Chapter 2

Fundamentals Tools and Methodology

In this chapter, we provide several useful tools of RDGs and rectangular duals that would be helpful for subsequent chapters of this thesis. Most of the presented tools are repeatedly used in the subsequent chapters of this thesis. We begin with general statements for plane graphs in Section 2.1 and proceed with various properties, and results of RDGs and rectangular duals in Sections 2.2-2.4. In Section 2.5, we describe the methodology, we adapt to derive the results in this thesis.

2.1 Planar And Plane Graphs

In this thesis, we consider simple and finite graphs. A graph is *simple* if it is independent of multiple edges (parallel edges) as well as loops. All the notations and terminologies are standard unless it is stated [68]. A graph is called *planar* if it can be drawn in the Euclidean plane without crossing its edges except endpoints. A *plane graph* is a planar graph with a fixed planar drawing. It splits the Euclidean plane into connected regions called *faces*; the unbounded region is the *exterior face* (the outermost face) and all other faces are interior faces. The vertices lying on the exterior face are exterior vertices and all other vertices are *interior vertices*. A vertex v_c of a graph is called a *cut-vertex* if the removal of v_c from the graph disconnects the graph. A graph is said to be $k-$connected if it has at least k vertices and the removal of fever than k vertices does not disconnect the graph. If a connected graph has a cut vertex, then it is called a *separable graph*, otherwise it is called a *nonseparable graph*. Since floorplans are concerned with connectivity, we only consider nonseparable (biconnected) and separable connected graphs in this thesis. A *plane block* is a biconnected graph. A maximal block of a graph G is a maximal biconnected subgraph. A plane graph is called *plane triangulated graph* (PTG) if it has triangular faces. The exterior face may not be triangular in a PTG. In case, the exterior face is also triangular, then the PTG is called *plane triangulation*. A *wheel graph* W_n is a graph in which a single vertex is adjacent to $n-1$ vertices lying on a cycle.

Theorem 2.1.1. [68] A graph G is 4-connected if and only if there exist at least 4 vertex-disjoint paths between any two vertices of G.

2.2 Rectangularly Dualizable Graphs

A graph H is called *dual* of a plane graph G if there is one-to-one correspondence between the vertices of G and the regions of H, and two vertices of G are adjacent if and only if the corresponding regions of H are adjacent.

Definition 2.2.1. A plane graph is a *rectangularly dualizable graph* (RDG) if its dual can be embedded as a *rectangular partition*.

Definition 2.2.2. A *rectangular dual* R of a plane graph G is a partition of a rectangle into n-rectangles such that (i) no four rectangles of R meet at a point, (ii) rectangles in R are mapped to vertices of G, and (iii) two rectangles in R share a common boundary segment if and only if the corresponding vertices are adjacent in G.

Definition 2.2.3. [38] The *block neighborhood graph* (BNG) of a plane graph G is a graph N whose vertex set is one-to-one corresponds with the set of biconnected components of G and two vertices of N are adjacent if and only if the corresponding biconnected components have a common vertex.

Definition 2.2.4. [38] A *shortcut* (chord) in a plane block G is an edge that is incident to two vertices of the outermost cycle C of G and is not a part of C. A corner implying path (CIP) in G is a $v_1 - v_k$ path on C such that it does not contain any vertices of a shortcut other than v_1 and v_k and then the shortcut (v_1, v_k) is called a critical shortcut. A critical CIP in a biconnected component H_k of a separable plane graph G is a CIP of H_k that contains no cut-vertex of G in its interior.

For instance, consider the graph shown in Fig. 2.2a. Edges (v_1, v_3), (v_4, v_9) and (v_6, v_8) are shortcuts. Paths $v_1v_2v_3$ and $v_6v_7v_8$ are CIPs while path $v_9v_1v_2v_3v_4$ is not a CIP because of containment of the endpoints of another shortcut (v_1, v_3) and hence (v_9, v_4) is not a critical shortcut. Each of shortcuts (v_1, v_3) and (v_6, v_8) is of length 2.

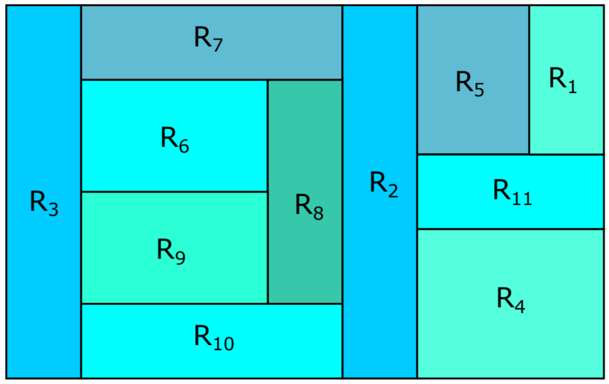

Figure 2.1: R_1 and R_4 are corner rectangles, R_3 is an end rectangle, and R_2 is a through rectangle.

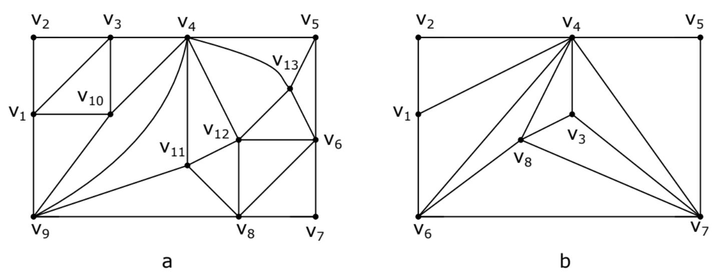

Figure 2.2: (a) Presence of CIPs $v_1v_2v_3$ and (b) a separating triangle $v_4v_6v_7$.

Definition 2.2.5. [52] A component rectangle of a rectangular dual is called a *corner rectangle* if its two adjacent sides share two common boundary segments with the exterior. A component rectangle of a rectangular dual is called an *end rectangle* if it shares the three boundary segments with the exterior. A through rectangle shares two sides to the exterior, but they are opposite sides (refer to Fig. 2.1).

Definition 2.2.6. A *separating cycle* is a cycle in a plane graph G that encloses vertices inside as well as outside. A separating cycle of length 3 is called a *separating triangle* (complex triangle) while a separating cycle of length 4 is called a *separating 4-cycle*. A 4-cycle is maximal if it is not contained in any other 4-cycle. We say a separating triangle in a plane graph is *critical separating triangle* if it does not contain any other separating triangle in its interior.

For instance, in Fig. 2.2b, the cycles $v_4v_6v_7v_4$, $v_1v_4v_7v_6v_1$ and $v_4v_8v_7v_4$ are a separating triangle, a separating 4-cycle, and a critical separating triangle respectively.

Theorem 2.2.1. [38, Theorem 3] A nonseparable plane graph G with triangular interior faces except exterior one is an RDG if and only if it has at most 4 CIPs and has no separating triangle.

Theorem 2.2.2. [38, Theorem 5] Let G be a separable connected planar graph with all triangular faces except the exterior one. Then G is an RDG if and only if

i. it has no separating triangle,

ii. BNG is a path,

iii. each of its maximal blocks corresponding to the endpoints of the BNG contains at most 2 critical CIPs,

iv. no other maximal block contains a critical CIP.

Theorem 2.2.3. [74, Theorem 1] If an RDG contains no complex 4-cycle, then it is slicible.

Theorem 2.2.4. [19, Theorem 1] An RDG G of n vertices $(n > 4)$ is slicible if it satisfies either of the following two conditions:

- the outermost cycle is the only complex 4-cycle in G and at least one of its four vertices is a non-distinct corner,
- each of the complex 4-cycles are maximal.

2.3 Concept of Regular Edge Labeling

An extended graph (4-completion) E(G) of an RDG is obtained by inserting a cycle of length 4 at the exterior of the RDG and then connecting vertices of the cycle to the exterior vertices of the RDG (see Fig. 2.3).

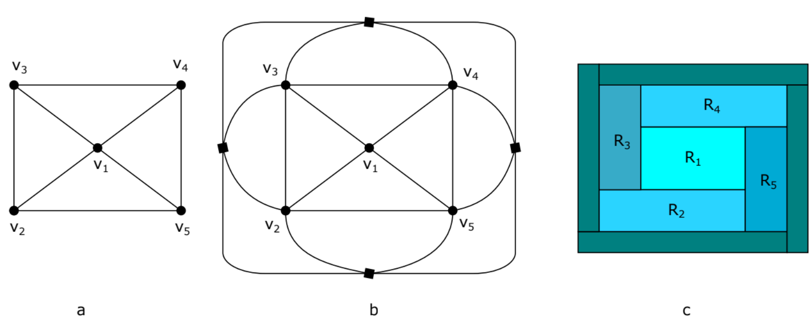

Figure 2.3: (a) An RDG G, (b) an extended graph E(G) and (c) a rectangular dual of E(G)

A rectangular dual R naturally induces a labeling of its extended dual graph E(G). If two rectangles of R share a vertical segment, then we assign blue color to the corresponding edge in E(G) and direct it from left to right otherwise if they share a horizontal segment, we assign red color to the corresponding edge in E(G) and direct it from bottom to top (see Fig. 2.4). If we consider the orientations of all edges incident to some vertex v_i of an RDG G, there is a clockwise sequence of these edges composed of four subsequences: vertical edges directed into v_i, followed by horizontal edges directed into v_i and then vertical and horizontal from v_i. Such labeling is called *regular edge labeling* and v_i is called a *well formed vertex*. If each vertex of G is well formed, then G is called a *well formed graph* (an oriented graph).

Definition 2.3.1. Two rectangular duals R' and R'' of an RDG G are said to be *combinatorially equivalent* (topologically equivalent) if they induce the same regular edge labeling of E(G), otherwise they are said to be topologically distinct.

For instance, the rectangular duals shown in Fig. 2.6 are topological distinct since they induce different regular edge labelings of their extended graph while the rectangular duals shown in Fig. 2.5 are combinatorially equivalent since they induce the same regular edge labeling where R_i is dualized to v_i. The only difference is of different dimensions of the component rectangles of the rectangular duals.

Note that every regular edge labelings of $E(G)$ generates an equivalence class of rectangular duals of the same graph G. In other words, there is a bijection between the regular edge labelings of $E(G)$ and equivalence classes rectangular duals of G. More precisely, regular edge labelings of $E(G)$ are in bijection with topologically distinct rectangular duals of G.

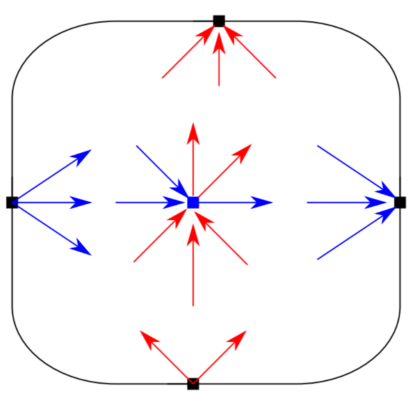

Figure 2.4: (a) Regular edge labeling of an extended graph E(G)

An interior face f of an RDG is towards a vertex v if two edges of f with the same orientation are incident to v. A corner vertex in an RDG is a vertex that corresponds a corner rectangle in the corresponding rectangular dual.

Theorem 2.3.1. [64, Theorem 2] An edge e or a block B formed by the four edges incident at an interior vertex of degree 4 of an oriented RDG is a *changeable edge set* if and if one of the following is true:

i. the four boundary edges of e or of B have alternating orientations,

ii. e is a boundary edge and the interior face containing e is towards a corner vertex.

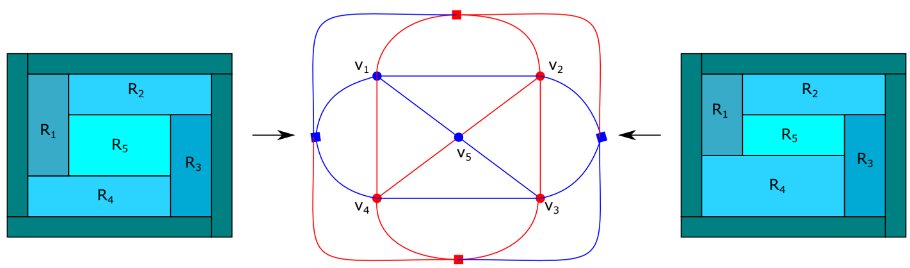

Figure 2.5: Two combinatorially equivalent rectangular duals induce the same regular edge labeling of their extended graph

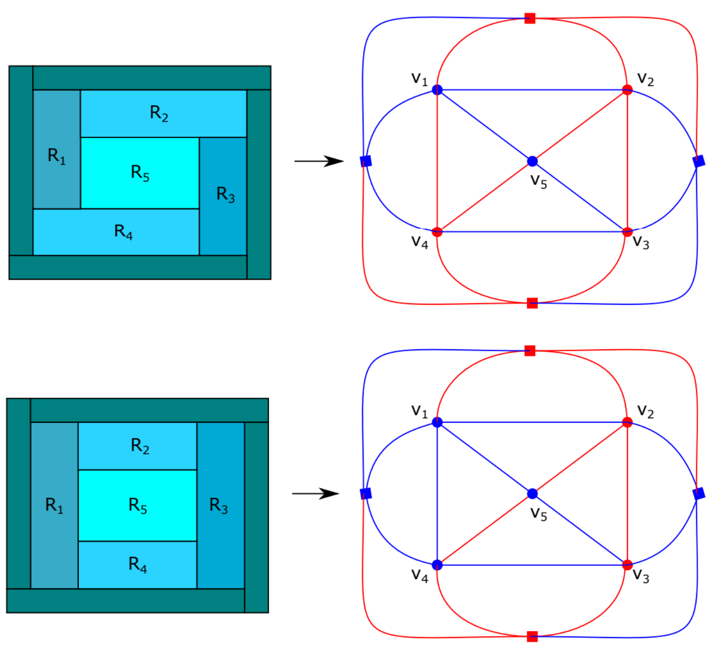

Figure 2.6: Two topologically distinct rectangular duals induce the different regular edge labelings of their extended graph

Definition 2.3.2. [37] A single edge in a rectangular dual R (or in its RDG) is a *turnable structure* (T-structure) if it occurs in any one of the four configurations shown in Fig. 2.7a-2.7d (or in Fig. 2.7e-2.7h). The red edges in Fig. 2.7 are T-structures. A T-structure may consist of more than one edge. A *simple T-structure* T in R is defined as a T-structure for which there exist 4 edges in R that do not belong to T, but share endpoints with T. Correspondingly in the RDG, a simple T-structure is a 4-cycle enclosing atleast one vertex. For more clarification, refer to Fig. 2.8.

Theorem 2.3.2. [37, Theorem 6] A necessary and sufficient condition for a set T of edges in a rectangular dual R to be a *simple T-structure* is that the subgraph T^* consisting of edges that are dual to T in the oriented extended graph E(G) representing R is the subgraph contained in the interior of a 4-cycle C.

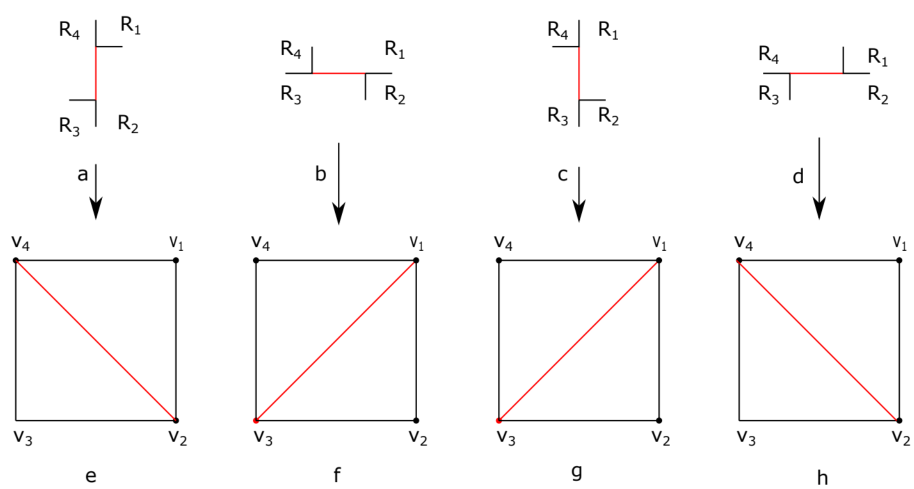

Figure 2.7: (a-d) Single edge simple T-structures in a rectangular dual and (e-h) corresponding single edge simple T-structures in its RDG

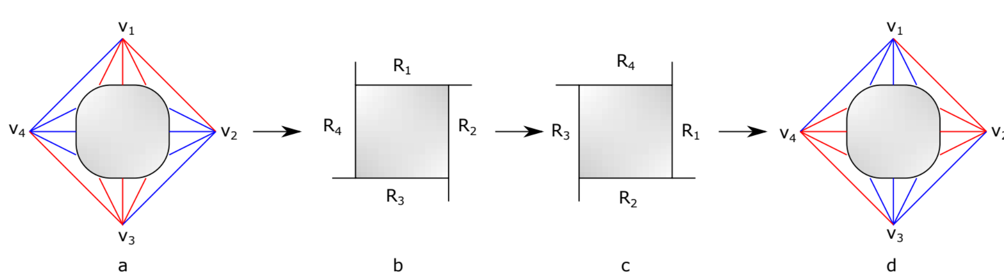

Figure 2.8: Multiple edge simple T-structures in G

2.4 Area-Universality

Definition 2.4.1. [22] A rectangular dual is called *area-universal* if each assignment of areas to its rectangles can be realized by a combinatorially equivalent rectangular dual.

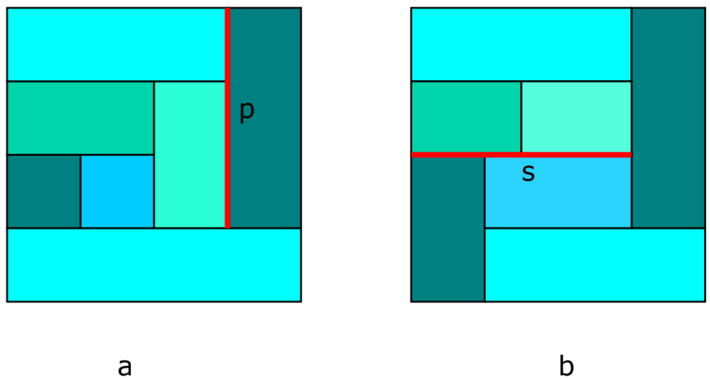

Figure 2.9: (a) An area-universal rectangular dual and (b) an rectangular dual that is not area-universal.

Definition 2.4.2. [22] A *line segment* in a rectangular dual is formed by a sequence of consecutive inner edges of the rectangular dual. A segment is *maximal* if it is not contained in any other segment.

Theorem 2.4.1. [22, Theorem 2] A rectangular dual is *area-universal* if and only if every maximal internal line segment is the side of atleast rectangle of the rectangular dual.

The rectangular dual shown in Fig. 2.9a is area-universal since every maximal internal line segment is the side of its rectangles (for example, the red color line segment p in Fig. 2.9a is the side of a rectangle) whereas the rectangular dual shown in Fig. 2.9b is not area-universal since the maximal line segment s (the red color line segment in Fig. 2.9b) is not the side of any of its rectangles.

2.5 Methodology

Methodology in this thesis we adapt, is the well known rectangular dualization method. Rectangular partitions are well studied in the context of graph notion. Such partitions are rectangular dual of planar graphs and the process of finding rectangular partitions is known as rectangular dualization method. To visualize this method, consider a plane graph G shown in Fig. 2.10a. An extended graph (Fig. 2.10b.) is constructed by inserting a 4-cycle at the exterior of G and then connecting vertices of the cycle to the exterior vertices of G. Then it is dualized in Fig. 2.10c where R_i is dualized to v_i. After assigning horizontal or vertical orientation to each of its edges, an embedding as shown in Fig. 2.10d is obtained. In fact, it is a rectangular dual of G. Thus G is rectangularly dualized to a rectangular partition (floorplan). This dualization method is known as the rectangular dualization method which is first theoretically studied by Kozminski and Kinnen [38].

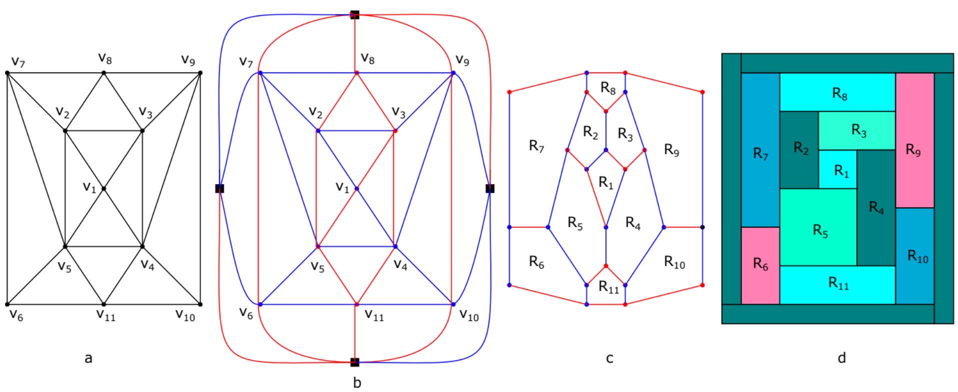

Figure 2.10: Rectangular dualization: (a) A plane graph G, (b) its extended graph, (c) dual of the extended graph, and (d) a rectangular dual of G.