22 (3), to

$$2\pi\phi^*(\omega).$$

Also, by 8·22 (3), $$\int_{-\infty}^{\infty}\Delta(\rho)\,e^{-i\omega\rho}\,d\rho=2\pi\,\Delta^*(-\omega) \qquad \text{......(2)}$$

and $$\int_{-\infty}^{\infty}b(x)\,e^{i\omega x}\,dx=2\pi\,b^*(\omega). \qquad \text{......(3)}$$

Hence (1) leads to $$b^*(\omega)=2\pi\,\Delta^*(-\omega)\,\phi^*(\omega). \qquad \text{......(4)}$$

Similarly, $$b^*(-\omega)=2\pi\,\Delta^*(\omega)\,\phi^*(-\omega). \qquad \text{......(5)}$$

Now, by 8·22 (4), $$\Delta(x)=\int_{-\infty}^{\infty}\Delta^*(\omega)\,e^{-i\omega x}\,d\omega,$$

whence, by (5), $$\Delta(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b^*(-\omega)}{\phi^*(-\omega)}\,e^{-i\omega x}\,d\omega$$

or $$\Delta(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b^*(\omega)}{\phi^*(\omega)}\,e^{i\omega x}\,d\omega. \qquad \text{......(6)}$$

Since $b(\omega)$ and $\phi(\omega)$ are presumed known, the conjugate functions $b^*(\omega)$ and $\phi^*(\omega)$ can be obtained; formula (6) then gives $\Delta(x)$, from which the density function $D(x)$ is easily deduced. An example of the process will be given later, in section 8·33.

8·24. *General solution of the two integral equations.*

We assume that the functions $b(x)$ and $a(x)$ are obtained from observations and it is required to find the functions $\Delta(x)$ and $\phi(x)$.

Consider $$a(x)=\int_{-\infty}^{\infty}e^{k\rho}\,\Delta(\rho)\,\phi(x+\rho)\,d\rho. \qquad \text{......(1)}$$

Multiply by $e^{i\omega x}\,dx$ and integrate between $-\infty$ and $+\infty$. Then

$$\int_{-\infty}^{\infty}a(x)\,e^{i\omega x}\,dx=\int_{-\infty}^{\infty}e^{-i\omega\rho+k\rho}\,\Delta(\rho)\,d\rho\int_{-\infty}^{\infty}e^{i\omega(x+\rho)}\,\phi(x+\rho)\,dx.$$

Hence $$2\pi\,a^*(\omega)=2\pi\,\phi^*(\omega)\int_{-\infty}^{\infty}e^{-i\rho(\omega+ik)}\,\Delta(\rho)\,d\rho.$$

Hence $$a^*(\omega)=2\pi\phi^*(\omega)\,\Delta^*(-\omega-ik). \qquad \text{......(2)}$$

Similarly, $$a^*(-\omega)=2\pi\phi^*(-\omega)\,\Delta^*(\omega-ik). \qquad \text{......(3)}$$

In deriving these relations we are assuming that the Fourier integrals hold for complex values of the argument.

But from 8·23 (5),

$$b^*(-\omega) = 2\pi \Delta^*(\omega)\, \phi^*(-\omega). \qquad \text{......(4)}$$

Hence, eliminating $\phi^*(-\omega)$ between (3) and (4), we derive

$$\frac{\Delta^*(\omega)}{\Delta^*(\omega - ik)} = \frac{b^*(-\omega)}{a^*(-\omega)}. \qquad \text{......(5)}$$

Let

$$\omega = ik(\xi + 1). \qquad \text{......(6)}$$

Then

$$\frac{\Delta^*\{ik(\xi+1)\}}{\Delta^*(ik\xi)} = \frac{b^*\{-ik(\xi+1)\}}{a^*\{-ik(\xi+1)\}}. \qquad \text{......(7)}$$

Since $b(x)$ and $a(x)$ are supposed to be known functions, the corresponding conjugate functions can be determined. Thus, the right-hand side is a known function of ξ and we shall write it in the form $e^{F(\xi)}$. Similarly, let

$$\Delta^*(ik\xi) = e^{G(\xi)}. \qquad \text{......(8)}$$

Then (7) becomes

$$G(\xi+1) - G(\xi) = F(\xi). \qquad \text{......(9)}$$

This is a functional difference equation, the solution of which gives us $\Delta^*(\xi)$. The function $\Delta(\xi)$ is then given by

$$\Delta(\xi) = \int_{-\infty}^{\infty} \Delta^*(\omega)\, e^{-i\omega\xi}\, d\omega. \qquad \text{......(10)}$$

This constitutes the formal solution of the problem.

8·25. *Solution of the difference equation.*

A particular solution of the difference equation

$$G(\xi+1) - G(\xi) = F(\xi) \qquad \text{......(1)}$$

can be written as

$$G(\xi) = -\sum_{s=0}^{\infty} F(\xi+s), \qquad \text{......(2)}$$

for then

$$G(\xi+1) = -\sum_{s=1}^{\infty} F(\xi+s) \qquad \text{......(3)}$$

and (1) is verified. It is assumed that the series in (2) and (3) are uniformly convergent.

It is to be noted that the most general solution will contain periodic terms of period unity; for such terms†

$$G_1(\xi+1) - G_1(\xi) = 0.$$

Hence

$$G(\xi) = G_1(\xi) - \sum_{s=0}^{\infty} F(\xi+s) \qquad \text{......(4)}$$

is a solution of (1).

† A simple example is $G_1(\xi) \equiv \sin 2\pi\xi$.

The series in (2) can be expressed as a closed expression as follows.

Consider the integral

$$I = \frac{1}{2\pi i}\int_{C_n} F(z+\zeta)\left\{\pi\cot\pi\zeta - \frac{1}{\zeta}\right\}d\zeta, \qquad \ldots\ldots(5)$$

taken, in the first instance, over the contour C_n, consisting of the imaginary axis between $+iR_n$ and $-iR_n$ and the semicircle of radius R_n, where $n < R_n < n+1$ (Fig. 49). Now

$$\pi\cot\pi\zeta - \frac{1}{\zeta} = \frac{1}{\zeta-1}+\frac{1}{\zeta+1}+\frac{1}{\zeta-2}+\frac{1}{\zeta+2}+\ldots+\frac{1}{\zeta-n}+\frac{1}{\zeta+n}+\ldots, \qquad \ldots\ldots(6)$$

so that

$$I = \frac{1}{2\pi i}\int_{C_n} F(z+\zeta)\sum_{n=1}^{\infty}\left(\frac{1}{\zeta-n}+\frac{1}{\zeta+n}\right)d\zeta. \qquad \ldots\ldots(7)$$

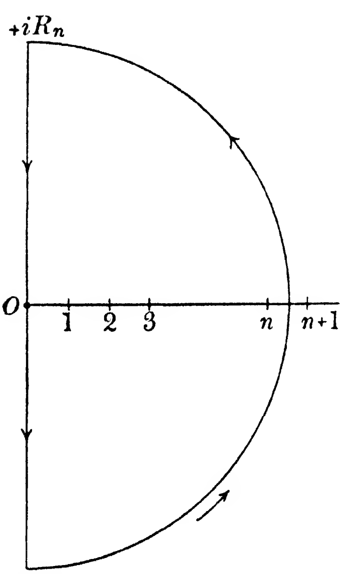

Fig. 49

Consider (7). The poles within the contour are at $\zeta = 1, 2, 3, \ldots n$ and the corresponding residues are $F(z+1)$, $F(z+2)$, $F(z+3)$, ... $F(z+n)$. Hence by Cauchy's residue theorem

$$\frac{1}{2\pi i}\int_{C_n} F(z+\zeta)\frac{d\zeta}{z+s} = 0, \qquad \ldots\ldots(8)$$

for all positive integral values of s, and

$$\frac{1}{2\pi i}\int_{C_n} F(z+\zeta)\frac{d\zeta}{z-s} = F(z+s), \quad \text{or} \quad = 0, \qquad \ldots\ldots(9)$$

according as s is one of the positive integers 1, 2, 3, ... n or as $s > n$.

For a semicircular contour C of infinite radius, we can write

$$I = \frac{1}{2\pi i}\int_{C} F(z+\zeta)\sum_{s=1}^{\infty}\left\{\frac{1}{\zeta-s}+\frac{1}{\zeta+s}\right\}d\zeta$$

$$= \sum_{s=1}^{\infty}\int_{C} F(z+\zeta)\left\{\frac{1}{\zeta-s}+\frac{1}{\zeta+s}\right\}d\zeta, \qquad \ldots\ldots(10)$$

and it follows from (8) and (9) that

$$I = \sum_{s=1}^{\infty} F(z+s). \qquad \ldots\ldots(11)$$

Assuming that $F(\zeta) \to 0$ as $\zeta \to \infty$ sufficiently rapidly for the integral in (10) along the semicircular arc to vanish, we have

$$\int_C = \int_{+i\infty}^{-i\infty}.$$

Hence, from (5) and (11),

$$\frac{1}{2i}\int_{+i\infty}^{-i\infty} F(z+\zeta)\left\{\cot\pi\zeta - \frac{1}{\pi\zeta}\right\}d\zeta = \sum_{s=1}^{\infty} F(z+s),$$

so that, from (3),

$$G(z+1) = \frac{i}{2}\int_{-i\infty}^{i\infty} F(z+\zeta)\left\{\cot\pi\zeta - \frac{1}{\pi\zeta}\right\}d\zeta. \qquad \ldots\ldots(12)$$

This is the required solution. Formula (12), or its equivalent, appears to have been given first by Schwarzschild.*

The general solution of the problem, as represented by (3) or (12), has up to the present been outside the possibility of practical application. This has been due, in great part, to the insufficiency of accurate observational material. So far as the author is aware, the only attempt to apply the Fourier solution of the integral equation to actual astronomical statistics was made by Eddington.†

8·31. *The distribution of apparent magnitudes for constant density.*

We assume that the absolute magnitudes are distributed in accordance with a frequency function $\phi(M_1)$ which is independent of distance from the sun.

From 8·15 (5) we have

$$b(m) = S\int_0^\infty r^2 D(r)\,\phi(m - 5\,\mathrm{Log}\,r)\,dr,$$

or, putting

$$r = e^{-c\rho},$$

where c is defined by 8·15 (8), and setting $D(r) = K$,

$$b(m) = cSK\int_{-\infty}^{\infty} e^{-3c\rho}\,\phi(m+\rho)\,d\rho.$$

Write $m+\rho = x$. Then

$$b(m) = cSKe^{3cm}\int_{-\infty}^{\infty} e^{-3cx}\,\phi(x)\,dx.$$

The integral on the right is a constant and we have

$$b(m) = Ae^{3cm}, \qquad \ldots\ldots(1)$$

in which A is a constant.

Let $B(m)$ denote the number of stars brighter than apparent magnitude m. Then

$$B(m) = \int_{-\infty}^{m} b(m)\,dm$$

or

$$B(m) = \frac{A}{3c}e^{3cm}. \qquad \ldots\ldots(2)$$

Similarly, if $B(m+1)$ is the number of stars brighter than apparent magnitude $(m+1)$,

$$B(m+1) = \frac{A}{3c}e^{3c(m+1)}. \qquad \ldots\ldots(3)$$

* *A.N.* **185**, 85, 1910. † *M.N.* **72**, 368, 1912.

Hence, (2) and (3) give

$$\frac{B(m+1)}{B(m)} = e^{3c} = 10^{\frac{3}{5}}$$

or

$$B(m+1) : B(m) = 3{\cdot}98 : 1. \qquad \ldots\ldots(4)$$

This result shows that, with the hypothesis concerning the density function, the total number of stars brighter than apparent magnitude $(m+1)$ should be nearly four times the number of stars brighter than apparent magnitude m, this ratio being independent of the frequency function of absolute magnitudes. Actually, the ratio is found from star-counts to be much less than 4 and the ratio is also found to diminish with increasing m. Hence the original assumption is incompatible with observation and the general inference is that the density function decreases with increasing distance. However, this conclusion must not be regarded as final inasmuch as we have not taken into consideration the effects of galactic absorption which we shall discuss in section 8·61.

8·32. *Seeliger's hypothesis.*

The density law used by Seeliger in his researches is

$$D(r) = D_0 r^{-s},$$

which gives a density diminishing as r increases if $s > 0$. We have as before

$$b(m) = SD_0 \int_0^\infty r^{2-s} \phi(m - 5\,\mathrm{Log}\,r)\,dr$$

$$= cSD_0 \int_{-\infty}^\infty e^{-(3-s)c\rho} \phi(m+\rho)\,d\rho,$$

so that on writing x for $(m+\rho)$,

$$b(m) = cSD_0 e^{(3-s)cm} \int_{-\infty}^\infty e^{-(3-s)cx} \phi(x)\,dx.$$

The integral on the right is a constant and so $b(m)$ can be expressed by

$$b(m) = Ce^{(3-s)cm}. \qquad \ldots\ldots(1)$$

Also

$$B(m) = \frac{C}{(3-s)c} e^{(3-s)cm} \qquad \ldots\ldots(2)$$

and

$$B(m+1) : B(m) = e^{(3-s)c} : 1$$

$$= 10^{\frac{1}{5}(3-s)} : 1. \qquad \ldots\ldots(3)$$

The ratio is thus a constant, being independent of m; it is less than the ratio $3{\cdot}98 : 1$ found in the previous section provided $s > 0$. Here again, the theoretical implications of Seeliger's hypothesis are not in accord with a diminishing ratio for fainter stars.

8·33. *Schwarzschild's density function.*

We investigate the density function† when the distributions of the apparent and absolute magnitudes are Maxwellian in form. We assume that

$$b(m) = ae^{-k^2(m-m_0)^2}, \qquad \text{......(1)}$$

$$\phi(M_1) = Ae^{-K^2(M_1-M_0)^2} \qquad \text{......(2)}$$

and that

$$K > k. \qquad \text{......(3)}$$

As regards (1), this is the form found to satisfy the star-counts for a given galactic latitude in the exhaustive investigation by Chapman and Melotte‡ on the number of stars of each photographic magnitude down to $17^{m}{\cdot}0$, the counts being based on the Franklin-Adams plates; and it is the form that represents, approximately at least, most of the later work in this department of stellar statistics. The parameters a, k and m_0 are to be regarded as functions of galactic latitude and possibly also of galactic longitude.

The formula (2) must be regarded as applicable only to a particular spectral type or subdivision of spectral type. In Strömberg's studies§ of the distribution of absolute magnitudes, the separation of the stars into the giant and dwarf classes is recognised, so that in dealing with a given spectral type we represent the function $\phi(M_1)$ as the sum of two Maxwellian expressions of the type (2). In the following analysis, we deal with a single Maxwellian function only.

The condition (3) implies that the dispersion about the mean is less for absolute magnitudes of a given spectral type than for the apparent magnitudes. This is in accordance with observation, for the giants or for the dwarfs.

From formula 8·15 (12) we have

$$b(m) = \int_{-\infty}^{\infty} \Delta(\rho)\, \phi(m+\rho)\, d\rho, \qquad \text{......(4)}$$

where

$$\Delta(\rho) = cSe^{-3c\rho} D(e^{-c\rho}) \qquad \text{......(5)}$$

and

$$m+\rho = M_1. \qquad \text{......(6)}$$

Since the functions $b(m)$ and $\phi(m+\rho)$ in (4) are given by (1) and (2), the function $\Delta(\rho)$ can be found by the method of section 8·23.

The solution is, by 8·23 (6),

$$\Delta(\rho) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{b^*(\omega)}{\phi^*(\omega)} e^{i\omega\rho}\, d\omega. \qquad \text{......(7)}$$

† Schwarzschild, *A.N.* **185**, 81, 1910.

‡ *Memoirs, R.A.S.* **60**, 145, 1914.

§ See for example *Mt Wilson Contribution*, No. 442, 1932, or *Ap. J.* **75**, 115, 1932.

Now
$$b^*(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} b(m)\, e^{i\omega m}\, dm$$
$$= \frac{a}{2\pi}\int_{-\infty}^{\infty} e^{-k^2(m-m_0)^2+i\omega m}\, dm,$$
or, on setting $m - m_0 = x$,
$$b^*(\omega) = \frac{a}{2\pi}\, e^{im_0\omega}\int_{-\infty}^{\infty} e^{-k^2x^2}\cos\omega x\, dx.$$
The value of this well-known integral is
$$\frac{\sqrt{\pi}}{k}\, e^{-\frac{\omega^2}{4k^2}}.$$
Hence
$$b^*(\omega) = \frac{a}{2k\sqrt{\pi}}\, e^{-\frac{\omega^2}{4k^2}+im_0\omega}. \qquad \text{......(8)}$$
Similarly,
$$\phi^*(\omega) = \frac{A}{2K\sqrt{\pi}}\, e^{-\frac{\omega^2}{4K^2}+iM_0\omega}. \qquad \text{......(9)}$$
Substituting (8) and (9) in (7), we obtain
$$\Delta(\rho) = \frac{aK}{2\pi Ak}\int_{-\infty}^{\infty} e^{-\frac{\omega^2}{4}\left(\frac{1}{k^2}-\frac{1}{K^2}\right)+i\omega(m_0-M_0+\rho)}\, d\omega$$
$$= \frac{aK}{2\pi Ak}\int_{-\infty}^{\infty} e^{-\frac{\omega^2}{4}\left(\frac{1}{k^2}-\frac{1}{K^2}\right)}\cos\omega(m_0-M_0+\rho)\, d\omega.$$
Hence
$$\Delta(\rho) = \frac{aK^2}{A\sqrt{\pi}(K^2-k^2)^{\frac{1}{2}}}\, e^{-\frac{K^2k^2(\rho+m_0-M_0)^2}{K^2-k^2}}. \qquad \text{......(10)}$$
Using (5) and remembering that $r = e^{-c\rho}$, we obtain for the density function
$$D(r) = Ce^{3c\rho-\frac{K^2k^2}{K^2-k^2}(\rho+m_0-M_0)^2}, \qquad \text{......(11)}$$
which, as regards ρ, is of the Maxwellian form
$$D(r) = C_1 e^{-\kappa^2(\rho-\rho_0)^2}. \qquad \text{......(12)}$$
This formula for the density function has been extensively used in statistical investigations.

It is to be noted that the maximum density occurs where $\rho = \rho_0$. With the values of the constants, as derived by Kapteyn† and Schwarzschild,‡ entering into the formula for the density function, the density is a maximum within a few parsecs of the sun.

† *A.J.* **24**, 115, 1906.
‡ *A.N.* **190**, 361, 1912.

8·34. *The mean parallax of stars of given apparent magnitude.*

We have from 8·15 (13)

$$b(m)\,.\,p(m) = \int_{-\infty}^{\infty} e^{c\rho}\,\Delta(\rho)\,\phi(m+\rho)\,d\rho,$$

where $p(m)$ is the mean parallax of stars of apparent magnitude m. Inserting the functions $b(m)$, $\Delta(\rho)$ and $\phi(m+\rho)$ in (1), (10) and (2) of the previous section, we obtain

$$p(m) = F\int_{-\infty}^{\infty} e^{c\rho - \frac{K^2k^2}{K^2-k^2}(\rho+m_0-M_0)^2 - K^2(\rho+m-M_0)^2 + k^2(m-m_0)^2}\,d\rho,$$

where F is a constant. Put

$$x = \rho + m_0 - M_0, \quad m_1 = m - m_0.$$

Then
$$p(m) = Fe^{-c(m_0-M_0)}\int_{-\infty}^{\infty} e^{cx - \frac{K^2k^2}{K^2-k^2}x^2 - K^2(x+m_1)^2 + k^2m_1^2}\,dx$$

$$= F_1 e^{-\frac{cm_1(K^2-k^2)}{K^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{K^2-k^2}\left\{K^2x+(K^2-k^2)\left(m_1-\frac{c}{2K^2}\right)\right\}^2}\,dx,$$

whence
$$p(m) = F_2 e^{-\lambda m}, \qquad \text{......(1)}$$

where
$$\lambda = \frac{c}{K^2}(K^2-k^2). \qquad \text{......(2)}$$

In these formulae F_1 and F_2 are constants.

Formula (1) may be written

$$\operatorname{Log} p(m) = a - bm, \qquad \text{......(3)}$$

where a and b are constants, b being positive; from (1) and (2) the value of b is easily seen to be $(K^2-k^2)/5K^2$.

This last formula (3) can be readily compared with observations; it has been used extensively in statistical studies.

8·35. *The mean proper motion of stars of given apparent magnitude.*

It follows from (1) of section 8·18 that the mean proper motion, $\mu(m)$, of stars of apparent magnitude m is given by

$$\mu(m) = F_3 e^{-\lambda m}, \qquad \text{......(1)}$$

F_3 being a constant.

8·36. *Kapteyn's formula for the mean parallax of stars of given apparent magnitude and proper motion.*

We denote, as in section 8·17, the number of stars with apparent magnitudes between m and $m+dm$, and total proper motions between μ and $\mu+d\mu$, by $b(m,\mu)\,dm\,d\mu$, and the mean parallax of such stars by $p(m,\mu)$. Kapteyn's formula* is

$$\operatorname{Log} p(m,\mu) = A + Bm + C\operatorname{Log}\mu.$$

This was originally derived empirically.

* *Groningen Publ.* No. 8, 1901.

We now show that the formula can be derived on the basis of (*a*) a Gaussian distribution for the absolute magnitudes, (*b*) Schwarzschild's density function, and (*c*) a form of the frequency function of the linear transverse velocities used by Schwarzschild. The latter function is

$$\psi(T) = A_0 e^{-j^2(\operatorname{Log} T - \operatorname{Log} T_0)^2}, \qquad \ldots\ldots(1)$$

where T is the transverse linear velocity μr relative to the sun and A_0, j and T_0 are constants. This velocity function is analogous in form to Schwarzschild's density function.

We have, from 8·17 (1),

$$b(m,\mu) = S\int_0^\infty r^3 D(r)\,\phi(m-5\operatorname{Log} r)\,\psi(\mu r)\,dr, \qquad \ldots\ldots(2)$$

and setting $$r = e^{-c\rho}, \quad \mu = e^{-c\tau}$$

and making use of (1), Schwarzschild's density function $D(r)$ and the form (2) of section 8·33 for the function $\phi(M_1)$, we can write (2) in the form

$$b(m,\mu) = P\int_{-\infty}^{\infty} e^{-4c\rho}\, e^{-\kappa^2(\rho-\rho_0)^2}\, e^{-l^2(\rho+m-M_0)^2}\, e^{-n^2(\tau+\rho-\tau_0)^2}\,d\rho, \qquad \ldots\ldots(3)$$

where P; κ, ρ_0; l, M_0; n, τ_0 are constants.

We further write (3) as

$$b(m,\mu) = P\int_{-\infty}^{\infty} e^{-a^2\rho^2 - 2\rho(l^2m+n^2\tau+2c+d)-q}\,d\rho,$$

where $a^2 \equiv \kappa^2+l^2+n^2$ and d, q are constants.

Similarly, (2) of section 8·17 becomes

$$b(m,\mu)\,.\,p(m,\mu) = P\int_{-\infty}^{\infty} e^{-a^2\rho^2-2\rho\left(l^2m+n^2\tau+\frac{3c}{2}+d\right)-q}\,d\rho.$$

Hence

$$p(m,\mu) = \frac{\displaystyle\int_{-\infty}^{\infty} e^{-a^2\rho^2-2\rho(l^2m+n^2\tau+c_1)}\,d\rho}{\displaystyle\int_{-\infty}^{\infty} e^{-a^2\rho^2-2\rho(l^2m+n^2\tau+c_2)}\,d\rho},$$

where c_1 and c_2 are constants. It is easily seen that

$$p(m,\mu) = e^{\frac{1}{a^2}(l^2m+n^2\tau+c_1)^2 - \frac{1}{a^2}(l^2m+n^2\tau+c_2)^2},$$

from which it follows that $\operatorname{Log} p(m,\mu)$ is of the form

$$\operatorname{Log} p(m,\mu) = A + Bm + C\operatorname{Log}\mu, \qquad \ldots\ldots(4)$$

in which A, B and C are constants. These constants, however, depend on galactic latitude and spectral type; their mean values for galactic latitudes numerically greater than 40° have been determined by Kapteyn and van

Rhijn* from an exhaustive analysis of the data relating to the bright stars, and they write (4) in this case as

$$\operatorname{Log} p(m, \mu) = -0{\cdot}691 - 0{\cdot}0682m + 0{\cdot}645 \operatorname{Log} \mu. \qquad \text{......(5)}$$

A more general form of (4) was later derived and employed by Seares.†

To secure more satisfactory agreement with the measured parallaxes of the very distant stars, Strömberg ‡ added a constant d to the proper motion μ, so that the formula becomes

$$\operatorname{Log} p(m, \mu) = A + Bm + C \operatorname{Log}(\mu + d). \qquad \text{......(6)}$$

Expressed in terms of absolute magnitude M, by means of

$$M = m + 5 + 5 \operatorname{Log} p(m, \mu),$$

the formula (6) assumes the form

$$M = a + bm + c \operatorname{Log}(\mu + k). \qquad \text{......(7)}$$

This is the form used by R. E. Wilson.§

With the numerical coefficients of (5), the formula (7) becomes

$$M = 1{\cdot}545 + 0{\cdot}659m + 3{\cdot}225 \operatorname{Log}(\mu + k),$$

in which the coefficient of Log $(\mu + k)$ is about five times the coefficient of m.

Luyten || has pointed out that the relation between absolute magnitude, proper motion and apparent magnitude for a given spectral class is equivalent (for the particular value of $k = 0$ in (7)) to

$$M = a_1 + b_1 H, \qquad \text{......(8)}$$

where

$$H = m + 5 + 5 \operatorname{Log} \mu. \qquad \text{......(9)}$$

A further generalisation¶ is

$$M = a_1 + b_1 H + c_1 H^2. \qquad \text{......(10)}$$

These formulae involving H have been used extensively by Luyten,** Prasad,†† Cecchini ‡‡ and others in deriving numerical relations between parallax (or absolute magnitude), proper motion and apparent magnitude.§§

* *Ap. J.* **52**, 23, 1920 (*Mt Wilson Contr.* No. 188, 1920).

† *Ap. J.* **59**, 310, 1924 (*Mt Wilson Contr.* No. 273, 1924). *Ap. J.* **74**, 320, 1931 (*Mt Wilson Contr.* No. 438, 1931).

‡ *Ap. J.* **47**, 9, 1918 (*Mt Wilson Contr.* No. 144, 1918).

§ *A.J.* **36**, 49, 1925.

|| *Lick Obs. Bull.* **11**, 39, 1923.

¶ *M.N.* **85**, 157, 1924.

** *P.A.S.P.* **34**, 156, 1922; **35**, 209, 1923.

†† *Loc. cit.*

‡‡ *Publ. R.O. Merate*, No. 5, 1931.

§§ For a complete discussion on the correlation between absolute magnitude, linear tangential velocity, distance, apparent magnitude and proper motion see R. A. Robb, *M.N.* **97**, 67, 1936.

8·4. *Eddington's method of deriving the velocity law.*

Consider the formula 8·16 (2)

$$b(\mu) = S\int_0^\infty r^3 D(r)\,\psi(\mu r)\,dr, \qquad \ldots\ldots(1)$$

$b(\mu)\,d\mu$ being the number of total proper motions between μ and $\mu+d\mu$ in a particular direction in the tangent plane for a given region, and $\psi(T)$ is the velocity function of the corresponding linear transverse velocities $T\ (\equiv \mu r)$.

We assume that the observations furnish the function $b(\mu)$. If we know the velocity law $\psi(T)$, the density function $D(r)$ can be determined from (1) according to the procedure of section 8·23.

We further assume that for the given region the direction of star-streaming is known and that at right angles to this direction the peculiar linear velocities are distributed according to the Maxwellian law

$$\frac{h}{\sqrt{\pi}} e^{-h^2u^2}.$$

The linear velocities, T, relative to the sun will consequently be distributed according to the law

$$\frac{h}{\sqrt{\pi}} e^{-h^2(T-V)^2},$$

where V is the component of the parallactic motion, in the tangent plane, perpendicular to the direction of the vertex. Hence by the method of section 8·23 the density function is derived.

Now consider any other direction in the tangent plane. The observations will furnish $b(\mu)$ as before and now we know the density function $D(r)$. By the method of section 8·23 we then derive the velocity function for the given direction.

Eddington* employed this method in 1912 and concluded that the observational material slightly favoured the two-streams theory rather than that of the ellipsoidal distribution of peculiar velocities.

As noted by Schwarzschild and Eddington,† the solution for ψ is indeterminate if Seeliger's density law

$$D(r) = D_0 r^{-s}$$

is used. From (1), we have

$$b(\mu) = SD_0\int_0^\infty r^{3-s}\,\psi(\mu r)\,dr,$$

which becomes, on setting $\mu r = T$, the transverse linear velocity,

$$b(\mu) = SD_0\,\mu^{s-4}\int_0^\infty T^{3-s}\,\psi(T)\,dT.$$

* *Stellar Movements*, 218, 1914; *M.N.* **72**, 368, 1912. † *M.N.* **72**, 371, 1912.

Since the integral is independent of μ, we have simply

$$b(\mu) = C\mu^{s-4},$$

so that the distribution of proper motions is independent of the form assumed by the velocity distribution.

8·51. *The density law.*

We consider here the components of the linear velocities and the proper motions of the stars, in a given region, perpendicular to the direction of the vertex of star-streaming. We assume as before that the distribution of the linear motions is independent of distance from the sun and that in particular, as in the previous section, the linear velocities T perpendicular to the direction of the vertex are distributed according to the law

$$\psi(T) = \frac{h}{\sqrt{\pi}} e^{-h^2(T-V)^2}, \qquad \ldots\ldots(1)$$

where V is the component of the parallactic motion in this direction.

From 8·16 (2),

$$b(\mu) = S\int_0^\infty r^3 D(r)\,\psi(\mu r)\,dr. \qquad \ldots\ldots(2)$$

Let $F(r)$ denote the number of stars within the cone of solid angle S up to the distance r from the sun. Then

$$F(r) = S\int_0^r r^2 D(r)\,dr. \qquad \ldots\ldots(3)$$

Integrate (2) by parts so as to make use of (3); then

$$b(\mu) = \left[r\psi(\mu r)\,S\int_0^r r^2 D(r)\,dr\right]_0^\infty - \int_0^\infty S\left[\int_0^r r^2 D(r)\,dr\right]\frac{\partial}{\partial r}\{r\psi(\mu r)\}\,dr$$

or

$$b(\mu) = \left[r\psi(\mu r)\,F(r)\right]_0^\infty - \int_0^\infty F(r)\frac{\partial}{\partial r}\{r\psi(\mu r)\}\,dr. \qquad \ldots\ldots(4)$$

Paying attention to the form of ψ in (1), we see that the integrated part of (4) vanishes at both limits, provided $\mu \neq 0$.

Also

$$\frac{\partial}{\partial r}\{r\psi(\mu r)\} = \psi(\mu r) + r\frac{\partial}{\partial r}\psi(\mu r)$$

$$= \psi(\mu r) + \mu r\frac{\partial}{\partial T}\psi(T).$$

But

$$\mu r\frac{\partial}{\partial T}\psi(T) = \mu\frac{\partial}{\partial \mu}\psi(\mu r).$$

Hence

$$\frac{\partial}{\partial r}\{r\psi(\mu r)\} = \frac{\partial}{\partial \mu}\{\mu\psi(\mu r)\}.$$

In consequence, (4) becomes

$$b(\mu) = -\int_0^\infty F(r)\frac{\partial}{\partial\mu}\{\mu\psi(\mu r)\}\,dr$$

or

$$b(\mu) = -\frac{\partial}{\partial\mu}\int_0^\infty \mu F(r)\,\psi(\mu r)\,dr. \qquad \text{......(5)}$$

Let $B(\mu)$ denote the total number of proper motions in the assigned direction with positive values between 0 and μ; we take the convention that μ is positive if it is in the same direction as the parallactic component V. Then, if N_1 is the total number of stars with positive values of μ between 0 and ∞,

$$N_1 - B(\mu) = \int_\mu^\infty b(\mu)\,d\mu.$$

Hence, by (5),

$$N_1 - B(\mu) = \mu\int_0^\infty F(r)\,\psi(\mu r)\,dr, \qquad \text{......(6)}$$

since, by (1), $\mu\psi(\mu r)\to 0$ as $\mu\to\infty$ for a given value of r.

In the same way, if N_2 denotes the total number of negative proper motions perpendicular to the direction of the vertex and $B(-\mu)$ the number of negative proper motions between 0 and $-\mu$, we have

$$N_2 - B(-\mu) = \mu\int_0^\infty F(r)\,\psi(-\mu r)\,dr. \qquad \text{......(7)}$$

We now assume that $F(r)$ is an even function of r. Then (7) becomes

$$N_2 - B(-\mu) = \mu\int_{-\infty}^0 F(r)\,\psi(\mu r)\,dr. \qquad \text{......(8)}$$

Hence, on adding (6) and (8),

$$N_1 + N_2 - B(\mu) - B(-\mu) = \mu\int_{-\infty}^\infty F(r)\,\psi(\mu r)\,dr. \qquad \text{......(9)}$$

These general formulae are due to Eddington.*

In (6), put $r = e^\rho$, $\mu = e^\alpha$. Then

$$B_1(\alpha) \equiv e^{-\alpha}\{N_1 - B(e^\alpha)\} = \int_{-\infty}^\infty F_1(\rho)\,\psi_1(\rho+\alpha)\,d\rho, \qquad \text{......(10)}$$

where

$$F_1(\rho) = e^\rho F(e^\rho),$$
$$\psi_1(\rho+\alpha) = \psi(e^{\rho+\alpha}).$$

Thus (10) is of the form (1) of section 8·22. If $B_1(\alpha)$ and $\psi_1(\rho+\alpha)$ are known, $F_1(\rho)$ is found by the method of section 8·23.

The solution of (7) is obtained in a similar manner. When $F_1(\rho)$ is derived, the density function $D(r)$ is found from (3), for

$$D(r) = \frac{1}{Sr^2}\frac{dF(r)}{dr}.$$

* *M.N.* 73, 346, 1913.

8·52. *Application to Dyson's density law.*

Eddington* has employed 8·51 (9) to test Dyson's law of density. Since, in this case,

$$D(r) = \frac{A}{r} e^{-h^2k^2r^2},$$

we obtain, from 8·51 (3),

$$F(r) = \frac{AS}{2h^2k^2}\{1 - e^{-h^2k^2r^2}\}.$$

If N is the total number of stars within the cone of solid angle S, this last formula is simply

$$F(r) = N\{1 - e^{-h^2k^2r^2}\}. \qquad \text{......(1)}$$

Inserting in 8·51 (9) the functions $F(r)$ and $\psi(\mu r)$, ψ being as in 8·51 (1), we find readily that

$$B(\mu) + B(-\mu) = \frac{N\mu}{(k^2+\mu^2)^{\frac{1}{2}}} e^{-\frac{h^2k^2V^2}{k^2+\mu^2}}, \qquad \text{......(2)}$$

or, if $R(\mu)$ denotes the number of proper motions greater than μ (irrespective of sign),

$$R(\mu) = N\left\{1 - \frac{\mu}{(k^2+\mu^2)^{\frac{1}{2}}} e^{-\frac{h^2k^2V^2}{k^2+\mu^2}}\right\}. \qquad \text{......(3)}$$

This is analogous to the general expression obtained in section 7·43 for a single drift and of course it applies only to components of proper motions perpendicular to the direction of the vertex.†

The test was applied to the proper motions of A and K type stars separately, in two regions, one in high galactic latitude and the other in low galactic latitude, and was considered to be successful for the K stars and less satisfactory for the A stars.

8·61. *Galactic absorption.*

Within the last few years there has come the realisation that galactic space is not perfectly transparent. Although the existence of dark nebulae in the Milky Way has long been known, it was formerly believed that these were isolated agglomerations of diffuse matter and that the remainder of the Milky Way was free from absorbing matter. The present position is that there is distinct evidence of an absorbing layer in the equatorial plane of the galaxy, variously estimated to be, at any rate so far as the denser parts are concerned, from 100 to 200 parsecs in thickness (that is, perpendicular

* *Loc. cit.*

† In 8·52 (3), V refers to the component of the parallactic motion at right angles to the direction of the vertex, whereas in 7·43 (4), V refers to the drift velocity.

to the galactic equator). This is, qualitatively, in conformity with the well-known fact that very distant objects such as globular clusters and extra-galactic nebulae are never observed in or near the galactic equator. Eddington's theoretical researches* and Plaskett and Pearce's observational investigations† on the stationary H and K lines of ionised calcium in O and B type stars leave little doubt as to the existence of interstellar matter, a conclusion strengthened subsequently by different lines of attack, notably by Trumpler's investigation‡ of the open clusters, to be discussed more fully in the next chapter.

Measuring the distance, r, of a star in parsecs and assuming that there is no loss of light during its passage through interstellar space, we have the relation between the apparent magnitude m and the absolute magnitude M,

$$M = m + 5 - 5 \operatorname{Log} r. \qquad \text{......(1)}$$

But, if there is an absorbing medium, the star's apparent brightness will be less than if its light traversed transparent space; accordingly the observed magnitude, m_0, will be greater than m by a quantity which will depend on the absorbing properties of the medium and on the length of path within the medium. We can thus write

$$m_0 = m + F(r),$$

the absorbing properties being regarded as a function of the distance r and the total effect being denoted by $F(r)$. The relation between the absolute magnitude, M, and the observed apparent magnitude (as affected by absorption) is then

$$M = m_0 + 5 - 5 \operatorname{Log} r - F(r), \qquad \text{......(2)}$$

where the function $F(r)$ depends on the characteristics of the absorbing medium, and is positive. For a first approximation it is generally assumed that the absorption, as expressed in magnitudes, is proportional to the distance, so that

$$M = m_0 + 5 - 5 \operatorname{Log} r - kr, \qquad \text{......(3)}$$

where k (a positive quantity) is defined to be the absorption constant (expressed here in terms of magnitude per parsec) and, strictly, should be regarded as pertaining to one direction in the Milky Way alone.

We can express the result (3) in another way by saying that the effect of the absorbing medium is to increase the apparent magnitude of a star by kr.

In accordance with physical principles, the loss of light we have been considering may be due to absorption by free atoms (or molecules) or due to scattering by electrons, or by atoms, or by small discrete particles. We shall discuss this aspect of the subject in the following chapter.

* *Bakerian Lecture, Proc. R.S.* **111**, A, 424, 1926.
† *M.N.* **90**, 243, 1930.
‡ *Lick Obs. Bull.* **14**, 154, 1930.

We further remark that k is a function of the wave-length of light and is larger for the photographic wave-lengths than for visual wave-lengths. We denote by k_p and k_v the values of k referring respectively to photographic and visual observations. According to Trumpler,* the mean values of k_p and k_v, the averages being taken over several directions in or near the galactic equator, are

$$k_p = +0^{\mathrm{m}}{\cdot}67 \text{ per 1000 parsecs},$$

$$k_v = +0^{\mathrm{m}}{\cdot}35 \text{ per 1000 parsecs}.$$

We have seen in section 8·31, formula (4), that the ratio $B(m+1):B(m)$ is numerically equal to 3·98, where $B(m)$ is the total number of stars brighter than the apparent magnitude m, on the assumption that the space-density of the stars is constant and that interstellar space is perfectly transparent. As the observed ratio is found to decrease with increasing m (that is, effectively, with increasing distance), it was early inferred that the star-density must decrease at increasing distances from the sun. But, as we have previously remarked, this apparent decrease in star-density may quite well be ascribed, in whole or in part, to the effect of interstellar absorption; on this assumption, Halm† and Schalén‡ obtained values of k of $2^{\mathrm{m}}{\cdot}1$ and $0^{\mathrm{m}}{\cdot}5$ per kiloparsec respectively; the former value must now be regarded as an upper limit.

8·62. *Galactic absorption and the statistical equations.*

From section 8·15, we have for the number, $b(m)\,dm$, of stars with apparent magnitudes between m and $m+dm$,

$$b(m) = S\int_0^\infty r^2 D(r)\,\phi(M_1)\,dr,$$

it being assumed that there is no absorption. In this formula

$$M_1 = m - 5\,\mathrm{Log}\,r,$$

where, M_1 being the modified absolute magnitude,

$$M_1 = M - 5.$$

Also, with an absorbing medium present, we have from 8·61 (2)

$$M_1 = m_0 - 5\,\mathrm{Log}\,r - F(r), \qquad \text{......(1)}$$

in which m_0 is now the apparent magnitude actually observed.

If $b(m_0)\,dm_0$ is the number of stars with observed apparent magnitudes between m_0 and m_0+dm_0,

$$b(m_0) = S\int_0^\infty r^2 D(r)\,\phi(M_1)\,dr, \qquad \text{......(2)}$$

in which M_1 is now given by (1).

* *Loc. cit.* † *M.N.* **80**, 162, 1919. ‡ *A.N.* **236**, 249, 1929.

Following the method of Seares,* write

$$F(r)+5\,\mathrm{Log}\,r = 5\,\mathrm{Log}\,r_0,$$

so that

$$r_0 = r\,.\,10^{\frac{1}{5}F(r)},$$

and, putting

$$c = \tfrac{1}{5}\log_e 10,$$

we have

$$r_0 = re^{cF(r)}. \qquad \ldots\ldots(3)$$

If $F(r)$ is known, we can invert this relation and obtain r as a function of r_0.

From (3)

$$r_0^2 dr_0 = r^2 e^{3cF(r)}\left\{1+cr\frac{dF(r)}{dr}\right\}dr \qquad \ldots\ldots(4)$$

or

$$r_0^2 dr_0 = G(r)\,r^2 dr, \qquad \ldots\ldots(5)$$

where

$$G(r) = e^{3cF(r)}\left\{1+cr\frac{dF(r)}{dr}\right\}. \qquad \ldots\ldots(6)$$

Now write

$$D_0(r_0) = \frac{1}{G(r)}D(r), \qquad \ldots\ldots(7)$$

the right-hand side of (7) being expressible in terms of r_0 by means of (3). Then (2) becomes

$$b(m_0) = S\int_0^\infty r_0^2 D_0(r_0)\,\phi(m_0-5\,\mathrm{Log}\,r_0)\,dr_0, \qquad \ldots\ldots(8)$$

which is of the same form as 8·15 (5) and leads to an integral equation similar to 8·15 (12).

If the functions $b(m_0)$ and $\phi(M_1)$ are known, the solution of (8) leads to the function $D_0(r_0)$; hence from (6) and (7) the density function $D(r)$ is given by

$$D(r) = D_0(re^{cF(r)})\,e^{3cF(r)}\left\{1+cr\frac{dF}{dr}\right\}. \qquad \ldots\ldots(9)$$

In the case of uniform absorption, $F(r) = kr$ and (9) becomes

$$D(r) = D_0(re^{kcr})\,e^{3kcr}\{1+kcr\}. \qquad \ldots\ldots(10)$$

The equation (9)—or (10)—constitutes a formal solution of the problem if the function $F(r)$—or the value of k—is known.

In the same way the other integral equations involving apparent magnitudes can be modified to take account of galactic absorption.

However, the application of these formulae must be a task for the future when observational material is more abundant and accurate. The integral equations, derived in earlier sections, must at present be confined to investigations of stars in regions at some distance from the Milky Way; in such directions, the apparent magnitudes of stars at distances exceeding $t\,\mathrm{cosec}\,g$ parsecs are all affected by a constant amount depending on the absorption by an effective depth of $t\,\mathrm{cosec}\,g$ parsecs of the interstellar cloud, t being half the thickness of the cloud and g the galactic latitude of the region

* *Ap. J.* **74**, 91, 1931.

concerned. If the characteristics of the cloud can be derived by other means, it will then be possible to make the proper magnitude allowance for the given direction.

8·71. *The frequency function of the space-velocities of the stars derived from radial velocities.*

We have remarked on several occasions that the observed characteristics of the stellar velocity distribution are represented equally well according to the assumptions inherent in the two-streams and ellipsoidal theories. The *modus operandi* has been to compare the implications of an assumed formula with observations. We now consider the inverse problem of deriving the frequency function of the space-velocities from the observations themselves, in the present instance the observed radial velocities freed from solar motion and the K term. In practical applications it will be generally convenient to deal with each spectral type separately. Also, the assumption is made that the frequency function is the same in all parts of the sky in our neighbourhood. The solution of the problem is due to Ambarzumian.*

8·72. *The two-dimensional problem.*

We consider first the distribution of stars in a plane with a view to the application of the method to stars strongly concentrated towards the galactic equator. The position of a star is then defined by its angular distance from any arbitrary point on the galactic equator. Denoting this angular distance by α—we refer to this simply as the longitude—and the radial velocity by V (freed from solar motion and the K term), we have that the number of stars with radial velocities between V and $V+dV$ and in longitudes between α and $\alpha+d\alpha$ can be written as

$$f(V,\alpha)\,dV\,d\alpha, \qquad \ldots\ldots(1)$$

where the function $f(V,\alpha)$ is supposed to be derived from observations. If $N(\alpha)\,d\alpha$ is the total number of stars observed between longitudes α and $\alpha+d\alpha$, we have

$$N(\alpha) = \int_{-\infty}^{\infty} f(V,\alpha)\,dV. \qquad \ldots\ldots(2)$$

Let $\phi(u,v)$ be the frequency function of the peculiar linear velocities whose components are (u,v) in the galactic plane, it being assumed that the components of velocity perpendicular to the galactic plane are negligible. We take the u-axis in the direction $\alpha = 0$.

* *M.N.* **96**, 172, 1936.

The proportion of stars with velocity components between (u, v) and $(u+du, v+dv)$ is

$$\phi(u,v)\,du\,dv,$$

the function ϕ satisfying the relation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(u,v)\,du\,dv = 1.$$

The number of stars between longitudes α and $\alpha+d\alpha$ and with velocity components between (u, v) and $(u+du, v+dv)$ is then

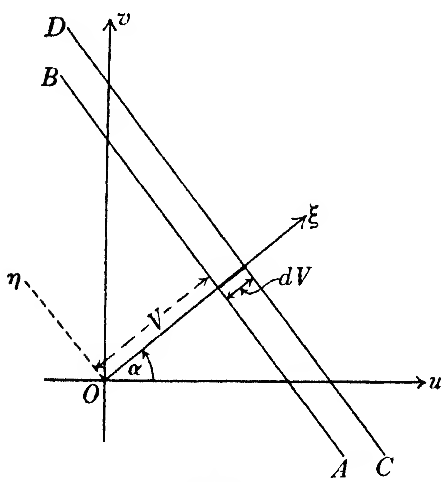

Fig. 50

$$N(\alpha)\,\phi(u,v)\,du\,dv\,d\alpha. \quad \text{......(3)}$$

Now the radial velocity is given by

$$V = u\cos\alpha + v\sin\alpha, \quad \text{......(4)}$$

which is the equation of a straight line in the (u, v) plane, the perpendicular from the origin to this line being V. Hence the number of stars in the longitude interval $(\alpha, \alpha+d\alpha)$ with radial velocities between V and $V+dV$ is obtained by summing the expression (3) over the strip between AB and CD in Fig. 50. But this number is also given by the expression (1). Hence

$$f(V,\alpha)\,dV = N(\alpha)\iint\phi(u,v)\,du\,dv, \quad \text{......(5)}$$

the integration being over the strip of width dV.

Let
$$\xi = u\cos\alpha + v\sin\alpha,$$
$$\eta = -u\sin\alpha + v\cos\alpha.$$

This transformation corresponds to a change of axes as indicated in Fig. 50.

We have $$du\,dv = d\xi\,d\eta,$$
and for the strip $$d\xi = dV.$$

Then over the strip

$$\iint\phi(u,v)\,du\,dv = dV\int_{-\infty}^{\infty}\phi(V\cos\alpha - \eta\sin\alpha,\, V\sin\alpha + \eta\cos\alpha)\,d\eta, \quad \text{......(6)}$$

the integration on the right-hand side being along the straight line AB.

Let
$$\frac{f(V,\alpha)}{N(\alpha)} = F(V,\alpha). \quad \text{......(7)}$$

As $f(V, \alpha)$ and $N(\alpha)$ are supposed known from the observations, we regard $F(V, \alpha)$ as a known function. From (5), (6) and (7) we obtain

$$F(V, \alpha) = \int_{-\infty}^{\infty} \phi(V\cos\alpha - \eta\sin\alpha, V\sin\alpha + \eta\cos\alpha)\, d\eta. \quad \ldots\ldots(8)$$

From this integral equation we have to deduce the function ϕ. It is to be added that the conditions of the problem require that $F(V, \alpha) \to 0$ as $V \to \infty$.

8·73. *Solution of the two-dimensional problem.*

In (8) of the previous section, write

$$V = x\cos\alpha + y\sin\alpha + W, \quad \ldots\ldots(1)$$

where x, y and W are arbitrary parameters, and introduce a new variable U by means of

$$\eta = U - x\sin\alpha + y\cos\alpha. \quad \ldots\ldots(2)$$

Then

$$V\cos\alpha - \eta\sin\alpha = x + W\cos\alpha - U\sin\alpha,$$
$$V\sin\alpha + \eta\cos\alpha = y + W\sin\alpha + U\cos\alpha,$$

and we obtain

$$F(x\cos\alpha + y\sin\alpha + W, \alpha)$$
$$= \int_{-\infty}^{\infty} \phi(x + W\cos\alpha - U\sin\alpha,\ y + W\sin\alpha + U\cos\alpha)\, dU.$$

Integrate both sides with respect to α between 0 and 2π, and set

$$X(x, y, W) \equiv \int_0^{2\pi} F(x\cos\alpha + y\sin\alpha + W, \alpha)\, d\alpha. \quad \ldots\ldots(3)$$

Since F is a known function, being derived from the observations, it follows that X is a known function; also $X(x, y, W) \to 0$ as $W \to \infty$. We have

$$X(x, y, W) = \int_{-\infty}^{\infty} dU \int_0^{2\pi} \phi(x + W\cos\alpha - U\sin\alpha,\ y + W\sin\alpha + U\cos\alpha)\, d\alpha.$$

Write

$$W = B\cos\beta, \quad U = B\sin\beta,$$

so that

$$U^2 = B^2 - W^2 \quad \ldots\ldots(4)$$

and β is independent of α. Then

$$X(x, y, W) = \int_{-\infty}^{\infty} dU \int_0^{2\pi} \phi\{x + B\cos(\alpha + \beta),\ y + B\sin(\alpha + \beta)\}\, d\alpha. \quad \ldots\ldots(5)$$

But, if we put $\gamma = \alpha + \beta$ in the integral with respect to α on the right of (5),

$$\int_0^{2\pi} \phi(\alpha)\, d\alpha = \int_\beta^{2\pi+\beta} \phi(\gamma)\, d\gamma$$
$$= \int_0^{2\pi} \phi(\gamma)\, d\gamma + \int_{2\pi}^{2\pi+\beta} \phi(\gamma)\, d\gamma - \int_0^{\beta} \phi(\gamma)\, d\gamma$$
$$= \int_0^{2\pi} \phi(\gamma)\, d\gamma.$$

Hence $$X(x,y,W) = \int_{-\infty}^{\infty} dU \int_0^{2\pi} \phi(x+B\cos\alpha,\, y+B\sin\alpha)\, d\alpha. \qquad \text{......(6)}$$

Let $$\Phi(x,y,B) = \int_0^{2\pi} \phi(x+B\cos\alpha,\, y+B\sin\alpha)\, d\alpha, \qquad \text{......(7)}$$

so that $$X(x,y,W) = \int_{-\infty}^{\infty} \Phi(x,y,B)\, dU. \qquad \text{......(8)}$$

But (4) gives $$dU = \frac{BdB}{(B^2-W^2)^{\frac{1}{2}}}. \qquad \text{......(9)}$$

Hence (8) becomes

$$X(x,y,W) = 2\int_W^{\infty} \Phi(x,y,B) \frac{BdB}{(B^2-W^2)^{\frac{1}{2}}}. \qquad \text{......(10)}$$

This integral equation can be transformed into Abel's form.* Consider the equation

$$X(x,y,W) - X(x,y,R) = 2\int_W^R \Phi(x,y,B) \frac{BdB}{(B^2-W^2)^{\frac{1}{2}}}, \qquad \text{......(11)}$$

where $R > W$.

Set $$h \equiv R^2 - W^2, \quad t \equiv R^2 - B^2$$

and $$X(x,y,W) - X(x,y,R) \equiv \theta(h),$$

$$\Phi(x,y,B) \equiv \psi'(t).$$

Then (11) becomes $$\theta(h) = \int_0^h \frac{\psi'(t)\, dt}{(h-t)^{\frac{1}{2}}}, \qquad \text{......(12)}$$

of which the solution is $$\psi(t) = \frac{1}{\pi}\int_0^t \frac{\theta(h)\, dh}{(t-h)^{\frac{1}{2}}}. \qquad \text{......(13)}$$

Integrating by parts, we have

$$\psi(t) = -\frac{2}{\pi}\Big[\theta(h)\,.\,(t-h)^{\frac{1}{2}}\Big]_{h=0}^{h=t} + \frac{2}{\pi}\int_0^t (t-h)^{\frac{1}{2}} \frac{d\theta(h)}{dh}\, dh. \qquad \text{......(14)}$$

The integrated part vanishes for $h = t$ and also for $h = 0$, since $\theta(0) = 0$, by (12). Accordingly, we have simply

$$\psi(t) = \frac{2}{\pi}\int_0^t (t-h)^{\frac{1}{2}} \frac{d\theta(h)}{dh}\, dh,$$

whence $$\psi'(t) = \frac{1}{\pi}\int_0^t \frac{\frac{d\theta(h)}{dh}\, dh}{(t-h)^{\frac{1}{2}}}$$

$$= -\frac{1}{\pi}\int_B^R \frac{\frac{\partial X}{\partial W}\, dW}{(W^2-B^2)^{\frac{1}{2}}}.$$

* *Crelle*, **1**, 1826.

This is the solution of (11). Making $R \to \infty$, we see that (11) takes the form (10), since $X(x, y, R) \to 0$. Thus the solution of (10) is

$$\Phi(x,y,B) = -\frac{1}{\pi}\int_B^\infty \frac{\frac{\partial X}{\partial W}\,dW}{(W^2-B^2)^{\frac{1}{2}}}. \qquad \ldots\ldots(15)$$

But from (7) $$\Phi(x,y,0) = 2\pi\phi(x,y),$$

and from (15) $$\Phi(x,y,0) = -\frac{1}{\pi}\int_0^\infty \frac{1}{W}\frac{\partial X}{\partial W}\,dW.$$

Hence $$\phi(x,y) = -\frac{1}{2\pi^2}\int_0^\infty \frac{1}{W}\frac{\partial X(x,y,W)}{\partial W}\,dW \qquad \ldots\ldots(16)$$

or $$\phi(u,v) = -\frac{1}{2\pi^2}\int_0^\infty \frac{1}{W}\frac{\partial X(u,v,W)}{\partial W}\,dW. \qquad \ldots\ldots(17)$$

Since the function X is known, the frequency function $\phi(u, v)$ is obtained by means of (17); this equation constitutes the solution of the two-dimensional problem.

8·74. *The three-dimensional problem.*

We consider a region, subtending a small solid angle $d\omega$, with galactic coordinates (G, g). We can set the number of radial velocities between V and $V+dV$ in the region to be

$$f(V, G, g)\,dV\,d\omega,$$

where f is the frequency function of the radial velocities. The total number, $N(G, g)\,d\omega$, of stars in the region is given by

$$N(G,g) = \int_{-\infty}^{\infty} f(V,G,g)\,dV. \qquad \ldots\ldots(1)$$

Let $\phi(u, v, w)$ denote the frequency function of the linear velocities. By analogy with the two-dimensional problem, the frequency of radial velocities between V and $V+dV$ is given by

$$f(V,G,g)\,dV = N(G,g)\iiint \phi(u,v,w)\,du\,dv\,dw, \qquad \ldots\ldots(2)$$

where the integration extends throughout the volume between two parallel planes, whose equations in the (u, v, w) coordinates are

$$V = lu+mv+nw, \qquad \ldots\ldots(3)$$

$$V+dV = lu+mv+nw,$$

in which (l, m, n) are the direction-cosines of the region; actually,

$$l = \cos G\cos g, \quad m = \sin G\cos g, \quad n = \sin g. \qquad \ldots\ldots(4)$$

Write $$F(V,G,g)=\frac{f(V,G,g)}{N(G,g)}. \qquad \text{......(5)}$$

Let $d\sigma$ denote the element of area of the plane (3). Then

$$dudvdw = d\sigma dV.$$

We thus obtain from (2)

$$F(V,G,g)=\iint \phi(u,v,w)\,d\sigma. \qquad \text{......(6)}$$

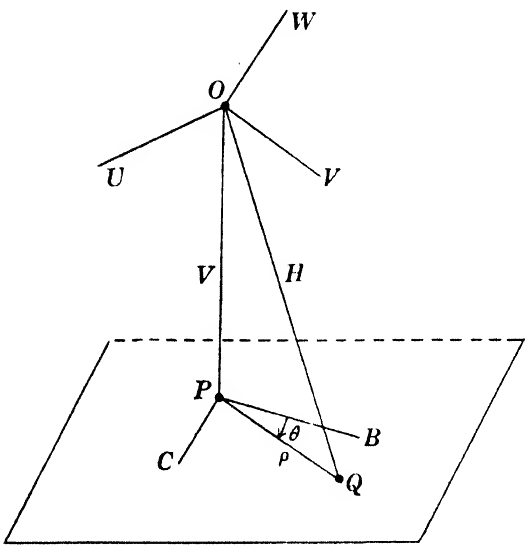

Fig. 51

In Fig. 51 OU, OV and OW are the u, v, w axes, which we take to be parallel to the usual galactic system; OP gives the direction of the region and the length of OP is V; the plane (3), which is perpendicular to OP, is shown in the figure. Referred to any rectangular axes PB, PC in this plane,

$$d\sigma = \rho d\rho d\theta,$$

(ρ,θ) being the polar coordinates of a point Q in the plane. Let $OQ = H$ and let the direction-cosines of OQ be (L, M, N). Then, in (6),

$$u = LH, \quad v = MH, \quad w = NH,$$

and hence $$F(V,G,g)=\int_0^\infty\int_0^{2\pi}\phi(LH,MH,NH)\rho d\rho d\theta. \qquad \text{......(7)}$$

As the integral on the right of (7) is independent of the choice of PB and PC as axes, we have

$$F(V,G,g)=2\pi\int_0^\infty \phi(LH,MH,NH)\rho d\rho. \qquad \text{......(8)}$$

Integrate both sides over the surface of a sphere, centre O, with unit radius. Then, $d\omega$ being an element of the surface of the unit sphere,

$$\int_0^{4\pi}F(V,G,g)\,d\omega = 2\pi\int_0^{4\pi}\int_0^\infty \phi(LH,MH,NH)\,d\omega\rho d\rho. \qquad \text{......(9)}$$

Set $$\Phi(H)=2\pi\int_0^{4\pi}\phi(LH,MH,NH)\,d\omega. \qquad \text{......(10)}$$

Then $$\int_0^{4\pi}F(V,G,g)\,d\omega=\int_0^\infty \Phi(H)\rho d\rho. \qquad \text{......(11)}$$

Now $$V^2+\rho^2 = H^2.$$

Hence, for a given V,

$$\int_0^{4\pi} F(V,G,g)\,d\omega = \int_V^{\infty} \Phi(H)\,H\,dH,$$

from which $$\Phi(V) = -\frac{1}{V}\frac{d}{dV}\int_0^{4\pi} F(V,G,g)\,d\omega. \qquad \text{......(12)}$$

But, from (10), $$\Phi(0) = 8\pi^2\phi(0,0,0), \qquad \text{......(13)}$$

and consequently (12) and (13) give

$$\phi(0,0,0) = -\frac{1}{8\pi^2}\operatorname*{Lim}_{V\to 0}\frac{1}{V}\frac{d}{dV}\int_0^{4\pi} F(V,G,g)\,d\omega. \qquad \text{......(14)}$$

In Fig. 51 the origin of the velocity system of axes, OU, OV and OW, is entirely arbitrary and the formula (14) will hold for any system of parallel axes. Hence, by a translation of the origin, we obtain from (14)

$$\phi(u,v,w) = -\frac{1}{8\pi^2}\operatorname*{Lim}_{W\to 0}\frac{1}{W}\frac{d}{dW}\int_0^{4\pi} F(lu+mv+nw+W,G,g)\,d\omega. \qquad \text{......(15)}$$

The formula (15) constitutes the solution of the problem, for the function F is obtainable from the observations and the right-hand side of (15) is thus determinable.

CHAPTER IX

STAR CLUSTERS

9·11. *The determination of the Convergent Point of a moving cluster.*

A moving cluster is an assembly of stars in a limited volume of space within the galactic system characterised by the parallelism and equality of their motions. Well-known examples are the Taurus Cluster and the Pleiades. In the former the stars belonging to the cluster are scattered over several hundred square degrees of the celestial sphere, whereas in the latter the cluster stars are much more concentrated in the sky.

Relative to the sun, each star has the same linear velocity components (X, Y, Z) with respect to the usual equatorial system of coordinates or, expressed somewhat differently, each star has the same velocity V in a particular direction (A, D), the corresponding point on the celestial sphere being called the *convergent point*, which we denote by C. The transverse linear velocity of each star gives rise to a proper motion along the great circle joining the position of the star to C. If the cluster stars are sufficiently well scattered in the sky, the point C can easily be determined.

As in Airy's method (section 3·31), we have for a star at (α, δ)

$$-X\sin\alpha + Y\cos\alpha = \frac{\kappa}{p}\mu_\alpha \cos\delta, \qquad \text{......(1)}$$

$$-X\cos\alpha\sin\delta - Y\sin\alpha\sin\delta + Z\cos\delta = \frac{\kappa}{p}\mu_\delta, \qquad \text{......(2)}$$

where μ_α, μ_δ are the components of proper motion, p is the parallax and $\kappa = 4{\cdot}74$. Also,

$$X\cos\alpha\cos\delta + Y\sin\alpha\cos\delta + Z\sin\delta = \rho, \qquad \text{......(3)}$$

where ρ is the radial velocity. Write

$$\mu_\alpha \cos\delta = \xi, \quad \mu_\delta = \eta. \qquad \text{......(4)}$$

Multiply (1) by η and (2) by ξ and subtract; we obtain

$$X(\eta\sin\alpha - \xi\cos\alpha\sin\delta) - Y(\eta\cos\alpha + \xi\sin\alpha\sin\delta) + Z\xi\cos\delta = 0, \quad \text{......(5)}$$

which we may write in the form

$$aX + bY + cZ = 0. \qquad \text{......(6)}$$

Each star contributes an equation of the form (6), the coefficients of X, Y and Z being supposed known. A least-squares solution then determines the ratios $X : Y : Z$.

Also, $X = V\cos A\cos D$, $Y = V\sin A\cos D$, $Z = V\sin D$,

so that $$\tan A = Y/X, \qquad \text{......(7)}$$

$$\tan D = \frac{Z}{(X^2+Y^2)^{\frac{1}{2}}}. \qquad \text{......(8)}$$

The formulae (7) and (8) thus determine the convergent point C from the proper motions alone. This method is due to Charlier.

In the same way, if there is a sufficient number of stars with observed radial velocities, the formula (3) may be used in a least-squares solution to give the values of X, Y and Z in km. per second. It is to be noted that if any of the stars are of type B, the observed radial velocity contains the K term whose value, being presumed known from other investigations, must be removed from ρ before the formula (3) is applied.*

9·12. *The determination of the Convergent Point (Bohlin's method).*†

This is essentially the same method as described in section 4·32; the procedure is, however, somewhat different. In Fig. 52, S is a star of the cluster with its proper motion directed towards C, and $Q(A', D')$ is a pole of the great circle SC. If θ is the position angle $(P\hat{S}C)$ of the proper motion of S, we have from the triangle PSQ (in which $P\hat{S}Q = \theta - 90°$ and $QS = 90°$)

$$\left.\begin{aligned} \cos D' \sin(A'-\alpha) &= -\cos\theta \\ \cos D' \cos(A'-\alpha) &= -\sin\delta\sin\theta \\ \sin D' &= \cos\delta\sin\theta \end{aligned}\right\}. \qquad \text{......(1)}$$

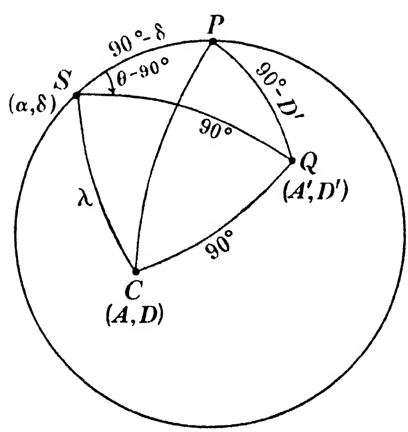

Fig. 52

Since θ is supposed known, these equations determine A' and D'.

From the triangle PCQ, in which $QC = 90°$, we have

$$\cos(A'-A) = -\tan D\tan D'. \qquad \text{......(2)}$$

Write $$\left.\begin{aligned} x &= \cot D\cos A \\ y &= \cot D\sin A \end{aligned}\right\}. \qquad \text{......(3)}$$

Then (2) becomes

$$x\cos A' + y\sin A' + \tan D' = 0. \qquad \text{......(4)}$$

Each star contributes an equation of the form (4) and by a least-squares solution we obtain the values of x and y, from which

$$\tan A = y/x, \qquad \cot D = (x^2+y^2)^{\frac{1}{2}}.$$

* See a remark by C. C. L. Gregory, *Observatory*, 59, 154, 1936.

† The formulae are given by Rasmuson, *Lund Medd.* Ser. II, No. 26, p. 6, 1921.

It may be remarked that the difference between the procedure in section 9·11 and in section 9·12 is merely in the arrangement of the computations. By definition $X:Y:Z$ in section 9·11 is the same as $x:y:z$ in section 9·12, where $z \equiv 1$ is the coefficient of $\tan D'$ in (4). Also, the coefficients in 9·11 (5) and 9·12 (4) are in the same ratio. For

$$\cos A' : \sin A' : \tan D' = \cos D' \cos A' : \cos D' \sin A' : \sin D'$$

$$= \cos D' \cos(A' - \alpha + \alpha) : \cos D' \sin(A' - \alpha + \alpha) : \sin D'$$

$$= \sin\alpha \cos\theta - \cos\alpha \sin\delta \sin\theta : -\cos\alpha \cos\theta - \sin\alpha \sin\delta \sin\theta : \cos\delta \sin\theta$$

by means of (1),

$$= \eta \sin\alpha - \xi \cos\alpha \sin\delta : -\eta \cos\alpha - \xi \sin\alpha \sin\delta : \xi \cos\delta.$$

9·13. *Determination of the parallaxes of the cluster stars.*

The angular distance, λ, of a star from the point of convergency, C, is found from

$$\cos\lambda = \sin\delta \sin D + \cos\delta \cos D \cos(A - \alpha).$$

Also, the radial velocity, ρ, of a star such as S (Fig. 52) is the projection of the common velocity, V, in the direction of S. Thus

$$\rho = V \cos\lambda. \qquad \text{......(1)}$$

If ρ is freed from such systematic effects as the K term, we obtain V from (1). Other stars ought to furnish the same value of V within, of course, the limits of observational error. A least-squares solution of (1) yields the cluster velocity relative to the sun.

Let μ denote the total annual proper motion of any star of the cluster. Then

$$V \sin\lambda = \frac{\kappa\mu}{p}, \qquad \text{......(2)}$$

from which the parallax, p, can now be computed.

The absolute magnitudes of the stars can then be found from the formula

$$M = m + 5 + 5 \operatorname{Log} p.$$

9·14. *The characteristics of the moving clusters.*

Rasmuson has made a detailed study* of thirteen clusters; in several, however, the proper motions do not satisfy very well the necessary criterion of convergency, to which, for example, the proper motions of the Taurus cluster conform so accurately.

When the effects of the solar motion are removed from the cluster velocity as derived by the methods of the previous sections, it is found that the clusters are moving almost parallel to the galactic equator.

* *Lund Medd.* Ser. II, No. 26, 1921.

The spatial distribution of the stars in any given cluster is found to be approximately ellipsoidal in form; as regards the Taurus cluster, the lengths of the axes of the ellipsoid are in the ratio 5·9 : 4·1 : 7·4; the corresponding results for the Ursa Major cluster are 4·5 : 5·9 : 2·7. As regards the Taurus cluster, the shortest axis is directed approximately to the galactic pole; and as regards the Ursa Major cluster, the longest axis is directed approximately to the galactic pole. R. E. Wilson* gives 8·5 : 5·7 : 5·0 as the ratio of the axes for the Taurus cluster, with the directions of the axes considerably different from those found by Rasmuson.

Important results concern the absolute magnitudes of stars of various spectral types. The derivation of individual parallaxes of stars in such clusters as the Taurus cluster is of a high order of accuracy and, consequently, the deduced absolute magnitudes carry great weight. Rasmuson's conclusions regarding mean absolute magnitudes† are summarised in the following table.

Table 43. *Mean absolute magnitude*, M_0

(Standard parallax 0″·1)

Spectral type	M_0
B0–B5	$-0^{m}{\cdot}45$
B8, 9	$+0{\cdot}92$
A	$+1{\cdot}57$

The mean dispersion is, in each case, about two-thirds of a magnitude, the distribution of absolute magnitudes being assumed to follow the law

$$\phi(M) = Ae^{-k^2(M-M_0)^2}.$$

9·21. *Open clusters.*

An open cluster is essentially of the same physical nature as a moving cluster but, because of its much greater distance and more compact apparent form, it has hitherto been found impossible to treat the open cluster by the methods of the previous sections. The proper motions of open clusters have been measured in only a small number of instances and radial velocity measures of the brighter cluster stars are also not very abundant. Photometric studies can be made, however, according to the usual methods, and it is on these that we have to depend for investigating the distances of the clusters. A knowledge of the clusters is important in investigations dealing with the rotation of the galaxy.

* *A.J.* **42**, 54, 1932.

† In Rasmuson's memoir the standard parallax for which absolute magnitude is defined is 0″·206; to convert Rasmuson's absolute magnitudes into the absolute magnitude system normally employed (standard parallax 0″·1), add $1^{m}{\cdot}57$.

9·22. *The distances of the clusters (Raab's method).*

As the distances of the open clusters are so great, we can assume that all the stars in a given cluster are at the same distance, r parsecs. We assume with Raab* that the frequency function of the absolute magnitudes of stars of a given spectral type is

$$\phi(M) = Ce^{-\frac{(M-M_0)^2}{2\sigma^2}}, \qquad \ldots\ldots(1)$$

in which M_0 is the mean absolute magnitude and σ is the dispersion, both of which are supposed known from the results of other investigations.

The relation between the apparent and absolute magnitudes of a star is

$$M = m + 5 - 5\operatorname{Log} r.$$

Since r is constant for all the stars of a cluster, the frequency function of the apparent magnitudes is

$$\phi(m) = Ce^{-\frac{1}{2\sigma^2}(m+5-5\operatorname{Log} r - M_0)^2}.$$

Consider all the stars brighter than a given apparent magnitude m_1. The mean magnitude $\overline{m}$ is given by

$$\overline{m} = \frac{\int_{-\infty}^{m_1} me^{-\frac{1}{2\sigma^2}(m+5-5\operatorname{Log} r - M_0)^2}\,dm}{\int_{-\infty}^{m_1} e^{-\frac{1}{2\sigma^2}(m+5-5\operatorname{Log} r - M_0)^2}\,dm}.$$

Let

$$\sigma x = m + 5 - 5\operatorname{Log} r - M_0,$$

$$\sigma x_1 = m_1 + 5 - 5\operatorname{Log} r - M_0. \qquad \ldots\ldots(2)$$

Then

$$m_1 - m = \sigma(x_1 - x).$$

Hence

$$\frac{m_1 - \overline{m}}{\sigma} = \frac{\int_{-\infty}^{x_1} (x_1 - x)\, e^{-\frac{x^2}{2}}\,dx}{\int_{-\infty}^{x_1} e^{-\frac{x^2}{2}}\,dx}$$

$$= x_1 + \frac{e^{-\frac{x_1^2}{2}}}{\int_{-\infty}^{x_1} e^{-\frac{x^2}{2}}\,dx}$$

or

$$\frac{m_1 - \overline{m}}{\sigma} = F(x_1). \qquad \ldots\ldots(3)$$

The values of $F(x_1)$ are readily computed and are given in the following table.†

* *Lund Medd.* Ser. II, No. 28, 1922.

† Raab (*ibid.*), p. 87.

Table 44. *Values of $F(x_1)$*

x_1	$F(x_1)$	x_1	$F(x_1)$	x_1	$F(x_1)$
−4·0	+0·226*	−1·6	+0·424	+0·8	+1·168
−3·8	0·235*	−1·4	0·454	+1·0	1·288
−3·6	0·246	−1·2	0·488	+1·2	1·419
−3·4	0·257	−1·0	0·525	+1·4	1·563
−3·2	0·270	−0·8	0·567	+1·6	1·717
−3·0	0·283	−0·6	0·615	+1·8	1·882
−2·8	0·298	−0·4	0·669	+2·0	2·055
−2·6	0·314	−0·2	0·729	+2·2	2·236
−2·4	0·332	0·0	0·798	+2·4	2·423
−2·2	0·351*	+0·2	0·875	+2·6	2·614
−2·0	0·373	+0·4	0·962	+2·8	2·808
−1·8	0·397	+0·6	1·059	+2·9	2·906
−1·6	+0·424	+0·8	+1·168	+3·0	+3·004

* Re-calculated by A. Fletcher.

In applying formula (3), Raab considers only the A type stars, for which he takes the dispersion in absolute magnitudes to be $0^m{\cdot}9$ and the mean absolute magnitude M_0 to be $1^m{\cdot}55$.

For any assigned value of m_1, the value of $\overline{m}$ is obtained from the observations; thus the value of $F(x_1)$ can be readily determined and from Table 44 the value of x_1 is obtained. The value of r is then calculated from (2).

9·23. *The influence of galactic absorption on the measured distances of the open clusters.*

Most of the open clusters are situated in or near the Milky Way, and as their distances are large the effect of the general galactic absorption on the apparent magnitudes of the cluster stars is considerable. If k now denotes the coefficient of absorption per kiloparsec, the observed magnitude m is given in terms of the absolute magnitude M—see 8·61 (3), where k was expressed in terms of magnitude per parsec—by

$$m = M - 5 + 5\,\mathrm{Log}\,r + \frac{kr}{1000}$$

or

$$m = M - 5 + 5\,\mathrm{Log}\,r_1, \qquad \text{......(1)}$$

where

$$\mathrm{Log}\,r_1 = \mathrm{Log}\,r + \frac{kr}{5000}. \qquad \text{......(2)}$$

The apparent magnitudes are generally measured photographically, and in these formulae we replace k by k_p, the absorption coefficient for photographically effective wave-lengths. Its value, according to Trumpler, may be taken to be $0^m{\cdot}67$ per 1000 parsecs.

Raab's investigation was undertaken before much was known of the galactic absorbing cloud and he implicitly assumed that galactic space is transparent. By referring to (1) it will be seen that the distances determined by Raab correspond to the values of r_1; the true distances, r, as is clear from formula (2) are much smaller. For example, if the true distance r is 1000 parsecs, the value of r_1, using the above value of k, is 1361 parsecs.

9·31. *Trumpler's research on open clusters; preliminary determination of the distances.*

Altogether a total of a hundred clusters form the subject of a systematic investigation by Trumpler.* On an average, the photographic magnitudes of 30 or 40 stars per cluster were obtained, together with the spectral types. The absolute magnitudes (visual and photographic) of the various spectral types and subdivisions as adopted by Trumpler are shown in Table 45. These values are mainly based on the results of Adams and Joy,† Lundmark,‡ Malmquist § and Hess.||

Table 45. *Mean absolute magnitudes according to spectral type (Trumpler)*

Spectral type	Dwarf branch		Giants	
	Visual	Photographic	Visual	Photographic
O	$-4^m{\cdot}0$	$-4^m{\cdot}3$		
B0	−3·1	−3·4		
B1	−2·5	−2·8		
B2	−1·8	−2·1		
B3	−1·2	−1·4		
B5	−0·8	−1·0		
B8	−0·2	−0·3		
B9	+0·3	+0·3		
A0	+0·9	+0·9		
A2	+1·7	+1·7		
A3	+2·0	+2·1		
A5	+2·3	+2·5		
F0	+2·9	+3·2	$+0^m{\cdot}5$	$+0^m{\cdot}9$
F2	+3·2	+3·5		
F5	+3·6	+4·0	+0·5	+1·0
F8	+4·2	+4·7		
G0	+4·5	+5·1	+0·5	+1·2
G5	+5·0	+5·7	+0·5	+1·4
K0	+6·2	+7·0	+0·5	+1·6

The value of $m-M$ is readily obtained for each observed star in the cluster and from the mean value of this quantity the distance, r, of the cluster is calculated from

$$\overline{m-M} = 5\,\mathrm{Log}\,r - 5. \qquad \text{......(1)}$$

* *Lick Obs. Bull.* No. 420, 1930.

† *Mt Wilson Contributions*, Nos. 199, 244, 262 or *Ap. J.* **53**, 13, 1921; **56**, 242, 1922; **59**, 294, 1923.

‡ *Publ. Astr. Society of the Pacific*, **34**, 150, 1922.

§ *Lund Medd.* Ser. II, No. 32, 1924.

|| *Seeliger Festschrift*, p. 265.

For example, in the open cluster M36 the thirty faintest stars observed, with photographic magnitudes between $9^{m}{\cdot}7$ and $13^{m}{\cdot}2$ and of spectral types ranging from B4 and A2, yielded

$$\overline{m-M} = 11^{m}{\cdot}12,$$

so that, by (1), r is found to be 1675 parsecs. On the other hand, the ten brightest stars, with photographic magnitudes between $8^{m}{\cdot}5$ and $9^{m}{\cdot}5$ and of spectral types B3 and B5, gave

$$\overline{m-M} = 10^{m}{\cdot}36.$$

It follows that, if we accept the results for the faintest stars, the ten stars of types B3, B5 are abnormally bright intrinsically by about three-quarters of a magnitude. Trumpler suggested that this discordance is due to selection and he applied an empirical correction to bring the distance as deduced from the bright stars into line with the distance derived from the faint stars.

The preliminary values of the distance, r, for the clusters, calculated by (1), take no cognisance, of course, of galactic absorption.

9·32. *Preliminary values of the linear diameters of the clusters.*

Let D denote the linear diameter of a cluster in parsecs, and d its angular diameter in minutes of arc, d being easily derived by measuring the extent of the cluster on a photographic plate. Then

$$D = r \sin d$$

or, with sufficient accuracy,

$$D = rd \sin 1'.$$

The preliminary values of D are then obtained from this formula.

9·33. *Classification of the clusters.*

Trumpler noted that clusters of similar constitution had similar linear diameters. Consequently, the clusters were classified in four main groups according to the concentration of the stars towards the centre; each group was further subdivided into three sections, the criterion used being the range in luminosity of the cluster stars.*

Trumpler introduced the assumption that clusters of similar constitution have actually the same linear dimensions, and it was on the basis of this assumption that it was found possible to determine the absorbing effects of the galactic cloud.

* For the details of the classification, see *Lick Obs. Bull.* No. 420, p. 159, 1930.

9·34. *The absorption of light.*

It is assumed that the absorption is uniform throughout that part of the galactic system with which the observations deal. We have the group of equations

$$m - M = 5 \operatorname{Log} r + \frac{kr}{1000} - 5, \qquad \ldots\ldots(1)$$

$$m - M = 5 \operatorname{Log} r_1 - 5, \qquad \ldots\ldots(2)$$

$$\operatorname{Log} r_1 = \operatorname{Log} r + \frac{kr}{5000}. \qquad \ldots\ldots(3)$$

Since the preliminary values of the linear diameters have been determined by way of equations of the form of (2), these values, which we denote by D_1, correspond to the unreduced distance r_1, so that

$$D_1 = r_1 d \sin 1'.$$

The true linear diameters, in terms of the true distances r, are given by

$$D = rd \sin 1'.$$

Hence, using (3),

$$\operatorname{Log} D_1 - \operatorname{Log} D = \frac{kr}{5000}, \qquad \ldots\ldots(4)$$

and, on taking means for the clusters of a subclass,

$$\overline{\operatorname{Log} D_1} - \overline{\operatorname{Log} D} = \frac{k\bar{r}}{5000}, \qquad \ldots\ldots(5)$$

or, since it is assumed that all the clusters of the subclass have the same linear diameter D,

$$\overline{\operatorname{Log} D_1} = \operatorname{Log} D + \frac{k\bar{r}}{5000}. \qquad \ldots\ldots(6)$$

Since the preliminary values, D_1, are known, the residual v_1 for a cluster, as defined by

$$v_1 = \operatorname{Log} D_1 - \overline{\operatorname{Log} D_1}, \qquad \ldots\ldots(7)$$

can be calculated; hence, from (4), (6) and (7),

$$v_1 = \frac{kr}{5000} - \frac{k\bar{r}}{5000}, \qquad \ldots\ldots(8)$$

which may be written

$$v_1 = a + br, \qquad \ldots\ldots(9)$$

where

$$a = -\frac{k\bar{r}}{5000}, \quad b = \frac{k}{5000}.$$

It is assumed further that $\bar{r}$ is the same for all sub-classes; consequently, (9) can be employed for all the clusters, a being regarded as a constant. Also, from (3),

$$\operatorname{Log} r_1 = \operatorname{Log} r + br. \qquad \ldots\ldots(10)$$

Equations (9) and (10) are solved for r by successive approximations. The first approximation is obtained as follows: for near clusters, br is regarded as a small quantity in comparison with a, so that a is given approximately by

$$a = \bar{v}_1,$$

where $\bar{v}_1$ is the mean value of v_1 for the near clusters. Also, from (9) and (10),

$$\operatorname{Log} r = \operatorname{Log} r_1 + a - v_1,$$

from which r is obtained for each cluster. Equation (9) is then solved by least squares to determine a and b. The process is repeated two or three times. In this way, Trumpler obtained the value of k, the value finally adopted being

$$k = 0^{m}{\cdot}67 \text{ per kiloparsec}$$

for photographic absorption.

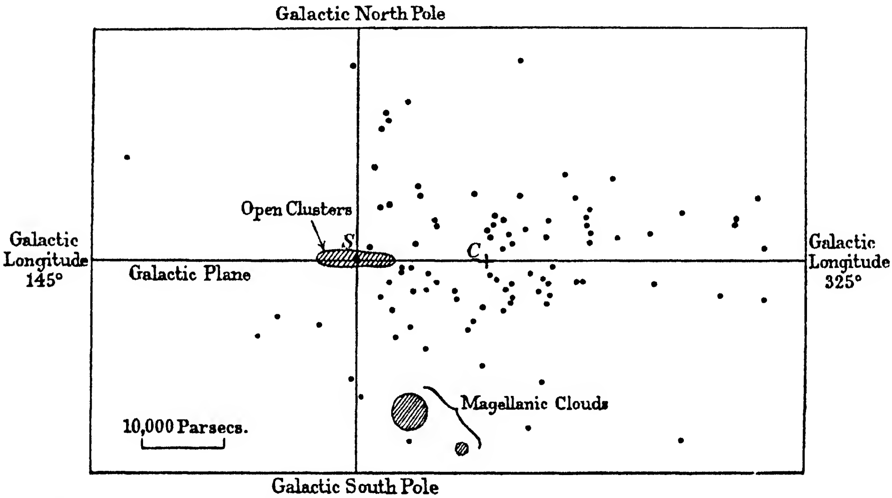

Fig. 53. *Trumpler's model of the Galaxy.* The Globular Clusters are represented by dots

With this value of k, the distances and the linear diameters of the clusters can then be calculated. It is found that the diameters range from 2 to 20 parsecs and the distances of the most remote clusters investigated are as great as 5000 parsecs.

In Fig. 53, reproduced from Trumpler's memoir,* the distribution of the open clusters, globular clusters and the Magellanic Clouds on a plane through the galactic poles and galactic longitude 325° is shown; the open clusters lie within the shaded elongated area around the sun at S; the globular clusters are represented by dots and the Magellanic Clouds by the two shaded circles. The direction along the galactic equator towards longitude 325° gives, according to Shapley, the direction of the centre, C, of the galactic system which is estimated to be about 15,000 parsecs† from the sun.

* *Lick Obs. Bull.* No. 420, p. 186, 1930.

† This figure is now believed to be somewhat exaggerated.

9·35. *Colour excess.*

In section 8·61 we briefly alluded to the fact that the coefficient of absorption is a function of wave-length. If, for example, the absorption is caused by scattering, the consequent loss of light is greater for short wave-lengths than for long wave-lengths. If m_p and m_v denote respectively the apparent photographic and visual magnitudes of a star, the normal colour index, I, is defined, in the absence of absorption, by

$$I = m_p - m_v \equiv M_p - M_v. \qquad \ldots\ldots(1)$$

The observed colour index, I', when absorption is present is given by

$$I' = m_p' - m_v', \qquad \ldots\ldots(2)$$

where m_p' and m_v' are the observed photographic and visual apparent magnitudes. But, with scattering, m_p is increased more than m_v is increased, so that I' is greater than I; in other words, the star appears redder, or less blue, than a star of the same spectral type unaffected by absorption. The difference between I' and I is called the *colour excess*, E, so that

$$E = I' - I. \qquad \ldots\ldots(3)$$

Denoting, as in section 8·61, the values of k for photographic and visual light by k_p and k_v, we have

$$m_p' = M_p + 5 \operatorname{Log} r + \frac{k_p r}{1000} - 5$$

and

$$m_v' = M_v + 5 \operatorname{Log} r + \frac{k_v r}{1000} - 5.$$

Hence, from (1), (2) and (3),

$$E = (k_p - k_v)\frac{r}{1000} \equiv \frac{cr}{1000}. \qquad \ldots\ldots(4)$$

Thus the colour excess increases linearly with distance.

Table 46 gives the results* obtained by several observers for seven clusters. The first column gives the number of the cluster in the *New General Catalogue*; the second, the distance as found by Trumpler; and the third, the observed colour excess.

Table 46. *Colour excess of open clusters*

Cluster *N.G.C.*	Distance r (parsecs)	Colour excess E	Number of stars	Residual
1647	610	$+0^m{\cdot}17$	33	$-0^m{\cdot}02$
2682	740	+0·26	81	+0·02
2099	820	+0·05	25	−0·21
1960	980	+0·05	40	−0·26
6705	1340	+0·65	46	+0·22
7654	1360	+0·49	43	+0·06
663	2170	+0·71	41	+0·02

* Trumpler, *loc. cit.* p. 165.

Solving the equation $E = cr/1000$ ......(5)

by least-squares, Trumpler finds that

$$c = 0^{m}{\cdot}32 \pm 0^{m}{\cdot}03 \text{ per kiloparsec.} \quad \text{......(6)}$$

The last column in Table 46 contains the residuals obtained from (5) with the value of c given by (6).

Hence, with the value of k_p (for photographic light) already found in section 9·34, we have the results

$$k_p = +0^{m}{\cdot}67 \text{ per kiloparsec,}$$
$$k_v = +0^{m}{\cdot}35 \quad \text{,,}$$

Shapley's earlier investigations dealing with globular clusters showed that the colour excess of cluster stars in the higher galactic latitudes was negligible; later,* he found some, but comparatively trifling, indications of the loss of light for globular clusters in the lower galactic latitudes. As the globular clusters appear to avoid the Milky Way, the conclusion we reach is that the absorbing cloud is extended along the galactic plane and is of comparatively small thickness. Van de Kamp† has made a determination of the thickness from a study of the absorption for distant objects presumably outside the layer and in different galactic latitudes; his estimate of the mean thickness is 175 parsecs. A later investigation‡ by Van de Kamp suggests a somewhat larger value for the thickness.

9·41. *Globular clusters.*

Nearly a hundred globular clusters are known. They are, apparently, dense aggregations of stars, many times more numerous than the stars of the richest open clusters. Shapley's researches give us reliable indications of their great distances. The most trustworthy method of estimating their distances is based on the period-luminosity law pertaining to Cepheids and cluster variables. The periods of the latter are, in general, much shorter than the periods of the galactic Cepheids but, assuming the whole to form a continuous sequence, the absolute luminosity of a cluster variable can be deduced simply from the period-luminosity relationship the characteristics of which are determined from a study of the galactic Cepheids. The observation of the apparent magnitude of a cluster variable in combination with the absolute magnitude so derived leads to the evaluation of the distance of the cluster. It is assumed that there is no light absorption; as the known clusters are situated in galactic latitudes well away from the Milky Way, this assumption involves but an almost negligible error resulting from the thickness of the galactic cloud which we have been considering in the

* *Harvard Bulletin*, No. 864, p. 9, 1929. † *A.J.* **40**, 145, 1930. ‡ *A.J.* **42**, 97, 1932.

previous sections. As the definitive existence of this cloud was unverified at the time of Shapley's investigations, it was assumed that interstellar space was transparent. Shapley's apparent magnitudes are thus too large by the amount of the equivalent absorption through this layer; consequently, his computed distances of the clusters are somewhat too large. We can obtain an estimate of the necessary correction roughly as follows. Let m be the apparent magnitude on the assumption of the transparency of space, and $m+dm$ the magnitude as affected by the absorbing cloud. Let r and $r+dr$ be the computed distances corresponding to m and $m+dm$ respectively. Then

$$M = m+5-5\,\mathrm{Log}\,r,$$

$$M = m+dm+5-5\,\mathrm{Log}\,(r+dr),$$

whence

$$\frac{dr}{r} = \frac{q}{5}\,dm,$$

where $q \equiv \log_e 10 = 2{\cdot}30$.

If we assume that the sun is in the centre of the absorbing layer and that h is half its thickness, the length of the path of light from a cluster, in galactic latitude g, through the cloud is $h\,\mathrm{cosec}\,g$. Taking dm to be the increase in photographic magnitude due to the absorption, we have

$$dm = \frac{kh\,\mathrm{cosec}\,g}{1000},$$

so that

$$\frac{dr}{r} = 0{\cdot}00046kh\,\mathrm{cosec}\,g.$$

For $k = 0^{\mathrm{m}}{\cdot}67$ and $h = 100$ parsecs,

$$\frac{dr}{r} = 0{\cdot}03\,\mathrm{cosec}\,g.$$

Hence the distances as computed by Shapley should be reduced by $3\,\mathrm{cosec}\,g$ %. Except for clusters in very low galactic latitudes, this is an almost insignificant amount and certainly much smaller than the uncertainties arising from the application of the period-luminosity law or of the other methods employed by Shapley.

Another line of attack on the investigation of cluster distances is based on the measurement of colour indices and their associated absolute magnitudes.

The distances found by Shapley* for the globular clusters range between 5000 and 50,000 parsecs. For example, the distance of M13 (the great cluster in Hercules—the brightest in the northern sky) is estimated to be 10,300 parsecs.

* For a description of Shapley's researches, together with a full bibliography, see Shapley's *Star Clusters* (McGraw-Hill), 1930.

9·42. *The stellar density-function for globular clusters.*

The distribution of stars in globular clusters is inferred from counts of the stellar images on photographic plates. Although several of the clusters appear to be somewhat ellipsoidal in form, we consider only the case of spherical symmetry; the analytical results can then be applied to spherical clusters and to those whose ellipticity is small. When the departure from the spherical form is considerable, the general problem of stellar distribution in such clusters becomes, in practice, almost intractable.

We take the origin of coordinates to be the centre of the cluster and the positive direction of the Z-axis in the line of sight and towards the observer. Owing to the assumed symmetry the X and Y axes can be chosen arbitrarily in the plane $z = 0$. The positions of the stellar images on the photographic plate will then be represented by the projections, parallel to the Z-axis, on the XY plane (Fig. 54).

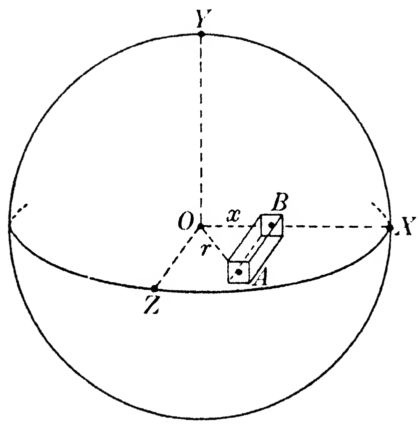

Fig. 54

Consider a cylinder of small rectangular cross-section with its axis parallel to OZ. Take the X-axis through B, the point of intersection of the axis of the cylinder with the plane $z = 0$. Let r denote the distance from the centre of an element of volume $dxdydz$ at A and let $\phi(r)$ be the density function. Then

$$\phi(r)\, dx\, dy\, dz$$

is the number of stars in the volume element. On the photographic plate, these stars will appear to be within the area $dxdy$ at B.

Let $f(x)\,dxdy$ be the number of stars observed in the element of the photographic plate corresponding to $dxdy$; $f(x)$ is the plate density of the stars at a distance x ($\equiv OB$) from the centre of the cluster as shown on the plate. This number is equal to the number of stars inside the complete cylinder through B terminated at the surface of the sphere. If the radius of the sphere is denoted by R, the z coordinates of the ends of the cylinder are $+\sqrt{R^2-x^2}$ and $-\sqrt{R^2-x^2}$. Accordingly, we have

$$f(x)\, dx\, dy = dx\, dy \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \phi(r)\, dz,$$

where

$$r^2 = x^2 + z^2. \qquad \ldots\ldots(1)$$

Hence $$f(x) = 2\int_0^{\sqrt{R^2-x^2}} \phi(r)\,dz. \qquad \ldots\ldots(2)$$

But from (1), x being constant for the cylinder,

$$r\,dr = z\,dz = (r^2-x^2)^{\frac{1}{2}}\,dz.$$

Hence $$f(x) = 2\int_x^R \frac{r\phi(r)\,dr}{(r^2-x^2)^{\frac{1}{2}}}. \qquad \ldots\ldots(3)$$

The result (3) is the integral equation, obtained by von Zeipel,* involving the plate density (which can be obtained from counts) and the cluster density function ϕ.

9·43. *Other expressions involving* ϕ.

(i) Let $\sigma(x)$ denote the number of images counted on the plate within a circle of radius x. Then the number of images in the ring defined by the radii x and $x+dx$ is $2\pi x f(x)\,dx$. Accordingly,

$$\sigma(x) = 2\pi\int_0^x x f(x)\,dx \qquad \ldots\ldots(1)$$

$$= 4\pi\int_0^x x\left\{\int_x^R \frac{r\phi(r)}{(r^2-x^2)^{\frac{1}{2}}}\right\}dx. \qquad \ldots\ldots(2)$$

Also $$f(x) = \frac{1}{2\pi x}\frac{d\sigma(x)}{dx}. \qquad \ldots\ldots(3)$$

Formula (1) has been used by von Zeipel for obtaining the function $f(x)$ from counts of stars on photographic plates.

(ii) Let $F(x)\,dx$ denote the number of stars on the plate in the strip bounded by the straight lines at distances x and $x+dx$ from the Y-axis. These stars are situated in the cluster between the planes parallel to YOZ at distances x and $x+dx$ from O (Fig. 55). Then

$$F(x)\,dx = dx\iint \phi(\rho)\,dy\,dz,$$

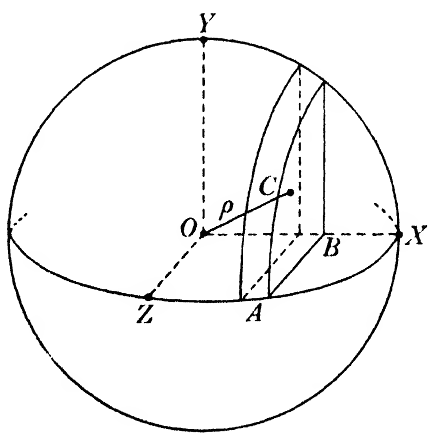

Fig. 55

the integration being taken over the circle, centre B and radius BA. Let (r_1, θ) be the polar coordinates of a point C on this circle. Then we obtain

$$F(x) = 2\pi\int_0^{\sqrt{R^2-x^2}} \phi(\rho)\,r_1\,dr_1.$$

* *Annales de l'Observatoire de Paris, Mémoires*, xxv, F 29, 1908.

But $$\rho^2 = x^2 + r_1^2,$$
so that, x being constant, $$\rho\, d\rho = r_1 dr_1.$$
Hence $$F(x) = 2\pi \int_x^R \rho\phi(\rho)\, d\rho. \qquad \text{......(4)}$$

(iii) Let $\Sigma(x)$ denote the number of stars on the plate between the diameter $x = 0$ and the straight line $x = x_1$. Then

$$\Sigma(x) = \int_0^{x_1} F(x)\, dx$$

or, using (4), $$\Sigma(x) = 2\pi \int_0^{x_1} \left\{ \int_x^R \rho\phi(\rho)\, d\rho \right\} dx. \qquad \text{......(5)}$$

The functions $\sigma(x)$, $F(x)$ and $\Sigma(x)$ can all be found from star counts and each of them is connected with the density function ϕ by an integral equation. It is therefore theoretically possible to deduce the function ϕ in several ways. These formulae were given by Plummer.*

9·44. *Solution of the integral equation in section* 9·42.

We had, 9·42 (3), $$f(x) = 2\int_x^R \frac{r\phi(r)\, dr}{(r^2 - x^2)^{\frac{1}{2}}}. \qquad \text{......(1)}$$

Von Zeipel† has shown that (1) can be reduced to an integral equation solved by Abel.‡

Put
$$h = R^2 - x^2,$$
$$\xi = R^2 - r^2,$$
and let
$$f(x) \equiv \theta(h),$$
$$\phi(r) \equiv \psi'(\xi) \equiv \frac{d\psi(\xi)}{d\xi}.$$

Then from (1) $$\theta(h) = \int_0^h \frac{\psi'(\xi)\, d\xi}{(h-\xi)^{\frac{1}{2}}}. \qquad \text{......(2)}$$

This is the usual form of Abel's integral equation of which the solution is

$$\psi(\xi) = \frac{1}{\pi} \int_0^\xi \frac{\theta(h)\, dh}{(\xi - h)^{\frac{1}{2}}}. \qquad \text{......(3)}$$

Integrate the right-hand side by parts; we obtain

$$\psi(\xi) = -\frac{2}{\pi}\Big[(\xi - h)^{\frac{1}{2}}\theta(h)\Big]_{h=0}^{\xi} + \frac{2}{\pi}\int_0^\xi (\xi - h)^{\frac{1}{2}} \frac{d\theta(h)}{dh}\, dh. \qquad \text{......(4)}$$

Consider now the boundary conditions. The plate density vanishes at the

* *M.N.* **71**, 460, 1911.

† *Loc. cit.* p. 29.

‡ *Crelle*, **1**, 1826.

periphery of the cluster corresponding to $x = R$, that is when $h = 0$; hence $\theta(h) = 0$ for $h = 0$. The integrated part of (4) thus vanishes. Hence

$$\psi(\xi) = \frac{2}{\pi}\int_0^{\xi}(\xi-h)^{\frac{1}{2}}\frac{d\theta(h)}{dh}\,dh, \qquad \ldots\ldots(5)$$

from which
$$\psi'(\xi) = \frac{1}{\pi}\int_0^{\xi}\frac{1}{(\xi-h)^{\frac{1}{2}}}\frac{d\theta(h)}{dh}\,dh,$$

or, in terms of the original functions and variables,

$$\phi(r) = -\frac{1}{\pi}\int_r^R\frac{f'(x)}{(x^2-r^2)^{\frac{1}{2}}}\,dx. \qquad \ldots\ldots(6)$$

On integrating by parts we obtain

$$\phi(r) = -\frac{1}{\pi}\left[(x^2-r^2)^{\frac{1}{2}}\frac{f'(x)}{x}\right]_{x=r}^R + \frac{1}{\pi}\int_r^R(x^2-r^2)^{\frac{1}{2}}\frac{d}{dx}\left\{\frac{1}{x}\frac{df(x)}{dx}\right\}dx.$$

The integrated part vanishes if $f'(x) = 0$ at $x = R$. Near the boundary of the cluster the star density is small, decreasing to zero at the boundary; hence we can write $f'(x) = 0$ at $x = R$. We then have

$$\phi(r) = \frac{1}{\pi}\int_r^R(x^2-r^2)^{\frac{1}{2}}\frac{d}{dx}\left\{\frac{1}{x}f'(x)\right\}dx. \qquad \ldots\ldots(7)$$

This is the form employed by von Zeipel in numerical applications to clusters.

9·45. *Alternative solution of the integral equation in section* 9·42.

The following analysis is due to Plummer.* From (1) of the previous section we have, applying the boundary conditions,

$$f(x) = 2\int_x^R\frac{d}{dr}\phi(r)\,(r^2-x^2)^{\frac{1}{2}}\,dr$$

$$= -2\int_x^R(r^2-x^2)^{\frac{1}{2}}\frac{d\phi(r)}{dr}\,dr.$$

Hence
$$\frac{df(x)}{dx} = 2x\int_x^R\frac{1}{(r^2-x^2)^{\frac{1}{2}}}\frac{d\phi(r)}{dr}\,dr.$$

In this formula, $x \equiv OB$ and $r \equiv OA$, where A is a representative point in BD (Fig. 56). Also

$$r^2 = x^2+z^2,$$

where $BA \equiv z$. Hence

$$f'(x) = 2x\int_0^{\sqrt{R^2-x^2}}\frac{\phi'(r)}{r}\,dz = 2x\int_0^{BD}\frac{\phi'(r)}{r}\,dz. \qquad \ldots\ldots(1)$$

* *M.N.* **71**, 461, 1911.

A formula of this type holds for any point such as C in the plane XOY; all we have to do is to rotate the X axis till it passes through C. Let $OC = q$ and let M be a representative point in the chord through C parallel to OZ; let $OM = \xi$. Then we have

$$\frac{1}{q}\frac{df(q)}{dq} = 2\int_0^{CJ} \frac{\phi'(\xi)}{\xi} dz.$$

Summing for all points on BE, drawn parallel to OY, we obtain

$$\int_0^{\sqrt{R^2-x^2}} \frac{1}{q} f'(q)\, dy = 2\int_0^{BE} dy \int_0^{CJ} \frac{\phi'(\xi)}{\xi} dz$$
$$= 2\iint \frac{\phi'(\xi)}{\xi} dS,$$

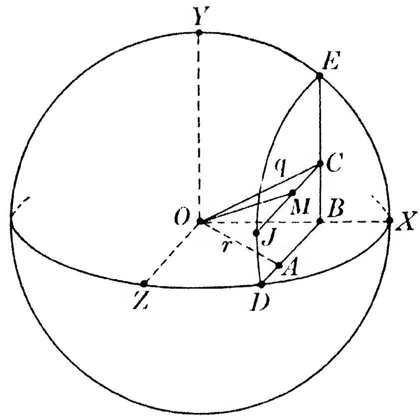

Fig. 56

where dS is an element of area of the quadrant BED and the double integration is taken over the quadrant.

Let (r_1, θ) be the polar coordinates of M in the plane BED. Then

$$\int_0^{\sqrt{R^2-x^2}} \frac{1}{q} f'(q)\, dy = 2\iint \frac{\phi'(\xi)}{\xi} r_1\, dr_1\, d\theta.$$

But $$\xi^2 = x^2 + r_1^2 \quad \text{and} \quad r_1 dr_1 = \xi d\xi.$$

Hence $$\int_0^{\sqrt{R^2-x^2}} \frac{1}{q} f'(q)\, dy = \pi \int_x^R \phi'(\xi)\, d\xi$$
$$= \pi\{\phi(R) - \phi(x)\}.$$

But, at the boundary of the cluster, $\phi(r) = 0$. Hence

$$\phi(x) = -\frac{1}{\pi}\int_0^{\sqrt{R^2-x^2}} \frac{1}{q} f'(q)\, dy.$$

Now $$q^2 = y^2 + x^2;$$

consequently, $$\frac{dy}{q} = \frac{dq}{y} = \frac{dq}{(q^2-x^2)^{\frac{1}{2}}}.$$

We then obtain $$\phi(x) = -\frac{1}{\pi}\int_x^R \frac{f'(q)}{(q^2-x^2)^{\frac{1}{2}}} dq. \qquad \text{......(2)}$$

This gives the density function at a distance x from the origin; changing the variables in (2) we have

$$\phi(r) = -\frac{1}{\pi}\int_r^R \frac{f'(x)}{(x^2-r^2)^{\frac{1}{2}}} dx, \qquad \text{......(3)}$$

which is the formula 9·44 (6). The form as used by von Zeipel—9·44 (7)—is obtained as before.

9·46. ***Evaluation of the density function from star counts (first method).***

This method is due to von Zeipel. The plate density $f(x)$ is obtained by means of 9·43 (3), namely,

$$f(x) = \frac{1}{2\pi x}\frac{d\sigma(x)}{dx}, \qquad \text{......(1)}$$

where $\sigma(x)$ is the number of stars within a circle of radius x on the plate.

The star density $\phi(\rho)$ at a distance ρ from the centre of the cluster is given by 9·44 (7), which is

$$\phi(\rho) = \frac{1}{\pi}\int_\rho^R (x^2-\rho^2)^{\frac{1}{2}}\frac{d}{dx}\left\{\frac{1}{x}f'(x)\right\}dx. \qquad \text{......(2)}$$

It is to be noticed that ϕ depends on f *only through* f', so that it is not altered by changing f to $f+$constant; hence, in calculating ϕ, it is unnecessary to determine and subtract the uniform density contributed by non-cluster stars.

The procedure as adopted by von Zeipel* is to calculate numerical values of $f(x)$ by means of (1) from the star counts $\sigma(x)$. Then by interpolation formulae the numerical values of the functions $f'(x)$, $\frac{1}{x}f'(x)$, $\frac{d}{dx}\left\{\frac{1}{x}f'(x)\right\}$ and $(x^2-\rho^2)^{\frac{1}{2}}\frac{d}{dx}\left\{\frac{1}{x}f'(x)\right\}$ are successively obtained for each value of x and ρ. Corresponding to a given value of ρ, we then have

$$\phi(\rho) = \frac{1}{\pi}\sum_{m=p}^{n}\left[(x^2-\rho^2)^{\frac{1}{2}}\frac{d}{dx}\left\{\frac{1}{x}f'(x)\right\}\right]_{x=x_m}\Delta x_m,$$

the radius of the cluster being divided into n parts Δx, and ρ being defined as $p\,.\,\Delta x$.

The following table† gives von Zeipel's results for the globular cluster M3.

Table 47. *Density function* $\phi(\rho)$ *for* M3

ρ	$\phi(\rho)$	ρ	$\phi(\rho)$	ρ	$\phi(\rho)$
1′	58·5	8′	4·83	19′	0·19
2	56·6	9	4·18	21	0·17
3	48·0	10	3·34	23	0·14
4	31·6	11	1·84	25	0·10
5	16·2	13	0·96	27	0·06
6	8·81	15	0·41	29	0·06
7	6·26	17	0·21	31	0·06

* *Kung. Svenska Vet. Akad. Handlingar*, Bd. **51**, No. 5, p. 9, 1913.

† *Annales de l'Observatoire de Paris, Mémoires*, xxv, F 31, 1908.

9·47. *Evaluation of the density function (second method).*

This method is due to Parvulesco.* A cluster is supposed to be built up of n concentric uniformly dense spheres of radii and density and numbers of stars according to the following scheme (the radius of the cluster is taken as unity):

Radii	$\frac{n}{n}$,	$\frac{n-1}{n}$,	$\frac{n-2}{n}$,	...,	$\frac{2}{n}$,	$\frac{1}{n}$.
Density	d_n,	d_{n-1},	d_{n-2},	...,	d_2,	d_1.
Number of stars	N_n,	N_{n-1},	N_{n-2},	...,	N_2,	N_1.

Consider one of these spheres, with radius R and density $\phi(\rho)\equiv d$; from 9·42 (2) we have

$$f(x) = 2d\int_0^{\sqrt{R^2-x^2}} dz$$

or

$$f(x) = 2d(R^2-x^2)^{\frac{1}{2}}. \qquad \text{......(1)}$$

Hence the number, $N(r_1, r_2)$, of stars, due to this sphere, between radii r_1 and r_2 on the plate is given by

$$N(r_1, r_2) = 2\pi\int_{r_1}^{r_2} xf(x)\,dx$$

$$= 4\pi d\int_{r_1}^{r_2} x(R^2-x^2)^{\frac{1}{2}}\,dx.$$

If N is the total number of stars in the sphere,

$$N = \frac{4\pi}{3}dR^3. \qquad \text{......(2)}$$

Hence

$$N(r_1, r_2) = N\left[\left(1-\frac{r_1^2}{R^2}\right)^{\frac{3}{2}} - \left(1-\frac{r_2^2}{R^2}\right)^{\frac{3}{2}}\right]. \qquad \text{......(3)}$$

Consider now the number of stars counted in the outermost ring on the plate between radii $(n-1)/n$ and n/n; the corresponding density is denoted by d_n. Since the remaining $(n-1)$ spheres do not contribute to this number we have, from (3), putting $R = 1$, $r_1 = (n-1)/n$ and $r_2 = n/n$,

$$N_n\left(\frac{n}{n}, \frac{n-1}{n}\right) \equiv C_n = N_n\left[1-\left(\frac{n-1}{n}\right)^2\right]^{\frac{3}{2}}. \qquad \text{......(4)}$$

If C_n is obtained from the star counts, N_n is given by (4) and hence d_n is found by (2).

The number of stars in the penultimate ring contains a number

$$N\left(\frac{n-1}{n}, \frac{n-2}{n}\right),$$

due to the outermost sphere of radius $R = 1$, given by

$$N\left(\frac{n-1}{n}, \frac{n-2}{n}\right) = N_n\left[\left\{1-\left(\frac{n-2}{n}\right)^2\right\}^{\frac{3}{2}} - \left\{1-\left(\frac{n-1}{n}\right)^2\right\}^{\frac{3}{2}}\right]. \qquad \text{......(5)}$$

* *Sur les amas globulaires d'étoiles et leur relations dans l'espace* (Gauthier-Villars, Paris), 1925.

The remainder of the stars in the ring come from the last ring pertaining to the sphere of radius $R \equiv (n-1)/n$; their number is given by

$$N_{n-1}\left(\frac{n-1}{n}, \frac{n-2}{n}\right) = N_{n-1}\left[1-\left(\frac{n-2}{n-1}\right)^2\right]^{\frac{3}{2}}. \qquad \ldots\ldots(6)$$

Hence, if C_{n-1} is the total number of stars in the penultimate ring, we obtain, from (5) and (6),

$$C_{n-1} = N_n\left[\left\{1-\left(\frac{n-2}{n}\right)^2\right\}^{\frac{3}{2}} - \left\{1-\left(\frac{n-1}{n}\right)^2\right\}^{\frac{3}{2}}\right] + N_{n-1}\left[1-\left(\frac{n-2}{n-1}\right)^2\right]^{\frac{3}{2}}. \qquad \ldots\ldots(7)$$

This equation gives N_{n-1}, since N_n is known; then, by (2), d_{n-1} is found.

This procedure is followed for the remaining rings. The density function ϕ corresponding to a ring m (that is, at a distance m from the centre) is found from

$$\phi = \sum_m^n d_s. \qquad \ldots\ldots(8)$$

This constitutes the numerical solution of the problem.

9·51. *Analogy with a spherical mass of gas.*

Plummer* has suggested that, if a globular cluster has originated by local condensations in a primordial gaseous nebula, it might be expected that the star density at different distances from the centre will bear an approximate similarity to the density in the nebula.

Consider a spherical mass of gas in equilibrium under its own gravitation. Let p and ρ denote the pressure and density at a distance r from the centre. For a perfect gas in convective equilibrium the relation between p and ρ is

$$p = \alpha\rho^\gamma, \qquad \ldots\ldots(1)$$

where α is a constant and γ is the ratio of the specific heats at constant pressure and at constant volume.

If $m(r)$ is the mass within a sphere of radius r,

$$m(r) = 4\pi\int_0^r \rho r^2 dr,$$

whence

$$\frac{dm}{dr} = 4\pi\rho r^2. \qquad \ldots\ldots(2)$$

Consider an element of volume with a base of area dS, lying on the surface of a sphere of radius r and between the spheres of radii r and $r+dr$. The hydrostatic equation gives

$$pdS-(p+dp)dS = \frac{Gm}{r^2}(\rho dS dr),$$

* *M.N.* **71, 462, 1911.**

where G is the constant of gravitation; consequently,

$$\frac{dp}{dr} = -\frac{Gm\rho}{r^2}, \qquad \ldots\ldots(3)$$

whence, by (1) and (2),

$$\frac{\alpha}{G}\frac{d}{dr}\left\{\frac{r^2}{\rho}\frac{d}{dr}(\rho^\gamma)\right\} = -4\pi\rho r^2. \qquad \ldots\ldots(4)$$

Let

$$\gamma = 1 + \frac{1}{n}, \qquad \ldots\ldots(5)$$

$$x^n = \rho, \qquad \ldots\ldots(6)$$

$$r = \frac{1}{u}, \qquad \ldots\ldots(7)$$

$$b^2 = \frac{4\pi G}{\alpha(n+1)}. \qquad \ldots\ldots(7a)$$

Then (4) becomes

$$u^4\frac{d^2x}{du^2} + b^2x^n = 0. \qquad \ldots\ldots(8)$$

For isothermal equilibrium, $\gamma = 1$, and we obtain from (4) in this case

$$u^4\frac{d^2}{du^2}(\log\rho) + b_1^2\rho = 0, \qquad \ldots\ldots(9)$$

where

$$b_1^2 = \frac{4\pi G}{\alpha}.$$

The equation (8) is Emden's equation for polytropic gas spheres.*

Two solutions of (8) are known—for the values $n = 1$ and $n = 5$. The first is Ritter's solution, namely

$$(n=1), \qquad x = A_1\frac{\sin br}{br}, \qquad \ldots\ldots(10)$$

and the second is the Schuster-Emden solution, namely

$$(n=5), \qquad x = \frac{\sqrt{3}c}{(3+b^2c^2r^2)^{\frac{1}{2}}},$$

which may be written in the alternative form

$$x = \frac{A}{(a^2+r^2)^{\frac{1}{2}}}, \qquad \ldots\ldots(11)$$

where

$$a = \frac{\sqrt{3}}{bc} \quad \text{and} \quad A^2 = \frac{\sqrt{3}a}{b}. \qquad \ldots\ldots(12)$$

Hence, from (6), we have the density ρ given by

$$\rho = \frac{B}{(a^2+r^2)^{\frac{5}{2}}}, \qquad \ldots\ldots(13)$$

where $B = A^5$.

* *Gaskugeln* (Leipzig), 1907.

9·52. *Plummer's law of density in a globular cluster.*

It is assumed, following the analogy with a spherical mass of gas, that the stellar density, $\phi(r)$, at a distance r from the centre of a globular cluster is given by

$$\phi(r) = \frac{B}{(a^2+r^2)^{\frac{5}{2}}}. \qquad \ldots\ldots(1)$$

This formula may be conveniently referred to as Plummer's density law.

It will be noticed that a cluster with this law of distribution is infinite in extent, but that the total number, N, of stars is finite. N is found from

$$N = 4\pi B \int_0^\infty \frac{r^2\,dr}{(a^2+r^2)^{\frac{5}{2}}}, \qquad \ldots\ldots(2)$$

whence

$$B = \frac{3Na^2}{4\pi}. \qquad \ldots\ldots(3)$$

From 9·43 (4)

$$F(x) = 2\pi B \int_x^R \frac{r\,dr}{(a^2+r^2)^{\frac{5}{2}}},$$

so that, when R tends to infinity,

$$F(x) = \frac{2\pi B}{3(a^2+x^2)^{\frac{3}{2}}} = \frac{Na^2}{2(a^2+x^2)^{\frac{3}{2}}}. \qquad \ldots\ldots(4)$$

Also

$$\Sigma(x) \equiv \int_0^x F(x)\,dx = \frac{2\pi B}{3}\int_0^x \frac{dx}{(a^2+x^2)^{\frac{3}{2}}}$$

or

$$\Sigma(x) = \frac{2\pi B}{3a^2}\cdot\frac{x}{(a^2+x^2)^{\frac{1}{2}}},$$

whence, by (3),

$$\Sigma(x) = \frac{N}{2}\cdot\frac{x}{(a^2+x^2)^{\frac{1}{2}}}. \qquad \ldots\ldots(5)$$

Again, by 9·42 (3),

$$f(x) = 2B\int_x^\infty \frac{r\,dr}{(a^2+r^2)^{\frac{5}{2}}(r^2-x^2)^{\frac{1}{2}}}.$$

Write

$$r^2-x^2 = \xi^2(a^2+x^2).$$

Then

$$f(x) = \frac{2B}{(a^2+x^2)^2}\int_0^\infty \frac{d\xi}{(1+\xi^2)^{\frac{5}{2}}},$$

from which and (3) we obtain

$$f(x) = \frac{Na^2}{\pi(a^2+x^2)^2}. \qquad \ldots\ldots(6)$$

Also

$$\sigma(x) = 2\pi\int_0^x xf(x)\,dx$$

and we find that

$$\sigma(x) = \frac{Nx^2}{a^2+x^2}. \qquad \ldots\ldots(7)$$

The formulae (4), (5), (6) and (7) can be used to test the assumed law of density by means of the observed distribution of the cluster stars on a photographic plate.

Plummer* has employed the formula (5) in connection with the star counts, made† under the direction of E. C. Pickering, of several globular clusters. For the cluster ω Centauri, for example, it is found that the formula

$$\Sigma(x) = 3540 \frac{x}{(1+x^2)^{\frac{3}{2}}}$$

represents the star counts very satisfactorily, the unit of distance, x, being taken to be 7·14 minutes of arc on the plate.

Von Zeipel‡ has also shown that the star counts of the globular clusters M2, M3, M13 and M15 are in accordance with Plummer's law.

In a later investigation Plummer§ used further statistical material|| for ten clusters to test the validity of the assumed density function; in some of the clusters, the counts showed an excess of stars, close to the centre, over the theoretical numbers; with this exception the density law represented the observed distribution with a close approach to accuracy.

It is to be remarked that the tests concern only the most luminous stars. By increasing the length of an exposure on a cluster, the number of images on the plate is increased, due to the inclusion of still fainter stars. But in practice there is a limit to the length of exposure for useful results to be obtained; beyond this limit, the central portions of the clusters become irresoluble into discrete images and, consequently, the density law cannot be examined.

Plummer's argument in favour of the stellar density law being a consequence of the adiabatic equilibrium of the primordial gaseous nebula has been criticised by von Zeipel on the grounds that during the condensation of stars, their movements would be taking place in a resisting medium with the result that the cluster would become more concentrated towards the centre as compared with the corresponding distribution of density in the original nebula.

To account for the density law in clusters, von Zeipel invoked the kinetic theory of gases, suggesting that the stars of a cluster behave like the molecules of a gas and interpreting γ, given by

$$\gamma \equiv 1 + \frac{1}{n} = 1{\cdot}2,$$

as the ratio of the specific heats in the sense of the kinetic theory. Eddington,¶ however, challenged von Zeipel's arguments and concluded that "instead of interpreting γ (or n) physically, we regard γ as a mathematical constant

* *M.N.* **71**, 464, 1911. † *Harvard Annals*, **26**, 213, 1897.
‡ *Kung. Svenska Vet. Akad. Handlingar*, Bd. **51**, No. 5, p. 9, 1913.
§ *M.N.* **76**, 107, 1915. || *Harvard Annals*, **76**, 43, 1915. ¶ *M.N.* **76**, 574, 1916.

in the equation of the star density", divorced entirely from any physical interpretation based on the equilibrium of a spherical mass of gas.

9·6. *Estimates of the maximum values of the proper motions and radial velocities of stars in a globular cluster.*

In a globular cluster the individual stars are necessarily in motion under the gravitational influence of the cluster stars as a whole. Taking spherical symmetry as the simplest case, we see that the field of force in the neighbourhood of a particular star consists, first, of the general attraction of a sphere in which the density is a function of the distance from the centre and, secondly, of the attraction of chance stars that are temporarily in the immediate neighbourhood of the given star. As will be shown in the next chapter, the second effect may be regarded as negligible.

Since the attraction is central, a star will move in a plane passing through the centre. Let (r, θ) be the polar coordinates of a star referred to the centre of the cluster and the star's orbital plane. If $M(r)$ denotes the mass within a sphere of radius r, the equations of motion for the star are

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM(r)}{r^2}, \qquad \text{......(1)}$$

where G is the constant of gravitation, and

$$\frac{d}{dt}(r^2\dot{\theta}) = 0. \qquad \text{......(2)}$$

From (2),

$$r^2\dot{\theta} = c, \qquad \text{......(3)}$$

where c is a constant.

Taking the units of length, time and mass to be respectively the astronomical unit of distance, the year and the mass of the sun, we have

$$G = 4\pi^2.$$

We consider two extreme possible orbits, (i) a rectilinear orbit and (ii) a circular orbit. It may be expected that the actual orbit will be intermediate in character.

For a rectilinear orbit, $c = 0$ and, from (1),

$$\ddot{r} \equiv \dot{r}\frac{d\dot{r}}{dr} = -4\pi^2\frac{M(r)}{r^2},$$

whence, on writing $\dot{r} \equiv v$,

$$v^2 = C - 8\pi^2\int^r \frac{M(r)}{r^2}\,dr.$$

If r_1 is the maximum distance of the star from the centre, we obtain, since $v = 0$ when $r = r_1$,

$$v^2 = 8\pi^2\int_r^{r_1} \frac{M(r)}{r^2}\,dr. \qquad \text{......(4)}$$

We clearly have $$v^2 < V^2,$$

where $$V^2 = 8\pi^2 \int_0^\infty \frac{M(r)}{r^2}\,dr. \qquad \text{......(5)}$$

Again, for a circular orbit of radius r, we have from (1)

$$v^2 = 4\pi^2 \frac{M(r)}{r}. \qquad \text{......(6)}$$

The proper motion for any given value of r will be a maximum when the rectilinear orbit and the plane of the circular orbit are perpendicular to the line of sight.

Let ρ, in seconds of arc, be the angular distance corresponding to r, so that

$$r = \frac{\rho}{p}, \qquad \text{......(7)}$$

where p is the parallax in seconds of arc. With the units adopted, the annual proper motion μ (in seconds of arc) is given by

$$\mu = Vp. \qquad \text{......(8)}$$

Then, from (5) and (6) and using (7) and (8), the proper motion cannot exceed

$$\pi(2p)^{\frac{3}{2}} \left\{\int_0^\infty \frac{M(\rho/p)}{\rho^2}\,d\rho\right\}^{\frac{1}{2}} \qquad \text{......(9)}$$

or $$2\pi p^{\frac{3}{2}} \left\{\frac{M(\rho/p)}{\rho}\right\}^{\frac{1}{2}} \qquad \text{......(10)}$$

for rectilinear or circular motion respectively.

Assume that the average mass of a star is m and that $N(\rho)$ is the number of stars within a sphere of angular radius ρ. Then

$$M(\rho/p) = mN(\rho).$$

Von Zeipel,* to whom the preceding arguments and the calculations below are due, has estimated from his counts of the cluster M3 that the maximum value of $N(\rho)/\rho$ is 4·2 for $\rho = 150''$ and that

$$\int_0^\infty \frac{N(\rho)}{\rho^2}\,d\rho = 10.$$

Hence, from (9) and (10), the maximum values of μ in the two cases (we write then μ_1 and μ_2) are given by

$$\mu_1 = \pi(2p)^{\frac{3}{2}}\,(10m)^{\frac{1}{2}} = 28(mp^3)^{\frac{1}{2}}, \qquad \text{......(11)}$$

$$\mu_2 = 2\pi p^{\frac{3}{2}}(4{\cdot}2m)^{\frac{1}{2}} = 13(mp^3)^{\frac{1}{2}}. \qquad \text{......(12)}$$

The brightest stars of a cluster (to which the earlier counts refer) are undoubtedly giant stars and as a reasonable estimate of mass of an average

* *Annales de l'Observatoire de Paris, Mémoires*, xxv, F 4, 1908.

star we can take the value of m to be 10. For a near cluster we can take p to be 0".0001. It is then found from (11) and (12) that

$$\mu_1 = 0''.00009, \quad \mu_2 = 0''.00004. \qquad \ldots\ldots(13)$$

As we can see from the modern long-exposure photographs of clusters, the estimates of the number of cluster stars with which we have been dealing are only a small proportion of the true number of stars; accordingly, the numerical values in (13) should be several times greater. Even if the factor were 10, the values of μ_1 and μ_2 are still too small to be determined successfully by present methods.

In the same way, the maximum radial velocities will occur when the rectilinear motion is in the line of sight and when the plane of the circular orbit contains the line of sight. The observable radial velocities in the two cases must then be not greater than V_1 and V_2 respectively, where

$$V_1 = 28(mp)^{\frac{1}{2}}, \quad V_2 = 13(mp)^{\frac{1}{2}}.$$

These velocities are expressed in astronomical units per annum; the values of V_1 and V_2 in km./sec. are accordingly

$$28\kappa(mp)^{\frac{1}{2}} \quad \text{and} \quad 13\kappa(mp)^{\frac{1}{2}}$$

respectively, where $\kappa = 4{\cdot}74$. Hence, with the previous values of m and p,

$$V_1 = 4{\cdot}2\,\text{km./sec.}, \quad V_2 = 1{\cdot}9\,\text{km./sec.} \qquad \ldots\ldots(14)$$

If we allow for the under-estimate of the number of cluster stars, these values might be expected to be increased several-fold. Radial-velocity measurements would then appear to hold out a certain amount of promise for determining the distribution of velocities of the brightest stars. But in view of existing telescopic equipment and technical methods of determining radial velocities, such an investigation could hardly be attempted at present with a reasonable prospect of success.

9·71. *The virial theorem as applied to a star cluster.*

We consider a stellar system in which the motions of the individual stars are governed by their mutual attractions and collisions are ignored.*

Let (x, y, z) be the coordinates of a star referred to the centroid of the cluster as origin. The x-equation of motion of a given star, of mass m_i, due to the attraction of a star of mass m_j with coordinates (x_j, y_j, z_j) is

$$m_i\ddot{x}_i = Gm_im_j\frac{x_j - x_i}{\Delta_{ij}^3}, \qquad \ldots\ldots(1)$$

where G is the gravitational constant and

$$\Delta_{ij}^2 = (x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2.$$

* H. Poincaré, *Hypothèses Cosmogoniques*, 90, 1913; A. S. Eddington, *M.N.* 76, 524, 1916.

From this last formula we have

$$\frac{x_j - x_i}{\Delta_{ij}^3} = \frac{\partial}{\partial x_i}\left(\frac{1}{\Delta_{ij}}\right). \quad \ldots\ldots(2)$$

From (1) and (2) and summing for all the stars, N in number, we have for the x-equation of motion of m_i

$$m_i \ddot{x}_i = \frac{\partial}{\partial x_i}\left\{\sum_{j=1}^{N} \frac{Gm_i m_j}{\Delta_{ij}}\right\}, \quad (j \neq i).$$

This equation is the same as

$$m_i \ddot{x}_i = \frac{\partial}{\partial x_i}\left\{\sum_{k=1}^{N}\sum_{j=1}^{N} \frac{Gm_k m_j}{\Delta_{kj}}\right\}, \quad (j \neq k).$$

Write
$$\Omega = -\sum_{k=1}^{N}\sum_{j=1}^{N} \frac{Gm_k m_j}{\Delta_{kj}}, \quad (j \neq k). \quad \ldots\ldots(3)$$

Here Ω is the total gravitational potential energy of the cluster. Let (X_i, Y_i, Z_i) be the components of the gravitational force on m_i. The equations of motion are then

$$m_i \ddot{x}_i = X_i = -\frac{\partial \Omega}{\partial x_i}, \quad \ldots\ldots(4)$$

with two similar equations in y and z.

The function Ω does not involve t explicitly, being a function of the coordinates only. Hence

$$\begin{aligned}\frac{d\Omega}{dt} &= \sum_{i=1}^{N}\left(\frac{\partial \Omega}{\partial x_i}\dot{x}_i + \frac{\partial \Omega}{\partial y_i}\dot{y}_i + \frac{\partial \Omega}{\partial z_i}\dot{z}_i\right)\\ &= -\sum_{1}^{N} m_i(\ddot{x}_i\dot{x}_i + \ddot{y}_i\dot{y}_i + \ddot{z}_i\dot{z}_i)\\ &= -\frac{1}{2}\frac{d}{dt}\sum_{1}^{N} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2). \quad \ldots\ldots(5)\end{aligned}$$

The internal kinetic energy, T, of the cluster stars is given by

$$T = \tfrac{1}{2}\sum_{1}^{N} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2). \quad \ldots\ldots(6)$$

Hence, from (5) and (6), $$T + \Omega = h, \quad \ldots\ldots(7)$$

where h is the constant of integration and is equal to the whole energy of the cluster with reference to its centre of mass.

Also, from the identity

$$\frac{d^2}{dt^2}(x^2) = 2\dot{x}^2 + 2x\ddot{x},$$

we have

$$m_i \frac{d^2}{dt^2}(x_i^2 + y_i^2 + z_i^2) = 2m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) + 2(x_i X_i + y_i Y_i + z_i Z_i). \quad \ldots(8)$$

Let C denote the moment of inertia of the cluster about its centre of mass. Then

$$C = \sum_1^N m_i r_i^2,$$

where $r_i^2 = x_i^2 + y_i^2 + z_i^2$.

Summing (8) for all the stars, we obtain

$$\frac{d^2C}{dt^2} = 4T + 2V, \qquad \ldots\ldots(9)$$

where

$$V = \sum_1^N (x_i X_i + y_i Y_i + z_i Z_i). \qquad \ldots\ldots(10)$$

The expression for V given in this last formula is called the *virial*, a function which has important applications in the kinetic theory of gases.

Now, from (4) and (10),

$$V = -\sum_1^N \left(x_i \frac{\partial \Omega}{\partial x_i} + y_i \frac{\partial \Omega}{\partial y_i} + z_i \frac{\partial \Omega}{\partial z_i} \right),$$

and since Ω is a homogeneous function of the coordinates of order -1, we have in the problem under consideration

$$V = \Omega.$$

Thus (9) becomes

$$\frac{d^2C}{dt^2} = 4T + 2\Omega, \qquad \ldots\ldots(11)$$

which can be written in the alternative forms by means of (7)

$$\frac{d^2C}{dt^2} = 2T + 2h, \qquad \ldots\ldots(12)$$

$$\frac{d^2C}{dt^2} = 4h - 2\Omega. \qquad \ldots\ldots(13)$$

If the cluster is in a steady state, C is independent of the time and in this case we have

$$\Omega = 2h, \quad T = -\tfrac{1}{2}\Omega = -h. \qquad \ldots\ldots(14)$$

9·72. *The rate of dissolution of a cluster.*

Eddington has employed the results of the previous section to investigate the rate of dissolution of a moving cluster.* As we have seen, the observed characteristics of a moving cluster are the substantial equality and parallelism of the motions of the individual constituent stars. As the cluster moves through galactic space it will encounter non-cluster stars which will produce changes in the magnitude and direction of the motions of the cluster stars through their gravitational attractions. We shall discuss this problem in greater detail in Chapter x; meanwhile it will be sufficient to consider some numerical results. For example, Jeans† has calculated that if the cluster is

* *M.N.* **76**, 527, 1916. † *M.N.* **74**, 111, 1913.

moving with the speed of 40 km./sec. relative to the non-cluster stars, a reasonable value of the space-density of these stars being assumed, the average deflection of the cluster stars will be 1′ in about one million years. Although this scattering appears small, yet it is evident that over long astronomical intervals of time the cluster would cease to maintain its compact form and would eventually be dissipated. The fact that clusters exist despite these disintegrating influences suggests that there is some counterbalancing force, such as the mutual attraction of the cluster, preventing its comparatively rapid dissolution. In the absence of such a force Eddington* has shown that the maximum age of the Taurus cluster (to which the preceding numerical details apply approximately) could not have exceeded 57 million years.

It will be shown in section 10·16 that the probable deflection of a cluster star due to the perturbations of non-cluster stars is proportional to the square root of the time during which the process continues; hence the kinetic energy, with reference to the centroid of the cluster, acquired through encounters is proportional to the time. Thus the rate of increase of internal kinetic energy is given by $\frac{1}{2}\Sigma m_i v_i^2$, where v_i is the linear component of velocity transverse to the direction of motion of the cluster. We write this expression just given as $\frac{1}{2}M\alpha^2$, in which M is the total mass of the cluster. From the previous section the rate of increase of potential energy will be twice $(\frac{1}{2}M\alpha^2)$, or $M\alpha^2$, if we assume that the steady state of the cluster is maintained. But, by 9·71 (3), the potential energy can be expressed as $-GM^2/2c$, in which c is related to the linear dimensions of the cluster and to the law of distribution of the cluster stars. Hence

$$\frac{d}{dt}\left(\frac{GM^2}{2c}\right) = -M\alpha^2,$$

from which

$$\frac{1}{c_0} - \frac{1}{c} = \frac{2\alpha^2}{GM}t. \qquad \ldots\ldots(1)$$

In this formula c_0 is the value of c at the beginning of an interval t. If $t \equiv \tau$ when $c_0 = \frac{1}{2}c$, we find that

$$\tau = \frac{GM}{2c\alpha^2}. \qquad \ldots\ldots(2)$$

Thus τ is the interval during which the linear dimensions of the cluster have been doubled.

We take α to be the linear transverse velocity corresponding to a deflection of 1′. Now, for the Taurus cluster, the velocity of 40 km./sec. is equivalent approximately to 800/19 parsecs per million years; accordingly,

$$\alpha = \tfrac{800}{19}\sin 1' = 0{\cdot}0123 \text{ parsecs per million years.}$$

* *Stellar Movements*, 254, 1914.

If we take the mass of the sun as unit and a million years as the unit of time, the unit of distance required to make the value of G in (2) unity is 0·165 parsec—the dimensions of G are $[L]^3[T]^{-2}[M]^{-1}$ and its value is $4\pi^2$ when $L = 1$ astronomical unit, $T = 1$ year and M = mass of sun. In terms of the new units, $\alpha = 0{\cdot}073$. The diameter of the Taurus cluster is estimated to be 10 parsecs and Eddington takes c as 3 parsecs, so that its value in terms of the new unit is 18. Hence, from (2),

$$\tau = 5M \text{ million years.} \qquad \ldots\ldots(3)$$

Forty highly luminous stars are known to belong to the cluster, on the average about 40 times brighter intrinsically than the sun,* that is to say about four magnitudes brighter on the absolute scale. As the sun's absolute magnitude is $+4^{\text{m}}{\cdot}6$ (bolometric), the absolute magnitude of the Taurus stars will, on the average, be close to 0; from Eddington's mass-luminosity relationship (Fig. 5, p. 21) their average mass will be about four times the mass of the sun. Thus M is approximately 160, so that, from (3), τ is of the order of one thousand million years. Additional members of the cluster will tend to increase this estimate still further. The mutual gravitational attraction of the cluster stars thus prolongs greatly the life of a cluster in spite of the disrupting influences of the general field of galactic stars.

* See *Stellar Movements*, 60, 1914.

CHAPTER X

THE DYNAMICS OF STELLAR SYSTEMS

10·11. *Introductory.*

We consider in this chapter an isolated stellar system. We have in view the dynamical and density characteristics of the Galaxy which we consider to be unaffected by the very remote extragalactic nebulae, themselves galaxies of the same order of magnitude as our stellar system.

The motion of a given star is determined by the gravitational attraction of all the other stars and matter in the system. This force may be regarded as arising in two ways: (*a*) from the "smoothed" attraction of the system as a whole and (*b*) from the accidental effects of stars temporarily in the neighbourhood of the given star. As regards (*b*) it is at once clear that the motion of a star may be radically changed by the close approach of another star and it is therefore necessary to investigate the frequency and magnitude of such effects. The term *encounter* is used to describe the close approach of one star to another, resulting in a change in direction of the velocity of each star. It will be shown that such encounters are so infrequent that we can ignore (*b*) in investigating the dynamics of a stellar system.

10·12. *The dynamics of stellar encounters.*

We consider two stars S_1 and S_2, of masses M_1 and M_2, moving independently and making a close approach. Let F_1 be the centre of mass, which we regard as being at rest. The orbit of each star will be a branch of a hyperbola with F_1 as focus. Fig. 57 shows the hyperbolic orbits of the two stars, the asymptotes with reference to S_1 being OK and OL. We shall investigate the deflection in the velocity of S_1 due to the gravitational attraction of S_2. At great distances from F_1, S_1 is moving parallel to KO before the encounter and parallel to OL after the encounter. The deflection is $\pi - KOL$, which will be denoted by ψ. It is clear that ψ is also the deflection suffered by S_2 during the encounter.

Let $F_1S_1 = r_1$ and $F_1S_2 = r_2$. Then from the definition of F_1,

$$M_1 r_1 = M_2 r_2,$$

and hence

$$r_1 + r_2 = \frac{(M_1 + M_2)\, r_1}{M_2}. \qquad \ldots\ldots(1)$$

The equations of motion of S_1 about F_1 are

$$M_1(\ddot{r}_1 - r_1\dot{\theta}^2) = -\frac{GM_1M_2}{(r_1+r_2)^2}, \qquad \ldots\ldots(2)$$

$$r_1^2\dot{\theta} = h, \qquad \ldots\ldots(3)$$

in which G is the gravitational constant and h is the "constant of areas".

Using (1) and (3), (2) becomes

$$\ddot{r}_1 - \frac{h^2}{r_1^3} = -\frac{\mu_1}{r_1^2}, \qquad \text{......(4)}$$

where

$$\mu_1 = \frac{GM_2^3}{(M_1+M_2)^2}. \qquad \text{......(5)}$$

Eliminating t between (3) and (4), we obtain the differential equation of the orbit, from which the polar equation of the orbit is found in the usual way to be

$$r_1 = \frac{l}{1+e\cos\theta_1},$$

where l is the semi-latus rectum and e is the eccentricity.

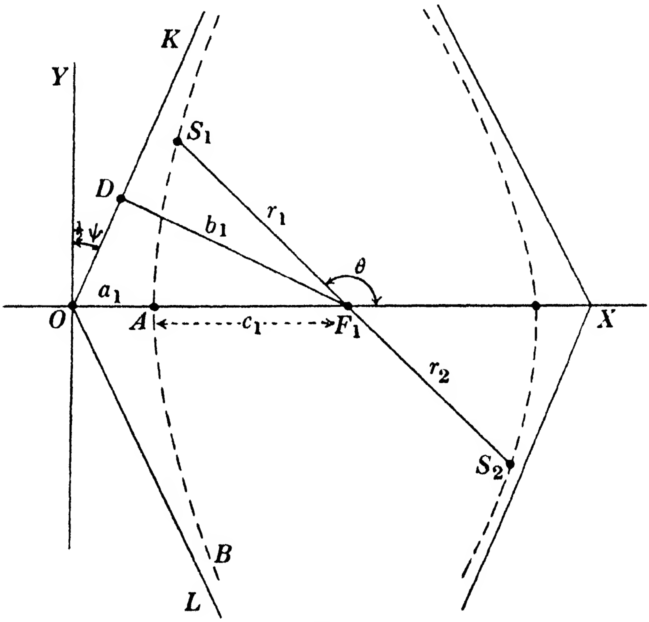

Fig. 57

Also, we have the following relation between h, μ_1 and l, namely,

$$h^2 = \mu_1 l. \qquad \text{......(6)}$$

Referred to the principal axes OX, OY the equation of the hyperbola is

$$\frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} = 1.$$

Then $\quad OA = a_1, \quad OF_1 = a_1 e \quad$ and $\quad AF_1 \equiv c_1 = a_1(e-1)$. ......(7)

Also, $\quad b_1^2 = a_1^2(e^2-1) \quad$ and $\quad l = \dfrac{b_1^2}{a_1}$. ......(8)

From (6) and (8), $\quad h^2 = \dfrac{\mu_1 b_1^2}{a_1}$. ......(9)

The equations of the asymptotes OK, OL are

$$b_1 x \pm a_1 y = 0.$$

Hence the perpendicular distance $F_1 D$ is $\frac{b_1 . a_1 e}{(a_1^2 + b_1^2)^{\frac{1}{2}}}$, since F_1 is $(a_1 e, 0)$. Thus

$$F_1 D = b_1.$$

If V_1 is the velocity of S_1 at a great distance from F_1, the constant h is given by

$$h = b_1 V_1. \qquad \ldots\ldots(10)$$

Also, from (9) and (10),

$$a_1 = \frac{\mu_1}{V_1^2}. \qquad \ldots\ldots(11)$$

The deflection ψ is given by

$$\cot\frac{\psi}{2} = \frac{b_1}{a_1}, \qquad \ldots\ldots(12)$$

whence, by (11),

$$\tan\frac{\psi}{2} = \frac{\mu_1}{b_1 V_1^2}. \qquad \ldots\ldots(13)$$

In the same way, if we consider the orbit of S_1 relative to S_2, the radial equation of motion of S_1 about F (Fig. 58) is

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2},$$

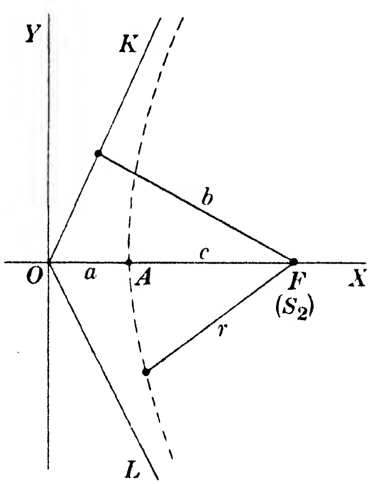

Fig. 58

where

$$\mu = G(M_1 + M_2) \qquad \ldots\ldots(14)$$

and r is the distance between M_1 and M_2. The relative orbit of S_1 is similar to the orbits in Fig. 57, with the same angle ψ between its asymptotes. If V denotes the velocity of S_1 relative to S_2 at a great distance from S_2 (or F), we have from (13)

$$\tan\frac{\psi}{2} = \frac{\mu}{bV^2}, \qquad \ldots\ldots(15)$$

in which b is the perpendicular distance of F (or S_2) from an asymptote. Thus b is the distance at which the stars would pass each other if their motions were unaffected by gravitation.

10·13. *Multiple encounters.*

Consider now the encounter of the star S_2 with a moving cluster consisting of equally massive stars, the common motion, relative to S_2, being V in a direction parallel to KO (Fig. 58). Let ν denote the number of cluster stars per unit volume. All stars which encounter S_2, so that the perpendicular distances from S_2 to the asymptotes of the relative orbits are less than a given length b, lie within a cylinder whose axis is parallel to KO and whose cross-section is πb^2. Hence the number of stars making such encounters with S_2 in unit time is

$$\pi\nu b^2 V, \qquad \text{......(1)}$$

and the deflection produced in the path of S_2 by each of these stars is, on the average, greater than ψ defined by 10·12 (15). Hence the number of encounters, per unit time, producing deflections greater than ψ is

$$\frac{\pi\nu\mu^2}{V^3}\cot^2\frac{\psi}{2}, \qquad \text{......(2)}$$

from which the average interval between two encounters is

$$\frac{V^3}{\pi\nu\mu^2}\tan^2\frac{\psi}{2}. \qquad \text{......(3)}$$

10·14. *Very close encounters.*

Following Jeans,* we define a very close encounter as one in which the deflection produced exceeds 90°. By 10·13 (3), the average interval between two such encounters is at least

$$\frac{V^3}{\pi\nu\mu^2}. \qquad \text{......(1)}$$

In evaluating this expression, Jeans expresses V, ν and μ in C.G.S. units. We shall use here the astronomical unit of distance, the sun's mass and the year as the units of length, mass and time. In these units, we have

$$G = 4\pi^2.$$

Also, putting $M_1 = M_2 = 1$ in 10·12 (14), we obtain $\mu = 8\pi^2$, or approximately,

$$\mu = 80.$$

We take $V = 20$ km./sec. as an average velocity of one star relative to another. Now

$$1 \text{ km./sec.} = \frac{1}{4{\cdot}74} \text{ astronomical unit per annum.}$$

In our units, we shall have $V = 20/4{\cdot}74$, or simply,

$$V = 4.$$

Jeans takes the density of stars near the sun to be approximately 1 per 10 cubic parsecs; in our units, this is

$$\nu = \tfrac{1}{8}\,.\,10^{-16}.$$

Inserting these values in (1), the average interval between two very close encounters is

$$3\,.\,10^{14} \text{ years.} \qquad \text{......(2)}$$

* *Astronomy and Cosmogony*, 310, 1928.

If c denotes the distance AF in Fig. 58 (the distance of closest approach), we have

$$c = b\left(\frac{e-1}{e+1}\right)^{\frac{1}{2}}. \qquad \text{......(3)}$$

For an encounter which produces a deflection of 90°, we have $a = b$ and $e = \sqrt{2}$, and by 10·12 (15)

$$b = \frac{\mu}{V^2}. \qquad \text{......(4)}$$

With the values of μ and V already used, we obtain from (3) and (4),

$$c = 2 \text{ astronomical units, approximately.}$$

For encounters within this distance, the direction of a star's motion will be altered by more than 90°.

10·15. *The frequency of collisions.*

If two stars, similar to the sun, have a grazing encounter, the value of c is given by the sun's diameter, so that in astronomical units $c = 2.7 . 10^5/15 . 10^7$ approximately, or roughly

$$c = \tfrac{1}{100}. \qquad \text{......(1)}$$

From the relations

$$c = a(e-1) \quad \text{and} \quad b^2 = a^2(e^2-1)$$

we obtain, by eliminating e,

$$b^2 = c^2 + 2ca.$$

Now, by analogy with 10·12 (11), $a = \mu/V^2$; hence

$$b^2 = c^2 + \frac{2c\mu}{V^2}.$$

With c given by (1) and with $\mu = 80$ and $V = 4$, we obtain, approximately,

$$b^2 = \tfrac{1}{10}.$$

From 10·13 (1), the average interval between two encounters is

$$\frac{1}{\pi\nu b^2 V},$$

so that, for grazing encounters, this interval is about

$$6 . 10^{16} \text{ years.}$$

For actual collisions, the interval will be greater; consequently, grazing encounters and collisions are such rare events that they can be ignored in discussing the dynamics of the stellar system. Even for globular clusters, with a star density one thousand times that under consideration, the interval for grazing encounters is of the order of 10^{14} years and, again, such encounters may be ignored.

10·16. *The cumulative effect of feeble encounters.*

Encounters between stars at comparatively great distances produce small deflections, but as such encounters must be very many times more frequent than the encounters just considered, their cumulative effect may not be negligible. Jeans's calculations* in this case are based on the following arguments.

From 10·13 (2), the number of encounters per annum producing deflections greater than ψ is

$$\frac{\pi\nu\mu^2}{V^3}\cot^2\frac{\psi}{2}.$$

Hence the number of encounters per annum producing deflections between ψ and $\psi+d\psi$ is

$$\frac{\pi\nu\mu^2}{V^3}\frac{\cos\frac{\psi}{2}}{\sin^3\frac{\psi}{2}}d\psi.$$

If ψ is small, this number is substantially

$$\frac{8\pi\nu\mu^2}{V^3}\frac{d\psi}{\psi^3}. \qquad \text{......(1)}$$

The encounters take place in haphazard directions and the corresponding small deflections are to be compounded according to the theory of errors. Thus the total probable deflection Ψ, compounded of small deflections $\psi_1, \psi_2, \ldots$, between two limits α and β and occurring during a time t, is given by

$$\Psi^2 = \psi_1^2+\psi_2^2+\ldots, \qquad \text{......(2)}$$

so that, by (1),

$$\Psi^2 = t\,.\frac{8\pi\nu\mu^2}{V^3}\int_\alpha^\beta\frac{d\psi}{\psi}$$

$$=\frac{8\pi\nu\mu^2}{V^3}t\log\frac{\beta}{\alpha}. \qquad \text{......(3)}$$

Formula (2) is derived on the assumption that the deflections $\psi_1, \psi_2, \ldots$ are independent. If the minimum value is extremely small, the corresponding minimum distance, c, of two stars during an encounter must be very large; if it is several times, say, the average distance between two adjacent stars in a normal distribution, the corresponding volume of space will, at a given instant, contain a certain number of stars each producing a minute deflection; but, in the aggregate, these random encounters may be expected to neutralise each other so far as their combined effect is concerned. Jeans accordingly chooses the lower limit α to correspond approxi-

* *Astronomy and Cosmogony*, 311, 1928.

mately to the distance between neighbouring stars in a normal distribution, that is, to $\nu^{-\frac{1}{3}}$. In this case $b \doteqdot c$ so that, by 10·12 (15),

$$\alpha = \frac{2\mu}{cV^2} = \frac{2\mu\nu^{\frac{1}{3}}}{V^2}.$$

With $\mu = 80$, $\nu = \frac{1}{8} . 10^{-16}$, $V = 4$, we find that

$$\alpha = 2 . 10^{-5} \doteqdot 4'' \qquad \text{......(4)}$$

and

$$\Psi^2 = \frac{22}{7} . 10^{-14} t \log \frac{\beta}{\alpha}.$$

If these feeble encounters eventually produce a deflection $\Psi = \pi/2$ in T years, we have

$$T = \frac{11}{14} . \frac{10^{14}}{\log(\beta/\alpha)}. \qquad \text{......(5)}$$

The formula (3) was derived on the supposition that α and β are small; however, T exceeds the value given by putting $\beta = \pi/2$ in (5). This minimum value of T is $7 . 10^{12}$ years.

10·17. *The fundamental principle of stellar dynamics.*

The preceding calculations indicate, owing to the immense intervals of time involved, that with the galactic system as at present constituted the effects of encounters are negligible; in general a star pursues its path which is not substantially affected by other stars temporarily in its neighbourhood.

Also, as we have seen in section 9·72, the mutual gravitation of the stars in such a cluster as the Taurus cluster ensures a long life for a formation of this kind and substantially counteracts the effectiveness of the ordinary galactic stars, through which the cluster must pass, to cause disruption.

A possible disintegrating influence to which we have not yet referred is that of galactic rotation and we briefly mention the results of a recent investigation* by B. J. Bok on the stability of moving clusters. It is assumed that the motions of the individual cluster stars are governed by (*a*) the attraction of the cluster as a whole, (*b*) the attraction of the galactic system as a whole and (*c*) the chance encounters of non-cluster stars. The principal influence tending to cause disruption of an extensive cluster is undoubtedly galactic rotation but, even so, Bok finds the effective life of a cluster to be, at least, of the order of 10^9 years. The numerical results in Table 48 for the Taurus cluster are taken from his paper;† it is assumed that the cluster is ellipsoidal in form with the present values of the semi-axes a, b and c respectively 6, 6 and 4 parsecs, and that the present stellar density, ν, of

* *Harvard Circular*, No. 384, 1934.

† *Ibid.* p. 30.

the cluster is 0·25 ⊙ per cubic parsec (the c-axis is perpendicular to the galactic equator). The table gives the values of these quantities at various times t in the future.

Table 48

t (unit = 10^9 years)	a (parsecs)	b (parsecs)	c (parsecs)	ν (⊙ per cubic parsec)
0·5	6·3	6·2	4·1	0·22
1·0	6·6	6·4	4·2	0·20
1·5	7·0	6·6	4·3	0·18
2·0	7·5	6·9	4·4	0·16
2·4	8·7	7·1	4·5	0·13
2·5	9·9	7·2	4·5	0·11
2·6	21·8	11·2	4·6	0·03

According to Bok, it may be anticipated that after $3 \,.\, 10^9$ years the cluster will be completely dispersed. The whole life of the cluster must of course be greater than the number just quoted and it may well be 10^{10} years.

So far as the motion of an individual galactic star is concerned, we conclude that its motion is, in general, determined only by the gravitational field of force of the galactic system as a whole (it is this feature that differentiates stellar dynamics from the dynamical theory of gases) and we take this statement as the fundamental principle of stellar dynamics.

For analytical purposes we replace the actual attraction of the system by a "smoothed" attraction of the system as a whole, in which we imagine the actual discontinuous gravitational field due to the stars, regarded as mass-points, to be replaced by a continuous gravitational field produced by a smoothed distribution of density.

10·21. *The fundamental equation of stellar dynamics.*

Consider a stellar system of N stars, referred to a rectangular system of axes. Let the number of stars with coordinates between (x, y, z) and $(x+dx,\, y+dy,\, z+dz)$ and with velocity components between (u, v, w) and $(u+du,\, v+dv,\, w+dw)$ be denoted by dN. If we assume the existence of a function

$$f(t;\, x, y, z;\, u, v, w)$$

expressing the space-velocity distribution of the stars, we have

$$dN = f(t;\, x, y, z;\, u, v, w)\, dx\, dy\, dz\, du\, dv\, dw \qquad \ldots\ldots(1)$$

or, denoting the space-velocity element by dQ, so that

$$dQ = dx\, dy\, dz\, du\, dv\, dw, \qquad \ldots\ldots(2)$$

we have

$$dN = f(t;\, x, y, z;\, u, v, w)\, dQ. \qquad \ldots\ldots(3)$$

Denote by V the gravitational potential at (x, y, z); V is a function of x, y and z. The components (X, Y, Z) of the gravitational force per unit mass are given by

$$X = \frac{\partial V}{\partial x}, \quad Y = \frac{\partial V}{\partial y}, \quad Z = \frac{\partial V}{\partial z}, \qquad \text{......(4)}$$

so that X, Y and Z are functions of the coordinates (x, y, z).

The equations of motion of an individual star are

$$\frac{du}{dt} = X, \quad \frac{dv}{dt} = Y, \quad \frac{dw}{dt} = Z. \qquad \text{......(5)}$$

After an interval dt, the (x, y, z) coordinates of a star become (x_1, y_1, z_1), where

$$x_1 = x + u\,dt, \quad y_1 = y + v\,dt, \quad z_1 = z + w\,dt, \qquad \text{......(6)}$$

and the (u, v, w) components of velocity become (u_1, v_1, w_1), where

$$u_1 = u + \frac{du}{dt}dt, \quad v_1 = v + \frac{dv}{dt}dt, \quad w_1 = w + \frac{dw}{dt}dt$$

or, by (5), $$u_1 = u + X\,dt, \quad v_1 = v + Y\,dt, \quad w_1 = w + Z\,dt. \qquad \text{......(7)}$$

The dN stars now occupy a space-velocity element dQ_1 given by

$$dQ_1 = dx_1 dy_1 dz_1 du_1 dv_1 dw_1. \qquad \text{......(8)}$$

Now $$dQ_1 = \frac{\partial(x_1, y_1, z_1, u_1, v_1, w_1)}{\partial(x, y, z, u, v, w)} dQ \qquad \text{......(9)}$$

and from (6) and (7) the value of the Jacobian is

$$\begin{vmatrix} 1, & 0, & 0, & dt, & 0, & 0 \\ 0, & 1, & 0, & 0, & dt, & 0 \\ 0, & 0, & 1, & 0, & 0, & dt \\ \frac{\partial X}{\partial x}dt, & \frac{\partial X}{\partial y}dt, & \frac{\partial X}{\partial z}dt, & 1, & 0, & 0 \\ \frac{\partial Y}{\partial x}dt, & \frac{\partial Y}{\partial y}dt, & \frac{\partial Y}{\partial z}dt, & 0, & 1, & 0 \\ \frac{\partial Z}{\partial x}dt, & \frac{\partial Z}{\partial y}dt, & \frac{\partial Z}{\partial z}dt, & 0, & 0, & 1 \end{vmatrix}$$

which reduces, to the first order in dt, to unity. Hence by (9)

$$dQ_1 = dQ. \qquad \text{......(10)}$$

After time dt, the number dN of stars in the space-velocity element dQ_1 is given by

$$dN = f(t+dt;\, x+u\,dt, y+v\,dt, z+w\,dt;\, u+X\,dt, v+Y\,dt, w+Z\,dt)\,dQ_1. \qquad \text{......(11)}$$

Expanding to the first order in dt and equating (11) to (1) we have, making use of (10),

$$\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}+w\frac{\partial f}{\partial z}+X\frac{\partial f}{\partial u}+Y\frac{\partial f}{\partial v}+Z\frac{\partial f}{\partial w}=0, \qquad \ldots\ldots(12)$$

or, by (4),

$$\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}+w\frac{\partial f}{\partial z}+\frac{\partial V}{\partial x}\frac{\partial f}{\partial u}+\frac{\partial V}{\partial y}\frac{\partial f}{\partial v}+\frac{\partial V}{\partial z}\frac{\partial f}{\partial w}=0. \qquad \ldots\ldots(13)$$

These formulae were given by Jeans.* They are the equivalent of Boltzmann's equation† in the kinetic theory of gases, with molecular collisions omitted.

Let the operator D/Dt denote differentiation with regard to the time, following a star throughout its motion; the operator thus represents the rate of change of a characteristic pertaining to a particular group of stars. In this sense, D/Dt is equivalent to

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+\frac{dx}{dt}\frac{\partial}{\partial x}+\frac{dy}{dt}\frac{\partial}{\partial y}+\frac{dz}{dt}\frac{\partial}{\partial z}+\frac{du}{dt}\frac{\partial}{\partial u}+\frac{dv}{dt}\frac{\partial}{\partial v}+\frac{dw}{dt}\frac{\partial}{\partial w}. \qquad \ldots\ldots(14)$$

Hence when the motions are governed by the gravitational potential, V, we can write (13) in the form

$$\frac{Df}{Dt}=0. \qquad \ldots\ldots(15)$$

This is the statement of Liouville's theorem.‡

We can write the theorem in a slightly different notation. Since dN is invariable with the time,

$$\frac{D}{Dt}(dN)=0, \qquad \ldots\ldots(16)$$

and hence, by (3) and (15),

$$\frac{D}{Dt}(dQ)=0, \qquad \ldots\ldots(17)$$

so that, as in (10),

$$dQ_1=dQ.$$

10·22. *Jeans's theorem.*

Since X, Y, Z are functions of x, y, z, the equation

$$\frac{Df}{Dt}\equiv\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}+w\frac{\partial f}{\partial z}+X\frac{\partial f}{\partial u}+Y\frac{\partial f}{\partial v}+Z\frac{\partial f}{\partial w}=0 \qquad \ldots\ldots(1)$$

is a partial differential equation, of the first order, in the variables t, x, y, z, u, v and w. The solution is found by Lagrange's method.§ Form the subsidiary equations

$$dt=\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}=\frac{du}{X}=\frac{dv}{Y}=\frac{dw}{Z}, \qquad \ldots\ldots(2)$$

which are seen to be the equations of motion.

* *M.N.* **76**, 70, 1915. † Jeans's *Dynamical Theory of Gases* (4th ed.), 208, 1925.

‡ Jeans's *Dynamical Theory of Gases* (4th ed.), 73, 1925.

§ See, for example, Forsyth's *Differential Equations* (6th ed.), 405, 1929.

If $I_1(t; x, y, z; u, v, w) = a_1$, $I_2 = a_2$, ... $I_6 = a_6$ are six independent integrals of (2), where $a_1, a_2, \dots a_6$ are constants, the general solution of (1) is

$$f = F(I_1, I_2, \dots I_6), \qquad \dots\dots(3)$$

where F is any function of $I_1, I_2, \dots I_6$.

Now, by (1), $Df/Dt = 0$, so that f is constant along the path of a star; hence f is a function of those quantities which remain constant along the orbit or path of the star.

Although (3) is the most general solution of the equation (1), the frequency function f is limited by the consideration that Poisson's equation must be satisfied at all points of the system when the potential V is due, as we are supposing, to the system itself. Poisson's equation is

$$\nabla^2 V = -4\pi\rho, \qquad \dots\dots(4)$$

where ρ is the mass per unit volume in the neighbourhood of the point (x, y, z).

We suppose that the stars are divided into groups according to mass, so that we may take all the stars within one group to have the same mass M. By 10·21 (1), the number of stars, of a given group, per unit volume and with velocity components between (u, v, w) and $(u+du, v+dv, w+dw)$ is

$$f(t; x, y, z; u, v, w)\, du\, dv\, dw. \qquad \dots\dots(5)$$

Hence the total number, ν, of such stars per unit volume is obtained by summing (5) for all possible values of u, v and w, with the limitation that a star with a given value of (u, v, w) shall not escape from the system. Thus

$$\nu = \iiint f\, du\, dv\, dw \qquad \dots\dots(6)$$

and ρ is then given by $\qquad \rho = \Sigma M \iiint f\, du\, dv\, dw,$

where the summation refers to the various groups of stars and the assumption is made that the same form for f applies to each group.

Poisson's equation is then

$$\nabla^2 V = -4\pi \Sigma M \iiint f\, du\, dv\, dw. \qquad \dots\dots(7)$$

In the sequel, we shall generally write (7) simply as

$$\nabla^2 V = -4\pi M \iiint f\, du\, dv\, dw, \qquad \dots\dots(8)$$

in which the summation is implied.

We refer to the results embodied in the formulae (3) and (8) as *Jeans's theorem.**

10·23. *The dynamical equations for a stellar system in a steady state.*

Rewriting 10·22 (6), we have for the number, ν, of stars per unit volume

$$\nu = \iiint f(t;\, x, y, z;\, u, v, w)\, du\, dv\, dw, \qquad \text{......(1)}$$

so that, in the general case, ν is a function of t, x, y and z. If the space-velocity function f is such that $\partial f/\partial t = 0$, then by (1)

$$\frac{\partial \nu}{\partial t} = 0;$$

consequently, the distribution of the stars in the neighbourhood of any point is independent of the time. The system is then said to be in a *steady state.* In the sequel we shall consider the possible steady states of a system, so that the function f has the analytical property

$$\frac{\partial f}{\partial t} = 0. \qquad \text{......(2)}$$

The subsidiary equations (which are now five in number) are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \frac{du}{X} = \frac{dv}{Y} = \frac{dw}{Z}, \qquad \text{......(3)}$$

and the general form of f is given by

$$f = F(I_1, I_2, \ldots I_5), \qquad \text{......(4)}$$

where $\qquad I_1(x, y, z;\, u, v, w) = a_1,\ \ldots,\ I_5 = a_5$

are first integrals of the equations of motion not involving the time explicitly. Thus f is a function of such quantities as do not involve the time explicitly and remain constant along the orbit of a star.

One such integral is the energy equation, which we write in the form

$$I_1 \equiv u^2 + v^2 + w^2 - 2V = a_1. \qquad \text{......(5)}$$

This follows simply from (3); for we have—writing $\partial V/\partial x$ for X, etc.—

$$u\,du = \frac{\partial V}{\partial x} dx, \quad v\,dv = \frac{\partial V}{\partial y} dy, \quad w\,dw = \frac{\partial V}{\partial z} dz,$$

whence, on adding, $\qquad \tfrac{1}{2} d(u^2 + v^2 + w^2) = dV,$

giving the result (5).

* *M.N.* **76**, 70, 1915. See also Jeans's *Astronomy and Cosmogony*, 364, 1929, and Charlier, *Lund Medd.* Ser. II, 19, 1918, in which the theory is developed to take account of encounters and collisions; the fundamental equation is written in the form

$$\frac{Df}{Dt} = \nabla f + \nabla f',$$

where ∇f and $\nabla f'$ denote the effects of encounters and collisions respectively.

Other possible integrals of (3) are conditioned by the general properties —spherical, uniaxial, etc.—of the system under investigation.

As before, if the potential is due to the system itself, Poisson's equation has to be satisfied at all points of the system.

10·24. *The ellipsoidal law of velocities.*

In the neighbourhood of the sun, the characteristic feature of stellar motions is the fact that the peculiar velocities have an axis of greatest mobility, and this characteristic is represented most conveniently on the basis of Schwarzschild's ellipsoidal law of velocity distribution. If we consider the ellipsoidal law to be associated in general and at all points with the steady state of a stellar system, the function f must be expressible in the form

$$f = F(x, y, z;\, au^2 + bv^2 + cw^2 + 2fvw + 2gwu + 2huv), \qquad \text{......(1)}$$

in which a, b, c, ... h are in general functions of x, y and z. In this general form the lengths and directions of the principal axes of the velocity ellipsoid vary from point to point of the system.

For example, if the system is heterogeneous, the only integral of the equations of motion that can be found is (5) of section 10·23 and the space-velocity function f takes the special form

$$f \equiv f(c^2 - 2V),$$

where

$$c^2 = u^2 + v^2 + w^2. \qquad \text{......(2)}$$

In this case the velocity-distribution is spherical but the space-distribution can be of any form, depending on the analytical expression for V in terms of x, y and z.

If we assume that $V = 0$ at infinity,

$$c^2 - 2V = \text{a negative quantity};$$

and if a star remains a member of the system its velocity must not exceed $\sqrt{2V}$; this latter is the *velocity of escape*. The density, ν, is given by

$$\nu = 4\pi \int_0^{\sqrt{2V}} f(c^2 - 2V)\, c^2 dc.$$

Hence ν is a function of V, namely $\nu(V)$, and the equi-density surfaces are the equi-potential surfaces $V =$ constant. Poisson's equation is then

$$\nabla^2 V = -4\pi M \nu(V).$$

10·31. *The frequency function for systems with spherical symmetry.*

In this case V is a function of the distance, r, from the centre of the system. Hence

$$X = \frac{x}{r}\frac{\partial V}{\partial r}, \quad Y = \frac{y}{r}\frac{\partial V}{\partial r}, \quad Z = \frac{z}{r}\frac{\partial V}{\partial r},$$

so that the group of subsidiary equations is

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \frac{du}{\frac{x}{r}\frac{\partial V}{\partial r}} = \frac{dv}{\frac{y}{r}\frac{\partial V}{\partial r}} = \frac{dw}{\frac{z}{r}\frac{\partial V}{\partial r}}.$$

From these, we have the pair

$$wdy = vdz,$$

$$ydw = zdv,$$

whence, on adding and integrating, a_2 being a constant,

$$I_2 \equiv yw - zv = a_2. \qquad \text{......(1)}$$

This is the expression of the constancy of the angular momentum, per unit mass, about the x-axis. Similarly,

$$I_3 \equiv zu - xw = a_3, \qquad \text{......(2)}$$

$$I_4 \equiv xv - yu = a_4. \qquad \text{......(3)}$$

The frequency function f is then

$$f \equiv f(I_1, I_2, I_3, I_4),$$

in which $I_1 = a_1$ is the energy integral.

Since V is a function of r, it follows formally from Poisson's equation—or simply from considerations of symmetry—that ν is a function of r. Hence by 10·22 (6), the function f must also be a function of r. Now the total angular momentum, I, defined by

$$I^2 = I_2^2 + I_3^2 + I_4^2$$

is a symmetrical function of r for, from (1), (2) and (3), we have

$$I^2 = (x^2+y^2+z^2)(u^2+v^2+w^2) - (xu+yv+zw)^2$$
$$= c^2r^2 - r^2\dot{r}^2,$$

where c is the velocity as in (2) of section 10·24.

The function f is then given by

$$f(I_1, I_2^2 + I_3^2 + I_4^2)$$

or by

$$f(c^2 - 2V, c^2r^2 - r^2\dot{r}^2). \qquad \text{......(4)}$$

10·32. *The fundamental equation in polar coordinates.*

Let (r, θ, ϕ) be the polar coordinates of a star with reference to the centre of the system (Fig. 59), ϕ being the azimuthal angle. The elements of a linear displacement are

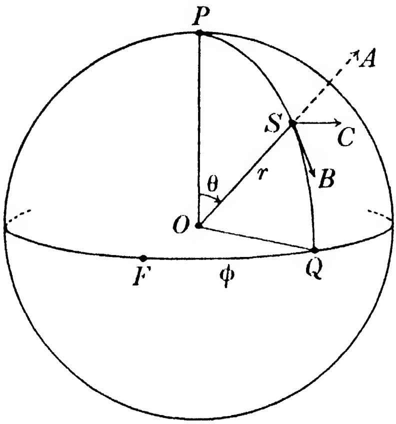

Fig. 59

$$(\delta r,\ r\,\delta\theta,\ r\sin\theta\,\delta\phi)$$

along SA, SB and SC respectively.

Let R, Θ, Φ denote the corresponding components of linear velocity. Then

$$R = \dot{r}, \quad \Theta = r\dot{\theta}, \quad \Phi = r\sin\theta\,\dot{\phi}. \quad \ldots(1)$$

The equation of continuity to be satisfied by the frequency function f is

$$\frac{Df}{Dt} = 0$$

or, in polar coordinates,

$$\frac{Df}{Dt}(r, \theta, \phi;\ R, \Theta, \Phi) = 0,$$

whence

$$\dot{r}\frac{\partial f}{\partial r} + \dot{\theta}\frac{\partial f}{\partial \theta} + \dot{\phi}\frac{\partial f}{\partial \phi} + \dot{R}\frac{\partial f}{\partial R} + \dot{\Theta}\frac{\partial f}{\partial \Theta} + \dot{\Phi}\frac{\partial f}{\partial \Phi} = 0$$

or

$$R\frac{\partial f}{\partial r} + \frac{\Theta}{r}\frac{\partial f}{\partial \theta} + \frac{\Phi}{r\sin\theta}\frac{\partial f}{\partial \phi} + \dot{R}\frac{\partial f}{\partial R} + \dot{\Theta}\frac{\partial f}{\partial \Theta} + \dot{\Phi}\frac{\partial f}{\partial \Phi} = 0. \quad \ldots\ldots(2)$$

Now, the components (α, β, γ) of the acceleration along SA, SB, SC respectively are

$$\alpha = \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,.\,\dot{\phi}^2,$$

$$\beta = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) - r\sin\theta\cos\theta\,.\,\dot{\phi}^2,$$

$$\gamma = \frac{1}{r\sin\theta}\frac{d}{dt}(r^2\sin^2\theta\,.\,\dot{\phi}),$$

whence, by (1),

$$\alpha = \dot{R} - \frac{\Theta^2}{r} - \frac{\Phi^2}{r},$$

$$\beta = \dot{\Theta} + \frac{R\Theta}{r} - \frac{\cot\theta}{r}\,.\,\Phi^2,$$

$$\gamma = \dot{\Phi} + \frac{R\Phi}{r} + \frac{\cot\theta}{r}\,.\,\Theta\Phi.$$

Also, expressing the gravitational potential in polar coordinates, we have

$$\alpha = \frac{\partial V}{\partial r}, \quad \beta = \frac{1}{r}\frac{\partial V}{\partial \theta}, \quad \gamma = \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}.$$

Hence
$$\dot{R}-\frac{1}{r}(\Theta^2+\Phi^2)=\frac{\partial V}{\partial r}, \qquad \text{......(3)}$$

$$\dot{\Theta}+\frac{R\Theta}{r}-\frac{\cot\theta}{r}.\Phi^2=\frac{1}{r}\frac{\partial V}{\partial \theta}, \qquad \text{......(4)}$$

$$\dot{\Phi}+\frac{R\Phi}{r}+\frac{\cot\theta}{r}.\Theta\Phi=\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}. \qquad \text{......(5)}$$

Insert the expressions for $\dot{R}$, $\dot{\Theta}$ and $\dot{\Phi}$, obtained from (3), (4) and (5), in (2); then

$$R\frac{\partial f}{\partial r}+\frac{\Theta}{r}\frac{\partial f}{\partial \theta}+\frac{\Phi}{r\sin\theta}\frac{\partial f}{\partial \phi}+\left\{\frac{\partial V}{\partial r}+\frac{1}{r}(\Theta^2+\Phi^2)\right\}\frac{\partial f}{\partial R}$$
$$+\left\{\frac{1}{r}\frac{\partial V}{\partial \theta}-\frac{R\Theta}{r}+\frac{\cot\theta}{r}.\Phi^2\right\}\frac{\partial f}{\partial \Theta}$$
$$+\left\{\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}-\frac{R\Phi}{r}-\frac{\cot\theta}{r}.\Theta\Phi\right\}\frac{\partial f}{\partial \Phi}=0. \qquad \text{......(6)}$$

This is the fundamental equation for the space-velocity distribution function f in polar coordinates, the system being in a steady state.

10·33. *Application of the fundamental equation in polar coordinates to spherical symmetry.*

For spherical symmetry, the functions f and V must be independent of θ and of ϕ, since the axes OP and OF ($\theta=0$ and $\phi=0$) in Fig. 59 can be chosen arbitrarily. Thus

$$\frac{\partial f}{\partial \theta}=\frac{\partial f}{\partial \phi}=0$$

and

$$\frac{\partial V}{\partial \theta}=\frac{\partial V}{\partial \phi}=0.$$

Then, (6) of the previous section becomes

$$R\frac{\partial f}{\partial r}+\left\{\frac{\partial V}{\partial r}+\frac{1}{r}(\Theta^2+\Phi^2)\right\}\frac{\partial f}{\partial R}-\frac{R\Theta}{r}\frac{\partial f}{\partial \Theta}-\frac{R\Phi}{r}\frac{\partial f}{\partial \Phi}$$
$$+\frac{\Phi}{r}\cot\theta\left\{\Phi\frac{\partial f}{\partial \Theta}-\Theta\frac{\partial f}{\partial \Phi}\right\}=0.$$

This equation must be true for all orientations of the θ-axis; hence

$$R\frac{\partial f}{\partial r}+\left\{\frac{\partial V}{\partial r}+\frac{1}{r}(\Theta^2+\Phi^2)\right\}\frac{\partial f}{\partial R}-\frac{R}{r}\left\{\Theta\frac{\partial f}{\partial \Theta}+\Phi\frac{\partial f}{\partial \Phi}\right\}=0 \qquad \text{......(1)}$$

and

$$\Phi\frac{\partial f}{\partial \Theta}-\Theta\frac{\partial f}{\partial \Phi}=0. \qquad \text{......(2)}$$

(2) can be written

$$\frac{\partial f}{\partial(\Theta^2)}=\frac{\partial f}{\partial(\Phi^2)}. \qquad \text{......(3)}$$

Let $$Q^2 = \Theta^2+\Phi^2. \qquad \ldots\ldots(4)$$

Then Q is the transverse linear velocity.

From (3) and (4) it is evident that so far as Θ and Φ are concerned f is a function of Q.

Also $$\Theta\frac{\partial f}{\partial\Theta}+\Phi\frac{\partial f}{\partial\Phi} = Q\frac{\partial f}{\partial Q}.$$

Hence (1) becomes

$$R\frac{\partial f}{\partial r}+\left\{\frac{\partial V}{\partial r}+\frac{Q^2}{r}\right\}\frac{\partial f}{\partial R}-\frac{RQ}{r}\frac{\partial f}{\partial Q}= 0. \qquad \ldots\ldots(5)$$

This is the fundamental equation in polar coordinates for spherical symmetry.

The solution of (5) is obtained by forming the subsidiary equations according to Lagrange's method; these are

$$\frac{dr}{R} = \frac{dR}{\dfrac{\partial V}{\partial r}+\dfrac{Q^2}{r}} = \frac{dQ}{-\dfrac{RQ}{r}}. \qquad \ldots\ldots(6)$$

The first and third of these give

$$\frac{dr}{r}+\frac{dQ}{Q}= 0,$$

so that a first integral is $$rQ = a_2, \qquad \ldots\ldots(7)$$

where a_2 is a constant. This equation expresses the fact that the total angular momentum about the centre is constant. Also

$$\frac{RdR+QdQ}{R\dfrac{\partial V}{\partial r}} = \frac{dr}{R},$$

whence $$R^2+Q^2-2V = a_1; \qquad \ldots\ldots(8)$$

this is the energy equation, which could of course have been written down at once. Thus the general integral of (5) is

$$f = f(R^2+Q^2-2V, rQ)$$

or $$f = f(c^2-2V, rQ), \qquad \ldots\ldots(9)$$

where $c \equiv (R^2+Q^2)^{\frac{1}{2}}$ is the linear velocity.

Also $$r^2Q^2 = r^2(c^2-R^2) = r^2(c^2-\dot{r}^2)$$

and so (9) is equivalent to (4) of 10·31.

This analysis is due to Shiveshwarkar.*

The star-density, ν, is given by

$$\nu = \iiint f(c^2-2V, rQ)\, dR\, d\Theta\, d\Phi.$$

If the velocity of escape is large, we may take the limits of integration in this formula to be $-\infty$ and $+\infty$ for each of R, Θ and Φ.

* *M.N.* 96, 751, 1936.

Write $$\Theta = Q\cos\beta, \quad \Phi = Q\sin\beta, \qquad \text{......(10)}$$

so that $$d\Theta d\Phi = QdQd\beta. \qquad \text{......(11)}$$

The range of β is from 0 to 2π and of Q from 0 to ∞. Also, f is an even function of R. Accordingly,

$$\nu = 4\pi\int_0^\infty\int_0^\infty Qf(c^2-2V, rQ)\,dRdQ. \qquad \text{......(12)}$$

Since, in polar coordinates, for spherical symmetry

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right), \qquad \text{......(13)}$$

Poisson's equation is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = -16\pi^2 M\int_0^\infty\int_0^\infty Qf(c^2-2V, rQ)\,dRdQ. \qquad \text{......(14)}$$

10·34. *Preferential motion in a stellar system with spherical symmetry.*

Star-streaming in accordance with Schwarzschild's ellipsoidal law will only occur if the frequency function f is a function of a quadratic expression of the velocity-components. In this case, the general expression for f must be of the form

$$f = F\{R^2+\Theta^2+\Phi^2-2V+kr^2(\Theta^2+\Phi^2)\} \qquad \text{......(1)}$$

and the number, $d\nu$, of stars per unit volume of space with velocity-components between (R, Θ, Φ) and $(R+dR, \Theta+d\Theta, \Phi+d\Phi)$ is given by

$$d\nu = F\,dRd\Theta d\Phi. \qquad \text{......(2)}$$

Let $$\Theta_1 = (1+kr^2)^{\frac{1}{2}}\,\Theta, \qquad \text{......(3)}$$

$$\Phi_1 = (1+kr^2)^{\frac{1}{2}}\,\Phi. \qquad \text{......(4)}$$

Then (2) becomes

$$d\nu = \frac{1}{1+kr^2}F(R^2+\Theta_1^2+\Phi_1^2-2V)\,dRd\Theta_1 d\Phi_1. \qquad \text{......(5)}$$

The function F in (5) is now spherical with regard to the velocity components R, Θ_1 and Φ_1. Define c_1 by

$$c_1^2 = R^2+\Theta_1^2+\Phi_1^2. \qquad \text{......(6)}$$

Then by (3) and (4) of section 2·21,

$$\nu = \frac{4\pi}{1+kr^2}\int_0^\infty c_1^2 F(c_1^2-2V)\,dc_1. \qquad \text{......(7)}$$

Let $\bar{R}$, $\bar{\Theta}_1$ and $\bar{\Phi}_1$ denote the mean values of R, Θ_1 and Φ_1, taken in each case without regard to sign. Then by (6) and (7) of section 2·21,

$$\bar{R} = \frac{\int_0^\infty c_1^3 F(c_1^2-2V)\,dc_1}{2\int_0^\infty c_1^2 F(c_1^2-2V)\,dc_1}. \qquad \text{......(8)}$$

By symmetry, $\bar{R} = \bar{\Theta}_1 = \bar{\Phi}_1$.

Hence, by (3) and (4),

$$\bar{R} : \bar{\Theta} : \bar{\Phi} = (1+kr^2)^{\frac{1}{2}} : 1 : 1. \qquad \text{......(9)}$$

$\bar{R}$, $\bar{\Theta}$ and $\bar{\Phi}$ are proportional to the semi-axes of the velocity ellipsoid (see section 5·11). If k is positive, the greatest axis of the ellipsoid is in the radial direction and consequently we shall have star-streaming at any point of the system with the axis of greatest mobility passing through the centre of the system. If k is negative, the previous solution will only apply to distances r less than $\sqrt{\frac{-1}{k}}$ and there will be star-streaming in all directions perpendicular to the radius vector.

It is to be noted, as pointed out by Shiveshwarkar,* that this investigation involves a generalised velocity function of which Schwarzschild's exponential law is a particular example.

Writing f (or F) as an exponential, we have

$$f = Ae^{p^2(2V-R^2-Q^2-kr^2Q^2)}.$$

The star-density, ν, is given by (5) to be

$$\nu = \frac{4\pi A}{1+kr^2}\int_0^\infty e^{p^2(2V-c_1{}^2)}\, c_1^2\, dc_1,$$

whence

$$\nu = \frac{\pi^{\frac{3}{2}} A e^{2p^2V}}{p^3(1+kr^2)}. \qquad \text{......(10)}$$

Poisson's equation is then

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} + \frac{4\pi^{\frac{5}{2}} MAe^{2p^2V}}{p^3(1+kr^2)} = 0. \qquad \text{......(11)}$$

The formulae (9) and (10) were found by Eddington by a method which will be described in the next section.

10·35. *Eddington's investigation of the dynamics of a globular stellar system.*

As we have just seen, Eddington's formulae for the density function of stars in a cluster, with spherical symmetry, in which Schwarzschild's law is obeyed is a particular case of the general solution derived from Jeans's theorem. Eddington's procedure† is as follows.

Consider the orbit of a star under the central attraction of the whole system; the orbit will then lie in a plane passing through the centre, C, of the system. Let r denote the distance of the star from C, R its radial velocity and T its transverse linear velocity. As before, the gravitational potential will be denoted by V. We assume spherical symmetry so that V is a function

* *Loc. cit.*

† *M.N.* **75**, 366, 1915. See also *M.N.* **74**, 5, 1913.

of r alone. We use a zero suffix to denote these quantities when the star is at its greatest distance from C (that is, when it is at apcentron). Then r_0 is the apcentric distance, T_0 is the transverse linear motion at apcentron, and the corresponding radial velocity, R_0, is zero; also V_0 is the gravitational potential corresponding to r_0.

The orbit being in a plane, the equations of motion are

$$\ddot{r} - r\dot{\theta}^2 = \frac{\partial V}{\partial r}, \qquad \text{......(1)}$$

$$rT = \text{a constant.} \qquad \text{......(2)}$$

In (2), the constant is $r_0 T_0$, so that

$$rT = r_0 T_0. \qquad \text{......(3)}$$

For a circular orbit of radius r_0, the transverse linear velocity, γ_0, is given from (1) by

$$\frac{\gamma_0^2}{r_0} = -\left(\frac{\partial V}{\partial r}\right)_0 = -\frac{\partial V_0}{\partial r_0}. \qquad \text{......(4)}$$

The actual velocity, T_0, at apcentron is less than γ_0.

Let P denote the orbital period; P is, of course, a function of r_0 and T_0. We can regard r_0 and T_0 as the elements of the orbit, ignoring the third element associated with the plane of the orbit, namely, the orientation of the line of apsides.

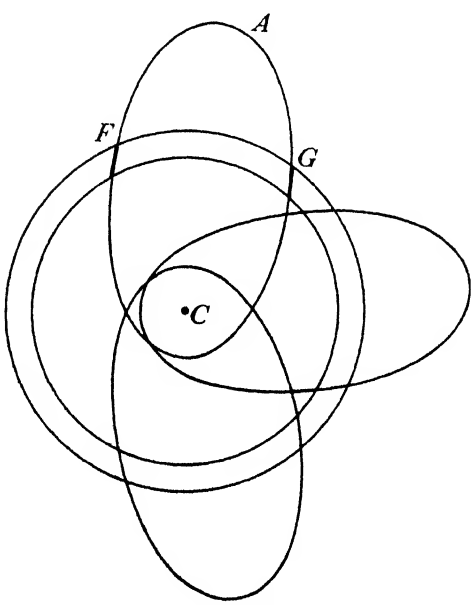

Fig. 60

Let $\quad Pf(r_0, T_0)\,dr_0\,dT_0 \quad$ ...(5)

denote the number of stars with apcentric distances between r_0 and $r_0 + dr_0$, and with apcentric velocities between T_0 and $T_0 + dT_0$.

In Fig. 60 several orbits are shown with the same apcentric distance r_0 and the same apcentric transverse velocity T_0. Consider a spherical shell defined by radii r and $r+dr$ with C as centre. The star whose orbit is A is within the shell at F and G; at F it is within the shell for a time dr/R, and similarly at G. Thus in each revolution the star is within the shell for the fraction $2dr/PR$ of the orbital period. Consequently, the stars defined in (5) contribute permanently a density $d\nu$ given by

$$d\nu = \frac{1}{4\pi r^2 dr} \cdot Pf(r_0, T_0)\, dr_0 dT_0 \cdot \frac{2dr}{PR},$$

since $4\pi r^2 dr$ is the volume of the shell. Hence

$$d\nu = \frac{1}{2\pi r^2 R} f(r_0, T_0)\, dr_0 dT_0. \qquad \text{......(6)}$$

It is to be noticed that in deriving (6) the radial velocity R is to be taken without regard to sign.

The density, ν, at a distance r from the centre for all the stars in the system is obtained by summing (6) for all values of r_0 and T_0, the radial velocity R being supposed to be expressed in terms of r_0 and T_0. Thus

$$\nu = \frac{1}{2\pi} \int\int \frac{1}{r^2 R} f(r_0, T_0)\, dr_0 dT_0. \qquad \text{......(7)}$$

We now change the variables in (6) from r_0, T_0 to R, T. We have the energy equation

$$\tfrac{1}{2}(R^2 + T^2) - V = \tfrac{1}{2}T_0^2 - V_0, \qquad \text{......(8)}$$

remembering that R_0 is zero.

Using (3) we obtain

$$R^2 = 2(V - V_0) - T_0^2\left(\frac{r_0^2}{r^2} - 1\right). \qquad \text{......(9)}$$

Now
$$dR\,dT = \frac{\partial(R, T)}{\partial(r_0, T_0)} dr_0 dT_0. \qquad \text{......(10)}$$

From (9) the following relations are easily found:

$$R\frac{\partial R}{\partial r_0} = -\frac{\partial V_0}{\partial r_0} - T_0^2 \frac{r_0}{r^2},$$

$$R\frac{\partial R}{\partial T_0} = -T_0\left(\frac{r_0^2}{r^2} - 1\right).$$

From (3), similarly,
$$r\frac{\partial T}{\partial r_0} = T_0, \quad r\frac{\partial T}{\partial T_0} = r_0.$$

Hence
$$rR\frac{\partial(R, T)}{\partial(r_0, T_0)} = \begin{vmatrix} -\dfrac{\partial V_0}{\partial r_0} - T_0^2 \dfrac{r_0}{r^2}, & T_0 \\ -T_0\left(\dfrac{r_0^2}{r^2} - 1\right), & r_0 \end{vmatrix}$$

$$= -r_0\frac{\partial V_0}{\partial r_0} - T_0^2$$

$$= \gamma_0^2 - T_0^2$$

by means of (4).

Hence (10) becomes

$$dRdT = \frac{\gamma_0^2 - T_0^2}{rR} dr_0 dT_0,$$

and we obtain from (6)

$$d\nu = \frac{f(r_0, T_0)}{2\pi r(\gamma_0^2 - T_0^2)} dRdT. \quad \text{......(11)}$$

This last equation gives the number of stars, per unit volume, with linear velocities between (R, T) and $(R+dR, T+dT)$. But if Schwarzschild's ellipsoidal law characterises the peculiar motions in the cluster, $d\nu$ is given by

$$d\nu = CTe^{-h_1^2R^2-h_2^2T^2} dRdT, \quad \text{......(12)}$$

in which the factor T is inserted owing to the fact that the transverse velocity T is a two-dimensional component (this is analogous to 10·33(11)). In the formula (12), C, h_1 and h_2 may be functions of r; in other words, the lengths of the axes of the velocity ellipsoid may be assumed to vary radially from one part of the system to another.

We can write (12) in the form

$$d\nu = CTe^{-h^2(R^2+T^2)} . e^{-k^2r^2T^2} dRdT, \quad \text{......(13)}$$

in which C, h and k may be functions of r. Using (8) and (3), we obtain

$$d\nu = \frac{C}{r} r_0 T_0 e^{-h^2T_0^2+2h^2V_0-2h^2V} . e^{-k^2r_0^2T_0^2} dRdT. \quad \text{......(14)}$$

Hence, from (11) and (14),

$$f(r_0, T_0) = 2\pi Ce^{-2h^2V} . r_0T_0(\gamma_0^2 - T_0^2) e^{2h^2V_0} . e^{-(h^2+k^2r_0^2)T_0^2}. \quad \text{......(15)}$$

Since $f(r_0, T_0)$ is independent of r, the right-hand side of (15) must not involve r. Since V and C are functions of r, this condition is satisfied if

$$2\pi Ce^{-2h^2V} = B, \quad \text{......(16)}$$

where B is independent of r, and

h and k are constants.

Hence, from (13) and (16),

$$d\nu = \frac{B}{2\pi} e^{2h^2V} . Te^{-h^2R^2-(h^2+k^2r^2)T^2} dRdT. \quad \text{......(17)}$$

The density, ν, at a distance r from the centre of the system is then given by integrating (17) for all values of R and T between 0 and ∞. Thus

$$\nu = \frac{B}{8\sqrt{\pi}\, h(h^2+k^2r^2)} e^{2h^2V}, \quad \text{......(18)}$$

which is of the same analytical form as (10) of section 10·34. Poisson's equation is found as before to be of the same form as (11) of section 10·34.

Choosing the units and the zero of potential so as to reduce the latter equation to its simplest algebraical form, we write it as

$$\frac{d^2V}{dr^2}+\frac{2}{r}\frac{dV}{dr}+\frac{e^{2V}}{1+r^2}=0. \qquad \text{......(19)}$$

Let $u = 1/r$. Then (19) becomes

$$\frac{d^2V}{du^2}+\frac{e^{2V}}{u^2(1+u^2)}=0 \qquad \text{......(20)}$$

and, setting $$V=-\log z,$$

(20) becomes $$z\frac{d^2z}{du^2}-\left(\frac{dz}{du}\right)^2=\frac{1}{u^2(1+u^2)}. \qquad \text{......(21)}$$

Also, in these units, $$4\pi\nu=\frac{e^{2V}}{1+r^2}=\frac{1}{z^2(1+r^2)}. \qquad \text{......(22)}$$

These equations have been studied by Eddington in the paper referred to.

10·41. *The fundamental equation in cylindrical coordinates.*

Let (ϖ, θ, z) be the cylindrical coordinates of a star S as shown in Fig. 61. The components of velocity parallel to OX, XY and XS are denoted respectively by Π, Θ and Z, a notation introduced by Jeans; we have

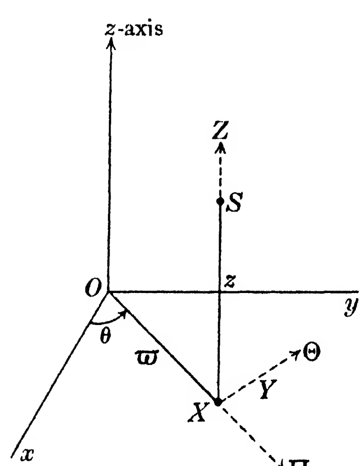

Fig. 61

$$\Pi=\dot{\varpi}, \quad \Theta=\varpi\dot{\theta}, \quad Z=\dot{z} \qquad \text{.......(1)}$$

The continuity equation $Df/Dt = 0$ becomes, for steady motion,

$$\dot{\varpi}\frac{\partial f}{\partial \varpi}+\dot{\theta}\frac{\partial f}{\partial \theta}+\dot{z}\frac{\partial f}{\partial z}+\dot{\Pi}\frac{\partial f}{\partial \Pi}+\dot{\Theta}\frac{\partial f}{\partial \Theta}+\dot{Z}\frac{\partial f}{\partial Z}=0, \qquad \text{......(2)}$$

f being expressed as a function of ϖ, θ, z; Π, Θ and Z.

From (1) and (2), we have

$$\Pi\frac{\partial f}{\partial \varpi}+\frac{\Theta}{\varpi}\frac{\partial f}{\partial \theta}+Z\frac{\partial f}{\partial z}+\dot{\Pi}\frac{\partial f}{\partial \Pi}+\dot{\Theta}\frac{\partial f}{\partial \Theta}+\dot{Z}\frac{\partial f}{\partial Z}=0. \qquad \text{......(3)}$$

If (α, β, γ) are the components of acceleration parallel to OX, XY and XS respectively, we have

$$\alpha \equiv \ddot{\varpi} - \varpi\dot{\theta}^2 = \frac{\partial V}{\partial \varpi},$$

$$\beta \equiv \frac{1}{\varpi}\frac{d}{dt}(\varpi^2\dot{\theta}) = \frac{1}{\varpi}\frac{\partial V}{\partial \theta},$$

$$\gamma \equiv \ddot{z} = \frac{\partial V}{\partial z},$$

whence

$$\dot{\Pi} = \frac{\partial V}{\partial \varpi} + \frac{\Theta^2}{\varpi}, \qquad \text{......(4)}$$

$$\dot{\Theta} = \frac{1}{\varpi}\frac{\partial V}{\partial \theta} - \frac{\Pi\Theta}{\varpi}, \qquad \text{......(5)}$$

$$\dot{Z} = \frac{\partial V}{\partial z}. \qquad \text{......(6)}$$

Formula (3) then becomes

$$\Pi\frac{\partial f}{\partial \varpi} + \frac{\Theta}{\varpi}\frac{\partial f}{\partial \theta} + Z\frac{\partial f}{\partial z} + \left(\frac{\partial V}{\partial \varpi} + \frac{\Theta^2}{\varpi}\right)\frac{\partial f}{\partial \Pi} + \left(\frac{1}{\varpi}\frac{\partial V}{\partial \theta} - \frac{\Pi\Theta}{\varpi}\right)\frac{\partial f}{\partial \Theta} + \frac{\partial V}{\partial z}\cdot\frac{\partial f}{\partial Z} = 0. \qquad \text{......(7)}$$

For uniaxial symmetry, the functions f and V are independent of θ; in this case we have

$$\Pi\frac{\partial f}{\partial \varpi} + Z\frac{\partial f}{\partial z} + \left(\frac{\partial V}{\partial \varpi} + \frac{\Theta^2}{\varpi}\right)\frac{\partial f}{\partial \Pi} - \frac{\Pi\Theta}{\varpi}\frac{\partial f}{\partial \Theta} + \frac{\partial V}{\partial z}\cdot\frac{\partial f}{\partial Z} = 0. \qquad \text{......(8)}$$

In these formulae Θ is not to be confused with the Θ of sections 10·32–10·34.

10·42. *Systems with uniaxial symmetry.*

We take the axis of symmetry to be the z-axis and we use cylindrical coordinates. From 10·41 (8), we can in theory find four independent integrals, I_1, ... I_4.

As before, we have the energy integral (in rectangular coordinates)

$$I_1 \equiv u^2 + v^2 + w^2 - 2V = a_1 \qquad \text{......(1)}$$

or, in cylindrical coordinates, defined as in the previous section,

$$I_1 = \Pi^2 + \Theta^2 + Z^2 - 2V = a_1. \qquad \text{......(2)}$$

Also, the constancy of angular momentum about the axis of symmetry gives a second integral of the equations of motion, namely,

$$I_2 \equiv \varpi\Theta = a_2. \qquad \text{......(3)}$$

These are the only integrals that can be found in general.

The frequency function f is then a function of I_1 and I_2 and we write it as

$$f \equiv f(I_1, I_2) \qquad \text{......(4)}$$

or, by means of (2) and (3),

$$f \equiv f(c^2 - 2V, \varpi\Theta), \qquad \text{......(5)}$$

where

$$c^2 = \Pi^2 + \Theta^2 + Z^2. \qquad \text{......(6)}$$

Owing to uniaxial symmetry, V is a function of ϖ and z only.

If the velocities are distributed according to a generalised ellipsoidal law, f must be a function of a quadratic expression of the velocity components, and the most general way of writing this, consistently with the form of f in (4) and (5), is

$$f = f(\xi), \qquad \text{......(7)}$$

where

$$\xi = I_1 + 2k_1 I_2 + k_2 I_2^2, \qquad \text{......(8)}$$

in which k_1 and k_2 are independent of Π, Θ and Z. In particular, Schwarzschild's ellipsoidal law in this notation is

$$f = Ae^{-B\xi}. \qquad \text{......(9)}$$

We can write ξ as follows:

$$\xi = \Pi^2 + \lambda^2(\Theta - \Theta_0)^2 + Z^2 - 2V_1, \qquad \text{......(10)}$$

where

$$\lambda^2 = 1 + k_2\varpi^2, \qquad \text{......(11)}$$

$$\Theta_0 = -\frac{k_1\varpi}{1 + k_2\varpi^2}, \qquad \text{......(12)}$$

$$V_1 = V + \frac{1}{2}\frac{k_1^2\varpi^2}{1 + k_2\varpi^2}. \qquad \text{......(13)}$$

In these equations, λ, Θ_0 and V_1 are functions of ϖ and z.

The formula (10) shows that Θ_0 is a rotational linear component. The residual velocity components (the rotational component being removed) are Π, Θ' and Z, where

$$\Theta' = \Theta - \Theta_0. \qquad \text{......(14)}$$

These components are distributed ellipsoidally and consequently there is star-streaming.

The star-density, ν, is given by

$$\nu = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)\, d\Pi\, d\Theta\, dZ. \qquad \text{......(15)}$$

On writing

$$\Theta_1 = \lambda(\Theta - \Theta_0), \qquad \text{......(16)}$$

$$c_1^2 = \Pi^2 + \Theta_1^2 + Z^2, \qquad \text{......(17)}$$

(15) becomes

$$\nu = \frac{1}{\lambda}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(c_1^2 - 2V_1)\, d\Pi\, d\Theta_1\, dZ. \qquad \text{......(18)}$$

The form of the function f in (18) implies a spherical velocity distribution with regard to the velocity-components Π, Θ_1 and Z. Hence by (3) and (4) of section 2·21, we obtain

$$\nu = \frac{4\pi}{\lambda}\int_0^\infty c_1^2 f(c_1^2 - 2V_1)\,dc_1. \qquad \text{......(19)}$$

Let $\bar{\Pi}$, $\bar{\Theta}_1$ and $\bar{Z}$ denote the mean values of Π, Θ_1 and Z, taken without regard to sign.

Then, by (6) and (7) of section 2·21, and as in section 10·34,

$$\bar{\Pi} = \bar{\Theta}_1 = \bar{Z} = \frac{\int_0^\infty c_1^3 f(c_1^2 - 2V_1)\,dc_1}{2\int_0^\infty c_1^2 f(c_1^2 - 2V_1)\,dc_1}. \qquad \text{......(20)}$$

Let $\bar{\Theta}'$ denote the mean value of $\Theta' \equiv \Theta - \Theta_0$, taken without regard to sign. Then, from (16) and (20), we have

$$\bar{\Pi} : \bar{\Theta}' : \bar{Z} = \lambda : 1 : \lambda. \qquad \text{......(21)}$$

With k_2 positive, λ is greater than unity, by (11). Since $\bar{\Pi}$, $\bar{\Theta}'$ and $\bar{Z}$ are proportional to the lengths of the axes of the velocity ellipsoid in the corresponding directions, the velocity ellipsoid in this case is an oblate spheroid with its greatest axes in the radial and in the z-directions.

10·51. *Stellar systems in general.*

It is assumed throughout the present treatment—except for a generalisation, due to Shiveshwarkar, which we interpolate in section 10·53—that the stellar velocities are distributed according to Schwarzschild's ellipsoidal law. Accordingly, we suppose that at any point of the stellar system a velocity ellipsoid is defined with its principal axes oriented in assigned directions. Consider one of the principal axes. At a neighbouring point it will be oriented in a slightly different direction and, tracing it from point to point, we see that the direction of this axis will be given at any point by the tangent to a three-dimensional curve which can be regarded as the intersection of two members of a family of surfaces with each of which is associated a parameter, the surfaces being envelopes of principal planes of the velocity ellipsoids. As the principal axes form an orthogonal set, the direction of a principal axis at a given point of the system is given by the normal to a surface which is one of a triply-orthogonal family. These surfaces are called by Eddington,* to whom the following analysis is due, the *principal velocity-surfaces*.

Let λ, μ and ν be the three parameters associated with the three families of principal velocity-surfaces, so that at any point of the system the coordinates are specified by λ, μ and ν which are, in effect, the curvilinear coordinates of the point.

* *M.N.* 76, 37, 1915.

The elements of length in the principal directions are

$$P d\lambda, \quad Q d\mu, \quad R d\nu,$$

where P, Q and R are, in general, functions of λ, μ, ν.

The components of velocity in the principal directions are

$$P\dot{\lambda}, \quad Q\dot{\mu}, \quad R\dot{\nu}.$$

The kinetic energy of a particle of unit mass is given by

$$T = \tfrac{1}{2}(P^2\dot{\lambda}^2 + Q^2\dot{\mu}^2 + R^2\dot{\nu}^2). \qquad \ldots\ldots(1)$$

As in previous sections, we suppose that the motion of an individual star is controlled by a gravitational potential, V, which will be in general a function of λ, μ, ν.

Using Lagrange's dynamical equations, a sample of which we write in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\lambda}}\right) - \frac{\partial T}{\partial \lambda} = \frac{\partial V}{\partial \lambda},$$

we obtain, from (1),

$$\frac{d}{dt}(P^2\dot{\lambda}) \equiv P^2\ddot{\lambda} + \dot{\lambda}\frac{dP^2}{dt} = \frac{1}{2}\left\{\dot{\lambda}^2\frac{\partial P^2}{\partial \lambda} + \dot{\mu}^2\frac{\partial Q^2}{\partial \lambda} + \dot{\nu}^2\frac{\partial R^2}{\partial \lambda}\right\} + \frac{\partial V}{\partial \lambda}.$$

But
$$\frac{dP^2}{dt} = \dot{\lambda}\frac{\partial P^2}{\partial \lambda} + \dot{\mu}\frac{\partial P^2}{\partial \mu} + \dot{\nu}\frac{\partial P^2}{\partial \nu}.$$

Hence

$$P^2\ddot{\lambda} = -\tfrac{1}{2}\dot{\lambda}^2\frac{\partial P^2}{\partial \lambda} + \tfrac{1}{2}\dot{\mu}^2\frac{\partial Q^2}{\partial \lambda} + \tfrac{1}{2}\dot{\nu}^2\frac{\partial R^2}{\partial \lambda} - \dot{\lambda}\dot{\mu}\frac{\partial P^2}{\partial \mu} - \dot{\lambda}\dot{\nu}\frac{\partial P^2}{\partial \nu} + \frac{\partial V}{\partial \lambda}. \qquad \ldots\ldots(2)$$

Similarly,

$$Q^2\ddot{\mu} = \tfrac{1}{2}\dot{\lambda}^2\frac{\partial P^2}{\partial \mu} - \tfrac{1}{2}\dot{\mu}^2\frac{\partial Q^2}{\partial \mu} + \tfrac{1}{2}\dot{\nu}^2\frac{\partial R^2}{\partial \mu} - \dot{\mu}\dot{\nu}\frac{\partial Q^2}{\partial \nu} - \dot{\mu}\dot{\lambda}\frac{\partial Q^2}{\partial \lambda} + \frac{\partial V}{\partial \mu}, \qquad \ldots\ldots(3)$$

$$R^2\ddot{\nu} = \tfrac{1}{2}\dot{\lambda}^2\frac{\partial P^2}{\partial \nu} + \tfrac{1}{2}\dot{\mu}^2\frac{\partial R^2}{\partial \nu} - \tfrac{1}{2}\dot{\nu}^2\frac{\partial R^2}{\partial \nu} - \dot{\nu}\dot{\lambda}\frac{\partial R^2}{\partial \lambda} - \dot{\nu}\dot{\mu}\frac{\partial R^2}{\partial \mu} + \frac{\partial V}{\partial \nu}. \qquad \ldots\ldots(4)$$

These are the equations of motion in curvilinear coordinates.

10·52. *The equation of continuity.*

At a given point of the stellar system, the axes of the velocity ellipsoid are oriented in particular directions, and taking these directions to define a system of rectangular axes, we can write the number of stars in an element of volume surrounding the point as $\sigma\, dx\, dy\, dz$, where we now denote the star-density by σ—and not by ν, as previously—to avoid confusion with the generalised coordinate ν. Accordingly, the number, dN, of stars with velocity components between (u, v, w) and $(u+du, v+dv, w+dw)$ in the element of volume is given by

$$dN = \sigma\, dx\, dy\, dz\, A e^{-a^2u^2 - b^2v^2 - c^2w^2}\, du\, dv\, dw. \qquad \ldots\ldots(1)$$

Hence, summing for all possible values of u, v and w,

$$\sigma\, dx\, dy\, dz = \sigma\, dx\, dy\, dz\, A \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a^2u^2-b^2v^2-c^2w^2}\, du\, dv\, dw,$$

from which $$A = \pi^{-3/2} abc. \qquad \ldots\ldots(2)$$

Hence (1) becomes

$$dN = \pi^{-\frac{3}{2}}\, dx\, dy\, dz\, \sigma abc\, e^{-a^2u^2-b^2v^2-c^2w^2}\, du\, dv\, dw. \qquad \ldots\ldots(3)$$

The lengths of the semi-axes of the velocity ellipsoid at the given point are $1/a$, $1/b$, $1/c$ and these are functions of λ, μ and ν.

In curvilinear coordinates, (3) becomes

$$dN = \pi^{-\frac{3}{2}}\, P d\lambda \,.\, Q d\mu \,.\, R d\nu \,.\, \sigma abc\, e^{-a^2P^2\dot{\lambda}^2-b^2Q^2\dot{\mu}^2-c^2R^2\dot{\nu}^2}\, P d\dot{\lambda} \,.\, Q d\dot{\mu} \,.\, R d\dot{\nu},$$

and, on writing $$e^{\kappa} = \sigma abc, \qquad \ldots\ldots(4)$$

$$\psi = -a^2P^2\dot{\lambda}^2 - b^2Q^2\dot{\mu}^2 - c^2R^2\dot{\nu}^2 + \kappa, \qquad \ldots\ldots(5)$$

we obtain $$dN = \pi^{-\frac{3}{2}}\, e^{\psi} P^2Q^2R^2\, d\lambda\, d\mu\, d\nu\, d\dot{\lambda}\, d\dot{\mu}\, d\dot{\nu}. \qquad \ldots\ldots(6)$$

The equation of continuity is

$$\frac{D}{Dt}(dN) = 0. \qquad \ldots\ldots(7)$$

Now, by Liouville's theorem, in the form given by 10·21 (17),

$$\frac{D}{Dt}(dQ) \equiv \frac{D}{Dt}(P d\lambda \,.\, Q d\mu \,.\, R d\nu \,.\, P d\dot{\lambda} \,.\, Q d\dot{\mu} \,.\, R d\dot{\nu}) = 0. \qquad \ldots\ldots(8)$$

Hence, from (6), (7) and (8), $$\frac{D}{Dt}\psi = 0$$

or $$\frac{D\kappa}{Dt} = \frac{D}{Dt}(a^2P^2\dot{\lambda}^2 + b^2Q^2\dot{\mu}^2 + c^2R^2\dot{\nu}^2). \qquad \ldots\ldots(9)$$

Now, the operator D/Dt for curvilinear coordinates is given by

$$\frac{D}{Dt} = \dot{\lambda}\frac{\partial}{\partial\lambda} + \dot{\mu}\frac{\partial}{\partial\mu} + \dot{\nu}\frac{\partial}{\partial\nu} + \ddot{\lambda}\frac{\partial}{\partial\dot{\lambda}} + \ddot{\mu}\frac{\partial}{\partial\dot{\mu}} + \ddot{\nu}\frac{\partial}{\partial\dot{\nu}}.$$

Hence, from (9), we obtain—κ being a function of λ, μ and ν alone—

$$\begin{aligned}
\dot{\lambda}\frac{\partial\kappa}{\partial\lambda} + \dot{\mu}\frac{\partial\kappa}{\partial\mu} + \dot{\nu}\frac{\partial\kappa}{\partial\nu} = {} & \dot{\lambda}^3\frac{\partial}{\partial\lambda}(a^2P^2) + \dot{\lambda}^2\dot{\mu}\frac{\partial}{\partial\mu}(a^2P^2) + \dot{\lambda}^2\dot{\nu}\frac{\partial}{\partial\nu}(a^2P^2) \\
& + \dot{\lambda}\dot{\mu}^2\frac{\partial}{\partial\lambda}(b^2Q^2) + \dot{\mu}^3\frac{\partial}{\partial\mu}(b^2Q^2) + \dot{\mu}^2\dot{\nu}\frac{\partial}{\partial\nu}(b^2Q^2) \\
& + \dot{\lambda}\dot{\nu}^2\frac{\partial}{\partial\lambda}(c^2R^2) + \dot{\mu}\dot{\nu}^2\frac{\partial}{\partial\mu}(c^2R^2) + \dot{\nu}^3\frac{\partial}{\partial\nu}(c^2R^2) \\
& + 2a^2\left\{-\tfrac{1}{2}\dot{\lambda}^3\frac{\partial P^2}{\partial\lambda} + \tfrac{1}{2}\dot{\lambda}\dot{\mu}^2\frac{\partial Q^2}{\partial\lambda} + \tfrac{1}{2}\dot{\lambda}\dot{\nu}^2\frac{\partial R^2}{\partial\lambda} - \dot{\lambda}^2\dot{\mu}\frac{\partial P^2}{\partial\mu} - \dot{\lambda}^2\dot{\nu}\frac{\partial P^2}{\partial\nu} + \dot{\lambda}\frac{\partial V}{\partial\lambda}\right\} \\
& + 2b^2\left\{\tfrac{1}{2}\dot{\lambda}^2\dot{\mu}\frac{\partial P^2}{\partial\mu} - \tfrac{1}{2}\dot{\mu}^3\frac{\partial Q^2}{\partial\mu} + \tfrac{1}{2}\dot{\mu}\dot{\nu}^2\frac{\partial R^2}{\partial\mu} - \dot{\lambda}\dot{\mu}^2\frac{\partial Q^2}{\partial\lambda} - \dot{\mu}^2\dot{\nu}\frac{\partial Q^2}{\partial\nu} + \dot{\mu}\frac{\partial V}{\partial\mu}\right\} \\
& + 2c^2\left\{\tfrac{1}{2}\dot{\lambda}^2\dot{\nu}\frac{\partial P^2}{\partial\nu} + \tfrac{1}{2}\dot{\mu}^2\dot{\nu}\frac{\partial Q^2}{\partial\nu} - \tfrac{1}{2}\dot{\nu}^3\frac{\partial R^2}{\partial\nu} - \dot{\lambda}\dot{\nu}^2\frac{\partial R^2}{\partial\lambda} - \dot{\mu}\dot{\nu}^2\frac{\partial R^2}{\partial\mu} + \dot{\nu}\frac{\partial V}{\partial\nu}\right\}, \qquad \ldots(10)
\end{aligned}$$

where, in the last three lines, the expressions for $\ddot{\lambda}$, $\ddot{\mu}$ and $\ddot{\nu}$, as given by the equations of motion (2), (3) and (4) of section 10·51, have been substituted.

The formula (10) holds for all points of the system and for all values of $\dot{\lambda}$, $\dot{\mu}$ and $\dot{\nu}$. Hence, for example, the coefficient of $\dot{\lambda}$ on the left-hand side of (10) is equal to the coefficient on the right-hand side. Accordingly,

$$\frac{\partial \kappa}{\partial \lambda} = 2a^2 \frac{\partial V}{\partial \lambda}.$$

Again, equating the coefficients of $\dot{\lambda}^3$, we have

$$0 = \frac{\partial}{\partial \lambda}(a^2 P^2) - a^2 \frac{\partial P^2}{\partial \lambda},$$

from which

$$\frac{\partial}{\partial \lambda} a^2 = 0. \qquad \text{......(11)}$$

Equating the coefficients of $\dot{\lambda}\dot{\mu}^2$, we find that

$$0 = \frac{\partial}{\partial \lambda}(b^2 Q^2) + a^2 \frac{\partial Q^2}{\partial \lambda} - 2b^2 \frac{\partial Q^2}{\partial \lambda},$$

from which

$$(a^2 - b^2) \frac{\partial Q^2}{\partial \lambda} = -Q^2 \frac{\partial b^2}{\partial \lambda}$$

or, using (11),

$$(a^2 - b^2) \frac{\partial Q^2}{\partial \lambda} = Q^2 \frac{\partial}{\partial \lambda}(a^2 - b^2).$$

Hence

$$\frac{\partial}{\partial \lambda}\left(\frac{a^2 - b^2}{Q^2}\right) = 0. \qquad \text{......(12)}$$

Proceeding in this way, we obtain the complete results as follows:

$$\frac{\partial}{\partial \lambda} a^2 = 0, \qquad \frac{\partial}{\partial \mu} b^2 = 0, \qquad \frac{\partial}{\partial \nu} c^2 = 0, \qquad \text{......(13)}$$

$$\frac{\partial}{\partial \mu}\left(\frac{b^2 - c^2}{R^2}\right) = 0, \quad \frac{\partial}{\partial \nu}\left(\frac{c^2 - a^2}{P^2}\right) = 0, \quad \frac{\partial}{\partial \lambda}\left(\frac{a^2 - b^2}{Q^2}\right) = 0, \quad \text{......(14)}$$

$$\frac{\partial}{\partial \nu}\left(\frac{b^2 - c^2}{Q^2}\right) = 0, \quad \frac{\partial}{\partial \lambda}\left(\frac{c^2 - a^2}{R^2}\right) = 0, \quad \frac{\partial}{\partial \mu}\left(\frac{a^2 - b^2}{P^2}\right) = 0, \quad \text{......(15)}$$

$$2a^2 \frac{\partial V}{\partial \lambda} = \frac{\partial \kappa}{\partial \lambda}, \qquad 2b^2 \frac{\partial V}{\partial \mu} = \frac{\partial \kappa}{\partial \mu}, \qquad 2c^2 \frac{\partial V}{\partial \nu} = \frac{\partial \kappa}{\partial \nu}. \qquad \text{......(16)}$$

It is to be remembered that P, Q and R are functions of λ, μ, ν and may be supposed known in accordance with the adopted system of curvilinear coordinates. Hence the nine equations in (13), (14) and (15) are sufficient to determine a, b and c in terms of the parameters λ, μ, ν; these functional values of a, b and c are independent of the potential, V, and the density σ.

The three equations in (16) determine κ if the potential V is known and consequently the density, σ, can be found; this last function must also satisfy Poisson's equation if the potential is due to the system itself.

10·53. *Shiveshwarkar's generalisation.**

Consider the frequency function $f(\psi)$, where

$$\psi = -a^2u^2 - b^2v^2 - c^2w^2 + K \qquad \text{......(1)}$$

and a, b, c and K may be functions of the coordinates (x, y, z) in a rectangular system. Then the number, dN, of stars with the volume-element $dx\,dy\,dz$ and with velocity components between (u, v, w) and $(u+du, v+dv, w+dw)$ is given by

$$dN = f(\psi)\,dx\,dy\,dz\,du\,dv\,dw.$$

The function f is of the generalised ellipsoidal type.

The star-density, σ, is given by

$$\sigma = \iiint f(\psi)\,du\,dv\,dw. \qquad \text{......(2)}$$

This equation connects σ and K in a way similar to that in which σ and κ are related in 10·52(4). Also, by Jeans's theorem, ψ must be a first integral—or a function of the first integrals—of the equations of motion.

Consider now the equations in curvilinear coordinates. Let p_1, p_2, p_3 denote the generalised momenta so that

$$p_1 = \frac{\partial T}{\partial \dot\lambda}, \quad p_2 = \frac{\partial T}{\partial \dot\mu}, \quad p_3 = \frac{\partial T}{\partial \dot\nu},$$

where T is given by (1) of section 10·51. We thus have

$$p_1 = P^2\dot\lambda, \quad p_2 = Q^2\dot\mu, \quad p_3 = R^2\dot\nu \qquad \text{......(3)}$$

and T becomes

$$T = \frac{1}{2}\left(\frac{p_1^2}{P^2} + \frac{p_2^2}{Q^2} + \frac{p_3^2}{R^2}\right). \qquad \text{......(4)}$$

The Hamiltonian function, H, is given by

$$H = T - V = \frac{1}{2}\left(\frac{p_1^2}{P^2} + \frac{p_2^2}{Q^2} + \frac{p_3^2}{R^2}\right) - V$$

and the equations of motion are

$$\dot\lambda = \frac{\partial H}{\partial p_1}, \quad \dot\mu = \frac{\partial H}{\partial p_2}, \quad \dot\nu = \frac{\partial H}{\partial p_3}, \qquad \text{......(5)}$$

$$\dot p_1 = -\frac{\partial H}{\partial \lambda}, \quad \dot p_2 = -\frac{\partial H}{\partial \mu}, \quad \dot p_3 = -\frac{\partial H}{\partial \nu}. \qquad \text{......(6)}$$

Since ψ = constant is an integral of the equations of motion, this equation must be satisfied by (5) and (6). Expressing ψ in terms of the generalised coordinates λ, μ, ν and the generalised momenta p_1, p_2, p_3, we write the integral as

$$\psi(\lambda, \mu, \nu; p_1, p_2, p_3) = \text{constant}. \qquad \text{......(7)}$$

* *M.N.* 96, 750, 1936.

Differentiate this last equation with respect to the time, and there results

$$\sum_{\lambda,\mu,\nu} \frac{\partial\psi}{\partial\lambda}\dot{\lambda} + \sum_{p_1,p_2,p_3} \frac{\partial\psi}{\partial p_1}\dot{p}_1 = 0.$$

Hence, by (5) and (6),

$$\Sigma \frac{\partial\psi}{\partial\lambda}\cdot\frac{\partial H}{\partial p_1} - \Sigma \frac{\partial\psi}{\partial p_1}\cdot\frac{\partial H}{\partial\lambda} = 0$$

or

$$\frac{\partial(\psi, H)}{\partial(\lambda, p_1)} + \frac{\partial(\psi, H)}{\partial(\mu, p_2)} + \frac{\partial(\psi, H)}{\partial(\nu, p_3)} = 0. \qquad \ldots\ldots(8)$$

By (1), ψ is defined in terms of the generalised coordinates by

$$\psi = -a^2P^2\dot{\lambda}^2 - b^2Q^2\dot{\mu}^2 - c^2R^2\dot{\nu}^2 + K,$$

in which a, b, c and K are functions of the coordinates λ, μ, ν. From (3), ψ is expressed in terms of the coordinates and generalised momenta by

$$\psi = -\frac{a^2}{P^2}p_1^2 - \frac{b^2}{Q^2}p_2^2 - \frac{c^2}{R^2}p_3^2 + K. \qquad \ldots\ldots(9)$$

Hence, from (8), we obtain

$$\Sigma \begin{vmatrix} p_1^2\frac{\partial}{\partial\lambda}\left(\frac{a^2}{P^2}\right) + p_2^2\frac{\partial}{\partial\lambda}\left(\frac{b^2}{Q^2}\right) + p_3^2\frac{\partial}{\partial\lambda}\left(\frac{c^2}{R^2}\right) - \frac{\partial K}{\partial\lambda}, & \frac{a^2}{P^2}p_1 \\ p_1^2\frac{\partial}{\partial\lambda}\left(\frac{1}{P^2}\right) + p_2^2\frac{\partial}{\partial\lambda}\left(\frac{1}{Q^2}\right) + p_3^2\frac{\partial}{\partial\lambda}\left(\frac{1}{R^2}\right) - 2\frac{\partial V}{\partial\lambda}, & \frac{1}{P^2}p_1 \end{vmatrix} = 0.$$

This equation holds for all values of p_1, p_2 and p_3. Equating coefficients of the various p's and combinations of the p's we arrive at the equations (13), (14), (15) and (16) of the previous section, K replacing κ in (16). If the potential V is known, K is determinable from the equations (16). Thus the function ψ is found. Poisson's equation remains to be satisfied and we must have

$$\nabla^2 V = -4\pi M \iiint f(\psi)\, PQR\, d\lambda\, d\mu\, d\nu. \qquad \ldots\ldots(10)$$

10·54. *Eddington's theorem.*

This theorem* states that the principal velocity-surfaces are confocal quadrics. The proof in the general case is long and we shall confine ourselves to the case of axial symmetry.†

In this case, two of the families of velocity-surfaces are surfaces of revolution and the third family consists of planes passing through the axis of revolution. Taking the ν parameter to be the azimuthal angle we see that P, Q, R, a, b and c are independent of ν.

* *M.N.* **76**, 54, 1915.
† *Ibid.* p. 42.

From (14) and (15) of section 10·52 we have

$$\frac{\partial}{\partial\lambda}\left(\frac{a^2-b^2}{Q^2}\right) = 0 \quad\text{and}\quad \frac{\partial}{\partial\mu}\left(\frac{a^2-b^2}{P^2}\right) = 0. \qquad \text{......(1)}$$

Hence $\dfrac{a^2-b^2}{Q^2}$ is independent of λ and must consequently be a function of μ alone; thus

$$\frac{a^2-b^2}{Q^2} = f_1(\mu).$$

Similarly, from (1),

$$\frac{a^2-b^2}{P^2} = f_2(\lambda).$$

We then have

$$a^2-b^2 = Q^2 f_1(\mu) = P^2 f_2(\lambda). \qquad \text{......(2)}$$

Take the axis of z to be the axis of revolution and consider cylindrical coordinates (ϖ, z, θ), θ being the azimuthal angle. The element of length corresponding to the λ parameter is $P\,d\lambda$, which is equivalent in cylindrical coordinates to $(d\varpi^2+dz^2)^{1/2}$. Thus

$$P^2 = \left(\frac{\partial\varpi}{\partial\lambda}\right)^2 + \left(\frac{\partial z}{\partial\lambda}\right)^2. \qquad \text{......(3)}$$

Similarly,

$$Q^2 = \left(\frac{\partial\varpi}{\partial\mu}\right)^2 + \left(\frac{\partial z}{\partial\mu}\right)^2. \qquad \text{......(4)}$$

Let λ_0, μ_0 be new parameters defined by

$$\lambda_0 = \int\frac{d\lambda}{\sqrt{f_2(\lambda)}} \quad\text{and}\quad \mu_0 = \int\frac{d\mu}{\sqrt{f_1(\mu)}}, \qquad \text{......(5)}$$

so that λ_0 is a function of λ alone and μ_0 is a function of μ alone. Now by (5)

$$\frac{\partial\varpi}{\partial\lambda_0} = \sqrt{f_2(\lambda)}\frac{\partial\varpi}{\partial\lambda} \quad\text{and}\quad \frac{\partial z}{\partial\lambda_0} = \sqrt{f_2(\lambda)}\frac{\partial z}{\partial\lambda} \qquad \text{......(6)}$$

and

$$\frac{\partial\varpi}{\partial\mu_0} = \sqrt{f_1(\mu)}\frac{\partial\varpi}{\partial\mu} \quad\text{and}\quad \frac{\partial z}{\partial\mu_0} = \sqrt{f_1(\mu)}\frac{\partial z}{\partial\mu}. \qquad \text{......(7)}$$

Hence, from (3) and (6),

$$\left(\frac{\partial\varpi}{\partial\lambda_0}\right)^2 + \left(\frac{\partial z}{\partial\lambda_0}\right)^2 = P^2 f_2(\lambda), \qquad \text{......(8)}$$

and from (4) and (7)

$$\left(\frac{\partial\varpi}{\partial\mu_0}\right)^2 + \left(\frac{\partial z}{\partial\mu_0}\right)^2 = Q^2 f_1(\mu). \qquad \text{......(9)}$$

We then obtain from (2)

$$\left(\frac{\partial\varpi}{\partial\lambda_0}\right)^2 + \left(\frac{\partial z}{\partial\lambda_0}\right)^2 = \left(\frac{\partial\varpi}{\partial\mu_0}\right)^2 + \left(\frac{\partial z}{\partial\mu_0}\right)^2. \qquad \text{......(10)}$$

Since, by definition, the surfaces ϖ = constant, z = constant are orthogonal, as are also the surfaces λ_0 = constant, μ_0 = constant, the formula

(10) is the sufficient condition that the transformation from the coordinates (ϖ, z) to the generalised coordinates (λ_0, μ_0) should be conformal. Hence we obtain, from (10),

$$\varpi + iz = F(\lambda_0 + i\mu_0), \qquad \text{......(11)}$$

where $i = +\sqrt{-1}$ or $-\sqrt{-1}$.

From (8)

$$P^2 f_2(\lambda) = \left(\frac{\partial\varpi}{\partial\lambda_0} + i\frac{\partial z}{\partial\lambda_0}\right)\left(\frac{\partial\varpi}{\partial\lambda_0} - i\frac{\partial z}{\partial\lambda_0}\right)$$

$$= F'(\lambda_0 + i\mu_0) \,.\, F'(\lambda_0 - i\mu_0). \qquad \text{......(12)}$$

Hence (2) and (12) give

$$a^2 - b^2 = F'(\lambda_0 + i\mu_0) \,.\, F'(\lambda_0 - i\mu_0). \qquad \text{......(13)}$$

Since, by (5), λ_0 is a function of λ alone and μ_0 is a function of μ alone, we can use λ_0 and μ_0 as the parameters of the corresponding surfaces, for the families of surfaces $\lambda_0 = \text{constant}$ and $\mu_0 = \text{constant}$ are the same as the families of surfaces $\lambda = \text{constant}$ and $\mu = \text{constant}$ respectively. We can accordingly drop the suffixes in (13), which we now write

$$a^2 - b^2 = F'(\lambda + i\mu) \,.\, F'(\lambda - i\mu), \qquad \text{......(14)}$$

from which

$$\frac{\partial^2}{\partial\lambda\,\partial\mu}(a^2 - b^2) = iF'''(\lambda + i\mu) \,.\, F'(\lambda - i\mu) - iF'(\lambda + i\mu) \,.\, F'''(\lambda - i\mu). \qquad \text{......(15)}$$

But, from (13) of section 10·52,

$$\frac{\partial(a^2)}{\partial\lambda} = \frac{\partial(b^2)}{\partial\mu} = 0;$$

hence the left-hand side of (15) vanishes and we obtain

$$\frac{F'''(\lambda + i\mu)}{F'(\lambda + i\mu)} = \frac{F'''(\lambda - i\mu)}{F'(\lambda - i\mu)}. \qquad \text{......(16)}$$

As $\lambda + i\mu$ and $\lambda - i\mu$ are different variables, each side of (16) must be equal to a constant, say p^2. Thus

$$F'''(\lambda + i\mu) = p^2 F'(\lambda + i\mu)$$

or, writing $\lambda + i\mu = Z$,

$$\frac{d^3F(Z)}{dZ^3} = p^2\frac{dF(Z)}{dZ},$$

the solution of which is

$$F(Z) = A\cosh(pZ + \epsilon) + B,$$

where A, B and ϵ are constants of integration and may, in general, be complex. The real and imaginary parts of ϵ only change the λ and μ origins, and the real and imaginary parts of B only change the ϖ and z origins; we can accordingly take $B = \epsilon = 0$. Hence

$$\varpi + iz = A\cosh p(\lambda + i\mu).$$

Taking A to be real and equating real and imaginary parts, we obtain

$$\varpi = A \cosh p\lambda \cos p\mu,$$

$$z = A \sinh p\lambda \sin p\mu.$$

The curves in the (ϖ, z) plane for $\lambda =$ constant and $\mu =$ constant are respectively

$$\frac{\varpi^2}{\cosh^2 p\lambda} + \frac{z^2}{\sinh^2 p\lambda} = A^2$$

and

$$\frac{\varpi^2}{\cos^2 p\mu} - \frac{z^2}{\sin^2 p\mu} = A^2.$$

These are confocal conics and the corresponding principal velocity-surfaces are formed by the revolution of these conics about the z-axis. If A is wholly imaginary, we obtain a similar result. If A is complex, we obtain systems of confocals unsymmetrical with respect to the ϖ and z axes.

Thus the theorem is established, in the case of axial symmetry, that the principal velocity-surfaces are confocal quadrics—the third family of quadrics in this case degenerating into the system of planes passing through the axis of revolution.

10·55. *The possibility of star-streaming.*

Taking the most general case, we have from Eddington's theorem that one family of principal velocity surfaces is a system of confocal ellipsoids. Assume that these ellipsoids correspond to the parameter λ. Then along a curve, at every point of which the tangent is a normal to the confocal ellipsoid through that point, λ varies but μ and ν are constant. Now from (13), (14) and (15) of section 10·52,

$$\frac{\partial a^2}{\partial \lambda} = 0, \quad \frac{\partial}{\partial \lambda}\left(\frac{a^2 - b^2}{Q^2}\right) = 0, \quad \frac{\partial}{\partial \lambda}\left(\frac{c^2 - a^2}{R^2}\right) = 0,$$

so that as we pass along the curve

$$a^2 = \text{constant} = A,$$

$$b^2 = A + BQ^2,$$

$$c^2 = A + CR^2,$$

A, B and C being constants.

Since the components, along the principal axes of the velocity ellipsoid, of the mean peculiar motion are inversely proportional to a, b and c, we must have B and C positive, otherwise b and c would vanish at some point of the system. Consequently, b and c are each larger than a, so that the greatest axis of the velocity ellipsoid is in the λ direction, that is normal to the principal velocity-surface. The ratios of the mean speeds along or parallel to the principal axes of the velocity ellipsoid are given by

$$A^{-\frac{1}{2}} : (A+BQ^2)^{-\frac{1}{2}} : (A+CR^2)^{-\frac{1}{2}}.$$

Unless the principal velocity surface departs very markedly from a sphere, the direction of the normal to the surface will be more or less in the direction of the centre of the system; in other words, the direction of the greatest mobility of the peculiar motions will be roughly radial.

If the equations (16) of section 10·52 can be satisfied, star-streaming in the radial direction is thus possible. We shall consider this point in detail in a later section.

10·56. *Integration of the equations for a, b and c.*

We consider particular cases of Eddington's theorem.

(i) *Spherical coordinates.*

If one of the systems of principal velocity-surfaces is a family of spheres, the orthogonal surfaces, in polar coordinates, are

$$r = \text{constant}, \quad \theta = \text{constant} \quad \text{and} \quad \phi = \text{constant},$$

where r, θ, ϕ correspond to λ, μ and ν respectively. Also,

$$dr, \quad r\,d\theta, \quad r\sin\theta\,d\phi$$

are the orthogonal components of a small displacement from one point to a neighbouring point. Hence

$$P = 1, \quad Q = r, \quad R = r\sin\theta.$$

The formulae (13), (14) and (15) of section 10·52 then become

$$\frac{\partial a^2}{\partial r} = 0, \qquad \frac{\partial b^2}{\partial \theta} = 0, \qquad \frac{\partial c^2}{\partial \phi} = 0, \qquad \text{......(1·1), (1·2), (1·3)}$$

$$\frac{\partial}{\partial \theta}\left(\frac{b^2-c^2}{r^2\sin^2\theta}\right) = 0, \quad \frac{\partial}{\partial \phi}(c^2-a^2) = 0, \quad \frac{\partial}{\partial r}\left(\frac{a^2-b^2}{r^2}\right) = 0, \quad \text{......(2·1), (2·2), (2·3)}$$

$$\frac{\partial}{\partial \phi}\left(\frac{b^2-c^2}{r^2}\right) = 0, \quad \frac{\partial}{\partial r}\left(\frac{c^2-a^2}{r^2\sin^2\theta}\right) = 0, \quad \frac{\partial}{\partial \theta}(a^2-b^2) = 0. \quad \text{......(3·1), (3·2), (3·3)}$$

Using (1·2) with (3·3), and (1·3) with (2·2), we have

$$\frac{\partial a^2}{\partial r} = \frac{\partial a^2}{\partial \theta} = \frac{\partial a^2}{\partial \phi} = 0, \qquad \text{......(4)}$$

so that

$$a^2 = \text{constant}. \qquad \text{......(5)}$$

From (1·3) and (3·1) we have

$$\frac{\partial b^2}{\partial \phi} = 0$$

or, using (4),

$$\frac{\partial}{\partial \phi}\left(\frac{a^2-b^2}{r^2}\right) = 0. \qquad \text{......(6)}$$

Similarly, (1·2) may be written

$$\frac{\partial}{\partial\theta}\left(\frac{a^2-b^2}{r^2}\right) = 0. \qquad \ldots\ldots(7)$$

The formulae (6), (7) and (2·3) show that $\dfrac{a^2-b^2}{r^2}$ is constant; hence we can write

$$b^2 = a^2(1+p^2r^2), \qquad \ldots\ldots(8)$$

where we put the constant equal to the positive quantity a^2p^2, since b must not vanish at any point of the stellar system.

Writing (2·3) as
$$\frac{\partial}{\partial r}\left(\frac{a^2-b^2}{r^2\sin^2\theta}\right) = 0$$

and adding it to (3·2), we have

$$\frac{\partial}{\partial r}\left(\frac{b^2-c^2}{r^2\sin^2\theta}\right) = 0,$$

which, together with (2·1) and (3·1), shows that

$$\frac{c^2-b^2}{r^2\sin^2\theta} = \text{constant} \equiv a^2C, \text{ say.}$$

Hence
$$c^2 = b^2 + a^2Cr^2\sin^2\theta$$

or, using (8),
$$c^2 = a^2(1+p^2r^2+Cr^2\sin^2\theta). \qquad \ldots\ldots(9)$$

(ii) *Prolate spheroidal coordinates.*

When one of the principal velocity-surfaces is a prolate spheroid, with the z-axis as axis of revolution, its equation can be put in the form

$$\frac{x^2+y^2}{\sinh^2\xi} + \frac{z^2}{\cosh^2\xi} = p^2.$$

Choose the unit of length so that $p = 1$; then the coordinates of any point on the surface can be expressed as

$$x = \sinh\xi\sin\eta\cos\gamma,$$
$$y = \sinh\xi\sin\eta\sin\gamma,$$
$$z = \cosh\xi\cos\eta.$$

We consider ξ, η, γ to be the generalised coordinates λ, μ, ν respectively. Then

$$P^2 = \left(\frac{\partial x}{\partial\xi}\right)^2 + \left(\frac{\partial y}{\partial\xi}\right)^2 + \left(\frac{\partial z}{\partial\xi}\right)^2,$$

from which and the similar equation for Q^2 we obtain

$$P^2 = Q^2 = \cosh^2\xi - \cos^2\eta. \qquad \ldots\ldots(10)$$

Similarly, it is easily found that

$$R^2 = \sinh^2\xi\sin^2\eta. \qquad \ldots\ldots(11)$$

Since $P^2 = Q^2$, we derive from (14) and (15) of section 10·52

$$\frac{\partial}{\partial \xi}\left(\frac{a^2-b^2}{P^2}\right) = \frac{\partial}{\partial \eta}\left(\frac{a^2-b^2}{P^2}\right) = 0.$$

Also, $$\frac{\partial}{\partial \gamma}\left(\frac{a^2-b^2}{P^2}\right) = -\frac{\partial}{\partial \gamma}\left(\frac{b^2-c^2}{Q^2}\right) - \frac{\partial}{\partial \gamma}\left(\frac{c^2-a^2}{P^2}\right) = 0.$$

Hence, $$\frac{a^2-b^2}{P^2} = \text{constant},$$

and we write this last result, using (10), as

$$b^2 - a^2 = BP^2 = B(\cosh^2 \xi - \cos^2 \eta), \qquad \text{......(12)}$$

in which B is a constant.

Also, $$\frac{\partial a^2}{\partial \xi} = 0 \quad \text{and} \quad \frac{\partial b^2}{\partial \eta} = 0. \qquad \text{......(13)}$$

Hence from (12) and (13) $$a^2 = A + B\cos^2 \eta, \qquad \text{......(14)}$$

$$b^2 = A + B\cosh^2 \xi, \qquad \text{......(15)}$$

in which A is positive (since $a^2 = A$ for $\eta = \pi/2$) and independent of ξ and η; also, since a and b must not vanish, B is by (15) a positive constant. A is also independent of γ—and is therefore a positive constant—as may readily be shown as follows. We have, from 10·52 (14),

$$\frac{\partial}{\partial \gamma}\left(\frac{c^2-a^2}{P^2}\right) = 0.$$

But, by (10), P is independent of γ. Hence, since $\partial c^2/\partial \gamma = 0$, by 10·52 (13), we obtain

$$\frac{\partial a^2}{\partial \gamma} = 0,$$

so that, by (14), $$\frac{\partial A}{\partial \gamma} = 0.$$

Accordingly, A is a positive constant.

Again, from (14) and (15) of section 10·52,

$$\frac{\partial}{\partial \eta}\left(\frac{c^2-b^2}{R^2}\right) = 0 \quad \text{and} \quad \frac{\partial}{\partial \xi}\left(\frac{c^2-a^2}{R^2}\right) = 0.$$

The latter equation can be written

$$\frac{\partial}{\partial \xi}\left(\frac{c^2-a^2-b^2}{R^2}\right) = -\frac{\partial}{\partial \xi}\left(\frac{b^2}{R^2}\right).$$

Writing b^2 in the form—from (15)—

$$b^2 = A + B + B\sinh^2 \xi,$$

we have, using (11),

$$\frac{\partial}{\partial \xi}\left(\frac{b^2}{R^2}\right) = \frac{\partial}{\partial \xi}\left(\frac{A+B}{R^2} + \frac{B}{\sin^2 \eta}\right) = \frac{\partial}{\partial \xi}\left(\frac{A+B}{R^2}\right).$$

Hence
$$\frac{\partial}{\partial \xi}\left(\frac{c^2-a^2-b^2+A+B}{R^2}\right) = 0. \qquad \text{......(16)}$$

Similarly,
$$\frac{\partial}{\partial \eta}\left(\frac{c^2-a^2-b^2+A+B}{R^2}\right) = 0. \qquad \text{......(17)}$$

Also, since
$$\frac{\partial c^2}{\partial \gamma} = 0,$$

and since a^2, b^2 and R^2 are all independent of γ, we have
$$\frac{\partial}{\partial \gamma}\left(\frac{c^2-a^2-b^2+A+B}{R^2}\right) = 0. \qquad \text{......(18)}$$

Hence (16), (17) and (18) give
$$c^2-a^2-b^2+A+B = CR^2,$$

where C is a constant. This last equation can be written
$$c^2 = A+B\cos^2\eta+B\sinh^2\xi+C\sinh^2\xi\,\sin^2\eta. \qquad \text{......(19)}$$

The results of the other cases have been given by Eddington* as follows.

(iii) *Rectangular coordinates* (x, y, z).

In this case the principal velocity-surfaces degenerate into orthogonal planes. Then
$$P = Q = R = 1$$
and a, b, c are all constant.

(iv) *Cylindrical coordinates* (ϖ, z, θ).

The principal velocity-surfaces are cylinders with the z-axis as axis, planes normal to the z-axis and planes passing through the z-axis. Then
$$P = Q = 1, \quad R = \varpi$$
and
$$a^2 = \text{constant},$$
$$b^2 = \text{constant},$$
$$c^2 = a^2(1+k\varpi^2).$$

(v) *Oblate spheroidal coordinates* (ξ, η, γ).

Here one of the principal velocity-surfaces is the oblate spheroid given by
$$x = \cosh\xi\cos\eta\cos\gamma, \quad y = \cosh\xi\cos\eta\sin\gamma, \quad z = \sinh\xi\sin\eta.$$

Then
$$P^2 = Q^2 = \cosh^2\xi-\cos^2\eta,$$
$$R^2 = \cosh^2\xi\cos^2\eta,$$
and
$$a^2 = A+B\cos^2\eta,$$
$$b^2 = A+B\cosh^2\xi,$$
$$c^2 = A+B\cosh^2\xi+B\cos^2\eta+C\cosh^2\xi\cos^2\eta.$$

* *M.N.* 76, 45, 1915.

(vi) *Ellipsoidal coordinates* (λ, μ, ν).

In this general case the principal velocity-surfaces are confocal quadrics. If the squares of the semi-axes of the confocals through (λ, μ, ν) are

$$\lambda^2, \lambda^2-\beta^2, \lambda^2-\gamma^2; \quad \mu^2, \mu^2-\beta^2, \mu^2-\gamma^2; \quad \nu^2, \nu^2-\beta^2, \nu^2-\gamma^2,$$

then

$$P^2 = \frac{(\lambda^2-\mu^2)(\lambda^2-\nu^2)}{(\lambda^2-\beta^2)(\lambda^2-\gamma^2)}, \quad Q^2 = \frac{(\mu^2-\nu^2)(\lambda^2-\mu^2)}{(\mu^2-\beta^2)(\gamma^2-\mu^2)}, \quad R^2 = \frac{(\lambda^2-\nu^2)(\mu^2-\nu^2)}{(\beta^2-\nu^2)(\gamma^2-\nu^2)}$$

and

$$a^2 = A + B\mu^2 + B\nu^2 + C\mu^2\nu^2,$$
$$b^2 = A + B\nu^2 + B\lambda^2 + C\nu^2\lambda^2,$$
$$c^2 = A + B\lambda^2 + B\mu^2 + C\lambda^2\mu^2.$$

10·57. *Evaluation of the density.*

The equations (16) of section 10·52 enable us to derive Eddington's function κ when the gravitational potential, V, is known. From these equations, we obtain

$$a^2\frac{\partial V}{\partial\lambda}d\lambda + b^2\frac{\partial V}{\partial\mu}d\mu + c^2\frac{\partial V}{\partial\nu}d\nu = \frac{1}{2}\left\{\frac{\partial\kappa}{\partial\lambda}d\lambda + \frac{\partial\kappa}{\partial\mu}d\mu + \frac{\partial\kappa}{\partial\nu}d\nu\right\} = \tfrac{1}{2}d\kappa. \quad \text{......(1)}$$

Since the right-hand side of this equation is a perfect differential, it follows that

$$\frac{\partial}{\partial\mu}\left(a^2\frac{\partial V}{\partial\lambda}\right) = \frac{1}{2}\frac{\partial^2\kappa}{\partial\lambda\,\partial\mu} = \frac{\partial}{\partial\lambda}\left(b^2\frac{\partial V}{\partial\mu}\right). \quad \text{......(2)}$$

Similarly

$$\frac{\partial}{\partial\nu}\left(a^2\frac{\partial V}{\partial\lambda}\right) = \frac{\partial}{\partial\lambda}\left(c^2\frac{\partial V}{\partial\nu}\right),$$

$$\frac{\partial}{\partial\mu}\left(c^2\frac{\partial V}{\partial\nu}\right) = \frac{\partial}{\partial\nu}\left(b^2\frac{\partial V}{\partial\mu}\right).$$

From (2),

$$\frac{\partial V}{\partial\lambda}\frac{\partial a^2}{\partial\mu} - \frac{\partial V}{\partial\mu}\frac{\partial b^2}{\partial\lambda} + (a^2-b^2)\frac{\partial^2 V}{\partial\lambda\,\partial\mu} = 0$$

or, since

$$\frac{\partial a^2}{\partial\lambda} = \frac{\partial b^2}{\partial\mu} = 0,$$

$$V\frac{\partial^2(a^2-b^2)}{\partial\lambda\partial\mu} + \frac{\partial}{\partial\mu}(a^2-b^2)\frac{\partial V}{\partial\lambda} + \frac{\partial}{\partial\lambda}(a^2-b^2)\frac{\partial V}{\partial\mu} + (a^2-b^2)\frac{\partial^2 V}{\partial\lambda\,\partial\mu} = 0,$$

whence

$$\frac{\partial^2}{\partial\lambda\,\partial\mu}\{(a^2-b^2)V\} = 0. \quad \text{......(3)}$$

Similarly,

$$\frac{\partial^2}{\partial\mu\,\partial\nu}\{(b^2-c^2)V\} = 0, \quad \text{......(4)}$$

$$\frac{\partial^2}{\partial\nu\,\partial\lambda}\{(c^2-a^2)V\} = 0. \quad \text{......(5)}$$

When the potential is given, it must satisfy these equations in which a, b and c are all expressed in terms of λ, μ and ν.

In the case of axial symmetry, all the variables concerned will be independent of ν (the azimuthal angle), and we obtain from (3)

$$V = \frac{F(\lambda)+G(\mu)}{a^2-b^2}, \qquad \ldots\ldots(6)$$

where F and G are arbitrary functions.

Now, from (1),
$$\frac{1}{2}\frac{\partial\kappa}{\partial\lambda} = a^2\frac{\partial V}{\partial\lambda}$$

or, since
$$\frac{\partial a^2}{\partial\lambda} = 0,$$
$$\frac{1}{2}\frac{\partial\kappa}{\partial\lambda} = \frac{\partial}{\partial\lambda}(a^2V).$$

Now, from (6), we can write
$$a^2V = \frac{a^2F(\lambda)+b^2G(\mu)}{a^2-b^2} + G(\mu).$$

Hence
$$\frac{1}{2}\frac{\partial\kappa}{\partial\lambda} = \frac{\partial}{\partial\lambda}\left\{\frac{a^2F(\lambda)+b^2G(\mu)}{a^2-b^2}\right\}.$$

Similarly
$$\frac{1}{2}\frac{\partial\kappa}{\partial\mu} = \frac{\partial}{\partial\mu}\left\{\frac{a^2F(\lambda)+b^2G(\mu)}{a^2-b^2}\right\},$$

Hence
$$\tfrac{1}{2}\kappa = \frac{a^2F(\lambda)+b^2G(\mu)}{a^2-b^2} + \tfrac{1}{2}\log C, \qquad \ldots\ldots(7)$$

where C is a constant.

From (4) of section 10·52, the density σ is given by

$$abc\sigma = e^{\kappa}. \qquad \ldots\ldots(8)$$

If the motions of the stars in the system are derived from the attraction of the system, Poisson's equation must also be satisfied; hence

$$\nabla^2\left\{\frac{F(\lambda)+G(\mu)}{a^2-b^2}\right\} = -\frac{4\pi C}{abc}e^{\frac{2a^2F(\lambda)+2b^2G(\mu)}{a^2-b^2}}. \qquad \ldots\ldots(9)$$

With spheroidal coordinates, it has been found impossible to separate the variables in the above equation—so as to determine the functions $F(\lambda)$ and $G(\mu)$. It therefore appears that a solution of (9) does not exist.

For spherical symmetry, $G(\mu) = 0$ if we associate λ with r. From (5), (8) and (9) of section 10·56—setting $C = 0$ in the last equation because of spherical symmetry—we have

$$a^2 = \text{constant}; \quad b^2 = c^2 = a^2(1+p^2r^2).$$

Then (9) becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left\{\frac{F(r)}{r^2}\right\}\right] = \frac{4\pi p^2}{a(1+p^2r^2)}e^{-\frac{2F(r)}{p^2r^2}}, \qquad \ldots\ldots(10)$$

from which $F(r)$ can theoretically be obtained; thus κ can be found and then the density is obtained by means of (8). It is to be remarked that formula (10) above is essentially (11) of section 10·34.

In Shiveshwarkar's generalised formulae κ is replaced by K which, in the case of spheroidal coordinates, is given by an equation similar to (7). The density is then given by (2) of section 10·53 and, if the system is moving under its own gravitational potential, Poisson's equation has to be satisfied.

There is one possibility that we have not yet considered. So far we have made the general assumption that the motions of the stars in a given system are derived from the gravitational attraction of the system itself. It may be that the stellar motions are controlled by a potential compared with which the potential of the system itself is negligible. The simplest case is where the motions are due principally or entirely to a dense aggregation of matter at the centre of the system; in this case we may assume that V is due to this central condensation and thus Poisson's equation does not require to be satisfied.

In the case of uniaxial symmetry the solution is complete if the potential V of the controlling system is given by (6), so that

$$V = \frac{F(\lambda)+G(\mu)}{a^2-b^2},$$

in which F and G are now supposed to be known functions; κ is then given by (7) and the density σ by (8).

10·58. Eddington dealt, in his paper, with further problems which we do not consider here in detail. The principal conclusions,* however, may be briefly stated, when the velocities are distributed according to Schwarzschild's ellipsoidal law.

(i) For systems moving solely under their own gravitational attraction, the only exact three-dimensional solution, for steady motion, is in the case of spherical symmetry.

(ii) For systems moving under forces other than their own gravitation, it is possible to obtain an oblate distribution of stars (as in the galactic system) either by taking prolate velocity-surfaces, or by assuming that the whole system is in rotation, or by a combination of these two causes.

We examine the question of rotation in Chapter XII.

* For a review of the problems with which we have been dealing see Eddington, *A.N.* **815**, 1921 (Jubilee Number); also G. L. Clark, *M.N.* **97**, 182, 1937; O. Heckmann and H. Strasl, *Gottingen Veröff.* Nos. 41 (1934), 43 (1935).

10·61. *The "hydrodynamical equations" for a stellar system.*

In rectangular coordinates the fundamental equation, from section 10·21, is

$$\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}+w\frac{\partial f}{\partial z}+\frac{\partial V}{\partial x}\frac{\partial f}{\partial u}+\frac{\partial V}{\partial y}\frac{\partial f}{\partial v}+\frac{\partial V}{\partial z}\frac{\partial f}{\partial w}=0. \qquad \text{......(1)}$$

Multiply (1) by $u\,du\,dv\,dw$ and integrate over the complete range of the variables.

The first term gives
$$\frac{\partial}{\partial t}\iiint fu\,du\,dv\,dw. \qquad \text{......(2)}$$

Since the star-density, ν, is given by

$$\nu=\iiint f\,du\,dv\,dw, \qquad \text{......(3)}$$

the expression (2) is $\frac{\partial}{\partial t}(\nu\bar{u})$, where $\bar{u}$ denotes the mean value of the velocity component u.

The second term gives

$$\frac{\partial}{\partial x}\iiint fu^2\,du\,dv\,dw,$$

which, by (3), is equivalent to $\frac{\partial}{\partial x}(\nu\overline{u^2})$, where $\overline{u^2}$ is the mean value of u^2. Similarly, the third and fourth terms give

$$\frac{\partial}{\partial y}(\nu\overline{uv}) \quad \text{and} \quad \frac{\partial}{\partial z}(\nu\overline{uw}).$$

The fifth term gives

$$\frac{\partial V}{\partial x}\iiint u\frac{\partial f}{\partial u}\,du\,dv\,dw \equiv \frac{\partial V}{\partial x}E, \text{ say.}$$

Integrating by parts, we have

$$E=\iint [uf]\,dv\,dw-\iiint f\,du\,dv\,dw, \qquad \text{......(4)}$$

where $[uf]$ denotes the value of uf after the limits of integration for u have been inserted. In the analytically simple case of Schwarzschild's ellipsoidal law, it is evident that

$$\lim_{u\to\infty}[uf(u,v,w)]=0 \qquad \text{......(5)}$$

and in the general case this condition must also be postulated from physical considerations. Hence, from (4),

$$E=-\iiint f\,du\,dv\,dw=-\nu$$

so that the fifth term yields $-\nu\frac{\partial V}{\partial x}$.

Again, we have, by arguments similar to that resulting in (5),

$$\iiint u\frac{\partial f}{\partial v}\,du\,dv\,dw = \iiint u\frac{\partial f}{\partial w}\,du\,dv\,dw = 0. \qquad \ldots\ldots(6)$$

We finally obtain

$$\frac{\partial}{\partial t}(\nu\overline{u})+\frac{\partial}{\partial x}(\nu\overline{u^2})+\frac{\partial}{\partial y}(\nu\overline{uv})+\frac{\partial}{\partial z}(\nu\overline{uw})=\nu\frac{\partial V}{\partial x}. \qquad \ldots\ldots(7)$$

There are two other equations of a similar nature.

Formula (7) is the hydrodynamical equation for stars, all of the same mass, in unit volume at the point (x, y, z) of the system, and moving under the gravitational field of the whole system.

Multiply (1) by $du\,dv\,dw$ and integrate; we obtain in the same way

$$\frac{\partial \nu}{\partial t}+\frac{\partial}{\partial x}(\nu\overline{u})+\frac{\partial}{\partial y}(\nu\overline{v})+\frac{\partial}{\partial z}(\nu\overline{w}) = 0, \qquad \ldots\ldots(8)$$

which is the hydrodynamical equation of continuity for stars of equal mass.

For steady motion, the equations are

$$\frac{\partial}{\partial x}(\nu\overline{u^2})+\frac{\partial}{\partial y}(\nu\overline{uv})+\frac{\partial}{\partial z}(\nu\overline{uw}) = \nu\frac{\partial V}{\partial x}, \qquad \ldots\ldots(9)$$

$$\frac{\partial}{\partial x}(\nu\overline{uv})+\frac{\partial}{\partial y}(\nu\overline{v^2})+\frac{\partial}{\partial z}(\nu\overline{vw}) = \nu\frac{\partial V}{\partial y}, \qquad \ldots\ldots(10)$$

$$\frac{\partial}{\partial x}(\nu\overline{uw})+\frac{\partial}{\partial y}(\nu\overline{vw})+\frac{\partial}{\partial z}(\nu\overline{w^2}) = \nu\frac{\partial V}{\partial z}, \qquad \ldots\ldots(11)$$

and

$$\frac{\partial}{\partial x}(\nu\overline{u})+\frac{\partial}{\partial y}(\nu\overline{v})+\frac{\partial}{\partial z}(\nu\overline{w}) = 0. \qquad \ldots\ldots(12)$$

10·62. *The hydrodynamical equations in cylindrical coordinates.*

For steady motion the fundamental equation in cylindrical coordinates is, from (7) of section 10·41,

$$\Pi\frac{\partial f}{\partial \varpi}+\frac{\Theta}{\varpi}\frac{\partial f}{\partial \theta}+Z\frac{\partial f}{\partial z}+\left(\frac{\partial V}{\partial \varpi}+\frac{\Theta^2}{\varpi}\right)\frac{\partial f}{\partial \Pi}+\left(\frac{1}{\varpi}\frac{\partial V}{\partial \theta}-\frac{\Pi\Theta}{\varpi}\right)\frac{\partial f}{\partial \Theta}+\frac{\partial V}{\partial z}\frac{\partial f}{\partial Z} = 0. \qquad \ldots\ldots(1)$$

Multiply by $\Pi d\Pi d\Theta dZ$, which we shall write as $\Pi d\alpha$, and integrate over the complete range of the velocity-components.

With a notation similar to that of the previous section,

$$\iiint \Pi^2\frac{\partial f}{\partial \varpi}d\alpha = \frac{\partial}{\partial \varpi}(\nu\overline{\Pi^2})$$

and

$$\iiint \frac{\Pi\Theta}{\varpi}\frac{\partial f}{\partial \theta}d\alpha = \frac{1}{\varpi}\frac{\partial}{\partial \theta}(\nu\overline{\Pi\Theta}).$$

Also,
$$\iiint \Pi\Theta^2 \frac{\partial f}{\partial \Pi} d\alpha = \iint [\Pi f]\,\Theta^2 d\Theta\, dZ - \iiint \Theta^2 f\, d\alpha$$
$$= -\nu\overline{\Theta^2},$$

by utilising the condition represented by (5) of the previous section.

Again,
$$\iiint \Pi \frac{\partial f}{\partial \Theta} d\alpha = \iint [f]\, \Pi\, d\Pi\, dZ = 0,$$

since we must suppose that f vanishes for infinite values of the component Θ.

Proceeding in this way with the remaining terms of (1), we finally obtain

$$\frac{\partial}{\partial \varpi}(\nu\overline{\Pi^2}) + \frac{1}{\varpi}\frac{\partial}{\partial \theta}(\nu\overline{\Pi\Theta}) + \frac{\partial}{\partial z}(\nu\overline{\Pi Z}) + \frac{\nu(\overline{\Pi^2} - \overline{\Theta^2})}{\varpi} = \nu\frac{\partial V}{\partial \varpi}. \quad \text{......(2)}$$

Similarly,

$$\frac{\partial}{\partial \varpi}(\nu\overline{\Pi\Theta}) + \frac{1}{\varpi}\frac{\partial}{\partial \theta}(\nu\overline{\Theta^2}) + \frac{\partial}{\partial z}(\nu\overline{\Theta Z}) + \frac{2}{\varpi}(\nu\overline{\Pi\Theta}) = \frac{\nu}{\varpi}\frac{\partial V}{\partial \theta}, \quad \text{......(3)}$$

$$\frac{\partial}{\partial \varpi}(\nu\overline{\Pi Z}) + \frac{1}{\varpi}\frac{\partial}{\partial \theta}(\nu\overline{\Theta Z}) + \frac{\partial}{\partial z}(\nu\overline{Z^2}) + \frac{1}{\varpi}(\nu\overline{\Pi Z}) = \nu\frac{\partial V}{\partial z}. \quad \text{......(4)}$$

Multiply (1) by $d\Pi\, d\Theta\, dZ$ and integrate; then

$$\frac{\partial}{\partial \varpi}(\nu\overline{\Pi}) + \frac{1}{\varpi}\frac{\partial}{\partial \theta}(\nu\overline{\Theta}) + \frac{\partial}{\partial z}(\nu\overline{Z}) + \frac{\nu}{\varpi}\overline{\Pi} = 0. \quad \text{......(5)}$$

10·63. *The equations for uniaxial symmetry.*

In this case, V is a function of ϖ and z alone, so that

$$\frac{\partial V}{\partial \theta} = 0. \quad \text{......(1)}$$

Also, from section 10·42, the distribution function, f, is given by

$$f \equiv f(\Pi^2 + \Theta^2 + Z^2 - 2V, \varpi\Theta) \quad \text{......(2)}$$

and the density, ν, by
$$\nu = \iiint f\, d\Pi\, d\Theta\, dZ.$$

Thus positive and negative values of Π and Z are equally probable; hence

$$\overline{\Pi\Theta} = \overline{\Pi Z} = \overline{\Theta Z} = 0. \quad \text{......(3)}$$

Let
$$p = \nu\overline{\Pi^2}, \quad \text{......(4)}$$
$$q = \nu\overline{\Theta^2}. \quad \text{......(5)}$$

Owing to the symmetry of Π and Z in (2), p is also given by

$$p = \nu\overline{Z^2}. \quad \text{......(6)}$$

The equations (2), (3) and (4) of section 10·62 become

$$\frac{\partial p}{\partial \varpi}+\frac{p-q}{\varpi}=\nu\frac{\partial V}{\partial \varpi}, \qquad \text{......(7)}$$

$$\frac{\partial q}{\partial \theta}=\frac{\partial V}{\partial \theta}=0, \qquad \text{......(8)}$$

$$\frac{\partial p}{\partial z}=\nu\frac{\partial V}{\partial z}. \qquad \text{......(9)}$$

The equation (8) simply expresses the fact that q is independent of θ.

By eliminating V between (7) and (9), we obtain

$$\frac{\partial}{\partial z}\left(\frac{q-p}{\nu\varpi}\right)=\frac{\partial p}{\partial \varpi}\frac{\partial}{\partial z}\left(\frac{1}{\nu}\right)-\frac{\partial p}{\partial z}\frac{\partial}{\partial \varpi}\left(\frac{1}{\nu}\right). \qquad \text{......(10)}$$

These equations were given by Jeans in an investigation* on the motions of the stars in a "Kapteyn Universe". From a study of direct parallax determinations and statistical relationships such as are treated in Chapter VIII, Kapteyn and van Rhijn† had obtained numerical values of the star-density at different points of the galaxy. Later, Kapteyn‡ showed that the stellar distribution could be represented tolerably well by means of spheroidal shells in each of which the density was constant; the equi-density surfaces were taken to be concentric, similar and similarly situated spheroids, the ratio of the axes being 5·1 : 5·1 : 1, with the shorter axis (the z-axis in our notation) perpendicular to the galactic equator. In the subsequent dynamical investigation Kapteyn assumed (i) that the system was in a steady state and (ii) that the distribution of velocities in the z-direction obeyed the Maxwellian law. In the paper referred to, Jeans showed that Kapteyn's second assumption was unnecessary, and on the basis of the equations of this section, together with Kapteyn's schematic representation of the galactic system, he was able to deduce some of the principal features of stellar motions.

* *M.N.* **82**, 122, 1922.
† *Ap. J.* **52**, 23, 1920 (*Mt Wilson Contr.* No. 188, 1920).
‡ *Ap. J.* **55**, 302, 1922 (*Mt Wilson Contr.* No. 230, 1922).

CHAPTER XI

GALACTIC ROTATION

11·11. *The principal characteristics of the galactic system.*

In the preceding chapter we considered certain general principles in the dynamics of a stellar system; in the present chapter our investigations will be devoted more particularly to one aspect of systematic motion in the galactic system as revealed by observation.

In a broad and general sense the galactic system is spheroidal, as was first noted by Sir William Herschel. It was later suggested by Sir John Herschel that this characteristic might be the result of rotation about an axis perpendicular to the plane of the Milky Way, and he further suggested that the centre of the system was in the direction of the rich star fields in or near Sagittarius. But it was not until nearly a century afterwards that the vast extent of the galactic system was revealed by the investigations of Shapley. Through the discovery of Cepheid variables in the nearer globular clusters, Shapley was enabled to deduce the distances of these objects and, on the assumption that the system of clusters bore a relation of symmetry with respect to the galactic system itself, he arrived at the first reliable conception of the dimensions of the Galaxy and of the direction and distance of the centre. In his earlier work Shapley* placed the centre in the galactic equator and in the direction of galactic longitude 325° (modified later† to 327°) and, in round figures, the distance of the sun from the centre was estimated to be 15,000 parsecs. This value of the longitude, namely 325° (or alternatively, 327°), of the galactic centre is now regarded as having great weight and we regard this result as expressing a definitive characteristic of the galactic system. As we shall see later, there are dynamical methods of estimating independently the direction and distance of the centre from the sun.

11·12. *The relation of the line of vertices to the direction of the centre.*

A second feature of the galactic system concerns the systematic character of the peculiar motions of the stars in general in the neighbourhood of the sun, which we describe as star-streaming. As we have seen in previous chapters, this feature may be expressed mathematically in terms of the two-streams theory, or more conveniently for analytical purposes, in terms of the ellipsoidal theory. Studies of the motions of faint stars show that this systematic feature extends up to distances of several hundred parsecs from the sun and, as it is hardly likely to be a purely local phenomenon, we are led

* *Ap. J.* **48**, 154, 1918 (*Mt Wilson Contr.* No. 152, 1918). † *Star Clusters*, 22, 1930.

to regard star-streaming as a general dynamical characteristic of the galactic system as a whole.

Now the analysis of stellar motions places the line of vertices in the galactic equator with one vertex in galactic longitude 343°. This result is Eddington's, obtained from the analysis of the proper motions of the Boss stars, and in consequence of the homogeneity and accuracy of these proper motions must, even now, be regarded as the best determination. Other investigations point to a similar value for the longitude of the vertex, and it is now agreed that the longitude of the vertex and the longitude of the centre of the system are not quite identical but differ by about 15° to 20°. It was suggested by Turner* in 1912 that the explanation of star-streaming might be found in the character of the orbits of stars described about a distant centre. If many of these orbits are highly elongated, analogous to ellipses with eccentricities near unity, then near the sun there will be a large number of such orbits in about half of which we shall observe the approach of stars on their way towards or from the centre and, in the remainder, the recession of stars on their way towards or from the centre. The general effect will be characterised by a preferential motion in a direction indicated by the position of the centre of the system. As we have already noted, the observed direction of the vertex differs by about 20° from the direction of the galactic centre and we regard this relation of the vertex to the centre as a feature of the galaxy that calls for explanation and interpretation.

11·13. *The asymmetric drift.*

Another important feature that has been studied only in recent years concerns what is called the *asymmetric drift* and, in particular, the phenomenon of the *high-velocity stars*. On the ellipsoidal theory the peculiar linear motions of the stars as a whole are distributed with equal numbers, in opposite directions, between any two assigned values of the speeds; in particular, if we consider peculiar speeds exceeding, say, 80 km./sec., we should expect from the theory that the number of such large speeds would be the same in opposite directions of the sky. It was however noted first by B. Boss† and independently by Adams and Joy‡ that a symmetry of this nature was definitely lacking and that the directions of motion of the *stars of high velocity*, as they are now called, avoided the greater part of one hemisphere of the sky. The work of Boss, Raymond and Wilson §, Strömberg ‖ and Oort¶ has left no doubt as to the reality of this phenomenon.

* *M.N.* **72**, 387, 474, 1912.

† *Dudley Observatory, Annual Report*, 1918; also, *Popular Astronomy*, **26**, 686, 1918.

‡ *Ap. J.* **49**, 179, 1919 (*Mt Wilson Contr.* No. 163, 1919).

§ *A.J.* **35**, 26, 1923.

‖ *Ap. J.* **59**, 228, 1924 (*Mt Wilson Contr.* No. 275, 1924); *Ap. J.* **61**, 363, 1925 (*Mt Wilson Contr.* No. 293, 1925).

¶ *Groningen Publ.* No. 40, 1926.

Fig. 62, taken from a paper by Oort,* shows the numbers of space-velocities exceeding 80 km./sec. plotted against the galactic longitudes towards which they are directed; the curve is obtained by smoothing the actual data in a simple way. The particular feature of the diagram, to which attention is to be called, is the region of avoidance of the velocity vectors between galactic longitudes 10° and 100°, the centre of which is given by Oort as 57°. Broadly speaking, we infer from this diagram and other like considerations that the directions of motion of by far the greatest proportion of high-velocity stars avoid the hemisphere whose centre is (57°, 0°) in galactic coordinates. This direction, it is to be noted, is approximately 90° from the direction of the galactic centre.

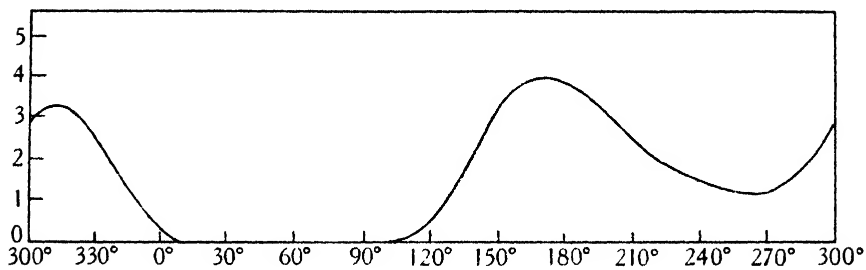

Fig. 62

The results of Strömberg's investigations may be best described by means of Fig. 63. The axes OX and OY refer to the components of the velocity vectors in the galactic plane and the scale in km. per second is indicated. Consider first the stars which give the normal solar velocity of 19·5 km./sec. The group-velocity, on the galactic plane, with respect to the sun is given by the vector joining O to the centre of the curve A, which represents the section of the velocity ellipsoid by the galactic plane; we may express the relation otherwise by saying that the solar motion with respect to these stars is the vector joining the centre of the curve A to the origin O.

The curve B refers to the short-period variables and the solar motion with respect to these stars is represented by the vector joining the centre of the curve B to the origin O. The curves C and D are interpreted in a similar way; these curves refer respectively to the high-velocity stars and the globular clusters. Strömberg's investigations included several other groups of stars and the complete observational data indicated, as is shown in Fig. 63, that the velocity vectors of the several groups B, C and D with respect to the sun lay nearly in the same direction, OP, and that the velocity dispersions, as indicated by the lengths of the axes of the curves, in the

* *Bulletin of the Astron. Institutes of the Netherlands*, **4**, 270, 1928.

several groups increased on the whole with the group-velocity. Also, it was found that the galactic longitude of the vector $\overrightarrow{PO}$ was 61°—again, approximately 90° away from the direction of the galactic centre.

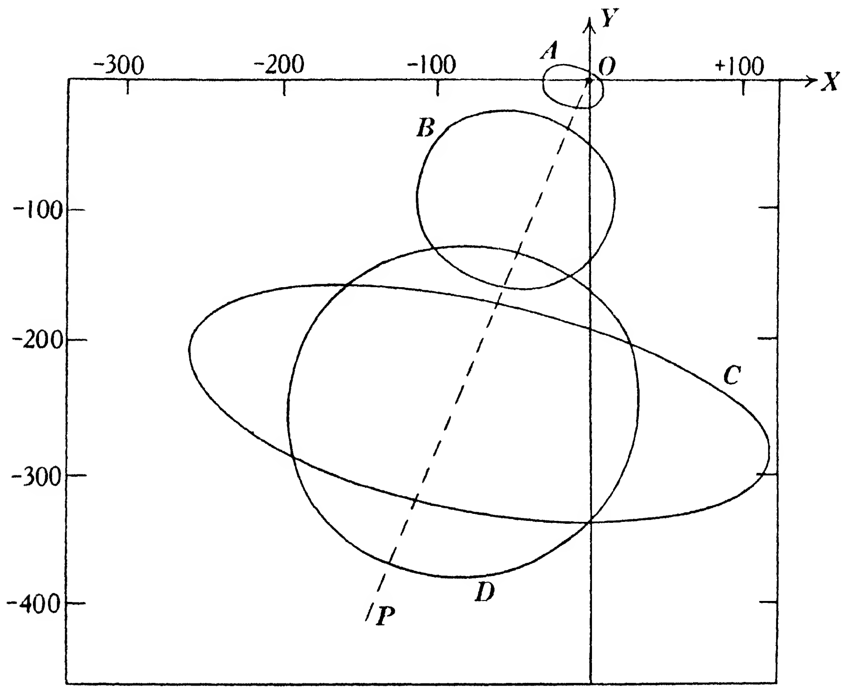

Fig. 63

11·14. *The suggestion of rotation.*

Proper motion investigations by several astronomers, notably Charlier,* Fotheringham† and Schilt,‡ had suggested a systematic motion which was interpreted as due to rotation of the galactic system about a distant centre. Charlier's values of this rotational effect in the two papers referred to were $-0''.0035$ and $-0''.0024$ per annum, while Fotheringham's and Schilt's values (the latter derived from a study of galactic Cepheids) were $-0''.0015$ and $-0''.0075$ respectively, the negative sign indicating that the supposed rotation is opposite to the direction in which galactic longitude is measured. These are all extremely small quantities and, as we shall see later, the detection and study of galactic rotation follow much more satisfactorily from radial velocity measures. However, assuming that the proper motion effect gives $-0''.004$ per annum as the average of the observed results, it follows

* *Lund Medd.* Ser. II, No. 9, p. 78, 1913; *Memoirs of the University of California*, **7**, 32, 1926.
† *M.N.* **86**, 414, 1926. ‡ *Ap. J.* **64**, 161, 1926.

that the local rotational period of the stars near the sun is, for circular orbits, about 320 million years. The motions of the stars, in the sense in which we regard galactic rotation, are analogous in many respects to those of the family of asteroids and comets around the sun.

Strömberg's work on the asymmetry of stellar motions led Lindblad* to seek a dynamical explanation. He supposed that the galactic system was composed of several subsystems, not actually separated spatially from one another, but differing according to the characteristics of their motions. Lindblad's picture of the subdivisions of the galactic system is represented in Fig. 64, where the Z-axis is the axis of rotation. The most flattened system is supposed to consist of the Milky Way clouds and the least flattened the system of globular clusters. It is further supposed that at any given point of the galaxy the rotation is greatest for the most flattened systems and smallest for the least flattened systems and that the dispersion of the internal velocities of a system increases from the most flattened to the least flattened systems. Taking the high-velocity stars to form one of these subsystems we see that, as the rotational velocity of such stars in the neighbourhood of the sun is less than the rotational velocity of the stars in the most flattened system, the high-velocity stars will appear to lag behind the stars of the latter system or, expressed otherwise, the group motion of the high-velocity stars with reference to the stars of the most flattened system near the sun will appear to be in the direction opposite to that of the general rotation. Thus we infer from Fig. 63 that the rotational direction is given by the vector PO, that is, towards galactic longitude 61°, which, as we have noted previously, is approximately at right angles to the direction of the galactic centre.

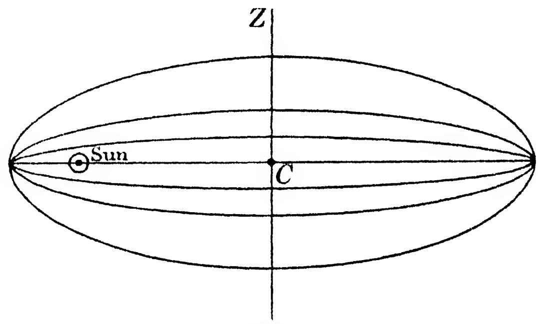

Fig. 64

In Oort's theory,† the stars are all considered to form one single system in rotation. His investigation of the high-velocity stars,‡ 250 in number, puts the lower limit at which the phenomenon becomes apparent at 65 km./sec. approximately. Let Θ_0 denote the linear velocity, due to rotation, for the centroid of the stars in the neighbourhood of the sun—it will be shown later

* *Upsala Medd.* No. 3, 1925; *Arkiv för Matematik*, **19**, A, No. 21, 1925.

† *B.A.N.* Nos. 120, 132, 133, 1927; 159, 1928.

‡ *Groningen Publ.* No. 40, 1926.

that Θ_0 is about 300 km./sec. Individual stars may have velocities greater or less than Θ_0, but there must be a limit for, otherwise, stars with velocities exceeding this limit would escape from the system. Let V_e denote the velocity of escape. Then, relative to the centroid of stars near the sun, the maximum velocity in the direction of the rotation must not exceed $V_e - \Theta_0$. Now we have seen (Fig. 62) that there is a large area of the sky, with its centre (57°, 0) in galactic coordinates, towards which no velocities exceeding 65 km./sec. are directed. With 300 km./sec. as the value of Θ_0, the velocity of escape is thus 365 km./sec. in the direction of rotation and this value must be the result of the particular physical characteristics of the galactic system as a whole. Stars moving with velocities exceeding 65 km./sec. in the direction opposite to that of rotation and relative to the centroid of the stars near the sun have velocities, relative to the centre of the system, less than 235 km./sec. and are, accordingly, to be regarded as slowly-moving stars, lagging behind the general field of stars near the sun. These are the so-called high-velocity stars, the observed distribution of which is to be expected on Oort's theory.

Oort's hypothesis is illustrated* in Fig. 65, which gives a representation of velocity vectors projected on the galactic equator. C corresponds to the centre of the galaxy and, relative to C, the rotational velocity of the centroid of the stars near the sun is represented by the vector CA, equivalent to 300 km./sec. drawn in the direction of longitude 55°. The solar motion of 19·5 km./sec. is represented similarly by the vector AS. The full-line circle with A as centre has a radius of 65 km./sec. and the large dotted circle with C as centre has a radius of 365 km./sec.

The dots represent the projections of the velocity vectors, measured from S, of all stars with parallaxes exceeding 0″·050 and brighter than $9^{m}\cdot5$ and with velocities, relative to the sun, exceeding 20 km./sec. The dots outside the full-line circle around A refer to part of the high-velocity stars tabulated by Oort, the small open circles to the remaining stars.

The diagram explains very clearly the reason why the velocity vectors of the high-velocity stars are mainly in one hemisphere of the sky only. With or without the solar motion removed from the individual velocities it will be seen that the area to the right of a straight line through A, or through S, perpendicular to CA and between the circles centred at C and A is almost infinitesimal compared with the area to the left of such a line. The velocity vectors of the high-velocity stars should accordingly be mostly distributed between longitudes 145° and 325°, and this we see from Fig. 65 is actually what the observational material shows.

For all stars belonging to the galactic system the representative points

* *B.A.N.* No. 159, p. 273, 1928.

should lie within the dotted circle with centre *C*. Oort mentions two possibly exceptional stars which many not fulfil this condition, namely the wide binary* Ci 2018–9 and RZ Cephei, a variable of the RR Lyrae type, but in

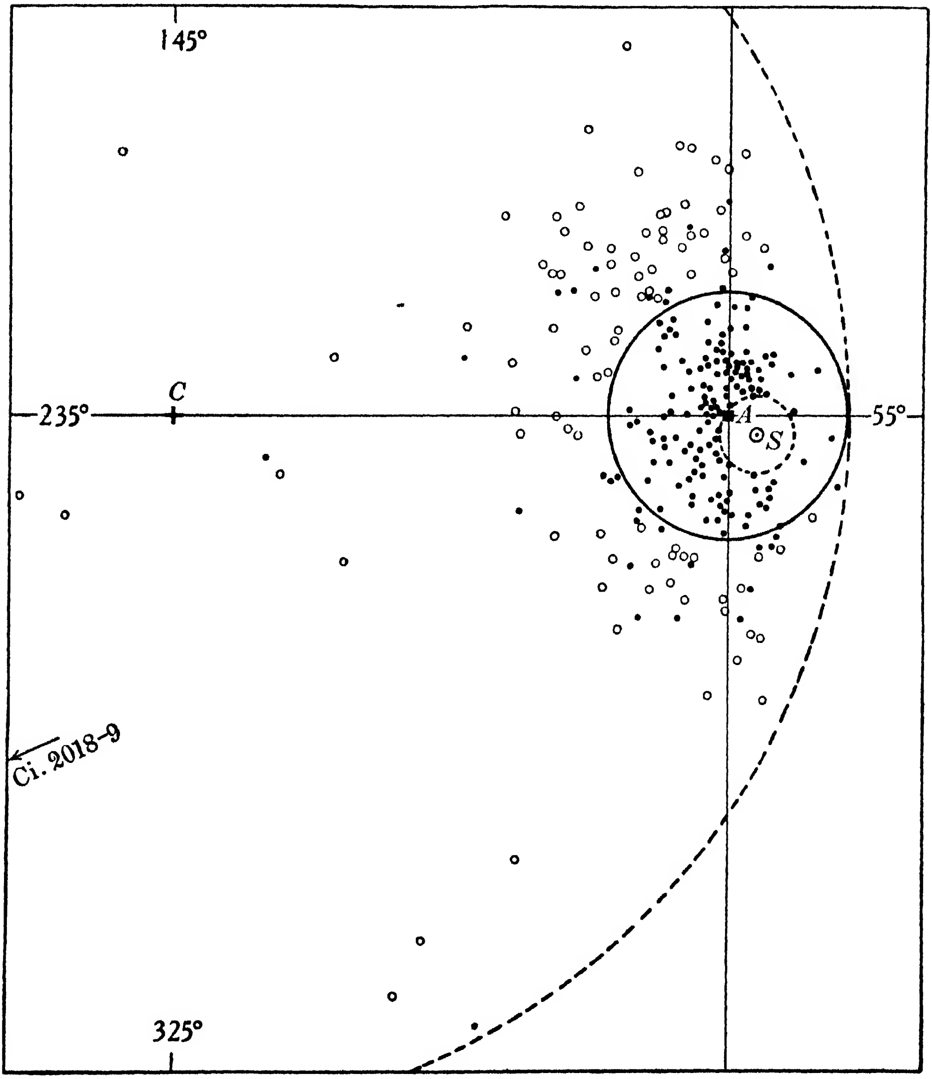

Fig. 65

each instance there is a large uncertainty as to the space-velocity relative to the sun. Considering the former star, the radial velocity is + 306 km./sec. and the annual proper motion is 3″.68; the spectroscopic and the trigonometric†

* The B.D. designation is −15° 4041, 4042.

† Schlesinger's *Catalogue of Parallaxes*, 1935 (No. 4505).

parallaxes are 0″.018 and 0″.040 respectively. The velocity relative to the sun, based on the spectroscopic parallax, is 1016 km./sec., while that based on the trigonometric parallax is 533 km./sec. The direction of motion (on the galactic plane) is indicated at the left-hand side of Fig. 65. With the smaller space-velocity the corresponding representative vectorial point falls just within the large dotted circle, in which case the star belongs to the galactic system and from the point of view suggested by Oort's diagram is not exceptional. With the larger space-velocity it must be concluded that the star is only a temporary member of the galactic system and that, in the absence of favourable close encounters with galactic stars, it will eventually escape from the system. If Shapley's estimate* of the distance (1170 parsecs) of RZ Cephei is correct, the velocity of this star, relative to the centre, C, of the system and based on proper motion data alone (the radial velocity is practically zero, -3 km./sec.), is found to be 1020 km./sec.—nearly three times the velocity of escape.

11·15. *The solar motion with respect to the globular clusters.*

Although there are about a hundred globular clusters known, radial velocities of only 26† have been determined; this paucity of information is due, of course, to the faintness of these objects in general, very long exposures being required to obtain measurable spectra. The procedure in finding the solar motion with respect to the system of clusters is the same as in section 3·41. It is convenient to make the solution in galactic coordinates.‡

If (X, Y, Z) denote the linear components of the solar motion with respect to the cluster system, the equation of condition is

$$X \cos G \cos g + Y \sin G \cos g + Z \sin g = -R, \qquad \ldots\ldots(1)$$

where (G, g) are the galactic coordinates of a cluster and R is its radial velocity.

The centroid of the stars in the neighbourhood of the sun is clearly a more convenient point of reference than the sun itself. Taking the solar motion with regard to the local stars to be 19·5 km./sec. towards the apex $(18^h, +30^\circ)$ in equatorial coordinates it is found that the linear components (X_1, Y_1, Z_1) of the solar motion in galactic coordinates are

$$X_1 = +17, \quad Y_1 = +7, \quad Z_1 = +7. \qquad \ldots\ldots(2)$$

A least-squares solution of (1) gives X, Y and Z. The linear components of motion of the centroid of the local stars are then $(X - X_1, Y - Y_1, Z - Z_1)$.

* *Harvard Bulletin*, No. 773, 1922.

† In 1936.

‡ The galactic coordinates of the individual clusters are given in Shapley's *Star Clusters*, 224, 1930.

For convenience and simplicity we shall use the term "solar motion" in the present connection to mean the motion of the centroid with respect to the system of globular clusters. From the values of $X - X_1$, $Y - Y_1$ and $Z - Z_1$ we derive in the usual way the velocity V_0 and the galactic longitude and latitude, G_0 and g_0, of the "solar motion".

The first determination of the "solar motion" was made by Lundmark.* Two years later, Strömberg† using the radial velocities of 18 clusters obtained the results (uncorrected for the local solar motion as in (2))

$$V_0 = 329 \pm 50 \text{ km./sec.}; \quad G_0 = 71°; \quad g_0 = +11°.$$

A further determination was made by Shiveshwarkar‡ using the radial velocities of 21 clusters and the equation of condition (1) with and without a K term; the value of the K term was found to be negligible. The values of X, Y and Z in km. per second, as found by Shiveshwarkar, are given, together with their probable errors:

$$X = +123 \pm 25; \quad Y = +285 \pm 45; \quad Z = +63 \pm 32,$$

which give, in conjunction with (2),

$$V_0 = 302; \quad G_0 = 69°; \quad g_0 = 10°. \qquad ......(3)$$

An almost simultaneous determination by Edmondson,§ from the radial velocities of 26 clusters, results in the values

$$V_0 = 274 \pm 40; \quad G_0 = 67° \pm 6°; \quad g_0 = +1° \pm 8°. \qquad ......(4)$$

It is to be noticed that in all these solutions, the apex of the "solar motion" is very close to the galactic equator and almost at right angles to the direction of the centre of the galaxy (longitude 325°).

If Π', Θ' and Z' denote the components of the "solar motion" in cylindrical galactic coordinates, Θ' being measured in the sense of increasing longitude, it is found|| that

$$\Pi' = +81 \pm 32; \quad \Theta' = -286 \pm 41; \quad Z' = +55 \pm 32. \qquad ......(5)$$

The values of Π' and Z' are not much larger than their probable errors and, in the absence of more adequate observational material, it is difficult to say how much weight should be attached to the definitive values of Π' and Z'. On the other hand, there can be little doubt about the reality and order of magnitude of the transverse component, Θ'. Accordingly, we arrive at the general conclusion that the centroid of the stars in the neighbourhood of the sun is moving, relative to the system of globular clusters, approximately in the galactic plane and nearly at right angles to the direction of the galactic centre, with a velocity of rather less than 300 km./sec.

* *Publ. Astr. Soc. Pacific*, **35**, 318, 1923.

† *Ap. J.* **61**, 353, 1925 (*Mt Wilson Contr.* No. 292, 1925).

‡ *M.N.* **95**, 555, 1935. § *A.J.* **45**, 1, 1935. || Shiveshwarkar, *loc. cit.*

This result is in accordance with Oort's hypothesis of galactic rotation. It is to be noted that the rotation is in a retrograde direction, that is, opposite to the direction in which galactic longitude is measured. Thus, the conclusions of the proper motion investigations by Charlier and others are, to this extent, in qualitative agreement with the direction of rotation as deduced from the radial velocities of the globular clusters.

Although the globular clusters have an approximately spherical distribution, symmetrical with the galactic equator, the previous value of the "solar motion" cannot be identified (without further consideration) with the velocity of the centroid of the stars near the sun *relative to the centre of the galactic system*, even if we assume that the centre is coincident with the centroid of the clusters; for it is reasonable to suppose that the clusters partake in the general galactic rotation. Consequently, it is necessary to interpret strictly the value of the velocity just found as the rotational linear velocity at the sun relative to the rotating system of clusters. Attempts* have been made to take into account a rotation of the clusters, but it is hardly likely that such attempts can lead to definitive results until much more observational data are available. In this sense, the values of V_0 found by the various investigators mentioned in this section are minimum values. The spherical distribution of the clusters suggests, however, that the linear rotational effect, so far as they are concerned, is small compared with the rotational effect near the sun, and in the absence of more definite information we shall regard the velocity of 300 km./sec. as the circular velocity at the sun relative to the centre of the galactic system.

11·21. *Oort's formula for the effect of galactic rotation on radial velocities.*

We consider here the simple case of the differential effect of galactic rotation on a star X, situated in the galactic plane at a distance r from the sun, S—or rather the centroid of the stars close to the sun—which we also suppose to lie in the galactic plane (Fig. 66).

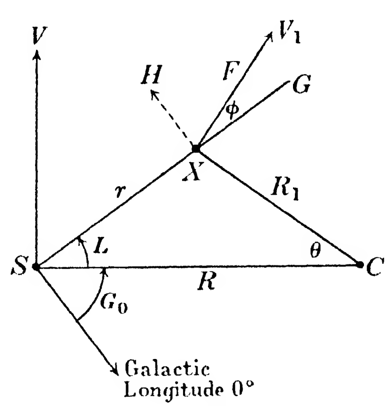

Fig. 66

Let C be the centre of the galaxy, at a distance R from S. We denote by G_0 the galactic longitude of the centre, C, measured in the direction of the arrow. It is assumed that the centroid of the stars near the sun moves in a circular orbit around C in the retrograde direction. Let V and V_1 be the

* H. Mineur, *M.N.* **96**, 61, 1935. See also comments by F. K. Edmondson, *M.N.* **96**, 636, 1936.

circular linear velocities at S and at X respectively; these are measured in the retrograde direction—the direction in which the rotation is observed to occur.

Let G be the galactic longitude of X. We write

$$L = G - G_0, \qquad \text{......(1)}$$

so that, in the figure, L is the angle CSX.

Let K be the gravitational attractive force per unit mass at S; it is assumed that K is a function of R.

Then we have
$$\frac{V^2}{R} = K(R) \qquad \text{......(2)}$$

and, similarly,
$$\frac{V_1^2}{R_1} = K(R_1). \qquad \text{......(3)}$$

Due to galactic rotation, the radial velocity, ρ, of X relative to S is given by
$$\rho = V_1 \cos\phi - V \sin L,$$

where ϕ is the angle between the radius vector SX and the direction of V_1.

Now
$$\phi = 90^\circ - L - \theta,$$

where θ is the angle XCS. Hence

$$\rho = V_1 \sin L \cos\theta + V_1 \cos L \sin\theta - V \sin L. \qquad \text{......(4)}$$

We now assume that r is small compared with R, and we shall neglect powers of r/R beyond the first.

Writing $R_1 = R + \Delta R$, we have from the relation

$$R_1^2 = R^2 - 2Rr\cos L + r^2$$

and neglecting $(r/R)^2$,
$$\Delta R = -r\cos L. \qquad \text{......(5)}$$

Now, by (3), V_1 is a function of R_1, so that

$$V_1 = f(R_1) = f(R + \Delta R)$$
$$= f(R) + \Delta R \frac{df}{dR}.$$

But, since $V = f(R)$, we obtain, using (5),

$$V_1 = V - r\frac{dV}{dR}\cos L. \qquad \text{......(6)}$$

Also,
$$\frac{\sin\theta}{r} = \frac{\sin L}{R_1},$$

and, to the approximation indicated, we have

$$\sin\theta = \frac{r}{R}\sin L.$$

It follows that we can write $\cos\theta = 1$. Using these results, together with (6), in (4) we have

$$\rho = \left(V - r\frac{dV}{dR}\cos L\right)\left(\sin L + \frac{r}{R}\sin L\cos L\right) - V\sin L,$$

whence
$$\rho = rA\sin 2L, \qquad \text{......(7)}$$

in which
$$A = \frac{1}{2}\frac{V}{R} - \frac{1}{2}\frac{dV}{dR}. \qquad \text{......(8)}$$

In terms of galactic longitudes, the formula (7) is

$$\rho = rA\sin 2(G - G_0). \qquad \text{......(9)}$$

It is to be noted that the radial velocity, ρ, due to differential rotation is proportional to r, the distance of the star at X from the sun.

The appropriate formula for a star in galactic latitude g and at a distance r from the sun is
$$\rho = rA\sin 2(G - G_0)\cos^2 g. \qquad \text{......(10)}$$

This formula is readily derivable from the more general investigation in section 11·32.

We remark that, since rA has the dimensions of a velocity, A is expressed in "km. per second per parsec".

11·22. *The formula for proper motion in galactic longitude.*

Let T denote the transverse linear velocity of X relative to S. The circular motion at X gives a component $V_1\sin\phi$ along XH (which is perpendicular to SX) and the circular motion at S gives a component $V\cos L$ parallel to XH. Hence
$$T = V_1\sin\phi - V\cos L.$$

From the formulae of the previous section we obtain

$$T = \left(V - r\frac{dV}{dR}\cos L\right)(\cos L - \sin L\sin\theta) - V\cos L$$

$$= -r\frac{V}{R}\sin^2 L - r\frac{dV}{dR}\cos^2 L$$

or
$$T = r\left\{\frac{1}{2}\frac{V}{R} - \frac{1}{2}\frac{dV}{dR}\right\}\cos 2L - r\left\{\frac{V}{2R} + \frac{1}{2}\frac{dV}{dR}\right\}.$$

Using (8) of the previous section, we have

$$T = rA\cos 2L + rB, \qquad \text{......(1)}$$

where
$$B = A - \frac{V}{R}. \qquad \text{......(2)}$$

The formula (1) shows that, due to differential galactic rotation, a star at X will have a systematic proper motion μ_G in galactic longitude, given by

$$\mu_G = A\cos 2(G - G_0) + B, \qquad \text{......(3)}$$

which is independent of the distance of the star concerned.

It will be shown later (section 11·33) that this formula holds also for a star in galactic latitude g, it being understood that μ_G refers to the increase of galactic longitude due to proper motion. In practice the component μ'_G, measured along the parallel of latitude, is more convenient. Since

$$\mu'_G = \mu_G \cos g,$$

the appropriate general formula is

$$\mu'_G = A \cos 2(G - G_0) \cos g + B \cos g. \qquad \text{......(4)}$$

From (2) the dimensions of B are the dimensions of V/R; if V is expressed in km./sec. and R in parsecs and if the proper motion μ'_G is expressed in the usual way in seconds of arc per annum, the formula (4) becomes

$$\kappa\mu'_G = A \cos 2(G - G_0) \cos g + B \cos g, \qquad \text{......(5)}$$

where $\kappa = 4{\cdot}74$.

The corresponding formula for μ_g (which will be proved in section 11·34) is

$$\kappa\mu_g = -\tfrac{1}{2}A \sin 2(G - G_0) \sin 2g. \qquad \text{......(6)}$$

It is to be noticed from Fig. 66 that the directions of T and of μ_G are in the sense of increasing longitudes. From (3), the mean proper motion of stars distributed uniformly round the galactic circle, due to the differential effects of galactic rotation, is B; thus the mean proper motion will be in the direct sense if B is positive and in the retrograde sense if B is negative. Thus the sign of B will determine the sense in which galactic rotation takes place.

11·23. *Oort's constants A and B.*

The expressions for A and B have been expressed in terms of V and R, which refer to the motion of S and its distance from the galactic centre.

Since, by 11·21 (2),

$$\frac{V^2}{R} = K, \qquad \text{......(1)}$$

we can express A and B in terms of K. From (1),

$$V\frac{dV}{dR} = \tfrac{1}{2}K + \tfrac{1}{2}R\frac{dK}{dR}.$$

Hence, A can be written

$$A = \frac{1}{4}\frac{V}{R}\left(1 - \frac{R}{K}\frac{dK}{dR}\right). \qquad \text{......(2)}$$

Now V/R is the angular rotation at S, and if we denote it by ω we have

$$A = \tfrac{1}{4}\omega\left(1 - \frac{R}{K}\frac{dK}{dR}\right). \qquad \text{......(3)}$$

The corresponding formula for B is

$$B = A - \omega. \qquad \text{......(4)}$$

We consider two simple cases.

(i) *Central mass.*

If the gravitational field at any point such as S in the galactic plane is due to a mass concentrated at the galactic centre, it follows that (K_1 being the value of K in this case)

$$K_1 = \frac{C}{R^2},$$

where C is a constant. Then, V_1 being the corresponding circular velocity* in the neighbourhood of the sun,

$$V_1^2 = \frac{C}{R},$$

and from (2) it is easily found that the corresponding value of A is given by

$$A_1 = \frac{3}{4}\frac{V_1}{R} = \tfrac{3}{4}\omega_1. \qquad \ldots\ldots(5)$$

Similarly,

$$B_1 = -\frac{1}{4}\frac{V_1}{R} = -\tfrac{1}{4}\omega_1. \qquad \ldots\ldots(6)$$

(ii) *Uniform ellipsoidal distribution of mass.*

If the gravitational field is due to an ellipsoidal system with constant star-density, or a series of such systems each with constant star-density, as Lindblad has suggested, the function K is then of the form (we write K in this case as K_2),

$$K_2 = DR,$$

where D is a positive constant. It follows that

$$A_2 = 0 \qquad \ldots\ldots(7)$$

and

$$B_2 = -\frac{V_2}{R} = -D^{\frac{1}{2}}, \qquad \ldots\ldots(8)$$

where V_2 is the corresponding circular velocity.

These formulae show that, in this case, there is no differential effect in the radial velocities and that the system would appear to rotate like a solid body.

11·24. *Numerical estimates.*

Assuming that the gravitational field is due to a central mass, we have, from 11·23 (5),

$$A = \frac{3}{4}\frac{V}{R}.$$

Taking V to be 300 km./sec. and R to be 15,000 parsecs, we find that when r = 1000 parsecs,

$$\rho = 15 \sin 2(G - G_0).$$

The Oort effect for radial velocities is thus considerable and, for the distant stars, easy to detect. With $G_0 = 325°$, the radial velocities due to galactic

* V_1 here is not to be confused with the V_1 of section 11·21.

rotation are greatest numerically for galactic longitudes 10°, 100°, 190° and 280° and are zero for longitudes 55°, 145°, 235° and 325°.

With the same assumption, $B = -\frac{1}{4}\frac{V}{R}$, so that the proper motion due to galactic rotation is given by

$$\kappa\mu_G = \frac{V}{4R}\{3\cos 2(G-G_0)-1\},$$

which is a maximum numerically when $G-G_0 = 90°$ or $270°$, and then

$$\kappa\mu_G = -\frac{V}{R},$$

whence, with the previous values of V and R,

$$\mu_G = -0''.0042.$$

Also, $B = -0''.0011$, which is somewhat smaller numerically than the estimates of Charlier and others, given in section 11·14.

Again, $\omega = 0''.0042$ per annum, leading to a period of revolution of about 300 million years for stars in the neighbourhood of the sun.

11·31. *The equations, for galactic rotation, with second-order terms.*

Oort's equations are sufficiently accurate for objects less than a thousand parsecs from the sun for, in this case, the fraction r/R is of the order 1/15 (assuming that the galactic centre is 15,000 parsecs distant from the sun); consequently, the inclusion of powers of r/R higher than the first will produce an unimportant increase of precision. But when the theory is applied to open clusters whose distances are several thousands of parsecs, the simple formulae are not quite sufficiently accurate. Formulae with terms of order higher than the first have been given by Miss P. Hayford* for objects situated on the galactic equator and by Bottlinger.†

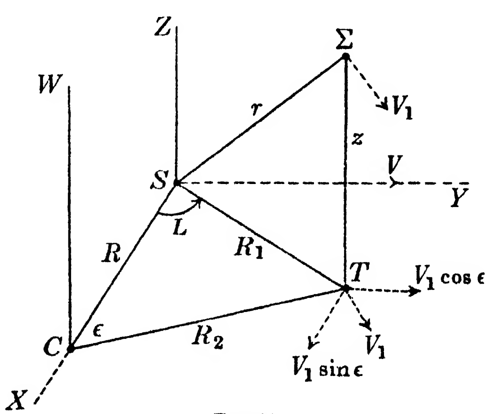

Fig. 67

Take the system of galactic axes (Fig. 67) with the sun, or rather the centroid of stars near the sun, at the origin S, the X-axis in the direction of the centre C (longitude G_0) and the Y-axis in the galactic plane and in longitude

* *Lick Obs. Bull.* **16**, 53, 1932. † *Veröff. Berlin-Babelsberg*, **10** (2), 4, 1933.

$G_0 + 90°$. We assume that the mean of the systematic motions is circular about an axis through C parallel to SZ (in the figure, this is CW). Since the motion is retrograde, the circular motion at S is along the positive direction of the Y-axis; we denote it by V. Consider a star Σ, with galactic coordinates (G, g), at a height z above the galactic plane and at a distance r from S; also, T is the projection of Σ on the galactic plane. With the usual notation we write

$$L = G - G_0,$$

where L is the angle CST.

Let V_1 denote the circular velocity of Σ about the axis CW. This is equivalent to

$$V_1 \sin \epsilon, \quad \text{parallel to } SX$$

and

$$V_1 \cos \epsilon, \quad \text{parallel to } SY,$$

where ϵ is the angle SCT.

Relative to S, the components of motion of Σ in the galactic plane are $V_1 \sin \epsilon$ and $V_1 \cos \epsilon - V$. We write

$$V' = V_1 \sin \epsilon, \qquad \text{......(1)}$$

$$V'' = V_1 \cos \epsilon - V. \qquad \text{......(2)}$$

To take account of a possible systematic motion perpendicular to the galactic plane, we denote the corresponding motion of Σ relative to S by V'''. The motion of Σ relative to S is then characterised by the components (V', V'', V''').

If (ξ, η, ζ) are the linear components corresponding to the proper motion components $\mu_G \cos g$, μ_g and to the radial velocity respectively, we have

$$\xi = -V' \sin L + V'' \cos L, \qquad \text{......(3)}$$

$$\eta = -(V' \cos L + V'' \sin L) \sin g + V''' \cos g, \qquad \text{......(4)}$$

$$\zeta = (V' \cos L + V'' \sin L) \cos g + V''' \sin g. \qquad \text{......(5)}$$

Let $SC = R$, $ST = R_1$ and $CT = R_2$.

The rotational velocity V_1 is assumed to be a function of R_2 and z. Then if

$$R_2 = R + \Delta R, \qquad \text{......(6)}$$

we write

$$V_1 = f(R_2, z) = f(R + \Delta R, z). \qquad \text{......(7)}$$

We regard r, ΔR and z as small quantities in comparison with R and we neglect powers of r/R, $\Delta R/R$ and z/R higher than the second.

From (7),

$$V_1 = f(R, 0) + \Delta R \left(\frac{\partial f}{\partial R}\right) + z\left(\frac{\partial f}{\partial z}\right) + \tfrac{1}{2}(\Delta R)^2 \left(\frac{\partial^2 f}{\partial R^2}\right) + \Delta R \,.\, z \left(\frac{\partial^2 f}{\partial R\, \partial z}\right) + \tfrac{1}{2} z^2 \left(\frac{\partial^2 f}{\partial z^2}\right), \qquad \text{......(8)}$$

the expressions for the various differential coefficients being evaluated at S.

Also

$$V = f(R, 0). \qquad \text{......(9)}$$

We set
$$A = \frac{V}{2R}\left\{1-\frac{R}{V}\left(\frac{\partial f}{\partial R}\right)\right\}, \qquad \text{......(10)}$$
$$B = A-\frac{V}{R}, \qquad \text{......(11)}$$
$$C = \frac{1}{2}\left(\frac{\partial f}{\partial z}\right), \qquad \text{......(12)}$$
$$D = \frac{1}{4}\left\{\left(\frac{\partial f}{\partial z}\right)-R\left(\frac{\partial^2 f}{\partial R\,\partial z}\right)\right\}, \qquad \text{......(13)}$$
$$E = \frac{1}{8}\left(\frac{\partial^2 f}{\partial R^2}\right), \qquad \text{......(14)}$$
$$F = \frac{1}{4}\left(\frac{\partial^2 f}{\partial z^2}\right), \qquad \text{......(15)}$$
$$a = \left(\frac{\partial f}{\partial R}\right). \qquad \text{......(16)}$$

A and B are Oort's coefficients. From (12) and (13)
$$\left(\frac{\partial^2 f}{\partial R\,\partial z}\right) = \frac{1}{R}(2C-4D). \qquad \text{......(17)}$$
The formula (8) becomes
$$V_1 = V+a\Delta R+2Cz+4E(\Delta R)^2+\frac{1}{R}(2C-4D)\Delta R\,.\,z+2Fz^2. \qquad \text{...(18)}$$

The expression for ΔR, to the second order of r/R, is obtained from the formulae
$$R_2 = (R^2-2RR_1\cos L+R_1^2)^{\frac{1}{2}}, \qquad \text{......(19)}$$
$$R_1 = r\cos g. \qquad \text{......(20)}$$
The result is
$$\frac{\Delta R}{R} = -\frac{r}{R}\cos L\cos g+\frac{1}{2}\left(\frac{r}{R}\right)^2\sin^2 L\cos^2 g. \qquad \text{......(21)}$$
Similarly,
$$\frac{R}{R_2} = 1+\frac{r}{R}\cos L\cos g+\frac{1}{2}\left(\frac{r}{R}\right)^2(3\cos^2 L-1)\cos^2 g. \qquad \text{......(22)}$$
We require the expansions for V' and V''.

(i) *Expansion for V'.*

We have
$$V' = V_1\sin\epsilon.$$
Also
$$R_2\sin\epsilon = R_1\sin L = r\sin L\cos g. \qquad \text{......(23)}$$
Hence
$$V' = V_1\cdot\frac{R}{R_2}\cdot\frac{r}{R}\sin L\cos g.$$

We thus require the expansions of V_1 and R/R_2 up to the first order only. From (18) and (22) we obtain
$$V' = \left(V-ar\cos L\cos g+V\frac{r}{R}\cos L\cos g+2Cz\right)\frac{r}{R}\sin L\cos g$$
or, using (10) and writing z as $r\sin g$,
$$V' = (V+2Ar\cos L\cos g+2Cr\sin g)\frac{r}{R}\sin L\cos g. \qquad \text{......(24)}$$

(ii) *Expansion for V''.*

$$V'' = V_1 \cos\epsilon - V$$

or

$$V'' = V_1 - V - 2V_1 \sin^2\frac{\epsilon}{2}. \qquad \ldots\ldots(25)$$

But, from (23),
$$\sin\epsilon = \frac{r}{R}\cdot\frac{R}{R_2}\sin L\cos g,$$

so that $\sin\epsilon$ is a quantity of the first order of magnitude; accordingly, to the approximation adopted,

$$V_1 \sin^2\frac{\epsilon}{2} = \frac{V}{4}\sin^2\epsilon.$$

Hence (25) becomes

$$V'' = V_1 - V - \frac{V}{2}\left(\frac{r}{R}\right)^2 \sin^2 L\cos^2 g. \qquad \ldots\ldots(26)$$

After some reduction we obtain

$$V'' = -ar\cos L\cos g + 2Cr\sin g - \frac{A}{R}r^2\sin^2 L\cos^2 g$$
$$+ 4Er^2\cos^2 L\cos^2 g - (C-2D)\frac{r^2}{R}\cos L\sin 2g + 2Fr^2\sin^2 g. \qquad \ldots\ldots(27)$$

11·32. *The formula for the radial velocity.*

Writing ρ for ζ (the radial velocity) in (5) of the previous section we obtain

$$\rho = rA\sin 2L\cos^2 g$$
$$+\sin L\left\{Cr\sin 2g - \frac{Ar^2}{4R}\cos^3 g + Er^2\cos^3 g + Fr^2\sin g\sin 2g\right\}$$
$$+\sin 2L\left\{\frac{Dr^2}{R}\sin 2g\cos g\right\}$$
$$+\sin 3L\left\{\frac{3}{4}A\frac{r^2}{R}\cos^3 g + Er^2\cos^3 g\right\}$$
$$+V'''\sin g. \qquad \ldots\ldots(1)$$

The first term in this formula gives the radial velocity effect, to the first order in r/R, for a star in latitude g, on the simple hypothesis of circular motion about the galactic axis, this motion being independent of the distance of the star from the galactic plane. If we include the possibility that the circular motion is a function of galactic latitude, the approximate Oort formula, to the first order in r/R, is, from (1),

$$\rho = rA\sin 2L\cos^2 g + rC\sin L\sin 2g. \qquad \ldots\ldots(2)$$

The complete formula (1) should be used in testing the hypothesis of galactic rotation by means of the radial velocities of very distant objects. For purposes of numerical analysis, (1) may be written in the form

$$\rho = rA\sin 2L\cos^2 g + rC\sin L\sin 2g + V'''\sin g$$
$$+ r^2(\alpha\sin L + \beta\sin 2L + \gamma\sin 3L). \qquad \ldots\ldots(3)$$

11·33. *The formula for the proper motion in galactic longitude.*

From (3) of section 11·31, ξ is the linear velocity corresponding to $\mu_G \cos g$. With r measured in parsecs, we have

$$\frac{\xi}{r} = \kappa\mu_G \cos g,$$

where $\kappa = 4{\cdot}74$. Also $\mu_G' = \mu_G \cos g,$

where μ_G' is the annual displacement along the parallel of latitude due to proper motion. Using the expressions for V' and V'' we find, after some reduction, that

$$\begin{aligned}\kappa\mu_G \equiv \kappa\mu_G' \sec g = {} & A\cos 2L + B \\ & + 2C\cos L \tan g - 2(C-D)\frac{r}{R}\sin g \\ & + r\cos L\left\{3E\cos g + 2F\sin g\tan g - \frac{3A}{4R}\cos g\right\} \\ & + r\cos 2L\left\{\frac{2D}{R}\sin g\right\} \\ & + r\cos 3L\left\{\frac{3A}{4R}\cos g + E\cos g\right\}. \qquad \text{......(1)}\end{aligned}$$

The terms in the first line are those already obtained when first-order terms in the galactic plane were alone considered. It is to be noticed that the non-periodic part of (1)—that is, the part independent of L—is

$$B - \frac{2r}{R}(C-D)\sin g, \qquad \text{......(2)}$$

which is of the form $B - Pr$ or $B + Pr$

for stars in north galactic latitudes or south galactic latitudes respectively.

11·34. *The formula for the proper motion in galactic latitude.*

In the same way, we obtain

$$\begin{aligned}\kappa\mu_g = {} & -\sin L\left\{2C\sin^2 g - \frac{rA}{4R}\sin g\cos^2 g + Er\sin g\cos^2 g + 2Fr\sin^3 g\right\} \\ & -\sin 2L\left\{\tfrac{1}{2}A\sin 2g + \frac{Dr}{R}\sin g\sin 2g\right\} \\ & -\sin 3L\left\{\frac{3rA}{4R}\sin g\cos^2 g + Er\sin g\cos^2 g\right\} \\ & + \frac{V'''}{r}\cos g. \qquad \text{......(1)}\end{aligned}$$

Except for the last term, this is a purely periodic expression in L.

If we keep only the first order terms and assume that the circular velocity at any point is independent of the distance z from the galactic plane, the simple formula for μ_g is

$$\kappa\mu_G = -\tfrac{1}{2}A \sin 2L \sin 2g, \qquad \text{......(2)}$$

as stated in section 11·22.

11·41. *Determination of Oort's constant A from radial velocities.*

We shall suppose that the rotational effect on the radial velocities is given, according to the simple theory, by the term

$$rA \sin 2(G - G_0) \cos^2 g. \qquad \text{......(1)}$$

As this term is proportional to r, the stars selected should be the most distant available and consequently O and B type stars and stars with the c-characteristic are mainly used in investigations of this kind. Moreover, as r is a variable, the stars selected for any particular solution should lie within narrow magnitude limits and in one or two sub-divisions of a spectral class; in this event, the range of variation of r may be expected to be comparatively slight and we can accordingly replace the coefficient rA in (1) by $\bar{r}A$, where $\bar{r}$ is the mean distance of the stars concerned. It is the quantity $\bar{r}A$ that will be determined from the radial velocities; with $\bar{r}$ supposed known from other sources, such as the statistical determination of parallaxes, the value of A is finally determined.

Let ρ_0 denote the observed radial velocity of a star relative to the sun. Then ρ_0 will include (i) the parallactic effect of the solar motion, (ii) the K term and (iii) the effect of galactic rotation as represented by (1). The equation of condition for a star is then written*

$$lX + mY + nZ + K + \bar{r}A \sin 2(G - G_0) \cos^2 g = \rho_0, \qquad \text{......(2)}$$

in which $(-X, -Y, -Z)$ are the components of the solar motion with respect to the centroid of the stars in the immediate neighbourhood of the sun and (l, m, n) are the direction-cosines of the star with regard to the galactic system of axes. In the usual notation,

$$l = \cos G \cos g, \quad m = \sin G \cos g, \quad n = \sin g. \qquad \text{......(3)}$$

Write

$$u = \bar{r}A \cos 2G_0, \qquad v = \bar{r}A \sin 2G_0, \qquad \text{......(4)}$$

$$p = \sin 2G \cos^2 g, \quad q = -\cos 2G \cos^2 g. \qquad \text{......(5)}$$

Then (2) becomes

$$lX + mY + nZ + K + pu + qv = \rho_0. \qquad \text{......(6)}$$

The quantities l, m, n; p and q are readily found for each star—or for the

* We omit the peculiar—or non-systematic—part of the radial velocity, which is to be regarded as an accidental error when the normal equations are formed.

mean of a group of stars in a small area of the sky. The equation (6) is solved by least-squares to determine X, Y, Z, K, u and v.

If the number of stars is small and the solar motion is known with tolerable accuracy for the type of stars concerned, the parallactic component of radial velocity can be first removed from the observed radial velocity and then the equation of condition becomes

$$K+pu+qv = \rho_1, \qquad \ldots\ldots(7)$$

where ρ_1 is the radial velocity referred to the centroid of stars near the sun.

In his earliest paper* on the subject, Oort first showed from some of the star groups utilised that the value of G_0, determined from the general solution of (7), was in accordance with Shapley's earlier value of 325°; assuming this value the equation of condition in this case is

$$K+\bar{r}A\sin 2(G-325^\circ) = \rho_1. \qquad \ldots\ldots(8)$$

In a subsequent investigation,† G_0 was determined from the solution of equations of the form of (7) and its value was found to be 324°. It is to be remarked that if u and v are derived from (6) or (7) two values of G_0 differing by 180° satisfy the formulae (4); the close agreement of one of these values with the value of the longitude of the galactic centre is to be expected on the theory of galactic rotation.

As an illustration, some of the details of Oort's solutions‡ are shown in Table 49.

Table 49. *Determination of A*

Spectrum	Magnitude	$\bar{m}$	Number of stars	$\bar{r}A$	$\bar{p}$	A
B0–B2	$3^m{\cdot}5$–$4^m{\cdot}9$	$4^m{\cdot}5$	23	+ 3	0″·0042	+0·010
,,	5·0–5·8	5·4	17	13	20	21
,,	5·9–6·9	6·1	7	15	17	21
B3–B5	3·5–4·9	4·5	75	4	66	21
,,	5·0–5·8	5·4	74	1	48	4
,,	5·9–6·9	6·3	10	16	30	39
c-stars	< 5·0	3·8	44	9	30	23
,,	5·0–5·8	5·4	26	14	19	23
,,	⩾ 5·9	6·8	23	35	09	28

The fifth column gives the value of $\bar{r}A$ as determined from the solutions of (7); the penultimate column contains the mean parallaxes of the several groups of stars, obtained by statistical or other methods. The last column contains the various values of A; the weighted mean, as given by Oort, of these and other determinations is

$$A = 0{\cdot}019 \text{ km./sec. per parsec.}$$

* *B.A.N.* No. 120, 1927. † *B.A.N.* No. 132, 1927. ‡ *Ibid.* p. 81.

The value of A published recently by Plaskett and Pearce* is

$$0{\cdot}0155 \pm 0{\cdot}0009 \text{ km./sec. per parsec,}$$

and, by J. M. Mohr,† 0·0177.

11·42. *Determination of Oort's constants from proper motions.*

The formula to be employed with the proper motions is 11·22 (5), namely,

$$\mu'_G = \frac{A}{\kappa}\cos 2(G - G_0)\cos g + \frac{B}{\kappa}\cos g. \qquad \text{......(1)}$$

If we assume that A has been determined from the radial velocities, the procedure in applying (1) is as follows. First, the effects of the solar motion are removed from the proper motions; second, the effects of errors in the precessional constants are also removed (the derivation of these errors will be discussed later in section 11·43); third, the value of G_0 may be assumed to be 325°.

Oort's first results‡ are shown in the following table:

Table 50. *Values of B/κ*

Stars	Spectrum	Number of stars	B/κ
Group I	B0–B2	105	−0″0075
II	B3–B5	336	− 42
III	c, O, N and δ Cephei stars	330	− 42

The second column shows the spectral sub-divisions etc. of the stars included in the several groups.

The weighted result is

$$B/\kappa = -0''0050 \text{ per annum} \qquad \text{......(2)}$$

or

$$B = -0{\cdot}024 \text{ km./sec. per parsec.} \qquad \text{......(3)}$$

Proceeding in a similar way, except that A and G_0 were not assumed, Oort obtained the result, from the stars of Boss's *P.G.C.*,

$$B/\kappa = -0''0023, \qquad \text{......(4)}$$

which is just about half of the value in (2).

In a recent memoir § Plaskett and Pearce determined the values of A, B and G_0, using (1), from the proper motions of 717 stars of types O to B7. The values of A and G_0 are in good agreement with the values of these con-

* *Publ. D.A.O.* (*Victoria*), **5**, 294, 1936.
† *M.N.* **92**, 592, 1932.
‡ *B.A.N.* No. 132, p. 86, 1927.
§ *Publ. D.A.O.* (*Victoria*), **5**, 299, 1936.

stants determined from the radial velocities. These latter values being now assumed, a further solution gave

$$B/\kappa = -0''.0025, \qquad \ldots\ldots(5)$$

which is close to Oort's value in (4) found from the Boss stars.

The proper motions of the stars of types B0–B5, B8–A0 and A2–A5 in Boss's catalogue have also been treated by Dyson,* by the method of harmonic analysis, but the presence of a third harmonic (in $3L$, according to our notation) with a large amplitude is disconcerting and not easily accounted for. Otherwise, the derived values of A, B and G_0 are in fair agreement with the results previously mentioned.

It is evident that further progress in determining the value of B with more satisfactory exactitude will not be possible until the proper motions of distant stars are available in larger numbers and less liable to systematic and accidental errors than at present. Meanwhile the best we can do under the circumstances is to take B, in round figures, to be given as follows:

$$B/\kappa = -0''.003, \qquad \ldots\ldots(6)$$

$$B = -0{\cdot}015 \text{ km./sec. per parsec.} \qquad \ldots\ldots(7)$$

11·43. *Precessional and equinox corrections.*

If we consider stars in low galactic latitudes, the effects of differential galactic rotation on the proper motions is wholly (or substantially so) in galactic longitude and by 11·22 (6) we can write

$$\mu_g = 0$$

so far as galactic rotation is concerned; in other words, we assume that the systematic motions in galactic latitude are negligible. This assumption is the basis of the method of deriving the precessional and equinox corrections.

Let Δp denote the correction to the value of the annual luni-solar precession (in the ecliptic) as used in determining the proper motions of the stars; in Oort's investigation, now to be described, Δp is the correction to the annual precession as used by Boss in the *P.G.C.*

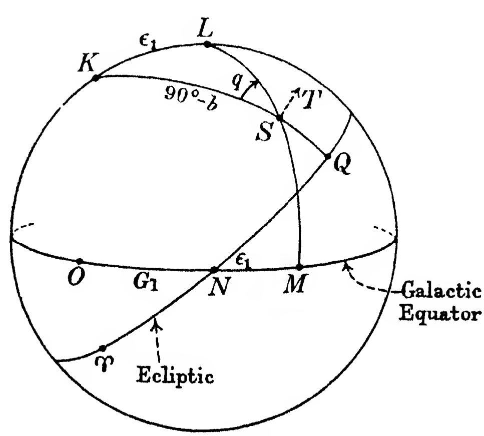

Fig. 68

In Fig. 68 let K and L be the poles of the ecliptic and of the galactic equator; let G_1 ($\equiv ON$) be the galactic longitude of the ascending node N of

* *M.N.* 90, 233, 1929.

the ecliptic on the galactic equator and ϵ_1 the inclination. Let (G, g) and (l, b) be the coordinates of a star S with reference to the usual galactic and ecliptic systems. From Fig. 68 it is easily seen that the true value of the component of proper motion of S measured along the ecliptic ΥNQ is greater than the catalogue value by the amount Δp. The true annual displacement of S along the small circle ST (parallel to the ecliptic) is thus greater than the catalogue value by $\Delta p \cos b$. Hence the component, $\Delta\mu_g'$, in galactic latitude is given by

$$\Delta\mu_g' = \Delta p \cos b \sin q, \qquad \text{......(1)}$$

where q is the angle LSK. But from the triangle KLS,

$$\cos b \sin q = \sin \epsilon_1 \sin KLS.$$

Now $$KLS = 90° + G - G_1.$$

Hence $$\Delta\mu_g' = \Delta p \sin \epsilon_1 \cos (G - G_1). \qquad \text{......(2)}$$

As in section 3·32, let $\Delta\lambda$ and Δe denote respectively the annual correction to the planetary precession and to Newcomb's adopted motion of the equinox, both measured along the celestial equator in the direction of increasing right ascension. If ϵ_2 and G_2 are respectively the inclination and the galactic longitude of the ascending node of the celestial equator on the galactic equator, the errors $\Delta\lambda$ and Δe give rise to a component of proper motion in galactic latitude given by

$$\Delta\mu_g'' = -(\Delta\lambda + \Delta e) \sin \epsilon_2 \cos (G - G_2). \qquad \text{......(3)}$$

If the effects of the solar motion are removed from the proper motions of the stars, the total systematic component, $\Delta\mu_g$, in galactic latitude due to Δp, $\Delta\lambda$ and Δe is given by

$$\Delta\mu_g = \Delta p \sin \epsilon_1 \cos (G - G_1) - (\Delta\lambda + \Delta e) \sin \epsilon_2 \cos (G - G_2). \qquad \text{......(4)}$$

This is the equation of condition used by Oort* for determining Δp and $(\Delta\lambda + \Delta e)$. With the coordinates of the galactic pole as found by Newcomb, the values of ϵ_1, ϵ_2, G_1, G_2 are as follows:

$$\epsilon_1 = 61.^\circ 2, \qquad \epsilon_2 = 63.^\circ 2,$$

$$G_1 = 153.^\circ 5, \qquad G_2 = 180.^\circ 0.$$

Using the proper motion data of the stars considered in Table 50 (section 11·42) and also of the stars in Boss's *P.G.C.*, Oort obtained the results

$$\Delta p = +0.''0113, \qquad \text{......(5)}$$

$$\Delta\lambda + \Delta e = +0.''0117. \qquad \text{......(6)}$$

* *B.A.N.* No. 132, p. 84, 1927.

From gravitational considerations* the value of $\Delta\lambda$ is taken to be $-0''.0020$ per annum; the value of Δe is then given by

$$\Delta e = +0''.0137. \qquad \ldots\ldots(7)$$

With the value of Δp in (5), the value of the annual luni-solar precession is, for epoch 1900·0,

$$50''.3821.$$

Plaskett and Pearce have made a similar investigation (*loc. cit.*).†

The corrections, $\Delta\mu_\alpha$ and $\Delta\mu_\delta$, to the components of the proper motion in right ascension and declination, when the proper motions have been calculated according to Newcomb's values of the fundamental precessional quantities, are readily found. In section 3·35 we have given Boss's results; we repeat these and add the results of Oort and of Plaskett and Pearce.

Boss:
$$\left.\begin{aligned}\Delta\mu_\alpha &= +0^s{\cdot}00021 - 0^s{\cdot}00015\sin\alpha\tan\delta\\ \Delta\mu_\delta &= -0''.0023\cos\alpha\end{aligned}\right\}. \qquad \ldots\ldots(8)$$

Oort:
$$\left.\begin{aligned}\Delta\mu_\alpha &= +0^s{\cdot}00009 - 0^s{\cdot}00030\sin\alpha\tan\delta\\ \Delta\mu_\delta &= -0''.0045\cos\alpha\end{aligned}\right\}. \qquad \ldots\ldots(9)$$

Plaskett and Pearce:
$$\left.\begin{aligned}\Delta\mu_\alpha &= +0^s{\cdot}00005 - 0^s{\cdot}00025\sin\alpha\tan\delta\\ \Delta\mu_\delta &= -0''.0038\cos\alpha\end{aligned}\right\}. \qquad \ldots\ldots(10)$$

These results are all consistent as regards the signs of the several coefficients and, considering the smallness of the quantities involved, in remarkably good agreement. It is to be remembered, however, that Boss's solution was made long before the study of galactic rotation had been begun; consequently, his results contain the effects of a systematic motion not allowed for in his equations of condition. For this reason and also because Oort's results for the Boss stars are in almost exact agreement with the results derived from the three groups of distant stars in Table 49, it is evident that the formulae (8) must be superseded by (9). The results of Plaskett and Pearce in (10) may be regarded as strongly confirmatory of Oort's numerical values. This conclusion is fortified by the accordant results of a lengthy investigation by Pariisky, Ogrodnikoff and Fessenkoff,‡ based on different material from that used by Oort.

* *B.A.N.* No. 132, p. 85, 1927. See also a note by Plaskett and Pearce (*Publ. D.A.O.* (*Victoria*), **5**, No. 4, 297, 1936) in which they quote the opinion of Fotheringham to the effect that this value of $\Delta\lambda$ is untrustworthy.

† See also Van de Kamp and Vyssotsky, *Leander McCormick Publ.* **7**, 11, 1937.

‡ "Study of the effect of known parallaxes and galactic rotation upon the determination of the constant of the luni-solar precession of Newcomb", *Publ. of the Sternberg State Astr. Institute*, **6** (1), 104, 1935 (with summary in English, p. 187). See also a paper by P. van de Kamp and A. N. Vyssotsky, *Proc. Nat. Acad. of Sciences*, **21**, 419, 1935.

11·44. *Interstellar matter.*

Within recent years definite evidence, both observational and theoretical, has been gathered to demonstrate that interstellar space (within the confines of the galaxy) is not empty but is filled with a very highly rarefied gas of substantially uniform density—at any rate up to distances of the order of a thousand parsecs from the sun. From this point of view the familiar diffuse nebulae, both luminous and dark, may be regarded as local concentrations of interstellar matter, as may also to a lesser extent the absorbing cloud considered in Chapter IX. As we shall see, it is possible to infer the presence of a galactic cloud from the principles of galactic rotation, for the interstellar matter must be expected to share in the rotation.

The observational evidence, so far as galactic rotation is concerned, is based on the behaviour of the "stationary" H and K lines of singly ionised calcium atoms (denoted by Ca II; the neutral atoms are denoted by Ca I). It was first noticed by Hartmann* that the H and K lines were present in the spectrum of δ Orionis, a spectroscopic binary of type B0, and that moreover they remained invariable in position whereas the hydrogen and helium lines, typical of a B0 star, oscillated in the period of the binary. It was subsequently shown by J. S. Plaskett† that the stationary lines—which are characterised by their narrowness and sharpness—were present in the spectra of single stars of type O and also of some Wolf-Rayet stars in the spectra of which absorption lines are normally absent. Also, the radial velocities deduced from the H and K lines differed appreciably in most instances from, and bore no relation to, the radial velocities deduced from the lines belonging normally to the types of stars concerned. The simplest explanation appeared to be that the stars were moving through a cloud in which ionised calcium atoms were present in such sufficient numbers as to produce the characteristic absorption lines of these atoms.

Later, Eddington,‡ by arguments into which we need not enter here, was led to the hypothesis that, except in the neighbourhood of the diffuse nebulae, space was filled with a highly rarefied gas with a density of $6 \, . \, 10^{-23}$ gm./cub. cm. and a kinetic temperature (as defined by the atomic or molecular speeds) of about 10,000°. Although most of the calcium atoms would be doubly ionised at this temperature, it was estimated that the number of Ca II atoms would be sufficient to produce unmistakable absorption lines provided the depth of the absorbing cloud was of the order of one or two hundred parsecs at least.

The hypothesis of interstellar matter, of substantially uniform distribution, leads to the two consequences that all distant stars, irrespective of

* *Ap. J.* **19**, 268, 1904. † *M.N.* **84**, 80, 1924. ‡ *Proc. Roy. Soc.* A, **111**, 424, 1926.

type, should show the stationary lines in their spectra and that the strength or intensity of the absorption should increase with increasing distance of the stars. In types later than B8, the stationary lines are not generally separable from the stellar H and K lines derived from the atmospheres of the stars themselves, and in types from F onwards they are invariably swallowed up in the broad stellar lines.

The detection of the interstellar lines depends then either on the complete absence of the H and K lines from the normal stellar spectrum or, if the stellar lines are present, on the sharpness of the latter and on an adequate Doppler displacement of about 50 km./sec. at least; in every instance the star concerned must be at a considerable distance from the sun.

The substantial uniformity of the interstellar cloud was first demonstrated by O. Struve* from intensity measures of the H and K lines of over 1700 stars of spectral types O to B3, the intensity being found to vary linearly with distance.

11·45. *The researches of Plaskett and Pearce on interstellar matter.*

Plaskett and Pearce† investigated the properties of the interstellar matter by considering the effects of galactic rotation; it was tacitly assumed, of course, that the cloud shared in the general rotation. The radial velocities deduced from the stationary lines for 261 stars, spread over about two-thirds of the galactic circle and mostly within 10° of the galactic equator, were first analysed for solar motion, K term and galactic rotation, the equation of condition being 11·41 (6). The solution yielded a negligible K term and a solar velocity of 19·9 km./sec., with the apex, however, about 20° (mainly in latitude) from the normal position. As the distribution of the stars is clearly unfavourable for the determination of the galactic latitude of the apex, the results were taken to indicate that the solar motion with respect to the interstellar matter is very much the same as the solar motion with respect to the stars in the neighbourhood of the sun. The radial velocities were then corrected for the usual solar motion and the equation of condition used in the solution is now 11·41 (7). The results were:

$$\bar{r}A: \quad +7{\cdot}9 \pm 0{\cdot}8 \text{ km./sec.},$$
$$K: \quad -0{\cdot}6 \pm 0{\cdot}6 \text{ km./sec.},$$
$$G_0: \quad 332^\circ \pm 6^\circ,$$

the galactic longitude, G_0, of the centre being in good agreement with Shapley's value of 327°.

The stars were then arranged in magnitude groups, an approximate way of arranging them according to distance. The solutions were then made

* *Ap. J.* **67**, 353, 1928. † *M.N.* **90**, 243, 1930.

separately for the stars and for the corresponding interstellar velocities in each individual group. Omitting the group of brightest stars, relatively few in number, we exhibit the results in Table 51.

Table 51

Group	Number of stars	$\bar{m}$	$\bar{r}A$	
			Stars	Cloud
(*a*)	45	5·6	+10·3	+5·0
(*b*)	79	6·0	+13·9	+7·7
(*c*)	119	7·1	+16·6	+8·3
(*d*)	69	7·3	+20·5	+10·1

If we adopt the value of 0·017 for A, the mean distances of the stars range from 600 parsecs in group (*a*) to 1200 parsecs in group (*d*), and the table shows that, over this range in distance, the mean distance for the stars in a group is twice the mean distance for the absorbing matter; the conclusion is that the interstellar matter is, more or less, uniformly distributed.

This result has an interesting application. Reliable estimates of the distances of novae are generally hard to obtain by ordinary methods; the measurement of the stationary H and K lines in their spectra enables us to find $\bar{r}A$ for the interstellar cloud and, with an assumed value of A (say 0·017), the distance, $2\bar{r}$, of the nova is then easily found. Estimates of the distance can also be obtained from the investigation of the intensities of the interstellar lines.*

11·51. *Dynamical results.*

Following Oort, we take as a simple working hypothesis that the gravitational field of the galaxy is due to a central mass M_1 and to a uniform distribution of matter, of total mass M_2, throughout a spheroid coextensive with the galactic system. This hypothesis can be regarded only as approximating very roughly to actual conditions, especially as regards the second part. It may be anticipated, however, that the deductions from such a hypothesis will give at least the order of magnitude of the quantities arising in the problem.

Let K_1 and K_2 be respectively the gravitational attractive forces per unit mass due to the central mass and to the spheroidal distribution. We then have

$$K = K_1 + K_2 \qquad \text{......(1)}$$

and by 11·23 (2)

$$A = \frac{1}{4}\frac{V}{R}\left(1 - \frac{R}{K}\frac{dK}{dR}\right). \qquad \text{......(2)}$$

* Cf. E. G. Williams, *M.N.* **95**, 573, 1935.

Now
$$K_1 = \frac{C}{R^2}, \qquad \text{......(3)}$$
$$K_2 = DR, \qquad \text{......(4)}$$
where C and D are constants. From these
$$\frac{dK_1}{dR} = -\frac{2}{R}K_1,$$
$$\frac{dK_2}{dR} = \frac{1}{R}K_2.$$
Hence, from (2),
$$A = \frac{3}{4}\frac{V}{R}\frac{K_1}{K}. \qquad \text{......(5)}$$
Also,
$$B = A - \frac{V}{R}. \qquad \text{......(6)}$$
Hence
$$B = -\frac{V}{4R}\left(\frac{K_1+4K_2}{K}\right). \qquad \text{......(7)}$$
From (1), (5) and (7) we readily obtain, by eliminating V/R,
$$\frac{K_1}{K} = \frac{4A}{3(A-B)}. \qquad \text{......(8)}$$
Thus, when the values of A and B are known, the ratio $K_1 : K$ can be derived from (8).

The formulae for ρ and μ_G can now be written as
$$\rho = \frac{3}{4}\frac{K_1}{K}\frac{V}{R} r \sin 2L, \qquad \text{......(9)}$$
$$\kappa\mu_G = \frac{3}{4}\frac{K_1}{K}\frac{V}{R}\left\{\cos 2L - \frac{1}{3} - \frac{4}{3}\frac{K_2}{K_1}\right\}. \qquad \text{......(10)}$$
Writing ω for V/R in (6), we find that (8) becomes
$$\omega = \frac{4}{3}\frac{K}{K_1}A. \qquad \text{......(11)}$$
Since, in the case under consideration, $K/K_1 > 1$, we have from (11)
$$\omega > \tfrac{4}{3}A. \qquad \text{......(12)}$$
We note that if the galactic rotation depends on a central mass alone, we have, clearly,
$$\omega = \tfrac{4}{3}A.$$

11·52. *The distance of the sun from the galactic centre.*

The formula 11·51 (6) shows that if A and B are found from observations as described in previous sections and the value of V is known, the distance, R, of the sun from the galactic centre can be easily found. For numerical purposes we have the following determinations of A and B by Oort* and by Plaskett and Pearce.†

Oort:	$A = 0{\cdot}019;$	$B = -0{\cdot}024.$
Plaskett and Pearce:	$A = 0{\cdot}0155;$	$B = -0{\cdot}0120.$

Taking V to be 275 km./sec. we easily find from 11·51 (6) the values of R to be, respectively,

$$6400 \text{ parsecs}, \quad \text{and} \quad 10{,}000 \text{ parsecs}. \qquad \ldots\ldots(1)$$

These values, Oort's in particular, are on the small side as compared with the distance estimated by Shapley from the study of the globular clusters even when the effect of absorption is taken into account.

11·53. *The value of K_1/K.*

Oort's value being given first as before, we derive from 11·51 (8) the two results:

$$K_1/K = 0{\cdot}59 \quad \text{and} \quad K_1/K = 0{\cdot}75. \qquad \ldots\ldots(1)$$

The inference is that the greater part of the central force is derived from the central mass.

Oort has also derived the value of K_1/K from the proper motion components μ_G, using 11·51 (10). We have

$$\kappa\mu_G = A\left\{\cos 2L - \frac{1}{3} - \frac{4}{3}\frac{K_2}{K_1}\right\}. \qquad \ldots\ldots(2)$$

In this formula, μ_G is supposed to have been freed from the effects of the solar motion. With A derived from the radial velocities, the formula (2) enables us to determine K_2/K_1; Oort's result‡ is $K_2/K_1 = 0{\cdot}29$, whence

$$K_1/K = 0{\cdot}78,$$

which points again to the predominance of the central mass in producing the rotational velocity.

11·54. *Formula for K_2.*

We consider first an ellipsoid of uniform density ρ, and of mass M_2, the semi-axes being a, b and c.

The components (X, Y, Z) of acceleration§ at any point $P(\xi, \eta, \zeta)$ within the ellipsoid are given by

$$X = -G\alpha\rho\xi, \quad Y = -G\beta\rho\eta, \quad Z = -G\gamma\rho\zeta,$$

where G is the gravitational constant and α, β and γ are defined by

$$\frac{\alpha}{2\pi abc} = \int_0^\infty \frac{du}{(a^2+u)\,Q}, \text{ etc.},$$

in which
$$Q^2 = (a^2+u)(b^2+u)(c^2+u).$$

* *B.A.N.* **4**, 79, 1927. † *Publ. D.A.O. (Victoria)*, **5**, 299, 1936. ‡ *B.A.N.* No. 120, p. 282, 1927.
§ See, for example, Routh, *Analytical Statics*, Vol. II, p. 106 (2nd ed. 1902).

If now $b = a$ and $a > c$, we find that

$$2\alpha + \gamma = 4\pi \qquad \ldots\ldots(1)$$

and

$$2\alpha a^2 + \gamma c^2 = \frac{4\pi a^2 c}{(a^2 - c^2)^{\frac{1}{2}}} \tan^{-1}\left\{\frac{(a^2 - c^2)^{\frac{1}{2}}}{c}\right\}. \qquad \ldots\ldots(2)$$

For the galaxy, a is several times larger than c, so that (2) becomes approximately

$$2\alpha a^2 + \gamma c^2 = 2\pi^2 ac. \qquad \ldots\ldots(3)$$

Hence from (1) and (3)

$$\alpha = \frac{\pi c(\pi a - 2c)}{a^2 - c^2}$$

or, with sufficient accuracy,

$$\alpha = \frac{\pi^2 c}{a}. \qquad \ldots\ldots(4)$$

The mass, M_2, is given by

$$M_2 = \tfrac{4}{3}\pi a^2 c\rho.$$

Hence, we obtain

$$\alpha\rho = \frac{3\pi M_2}{4a^3}. \qquad \ldots\ldots(5)$$

Suppose now that the ξ-axis passes through the sun, which we assume to be in the galactic plane; the coordinates of the sun are now $(R, 0, 0)$ and

$$X = -G\alpha\rho R = -\frac{3\pi G M_2 R}{4a^3}$$

or, in the notation of section 11·23,

$$K_2 = \frac{3\pi G M_2 R}{4R_1^3}, \qquad \ldots\ldots(6)$$

in which we have written R_1 for a.

The most plausible estimate of the diameter of the galactic system is about 30,000 parsecs. We accordingly take R_1 to be given by

$$R_1 = 15{,}000 \text{ parsecs}, \qquad \ldots\ldots(7)$$

so that, if $R = 10{,}000$ parsecs,

$$\frac{R_1}{R} = \tfrac{3}{2}. \qquad \ldots\ldots(8)$$

11·55. *An estimate of the mass of the galactic system.*

The formulae in the preceding sections can be used to give some idea of the mass of the galactic system if we base our hypothesis on a central mass and a uniform spheroidal distribution.

From 11·23 (1), we have

$$V^2 = KR, \qquad \ldots\ldots(1)$$

and from 11·23 (3),

$$A = \frac{\omega}{4}\left(1 - \frac{R\,dK}{K\,dR}\right), \qquad \ldots\ldots(2)$$

where $\omega = V/R$.

Using (1), we find that (2) becomes

$$A = \frac{\omega}{4}\left(1 - \frac{1}{\omega^2}\frac{dK}{dR}\right). \qquad \ldots\ldots(3)$$

Also $$K \equiv K_1 + K_2 = \frac{GM_1}{R^2} + pGM_2 R, \qquad \text{......(4)}$$

in the second term of which we have written p for $3\pi/4R_1^3$ in the expression for K_2 given by 11·54 (6).

From (4), $$\frac{dK}{dR} = -\frac{2GM_1}{R^3} + pGM_2, \qquad \text{......(5)}$$

and from (1) and (4), $$V^2 = \frac{GM_1}{R} + pGM_2 R^2,$$

from which $$pGM_2 = \omega^2 - \frac{GM_1}{R^3}. \qquad \text{......(6)}$$

Consequently, from (5) and (6),

$$\frac{dK}{dR} = \omega^2 - \frac{3GM_1}{R^3},$$

so that (3) gives $$A = \frac{3GM_1}{4\omega R^3}, \qquad \text{......(7)}$$

whence $$M_1 = \frac{4\omega R^3 A}{3G}. \qquad \text{......(8)}$$

From (6) and (8), we obtain, on inserting the expression for p,

$$M_2 = \frac{4\omega(3\omega - 4A) R_1^3}{9\pi G}. \qquad \text{......(9)}$$

Hence $$\frac{M_2}{M_1} = \frac{1}{\pi}\left(\frac{R_1}{R}\right)^3 \left(\frac{\omega}{A} - \frac{4}{3}\right). \qquad \text{......(10)}$$

It may be noted that (10) is consistent with the inequality in 11·51 (12).

The formula (10) may be written in a slightly different form; from 11·51 (11),

$$\frac{\omega}{A} = \frac{4}{3}\left(\frac{K_1 + K_2}{K_1}\right).$$

Hence $$\frac{M_2}{M_1} = \frac{4}{3\pi}\left(\frac{R_1}{R}\right)^3 \frac{K_2}{K_1}. \qquad \text{......(11)}$$

For the purpose of making an estimate of the mass of the galactic system, we shall use Plaskett and Pearce's values for A and R which are respectively 0·0155 km./sec. per parsec and 10,000 parsecs. We also take R_1/R to be 3/2 as given in 11·54 (8).

We adopt the year, the sun's mass and the astronomical unit of distance as the units of time, mass and length respectively; in this system

$$G = 4\pi^2.$$

Taking V to be 275 km./sec., we have in the new units

$$V = \frac{275}{\kappa},$$

where $\kappa = 4{\cdot}74$; and

$$R = 10^4 \operatorname{cosec} 1''$$
$$= 2\,.\,10^9 \text{ astr. units, approximately.}$$

Also $$A = \frac{0{\cdot}0155}{\kappa} \text{ astr. units per annum per parsec}$$

$$= \frac{0{\cdot}0155}{2\kappa\,.\,10^5} \text{ astr. units per annum per astr. unit} \quad \text{......(12)}$$

and $$\omega = \frac{275}{2\kappa\,.\,10^9}. \quad \text{......(13)}$$

From (8), we find $$M_1 = 13\,.\,10^{10}, \quad \text{......(14)}$$

and, from (10) and (14), $$M_2 = 6\,.\,10^{10}. \quad \text{......(15)}$$

This last result may also be derived from (11) and the second value of K_1/K given in 11·53 (1); it follows that

$$\frac{K_2}{K_1} = \frac{1}{3}.$$

Thus the total mass of the galactic system is approximately

$$2\,.\,10^{11} \quad \text{......(16)}$$

in terms of the solar mass as unit.

We conclude that, on the hypotheses stated, the observed features of galactic rotation may be ascribed to a highly concentrated central mass together with a uniform spheroidal distribution of matter (including stars and the cosmic cloud) whose total mass, as given in (16), may be regarded as a rough estimate; it is to be remembered that the value of R_1 adopted in the calculations is somewhat uncertain.

The period, P, of revolution in circular orbits for stars in the neighbourhood of the sun, is given by $$P = 2\pi/\omega;$$

with the value of ω adopted in (13), we find that, approximately,

$$P = 2\,.\,10^8 \text{ years.}$$

CHAPTER XII

THE DYNAMICS OF THE GALAXY

12·11. *Formulae for Oort's constants.*

In the previous chapter we considered the phenomenon of galactic rotation and, in particular, we derived the theoretical expressions for Oort's constants A and B—11·21 (8) and 11·22 (2)—in the form

$$A = \frac{1}{2}\frac{V}{R} - \frac{1}{2}\frac{dV}{dR}, \qquad \ldots\ldots(1)$$

$$B = A - \frac{V}{R} = A - \omega, \qquad \ldots\ldots(2)$$

in which the angular velocity ω, at the distance R from the galactic centre, is given by V/R. In these formulae V and ω refer to the direction in which the rotation actually occurs, that is, in the retrograde direction—or opposite to that in which galactic longitude is measured.

We shall assume that the galaxy is a system with uniaxial symmetry. In terms of cylindrical coordinates and in the notation of section 10·41, we represent the systematic transverse linear velocities at S and X (Fig. 69) by Θ_0 and Θ_1 and the distances of S and X from the galactic centre, C, by ϖ and ϖ_1, where S refers to the centroid of the stars in the neighbourhood of the sun and X is any star more remote. The figure is supposed to be in the plane of the galactic equator with the positive direction of the axis of symmetry through C and perpendicular (upwards) to the plane of the paper. The directional sense of Θ_0, and of Θ_1, is thus the same as that in which the azimuthal angle, θ, of the cylindrical coordinates is measured; also, the sense in which θ is measured is the same as that of increasing galactic longitude.

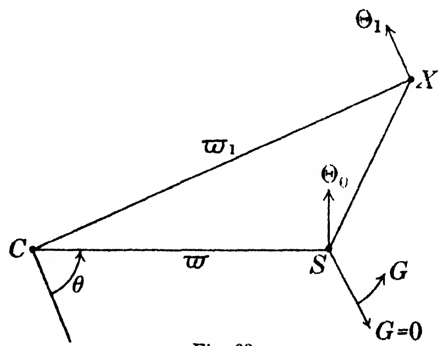

Fig. 69

We denote the angular velocity at S by ω_0, so that

$$\omega_0 = \frac{\Theta_0}{\varpi}. \qquad \ldots\ldots(3)$$

Also, Θ_0 and V in (1) are the transverse linear velocities at S but measured in opposite directions; hence

$$\Theta_0 = -V, \quad \text{and} \quad \omega_0 = -\omega. \qquad \text{......(4)}$$

We then have from (1) and (2)

$$A = \frac{1}{2}\frac{d\Theta_0}{d\varpi} - \frac{1}{2}\frac{\Theta_0}{\varpi}, \qquad \text{......(5)}$$

$$B = A + \omega_0 = \frac{1}{2}\frac{d\Theta_0}{d\varpi} + \frac{1}{2}\frac{\Theta_0}{\varpi}. \qquad \text{......(6)}$$

As before, we denote by K the gravitational attractive force per unit mass at S; if, as in Oort's theory, the systematic transverse velocity is due to the gravitational force K, then

$$K = \frac{\Theta_0^2}{\varpi}. \qquad \text{......(7)}$$

It is easily seen from (5), (6) and (7) that, in this case,

$$A = -\frac{\Theta_0}{4\varpi}\left(1 - \frac{\varpi}{K}\frac{dK}{d\varpi}\right), \qquad \text{......(8)}$$

$$B = \frac{\Theta_0}{4\varpi}\left(3 + \frac{\varpi}{K}\frac{dK}{d\varpi}\right), \qquad \text{......(9)}$$

or, by (3),

$$A = -\frac{\omega_0}{4}\left(1 - \frac{\varpi}{K}\frac{dK}{d\varpi}\right) = -\frac{1}{4\omega_0}\left(\omega_0^2 - \frac{dK}{d\varpi}\right), \qquad \text{......(10)}$$

$$B = \frac{\omega_0}{4}\left(3 + \frac{\varpi}{K}\frac{dK}{d\varpi}\right). \qquad \text{......(11)}$$

Another expression for A is obtained as follows:

$$2A = \frac{d\Theta_0}{d\varpi} - \frac{\Theta_0}{\varpi} = \varpi\frac{d}{d\varpi}\left(\frac{\Theta_0}{\varpi}\right) = \varpi\frac{d\omega_0}{d\varpi}, \qquad \text{......(12)}$$

so that

$$A = \frac{\varpi}{4\omega_0}\frac{d}{d\varpi}(\omega_0^2) = \frac{\varpi}{4\omega_0}\frac{d}{d\varpi}\left(\frac{K}{\varpi}\right). \qquad \text{......(13)}$$

It is to be remarked—from (11)—that $B \neq 0$ unless $K(\varpi)$ varies as ϖ^{-3}.

Also, from (13), since ω_0 is negative, the actual rotation being in the retrograde direction, A is positive if $\frac{d}{d\varpi}\left(\frac{K}{\varpi}\right)$ is negative. If $K(\varpi)$ is given by

$$K(\varpi) = \frac{C}{\varpi^2} + D\varpi, \qquad \text{......(14)}$$

as for a central condensation combined with a spheroidal distribution of matter,

$$\frac{d}{d\varpi}\left(\frac{K}{\varpi}\right) = -\frac{3C}{\varpi^4}. \qquad \text{......(15)}$$

Accordingly, in this case, A is positive.

12·12. *Relation between galactic rotation and star-streaming.*

We have seen in section 10·42 that, for a system with uniaxial symmetry in a steady state, the space-velocity frequency function f is given by

$$f\{\Pi^2+\lambda^2(\Theta-\Theta_0)^2+Z^2-2V_1\}, \qquad \text{......(1)}$$

where Π, Θ, Z are the velocity components in cylindrical coordinates and

$$\lambda^2 = 1+k_2\varpi^2, \qquad \text{......(2)}$$

$$\Theta_0 = -\frac{k_1\varpi}{1+k_2\varpi^2}, \qquad \text{......(3)}$$

$$V_1 = V+\frac{1}{2}\frac{k_1^2\varpi^2}{1+k_2\varpi^2}, \qquad \text{......(4)}$$

V being the potential.*

The distribution of velocities in (1) is spheroidal with the equal and longer axes in the ϖ and z directions, k_2 being positive. Now Θ_0 is the systematic linear rotational velocity at the point of the system under consideration, and if this point be taken at S in Fig. 69, we have from 12·11 (5)

$$2A = \frac{d\Theta_0}{d\varpi}-\frac{\Theta_0}{\varpi}.$$

Also
$$2B = \frac{d\Theta_0}{d\varpi}+\frac{\Theta_0}{\varpi}.$$

From (3) and (2),
$$\frac{d\Theta_0}{d\varpi} = \frac{1}{\lambda^4}(k_1k_2\varpi^2-k_1).$$

Hence
$$2A = \frac{2k_1k_2\varpi^2}{\lambda^4}, \quad 2B = -\frac{2k_1}{\lambda^4},$$

from which
$$\frac{B}{B-A} = \frac{1}{\lambda^2},$$

and in consequence of 12·11 (6),

$$\frac{1}{\lambda^2} = 1+\frac{A}{\omega_0}. \qquad \text{......(5)}$$

Now in (1), $1:\lambda$ is the ratio of the semi-minor axis to the semi-major axis for the velocity spheroid; this ratio can, accordingly, be derived from (5).

From 11·55 (12) and 11·55 (13) the values of A and ω_0—we remember that $\omega_0 = -\omega$ by 12·11 (4)—have been given as

$$A = \frac{0{\cdot}0155}{2\kappa\,.\,10^5}, \quad \omega_0 = -\frac{275}{2\kappa\,.\,10^9}. \qquad \text{......(6)}$$

* V is not to be confused with the velocity V occurring in the definition of the Oort constants.

These may be regarded as typical numerical results with a substantial claim to reliability. Inserting these in (5) we obtain

$$\frac{1}{\lambda} = 0{\cdot}66, \qquad \text{......(7)}$$

which is in good agreement with the usual values of the ratio of the semi-axes found by the methods of Chapter V (compare, for example, the value $K/H = 0{\cdot}63$ mentioned in 5·4).

This accordance must not, however, be unduly stressed, for (5) is derived from a model of a spheroidal galaxy in a dynamically steady state embodying the mathematical implication that the semi-axes of the velocity ellipsoid are equal in the direction of the vertex (the ϖ-direction) and in the direction perpendicular to the plane of symmetry (the z-direction). So far, the bulk of the observational evidence is contrary to this equality. If we adopt a short time scale for the age of the galaxy, stars in the neighbourhood of the sun will have made but a few revolutions round the galactic centre and it may be anticipated that the steady state contemplated in the theory and leading to (1) is not quite realised in the actual galactic system. The model which we are now considering must then be regarded as a kind of first approximation to the galaxy itself.

12·13. *Formulae for Schwarzschild's velocity-law.*

If the function in 12·12(1) is exponential as in Schwarzschild's law, we can write

$$f = Pe^{-\left\{\frac{\Pi^2+Z^2-2V_1}{a^2}+\frac{(\Theta-\Theta_0)^2}{b^2}\right\}}, \qquad \text{......(1)}$$

where P and a are constants, or

$$f = f_1 e^{-\left\{\frac{\Pi^2+Z^2}{a^2}+\frac{(\Theta-\Theta_0)^2}{b^2}\right\}}, \qquad \text{......(2)}$$

where

$$f_1 = Pe^{2V_1/a^2}. \qquad \text{......(3)}$$

Also,

$$b = \frac{a}{\lambda}. \qquad \text{......(4)}$$

If $\overline{\Pi}$ denotes the mean of the peculiar *speeds* in the ϖ direction we have, in the notation of section 10·42,

$$\overline{\Pi} = \frac{\int_0^\infty c_1^3 e^{-c_1^2/a^2}\, dc_1}{2\int_0^\infty c_1^2 e^{-c_1^2/a^2}\, dc_1},$$

whence

$$\overline{\Pi} = \frac{a}{\sqrt{\pi}}. \qquad \text{......(5)}$$

Similarly,

$$\overline{\Theta - \Theta_0} = \frac{b}{\sqrt{\pi}} \quad \text{and} \quad \overline{Z} = \frac{a}{\sqrt{\pi}}. \qquad \text{......(6)}$$

By 10·42 (19) the star-density, ν, is given by

$$\nu = \frac{4\pi f_1}{\lambda}\int_0^\infty c_1^2 e^{-c_1^2/a^2}\,dc_1,$$

from which
$$\nu = \frac{P}{\lambda}(\pi a^2)^{\frac{3}{2}}\, e^{2V_1/a^2}. \qquad \ldots\ldots(7)$$

We shall require these formulae later.

12·14. *Lindblad's derivation of the ellipsoidal distribution of stellar velocities.*

Relative to the centroid of stars in the neighbourhood of the sun the peculiar stellar velocities, in one coordinate, have an average value, taking all spectral types together, of the order of 10 km./sec. As this is but a very small fraction of the linear rotational velocity near the sun, the orbits of the stars as viewed from the galactic centre do not differ very markedly from circles, on the average. Lindblad's investigation,* which will now be described, is concerned with these orbits.

We consider motions in the galactic plane only. Consider a point, S, at a distance ϖ from the galactic centre, C (Fig. 70). The angular rotational velocity at S, relative to fixed axes CX and CY, will be denoted by ω, measured in the direction of increasing galactic longitude. As before, we denote by $K(\varpi)$ the attractive force per unit mass at a distance ϖ from C; then
$$\varpi\omega^2 = K(\varpi). \qquad \ldots\ldots(1)$$

Let P be a star near S with coordinates (ξ, η) relative to the axes SB and SD as shown in Fig. 70. The coordinates of P with respect to the axes CB and CE are $(\varpi+\xi, \eta)$; these axes form a rotating system, the angular motion being ω.

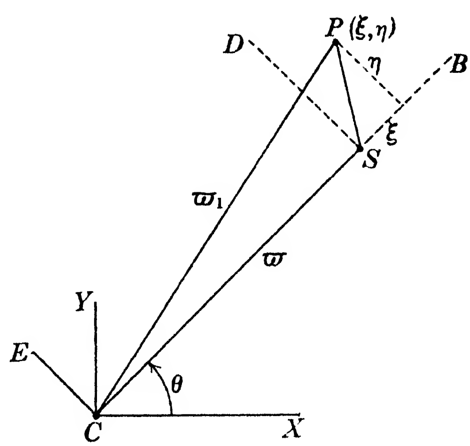

Fig. 70

Let ϖ_1 denote the distance CP. In general, the motion of P about C will be somewhat different from that in a circular orbit of radius ϖ_1. The attractive force at P, per unit mass, has components

$$-\frac{\varpi+\xi}{\varpi_1}K(\varpi_1) \quad \text{and} \quad -\frac{\eta}{\varpi_1}K(\varpi_1)$$

parallel to CB and CE respectively.

* *Arkiv för Matematik, Astronomi och Fysik*, Bd. **20**, A, No. 17, 1927; *Upsala Medd.* No. 26, 1927.

We accordingly have the following equations for the motion of P relative to C:

$$\frac{d^2}{dt^2}(\varpi+\xi)-2\omega\dot{\eta}-\omega^2(\varpi+\xi) = -\frac{\varpi+\xi}{\varpi_1}K(\varpi_1), \qquad \text{......(2)}$$

$$\ddot{\eta}+2\omega\frac{d}{dt}(\varpi+\xi)-\omega^2\eta = -\frac{\eta}{\varpi_1}K(\varpi_1). \qquad \text{......(3)}$$

We assume that ξ/ϖ and η/ϖ are small and we shall neglect squares and higher powers of these quantities.

In (2) and (3), ϖ and ω are constants, since we assume circular motion for S.

Now $$\varpi_1^2 = (\varpi+\xi)^2+\eta^2,$$

from which, with the approximation in view,

$$\varpi_1 = \varpi+\xi.$$

Then $$K(\varpi_1) = K(\varpi)+\xi\frac{dK(\varpi)}{d\varpi}.$$

Making use of this result and also of (1), we find that (2) and (3) become

$$\ddot{\xi}-2\omega\dot{\eta}-\xi\left\{\omega^2-\frac{dK(\varpi)}{d\varpi}\right\} = 0, \qquad \text{......(4)}$$

$$\ddot{\eta}+2\omega\dot{\xi} = 0. \qquad \text{......(5)}$$

Now, by 12·11 (10), Oort's coefficient A, as defined for the point S, is given by

$$A = -\frac{1}{4\omega}\left\{\omega^2-\frac{dK(\varpi)}{d\varpi}\right\}. \qquad \text{......(6)}$$

Hence (4) can be written $$\ddot{\xi}-2\omega\dot{\eta}+4\omega A\xi = 0. \qquad \text{......(7)}$$

From (5), $$\dot{\eta} = 2\omega(\xi_1-\xi), \qquad \text{......(8)}$$

where ξ_1 is a constant of integration. Hence (7) becomes

$$\ddot{\xi}+4\omega(\omega+A)\xi = 4\omega^2\xi_1. \qquad \text{......(9)}$$

If $K(\varpi)$ is defined as in 12·11 (14), we have by 12·11 (13)

$$\omega(\omega+A) = \omega^2+\frac{\varpi}{4}\frac{d}{d\varpi}\left(\frac{K}{\varpi}\right) = \frac{K}{\varpi}+\frac{\varpi}{4}\frac{d}{d\varpi}\left(\frac{K}{\varpi}\right)$$
$$= \frac{C}{4\varpi^3}+D.$$

The coefficient of ξ in (9) is thus positive. Let

$$g^2 = 4\omega(\omega+A). \qquad \text{......(10)}$$

Then (9) becomes $$\ddot{\xi}+g^2\xi = 4\omega^2\xi_1. \qquad \text{......(11)}$$

The solution of this equation is

$$\xi = c\sin g(t-t_0)+\frac{4\omega^2}{g^2}\xi_1, \qquad \text{......(12)}$$

in which c and t_0 are constants of integration.

12·15. In Fig. 70 let θ denote the angle which the rotating axis CSB makes with the fixed axis CX. If (X, Y) are the coordinates of P with respect to the fixed axes,

$$\left.\begin{aligned} X &= (\varpi+\xi)\cos\theta-\eta\sin\theta \\ Y &= (\varpi+\xi)\sin\theta+\eta\cos\theta \end{aligned}\right\}, \qquad \text{......(1)}$$

from which, since $\dot{\theta}=\omega$,

$$\left.\begin{aligned} \dot{X} &= \dot{\xi}\cos\theta-\dot{\eta}\sin\theta-\omega Y \\ \dot{Y} &= \dot{\xi}\sin\theta+\dot{\eta}\cos\theta+\omega X \end{aligned}\right\}. \qquad \text{......(2)}$$

Since the force is central, the radius vector CP will sweep out areas at a constant rate, say $\frac{1}{2}h$. Now

$$h = X\dot{Y}-\dot{X}Y,$$

which is readily found by (1) and (2) to become

$$h = (\varpi+\xi)\dot{\eta}-\dot{\xi}\eta+\omega\varpi_1^2.$$

But h is also given, by considering the radius vector CS, by

$$h = \omega\varpi^2.$$

Hence, to the first order of ξ/ϖ and η/ϖ,

$$(\varpi+\xi)\dot{\eta}-\dot{\xi}\eta+2\omega\varpi\xi = 0. \qquad \text{......(3)}$$

If $(\dot{\eta})$ denotes the value of $\dot{\eta}$ when $\xi=\eta=0$, we have, from (3),

$$(\dot{\eta}) = 0.$$

It follows from 12·14 (8) that $\qquad \xi_1 = 0.$

Hence $\qquad \dot{\eta}+2\omega\xi = 0 \qquad$ ......(4)

and the solutions of 12·14 (11) and 12·14 (8) are

$$\xi = c\sin g(t-t_0), \qquad \text{......(5)}$$

$$\eta = \frac{2\omega c}{g}\cos g(t-t_0). \qquad \text{......(6)}$$

Thus the orbit of P relative to S is the ellipse, with its centre at S,

$$\xi^2+\frac{g^2}{4\omega^2}\eta^2 = c^2 \qquad \text{......(7)}$$

and the relative velocity components $\dot{\xi}$ and $\dot{\eta}$ satisfy the relation

$$\dot{\xi}^2+\frac{g^2}{4\omega^2}\dot{\eta}^2 = g^2c^2. \qquad \text{......(8)}$$

12·16. We consider now the orbits passing through a small volume of space surrounding the sun, S_0 (Fig. 71). Referred to S and the axes SB and SD, the coordinates of a star at, or very close to, S_0 are (ξ, η) and, to the order of approximation adopted,

$$\varpi_0 = \varpi+\xi, \qquad \text{......(1)}$$

where ϖ_0 is the distance CS_0.

Let ξ_0, $\dot{\eta}_0$ denote the components of velocity of a star with respect to the axes S_0B_0 and S_0D_0. Since ξ is the component of velocity parallel to CB we have—neglecting small quantities of the second order—

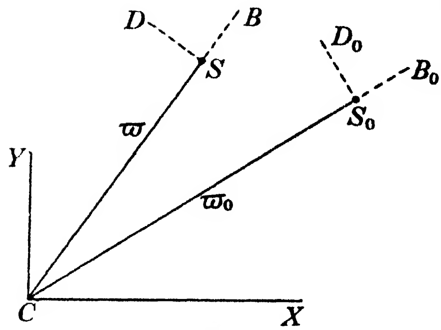

Fig. 71

$$\xi_0 = \xi. \qquad \text{......(2)}$$

Also, referred to a fixed axis with which CB is momentarily coincident, the velocity of the star perpendicular to this fixed axis is $\dot{\eta} + \omega\varpi$. Similarly, referred to a fixed axis with which CB_0 is momentarily coincident, the velocity of the star perpendicular to this fixed axis is $\dot{\eta}_0 + \omega_0\varpi_0$, where ω_0 is the value of ω at S_0. Since CB and CB_0 are inclined at a small angle we obtain, on neglecting small quantities of the second order as before,

$$\dot{\eta} + \omega\varpi = \dot{\eta}_0 + \omega_0\varpi_0,$$

which can be written with sufficient accuracy

$$\dot{\eta}_0 = \dot{\eta} + \varpi(\omega - \omega_0). \qquad \text{......(3)}$$

Now, ω is a function of ϖ and ω_0 is a function of ϖ_0; hence

$$\omega - \omega_0 = (\varpi - \varpi_0)\frac{d\omega}{d\varpi}.$$

Hence, from (1) and 12·15 (4), the formula (3) becomes

$$\dot{\eta}_0 = \dot{\eta}\left(1 + \frac{\varpi}{2\omega}\frac{d\omega}{d\varpi}\right). \qquad \text{......(4)}$$

But from 12·11 (12), A is defined for S by

$$A = \frac{\varpi}{2}\frac{d\omega}{d\varpi}. \qquad \text{......(5)}$$

Hence (4) becomes
$$\dot{\eta}_0 = \dot{\eta}\left(1 + \frac{A}{\omega}\right). \qquad \text{......(6)}$$

Inserting the values of ξ and $\dot{\eta}$, given by (2) and (6), in 12·15 (8) and remembering that g is given by 12·14 (10), we find that ξ_0 and $\dot{\eta}_0$ satisfy the relation

$$\frac{\xi_0^2}{g^2} + \frac{\omega}{g^2(\omega + A)}\dot{\eta}_0^2 = c^2. \qquad \text{......(7)}$$

In this formula ω and A, together with g, refer to the point S in Fig. 71. To the order of accuracy with which we are concerned we can replace ω, A and g by the values of these quantities at S_0. The formula (7) is then

$$\frac{\dot{\xi}_0^2}{g_0^2}+\frac{\omega_0}{g_0^2(\omega_0+A)}\dot{\eta}_0^2=c^2, \qquad \text{......(8)}$$

in which A now denotes Oort's constant for S_0 and

$$g_0^2=4\omega_0(\omega_0+A). \qquad \text{......(9)}$$

The quantities ω_0, A and g_0 are now constants independent of the position of S relative to S_0, subject of course to the degree of approximation already indicated.

It is to be remarked that $\dot{\xi}_0$ and $\dot{\eta}_0$ are the components of velocity to be associated with the stellar motions as observed at S_0. Further, the equation (8) has been derived by considering the orbit of a star relative to a given point S; the orbit is found to be an ellipse with the radial axis of length c. When c is given, the components $\dot{\xi}_0$ and $\dot{\eta}_0$, referred to axes through the sun, satisfy (8), and for a given c and varying positions of S there will be a family of orbits for each of which (8) will be satisfied.

12·17. We have to consider the continuity condition at S_0. Referred to fixed axes CX, CY the kinetic energy $T\equiv\frac{1}{2}(\dot{X}^2+\dot{Y}^2)$ is given—using (1) and (2) of 12·15 and applying these formulae to the coordinate system at S_0—by

$$2T=\dot{\xi}_0^2+\dot{\eta}_0^2-2\omega_0\eta_0\dot{\xi}_0+2\omega_0(\varpi_0+\xi_0)\dot{\eta}_0+\omega_0^2(\overline{\xi_0+\varpi_0}^2+\eta_0^2).$$

Let $$q_1=\varpi_0+\xi_0,\quad q_2=\eta_0. \qquad \text{......(1)}$$

The generalised momenta p_1 and p_2 (per unit mass) are given by

$$p_1=\frac{\partial T}{\partial\dot{q}_1}=\frac{\partial T}{\partial\dot{\xi}_0},$$

$$p_2=\frac{\partial T}{\partial\dot{q}_2}=\frac{\partial T}{\partial\dot{\eta}_0}.$$

Hence $$\left.\begin{aligned}p_1&=\dot{\xi}_0-\omega_0\eta_0\\ p_2&=\dot{\eta}_0+\omega_0(\varpi_0+\xi_0)\end{aligned}\right\}. \qquad \text{......(2)}$$

The equations of motion are then

$$\dot{q}_1=\frac{\partial H}{\partial p_1},\quad \dot{q}_2=\frac{\partial H}{\partial p_2},$$

$$\dot{p}_1=-\frac{\partial H}{\partial q_1},\quad \dot{p}_2=-\frac{\partial H}{\partial q_2},$$

where $H\equiv T-V(\varpi)$ is the Hamiltonian function. The q's and p's are canonical variables and by Liouville's theorem

$$\frac{D}{Dt}(dq_1dq_2dp_1dp_2)=0.$$

Now $$dq_1 dq_2 dp_1 dp_2 = \frac{\partial(q_1, q_2, p_1, p_2)}{\partial(\xi_0, \eta_0, \dot{\xi}_0, \dot{\eta}_0)} d\xi_0 d\eta_0 d\dot{\xi}_0 d\dot{\eta}_0.$$

It is easily seen from (1) and (2) that the value of the Jacobian in the previous expression is unity. Hence

$$\frac{D}{Dt}(d\xi_0 d\eta_0 d\dot{\xi}_0 d\dot{\eta}_0) = 0 \qquad ......(3)$$

or $$\frac{D}{Dt}(dP) = 0, \qquad ......(4)$$

where $dP \equiv d\xi_0 d\eta_0 d\dot{\xi}_0 d\dot{\eta}_0$ is the phase-element.

If we assume that the number of stars per unit phase-element is a function $\phi(c^2)$ of the parameter c (the semi-major axis of the relative orbit), the number of stars in the phase-element dP is given by

$$dN = \phi(c^2)\, dP.$$

Hence, by (4), $$\frac{D}{Dt}(dN) = 0,$$

so that the necessary condition is satisfied.

Thus, using 12·16 (8), it is seen that the number of stars, per unit area of the galactic plane, with velocities between $(\dot{\xi}_0, \dot{\eta}_0)$ and $(\dot{\xi}_0 + d\dot{\xi}_0, \dot{\eta}_0 + d\dot{\eta}_0)$ is

$$\phi\left\{\frac{\dot{\xi}_0^2}{g_0^2} + \frac{\omega_0}{g_0^2(\omega_0 + A)}\dot{\eta}_0^2\right\} d\dot{\xi}_0 d\dot{\eta}_0. \qquad ......(5)$$

If ϕ takes the form $e^{-\alpha c^2}$, (5) is equivalent to Schwarzschild's ellipsoidal law applied to two-dimensional motion. The major axis of the velocity ellipsoid points towards the galactic centre and the ratio of the axes in the galactic plane is

$$\left(\frac{\omega_0}{\omega_0 + A}\right)^{\frac{1}{2}},$$

agreeing with the result of 12·12.

12·21. *Asymmetry of stellar motions.*

We begin with the formulae in section 12·13. It is to be remembered that we have defined the systematic rotational velocity Θ_0 near the sun as positive in the sense of increasing galactic longitude.

From 12·13 (7) we have

$$\frac{d}{d\varpi}(\log \nu) = -\frac{1}{\lambda}\frac{d\lambda}{d\varpi} + \frac{2}{a^2}\frac{dV_1}{d\varpi}$$

or, using 12·12 (4) and 12·12 (2),

$$\frac{1}{\nu}\frac{d\nu}{d\varpi} = -\frac{1}{\lambda}\frac{d\lambda}{d\varpi} + \frac{2}{a^2}\left\{\frac{dV}{d\varpi} + \frac{k_1^2}{k_2 \lambda^3}\frac{d\lambda}{d\varpi}\right\}.$$

Now, since $\lambda^2 = 1+k_2\varpi^2, \quad \Theta_0 = -\frac{k_1\varpi}{\lambda^2} \quad \text{and} \quad \frac{1}{\lambda} = \frac{b}{a},$

we obtain, on eliminating k_1 and k_2,

$$\frac{1}{\nu}\frac{d\nu}{d\varpi} = -\frac{1}{\varpi\lambda^2}(\lambda^2-1)+\frac{2}{a^2}\frac{dV}{d\varpi}+\frac{2}{a^2}\frac{\Theta_0^2}{\varpi}$$

$$= -\frac{1}{\varpi}\left(1-\frac{b^2}{a^2}\right)-\frac{2}{a^2\varpi}(\Theta_c^2-\Theta_0^2), \qquad \text{......(1)}$$

where
$$\Theta_c^2 = -\varpi\frac{dV}{d\varpi}. \qquad \text{......(2)}$$

Here, Θ_c is the circular velocity at a distance ϖ from the galactic centre arising from the potential V of the field.

We write
$$S = \Theta_c - \Theta_0. \qquad \text{......(3)}$$

Then
$$\Theta_c^2 - \Theta_0^2 = S^2 + 2S\Theta_0$$

and, if S is small compared with Θ_0, we can write with sufficient accuracy

$$\Theta_c^2 - \Theta_0^2 = 2S\Theta_0. \qquad \text{......(4)}$$

Now by (3) and (6) of section 12·11,

$$\Theta_0 = -\varpi(A-B). \qquad \text{......(5)}$$

Hence, by (4) and (5), the equation (1) becomes

$$S = \frac{a^2}{4(A-B)}\left\{\frac{1}{\nu}\frac{d\nu}{d\varpi}+\frac{1}{\varpi}\left(1-\frac{b^2}{a^2}\right)\right\}. \qquad \text{......(6)}$$

We see from 12·13 (5) that a is $\sqrt{\pi}$ times the mean speed in the ϖ-direction (and also in the z-direction). We can thus regard a as a measure of the peculiar motions of stars in the system under consideration. If the peculiar motions are small, as for the majority of stars in the neighbourhood of the sun, a is small and, by (6), S is small; accordingly, Θ_c approaches Θ_0 in magnitude. The stars in such a system describe orbits about the galactic centre not differing greatly from circles, and the systematic rotational velocity Θ_0 is close to the circular velocity Θ_c arising from the potential of the field.

But for stars with large peculiar motions, such as the high velocity stars, the deviation between Θ_0 and Θ_c for this particular system is very much greater owing to the larger value of a. Accordingly, the high-velocity stars have a systematic velocity $(\Theta_0 - \Theta_c)$ relative to the stars with small peculiar velocities, that is to say, relative to the great majority of stars in the neighbourhood of the sun.

According to Oort,* who has made extensive calculations based on the observational material treated by Strömberg,† the numerical value of the

* *B.A.N.* No. 159, p. 283, 1928. † *Ap. J.* **61**, 363, 1925.

first term in (6) is greater than that of the second term; also, if we make the reasonable assumption that ν increases as we approach the galactic centre, $d\nu/d\varpi$ is negative. Hence the value of $S \equiv \Theta_c - \Theta_0$ is negative, or the systematic relative velocity of the high-velocity stars, namely $\Theta_0 - \Theta_c$, is positive.

Now Θ_0 and Θ_c are measured in the sense of increasing galactic longitude and, as the actual rotation is retrograde, we write

$$V_0 = -\Theta_0, \quad V_c = -\Theta_c,$$

where V_0 and V_c are the speeds concerned in the direction in which the rotation is observed to take place. We then have that $V_0 - V_c$ is negative; in other words, the systematic transverse speed, V_0, of the high-velocity stars is less than the systematic transverse speed, V_c, of the majority of stars near the sun. Thus, the high-velocity stars appear to lag behind the normal stars and this is in conformity with the conclusions as summarised in section 11·14.

Assuming that the quantity within the brackets on the right of (6) is substantially the same for various groups of stars with different velocity-dispersions σ (where σ is proportional to a), we can write (6) in the form

$$S = c\sigma^2,$$

where c is a constant. This is a parabolic relation between the systematic motion of a group (relative to the normal stars near the sun) and the dispersion. This parabolic relation was first obtained by Strömberg* from the observational material to which reference has already been made. The dynamical theory which we have just outlined is thus in accordance—at least, qualitatively—with the observational evidence.

It is of interest to add that the average value of $\frac{1}{\nu}\frac{d\nu}{d\varpi}$ as determined by Oort† from Strömberg's data is $-0{\cdot}00019$.

12·22. Lindblad has arrived at a similar conclusion by a slightly different procedure,‡ substantially as follows.

From 12·13 (7),

$$\log \nu = \log\{P(\pi a^2)^{\frac{3}{2}}\} - \log \lambda + \frac{2}{a^2} V_1(\varpi).$$

Similarly, if ν_1 is the density at a distance ϖ_1 from the galactic centre,

$$\log \nu_1 = \log\{P(\pi a^2)^{\frac{3}{2}}\} - \log \lambda_1 + \frac{2}{a^2} V_1(\varpi_1),$$

where $$\lambda_1 = (1 + k_2 \varpi_1^2)^{\frac{1}{2}}.$$

Writing $$\nu_1 = \nu e^{-\chi}, \qquad \text{......(1)}$$

we have

$$\chi = \tfrac{1}{2}\log\left(\frac{1 + k_2\varpi_1^2}{1 + k_2\varpi^2}\right) + \frac{2}{a^2}\left\{V(\varpi) - V(\varpi_1) - \frac{k_1^2(\varpi_1^2 - \varpi^2)}{2(1 + k_2\varpi^2)(1 + k_2\varpi_1^2)}\right\}. \qquad \text{...(2)}$$

* *Ap. J.* **61**, 379, 1925. † *B.A.N.* No. 159, p. 282, 1928.

‡ *Stockholms Obs. Annaler*, **12**, No. 4, p. 15, 1936.

We now assume that ϖ_1 has been chosen so that $(\varpi_1 - \varpi)$ is small compared with ϖ. It is easily found, by means of the reductions of the previous section, that

$$S = \frac{a^2}{4\Theta_0}\left\{\frac{\varpi\chi}{\varpi_1 - \varpi} - \left(1 - \frac{b^2}{a^2}\right)\right\}. \qquad \ldots\ldots(3)$$

We suppose ϖ and ϖ_1 definitely known, the former being the sun's distance from the galactic centre and the latter having the limitation already indicated. As the density, ν, is supposed to be known as a function of ϖ, the value of χ can then be determined from (1).

We may proceed in a slightly different way by choosing χ to be unity; then ϖ_1 can be determined from (1); in this case, we may define ϖ_1 as the "effective limit", its value depending of course on the density-gradient $d\nu/d\varpi$. If the limitation regarding $(\varpi_1 - \varpi)$ still holds, the formula (3) is the parabolic relation connecting S and a already found in the previous section.

It may be added that, from Strömberg's observational data already referred to, Lindblad* finds that, for $\chi = 1$,

$$\frac{\varpi_1 - \varpi}{\varpi} = 0{\cdot}23.$$

12·31. *General kinematical considerations.*

In Oort's theory, as we have seen, it is assumed that the galaxy is rotating about the galactic centre and that the consequences of this assumption are expressed in terms of equations giving the differential effects for radial velocities and for proper motions. It has been shown independently by Pilowski,† Ogrodnikoff ‡ and Milne§ that the *forms* of these equations do not depend uniquely on the particular dynamical theory envisaged but are deducible from the simple hypothesis of the existence of a space-velocity frequency function for the assembly of stars concerned, provided that this function is continuous with respect to the coordinate system by which it is described. As Milne's investigation is the most general of those just mentioned, we shall consider it here.

Consider an assembly of stars whose positions are referred to non-rotating axes OX, OY, OZ (Fig. 72). Let the coordinates of the sun, S_0, and of a star, S, be respectively (X_0, Y_0, Z_0) and (X, Y, Z).

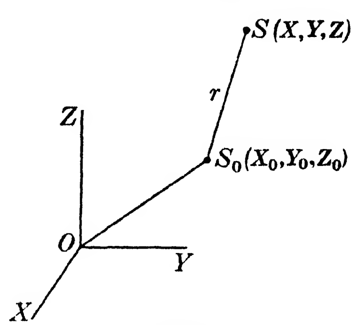

Fig. 72

* *Handbuch der Astrophysik*, 5 (2), 1051, 1933.
† *Zeits. f. Astrophysik*, 3, 53, 279, 291, 1931; 4, 396, 1932.
‡ *Ibid.* 4, 190, 1932. § *M.N.* 95, 560, 1935.

Let the linear velocity components of S_0 and of S be (U_0, V_0, W_0) and (U, V, W). We assume that a space-velocity function f exists so that, if dN denotes the number of stars in the volume-element at S, dN is given by

$$dN = dXdYdZ\iiint f(t;\, X, Y, Z;\, U, V, W)\,dU\,dV\,dW,$$

where the integration is taken over all possible values of U, V and W.

If ν denotes the number of stars per unit volume at S, we have

$$\nu = \iiint f(t;\, X, Y, Z;\, U, V, W)\,dU\,dV\,dW. \qquad \text{......(1)}$$

In (1) we have written the frequency function as depending explicitly on the time, so that a steady state is not assumed.

The mean velocity component, $\overline{U}$, of stars in the immediate neighbourhood of the sun is given by

$$\nu\overline{U} = \iiint Uf\,dU\,dV\,dW. \qquad \text{......(2)}$$

There are two similar equations giving $\overline{V}$ and $\overline{W}$.

In the same way we can derive the mean components $(\overline{U}_0, \overline{V}_0, \overline{W}_0)$ for the stars near the sun.

If (ξ, η, ζ) are the components of the solar motion with respect to the centroid of the nearby stars, we have

$$\xi = U_0 - \overline{U}_0, \quad \eta = V_0 - \overline{V}_0, \quad \zeta = W_0 - \overline{W}_0. \qquad \text{......(3)}$$

12·32. *Radial velocities.*

Let ρ denote the radial velocity of a star at S relative to the sun. If (x, y, z) are the coordinates of S with respect to axes through S_0 parallel to OX, OY, OZ, we have

$$\rho = (U - U_0)\frac{x}{r} + (V - V_0)\frac{y}{r} + (W - W_0)\frac{z}{r},$$

where r is the heliocentric distance of S.

Averaging over all the stars near S we obtain the mean radial velocity $\overline{\rho}$, relative to the sun, given by

$$\nu\overline{\rho} = \frac{1}{r}\iiint \{(U - U_0)x + (V - V_0)y + (W - W_0)z\}f\,dU\,dV\,dW.$$

It follows from (2) and (1) of the previous section that

$$\overline{\rho} = \frac{1}{r}\{(\overline{U} - U_0)x + (\overline{V} - V_0)y + (\overline{W} - W_0)z\}. \qquad \text{......(1)}$$

Now $$\overline{U} - U_0 = \overline{U} - \overline{U}_0 - (U_0 - \overline{U}_0),$$

whence, by 12·31 (3), $$\overline{U} - U_0 = \overline{U} - \overline{U}_0 - \xi.$$

Then (1) becomes

$$\bar{\rho}+\frac{1}{r}(\xi x+\eta y+\zeta z) = \frac{1}{r}\{(\bar{U}-\bar{U}_0)x+(\bar{V}-\bar{V}_0)y+(\bar{W}-\bar{W}_0)z\}.$$

Now $$\xi\frac{x}{r}+\eta\frac{y}{r}+\zeta\frac{z}{r}$$

is the component of the solar motion in the direction of S_0S; denoting it by ρ_0, we have

$$\bar{\rho}+\rho_0 = \frac{1}{r}\{(\bar{U}-\bar{U}_0)x+(\bar{V}-\bar{V}_0)y+(\bar{W}-\bar{W}_0)z\}. \qquad \ldots\ldots(2)$$

Now $\bar{U}$ is a function of X, Y, Z and t, and similarly $\bar{U}_0$ is a function of X_0, Y_0, Z_0 and t; to the first order* in x, y and z we can write

$$\bar{U} = \bar{U}_0+x\left(\frac{\partial\bar{U}}{\partial X}\right)_0+y\left(\frac{\partial\bar{U}}{\partial Y}\right)_0+z\left(\frac{\partial\bar{U}}{\partial Z}\right)_0, \qquad \ldots\ldots(3)$$

where the zero suffix indicates that $(\partial\bar{U}/\partial X)$, etc. are to be evaluated for the coordinates of S_0. It is assumed that these differential coefficients exist. We obtain similar equations for $\bar{V}$ and $\bar{W}$.

Let $\bar{\rho}_1$ denote the mean radial velocity of stars at S with the solar motion component removed. Then (2) becomes

$$\bar{\rho}_1 = \frac{1}{r}(ax^2+by^2+cz^2+2fyz+2gzx+2hxy), \qquad \ldots\ldots(4)$$

where $$a = \left(\frac{\partial\bar{U}}{\partial X}\right)_0, \quad b = \left(\frac{\partial\bar{V}}{\partial Y}\right)_0, \quad c = \left(\frac{\partial\bar{W}}{\partial Z}\right)_0, \qquad \ldots\ldots(5)$$

$$2f = \left(\frac{\partial\bar{V}}{\partial Z}\right)_0+\left(\frac{\partial\bar{W}}{\partial Y}\right)_0, \quad 2g = \left(\frac{\partial\bar{W}}{\partial X}\right)_0+\left(\frac{\partial\bar{U}}{\partial Z}\right)_0, \quad 2h = \left(\frac{\partial\bar{U}}{\partial Y}\right)_0+\left(\frac{\partial\bar{V}}{\partial X}\right)_0 \ldots(6)$$

or, if (l, m, n) are the direction-cosines of the vector S_0S,

$$\bar{\rho}_1 = r(al^2+bm^2+cn^2+2fmn+2gnl+2hlm). \qquad \ldots\ldots(7)$$

We can define (l, m, n) in terms of angles λ and β regarded as analogues of longitude and latitude with respect to the system of axes OX, OY, OZ; then

$$l = \cos\lambda\cos\beta, \quad m = \sin\lambda\cos\beta, \quad n = \sin\beta \qquad \ldots\ldots(8)$$

and (7) becomes

$$\bar{\rho}_1 = \frac{r}{4}[(a+b+2c)+(a+b-2c)\cos 2\beta+4(f\sin\lambda+g\cos\lambda)\sin 2\beta$$
$$+\{(a-b)\cos 2\lambda+2h\sin 2\lambda\}\{1+\cos 2\beta\}] \ldots\ldots(9)$$

The expression for $\bar{\rho}_1$ in this equation thus consists of a constant part $r(a+b+2c)/4$ independent of the position of S on the celestial sphere, and a part depending on the coordinates of S with respect to the system of axes

* An extension of Milne's analysis to second-order terms has been recently made by Edmondson, *M.N.* 97, 473, 1937.

chosen. If the radial velocities in different parts of the sky are analysed according to (9), we should regard

$$\frac{r}{4}(a+b+2c)$$

as the "K term" which, we notice, varies as the distance, up to the order of approximation adopted.

If we obtain the values of $\bar{\rho}_1$ for all small areas of the celestial sphere and take the mean—which we denote by $\bar{\rho}_2$—we obtain

$$4\pi\bar{\rho}_2 = \int_{\lambda=0}^{2\pi}\int_{\beta=-\pi/2}^{\pi/2} \bar{\rho}_1 \cos\beta\, d\beta\, d\lambda.$$

Now $$\iint l^2 \cos\beta\, d\beta\, d\lambda = 8\int_0^{\pi/2}\int_0^{\pi/2} \cos^3\beta \cos^2\lambda\, d\beta\, d\lambda = \frac{4\pi}{3},$$

with similar results for the integrals involving m^2 and n^2. Also, over the sphere,

$$\iint lm \cos\beta\, d\beta\, d\lambda = 0,$$

with similar results for the integrals involving mn and nl. Hence

$$\bar{\rho}_2 = \frac{r}{3}(a+b+c). \qquad \text{......(10)}$$

Milne regards this last expression as the "K term".

If we are analysing the radial velocities of stars situated on a great circle of the celestial sphere—we have in mind the concentration of the O and B type stars towards the galactic equator—we can choose our axes so that the plane of the great circle is the plane $z = 0$. For such stars we have from (9), on putting $\beta = 0$,

$$\bar{\rho}_1 = \frac{r}{2}\{a+b+(a-b)\cos 2\lambda + 2h \sin 2\lambda\}, \qquad \text{......(11)}$$

in which a, b and h now refer to the new system of axes.

In (11) write

$$2h = 2P\cos 2\lambda_0, \quad a-b = -2P\sin 2\lambda_0, \qquad \text{......(12)}$$

from which $$\tan 2\lambda_0 = -\frac{a-b}{2h}, \quad P = \left\{\left(\frac{a-b}{2}\right)^2 + h^2\right\}^{\frac{1}{2}}. \qquad \text{......(13)}$$

Hence (11) becomes

$$\bar{\rho}_1 = \frac{r}{2}(a+b) + r\left\{\left(\frac{a-b}{2}\right)^2 + h^2\right\}^{\frac{1}{2}} \sin 2(\lambda-\lambda_0). \qquad \text{......(14)}$$

This is of the same form as Oort's equation for the radial velocities of stars lying in the galactic equator, with the addition of a "K term", equal to $r(a+b)/2$, varying with the distance of the stars concerned. It follows that the observational verification of (14) is not sufficient evidence in favour of any particular dynamical theory, for the equation, as we have seen, follows from the assumption of the existence of a space-velocity frequency function, whether the system is in a steady state or not.

12·33. *Proper motions.*

We consider, as before, the general system of axes in Fig. 72 and we denote by μ_λ, μ_β the proper motion components in "longitude" and "latitude". If u, v, w are the linear components of velocity of S relative to S_0, we have

$$u = U - U_0 = U - \overline{U}_0 - \xi. \qquad \ldots\ldots(1)$$

Similarly

$$v = V - \overline{V}_0 - \eta,$$

$$w = W - \overline{W}_0 - \zeta,$$

where (ξ, η, ζ) are the components of the solar motion.

From (3) and (4) of section 1·33 we obtain, in the usual system of units with r in parsecs and $\kappa \equiv 4{\cdot}74$,

$$\kappa r \mu_\lambda \cos\beta = -u \sin\lambda + v \cos\lambda, \qquad \ldots\ldots(2)$$

$$\kappa r \mu_\beta = -u \cos\lambda \sin\beta - v \sin\lambda \sin\beta + w \cos\beta. \qquad \ldots\ldots(3)$$

From (1) and (2)

$$\kappa r \mu_\lambda \cos\beta = -(U - \overline{U}_0) \sin\lambda + (V - \overline{V}_0) \cos\lambda + Q,$$

where

$$Q = \xi \sin\lambda - \eta \cos\lambda.$$

Here Q is the effect produced by the solar motion. If we assume now that μ_λ and μ_β have been corrected for the solar motion, we write

$$\kappa r \mu_\lambda \cos\beta = -(U - \overline{U}_0) \sin\lambda + (V - \overline{V}_0) \cos\lambda, \qquad \ldots\ldots(4)$$

$$\kappa r \mu_\beta = -(U - \overline{U}_0) \cos\lambda \sin\beta - (V - \overline{V}_0) \sin\lambda \sin\beta + (W - \overline{W}_0) \cos\beta. \quad \ldots(5)$$

From (4), by averaging for all the stars in the neighbourhood of S, we obtain

$$\kappa r \overline{\mu}_\lambda \cos\beta = -(\overline{U} - \overline{U}_0) \sin\lambda + (\overline{V} - \overline{V}_0) \cos\lambda$$

and, expanding $\overline{U}$, $\overline{V}$ to the first order in x, y and z, we find

$$\kappa r \overline{\mu}_\lambda \cos\beta = -\sin\lambda \left\{ x \left(\frac{\partial \overline{U}}{\partial X}\right)_0 + y \left(\frac{\partial \overline{U}}{\partial Y}\right)_0 + z \left(\frac{\partial \overline{U}}{\partial Z}\right)_0 \right\}$$
$$+ \cos\lambda \left\{ x \left(\frac{\partial \overline{V}}{\partial X}\right)_0 + y \left(\frac{\partial \overline{V}}{\partial Y}\right)_0 + z \left(\frac{\partial \overline{V}}{\partial Z}\right)_0 \right\}.$$

Since $x = r \cos\lambda \cos\beta$, $y = r \sin\lambda \cos\beta$, $z = r \sin\beta$ we obtain, using the notation in (5) and (6) of section 12·32,

$$\kappa \overline{\mu}_\lambda = \frac{1}{2} \left\{ \left(\frac{\partial \overline{V}}{\partial X}\right)_0 - \left(\frac{\partial \overline{U}}{\partial Y}\right)_0 \right\} + h \cos 2\lambda - \tfrac{1}{2}(a - b) \sin 2\lambda$$
$$+ \tan\beta \left\{ \cos\lambda \left(\frac{\partial \overline{V}}{\partial Z}\right)_0 - \sin\lambda \left(\frac{\partial \overline{U}}{\partial Z}\right)_0 \right\}. \qquad \ldots\ldots(6)$$

If we confine ourselves to observations made on the great circle $\beta = 0$, (6) is reduced to

$$\kappa\bar{\mu}_\lambda = \omega_3 + \left\{\left(\frac{a-b}{2}\right)^2 + h^2\right\}^{\frac{1}{2}} \cos 2(\lambda - \lambda_0), \qquad \text{......(7)}$$

where
$$\omega_3 = \frac{1}{2}\left\{\left(\frac{\partial \bar{V}}{\partial X}\right)_0 - \left(\frac{\partial \bar{U}}{\partial Y}\right)_0\right\}$$

and λ_0 is defined as in 12·32 (13).

Again, (7) is similar to Oort's equation for proper motions of stars in the galactic equator.

In the same way we derive from (5)

$$\kappa\bar{\mu}_\beta = -\frac{a-b}{4}\cos 2\lambda \sin 2\beta - \frac{h}{2}\sin 2\lambda \sin 2\beta - \frac{a+b-2c}{4}\sin 2\beta$$
$$+ g\cos\lambda\cos 2\beta + f\sin\lambda\cos 2\beta - \frac{1}{2}\left\{\left(\frac{\partial \bar{U}}{\partial Z}\right)_0 - \left(\frac{\partial \bar{W}}{\partial X}\right)_0\right\}\cos\lambda$$
$$-\frac{1}{2}\left\{\left(\frac{\partial \bar{V}}{\partial Z}\right)_0 - \left(\frac{\partial \bar{W}}{\partial Y}\right)_0\right\}\sin\lambda. \qquad \text{......(8)}$$

For stars situated on the great circle $\beta = 0$ we have from (8)

$$\kappa\bar{\mu}_\beta = \left(\frac{\partial \bar{W}}{\partial X}\right)_0 \cos\lambda + \left(\frac{\partial \bar{W}}{\partial Y}\right)_0 \sin\lambda. \qquad \text{......(9)}$$

Unless $\left(\frac{\partial \bar{W}}{\partial X}\right)_0$ and $\left(\frac{\partial \bar{W}}{\partial Y}\right)_0$ are both zero, the general frequency function will give rise to a component $\bar{\mu}_\beta$ depending on the "longitude" in the great circle $\beta = 0$.

12·34. *Application to the galactic system.*

We identify the origin O in Fig. 72 with the galactic centre and the plane XOY with that of the galactic equator. We assume that the galactic equator is a plane of symmetry so that $\bar{W} = 0$ and that $\bar{U}$ and $\bar{V}$ are independent of Z. It follows that

$$c = f = g = 0.$$

We then have from 12·32 (9), expressing our results in galactic coordinates,

$$\bar{p}_1 = \frac{r}{2}\cos^2 g\{a + b + (a-b)\cos 2G_1 + 2h\sin 2G_1\} \qquad \text{......(1)}$$

and from (6) and (8) of section 12·33,

$$\kappa\bar{\mu}_G = \omega_3 + h\cos 2G_1 - \tfrac{1}{2}(a-b)\sin 2G_1, \qquad \text{......(2)}$$
$$\kappa\bar{\mu}_g = -\tfrac{1}{4}\sin 2g\{a + b + (a-b)\cos 2G_1 + 2h\sin 2G_1\}, \qquad \text{......(3)}$$

where G_1 is the "longitude" measured from an arbitrary position of the X-axis.

Transform to cylindrical coordinates so that with respect to O the coordinates of S are (ϖ, θ, z) and the linear components of velocity are (Π, Θ, Z). In Fig. 73 S_1 is the projection of S on the plane of symmetry; also, θ is the angle between OS_0 and OS_1.

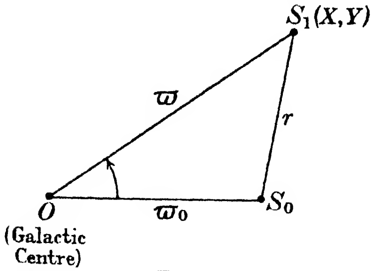

Fig. 73

We have to evaluate $\partial\bar{U}/\partial X$, etc. for S_0 in terms of $\partial\bar{\Pi}/\partial\varpi$, etc. We choose the X-axis to pass through S_0. Then

$$X = \varpi\cos\theta, \quad Y = \varpi\sin\theta, \qquad \text{......(4)}$$

$$U = \Pi\cos\theta - \Theta\sin\theta, \quad V = \Pi\sin\theta + \Theta\cos\theta. \qquad \text{......(5)}$$

From (4),

$$\frac{\partial\varpi}{\partial X} = \cos\theta, \qquad \frac{\partial\varpi}{\partial Y} = \sin\theta,$$

$$\frac{\partial\theta}{\partial X} = -\frac{\sin\theta}{\varpi}, \qquad \frac{\partial\theta}{\partial Y} = \frac{\cos\theta}{\varpi}.$$

From (5),

$$\frac{\partial U}{\partial X} = \left(\frac{\partial\Pi}{\partial\varpi}\frac{\partial\varpi}{\partial X} + \frac{\partial\Pi}{\partial\theta}\frac{\partial\theta}{\partial X}\right)\cos\theta - \Pi\sin\theta\frac{\partial\theta}{\partial X} - \sin\theta\frac{\partial\Theta}{\partial X} - \Theta\cos\theta\frac{\partial\theta}{\partial X}.$$

The value of $\partial U/\partial X$ at S_0 is obtained by putting $\theta = 0$ and $\varpi = \varpi_0$ in this expression. We thus obtain

$$\left(\frac{\partial U}{\partial X}\right)_0 = \left(\frac{\partial\Pi}{\partial\varpi}\right)_0.$$

Similarly

$$\left(\frac{\partial U}{\partial Y}\right)_0 = \frac{1}{\varpi_0}\left(\frac{\partial\Pi}{\partial\theta}\right)_0 - \frac{\Theta_0}{\varpi_0},$$

$$\left(\frac{\partial V}{\partial X}\right)_0 = \left(\frac{\partial\Theta}{\partial\varpi}\right)_0,$$

$$\left(\frac{\partial V}{\partial Y}\right)_0 = \frac{1}{\varpi_0}\left(\frac{\partial\Theta}{\partial\theta}\right)_0 + \frac{\Pi_0}{\varpi_0}.$$

We then have the following relations:

$$\left.\begin{aligned} a+b &= \left(\frac{\partial\bar{\Pi}}{\partial\varpi} + \frac{1}{\varpi}\frac{\partial\bar{\Theta}}{\partial\theta} + \frac{\bar{\Pi}}{\varpi}\right)_0 \\ a-b &= \left(\frac{\partial\bar{\Pi}}{\partial\varpi} - \frac{1}{\varpi}\frac{\partial\bar{\Theta}}{\partial\theta} - \frac{\bar{\Pi}}{\varpi}\right)_0 \\ 2h &= \left(\frac{1}{\varpi}\frac{\partial\bar{\Pi}}{\partial\theta} + \frac{\partial\bar{\Theta}}{\partial\varpi} - \frac{\bar{\Theta}}{\varpi}\right)_0 \\ 2\omega_3 &= \left(\frac{\partial\bar{\Theta}}{\partial\varpi} - \frac{1}{\varpi}\frac{\partial\bar{\Pi}}{\partial\theta} + \frac{\bar{\Theta}}{\varpi}\right)_0 \end{aligned}\right\} \qquad \text{......(6)}$$

12·35. *Galactic rotation.*

For simple galactic rotation, we have $\overline{\Pi} = 0$ and $\overline{\Theta}\ (\equiv \Theta_0)$ is independent of θ. Then

$$a+b = a-b = 0$$

and we obtain from 12·34 (1)

$$\overline{\rho}_1 = rh\cos^2 g \sin 2G_1. \qquad \text{......(1)}$$

In this formula G_1 is the "longitude" measured from the X-axis, but as we have now specified the X-axis as passing through S_0, (1) becomes

$$\overline{\rho}_1 = rh\cos^2 g \sin 2(G-G_0), \qquad \text{......(2)}$$

where G is the galactic longitude of S, as usually defined, and G_0 is the galactic longitude of the galactic centre.

Also, since $\overline{\Theta} = \Theta_0$, we observe that h is simply Oort's constant A. Thus (2) is Oort's equation for the radial velocities.

Similarly, $$\kappa\overline{\mu}_G = B + A\cos 2(G-G_0), \qquad \text{......(3)}$$

where B is identified with ω_3.

Also, $$\kappa\overline{\mu}_g = -\tfrac{1}{2}A\sin 2g \sin 2(G-G_0). \qquad \text{......(4)}$$

The formula (3) is Oort's equation for the longitude component of the proper motions.

12·36. *Galactic rotation and expansion.*

The general equations for $\overline{\rho}_1$ and μ_G are given by (1) and (2) of section 12·34, namely,

$$\overline{\rho}_1 = \frac{r}{2}\cos^2 g\{(a+b)+(a-b)\cos 2(G-G_0)+2h\sin 2(G-G_0)\}, \qquad \text{......(1)}$$

$$\kappa\overline{\mu}_G = \omega_3 + h\cos 2(G-G_0) - \tfrac{1}{2}(a-b)\sin 2(G-G_0), \qquad \text{......(2)}$$

in which we have replaced G_1 by $(G-G_0)$, as in the previous section. Introducing λ_0 and P as defined in 12·32 (13), we have

$$\overline{\rho}_1 = \tfrac{1}{2}r(a+b)\cos^2 g + Pr\cos^2 g \sin 2(G-G_0-\lambda_0), \qquad \text{......(3)}$$

$$\kappa\overline{\mu}_G = \omega_3 + P\cos 2(G-G_0-\lambda_0). \qquad \text{......(4)}$$

In Oort's formula for the radial velocities, $\overline{\rho}_1$ vanishes when

$$G = G_0 + n\pi/2 \quad (n = 0, 1, 2, 3)$$

and the K term depending on r is absent. If the galaxy is rotating and expanding (or contracting), it is seen from (3) that the variable part of $\overline{\rho}_1$ vanishes when

$$G = G_0 + \lambda_0 + m\pi/2;$$

one effect of expansion is thus to increase the apparent longitude of the centre, as determined by Oort's method, algebraically by λ_0. A similar conclusion follows from a consideration of the formula (4) for the proper motions in longitude.

To form the equation of condition to be used in the analysis of the radial velocities we write (3) as follows:

$$\bar{\rho}_1 = K_1 + K_2 r \cos^2 g + r \cos^2 g \sin 2(G - G_0) \,.\, P \cos 2\lambda_0$$
$$- r \cos^2 g \cos 2(G - G_0) \,.\, P \sin 2\lambda_0, \quad \ldots\ldots(5)$$

where $K_2 \equiv \frac{1}{2}(a+b)$ and K_1 is a possible systematic effect arising, for example, from an inaccurate knowledge of wave-lengths; K_1 is thus equivalent to the K term originally introduced by Campbell.

In (5) we suppose that the value of G_0—the longitude of the galactic centre—is known from considerations, independent of dynamical theory, as in Shapley's investigations. Also, it is assumed that the distances, r, of the stars are known with sufficient accuracy; the O and early B type stars will be mainly used in investigations of this type, arranged in groups according to distance.

The quantities to be determined from a numerical application of (4) are

$$K_1, \quad K_2, \quad P \cos 2\lambda_0, \quad P \sin 2\lambda_0.$$

Similarly, the quantities to be determined from an analysis of (5) are

$$\omega_3, \quad P \cos 2\lambda_0, \quad P \sin 2\lambda_0.$$

From 12·34 (6), we obtain

$$\left(\frac{\partial \bar{\Pi}}{\partial \varpi}\right)_0 = K_2 - P \sin 2\lambda_0, \quad \ldots\ldots(6)$$

$$\left(\frac{\bar{\Pi}}{\varpi} + \frac{1}{\varpi}\frac{\partial \bar{\Theta}}{\partial \theta}\right)_0 = K_2 + P \sin 2\lambda_0, \quad \ldots\ldots(7)$$

$$\left(\frac{\partial \bar{\Theta}}{\partial \varpi}\right)_0 = \omega_3 + P \cos 2\lambda_0, \quad \ldots\ldots(8)$$

$$\left(\frac{\bar{\Theta}}{\varpi} - \frac{1}{\varpi}\frac{\partial \bar{\Pi}}{\partial \theta}\right)_0 = \omega_3 - P \cos 2\lambda_0. \quad \ldots\ldots(9)$$

A full analysis of the observational material will thus give the values of the quantities on the left-hand sides of (6) to (9).

An attempt has been made by Pilowski* to apply these formulae to the results of Plaskett and Pearce† obtained from their investigation of the O and B type stars; but it would appear that Pilowski's conclusions must be considerably vitiated owing to Plaskett and Pearce's erroneous procedure in dealing with the K term.‡

* *A.N.* **257**, 225, 1935.
† *M.N.* **94**, 679, 1934; see also *Publ. D.A.O.* (*Victoria*), **5**, No. 4, 1936.
‡ W. M. Smart, *M.N.* **96**, 568, 1936.

It is very doubtful if we have at the present time adequate observational material (radial velocities and proper motions), covering a sufficient range in distance and distributed over all galactic longitudes, to justify the somewhat elaborate analysis required in the problem discussed in this section.

12·41. *The direction of star-streaming.*

We have seen that in the theory of galactic rotation the axis of greatest mobility on the galactic plane is directed towards the galactic centre. If, however, the system of stars is expanding (or contracting) with reference to the galactic centre, the projection of the velocity ellipsoid on the galactic plane may have its major axis at an angle ψ with the axis defined by joining the galactic centre and the sun. This is illustrated in Fig. 74, which is drawn for the galactic plane, the direction in which galactic longitude is measured being indicated by the arrow. If ψ is positive (as in the figure), the direction of the vertex of preferential motions is given by SV, so that the longitude of the vertex exceeds the longitude of the galactic centre, C. Eddington's discussion of the Boss stars (of all types) places the longitude of the vertex at 346°, just about 20° greater than Shapley's value for the longitude of the galactic centre; several other investigations are in support, at least qualitatively. On the other hand, if we consider only the evidence from the B type stars, from which the characteristics of galactic rotation—deduced from radial velocity measures—can only be satisfactorily determined, the results are conflicting. For example, the longitude of the vertex as determined by Lindblad* for stars of types B0 to B7 is 289° (it is to be remarked, however, that, as the projection of his velocity ellipsoid on the galactic plane is nearly circular, the longitude of the vertex can be but uncertainly defined); more recent values† for B type stars are 336° and 301°, the former for stars close to the galactic equator and the latter for stars within a fairly narrow zone centred at 20° north or south galactic latitude, and Nordström's value‡ of 315° for stars of types O5 to B5 brighter than the sixth

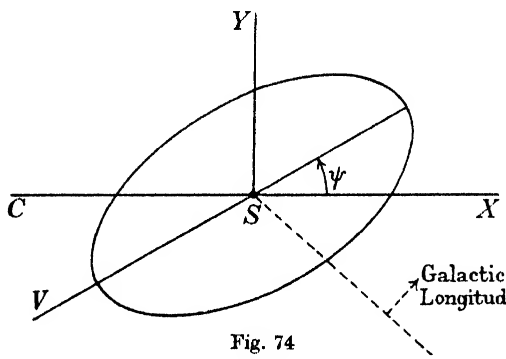

Fig. 74

* *M.N.* **90**, 509, 1930.
† W. M. Smart and H. E. Green, *M.N.* **96**, 479, 1936.
‡ *Lund Medd.* Ser. II, No. 79, p. 137, 1936.

magnitude. In any event, all these results carry a considerable probable error (of the order of $\pm 5°$), and it is fairly evident that in this connection theoretical investigations must not be strained too far in alliance with observational results. The following theoretical discussion is based on work by Shiveshwarkar.*

12·42. *The frequency function.*

We consider only the motions of stars in the galactic plane. With the usual notation the equations of motion are

$$\dot{\Pi} - \frac{\Theta^2}{\varpi} + K = 0, \qquad \text{......(1)}$$

$$\dot{\Theta} + \frac{\Pi\Theta}{\varpi} = 0, \qquad \text{......(2)}$$

in which K is the attractive force per unit mass; it is assumed that K is a function of ϖ only. Also, the direction of Θ is in the sense of increasing galactic longitude.

The equation of continuity for the frequency function f is, in two dimensions,

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \dot{\varpi}\frac{\partial f}{\partial \varpi} + \dot{\theta}\frac{\partial f}{\partial \theta} + \dot{\Pi}\frac{\partial f}{\partial \Pi} + \dot{\Theta}\frac{\partial f}{\partial \Theta} = 0$$

or, by (1) and (2),

$$\frac{\partial f}{\partial t} + \Pi\frac{\partial f}{\partial \varpi} + \frac{\Theta}{\varpi}\frac{\partial f}{\partial \theta} + \left(\frac{\Theta^2}{\varpi} - K\right)\frac{\partial f}{\partial \Pi} - \frac{\Pi\Theta}{\varpi}\frac{\partial f}{\partial \Theta} = 0. \qquad \text{......(3)}$$

Let Π_0, Θ_0 denote the values of the radial and transverse velocities for the mean of the stars close to the sun; for stars at some distance from the sun, the observed quantities will be $\Pi - \Pi_0$ and $\Theta - \Theta_0$ (the effects of the ordinary solar motion being supposed removed). If, as in Fig. 74, the major axis of the velocity ellipse is inclined at an angle ψ to the axis defined by the galactic radius through the sun, Schwarzschild's ellipsoidal distribution of velocities will be represented in two dimensions by the velocity ellipse

$$h(\Pi - \Pi_0)^2 + m(\Pi - \Pi_0)(\Theta - \Theta_0) + k(\Theta - \Theta_0)^2 = 1, \qquad \text{......(4)}$$

and the frequency function may be written as

$$f = f_0 e^{-h(\Pi-\Pi_0)^2 - m(\Pi-\Pi_0)(\Theta-\Theta_0) - k(\Theta-\Theta_0)^2}, \qquad \text{......(5)}$$

in which it is assumed by Shiveshwarkar that f_0, h, m and k are functions of the coordinates alone. Writing x and y for $\Pi - \Pi_0$ and $\Theta - \Theta_0$ respectively, we have the velocity ellipse given by

$$hx^2 + mxy + ky^2 = 1, \qquad \text{......(6)}$$

for which the X and Y axes are as shown in Fig. 74.

* *M.N.* **95**, 655, 1935.

The orientation of the major axis of this ellipse with respect to the x-axis is given by

$$\tan 2\psi = \frac{m}{h-k}. \qquad \ldots\ldots(7)$$

Differentiate (5) logarithmically with respect to ϖ, θ, Π and Θ in succession and substitute the resulting expressions for $\frac{\partial f}{\partial \varpi}$, $\frac{\partial f}{\partial \theta}$, $\frac{\partial f}{\partial \Pi}$ and $\frac{\partial f}{\partial \Theta}$ in (3); we obtain

$$\Pi\left[\frac{1}{f_0}\frac{\partial f_0}{\partial \varpi} - \Pi^2\frac{\partial h}{\partial \varpi} - \Pi\Theta\frac{\partial m}{\partial \varpi} - \Theta^2\frac{\partial k}{\partial \varpi} + \Pi\frac{\partial}{\partial \varpi}(2h\Pi_0 + m\Theta_0)\right.$$
$$\left. + \Theta\frac{\partial}{\partial \varpi}(2k\Theta_0 + m\Pi_0) - \frac{\partial}{\partial \varpi}(h\Pi_0^2 + m\Pi_0\Theta_0 + k\Theta_0^2)\right]$$
$$+ \frac{\Theta}{\varpi}\left[\frac{1}{f_0}\frac{\partial f_0}{\partial \theta} - \Pi^2\frac{\partial h}{\partial \theta} - \Pi\Theta\frac{\partial m}{\partial \theta} - \Theta^2\frac{\partial k}{\partial \theta} + \Pi\frac{\partial}{\partial \theta}(2h\Pi_0 + m\Theta_0)\right.$$
$$\left. + \Theta\frac{\partial}{\partial \theta}(2k\Theta_0 + m\Pi_0) - \frac{\partial}{\partial \theta}(h\Pi_0^2 + m\Pi_0\Theta_0 + k\Theta_0^2)\right]$$
$$- \left(\frac{\Theta^2}{\varpi} - K\right)[2h\Pi + m\Theta - (2h\Pi_0 + m\Theta_0)]$$
$$+ \frac{\Pi\Theta}{\varpi}[2k\Theta + m\Pi - (2k\Theta_0 + m\Pi_0)] = 0. \qquad \ldots\ldots(8)$$

This equation must be satisfied identically for all values of Π and Θ. We obtain, on equating to zero the coefficients of Π^3, $\Pi^2\Theta$, Θ^3, $\Pi\Theta^2$, Π^2, $\Pi\Theta$, Θ^2, Π, Θ and the independent term in the order indicated,

$$\frac{\partial h}{\partial \varpi} = 0, \qquad \ldots\ldots(9)$$

$$\varpi\frac{\partial m}{\partial \varpi} - m + \frac{\partial h}{\partial \theta} = 0, \qquad \ldots\ldots(10)$$

$$\frac{\partial k}{\partial \theta} + m = 0, \qquad \ldots\ldots(11)$$

$$\varpi\frac{\partial k}{\partial \varpi} + \frac{\partial m}{\partial \theta} + 2h - 2k = 0, \qquad \ldots\ldots(12)$$

$$\frac{\partial}{\partial \varpi}(2h\Pi_0 + m\Theta_0) = 0, \qquad \ldots\ldots(13)$$

$$\varpi\frac{\partial}{\partial \varpi}(2k\Theta_0 + m\Pi_0) + \frac{\partial}{\partial \theta}(2h\Pi_0 + m\Theta_0) - (2k\Theta_0 + m\Pi_0) = 0, \qquad \ldots\ldots(14)$$

$$\frac{\partial}{\partial \theta}(2k\Theta_0 + m\Pi_0) + (2h\Pi_0 + m\Theta_0) = 0, \qquad \ldots\ldots(15)$$

$$\frac{\partial}{\partial \varpi}(\log f_0 - h\Pi_0^2 - m\Pi_0\Theta_0 - k\Theta_0^2) + 2hK = 0, \qquad \ldots\ldots(16)$$

$$\frac{\partial}{\partial \theta}(\log f_0 - h\Pi_0^2 - m\Pi_0\Theta_0 - k\Theta_0^2) + m\varpi K = 0, \qquad \ldots\ldots(17)$$

$$2h\Pi_0 + m\Theta_0 = 0. \qquad \ldots\ldots(18)$$

Having regard to (18), we observe that (14) becomes

$$\varpi \frac{\partial}{\partial \varpi}(2k\Theta_0 + m\Pi_0) - (2k\Theta_0 + m\Pi_0) = 0 \qquad \ldots\ldots(19)$$

and that (15) becomes

$$\frac{\partial}{\partial \theta}(2k\Theta_0 + m\Pi_0) = 0 \qquad \ldots\ldots(20)$$

The equations (19) and (20) give immediately

$$2k\Theta_0 + m\Pi_0 = p\varpi, \qquad \ldots\ldots(21)$$

where p is a constant of integration.

12·43. *Application to observations.*

In Fig. 75 S denotes the mean of the stars in the immediate neighbourhood of the sun and X a star at a heliocentric distance r and distant $\varpi + d\varpi$ from the galactic centre C. Let L denote the difference $(G - G_0)$ of the galactic longitudes of X and C; then the angle between CS produced and SX is $L + \pi$. The systematic motions at S are Π_0 and Θ_0—the latter in the sense of increasing longitude—and at X they are Π_1 and Θ_1. We assume that Π_1 and Θ_1 are functions of the coordinates at X.

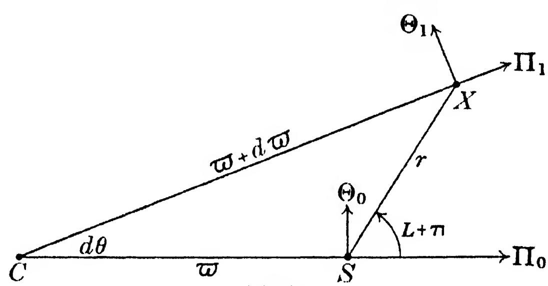

Fig. 75

Then

$$\Pi_1 = \Pi_0 + \frac{\partial \Pi_0}{\partial \varpi} d\varpi + \frac{\partial \Pi_0}{\partial \theta} d\theta,$$

$$\Theta_1 = \Theta_0 + \frac{\partial \Theta_0}{\partial \varpi} d\varpi + \frac{\partial \Theta_0}{\partial \theta} d\theta,$$

to the first order in $d\varpi$ and $d\theta$.

Also, from the figure,

$$d\varpi = r\cos(L+\pi) = -r\cos L, \qquad \ldots\ldots(1)$$

$$\varpi\, d\theta = r\sin(L+\pi) = -r\sin L. \qquad \ldots\ldots(2)$$

Let ρ and T denote the radial and transverse linear velocities relative to S (the latter in the sense of increasing longitude). Then

$$\rho = \Pi_1 \cos(L+\pi-d\theta) + \Theta_1 \sin(L+\pi-d\theta) - \Pi_0 \cos(L+\pi) - \Theta_0 \sin(L+\pi), \qquad \ldots\ldots(3)$$

$$T = \Theta_1 \cos(L+\pi-d\theta) - \Pi_1 \sin(L+\pi-d\theta) - \Theta_0 \cos(L+\pi) + \Pi_0 \sin(L+\pi). \qquad \ldots\ldots(4)$$

Simplifying (3) and (4) and using (1) and (2) we obtain, to the first order,

$$\rho = r(A + B\sin 2L + C\cos 2L), \qquad \text{......(5)}$$

$$T = r(A_1 - C\sin 2L + B\cos 2L), \qquad \text{......(6)}$$

where

$$2A = \frac{\Pi_0}{\varpi} + \frac{\partial \Pi_0}{\partial \varpi} + \frac{1}{\varpi}\frac{\partial \Theta_0}{\partial \theta}, \qquad \text{......(7)}$$

$$2B = \frac{\partial \Theta_0}{\partial \varpi} - \frac{\Theta_0}{\varpi} + \frac{1}{\varpi}\frac{\partial \Pi_0}{\partial \theta}, \qquad \text{......(8)}$$

$$2C = \frac{\partial \Pi_0}{\partial \varpi} - \frac{\Pi_0}{\varpi} - \frac{1}{\varpi}\frac{\partial \Theta_0}{\partial \theta}, \qquad \text{......(9)}$$

$$2A_1 = \frac{\Theta_0}{\varpi} + \frac{\partial \Theta_0}{\partial \varpi} - \frac{1}{\varpi}\frac{\partial \Pi_0}{\partial \theta}. \qquad \text{......(10)}$$

The formula (6) can be expressed in a form suitable for use with the component, μ_G, of the proper motion by writing $T = \kappa r \mu_G$.

The analysis of radial velocities and proper motions will furnish the numerical values of A, B, C and A_1.

The equations of motion at S can be expressed in terms of the coefficients A, B, C, A_1 as follows:

$$\frac{d^2\varpi}{dt^2} - \varpi\dot{\theta}^2 + K \equiv \frac{d\Pi_0}{dt} - \frac{\Theta_0^2}{\varpi} + K = 0. \qquad \text{......(11)}$$

Now

$$\frac{d\Pi_0}{dt} = \Pi_0\frac{\partial \Pi_0}{\partial \varpi} + \frac{\Theta_0}{\varpi}\frac{\partial \Pi_0}{\partial \theta}.$$

Hence (11) becomes

$$\Pi_0\frac{\partial \Pi_0}{\partial \varpi} + \frac{\Theta_0}{\varpi}\left(\frac{\partial \Pi_0}{\partial \theta} - \Theta_0\right) + K = 0. \qquad \text{......(12)}$$

Similarly, the second equation of motion

$$\frac{d}{dt}(\varpi^2\dot{\theta}) \equiv \frac{d}{dt}(\varpi\Theta_0) = 0$$

can be written

$$\Pi_0\frac{\partial \Theta_0}{\partial \varpi} + \Theta_0\left(\frac{\Pi_0}{\varpi} + \frac{1}{\varpi}\frac{\partial \Theta_0}{\partial \theta}\right) = 0. \qquad \text{......(13)}$$

Hence (12) and (13) become, with the help of (7) to (10),

$$(A + C)\Pi_0 + (B - A_1)\Theta_0 + K = 0, \qquad \text{......(14)}$$

$$(B + A_1)\Pi_0 + (A - C)\Theta_0 = 0. \qquad \text{......(15)}$$

12·44. *The formula for ψ in terms of $A, \ldots A_1$.*

From 12·42 (21), by differentiating with respect to ϖ, we have

$$p = 2k\frac{\partial \Theta_0}{\partial \varpi} + 2\Theta_0\frac{\partial k}{\partial \varpi} + \Pi_0\frac{\partial m}{\partial \varpi} + m\frac{\partial \Pi_0}{\partial \varpi}. \qquad \text{......(1)}$$

Also, since by 12·42 (18) $$m\frac{\Theta_0}{\Pi_0} = -2h,$$

we have, on differentiating with respect to ϖ and using 12·42 (9),

$$\Pi_0\Theta_0\frac{\partial m}{\partial\varpi}+m\Pi_0\frac{\partial\Theta_0}{\partial\varpi}-m\Theta_0\frac{\partial\Pi_0}{\partial\varpi} = 0. \qquad ......(2)$$

Eliminate $\frac{\partial m}{\partial\varpi}$ from (1) and (2); there results

$$p = 2\Theta_0\frac{\partial k}{\partial\varpi}+2k\frac{\partial\Theta_0}{\partial\varpi}+m\left(2\frac{\partial\Pi_0}{\partial\varpi}-\frac{\Pi_0}{\Theta_0}\frac{\partial\Theta_0}{\partial\varpi}\right) \qquad ......(3)$$

or, in terms of the coefficients $A, \dots A_1$,

$$p = 2\Theta_0\frac{\partial k}{\partial\varpi}+2k(B+A_1)+m(3A+C). \qquad ......(4)$$

Again, differentiating 12·42 (21) with respect to θ, we have

$$2k\frac{\partial\Theta_0}{\partial\theta}+2\Theta_0\frac{\partial k}{\partial\theta}+\Pi_0\frac{\partial m}{\partial\theta}+m\frac{\partial\Pi_0}{\partial\theta} = 0. \qquad ......(5)$$

But, by 12·42 (11), $$\frac{\partial k}{\partial\theta} = -m,$$

and, by (12) and (18) of 12·42,

$$\frac{\partial m}{\partial\theta} = -\varpi\frac{\partial k}{\partial\varpi}+m\frac{\Theta_0}{\Pi_0}+2k.$$

Hence (5) becomes

$$\varpi\frac{\partial k}{\partial\varpi} = \frac{2k}{\Pi_0}\left(\Pi_0+\frac{\partial\Theta_0}{\partial\theta}\right)-\frac{m}{\Pi_0}\left(\Theta_0-\frac{\partial\Pi_0}{\partial\theta}\right),$$

from which $$\Pi_0\frac{\partial k}{\partial\varpi} = 2k(A-C)-m(A_1-B). \qquad ......(6)$$

Substitute the expression for $\partial k/\partial\varpi$, given in (6), in (4); then

$$p = 2k(B+A_1)+m(3A+C)+\frac{2\Theta_0}{\Pi_0}\{2k(A-C)-m(A_1-B)\}. \qquad ......(7)$$

Also, by 12·42 (21), $$p = 2k\frac{\Theta_0}{\varpi}+m\frac{\Pi_0}{\Theta_0}\frac{\Theta_0}{\varpi}. \qquad ......(8)$$

Equate (7) and (8) and write ω for $\frac{\Theta_0}{\varpi}$ and $\frac{C-A}{A_1+B}$ for $\frac{\Pi_0}{\Theta_0}$ as derived from 12·43 (15). After some reduction we obtain

$$2k(A_1+B+\omega) = m\left\{3A+C+\frac{2(A_1^2-B^2)}{A-C}+\frac{A-C}{A_1+B}\omega\right\}. \qquad ......(9)$$

Also, $$2h\Pi_0+m\Theta_0 = 0$$

or $$2h(A-C) = m(A_1+B).$$

Hence the deviation, ψ, defined in 12·42 (7) is given by

$$2\cot 2\psi = \frac{A_1+B}{A-C} - \frac{\left\{3A+C+\frac{2(A_1^2-B^2)}{A-C}+\frac{A-C}{A_1+B}\omega\right\}}{A_1+B+\omega} \quad\text{......(10)}$$

Ogrodnikoff* has summarised the values of A, B, C and A_1 obtained by several investigators from the analysis of the radial velocity and proper motion material then available. For purposes of calculation in (10) we take the set

$$A = 0{\cdot}0301, \quad B = 0{\cdot}0153, \quad C = -0{\cdot}0051, \quad A_1 = -0{\cdot}026.$$

Also, if $\Theta_0 = -300$ km./sec. and $\varpi = 10{,}000$ parsecs, we have $\omega = -0{\cdot}03$.

Substituting these values in (10), we find

$$\psi = +11^\circ.$$

Thus the longitude of the vertex exceeds the longitude of the centre by 11°.

This result is in qualitative agreement with conclusions drawn from the analysis of proper motions of stars of all spectral types according to the two-stream or ellipsoidal hypothesis. It is to be remembered, however, that the information concerning the values of A, B and C comes from the radial velocities of B type stars alone, and the evidence is somewhat conflicting and inconclusive as to the longitude of the vertex derived from these stars. For example, Lindblad† and Nordström‡ obtained values for the longitude considerably less than 325°. On the other hand, the values found by Smart and Green§ are 336° from 281 B type stars within 10° of the galactic equator, 301° for 253 stars between north and south galactic latitudes 10° to 30°, and 330° for 630 stars lying within 60° of the galactic equator; the probable error of the last result is ± 3°.

12·45. It is easily seen from the expression in 12·42 (5), defining the frequency function f, that the star density ν is given by

$$\nu \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\,d\Pi\,d\Theta = \frac{2\pi f_0}{(4hk-m^2)^{\frac{1}{2}}}.$$

If the potential is due entirely to the system of stars under consideration, Poisson's equation must be satisfied, namely,

$$\frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi K) = \frac{8\pi^2 G f_0}{(4hk-m^2)^{\frac{1}{2}}}. \quad\text{......(1)}$$

If, on the other hand, we regard the system of stars under observation—for example, the B type stars—as a subsystem, the potential under which such

* *Zeits. f. Astrophysik*, **4**, 190, 1932; see also Shiveshwarkar, *M.N.* **95**, 662, 1935.

† *M.N.* **90**, 503, 1930.

‡ *Lund Medd.* Ser. II, No. 79, p. 162, 1936.

§ *M.N.* **96**, 479, 1936.

stars move will be that produced by the entire galactic system and in this case the formula (1) does not require to be satisfied.

The group of equations (9) to (21), however, impose certain conditions on the form of the function K, as first shown independently by Heckmann* and Lindblad.†

By 12·42 (9), h is a function of θ alone. Hence, from 12·42 (10),

$$\frac{\partial^2 m}{\partial \varpi^2} = 0.$$

Consequently,
$$m = C\varpi + D, \qquad \text{......(2)}$$

where C and D are, at present, arbitrary functions of θ.

Also, from 12·42 (10) and (2),

$$\frac{\partial h}{\partial \theta} = D. \qquad \text{......(3)}$$

From 12·42 (11),
$$\frac{\partial^2 k}{\partial \theta \, \partial \varpi} = -C \qquad \text{......(4)}$$

and from 12·42 (12)
$$\varpi \frac{\partial^2 k}{\partial \theta \, \partial \varpi} + \frac{\partial^2 m}{\partial \theta^2} + 2\frac{\partial h}{\partial \theta} + 2m = 0,$$

whence, by (2), (3) and (4),

$$\varpi\left(\frac{\partial^2 C}{\partial \theta^2} + C\right) + \left(\frac{\partial^2 D}{\partial \theta^2} + 4D\right) = 0.$$

We must then have

$$\frac{\partial^2 C}{\partial \theta^2} + C = 0 \quad \text{and} \quad \frac{\partial^2 D}{\partial \theta^2} + 4D = 0,$$

so that
$$C = a\cos\theta + b\sin\theta, \qquad \text{......(5)}$$
$$D = -2\alpha \sin 2\theta + 2\beta \cos 2\theta, \qquad \text{......(6)}$$

where a, b, α and β are constants.

Hence
$$m = \varpi(a\cos\theta + b\sin\theta) - 2\alpha\sin 2\theta + 2\beta\cos 2\theta \qquad \text{......(7)}$$

and
$$h = \alpha\cos 2\theta + \beta\sin 2\theta + \gamma, \qquad \text{......(8)}$$

in which γ is a constant.

Also from 12·42 (12),
$$\frac{\partial}{\partial \varpi}\left(\frac{k}{\varpi^2}\right) = -\frac{1}{\varpi^3}\frac{\partial m}{\partial \theta} - \frac{2h}{\varpi^3},$$

from which it is readily found that

$$k = \varpi(b\cos\theta - a\sin\theta) - \alpha\cos 2\theta - \beta\sin 2\theta + \gamma + \kappa\varpi^2,$$

where κ is a constant.

From (16) and (17) of section 12·42, we have

$$2K\frac{\partial h}{\partial \theta} = \frac{\partial}{\partial \varpi}(m\varpi K). \qquad \text{......(9)}$$

* *M.N.* **96**, 67, 1935. † *M.N.* **96**, 69, 1935.

This becomes, on writing $F(\varpi) \equiv \frac{\varpi}{K}\frac{\partial K}{\partial \varpi}$,

$$\varpi\{F(\varpi)+2\}(a\cos\theta+b\sin\theta)-2\{F(\varpi)-1\}(\alpha\sin 2\theta-\beta\cos 2\theta) = 0. \quad \text{......(10)}$$

This equation must hold for all values of ϖ and θ. We then have the following cases.

(i) $a = b = \alpha = \beta = 0.$

The function $F(\varpi)$—and consequently the function K—is accordingly arbitrary.

We have $h = \gamma;\quad m = 0;\quad k = \gamma+\kappa\varpi^2,$

and from 12·42 (18) and (21),

$$\Pi_0 = 0 \quad \text{and} \quad \Theta_0 = \frac{p\varpi}{2(\gamma+\kappa\varpi^2)}.$$

These results are the same as those obtained in the theory of galactic rotation discussed previously.

(ii) $a \neq 0$ or $b \neq 0$; $\alpha = \beta = 0$.

Then $$F(\varpi)+2 = 0,$$

whence $$K(\varpi) = \frac{q}{\varpi^2}.$$

The attraction in this case is due to a central mass.

(iii) $a = b = 0$; $\alpha \neq 0$ or $\beta \neq 0$.

Then $$F(\varpi) = 1,$$

whence $$K(\varpi) = s\varpi.$$

The attraction in this case is due to a uniform spheroidal distribution of mass.

The conclusion follows that if there is expansion (or contraction) Shiveshwarkar's theoretical derivation of a vertex deviation is legitimate only if the attractive force is due either to a central mass or to a uniform spheroidal distribution.

Lindblad's derivation* of the preceding results follows simply from (2), (3) and (9). We have at once

$$\frac{\varpi}{K}\frac{\partial K}{\partial \varpi} = \frac{D-2\varpi C}{D+\varpi C}. \quad \text{......(11)}$$

The left-hand side of (11) is a function of ϖ alone. Hence, if (11) is to be valid for all values of ϖ and θ, either $D = 0$ or $C = 0$, so that

$$\frac{\varpi}{K}\frac{\partial K}{\partial \varpi} = -2 \quad \text{or} \quad 1.$$

The results for K follow as in cases (ii) and (iii).

* *M.N.* **96**, 71, 1935.

12·46. *The analytical forms of f_0.*

In (16) and (17) of section 12·42 we write

$$\log f_0 - h\Pi_0^2 - m\Pi_0\Theta_0 - k\Theta_0^2 \equiv \log f_0 - P \equiv Q.$$

Now, by (18) and (21) of section 12·42,

$$\Pi_0 = -\frac{pm\varpi}{4hk-m^2}, \quad \Theta_0 = \frac{2ph\varpi}{4hk-m^2}.$$

Hence $$P = \frac{p^2h\varpi^2}{4hk-m^2}. \qquad \text{......(1)}$$

Considering case (ii) of the previous section ($K = q/\varpi^2$), we have

$$h = \gamma; \quad m = \varpi(a\cos\theta + b\sin\theta),$$

$$k = \varpi(-a\sin\theta + b\cos\theta) + \gamma + \kappa\varpi^2.$$

We then write (16) and (17) of section 12·42 as follows (using (11)):

$$\frac{\partial}{\partial\varpi}\left(Q - \frac{2\gamma q}{\varpi}\right) = 0,$$

$$\frac{\partial}{\partial\theta}\left(Q - \frac{2\gamma q}{\varpi} - \frac{qk}{\varpi}\right) = 0$$

or $$\frac{\partial}{\partial\varpi}\left\{Q - \frac{2\gamma q}{\varpi} + q(a\sin\theta - b\cos\theta)\right\} = 0,$$

$$\frac{\partial}{\partial\theta}\left\{Q - \frac{2\gamma q}{\varpi} + q(a\sin\theta - b\cos\theta)\right\} = 0.$$

Hence $$Q = \frac{2\gamma q}{\varpi} - q(a\sin\theta - b\cos\theta) + c,$$

where c is a constant. Thus

$$\log f_0 = \frac{p^2h\varpi^2}{4hk-m^2} + \frac{2\gamma q}{\varpi} - q(a\sin\theta - b\cos\theta) + c.$$

In the first term on the right the expressions for k and m are to be substituted.

In case (iii), $K = s\varpi$ and

$$h = \alpha\cos 2\theta + \beta\sin 2\theta + \gamma,$$

$$m = -2\alpha\sin 2\theta + 2\beta\cos 2\theta,$$

$$k = -h + 2\gamma + \kappa\varpi^2.$$

We have $$\frac{\partial}{\partial\varpi}(Q + hs\varpi^2) = 0,$$

$$\frac{\partial}{\partial\theta}(Q - ks\varpi^2) = 0.$$

This last equation is equivalent to

$$\frac{\partial}{\partial\theta}(Q+hs\varpi^2) = 0.$$

Hence
$$\log f_0 = \frac{p^2h\varpi^2}{4hk-m^2} - hs\varpi^2 + c_1,$$

where c_1 is a constant, the expressions for h, k and m being supposed substituted in the first term of the right-hand side of this formula.

12·51. *Oort's investigation of the acceleration perpendicular to the galactic plane.*

With the usual notation we take z to be the perpendicular distance of a star from the galactic plane and Z the corresponding component of velocity. The acceleration in the z-direction due to the gravitational field of the galaxy will be denoted by $K(z)$. From the observed fact that the distribution of stars, especially of the later types, is fairly symmetrical on either side of the galactic plane, it is reasonable to assume that the function $K(z)$ is such as to be capable of maintaining, at least approximately, the present observed distribution and that, so far as the z-direction is concerned, the stars are, on the whole, well mixed. As we shall see later, the distribution of the Z-components of velocity is also symmetrical about the galactic plane. The following analysis is due to Oort.*

The equation of motion of a star in the z-direction is

$$\dot{Z} \equiv Z\frac{dZ}{dz} = K(z),$$

whence
$$Z^2 = 2\int_0^z K(z)\,dz + \text{constant}.$$

If Z_0 is the value of Z for $z = 0$, this last equation gives

$$Z^2 = Z_0^2 + 2\int_0^z K(z)\,dz. \qquad \text{......(1)}$$

From (1), $Z \equiv dz/dt$ is a function of Z_0 and z; call it $F(Z_0, z)$. Hence

$$dt = \frac{dz}{F(Z_0, z)}. \qquad \text{......(2)}$$

The period of a star's oscillation in the z-direction is then obtained by integrating (2) between the appropriate limits of z. Thus the period is a function of Z_0; we denote it by $T(Z_0)$.

The fraction of the period spent in the layer between z and $z+dz$ is evidently
$$\frac{2dz}{ZT(Z_0)},$$
the factor 2 accounting for the upward and downward passage through the layer.

* *B.A.N.* **6**, 249, 1932.

Let $f(Z_0)\,dZ_0$ denote the total number of stars, above or below unit area of the galactic plane, with components of velocity at $z = 0$ between Z_0 and Z_0+dZ_0; let $\phi(z, Z)\,dz\,dZ$ denote the average number of stars in the element of volume between z and $z+dz$ with velocity components between Z and $Z+dZ$. We then have

$$\phi(z, Z)\,dz\,dZ = f(Z_0)\,dZ_0\,\frac{2dz}{ZT(Z_0)}.$$

But, from (1), $$dZ/Z_0 = dZ_0/Z.$$

Hence $$\phi(z, Z) = \frac{2f(Z_0)}{Z_0\,T(Z_0)}. \qquad \text{......(3)}$$

Accordingly, $\phi(z, Z)$ is a function of Z_0 alone or, by (1), of

$$\left\{Z^2 - 2\int_0^z K(z)\,dz\right\}.$$

If the distribution of the Z-components for stars on the galactic plane is Gaussian, it follows that the distribution of velocities at a distance z is also Gaussian. For, if $\nu(0)$ denotes the stellar density for $z = 0$, we have

$$\phi(0, Z_0) = \nu(0)\frac{l}{\sqrt{\pi}}\,e^{-l^2 Z_0^2}, \qquad \text{......(4)}$$

since $\nu(0)$ is the value obtained by integrating $\phi(0, Z_0)$ over all possible values of Z_0. Also, since $\phi(z, Z)$ is a function of Z_0 alone, we must have, by means of (1) and (4),

$$\phi(z, Z) = \nu(0)\,e^{2l^2\int_0 K(z)dz}\cdot\frac{l}{\sqrt{\pi}}\,e^{-l^2 Z^2}, \qquad \text{......(5)}$$

which is a Gaussian distribution for the Z-components. By integrating (5) over all possible values of Z, we find that the density, $\nu(z)$, at a distance z from the galactic plane is given by

$$\nu(z) = \nu(0)\,e^{2l^2\int_0^z K(z)dz}. \qquad \text{......(6)}$$

If the observed velocity distribution is not quite Gaussian it can, according to Oort, be represented satisfactorily, as a rule, by the sum of two Gaussian distributions. The relevant formulae are then

$$\phi(0, Z_0) = \nu_0\left\{\theta_1\cdot\frac{l_1}{\sqrt{\pi}}\,e^{-l_1^2 Z_0^2} + \theta_2\cdot\frac{l_2}{\sqrt{\pi}}\,e^{-l_2^2 Z_0^2}\right\} \qquad \text{......(7)}$$

and $$\nu(z) = \nu_0\left\{\theta_1 e^{2l_1^2\int_0^z K(z)dz} + \theta_2 e^{2l_2^2\int_0^z K(z)dz}\right\} \qquad \text{......(8)}$$

in which $0 < \theta_1 \leqslant 1$ and $\theta_1 + \theta_2 = 1$.

For an arbitrary distribution of the Z-components the appropriate relation between ν, $K(z)$ and the mean, $\overline{Z^2}$, of the squared velocities is

$$K(z) = \frac{1}{\nu}\frac{\partial(\nu\overline{Z^2})}{\partial z}. \qquad \text{......(9)}$$

If $K(z)$ is expressed in cm./sec.^{2}, and Z and z, as usual, in km./sec. and parsecs respectively, (9) can be written*

$$K(z) = 7{\cdot}48 \times 10^{-9}\overline{Z^2}\frac{\partial}{\partial z}\text{Log}\,(\nu\overline{Z^2}). \qquad \text{......(10)}$$

12·52. Oort's application of the formulae in the previous section will now be briefly noted.

The distribution of the Z-components of linear velocity was first investigated by means of radial velocities of stars within 50° of the galactic poles—in this connection the radial velocities are assumed to be corrected for the effects of the solar motion. For a star situated at one or other of the galactic poles the radial velocity and the Z-component are evidently identical. For stars in the galactic zones considered, a suitable factor was applied to the radial velocities to transform them to the Z-components. Generally, the Z-distribution was found to be represented satisfactorily by a sum of two Gaussian distributions and the values of l_1, l_2 and θ_1, of formulae (7) and (8), were obtained for each spectral type. A subsidiary investigation showed that the Z-distributions were substantially symmetrical above and below the galactic plane, which supports the assumption on which the formulae of the previous section are based.

To determine the variation of density (ν) with distance from the galactic plane, Oort utilised van Rhijn's results† concerning the numbers of stars between given limits of parallax and absolute magnitude. Thus the values of Log $\nu(z)$ and d/dz Log $\nu(z)$ were obtained for values of z up to 600 parsecs in four groups of absolute magnitude ranging from $-1^{m}{\cdot}5$ to $+5^{m}{\cdot}5$.

With the data indicated and values of $\overline{Z^2}$ for different distances, numerical estimates of $K(z)$ for the four magnitude groups were calculated by means of 12·51 (10). Table 52 shows the values of $K(z)$ averaged for the four groups.

Table 52. *Values of K(z)*

z (parsecs)	$K(z)$ (cm./sec.^{2})	z (parsecs)	$K(z)$ (cm./sec.^{2})
50	$-0{\cdot}77 \times 10^{-9}$	250	$-3{\cdot}78 \times 10^{-9}$
100	$-1{\cdot}55$	300	$-3{\cdot}86$
150	$-2{\cdot}59$	400	$-3{\cdot}68$
200	$-3{\cdot}52$	600	$-4{\cdot}44$

* *B.A.N.* **6**, 261, 1932. † *Groningen Publ.* **38**, 1930.

In particular, the data of the table enable us to derive a value (at least approximate) of $\partial K(z)/\partial z$ for $z = 0$—that is, for the neighbourhood of the sun; this value is $5 \,.\, 10^{-30}$ sec.$^{-2}$.

Oort then attempted to obtain some idea of the density of dark matter near the sun by considering several models of the galaxy. For example, if the model consists of a central mass and a uniform spheroidal distribution, with semi-axes a and c, we have

$$\frac{\partial K(z)}{\partial z} = -4\pi G\gamma\Delta, \qquad \text{......(1)}$$

in which G is the gravitational constant, γ is a number depending on the ratio of $a:c$—the value of γ can be easily found from the formulae (1) and (2) of section 11·54—and Δ is the total density, including stars and dark matter. Formula (1) then enables us to calculate the value of Δ near the sun. Oort investigated four different models, the total masses being conditioned by the consideration that the forces exerted in the galactic plane balance the centrifugal force arising from a rotational velocity of 300 km./sec. at a distance of 10,000 parsecs from the galactic centre. It was found that Δ varied between 0·079 and 0·108 solar masses per cubic parsec, with an average value of 0·092. The stellar density near the sun is estimated to be 0·038 ⊙ per cubic parsec for stars of absolute magnitude brighter than + 13·5. Making an allowance for intrinsically fainter stars, Oort concluded that the density of dark matter near the sun is not likely to exceed 0·05 ⊙ per cubic parsec or 3×10^{-24} gm./cm.^{3}

APPENDIX

ASTRONOMICAL CONSTANTS

[Coordinates are for epoch 1900·0; T is measured in Julian centuries from 1900·0.]

Constant of gravitation: $6{\cdot}658 \times 10^{-8}$ c.g.s. units.

Velocity of light in vacuo: 299,774 km./sec. or 186,271 miles/sec.

1 Astronomical unit: $149{\cdot}5 \times 10^{6}$ km. or $92{\cdot}9 \times 10^{6}$ miles.

1 Parsec: $30{\cdot}84 \times 10^{12}$ km. or $19{\cdot}16 \times 10^{12}$ miles.

1 Light-year: $9{\cdot}463 \times 10^{12}$ km. or $5{\cdot}880 \times 10^{12}$ miles.
  1 Parsec = 3·26 light-year.
  1 Light-year = 0·307 parsec.

Mass of Earth: $5{\cdot}98 \times 10^{27}$ gm.

Mass of Sun: $2{\cdot}00 \times 10^{33}$ gm.

Annual general precession (*Newcomb*): $50''{\cdot}2564 + 0''{\cdot}0222\,T$.

Annual precession in R.A. (*Newcomb*): $m = 46''{\cdot}0850 + 0''{\cdot}0279\,T$
$= 3^{s}{\cdot}07234 + 0^{s}{\cdot}00186\,T$.

Annual precession in Dec. (*Newcomb*): $n = 20''{\cdot}0468 - 0''{\cdot}0085\,T$.

Obliquity of ecliptic (*Newcomb*): $\epsilon = 23^\circ 27' 8''{\cdot}26 - 46''{\cdot}85\,T$.

Galactic equator:—
  Pole: $\alpha = 190^\circ$, $\delta = +28^\circ$.
  Longitude of ascending node on ecliptic: $266^\circ{\cdot}96 + 1^\circ{\cdot}40\,T$.
  ,, ,, ,, ,, ,, equator: $280^\circ{\cdot}00 + 1^\circ{\cdot}23\,T$.
  Inclination to ecliptic: $60^\circ{\cdot}55$.
  ,, ,, equator: $62^\circ{\cdot}00 + 0^\circ{\cdot}55\,T$.

Coordinates of galactic centre: $G = 327^\circ$, $g = 0^\circ$.

Sun's distance from galactic centre: 10,000 parsecs.

	α	δ
Solar apex (*Boss*)	270°	+ 34°
(*Eddington*)	267°	+ 36°
(*Campbell and Moore*)	271°	+ 29°
(*Smart and Green*)	267°	+ 32°

Solar motion (*Campbell and Moore*): 19·7 km./sec.
(*Smart and Green*): 19·5 km./sec.

Apex of Drift I (*Eddington*): $\alpha = 91°$, $\delta = -15°$.

,, ,, ,, II ,, $\alpha = 288°$, $\delta = -64°$.

Vertex of star-streaming (*Eddington*): $\alpha = 274°$, $\delta = -12°$.
$G = 347°$, $g = 0°$.

Solar motion with respect to clusters (*Edmondson*):—
$V_0 = 274$ km./sec.; Apex: $G = 67°$, $g = +1°$.

Oort's constants: $A = 0{\cdot}017$ km./sec./parsec.
$B = -0{\cdot}015$ km./sec./parsec.

Mass of galaxy: $2 \times 10^{11} \odot = 4 \times 10^{44}$ gm.

Orbital period at sun's distance from galactic centre: 2×10^8 years

Galactic absorption (*Trumpler*):
Visual: $0^{m}{\cdot}35$ per kiloparsec.
Photographic: $0^{m}{\cdot}67$,, ,,

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