# Chapter 6

# Uniqueness of Rectangular Duals

So far we have studied the methods of transformation of a rectangular dual to another rectangular dual. In this chapter, we derive a class of RDGs wherein each RDG admits a unique rectangular dual upto combinatorial equivalence. Then we show that such RDGs are slicible as well as area-universal.

## 6.1 Introduction

Our motivation to find the class of RDGs wherein each RDG admits a unique rectangular dual stems from slicibility and area-universality characteristics of the unique rectangular duals upto combinatorial equivalence. A slicible rectangular dual is always desirable due to its simplicity and efficiency [19]. In an area-universal rectangular dual, assignments of areas to its component rectangles can be specified at later design stages. Thus, we see that the ability of finding an area-universal rectangular dual at the early design stage will greatly simplify the design process at later stages. In fact, a slicible rectangular dual generates strong equivalence class whereas an area-universal rectangular dual generates weak equivalence class. Thus in VLSI circuit and architectural floorplanning, such a rectangular dual is always desirable.

Previous attempts [37, 40, 64] shows that a number of topologically distinct rectangular duals can be realized from an RDG. Yet, there exist a lot of RDGs which can be uniquely dualized (refer to Fig. 6.1) but the class of RDGs wherein each RDG can be uniquely dualized is still lacking.

In this chapter, we derive a class of RDGs in which each RDG admits a unique rectangular dual upto combinatorial equivalence. Then we show that such RDGs are slicible as well as area-universal. Mathematically such class is interesting to study since each of its RDGs has no alternative solutions, i.e., there is no need to recursively improve the solution.

The chapter is structured as follows. In Section 6.2, we first present a necessary and sufficient condition for an RDG to admit a unique rectangular dual upto combinatorial equivalence. Then we prove that an RDG representing a unique rectangular dual is slicible as well as area-universal. Finally, we conclude our contributions in Section 6.3.

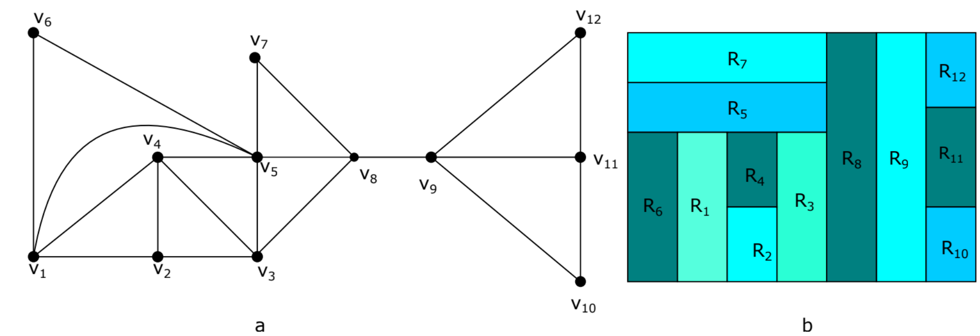


**Figure 6.1:** (a) An RDG that admits (b) a unique rectangular dual upto combinatorial equivalence.

## 6.2 The Class of Unique Rectangular Duals

In this section, we derive a class of RDGs wherein each RDG can be realized by a unique rectangular dual upto combinatorial equivalence. We also show that each of its RDGs in the class is slicible as well as area-universal.

**Theorem 6.2.1.** Let $G$ be an RDG having atleast four vertices. A necessary condition for $G$ to admit a unique rectangular dual is that $G$ has exactly 4 vertices of degree 2.

*Proof.* Assume that $G$ admits a unique rectangular dual $R$. Denote the degree of a vertex $v_k$ by $d(v_k)$. Let $v_c$ be a vertex in $G$ corresponding to a corner rectangle $R_c$ in $R$. Since $R_c$ is a corner rectangle in $R$, its two sides are adjacent to the exterior. We claim that $d(v_c) = 2$. To the contrary, suppose that $d(v_c) > 2$. This implies that there exist rectangles $R_1, R_2, \ldots, R_n$ $(n \geq 2)$ that are adjacent to the same side of $R_c$ and one of them is an exterior rectangle. Denote it by $R_e$. Now the edges that have an endpoint incident to $R_1, R_2, \ldots, R_n$ and the other endpoint incident to $v_c$ have the same orientations (horizontal or vertical). Consequently, the inner face containing the boundary edge $e_b$ joining $v_c$ and the vertex dual to $R_e$ is towards $v_c$. By Theorem

2.3.1, the orientation of $e_b$ is changeable, which contradicts the fact that $G$ admits a unique rectangular dual. This proves our claim. Similarly, the degree of vertices, that are duals to the remaining three corner rectangles of $R$, can be shown to be equal to two. Hence the theorem. □

Consider an RDG $G$ shown in Fig. 6.2b. Although it has 4 vertices of degree 2, it admits more than one topological distinct rectangular duals as shown in Fig. 6.2a and 6.2c (their regular edge labelings shown in 6.2d and 6.2e are distinct). Hence, the converse of Theorem 6.2.1 is not true.

It can be seen from Theorem 6.2.1 that the orientations of both boundary edges incident to a 2 degree vertex of an RDG is not changeable, i.e., the orientations of edges of four corner rectangles in its rectangular dual are fixed and hence the orientations of edges of all its exterior rectangles with the exterior are also fixed. Now we make the necessary condition more stronger such that it can be a sufficient condition for an RDG to admit a unique rectangular dual. We first need to prove the following Lemma.

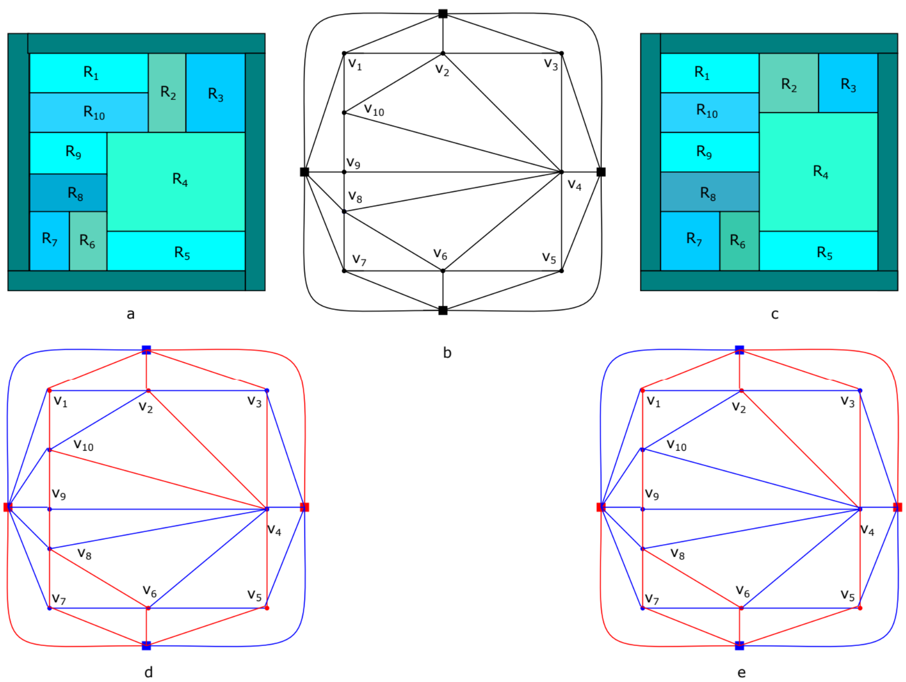


**Figure 6.2:** (b) An RDG with exactly 4 vertices of degree 2, (a) and (c) the corresponding more than one topological distinct rectangular duals, (d) and (e) respective regular edge labelings of these rectangular duals in (c) and (d).

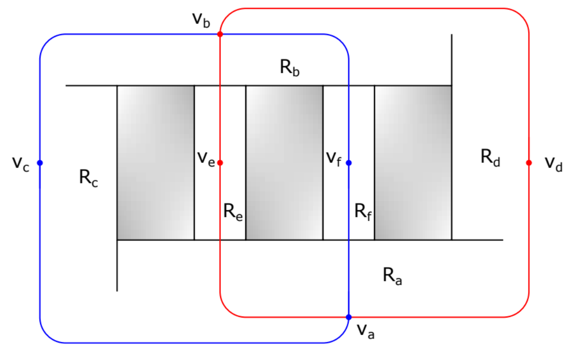


**Figure 6.3:** Two 4-cycles intersecting at two non-adjacent vertices of each other.

**Lemma 6.2.1.** Let $G$ be an RDG. If $C_1$ and $C_2$ are 4-cycles in $G$ intersecting each other at vertices $v_a$ and $v_b$ such that two edges of $C_1$ lie in the interior of $C_2$ and $v_a, v_b$ are non-adjacent in both $C_1$ and $C_2$, then $C_1$ and $C_2$ never bound a T-structure.

*Proof.* The statement can be visualized using Fig. 6.3, where $C_1$ (shown by red edges) is intersected by $C_2$ (shown by blue edges) at two non-adjacent vertices $v_a$ and $v_b$. Shaded areas contain other component rectangles. Note that the rectangles dual to a 4-cycle of an RDG enclosing some vertices, bound a rectangular area. Therefore the rectangles dual to the four vertices of $C_2$ bounds a rectangular area enclosing the rectangle $R_e$ which is dual to a vertex $v_e$ of $C_1$. This implies that the edges incident to $v_e$ and lying on $C_2$ have the same orientations which leads to a directed path consisting of only red edges (or blue edges) in the regular edge labeling of $G$ joining $v_a$ and $v_b$ on $C_1$. This implies that $C_1$ is not a cycle of alternating edges in the regular edge labeling of $G$ and hence by Theorem 2.3.1 or 2.3.2, $C_1$ never bounds a T-structure. Applying the same argument, we can show that $C_2$ never bounds a T-structure. Hence the result. □

**Theorem 6.2.2.** Let $G$ be an RDG. A necessary and sufficient condition for $G$ to admit a unique rectangular dual is that $G$ has four vertices of degree 2 and for any 4-cycle $C_1$ in $G$, there is another 4-cycle $C_2$ intersecting $C_1$ at vertices $v_a$ and $v_b$ of $G$ such that two edges of $C_1$ lie in the interior of $C_2$ where $v_a$ and $v_b$ are non-adjacent in both $C_1$ and $C_2$.

*Proof.* **Necessary Condition:** Assume that $G$ admits a unique rectangular dual $R$. By Theorem 3.4.1, it has four vertices of degree 2.

By Lemma 5.3.3, each regions of $G$ is triangular. Then for $n > 3$, there always exists atleast one 4-cycle in $G$ consisting of a single edge or atleast one vertex in its interior. To the contrary, assume that for a 4-cycle $C_1$ in $G$, there is no another 4-cycle $C_2$ intersecting $C_1$ at vertices $v_a$ and $v_b$ of $G$ such that two edges of $C_1$ lie in the interior of $C_2$ where $v_a$ and $v_b$ are non-adjacent in both $C_1$ and $C_2$. Then there are the following six possibilities for the occurrence of 4-cycles in $G$:

i. a 4-cycle enclosing a single edge (see Fig. 6.4a),

ii. two 4-cycles $C_p$ and $C_q$ intersecting at two vertices $v_a$ and $v_b$ where $v_a$ and $v_b$ are non-adjacent in $C_p$ and are adjacent in $C_q$ (see Fig. 6.4d),

iii. a 4-cycle enclosing atleast one vertex (see Fig. 6.4g),

iv. two 4-cycles intersecting at vertex $v_a$ such that one completely lies inside the other (see Fig. 6.4j),

v. two 4-cycles sharing an edge such that one completely lies inside the other (see Fig. 6.4k),

vi. two 4-cycles sharing two edges such that one completely lies inside the other (see Fig. 6.4l).

As shown in Fig. 6.4, the first three cases have T-structures. In Fig. 6.4j, the interior 4-cycle has no T-structure due to the well formedness of the outer 4-cycle. Since the outer 4-cycle encloses atleast one vertex, therefore it is similar to the 4-cycle shown in Fig. 6.4g and hence it has a T-structure.

If there exists another 4-cycle $C$ containing this outer 4-cycle in its interior sharing a vertex, then $C$ is similar to the 4-cycle shown in Fig. 6.4g and hence $C$ has a T-structure and the interior two 4-cycles enclosed by it has no T-structure. If there is a chain of such 4-cycles with the property that the one which lies inside other shares a vertex, then the outermost 4-cycle in such chain has always a T-structure. Similarly, the outermost 4-cycle has a T-structure in Fig. 6.4k and 6.4l. Thus we have seen that all the six possibilities have T-structure which is a contradiction to the fact that $G$ admits a unique rectangular dual $R$. This proves the necessary part.

**Sufficient Condition:** Assume that the given conditions hold. Note that none of the boundary edges incident to a vertex $v_t$ of degree 2 can be towards $v_t$. By Theorem 2.2.1, the orientations of both boundary edges incident to $v_t$ are not changeable. Also by Lemma 6.2.1, $C_1$ and $C_2$ never bound a T$-$structure. Consequently, $G$ admits a unique rectangular dual. Hence the proof. □

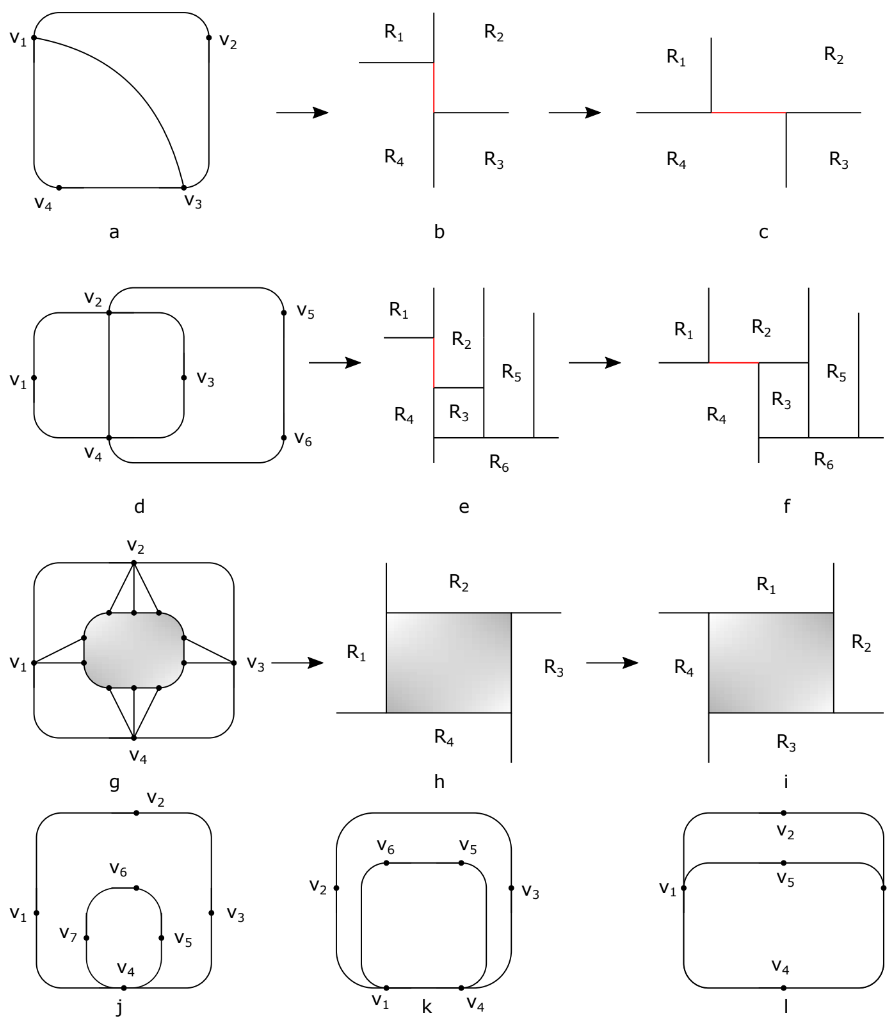


**Figure 6.4:** Demonstrations of all possible T-structures upto isomorphism.

It has been observed but not yet proved that if a given nonseparable RDG has 4 vertices of degree 2, then there is always T-structure in its rectangular dual and hence the following result is conjectured.

**Conjecture 6.2.1.** If $G$ is a nonseparable RDG, it always admits more than one topologically distinct rectangular dual.

**Theorem 6.2.3.** If a nonseparable RDG $G$ satisfying the condition of Lemma 6.2.1, then $G$ admits a unique rectangular dual upto the fixed orientations of edges of its corner rectangles.

*Proof.* Let $G$ admit a rectangular dual $R$. By given condition, $R$ is a rectangular dual upto the fixed orientations of edges of its corner rectangles. Then the orientations of the boundary edges of $R$ is not changeable. Also given that $G$ satisfies Lemma 6.2.1. Thus there is no T-structure in $G$. Hence the proof. □

Consider an RDG $G$ shown in Fig. 6.5a. Its extended RDG is constructed in Fig. 6.5b in order to fix corner vertex assignments. A rectangular dual constructed for $G$ is shown in Fig. 6.5c. There are two 4-cycles in $G$ intersecting each other at vertices $v_4$ and $v_6$. Therefore, these 4-cycles do not enclose T-structure. Also, there are two 4-cycles passing through vertices $v_4$, $v_5$, $v_7$ and $v_8$, and $v_6$, $v_5$, $v_7$ and $v_9$. These 4-cycles do not enclose a T-structure since they are passing through corner vertices. In fact, the orientations of edges of corner rectangles of the rectangular dual are fixed. Thus, $G$ admits a unique rectangular dual upto the fixed orientations of its corner rectangles.

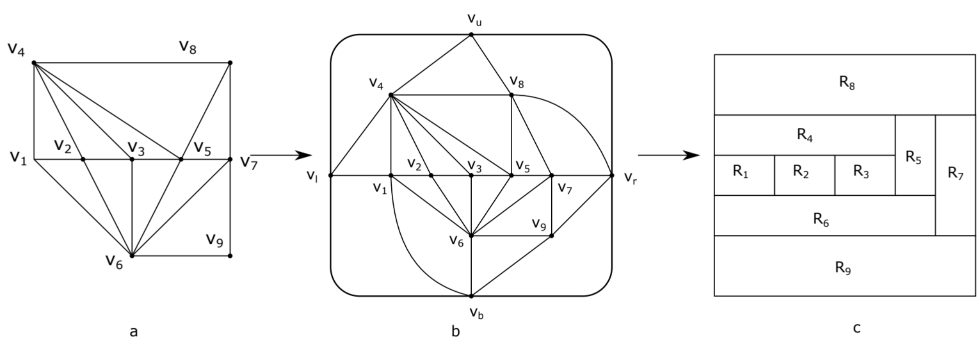


**Figure 6.5:** (a) A nonseparable RDG, (b) its extended RDG and (c) a unique rectangular dual for the RDG upto the fixed orientations of its corner rectangles.

**Theorem 6.2.4.** If an RDG admits a unique rectangular dual, then the rectangular dual is slicing.

*Proof.* Assume that $G$ is an RDG admitting a unique rectangular dual. If $G$ has no 4-cycle, by Theorem 2.2.3, $G$ is a slicing RDG. If $G$ has 4-cycles, then by Theorem 6.2.2 there only exist pairs of 4-cycles in $G$ intersecting each other at vertices $v_a$ and $v_b$ of $G$ such that the two edges of one of the two cycles (forming a pair) lie in the interior of the other where $v_a$ and $v_b$ are non-adjacent in both cycles. Clearly, none of them is contained in each other and hence both 4-cycles are maximal. Since this is an arbitrary pair of 4-cycles, each 4-cycle of $G$ is maximal. By Theorem 2.2.4, $G$ can be realized by a slicing rectangular dual. □

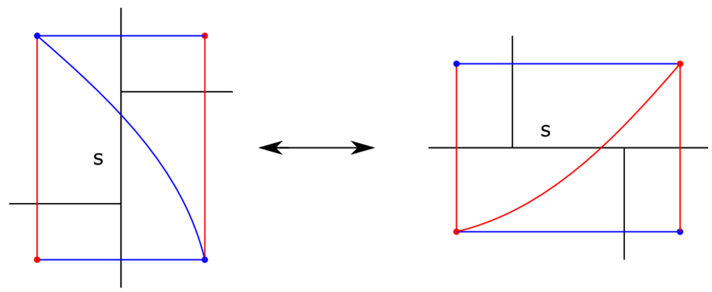


**Figure 6.6:** Multiple rectangles on both sides of a maximal line segment $s$.

**Theorem 6.2.5.** If an RDG admits a unique rectangular dual, then it is area-universal.

*Proof.* Suppose that an RDG $G$ admits a unique rectangular dual $R$. To the contrary, suppose that $R$ is not area-universal. This implies that there is a maximal internal line segment $s$ in $R$ which is not the side of any of its rectangles, i.e., $s$ is a maximal internal line segment of $R$ with multiple rectangles on both of its sides. Then an edge $e$ of $R$ (from which $s$ is formed) must have one of its endpoints as a T-junction (a point where three rectangles meet) formed by the corners of two rectangles on one side of $s$, and on its other endpoint, it must have a T-junction formed by the corners of two rectangles on the other side of $s$, as shown in Fig. 6.6. Vertices dual to these four rectangles form a 4-cycle of the alternating orientations in $G$. By Theorem 2.3.1, this 4-cycle is a changeable set which contradicts to the fact that $G$ admits a unique rectangular dual $R$. Hence the theorem. □

## 6.3 Concluding Remarks

In general the rectangular dual solution space of RDGs is very large. There is some work on generation of different rectangular duals of the oriented RDGs, but we derived a necessary and sufficient condition for an unoriented RDG to admit a unique rectangular dual. We also characterized such RDGs are slicing as well as area-universal which play an important role in the floorplanning.

It remains to investigate whether a non-separable RDG always has more than one topologically distinct rectangular duals and to find the exact number of such rectangular duals.