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ANALYTIC AND VECTOR MECHANICS

BY

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PREFACE

This book is intended to be used as a text for students in physics and mathematics. It contains essentially the subject matter as presented in a three-hour course, extending throughout the year, to a class of junior and senior students in the University of California at Los Angeles. The material which is included has been selected because, in the opinion of the author, it is peculiarly fitted to serve the purpose of an introduction to the study of mechanics. A thorough knowledge of calculus and at least one elementary course in college physics have been prerequisites.

The range of topics has been limited to those which are considered to be fundamental and which best serve the purpose of illustrating the methods and procedure vital in an introductory course. Vector methods have been used freely and parallel, in many cases, the analytical treatment. It is thought that such a method of presentation not only may make the subject more interesting but will enable the student to gain a more secure understanding of mechanics and at the same time prepare the way for advanced work in which vector methods are commonly used.

The author has tried to teach the student to regard mathematical expressions as representations of physical relations rather than as aggregations of symbols which are to be manipulated according to prescribed rules. Each term of an equation should be examined so that the physical quantity represented is clearly recognized. In several types of expressions, such as those associated with simple harmonic motion, geometrical relations are used to give reality to the quantities represented. This viewpoint is essential to a mastery of the subject.

The study of such a subject as mechanics has for one of its objectives the presentation of certain concepts and principles. The knowledge of such subject matter is in itself sufficiently valuable to justify its acquisition. The student should be sure that his understanding of any process or principle is secure.

It is particularly important to recognize the limitations or restricting conditions which have been prescribed in the derivations of a particular principle and to realize that a fundamental equation may be applied to the whole or to only a part of some configuration. Consistency in the application of a principle is absolutely necessary. The author would like to suggest that facility in the use of applying the principles comes from keeping a knowledge of them in a state of suspension.

The goal for every student should be to learn to think analytically. Information is useful and necessary but the methods and means by which new information may be obtained are potent tools which make progress not only certain but rapid. The natural method of thinking, if there be such a thing, is an empirical one. Such a process is essentially a "cut and try" process. It is practiced extensively but seldom makes progress. To learn to think analytically requires real intelligent persistence.

There are, fortunately, devices which are helpful. Practical suggestions and concrete illustrations of procedure are given in this text for the purpose of helping the student to achieve this objective.

Progress toward this purpose may be obtained by a proper attitude of mind in the problem solving. After all it is not so much the answer to a particular problem that is important, but rather the knowledge of how to attain that answer. It is always an excellent procedure to look at the problem first from a more general point of view in order to lay out a plan and then to approach the details only when the proposed plan seems possible. This idea may be extended with advantage. Frequently the time element prevents one from covering as much ground as he would like. In such cases it is recommended that the details of calculation or of integration may be omitted, if the student feels sufficiently sure of his ability to warrant such a procedure.

The author wishes to express his appreciation of the criticisms and suggestions of Professor W. J. Raymond and Professor V. O. Lenzen, both of the University of California at Berkeley. He also wishes to extend his cordial thanks to his students, particularly Mr. Reed Lawlor, who have contributed in various ways in the development of this work.

HIRAM W. EDWARDS.

LOS ANGELES, CALIFORNIA,
May, 1933.

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CHAPTER I

VELOCITY

1-1. The Reference System.—In describing the position or motion of a body it is necessary to select a definite reference system. Without a reference system it is impossible to specify just where that body is or how it may be moving. Positions and motions are relative matters. Whether a body is to be regarded as at rest or in motion depends entirely upon how the particular reference system is selected. Is there any body which is actually at rest? We may say that a person sitting in a street car is at rest. He is at rest with respect to the street car even though it may be in motion. While the car is in motion, the man is not at rest with respect to the ground. But even if the car had stopped, our statement of the condition of rest of the man must again be qualified if we are to include a consideration of the earth's motion. Certainly he is not at rest with respect to the sun, for the earth is moving through space, relatively to the sun, with a speed of about eighteen miles per second. We may, therefore, say that a particle is "at rest" if its coordinates with respect to a chosen reference system are not changing.

To describe the motions of a body and the path along which the body moves, we are at liberty to select any frame of reference, but we must be careful to specify the exact location of this selected reference system. If we are concerned only with what happens in the street car and are at liberty to disregard external influences, then a frame of reference which is attached to the street car will best serve our purpose. If, however, we wish to describe this motion with reference to the surface of the earth, then our reference system must be fixed to that surface. It is inherent in the reference system that it is fixed to something which, as far as we are concerned, is not moving, or at least whose motion is not a part of, or does not influence, the phenomena under consideration. Later we shall have occasion to use moving coordinate systems. Even though the moving coordinate system and the stationary reference systems are mechanically similar, in so far as each system may be represented by a set of three mutually perpendicular lines and both use coordinates which are expressed by similar letters of the alphabet, there is the described essential difference which has warranted this terminology.

1-2. The Standard Reference System.—The standard reference system may utilize any system of space representation which is commonly employed. Use will be made here of rectangular, spherical, and polar systems of coordinates. The rectangular system is most commonly used. Frequently the polar system of coordinates will be found convenient. It will be of value to the student to have some drill in the use of the various systems of coordinates and in transforming expressions from one system to another. Work of this kind is given below. Frequently an aspect of a phenomenon is made prominent in one system where its expression in another leads to obscurity.

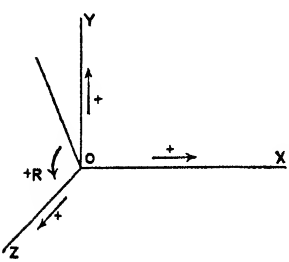

Fig. 1.

The convention of signs which is to be used as standard in this work is partially expressed in the accompanying figure (Fig. 1). The axes, mutually perpendicular to each other, are OX, OY, and OZ, with origin at O. Positive distances are measured outward from O along or parallel to an axis. Rotation of a line about the X-axis is positive if the line moves in or parallel to the YZ plane from OY toward OZ, as shown by the arrow R in the figure. Positive rotations about the Y-axis would move points in the ZX plane from OZ toward OX. This may be expressed in another manner. Suppose that a standard right-handed thread be cut upon the X-axis and a nut be placed upon it. A positive rotation of the nut would make it advance in a positive linear direction along OX.

1-3. Motion of a Point along a Line.—Consider a particle P which is moving along a definite path AB (Fig. 2). Suppose that the distance of P from A is measured by the coordinate s. Let the direction from A toward B be considered positive. At each instant of time, P occupies the position s_1 at the time t_1 and s_2 at the time t_2; then the ratio $(s_2 - s_1)/(t_2 - t_1)$ represents the average speed for either the time or space interval indicated.

The ratio of distance to time which gives a value for the average speed in any particular case supplies only a limited amount of information. It may be necessary to know more about the behavior of the particle than could be learned from a knowledge of its average speed during a selected time interval.

The speed of a particle may be varying continuously. This fact could be ascertained by determining the average speeds in a large number of consecutive and very small time intervals. The smaller the time interval selected, the more complete would the description of the motion of the particle be. This idea leads us to a conception of *instantaneous speed*, which may be defined in the following manner. Suppose in a particular interval of time Δt the particle passes over a distance Δs. The ratio $\Delta s/\Delta t$ will

FIG. 2.

approach a limiting value in any particular case if we let Δt approach zero as a limit. In the language of calculus, if we write

$$\frac{ds}{dt} = \lim_{\Delta t = 0} \frac{\Delta s}{\Delta t}$$

ds/dt becomes the limiting value of the average speed over an extremely small time interval. We may therefore call ds/dt the instantaneous speed for the particular position, defined by the value of the coordinate or time instant.

In a given case it may be possible to express the coordinate s as some definite function of the time. By differentiating this function with respect to the time, an equation is thereby obtained which expresses the instantaneous speed as a function of the time. By eliminating the time factor from the coordinate-time and speed-time relations, an equation may be obtained which expresses the instantaneous speed as a function of the coordinate. From these two relations, instantaneous speed expressed in terms of the time and in terms of the coordinate, a more complete description of the motion of the particle is obtained.

The algebraic sign of ds/dt depends upon the sign of ds, for dt is always positive. If ds designates an increase of distance in the positive direction of the path, the sign of ds is then positive. If ds represents a displacement along the path in the opposite direction, then the sign of ds is negative. The sign of the speed ds/dt is therefore positive when P moves in the direction assigned as positive. The selection of the point to be origin in no way affects the value of the speed, provided only that the point (the origin) be fixed. The speed may be constant or variable. If the speed is constant, then the magnitude and the sign of ds/dt must remain unchanged during the interval under consideration. The speed may be constant even though the direction of motion be changing.

The units in which the speed is to be measured are given by the choice of units used in expressing the distance and time factors. The distance is usually expressed in centimeters, feet, or miles, with the time in seconds or hours.

1-4. Determination of the Speed.—It is usually possible to express in terms of the time the distance a body or a particle moves along some assigned path. For example, in the case of a body falling from rest in a vacuum, the simple relation $s = \frac{1}{2}at^2$ is valid for short distances, where s represents the distance, a is a constant (acceleration), and t expresses the time. The general expression for the speed may be found by differentiating this equation with respect to the time, which gives $ds/dt = at$. If the constant a is known, values of the speed, $V = ds/dt$, may be found for any particular instant by substituting the corresponding values of the time for t in this equation. Since the speed V is a function of the time, V is constantly changing and hence the values of V are instantaneous values.

In other cases it may not be possible to express the speed in terms of the time by any simple relation. The instantaneous values of the speed may be determined graphically, however, if the distance-time relation for the motion of the body is known so that a plot may be made of these values. In such cases the slope of the tangent to the curve at the point for which the speed is desired may be measured from the plot. The value of the slope of the tangent gives the desired speed.

Suppose that P_1 and P_2 are two points on the curve in Fig. 3 and that they correspond to the two distances s_1 and s_2 and the instants t_1 and t_2, respectively. Suppose also that these two points are very close together so that we may write Δs for $s_2 - s_1$ and Δt for $t_2 - t_1$. It is evident that the speed $\Delta s/\Delta t$ is given by the tangent of the angle which the line P_1P_2 makes with the *time* axis. The magnitude of the tangent of this angle is given by $\Delta s/\Delta t$. If P_2 is made to approach P_1, then the line P_1P_2 becomes the tangent to the curve at the point P_1 in the limiting position. The speed of the moving body or particle at any given time t is therefore found by the slope of the tangent drawn to the distance-time curve at the point on the curve corresponding to the given time t. If the direction of the speed is desired, it is only necessary to observe whether the coordinate s is increasing or decreasing at the time point selected. As shown in the figure, s is increasing at the time t_1; hence the speed is

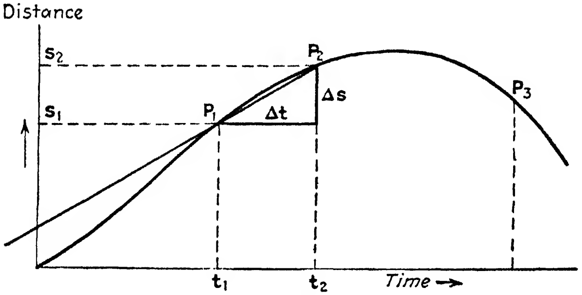

FIG. 3.

positive. At some other point on the curve, such as P_3, the speed is negative.

Problems.—1. Plot the curves $s = \frac{1}{2}at^2$ for some selected value of a and determine the corresponding speed-time curve graphically and analytically.

2. If the displacement of a body be expressed in terms of a sine function of the time, what would be the corresponding speed curve?

1-5. Velocity—a Vector.—The term velocity includes all that is implied by the term speed, with an additional element, *viz.*, a direction, associated with it. Velocity is a directed quantity or, in other words, is a quantity that has both magnitude and direction. Such a quantity is a typical vector. Any quantity which requires both a direction and magnitude for its complete description is a vector. Force, momentum, acceleration, and force moment are vectors. Such quantities permit of a unique form of expression known as graphical representation, and for convenience in studying them a certain form of analysis has been devised which is called vector analysis. This system of expression and manipulation is the same for any vector, but the physical interpretation of the results obtained depends upon the particular physical vector employed. Some of the commoner processes used in vector analysis will be illustrated by the use of the velocity vector.

When a particle is moving along a straight line, its velocity may be represented by a vector drawn parallel to that line. The convention for this representation may best be shown by means of a concrete example. Suppose that a certain particle has a speed of 60 cm. per second in a horizontal line and is moving due east. The magnitude of the speed is represented by the length of the vector. To do this, a suitable scale of representation must first be selected. We may, in this case, let 1 cm. of length along the vector represent a speed of 15 cm. per second; hence the vector AB (Fig. 4) is drawn 4 cm. long in order to represent the given speed of 60 cm. per second. The direction of the vector AB is parallel to the direction of the velocity (following the usual convention in representing geographical directions).

FIG. 4.

If the particle is moving in a curved path, the vector representing the velocity of the particle is parallel to the tangent drawn through that point of the path at which the velocity is to be represented.

A velocity is constant only when both of its elements are constant. This implies that, when the velocity is constant, the motion is of constant speed in a direction which is not changing. A particle moving with a constant speed of 20 m.p.h. along a straight line has a constant velocity. The velocity of a particle moving along the circumference of a circle is not constant, even though its speed may be constant.

1-6. Angular Velocity.—There are, in general, two kinds of motion: translation and rotation. In *pure translational motion*, any line fixed in the body will remain parallel to its original position during the course of the motion. In *pure rotational motion*, all points of the body, except those on the axis of rotation, describe paths which are circumferences of circles. Displacements and velocities in translation are analogous to angular displacements and angular velocities in rotation. The angle in rotation corresponds to the linear coordinate in translation. The position of a point on a line is given by its linear distance from some reference point in that line. Similiarly in rotation the position of a line is given by the angle through which the line has rotated from the reference position.

While angles are commonly measured in degrees, it has been found more convenient in the study of rotating bodies to use another unit, which has been called the *radian*. The radian is the angle subtended by an arc the length of which is equal to the radius of that arc. The value of the angle γ (Fig. 5) is given in radians by the simple relation $\gamma = s/r$, where s is the length of the arc included between the sides of the angle γ and r is the radius of the arc. Hence, if $s = r$, the value of the angle γ is 1 radian. If the above relation is put in the form

$$s = r\gamma \tag{1-1}$$

we obtain an expression for the length of the arc which is useful for transforming linear displacements in circular paths into the corresponding angular displacements or *vice versa*.

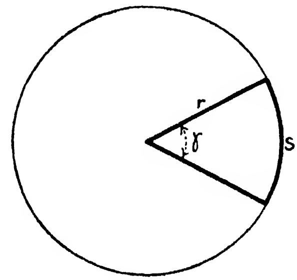

Fig. 5.

The time derivative of the angle which specifies the position of a line gives the angular speed of that line. When a direction is associated with angular speed, the resulting quantity becomes an angular velocity. It is customary to represent the angular velocity by a vector drawn along the axis of rotation. The length of the vector represents, to some selected scale, the angular speed. The sense of the vector indicates the direction of rotation in accord with the convention described in Sec. **1-2** above.

If we differentiate Eq. (1-1) with respect to the time, assuming r to be constant, we obtain the expression

$$\frac{ds}{dt} = r\frac{d\gamma}{dt}$$

usually written

$$V = r\omega \tag{1-2}$$

in which the symbol V is used for ds/dt and ω for the angular speed $d\gamma/dt$. This equation is useful for expressing the linear speed of a particle on the circumference of a circle of radius r in terms of the angular speed.

1-7. Vector Additions.—The process of addition as applied to vectors is an extension of the idea from simple cases of arithmetical and algebraic processes to the case where the addition deals with quantities having directions as well as plus and minus magnitudes. To illustrate this process, let us consider the case of a man rowing a boat directly across a river. The man can row with a definite velocity in still water. In a case of this sort, however, we are concerned with how the man actually moves—the path and the speed, both with respect to a reference system fixed to the ground. His velocity in the fixed reference system is made up of a combination (vector sum) of two velocities. The process of the vector addition of these two velocities gives the desired velocity.

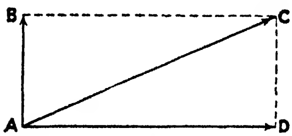

FIG. 6.

If AB (Fig. 6) represents the man's velocity with respect to the water, a moving system, and AD the velocity of the stream with respect to the reference system fixed to the ground, then AC represents the velocity of the man with respect to the fixed reference system. A definite scale of representation is to be used; 1 cm. might represent 1 m.p.h. The diagonal AC of the parallelogram drawn on AB and AD as sides gives the desired velocity.

The process of adding two or more vectors may be regarded as that of drawing a broken line, each segment of which represents

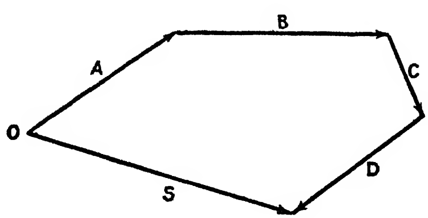

FIG. 7

(to some selected scale) one of the vectors. The sum of these vectors is represented by the line which converts the broken line into a closed polygon, the direction of the closing line to be away from the starting point. In Fig. 7 such a sum is given in which $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ are the vectors to be added and $\boldsymbol{S}$ is the vector representing their sum. In symbols this is written as follows: $\boldsymbol{S} = \boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C} + \boldsymbol{D}$. Notice the direction assigned to $\boldsymbol{S}$ in the diagram. The order in which the vectors are arranged is immaterial to the final result. If this fact is not obvious, the student should add the same vectors but take them in a different order.

1-8. The Components of a Vector.—Any two or more vectors whose sum is equal to a given vector may be regarded as *com-*

ponents of the given vector. For example, in Fig. 7, the vector ***S*** represents the sum of the vectors ***A, B, C,*** and ***D;*** hence ***A, B, C,*** and ***D*** are components of ***S***. In Fig. 6, ***AD*** and ***AB*** are components of ***AC***.

The process of finding certain desired components of a given vector is spoken of as decomposing, or resolving, the vector into its components.

The number of components into which a given vector is to be resolved is entirely an arbitrary matter. An inspection of the geometrical arrangement of the particular situation under consideration will usually be sufficient to indicate how many components will be needed and, what is more important, will show the directions along which the components are to be taken. In those problems in which all of the vectors are coplanar, it may be found that each vector should be resolved into two components which are respectively parallel to some selected reference axes. In problems involving three dimensions, it is often desirable to resolve each vector into three mutually perpendicular components, each being parallel to one of the reference axes.

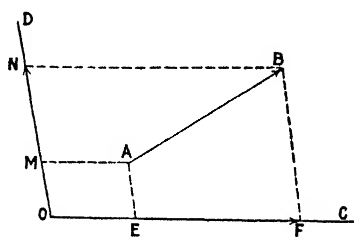

FIG. 8.

For a given vector, the magnitude of a component in any selected direction is definite and is determined by the magnitude of the given vector, together with the angle between direction of the vector and that of the component.

The graphical method for determining the magnitudes of a pair of components, which are to be taken along two given lines, consists in finding the intercepts formed on these lines by drawing two pairs of constructional lines; the lines of each pair are parallel to each other and to one of the given lines and pass through the ends of the given vector. The intercepts formed on the given lines are the desired components. The directions of the components are taken so that their vector sum gives the original vector. For example, it is desired to find the two components of a given vector AB (Fig. 8) and to have these components parallel to the lines OC and OD, respectively. The constructional lines AE and BF are drawn through the end points of the given vector as shown and are parallel to OD. The intercept on OC, *viz.*, EF, is the desired component parallel to OC. The direction to be assigned to EF is shown by the arrow. The component parallel to OD, *viz.*, MN, is found in a similar manner.

The trigonometric expressions which give the magnitudes of the components of a given vector along two lines not perpendicular to each other may be readily written by the use of the law of sines. Given the vector C (Fig. 9) and the lines OM and ON along which the components are to be found. If α and β are the angles between the vector C and the lines ON and OM, respectively, then the magnitude of A, the component along OM, is given by the equation

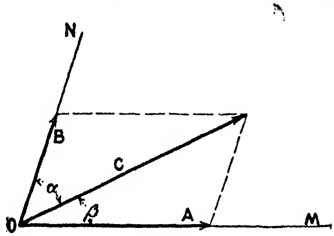

FIG. 9.

$$A = \frac{C \sin \alpha}{\sin (\alpha + \beta)} \tag{1-3}$$

and, since the $\sin [\pi - (\alpha + \beta)] = \sin (\alpha + \beta)$, similarly the magnitude of B, the component in the line ON, is

$$B = \frac{C \sin \beta}{\sin (\alpha + \beta)} \tag{1-4}$$

Ordinarily a pair of mutually perpendicular components (orthogonal components) is found more convenient to use than a pair of oblique components. In many cases the nature of the problem is such that only the component in some particular direction is needed. Unless specified to the contrary, that component is to be considered as one of a pair of orthogonal components.

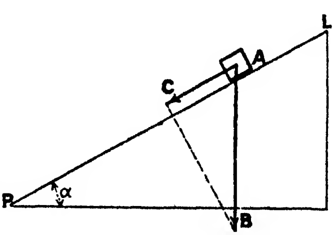

FIG. 10.

For example, we may be considering the motion of a body which is sliding down an inclined plane and for this purpose need to know only that component of the weight of the body which is parallel to the inclined plane. Suppose PL (Fig. 10) is the inclined plane and AB represents the weight of the body. The component of AB parallel to the plane PL is found by drawing a line through A parallel to PL and then dropping a perpendicular from B to this line, thus locating the point C. The segment AC gives the desired component. If α is the angle of inclination of the plane, then the magnitude of AC may also be found by the equation

$$AC = AB \sin \alpha$$

The other component in this particular case is, of course, perpendicular to AC and its magnitude can be readily found. The fact that it is not needed in the particular problem should not lead one to assume its nonexistence.

The advantage of studying the components of a vector, instead of the vector itself, may be illustrated by examining the components of the force used on a lawn mower. The push exerted upon the handle of the lawn mower does two things. In the first place, it is responsible for motion along the ground and it also pushes the mower down on the ground, thereby increasing the friction and lessening the chance of the wheels slipping. The two components here are perpendicular to each other: one is parallel to the ground and the other is perpendicular to the ground. By the process of resolving the applied force into the two components, we are able to assign to each component one of the functions of the applied force. Because of this analysis, the effect of each component may be expressed in terms of the applied force. The magnitude of each component is found by multiplying the magnitude of the applied force by the cosine of the angle between the direction of the applied force and the direction of the desired component. In Fig. 11, P stands for the applied force and H and V are the horizontal and vertical components, respectively. We may therefore write

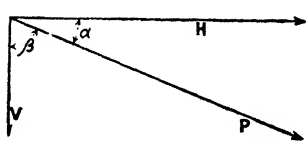

FIG. 11.

$$H = P \cos \alpha \quad \text{and} \quad V = P \cos \beta = P \sin \alpha \tag{1-5}$$

1-9. Component Velocities.—Given any velocity V. It is required to express this vector in terms of its components along the coordinate axes in each of three systems of coordinates and to write the analytical relations for determining the speed and the direction of the velocity in terms of the various components. The systems of coordinates to be used are the rectangular, spherical, and the polar systems.

a. The Rectangular System.—If V makes the angles α, β, and γ with the X-, Y-, and Z-axes, respectively (Fig. 12), then

$$V_x = V \cos \alpha, \qquad V_y = V \cos \beta, \qquad V_z = V \cos \gamma \tag{1-6}$$

where V_x, V_y, and V_z are the components of V parallel to the X-, Y-, and Z-axes, respectively.

To express the magnitude of V in terms of its components, we may square and add Eqs. (1-6), which gives the following desired equation:

$$V^2 = V_x^2 + V_y^2 + V_z^2 \tag{1-7}$$

The direction of V may be found by Eqs. (1-6) or by any two of the following equations:

$$\tan \alpha = \frac{\sqrt{V_y^2 + V_z^2}}{V_x} \qquad \tan \beta = \frac{\sqrt{V_z^2 + V_x^2}}{V_y}$$
$$\tan \gamma = \frac{\sqrt{V_x^2 + V_y^2}}{V_z} \tag{1-8}$$

In case V lies in the XY plane, then $\gamma = 90°$, $V_z = 0$, and

$$V^2 = V_x^2 + V_y^2 \qquad \tan \alpha = \frac{V_y}{V_x}$$

In the foregoing equations we are dealing with orthogonal projection. It should be understood that orthogonal projection

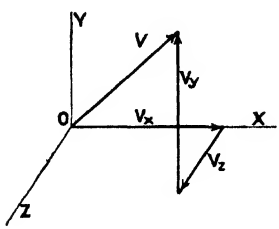

FIG. 12.

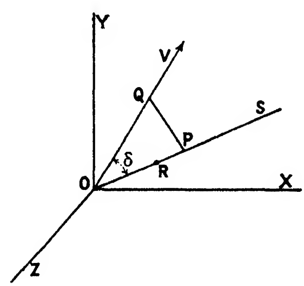

FIG. 13.

is meant when the unqualified expression "projection" or component is used.

It is important at this point to derive some general relations between the time derivatives of the coordinates and the velocity components. Let S (Fig. 13) be a line which makes an angle δ with V, the velocity of a particle at Q. Then the component of V in the line of S, *viz.*, V_s, is expressed by the relation

$$V_s = V \cos \delta$$

We may also express V_s in terms of V_x, V_y, and V_z. This expression is obtained by equating V_s to the sum of the projections of the components V_x, V_y, and V_z into the line of S. The validity of this equation will be left to the student to substantiate. In symbols, then, we may write

$$V_s = V_x \cos \alpha' + V_y \cos \beta' + V_z \cos \gamma' \tag{1-9}$$

where α', β', and γ' are the direction angles of the line S.

Let P be the projection of Q upon the line S. (The process of projecting a point upon a given line is one of drawing a line through the given point so that the constructional line intersects the given line perpendicularly. The point of intersection of the two lines is the desired projection of the given point.) Let s (RP in the diagram) be the distance of P from any other point R (x_0, y_0, z_0), another point in S. Designate the coordinates of P as x, y, and z. We may then express the distance s in terms of the coordinates of R and P and the direction angles of S as follows:

$$s = (x - x_0) \cos \alpha' + (y - y_0) \cos \beta' + (z - z_0) \cos \gamma' \tag{1-10}$$

The time derivative of s is ds/dt. The value of this derivative may be found by differentiating the right-hand member of Eq. (1-10). If ds/dt is to be equal to V_s as given by Eq. (1-9), the time derivatives of x_0, y_0, z_0, $\cos \alpha'$, $\cos \beta'$, and $\cos \gamma'$ must all be equal to zero. In order that all of these six derivatives may be equal to zero, both the point R and the line S must be stationary in the $X\ Y\ Z$ system. If the point R were moving along S or the line S were changing its position in the reference system, then either of such motions would, in effect, introduce a moving coordinate system. We should not expect, therefore, that ds/dt, which gives the velocity of P with respect to R in the line S, would be equal to V_s, the component of V in S, unless the S system were fixed in the $X\ Y\ Z$ system.

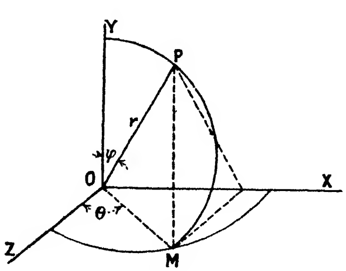

FIG. 14.

b. Spherical Coordinates.—In the spherical system of coordinates we make use of three coordinates: a radius and two angles. The coordinates are designated by the symbols r, φ, and θ and are measured from a fixed reference point, a reference line, and a reference plane, respectively. In order to describe the relative positions of the reference point, line, and plane, it is convenient to locate them in a standard rectangular coordinate system. Let X Y Z be the rectangular system with origin at O (Fig. 14). Let P be any point whose coordinates are x, y, and z in the rectangular system and r, φ, and θ in the spherical system.

The origin of the spherical system is taken at O, which is the center of a sphere with radius r. The point P is therefore on the surface of the sphere. The linear coordinate of P in the spherical system is the distance of P from the reference point O. The second coordinate φ is to be measured from a fixed reference line, which is to be a radius of the sphere (and, since its selection is arbitrary, we shall let it coincide with OY), to the radius r. The third coordinate θ will measure the angle between the reference plane, which we let coincide with the YZ plane, and the plane containing OY and r.

The relations between the two sets of coordinates of P may be written as follows:

$$\begin{aligned} x &= r \sin \varphi \sin \theta \\ y &= r \cos \varphi \\ z &= r \sin \varphi \cos \theta \end{aligned} \qquad (1\text{-}11) \qquad \begin{aligned} r^2 &= x^2 + y^2 + z^2 \\ \cos \varphi &= y(x^2 + y^2 + z^2)^{-\frac{1}{2}} \\ \sin \theta &= x(x^2 + z^2)^{-\frac{1}{2}} \end{aligned} \qquad (1\text{-}12)$$

Expressions for the component velocities in the X Y Z system V_x, V_y, and V_z in terms of variables in the spherical system may be found by differentiating Eqs. (1-11) with respect to the time. If we write $\dot{\varphi}$ for $d\varphi/dt$ and $\dot{\theta}$ for $d\theta/dt$, we obtain the following equations:

$$V_x = \frac{dr}{dt} \sin \varphi \sin \theta + r\dot{\varphi} \cos \varphi \sin \theta + r\dot{\theta} \sin \varphi \cos \theta$$

$$V_y = \frac{dr}{dt} \cos \varphi - r\dot{\varphi} \sin \varphi$$

$$V_z = \frac{dr}{dt} \sin \varphi \cos \theta + r\dot{\varphi} \cos \varphi \cos \theta - r\dot{\theta} \sin \varphi \sin \theta$$

The right-hand member of each of these equations is a sum of the projections of the three spherical components of the resultant velocity on to the X-, Y-, or Z-axis. Let us examine the first of these equations in order that we may identify both the magnitudes and the directions of the spherical components. We must expect to find a velocity component in each term multiplied by the proper factor which projects it into that reference axis along which the orthogonal velocity component is being expressed.

In the first term, dr/dt occurs. This quantity is the velocity component in the line of r, because, in the equation for V_x,

dr/dt occurs with the factors $\sin \varphi \sin \theta$ which collectively are equivalent to the cosine of the angle between r and X. The second term contains the spherical component $r\dot{\varphi}$. This component may be regarded as being due to the angular velocity $\dot{\varphi}$ which, together with the factor r, gives a linear velocity in the YOP plane and perpendicular to r. The factors $\cos \varphi$ and $\sin \theta$ project it into the line of X.

The third component is $r\dot{\theta} \sin \varphi$. This component is due to the angular velocity $\dot{\theta}$. Since θ is always measured in the ZX plane, the Y-axis is the axis of rotation for $\dot{\theta}$. Since $r \sin \varphi$ is the perpendicular distance of P from the Y-axis, this component of the velocity is perpendicular to the YOP plane.

The three spherical components of the velocity are mutually perpendicular to each other. The resultant velocity of P may therefore be expressed by the following relation:

$$V^2 = \left(\frac{dr}{dt}\right)^2 + (r\dot{\varphi})^2 + (r\dot{\theta} \sin \varphi)^2 \tag{1-13}$$

If the three components are known, then the magnitude of V may be obtained from the preceding relation. The direction of V may be determined from the relative magnitudes of the three components.

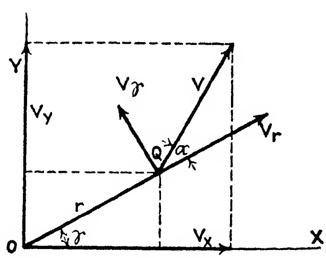

Fig. 15.

c. The Polar-coordinate System.—To find the relations between the velocity V and its components in a plane polar-coordinate system, it will be convenient to use a rectangular system as an auxiliary. Let this latter system be XOY with origin at O (Fig. 15). Let OX be the reference line for the polar system, with the radius vector r drawn to any point Q, and making the angle γ with OX.

Let V be the velocity of the particle at Q with coordinates x and y in the rectangular system at the instant under consideration and let α be the angle which V makes with r. From geometrical relations,

$$r^2 = x^2 + y^2 \tag{1-14}$$

$$\tan \gamma = \frac{y}{x} \tag{1-15}$$

The component of the velocity parallel to r is V_r and is equal to $V \cos \alpha$. V_r may be expressed by the equation

$$V_r = V_x \cos \gamma + V_y \sin \gamma$$

where V_x and V_y are the components of V parallel to OX and OY, respectively. This equation might have been derived by differentiating Eq. (1-14) with respect to the time, since the component of the velocity parallel to r is dr/dt.

The velocity component perpendicular to r is called V_γ. Its value is $V \sin \alpha$. It may also be expressed in terms of V_x and V_y by differentiating Eq. (1-15) with respect to the time.

$$\frac{d\gamma}{dt} = \frac{x\,V_y - y\,V_x}{r^2}$$

$$V_\gamma = r\frac{d\gamma}{dt} = V_y \frac{x}{r} - V_x \frac{y}{r}$$

$$= V_y \cos \gamma - V_x \sin \gamma \qquad (1\text{-}16)$$

This equation gives the velocity perpendicular to r even though r is moving, provided Q is contained on r.

Since $d\gamma/dt$ is the time rate of change of an angle, it is called an angular velocity and is designated by the letter ω. The angular velocity ω describes the rate of change of position of r and is expressed in terms of radians per second.

It is instructive to observe also that

$$V^2 = V_r^2 + V_\gamma^2 \qquad \tan \alpha = \frac{V_\gamma}{V_r}$$

$$V_x = V_r \cos \gamma - V_\gamma \sin \gamma \qquad (1\text{-}17a)$$

$$V_y = V_r \sin \gamma + V_\gamma \cos \gamma \qquad (1\text{-}17b)$$

In this section we have used two pairs of components of the velocity vector V. Either pair may be used, in studying the velocity, with equally valid results, but sometimes discrimination in selection is to the students' advantage.

1-10. Change of Origin for Velocities in Translation.—Frequently the velocity of a particle may advantageously be described in terms of a geometric combination of its velocity in a moving coordinate system and the velocity of the moving system referred to some fixed reference system. A general principle for this purpose will be developed.

Let the fixed reference system be XYZ and the moving system be $X_1Y_1Z_1$ (Fig. 16). The only limitation on the movement of the moving system is that its axes shall always be parallel to those of the reference system.

The coordinates of any point Q in the fixed reference system will be xyz and in the moving system $x_1y_1z_1$. The coordinates of O_1 in the fixed reference system will be called $x_0y_0z_0$. The relations between these coordinates are expressed by the equations

$$x = x_0 + x_1, \qquad y = y_0 + y_1, \qquad \text{and} \qquad z = z_0 + z_1$$

If Q and O_1 are both moving, then

$$\frac{dx}{dt} = \frac{dx_0}{dt} + \frac{dx_1}{dt}, \qquad \frac{dy}{dt} = \frac{dy_0}{dt} + \frac{dy_1}{dt}, \qquad \frac{dz}{dt} = \frac{dz_0}{dt} + \frac{dz_1}{dt} \qquad (1\text{-}18)$$

If we take the vector sum of dx/dt, dy/dt, and dz/dt, we obtain a vector which is Q's velocity in the reference system. We shall designate this velocity by the symbol $V_Q(O)$. The vector sum of the components dx_0/dt, dy_0/dt, and dz_0/dt gives a vector expressing the velocity of O_1 in the reference system. We shall call this velocity $V_{O_1}(O)$. Similarly the vector sum of the last terms in Eq. (1-18) gives the velocity of Q in the moving system. We shall designate this velocity by the symbol $V_Q(O_1)$. Combining these results into one equation gives

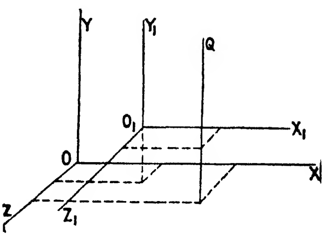

FIG. 16.

$$V_Q(O) = V_Q(O_1) + V_{O_1}(O) \qquad (1\text{-}19)$$

which expresses the velocity of Q with respect to the reference system as equal to the velocity of Q in the moving system, plus the velocity of the moving system with respect to the reference system.

This important equation is particularly convenient if we wish to determine any one of the three velocities when the two others are known. The technique of the graphical solution is not so obvious in those cases where one of the terms of the right-hand member is the unknown. Suppose that $V_Q(O_1)$ is to be determined and that the two other vectors of Eq. (1-19) are known. Writing the equation so that it is explicit for $V_Q(O_1)$ gives

$$V_Q(O_1) = V_Q(O) + [-V_{O_1}(O)]$$

The ordinary rules of vector addition may be applied to this equation but we must reverse the direction of $V_{O_1}(O)$ because it appears with the minus sign. The vector addition is shown in Fig. 17.

In a limited number of special cases it might be convenient to have two or more moving systems serve in expressing the velocity of a point. In such cases the restriction should again be made that the axes of the moving systems remain parallel to the axes of the reference system. The velocities of the origins of the moving systems are not otherwise limited. To each would be assigned the value which the nature of the particular problem indicated. The general equation is

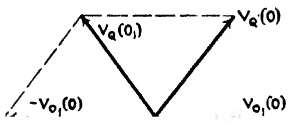

FIG. 17.

$$V_Q(O) = V_Q(O_1) + V_{O_1}(O_2) + V_{O_2}(O_3) + \cdot \cdot \cdot V_{O_n}(O)$$

The polygon method of velocity addition would be applicable here. It must be obvious that a solution is always possible if all but one of the velocities are known. It will be shown in the section below that a solution may be obtained if not more than two quantities, a magnitude and a direction or two directions, are unknown. The same device which is shown above would serve here if the sign of one or more of the vectors is minus. The result obtained is independent of the order in which the elements are taken.

1-11. How to Solve Problems.—The method which the student uses in solving problems is important. He should be distinctly conscious of the way he proceeds, so that the solution is obtained as a logical step-by-step process rather than a haphazard one. The first step in any solution is to identify the known and the unknown quantities and to express the given quantities in a single set of consistent units. Next he should select the principle or principles which are pertinent to the problem. These should be written down together with a diagram and the data of the problem. In general, there will be required as many algebraic equations as there are unknown quantities. If, however, the solution of the problem is to be found by the use of vector equations, as in the illustrative problems given below, only one equation will be required for two unknowns, *i.e.*, two magnitudes, a magnitude and a direction, or two directions. The solution of a vector equation is usually obtained by a vector diagram. Each vector equation may, however, be transformed into two or three algebraic equations. The graphical or vector method of solution is more concise than the algebraic and is usually the easier method.

The solution of a vector equation involves simply the geometrical construction of a triangle, where only three vectors are involved, or a polygon, if more than three vectors are involved. The details of the vector method of solving a problem are shown by two selected illustrations below; the first involves three vectors and the second requires four vectors. The student is urged to pay particular attention to the method shown in these two illustrations and then to use this method in the solutions of the other problems given in this section.

Problems.—1. A train is going due north with a speed of 60 m.p.h. A bandit near the track shoots due east at an object in the train. If the speed of the bullet with respect to the bandit is 1,000 ft. per second, find the velocity of the bullet with respect to the train.

Identification of quantities	Symbol	Units		Known elements
		Given	Consistent	
Velocity of train with respect to a system fixed to the ground.	$V_T(E)$	60 mi. per hr.	88 ft. per sec.	Magnitude direction
Velocity of bullet with respect to fixed system.	$V_B(E)$	....	1,000 ft. per sec.	Magnitude direction
Velocity of bullet with respect to the train.	$V_B(T)$	....	?	None

General principle applied to the particular problem:

$$V_B(E) = V_B(T) + V_T(E)$$

Solution.—Since we have only two unknowns, the magnitude and the direction of $V_B(T)$, the one vector equation is sufficient to obtain the desired quantities. We must then construct a triangle with the two known vectors as sides and the angle between them equal to 90 deg. Rearranging the equation to make it explicit for the unknown vector gives

$$V_B(T) = V_B(E) - V_T(E)$$

In the diagram (Fig. 18) the vectors are not drawn to scale. Both magnitude and direction of $V_B(T)$ would be correctly given if the diagram had been made to scale. The magnitude of $V_B(T)$ should be expressed in the selected consistent units, *viz.*, feet per second.

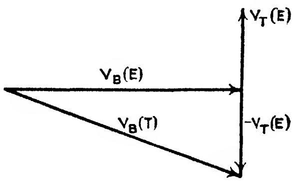

FIG. 18.

2. Two automobiles, M and N, are going in the same direction along parallel lines with speeds of 50 and 30 m.p.h., respectively. At the instant when a boy in N is directly opposite a man in M, the boy throws a ball (horizontally) at the man and hits him. If the speed of the ball with respect to the boy is 100 ft. per second, what is the direction of the velocity of the ball with respect to the boy in N? Find also the velocity of the ball with respect to the man in M and the velocity of the ball with respect to the ground.

Identification of quantities	Symbol	Units		Known elements
		Given	Consistent	
Velocity of M with respect to a fixed system on the ground.	$V_M(G)$	50 mi. per hr.	73.3 ft. per sec.	Magnitude direction
Velocity of N with respect to the fixed system.	$V_N(G)$	30 mi. per hr.	44 ft. per sec.	Magnitude direction
Velocity of M with respect to N.......	$V_M(N)$	....	....	None
Velocity of the ball with respect to N..	$V_B(N)$	....	100 ft. per sec.	Magnitude
Velocity of the ball with respect to M..	$V_B(M)$	....	....	Direction
Velocity of the ball with respect to fixed system...........	$V_B(G)$	....	....	None

The *general principle* applied to this problem gives

$$V_M(G) = V_M(N) + V_N(G) \qquad (a)$$
$$V_B(G) = V_B(M) + V_M(N) + V_N(G) \qquad (b)$$
$$V_B(N) = V_B(M) + V_M(N) \qquad (c)$$

Solution.—Using Eq. (*a*), we may find $V_M(N)$, since the two other vectors in this equation are known. In this case the vector solution degenerates into an algebraic equation because the two known vectors are parallel to each other. The magnitude of $V_M(N)$ is found to be equal to 29.3 ft. per second and the direction is parallel to that of $V_M(G)$.

We may now use Eq. (*c*) and find the direction of $V_B(N)$ and the magnitude of $V_B(M)$, since in this equation these two quantities are the only unknowns. There may be a doubt in the student's mind that the direction of $V_B(M)$ is known, but he must remember that the ball was thrown at the instant when the line drawn from the boy to the man was perpendicular to the velocities of the two automobiles. To the man, the ball would appear to be coming directly toward him; hence the direction of $V_B(M)$ is perpendicular to $V_M(G)$. The vector diagram (Fig. 19), then, consists of a triangle with the lengths of two sides and an angle opposite one of the given sides known. This solution gives the magnitude of $V_B(M)$ and the direction of $V_B(N)$.

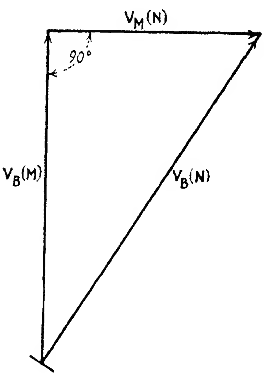

Fig. 19.

The velocity of the ball with respect to the ground, $V_B(G)$, may now be found by using Eq. (*b*). All three velocities on the right-hand side of this equation are now known; hence the unknown $V_B(G)$ may be found.

Problems.—1. An automobile is going at a speed of 40 ft. per second. A man in the automobile throws a ball with a speed of 100 ft. per second with respect to himself along a horizontal line at right angles to the velocity of the automobile. Find the velocity of the ball with respect to the ground.

2. A person on the ground throws a ball (80 ft. per second) at an automobile which has a speed of 50 ft. per second. The horizontal direction of the ball makes an angle of 45 deg. with the velocity of the automobile. Find the velocity of the ball with respect to the automobile.

3. An automobile is going at a speed of 75 ft. per second in a rain. The wind is blowing the rain in a direction parallel to the car's motion. The rain drops fall at an angle of 30 deg. with the vertical and at a rate of 25 ft. per second. Find the velocity of the rain drops with respect to the automobile.

4. A train is going due north at a speed of 80 ft. per second. The smoke from the train trails off in a line toward the southeast. What is the speed of the wind if its direction is toward the east?

5. An airplane has an air speed of 100 m.p.h. In what direction must the fuselage point if the plane is to go due south in a wind blowing 30 m.p.h. toward the west?

6. A stream has a speed of 3 m.p.h. A man can row a boat 4 m.p.h. in still water. What direction must he point his boat if he is to go straight across? What would be his direction if he wishes to go across in the shortest time?

7. A person walking east at a rate of 4 m.p.h. finds that the wind appears to be from the south. If he increases his pace to 6 m.p.h., he finds that the wind appears to come from the southeast. What is the velocity of the wind?

8. Two airplanes, A and B, are flying horizontally in straight lines perpendicular to each other at the same elevation and at the same speed of 100 m.p.h. A crosses B's line of flight when they are 500 ft. apart. B fires at A when he is 200 ft. from A's line of flight. If the speed of the bullet is 2,000 ft. per second, what must be B's line of sight if he is to hit A? Neglect gravitational effects. What would be the velocity of the bullet with respect to A?

1-12. Change of Origin for Velocities—Rotation.—One occasionally encounters motions in which a rotating coordinate system is a better auxiliary system for describing or analyzing the velocities than the system moving with translation. Although such cases are not numerous, the few encountered are so adequately handled by this type of moving system that it is worth while to develop the general expression. The velocity of any moving particle is to be expressed in terms of its velocity in a rotating system and the angular velocity of the moving system.

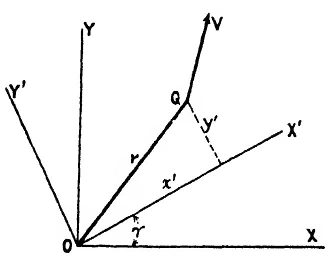

Fig. 20.

Let the reference system be XOY (Fig. 20) and the moving system be $X'OY'$ with its origin coincident with O of the fixed system and coplanar with XOY. Designate the angle XOX' by γ. The coordinates of Q, any point at which the moving particle is situated, are xy and $x'y'$ in the fixed and moving systems, respectively. It follows from these assignments that

$$x = x' \cos \gamma - y' \sin \gamma$$
$$y = x' \sin \gamma + y' \cos \gamma \qquad (1\text{-}20)$$

If we differentiate both of these equations with respect to the time, the component velocities are obtained directly. Writing

V_x, V_y, $V_{x'}$, and $V_{y'}$ for the derivatives dx/dt, dy/dt, dx'/dt, and dy'/dt, respectively, and ω for $d\gamma/dt$ gives

$$\begin{aligned} V_x &= V_{x'} \cos \gamma - x'\omega \sin \gamma - V_{y'} \sin \gamma - y'\omega \cos \gamma \\ V_y &= V_{x'} \sin \gamma + x'\omega \cos \gamma + V_{y'} \cos \gamma - y'\omega \sin \gamma \end{aligned} \quad (1\text{-}21)$$

Substituting x and y from Eq. (1-20) for the coefficients of ω in Eq. (1-21) gives

$$\begin{aligned} V_x &= V_{x'} \cos \gamma - V_{y'} \sin \gamma - y\omega \\ V_y &= V_{x'} \sin \gamma + V_{y'} \cos \gamma + x\omega \end{aligned} \quad (1\text{-}22)$$

The two equations of Eq. (1-22) may be combined into a single equation which will be more useful. The vector sum of V_x and V_y is V, the resultant velocity of the particle at Q, referred to the reference system. We may call this velocity $V_Q(O)$ in harmony with the convention adopted above. The first two terms of the right-hand members may be similarly combined. The resultant of these four terms gives the velocity of Q in the rotating system, which velocity may be called $V_Q(O')$. The last terms of the two equations may also be combined. It will be observed (Fig. 21) that ωx and $-\omega y$ are linear velocities and that they are perpendicular to OX and OY, respectively. If these velocities are added geometrically, a linear velocity is obtained which is equal to $\omega\ r$ [Eq. (1-2)] where r is the line drawn from O to Q. The direction of $\omega\ r$ is perpendicular to r.

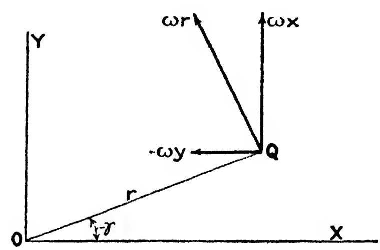

FIG. 21.

It will be convenient to use the symbol $\boldsymbol{\omega r}$, in this and the two following sections, to designate the velocity of the terminal point of a line segment, such as OQ (or r, Fig. 21), which is rotating about an axis through its initial point. In the next chapter a more conventional method of describing such a velocity is given.

We may now express the velocity of Q in terms of its components as follows:

$$V_Q(O) = V_Q(O') + \boldsymbol{\omega r} \quad (1\text{-}23)$$

An interpretation of this equation leads us to see that the resultant velocity of Q as expressed in the fixed system is made up of two parts: the velocity of Q in the moving system and a velocity which depends upon the angular velocity of the moving system and the distance of Q from the origin. Ordinarily, ω refers to the angular velocity of the radius vector r. It is to be noticed that here ω expresses the angular velocity of the axes of the rotating system about an axis drawn through O perpendicular to the common plane of the two systems. The velocity vector which expresses the difference between the velocities of Q in the two systems is $\boldsymbol{\omega r}$.

An illustration of the use of the general formula will be given to show the type of problem to which it may be applied. It is required to determine the resultant velocity of a small particle at any instant as it moves with constant velocity outward along the radius of a wheel. The wheel is turning with a constant angular velocity about a fixed axis.

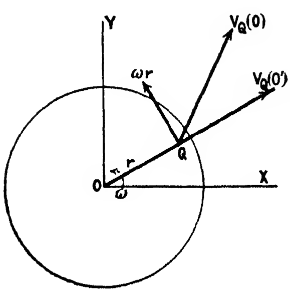

FIG. 22.

Let XOY (Fig. 22) be the fixed reference system with the center of the wheel at O, ω the angular velocity of the wheel and r the distance from O to the moving particle at Q. Also let the rotating system be fixed to the wheel. The velocity of Q in the rotating system $V_Q(O')$ will be directed outward along r, as shown in the diagram. The velocity expressed by $\boldsymbol{\omega r}$ is drawn perpendicular to r. The vector sum of these two velocities gives the desired velocity $V_Q(O)$.

1-13. Uniplanar Motion.—A body is considered *rigid* if the line segments connecting any two pairs of points and the angles between these line segments remain constant when the body is under the influence of external forces. No body is perfectly rigid but there are many bodies in which the deformations which do occur are so small that they may be neglected. Bodies in which the deformations are large enough to be taken into consideration are classified as *elastic bodies*.

A rigid body may have two types of motion: translation and rotation. In translational motion, all points of the rigid body describe equal and parallel curves. If a rigid body has any two of its points fixed, any motion which it may have is a rotation about the line passing through the two fixed points. The line in this case is called the axis of rotation. All points of a rigid body, except those on the axis, describe circular paths, the centers of which lie on the axis and whose planes are perpendicular to the axis.

Many cases of motions of rigid bodies are neither pure translation nor pure rotation but are combinations of the two motions. Such motions are called uniplanar motions if the velocities of all points of the body are continuously parallel to a fixed plane, which is called the guide plane. The velocity of any point of a body, which is in uniplanar motion, may be expressed as a vector sum of two component velocities, one of which depends upon a rotation about an arbitrarily selected axis (perpendicular to the guide plane) and the other consists of the translational velocity of the selected axis with respect to the fixed reference system.

Let $V_P(O)$ represent the velocity of any point P of the body, ω the angular velocity of the body about the selected axis A, and AP the distance from P to A. The general expression for the velocity of P is

$$V_P(O) = V_A(O) + \omega AP \tag{1-24}$$

in which $V_A(O)$ is the velocity of A in the reference system.

The velocity of any point may be expressed by an indefinite number of such combinations, since the position of the axis is not limited. The angular velocity of the motion is instantaneously the same for all points of the body regardless of which line may be selected as the axis, but the translational element will, in general, be dependent upon the position of the axis in the body. This statement is true even though the axis is not strictly a line of the body, for the axis may be considered as being attached to the body by a massless frame.

In order to illustrate just how the velocity of any point of a body in uniplanar motion may be described in terms of a translational element together with a rotational element, and also to show that the selection of the position of the axis of rotation is immaterial to the process of description or to the final result, let us express the velocity of a selected point of a body in uniplanar motion by using axes in two different positions.

Let P be the selected point (Fig. 23) on the rim of a wheel of radius r which is rolling with its rim in contact with a straight line OX. The wheel is to roll with its plane always parallel to the guide plane XOY. Let ω be the angular velocity of the wheel at the instant at which the velocity of P is to be described. We shall select R and C as the two points in the XOY plane through which the two axes are to pass. Also, for the sake of simplicity, let C be the center of the wheel and R be on the rim of the wheel and in the line SCR, where S is the instantaneous point of contact between the wheel and the line OX.

We shall first determine the velocity of P with respect to XOY by using the axis through C. The velocity of C with respect to XOY, $V_C(O)$, is parallel to OX and equal to ωr. The velocity of P with respect to C is perpendicular to CP and is also equal to ωr. The vector sum of these two velocities gives the desired velocity of P, $V_P(O)$, as shown in the diagram.

In a similar manner we may find $V_P(O)$ by using the axis through R. The velocity of R with respect to O, $V_R(O)$, is

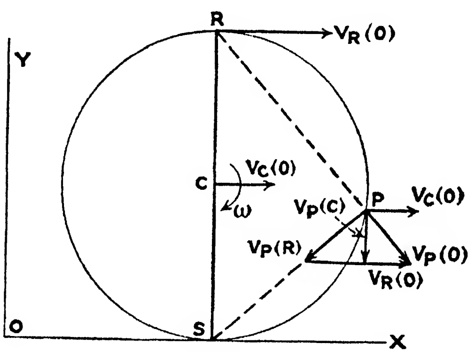

FIG. 23.

parallel to OX and equal to $2\omega r$. The velocity of P with respect to R, $V_P(R)$, is perpendicular to the line RP and is equal to ωRP. The vector sum of these two velocities again gives $V_P(O)$, as may be verified by computation.

1-14. The Fixed and Moving Centrodes.—The description of the velocity of any point of a body which is in uniplanar motion may usually be simplified by a judicious selection of the axis about which the rotational element is to be taken. It may be, at the instant at which the velocities of various points of the body are to be expressed, that there is a particular line of the body which is instantaneously at rest with respect to the fixed reference system. If this line is chosen to be the axis of the rotational element, then the translational term will be zero. The velocities of all points of the body could then be regarded as due to a pure rotational motion about the selected axis for that instant. Because of its importance, such an axis is called the *instantaneous axis* of rotation. The instantaneous axis is always perpendicular to the guide plane. The point of its intersection with the guide plane is called the *instantaneous center*.

The convenience of such a device for describing a uniplanar motion lies in the fact that the velocity of any point of the moving body may be expressed as the product of the angular velocity about the instantaneous axis and the distance of the point from that axis. The direction of the velocity is perpendicular to the radius vector drawn from the instantaneous center to the point whose velocity is to be expressed.

To illustrate the use of the instantaneous axis in expressing the velocity of any point of the moving body, let us consider again the illustration given in the preceding section (Fig. 23). As the wheel is rolling along the line OX, the point S, on the rim of the wheel, is instantaneously at rest in the position shown. If then we select S as the point in the XOY plane through which the instantaneous axis is to pass, *i.e.*, the instantaneous center, the magnitude of the velocity of any other point, say R, will be equal to the angular velocity ω multiplied by the distance SR. The direction of this velocity is given by the vector $V_R(O)$. Similarly, the velocity of P is equal to ωSP and is perpendicular to SP.

It must be borne in mind that the particular axis is only instantaneously at rest; hence the expressions for the velocities of the various points of the body are only true, in general, for that particular instant. For succeeding instants there will be other points which in turn may be regarded as the instantaneous centers of rotation. The aggregate of the series of instantaneous centers (points fixed in the reference system) is known as the *fixed centrode*. In the illustration used above (Fig. 23), OX is the fixed centrode. Corresponding to the fixed centrode, a locus fixed in the reference system, there is another locus of the instantaneous center. This latter locus is a curve fixed to the moving body and is called the *moving centrode*. In the rolling wheel of Fig. 23, the circumference of the wheel is the moving centrode. As the motion proceeds, the moving centrode rolls upon the fixed centrode in such a way that the particular point which is the instantaneous center for the given instant is the pair of points, one in each curve, which are instantaneously coincident.

If the velocities of any two points of the body are known, then the instantaneous axis may be located by drawing lines from the points respectively perpendicular to the velocities. The point of intersection of these two lines is the instantaneous center for that instant. The proof of this construction may be established in the following manner. Let P and R (Fig. 24) be any two points of the moving body whose velocities are known. Also let I be the instantaneous center, located by the intersection of the lines PI and RI drawn perpendicular to the velocities of P and R, respectively. Making use of Eq. (1-24), we may write

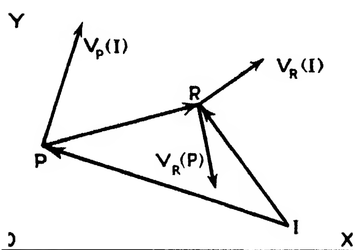

FIG. 24.

$$V_R(I) = V_P(I) + \omega PR \quad (1\text{-}25)$$

in which $V_R(I)$ and $V_P(I)$ are the velocities of R and P, respectively, in the reference system XOY. This equation expresses the velocity of R in terms of the velocity of P in the reference system and the velocity of R relative to P due to a rotation about an axis through P. Now to prove that

$$V_R(I) = \omega IR \quad \text{and} \quad V_P(I) = \omega IP$$

we may use the triangle IPR. Each of the velocities given in Eq. (1-25) is perpendicular to one of the sides of the triangle IPR, so that, by rotating all three of the velocity vectors through an angle of +90 deg., each velocity will then be parallel to one of the sides of the triangle; $V_R(I)$ will be parallel to IR, $V_P(I)$ will be parallel to IP, and ωPR to PR, with the directions as indicated in the diagram. Since the velocity ωPR is proportional to PR, then $V_R(I)$ and $V_P(I)$ must be equal to ωIR and ωIP, respectively, because of the vector relation expressed in Eq. (1-25). This development may be extended to include any other point of the body and it may be shown that the velocity of the selected point will be equal to the product of the common instantaneous angular velocity by the distance of that point from the instantaneous center.

A special case is next to be considered in which the above scheme for determining the instantaneous center fails. If the two known velocities are parallel to each other, there will, obviously, be no point of intersection of the two lines drawn through the points (whose velocities are given) perpendicular to the velocities. The instantaneous center may, however, be located by an algebraic solution. If both velocities are in the same sense, then (Fig. 25)

$$V_R = \omega\, IR \qquad \text{and} \qquad V_P = \omega\,(IR + RP)$$

where V_R and V_P are the velocities of the points R and P and ω is the instantaneous angular velocity. Hence

$$\frac{V_R}{V_P} = \frac{IR}{IR + RP}$$

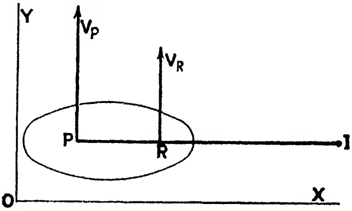

FIG. 25.

from which IR may be determined. If $V_P = V_R$, the motion is one of pure translation. If V_P and V_R are in opposite sense, then I must lie between P and R. In this case $V_R/V_P = IR/IP$.

1-15. General Solution for the Equations of the Fixed and Moving Centrodes.—We have seen that uniplanar motion of a body may be described in terms of two parts: a translational and a rotational part. It has also been shown that uniplanar motion may be described in terms of pure rotation about an instantaneous axis which is, in general, moving so that its point of intersection with the guide plane, the instantaneous center, describes a curve in space called the fixed centrode. An expression is to be found which gives the equation of the fixed centrode (sometimes called the space centrode) and a similar equation for the moving (or body) centrode.

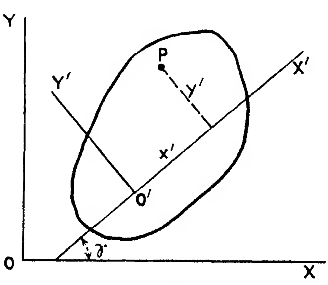

FIG. 26.

Given a fixed reference system (Fig. 26) XOY and a system $X'O'Y'$ which is attached to the moving body. The coordinates of P, a point of the moving body, are xy and $x'y'$ in the fixed and moving systems, respectively. The axis $O'X'$ makes an angle γ with OX. The coordinates of O' are x_0y_0 in the fixed system.

The following equations express the relations among the coordinates:

$$\begin{aligned} x &= x_0 + x' \cos\gamma - y' \sin\gamma \\ y &= y_0 + x' \sin\gamma + y' \cos\gamma \end{aligned} \qquad (1\text{-}26)$$

The velocity relations may now be found by differentiating these equations with respect to the time. We must remember that dx'/dt and dy'/dt are both equal to zero, for the moving system contains P as a point fixed in it. For brevity, ω is written for $d\gamma/dt$. Hence,

$$\frac{dx}{dt} = \frac{dx_0}{dt} - \omega(x' \sin \gamma + y' \cos \gamma)$$
$$\frac{dy}{dt} = \frac{dy_0}{dt} + \omega(x' \cos \gamma - y' \sin \gamma) \tag{1-27}$$

Even though no limitation is imposed upon the motion of the moving system, it is to be noticed that the velocity of P may be expressed in terms of two parts: a translation of the moving system and a rotation about an axis through O'.

We may write Eq. (1-27) in another form if we replace the quantities in the parentheses by their equivalents $y - y_0$ and $x - x_0$, respectively. This gives

$$\frac{dx}{dt} = \frac{dx_0}{dt} - \omega(y - y_0)$$
$$\frac{dy}{dt} = \frac{dy_0}{dt} + \omega(x - x_0) \tag{1-28}$$

An inspection of the figure will show the validity of these equivalences.

If we had selected another point of the body as origin for the moving system, say Q in place of O', the resulting expressions for dx/dt and dy/dt would have been similar to those written in Eq. (1-28) and would have contained the same value of ω. This means that the angular velocity of the body must be independent of the point through which the axis of rotation is supposed to pass, as indeed it should be, for the body is supposed to be rigid.

If we put dx/dt and dy/dt both equal to zero in Eq. (1-27), we imply that the point P, whose coordinates are x and y in the reference system, is stationary. By this evaluation we cannot make any point stationary but we can obtain relations from Eq. (1-27) which express the locus of those points which will be stationary at some time in the motion of the body. The resulting expressions

$$\frac{dx_0}{dt} - \omega\,(x' \sin \gamma + y' \cos \gamma) = 0$$
$$\frac{dy_0}{dt} + \omega\,(x' \cos \gamma - y' \sin \gamma) = 0 \tag{1-29}$$

which contain x' and y', give us the coordinates $(x'y')$ of the instantaneous center of rotation referred to the moving system. For the purpose of distinguishing these coordinates of the instantaneous center in the moving system, let us write ξ' and η' for x' and y', respectively, and, if we make the equations explicit for these symbols, we obtain

$$\begin{aligned} \xi' &= \frac{1}{\omega}\left(\frac{dx_0}{dt}\sin\gamma - \frac{dy_0}{dt}\cos\gamma\right) \\ \eta' &= \frac{1}{\omega}\left(\frac{dx_0}{dt}\cos\gamma + \frac{dy_0}{dt}\sin\gamma\right) \end{aligned} \tag{1-30}$$

These two equations may be combined by an elimination of the angle γ. Squaring both Eqs. (1-30), adding, and rearranging gives

$$\omega^2\,(\xi'^2 + \eta'^2) = \left(\frac{dx_0}{dt}\right)^2 + \left(\frac{dy_0}{dt}\right)^2$$

or

$$\omega\sqrt{\xi'^2 + \eta'^2} = \left[\left(\frac{dx_0}{dt}\right)^2 + \left(\frac{dy_0}{dt}\right)^2\right]^{\frac{1}{2}} \tag{1-31}$$

An interpretation of Eq. (1-31) affords an opportunity to check its validity. It will be readily observed that the right-hand member represents the velocity of the origin of the moving system. In the left-hand member the coefficient of ω is the radius vector drawn from the origin of the moving system to the point which is instantaneously at rest, because ξ' and η' are the coordinates of this stationary point. The left-hand member therefore also expresses the velocity of the origin of the moving system. This analysis of Eq. (1-31) gives confidence, perhaps, in accepting the fact that the point whose coordinates are ξ' and η' in the moving system is instantaneously at rest.

Equation (1-31) is the general expression of the locus of the instantaneous center referred to the moving system. Such a locus, *i.e.*, the moving centrode, must be independent of the angular velocity (ω) and the linear velocity (dx_0/dt and dy_0/dt) of the origin of the moving system. When the general equation is applied to any particular case, the information obtained from a selection of the particular moving system, together with other pertinent data, will supply the necessary relations for an elimination of these velocities. An illustration of the procedure is given in the following section.

In a similar manner we may find the equations for the fixed centrode by putting dx/dt and dy/dt of Eqs. (1-28) equal to zero. The following expressions are then obtained:

$$\frac{dx_0}{dt} - (y - y_0)\omega = 0$$

$$\frac{dy_0}{dt} + (x - x_0)\omega = 0$$

In order to avoid confusion, we may write ξ and η for x and y, respectively. Hence we have the following equations for the coordinates of the instantaneous center expressed in the fixed reference system.

$$\xi = x_0 - \frac{1}{\omega}\frac{dy_0}{dt}$$

$$\eta = y_0 + \frac{1}{\omega}\frac{dx_0}{dt} \qquad (1\text{-}32)$$

As in the case of the moving centrode when data for a particular case are presented, other equations may be written which may be used with Eq. (1-32) in obtaining an expression for the locus of the instantaneous center in space, *i.e.*, the fixed centrode.

In any given case we are at liberty to select the positions of the fixed and moving systems so that the resulting analytical expressions may be simplified. The equations for the two centrodes are then obtained by expressing x_0, y_0, and γ and their derivatives in terms of their values as determined by the characteristics of the problem. In order to make the details of the process of obtaining the centrodes clear, the following illustrations have been selected.

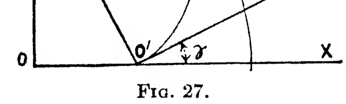

Fig. 27.

1-16. The Centrodes of a Falling Ladder.—To find the moving and fixed centrodes of a ladder as it falls, with its lower end sliding along the horizontal ground and its upper end in contact with a vertical wall. Let BO' (Fig. 27) be the ladder, XOY the fixed reference system with OY the vertical wall and OX the ground, and let the moving system $X'O'Y'$ be fixed to the ladder with O' the foot of the ladder, $O'Y'$ along the ladder, and $O'X'$ perpendicular to the ladder, as shown in the diagram. Let γ be the angle which $O'X'$ makes with OX. We may let O' (with coordinates x_0 and y_0 in the fixed system) move along OX so that

$$\frac{dx_0}{dt} = C \qquad \text{and} \qquad \frac{dy_0}{dt} = O$$

where C is the speed of O' along OX. The positions of the centrodes, however, must be independent of the speed of falling; hence we should not expect the equations of the centrodes to contain C. The speed C does not appear in the final equations as will be observed. If L is the length of the ladder, then

$$\sin\gamma = \frac{x_0}{L}, \quad \cos\gamma = \frac{\sqrt{L^2 - x_0^2}}{L}, \quad \text{and} \quad \omega = \frac{C}{(L\cos\gamma)} \tag{1-33}$$

If we substitute these values in Eq. (1-30), the following equations are obtained:

$$\xi' = L\sin\gamma\cos\gamma \qquad \eta' = L\cos^2\gamma$$

Eliminating γ gives

$$\xi'^2 + \eta'^2 = L\eta' \tag{1-34}$$

which is the equation of the moving centrode referred to the moving axes. This centrode is obviously a circle constructed about the ladder as a diameter.

By using Eq. (1-32) the fixed centrode may be determined. If we substitute in these equations the values of the quantities as indicated above [Eq. (1-33)], we obtain the expressions for the coordinates of the instantaneous center referred to the fixed axes.

$$\xi = x_0 = L\sin\gamma \qquad \eta = L\cos\gamma$$

Eliminating γ gives

$$\xi^2 + \eta^2 = L^2 \tag{1-35}$$

which is a circle of diameter $2L$ constructed about the point O as center. This circle is the fixed centrode. It is interesting and instructive to observe how the moving centrode rolls upon the fixed centrode as the ladder falls. There is instantaneously only one point of contact between the two centrodes, *viz.*, the point of tangency (I in Fig. 27) which is the instantaneous center. It is instructive for the student to determine the centrodes for the falling ladder by a completely graphical method.

1-17. The Centrode of a Connecting Rod—Problem.—Let the connecting rod PO' (Fig. 28) be of fixed length L. Let the fixed reference system be XOY and the moving system be $X'O'Y'$, as shown in the diagram. One end of the rod moves about the circle, of radius r and center at O, with angular velocity $d\alpha/dt$. The other end O' of the rod moves along OX. The line OP makes the angle α with OX, and $O'X'$ makes the angle γ with OX.

The details of the process of determining the two centrodes are left for the student to work out. The equations for the centrodes are given below [Eqs. (1-36) and (1-37)], for the purpose of providing a check upon the results obtained. If desired, the

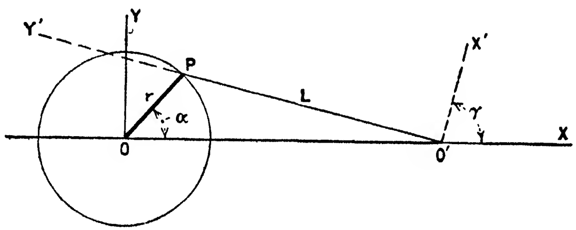

FIG. 28.

two centrodes may be obtained by a purely graphical method. For the fixed centrode:

$$\eta(\xi^2 - L^2 + r^2) = \xi[4L^2\xi^2 - (\xi^2 + L^2 - r^2)^2]^{\frac{1}{2}} \qquad (1\text{-}36)$$

For the moving centrode:

$$\eta^2 + \xi^2 = L\eta\left[\frac{L\xi}{\sqrt{r^2(\xi^2 + \eta^2) - L^2\eta^2}} + 1\right] \qquad (1\text{-}37)$$

Problems.—1. A wheel rolls with angular velocity ω along a straight line but the point of contact slips with a velocity V as the wheel rolls. The velocity of slip is oppositely directed with respect to the forward motion of the wheel. Find the instantaneous center.

2. A circle of radius r rolls inside a larger circle of radius $2r$. Prove that P, any point of the circumference of the smaller circle, moves along a straight line.

3. If a point P moves along a straight line with a constant linear velocity, prove that its angular velocity about any fixed point Q which is not on the line of P's motion varies inversely as $(PQ)^2$.

4. A rigid plane body containing the two points P and Q moves in such a manner that the two points P and Q are always guided by two intersecting straight lines, so that P is on one of the lines and Q is on the other. Prove that the centrodes are circles.

5. A plane figure rotates with constant angular velocity, while it moves in such a manner that one of its points is guided around a fixed circle with constant speed. Find the centrodes.

6. The coordinates of a point in an orthogonal system are 4 and -15. Find its coordinates in a plane polar system. Also find its coordinates in an oblique system in which the angle between the X- and Y-axes is 70 deg.

7. A boy (A) is riding in a power boat (speed in still water is 20 m.p.h.) and is headed directly across a stream (speed 10 m.p.h.). He throws a ball with a speed of 100 ft. per second at another boy (B) who is running along the shore with a speed of 8 ft. per second. The ball is thrown at the instant when B is in a direct line with the keel of the boat and at that instant A is 40 ft. from B. The ball arrives at the shore line 2 ft. behind B. Find the velocity of the ball with respect to the shore, the water, and the boat.

8. A wheel 4 ft. in diameter is rolling in a vertical plane along level ground with a speed of 3 r.p.s. A pebble is at the highest point of the wheel. The velocity of the pebble is 10 ft. per second relative to the wheel and its direction makes an angle of 45 deg. (forward) with the radius drawn to the pebble. Find the velocity of the pebble with respect to the ground.

CHAPTER II

VECTORS

2-1. Definitions.—In mathematical physics we are concerned with two classes of quantities. The quantities of one class are characterized by the fact that they possess magnitude only. A number together with a selected dimensional unit is sufficient to completely specify the magnitude of a quantity of this class. Such quantities are called *scalar* quantities. Some of the scalar quantities which are frequently encountered are time, mass, energy, work, temperature, potential, and moment of inertia. Manipulation of scalar quantities is conducted according to the laws of ordinary algebraic analysis.

The other class of quantities with which we deal in physics requires a direction as well as a magnitude to describe them completely. Such quantities are called *vector* quantities. The group of vector quantities includes displacement, velocity, force, acceleration, momentum, force moment, and angular velocity. Vector quantities are more complex than scalar quantities because they possess direction as an additional element. Except in a limited number of special cases, vector quantities are not subject to the rules of algebraic analysis but must be manipulated according to a different code of rules which is called vector analysis. Two of the processes of vector analysis, vector addition and projection of vectors, have been described in the preceding chapter. Because other processes of vector analysis are useful in the study of mechanics, descriptions and illustrations of them are introduced in this chapter. The principal advantage which vector descriptions have in comparison with algebraic descriptions is brevity. Two or more algebraic equations may be required to present a description which can be given by a single vector equation.

Vector quantities are represented by *vectors*. A *vector* is a directed straight line segment. The length of the line segment is determined by the scale of representation selected, together with the magnitude of the quantity represented. The magnitude and direction of the vector correspond to the magnitude and direction of the vector quantity represented.

There are, in general, two types of vectors: *polar* and *axial* vectors.

Polar vectors are those having directions which are directly determined by the single direction of the quantity. Examples of the polar vectors are force, momentum, acceleration, and translational velocity.

Axial vectors represent those vector quantities which are composite in nature, consisting of at least two elements, each of which requires a direction to determine its position. The direction of the axial vector is conventional, but it is associated with a line or axis which is prominently connected with some physical aspect of the quantity, like the axis of rotation in the case of force moment. Examples of this type are force moment, angular velocity, and moment of momentum.

The processes used in vector analysis may be applied alike to polar and axial vectors as long as the standard right-hand reference system only is used.

Because there is a direction associated with a magnitude in vector quantities, this group of physical quantities is conveniently subject to graphical representation. Such a scheme of representation is exceedingly helpful to the student, for in the process of graphical representation a means of mentally visualizing these quantities is at hand which is of valuable assistance in the better understanding of the physical relations that are fundamental to them. Because this is true and because of the brevity in expression, vectors and vector equations are too valuable as tools to neglect. In this work, therefore, use will be made of them, along with the more familiar analytical expressions. In the following pages vectors are expressed in bold-faced type in order to distinguish them from the scalar quantities which are printed in light-faced type. When writing vectors, the student should adopt some other scheme for distinguishing vectors from scalars such as drawing a bar above the letter, *i.e.*, $\bar{A}$.

2-2. The Unit Vector.—Since the vector consists of a magnitude and a direction, we may express the vector in terms of these two parts. The direction may be expressed by a vector of unit length which has the same direction as the given vector. The directional part is spoken of as the *unit vector*. It will be written by putting the subscript 1 after the letter standing for the vector. Thus, if the vector is V, its unit vector is V_1. We may then represent the vector V by VV_1 where V gives the magnitude and V_1 the direction of the vector.

2-3. The Rectangular Components of a Vector.—It is often convenient to express a vector in terms of its components, projected on the three axes of an XYZ reference system. The relation employed for this purpose utilizes the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$, which are always associated with the X-, Y-, and Z-axes, respectively. Given the vector $\boldsymbol{A}$ (Fig. 29) in the orthogonal system XYZ with α, β, and γ the direction angles of $\boldsymbol{A}$. The components of A upon the axes are $A \cos \alpha$, $A \cos \beta$, and $A \cos \gamma$.

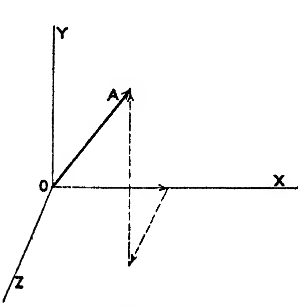

FIG. 29.

The components are more conveniently expressed by using the unit vectors $\boldsymbol{ijk}$. If A_x, A_y, A_z are the magnitudes of the components upon the axes and are therefore the scalar factors, we may write $A_x\boldsymbol{i}$, $A_y\boldsymbol{j}$, $A_z\boldsymbol{k}$ for the vector components of $\boldsymbol{A}$ and hence the following vector equation is valid:

$$\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$$

This equation represents a very important type of equation. It exhibits the link which connects vector algebra with the Cartesian relations.

The vector sum of two (or more) vectors is expressed by the vector equation $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$ or one similar. If

$$\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$$
$$\boldsymbol{B} = B_x\boldsymbol{i} + B_y\boldsymbol{j} + B_z\boldsymbol{k}$$

we may add these equations and obtain the expressions

$$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B} = (A_x + B_x)\boldsymbol{i} + (A_y + B_y)\boldsymbol{j} + (A_z + B_z)\boldsymbol{k}$$
$$= C_x\boldsymbol{i} + C_y\boldsymbol{j} + C_z\boldsymbol{k}$$

This is a vector equation which indicates that the components of the vector $\boldsymbol{C}$, which represents the vector sum of the two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, may be found by adding algebraically the corresponding components of the given vectors.

Example.—Find the vector sum of the three vectors graphically and also by adding the components by the method indicated above.

$$\begin{aligned} A &= 5i - 3j + 4k \\ B &= -2i + 4j + 3k \\ C &= 3i - 2j \end{aligned}$$

2-4. The Multiplication of Vectors.—In the preceding section it was shown that a vector may be expressed in terms of the unit vectors which are associated with some assigned rectangular system of coordinates. We wish next to show the development of the two kinds of products of vectors, *viz.*, the *scalar* and *vector* products of two vectors. This is to be done by first multiplying algebraically the three rectangular components of the two vectors together in the usual algebraic manner and then interpreting the nine resulting terms. Given the two vectors A and B, expressed in terms of their components in some reference system. Let

$$\begin{aligned} A &= a_1i + a_2j + a_3k \\ B &= b_1i + b_2j + b_3k \end{aligned}$$

where the coefficients of the unit vectors are the ordinary components of the given vectors on the reference axes. Multiplying these two equations together in the ordinary algebraic manner gives

$$\begin{aligned} AB = {} & a_1b_1ii + a_2b_1ji + a_3b_1ki \\ & a_1b_2ij + a_2b_2jj + a_3b_2kj \\ & a_1b_3ik + a_2b_3jk + a_3b_3kk \end{aligned} \tag{2-1}$$

In writing the terms of the right-hand member, the order of the factors in each term has been carefully preserved. In this expression there are three kinds of terms: those with similar unit vectors (ii, jj, kk), those with unit vectors which occur in the cyclic order (ij, jk, ki) and those with the inverse cyclic order (ji, kj, ik).

In order to interpret the right-hand member of Eq. (2-1), let us first consider collectively those terms having similar unit vectors and then later the remaining six terms. The first group may be put into another form which is useful for our purpose if we designate by $l_1m_1n_1$ and $l_2m_2n_2$ the direction cosines of the vectors A and B, respectively.

It is readily seen that

$$\begin{aligned} Al_1 &= a_1 \qquad & Am_1 &= a_2 \qquad & An_1 &= a_3 \\ Bl_2 &= b_1 & Bm_2 &= b_2 & Bn_2 &\doteq b_3 \end{aligned}$$

If now we put the products $\boldsymbol{ii}$, $\boldsymbol{jj}$, and $\boldsymbol{kk}$ each equal to unity, an assumption which is not inconsistent with the results developed below, we may then write

$$\begin{aligned} a_1b_1\boldsymbol{ii} + a_2b_2\boldsymbol{jj} + a_3b_3\boldsymbol{kk} &= AB\,(l_1l_2 + m_1m_2 + n_1n_2) \\ &= AB \cos \alpha \end{aligned} \tag{2-2}$$

where α is the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$.

We have then reduced the three terms containing similar unit vectors to a simple expression which involves the magnitudes of the vectors A and B and the cosine of the angle between them. It is customary to use this result as a means for defining the so-called *scalar product* of two vectors.

The scalar product of any two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is then defined by the following equation:

$$\boldsymbol{A} \cdot \boldsymbol{B} = A\ B \cos \alpha \tag{2-3}$$

in which the dot, written between the two vectors as shown, signifies that the product of $\boldsymbol{A}$ and $\boldsymbol{B}$ is to be a scalar product. Because of the use of the dot in this connection, this product is sometimes called the dot product. This expression is taken to be a scalar quantity because of the fact that in mechanics there are occasions when the product of two vector quantities yields a scalar quantity whose magnitude is given by the result obtained above. For example, in the case of the work done by a force $\boldsymbol{F}$ acting upon a body which has a displacement $\boldsymbol{S}$ even though $\boldsymbol{S}$ is not parallel to $\boldsymbol{F}$, we may write in vector notation

$$\boldsymbol{F} \cdot \boldsymbol{S} = F\ S \cos \alpha$$

where α is the angle between $\boldsymbol{F}$ and $\boldsymbol{S}$. In this expression we have a general equation for the work done. Since work is a scalar quantity, the left-hand member of the equation must also be a scalar quantity, for it would be incorrect to equate a scalar quantity to a vector quantity.

Consistent with this definition, the scalar product of two similar unit vectors, such as $\boldsymbol{i} \cdot \boldsymbol{i}$, is equal to unity. The left-hand member of Eq. (2-2) is therefore a scalar quantity. A further description of the meaning and use of the scalar product will be reserved for a section below [Sec. (**2-5**)].

It is next to be shown that the remaining six terms may be reduced to another simple expression, to which the name *vector product* has been applied. The adjective "vector" has been used here to indicate that the result obtained is a vector quantity instead of a scalar quantity. In order to carry out the development, it is necessary to interpret the meaning of the product of two unit vectors which are at right angles to each other. Consider the product $\boldsymbol{ij}$ as an illustration. By multiplying together two lengths which are perpendicular to each other, we obtain the area of a rectangle constructed upon these lengths as adjacent sides. We may use here the idea of the axial type of vector and represent this area by an axial vector drawn mutually perpendicular to the two elements of the product. In the case selected, the axial vector representing $\boldsymbol{ij}$ would therefore be directed along the $\boldsymbol{k}$-axis. Since $\boldsymbol{i}$ and $\boldsymbol{j}$ are both unit vectors, their product gives unit area and hence $\boldsymbol{k}$ is also of the proper magnitude to represent this product. Whether $\boldsymbol{k}$ is to be taken in the positive or negative direction is a matter for convention. Consistent with our standard system, we should put

$$\boldsymbol{k} = \boldsymbol{i} \times \boldsymbol{j} \qquad \text{and} \qquad -\boldsymbol{k} = \boldsymbol{j} \times \boldsymbol{i}$$

The cross ($\times$) is introduced to call attention to the fact that we intend this product to represent a vector product. It is in general use for this purpose.

Returning now to the consideration of the six terms of Eq. (2-1), which contain unlike unit vectors, we may collect these terms into the following expression:

$$(a_1b_2 - a_2b_1)\,\boldsymbol{k} + (a_2b_3 - a_3b_2)\,\boldsymbol{i} + (a_3b_1 - a_1b_3)\,\boldsymbol{j}$$

If we now let λ, μ, ν be the direction angles of the normal to the plane containing the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, the following expressions may be written:

$$\cos\lambda = \frac{m_1n_2 - m_2n_1}{\sin\alpha}$$

$$\cos\mu = \frac{n_1l_2 - n_2l_1}{\sin\alpha}$$

$$\cos\nu = \frac{l_1m_2 - l_2m_1}{\sin\alpha} \qquad (2\text{-}4)$$

Using the first of Eqs. (2-4) as a type and multiplying both sides by AB gives

$$(AB \sin \alpha) \cos \lambda = AB\ (m_1n_2 - m_2n_1)$$

$AB \sin \alpha$ is evidently an area which is projected upon the YZ plane when multiplied by $\cos \lambda$, since λ is also the angle between the plane containing $\boldsymbol{A}$ and $\boldsymbol{B}$ and the YZ plane.

We may represent the area $AB(m_1n_2 - m_2n_1)$, which is in the YZ plane, by an axial vector drawn along the X-axis. But

$$AB\ (m_1n_2 - m_2n_1) = a_2b_3 - a_3b_2$$

hence $(a_2b_3 - a_3b_2)\boldsymbol{i}$ is the vector along the X-axis which represents the X component of the area $AB \sin \alpha$. In a similar manner it may be shown that

$$(a_3b_1 - a_1b_3)\boldsymbol{j} \qquad \text{and} \qquad (a_1b_2 - a_2b_1)\boldsymbol{k}$$

are the Y and Z components, respectively, of the vector which represents the area $AB \sin \alpha$.

Hence we may regard the six terms of Eq. (2-1), which contain unlike unit vectors, as collectively representing the vector sum of the three component vectors, each component representing the projection of the area $AB \sin \alpha$ upon the particular reference plane which is perpendicular to the axis along which the component is taken.

This part of the general product is to be represented symbolically by the expression

$$\boldsymbol{A} \times \boldsymbol{B} = (AB \sin \alpha)\ \boldsymbol{n}_1 \tag{2-5}$$

in which the left-hand member is to be called the *vector product* of the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ and is a *vector* quantity. The magnitude of the vector product is given by the expression $AB \sin \alpha$ and the direction by the unit vector $\boldsymbol{n}_1$, which is mutually perpendicular to both of the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. A more complete description of this vector product and some of its applications are included in a section below.

2-5. The Scalar Product of Two Vectors.—The scalar product of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ gives a scalar quantity whose magnitude is $AB \cos \alpha$, where α is the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$. Either we may interpret this result to mean that $\boldsymbol{A}$ is projected into the line of $\boldsymbol{B}$ through the angle α and the result then multiplied by B or we may regard $\boldsymbol{B}$ as the vector which is projected into the line of A and is then multiplied by A. Either interpretation is valid. It must not be forgotten that the scalar product gives a scalar quantity even though it is obtained from two vectors.

As a special case of this process we have $A \cdot B = 0$ if $\alpha = 90$ deg. If $\alpha = 0$, as in the case of the square of a vector then $A \cdot A = A^2$. The scalar products of the unit vectors may be written:

$$i \cdot i = j \cdot j = k \cdot k = 1$$
$$i \cdot j = j \cdot k = k \cdot i = 0$$

Illustration.—It is required to find the simple trigonometric relation which exists between the squares of the sides of an oblique triangle. Any side of the triangle may be regarded as the vector sum of the two other sides; hence $C = A + B$, where A, B, and C are the sides of the triangle as shown in Fig. 30. The commutative and distributive laws of ordinary algebra hold for scalar multiplication of vectors. The proofs for this statement will be left to the student. The validity of these laws being assumed, the following results are easy to obtain:

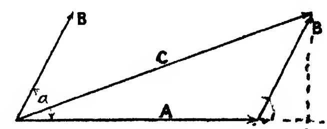

FIG. 30.

$$C \cdot C = (A + B) \cdot (A + B)$$
$$= A \cdot A + 2A \cdot B + B \cdot B$$

Hence

$$C^2 = A^2 + 2AB \cos \alpha + B^2$$

Work, a Scalar Quantity.—In the scalar quantity work, we find an application of the scalar product of two vectors. The

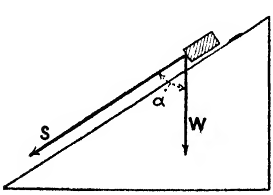

FIG. 31.

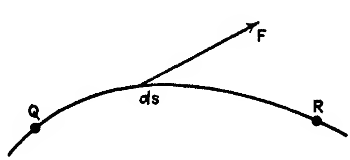

FIG. 32.

work done by the force W in moving a body a distance S down an inclined plane (Fig. 31) is expressed by $W \cdot S = WS \cos \alpha$. The scalar product takes care of the necessity for either projecting W into the line of S or *vice versa*.

2-6. Other Illustrations of the Scalar Product.—It is desired to express the work done by a variable force F as it moves an object along a definite path, such as $Q\ R$ in Fig. 32. If s is a coordinate measured along the path from Q, as origin, toward R and $\boldsymbol{ds}$ is a differential displacement in this path, then the element of work done by moving the object a distance ds is $F \cos \alpha \, ds$, where α is the angle between $\boldsymbol{F}$ and $\boldsymbol{ds}$. The total work done in moving the object from Q to R may then be expressed by the following integral:

$$\text{Work} = \int_Q^R F \cos \alpha \, ds$$

The form of the expression may be abbreviated by the use of the scalar product of the vectors $\boldsymbol{F}$ and $\boldsymbol{ds}$; hence

$$\text{Work} = \int_Q^R \boldsymbol{F} \cdot \boldsymbol{ds} \tag{2-6}$$

The integral may be evaluated if the law of the variation of F with s is known.

An integral of this type [Eq. (2-6)] is called a line integral of the vector quantity. It is used for determining several scalar quantities.

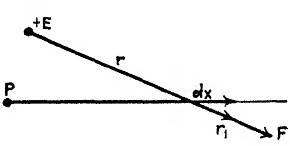

Fig. 33.

To illustrate further the use of this integral, let us determine the potential, a scalar quantity, at a point P (Fig. 33) in an electric field due to charge E which is, for simplicity, to be regarded as concentrated at one point. The field strength $\boldsymbol{F}$ at any point is a vector which gives the force exerted by the field on a unit positive charge placed at the point in question. The magnitude of the field is expressed by the relation

$$\boldsymbol{F} = \frac{E}{k} \frac{\boldsymbol{r}_1}{r^2}$$

where k is the specific inductive capacity of the medium, r is the distance from the charge $+E$ to the point at which the field strength is expressed, and $\boldsymbol{r}_1$ is the unit vector in the line of r and directed away from E. The potential at P is determined by the work done against the field force $\boldsymbol{F}$ in bringing a unit positive charge from infinity up to P. This is expressed by the integral

$$\text{Potential} = \int_\infty^P \boldsymbol{F} \cdot \boldsymbol{dx} \tag{2-7}$$

which is to be taken from infinity to P, and in which the coordinate x is to be measured outward from P along the path over which the unit charge is moved. By introducing the value of F and expressing r in terms of x, the value of the potential may be determined. Attention is to be directed to the scalar product of the two vectors in the integrand.

If the vector form of the expression had not been used, it would have been necessary to have used the component of F which is in the line of x.

2-7. The Surface Integral of a Vector.—A *vector field* is a region in which there is a definite value of some vector at every point of the field. In a moving fluid a velocity vector V may be used to express the velocity of each moving particle. In general, V will be variable. To each and every particle of the fluid there will be some definite value of V. The aggregate of these vectors, one assigned to each particle, is spoken of as a vector field.

We are to imagine a closed surface, such as a spherical surface, surface of a cube, or the like, to be placed in a vector field. The presence of the imaginary surface in the field is in no way to produce any alteration of the vector field. At every point on the selected surface, the vector will have a determinate value. In general, however, the direction of the vector will not be perpendicular to the surface element surrounding the particular point on the surface. If n_1 is the unit vector drawn outward and perpendicular to the surface element, then $V \cdot n_1$ gives the magnitude of the component of V which is perpendicular to the surface element, and the quantity $V \cdot n_1 ds$ gives the volume of fluid going out through the elemental area ds in unit time. If we integrate the latter quantity over the entire closed surface, the result obtained gives the total volume of fluid passing out through the surface in unit time. In symbols this may be written as follows:

$$\text{Volume of fluid per second} = \iint_S V \cdot n_1 ds \qquad (2\text{-}8)$$

The use of the double integration sign indicates that the integration is to be taken over a surface. The subscript S written to the right of the integration signs shows that the integration is to be extended over the entire closed surface. If the value of this particular integration yields a positive quantity, the closed surface is said to contain a "source." If, however, the result is negative, then the surface contains a "sink." A zero value for the integral indicates that there is neither source nor sink within the closed surface.

This type of integral [Eq. (2-8)], containing the scalar product of two vectors, is a scalar quantity. It has an extensive application and is used in determining the sources of electricity in electric fields, masses in gravitational fields, and sources of heat in regions of flowing heat.

The following illustration shows how the details of evaluating a surface integral may be carried out. Let us suppose the vector field is one in which the vectors give the velocities of the particles of a moving fluid and that throughout the region under consideration the velocity is everywhere the same. We shall determine the surface integral of the velocity V over a spherical surface of radius r. Let an XYZ reference system, with origin at the center of the sphere, be placed with the X-axis parallel to V, and let P be any point on the surface of the sphere. We shall use the spherical coordinates r, φ, and θ, as shown in Fig. 34.

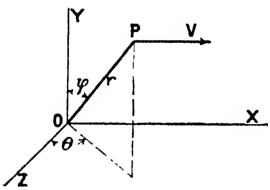

Fig. 34.

Since r is everywhere perpendicular to the surface element ds to which it may be drawn, the unit vector $\boldsymbol{n}_1$ is always parallel to r and directed outward along r. The scalar product $\boldsymbol{V} \cdot \boldsymbol{n}_1$ in the integral of Eq. (2-8) is to be replaced by the component of $\boldsymbol{V}$ which is in the line of r, which is $V \sin\theta \sin\varphi$. The area ds in spherical coordinates is $rd\varphi \cdot r \sin\varphi\, d\theta$. With these evaluations of the quantities, we may write the surface integral as follows:

$$\begin{aligned}\iint_S \boldsymbol{V} \cdot \boldsymbol{n}_1 ds &= \iint_S (V \sin\theta \sin\varphi)\, r\, d\varphi\, r \sin\varphi\, d\theta \\ &= V\, r^2 \int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta \sin^2\varphi\, d\varphi \\ &= V\, r^2 \Big[-\cos\theta\Big]_0^{2\pi}\Big[-\tfrac{1}{2}\cos\varphi \sin\varphi + \tfrac{1}{2}\varphi\Big]_0^{\pi} \\ &= 0 \end{aligned} \tag{2-9}$$

It should be observed that the limits of integration for θ are from 0 to 2π and for φ are from 0 to π. The zero result for the integral indicates that the amount of fluid flowing out of the surface in unit time is equal to that which enters.

2-8. The Vector Product of Two Vectors.—Some physical quantities are measured by the product of two vectors and yet are vector quantities. Those quantities are adequately expressed by the vector product. Such a case is found in the quantity force moment, or "torque," or in linear velocity when the linear velocity is expressed in terms of an angular velocity and a radius vector. The vector product of the two vectors A and B is usually written as the cross ($\times$) product $A \times B$ and has for its magnitude the value $AB \sin \alpha$, where α is the angle between them. It is to be noticed that the presence of the sine of the angle projects one of the vectors into a line which is perpendicular to the other. It is immaterial which vector is considered as being projected. The direction of the vector which expresses the product is perpendicular to the plane determined by the two vectors. In the illustration (Fig. 35) if $C = A \times B$, the direction of C would be drawn as shown in the diagram. This representation is in accord with the standard right-hand convention of signs. On the other hand, the product $B \times A$ would be represented by $-C$. Hence $A \times B = -B \times A$. The magnitude of $A \times B$ is represented by the area of the parallelogram formed upon A and B as sides. In a scalar product no such representation is to be made.

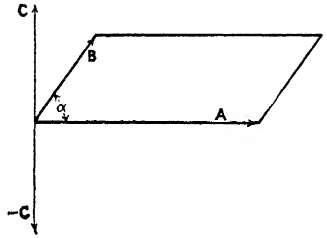

FIG. 35.

In the expression $V = \omega \times r$, the vector V expresses the linear velocity of a point which is at a distance given by the vector r from the rotation axis, and ω is the angular velocity about the rotation axis. The vector ω is drawn along the rotation axis.

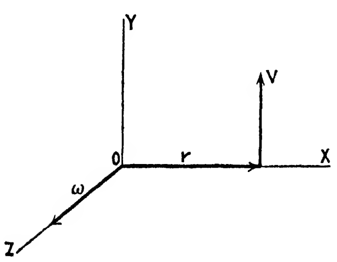

FIG. 36.

In the diagram (Fig. 36), if ω is positive and is measured along the Z-axis and r is parallel to X, then V is parallel to Y.

Illustrations—The Unit Vector. From the statements already made, it should be obvious that

$$i \times i = j \times j = k \times k = 0$$
$$i \times j = k; \quad j \times k = i; \quad k \times i = j$$
$$j \times i = -k; \quad k \times j = -i; \quad i \times k = -j \qquad (2\text{-}10)$$

Product of Two Vectors in Terms of Their Components.—If any two vectors A and B are given in terms of their components

A_x, A_y, A_z and B_x, B_y, B_z along a set of rectangular axes OX, OY, and OZ, respectively, as, for example,

$$A = A_x i + A_y j + A_z k$$
$$B = B_x i + B_y j + B_z k$$

we may expand the vector product of their components by observing the above relations for the vector products of the unit vectors; hence,

$$A \times B = (A_yB_z - A_zB_y)i + (A_zB_x - A_xB_z)j + (A_xB_y - A_yB_x)k \quad (2\text{-}11)$$

The right-hand member represents the vector sum of the components of a new vector. The magnitude of the product $A \times B$ is the magnitude of the vector represented by the right-hand member. This magnitude may be found by taking the square root of the sum of the squares of its components.

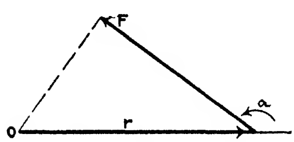

Fig. 37.

Parallel Vectors.—If any two vectors are parallel to each other, their vector product must be equal to zero. That this is true may be seen either from the fact that the sine of the angle between them is zero or from the fact that the coefficients of the unit vectors in the foregoing equation would separately vanish. In particular, the vector square of any vector is zero.

The Moment of a Force.—One of the simplest illustrations of the use of the vector product is in the expression for the moment of force. In the analytical expression for the moment of a force, the lever arm must be perpendicular to the line of the force and is measured from the axis of rotation to the line of the force. If r, the radius vector (Fig. 37), is not perpendicular to F, the force, then it must be projected into a line which is perpendicular to F. This process is automatically taken care of when the moment is expressed as the vector product of r and F. Hence in the vector equation

$$\text{Moment} = r \times F \quad (2\text{-}12)$$

we have a simplified expression for the moment of the force F.

2-9. The Angle between Two Vectors.—It is frequently necessary to determine the magnitude of the angle between two vectors when the components of the vectors along the axes of a given reference system are known. Two expressions are developed in this section from which the desired angle may be found. One expression is obtained from the relations that hold for the vector product of two vectors and the other is based upon the scalar product of the two vectors.

Given the vectors $\mathbf{A}$ and $\mathbf{B}$ with α the angle between them and measured from $\mathbf{A}$ to $\mathbf{B}$. Let $\mathbf{A}_1$ and $\mathbf{B}_1$ be the unit vectors in the lines of $\mathbf{A}$ and $\mathbf{B}$, respectively, and $\mathbf{C}_1$ the unit vector in the direction of the vector which represents the vector product of $\mathbf{A}$ and $\mathbf{B}$. We may therefore write

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (AB)\mathbf{A}_1 \times \mathbf{B}_1 \\ &= (AB \sin \alpha)\mathbf{C}_1 \end{aligned} \tag{2-13}$$

Writing $\mathbf{A}$ and $\mathbf{B}$ in terms of their components along the lines of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, we may express the vector product of $\mathbf{A}$ and $\mathbf{B}$ as was done in Eq. (2-11). The expression $\mathbf{A} \times \mathbf{B}$ in Eq. (2-11) may be replaced by its equivalent expression $(AB \sin \alpha)\mathbf{C}_1$. If we now square both members of the resulting equation, we obtain

$$(AB \sin \alpha)^2 = (A_yB_z - A_zB_y)^2 + (A_zB_x - A_xB_z)^2 + (A_xB_y - A_yB_x)^2$$

Hence

$$\sin \alpha = \frac{1}{AB}[(A_yB_z - A_zB_y)^2 + (A_zB_x - A_xB_z)^2 + (A_xB_y - A_yB_x)^2]^{\frac{1}{2}} \tag{2-14}$$

If any two vectors are given in terms of their components, we may determine the angle between them by the use of the foregoing equation, for all of the quantities are known except the magnitudes A and B and these may be easily found by the simple relation

$$A^2 = A_x^2 + A_y^2 + A_z^2$$

and by a similar expression for B.

Another expression for the angle α may be found by using the *scalar* product of the given vectors. For example,

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= A\ B \cos \alpha \\ &= A_xB_x + A_yB_y + A_zB_z \end{aligned}$$

Hence

$$\cos \alpha = \frac{A_xB_x + A_yB_y + A_zB_z}{A\ B} \tag{2-15}$$

This expression is the simpler and hence is easier to evaluate than the relation given in Eq. (2-14).

Problems.—1. Prove that the following vectors are perpendicular to each other: $\boldsymbol{A} = 9\boldsymbol{i} + \boldsymbol{j} - 6\boldsymbol{k}$ and $\boldsymbol{B} = 4\boldsymbol{i} - 6\boldsymbol{j} + 5\boldsymbol{k}$.

2. Find the angle between the vectors $\boldsymbol{A} = 3\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{B} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$.

3. Find the vector product of the vectors $\boldsymbol{A} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{B} = 3\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$ and prove that the vector representing this product is perpendicular to each of the given vectors.

4. Prove that the following vectors are parallel: $\boldsymbol{A} = 7.5\boldsymbol{i} + 3\boldsymbol{j} + 6\boldsymbol{k}$ and $\boldsymbol{B} = 5\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}$.

5. What are the necessary relations between the components of any two vectors in order that they be parallel to each other?

6. Find the area of the triangle which is bounded by the two vectors $\boldsymbol{A} = 2\boldsymbol{i} - 4\boldsymbol{j}$ and $\boldsymbol{B} = 3\boldsymbol{i} - 2\boldsymbol{j}$.

7. Find the area of the parallelogram determined by the two vectors $\boldsymbol{A} = 2\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{B} = -4\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k}$.

8. Find the vector and also the scalar product of the diagonals of the parallelogram determined by the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$.

2-10. The Product of Three Vectors.—There are three kinds of triple products of vectors which we wish to investigate: (*a*) $\boldsymbol{A}(\boldsymbol{B} \cdot \boldsymbol{C})$, the product of a vector into the scalar product of two other vectors, (*b*) $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$, the scalar product of a vector into the vector product of two others, and (*c*) $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})$, which is the vector product of a vector into the vector product of two other vectors. These three triple vector products will be discussed below in the order given.

Such triple vector products as have just been written may be evaluated; *i.e.*, they may be expressed as either a single vector quantity or a single scalar quantity. The value of a triple vector product will depend, in general, upon the order of the steps taken in the evaluating process. In order to avoid uncertainty as to the procedure in expanding a triple product of vectors, it is customary to enclose within the parenthesis that pair of vectors which are to be multiplied together first. The procedure to be followed in expanding a triple product of vectors is therefore to determine first the product (vector or scalar as indicated) of the two vectors within the parenthesis and then to multiply this result by the third (unenclosed) vector in the manner indicated by the dot or cross symbol.

a. $\boldsymbol{A}(\boldsymbol{B} \cdot \boldsymbol{C})$.—In expanding this triple product, we must first determine the scalar product $\boldsymbol{B} \cdot \boldsymbol{C}$. The product $\boldsymbol{B} \cdot \boldsymbol{C}$ yields the scalar quantity $BC \cos bc$, where bc is the angle between $\boldsymbol{B}$ and $\boldsymbol{C}$ (Fig. 38*a*). The next step is to multiply the vector $\boldsymbol{A}$ by the scalar quantity $BC \cos bc$. Such a product, that of a vector by a scalar quantity, yields a vector quantity, the direction of which is the same as the direction of the vector quantity, in this case $\boldsymbol{A}$, and the magnitude is $ABC \cos bc$. Hence we may write

$$\boldsymbol{A}\,(\boldsymbol{B}\cdot\boldsymbol{C}) = (ABC \cos bc)\,\boldsymbol{A}_1 \tag{2-16}$$

where $\boldsymbol{A}_1$ is the unit vector drawn in the direction of $\boldsymbol{A}$. It is important to notice that, since $\boldsymbol{B}\cdot\boldsymbol{C}$ is a scalar quantity, it is

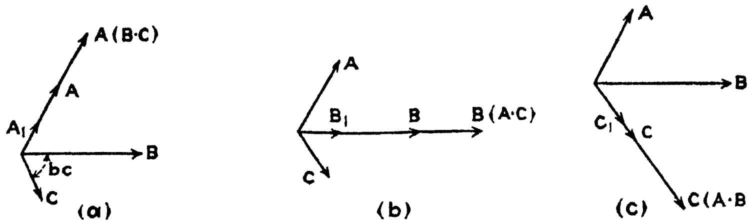

FIG. 38.

incorrect to write a dot (or cross for that matter) immediately following $\boldsymbol{A}$ in the expression $\boldsymbol{A}\,(\boldsymbol{B}\cdot\boldsymbol{C})$, for there is only one way to multiply a vector by a scalar quantity. The expression $\boldsymbol{A}(\boldsymbol{B}\cdot\boldsymbol{C})$ may be written $(\boldsymbol{B}\cdot\boldsymbol{C})\boldsymbol{A}$, $\boldsymbol{A}(\boldsymbol{C}\cdot\boldsymbol{B})$, or $(\boldsymbol{C}\cdot\boldsymbol{B})\boldsymbol{A}$ without change in the meaning or value of the quantity. The vector expressed by the triple product $(\boldsymbol{A}\cdot\boldsymbol{B})\,\boldsymbol{C}$ is obviously not the same as $\boldsymbol{A}\,(\boldsymbol{B}\cdot\boldsymbol{C})$ for the direction of $(\boldsymbol{A}\cdot\boldsymbol{B})\,\boldsymbol{C}$ is that of $\boldsymbol{C}$ and its magnitude is $ABC \cos ab$, where ab is the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$ (Figs. 38*b* and 38*c*).

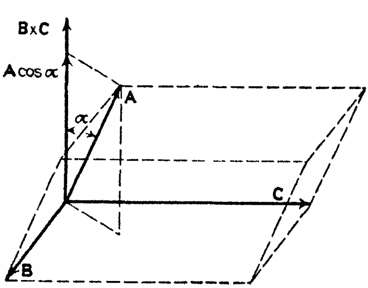

FIG. 39.

b. $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})$.—In expanding this triple vector product, we must first obtain the vector product of $\boldsymbol{B}$ by $\boldsymbol{C}$. The magnitude of $\boldsymbol{B}\times\boldsymbol{C}$ is $BC \sin bc$, where bc is the angle between $\boldsymbol{B}$ and $\boldsymbol{C}$. The direction of the vector representing $\boldsymbol{B}\times\boldsymbol{C}$ is perpendicular to the plane containing $\boldsymbol{B}$ and $\boldsymbol{C}$ (Fig. 39). The next step involves the scalar product of the vector $\boldsymbol{A}$ by the vector representing $\boldsymbol{B}\times\boldsymbol{C}$. The resulting quantity is a scalar quantity, the magnitude of which is $ABC \cos \alpha \sin bc$, where α is the angle between A and the vector representing $\boldsymbol{B} \times \boldsymbol{C}$. Hence

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = ABC \cos \alpha \sin bc \qquad (2\text{-}17)$$

It is profitable to examine the meaning of this triple product.

The product $\boldsymbol{B} \times \boldsymbol{C}$ gives the area of the parallelogram determined by $\boldsymbol{B}$ and $\boldsymbol{C}$ as sides. This area is represented by a vector perpendicular to the plane of $\boldsymbol{B}$ and $\boldsymbol{C}$. Its scalar product with $\boldsymbol{A}$ may be taken to mean that the area $BC \sin bc$ is multiplied by the component of $\boldsymbol{A}$ which is in the line of the vector representing $\boldsymbol{B} \times \boldsymbol{C}$. We have then a product yielding a scalar quantity which expresses the volume of the parallelopiped constructed upon the three vectors as sides. Since this is the case, it is easy to see the following equalities:

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B} \cdot (\boldsymbol{C} \times \boldsymbol{A}) = \boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B}) \qquad (2\text{-}18)$$

If the order of the factors in any one of the vector products is changed, a negative sign must be introduced. The order of the factors expressed in the scalar product may be changed without changing the sign thus:

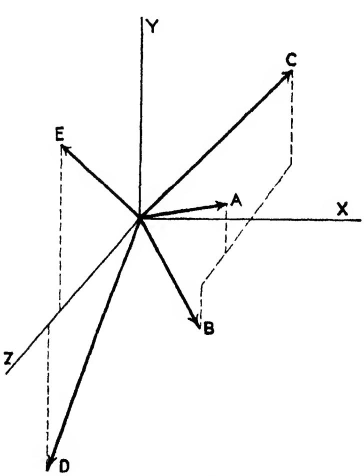

FIG. 40.

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{B} \times \boldsymbol{C}) \cdot \boldsymbol{A}$$

c. $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})$.—The vector product of a vector by the vector product of two other vectors may be expanded in the following manner: the vector product of $\boldsymbol{B}$ and $\boldsymbol{C}$ is to be found first and then the resulting vector is to be multiplied by $\boldsymbol{A}$. In order to simplify such an expression, we shall make use of the method of expansion, in which the vectors first are expressed in terms of their components and then are multiplied together according to the indicated processes.

Let us put

$$\boldsymbol{D} = \boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) \qquad \text{and } \boldsymbol{E} = \boldsymbol{B} \times \boldsymbol{C} \qquad (2\text{-}19)$$

If we now select an XYZ reference system (Fig. 40), we may write the components of $\boldsymbol{E}$, $E_x\boldsymbol{i}$, $E_y\boldsymbol{j}$, and $E_z\boldsymbol{k}$ in terms of the x, y, and z components of the vectors $\boldsymbol{B}$ and $\boldsymbol{C}$ as follows:

$$E_x i = (B_y C_z - B_z C_y) i$$
$$E_y j = (B_z C_x - B_x C_z) j$$

and

$$E_z k = (B_x C_y - B_y C_x) k$$

where B_x, B_y, B_z and C_x, C_y, C_z are the components of $\boldsymbol{B}$ and $\boldsymbol{C}$. If A_x, A_y, and A_z are the components of $\boldsymbol{A}$, then the x component of $\boldsymbol{D}$ may be expressed in terms of the components of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ by the use of the following steps:

$$\begin{aligned} D_x i &= (A_y E_z - A_z E_y) i \\ &= A_y (B_x C_y - B_y C_x) i - A_z (B_z C_x - B_x C_z) i \end{aligned}$$

If we add and subtract the term $A_x B_x C_x i$ and rearrange the order of the terms, we get

$$\begin{aligned} D_x i &= B_x (A_x C_x + A_y C_y + A_z C_z) i - C_x (A_x B_x + A_y B_y + A_z B_z) i \\ &= B_x (\boldsymbol{A} \cdot \boldsymbol{C}) i - C_x (\boldsymbol{A} \cdot \boldsymbol{B}) i \end{aligned}$$

In a similar manner, expressions may be obtained for $D_y j$ and $D_z k$, the two other components of $\boldsymbol{D}$. Adding vectorially the three expressions for the components gives the following important relations:

$$\boldsymbol{D} = \boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}\ (\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}\ (\boldsymbol{A} \cdot \boldsymbol{B}) \qquad (2\text{-}20)$$

in which the vector $\boldsymbol{D}$, which is equivalent to the triple vector product, is expressed in terms of $\boldsymbol{B}$ and $\boldsymbol{C}$ each multiplied by a scalar quantity. The right-hand member of Eq. (2-20) may be represented by a single vector $\boldsymbol{D}$, which is equal to the vector sum of two vectors one of which is parallel to $\boldsymbol{B}$ and the other parallel to $\boldsymbol{C}$. The vector $\boldsymbol{D}$ lies in the plane containing $\boldsymbol{B}$ and $\boldsymbol{C}$ and is perpendicular to $\boldsymbol{A}$ and also to $\boldsymbol{E}$ (Fig. 40).

Using the formula of Eq. (2-20) for expanding the triple vector product, we may now see whether the changed grouping of factors in the expression $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})$ will alter the value of the expression. If we use the following expression, in which the grouping of the factors is altered, and then expand this expression according to Eq. (2-20), we obtain

$$\begin{aligned} (\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C} &= -\boldsymbol{C} \times (\boldsymbol{A} \times \boldsymbol{B}) \\ &= -\boldsymbol{A}\ (\boldsymbol{C} \cdot \boldsymbol{B}) + \boldsymbol{B}\ (\boldsymbol{C} \cdot \boldsymbol{A}) \qquad (2\text{-}21) \end{aligned}$$

The right-hand member of Eq. (2-21) is not equivalent to that in Eq. (2-20) except in the one special case in which $\boldsymbol{B}$ is perpendicular to both $\boldsymbol{C}$ and $\boldsymbol{A}$. That this is true may be seen by applying the indicated limitation to both expressions of Eqs. (2-20) and (2-21). In this case the two triple products reduce to a common value, *viz.*, $\boldsymbol{B}\,(\boldsymbol{C}\cdot\boldsymbol{A})$, because the scalar products $\boldsymbol{C}\cdot\boldsymbol{B}$ and $\boldsymbol{A}\cdot\boldsymbol{B}$ are then both equal to zero.

It is also to be noticed that the right-hand member of Eq. (2-20) reduces to zero if $\boldsymbol{A}$ is perpendicular to both $\boldsymbol{C}$ and $\boldsymbol{B}$. This fact may be used in determining whether a vector is perpendicular to the plane determined by two other vectors.

Problem.—Given the vectors $\boldsymbol{A} = \boldsymbol{i} + 0.5\boldsymbol{j} + 0.5\boldsymbol{k}$, $\boldsymbol{B} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$, and $\boldsymbol{C} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$. Find the vectors $\boldsymbol{D}$ and $\boldsymbol{E}$ by Eqs. (2-19) and check the values of $\boldsymbol{D}$ by the use of Eq. (2-20). Prove that $\boldsymbol{B}$ and $\boldsymbol{C}$ are perpendicular to $\boldsymbol{E}$ and that $\boldsymbol{A}$ and $\boldsymbol{E}$ are perpendicular to $\boldsymbol{D}$. Prove also that $\boldsymbol{D}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are in the same plane. (It is very instructive for the student to locate all of the above vectors in space by the use of space reference system. The latter may be improvised by three straight pieces of wire thrust into a cork. Other wires, each labeled, may be used to represent the vectors).

2-11. The Differentiation of a Vector with Respect to a Scalar.—There are many occasions when it is desired to find the rate of change of a vector with respect to some scalar variable, as, for example, when expressing the velocity of a moving point in terms of the time rate of change of the radius vector drawn to that point. The expressions obtained from differentiating vectors are more abbreviated in form than those obtained from differentiating scalar quantities and hence more care is required in an interpretation of them.

a. Unit Vector.—Let $\boldsymbol{n}_1$ be any unit vector which is varying with respect to some scalar quantity such as time, which we may represent by t. Also let $\boldsymbol{n}_1'$ be the new value of $\boldsymbol{n}_1$ after the small time interval Δt. The time rate of change of $\boldsymbol{n}_1$ is therefore $(\boldsymbol{n}_1' - \boldsymbol{n}_1)/\Delta t$, or for brevity $\Delta\boldsymbol{n}_1/\Delta t$. The quantity $\Delta\boldsymbol{n}_1$ is that vector which expresses the change in $\boldsymbol{n}_1$ in the interval Δt. We define the differential quotient $d\boldsymbol{n}_1/dt$ by the equation

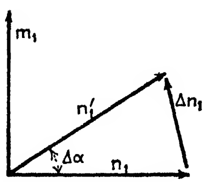

Fig. 41.

$$\frac{d\boldsymbol{n}_1}{dt} = \lim_{\Delta t \doteq 0} \frac{\Delta\boldsymbol{n}_1}{\Delta t}$$

Since a unit vector can change only in direction, the vector $\Delta\boldsymbol{n}_1$ must be perpendicular to $\boldsymbol{n}_1$ at the limit where $\Delta t = 0$. The direction of $\Delta\boldsymbol{n}_1/\Delta t$, when $t = 0$, is represented by the unit vector $\boldsymbol{m}_1$ which is drawn parallel to $\boldsymbol{\Delta n}_1$. If we let $\Delta\alpha$ be the angle between $\boldsymbol{n}_1$ and $\boldsymbol{n}_1'$, then, since $\boldsymbol{n}_1$ is of unit length, $\Delta\alpha$ is equal to the scalar magnitude of $\boldsymbol{\Delta n}_1$. Hence it follows that

$$\frac{d\boldsymbol{n}_1}{dt} = \left(\lim_{\Delta t \doteq 0} \frac{\Delta\alpha}{\Delta t}\right)\boldsymbol{m}_1 = \omega \boldsymbol{m}_1 \qquad (2\text{-}22)$$

in which ω is written for $\Delta\alpha/\Delta t$, since the latter is an angular velocity.

b. Ordinary Vector.—Let $\boldsymbol{A}$ be any ordinary vector, $\boldsymbol{A}'$ its value after a small time interval Δt, and let $\boldsymbol{\Delta A}$ be the change in $\boldsymbol{A}$ in the time Δt; hence $\boldsymbol{\Delta A} = \boldsymbol{A}' - \boldsymbol{A}$ (Fig. 42). We may regard $\boldsymbol{\Delta A}$ as equivalent to two components, one of which is in the line of $\boldsymbol{A}$ and is equal to $\Delta A \boldsymbol{A}_1$ (ΔA is to represent the magnitude change of A) and the other perpendicular to $\boldsymbol{A}$ and equal to $A\boldsymbol{\Delta A}_1$ ($\boldsymbol{\Delta A}_1$ gives the direction change of $\boldsymbol{A}$). Using these symbols, we define $d\boldsymbol{A}/dt$ by the following equations:

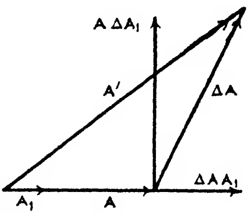

Fig. 42.

$$\begin{aligned} \frac{d\boldsymbol{A}}{dt} &= \lim_{\Delta t \doteq 0} \frac{\boldsymbol{\Delta A}}{\Delta t} \\ &= \lim_{\Delta t \doteq 0} \left(\frac{\Delta A}{\Delta t} \boldsymbol{A}_1 + A \frac{\boldsymbol{\Delta A}_1}{\Delta t}\right) \\ &= \frac{dA}{dt} \boldsymbol{A}_1 + A \frac{d\boldsymbol{A}_1}{dt} \qquad (2\text{-}23) \end{aligned}$$

This equation conveniently expresses the time rate of change of a vector in terms of the time rates of change of its magnitude and its direction.

If the vector $\boldsymbol{A}$ be expressed in terms of the unit vectors of some selected reference system,

$$\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$$

then, applying the method used in obtaining Eqs. (2-22) and (2-23), we may write

$$\frac{d\boldsymbol{A}}{dt} = \frac{dA_x}{dt}\boldsymbol{i} + \frac{dA_y}{dt}\boldsymbol{j} + \frac{dA_z}{dt}\boldsymbol{k} \qquad (2\text{-}24)$$

Since the unit vectors $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ are fixed in the reference system, their derivatives are zero and therefore do not appear in this equation.

c. Other Formulas.—By using the relation given in Eq. (2-24) and expressing the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ in terms of their unit vectors, the following formulas may be easily established:

$$\frac{d\,(\boldsymbol{A} + \boldsymbol{B})}{dt} = \frac{d\boldsymbol{A}}{dt} + \frac{d\boldsymbol{B}}{dt} \tag{2-25}$$

$$\frac{d\,(\boldsymbol{A} \cdot \boldsymbol{B})}{dt} = \boldsymbol{A} \cdot \frac{d\boldsymbol{B}}{dt} + \boldsymbol{B} \cdot \frac{d\boldsymbol{A}}{dt} \tag{2-26}$$

$$\frac{d\,(\boldsymbol{A} \times \boldsymbol{B})}{dt} = \boldsymbol{A} \times \frac{d\boldsymbol{B}}{dt} + \frac{d\boldsymbol{A}}{dt} \times \boldsymbol{B} \tag{2-27}$$

d. Illustration.—As an illustration of the use of Eq. (2-23), we shall express the velocity $\boldsymbol{V}_P$ of a moving particle P (Fig. 43) whose position in the reference system XOY is given by the vector $\boldsymbol{r}$. Then for the velocity of P we may write

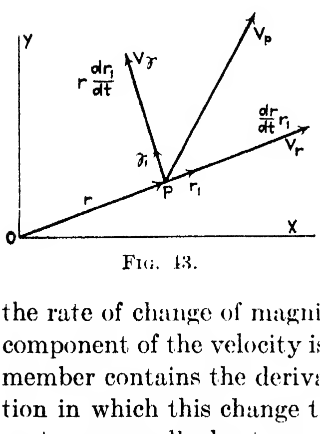

FIG. 43.

$$\boldsymbol{V}_P = \frac{d\boldsymbol{r}}{dt}$$

$$= r\frac{d\boldsymbol{r}_1}{dt} + \frac{dr}{dt}\boldsymbol{r}_1 \tag{2-28}$$

The last term of this equation gives the rate of change of magnitude of $\boldsymbol{r}$ and is in the line of $\boldsymbol{r}_1$. This component of the velocity is $\boldsymbol{V}_r$. The first term of the right-hand member contains the derivative of the unit vector $\boldsymbol{r}_1$. The direction in which this change takes place is in the line of γ_1, the unit vector perpendicular to $\boldsymbol{r}_1$ and making an angle of $+90$ deg. with $\boldsymbol{r}_1$. The magnitude of the rate of change of direction of $\boldsymbol{r}_1$ is ω, the angular velocity of $\boldsymbol{r}$ or $\boldsymbol{r}_1$. Hence

$$\boldsymbol{V}_P = \omega r \gamma_1 + \frac{dr}{dt}\boldsymbol{r}_1$$

$$= \boldsymbol{V}_\gamma + \boldsymbol{V}_r \tag{2-29}$$

where $\boldsymbol{V}_\gamma$ is the component of $\boldsymbol{V}_P$ which is perpendicular to $\boldsymbol{r}$.

In the work which follows there will be occasions where it is necessary to take the second derivative of a unit vector. Using the unit vector $\boldsymbol{r}_1$ in illustration, we may first write the derivative of $\boldsymbol{r}_1$ as it was just shown to be:

$$\frac{d\boldsymbol{r}_1}{dt} = \omega\gamma_1 \tag{2-30}$$

The second derivative of $\boldsymbol{r}_1$ gives

$$\frac{d^2\boldsymbol{r}_1}{dt^2} = \frac{d\omega}{dt}\boldsymbol{\gamma}_1 + \omega\frac{d\boldsymbol{\gamma}_1}{dt}$$

The derivative of $\boldsymbol{\gamma}_1$ yields a result similar to that of $d\boldsymbol{r}_1/dt$. Its magnitude is again ω and its direction makes an angle of $+90$ deg. with $\boldsymbol{\gamma}_1$, which is $-\boldsymbol{r}_1$. Hence we may write

$$\frac{d^2\boldsymbol{r}_1}{dt^2} = \frac{d\omega}{dt}\boldsymbol{\gamma}_1 - \omega^2\boldsymbol{r}_1 \qquad (2\text{-}31)$$

Problems. 1. Prove that an angular displacement is not a vector.

2. Prove that $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) + \boldsymbol{B} \times (\boldsymbol{C} \times \boldsymbol{A}) + \boldsymbol{C} \times (\boldsymbol{A} \times \boldsymbol{B}) = 0$.

3. Prove that $(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) = (\boldsymbol{A} \cdot \boldsymbol{C})(\boldsymbol{B} \cdot \boldsymbol{D}) - (\boldsymbol{B} \cdot \boldsymbol{C})(\boldsymbol{A} \cdot \boldsymbol{D})$.

4. Prove that $(\boldsymbol{A} \times \boldsymbol{B}) \times (\boldsymbol{C} \times \boldsymbol{D}) = \boldsymbol{B}\,(\boldsymbol{C} \times \boldsymbol{D}) \cdot \boldsymbol{A} - \boldsymbol{A}\,(\boldsymbol{C} \times \boldsymbol{D}) \cdot \boldsymbol{B}$.

5. A vector makes an angle of 40 deg. with the X-axis and 60 deg. with the Y-axis. Find the angle it makes with the Z-axis.

6. The coordinates of the terminal points of a certain vector are 2, 3, and -4 and 5, 6, and -8. Find the magnitude of the vector and the angles it makes with each of the axes.

7. A certain force, $3\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$ lb., pushes an object a distance which is expressed by the vector $2\boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k}$ ft. Find the work done and the magnitude of both force and displacement. What is the angle between the two vectors?

8. Find the value of $(\boldsymbol{A} - \boldsymbol{B}) \times (\boldsymbol{A} + \boldsymbol{B})$. Interpret geometrically.

9. Find the value of $(\boldsymbol{A} - \boldsymbol{B}) \cdot (\boldsymbol{A} + \boldsymbol{B})$.

10. By vector methods prove that the diagonals of a parallelogram bisect each other.

11. Show by vector methods that the diagonals of a parallelogram are not necessarily perpendicular to each other.

CHAPTER III

ANGULAR VELOCITY

3-1. Angular Displacement.—It is often advantageous to study rotational quantities by comparing them with the more familiar translational quantities, and to observe the points of similarity and dissimilarity. In comparing a finite angular displacement with a finite linear displacement, one should observe that the former is not a vector quantity (in the ordinary sense), while the latter is. That a finite angular displacement is a scalar quantity is not obvious, for it has some of the characteristics which might easily mislead one into supposing that it could be regarded as a vector of the axial type. This statement may be readily proved by showing that a finite angular displacement fails to meet one of the most fundamental requirements of a vector. It is not subject to vector addition. A simple illustration will prove this point.

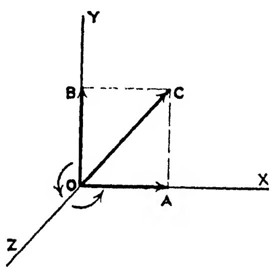

FIG. 44.

Suppose that an angular displacement of $\pi/2$ radians is made from OY to OZ (Fig. 44) and that this displacement could be represented by the line segment A along the X-axis. Similarly, an angular displacement of equal magnitude could be made from OZ to OX and it might be represented by B along OY. Now if A and B were vectors, their sum would be along the line C and would then have the value 0.707π radian. An angular displacement of π radians, however, is required to rotate the line OB into the final position OA about the axis OC. From this inconsistency it may be concluded that angular displacement is not a vector.

Consistent with this result, it is to be seen that the time derivative of γ, the angular displacement, gives angular speed ω, a scalar quantity. In order to convert $d\gamma/dt$(or ω) into a vector quantity, it is therefore necessary to multiply it by the unit vector having the direction of ω.

3-2. Angular Velocity, a Vector.—Angular velocity is an axial type of vector as may be readily appreciated from the analysis given in this and the following sections. Let us imagine a body, such as a wheel, mounted so that it may rotate on a fixed shaft and not slide along the shaft—in other words, so that the body has pure rotational motion (Sec. 1-6). The rate at which the body is rotating may be changing. If such is the case, the instantaneous angular speed ω may be defined by the equation

$$\omega = \lim_{\Delta t \doteq 0} \frac{\Delta\gamma}{\Delta t} \tag{3-1}$$

if we let $\Delta\gamma$ be the angular displacement in the small time interval Δt. The average angular speed for a certain time interval $t_2 - t_1$ is equal to the angular displacement $(\gamma_2 - \gamma_1)$ (through which the body rotates in the time $t_2 - t_1$), divided by the time, or in symbols

$$\omega(\text{av.}) = \frac{\gamma_2 - \gamma_1}{t_2 - t_1} \tag{3-2}$$

Either quantity, instantaneous or average angular speed, may be converted into the corresponding velocity by multiplying that speed by the unit vector which is to designate its direction. The direction of an instantaneous angular velocity is to be taken in the proper sense along the axis of rotation. The convention selected for angular velocities is consistent with that described for angular displacements (Sec. 1-2). The "right-hand rule" is a convenient one for identifying the proper sense along the axis. If one imagines the palm of the right hand to be placed on the axis with the fingers extending around the axis in the direction of motion, the thumb, placed along the axis, will point in the direction of the angular velocity.

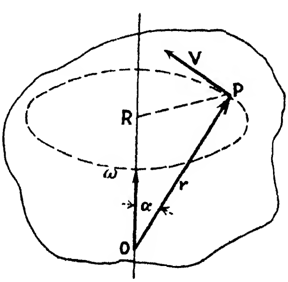

FIG. 45.

The vector equation for expressing the linear velocity of a point of a body in pure rotation is next to be developed. Let P be any point of the body (not on the rotation axis) and let its position be given by the vector $\boldsymbol{r}$ which is to be drawn from O, selected as origin on the axis of rotation, to P. Also let α be the angle between $\boldsymbol{r}$ and the axis. In Fig. 45 the dotted curve is intended to represent P's circular path and R is the center of this path. If the angular velocity is $\boldsymbol{\omega}$ at the instant under consideration, the linear speed (V) of P is $\omega(RP)$. The direction of the velocity ($\boldsymbol{V}$) is continually changing; hence the scalar equation $V = \omega(RP)$ is not particularly useful if one desires also to express the direction of $\boldsymbol{V}$. Since $RP = r \sin \alpha$, then it follows that

$$\boldsymbol{V} = \boldsymbol{\omega} \times \boldsymbol{r} \tag{3-3}$$

gives a more complete description of $\boldsymbol{V}$. It will be noticed that the magnitude of $\boldsymbol{V}$, as given by this equation, is $\omega r \sin \alpha$ and also that the direction of $\boldsymbol{V}$ is perpendicular to the plane containing $\boldsymbol{\omega}$ and $\boldsymbol{r}$ and is consistent with the convention of signs for the quantities as indicated in the diagram. One must be careful, however, of the order in which the two factors $\boldsymbol{\omega}$ and $\boldsymbol{r}$ are written. In the particular illustration it would be incorrect to write $\boldsymbol{r} \times \boldsymbol{\omega}$, although $-\boldsymbol{r} \times \boldsymbol{\omega}$ is correct.

3-3. The Components of Angular Velocity.—Given a body in a state of rotational motion about some axis which is fixed in an XYZ reference system selected so that the origin is on the axis of rotation. If $\boldsymbol{\omega}$ is the instantaneous value of the angular velocity, then, since $\boldsymbol{\omega}$ is a vector quantity,

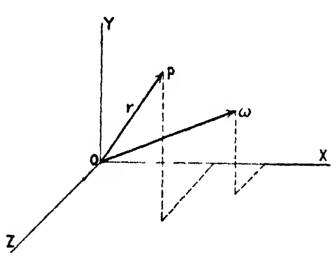

FIG. 46.

$$\boldsymbol{\omega} = \omega_x \boldsymbol{i} + \omega_y \boldsymbol{j} + \omega_z \boldsymbol{k} \tag{3-4}$$

where $\omega_x \boldsymbol{i}$, $\omega_y \boldsymbol{j}$, and $\omega_z \boldsymbol{k}$ are the components of the angular velocity along the X-, Y-, and Z-axis, respectively.

Let P be any point of the body. The position of P in the reference system (Fig. 46) is to be given by the vector $\boldsymbol{r}$. If the coordinates of P are x, y, and z, then

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \tag{3-5}$$

It is desired to obtain the relations which express the linear velocity of P in terms of the coordinates of P and the components of the angular velocity.

If $\boldsymbol{V}$ is the linear velocity of P, then we may write

$$\boldsymbol{V} = \boldsymbol{\omega} \times \boldsymbol{r} \tag{3-6}$$

Substituting the expressions written above for ω and r and expanding the vector product gives

$$\begin{aligned} V &= (\omega_x \boldsymbol{i} + \omega_y \boldsymbol{j} + \omega_z \boldsymbol{k}) \times (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \\ &= (\omega_y z - \omega_z y)\boldsymbol{i} + (\omega_z x - \omega_x z)\boldsymbol{j} + (\omega_x y - \omega_y x)\boldsymbol{k} \quad (3\text{-}7) \end{aligned}$$

This equation expresses the linear velocity of P in terms of the components of the angular velocity and the coordinates of P. The magnitudes of the components of V taken along the X-, Y-, and Z-axes of the reference system are given by the coefficients of the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$, respectively.

It is to be noticed that a component *linear* velocity parallel to an axis does not contain a component of the *angular* velocity which is parallel to that axis, nor does it contain a coordinate parallel to that axis. Each component of the linear velocity is dependent upon two components of the angular velocity; for example, V_x is dependent upon ω_y and ω_z. Both of the quantities $\omega_y z$ and $-\omega_z y$ will produce linear velocities along the X-axis and hence their algebraic sum will give V_x.

The contributions of each component of the angular velocity to the components of the linear velocity are given in the following tables:

Contributions by	Component velocities in the line of		
	X	Y	Z
ω_x	0	$-\omega_x z$	$\omega_x y$
ω_y	$\omega_y z$	0	$-\omega_y x$
ω_z	$-\omega_z y$	$\omega_z x$	0

The fact that the linear velocity of a particle may be expressed either in terms of the components of the angular velocity or in terms of the resultant angular velocity may be illustrated by the following consideration. Suppose that a particle P (Fig. 47) with coordinates x and y is in the XY plane of a reference system and that it is moving parallel to the Z-axis with such a linear velocity as may be described as due to angular velocities ω_x and ω_y about the X- and Y-axes, respectively. The resultant linear velocity of P, V_P, consists therefore of two com-

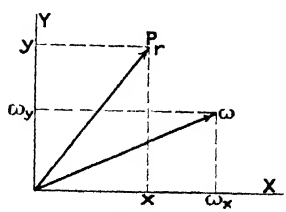

FIG. 47.

ponent linear velocities, the magnitude of one is equal to $\omega_x y$ and of the other is equal to $-\omega_y x$. Both components are parallel to the Z-axis. Hence

$$V_P = (\omega_x y - \omega_y x)\boldsymbol{k} \tag{3-8}$$

This equation may be established by a vector product. If the resultant angular velocity $\boldsymbol{\omega} = \omega_x \boldsymbol{i} + \omega_y \boldsymbol{j}$ and the vector $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$, which denotes the position of P, are in the XY plane, then the linear velocity of P may be expressed in terms of the vector product as follows:

$$\begin{aligned} V_P &= \boldsymbol{\omega} \times \boldsymbol{r} \\ &= (\omega_x \boldsymbol{i} + \omega_y \boldsymbol{j}) \times (x\boldsymbol{i} + y\boldsymbol{j}) \\ &= \omega_x y\, \boldsymbol{i} \times \boldsymbol{j} + \omega_y x\, \boldsymbol{j} \times \boldsymbol{i} \\ &= (\omega_x y - \omega_y x)\boldsymbol{k} \end{aligned} \tag{3-9}$$

That this result agrees with the one given in Eq. (3-8) indicates the validity of either form of expression.

3-4. Composition of Parallel Angular Velocities.—In the preceding section we have seen that angular velocities, being vector

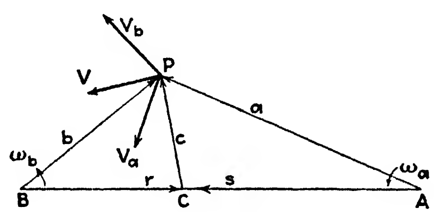

FIG. 48.

quantities, are subject to the ordinary rules of vectors. When the resultant of two parallel angular velocities is to be found, vector methods are no longer necessary. The process of combining parallel vectors is an algebraic process. The *magnitude* of the resultant of two parallel angular velocities is equal to the algebraic sum of the two angular velocities. The *position* of the resultant is not so simply expressed but may be found by the method given below. There are two cases to be considered, one in which the directions of the two angular velocities are alike and the other in which the directions are opposite. The former will be considered first.

Let the two parallel angular velocities, of magnitudes ω_a and ω_b, have directions which are perpendicular to the plane of the diagram (Fig. 48) and let the axes of the angular velocities pass through the fixed points A and B, respectively, of the diagram. The directions of rotation are, in this case, similar and are indicated by the arrows in the diagram. We are to determine the magnitude and position of the resultant of these two angular velocities.

Let us consider the linear velocity of a particle P, in the plane of the diagram, as caused by the combination of the two angular velocities. Owing to the angular velocity $\boldsymbol{\omega}_a$ about the center A, P will have a linear velocity $\boldsymbol{V}_a$, which is perpendicular to the line AP.

The linear velocity $\boldsymbol{V}_b$ of P due to rotation about B will be perpendicular to the line BP. The values of these two linear velocities are given by the equations

$$\boldsymbol{V}_a = \boldsymbol{\omega}_a \times \boldsymbol{a} \qquad \text{and} \qquad \boldsymbol{V}_b = \boldsymbol{\omega}_b \times \boldsymbol{b} \tag{3-10}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are the vectors which represent the position of P from the rotation axes passing through A and B, respectively.

The resultant of the two velocities $\boldsymbol{V}_a$ and $\boldsymbol{V}_b$ is $\boldsymbol{V}$, as shown in the diagram. The resultant linear velocity $\boldsymbol{V}$ may be regarded as due to an angular velocity $\boldsymbol{\omega}$ about the axis of the desired resultant angular velocity. The point in the plane of the diagram through which the axis of the resultant angular velocity passes must be somewhere in the line PC which is drawn perpendicular to $\boldsymbol{V}$. The particular point of the line PC through which the axis of the resultant angular velocity passes could be found graphically by first determining the resultant linear velocity $\boldsymbol{V}'$ of some other particle P' in the plane of the diagram and then by locating the point of intersection which a line passing through P' and perpendicular to $\boldsymbol{V}'$ would make with PC.

In place of the suggested graphical solution, it is desired to locate the position of the axis of the resultant angular velocity by an examination of the point C, the intersection of AB with PC. If C is not the center of the resultant angular velocity, then it (C) must have a linear velocity in the plane of the diagram. If C has a linear velocity, then it must have two component linear velocities, one due to $\boldsymbol{\omega}_a$ and the other due to $\boldsymbol{\omega}_b$. These linear velocities would be equal to $\boldsymbol{\omega}_a \times \boldsymbol{s}$ and $\boldsymbol{\omega}_b \times \boldsymbol{r}$ if $\boldsymbol{s}$ and $\boldsymbol{r}$ are the vectors indicating the position of C with respect to A and B, respectively. It is readily seen that it is only some point in the line AB that could possibly have no resultant linear velocity due to rotations about A and B, because only points in the line AB could have component linear velocities which are parallel to each other. The point C is therefore the desired center of the resultant angular velocity, for it satisfies both requirements, *viz.*, of being in the lines PC and AB. The resultant angular velocity must therefore pass through C. Since the directions of $\boldsymbol{\omega}_a$ and $\boldsymbol{\omega}_b$ are alike, the point C must lie between A and B.

As a further consequence of the foregoing consideration, we may write

$$\boldsymbol{\omega}_a \times \mathbf{s} = -\boldsymbol{\omega}_b \times \mathbf{r} \tag{3-11}$$

The magnitude of the angular velocity $\boldsymbol{\omega}$ about the axis through C is next to be found. To find the expression for $\boldsymbol{\omega}$ in terms of $\boldsymbol{\omega}_a$ and $\boldsymbol{\omega}_b$, we may use a vector method. The resultant velocity of P is given by the following equations:

$$\begin{aligned} \mathbf{V} &= \mathbf{V}_a + \mathbf{V}_b \\ &= \boldsymbol{\omega}_a \times \mathbf{a} + \boldsymbol{\omega}_b \times \mathbf{b} \end{aligned} \tag{3-12}$$

By referring to the diagram, if we let $\mathbf{c}$ represent the line CP, it is readily seen that

$$\mathbf{a} = \mathbf{s} + \mathbf{c} \qquad \text{and} \qquad \mathbf{b} = \mathbf{r} + \mathbf{c} \tag{3-13}$$

Substituting these vector relations in Eq. (3-12) and simplifying the resulting expression gives

$$\begin{aligned} \mathbf{V} &= \boldsymbol{\omega}_a \times (\mathbf{s} + \mathbf{c}) + \boldsymbol{\omega}_b \times (\mathbf{r} + \mathbf{c}) \\ &= (\boldsymbol{\omega}_a + \boldsymbol{\omega}_b) \times \mathbf{c} + \boldsymbol{\omega}_a \times \mathbf{s} + \boldsymbol{\omega}_b \times \mathbf{r} \\ &= (\boldsymbol{\omega}_a + \boldsymbol{\omega}_b) \times \mathbf{c} \end{aligned} \tag{3-14}$$

The last step in the simplification is validated by the use of Eq. (3-11).

The velocity $\mathbf{V}$ must be equal to $\boldsymbol{\omega} \times \mathbf{c}$; hence it follows from Eq. (3-14) that

$$\boldsymbol{\omega} = \boldsymbol{\omega}_a + \boldsymbol{\omega}_b \tag{3-15}$$

Hence we may conclude that the magnitude of the resultant of two parallel angular velocities is equal to the algebraic sum of the two angular velocities [Eq. (3-15)] and the position of the axis of the resultant of two similarly directed component angular velocities is in the plane containing the two components and is at distances from the axis of the two components which are inversely proportional to the magnitude of the two components [Eq. (3-11)].

In case the directions of the two given parallel angular velocities are oppositely directed, the results given above are still valid, but the point C would, obviously, lie outside the line segment AB and would be on the side of the greater angular velocity.

If one of the angular velocities is equal but opposite in direction to the other, the equation $\mathbf{V} = \boldsymbol{\omega} \times \mathbf{c}$ indicates that the position of C is at infinity, for, since $\boldsymbol{\omega}_a + \boldsymbol{\omega}_b = 0$,

$$\mathbf{V} = \mathbf{0} \times \mathbf{c}$$

and $\mathbf{c}$ is obviously equal to infinity if $\mathbf{V}$ is finite. In this special case the motion of P is one of translation with the direction of motion perpendicular to the line AB.

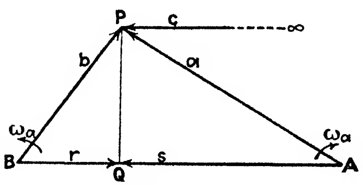

FIG. 49.

The magnitude of P's velocity in this special case may be determined in the following manner. Since the velocity of P is perpendicular to AB, we may regard P as being at Q (Fig. 49). Writing the following scalar equations,

$$V_a = \omega_a s \qquad V_b = \omega_a r$$

where ω_a is the common magnitude of the two given angular velocities, since V_a and V_b are parallel, we have

$$V = V_a + V_b$$

Hence,

$$\begin{aligned} V &= \omega_a s + \omega_a r \\ &= \omega_a (r + s) \qquad (3\text{-}16) \end{aligned}$$

From this it is seen that the magnitude of the velocity of P is equal to $\omega_a(r + s)$ which is independent of the position of P. Therefore any point in the plane of the diagram which has straight-line motion may be considered as rotating about two points which are in a line perpendicular to the line of motion, the angular velocities about these two points being equal in magnitude but opposite in direction. This statement is proved geometrically in the following manner:

Since

$$V_a = \omega_a a$$
$$V_b = \omega_a b$$
$$\frac{V_a}{V_b} = \frac{a}{b}$$

It is easily seen that the two triangles BPA and PSR (Fig. 50) are similar, for $V_a/V_b = a/b$ and the angle BPA equals the angle PSR. Hence it follows that the angle RPS equals the angle PAB and therefore RP is perpendicular to AB. Since the triangles are similar, it follows that

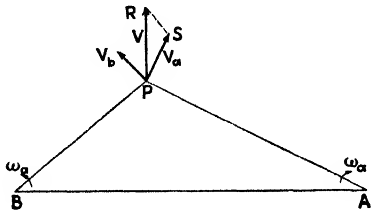

FIG. 50.

$$\frac{V_a}{V} = \frac{a}{AB}$$

Then,

$$V = \omega_a AB \qquad (3\text{-}17)$$

The converse theorem is also true. Any translational motion may be regarded as due to two rotations of equal magnitude but of opposite directions, the axes of rotation being selected so that their plane is perpendicular to the direction of the translation and the distance between the axes being equal to the linear speed divided by the selected value for angular speed.

3-5. Composition of Rotation and Translation.—When any rigid body is moving, regardless of the character of the motion, the motion of all of its points at any instant may be completely described as consisting of a linear velocity common to all points and an angular velocity about an axis through a selected point of the body. If V_p is the resultant velocity of a particular point P of the body and at the particular instant there is a moving coordinate system attached to the body with an axis, say OX, of the system as a rotation axis passing through the selected point P, then instantaneously all other points of the body may be considered as having angular motion about the axis OX in the moving system. Hence the motion of any other point Q of the body, and therefore all points, may be described in terms of the linear velocity of the moving system (which is equal to the linear velocity of P) and an angular velocity ω in the coordinate system. In symbols this may be expressed as follows:

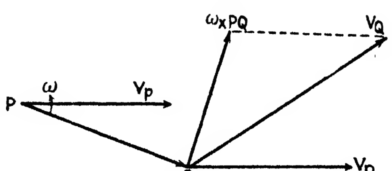

FIG. 51.

$$V_Q = V_P + \omega \times PQ \qquad (3\text{-}18)$$

In vectors this is shown by the diagram of Fig. 51 (where the vectors ω and V_P need not be at right angles to each other).

From the general relation [Eq. (3-18)] we may proceed to express another relation which describes the motion of a body in terms of a rotation about some axis PY (Fig. 52), together with a translation parallel to the axis PY. Select a coordinate system so that its origin is at the point P and so that V_P is in the XY plane. If the angle between V_P and ω is not a right angle, we may resolve V_P into two components, V_P' parallel to PY and V_P'' perpendicular to PY and in the plane of V_P and PY. (Suppose this plane is the XY plane.)

Let us imagine that the body has angular velocities of $+\omega$ and $-\omega$ about some other axis OY', parallel to PY, which cuts the $-PZ$ reference axis at some distance PO from P. From the result obtained in the preceding section we may combine ω (about PY) with $-\omega$ (about OY') to give us a translational velocity which is perpendicular to PO. If we select PO of the proper magnitude, the value of the translational velocity may be made equal to $-V_P''$. The pair of angular velocities (ω and $-\omega$) so selected will give the body a velocity which just neutralizes the part of V_P which is parallel to X (*viz.*, V_P''). We have left then V_P', which is the velocity parallel to PY, and the angular velocity ω about OY'. This motion is sometimes called an instantaneous screw motion. In general, both ω and V_P' will change as the motion progresses. The position of OY' will, in general, also change.

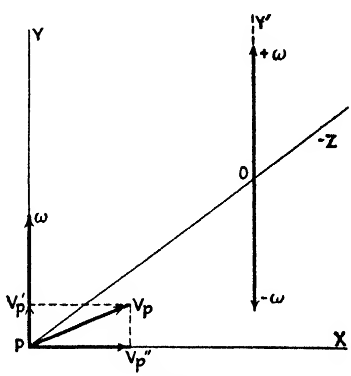

FIG. 52.

A special case of this is of interest. If V_P is parallel to PX in the above diagram, then V_P' is equal to zero and obviously the resulting motion will be uniplanar. The axis OY' becomes the instantaneous axis of rotation, the locus of which gives the fixed and moving centrodes. The method just described may be used therefore as a device for locating the instantaneous velocity axis.

It is also instructive to observe that the results of the foregoing analysis may be obtained by replacing the vector V_P'' by its equivalent "rotor couple" (ω and $-\omega$). In this case we may regard V_P'' as made up of $-\omega$ about the axis PY and ω about OY', together with the proper spacing distance PO between the axes, so that the magnitudes are alike. The angular velocity $-\omega$ about OY neutralizes $+\omega$ which leaves only V_P' and ω about OY' as before.

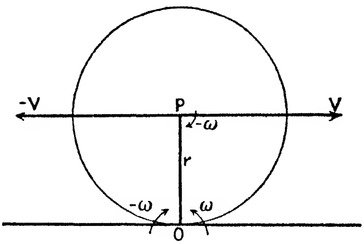

FIG. 53.

An illustration will be of value in this connection. Let us consider the motion of a wheel (Fig. 53) rolling along a straight line. Let the angular velocity of the wheel be $-\omega$ about an axis through its center P and the linear velocity of P be V. The axis PY will be perpendicular to the diagram through P. The second axis OY', which is to be the instantaneous axis, will also be perpendicular to the plane of the diagram and intersects the line PO, which is perpendicular to V, at O. The vector $-V$, which is to neutralize V, is then equivalent to the rotor couple $-\omega$ about PY and ω about the axis through O. From the relation $-V = \omega \times PO$ the magnitude of PO can be determined. It is easy to see that $PO = r$ (the radius of the wheel), for $V = \omega \times r$. It is also to be noticed that O is below P in the diagram and not above it. With V neutralized by $-V = \omega \times r$, there remains $-\omega$ about the axis through O. This axis at O is therefore the instantaneous axis.

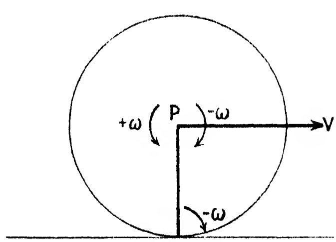

FIG. 54.

Or, by the second method suggested above, we may replace V by the proper rotor couple, which yields the same final result.

Problems.—1. A rigid wheel is rotating about a fixed axis with an angular velocity of 10 r.p.s. Find the linear velocity of a point 1 ft. from the axis of rotation.

2. A rigid body is in a steady state of rotation with an angular velocity which is expressed by the equation $\omega = 2i + 3j - 4k$ radians per second, referred to a selected reference system. The position of a certain point P is given by the vector $r = 4i - 2j + 5k$. Find the vector which expresses the linear velocity of P.

3. A rigid body has an angular velocity of $\boldsymbol{\omega} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ radians per second in a given reference system. If the linear velocity of a certain point of the body is $\boldsymbol{V} = -4\boldsymbol{i} + 5\boldsymbol{j} + 3\boldsymbol{k}$, find the vector $\boldsymbol{r}$ which defines the position of the point, if $\boldsymbol{r}_1 = \frac{1}{3}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} - \frac{2}{3}\boldsymbol{k}$.

4. Find the resultant of the three following angular velocities:

$$\boldsymbol{\omega}_1 = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k},\ \boldsymbol{\omega}_2 = -3\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k},\ \text{and}\ \boldsymbol{\omega}_3 = 4\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}.$$

5. What are the necessary relations between the components of two angular velocities in order that they may be parallel to each other?

6. An automobile is going with a speed of 30 m.p.h. The diameter of a wheel is 28 in. Find the linear velocity of the highest point of a wheel and also that of the point which is foremost on the wheel.

7. A ladder of length L stands on a horizontal floor and leans against a vertical wall. If the ladder slides down with its ends in contact with the floor and wall, respectively, the ensuing motion may be described as a uniplanar motion in terms of the linear velocity of any selected point of the ladder and an angular velocity about an axis through that point and perpendicular to the guide plane. Assuming a suitable linear velocity for any selected point of the ladder, find the corresponding angular velocity of the ladder for that instant and the position of the instantaneous axis. Use one of the methods outlined in Sec. **3-5**. Check your result by using a second point.

CHAPTER IV

ACCELERATION

4-1. Acceleration—a Vector Quantity.—The kinematical quantity velocity is a vector quantity. Constancy of a velocity therefore implies motion with unchanging speed in a straight line. On the other hand, a change in velocity may mean either a change in speed or a change in direction or a change in both speed and direction simultaneously. All changes in velocity may be collectively expressed by a single vector equation. If we let U and V be the velocities of a particle at the beginning and end, respectively, of a given time interval, then the change in velocity during this interval will be given by the vector expression $V - U$, in which no limitation is imposed upon the way in which the velocity may change. That this velocity change is a vector quantity is obvious.

Acceleration has been defined as the quantity which expresses the change in velocity in a unit interval of time. If we divide the vector $V - U$ by the time (a scalar quantity) in which this change occurred, we still have a vector which has the direction of the vector expressing the change of velocity but whose magnitude has been changed, unless the particular value of the time interval happened to be unity. In general, then, we may define the *average acceleration* over a given time interval by the equation

$$J = \frac{V - U}{t}$$

in which the vector J shall be used to represent the acceleration, and t the time interval. This equation is general and includes all possible changes of the velocity. The rate of change of the velocity need not be constant. This possibility necessitates the inclusion of the term "average" in the above definition. It will be recalled that average velocity was defined in an analogous manner.

When the variations in the acceleration are small or a knowledge of them is not germane to the particular consideration, average acceleration may supply all of the information which is needed. There are many considerations, however, which require a knowledge of the exact value of the acceleration at definite positions of the particle or at definite time instants. We shall use the term *instantaneous* acceleration to describe the value of the acceleration at any given position or instant. Instantaneous acceleration may be defined in the following manner. Let the change in the velocity during a small interval of time Δt be expressed by the vector ΔV. The average rate of change of the velocity in the time interval Δt is therefore $\Delta V/\Delta t$. If the time interval approaches zero as a limit, the value of the ratio $\Delta V/\Delta t$ at the limit will be the acceleration at the instant selected. Using the notation of calculus, we may express the instantaneous acceleration as follows:

$$J = \frac{dV}{dt} = \lim_{\Delta t = 0} \frac{\Delta V}{\Delta t} \tag{4-1}$$

4-2. Acceleration and the Reference System.—Let a particle P move in a fixed reference plane with a variable velocity (Fig. 55). Let OM be a fixed line in the reference system. If Q is the projection of P upon the line OM, then the component of P's velocity along this line is the velocity of Q. We shall designate the component velocity of P in the line of OM by the symbol V_m. The velocity V_m will also, in general, be variable.

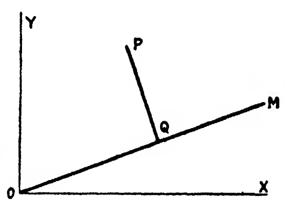

Fig. 55.

Let us designate the acceleration of P by J, the absence of the subscript indicating that it is a resultant acceleration. The acceleration of Q will be identified by J_m, in which the subscript again indicates the line in which the component acceleration is to be taken. The sign of J_m will be positive if the additions to the velocity are positive. A decreasing value of the magnitude of $-V_m$ indicates positive acceleration. Negative additions to the velocity give a negative acceleration regardless of the sign of the velocity. This is seen directly from a consideration of the signs of the numerator and denominator of dV_m/dt, which is the differential form of the acceleration along OM. Since dt is always positive, the sign of the fraction depends only upon dV_m, which represents the change in velocity in the time dt.

It is sometimes convenient to express J_m in other *equivalent forms*. Obviously,

$$J_m = \left(\frac{d^2m}{dt^2}\right)\boldsymbol{m}_1 \tag{4-2}$$

in which m is a coordinate which measures the displacement of Q in the line OM and $\boldsymbol{m}_1$ is a unit vector in the line OM. Also,

$$J_m = \frac{dm}{dt}\,\frac{d\left(\frac{dm}{dt}\right)}{dm}\boldsymbol{m}_1$$

$$= V_m \frac{dV_m}{dm}\boldsymbol{m}_1 \tag{4-3}$$

where $V_m = dm/dt$, the speed of Q in the line OM.

In a similar manner we may express the acceleration along any line, *i.e.*, OX, in the following three alternative forms:

$$J_x = \frac{d^2x}{dt^2}\boldsymbol{i} = \frac{dV_x}{dt}\boldsymbol{i} = V_x\frac{dV_x}{dx}\boldsymbol{i} \tag{4-4}$$

The acceleration in any fixed line OX may be expressed in terms of the resultant velocity V if the angle between the resultant velocity and OX be given. Let α be this angle. Then $V_x = V\cos\alpha$, and

$$J_x = \frac{dV_x}{dt}\boldsymbol{i} = \left(\frac{dV}{dt}\cos\alpha - V\frac{d\alpha}{dt}\sin\alpha\right)\boldsymbol{i} \tag{4-5}$$

It is of interest to interpret this equation. All of the acceleration which expresses the rate of change of speed (dV/dt) is contained in the first term of the right-hand member. The direction of dV/dt is in the line of V (Fig. 56). It is projected into the line of OX by being multiplied by the $\cos\alpha$. The part of the acceleration which expresses the magnitude of the rate of change of direction of the velocity is $V\,d\alpha/dt$ and is in a line which is perpendicular to V. It is projected into OX by the factor $\sin\alpha$. The sum of these two projections on OX gives the x component of the resultant acceleration.

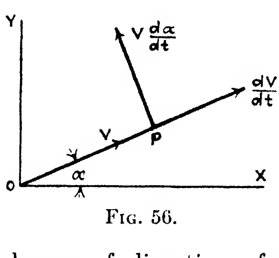

FIG. 56.

4-3. Tangential and Normal Acceleration.—It is frequently desirable to express the resultant acceleration in terms of two components, one of which is parallel to and the other perpendicular to the resultant velocity. To obtain the acceleration parallel to the resultant velocity, Eq. (4-5) may be used if we put $\alpha = 0$. This may be done, provided that we have the line OX fixed in the reference system. If V is continually changing in direction, it is obvious that OX may be regarded as fixed but only instantaneously so. The value of the result is not impaired by this limitation, for this selection may be repeatedly made for each point of the path. Putting $\alpha = 0$ in Eq. (4-5) gives

$$J_x = \frac{dV}{dt}\boldsymbol{i}$$

Since this acceleration is always parallel to the resultant velocity and hence is in the tangent to the curve which P is describing, it is customary to call this component the *tangential* acceleration.

Consistent with the foregoing use of the subscript, J_t may be written for the tangential acceleration. The other forms of J_t are

$$J_t = \frac{dV}{dt}\boldsymbol{t}_1 = \frac{d^2s}{dt^2}\boldsymbol{t}_1 = V\frac{dV}{ds}\boldsymbol{t}_1 = \frac{d}{ds}\left(\frac{V^2}{2}\right)\boldsymbol{t}_1 \qquad (4\text{-}6)$$

where s is the coordinate which measures the displacement of P in its path, and $\boldsymbol{t}_1$ is a unit vector parallel to V.

The acceleration perpendicular to the tangent may be found in a similar manner but by putting $\alpha = 90°$ in Eq. (4-5). This means that OX is to be successively selected in positions perpendicular to the resultant velocity. By this selection the velocity in OX is continually zero. Designating this component of the resultant acceleration by J_n and calling it the *normal* acceleration, we have

$$J_n = \left(-V\frac{d\alpha}{dt}\right)\boldsymbol{n}_1 \qquad (4\text{-}7)$$

where $\boldsymbol{n}_1$ is the unit vector perpendicular to V.

An inspection of this result leads us to see that the magnitude of the normal acceleration has for its value the product of the resultant speed by the rate of change of direction of the velocity. The normal acceleration will be zero when either of its factors is zero. Any motion in which the velocity is not changing direction is obviously rectilinear motion. There may, however, still be acceleration in the line of the velocity. It is to be noticed that J_n is the acceleration which measures the rate of the change of direction of motion.

The direction of J_n will depend upon the signs of both factors (V and $d\alpha/dt$) [Eq. (4-7)]. If one of the factors is negative, the sign of J_n is positive. If both are either negative or positive, then J_n will be negative. It is to be observed from the diagram (Fig. 57) that, if P is moving with a velocity $+V$, then ω (written for $d\alpha/dt$) is also positive according to convention as indicated. In this case J_n is negative. If P has a velocity of $-V$, then ω is also negative and J_n is still negative.

Considering the motion of P along the curve shown in Fig. 58, we observe that J_n is positive regardless of whether the velocity of P is positive or negative. As a general statement, then, we may say that the direction of J_n is always toward the concave side of the curve.

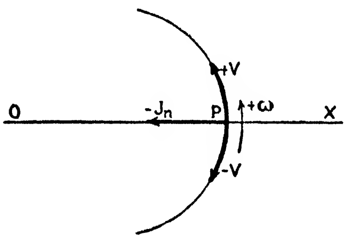

FIG. 57.

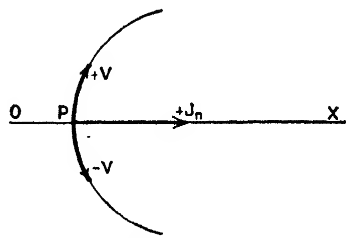

FIG. 58.

As a special case we may consider circular motion. The speed may remain constant, in which case J_t vanishes but J_n remains. J_n may be put into a more useful form for this purpose if we introduce the radius r of the circle in which the particle P is moving. In this case the radius is perpendicular to the resultant velocity and hence J_n is always parallel to r. Since $V = r\,d\alpha/dt$, we may replace $d\alpha/dt$ by V/r in Eq. (4-7) and obtain

$$J_n = \left(\frac{-V^2}{r}\right)n_1 \tag{4-8}$$

The two components of the resultant acceleration (J_t and J_n) are usually more convenient to use than the resultant acceleration, for J_t and J_n are always parallel and perpendicular, respectively, to the resultant velocity. The resultant acceleration may be more easily found by combining these two rectangular components, for they are easily expressed in terms of V, ω, and the time derivative of V.

Any other pair of rectangular components of J will not permit of expressions as simple as J_t and J_n. Considering any other such pair of components, as, for example, those along the coordinate axes X and Y, it is to be observed that the component accelerations J_x and J_y will each consist of two terms, *viz.*, the projections of both J_t and J_n upon the axis along which the desired component is to be found. Such a component, *i.e.*, J_x, will have one term which expresses the change of direction of V in the line of X (the projection of J_n) and the other which gives the rate of change of the magnitude of V in that line.

4-4. The Three Pairs of Components of Acceleration.—In our study of acceleration we shall have occasion to use three different pairs of orthogonal components. One of these, the tangential and normal pair, was discussed in the preceding section. Of the remaining two pairs, one consists of the components along the X- and Y-axes, and the other consists of the components along the radius vector and a line perpendicular to it. The last pair of components is to be referred to as the r and γ pair and will be studied in detail in Sec. **4-5.**

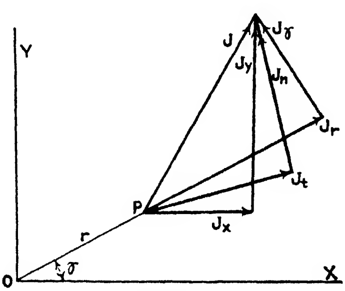

FIG. 59.

The vector sum of the two components forming any pair is obviously equal to the resultant acceleration. This is shown graphically in Fig. 59 and is written symbolically as follows:

$$J = J_x + J_y = J_r + J_\gamma = J_t + J_n \tag{4-9}$$

Any one pair of components may be expressed in terms of the components of any other pair by projecting both components of the one pair into the two lines of the other pair. For example, we may express J_x and J_y in terms of J_t and J_n by projecting both J_t and J_n into the lines of X and Y. If α is the angle between J_n and X, then

$$\begin{aligned} J_x &= (J_n \cos \alpha + J_t \sin \alpha)i \\ J_y &= (J_n \sin \alpha - J_t \cos \alpha)j \end{aligned} \tag{4-10}$$

Similarly, if we wish to express J_t and J_n in terms of J_x and J_y, we must project both J_x and J_y into the lines of the tangent and its normal, which would give

$$J_n = (J_x \cos\alpha + J_y \sin\alpha)\boldsymbol{n}_1$$
$$J_t = (J_y \cos\alpha - J_x \sin\alpha)\boldsymbol{t}_1 \tag{4-11}$$

Equations (4-11) might have been obtained analytically by first reducing Eqs. (4-10) to scalar equations and then simply solving for J_n and J_t.

4-5. Components Parallel and Perpendicular to the Radius Vector.—The device, introduced above, of using a pair of orthogonal lines, which are instantaneously fixed and coincident with the tangent and the normal, will be used again here, except that in this case these reference lines are to be instantaneously coincident with the radius vector and the line perpendicular to it. In order to study the results of successive differentiation, we shall start with the coordinate relations and, by differentiation with respect to the time, obtain the components of the acceleration along the X- and Y-axes and then find the J_r and J_γ components by projecting J_x and J_y into the line of r and that of γ, perpendicular to r. Scalar equations will first be written in order to reveal all of the illuminating details, for the vector expressions are too condensed for present purposes.

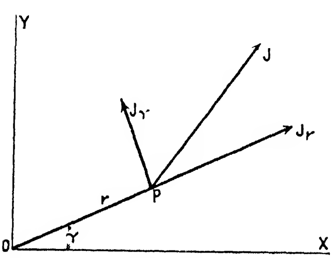

FIG. 60.

The coordinate relations are (Fig. 60)

$$x = r\cos\gamma$$

and

$$y = r\sin\gamma$$

The magnitudes of r and γ are both variable; hence the first derivative with respect to the time gives

$$\frac{dx}{dt} = \frac{dr}{dt}\cos\gamma - r\omega\sin\gamma$$
$$\frac{dy}{dt} = \frac{dr}{dt}\sin\gamma + r\omega\cos\gamma \tag{4-12}$$

in which ω is written for $d\gamma/dt$. Both of the foregoing equations, giving the component velocities along X and Y, respectively, consist of the sums of the component velocities parallel and perpendicular to r, both projected into each of the lines of X and Y.

Differentiating both equations of Eqs. (4-12) with respect to the time gives

$$J_x = \cos\gamma\left(\frac{d^2r}{dt^2} - r\omega^2\right) - \sin\gamma\left(\omega\frac{dr}{dt} + r\frac{d\omega}{dt} + \omega\frac{dr}{dt}\right)$$

$$J_y = \sin\gamma\left(\frac{d^2r}{dt^2} - r\omega^2\right) + \cos\gamma\left(\omega\frac{dr}{dt} + r\frac{d\omega}{dt} + \omega\frac{dr}{dt}\right) \quad (4\text{-}13)$$

We may now write the expressions for J_r and J_γ by projecting both J_x and J_y into each of the lines r and γ. The student should carry through the details of the process and simplify the resulting equations. The results are

$$J_r = J_x \cos\gamma + J_y \sin\gamma$$

$$= \frac{d^2r}{dt^2} - r\omega^2 \quad (4\text{-}14)$$

$$J_\gamma = J_y \cos\gamma - J_x \sin\gamma$$

$$= 2\omega\frac{dr}{dt} + r\frac{d\omega}{dt} \quad (4\text{-}15)$$

These equations for J_r and J_γ may be combined into a single vector equation to give the resultant acceleration J by introducing the unit vectors r_1 and γ_1 which are in the lines of r and γ, respectively, giving

$$J = \left(\frac{d^2r}{dt^2} - r\omega^2\right)r_1 + \left(2\omega\frac{dr}{dt} + r\frac{d\omega}{dt}\right)\gamma_1 \quad (4\text{-}16)$$

Equations (4-14) and (4-15) give the values of the magnitudes of J_r and J_γ in terms of r and γ and their derivatives. The direction which the resultant acceleration makes with r is expressed by the angle whose tangent is J_γ/J_r.

The expressions for J_r and J_γ will bear further investigation. The signs of these accelerations obviously depend upon the signs and relative magnitudes of the terms of their right-hand members. The convention of signs gives a positive direction of J_r outward along r and for J_γ an angle of +90 deg. from r. Since r is always positive, $-r\omega^2$ will always be negative. The sign of d^2r/dt^2 will depend upon whether dr/dt is increasing or decreasing. J_r will be positive, therefore, only when d^2r/dt^2 is positive and greater than $r\omega^2$. The quantity $\omega dr/dt$ will be positive when the signs of ω and dr/dt are alike. The quantity $r(d\omega/dt)$ will be positive or negative according to the sign of $d\omega$.

The magnitudes of the component velocities parallel and perpendicular to the radius vector are $V_r = dr/dt$ and $V_\gamma = \omega r$. These components are always taken so that they are instantaneously parallel and perpendicular to the radius vector. The expressions for the magnitudes of these component velocities remain unchanged. Their resultant must be the resultant velocity. It is legitimate to consider the resultant velocity as made up of two separate motions: circular motion and motion along the radius.

While it is also correct to obtain the resultant acceleration by combining as vectors the two component accelerations J_r and J_γ, still it is not correct to regard one component as being responsible for the rate of change of the radial velocity alone and the other as being responsible for the rate of change of the velocity perpendicular to the radius vector. The acceleration along r is made up of two parts: one part (d^2r/dt^2) which represents the rate of change of magnitude of V_r and the other $(-r\omega^2)$ which gives the rate of change of direction of V_γ.

Let us next consider the component perpendicular to J_r, *viz.*, J_γ. If the general statement regarding the two parts of the component acceleration J_r is true, we should expect that the right-hand member of Eq. (4-15) would consist of two parts, of which one should express the rate of change of magnitude of the component velocity V_γ and the other the rate of change of the direction of V_r. If Eq. (4-15) is arranged differently, this point is apparent.

$$
\begin{aligned}
J_\gamma &= \omega V_r + \left(\omega V_r + r\frac{d\omega}{dt}\right) \\
&= \omega V_r + \frac{d}{dt}V_\gamma \qquad (4\text{-}17)
\end{aligned}
$$

in which dV_γ/dt is the rate of change of magnitude of the velocity in the line of γ, since $V_\gamma = \omega r$. The other term ωV_r gives the rate of change of the direction of V_r.

As a direct conclusion, we may say that, while the first time derivative of a coordinate does give the component velocity parallel to that line along which the coordinate is measured, the second time derivative of the coordinate does not give the component acceleration parallel to that line unless the line is permanently fixed.

Referring again to the expression for J_n [Eq. (4-7)], which gives the acceleration for the *directional* change of *V*, and comparing the form of expression with the corresponding parts of J_r and J_γ, we may see that in this type of acceleration (rate of directional change of a velocity) the magnitude is expressed by the product of a linear speed and an angular speed. The angular speed is the rate of change of the direction of the linear speed. The other type of acceleration has for its magnitude the rate of change of the magnitude of a velocity, and its direction is that of the velocity.

The results of this analysis for J_r and J_γ may be summarized in the following manner:

Symbol	Change of magnitude	Change of direction
J_r	dV_r/dt	$-\omega V_\gamma$
J_γ	dV_γ/dt	ωV_r

4-6. Vector Determination of the Resultant Acceleration.—An expression for the resultant acceleration of a particle in terms of r and γ (the polar coordinates of the particle) and their derivatives may be obtained by purely vector methods. If the position of the particle P is given by the radius vector $\boldsymbol{r}$, then the resultant velocity of P as found by Eq. (2-28) is

$$\boldsymbol{V} = \frac{dr}{dt}\boldsymbol{r}_1 + r\frac{d\boldsymbol{r}_1}{dt}$$

Since the resultant acceleration is $d\boldsymbol{V}/dt$, we may write for $\boldsymbol{J}$:

$$\boldsymbol{J} = \frac{d^2r}{dt^2}\boldsymbol{r}_1 + r\frac{d^2\boldsymbol{r}_1}{dt^2} + 2\frac{dr}{dt}\frac{d\boldsymbol{r}_1}{dt}$$

Substituting the values for $d\boldsymbol{r}_1/dt$ and $d^2\boldsymbol{r}_1/dt^2$ as given by Eqs. (2-30) and (2-31), respectively, gives

$$\boldsymbol{J} = \left(\frac{d^2r}{dt^2} - r\omega^2\right)\boldsymbol{r}_1 + \left(2\omega\frac{dr}{dt} + r\frac{d\omega}{dt}\right)\boldsymbol{\gamma}_1 \tag{4-18}$$

which is identical with Eq. (4-16). It is obvious that the vector method of obtaining an expression for $\boldsymbol{J}$ is much briefer than the analytical method used above.

Problems.—1. Obtain a vector expression for the resultant acceleration in terms of the tangential and normal components by differentiating the equation $V = V V_1$.

2. Examine each term of Eqs. (4-13) and identify those which express a rate of change of the magnitude of a velocity and also those which express a rate of change of the direction of a velocity.

4-7. Differential Equations in Accelerated Motion.—There are three important relations between the quantities acceleration (a), time (t), the initial velocity (U), the final velocity (V), and a coordinate (s), which describe the motion of a particle having some stipulated acceleration. These relations are usually of the following forms:

$$V = f_1(U,\ a,\ t), \quad s = f_2(U,\ a,\ t), \quad \text{and} \quad V = f_3(U,\ a,\ s) \qquad (4\text{-}19)$$

It is important for the student to observe the procedure used in deriving these equations in uniplanar motion. A general method of solution is discussed in this section and is illustrated in each of the four following sections.

The statement of the problem includes the so-called initial conditions, such as the initial velocity at some given position or at some instant of time (usually zero) and an expression describing the acceleration. This information, together with the alternative forms of the acceleration, comprises the working material.

The first step involves a decision as to which of the three pairs of acceleration components (the x and y, the r and γ, or the t and n pair) is to be used. Obviously the selection will depend upon the particular data presented. For example, the acceleration of a particle may be given as varying inversely with the distance r of the particle from some given fixed point. In this case a polar coordinate system with the origin at the fixed point seems to recommend itself as giving convenient service. The r and γ pair of acceleration components is then to be used, for the resultant acceleration will, under such circumstances, always be along the line of r.

The next step would be to write down the general expressions for the selected components of the acceleration and to put each equal to the particular value given by the data. Using the given example, we would write the following scalar relations:

$$J_r = \frac{dV_r}{dt} - r\omega^2 = \frac{K}{r} \qquad J_\gamma = 2\omega V_r + r\frac{d\omega}{dt} = 0$$

where K is some constant. Even though one of the components is zero we must include the expression for it as above, for its solution will give a relation which is usually useful in eliminating one of the variables in the other equation.

In order to integrate either equation, we must reduce the number of variables to two by some method or other. It is difficult to give a general rule for this procedure. In the illustration the second equation may be integrated if we replace V_r by dr/dt and then multiply through by dt. This process would leave only two variables and the resulting equation could then be integrated. Using the result obtained by this integration, we may eliminate one of the variables (ω, in this case) from the first equation and then, if we replace dV_r/dt by $V_r dV_r/dr$, this equation may also be integrated. The constants could be determined by the use of the initial conditions.

In nearly all cases the first equation obtained will express the velocity in terms of the time or a coordinate. In either case the next step is to express the velocity in its differential form and then substitute this expression in the equation which gives the velocity as a function of the time or coordinate. Integration of the resulting equation would give the second desired expression. This equation will also contain constants of integration which may be evaluated as before. The third desired relation may now be found by an algebraic elimination of one of the variables.

The important thing to observe in this process is that, when the most convenient pair of acceleration components has been selected, such devices as are available are to be used in manipulating the equations into integrable forms. The procedure just described is general but should prove to be of value if carefully analyzed.

4-8. Translational Motion of a Particle with Constant Acceleration.—The equations are now to be derived which express the motion of a particle subject to acceleration that is constant and which is always parallel to the resultant velocity. Such limitations restrict the motion of the particle to a straight line which is parallel to the direction of the acceleration. Since in this case the normal component of the acceleration is zero, the three alternative forms of the tangential component of acceleration [Eq. (4-6)] may serve as a starting point. Putting each of the three alternative forms of the acceleration equal to the given value of the magnitude of the acceleration gives three differential equations whose integrations yield the desired expressions. Let a represent the given magnitude of the acceleration and let the positive direction of all directed quantities be measured parallel to the positive direction of the acceleration. For initial conditions we may put

$$t = 0, \qquad s = 0, \qquad \text{and} \qquad V = U$$

If we use the form dV/dt for the acceleration, the first differential equation is

$$\frac{dV}{dt} = a$$

Integration of this equation gives

$$V = at + C$$

The integration constant C is evaluated by the use of the initial conditions. Since $V = U$ at the time $t = 0$, then $C = U$; hence

$$V = at + U \tag{4-20}$$

which is the first of the desired equations.

Using the second alternative form of the tangential acceleration gives the differential equation

$$\frac{d^2s}{dt^2} = a$$

Integration of this equation gives

$$\frac{ds}{dt} = at + D$$

Applying the values of the quantities at the initial position makes $D = U$. Making this substitution and integrating again gives

$$s = \tfrac{1}{2}at^2 + Ut + E$$

Since $s = 0$ at the time $t = 0$, the value of E is seen to be zero. Hence

$$s = \tfrac{1}{2}at^2 + Ut \tag{4-21}$$

The third differential equation is

$$\frac{VdV}{ds} = a$$

Separating and integrating gives the equation

$$\tfrac{1}{2}V^2 = as + F$$

in which $F = \tfrac{1}{2}U^2$ for $V = U$ where $s = 0$; hence

$$V^2 = U^2 + 2as \tag{4-22}$$

Equations (4-20), (4-21), and (4-22) give the three fundamental expressions for translational motion in which the acceleration is constant and the initial velocity is parallel to the acceleration.

In gravitational fields of force the acceleration has a special value which is usually designated by the symbol g. Making this substitution for the value of the acceleration gives the following three equations for gravitational motion in a vacum:

$$V = U + gt$$
$$s = Ut + \tfrac{1}{2}gt^2 \qquad V^2 = U^2 + 2gs \tag{4-23}$$

These equations are subject to the limitation stated above and hence apply only to vertical motion with positive values of the velocities, distance, and acceleration measured downward.

4-9. Pure Rotational Motion with Constant Angular Acceleration.—If the motion of a particle is such that it moves constantly in a circular path, pure rotational motion results. When the angular acceleration is constant, the rotational motion may be conveniently described by three equations which are analogous to those derived above for translational motion. The procedure for obtaining these equations is similar to that used in the above section. Angular acceleration may be expressed by any one of the three alternative forms $d\omega/dt$, $d^2\gamma/dt^2$, and $\omega d\omega/d\gamma$. If we put each of these expressions equal to the constant A, which we may use to represent the value of the magnitude of the angular acceleration, and if we assume as initial conditions that at the time $t = 0$, $\gamma = 0$, and $\omega = \omega_1$, integration of the three equations gives the following scalar expressions:

$$\omega = At + \omega_1$$
$$\gamma = \tfrac{1}{2}At^2 + \omega_1 t$$
$$\omega^2 = \omega_1^2 + 2A\gamma \tag{4-24}$$

These equations might have been obtained directly from those for translational motion by substituting in Eqs. (4-20), (4-21), and (4-22) those quantities in rotation which correspond to the quantities in translation.

4-10. Motion of a Particle with Constant Acceleration. *Initial Velocity Making an Angle α_1 with the Acceleration.*—When the linear acceleration is constant and makes an angle α_1, not equal to zero, with the initial velocity, the resulting motion will be curvilinear. It is required to determine the scalar equations which express the coordinate s (measured along the curved path) and the velocity V in terms of the time and the initial conditions. This problem is solved by making use of the expressions for the pair of rectangular components of acceleration J_t and J_n.

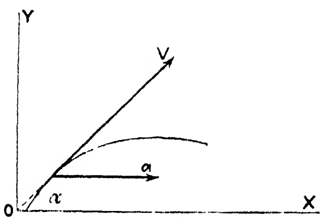

FIG. 61.

Let the acceleration be of magnitude a and always directed parallel to OX (Fig. 61). Also let α_1 be the angle between the initial velocity U and a and α the angle between the velocity V in any position and a.

The initial conditions are taken to be $t = 0$, $s = 0$ and $V = U$, $\alpha = \alpha_1$. The component accelerations are expressed as follows:

$$J_t = \frac{dV}{dt} = a \cos \alpha$$

$$J_n = \frac{-V d\alpha}{dt} = a \sin \alpha \qquad (4\text{-}25)$$

Dividing the first equation by the second gives

$$-\frac{dV}{V} = \cot \alpha \, d\alpha$$

Integrating this equation gives

$$\log \sin \alpha + \log V = C$$
$$\log (V \sin \alpha) = C \qquad (4\text{-}26)$$

in which C is the constant of integration. C may be evaluated by putting $V = U$, which gives $C = \log (U \sin \alpha_1)$. Hence

$$V \sin \alpha = U \sin \alpha_1 \qquad (4\text{-}27)$$

Examining this equation for its physical meaning, we see that the component of the velocity perpendicular to the line of a is a constant and has for its magnitude $U \sin \alpha_1$. This result might have been anticipated, for there cannot be any change in the magnitude of the velocity component ($V \sin \alpha$) in a line which is perpendicular to the acceleration.

From Eq. (4-27) we may evaluate sin α and hence find a value for cos α. Substituting this value for cos α in Eq. (4-25) gives

$$\frac{dV}{dt} = \frac{a}{V}\sqrt{V^2 - U^2 \sin^2 \alpha_1}$$

Separating the variables and integrating gives

$$\sqrt{V^2 - U^2 \sin^2 \alpha_1} = a\,t + C_1$$

where C_1 is the constant of integration. From the initial conditions $t = 0$, $V = U$, we find $C_1 = U \cos \alpha_1$. Hence we obtain one of the desired relations

$$V^2 = U^2 + 2\,a\,t\,U \cos \alpha_1 + a^2\,t^2 \qquad (4\text{-}28)$$

It is desired to find next the relation between s and t. As a starting point we may use Eq. (4-28). This may be written as follows:

$$\left(\frac{ds}{dt}\right)^2 = a^2\,t^2 + U^2 + 2\,a\,t\,U \cos \alpha_1$$

Separating the variables and integrating gives the desired equation.

$$s = \frac{at + U \cos \alpha_1}{2a}\sqrt{K} + \frac{U^2 \sin^2 \alpha_1}{2a} \log\,(\sqrt{K} + at + U \cos \alpha_1) + C_2 \qquad (4\text{-}29)$$

where $K = a^2t^2 + U^2 + 2at\,U \cos \alpha_1$ and C_2 is the integration constant.

From the initial conditions $s = 0$ and $t = 0$ we may evaluate C_2, which is

$$C_2 = \frac{-U^2}{2a}[\cos \alpha_1 + \sin^2 \alpha_1 \log\,(U + U \cos \alpha_1)]$$

Problems.—1. Starting with Eq. (4-26), find an expression for V in terms of the angle α.

2. Using the coordinate system indicated in Fig. 61, find the equation for the path of the moving point. What kind of a curve is this? *Hint:* Write the expression for the component accelerations along X and Y.

3. Show that Eqs. (4-28) and (4-29) reduce to Eqs. (4-22) and (4-21), respectively, if α_1 is put equal to zero.

4-11. Rectilinear Motion with Acceleration Not Constant.—There are several important cases of translational motion in which the acceleration is not constant. The following illustration has been selected from the universal law of gravitation. Two equations are to be derived, one expressing the velocity in terms of the coordinate and the other giving the time as a function of the coordinate.

A particle in space under the attraction of the earth alone will have an acceleration which is inversely proportional to the square of the distance from it to the center of the earth.

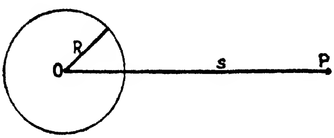

FIG. 62.

Let there be a small particle at P (Fig. 62), whose distance from O, the reference point which is taken as the center of mass of the earth, is given by the coordinate s. Assuming for initial conditions that $t = 0$, $s = s_0$, and $V = U$, with U parallel to the line along which the coordinate s is measured, we may first find an expression for the velocity in terms of s. Since the resultant acceleration is in the line of s, and the magnitude of the acceleration is inversely proportional to the square of s, the differential equation is

$$\frac{d^2s}{dt^2} = \frac{-k}{s^2} \tag{4-30}$$

where k is a proportionality constant.

The minus sign appears here because the direction of the acceleration is inward along s. To integrate this equation we may first multiply both sides by ds/dt, which gives

$$\frac{ds}{dt}\frac{d^2s}{dt^2} = \frac{-k}{s^2}\frac{ds}{dt}$$

or

$$\frac{1}{2}\frac{d}{dt}\left(\frac{ds}{dt}\right)^2 = k\frac{d}{dt}\left(\frac{1}{s}\right)$$

Integration gives

$$\frac{1}{2}\left(\frac{ds}{dt}\right)^2 = \frac{k}{s} + C$$

If we apply the initial conditions, we find the constant of integration C:

$$C = \frac{1}{2}U^2 - \frac{k}{s_0}$$

Hence the final expression is

$$\frac{1}{2}(V^2 - U^2) = k\left(\frac{1}{s} - \frac{1}{s_0}\right) \tag{4-31}$$

An examination of this equation shows that, if a particle moved from infinity, so that $1/s_0 = 0$, and started from rest ($U = 0$), the final velocity V would be equal to $\sqrt{2k/s}$. Hence, as the particle approached the gravitating mass, s would decrease and V would increase.

The limitations imposed upon this equation are those given by the initial conditions and an additional condition, which comes from the law of gravitation. The equation is valid only for values of s which are greater than the radius of the earth, for when s is less than the radius of the earth the quantity k is no longer constant. This last condition is to be found in the derivation of the expression for the value of the acceleration at points which are outside and also inside the gravitating body. We shall consider the case of the particle moving within the gravitating body in a later section. We cannot therefore put $s = 0$ in Eq. (4-31) and assume that V would become infinite at that point.

Equation (4-31) reduces to the equation for V^2 of bodies falling in a vacuum at the surface of the earth if we put $k = gR^2$ where g is the acceleration on the earth's surface and R is the radius of the earth. Making this substitution we have

$$\begin{aligned} V^2 - U^2 &= 2gR^2\frac{s_0 - s}{ss_0} \\ &= 2gh \end{aligned} \tag{4-32}$$

in which $h = s_0 - s$, where h is small in comparison with either s or s_0, and we may also put $ss_0 = R^2$ as an approximation.

Returning to the more general case, we shall proceed to obtain an equation expressing the time t as a function of s. By writing Eq. (4-31) explicit for V and putting $V = ds/dt$, we have

$$\begin{aligned} \frac{ds}{dt} &= \pm\sqrt{U^2 + 2k\left(\frac{1}{s} - \frac{1}{s_0}\right)} \\ &= -\sqrt{k' + \frac{2k}{s}} \end{aligned}$$

The minus sign is taken because we are interested in the motion of the particle P toward the gravitating body. For the sake of brevity, k' is written for $U^2 - \frac{2k}{s_0}$. Separating the variables, writing C' for the constant of integration and integration gives

$$t = -\int \sqrt{\frac{s}{2k + k's}}\, ds + C'$$

$$= -\sqrt{\frac{s(2k+k's)}{k}} + \frac{k}{k'\sqrt{k'}} \log\left(\sqrt{s(2k+k's)} + s\sqrt{k'} + \frac{k}{\sqrt{k'}}\right) + C' \quad (4\text{-}33)$$

in which it was assumed that k' was positive. If, however, we assume that k' is negative, then the integration would yield

$$t = -\frac{\sqrt{s(2k + k's)}}{k'} + \frac{k}{k'\sqrt{-k'}} \sin^{-1}\left(\frac{-k's - k}{k}\right) + C'' \quad (4\text{-}34)$$

The constants C' and C'' may be evaluated by the use of initial conditions.

Problem.—Determine the numerical value of the speed which a small meteorite would have on reaching the earth's surface if it started from rest at an infinite distance.

4-12. The Hodograph.—The *hodograph* is an auxiliary curve drawn to facilitate the identification of the acceleration at any point in the path of a moving particle. An illustration will serve the purpose of making clear the details of construction of the auxiliary curve and of showing how the acceleration vector may be obtained from it.

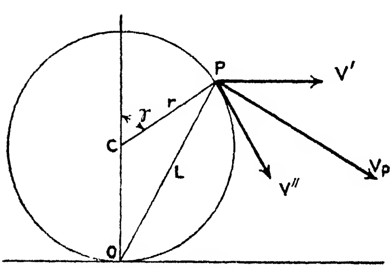

FIG. 63.

Suppose that we consider the velocity of a particle P which is on the rim of a wheel (Fig. 63). The wheel rolls along a straight line in the plane of the diagram with a constant angular velocity ω. At any instant the velocity of P may be expressed as the vector sum of two velocities, one the constant forward velocity V' of the center C of the wheel and the other V'' which is the velocity of P relative to the center of the wheel. The magnitude of V'' will always be equal to that of V' and will remain constant but the direction of V'' will be constantly changing. The acceleration of P will therefore be perpendicular to V'' and will be directed inward along the radius of the wheel r which is drawn to P and will be of magnitude $-\omega^2 r$. The resultant velocity of P (*i.e.*, V_P) will always be perpendicular to the line PO connecting P to the point O of contact of the wheel to the guide line. The point O is the instantaneous center.

Let us now draw from any point Q (Fig. 64) lines which shall respectively represent the velocities of P in its various positions, always drawing these lines parallel to V_P and of such lengths that they may represent the velocity to some arbitrary scale. In the particular case selected for illustration, if we write L for PO, the magnitude of V_P is equal to ωL for any position and, since ω is constant, then the auxiliary lines will be proportional to L.

If we measure the angle γ (Fig. 63), through which the wheel turns, from a line constantly vertical and passing through C, then

$$L = 2r \cos \tfrac{1}{2}\gamma \qquad (4\text{-}35)$$

so that the length of the auxiliary lines in Fig. 64 will be proportional to $2r \cos \frac{1}{2}\gamma$, and each line will be perpendicular to L for the particular position.

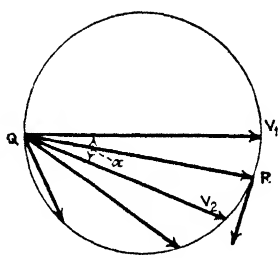

FIG. 64.

Starting with $\gamma = 0$ and $L = 2r$, we may draw V_1 from any arbitrary point Q (Fig. 64) and proportional to $2r$ in length—and also perpendicular to L.

As the wheel rolls through an angle γ, the line L and hence V_P will move through the angle α. It is readily seen that $\alpha = \frac{1}{2}\gamma$. The angle α is the constructional angle of Fig. 64. At the position $\gamma = 180°$ the constructional angle passes through a tangent and hence for values of γ from 180 to 360 deg.

$$\alpha = \tfrac{1}{2}\gamma + 180°.$$

By selecting values for γ the corresponding values of L and α may be found. A sufficient number of positions should be used in order that the resulting curve may be an accurate representation of the relations.

In further illustration of the constructional processes, we find for the position $\gamma = 45°$ that V_2 must be drawn proportional to $1.8r$ along a line making an angle of -22.5 deg. with V_1.

Similar constructions follow for other values. The *curve* passing through the ends of such constructional lines V_1, V_2, etc., is the hodograph for the particular motion described. Since in this case the motion is cyclic, the single curve represented in the diagram as a circle would be duplicated for any other cycle. The value of the velocity for any position of P may be readily obtained from the hodograph.

The acceleration of P is next to be expressed by using the hodograph. If the point R (Fig. 64) is constrained to move on the hodograph so that its position instantaneously gives $\mathbf{QR}$ as the proper vector for expressing the instantaneous velocity of P, then it may be shown that the tangent to the curve at R gives the direction of the acceleration of P, and the speed of R along the curve will give the magnitude of the acceleration of P. If we take any other point R' in the hodograph (not shown in the diagram) which is close to R, then the velocity vector $\Delta \mathbf{RR'}$ will be the vector which must be added to $\mathbf{QR}$ to give $\mathbf{QR'}$. Let Δt be the time interval between the positions QR and QR'. The value of $\Delta \mathbf{RR'}/\Delta t$ when Δt approaches zero will give the rate of change of $\mathbf{QR}$ which is the velocity of the point R and is therefore the acceleration of P. It is to be observed that the direction of the velocity of R and hence the direction of the acceleration of P (Fig. 63) is always parallel to the radius of the wheel which connects P to the center of the wheel.

In the particular case selected for illustration, it is interesting to show that the hodograph is a circle. To prove that this is true, we may use polar coordinates to advantage. If Q (Fig. 64) is the origin, then the magnitude of the velocity vector at any position represents the radius vector to the selected scale. Let ρ and α be the coordinates. Since the magnitude of V_P is ωL and $L = 2r \cos \frac{1}{2}\gamma$, we may put

$$\rho = K \cos \alpha \tag{4-36}$$

where $K = 2\omega r$ and $\alpha = \frac{1}{2}\gamma$. This is the equation of a circle in polar coordinates where the origin is on the circumference and the diameter of the circle is K.

Problem.—Find the hodograph for the motion of a particle P which moves so that the initial velocity $\mathbf{U}$ makes an angle α_1 with the direction of the constant acceleration a.

The equations for the motion of P, with the limitations indicated, were derived above in Sec. 4-10. Equation (4-27) expresses the magnitude of the velocity of the particle as follows:

$$V = \frac{U \sin \alpha_1}{\sin \alpha}$$

where α, as above, is the angle between V and the acceleration $\mathbf{a}$.

We may determine the hodograph by calculating the value of V for different values of α and then draw the corresponding vectors from a common point Q in the manner indicated above or we may use polar coordinates and determine the equation of the hodograph which is the locus of the terminal point of the velocity vector. Selecting the latter method we see that we should take Q (Fig. 65) the origin of the polar coordinate system at the starting point of P and with the line of the direction of $\mathbf{a}$ as the reference line. V then becomes the radius vector ρ. The resulting equation is

$$\rho = \frac{U \sin \alpha_1}{\sin \alpha}$$

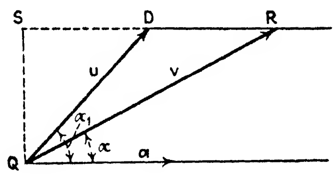

Fig. 65.

which is the polar equation of a straight line. The line is evidently parallel to $\mathbf{a}$ and is situated at a distance of $U \sin \alpha_1$ from $\mathbf{a}$. For the particular problem, only that portion of the line which extends from D, the terminal point of U, to infinity is the hodograph. It is also evident that the velocity increases indefinitely and never becomes parallel to the line of $\mathbf{a}$ except at infinity.

One would expect that the velocity of R, the terminal point of V in the hodograph, would be constant, since the velocity of R is the acceleration of the particle P in the original motion and is constant and is equal to $\mathbf{a}$. That this is true may be shown by expressing the velocity of R (*i.e.*, V_R). For this purpose let us measure the distance SR by the coordinate s taken from S as reference point, where S is a point in the line DR and so situated that QS is perpendicular to DR. The magnitude of s is given by the equation $s = V \cos \alpha$. Hence, since $V_R = ds/dt$ and $\omega = d\alpha/dt$,

$$V_R = \frac{dV}{dt} \cos \alpha - V\omega \sin \alpha$$

Substituting the values of dV/dt and V as given respectively by the equations

$$\frac{dV}{dt} = a \cos \alpha \qquad -V\omega = a \sin \alpha$$

gives

$$V_R = a \cos^2 \alpha + a \sin^2 \alpha$$
$$= a$$

The velocity of R in the hodograph is therefore constant and has for its magnitude the acceleration of the point P.

Problems.—1. By using the velocity components of R (Fig. 64) which are parallel and perpendicular to the line V_1, show that the speed of R in the hodograph is $\omega^2 r$.

2. By the use of the velocity components of R (Fig. 64) which are parallel and perpendicular to the line QR, find the speed of R.

3. Find the hodograph for the falling motion of a small bead which is guided by a vertical wire loop bent in the form of a circular arc. Disregard friction and assume that the bead starts from a position which is 45 deg. from the vertical.

4-13. The Instantaneous Acceleration Center.—In a previous section it was shown that any body in uniplanar motion has an instantaneous center of rotation, about which the velocity of any point of the body may be expressed. The magnitude of the velocity of any point of the body was shown to be equal to the product of the line segment joining the point to the instantaneous center by the common angular velocity. It is now proposed to show that an *instantaneous center of acceleration* exists with which the acceleration of any point may be expressed.

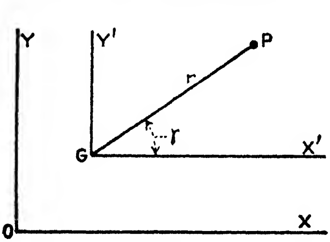

FIG. 66.

Let G (Fig. 66) be any point of a body having uniplanar motion and let x_1 and y_1 be the coordinates of G in the reference system XOY. The point P, with coordinates x and y in the reference system, is any other point of the body situated at a distance r from G. The point G is to be the origin of a moving system $X'\,GY'$ whose axes are always parallel to those of the reference system. Let γ be the angle between GP and GX'. Then the coordinates of P may be expressed by the equations

$$x = x_1 + r \cos \gamma$$
$$y = y_1 + r \sin \gamma \tag{4-37}$$

Since r is of constant magnitude, a differentiation of these equations with respect to the time gives

$$\frac{dx}{dt} = \frac{dx_1}{dt} - r\omega \sin \gamma$$
$$\frac{dy}{dt} = \frac{dy_1}{dt} + r\omega \cos \gamma \qquad (4\text{-}38)$$

where ω is written for $d\gamma/dt$.

The position of the instantaneous velocity axis of rotation may be expressed analytically by using Eqs. (4-38) if we select P as the trace of this axis in the guide plane and then put the velocity of P equal to zero. Putting dx/dt and dy/dt of Eqs. (4-38) equal to zero gives

$$\frac{dx_1}{dt} = r\omega \sin \gamma \qquad \frac{dy_1}{dt} = -r\omega \cos \gamma \qquad (4\text{-}39)$$

The coordinates x and y of the instantaneous velocity center may now be expressed by solving Eqs. (4-39) for $r \cos \gamma$ and $r \sin \gamma$ and substituting the resulting expressions for $r \cos \gamma$ and $r \sin \gamma$ in Eqs. (4-37), which gives

$$x = x_1 - \frac{1}{\omega}\frac{dy_1}{dt} \qquad y = y_1 + \frac{1}{\omega}\frac{dx_1}{dt} \qquad (4\text{-}40)$$

These equations [Eqs. (4-40)] may be used for determining the coordinates (x and y) of the instantaneous velocity center.

Since ω [and also r of Eqs. (4-39)] is unknown, it is necessary to know the positions of the two points ($x_1\, y_1$, and $x_2 y_2$, say) of the body and their component velocities (dx_1/dt, dy_1/dt and dx_2/dt, dy_2/dt) in order to determine the coordinates of the instantaneous velocity center. With these data, in any given case, the unknown ω may be eliminated by using Eqs. (4-40) together with similar equations containing $x_2 y_2$ and their derivatives. When the instantaneous velocity center has been determined, the instantaneous velocities of all other points of the body may be determined by the method described in a preceding chapter.

Proceeding in a similar manner we may now determine whether there is an instantaneous center of acceleration and, if it exists, we may determine its position. By using the coordinate relations which are given by Eqs. (4-37), the second time derivatives of x and y give the component accelerations of P as follows:

$$\frac{d^2x}{dt^2} = \frac{d^2x_1}{dt^2} - \omega^2 r \cos\gamma - r\frac{d\omega}{dt}\sin\gamma$$
$$\frac{d^2y}{dt^2} = \frac{d^2y_1}{dt^2} - \omega^2 r \sin\gamma + r\frac{d\omega}{dt}\cos\gamma \qquad (4\text{-}41)$$

If P is to be a point with no acceleration, then d^2x/dt^2 and d^2y/dt^2 must both be equal to zero. By equating the right-hand members of Eqs. (4-41) to zero and then solving for $r\cos\gamma$ and $r\sin\gamma$, the coordinates of P in the moving system, the following equations are obtained:

$$r\sin\gamma = \frac{\omega^2\ddot{y}_1 + \dot{\omega}\ddot{x}_1}{\omega^4 + \dot{\omega}^2} \qquad r\cos\gamma = \frac{\omega^2\ddot{x}_1 - \dot{\omega}\ddot{y}_1}{\omega^4 + \dot{\omega}^2} \qquad (4\text{-}42)$$

in which $\ddot{x}_1$, $\ddot{y}_1$, and $\dot{\omega}$ are written for d^2x_1/dt^2, d^2y_1/dt^2, and $d\omega/dt$, respectively. But $r\cos\gamma$ and $r\sin\gamma$ are respectively equal to x' and y', the coordinates of P in the moving system. Hence we have obtained a point P with definite coordinates which has no acceleration. There is therefore an acceleration center in the reference system.

Let us designate the instantaneous acceleration center by the letter I. The coordinates of I in the reference system may now be expressed as follows:

$$x = x_1 + \frac{\omega^2\ddot{x}_1 - \dot{\omega}\ddot{y}_1}{\omega^4 + \dot{\omega}^2} \qquad y = y_1 + \frac{\omega^2\ddot{y}_1 + \dot{\omega}\ddot{x}_1}{\omega^4 + \dot{\omega}^2} \qquad (4\text{-}43)$$

These equations may be used for finding the coordinates x and y of the instantaneous acceleration center for any instantaneous position of the body in uniplanar motion, provided that we know the coordinates and component accelerations of at least two points of the body. By substituting the coordinates (x_1y_1 and x_2y_2) and the acceleration components ($\ddot{x}_1$, $\ddot{y}_1$ and $\ddot{x}_2$, $\ddot{y}_2$) of any two selected points of the body, the unknown quantities ω and $\dot{\omega}$ may be eliminated and the coordinates of the instantaneous acceleration center determined. It is well to remember that ω and $\dot{\omega}$ are the instantaneous angular velocity and acceleration, respectively, of the line drawn from the instantaneous center to any point of the body.

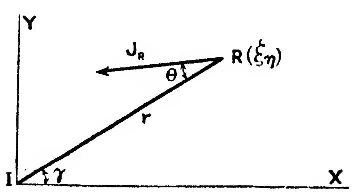

FIG. 67.

In order to find expressions which are to give the acceleration of any point R (Fig. 67) of the body in terms of the distance of R from I, it will be convenient to select the point I as origin of a reference system XIY which is instantaneously at rest. The coordinate relations in the new system will be

$$\xi = r \cos \gamma \qquad \eta = r \sin \gamma$$

where ξ and η are the coordinates of R, r is the distance of R from I, and γ is the angle between r and IX. Since r is of constant length, the second time derivatives of these equations give

$$\frac{d^2\xi}{dt^2} = -r\omega^2 \cos \gamma - r\dot{\omega} \sin \gamma$$

$$\frac{d^2\eta}{dt^2} = r\dot{\omega} \cos \gamma - r\omega^2 \sin \gamma \qquad (4\text{-}44)$$

If J_R is the resultant acceleration of R, then

$$J_R = \sqrt{\left(\frac{d^2\xi}{dt^2}\right)^2 + \left(\frac{d^2\eta}{dt^2}\right)^2}$$

Substituting the right-hand members of Eqs. (4-44) for the component accelerations in this equation gives, after simplification,

$$J_R = r\sqrt{\omega^4 + \dot{\omega}^2} \qquad (4\text{-}45)$$

It is evident from this equation that the magnitude of the acceleration of any point R of the body is proportional to its distance from the instantaneous acceleration center, since the coefficient of r is common to all points of the body if the body is rigid.

The direction of J_R is next to be determined. For this purpose, Eqs. (4-44) together with the general expressions of the components of J_R [J_r of Eq. (4-14) and J_γ of Eq. (4-15)] may be used to advantage. In the case under consideration, r is instantaneously of fixed magnitude; hence the first and second derivatives of r with respect to the time are zero. The following magnitude relations then hold:

$$J_r = r\omega^2 \qquad J_\gamma = r\dot{\omega}$$

If we now examine Eq. (4-44), we may see that both component accelerations $d^2\xi/dt^2$ and $d^2\eta/dt^2$ could be regarded as being formed by the sum of the projections of J_r and J_γ into the lines IX and IY.

Let us designate the angle between J_R and r by θ (Fig. 67). The direction of J_R may then be expressed as follows:

$$\tan\theta = \frac{J_\gamma}{J_r} = \frac{\dot{\omega}}{\omega^2} \tag{4-46}$$

From this expression it follows that

$$\cos\theta = \frac{\omega^2}{B} \qquad \sin\theta = \frac{\dot{\omega}}{B} \tag{4-47}$$

in which for brevity B is written for $\sqrt{\omega^4 + \dot{\omega}^2}$. The component accelerations of R along the reference axes may then be expressed in different forms by substituting in Eq. (4-44) the values of ω^2 and $\dot{\omega}$ given by Eqs. (4-47), which gives

$$J_\xi = -B\ r\ \cos\ (\gamma - \theta) \qquad J_\eta = -B\ r\ \sin\ (\gamma - \theta) \tag{4-48}$$

Since Br is the acceleration J_R [see Eq. (4-45)], it is evident that the component accelerations along IX and IY could be obtained by projecting J_R into the axes of reference by multiplying J_R by $\cos(\gamma - \theta)$ and $\sin(\gamma - \theta)$, respectively. Hence $\gamma - \theta$ is the angle which the acceleration of R makes with the X-axis as is apparent from Fig. 67. Assuming that γ is measured in the standard positive sense, then θ will be drawn as shown in Fig. 67. If, however, $\dot{\omega}$ is negative, then θ will be negative and J_R will be on the other side of the line r.

As a special case in this connection, suppose that the angular acceleration of the body is zero. The angular velocity would, in this case, be constant and J_R would be equal to $\omega^2 r$. The angle θ would also be zero. The acceleration of any point of the body would then be directed along the line which connects the point with the instantaneous acceleration center.

The instantaneous acceleration center may be located by a graphical method when the positions and accelerations of any two points of the body are known. Given the two points A and B (Fig. 68) and their accelerations J_A and J_B, respectively. Suppose that the lines giving the directions of J_A and J_B intersect in the point O. Let I be the instantaneous acceleration center. The circle drawn through the three points A, O, and B must pass through I. That this is true may be readily seen from the geometry of the relations, since the angle θ between J_A and AI must be equal to the angle between J_B and BI as was shown above.

To determine the particular position of I on the circle drawn through A, O, and B, we have another relation to use. The ratio of AI to BI is equal to the ratio of J_A to J_B [Eq. (4-45)]. The locus of the point which moves so that the ratio of its distances to A and B (AI/BI) is constant is a circle which may be readily drawn. If we designate the ratio AI/BI by n and the distance from A to B by d, then it may be shown that the radius of this circle is equal to $nd/(1 - n^2)$ and that the center of the circle is situated at a distance equal to $n^2d/(1 - n^2)$ from A on the line drawn through the points A and B. The center will be beyond A in the direction from B to A, if n is less than 1. The point of intersection of the two circles is the acceleration center for the particular instant under consideration.

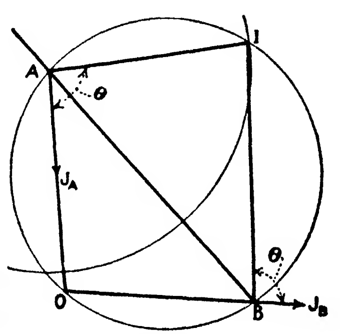

FIG. 68.

Since there is an instantaneous acceleration center, which is in general not fixed, there will be a locus of such points in the reference system. This locus is called the fixed acceleration centrode. The locus of the instantaneous acceleration center in a moving system attached to the body is the moving acceleration centrode. These centrodes are not to be confused with the velocity centrodes. In the case of the velocity centrodes it was shown that, during the motion of the body, the moving centrode rolls upon the fixed centrode in such a way that the instantaneous velocity center for any given position of the body is the pair of points, one from each centrode, which are instantaneously coincident. The velocity centrodes are tangent to each other at the point which is the instantaneous velocity center for that position. The instantaneous acceleration center is a point of intersection of the two acceleration centrodes. In general, there will be at least two points of intersection of the two acceleration centrodes and only one point of intersection can be the instantaneous acceleration center for that position. It is not difficult to determine which point of intersection is the acceleration center. If ω and $\dot{\omega}$ are known, then θ [Eq. (4-46)] may be used in the decision. If the accelerations of any two points of the body are known, then the correct selection may be made, since the ratio of the accelerations of any two points of the body is equal to the ratio of the corresponding distances of these two points from the instantaneous acceleration center.

Problems.—The solutions of the first two problems are included below in order to serve as guides in solving problems involving the instantaneous acceleration center. The student should attempt to obtain the solutions without referring to the text. After a reasonable effort the procedure given here should be consulted if the attempt was unsuccessful. The purpose of problem solving is to develop ability in the successful application of the principles and to assist in the better understanding of the principles.

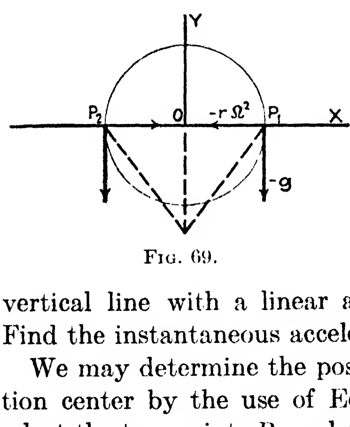

Fig. 69.

1. *Falling Disk.*—A circular disk is rotating with constant angular velocity Ω about a horizontal axis and falls freely in a vertical line with a linear acceleration of its center equal to g. Find the instantaneous acceleration center for any position.

We may determine the position of the instantaneous acceleration center by the use of Eq. (4-43). For this purpose let us select the two points P_1 and P_2 in the positions shown in Fig. 69. Let the reference system XOY be selected at an instant when the origin O coincides with the center of the disk and with axes horizontal and vertical. If the radius of the disk is a, then coordinates of P_1 and P_2 are

$$x_1 = a, \qquad y_1 = 0, \text{ and } x_2 = -a, \qquad y_2 = 0$$

The acceleration components of P_1 and P_2 are

$$\ddot{x}_1 = -a\Omega^2, \qquad \ddot{y}_1 = -g, \text{ and } \ddot{x}_2 = a\Omega^2, \qquad \ddot{y}_2 = -g$$

Substituting the values for the coordinates and acceleration components in Eq. (4-43) gives the following four equations:

P_1:

$$x = a - \frac{\omega^2 A - \dot{\omega} g}{B} \qquad y = -\frac{\omega^2 g + \dot{\omega} A}{B}$$

P_2:

$$x = -a + \frac{\omega^2 A + \dot{\omega} g}{B} \qquad y = \frac{-\omega^2 g + \dot{\omega} A}{B} \tag{4-49}$$

where B is written for $\omega^4 + \dot{\omega}^2$ and A for $a\Omega^2$. Eliminating $\dot{\omega}/B$ and ω^2/B from these equations and then solving for x and y, the coordinates of the instantaneous acceleration center, gives

$$x = 0 \qquad y = \frac{-g}{\Omega^2}$$

This point is obviously located on the vertical line through the center of the disk and at a distance $-g/\Omega^2$ below the center of the disk.

The values of $\dot{\omega}$ and ω^2 may also be found from Eqs. (4-49) and are 0 and Ω^2, respectively. These results might have been anticipated from the fact that any line of the body, and hence the line connecting P_1 (or P_2) to the instantaneous center, has instantaneously the same angular velocity and angular acceleration that any other line of the body has about the instantaneously fixed axis.

Since $\dot{\omega}$ is zero, then, and $\tan \theta = \dot{\omega}/\omega^2$, the angle θ is also equal to zero and hence the instantaneous acceleration center lies at the point of intersection of the lines of acceleration of any two points of the body.

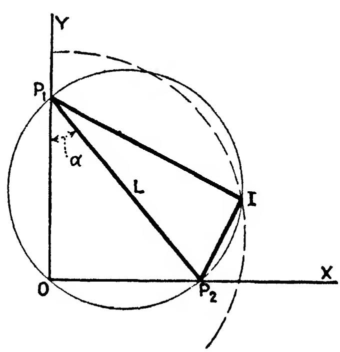

FIG. 70.

2. *Sliding Ladder*.—Suppose that a ladder of length L slides down between a vertical wall and a horizontal floor in such a manner that, if α is the acute angle which the upper end of the ladder makes with the wall, the motion is subject to the relation $\ddot{\alpha} = K \sin \alpha$. Find the instantaneous acceleration center for any position of the ladder and also the acceleration centrodes.

Let the reference system be XOY with OX the horizontal floor and OY the vertical wall. Let the terminal points of the ladder be P_1 and P_2, as shown in Fig. 70. The motion of P_1 will be along the OY-axis and that of P_2 will be along the OX-axis. In order to simplify the resulting expressions, we shall select P_1 and P_2 for the two points to be used in connection with Eqs.

(4-43) in determining the expressions for the instantaneous acceleration center.

The coordinates of P_1 and P_2 are x_1y_1 and x_2y_2, respectively; hence

$$x_1 = 0, \qquad y_1 = L \cos \alpha \text{ and } x_2 = L \sin \alpha, \qquad y_2 = 0$$

The acceleration components of P_1 and P_2 are

$$\begin{aligned} \ddot{x}_1 &= 0, \qquad \ddot{y}_1 = -L(\dot{\alpha}^2 \cos \alpha + \ddot{\alpha} \sin \alpha) \\ \ddot{y}_2 &= 0, \qquad \ddot{x}_2 = L(-\dot{\alpha}^2 \sin \alpha + \ddot{\alpha} \cos \alpha) \end{aligned} \tag{4-50}$$

The value of $\dot{\alpha}^2$ may be found from the given relation

$$\ddot{\alpha} = K \sin \alpha \tag{4-51}$$

provided that we replace $\ddot{\alpha}$ by its equivalent expression $\dot{\alpha}\, d\dot{\alpha}/d\alpha$ and then integrate the resulting equation. If we put $\alpha = \beta$ when $\dot{\alpha} = 0$ (initial conditions), we find

$$\dot{\alpha}^2 = 2K\,(\cos \beta - \cos \alpha) \tag{4-52}$$

Substituting the values of the coordinates and acceleration components of P_1 and also the values of $\dot{\alpha}^2$ and $\ddot{\alpha}$ in Eqs. (4-43) gives the following expressions for the coordinates of the acceleration center:

$$\begin{aligned} x &= \frac{\dot{\omega} L\, K}{B}(2 \cos \beta \cos \alpha - 2 \cos^2 \alpha + \sin^2 \alpha) \\ y &= L \cos \alpha - \frac{\omega^2 L\, K}{B}(2 \cos \beta \cos \alpha - 2 \cos^2 \alpha + \sin^2 \alpha) \end{aligned} \tag{4-53}$$

Two similar equations may be written by using the corresponding data for P_2:

$$\begin{aligned} x &= L \sin \alpha + \frac{\omega^2 L\, K}{B}(3 \sin \alpha \cos \alpha - 2 \cos \beta \sin \alpha) \\ y &= \frac{\dot{\omega}^2\, L\, K}{B}(3 \sin \alpha \cos \alpha - 2 \cos \beta \sin \alpha) \end{aligned} \tag{4-54}$$

If we now eliminate $\dot{\omega}/B$ and ω^2/B from the four equations of Eqs. (4-53) and (4-54), the resulting expressions are:

$$\begin{aligned} x &= \frac{L \sin \alpha\,(2 \cos \beta \cos \alpha - 2 \cos^2 \alpha + \sin^2 \alpha)}{4(\cos \beta - \cos \alpha)^2 + \sin^2 \alpha} \\ y &= \frac{L \sin^2 \alpha\,(3 \cos \alpha - 2 \cos \beta)}{4(\cos \beta - \cos \alpha)^2 + \sin^2 \alpha} \cdot \end{aligned} \tag{4-55}$$

From these equations the coordinates of the instantaneous acceleration center may be found for any position of the ladder, provided that the initial angle β is known.

The fixed acceleration centrode may be found by eliminating the angle α from Eqs. (4-55). It is easier, however, to obtain a graphical solution. Putting $\beta = 0$ and using several values for α gives data for obtaining the fixed centrode. The dotted curve in Fig. 70 is the fixed centrode.

The moving acceleration centrode is a circle drawn about the ladder P_1P_2 as a diameter. The proof for this statement will be left to the student.

Problems.—1. A cylinder rolls down an inclined plane under the influence of gravity. Assuming that the acceleration of its center is $kg \sin \alpha$, where α is the angle which the plane makes with the horizontal, find the coordinates of the instantaneous acceleration center for any position of the cylinder.

2. A circular disk is rotating with constant angular acceleration about a fixed horizontal axis. Find the position of the instantaneous acceleration center.

3. A crank is rotating about a fixed axis with constant angular velocity. One end of a connecting rod is attached to the crank pin and the other is guided along a straight line which passes through the axis of rotation of the crank. Find the instantaneous acceleration center for any position of the connecting rod.

4. If the acceleration of a particle P is always directed toward a fixed point O, show that the vector OP describes equal areas in equal intervals of time (Kepler).

5. Two bodies A and B are released at the same instant. B falls in a vertical line from rest. A has an initial velocity U, the direction of which passes through the initial position of B and makes angle α with the horizontal line. Neglecting friction, prove that the two bodies will meet.

6. A cord is unwound at a constant rate from a fixed circle. If kept tight constantly, find the velocity and the acceleration of its moving end.

7. A particle is moving along a curved path. Its position in the path is given by a coordinate s measured from some fixed point in the curve. The coordinate-time relation is $s = 2t^4 + 3t^3 - t + 5$. Find the tangential acceleration when $t = 4$ sec.

8. A particle is moving with constant acceleration along a straight line. At a certain instant its speed is $+ 1{,}500$ cm. per second and 1 min. later its speed is $-1{,}500$ cm. per second. What is its acceleration?

9. A particle has constant acceleration and moves in a straight line. Its speed is $+1{,}000$ cm. per second at a certain place, and $+200$ cm. farther on in the path its speed is 350 cm. per second. Find its acceleration.

10. A particle is being whirled in a circular path of constant radius with an angular acceleration. By differentiating the equation $V = \omega \times r$ with respect to the time, find expressions for J_r and J_γ. Is either of these components constant?

11. Assuming that the period of the moon's motion about the earth is 27.3 days and that its distance from the earth is 240,000 miles, find the acceleration of the moon.

12. A particle is moving along a straight line. Its position is given by the vector equation $\boldsymbol{r} = \boldsymbol{d} + \frac{1}{2} at^2 \boldsymbol{s}_1$, where $\boldsymbol{r}$ is the vector which defines the position of the particle, $\boldsymbol{d}$ is a constant vector, a is a constant scalar, t is the time, and $\boldsymbol{s}_1$ is a unit vector in the line of motion. Interpret the equation geometrically. Find the velocity and the acceleration of the particle.

13. A projectile leaves the gun with a speed V. Prove that the angle which the direction of the velocity makes with the horizontal is 45 deg. for a maximum horizontal range. (Neglect friction.)

14. A golf ball is driven from a tee with a speed of 200 ft. per second. The initial velocity of the ball makes an angle of 20 deg. with the horizontal. The ground slopes down from the tee at an angle of 5 deg. from the horizontal plane. Find the distance from the tee to the point where the ball hits the ground. (Neglect friction and assume $g = 32.2$ ft. sec.)

15. A wheel 4 ft. in diameter is rolling down an inclined plane which makes an angle of 30 deg. with the horizontal. Find the acceleration of a particle which is at the highest point on the wheel 2 sec. after the wheel commenced to roll.

CHAPTER V

HARMONIC MOTION

5-1. Definition of Simple Harmonic Motion.—In the preceding chapter we have developed the equations of motion for a number of cases in which the value of the acceleration was constant or was expressed as some simple function of a coordinate. Because of its relative importance, one such case has been reserved for special consideration.

We find throughout the entire field of physics many arrangements in which bodies are so bound by elastic forces that, if a displacement is produced in some way or other, a vibratory motion follows. Forces of resistance, which are present, cause the initial endowment of energy to be gradually converted into heat, except for a portion of the energy which may be radiated in some form or other. The vibrations or oscillations take place about the equilibrium position as a mean position. Such motions are to be found in a vibrating tuning fork, a simple pendulum, a mass hung by a light spring, and in others of a similar nature.

The acceleration in such motions is not constant either in magnitude or in direction but changes in a cyclic manner as the motion progresses. The acceleration is expressible in terms of a coordinate, which measures the displacement of the body from the equilibrium position, together with a constant that includes inertia factors and quantities involving the elastic forces. The direction of the acceleration is always toward the equilibrium or rest position.

If we let x be a coordinate which measures the displacement of a body from its rest position, and let K^2 be a constant, then the acceleration J_x in the type of motion under consideration may be expressed as follows:

$$J_x = -K^2 x \mathbf{x}_1 \tag{5-1}$$

where $\mathbf{x}_1$ is a unit vector in the line of x. A motion in which the acceleration may be expressed by the foregoing equation is called *simple harmonic motion.*

As found in nature, such motions are nearly always accompanied by resistance factors which sooner or later bring the motion to rest. As a preliminary study to these more important cases the resistance effects are to be excluded from present considerations but they will be taken up at a more appropriate place later. This procedure will make the study of these motions less difficult.

5-2. The Fundamental Equations in Simple Harmonic Motion. The simplest approach to the study of simple harmonic motion is to be obtained from a consideration of the projection of the uniform motion of a particle, moving in a circular path, upon a diameter of the circle. Given the particle P (Fig. 71) which is moving along the circumference of a circle with constant linear speed V. We wish to prove that the motion of the projection of P, upon a diameter of this circle, *viz.*, the motion of Q, is simple harmonic motion.

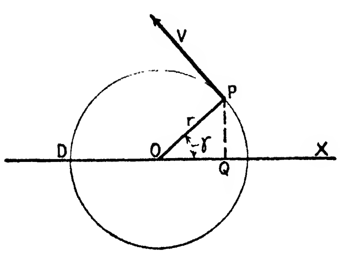

FIG. 71.

The radius r connecting P to the center of the circle O will rotate with constant angular velocity ω about O; hence $V = \omega r$. The speed of $Q(V_Q)$ will be the projection of V upon the diameter DX along which Q moves. Hence the magnitude of V_Q is $-V \sin \gamma$, where γ is the angle which r makes with OX, and its direction will always be parallel to OX. Obviously V_Q will be negative if γ lies between O and π and positive if γ has a value greater than π but less than 2π.

The acceleration of $Q(J_Q)$ may be found by differentiating the expression for the speed of Q. Since V is constant, this gives $-\omega V \cos \gamma$, where ω is written for $d\gamma/dt$. But the magnitude of V is ωr; hence

$$J_Q = -r\omega^2 \cos \gamma$$

Obviously this result could have been obtained by projecting the acceleration of P, the magnitude of which is $-r\omega^2$ and whose direction is inward along r, on to the line of OX.

The magnitude of J_Q changes with $\cos \gamma$. It will be a maximum when $\gamma = 0$ or π, and a minimum for $\gamma = \pi/2$ or $3\pi/2$.

The direction of J_Q will also depend upon the cos γ, since ω^2 and r are always positive.

If the distance of Q from O is designated by x, then the equation for this motion is typified by the following form:

$$\begin{aligned}\frac{d^2x}{dt^2} &= -r\omega^2 \cos \gamma \\ &= -K^2x \qquad (5\text{-}2)\end{aligned}$$

since $x = r \cos \gamma$. This equation is a general expression and is the best and most concise definition of simple harmonic motion. The factor K is introduced here for the sake of generality although in the particular case it is equal to ω.

The equations for the velocity of the point Q in terms of x and for the displacement x in terms of the time may be obtained by integration of the general equation. The general solution of this type of differential equation is of the form

$$x = A \cos Kt + B \sin Kt \qquad (5\text{-}3)$$

since the second time derivative of this equation yields

$$\frac{d^2x}{dt^2} = -K^2 (A \cos Kt + B \sin Kt)$$

where A and B are constants of integration which may be determined in any particular case by the use of the initial conditions.

If at the time $t = 0$, $x = x'$, and $V_x = U$, where V_x is the speed of Q in the line OX, we could evaluate A and B provided we had an equation expressing V_x as a function of t. Such an equation is obtained by a differentiation of Eq. (5-3) which gives

$$V_x = -AK \sin Kt + BK \cos Kt$$

From this equation together with Eq. (5-3) and the initial conditions, we find that

$$A = x' \text{ and } B = \frac{U}{K}$$

Hence

$$x = x' \cos Kt + \left(\frac{U}{K}\right) \sin Kt \qquad (5\text{-}4)$$

$$V_x = -x'K \sin Kt + U \cos Kt \qquad (5\text{-}5)$$

Since the simple harmonic motion of Q may be regarded as the projection of the uniform circular motion of some particle

P upon a diameter of the circle, it is convenient to use the circular motion of P to interpret some of the constants of Eqs. (5-4) and (5-5). For the time $t = 0$ we may write γ_0 for the particular value of the angle γ. Hence we may put

$$x' = r \cos \gamma_0$$

from which it is seen that r not only is the radius of the circle but also is the amplitude of the motion of Q. The angle γ_0 is called the *epoch* angle.

The other constant $B = U/K$ may also be expressed in terms of r, the amplitude of Q, and the epoch angle. Since U is the velocity of Q at the time $t = 0$, then

$$U = -\omega r \sin \gamma_0 \qquad (5\text{-}6)$$

But ω is equal to K; hence we have

$$\frac{U}{K} = -r \sin \gamma_0$$

Substituting the value of x' and of U/K, as expressed in terms of r and the epoch angle γ_0, in Eq. (5-4) gives

$$x = r \cos \gamma_0 \cos Kt - r \sin \gamma_0 \sin Kt \qquad (5\text{-}7)$$
$$= r \cos (Kt + \gamma_0) \qquad (5\text{-}8)$$

By making similar substitutions in Eq. (5-5), we obtain

$$V_x = -rK \sin (Kt + \gamma_0) \qquad (5\text{-}9)$$

The angle $(Kt + \gamma_0)$ is called the *phase* angle. Equation (5-8) again reveals the fact that x may vary from $+r$ to $-r$. It is also to be noticed that Kt (which is equal to ωt) represents the angle γ through which r turns in the time t.

An equation which expresses V_x as a function of x is next to be obtained. This may be found by eliminating t from Eqs. (5-8) and (5-9) or it may be obtained by an integration of the general equation [Eq. (5-2)] in which we may substitute for d^2x/dt^2 its alternative from $V_x \, dV_x/dx$. Making this substitution gives

$$V_x \frac{dV_x}{dx} = -K^2 x$$

Integrating this equation and evaluating the constant of integration gives

$$V_x^2 = K^2 (r^2 - x^2) \qquad (5\text{-}10)$$

This equation and Eqs. (5-8) and (5-9) are very useful in mathematical descriptions of simple harmonic motion.

5-3. Geometrical Interpretation of the Fundamental Equations.—It is important for the student in physics to learn to regard mathematical equations in physics as symbolic expressions for physical relations. There is always the tendency to look at such equations as mere algebraic expressions rather than to recognize in them a more important meaning. The habit of reading physical meaning into the equations where possible is one which affords a check upon the validity or accuracy of the expressions. Errors in writing equations, especially in selecting incorrect signs for the quantities, are frequently avoided by keeping the physical picture constantly in mind.

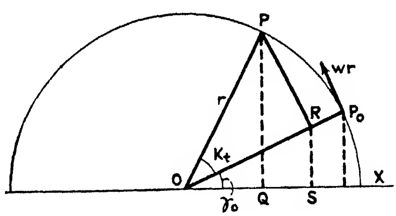

FIG. 72.

To illustrate the value of these statements, the material in the preceding section may be used to advantage. Consider Eq. (5-6) which expresses the velocity of Q at the position where $\gamma = \gamma_0$. It is evident that ωr is the velocity of the point P in its circular path as shown in the diagram (Fig. 72) at P_0. The projection of this velocity into the line of OX gives the velocity U of Q. But it is possible to overlook the fact that the sign of U must be negative. Hence we see that

$$U = -\omega r \sin \gamma_0$$

and it is to be noticed that the expression is correct for all possible values of γ_0.

Next let us consider the geometrical meaning of Eq. (5-7); *viz.*,

$$x = r \cos Kt \cos \gamma_0 - r \sin Kt \sin \gamma_0$$

This equation shows that x, the coordinate of Q, is made up of two parts, whose difference is equal to x. Since this is a scalar equation, each term of the right-hand member must be in the line of OX. The first term of the right-hand member shows that r is projected twice, first into the line of OP_0 by multiplying by $\cos Kt$ and then the resulting line (OR in the diagram) is projected into the line of OX by multiplying by $\cos \gamma_0$, which finally gives OS. Similarly the second term may be shown to be a projection of r, first into a line perpendicular to OP_0, *viz.*, PR, and then this line is projected into OX giving QS. Obviously

$$x \text{ (or } OQ) = OS - QS$$

In a similar manner the significance of Eqs. (5-8) and (5-9) may be made apparent. These will be left to the student as well as the equation

$$V_x = -Kr\,(\sin Kt \cos \gamma_0 + \cos Kt \sin \gamma_0)$$

from which Eq. (5-9) was derived. Equation (5-10) should also be examined for its geometrical meaning.

5-4. An Illustration—the Simple Pendulum.—The simple pendulum consists of a heavy bob suspended by a light wire from a rigid support. The component of the acceleration, due to gravity (g), which is responsible for the motion is $-g \sin \alpha$ and is tangent to the circular path (Fig. 73) of the bob, where α is the angle of displacement of the bob from its rest position. The direction of $-g \sin \alpha$ is always toward O, the rest position. If the amplitude of the motion is small, $\sin \alpha$ may be replaced by x/L, where x is the displacement in the circular path and is measured from O, and L is the "length" of the simple pendulum.

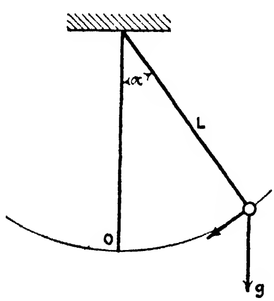

FIG. 73.

The differential equation is therefore

$$\frac{d^2x}{dt^2} = -\left(\frac{g}{L}\right)x \tag{5-11}$$

Hence for the limitation imposed the motion is simple harmonic. The other equations which describe the motion, *i.e.*, those corresponding to Eqs. (5-8), (5-9), (5-10), may be readily found from assigned initial conditions by the methods illustrated in Sec. **5-2.**

5-5. The Period in Simple Harmonic Motion.—The period of motion of the kind under consideration is the time required for an entire cycle of the motion to be completed. This is the time required for Q (Fig. 71) to go from any given position back to its starting point after having traversed the entire path of its oscillation. The period of the motion may be determined by an examination of the expressions for x or V_x. For the sake of brevity let us assume that the epoch angle in these equations is equal to zero. Equations (5-5) and (5-7) contain either cos (Kt) or sin (Kt). Since both of these functions of Kt are periodic and contain the same angle (Kt), we may determine the period of the motion by an examination of only one of them.

The sin (Kt) will have the same value when Kt is increased by 2π or any integral multiple of 2π, *i.e.*, $2\pi n$. Hence

$$\sin Kt = \sin (Kt + 2\pi n)$$
$$= \sin K\left(t + \frac{2\pi n}{K}\right)$$

It is obvious that $2\pi/K$ is the increase of t necessary to effect a return of x or V_x to the same value. Hence the period T of the motion will be expressed by

$$T = \frac{2\pi}{K} \tag{5-12}$$

This result could have been obtained by an examination of P's motion in the circle (Fig. 71), since the period of the motion of P is identical with that of Q. The period of P's motion is the time required for it to go once around the circle; hence

$$T = \frac{2\pi r}{V}$$
$$= \frac{2\pi}{K}$$

since $V = \omega r$ and $\omega = K$.

5-6. Circular Harmonic Motion.—The motion of the particle Q of Sec. **5-2** was confined to a straight line DX and may be described therefore more completely by the term translational simple harmonic motion. Corresponding to the case of simple harmonic motion in translation, we have in rotation an analogous motion in which the particle moves along a circular path with angular acceleration which is proportional to and oppositely directed to the angular displacement γ, measured from the rest position. This condition is appropriately expressed in differential form by the equation

$$\frac{d^2\gamma}{dt^2} = -K^2\gamma \tag{5-13}$$

in which K^2, as before, is the proportionality factor. It is to be noticed that the magnitude of K^2 is that of the angular acceleration at the position where the angle γ is equal to 1 radian.

Motion of this sort is seen to exist in the torsional pendulum in which a disk is supported by a stiff spring steel wire so that the rotation may take place about an axis coincident with that of the wire. The upper end of the wire is clamped to some rigid support.

The three equations which express the motion may be obtained by direct integration of Eq. (5-13). The solution of this equation is of the form

$$\gamma = C \cos Kt + D \sin Kt \tag{5-14}$$

in which C and D are constants of integration.

The expression for the angular velocity ω may be obtained by differentiation of Eq. (5-14), which gives

$$\omega = -CK \sin Kt + DK \cos Kt \tag{5-15}$$

The constants C and D may be evaluated by putting $\gamma = \gamma_0$ and $\omega = \omega_0$ at the time $t = 0$, which gives

$$C = \gamma_0 \qquad \text{and} \qquad D = \frac{\omega_0}{K}$$

Substituting these values in the foregoing equations gives

$$\gamma = \gamma_0 \cos Kt + \left(\frac{\omega_0}{K}\right) \sin Kt \tag{5-16}$$

$$\omega = -\gamma_0 K \sin Kt + \omega_0 \cos Kt \tag{5-17}$$

The third equation of the motion, *i.e.*, that which expresses the relation between γ and ω, may be obtained by eliminating the time from the two equations already obtained or by replacing $d^2\gamma/dt^2$ in Eq. (5-13) by its alternative form $\omega d\omega/d\gamma$ and then integrating. Using the latter method gives

$$\omega \frac{d\omega}{d\gamma} = -K^2\gamma \tag{5-18}$$

$$\begin{aligned} \omega^2 &= -K^2\gamma^2 + 2E \\ &= \omega_0^2 + K^2(\gamma_0^2 - \gamma^2) \end{aligned} \tag{5-19}$$

where E is a constant of integration and is evaluated by using initial conditions.

5-7. Compound Harmonic Motion—Periods Alike.—Many of the periodic motions encountered in various fields of physics, especially in the wave motions found in the study of mechanics, sound, light, and electricity, are not simple harmonic motion but are related to simple harmonic motion in that they may be expressed in terms of two or more simple harmonic motions combined in various ways. The study of wave motion particularly is best approached by seeing just what sort of result is obtained when two linear simple harmonic motions are superimposed. The directions of the two components may be parallel to each other or may make any angle with each other. By commencing with the simplest arrangements and then by making the combinations more general we may gain a better appreciation of the more complex periodic motions and their equivalent composition.

A linear simple harmonic motion is completely described when the *amplitude, period, epoch angle, and rest position* are given together with the *line* along which the motion takes place. If two such motions are to be combined, we must specify all of these factors for each component in order to determine the resulting motion. It is not to be inferred that the motion obtained from any or all arrangements will necessarily produce a simple harmonic motion. The developments given below are selected to show what relations are necessary in order that linear harmonic motion may result from the combination of two component harmonic motions and also to show the conditions of combination which do not yield a resulting harmonic motion.

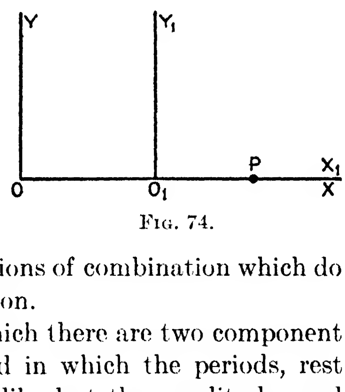

FIG. 74.

The first case selected is one in which there are two component linear simple harmonic motions and in which the periods, rest positions, and lines of motion are alike but the amplitudes and epoch angles are different. In order to visualize the process of combination, let us select a moving coordinate system $X_1O_1Y_1$ (Fig. 74) and assign to it a simple harmonic motion subject to the given characteristics of one of the two given components. The system $X_1O_1Y_1$ must always be coplanar with the fixed reference system XOY and must move so that one of its lines, say the O_1X_1-axis, shall be coincident with the OX-axis of the fixed reference system. Since all points of the moving system will then have simple harmonic motion in the fixed reference system, we may select the origin O_1 to be the particular point upon which attention is directed, and let it have simple harmonic motion about the point O as center. Let us now select another point P and assign to it a simple harmonic motion in the moving system subject to the values of the second component and have it move along the line O_1X_1 with O_1 as center. The motion of P referred to the fixed reference system will then represent the combination of the motions of the two components.

To determine the nature of this motion, we may first express the coordinate x of P in the reference system in terms of x_1, the coordinate of O_1 in the reference system, and x_2 the coordinate of P in the moving system. Obviously

$$x = x_1 + x_2 \tag{5-20}$$

Using Eq. (5-8) as a type form, we may express x_1 and x_2 in terms of the amplitudes and epoch angles, which may be called r_1, r_2 and γ_1, γ_2, respectively. Hence

$$x_1 = r_1 \cos (Kt + \gamma_1) \tag{5-21}$$

$$x_2 = r_2 \cos (Kt + \gamma_2) \tag{5-22}$$

in which $2\pi/K$ is the common period of the two motions. An expression for x may now be obtained by adding x_1 and x_2, which gives

$$x = r_1 \cos (Kt + \gamma_1) + r_2 \cos (Kt + \gamma_2) \tag{5-23}$$

$$= (r_1 \cos \gamma_1 + r_2 \cos \gamma_2) \cos Kt - (r_1 \sin \gamma_1 + r_2 \sin \gamma_2) \sin Kt \tag{5-24}$$

In order to simplify this equation it is best first to determine the meaning of the right-hand member. This information may be obtained by using the geometrical relations given by the auxiliary circular motions. The relations are shown in Fig. 75. In the reference system XOY, let P_1 and P_2 move with constant angular speed K in circular paths whose centers are both at O and whose radii are r_1 and r_2, respectively. The auxiliary points P_1 and P_2 are shown in their positions for the time $t = 0$; hence γ_1 and γ_2 are the angles which r_1 and r_2 make, respectively, with OX. If we put $t = 0$ in Eq. (5-23), it is readily seen that x is the projection upon OX of the diagonal OP of the parallelogram formed upon r_1 and r_2 as sides. The length of the diagonal is designated by the letter r in the figure.

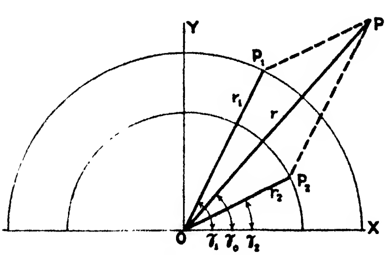

FIG. 75.

The angular velocities of P_1 and P_2 about O are the same, *viz.*, K; hence the angle between r_1 and r_2 will remain constant during the entire motion. The line r will therefore be constant in length and will rotate about O with r_1 and r_2 as though attached to them. If we write γ_0 for the angle which r makes with OX at the time $t = 0$, it is obvious that

$$r \cos \gamma_0 = r_1 \cos \gamma_1 + r_2 \cos \gamma_2 \tag{5-25}$$

and is a constant. Similarly it is readily seen that

$$r \sin \gamma_0 = r_1 \sin \gamma_1 + r_2 \sin \gamma_2 \tag{5-26}$$

and is the projection of r upon OY.

By substituting these expressions for $r \cos \gamma_0$ and $r \sin \gamma_0$ in Eq. (5-24) the desired expression for x may be obtained. Hence

$$\begin{aligned} x &= r \cos \gamma_0 \cos Kt - r \sin \gamma_0 \sin Kt \\ &= r \cos (Kt + \gamma_0) \end{aligned} \tag{5-27}$$

This result expresses x, the coordinate giving the sum of the coordinates of the two component simple harmonic motions, in a form which shows that the combination gives a resulting motion which, under the imposed limitations, is also simple harmonic. Since x is the projection of P upon OX and P moves in a circular path of radius r with an angular velocity K, it is obvious from the geometrical relations alone that the motion given by Eq. (5-27) is simple harmonic. The period of this compound motion is $2\pi/K$, the same as that of the two components.

As a general conclusion it may be seen that this result may be extended to include any number of components as long as their periods, rest positions, and lines of motion are the same. The amplitude of the motion of the resulting simple harmonic motion will be dependent upon the amplitudes and epoch angles of the several components.

A further conclusion may be drawn. Any simple harmonic motion may be regarded as made up of several component simple harmonic motions, provided that the periods, rest positions, and lines of motion are the same as those of the given simple harmonic motion. The epoch angles and amplitudes of the components may be selected in so far as the selected values satisfy equations similar to Eqs. (5-25) and (5-26).

Problems.—1. Under what circumstances would it be possible to combine two linear harmonic motions having the same periods, lines of motion, and centers so that the resulting motion would have an amplitude which would be less than the amplitude of either component?

2. Make diagrams indicating the geometrical relations shown by Eqs. (5-23) and (5-27).

3. Express the velocity of the projection of point P (Fig. 75) in terms of the velocities of the projections of P_1 and P_2, all of the projections being upon OX.

4. If the rest positions of the two simple harmonic motions are not coincident but the periods and lines of motion are the same, would the combination of these motions be simple harmonic?

5. Find the amplitude of the harmonic motion which results from combining three simple harmonic motions having the same period ($2\pi/K = 10$ sec.) and the same line of motion but the amplitudes are 3, 4, 5 cm., respectively. The phase difference between the first and second is 45 deg. and that between the second and third is 30 deg.

6. Given a simple harmonic motion in which the displacement is expressed by the equation $x = 5 \cos (Kt + 30°)$. Find the two components which when combined will satisfy the given equation if the phase difference is 45 deg. and one of them has an epoch angle of 45 deg.

5-8. Combination of Simple Harmonic Motions—Periods Unlike.—The motion resulting from the combination of two simple harmonic motions having the same center and line of motion, but with different periods, epoch angles, and amplitudes, is to be expressed and its meaning examined. The general procedure is similar to that used in the preceding section.

The displacements along the line of motion, measured from the rest position, may be expressed as follows:

$$x_1 = r_1 \cos (K_1t + \gamma_1); \qquad x_2 = r_2 \cos (K_2t + \gamma_2) \qquad (5\text{-}28)$$

in which the symbols are used as in Sec. **5-7** above and the selection of subscripts serves to identify the component to which the quantities belong. The resultant displacement x of the combination is

$$x = r_1 \cos (K_1t + \gamma_1) + r_2 \cos (K_2t + \gamma_2) \qquad (5\text{-}29)$$

Since the periods of the components are unlike, the lines r_1 and r_2 of the auxiliary motions will rotate at different angular velocities and hence the angle between them, which we may call β, is not constant. The angle β gives the phase difference between the two motions. It may be expressed in terms of the epoch angles and the periods by the following equation:

$$\beta = (K_2t - K_1t) + (\gamma_2 - \gamma_1) \qquad (5\text{-}30)$$

By means of this equation we may eliminate from Eq. (5-29) either phase angle. Since it makes no difference in the interpretation of the final result which is eliminated, we may write

$$K_2t + \gamma_2 = \beta + (K_1t + \gamma_1) \tag{5-31}$$

and substitute this value in Eq. (5-29), which gives

$$x = r_1 \cos (K_1t + \gamma_1) + r_2 \cos (K_1t + \gamma_1 + \beta) \tag{5-32}$$

Expanding this expression and rearranging terms gives

$$x = (r_1 + r_2 \cos \beta) \cos (K_1t + \gamma_1) - r_2 \sin \beta \sin (K_1t + \gamma_1) \tag{5-33}$$

In order to simplify this expression, we may use r to represent the diagonal of the parallelogram formed upon r_1 and r_2 as sides and express the angle between r_1 and r by γ_0. We may therefore write

$$r_1 + r_2 \cos \beta = r \cos \gamma_0 \qquad r_2 \sin \beta = r \sin \gamma_0 \tag{5-34}$$

Introducing these expressions into Eq. (5-33) gives

$$\begin{aligned} x &= r \cos \gamma_0 \cos (K_1t + \gamma_1) - r \sin \gamma_0 \sin (K_1t + \gamma_1) \\ &= r \cos (K_1t + \gamma_1 + \gamma_0) \end{aligned} \tag{5-35}$$

An examination of this equation shows that the resulting motion is not simple harmonic motion. In order for the motion to be simple harmonic, it must satisfy the requirements expressed by the fundamental equation [Eq. (5-2)]. In the expression, γ_0 is not a constant and hence its derivative will appear in the expression giving the acceleration d^2x/dt^2. Furthermore, r is not constant, for its magnitude is the length of the diagonal of the parallelogram formed on r_1 and r_2 with the variable angle β between them. The shape of the parallelogram changes even though the lengths of the sides do not.

Problems.—1. Assign values to the amplitudes, periods, and epoch angles for two simple harmonic motions and, using Eq. (5-29), make a plot of x as a function of the time. Is the motion periodic, and, if so, what is the period?

2. Given two simple harmonic motions, one of which has a period that is twice the other, with the centers and lines of motion the same. Will the combination of the two be periodic? Assuming that the amplitudes are alike, what will be the resulting wave form (the curve giving the resultant displacement as a function of the time) if their phase difference is 30 deg.? Determine also the wave form for phase differences of 45, 60, and 90 deg.

5-9. Combination of Two Simple Harmonic Motions at Right Angles—Periods Alike.—The motion of a particle is to be investigated when obtained by combining two simple harmonic motions whose period and center are the same but whose lines of motion are at right angles to each other and whose amplitudes are, in general, unlike. The instantaneous displacements of the particles P_1 and P_2, having the assigned simple harmonic motions, are to be combined vectorially. The chief interest in such a combination lies in the nature of the path of the point situated at the end of the vector resulting from the vector sum of the displacements of the two components. In order to make the consideration one of general interest, a phase difference between the components is to be assigned. It will not be necessary to assume an epoch angle for each component.

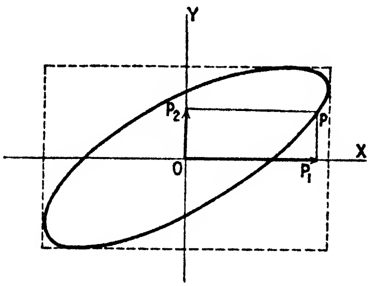

FIG. 76.

Given the reference system XOY (Fig. 76) with O the center of the component simple harmonic motions. Let

$$x = r \cos Kt$$
$$y = s \cos (Kt + \beta) \quad (5\text{-}36)$$

express the displacements of the two perpendicular components along the axes OX and OY, respectively, with the common period $2\pi/K$ and with r and s the amplitudes. Let β be the epoch angle for the component along OY, and let the epoch angle for the component OX be zero.

The coordinates of the point P, whose path is to be found, may be determined for any instant of time by a direct substitution of numerical values for the quantities in Eqs. (5-36) and from these values the path may be plotted. The character of the curve may be determined by finding the analytical expression for x and y. Such an analytical expression may be obtained by eliminating the time, or Kt, from the given equations [Eqs. (5-36)]. To do this we may solve the first equation for cos Kt and substitute its value in the second equation after it has been expanded in terms of the functions of single angles. The following steps indicate the process:

$$\cos Kt = \frac{x}{r}$$

$$\frac{y}{s} = \cos Kt \cos \beta - \sin Kt \sin \beta$$
$$= \left(\frac{x}{r}\right) \cos \beta - \sqrt{1 - \frac{x^2}{r^2}} \sin \beta$$

Rearranging the terms and squaring to remove the radical gives

$$\frac{x^2}{r^2} + \frac{y^2}{s^2} - \left(\frac{2xy}{rs}\right) \cos \beta = \sin^2 \beta \tag{5-37}$$

The nature of the curve represented by this equation is to be investigated. The fact that it contains no first-degree terms shows that the center of the curve is at the origin. The form alone does not indicate whether it is an ellipse or a hyperbola. The curve must be confined within a rectangle having its center at O and sides parallel to the axes and equal to $2r$ and $2s$. This fact prohibits the curve from being a hyperbola. It must therefore be an ellipse. It is possible for the conditions to be such that the curve degenerates into a straight line. This case will be discussed below.

An analytical proof that the curve under consideration is an ellipse is to be obtained. The presence of the term in xy indicates that the axes of the curve make some angle, say θ, with the reference axis OX. From a well-known theorem in analytical geometry, if the equation has the general form

$$Ax^2 + 2h\,xy + By^2 = C$$

the angle θ may be expressed in terms of the coefficients by the following equation:

$$\tan 2\theta = \frac{2h}{(A - B)} \tag{5-38}$$

Hence for any particular case the angle θ may be found. If now Eq. (5-37) is transformed to a new set of axes having the same origin but inclined to the given axis by an angle θ, the term in xy will vanish.

If the coefficients of x^2 and y^2 in the transformed equation are designated by A' and B', respectively, then by means of the relations

$$\begin{aligned} A' + B' &= A + B \\ A'B' &= AB - h^2 \end{aligned} \tag{5-39}$$

the coefficients in the new equation may be obtained. If the coefficients of x^2 and y^2 in the new equation are both positive, the equation is an ellipse; if one of them is negative, the equation represents a hyperbola. The sign of the quantity $A'B'$ is therefore a criterion. From the equation for the curve under consideration [Eq. (5-37)] we may determine the sign of $A'B'$ by using the relation given by Eq. (5-39). Substituting the values of the coefficients gives

$$
\begin{aligned}
A'B' &= \frac{1}{r^2s^2} - \frac{\cos^2 \beta}{r^2s^2} \\
&= \frac{\sin^2 \beta}{r^2s^2} \qquad (5\text{-}40)
\end{aligned}
$$

which must always be positive and hence the equation is an ellipse, as was shown above by other considerations. If desired, the equation for the ellipse referred to the new axes could be determined but it would be a cumbersome equation to write and no particular advantage would accrue from an examination of it.

Under the given conditions the resulting curve is an ellipse and therefore the motion resulting from the combination of the two component harmonic motions, which have the imposed limitations, is called *elliptical harmonic motion.*

If the angle which denotes the phase difference between the components is zero, then, by referring to the diagram, it may be seen that the points P_1 and P_2 will be at their extreme positions at the same instant and will move toward O at such rates that the point P must move along a diagonal of the rectangle. That this is true may be shown analytically by observing that under this limitation the ratio of x/y will be constant and equal in magnitude to r/s [see Eqs. (5-36)].

Problems.—1. Using the notation of the preceding section, make a plot which will show the curves for the elliptical motion of a point when the amplitudes of the components are $r = 3$ and $s = 2$ and when the phase difference has the following values: 0, 30, 60, and 90 deg. Find the position for the axes of the ellipse in each case.

2. Given two perpendicular simple harmonic motions with the same period and center but with different amplitudes and with a phase difference not zero. Express the path in polar coordinates.

3. Under what conditions could the ellipse of the preceding section become a circle?

4. How may the direction of motion of the point around the ellipse in elliptical harmonic motion be determined?

5-10. Lissajous Curves.—A necessary limitation in elliptical harmonic motion is that the periods of the components must be the same. If two perpendicular component simple harmonic motions have the same center but their amplitudes, epoch angles, and periods are not alike, the path of the point which is found from the vector combination of the displacements of the two components is usually complicated. The curves formed by such combinations were first studied by Lissajous and have received his name as a consequence.

A rather simple apparatus for producing these curves experimentally with but little error is the "Blackburn's" pendulum. The apparatus may be arranged as shown in the diagram (Fig. 77). A rigid horizontal bar supports the heavy bob D by means of light wires AC and BC which are fastened to the bar at the points A and B. The wire CD is fastened to the wire ACB at C. The bob has a smooth vertical hole drilled in it so that some marking device such as a glass inking pen or soft-leaded pencil may slide in the hole with but little friction and thereby leave a trace of the motion upon a piece of paper placed horizontally below

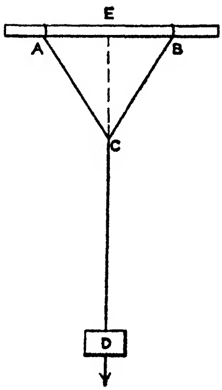

Fig. 77.

With this arrangement two simple harmonic motions at right angles to each other and with different periods may be compounded. One component is obtained by allowing the bob to swing in the plane of the diagram. Its period will be $2\pi\sqrt{CD/g}$. Its amplitude may be varied but, if made too large, the motion is not simple harmonic. The other component is perpendicular to the plane of the diagram and has a period equal to $2\pi\sqrt{DE/g}$. In addition to the fact that the motion of a pendulum is only approximately simple harmonic, as was shown in Sec. **5-4**, there is a damping factor in the motion due to unavoidable resistance. This resistance shows its effect in the slowly decreasing amplitudes of both components and in the fact that the quadrilateral which encloses the curve is a distorted rectangle. Even with these defects, which are not serious, the resulting curves are very good illustrative material for a study of this sort of motion.

The curve shown in Fig. 78 was drawn by a compound pendulum similar to the one described. In making this curve the lengths of the pendulums were 100 and 90 cm., which would make the periods 2.01 and 1.90 sec., respectively.

From the curve itself it is not possible to determine the periods of the components. The ratio of the periods, however, may be found by a graphical method. This may be done in the following manner. Any two points on the curve, such as N and

M, are to be selected. These must not be too far apart or difficulties will be encountered. The lines *AB* and *CD*, which are perpendicular to each other and parallel to the lines of the component simple harmonic motions, are to be drawn as shown in the diagram. The line *AB* represents the diameter of the auxiliary circle (with center at *Q*) along which the point *P* moves subject to the characteristics of one component. The other line

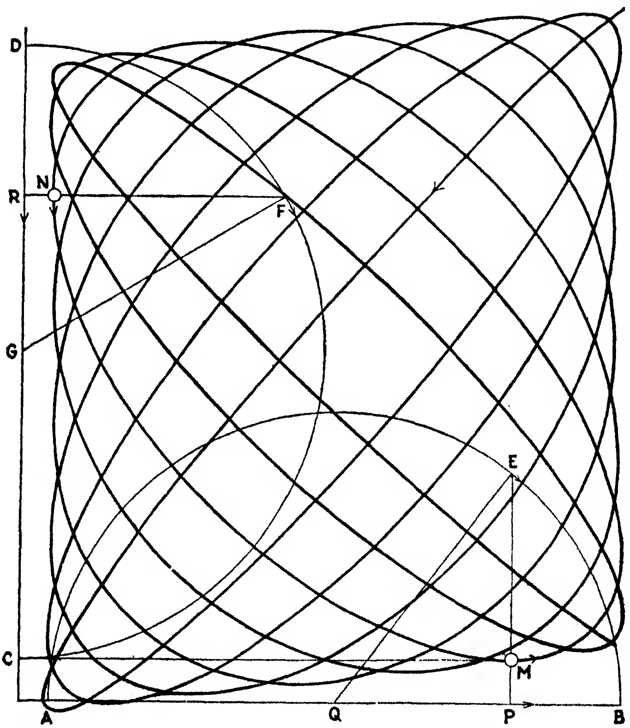

FIG. 78.

CD is the diameter of the auxiliary circle (with center at *G*) with the point *R* describing the motion of the second component. *E* and *F* are the points in the auxiliary circles whose projections upon *AB* and *CD*, respectively, are *P* and *R*.

At the particular instant when the point describing the Lissajous curve is at the point *N*, the corresponding positions of the points in the four other paths *CD*, *DFC*, *AB*, and *AEB* are *R*, *F*, *A*, and *A*, respectively. While *N* is moving to *M*, *R* moves to *C*, *F* to *C*, *A* along *AB* to *P*, and *A* along the arc of its circle to *E*. The time interval required for each of the five displacements is the same. Now, since the angular velocities of F and E are inversely proportional to the periods of the two components of the simple harmonic motions, the ratio of the angles CGF and AQE must also be the same as the inverse ratio of the periods. In the particular case these angles were found by measurement to be 121 and 128 deg., respectively. Their ratio is 0.95. The inverse ratio of the periods is likewise 0.95, which is a closer agreement than one would expect, for the error in the measurement of the angles might very well be 1 or 2 per cent.

If the ratio of the periods of the two components may be expressed as the ratio of two whole numbers, then the curve will be closed. If this ratio is assumed to be n/m, where n and m are prime to each other, then there will be n complete cycles of one component to m complete cycles of the other. For example, if the two periods are 8 and 2 sec., then there will be one oscillation of the first to four of the second.

Problems.—1. Determine the character of the curve made by combining two simple harmonic motions which are expressed by the equations $x = r \cos Kt$ and $y = r \cos 2Kt$.

2. Devise a graphical method for obtaining a Lissajous curve by using the auxiliary circular motions.

5-11. Fourier Series.—A Fourier series is a sum of sine and cosine terms whose amplitudes and periods are of such values that the series accurately represents some periodic function. It was first shown by Fourier in 1822 that, if a periodic function is everywhere finite and continuous, or if not continuous at least has only finite discontinuities and is single valued, the function could be expressed in a series of sine and cosine terms. The terms were called harmonics because the series was used very largely in the study of sound waves and because the frequencies were such that the components were one, two, three, or more octaves above the fundamental frequency of the wave being studied.

Any function is periodic if all of its values reoccur when the variable is increased by some constant (the period) or some integral multiple of the constant. Expressed symbolically, if

$$f(x + nP) = f(x)$$

where n is any integer and P is a constant, then $f(x)$ is a periodic function and P is its period. For example,

$$\sin(x + 2\pi n) = \sin x$$

hence $\sin x$ is a periodic function of x.

The variable may be a displacement, time, or the like. In the case of vibrating bodies we are concerned with the variation of displacement with time. In dealing with stationary waves the amplitude of the wave is to be expressed as a function of some coordinate. In either case, periodicity implies an exact repetition of a complete cycle at regular intervals of some variable.

If we had a number of tuning forks whose frequencies could be expressed in terms of n, $2n$, $3n$, etc., and we should strike them all at once, the resulting sound would have a wave form which would be periodic and which would depend upon the number of tuning forks, their amplitudes, and frequencies, and phase differences. The resulting wave form may be expressed in terms of a series of simple harmonic motions, provided that the characteristics of the components are known. Now when we know the graphical form of a single period of a wave and know also its period and can express this wave form mathematically as a function of the time or of the abscissa, we may then analyze the function by the method outlined by Fourier and determine the component harmonics.

Fourier expressed the general equation in the following terms:

$$\begin{aligned} f(x) = A_0 &+ A_1 \cos x + A_2 \cos 2x + \cdots + A_n \cos n x \\ &+ B_1 \sin x + B_2 \sin 2x + \cdots + B_n \sin n x \end{aligned} \tag{5-41}$$

in which

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \tag{5-42}$$

$$A_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx \tag{5-43}$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx \tag{5-44}$$

It will be noted that the periods of the trigonometric terms are 2π, $2\pi/2$, $2\pi/3$, . . . $2\pi/n$. In order to apply this series to any special case, the period of the given wave form must be made equal to 2π, if the components or the terms of the series are to be harmonics of the fundamental. This presents no difficulty, for it is merely a matter of selecting a suitable scale factor for the variable.

To show how the constants are evaluated, a concrete illustration will be used. Suppose the wave form consists of the isosceles triangle as shown in Fig. 79. The single triangle represents a complete cycle and is repeated at regular intervals. Now we must put $x = 2\pi$ at the point C. For the sake of simplicity in the illustration, let the angles opposite the equal sides in the triangle be 45 deg. This assignment makes the altitude of the triangle equal to π. From these relations it follows that from $x = 0$ to $x = \pi$, $f(x) = x$, and from $x = \pi$ to $x = 2\pi$, $f(x) = 2\pi - x$.

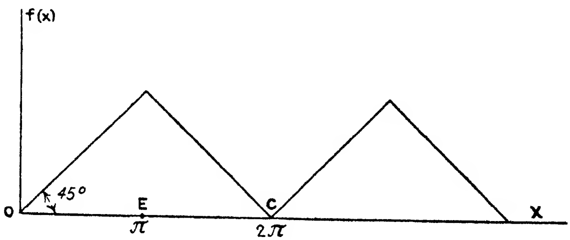

Fig. 79.

Because the expression for $f(x)$ changes at the point $x = \pi$, it will be necessary to replace the single integrals, used in the evaluation of the constants, by two integrals, one extending from 0 to π and the other from π to 2π, each to be used with the corresponding value for $f(x)$. The constants are determined by the following integrations:

$$\begin{aligned}
A_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \\
&= \frac{1}{2\pi}\int_0^{\pi} x dx + \frac{1}{2\pi}\int_{\pi}^{2\pi} (2\pi - x)\,dx \\
&= \frac{\pi}{2} \\
A_n &= \frac{1}{\pi}\int_0^{\pi} x \cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} (2\pi - x)\cos nx\,dx \\
&= -\frac{4}{\pi n^2} \text{ (and } n \text{ can only be odd)} \\
B_n &= \frac{1}{\pi}\int_0^{\pi} x \sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} (2\pi - x)\sin nx\,dx \\
&= 0 \text{ for all values of } n
\end{aligned}$$

If we substitute these values for the coefficients in Eq. (5-41), the resulting equation is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x \cdot \cdot \cdot - \frac{4}{\pi n^2}\cos nx \qquad (5\text{-}45)$$

in which n may have only odd values of the integer.

It is instructive to determine how closely the final equation represents the given function. This may be done in an illuminating fashion by plotting the terms and then adding the ordinates for several values of the abscissa and comparing the curve, drawn through these resulting summation points, with the original wave form. In Fig. 80, the first three terms are plotted but the summation points for four terms are indicated by the circles. It is evident that the series is rapidly convergent and,

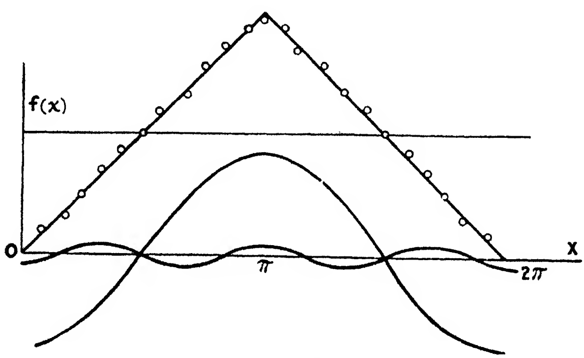

Fig. 80.

if a sufficient number of terms is taken, the function is accurately represented by the series.

Problems.—1. Determine the constants for expressing each of the following functions in a Fourier series.

	From $x = 0$ to $x = \pi$	From $x = \pi$ to $x = 2\pi$
a.	$f(x) = \pi/4$	$f(x) = -\pi/4$
b.	$f(x) = x$	$f(x) = x - 2\pi$
c.	$f(x) = \pi/2$	$f(x) = 0$

2. Plot the first three terms of each of the series obtained in the foregoing problem and compare the summation curves with the given functions.

3. Draw curves which show the following relations in simple harmonic motion:

a. Acceleration as a function of displacement from the rest position.

b. Acceleration as a function of time.

c. Speed as a function of displacement from the rest position.

4. A particle moving with simple harmonic motion has speed of 20 and 10 cm. per second when having displacements of 6 and 12 cm., respectively. Find the period and amplitude of the motion.

5. Find the maximum speed of a particle in simple harmonic motion if the amplitude is 2 ft. and the period 4 sec.

6. If the period of a certain simple harmonic motion is 3 sec. and the amplitude is 25 cm., find the time required for the particle to go from the rest position to a position at which the displacement is 15 cm.

CHAPTER VI

INERTIA AND MASS

6-1. Foundations of Mechanics.—The present-day student of mechanics can only with considerable study appreciate the darkness which enveloped the subject in the seventeenth century. Ideas and concepts, which are our common tools today, were then in a very nebulous stage. To become familiar with these early conceptions and to view the progress of their evolution, one should consult the historical literature. Prior to the seventeenth century, contributions to mechanics dealt with the statics of solids and of fluids. The foundations of dynamics were laid by Galileo (1564–1642). Further contributions were made by Huyghens (1629–1695). To these two men may be given the credit for paving the way for the epoch-making formulations of Newton (1642–1726).

Prior to the advent of Galileo, the Aristotelian idea of motion had been accepted for approximately two thousand years. Galileo quoted these doctrines and, it will be remembered, openly demonstrated to an incredulous group of Aristotelian disciples, who had assembled before the famous leaning tower of Pisa, that bodies dropped from a height arrived at the ground in (very nearly) equal times rather than with time intervals which were, according to Aristotle, proportional to their weights. Encouraged by the success of his experiment, Galileo proceeded to discover *how* bodies fall. Unlike his philosophical predecessors who were interested mainly in the *why* of things, Galileo's attitude was guided by a desire to obtain first an accurate description, the manner in which the phenomena occurred, and then to consider the probable reason for the occurrence. This attitude, at that time new in the scientific world, was continued by the men who followed Galileo, with the result that science has achieved successes which would have been utterly impossible with the older attitude of Aristotle. Progress in the search for the truth concerning the nature of things is more rapid when the experimental method of Galileo is combined with the philosophical attitude of Aristotle.

Galileo's introduction of the inclined plane as a means of "diluting" gravity clearly shows his ingenuity. By having at his command motion in which the acceleration was sufficiently small to permit reasonably accurate observations, Galileo was able to give the world a new concept—acceleration—and at the same time to establish the fact that bodies, near the earth's surface, fall with constant acceleration (neglecting friction). From this work, then, resulted two of the equations which describe the motion of a particle moving from rest with constant acceleration. In terms of the symbols familiar to us, these equations may be written as follows:

$$V = at \qquad s = \tfrac{1}{2}\, at^2$$

Further experimentation with inclined planes led Galileo to observe that, if a body falls down an inclined plane through a definite vertical height, it would be able to return to approximately the same height by means of a second inclined plane regardless of the angles of inclination of the planes. He also saw that a body, in falling through a certain vertical height (on an inclined plane), would acquire the same velocity as any other body falling the same vertical height.

More important, perhaps, than these results was the observation which he made concerning motion without acceleration. Measurements were made upon the horizontal motion of bodies moving with velocities acquired by falling down an inclined plane. As a conclusion from these observations, Galileo wrote in his "Discourses," "but in the horizontal plane *GH* its [the moving body's] equable motion, according to its velocity as acquired in the descent from *A* to *B*, will be continued *ad infinitum.*" In this statement we find the nucleus of the concept *inertia.*

Other important contributions to mechanics by Galileo were the parallelogram of velocities, which led to a clearer idea of motion of projectiles, the possible use of a pendulum for time measurement, and the conception of force as the time rate of momentum.

Huyghens took up the study of mechanics where Galileo left off. He undoubtedly had a better idea of the concept inertia than did Galileo. The following quotation from the "Horologium Oscillatorium" (1673) clearly indicates his understanding of inertia, "If gravity did not exist, nor the atmosphere obstruct the motions of bodies, a body would keep up forever the motion once impressed upon it, with equable velocity, in a straight line." Huyghens determined the length of the seconds pendulum, discovered the center of oscillation, invented the pendulum clock, and obtained a fairly accurate value (32.16 ft. per second squared) of g. He derived the third equation ($V^2 = 2\,as$) of motion for a particle moving from rest with constant acceleration, studied the uniform motion of a particle in a circular path, and shared with two other "geometers" the honor of establishing the principle of the conservation of momentum. Apparently neither Huyghens nor Galileo understood the distinction to be made between mass and weight. It remained for Sir Isaac Newton to straighten out this tangle, and otherwise to render invaluable assistance to the progress of science by his clear enunciation of several fundamentally important concepts and by a careful formulation of the so-called "laws of motion."

6-2. Newton's Contributions.—A great deal of credit is due Newton for the wonderfully clear and accurate expressions of the principles which to this day form the basis of the formal development of mechanics and for the idea of universal gravitation and its concise formulation. Newton also gave us a number of definitions of fundamental quantities expressed in exquisitely clear language.

It was he who was able to generalize from current views and to consolidate these ideas into concise principles and concepts. In order to do this, it was necessary for him to invent new mathematical processes, particularly the "inverse method of fluxions" or infinitesimal calculus, as we call it today. His own statement concerning his work (1714) is quoted here to give the student some idea of the volume and character of the work which he accomplished in a two-year period.

In the beginning of the year 1665 I found the method of approximating Series and the Rule for deducing any dignity of any Binomial into such a Series. The same year in May I found the method of tangents of Gregory and Slusius, and in November had the direct method of fluxions, and in the next year in January had the Theory of Colours, and in May following I had intrance into ye inverse method of fluxions. And the same year I began to think of gravity extending to ye orb of the Moon, and having found out how to estimate the force with which [a] globe revolving within a sphere presses the surface of the sphere, from Kepler's Rule of the periodical times of the Planets being in a sesquialterate proportion of their distances from the centre of their Orbs, I deduced that the forces which keep the Planets in their Orbs must [be] reciprocally as the squares of their distances from the centres about which they revolve: and thereby compared the force requisite to keep the Moon in her Orb with the force of gravity at the surface of the earth, and found them answer pretty well. All this was in the two plague years of 1665 and 1666, for in those days I was in the prime of my age for invention, and minded Mathematicks and Philosophy more than at any time since.[1]

The three laws of motion as given in the "Axiomata sive Leges Motus" are as follows:

Law I. Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it may be compelled by impressed forces to change that state.

Law II. Change of motion is proportional to the impressed force and takes place in the direction of the straight line in which the force acts.

Law III. To every action there is always an equal and contrary reaction; or, the mutual actions of any two bodies are always equal and oppositely directed.

Some of Newton's definitions as given in "Philosophiae Naturalis Principia Mathematica" (1686) together with a few of his comments are given below:

Definition 1. Quantity of matter is the measure of it arising from its density and bulk conjointly.

This quantity of matter is, in what follows, sometimes called the body, or mass. It is known for each body by means of its weight; for it has been found, by very accurate experiments with pendulums, to be proportional to the weight.

Definition 2. The quantity of motion of a body is the measure of it, arising from its velocity and the quantity of matter conjointly.

Definition 3. The innate force of matter is its power of resisting, whereby every body, so far as depends on itself, perseveres in its state, either of rest, or of uniform motion in a straight line.

This is always proportional to the body and differs in no respect from the inertia of the mass, except in the manner of viewing it. To the inertia of matter is due the difficulty of disturbing bodies from their state of rest or motion; on which account the innate force may be called by the very suggestive name, force of inertia.

[1] BUCKLEY, "A Short History of Physics," p. 33.

Definition 4. An impressed force is an action exerted on a body, tending to change its state either of rest or of uniform motion in a straight line.

6-3. Inertia.—We cannot, at the present time at least, offer any better definition of inertia than that given by Newton in his third definition or as described in his first law of motion. Some familiar illustrations of this important property of bodies may serve to clarify this concept.

It is a common observation that a heavily loaded automobile is not so readily brought up to a given speed as when empty, even though the conditions are otherwise alike. The greater inertia of the loaded automobile is perhaps more noticeable when it is desired to bring it to a stop.

Through our experiences with objects which have different amounts of inertia we have learned that it is not easy, for example, to stop a rapidly rolling barrel which is filled with cast iron though we would not hesitate to make the attempt with an empty barrel.

The hydraulic ram gives us a splendid illustration of a practical use of the inertia of a relatively large amount of moving water to lift a much smaller quantity to a desired position.

Inertia effects are as important when bodies are in rotational motion as when they are in translation. The flywheel of an engine, when in rotation, tends to continue in rotation. Use is made of this fact to produce a steadier motion than would be obtainable without it.

Another common effect of the property of inertia is to be found in the phenomena of directional changes in the motions of bodies. According to Newton "every body continues in its state . . . of uniform motion in a straight line" unless acted upon by external forces so applied to the body that the direction of motion may be changed. Forces whose directions are perpendicular to the resultant velocity are needed to change the direction of motion of a body. This inertial tendency of a body to continue in a straight line is common observation. The occasional skidding of an automobile when attempting to round a corner, the tangential flying of mud or water from a rotating wheel, and other similar phenomena are illustrations of inertia.

6-4. Mass.—Inertia is a qualitative property of matter. This property has been made quantitative in mass. Mass may properly be defined as the measure of inertia. The relative masses of two bodies may be determined by comparing, in some way, their inertias. There are several ways in which this may be done.

One of the simplest pieces of apparatus which has been designed for comparing the relative masses of two bodies consists of a horizontal rod, or guide, upon which the masses may slide with negligible friction. A suitable spring may be placed between the two masses and held in a compressed state by a light cord which is to be tied to each of the two masses. When the string is cut the two masses, initially at rest, are acted upon by the spring which forces them apart. The force of the spring is variable, decreasing as the masses move apart. The spring will exert instantaneously equal, but oppositely directed, forces upon the two masses and will act upon both masses for the same length of time. An attachment is provided by means of which it is possible to measure the velocity of each mass at the instant at which the force of the spring becomes zero.

The results of experiments performed with apparatus of this sort indicate very clearly that, when two masses of equal weights are used, the velocities given to the two masses are equal (within the limits of experimental error) and, when masses of unequal weights are used, the velocities produced are unequal with the lighter body receiving the greater velocity.

With the apparatus arranged as described, a means is provided for comparing the inertias of two given bodies by measuring the velocities produced. We may use the experimental relation as a means for a quantitative evaluation of the mass of a body. The first step would be to select some particular body as a standard of mass (such as the standard kilogram or pound), and then to adopt the statement that the masses of two bodies are inversely proportional to the velocities produced in the inertia apparatus. If m_1 and m_2 represent the masses of the two bodies and V_1 and V_2 are the velocities produced, then the equation

$$m_1V_1 = -m_2V_2 \qquad \text{or} \qquad \frac{m_1}{m_2} = -\frac{V_2}{V_1}$$

would serve to express this conception in terms of the symbols used.

For purposes of establishing a logical foundation for the concept of mass as a measure of inertia, the foregoing procedure is entirely satisfactory but in practice the evaluation of mass by the process of weighing is experimentally easier to perform and is much more accurate. The validity of using the weighing process in place of the other more fundamental procedure has been repeatedly established by the fact that the two processes yield results which are identical within the limits of experimental error.

Unfortunately, some confusion has arisen in the minds of students over the distinction between mass and weight because the process of weighing is used for determining mass. The two quantities mass and weight are fundamentally very different, mass being a measure of inertia which is independent of the earth's attraction and weight is a gravitational force which changes from place to place, although on the earth's surface the change is not large. The units in which these two quantities are expressed are dimensionally unlike. Mass may be expressed in grams or pounds, while, in the corresponding systems of units, weights are given as gram centimeters per second squared or pounds feet per second squared, respectively. The weight units are combinations of a mass unit and an acceleration unit.

6-5. Mass a Function of Speed.—For a long time it was supposed that the mass of a particular body depended solely upon the "amount of matter" of which the body is composed. Moving the body to some remote corner of cosmic space was thought to have no effect upon the amount of its mass. Probably the position of the body does not affect its mass, but while being moved to some other place it does possess a different mass. That there is an increase in mass of a body when that body is in motion is a result of modern physics.

Sir J. J. Thomson first showed that because of the probable electromagnetic nature of matter one should expect an increase of its mass with speed, especially if the speed is large. His hypothesis was later confirmed by experimental measurements of the ratio of the electric charge to a mass of an electron. These tests were made independently by Kaufman and Bucherer who experimented with high-speed electrons ejected from radioactive substances. As a development from the principle of relativity, it was shown that a measurable increase of mass was to be expected under conditions of high speeds. The quantitative formulation of this relation is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \qquad (6\text{-}1)$$

where m is the mass of the body when in motion, m_0 is the "rest" mass, v the speed of the body, and c the speed of light. This expression is somewhat different from the one deduced by Thompson who used an electromagnetic hypothesis.

Since the speed of light is very large, m will be nearly equal to m_0 for all ordinary velocities. It is only for speeds that are more than one-tenth of that of light that the correction need be made for ordinary accuracies. When $v = 0.1\ c$, the increase in the mass is about one-half of 1 per cent.

As a consequence of this relation, the speed of light becomes a limiting barrier beyond which the speeds of a material body may not pass.

6-6. Moment of Mass.—For certain conveniences which will be apparent in a later section it is advisable to introduce at this point the physical quantity moment of mass. This quantity is defined as the product of a mass by a distance. If a certain mass m be regarded as concentrated at a point which is at a distance L from an arbitrary reference point, then mL is the moment of that mass with reference to the selected point. A particle may be regarded as having a moment with reference to a given fixed line or plane and its value in these two cases would be mL, as before, if L is the distance of the particle from the line or the plane.

If the collective moment of several masses, each regarded as concentrated at a definite point, is desired, the magnitude of the moment of the group of particles is found by merely effecting an algebraic sum of the individual moments. Moment of mass is a scalar quantity. The algebraic sign to be used with a particular moment is entirely dependent upon the sign of L, the distance factor. In general, then

$$ML = m_1L_1 + m_2L_2 + m_3L_3 + \cdots \qquad (6\text{-}2)$$

where M is sum of the masses and L is the distance from the reference line or plane to the point at which the total mass M may be regarded as being concentrated, and L_1, L_2, L_3, etc., are the distances of the particles m_1, m_2, m_3, etc., respectively, from the same reference line or plane.

6-7. Center of Mass and Centroid.—The center of mass of a system of particles may be defined as that point at which the total mass may be regarded as being concentrated in order that the moments of the total mass with reference to the three planes of a reference system may be equal, respectively, to the sums of the mass moments of the individual particles with respect to the same reference planes. The convenience of the center of mass is particularly noticeable in the description of the motion of a body or group of bodies under the influence of applied external forces. Center of mass is a very important conception, as will be prominently revealed in the work below.

There are occasions when it is convenient to determine that point in a massless line, surface, or volume which corresponds to the center of mass. This point is called the *centroid* of the particular configuration. The centroid of a geometric volume is that point which would coincide with the center of mass if the given volume were filled with some material of uniform density. The centroid of a surface is the point which would coincide with the center of mass of a thin homogeneous sheet, bent if necessary to fit the given surface, when the thickness of the sheet is made to approach zero as a limiting value. A similar definition could be written for the centroid of a line.

The method of determining the center of mass for a given distribution of particles consists in finding the coordinates of the center of mass by calculating the moment of the group of mass points with reference to each of the three planes of a given reference system. Dividing each of the sums of the moments of mass thus found by the total mass gives the desired coordinates of the center of mass. The process is made obvious by a consideration of the three following equations:

$$\begin{aligned} M\bar{x} &= m_1x_1 + m_2x_2 + m_3x_3 + \cdots \\ M\bar{y} &= m_1y_1 + m_2y_2 + m_3y_3 + \cdots \\ M\bar{z} &= m_1z_1 + m_2z_2 + m_3z_3 + \cdots \end{aligned} \tag{6-3}$$

where $m_1m_2m_3$, etc., are the masses of the particles $x_1y_1z_1$, $x_2y_2z_2$, and $x_3y_3z_3$ coordinates respectively of the particles, M the total mass and $\bar{x}$, $\bar{y}$, $\bar{z}$, the coordinates of the center of mass of the system of particles.

Hence the coordinates of the center of mass are given by the equations

$$\bar{x} = \frac{\sum_0^n m_ix_i}{M} \qquad \bar{y} = \frac{\sum_0^n m_iy_i}{M} \qquad \bar{z} = \frac{\sum_0^n m_iz_i}{M} \tag{6-4}$$

In general, however, the individual mass points will not be separated but will be the constituent differential mass elements of which the body is composed. We must, therefore, replace the summation expressions of Eq. (6-4) by integrals which are written as follows:

$$\bar{x} = \frac{1}{M}\int_M x\,dm \qquad \bar{y} = \frac{1}{M}\int_M y\,dm \qquad \bar{z} = \frac{1}{M}\int_M z\,dm \tag{6-5}$$

the integrations being taken over the entire mass M. The centroid for volumes, surfaces, and lines may be determined by equations similar to those of Eq. (6-5) except that volumes, surfaces, and lengths are introduced in place of the corresponding masses.

The particular reference system selected will not affect the position of the center of mass in the body but discretion in its selection may make the details of the integration process easier. When the shape of a body of uniform density is such that the entire body may be regarded as being made up of pairs of equal mass particles so situated that a certain plane bisects perpendicularly the lines joining the particles of each pair, then such a plane is called a *plane of symmetry* and must contain the center of mass of the body. Suppose the YZ plane of a selected reference system were such a plane of symmetry; then the integral $\int x\,dm$ taken over the entire body must be equal to zero and hence $\bar{x} = 0$. In this case the integral consists of a sum of self-canceling pairs of mass moments. Usually planes of symmetry may be found by inspection. If the nature of the body is such that there are two planes of symmetry, then the center of mass must lie in the line of intersection of these two planes. In determining the center of mass of a given body or configuration of bodies, therefore, it is advantageous to select a reference system so that one or two, if possible, of the reference planes may become planes of symmetry. For example, in determining the center of mass of a right circular cone of uniform density, it is to be observed that, if the reference system is so selected that the x-axis coincides with the geometrical axis of the cone, then the XY and the ZX planes are planes of symmetry; hence the center of mass of the cone must lie on the X-axis.

After a reference system has been selected, the next consideration to receive attention should be the selection of a convenient mass element. It is not always necessary to use a mass element of which all three dimensions are of differential magnitude. In some cases the differential mass may be of a rodlike shape with one dimension of finite length. In other cases it may be a thin section, sheetlike, of differential thickness. In all cases, however, all parts of the differential mass must be at equal distances from the reference plane from which the particular coordinate is measured. If the differential mass has three differential dimensions, then each mass integral is to be converted to a triple volume integral; *viz.*,

$$\int_M x\,dm = \iiint \rho\, x\, dx\, dy\, dz$$

with the limits of the integrations to be determined by the nature of the configuration and the reference system. By an appropriate selection of the differential mass, then, one or two steps in the integration may be avoided in certain special cases.

In the case of the right circular homogeneous cone the differential mass may be a thin section taken perpendicular to the axis of the cone. This selection gives a differential mass having only one differential dimension and hence the evaluation of the mass integral is reduced to a single integration.

To find the center of mass of a given body or group of bodies it is wise, therefore, to look first for planes of symmetry, next for a suitable reference system, and finally for a convenient differential mass. Simplicity in work of this character is obtained by skill in planning the procedure.

A number of problems are given in the section below to illustrate the details of the process for determining the center of mass of typical cases.

6-8. Determination of the Center of Mass—Problems.— 1. Find the center of mass of a straight rod of length L having uniform cross-sectional area. The density of the rod is to vary uniformly from zero at one end to a value K at the other.

Since the rod has a uniform cross section, there are two planes of symmetry, and these planes are perpendicular to each other and intersect in the axis of the rod. The center of mass of the rod must therefore lie in the axis of the rod. The reference system may be taken with the YZ plane perpendicular to the rod, the X-axis being coincident with the axis of the rod and the origin at one end of the rod, say the end with the zero density. By this selection two coordinates of the center of mass, $\bar{y}$ and $\bar{z}$, are then known to be equal to zero. For the differential mass we may select a thin section perpendicular to the axis of the rod and of thickness dx.

The density ρ of the rod at any point, which is at a distance x from the end of the rod having the zero density, is $\rho = Kx/L$. Since the density varies uniformly from one end to the other, we may obtain the total mass M of the rod by multiplying the average density $(K/2)$ by the volume, which gives $M = K\,A\,L/2$ if A is the cross-sectional area. The differential mass may now be expressed in terms of the symbols as follows:

$$dm = K\;A\;x\frac{dx}{L}$$

Using these values with the first equation of Eqs. (6-5) gives:

$$\begin{aligned}\bar{x} &= \frac{1}{M}\int_M x\,dm = \frac{KA}{ML}\int_0^L x^2dx \\ &= \tfrac{2}{3}L\end{aligned}$$

2. Find the center of mass of a thin plane uniform sheet of metal which is in the form of the right-angle triangle having sides 3, 4, and 5 cm.

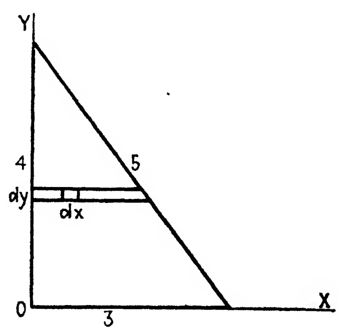

FIG. 81.

There is only one plane of symmetry in this case and it is midway between the faces of the sheet. The center of mass must be in this plane; hence we need to integrate expressions for two coordinates only. If we select a reference system so that the axes OX and OY form two sides of the triangle as shown in Fig. 81, then $\bar{z} = 0$.

It is convenient here to put the differential mass $dm = dx\,dy\,\rho\,t$ in which ρ is the density and t is the thickness of the sheet. The total mass M of the sheet is therefore $6\rho t$. It is to be noticed in the details of the integration as given below that each integral becomes a double integral. In the expression for $\bar{x}$ we integrate first with respect to dx, while dy is treated as a constant. Geometrically this integration gives the moment of the area of a strip of width dy and of a length which depends upon the position of the strip and hence upon y. The upper limit of the variable x is determined by the slope of the hypotenuse of the triangle.

Since the equation of the hypotenuse is $4x = 12 - 3y$, then the upper limit to be used for x is

$$x = \tfrac{1}{4}(12 - 3y)$$

The second step consists of an integration with respect to the variable y. This process, in reality, determines the sum of the moments of an infinite number of strips parallel to OX with respect to the YZ plane. The coordinates $\bar{x}$ and $\bar{y}$ of the center of mass are obtained by integration as shown in the following steps:

$$\begin{aligned}
\bar{x} &= \frac{1}{M}\int_M x\,dm & \bar{y} &= \frac{1}{M}\int_M y\,dm \\
&= \frac{\rho t}{M}\int_0^y\int_0^x x\,dy\,dx & &= \frac{\rho t}{M}\int_0^x\int_0^y y\,dx\,dy \\
&= \frac{1}{6}\int_0^4\int_0^{\frac{12-3y}{4}} x\,dy\,dx & &= \frac{1}{6}\int_0^3\int_0^{\frac{12-4x}{3}} y\,dx\,dy \\
&= \frac{1}{6}\int_0^4 \frac{(12-3y)^2}{32}\,dy & &= \frac{1}{6\times 18}\int_0^3 (12-4x)^2 dx \\
&= 1 & &= \tfrac{4}{3}
\end{aligned}$$

3. Find the center of mass of a solid homogeneous right pyramid having a square base.

Let the altitude of the pyramid be h, each edge of the base b, and let the density be designated by ρ. Because there are two planes of symmetry perpendicular to each other, which intersect in the axis of the pyramid, the center of mass must lie in the axis. The reference system may be selected with the YZ plane parallel to and containing the base of the pyramid and with the X-axis coinciding with the axis of the pyramid. The $\bar{x}$ coordinate of the center of mass is all that remains to be found.

If we now select as a differential mass a thin section of the pyramid perpendicular to the X-axis, then

$$dm = \rho\,(h - x)^2\,b^2\,\frac{dx}{h^2}$$

since the section is a square with each edge equal to $(h - x)b/h$ and of thickness dx.

The mass M of the pyramid may be found by direct integration of the expression

$$M = \int_M dm$$

Hence we may write

$$\bar{x} = \frac{\int_M x\,dm}{\int_M dm}$$

$$= \frac{\int_0^h x(h - x)^2 dx}{\int_0^h (h - x)^2 dx}$$

$$= \frac{h}{4}$$

Problems.—1. Find the center of mass of a system of three particles, 15, 20, and 6 g. The particles are in a straight line with spacing distances of 10 and 25 cm. between the first and second and between the second and third, respectively.

2. Find the center of mass of the system of particles, 2, 3, 4, and 5 g., one placed at each of the corners of a 10-cm. square in the order given.

3. Find the center of mass of the system of eight particles, 1, 2, 3, 4, 5, 6, 7, and 8 g., one placed at each corner of a cube 10 cm. on each edge, in such a manner that, when the cube is placed with sides horizontal and vertical, the upper face contains the first four masses placed cyclically in numerical sequence and the lower face contains the other four masses with the 5-g. mass below the 1 g., the 6-g. mass below the 2-g. mass, etc.

4. Locate the center of mass of a right circular cone having uniform density.

5. Find the center of mass of a homogeneous hemisphere.

6. Determine the centroid of a circular arc when the angle subtended at the center of the arc is 180, 90, and 40 deg.

7. Locate the centroid of the sector of a circle where α is the angle and r the radius.

8. Find the center of mass of two homogeneous spheres, one of radius 5 cm. and the other 8 cm., when the distance between the centers is 20 cm. and the density of the smaller sphere is twice that of the larger.

9. Prove that the medians of a triangle intersect at the centroid.

10. Find the center of mass of a thin circular disk which has a diameter of 10 cm. and which has a 2-cm. circular hole cut out. The center of the hole is 3 cm. from the center of the disk.

11. Using the formula of Eq. (6-1), make a plot showing the variation of m/m_0 when expressed as a function of v/c.

12. Express Eqs. (6-3), (6-4), and (6-5) as vector equations.

CHAPTER VII

THE FUNDAMENTAL EQUATIONS IN TRANSLATION

7-1. Definitions.—Up to this point in the study of mechanics we have been concerned with kinematical quantities, with the exception of the quantity mass which was discussed in the preceding chapter. In the present chapter we shall study some fundamental dynamical relations. As a preliminary to this study it is necessary to define the two quantities momentum and force.

Momentum.—We have already defined the two quantities mass and velocity, both of which are included in the quantity momentum. We may, with Newton, define the momentum of a body as the "quantity of motion" possessed by that body.

It is measured by the product of the mass of the body by its velocity. The term "quantity of motion" is not commonly used in this country. The French term for momentum, however, is *quantité de mouvement*, literally, quantity of motion. The momentum (of a body) is more commonly defined as the product of the mass by the velocity.

Momentum is a vector quantity. It has both direction and magnitude. The direction of the momentum is that of the velocity factor. The magnitude is equal to the product of the mass by the speed. If m is the mass of a particle and $\boldsymbol{V}$ is its velocity, then the momentum of the particle may be expressed in terms of its components along the axes of some selected reference system as follows:

$$m\boldsymbol{V} = m\frac{dx}{dt}\boldsymbol{i} + m\frac{dy}{dt}\boldsymbol{j} + m\frac{dz}{dt}\boldsymbol{k}$$

The units in which momentum may be expressed are combinations of the units used for mass and velocity, such as gram centimeter per second, pound foot per second, etc.

Force.—The term force is in common usage and undoubtedly all students of physics have a correct conception of it, but for the sake of being explicit we may define force as that physical quantity which, when acting upon a particle that is free to move, will produce a change in the momentum of the particle and, quantitatively, the force is equal to the time rate of change of the momentum (see Newton's second law of motion). Expressed in symbols, if F is the force acting upon a particle of mass m, we may write

$$F = \frac{d}{dt}(mV)$$

It is to be noticed this is a vector equation and therefore, to be valid, both sides of the equation must have the *same direction* as well as the same magnitude. This statement does not mean that the *momentum* must have the same direction as that of the force but rather that the direction of the quantity which expresses the *time rate of change of the momentum* must be that of the force.

In nearly all of the problems which the student will study, the mass or masses of the bodies involved will be constant. In such cases it is legitimate to equate the force to the product of the mass by the acceleration of that mass. Because of the predominance of such problems, the equations developed below are subject to this limitation. It is felt that no handicap will be imposed upon the student because of this limitation because it is not difficult to rewrite any of the equations to include the more general case if there be a need for so doing.

The units in which force may be expressed are the dyne, poundal, gram weight, pound weight, etc. The dimensions of force are those of mass times acceleration, for example, gram centimeter per second squared.

7-2. The Force Equation.—It has been demonstrated repeatedly that, whenever a force acts upon a body which is unrestrained so that it responds freely to the force, the body experiences an acceleration. Within the limits of error due to measurement, such experiments show that for any given body with mass constant the magnitude of the acceleration is directly proportional to the force and inversely proportional to the mass of the body. This relation is expressed in the force equation.

For purposes of obtaining a general relation we may imagine that the external force applied to a body is so distributed to all differential mass elements that each elemental mass has acting upon it the differential force which would independently produce the particular acceleration which it actually has when moving with the body. If the differential force is taken parallel to some reference line, say OX, then we may write

$$dF_x = dm\frac{d^2x}{dt^2} \tag{7-1}$$

in which dF_x is the differential force, parallel to OX, acting on the differential mass element dm and the second derivative of x with respect to the time is the acceleration of that mass element.

The force equation for the entire body (mass $= m$) may be found by integration. If F_x is the component of the resultant force acting on the body parallel to the X-axis, then

$$F_x = \int_0^m dm\frac{d^2x}{dt^2} \tag{7-2}$$

where the integration is to extend over the entire body.

It is possible to effect the integration even in this general expression if we make use of the first equation of Eqs. (6-5) which expresses the $\bar{x}$ coordinate of the center of mass of the body. This equation is rewritten here for the sake of convenience and is

$$\bar{x} = \frac{1}{m}\int_0^m x\, dm$$

If we multiply both sides of this equation by m and then differentiate twice with respect to the time, assuming that m is constant, we obtain

$$m\frac{d^2\bar{x}}{dt^2} = \int_0^m dm\frac{d^2x}{dt^2} \tag{7-3}$$

Substituting this value for the integral in Eq. (7-2) gives the desired expression

$$F_x = m\frac{d^2\bar{x}}{dt^2} \qquad \text{or} \qquad F_x\mathbf{i} = m\frac{d^2\bar{x}}{dt^2}\mathbf{i} \tag{7-4}$$

This equation shows that the effect of the external force F_x may be measured by the product of the mass of the body and the acceleration of the center of mass. This equation expresses an important general relation and is valid for any position of the force, *i.e.*, whether the line of the force actually passes through the center of mass or not. It will be shown later that, when the line of the force does not pass through the center of mass, an additional effect, *viz.*, rotational acceleration, is also produced, but the acceleration of the center of mass remains as given by Eq. (7-4).

Equation (7-4) is a vector equation. Since mass is always a scalar quantity and both force and acceleration are vectors, it follows that the acceleration is always parallel to the force which produces it.

The relation expressed in Eq. (7-4) is an instantaneous one only. Whether the force remains constant or not in any particular case is a matter which is usually expressed in the given data. Two similar equations may be written for the component effects parallel to the two other axes of reference. The equation for the resultant force could be obtained from the three component equations by adding, as vectors, the members of both sides of the equations separately and then equating the results. Conversely, if the equation for the resultant force were known and any or all three of the components were desired, these could be obtained in the usual manner by projection.

In a previous chapter we learned that there were several types of acceleration, such as tangential, normal, and radial. For each type of acceleration we have a corresponding force which bears the same name as the type of acceleration for which it may be regarded as being directly responsible.

7-3. The Impulse Equation.—Whenever a force is permitted to act on a body for a definite time and during that interval there is a freedom from constraints, then there is always a change in the velocity of the body. The quantity which measures the effect of the force upon the body, when this effect is accumulated over a period of time, is called *change of momentum.* The quantity which produces the change in momentum is called the *impulse* of the force. The magnitude of the impulse is the integrated sum of the particular values of the force (generally considered as varying) multiplied by the corresponding time elements. If the force F, in a particular case, is constant and acts for a time t, then the impulse of the force is expressed by the product Ft.

To derive the general form of the impulse equation, we may use the differential force equation (parallel to OX) as a starting point. In order to have the velocity factor prominent, we shall use dV_x/dt as an alternative form of the acceleration, where V_x is the velocity parallel to the line OX. The starting point is then

$$dF_x = dm\frac{dV_x}{dt}$$

This equation is readily converted into an impulse equation if we multiply through by the time dt. The impulse of the force dF_x for a finite time t on a differential mass dm is then given as follows:

$$\int_0^t dF_x \, dt = \int_0^t dm \, dV_x$$

Since dm is a constant, we may put it outside the integration sign in the right-hand member. If V_x and V_{x_0} be the velocities of dm at the times t and 0, respectively, then the equation becomes

$$\int_0^t dF_x \, dt = dm \, (V_x - V_{x_0}) \qquad (7\text{-}5)$$

The equation shows that the effect of the impulse on dm is to change its momentum. Consistent with their vector nature, both quantities, impulse and change of momentum, are in the same direction along OX. The direction of the change of momentum depends upon the signs and relative magnitudes of both V_x and V_{x_0}.

If the total impulse and total change of momentum parallel to OX are required, then it is necessary to integrate both quantities over the entire mass. This is expressed as follows:

$$\int_0^m \int_0^t dF_x \, dt = \int_0^m dm \, (V_x - V_{x_0}) \qquad (7\text{-}6)$$

Since dt is common to all force elements, the first integration, *i.e.*, of dF_x over the entire mass, is readily effected and gives F_x, the x component of the resultant force.

The right-hand member may be reduced to a simpler form by introducing the velocities (parallel to OX) of the center of mass. If we differentiate the expressions for the $\bar{x}$ coordinate of the center of mass with respect to the time for the initial and final positions of the body corresponding to the times 0 and t, the following result is obtained:

$$m \, (\bar{V}_x - \bar{V}_{x_0}) = \int_0^m dm \, (V_x - V_{x_0}) \qquad (7\text{-}7)$$

where $\bar{V}_x$ and $\bar{V}_{x_0}$ are the velocities of the center of mass at the times t and 0, respectively. By substituting this result in Eq. (7-6) it follows that

$$\int_0^t F_x \, dt = \int_0^m dm \, (V_x - V_{x_0})$$
$$= m \, (\bar{V}_x - \bar{V}_{x_0}) \qquad (7\text{-}8)$$

This expression is the general form of the impulse equation.

If, in a particular case, F_x is not a function of the time but remains constant, then

$$F_x t = m(\bar{V}_x - \bar{V}_{x_0}) \tag{7-9}$$

It is possible to effect such integrations as are indicated above even though the direction of the force is not constant. In the case of the impulse of the resultant force dF, Eq. (7-5) takes the following form

$$\int_0^t dF\,dt = dm\,(V - V_0) \tag{7-10}$$

where the velocities V and V_0 of the differential mass are its resultant velocities at the instants of time corresponding to t and 0. The single vector giving the change of momentum is not necessarily parallel to either velocity, initial or final, but is parallel to the force which produces the change of momentum.

7-4. The Work Equation.—We may derive the elemental work equation from the differential force equation, provided that we select the alternative form of the acceleration which contains the expression for the coordinate rather than the time derivative. This selection gives the following form to the force equation for the differential force dF_x acting upon dm:

$$dF_x = dm\frac{V_x\,dV_x}{dx} \tag{7-11}$$

where V_x is the velocity of dm, and dV_x the change in V_x as dm is displaced a distance dx. Multiplying both sides of this equation by dx gives

$$dF_x\,dx = dm\,V_x\,dV_x \tag{7-12}$$

which is the work equation in its differential form. If the differential mass is displaced a distance measured from 0 to x and has velocities V_0 and V_x at the beginning and end, respectively, of this displacement, then upon integrating Eq. (7-12) from 0 to x we obtain

$$\int_0^x dF_x\,dx = \tfrac{1}{2}\,dm\,V_x^2 - \tfrac{1}{2}\,dm\,V_0^2 \tag{7-13}$$

The left-hand member gives the work of the differential force over the displacement. The integral could be evaluated if the expression giving the variation of dF_x with the displacement were known. The right-hand member is a characteristic expression for the change of *kinetic energy*. Obviously V_x^2 and V_0^2 of this member are positive regardless of the sign of the speed.

Since work is a scalar quantity, the actual direction of the force is immaterial in the process of integration. So long as each element of the displacement is taken parallel to the force at the corresponding position in the path, it does not matter whether the direction of the force is changing or not.

As an illustration of the work done by a force whose direction is continually changing, we may write the work equation for the radial force. The force equation for the force dF_r in the line of r, the radius vector, is

$$dF_r = dm\left(\frac{V_r\,dV_r}{dr} - r\omega^2\right)$$

where the quantity within the parenthesis is the component acceleration of dm in the line of r.

This equation is converted into a work equation by multiplying both sides by dr. If the work is desired for a displacement which is expressed by a variation of the radius vector from 0 to r, then we may write

$$\int_0^r dF_r\,dr = dm\frac{V^2_r - V^2_{r_0}}{2} - \int_0^r dm\,r\omega^2\,dr \qquad (7\text{-}14)$$

in which V_r and V_{r_0} are the velocities of dm in the line of r at the beginning and end of the displacement interval. If dm were restricted to move in a circular path, then the magnitude of r would remain constant and the force dF_r would always be perpendicular to the resultant velocity. In this case the velocity along r must remain equal to zero and hence there could be no change of kinetic energy in this line. Furthermore, since dr would be equal to zero, there would be no work done in this line.

The conclusion just drawn is always true for the normal force; since the normal force is permanently perpendicular to the resultant displacement, there is no component displacement in the line of the normal force and hence the normal force never does any work.

The tangential force acting on a differential mass produces an acceleration which may be written in the form $V\,dV/ds$, where s is the coordinate measured along the path in which the particle moves.

The work equation for the tangential force may then be written as follows:

$$dF_t\,ds = dm\;V\,dV$$

$$\int_0^s dF_t\;ds = \tfrac{1}{2}\,dm\,(V^2 - V_0^2) \qquad (7\text{-}15)$$

Since the normal force does not work, one would expect as a conclusion from this fact that the tangential force must do all the work, that it does as much work as the resultant force. This is readily proved in the following manner. If dF (Fig. 82) is the resultant force on dm, and du the displacement in the line of dF, and if α is the angle between dF and the tangential force dF_t, then

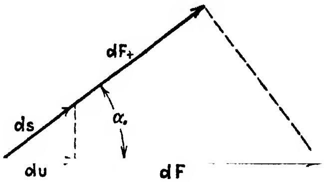

FIG. 82.

$$dF_t = dF\cos\alpha$$

for dF_t is a component of dF in the line of the tangent. The resultant displacement ds, however, is in the line of the tangent, therefore

$$du = ds\cos\alpha$$

Hence if we multiply these two equations together so that the work of each force is expressed, we obtain the important result that

$$dF_t\,ds = dF\,du \qquad (7\text{-}16)$$

From this we may conclude that the work of the tangential force is equal to the work of the resultant force. Since this is true, it follows that the normal force, the other component of the resultant force, can do no work.

In the case of either the force or impulse equations, if the expressions for the effect of the components along the axes of any reference system be given and the equation for the resultant force or impulse is desired, a vector sum must be made of the three equations. However, in the case of the work, expressed in terms of the x, y, and z components, a scalar sum must be taken if the work of the resultant force is to be obtained. That this is true follows from the fact that work and energy are scalar quantities. The statement may be proved, however, by writing the three equations which express the work of the component forces along the three reference axes and showing that the algebraic (not vector) sum of the equations gives the proper result. If all of the components are acting on the same differential mass, then the following equations give the elements of work and corresponding changes of energy:

$$\int_0^x dF_x\,dx = \tfrac{1}{2}\,dm\,(V^2{}_x - V^2{}_{x_0})$$
$$\int_0^y dF_y\,dy = \tfrac{1}{2}\,dm\,(V^2{}_y - V^2{}_{y_0})$$
$$\int_0^z dF_z\,dz = \tfrac{1}{2}\,dm\,(V^2{}_z - V^2{}_{z_0})$$

Adding these equations algebraically gives

$$\int_0^x dF_x\,dx + \int_0^y dF_y\,dy + \int_0^z dF_z\,dz = \tfrac{1}{2}\,dm\,(V^2 - V_0^2)$$
$$= \int_0^s dF_t\,ds \qquad (7\text{-}17)$$

Referring to Eq. (7-15) which expresses the equivalence between the change in kinetic energy, obtained here by the sum of the three equations giving the work of the component forces, and the work of the tangential force, we may conclude that the work of the resultant force may be found by adding algebraically the work of the component forces.

It is of interest here to use the rule for the scalar product of two vectors and show that its result in this case is consistent with the results obtained above. The scalar product of the two vectors (force $\boldsymbol{dF}$ and displacement $\boldsymbol{ds}$) is expressed as follows:

$$\boldsymbol{dF} \cdot \boldsymbol{ds} = (\boldsymbol{i}\,dF_x + \boldsymbol{j}\,dF_y + \boldsymbol{k}\,dF_z) \cdot (\boldsymbol{i}\,dx + \boldsymbol{j}\,dy + \boldsymbol{k}\,dz)$$
$$= dF_x\,dx + dF_y\,dy + dF_z\,dz \qquad (7\text{-}18)$$

To determine the total work done by the force parallel to any line OX, it is necessary to integrate over the entire mass of which dm is a part. Hence

$$\int_0^m \int_0^x dF_x\,dx = \tfrac{1}{2} \int_0^m dm\,(V^2{}_x - V^2{}_{x_0}) \qquad (7\text{-}19)$$

In order to integrate these expressions, we must know the law of distribution of the forces and velocities throughout the mass. In the general case all elements of mass may not have the same displacement. This is true in the case of motions which include rotations but in the case of pure translation the displacement is common to all elements of mass if the body is rigid.

7-5. The Three Fundamental Equations.—The three fundamental equations of motion which have been developed above may be conveniently brought together for certain important considerations. They are written below in forms which will serve best our present purpose.

Force:

$$\int_0^m dF_x(e) = \int_0^m dm \frac{d^2x}{dt^2} \tag{7-20}$$

Impulse:

$$\int_0^t F_x(e)\, dt = \int_0^m dm\,(V_x - V_{x_0}) \tag{7-21}$$

Work:

$$\int_0^m \int_{x_0}^x dF_x\, dx = \tfrac{1}{2} \int_0^m dm\,(V^2_x - V^2_{x_0}) \tag{7-22}$$

In both the force and impulse equations we are concerned with vectors, while in the work equation the quantities are scalar. Because of this difference it is necessary to consider only external forces in the first two equations and hence the (e) is written after the force F to emphasize this fact. The reason for this emphasis is because in our development of these equations we have not imposed the condition that the mass was rigid. In a nonrigid body there may be displacement of one part of the body with respect to another part. If we were to include internal forces in an integral in which the vector character of the quantity was involved, the opposite signs connected with any pair of such internal vectors would cause the quantities to cancel each other in the sum and hence drop out of the final result. In the work equation we may have internal forces contributing to the total work if the parts of the mass, upon which these forces are acting, have relative displacement.

To illustrate this point let us consider two small masses joined to each other by a stretched elastic massless spring (Fig. 83). The spring exerts internal forces $+T$ and $-T$ upon the system taken as a whole and, while the

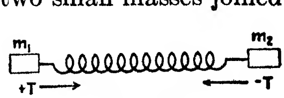

FIG. 83.

masses may move relatively to each other because of the spring, the center of mass is not affected by it. In a consideration of the acceleration of the center of mass of this system due to an external force, the force equation should not include the forces of the spring, for the acceleration of the center of mass of the system is not affected by $+T$ and $-T$. In the force integral $+T$ would cancel $-T$. Similarly in the impulse equation the impulse of $+T$, being a vector, would cancel the impulse of $-T$. In the work equation, however, the scalar quantities would not cancel each other, because both elements would be positive and hence would contribute to the final result.

7-6. Position of the Force and Its Effect.—In the forms given in the preceding section for the three fundamental equations, the right-hand members of the force and impulse equations may be reduced to indicate the acceleration and momentum of the total mass in terms of the center of mass. This is not the case in the work equation. It is in the work equation that the effect of the position of the line of the force with respect to the center of mass becomes important. In the form in which the work equation is given above, this characteristic is not expressed but, by recasting the equation, prominence may be given to this difference between the three equations. By so doing we may express an important property of the center of mass and hence bring all three equations to such terms of similarity as involve the center of mass.

We have seen in our study of the velocity that the resultant velocity of a point may be expressed in terms of its velocity relative to some moving system (here we shall use a moving system attached to the center of mass) and the velocity of the moving system with respect to the fixed system. This relation was expressed as follows:

$$V_P\,(O) = V_P\,(Q) + V_Q\,(O) \qquad (7\text{-}23)$$

If V be the velocity of dm in the reference system and U be its velocity with reference to the moving system connected to the center of mass and W be the velocity of the center of mass, then

$$V = U + W$$

If α is the angle between W and U, then

$$V^2 = W^2 + 2WU \cos \alpha + U^2$$

The kinetic energy of dm is therefore

$$\tfrac{1}{2}\, dm\, V^2 = \tfrac{1}{2}\, dm\, W^2 + \tfrac{1}{2}\, dm\, U^2 + dm\, WU \cos \alpha \qquad (7\text{-}24)$$

Since these quantities are scalar, we may integrate over the entire mass and have the following scalar sum:

$$\int_0^m \tfrac{1}{2}\,dm\,V^2 = \int_0^m \tfrac{1}{2}\,dm\,W^2 + \int_0^m \tfrac{1}{2}\,dm\,U^2 + \int_0^m dm\,WU\cos\alpha \tag{7-25}$$

In the last term, W is common to all elements and may therefore be put outside the integration sign, leaving

$$\int_0^m dm\,U\cos\alpha$$

This quantity is evidently the momentum parallel to W (at the particular instant) which is due to U. But the total momentum parallel to W is

$$\int_0^m dm\,(W + U\cos\alpha)$$

and is equal to mW. Therefore the integral which expresses the momentum due to U must be equal to zero.

That this is true may be seen from another point of view. For this purpose we may use the equation for one coordinate of the center of mass; *viz.*,

$$\bar{x}_1 = \frac{1}{m}\int_0^m dm\,x_1$$

We will select a moving system with origin at the center of mass and with the X_1-axis parallel to W. Differentiating the equation for $\bar{x}_1$ with respect to the time and assuming that the mass is constant gives

$$\frac{d\bar{x}_1}{dt} = \frac{1}{m}\int_0^m dm\frac{dx_1}{dt}$$

Since

$$\frac{dx_1}{dt} = U\cos\alpha \qquad \text{and} \qquad \frac{d\bar{x}_1}{dt} = 0$$

it follows that

$$\int_0^m dm\,U\cos\alpha = 0$$

Equation (7-25) may then be written in the form

$$\int_0^m \tfrac{1}{2}\,dm\,V^2 = \tfrac{1}{2}\,mW^2 + \tfrac{1}{2}\int_0^m dm\,U^2 \tag{7-26}$$

We may conclude from this result that the total kinetic energy of a body may be regarded as consisting of two parts, one, given by $\frac{1}{2}mW^2$, represents the kinetic energy of the entire mass moving with a speed equal to that of the center of mass and the other, given by the last term, is the kinetic energy due to velocities relative to the center of mass. In rigid bodies the first term gives the translational kinetic energy and the other term expresses the rotational kinetic energy. This division is a natural and convenient one from the standpoint of both mathematics and dynamics.

With this analysis of the kinetic energy term, the work of the forces parallel to OX, a reference line, may now be expressed as follows:

$$\int_0^m \int_0^x dF_x dx = \tfrac{1}{2}m(W^2_x - W^2_{x_0}) + \int_0^m \tfrac{1}{2}dm(U^2_x - U^2_{x_0}) \quad (7\text{-}27)$$

In the equation the differential force dF_x displaces the differential mass dm from x_0 to x and changes its velocity from U_{x_0} to U_x as measured in the moving system. The velocities W_{x_0} and W_x are the initial and final velocities, respectively, of the center of mass. All velocities are component velocities parallel to the reference line OX.

A similar analysis may be made for the left-hand member of the work equation. Using a coordinate system which is moving with its axes parallel to those of the reference system and which is attached to the center of mass of the body, so that the displacement dx of the differential mass is expressed in terms of the displacement of the moving system, we may write $dx = d\bar{x} + dx'$ where $d\bar{x}$ is the displacement of the center of mass (parallel to OX) and dx' is the displacement of the differential mass in the moving coordinate system, *i.e.*, relative to the center of mass. Substituting the equivalent expression for dx in the left-hand member of the work equation [Eq. (7-27)] gives

$$\int_0^m \int_{x_0}^x dF_x\, dx = \int_0^m \int_{x_0}^x dF_x\, d\bar{x} + \int_0^m \int_{x_0}^x dF_x\, dx' \quad (7\text{-}28)$$

But $d\bar{x}$ is common to all elements of the force; hence one step in the double integration of the first term of the right-hand member may be carried out, which gives the following equivalence:

$$\int_0^m \int_{x_0}^x dF_x\, d\bar{x} = \int_{x_0}^x F_x\, d\bar{x}$$

A substitution of this expression together with that expressed in Eq. (7-28) in the general equation [Eq. (7-27)] gives

$$\int_{x_0}^{x} F_{\bar{x}} d\bar{x} + \int_0^m \int_{x_0}^{x} dF_x dx' = \tfrac{1}{2} m(W^2_x - W^2_{x_0}) + \int_0^m \tfrac{1}{2} dm(U^2_x - U^2_{x_0}) \tag{7-29}$$

It is easy to show that this equation may be regarded as a sum of two separate equations, which are

$$\int_{x_0}^{x} F_x \, d\bar{x} = \tfrac{1}{2} \, m \, (W^2_x - W^2_{x_0}) \tag{7-30}$$

$$\int_0^m \int_{x_0}^{x} dF_x \, dx' = \int_0^m \tfrac{1}{2} \, dm \, (U^2_x - U^2_{x_0}) \tag{7-31}$$

This may be proved in the following manner. From the force equation we know that the resultant force may be regarded as acting on the center of mass and will produce an acceleration of the center of mass expressible in the following manner:

$$F_x = m \, W_x \, \frac{dW_x}{dx}$$

Separating the variables and integrating gives

$$\int_{x_0}^{x} F_x \, d\bar{x} = \tfrac{1}{2} \, m \, (W^2_x - W^2_{x_0}) \tag{7-32}$$

This establishes the truth of the character of Eq. (7-29) by proving the validity of Eq. (7-30). From this consideration, Eq. (7-31) must be accepted.

Since this equivalence may be similarly established for the work done parallel to each of the two other axes, it is correct to conclude that the work done by the force which is applied along a line which does not pass through the center of mass of a body may be regarded as a superposition of two independent processes. The off-center force does work expressed in terms of the displacement of the center of mass and measured by the change of energy of the whole mass moving with the speed of the center of mass and in addition does work measured by displacements and energy changes relative to the center of mass. Both work and energy are divided into two parts, *viz.*, *translational* and *rotational.*

In cases of pure translational motion Eq. (7-31) drops out and only the equation expressing the motion in terms of the center of mass remains.

We are now able to see that all three fundamental equations of motion apply to the motion of the entire mass expressed in terms of the motion of its center of mass, and also that the work equation alone is concerned with expressions dealing with motions relative to the center of mass. Furthermore, the conception of the idea of center of mass gains added meaning, for it carries with it the ideas of center of force and momentum as well. There are certain conveniences of description to be gained from the expressions which do not give any indication of the position of the force, but at the same time these same expressions fail to describe any but average values. The work equation alone can furnish us with a knowledge of the values of the accelerations or of the velocities of the different parts of the body.

7-7. Conservation of Momentum.—In the present section we shall derive those equations which are fundamental to and descriptive of that group of phenomena known as collisions or impacts. A *collision* between two bodies takes place when the two bodies strike each other or come together in the course of their motion in such a way that their velocities are altered by the mutual effect of one body upon the other. In general, both bodies will have velocities, before and after the impact, which are different from zero, but there are many cases in which one body may have no velocity before the impact. The directions of the velocities need not be parallel to a fixed line. If the velocities are all parallel to a fixed line before and after impact, the case is a special one in which the term "central impact" is used for identification.

In general, the duration of time during which the two bodies are in actual contact with each other is comparatively short. In fact in many cases the actual time is so short that it is difficult, if not impossible, to measure the stresses brought into action by the collision. During the collision the bodies are deformed by the internal stresses but they return to their original forms if the materials are elastic. If the materials are inelastic, the deformations will be permanent at least as far as the particular collision is concerned. We may regard the time interval during which the two bodies are in contact as embracing two periods, one in which the deformations are taking place and the other in which the bodies are returning to their final form. At the instant of time which separates the two intervals, the two bodies will have a common velocity.

In a consideration of a collision we exclude all forces external to the system. The only forces which are to be included are internal forces or those which are caused by the collision. In dealing with only two bodies, the forces occur as a pair of equal and oppositely directed forces. During the impact the magnitudes of the forces increase from zero up to a maximum value and then decrease to zero as the terminal value.

We may let $+F$ represent the instantaneous value of one of the pair of stresses and $-F$ will then be the other. The force $+F$ acts on one body and $-F$ upon the other. Let m_1 and m_2 represent the masses of the two bodies and let U_1, U_2 and V_1, V_2 be the velocities before and after the impact of m_1 and m_2, respectively. If t is the time of contact of the two bodies, then the impulse of each force may be written as follows:

$$\int_0^t -F\,dt = m_1(V_1 - U_1) \qquad \int_0^t F\,dt = m_2(V_2 - U_2) \qquad (7\text{-}33)$$

The two equations are written in vector form. The right-hand members express the change of momentum of the masses. It is not necessary to assume that U_1 is parallel to V_1 or that U_2 is parallel to V_2.

The impulses of the two forces may be eliminated by adding the two equations, which gives the following important result:

$$m_1U_1 + m_2U_2 = m_1V_1 + m_2V_2 \qquad (7\text{-}34)$$

This equation gives rise to the expression *conservation of momentum.* The momentum of the system before impact is equal to the momentum of the system after impact.

It will be convenient at this point to introduce and define the term *coefficient of restitution.* Attention was directed above to a division of the time of impact into two intervals. The impulse of either force during the interval of deformation is greater than the impulse of the force during the interval of reformation except in those cases where the constituent material is perfectly elastic. The ratio of the latter impulse to the former is called the coefficient of restitution and is designed by the letter ϵ. The numerical value of ϵ varies from zero in the case of inelastic bodies to unity in the case of perfectly elastic bodies.

Let V be the common velocity of the two bodies at the instant of maximum deformation. Since there are no external forces acting, the momentum of the system is constant [Eq. (7-34)]; hence we may write

$$(m_1 + m_2)V = m_1U_1 + m_2U_2 \qquad (7\text{-}35)$$

But V must be the velocity of the center of mass of the system. Furthermore, the velocity (V) of the center of mass of the system must remain constant.

We shall digress at this point to establish Eq. (7-35) by another mode of reasoning. Let $x_1\ y_1$ and $x_2\ y_2$ be the coordinates of the centers of mass of the two bodies in a selected reference system. In a later chapter it will be shown that the motions of two bodies, such as we are now considering, are confined to one plane. We shall select the reference system to contain this plane of motion. Also let $\bar{x}\ \bar{y}$ be the coordinates of the center of mass of the system. By definition of center of mass it follows that

$$(m_1 + m_2)\bar{x} = m_1x_1 + m_2x_2 \qquad (m_1 + m_2)\bar{y} = m_1y_1 + m_2y_2$$

Differentiating both of these equations with respect to the time (masses are constant) and writing in vector form gives

$$(m_1 + m_2)\frac{d\bar{x}}{dt}\,\boldsymbol{i} = \left(m_1\frac{dx_1}{dt} + m_2\frac{dx_2}{dt}\right)\boldsymbol{i}$$

$$(m_1 + m_2)\frac{d\bar{y}}{dt}\,\boldsymbol{j} = \left(m_1\frac{dy_1}{dt} + m_2\frac{dy_2}{dt}\right)\boldsymbol{j}$$

Combining into a single vector equation, we obtain

$$(m_1 + m_2)\ V = m_1U_1 + m_2U_2$$

which is identical with Eq. (7-35) above.

Returning now to the development of the fundamental equations and more particularly to the writing of expressions which involve the coefficient of restitution ϵ, we shall first introduce a limitation in order to simplify the work which is to follow. Up to this point the assigned velocities have been general but now we shall limit the velocities to parallelism with a fixed line—the case of central impact. The preceding equations are still valid but the quantities in them, such as impulse, momentum, and velocity, become ordinary scalar quantities under this restriction.

We may equate the change of momentum of the mass m_1 during that part of the interval of impact which follows the instant of maximum deformation to ϵ times the change of m_1's momentum during the time which precedes the instant of maximum deformation. The validity of the equality is based upon the definition of ϵ and the impulse—momentum equation. In symbols for both m_1 and m_2 we may write

$$m_1(V_1 - V) = \epsilon m_1(V - U_1) \qquad m_2(V_2 - V) = \epsilon m_2(V - U_2) \tag{7-36}$$

A relation between the velocities may now be obtained by canceling the mass factors in each equation and eliminating V. Hence

$$V_1 - V_2 = -\epsilon\,(U_1 - U_2) \tag{7-37}$$

From this equation, additional information may be obtained as to the meaning of the coefficient of restitution. This detail will be left to the student. It is also profitable to convert Eq. (7-37) into one which shows that ϵ is the ratio of the impulses as given in the definition.

There are two other equations needed in order to complete the present development. These are equations expressing the velocities after impact in terms of the masses, coefficient of restitution and the velocities before impact. Substituting the value of V as given by Eq. (7-35) in Eqs. (7-36) and writing the resulting expressions so that they are explicit for the velocities after impact gives

$$V_1 = \frac{m_1U_1 + m_2U_2}{m_1 + m_2} - \epsilon\frac{m_2(U_1 - U_2)}{m_1 + m_2} \tag{7-38}$$

$$V_2 = \frac{m_1U_1 + m_2U_2}{m_1 + m_2} + \epsilon\frac{m_1(U_1 - U_2)}{m_1 + m_2} \tag{7-39}$$

7-8. Kinetic Energy Changes during Impact.—While there is no change of momentum during a collision, there is usually a change in the kinetic energy of the system. If the two bodies are not perfectly elastic, some of the kinetic energy which the system possesses before impact is converted into other forms of energy, such as heat, sound, and mechanical work. If E_1 and E_2 are used to express the kinetic energy of the system before and after impact, then

$$E_1 = \tfrac{1}{2}\,m_1U_1^2 + \tfrac{1}{2}\,m_2U_2^2 \qquad E_2 = \tfrac{1}{2}\,m_1V_1^2 + \tfrac{1}{2}\,m_2V_2^2$$

The change in kinetic energy is therefore

$$E_1 - E_2 = \tfrac{1}{2}\,m_1\,(U_1^2 - V_1^2) + \tfrac{1}{2}\,m_2\,(U_2^2 - V_2^2) \tag{7-40}$$

In order to see just how the change in kinetic energy depends upon the coefficient of restitution, it is desirable to eliminate V_1 and V_2 from Eq. (7-40) by the use of Eqs. (7-34) and (7-37). The details of the algebraic process will be left to the student. Equation (7-34) should be written as a scalar equation, *i.e.*, for the case of a central impact, since Eq. (7-37) is written with this limitation. The final result is

$$E_1 - E_2 = \frac{1}{2}\frac{m_1 m_2}{(m_1 + m_2)}(U_1 - U_2)^2(1 - \epsilon^2) \qquad (7\text{-}41)$$

An inspection of this equation will show that, if the bodies are perfectly elastic ($\epsilon = 1$), the right-hand member becomes equal to zero and hence we may conclude for this special case that there will be no change in the kinetic energy. In other words, no kinetic energy is converted into other forms of energy for collisions between perfectly elastic bodies. It is also to be noticed that, if the bodies are inelastic ($\epsilon = 0$), a maximum amount of kinetic energy will be converted into other forms of energy.

Problems.—1. Two spheres of masses 50 and 100 g. are moving parallel to a fixed line and toward each other with velocities of 30 and 20 cm. per second, respectively. If the coefficient of restitution is 0.6 and the impact is central, find the velocities after impact and the change in energy due to impact.

2. Two bodies having masses of 2 and 5 lb. are moving with centers constantly in the same straight line. The larger mass is in front of the smaller and has a velocity of 10 ft. per second. The velocity of the smaller body is 15 ft. per second. The coefficient of restitution is 0.5. Find the velocities after impact and the change in kinetic energy.

7-9. Consistent Units.—In using any of the three dynamical equations in the solution of numerical equations, it is necessary to express all of the symbols involved in units of a single consistent set. There are four consistent sets of units which are in common usage. In both the c. g. s. and English systems there are two sets of units—the absolute units and the weight units. Some of the units have received special names; others have not. The four sets of consistent units are listed in the table below.

TABLE OF CONSISTENT UNITS

Quantity	C. g. s.		English	
	Absolute	Weight	Absolute	Weight
Displacement	cm.	cm.	ft.	ft.
Velocity	cm./sec.	cm./sec.	ft./sec.	ft./sec.
Acceleration	cm./sec.^{2}	cm./sec.^{2}	ft./sec.^{2}	ft./sec.^{2}
Force	dyne	gram weight ($g._w$)	poundal	pound weight ($lb._w$)
Mass	g.	$g._w/g$ (980)	pound	$lb._w/g$ (32)
Time	sec.	sec.	sec.	sec.
Momentum	g. cm./sec.	$g._w$. cm./g sec.	lb. ft./sec.	$lb._w$. ft./g ft.
Energy	g. cm.^{2}/sec.^{2}	$g._w$. cm.^{2}/g sec.^{2}	lb. ft.^{2}/sec.^{2}	$lb._w$.ft.^{2}/g ft.^{2}

In the foregoing table the symbol g is used to express the acceleration due to gravity and has a value which depends upon the particular place at which the weight forces are to be used. The given values, 980 cm. per second squared and 32 ft. per second squared, are approximate only.

Problems.—Certain aspects of the use of the fundamental equations in translational motion may be illustrated by a few numerical problems.

1. Two masses, 200 and 250 g., are hung, by means of a string, over a massless pulley. If the friction opposing the motion is constant and is equal to a 5-g. weight and acts only at the pulley, find the acceleration of the system and the tension of each cord.

The two masses must move with the same acceleration because they are fastened together with the cord. The fact that the lighter body moves up and the heavier one goes down does not present any difficulties. We may select a convention of signs which will take care of this peculiarity.

Let the signs of the forces, accelerations and velocities be taken positive when acting downward on the side of the heavier body and upward on the side of the lighter body. The assigned positive direction does not need to be in the actual direction of motion but it always seems easier to take it so. If a selection of positive direction is made and then after solving the equations for the acceleration, say, it turns out to be a negative quantity, this merely means that the acceleration is opposite to our assigned direction.

To express the resultant force F in the given case, we must add all the existing forces which are in the line of motion. If we express the force in dynes, we may write

$$\begin{aligned} F &= 250g + T - T - 200g - 5g + T' - T' \\ &= 45g \text{ dynes} \end{aligned}$$

where T and T' are the tensions in the cord between the heavier body and the pulley and between the lighter body and the pulley, respectively, and g is the gravitation constant. If there were no friction, then the tension in the cord would be the same on both sides of the pulley. In this case, however, the tensions must differ by 5 g. weight. We are at liberty to regard the cord on either side of the pulley as producing two tensions; *e.g.*, on the side of the heavier body, the cord pulls up with a force of $-T$ on the 250-g. mass and also pulls on the pulley with a force of $+T$. In such a case as this the tensions are internal forces in the system taken as a whole and therefore cannot affect the acceleration of the system as a whole. They cancel out in the expression for the resultant force as they should.

The total mass of the system is the sum of the two masses. Since we now know two quantities of the force equation, the third may be determined. By expressing the force in dynes, the masses in grams, and the acceleration in centimeters per second squared, the units are consistent; hence

$$\begin{aligned} a &= \frac{F}{M} \\ &= 45 \times \frac{980}{450} \\ &= 98 \text{ cm. per second squared} \end{aligned}$$

Having found the acceleration of the system and hence of either mass, we may now determine the tension in the cord on either side of the pulley. This may be done only by applying the force equation to a part of the system in such a way that the desired tension becomes an external force. If we now write the force equation for the 250-g. mass alone, the tension T becomes an external force and may therefore be determined. The resultant force on the 250-g. mass is $250g - T$; hence by substituting in the force equation, since the mass and acceleration are known, we have

$$\begin{aligned} 250g - T &= 250 \times 98 \\ T &= 220{,}500 \text{ dynes or } 225 \text{ g.}_{\text{w}}. \end{aligned}$$

In a similar manner the tension on the other side of the pulley may be found and it comes out to be 220 g._{w}.

2. A variable force of magnitude $(50 - 10\,t)$ poundals is acting on a mass of 10 lb. for 5 sec. If the body is initially at rest, what will be its final velocity?

In this problem the force, time, initial velocity, and mass are given and the final velocity is to be determined; hence the impulse equation may be used. Since the force is variable but is expressed as a function of the time, we may use the general expression

$$\int_0^t F\,dt = m(V - U)$$

Substituting the given values and integrating gives

$$\int_0^5 (50 - 10t)dt = 10V \qquad \text{or} \qquad V = 12.5 \text{ ft. per second}$$

3. A variable force is acting upon a 320-lb. mass. The force is expressed by the equation $F = (10 - 0.01\ s)$ lb.$_{w.}$, in which s is a coordinate measured in feet and parallel to the direction of the force. If the body has an initial velocity of 50 ft. per second in a direction parallel to the force, find the velocity which the body will have after it has moved 200 ft.

The data given introduce those quantities which are contained in the work equation; hence it is to be used here. Since the force is variable but is expressed in terms of the displacement, the general equation to be used is

$$\int_0^s F\,ds = \tfrac{1}{2}\,m\,(V^2 - U^2)$$

Substituting the given quantities for the symbols, and converting the given mass unit to that consistent with the other quantities, *i.e.*, $m = \frac{320}{32}$, and then integrating gives

$$\int_0^{200} (10 - 0.01\,s)ds = \tfrac{1}{2} \times \tfrac{320}{32}(V^2 - 50^2) \qquad V = 53.5 \text{ ft. per second}$$

Problems.—1. Two masses, 150 and 100 g., are supported by a light cord over a massless pulley. Find the acceleration of the system and the tension in the cord.

2. A 50-g. mass is restricted to horizontal motion on a table. It is connected by a cord to a 100-g. mass. The cord passes over a massless pulley at the edge of the table and supports the 100-g. mass at some point vertically below the pulley. If the friction is 1,000 dynes, find the acceleration and the tension in the cord.

3. Three masses, 50, 100, and 75 lb., are arranged to move in the same vertical plane. The 100-lb. mass can move only on a smooth horizontal table top. Strings attached to the mass support the other masses over opposite edges of the table, so that vertical motion only of the 50- and 75-lb. masses is possible. Neglecting all friction and masses of pulleys, find the acceleration and tensions.

4. A mass of 25 g. is made to rotate in a circular path having a radius of 50 cm. at the rate of 2 r.p.s. What force is required to keep the mass in the path?

5. A 100-g. mass, initially at rest, is subjected to a variable force for 10 sec. The magnitude of the force is given by the expression $50\sqrt{t}$ dynes. A force of resistance expressed by $(15 - t)$ dynes is also acting. What is the velocity at the end of 10 sec.?

6. Two electrically charged bodies are arranged so that one of them is fixed and the other (mass, 2 g.) may move in a horizontal line without

friction. Initially they are 20 cm. apart and the force between them is one of attraction and is equal to 100 dynes. The force varies inversely as the square of the distance. Determine the velocity of the movable body after it has moved a distance of 15 cm.

7. A 3,000-lb. automobile can be accelerated from 5 to 30 m.p.h. in 5 sec. What force is necessary?

8. A 3,000-lb. automobile is moving at a speed of 30 m.p.h. What braking force is required to stop the automobile in a distance of 40 ft.?

9. A 1,000-lb. elevator starts upward with an acceleration of 5 ft. per sec.^{2} Find the tension in the supporting cable. What would be the force exerted by the floor of the elevator upon a 100-lb. person standing in the elevator?

10. A spring, which is normally 3 ft. long when in an unstretched position, has a stiffness such that a force of 10 lb. weight will stretch it a distance of 1 ft. One end of the spring is attached to a rigid support and the other end is fastened to a 5-lb. mass, the arrangement being such that horizontal motion only is possible. If the spring is stretched so that its normal length is increased to 5 ft. and the mass is initially at rest when the spring is released, what will be the velocity of the mass when the spring returns to its normal length? (Neglect the mass of the spring and friction and assume that the force exerted upon the 5-lb. mass is proportional to the elongation of the spring.)

11. Two spheres of the same size but of unequal masses are dropped simultaneously from some point above the ground. Assuming that both are subjected to the same force of friction, prove that the heavier mass will reach the ground first.

12. A mass of 200 g. is hung from a massless spring, the upper end of which is attached to a rigid support. If the stiffness of the spring is such that a load of 50 g. produces a displacement of 10 cm., what would be the period of the simple harmonic motion which will take place when the 200-g. mass is released from a small vertical displacement? (Assume $g = 980$ cm. per second squared.)

13. Two masses, 200 and 250 g., are suspended by a cord which passes over a massless pulley as in the Atwood's machine. An arrangement is provided so that the moving system picks up an additional mass of 60 g. (which is at rest before the impact) after the two masses have gone a distance of 40 cm. If the system is initially at rest, what will the velocities be just before and after impact?

14. A flexible heavy chain is hung over a massless pulley. Its linear mass is 10 lb. per foot. It is at rest initially with one side slightly longer than the other. Write an expression for its acceleration in any position.

15. A leaky bucket filled with water is held initially in a position of equilibrium by a cord which passes over a massless pulley to a counterpoise on the other side. If the leak is m g. per second, what is the acceleration in any position?

CHAPTER VIII

THE DYNAMIC EQUATIONS FOR PURE ROTATION

8-1. Introduction.—In order to introduce the student to those dynamical quantities which are fundamental in rotational motion, we shall consider the effect which an external force will produce in the motion of a rigid body mounted on a fixed axis. Let us select a reference system with OZ the axis of rotation and perpendicular to the plane of the diagram (Fig. 84). An external force F is to be supplied to the body. For convenience, let F

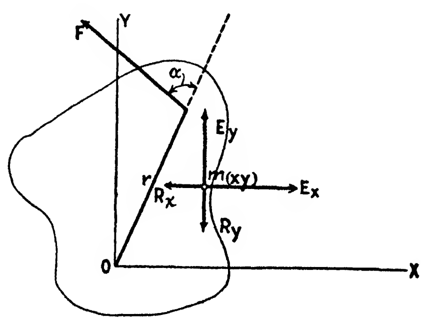

FIG. 84.

be in the XY plane and let it not intersect the Z-axis. Experience teaches us that, under the assigned conditions, the body will have an angular acceleration.

Now we may consider that the body consists of a large number of particles and that to each particle is allotted its particular share of the *external* force F. Let the components of the force, assigned to the particle of mass m at the point x y z, be E_x and E_y. In addition to the external force upon each particle there will be an *internal* force which, for the particle at x y z, may be represented by the components R_x and R_y. The internal forces are supplied by cohesion.

The force equations for the selected particles are

$$E_x + R_x = m\frac{d^2x}{dt^2} \qquad E_y + R_y = m\frac{d^2y}{dt^2} \tag{8-1}$$

We may sum up such equations as may be written for all particles of the body with the following results:

$$\sum E_x = \sum\left(m\frac{d^2x}{dt^2}\right) \qquad \sum E_y = \sum\left(m\frac{d^2y}{dt^2}\right) \tag{8-2}$$

The sums of the internal forces ΣR_x and ΣR_y taken over the entire body must be zero. The proof for this statement is left to the student.

Up to this point we have used only translational quantities. To introduce the rotational quantities let us examine the effect of changing the position of the external force $\boldsymbol{F}$. Experience teaches us that not only does the magnitude of the force influence the motion of the body under consideration, but the perpendicular distance (the lever arm) from the axis of rotation to the line of the force is also important. A single quantity, the force moment or torque, has been designed to combine these two factors. We may define the *force moment* ($\boldsymbol{M}$) by the vector equation

$$\boldsymbol{M} = \boldsymbol{r}_F \times \boldsymbol{F} \tag{8-3}$$

in which $\boldsymbol{r}_F$ is a vector which is drawn from the origin of the reference system to the point of application of the force $\boldsymbol{F}$. The moment of force is a vector quantity whose direction is perpendicular to the plane containing $\boldsymbol{r}_F$ and $\boldsymbol{F}$ and whose magnitude is $r_F F \sin \alpha$ if α is the angle measured from $\boldsymbol{r}_F$ to $\boldsymbol{F}$. The vector $\boldsymbol{r}_F$ becomes the so-called lever arm when α is a right angle. A more detailed discussion of the force moment is reserved for Sec. **8-13** (below).

To introduce force moments into Eq. (8-2) and to obtain, thereby, equations which will serve to supply us with information about other rotational quantities, we shall multiply both members of the first equation by $-y$ and the second by x and then by adding obtain

$$\sum (x\,E_y - y\,E_x) = \sum\left[m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right)\right] \tag{8-4}$$

The left-hand member of this equation is equivalent to the moment of the external force $\boldsymbol{F}$ about the Z-axis. The right-hand member must therefore express the effect of the force moment upon the body as a whole. To effect the summation of this member some manipulation is necessary. The clue to the transformation is to be found in introducing the angular velocity

(ω) of the body, since it is common to all particles of the body. This may be done by means of the relations between angular velocity and linear velocity, of the type form $V = \omega \times r$, where r is the vector giving the position of the particle whose velocity is V.

Since the mass m of Eq. (8-4) is constant, we may write

$$\sum\left[m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right)\right] = \frac{d}{dt}\sum\left[m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)\right] \quad (8\text{-}5)$$

$$= \frac{d}{dt}\sum[m(r \times V)]_z \quad (8\text{-}6)$$

In the vector expression $r = ix + jy$. The subscript z is used to indicate that we are using here only the z component of the more general quantity $m\ (r \times V)$.

We may put $V = \omega \times r$ in Eq. (8-6) and then equate this expression to the first member of Eq. (8-4), which is in reality the z component of the force moment $r_F \times F$, written here as $[r_F \times F]_z$.

$$[r_F \times F]_z = \frac{d}{dt}\sum[m(r \times (\omega \times r))]_z \quad (8\text{-}7)$$

$$= \frac{d}{dt}\sum(mr^2\omega)_z$$

$$= \frac{d}{dt}\,(I\omega)_z \quad (8\text{-}8)$$

In obtaining the last expression we must remember that ω is common to all particles (since the body is rigid) and we have put

$$I = \Sigma(mr^2) \quad (8\text{-}9)$$

The student should carefully go over the derivation of Eq. (8-8) to make sure that he understands why the introduction of the z component in the vector expressions [Eq. (8-6) and following] is necessary.

From Eq. (8-8) we may write the following more general expression

$$M = r_F \times F = \frac{d}{dt}\,(I\omega) \quad (8\text{-}10)$$

in which M is the resultant force moment.

The quantity I is a very important rotational quantity. As shown by the defining equation, it is obtained by summing up the products formed by multiplying the mass of each particle by the square of its distance from the rotation axis. It is called the *moment of inertia.* In place of the summation sign we may use the integration symbol, which gives

$$I = \int_0^m r^2 dm \tag{8-11}$$

which means that each mass element dm is to be multiplied by the square of the distance of the element from a selected axis. The axis from which r is to be measured may be any line fixed in the body. In any particular case the position of the axis must be specified.

Moment of inertia is a scalar quantity. It occupies a position in rotational motion which is analogous to mass in translational motion. It is a quantitative expression for what we may call rotational inertia, or a measure of the tendency of a body, which is rotating, to continue to rotate with no change in its angular velocity.

Another rotation quantity is introduced by Eq. (8-10). This is the quantity $I\omega$ which is called the *rotational* or *angular momentum* or sometimes *moment of momentum.* It is a vector quantity and the direction of it is that of the angular velocity.

Equation (8-10) is one of the three important dynamical equations in rotational motion. It is called the *force-moment* equation. It expresses the important fact that the effect of a force moment applied to a body is measured by the time rate of change of the rotational momentum of that body. In many of the cases encountered, the moment of inertia of the body is constant. In such a case the force-moment equation may be written

$$M = I \frac{d\omega}{dt} \tag{8-12}$$

introducing the *angular acceleration* $d\omega/dt$, which is the time rate of change of the angular velocity.

All of these rotational quantities are to be examined in more detail in the work which follows.

8-2. The Moment of Inertia.—The particular value of the moment of inertia for any given case depends upon the distribution of the mass of the body with reference to some given line about which the moment of inertia is desired. If the body is a regular solid or has a form which may permit an analytical expression of the distribution of its mass, then the moment of inertia I may be determined by a direct integration of the formula (repeated here for convenience)

$$I = \int_0^m r^2 dm \qquad (8\text{-}11)$$

where r is the distance of the particle dm from the selected axis.

This integral may be converted into a volume integral by replacing dm by its equivalent expression $\rho dx\ dy\ dz$, if ρ designates the density and $dx\ dy\ dz$ the volume of the differential particle. Now if ρ is constant throughout the entire volume, then it may be placed outside the integration sign. If, however, the density of the body is not constant but may be expressed as a function of the coordinates of dm referred to some reference system, then this function must be written for ρ. If the density cannot be expressed as a function of the coordinates of dm or if the body is too irregular to permit integration, then the only way in which the moment of inertia may be found is by the use of some experimental method, one of which is described later (Sec. **8-19**).

In general, the volume integral used for finding the moment of inertia involves a triple integration. In case one dimension of the body is small in comparison with the two others, as in the case of a sheetlike body, the integration may be simplified by confining it to two dimensions. If the body is essentially a one-dimensional solid, a single integration is usually sufficient to give a result which is accurate enough for practical purposes.

Even though all the dimensions of the body must be taken into account, it is frequently possible to select a differential mass or volume in such a manner that one or two integrations may be eliminated. For a good working rule, applicable in many cases, it is best to select the mass element as large as possible, the selection being subject to one limitation, *viz.*, that all parts of the mass element must be situated at one common distance from the axis of rotation. This may be stated in another, perhaps clearer, way. The differential mass may be the locus of all mass points which are equally distant from the axis of rotation.

Illustration.—a. Find the moment of inertia of a thin rod of length L and of uniform cross section, about an axis which passes perpendicularly through one end. The density of the rod varies uniformly from zero at one end to a value K at the other.

Let the axis about which the moment of inertia is desired pass through the less dense end of the rod. Also let x (Fig. 85) be a coordinate measured along the rod from the less dense end O as the origin. The density at any point of the rod may therefore be written

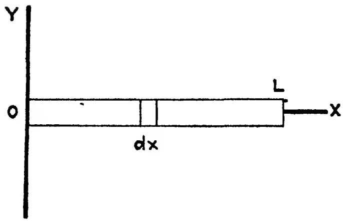

FIG. 85.

$$\rho = \frac{Kx}{L}$$

If A is the uniform area of cross section of the rod, then

$$\begin{aligned} dm &= \rho\, A\, dx \\ &= KA\, x\, \frac{dx}{L} \end{aligned} \tag{8-13}$$

Substituting this value in the general expression for the moment of inertia and replacing the mass limit by the limits for the coordinate gives

$$I = \frac{KA}{L}\int_0^L x^3 dx \tag{8-14}$$

$$= \frac{KA}{L}\frac{L^4}{4}$$

$$= \tfrac{1}{2}\, ML^2 \tag{8-15}$$

in which M is the mass of the rod and is equal to $\frac{1}{2}$ KLA.

If the axis of rotation passes through the other end, then by the use of the same figure, the density would be $(L - x)\, K/L$. With this change the details could be carried through as indicated above. The final result would be $ML^2/6$.

b. Find the moment of inertia of a homogeneous right circular cylinder about the longitudinal axis.

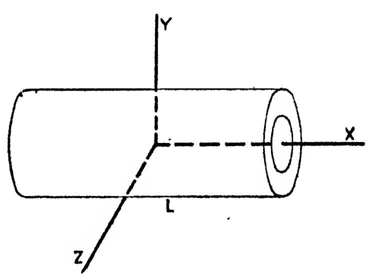

FIG. 86.

In this case one might select $\rho dx\ dy\ dz$ as the differential mass and, with a reference system having the origin situated at the center of mass as shown in Fig. 86, perform the triple integration indicated by the expression

$$I = \iiint (y^2 + z^2)\, \rho dx\, dy\, dz \tag{8-16}$$

A much simpler method would be to select a larger mass element. We may take a cylindrical shell concentric with the cylinder, having thickness dr, length L, and base of variable radius r, for the differential mass, because all parts of such a shell are at the same distance from the axis. If the radius of the cylinder is a, then the general expression and final result may be written as follows:

$$I = \rho \int_0^a r^2 \, 2\pi r \, L \, dr$$
$$= \tfrac{1}{2} M a^2 \qquad (8\text{-}17)$$

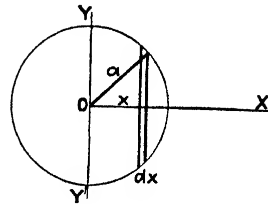

FIG. 87.

c. Find the moment of inertia of a thin, uniform, circular lamina about a diameter.

Let the selected axis be $Y'OY$ (Fig. 87) and let a be the radius of the disk. For the differential mass we may use a strip of width dx situated parallel to the axis $Y'OY$ and at a distance x from it. If we represent the thickness of the lamina by t, then the mass of the strip will be

$$dm = 2\rho t\sqrt{a^2 - x^2}\, dx \qquad (8\text{-}18)$$

The general expression then becomes

$$I = 2\rho t \int_{-a}^{a} x^2\sqrt{a^2 - x^2}\, dx$$
$$= \tfrac{1}{4} M a^2 \qquad (8\text{-}19)$$

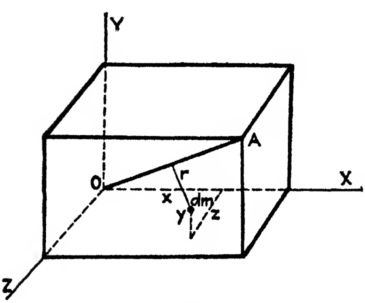

FIG. 88.

d. Find the moment of inertia of a homogeneous rectangular parallelepiped about a diagonal.

Let the dimensions of the parallelepiped be $2a$, $2b$, and $2c$. Place the reference system as shown in the Fig. 88. In this case a triple volume integration will be necessary. The differential mass is $\rho dx\, dy\, dz$ and its coordinates are x, y, z. The distance r of this differential element from the axis OA, about which the moment of inertia is to be found, is given by the expression

$$r^2 = x^2 + y^2 + z^2 - (l^2x^2 + m^2y^2 + n^2z^2 + 2lmxy + 2lnxz + 2mnyz) \qquad (8\text{-}20)$$

in which l, m, and n are the direction cosines of the axis OA. The values of l, m, and n are given by the equations

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}} \qquad m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}} \tag{8-21}$$

The expression for the moment of inertia about OA is then

$$I = \rho \int_0^{2a} \int_0^{2b} \int_0^{2c} r^2 \, dx \, dy \, dz \tag{8-22}$$

If we substitute in this equation the expression for r^2 and perform the integration, the following result is obtained:

$$I = \frac{2M}{3} \frac{(a^2b^2 + a^2c^2 + b^2c^2)}{(a^2 + b^2 + c^2)} \tag{8-23}$$

8-3. Radius of Gyration.—In the preceding section, formulas were developed which express the moments of inertia of a few selected bodies about certain specified axes in terms of the masses of the bodies and one or more of their dimensional quantities. It is sometimes more convenient to express the moment of inertia by the simple relation,

$$I = M\,G^2$$

in which I is the moment of inertia about a specified axis, M is the mass of the body, and G is the so-called radius of gyration.

The quantity G^2 is simply a symbol for expressing collectively the geometrical elements (together with numerical factors) of the more fundamental expressions for the moment of inertia. For example, the moment of inertia of a homogeneous sphere of radius r and of mass M about a diameter is $(\frac{2}{5})Mr^2$. In this case the square of the radius of gyration is equal to $(\frac{2}{5})r^2$. For the rectangular parallelepiped with the diagonal as axis, the square of the radius of gyration is the coefficient of the mass M in the right-hand member of Eq. (8-23).

8-4. Moment of Inertia about Parallel Axes.—A theorem is to be developed which expresses the moment of inertia of a body about any axis in terms of the moment of inertia about a parallel axis passing through the center of mass of the body. This relation is a very convenient one for simplifying some determinations which might otherwise be troublesome.

Let a cross section of the body be represented as shown in Fig. 89 with O the center of mass of the body. The moment of inertia is to be expressed for an axis which passes through any point P, and which is perpendicular to the plane of the diagram. Let dm be the differential mass situated at distances b and r from O and P, respectively, and let R be the distance between O and P.

The moment of inertia I_P of the body about the axis through P is expressed by the equation

$$I_P = \int_0^M r^2 \, dm \qquad (8\text{-}24)$$

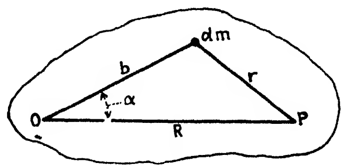

FIG. 89.

If we replace r^2 by its value

$$r^2 = b^2 + R^2 - 2bR \cos \alpha$$

where α is the angle between the lines b and R, the expression becomes

$$I_P = \int_0^M b^2 \, dm + \int_0^M R^2 \, dm - 2\int_0^M (bR \cos \alpha) dm \qquad (8\text{-}25)$$

The first integral of the right-hand member expresses the moment of inertia (I_0) about an axis through O, the center of mass, if the axis is perpendicular to the plane of the diagram. The second integral becomes MR^2, where M is the mass of the body, since R is constant.

The third integral reduces to zero. This is readily seen if R is placed outside the integration sign and the integration of the remaining quantity considered. If we select a reference system XOY in the plane of the diagram with the origin at O, the center of mass, and with the axis OX coincident with OP, then it may be readily seen that

$$\frac{1}{M}\int_0^m b \cos \alpha \, dm = \frac{1}{M}\int_0^m x \, dm \qquad (8\text{-}26)$$

which expresses the x coordinate of the center of mass in the selected reference system. Since O is the center of mass, the integral must be equal to zero. Equation (8-25) therefore reduces to

$$I_p = I_0 + MR^2 \qquad (8\text{-}27)$$

By means of this equation we may determine the moment of inertia about an axis through any point, provided that we know the moment of inertia about a parallel axis which passes through the center of mass, the distance between the two axes, and the mass of the body.

In case the point O is not the center of mass, then the last integral in Eq. (8-25) is not equal to zero. In any such case the value of the third integral might be determined from a knowledge of the distribution of the mass of the body. The general expression would then not reduce to the simple expression given by Eq. (8-27) and hence would probably be of little value. The moment of inertia I_Q for any other line parallel to the selected axis through O and passing through some point Q could be expressed as follows:

$$I_Q = I_0 + MR'^2 \tag{8-28}$$

if R' is the distance between the two axes. Combining this equation with Eq. (8-27) by eliminating I_0 gives

$$I_P = I_Q + M(R^2 - R'^2) \tag{8-29}$$

An examination of this equation shows that, if $R = R'$, then $I_p = I_Q$. We may conclude, therefore, that the moments of inertia about all axes, equally distant from the axis through the center of mass and parallel to it, are equal.

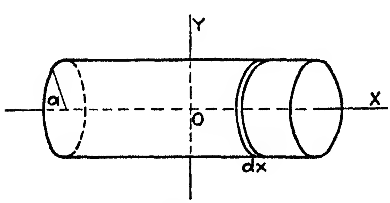

FIG. 90

If one returns to Eq. (8-27), it is to be noticed that, since all the terms of this equation are always positive, the moment of inertia about a given axis passing through the center of mass is a minimum for all moments of inertia of that body about axes which are parallel to the given axis.

Illustration.—Find the moment of inertia of a homogeneous right circular cylinder about an axis passing through its center of mass and perpendicular to the geometric axis.

Let the length of the cylinder be $2L$, its mass M, its density ρ, and its radius a.

Select the reference system as shown in the diagram (Fig. 90) with OY the axis about which the moment of inertia is to be found.

We may imagine that the cylinder is made up of a very large number of thin circular disks all perpendicular to the OX axis. The moment of inertia of a thin disk about a diameter is given by Eq. (8-19). Using the theorem expressed by Eq. (8-27), we may write the moment of inertia, of any circular disk, about the axis OY as follows:

$$dI = dI_c + dm\,x^2 \tag{8-30}$$

in which dI_c is the moment of inertia of the disk about an axis which passes through its center of mass and is parallel to OY and x is the distance between the parallel axes.

The moment of inertia of the cylinder may now be found by integrating the foregoing expression over the entire body. Hence

$$\begin{aligned} I &= \int_0^M \tfrac{1}{4} dm\,a^2 + \int_0^M dm\,x^2 \\ &= \frac{\pi\rho a^4}{4}\int_{-L}^{L} dx + \pi a^2\rho \int_{-L}^{L} x^2\,dx \\ &= M\left(\frac{a^2}{4} + \frac{L^2}{3}\right) \end{aligned} \tag{8-31}$$

8-5. Perpendicular Axes and a Lamina.—The moment of inertia for a lamina about any axis which is perpendicular to the plane of the lamina is equal to the sum of the moments of inertia about two mutually perpendicular lines which are in the plane of the lamina and which intersect the perpendicular axis at a common point. This theorem may be proved in the following manner.

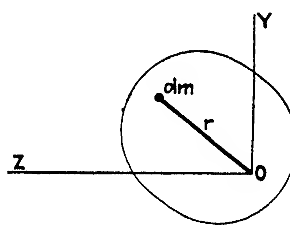

FIG. 91.

Suppose that the lamina lies in the YZ plane (Fig. 91) of the reference system XYZ. It is desired to express the moments of inertia I_x, about the axis OX, in terms of the moments of inertia I_y and I_z, about OY and OZ, respectively. The coordinates of any differential particle may be taken as y and z. If dm is situated at a distance r from the axis OX and M is the mass of the lamina, then

$$I_x = \int_0^M r^2\,dm$$

Since

$$r^2 = y^2 + z^2$$

$$\begin{aligned} I_x &= \int_0^M y^2\,dm + \int_0^M z^2\,dm \\ &= I_z + I_y \end{aligned} \tag{8-32}$$

This relation is obviously also true for any other pair of axes, Y' and Z', in the YZ plane as long as they intersect at the point O. Hence we may write

$$I_y + I_z = I_{y'} + I_{z'} \tag{8-33}$$

Illustration.—Find the moment of inertia of a homogeneous sphere of radius a about any diameter.

Select a reference system XYZ with the origin at the center of the sphere and the axis OX about which the moment of inertia is to be found.

We may imagine that the entire sphere is divided into a large number of circular disks each of differential thickness (dx) and all perpendicular to OX (Fig. 92).

Since each disk is a circular lamina, the moment of inertia of a disk about any of its diameters is equal to $\frac{1}{4}\,dm\,r^2$, where r is the radius of the disk. Applying the theorem expressed by Eq. (8-32), we may write the expression for the moment of inertia (dI_x) of any disk of differential thickness dx about the OX-axis as follows, if $dI_{y'}$ and $dI_{z'}$ are the differential moments of inertia of the disk about two mutually perpendicular axes which are in the plane of the disk and the intersect on OX-axis:

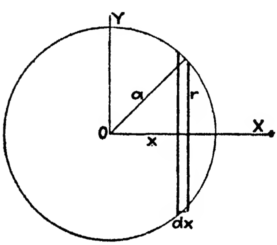

Fig. 92.

$$\begin{aligned} dI_x &= dI_{y'} + dI_{z'} \\ &= \tfrac{1}{2}\,dm\,r^2 \end{aligned} \tag{8-34}$$

If r is the radius of any disk and ρ is the density, then

$$\begin{aligned} dm &= \pi r^2 \rho\, dx \\ &= \pi\rho(a^2 - x^2)\, dx \end{aligned}$$

We may substitute this value of dm in Eq. (8-34) and integrate over the entire mass because the axes of all such laminas coincide. Hence

$$\begin{aligned} I_x &= \int_{-a}^{a} \frac{\pi\rho}{2}(a^2 - x^2)^2\, dx \\ &= \frac{2Ma^2}{5} \end{aligned} \tag{8-35}$$

Problems.—Determine the moment of inertia for each of the following homogeneous bodies about the axis indicated:

1. Sphere of radius a about a diameter, using the method which involves the theorem of parallel axes.
2. Sphere of radius a about a tangent.
3. Right circular cone of height h and radius of base a about the axis of the figure.

4. Right circular cone of height h and radius of base a about a diameter of its base.
5. Hollow circular cylinder, length L, external radius r_1 and internal radius r_2 about the geometrical axis.
6. An ellipsoid having axes $2a$, $2b$, and $2c$ about the $2a$-axis.
7. Rectangular parallelepiped having sides a, b, and c about an axis through the center perpendicular to the face ab.
8. Spherical shell, external radius r_1 and internal radius r_2 about a diameter.
9. An elliptical lamina having axes $2a$ and $2b$ about a normal to the lamina and through the center.

8-6. The Principal Axes of a Body.—It is to be seen in the foregoing theorems and problems that the lines through the center of mass are of peculiar value in expressing the transfer of moment of inertia from one line to a parallel line. By a judicious selection of three mutually perpendicular lines (the principal axes) which pass through the center of mass of the body, the moment of inertia about any other line through the center of mass may be expressed in terms of the moments of inertia about these three principal axes.

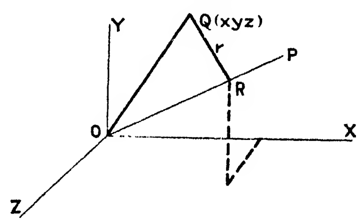

FIG. 93.

Given the reference system XYZ (Fig. 93) with origin at O, the center of mass, and any line OP having direction cosines l, m, and n. Let a differential mass be located at Q $(x\ y\ z)$ and let R be a point in OP situated so that QR is perpendicular to OP. If $QR = r$, then

$$\begin{aligned} r^2 &= OQ^2 - OR^2 \\ &= x^2 + y^2 + z^2 - (lx + my + nz)^2 \end{aligned} \qquad (8\text{-}36)$$

since OR is the projection of OQ upon OP.

Since

$$l^2 + m^2 + n^2 = 1$$

we may introduce it as a factor into the equation above and then rearrange the terms as shown.

$$r^2 = (l^2 + m^2 + n^2)(x^2 + y^2 + z^2) - (lx + my + nz)^2 \qquad (8\text{-}37)$$

$$= l^2\,(y^2 + z^2) + m^2\,(z^2 + x^2) + n^2\,(x^2 + y^2) - 2lmxy - 2mnyz - 2nlzx \qquad (8\text{-}38)$$

The moment of inertia of the body about the axis OP is therefore

$$\begin{aligned}
I_p &= \int_0^M r^2\,dm \\
&= l^2\int_0^M (y^2 + z^2)dm + m^2\int_0^M (z^2 + x^2)dm + n^2\int_0^M (x^2 + y^2)dm \\
&\qquad - 2mn\int_0^M yzdm - 2nl\int_0^M zx\,dm - 2lm\int_0^M xydm \\
&= l^2A + m^2B + n^2C - 2mnD - 2nlE - 2lmF \qquad (8\text{-}39)
\end{aligned}$$

in which the letters A, B, C, D, E, and F are used to designate the integrals in the right-hand members in the order in which they are written.

The quantities A, B, and C are moments of inertia about the axes OX, OY, and OZ, respectively. The quantities D, E, and F are spoken of as products of inertia. These are dimensionally similar to moments of inertia but possess the distinguishing characteristic of having a product of two different coordinates in place of the square of a single coordinate.

If I_p is to be expressed in terms of A, B, and C only, then it will be necessary to select the positions of the reference axes in the body in such a manner that the products of inertia are each equal to zero. This may be done by inspection if the solid possesses a sufficient degree of symmetry. If the XY plane is one of symmetry, then D and E will both be equal to zero, because each contains the coordinate z. If either of the two other principal planes is one of symmetry, then F will also be equal to zero. For any selection of the axes which makes two of the principal planes occupy symmetrical positions in the body, all three products of inertia reduce to zero and the three coordinate axes are then called principal axes of the body.

Equation (8-39) is valid for a reference system situated anywhere in the body and may be used for determining the moment of inertia about any line, such as OP, which passes through the origin, provided that the direction cosines of OP, the moments of inertia A, B, and C, and the products of inertia D, E, and F may be determined.

Problem.—Using Eq. (8-39), determine the moment of inertia of a rectangular parallelepiped about a diagonal. The sides of the figure are to be taken equal to $2a$, $2b$, and $2c$. Two methods are to be used, one making use of a reference system with origin at the center of mass and axes parallel to the edges of the body and the other with the origin at a vertex and axes forming the sides of the solid. Obviously the results should be independent of the position of the axes.

8-7. The Momental Ellipsoid.—From the foregoing results it is seen that the moment of inertia I_p about any line such as OP which passes through the point O may be expressed in terms of the six constants A, B, C, D, E, F and the direction cosines of the line OP referred to the axes of reference. For every line OP there is a definite value for I_p. If now we draw a radius vector along OP from O as origin and make the length such that its square is inversely proportional to I_p for the particular position, the locus of the end points of all such vectors drawn for all possible positions of OP will describe a surface which can be shown to be an ellipsoid for any particular body. This surface must be a closed surface, for I_p cannot be zero or infinite for any rigid body.

Let the length of the radius vector be expressed by r. Since r^2 is to be inversely proportional to I_p, then we may write

$$I_p r^2 = K \text{ (a constant)}$$

If we now multiply each term of Eq. (8-39) by r^2 and replace r^2l^2 by x^2, etc., and r^2mn by yz, etc., where x, y, z are the coordinates of the end point of the vector r, the equation becomes

$$K = Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy \qquad (8\text{-}40)$$

The locus of the terminal point of the radius vector r is given by this equation which is a closed quadric and must be the surface of an ellipsoid.

This ellipsoid is called the *momental ellipsoid*. The center of the ellipsoid is the point O. This point may be any point of the body, but it is usually taken at the center of mass for convenience. The axes of the ellipsoid are called the *principal axes* at the point O. The quantities A, B, and C are the principal moments of inertia at the point O. If the geometrical axes of the ellipsoid are taken as the axes of the reference system, then the equation becomes simplified by that choice and reduces to

$$K = Ax^2 + By^2 + Cz^2 \qquad (8\text{-}41)$$

If the equation of the ellipsoid, referred to a coordinate system with origin at the center of the ellipsoid, is given, the positions of the axes of the ellipsoid may be determined by the following process. Since each axis of the ellipsoid must be perpendicular to the surface of the ellipsoid at the point where it cuts the surface, the direction cosines of that axis must be proportional to the corresponding partial derivatives of the equation of the ellipsoid. Now if r_1 is a radius vector which is one of the semi-axes, its direction cosines l_1, m_1, n_1 must be proportional to the corresponding partial derivatives. If $F(xyz)$ is the equation of the ellipsoid, then this condition may be expressed as follows·

$$\frac{\dfrac{\partial F(xyz)}{\partial x}}{l_1} = \frac{\dfrac{\partial F(xyz)}{\partial y}}{m_1} = \frac{\dfrac{\partial F(xyz)}{\partial z}}{n_1} \tag{8-42}$$

If $F\ (x\ y\ z)$ is the equation for the ellipsoid as given in Eq. (8-40), then taking the indicated partial derivatives and substituting in Eq. (8-42) gives

$$\frac{Ax - Fy - Ez}{l_1} = \frac{By - Dz - Fx}{m_1} = \frac{Cz - Dy - Ex}{n_1} \tag{8-43}$$

Since $x/r_1 = l_1$; $y/r_1 = m_1$; $z/r_1 = n_1$, if we divide each member of Eq. (8-43) by r_1. we may substitute these equivalents and thereby obtain

$$\frac{Al_1 - Fm_1 - En_1}{l_1} = \frac{Bm_1 - Dn_1 - Fl_1}{m_1} = \frac{Cn_1 - Dm_1 - El_1}{n_1} \tag{8-44}$$

In anticipation of the value of these fractions, let us put each member equal to I and write

$$\begin{aligned} Il_1 &= Al_1 - Fm_1 - En_1 \\ Im_1 &= Bm_1 - Dn_1 - Fl_1 \\ In_1 &= Cn_1 - Dm_1 - El_1 \end{aligned} \tag{8-45}$$

Multiplying the first of these equations by l_1, the second by m_1, and the third by n_1, adding, and collecting similar terms gives

$$I = Al_1^2 + Bm_1^2 + Cn_1^2 - 2Dm_1n_1 - 2El_1n_1 - 2Fl_1m_1 \tag{8-46}$$

which shows that I is the moment of inertia for the line whose direction cosines are $l_1\ m_1\ n_1$, and hence the suggested test is valid for determining the axes of the ellipsoid. The student must not confuse the axes of the ellipsoid with the axes (XYZ) of the reference system. It is to be remembered here that A, B, and C

are moments of inertia about the XYZ-axes and D, E, and F products of inertia for those axes.

If the moment of inertia for an axis of the ellipsoid is to be determined in terms of A, B, C, D, E, and F, then l_1, m_1, and n_1, the direction cosines of that axis, must be eliminated from the three equations of Eqs. (8-45). These equations may be written again in the form which makes the use of the determinant easy to apply:

$$\begin{aligned}(I - A)\,l_1 + \quad & Fm_1 + \quad & En_1 = 0\\ Fl_1 + (I - B)\ & m_1 + \quad & Dn_1 = 0\\ El_1 + \quad & Dm_1 + (I - C)\ & n_1 = 0\end{aligned}$$

On eliminating l_1, m_1, and n_1 we have

$$\begin{vmatrix} I - A & F & E \\ F & I - B & D \\ E & D & I - C \end{vmatrix} = 0 \qquad (8\text{-}47)$$

This gives a cubic equation in I, the roots of which we may call I_1, I_2, and I_3. These three values of I are the moments of inertia about the three axes of the ellipsoid.

After obtaining a solution of Eq. (8-47), if the positions of these axes are desired, their direction cosines may be determined by substituting I_1 (or I_2 and I_3) in Eq. (8-45) and solving for l_1, m_1, and n_1.

In general, these three moments of inertia will be all different. If two are alike, then the ellipsoid is one of revolution. If three are alike, the ellipsoid is a sphere, with the consequence that all lines through the center have equal moments of inertia.

Problems.—1. Find the principal axes and the equation of the momental ellipsoid for a rectangular parallelepiped, taking the center of the parallelepiped as the origin of the axis of reference, and $2a$, $2b$, $2c$ the edges of the solid. Take reference axes perpendicular and parallel to the faces of the figure.

2. Determine the momental ellipsoid and the principal axes of a cube whose edge is a, taking the origin of the reference system at one vertex, and three intersecting edges of the cube as reference axes.

3. Find the momental ellipsoid for a right circular cone of height h and radius of base a, taking the origin of the reference system at the center of mass, with the geometric axes of the cone one of the reference axes.

8-8. The Principal Axes of a Thin Lamina.—Given a lamina with the reference axes OX and OY (Fig. 94) about which the moments of inertia I_x and I_y are known. Let a second pair of axes $X'OY'$, also in the plane of the lamina with O the common origin, make an angle α with XOY. To find the relation between I_x, I_y; $I_{x'}$, $I_{y'}$; and α.

Let the differential mass dm (or $\rho\, dx\, dy$) be situated at P with coordinates x and y. Draw the line PQ perpendicular to OX' with Q on the line OX'; then we may put

$$PQ = y \cos \alpha - x \sin \alpha$$

Using this relation, the value of $I_{x'}$, may be expressed as follows:

$$\begin{aligned} I_{x'} &= \int_0^M (y \cos \alpha - x \sin \alpha)^2\, dm \\ &= I_x \cos^2 \alpha + I_y \sin^2 \alpha - 2\int_0^M xy \sin \alpha \cos \alpha\, dm \\ &= I_x \cos^2 \alpha + I_y \sin^2 \alpha - F \sin 2\alpha \qquad (8\text{-}48) \end{aligned}$$

where F is the product of inertia with respect to the axes OX and OY.

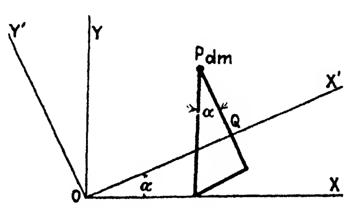

Fig. 94.

By replacing α by $\alpha + \frac{\pi}{2}$, we may write the moment of inertia about OY'. Hence

$$I_{y'} = I_x \sin^2 \alpha + I_y \cos^2 \alpha + F \sin \alpha \qquad (8\text{-}49)$$

Subtracting $I_{y'}$ from $I_{x'}$ gives

$$I_{x'} - I_{y'} = (I_x - I_y) \cos 2\alpha - 2F \sin 2\alpha \qquad (8\text{-}50)$$

Obviously the lamina is a special case of a solid body, *viz.*, one in which one of the dimensions is made very small. Both Eqs. (8-48) and (8-49) could have been found from the more general equation [Eq. (8-39)] by putting z, say, equal to zero and writing $\cos \alpha$ for l, $\sin \alpha$ for m, and zero for n.

Equations (8-48) and (8-49) may be changed into slightly different forms by replacing the functions of a single α by ones containing 2α. The altered expressions are

$$I_{x'} = \tfrac{1}{2}(I_x + I_y) + \tfrac{1}{2}(I_x - I_y) \cos 2\alpha - F \sin 2\alpha \qquad (8\text{-}51)$$
$$I_{y'} = \tfrac{1}{2}(I_x + I_y) - \tfrac{1}{2}(I_x - I_y) \cos 2\alpha + F \sin 2\alpha \qquad (8\text{-}52)$$

8-9. Maximum and Minimum Moments of Inertia of a Lamina.—It is sometimes necessary to find the axis for a lamina about which there is a maximum or a minimum value of the moment of inertia. By using the notation and figure of the preceding section it may be seen that, since $I_{x'}$ or $I_{y'}$ varies with α, the first differential of these quantities with respect to α gives the value of α for which the corresponding moment of inertia is maximum or minimum. Differentiating Eq. (8-51) with respect to α gives

$$\frac{dI_{x'}}{d\alpha} = (I_y - I_x) \sin 2\alpha - 2F \cos 2\alpha$$

If we designate by α' the particular value which the angle has for a maximum or minimum position, then

$$(I_y - I_x) \sin 2\alpha' - 2F \cos 2\alpha' = 0$$

$$\tan 2\alpha' = \frac{2F}{(I_y - I_x)} \qquad (8\text{-}53)$$

By substituting the values of I_y, I_x, and $2F$ in this equation the desired values of α' are obtained. The two values of the angle which satisfy this equation are $2\alpha'$ and $2\alpha' + \pi$. The significance of the two values of α' is to be found in the fact that, when $I_{x'}$ is a maximum, $I_{y'}$ is a minimum or *vice versa*, for $I_{x'} + I_{y'} =$ constant. From this it is obvious that, if $I_{x'}$ is a maximum, then $I_{y'}$ which is perpendicular to OX' must be a minimum.

Problems.—1. Find the position of the axis for maximum and minimum moment of inertia of a right triangle whose sides are 5 and 10 cm. Find also the equation for the momental ellipse and determine the position of the axis.

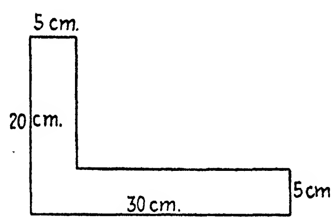

FIG. 95.

2. Find the momental ellipse of the unsymmetrical lamina given in the figure (Fig. 95).

3. Show that the momental ellipse of any regular polygon is a circle.

8-10. Theorem of the Product of Inertia of a Lamina and Center of Mass.—The theorem for product of inertia of a lamina is analogous to the theorem of parallel axes for moments of inertia. If the product of inertia for any two intersecting perpendicular axes in the plane of a uniform lamina is known, then the product of inertia for any other pair of perpendicular axes which are respectively parallel to the axes of the given pair may be determined.

Given the axes OX and OY (Fig. 96) with origin at O, the center of mass of the lamina, and the product of inertia F with reference to these axes. Let $O'X'$ and $O'Y'$ be any other pair of axes parallel to OX and OY, respectively. Let the coordinates of O in the coordinate system $X'O'Y'$ be p and q and let dm be any differential mass of the lamina. If F' is the product of inertia of the lamina with respect to the $X'O'Y'$ system, and M is the mass of the lamina, then

$$\begin{aligned} F' &= \int_0^M (p + x)(q + y)\, dm \\ &= pq\int_0^M dm + q\int_0^M x\, dm + p\int_0^M y\, dm + \int_0^M xy\, dm \qquad (8\text{-}54) \end{aligned}$$

Since O is the center of mass of the lamina, then, because of symmetry, the second and third integrals of the right-hand member are equal to zero, and hence

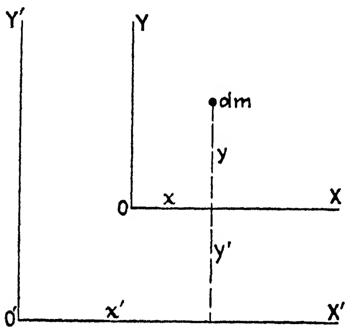

FIG. 96.

$$F' = pq\, M + F \qquad (8\text{-}55)$$

The equation is similar to that for parallel axes in moments of inertia except that in place of the square of the distance between the parallel axes we have the product of the coordinates of the origin of the new axes.

8-11. Rotational Momentum.—The quantity rotational momentum is equal to the product of the two fundamental quantities moment of inertia and angular velocity. If I is used to represent the moment of inertia and ω the angular velocity, then the product $I\omega$ expresses the rotational momentum.

The term moment of momentum is frequently used to designate this quantity. The reason for using this name becomes apparent if we develop the expression $I\omega$ for the rotational momentum. Suppose there is a rigid body in a state of rotation about some fixed axis. Each differential particle dm of the body will have a linear velocity (V) which may be expressed in terms of the angular velocity (ω) of the body and the distance (r) of the particle from the axis of rotation by the relation $V = \omega r$. The momentum of the particle is $dm\omega r$ and its moment of momentum is $dm\omega r^2$. To express the total moment of momentum of the body, it is necessary to integrate the expression for the moment of momentum of the particle over the entire body. This may be written as follows:

$$\int_0^m r^2\,dm\,\omega = \omega \int_0^m r^2\,dm = I\omega \tag{8-56}$$

The angular velocity ω is common to all differential particles because the body is rigid and hence may be placed outside the integration sign.

The term angular momentum may be applied to a nonrigid body. If the body is not rigid, the angular velocity of the differential element would not be common to all other elements and hence in this case the simple expression $I\omega$ could not be used. The integral given in the left-hand member of the foregoing equation would, however, be the angular momentum of the body. The integration could not be effected unless ω could be expressed in terms of the coordinates of a selected reference system.

The units in which rotational momentum may be expressed are grams centimeter squared per second or pounds foot squared per second.

8-12. Rotational Kinetic Energy.—The kinetic energy of a body rotating about some axis may be expressed in terms of its moment of inertia and the square of its angular velocity. Using the symbols introduced above, we may write

$$KE = \tfrac{1}{2}I\omega^2 \tag{8-57}$$

In this expression the angular velocity and moment of inertia are to be referred to the same axis.

The validity of the expression given in Eq. (8-57) for the rotational kinetic energy may be established in a manner somewhat similar to that used in the preceding section. If we again consider a rigid body rotating about a fixed axis, the kinetic energy of a differential mass may be expressed in the form $\frac{1}{2}\,dm\,V^2$ or $\frac{1}{2}\,dmr^2\omega^2$. Since ω is common to all differential elements of the body, the total rotational kinetic energy of the body is $\frac{1}{2}\,I\omega^2$.

The term rotational kinetic energy may also be applied to a nonrigid body, but the simple expression used above for the rotational kinetic energy would not be obtained.

The units in which the kinetic energy is to be expressed will depend upon those of its constituent elements. Ordinarily, in the metric system, the moment of inertia is expressed in grams centimeter squared and angular velocity in radians per second. In this case the energy is expressed in grams centimeter squared per second squared or ergs.

8-13. The Force Moment.—A force moment consists of two elements, a force and a distance factor. If the force is designated by the symbol $\boldsymbol{F}$ and its point of application from some reference point is given by the radius vector $\boldsymbol{r}$, then the moment of the force with respect to the reference point is $\boldsymbol{r} \times \boldsymbol{F}$. The axis about which rotational effects might be produced is perpendicular to the plane determined by $\boldsymbol{r}$ and $\boldsymbol{F}$.

A force moment is a vector quantity of the axial type. Graphically, then, the moment of a force may be represented by a line segment, drawn to some convenient scale, which is perpendicular to the plane containing the force and the distance factor. Its direction is determined by the convention of signs for axial vectors. It is convenient to use the rotation axis for this purpose, provided that the rotation axis is perpendicular to the plane of the moment. There is a tendency for the student to forget that the moment of force is an axial vector and hence to confuse the direction of the force with that of its moment.

In translational motion it is frequently desirable to express the motion of a body in terms of a component of the applied force, as in the case of a body sliding without friction down an inclined plane under the influence of its weight. In such a case, owing to the constraint offered by the inclined plane, the motion is best determined by the component of the weight which is parallel to the inclined plane. So, also, in rotational motion a body may have a fixed rotational axis, and the moment may be applied to the body so that the direction of the moment is not parallel with the rotation axis. In such a case it would be convenient to project the moment into the rotation axis in order to determine the resulting motion of the body. The other component of the moment in this case is neutralized by moments exerted by the supporting frame which holds the bearings. This is entirely analogous to the fact that in the translational case the other component of the weight is neutralized by the inclined plane.

8-14. The Couple.—It frequently happens that the total or resultant moment consists of several forces together with their respective lever arms. If there be but two forces and these are equal in magnitude, parallel but not collinear, the sum of their moments about an axis perpendicular to the plane containing the forces is constant as long as the distance between the two forces is constant. Because of this fact and because such cases are frequently encountered, the name couple has been applied to the sum of the separate moments. The couple is then merely a special case of moments.

The magnitude of the couple is the magnitude of either force multiplied by the perpendicular distance between the lines of the forces. If each force has a magnitude F and the distance between them is h, then hF is the magnitude of the couple.

There are three theorems describing the characteristics of a couple which are of use in some of the later work. The student should prove the validity of each.

a. The moment of a couple is independent of the position of the axis of rotation as long as it is perpendicular to the plane of the couple.

b. The moment of a couple is not altered by rotating the couple to some new position in its plane.

c. The moment of a couple remains unchanged if the magnitude of the forces is changed and at the same time the distance between the lines of forces is also changed to such a value that the product hF remains constant.

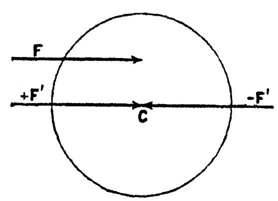

Fig. 97.

8-15. The Off-center Force.—We have seen in Sec. **7-6** of the preceding chapter that the effect of a force which is applied to a body along a line which does not pass through the center of mass of the body is to produce a resulting uniplanar motion which may be regarded as consisting of two distinct parts, one of which is translational and is described by the motion of the center of mass and the other is rotational and is motion relative to the center of mass.

The double effect of an off-center force may be analyzed by the device of introducing a pair of forces which are each equal to the existing force, and parallel to it, oppositely directed to each other and acting through the center of mass. In the diagram (Fig. 97) F is the off-center force, C is the center of mass, and $+F'$ and $-F'$ constitute the canceling pair of forces. This addition does not change the character of the motion, because the resultant force and its position remain the same. The translational part of the motion may now be described in terms of the effects of $+F'$. Since $+F'$ is applied to the center of mass, the translational acceleration of the body will be exactly the same as though under the influence of F, for $+F'$ cannot produce any rotational effects. The remaining forces F and $-F'$ constitute a couple which produces the rotational acceleration of the body and nothing else. The magnitude of this couple, produced by F and $-F'$, is equal to the moment of F about an axis drawn through C and perpendicular to the diagram.

Using this device, we may determine the complete motion of the body by means of two separate sets of equations, the translational equations developed in the preceding chapter and the rotational equations which are to be developed in this chapter. The two sets of equations may be applied independently of each other. In fact, we are at liberty to regard the motion as though the translational part took place first and then the rotational part followed it, or *vice versa.*

8-16. Analogies in Dynamics.—In studying the dynamical quantities and their equations in rotational motions we are assisted by the similarities which exist between these quantities and the corresponding quantities and equations in translational dynamics. This correspondence has already been utilized in developing rotational kinematics from translational kinematics. Many of the expressions for rotational motion may be written directly by substituting in the translational equations the corresponding rotational factors. For example, we may write the force-moment equation by replacing the force (in the force equation) by the moment of force, the mass by moment of inertia, and the linear acceleration by its corresponding quantity, angular acceleration. This substitution would give the equation

Force moment = moment of inertia × angular acceleration

which is a proper expression for the rotational effect of a constant force moment upon a rigid body.

While this procedure is a valuable short cut for remembering the fundamental equations in pure rotation, it is not sufficiently illuminating to reveal those characteristics which are essential to a complete understanding of the subject. The fundamental translational and rotational quantities are arranged in the following table so that the correspondence may be readily observed. The symbols used for these quantities are also introduced.

Translation	Rotation
Force (F)	Force moment (M)
Mass (M)	Moment of inertia (I)
Displacement (s)	Angular displacement (γ)
Velocity (V or U)	Angular velocity (ω or ω_0)

Translation	Rotation
Acceleration (J)	Angular acceleration ($d\omega/dt$)
Momentum (mV)	Angular momentum ($I\omega$)
Kinetic energy ($\frac{1}{2}mV^2$)	Rotational kinetic energy ($\frac{1}{2}I\omega^2$)

It is of interest to observe that all of the rotational quantities may be obtained from the corresponding translational quantities by some use of a distance factor. In other words we may express mathematically each translational quantity in terms of the corresponding rotational quantity by introducing a length usually designated by the symbol r. For example, as already shown, $\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F}$ and $I = mG^2$ where G is the radius of gyration (Sec. **8-3**).

The physical basis for the relations between the two sets of quantities is to be found in the motion of a particle moving in a circular path. We may accurately describe such a motion from either viewpoint. It is well to observe that these equations, which we may speak of as *transformation equations*, mathematically depend upon the fact that the distance factor remains constant. This condition is expressed physically by stating that the path of the particle is a circle or the motion is pure rotational. The student is asked to write the transformation equations for the last five pairs of quantities listed above.

8-17. The Impulse of the Force Moment.—The three dynamical equations in rotation and their corresponding equations in translation are

Rotation	Translation
a. Force moment	Force
b. Impulse of the force moment	Impulse
c. Work of the force moment	Work

The force-moment equation was derived in Sec. **8-1** above [see Eqs. (8-10) and (8-12)]. In this and the following section we shall develop the two other equations.

The *impulse of a force moment* is a quantity which is measured by the product of the moment and the time during which it is acting. To obtain an expression for the effect of the impulse of the force moment, we may multiply Eq. (8-10) by the time dt. If we desire the effect of the resultant impulse M for any finite time t, the equation becomes

$$\int_0^t M\,dt = \int_0^t d(I\omega) \tag{8-58}$$

If the moment remains constant during the time over which the integration is extended and if the body is rigid, the equation may be written in the form

$$\boldsymbol{M}t = I\ (\boldsymbol{\omega} - \boldsymbol{\omega}_0) \tag{8-59}$$

in which $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_0$ are respectively the final and initial values of the angular velocity. The left-hand member of Eq. (8-59) is the impulse of the resultant moment. The other member of the equation expresses the change in angular momentum. Both members of Eq. (8-59) are vector quantities and hence the direction of $\boldsymbol{M}t$ must be the same as that resulting from the vector sum of $I\boldsymbol{\omega}$ and $-I\boldsymbol{\omega}_0$. It is not necessary for the direction of the resultant moment to be parallel to either $\boldsymbol{\omega}$ or $\boldsymbol{\omega}_0$. If the direction of $\boldsymbol{M}$ coincides with that of $\boldsymbol{\omega}_0$, then the direction of $\boldsymbol{\omega}$ will remain the same as that of $\boldsymbol{\omega}_0$. It is only when there is a component of $\boldsymbol{M}$ in a line which is perpendicular to $\boldsymbol{\omega}_0$ that the direction of the angular velocity may change.

8-18. The Work of the Force Moment.—The quantity in rotation which corresponds to displacement in translation is the angle. To obtain the work done by a moment of force, we must therefore multiply the moment of force by the angular displacement which it produces.

In order to develop an expression for the work of the force moment we may use Eq. (8-10) as a starting point.

For the sake of generality we may assume that the body is a particle, attached by a massless frame to a rotation axis. To extend the equations, obtained on the basis of this assumption, to the case of a body or system of particles, we may regard the particle as a differential mass element and then integrate over the entire system. With this interpretation of Eq. (8-10), in mind, we may regard I as constant and may introduce the alternative form $\omega d\omega/d\gamma$ for the angular acceleration $d\omega/dt$. After changing the altered equation to a scalar equation, we have

$$M = I\,\omega\,\frac{d\omega}{d\gamma} \tag{8-60}$$

Multiplying both sides by $d\gamma$ and indicating the integrations gives

$$\int_{\gamma_0}^{\gamma} M\,d\gamma = \int_{\omega_0}^{\omega} I\omega\,d\omega \tag{8-61}$$

if we assume that $\omega = \omega_0$ and $\gamma = \gamma_0$ in the initial position.

If the moment remains constant and the body is rigid, the equation may be written as follows:

$$M\gamma = \tfrac{1}{2} I (\omega^2 - \omega_0{}^2) \tag{8-62}$$

This equation expresses the work done by the moment as it turns the body through the angle γ and indicates a measure of the work in the change of rotational energy produced.

8-19. The Torsional Pendulum.—For the sake of simplicity we may regard the torsional pendulum as consisting of a cylindrical-shaped disk, hung by a steel wire from some rigid support. If the disk is turned from rest through an angle γ by means of an external force moment acting about an axis containing the wire, the wire is twisted and will therefore exert a restoring moment or torque upon the disk. The value of the restoring moment will depend upon the angular displacement, the coefficient of elasticity, and the dimensions of the wire. If the wire is not twisted beyond its elastic limit, the restoring moment M may be put equal to some constant, say C, times the angle of displacement; *i.e.*, $M = -C\gamma$. The minus sign must be included because the restoring moment tends to reduce the angle. From this equation it is obvious that C is the torque required to produce a displacement of 1 radian.

If the disk is rotated through an angle and then released from the external moment, the elastic moment will produce an angular acceleration of the disk which is expressible as follows:

$$M = -C\gamma = I \frac{d^2\gamma}{dt^2} \tag{8-63}$$

Under the conditions of no external force moment this equation shows that angular acceleration is proportional to the displacement angle γ. We have seen above [Eq. (5-13)] that under these conditions the motion will be simple harmonic. The condition for simple harmonic motion in pure rotation

$$\frac{d^2\gamma}{dt^2} = -K^2\gamma$$

is rewritten here for convenience. Evidently, in this case,

$$K = \sqrt{\frac{C}{I}}$$

The period for the motion is therefore

$$T = \frac{2\pi}{K} = 2\pi\sqrt{\frac{I}{C}} \tag{8-64}$$

We may use this equation as a means for determining the moment of inertia of any body. In such cases, where the integration is difficult or impossible to effect, the torsional pendulum affords a way for an experimental determination of the moment of inertia. The procedure may be varied but the usual course is to attach some body like a homogeneous cylinder to a spring steel wire so that it may execute rotational simple harmonic motion about an axis for which the moment of inertia is known. If the period of the motion is determined, then the constant C may be calculated. The solid for which the moment of inertia is desired is then attached to the wire alone or together with the cylinder in such a way that the axis about which the moment of inertia is desired coincides with that of the wire. The period is again obtained by measurement and the moment of inertia may then be calculated.

Problems.—1. State two consistent sets of units for each of the three fundamental equations [Eqs. (8-10), (8-59), and (8-62)] one of these in the c. g. s. system and the other in the English system of units.

2. A homogeneous right circular cylinder rolls down an inclined plane (angle 30 deg. with the horizontal) under the influence of its weight. The cylinder rolls without slipping. If the mass of the cylinder is 500 g. and the radius of its base is 5 cm., determine its linear acceleration and angular acceleration. If the center of mass moves a distance of 100 cm., starting from rest, what will be its linear velocity, angular velocity, and total kinetic energy?

3. A flywheel, mounted to turn freely on its axis, is subjected to a force moment of 100 lb._w ft. If the moment of inertia of the wheel is 500 lb. ft.^2 what will be the angular velocity at the end of 5 sec. if the wheel is initially at rest?

4. A cylinder of mass 100 g. and radius 5 cm. is arranged to rotate about its geometric axis. A massless cord is wrapped around the cylinder. (*a*) If a constant force of 1,000 dynes is applied to the cord, what will be the angular acceleration of the cylinder? (*b*) If the force is variable and is expressed by the equation $F = (1{,}000 + 100t)$ dynes, where t is the time in seconds, find the angular velocity at the end of 10 sec., assuming that the cylinder is initially at rest. (*c*) If the force is expressed by the equation $F = 100\,x$ dynes, where x is a coordinate measured in centimeters along the line of the force, find the angular velocity when the cylinder has turned through five complete revolutions.

5. A certain governor-like body (Fig. 98) consists of a cylinder (mass 10 lb., radius of base 2 in.) and two 5-lb. balls (radius 1 in.) which are attached by thin uniform rods (mass 1 lb. each). The rods, each 4 in. long, measured from surface of cylinder to surface of ball, are connected by hinges to points on opposite sides of the cylinder so that the balls and rods must move in a plane which contains the axis of rotation of the cylinder.

Initially the rods make an angle of 90 deg. with the axis of the cylinder and the angular velocity of the body is 4π radians per second. By some internal mechanism the balls are brought into contact with the surface of the cylinder without the application of any external force moment. Find the resulting angular velocity. (*Hint:* If no external force moment is applied, the angular momentum must remain constant.)

6. A sphere of mass 150 g. and radius 3 cm. is supported by a wire which passes through its center. The period of the simple harmonic motion produced was found to be 5 sec. A second object attached to the sphere gave harmonic motion with a period of 8 sec. Find the moment of inertia of the second body. What was the value of the constant C in this case? In what units is it expressed?

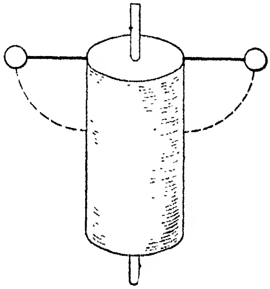

FIG. 98.

7. A uniform meter stick (mass 80 g.) is clamped at one end in horizontal position in a vise. Five centimeters of its length are held in the vise. Horizontal motion in the direction of the smallest dimension only is to be considered. A force of 100 g. weight applied perpendicularly to the stick at its free end produces a displacement of 2 cm. of that end from the rest position. Find the period of the simple harmonic motion which takes place when the free end of the stick is released from a small initial displacement. (Assume that Hooke's law applies.)

8. A rectangular block having a mass of 1,200 g. and dimensions 5 by 10 by 30 cm. is hung, with its largest surfaces horizontal, by two parallel strings 5 m. long. The strings are 20 cm. apart and are attached to symmetrically situated points on the upper surfaces. If the block is rotated through a small angle about a vertical axis which passes through the center of mass and is then released, is the motion strictly simple harmonic? What approximations, if any, may be made in order to regard the motion as simple harmonic? What would the period of the motion be?

9. A 50-g. mass is hung by a string which is wrapped several times around the rim of a wheel. The axis of the wheel is fixed in a horizontal position. If the moment of inertia of the wheel is 200 g. cm.^{2} and its radius of 5 cm., find the acceleration of wheel and of the falling mass and the tension of the string.

CHAPTER IX

STATICS

9-1. Introduction.—We have already seen that the motion of a body may, in general, be described in terms of a combination of translational and rotational motions. Either type of motion may exist in a particular body without the other. A body may have translational motion only or it may be in a state of pure rotational motion. We have also learned that linear acceleration is produced by a force and that angular acceleration is produced by a force moment. If the resultant force, in a particular case, is zero, then the linear acceleration must be also zero; a corresponding statement may be made for the resultant force moment.

There are many important relations which exist between those forces and force moments which may be applied to a body or system of bodies in those special cases where the resultant force and force moment are zero. These relations are presented in this chapter.

9-2. Definitions.—There are a number of technical terms to be introduced in this chapter which, to avoid inaccurate conceptions, require definition.

The term *statics* is applied to that part of mechanics which deals with bodies at rest or in a state of motion with constant velocity. It is concerned with those relations which must exist between the forces or force moments or both quantities in order that the particular body or bodies may remain with no acceleration. The time factor does not enter into considerations dealing with statics.

Equilibrium is a state in which a body exists if it has no acceleration. Consistent with this definition a body may be at rest or it may be moving with constant velocity.

A body is said to be *rigid* if under the influence of applied forces, the relative distances between the various particles of which the body is composed remain unaltered. We do not know of any body which is perfectly rigid. If, however, the relative displacements of the particles of the body are small in comparison with other essential dimensions, we may say that the body is rigid.

In statics as well as in dynamics we may be concerned with internal as well as external forces. When dealing with a system of bodies joined together in some way or other, a certain force may be internal as far as the system as a whole is concerned but at the same time it may be external to a given part of the system. Internal forces in a rigid body always occur in pairs of equal but oppositely directed forces. In studying the statics of rigid bodies, it is important for the student to distinguish between the two forces of a given pair which may be brought into existence at the point of contact of one body with another. If the interest is centered specifically upon one of the bodies, the particular force of the pair which acts upon that body is to be used and not its oppositely directed "twin."

9-3. Equilibrium of a Particle.—It is to be remembered that a particle is a body or part of a body the dimensions of which may be neglected in comparison with other dimensions or distances which may be involved. When this limitation is imposed, all the forces applied are regarded as passing through a common point of the body usually considered to be the center of mass, or, in other words, the forces are concurrent. Under such circumstances, in order for a particle to be in equilibrium, the forces applied to the particles must be of such magnitudes and have such directions as are necessary to prevent linear (or translational) acceleration.

a. Case of Two Forces.—When two forces are applied to a particle and the particle is in equilibrium, the only possible arrangement is that the two forces must be equal in magnitude and oppositely directed. Any other relation would give a resultant force which would produce an acceleration of the particles.

b. Case of Three or More Forces.—In order for the particle to be in equilibrium, the acceleration must be zero; hence we may conclude that the vector sum of the applied forces must be zero. This is the one necessary and sufficient condition for the equilibrium of a single particle.

Suppose that there are n forces acting upon the particles $\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3 \ldots \boldsymbol{F}_n$. Symbolically, the condition for equilibrium may then be written

$$F_1 + F_2 + F_3 + \cdots + F_n = 0 \tag{9-1}$$

This condition may be expressed in terms of the components of the several forces taken along the axis of any selected reference system. If the components of F_1 are designated by the symbols F_{1x}, F_{1y}, and F_{1z} and a similar form of expression is used for the other forces, then we may write:

$$(F_{1x} + F_{2x} + \cdots F_{nx})\boldsymbol{i} + (F_{1y} + F_{2y} + \cdots F_{ny})\boldsymbol{j} + (F_{1z} + F_{2z} + \cdots F_{nz})\boldsymbol{k} = 0$$

In order for this equation to be satisfied it is necessary for the coefficients of the unit vectors to be separately equal to zero; hence

$$\sum_n F_{ix} = 0, \qquad \sum_n F_{iy} = 0, \qquad \text{and} \qquad \sum_n F_{iz} = 0 \tag{9-2}$$

In case there are only three forces acting upon the particles, a corollary may be deduced from the preceding relation. Suppose the plane determined by any two of the given forces is selected as the YZ plane of the reference system, then these two forces will have no components along the X-axis. The third force must therefore also lie in the YZ plane in order for the first of Eqs. (9-2) to be satisfied. From this we may conclude that, if there are three and only three forces acting upon a particle, the three forces must be coplanar.

c. Lami's Theorem.—Lami's theorem includes the preceding corollary as well as the familiar trigonometric relation which exits between the three sides of a triangle and the sines of the opposite angles. Since the vector sum of the three forces is equal to zero, the graphical representation of the vector sum is a triangle, the sides of which represent the three given forces. If the magnitudes of the three forces are designated by the symbols A, B, and C and the interior angles of triangle opposite the sides are expressed by the corresponding small letters, then we may write

$$\frac{A}{\sin a} = \frac{B}{\sin b} = \frac{C}{\sin c} \tag{9-3}$$

9-4. Strings.—In mechanics it is frequently convenient to employ strings, ropes, or chains as a means of applying a force to a particle or a body. A digression will be made at this point in order to introduce some of the essentials characteristic of these mechanical tools.

Ordinarily the weight of a string or rope is sufficiently small, in comparison with the other forces involved, to permit one to neglect it. Unless explicitly stated to the contrary we shall omit the weights of strings or ropes in the following considerations. The weights of chains, on the other hand, are not usually small enough to neglect.

Strings, ropes, and chains are considered to be perfectly flexible; *i.e.*, they can be drawn around a pulley or some other similar object without requiring the application of a force to effect the change in linear form. This is not to be confused with the fact that tangential forces of resistances may be produced when the strings are in contact with surfaces of pulleys and the like.

If forces are applied to the ends of a string so that there is a tendency to stretch it, the string is in a state of tension. With strings of negligible weight the forces applied to the two ends are of equal magnitude but are oppositely directed. It is legitimate to consider the string to be made of a large number of particles joined together like links of a fine chain. Any single particle of the string, like a single link of the chain, is under the influence of a pair of equal and oppositely directed forces. In speaking of the tension of a string we refer to the magnitude of the force applied to either end of the string.

Strings are usually classified as *elastic* (extensible) or *inelastic* (or inextensible). Elastic strings become elongated when subjected to tension. Inelastic strings do not have a change in length when in a state of tension. Perfectly inelastic strings do not exist but in many cases the elongation is small enough to neglect.

If not stretched beyond their elastic limit, the elongations of strings may be expressed quantitatively by Hooke's law, which states that the strain is proportional to the stress. Strain is the increase of length per unit of length. Stress is the tension per unit of cross-sectional area. If F is the applied force, A the cross-sectional area, e the increase of length, and L the length of the unstretched string, then

$$\text{Strain} = \frac{e}{L} \qquad \text{Stress} = \frac{F}{A}$$

Hooke's law may then be written·

$$\text{Young's modulus } (E) = \frac{\text{stress}}{\text{strain}} = \frac{F/A}{e/L}$$

In the case of strings, wires, etc., the terms tensile stress and tensile strain are frequently used.

It may be readily shown that for any given string the ratio of the applied force (F) to the increase in length per unit length (e/L) is a constant within the elastic limit. It has been proposed to call this ratio the modulus of stiffness, or in symbols, if we let λ represent the modulus of stiffness,

$$\lambda = \frac{FL}{e} \tag{9-4}$$

Illustrations.—Two problems will be introduced in this section to show uses of the principles given above. One of these deals with inelastic strings and the other with elastic strings.

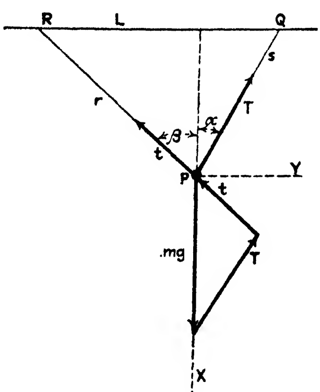

Fig. 99.

a. Inelastic Strings.—A particle P of mass m is suspended by two inelastic strings which are attached to a rigid horizontal support at the points R and Q. Find the tensions of the strings.

Let the distance from R to Q (Fig. 99) be L and the lengths of the strings $P\,R$ and $P\,Q$ be r and s, respectively.

The particle is in equilibrium under the influence of three forces, the weight of the particle W ($m\,g$) and the tensions of the two strings which we may call t and T taken along $P\,R$ and $P\,Q$, respectively. The simplest way to solve such a problem is by a graphical method. If the distances L, r, and s are known, then the triangle $P\,Q\,R$ may be laid off to a convenient scale. From this diagram the angles between any pair of forces may be measured. Since the particle is in equilibrium, the vector sum of the three forces is zero and hence we may construct a second triangle which we may call the force triangle. The angles of the force triangle are the angles between the forces as found from the triangle $P\,Q\,R$ and the sides of the triangle are proportional (to a second selected scale) to the magnitudes of the forces. The construction of the force triangle is possible because the length of one side and the two adjacent angles are known. The force triangle is shown in the diagram. The accuracy of the results obtained depends upon the skill involved in mechanical drawing.

A trigonometric solution may be made by the use of Lami's theorem. The angles between any pair of forces in the force triangle are first to be determined from the data which give the lengths of the sides of the triangle $P\ Q\ R$. With this information available, Lami's theorem may be used from which two trigonometric equations can be written. The solution of these questions will yield the values of the tensions.

A third method of solution of this problem may be used. The procedure involved makes use of Eqs. (9-2). For this purpose we may introduce a reference system $X\ P\ Y$ with origin at P and with the $P\ X$-axis vertically downward as shown in the diagram.

With the angles of triangle $P\ Q\ R$ determined, one may readily calculate the values of the angles α and β (see diagram). Using the principle expressed in Eqs. (9-2), we may write the following force relations:

$$\begin{aligned} mg - T\cos\alpha - t\cos\beta &= 0 \\ T\sin\alpha - t\sin\beta &= 0 \end{aligned} \tag{9-5}$$

Since these two equations contain the two only unknown quantities t and T, the tensions may be found.

b. Elastic Strings.—In this problem we shall use the same arrangement as that given in the preceding problem but with elastic strings. Suppose that the unstretched lengths of the strings are r and s and that the modulus of stiffness λ is known and is the same for both strings. We shall assume that α and β are the angles, as shown in the diagram in the position of the equilibrium, between the strings $P\ Q$ and $P\ R$, respectively, and the line of the weight of the particle (*i.e.*, the vertical line). The problem is to find the tensions as before. The lengths of the strings (r and s) have been increased by the tensions to values which we shall call r' and s', respectively. There are now six unknown quantities t, T, α, β, r' and s'. We shall need, therefore, six independent equations in order to find the tensions. We are at liberty to use two equations written by the use of Lami's theorem or two derived from the principle expressed by Eqs. (9-2) but we cannot use all four equations, because the latter could be derived from Lami's theorem. We select those obtained from Eqs. (9-2) because they are already written [Eq. (9-5)]. The values of t, T, α, and β of the present problem are obviously not those of the preceding problem.

Two more equations may be written by the use of Eq. (9-4), which are

$$T\,s = \lambda\,(s' - s) \qquad \text{and} \qquad tr = \lambda\,(r' - r) \tag{9-6}$$

It is to be noticed that if the moduli of stiffness of the strings were not alike, this difference would modify one of the preceding equations. If both moduli are assumed to be known no difficulty is introduced, but if one were unknown the problem could not be solved.

Two other equations may be written by using the trigonometric relations to be found from the triangle $P\,Q\,R$. These relations are

$$s' \cos \alpha = r' \cos \beta, \qquad \text{and} \qquad s' \sin \alpha = L - r' \sin \beta \tag{9-7}$$

By the use of the six equations of Eqs. (9-5), (9-6), and (9-7) it is possible to evaluate the tensions in any given problem. The algebraic details necessary to express the tensions in the general case are uninteresting and will be omitted. The method of analysis of the problems is our present interest.

Problems.—1. Show that Eqs. (9-5) may be derived by using Lami's theorem.

2. A 100-lb. shot is suspended by an inelastic cord which is 10 ft. long, *i.e.*, measured from the point of support to the surface of the shot. How large a horizontal force is needed to hold the shot so that the upper part of the cord makes an angle of 30 deg. with the vertical, if the horizontal force is applied at a point in the cord 9 ft. below the point of support? What particle of the system is in equilibrium under three forces?

3. The following forces are applied to a single free particle:

$$\begin{aligned} \boldsymbol{F}_1 &= 3\,\boldsymbol{i} + 4\,\boldsymbol{j} + 5\,\boldsymbol{k} \\ \boldsymbol{F}_2 &= -2\,\boldsymbol{i} + 5\,\boldsymbol{j} - 6\,\boldsymbol{k} \\ \boldsymbol{F}_3 &= 5\,\boldsymbol{i} - 8\,\boldsymbol{j} - 2\,\boldsymbol{k} \end{aligned}$$

What single force will produce equilibrium?

4. A 500-g. particle is to be suspended vertically by an elastic cord which is 1 m. long when under no tension. If the modulus of stiffness of the cord is 10^7 dynes, what will be the stretched length of the cord?

What horizontal force applied at the particle would be needed to produce equilibrium with the cord making an angle of 20 deg. with the vertical?

9-5. Particle on a Plane.—When a particle is placed upon a horizontal plane and it remains at rest, two forces will act upon the particle. One of these forces is the weight of the particle acting vertically downward and the other is the reaction of the plane. Since the particle is in equilibrium, the two forces are equal in magnitude and oppositely directed. The reaction of the plane is therefore directed vertically upward. The origin of the force supplied by the plane is to be found in a deflection of the plane, although the deflection is usually very small. Such a deflection produces an upward force because of the elastic nature of the plane.

If a third force, which is so directed that it has a horizontal component, is applied to the particle and the particle remains at rest, the direction of the force exerted by the plane must change in order to neutralize the horizontal component of the applied force. The vertical component of the plane's reaction will also change to the extent which is necessitated by the introduction of an additional component in the vertical line. The value of the force exerted by the plane must be equal and opposite to the vector sum of the weight of the particle and the third force. This is, of course, consistent with the fundamental principle that the vector sum of the three forces must be zero in the case of equilibrium.

The source of the horizontal component of the reaction of the plane is to be found in the friction between the two surfaces in contact with each other. If the surfaces were "smooth," *i.e.*, with no friction, then equilibrium could not be produced with the introduction of a single force having a horizontal component. In those cases in which there is no friction between the particle and the plane, the reaction of the plane is always perpendicular to the plane, whether the plane is horizontal or not. The term *smooth* is usually used to indicate the absence of friction, while *rough* indicates that friction is present.

Suppose a particle is in equilibrium on a horizontal plane and that a variable horizontal force (H) is introduced. If the magnitude of H is steadily increased from a zero value, simultaneously with this increase the neutralizing friction (F) increases so that $H = F$ continuously. There is, however, a limiting value of F in any particular case, the value of which depends upon the character of the two surfaces, such as degree of smoothness and nature of materials, and upon the magnitude of the force perpendicular to the plane, *i.e.*, the vertical component of the reaction of the plane. The ratio of the magnitude of the maximum value of friction (F) to the magnitude of the vertical component (N) of the plane's reactions is approximately a constant and has been called the coefficient of friction. If we let μ represent the coefficient of friction, then

$$\mu = \frac{F}{N} \tag{9-8}$$

Numerical values have been determined experimentally for combinations of surfaces of various materials.

It must be remembered that, when a particle is in equilibrium on a horizontal plane and there is a horizontal force applied to the particle, the term reaction (**R**) of the plane includes the friction (a force parallel to the plane) as well as the normal component. The direction which **R** makes with the normal to the plane is dependent upon the magnitudes of the two components of **R**. In those cases in which the friction **F** has its maximum or limiting value, the angle which **R** makes with the normal to the surface is a constant for any given pair of surfaces. The tangent of this angle, usually designated ϵ, is equal to the coefficient of friction.

If a particle is in equilibrium when on an inclined plane with no forces other than the weight of the particle and the reaction of the plane, then the reaction of the plane must be vertical. The component of **R** parallel to the plane (friction) is equal to the component of the weight in the same line but its direction is opposite. The component of **R** normal to the plane is equal and opposite to the component of the weight in this line. As the angle of inclination of the plane is increased, the component of the weight parallel to the plane increases, which requires a correspondingly larger value of the friction in order to preserve equilibrium. At the same time the normal component of the reaction of the plane decreases, which reduces the limiting value of friction. It is evident that there is a limiting angle of inclination of the plane at which equilibrium will serve unless an additional force parallel to the plane is added. This is the angle ϵ whose tangent equals the coefficient of friction. The fact just stated suggests a simple experimental method of determining the coefficient of friction.

9-6. Equilibrium of a Rigid Body.—While a particle may have translational acceleration only, a rigid body may move with a combination of translational and rotational acceleration. To maintain a particle in a state of equilibrium, all that is necessary is to prevent translational acceleration. This may be done by properly applying a single force. Since a rigid body may have rotational as well as translational acceleration, a force moment as well as a force must be used to produce equilibrium. It is possible in some cases to have a single force supply both needs by selecting a proper magnitude, direction, and point of application for the force. This matter will be discussed in more detail below.

From the preceding statements and the definition of equilibrium it follows that there are two general conditions which are necessary and sufficient to maintain a rigid body in a state of equilibrium. These conditions may be written as follows:

First Condition.—The vector sum of the forces must be equal to zero.

Second Condition.—The vector sum of the moments of the forces about any axis must be equal to zero.

The first condition for equilibrium of a rigid body is identical with the only condition for the equilibrium of a particle. In the case of the particle, all forces were regarded as passing through its center of mass. In dealing with a rigid body the forces may or may not pass through the center of mass. Each force applied to a rigid body has what has been called a point of application. This is usually the particular point of the body at which some other body makes contact with the body under consideration, such as the point to which a cord may be attached to the body, or it may be the central point of an area of contact. As far as the first condition is concerned, it does not matter where the forces are applied to the body; if the vector sum remains equal to zero, translational acceleration is prevented.

The second condition of equilibrium is used in connection with extensive bodies (not particles) only.

In order that there may be no misunderstanding about this condition, there are several aspects concerning the principle which will be described in detail.

To determine the vector sum of the moments in any given problem, one is not at liberty to compute the sum of the moments about just one selected axis unless the case is a special one in which the body is constrained to rotate about some fixed axis and the selected axis is the fixed axis. At present we are interested in a more general case, that of a free body. In such cases it is necessary to determine the moments about any three mutually perpendicular axes, such as the axis of a reference system. It is immaterial where the reference system is located in the body, although the geometry of the arrangement may make one selection simpler to use than many others. The following development will indicate the validity of the indicated procedure.

Give any reference system $X\ Y\ Z$ and $\boldsymbol{M}$ the moment of any force. $\boldsymbol{M}$ may be expressed in terms of its components along the reference axes as follows:

$$\boldsymbol{M} = M_x\boldsymbol{i} + M_y\boldsymbol{j} + M_z\boldsymbol{k} \tag{9-9}$$

when M_x, M_y, and M_z are the components of $\boldsymbol{M}$ about the X-, Y-, and Z-axis, respectively. If there are n moments and they are designated $\boldsymbol{M}_1$, $\boldsymbol{M}_2$, $\boldsymbol{M}_3$ . . . $\boldsymbol{M}_n$, and M_{1x}, M_{1y}, M_{1z} etc., are their components, then the second condition of equilibrium may be written

$$(M_{1x} + M_{2x} + \cdots M_{nx})\boldsymbol{i} + (M_{1y} + M_{2y} + \cdots M_{ny})\boldsymbol{j} + (M_{1z} + M_{2z} + \cdots M_{nz})\boldsymbol{k} = 0 \tag{9-10}$$

For this equation to be satisfied it is necessary that the coefficients of the unit vectors be separately equal to zero.

In place of expressing the moments with reference to axes it is sometimes more convenient to express them with reference to a selected point. It will be remembered that the moment of a force with reference to a point is given by the vector product of the radius vector, drawn from the reference point to the point of application of the force, by the applied force. If $\boldsymbol{r}$ is the radius vector, $\boldsymbol{F}$ the force, and $\boldsymbol{M}$ the moment, then

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F}$$

If the reference point is the origin of an $X\ Y\ Z$ reference system and x, y, and z are the coordinates of the point of application of the force, then

$$\begin{aligned}\boldsymbol{M} &= (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \times (F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k}) \\ &= (yF_z - zF_y)\boldsymbol{i} + (zF_x - xF_z)\boldsymbol{j} + (xF_y - yF_x)\boldsymbol{k}\end{aligned} \tag{9-11}$$

Written in the determinant from this becomes

$$\boldsymbol{M} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} \tag{9-12}$$

The components of $\boldsymbol{M}$ are the coefficients of the unit vectors; hence

$$M_x = yF_z - zF_y, \quad M_y = zF_x - xF_z, \quad \text{and} \quad M_z = xF_y - yF_x \tag{9-13}$$

It is worth while to examine each of the preceding expressions for the components of $\boldsymbol{M}$ with the information given in Fig. 100. The direction of $\boldsymbol{M}$ is perpendicular to the plane determined by $\boldsymbol{r}$ and $\boldsymbol{F}$.

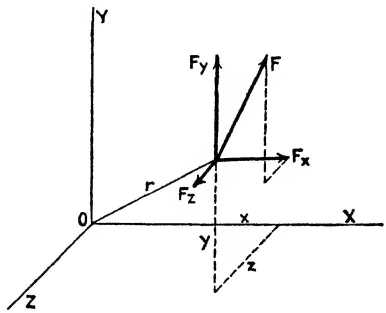

Fig. 100.

Problem.—If $\boldsymbol{r} = 8\boldsymbol{i} + 7\boldsymbol{j} + 5\boldsymbol{k}$ and $\boldsymbol{F} = 3\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$, find the direction and magnitude of $\boldsymbol{M}$ using Eq. (9-11).

9-7. Shifting the Point of Application of Forces.—Consider a force F which is one of several forces applied to a body in equilibrium. We are to determine the extent to which the point of application of F may be moved without destroying equilibrium. Obviously the direction of F could not be altered without producing a change in the vector sum of the forces. If F is moved parallel to itself, then its moment with reference to any selected point would be changed and hence the second condition of equilibrium would not be satisfied. There remains only the possibility of moving the point of application to some other point in the line of F. Such a shift would not alter the magnitudes or signs of the components of F in any selected reference system and hence, by Eq. (9-2), would be allowable as far as the first condition of equilibrium is concerned. This conclusion may be obtained from another point of view. Suppose F (Fig. 101) is applied at the point P and it is desired to shift the point of application to Q, any other point in the line of F. We may introduce a pair of equal and opposite forces $-F$ and $+F$ at Q with both forces parallel to the given F. This addition could not affect the vector sum of the forces. We may now regard $-F$ at Q canceling F at P, which leaves $+F$ at Q.

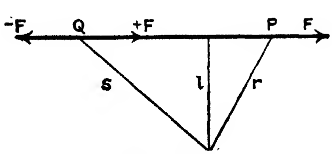

Fig. 101.

The shifting of $\boldsymbol{F}$ from P to Q will also produce no change in the moment of $\boldsymbol{F}$ with reference to any point O. The moment of $\boldsymbol{F}$ (at P) is $\boldsymbol{r} \times \boldsymbol{F}$, which is equal to $\boldsymbol{l} \times \boldsymbol{F}$ where $\boldsymbol{l}$ is the vector drawn from O perpendicular to the line Q P. Similarly the moment of $\boldsymbol{F}$ at Q is equal to $\boldsymbol{l} \times \boldsymbol{F}$. Hence we may conclude that shifting the point of application of a force to any other point in the line of that force does not affect the state of equilibrium.

9-8. Rigid Body in Equilibrium with Three Forces.—Because of the fact that there are simplifications made possible in the general condition for equilibrium by limiting the number of applied forces to three, special attention is to be directed to this case.

a. The Three Forces Are Coplanar.—This fact is readily deduced from Eqs. (9-2). The proof is identical with that used in proving that three forces applied to a particle must be coplanar (Sec. **9-3**) and will therefore be omitted.

b. The Three Nonparallel Forces Are Concurrent.—We may use the second condition for equilibrium to prove this statement. Since the forces are coplanar, the line of action of each force will intersect the lines of action of the two others. Imagine an axis to be drawn through the point of intersection of the lines of any two of the forces. Neither of the selected pair of forces will have a moment about this axis. The third force would have a moment about this axis if it did not pass through the point of intersection of the first two forces. But the moment of the third force about the selected axis must be zero; otherwise the second condition would not be satisfied. Hence we may conclude that the three forces must be concurrent.

9-9. Bodies with Parallel Forces.—Problems dealing with bodies in equilibrium under the influence of parallel forces present certain simplified procedures which warrant a separate treatment. Because of the fact that there are two conditions of equilibrium which must be satisfied it is possible to have two, but not more than two, unknown quantities. One of the unknowns must be a force, since the first condition is independent of the positions of the forces. The first step, then, is to ascertain that sufficient data are presented to make the solution possible.

As an illustration, we may consider a uniform straight horizontal bar, of weight W, in equilibrium under the influence of the four parallel forces F, F_1, F_2, and F_3 (Fig. 102) which with W make an angle θ with the bar. Let the length of the bar be 4 L, with F_1 and F_3 applied at the ends of the bar and F_2 at a distance L from A. Find the magnitude and point of application of F if F is the only unknown force.

Since the bar is uniform, W acts at a distance 2 L from A. Applying the first condition of equilibrium and letting the upward direction be positive gives

$$-F_1 - W - F_3 + F_2 + F = 0 \tag{9-14}$$

Since there are two unknown quantities, two equations will be needed. The second condition may be used to obtain the second equation. Before one attempts to apply this equation, it is necessary to select an axis about which the moments of the forces are to be taken. It is immaterial where the axis is to be placed but it must be specified in order to write the equation. Let the axis be taken through A, perpendicular to the plane of the forces. If the point of application of F is designated by the distance x from A, the equation is

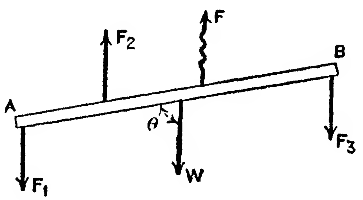

FIG. 102.

$$(F_2L + Fx - 2WL - 4F_3L) \sin \theta = 0 \tag{9-15}$$

Notice that the signs of the terms are written with positive moments taken counterclockwise. The sign of the F is also written plus even though we do not know its direction. The reason for this is to make the equation consistent with Eq. (9-14). With numerical values given for the known quantities, the two unknowns may be evaluated.

If $\sin \theta$ is not zero, it may be taken out of the second equation. This fact indicates that the relation is independent of the particular value of θ. If θ were equal to zero, the forces would be parallel to the bar and, by writing $\sin \theta$ with each distance factor, it is apparent that the moment of each force would be zero. In this case the point of application of F is immaterial to the equilibrium.

The magnitude and direction (whether + or −) of F are obtained from Eq. (9-14). If it is found that F is negative, this means that F is directed downward. The distance factor x is then found from the second equation. If x turns out to be negative, then the position of F must be to the left of the point A.

Problems.—1. With the following values: $F_1 = 500$ g. weight, $F_2 = 200$ g. weight, $W = 1$ kg. weight, $F_3 = 150$ g. weight, and $4L = 100$ cm., find F and x.

2. Considering the lengths L, $2L$, etc., as vectors from A, write Eq. (9-15) as a vector equation.

3. Show how the position of the resultant of a system of parallel forces may be found from the preceding development.

9-10. Rigid Bodies with a Fixed Axis.—Motions of bodies constrained to move about a fixed axis are limited to pure rotation. Some of the common appliances which are to be included in this classification are levers, pulley mechanisms, wheel and axle, capstans, and others of like nature. Two illustrations will be given in order to show the details of the procedure.

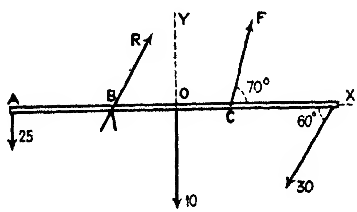

FIG. 103.

a. The Lever.—A straight uniform bar, weighing 10 lb. and 6 ft. long, is in equilibrium in a horizontal position with the fixed axis at B (Fig. 103), 2 ft. from one end. Besides the force which may be exerted by the axis, there are four other forces: 25 lb. vertically downward at A, the weight of the bar, 10 lb. at O the center of the bar, F, an unknown force at C 4 ft. from A and making an angle of 70 deg. with the bar, and 30 lb. at X, the other end of the bar and making an angle of 60 deg. with the bar as shown in the diagram. All forces are coplanar. Find the magnitude of F and the direction and magnitude of R, the force exerted by the axis upon the bar.

The magnitude of F may first be found by using the second condition of equilibrium. Taking moments about the fixed axis, since R is unknown and is thereby eliminated, we may write the following equation:

$$2 \times 25 - 1 \times 10 + (2 \sin 70) \times F - (4 \sin 60) \times 30 = 0 \tag{9-16}$$

The units used to express each moment are pound weight feet. Solving this equation, we find that $F = 34$ lb.

The magnitude and direction of R may be determined by equating to zero the sum of the components of all forces taken along the axes of any selected reference system. Let the reference system be XOY with origin at the center of the bar and OX parallel to the bar. Also let θ be the angle which R makes with the X-axis. By using the first condition of equilibrium, the following equations may be written:

$$R \cos \theta + 34 \cos 70 - 30 \cos 60 = 0$$
$$-25 + R \sin \theta - 10 + 34 \sin 70 - 30 \sin 60 = 0 \quad (9\text{-}17)$$

From this we find that

$$R \cos \theta = 3.40 \qquad \text{and} \qquad R \sin \theta = 29.03$$

The magnitude of R^2 is readily found by adding the squares of the equations, from which we find that $R = 29.2$ lb. Dividing the second of the preceding equations by the first gives $\tan \theta = 8.55$, from which $\theta = 83° 20'$. The direction of R makes an angle of 83° 20′ with the X-axis. Attention should be called to the fact that, for the purpose of writing Eqs. (9-17), R was placed in the diagram with θ in the first quadrant.

b. Bar Hinged at One End.—A uniform bar $A\ B$, weighing 1,000 lb., is hinged at one end (A) to a rigid support (Fig. 104). The bar is held by a rope attached to the other end (B) so that the bar makes an angle of 70 deg. with the vertical support. The angle between the rope and the support is 50 deg. Find the tension ($\boldsymbol{T}$) of the rope and the reaction ($\boldsymbol{R}$) of the support.

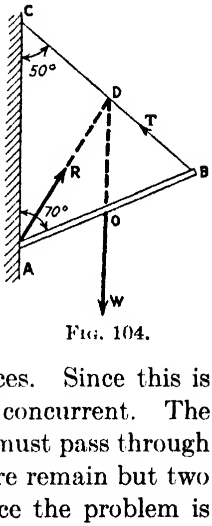

Fig. 104.

The bar is under the influence of three forces. Since this is the case, the forces must be coplanar and concurrent. The latter fact establishes the direction of $\boldsymbol{R}$, for $\boldsymbol{R}$ must pass through D, the point of intersection of $\boldsymbol{T}$ and $\boldsymbol{W}$. There remain but two unknowns, the magnitudes of $\boldsymbol{T}$ and $\boldsymbol{R}$; hence the problem is solvable.

The first step in the solution is to determine the angle which $\boldsymbol{R}$ makes with the vertical support. The geometric relations may be used for this purpose. If the length of the bar is taken as L, then the length of the rope $C\ B$ may be found. From this point one may equate to zero the sum of the moments of $\boldsymbol{W}$ and $\boldsymbol{T}$ about an axis through A. The axis is taken through A in order to eliminate one unknown ($\boldsymbol{R}$). From the equation of moments the magnitude of $\boldsymbol{T}$ may be found. Then by writing another equation of moments about an axis through B or O, the magnitude of R is readily found.

Another procedure is to introduce a reference system and write the two equations which express the sum of the components along the two axes. Either method is equally serviceable. The details of the solution will be left to the student.

9-11. Center of Gravity.—The center of gravity of a body is a point of the body through which the resultant of the weights of the various particles, of which the body is composed, may be regarded as acting. The center of gravity of a body may be found by using the principles of equilibrium. A single illustration is given to show the method used.

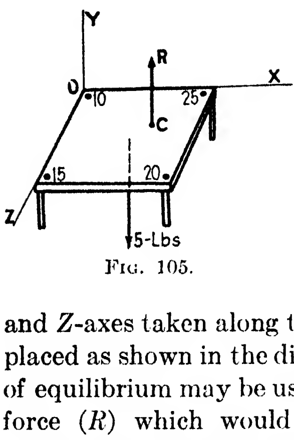

FIG. 105.

A square table, the length of each edge being L, weighs 5 lb. and carries four masses, one at each corner. The masses are 10, 15, 20, and 25 lb. and are placed in the order given around the table. Find the vertical line which contains the center of gravity, assuming that the plane of the table is horizontal.

Let us select a reference system $X\,Y\,Z$ with origin at one corner, with the X- and Z-axes taken along two edges of the table and with the masses placed as shown in the diagram (Fig. 105). The second condition of equilibrium may be used to find the position (C) of the vertical force (R) which would produce equilibrium if the table were suspended by a cord attached at that point. Let the coordinates of C be x and z.

The equations for the sum of the moments about the X- and Z-axes are

$$-Rz + 5\frac{L}{2} + 15L + 20L = 0$$

$$Rx - 5\frac{L}{2} - 20L - 25L = 0 \qquad (9\text{-}18)$$

Since the forces are all parallel to each other, the magnitude of R is 75 lb. Introducing this value for R and then solving the equations for x and z gives $x = 0.634\,L$ and $z = 0.5\,L$.

If the third coordinate of the center mass is desired, the procedure is readily extended by first rotating the arrangement about the X-axis, say, through 90 deg. and then writing the sum of the moments about the Y-axis.

9-12. Illustrations.—The following problems have been selected as being typical of those problems in which a rigid body is in equilibrium under the influence of three or more forces.

a. Man on a Ladder.—A uniform ladder of length L and of weight mg is placed on a rough horizontal floor and leans against a smooth vertical wall so that it makes an angle α with the wall. A man of weight Mg stands on a rung of the ladder situated at a point $\frac{3}{4}L$ from the lower end. Find the forces exerted by the wall and floor upon the ladder. How does the position of the man affect the tendency of the foot of the ladder to slip?

Given the ladder $A\ B$ as shown in the diagram (Fig. 106). The two known forces mg and Mg are directed vertically downward and act through the points C and P which are one-half and three-fourths of L, respectively, from the foot of the ladder. The force F exerted by the wall upon the ladder must be along a line perpendicular to the wall because the wall is assumed to be smooth (no friction). The magnitude of F is unknown. The reaction R of the floor upon the ladder is completely unknown.

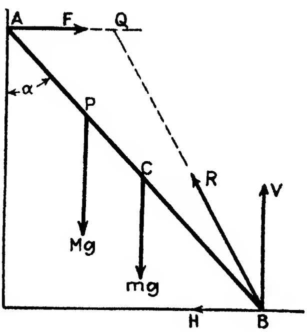

FIG. 106.

The problem may be reduced to an equivalent problem in which there are but three forces, if we replace the two weights by a single equivalent force. If numerical values of the two weights were given, the magnitude and position of a single equivalent force could then be found. With only three forces to consider, the direction of R could be found, since it is known that three forces must be concurrent. The two remaining magnitudes could then be readily found by the procedure indicated in a preceding illustration (Sec. **9-10***b*).

Another approach to the problem may be obtained by considering the components of all forces along a vertical and a horizontal line. Let the vertical component of R be V and its horizontal component be H. Since the sum of all components along any line must be equal to zero, it follows that

$$F - H = 0 \qquad V - Mg - mg = 0 \tag{9-19}$$

The second of these equations gives the magnitude of V. There remains only one unknown quantity.

To find the magnitude of F, another equation is needed. Using the second condition of equilibrium, we may write a moment equation. Selecting the point B as axis for the moments, we may write

$$\tfrac{3}{4}MgL \sin \alpha + \tfrac{1}{2}mgL \sin \alpha - FL \cos \alpha = 0 \qquad (9\text{-}20)$$

from which F and hence H may be determined.

There still remains to be discussed the question of the effect of the position of the man upon the tendency of the foot of the ladder to slip. The answer to this question is to be found in the variation of the magnitude of H as the position of the man is altered. The component H is supplied by the friction between the foot of the ladder and the floor. An equation expressing H in terms of the position of the man is needed. This could be obtained by replacing F in Eq. (9-20) by its equal quantity H but the equation so obtained, while mathematically correct, is not satisfactory from the standpoint of its physical meaning. Hence we choose to write a more suitable equation by expressing the moments about an axis through the point A. This equation is

$$VL \sin \alpha - HL \cos \alpha - s\, MgL \sin \alpha - \tfrac{1}{2}mgL \sin \alpha = 0 \qquad (9\text{-}21)$$

in which the symbol s is introduced (in the third term) to provide a means for expressing a variation of the position of the man. The factor s indicates, in terms of the fractional length of the ladder, the position of the man from the point A. For present purposes this equation contains only two variables H and s. To preserve the equality, if s increases, H must decrease and *vice versa.* Hence H will have its maximum value when s is minimum. We may conclude, therefore, that as the man climbs higher on the ladder the tendency to slip increases.

b. Stepladder.—A uniform symmetrical stepladder is placed upon a smooth horizontal surface. The two equal halves of the ladder are hinged at the top and the feet are kept from slipping by a light inelastic rope which is attached at points which are one-third of the distance from the feet to the top of the ladder. The angle between the two parts of the ladder is α. Find the tension in the rope and the force exerted at the hinge by one side of the ladder against the other.

Let the arrangement be as shown in Fig. 107, with W, the weight of either side of the ladder, acting vertically downward through the centers of the sides.

In a problem of this sort, when it is necessary to find the forces which act upon a certain part of the mechanism, the student will find it a great help to draw a boundary line completely enclosing that part of the apparatus to which attention is to be directed. Because of symmetry in the present problem we need consider only the side AC. By drawing a dotted line about this side we are to take account only of those forces which are acting upon it. This device makes the selection of the proper force of a given pair of forces less liable to be incorrect. For example, at the point A there is a pair of equal and opposite forces; one of this pair contributes to the group of forces needed to maintain the equilibrium of the side AC and the other (not shown on the diagram) is to be used when dealing with the equilibrium of the side AB.

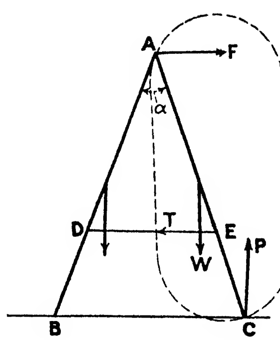

Fig. 107.

The forces applied to AC are to be identified. The weight W has already been mentioned. The tension T is horizontal, is applied one-third of the distance up on AC, and is directed as shown. Since the floor is smooth, the force P which the floor exerts at the foot of the ladder is directed vertically upward. The only remaining force is located at the upper end of the ladder and is horizontal. The latter fact may not be obvious. But when one remembers that the arrangement is symmetrical and that it is assumed that the hinge is frictionless, it is seen that no other direction can satisfy these conditions.

There are apparently three unknown quantities and these are the magnitudes of F, T, and P. This number is readily reduced to one when it is observed that F and T are horizontal and W and P are vertical. Hence $P = W$, and $F = T$, as far as the magnitudes of the forces are concerned.

To evaluate T we may use the second condition of equilibrium. With the point A as axis of the moments the following equation may be written, if we let L represent the length of AC,

$$PL \sin \frac{\alpha}{2} - \frac{1}{2}WL \sin \frac{\alpha}{2} - \frac{2}{3}TL \cos \frac{\alpha}{2} = 0 \qquad (9\text{-}22)$$

From this equation the magnitude of T may be found.

Problem.—In the case of the stepladder arrangement of the preceding illustration, discuss the effect upon the results there obtained if the two sides of the stepladder were of unequal lengths or of unequal weights. What change in the arrangement would be necessary in order that the force F would not be horizontal?

c. The Cube, with Concurrent Forces.—A cube has several forces applied to its vertices as shown in Fig. 108. The directions of the forces are all parallel and perpendicular to the sides of the cube. Find the single force necessary to produce equilibrium, and its position.

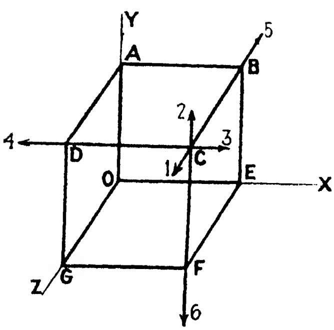

Fig. 108.

It is immaterial to the method of the solution whether the forces are parallel to the edges of the cube or not. If a given force were directed so that it was not either perpendicular or parallel to the edges meeting at a vertex, then we could simply use the components of the force parallel to the three mutually perpendicular edges.

We shall first find the force needed to produce equilibrium. If we select a reference system with origin at one vertex and with axes parallel to the three edges which meet at that vertex, we may find the resultant of the applied forces by finding the sums of the components along each of the three reference axes. If we let R_x, R_y, and R_z be the components of the resultant force, then it is readily found that

$$R_x = -1 \text{ lb.}, \qquad R_y = -4 \text{ lb.}, \qquad \text{and} \qquad R_z = -4 \text{ lb.}$$

and that

$$\boldsymbol{R} = -\boldsymbol{i} - 4\boldsymbol{j} - 4\boldsymbol{k}$$

The force ($\boldsymbol{F}$) needed to neutralize $\boldsymbol{R}$ is

$$\boldsymbol{F} = \boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k} \qquad (9\text{-}23)$$

In order to find the position of $\boldsymbol{F}$ it is necessary to find the resultant force moment $\boldsymbol{M}$ of the applied forces. If $\boldsymbol{M}_x$, $\boldsymbol{M}_y$, and $\boldsymbol{M}_z$ are the components of the resultant force moment about the X-, Y-, and Z-axes and if L is the length of one edge of the cube, the values of the components are readily found to be

$$M_x = L - 2L - 5L + 6L = 0$$
$$M_y = 3L - 4L - L + 5L = 3L$$
$$M_z = 4L - 3L + 2L - 6L = -3L$$

Hence

$$\boldsymbol{M} = 3L\boldsymbol{j} - 3L\boldsymbol{k}$$

The moment needed to neutralize $\boldsymbol{M}$ may be called $\boldsymbol{N}$ and is

$$\boldsymbol{N} = -3L\boldsymbol{j} + 3L\boldsymbol{k} \tag{9-24}$$

Whether this moment may be supplied by the force $\boldsymbol{F}$, when $\boldsymbol{F}$ is properly positioned, depends upon the angle between $\boldsymbol{F}$ and $\boldsymbol{N}$. If this angle, which we may call θ, is 90 deg., then it is possible for $\boldsymbol{F}$ to serve the double function of neutralizing the applied forces for the prevention of translational acceleration and at the same time to supply the needed moment which is necessary to prevent rotational acceleration. If the angle θ is not a right angle, then $\boldsymbol{F}$ may be used to supply only that component of $\boldsymbol{N}$ which is perpendicular to $\boldsymbol{F}$. The component of $\boldsymbol{N}$ which is parallel to $\boldsymbol{F}$ must be supplied by the addition of a couple, the plane of which is perpendicular to $\boldsymbol{F}$.

In order to find the value of θ, we may use the following theorem

$$\cos\theta = \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 \tag{9-25}$$

in which θ is the angle between two lines having the direction cosines $\alpha_1\,\beta_1\,\gamma_1$ and $\alpha_2\,\beta_2\,\gamma_2$. If we let $\alpha_1\,\beta_1\,\gamma_1$ be the direction cosines of $\boldsymbol{F}$ and $\alpha_2\,\beta_2\,\gamma_2$ be the direction cosines of $\boldsymbol{N}$, then we find that

$$\cos\theta = +\frac{1}{\sqrt{33}} \times 0 - \frac{4 \times 3}{\sqrt{33}\,\sqrt{18}} + \frac{4 \times 3}{\sqrt{33}\,\sqrt{18}} = 0$$

and

$$\theta = 90^\circ$$

We may, therefore, use $\boldsymbol{F}$ to supply the moment needed to prevent rotational acceleration.

The position of $\boldsymbol{F}$ is next to be determined. Let us suppose that the point of application of $\boldsymbol{F}$ has the coordinates x, y, and z and that $\boldsymbol{r} = \boldsymbol{i}\,x + \boldsymbol{j}\,y + \boldsymbol{k}\,z$. Then we may put

$$\begin{aligned}\boldsymbol{N} &= \boldsymbol{r} \times \boldsymbol{F} \\ &= \boldsymbol{i}\,(y\,F_z - z\,F_y) + \boldsymbol{j}\,(z\,F_x - x\,F_z) + \boldsymbol{k}\,(x\,F_y - y\,F_x)\end{aligned}$$

The components of F are given by Eq. (9-23), and the components of N are expressed in Eq. (9-24); hence, if L is the length of an edge of the cube,

$$\begin{aligned} N_x &= 4y - 4z = 0 \\ N_y &= \ \ z - 4x = -3L \\ N_z &= 4x - \ \ y = \ \ 3L \end{aligned} \tag{9-26}$$

From these equations we may determine the position of F by finding the coordinates of two points through which F passes. The first equation indicates that $y = z$ for all points through which F passes and hence F must be in the plane containing the X-axis and making equal angles (45 deg.) with the Y- and Z-axes. Since the three equations of Eq. (9-26) are not independent equations, because any point in the line of F will satisfy them, we may select one coordinate. Let us put $y = L$, then $x = z = L$. The line of F will, therefore, pass through the vertex C of the cube. In a similar manner it may be shown that F also passes through the point, $x = 3L/4$, $y = z = 0$.

The line of action of F may be determined in another way. If, as above, we let r be the radius vector drawn from the origin to any point on the line of F, then all that is necessary to do is to find an expression for r. Since the terminal point of r may be any point in the line of F, the vector expression for r must contain one scalar variable. A vector solution of the equation $N = r \times F$ for r is

$$r = \frac{F \times N}{F^2} + s' F$$

in which s' is the scalar variable. The validity of this equation may be established by multiplying each term by $\times F$ and then expanding the resulting expression. The details of the verification will be left to the student.

Introducing the values of F and N [Eqs. (9-23) and (9-24)] and writing sL for s' gives

$$\begin{aligned} r &= \frac{L}{33}(24\,i - 3\,j - 3\,k) + sL\,(i + 4\,j + 4\,k) \\ &= \frac{L}{11}[(8 + 11\,s)\,i + (44\,s - 1)\,j + (44\,s - 1)\,k] \end{aligned}$$

This is the desired expression for r. From it we may determine two points through which F passes by locating the terminal points for two positions of $\boldsymbol{r}$. To secure the latter we may arbitrarily assign numerical values to s, since s may have any numerical values. If s is put equal to $\frac{8}{11}$, we find that

$$\boldsymbol{r} = L\,\boldsymbol{i} + L\,\boldsymbol{j} + L\,\boldsymbol{k}$$

Hence $\boldsymbol{F}$ passes through the point whose coordinates are L,L,L. Similarly if $s = \frac{1}{44}$, we find that $\boldsymbol{F}$ passes through the point $3L/4$, O, O.

Problem.—If two nonparallel and nonintersecting forces are applied to a rigid body, is it possible to obtain equilibrium by the use of a force alone (*i.e.*, without the use of a couple)?

d. Cube with Nonconcurrent Forces.—Given the cube with three nonconcurrent forces as shown in Fig. 109. Find the force and couple needed to produce equilibrium.

Let the length of each edge of the cube be L ft. Also let the $X\,Y\,Z$ reference system be placed as shown in the diagram.

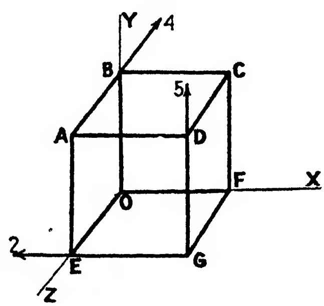

FIG. 109.

The resultant of the forces acting is $\boldsymbol{R} = -2\,\boldsymbol{i} + 5\,\boldsymbol{j} - 4\,\boldsymbol{k}$, and the force $\boldsymbol{F}$ needed to produce translational equilibrium is

$$\boldsymbol{F} = 2\,\boldsymbol{i} - 5\,\boldsymbol{j} + 4\,\boldsymbol{k} \qquad (9\text{-}27)$$

The resultant moment of the applied forces is readily found to be

$$\boldsymbol{M} = -9\,L\,\boldsymbol{i} - 2\,L\,\boldsymbol{j} + 5\,L\,\boldsymbol{k}$$

The moment $\boldsymbol{N}$, which will prevent rotational acceleration, is therefore

$$\boldsymbol{N} = 9\,L\,\boldsymbol{i} + 2\,L\,\boldsymbol{j} - 5\,L\,\boldsymbol{k} \qquad (9\text{-}28)$$

Whether the force $\boldsymbol{F}$ may be so placed as to supply $\boldsymbol{N}$ or not may be determined by determining θ, the angle between $\boldsymbol{F}$ and $\boldsymbol{N}$. If, as in the preceding problem, we let $\alpha_1\,\beta_1\,\gamma_1$ and $\alpha_2\,\beta_2\,\gamma_2$ be the direction cosines of $\boldsymbol{F}$ and $\boldsymbol{N}$, respectively, we find that

$$\begin{aligned}\cos\theta &= \alpha_1\,\alpha_2 + \beta_1\,\beta_2 + \gamma_1\,\gamma_2 \\ &= \frac{1}{\sqrt{45}} \times \frac{1}{\sqrt{110}}(2 \times 9 - 5 \times 2 - 4 \times 5) \\ &= -0.1707\end{aligned}$$

From this result we learn that N is not perpendicular to F and that the angle θ is 99 deg. and 50 min.

The next step is to find the components of N which are perpendicular and parallel to F. If we let S and T be the magnitudes of the components of N, perpendicular and parallel, respectively, to F, then, since the magnitude of N is $\sqrt{110}\, L$ lb.$_w$ ft.,

$$S = N \sin\theta = 10.31\, L \text{ lb.}_{\text{w}} \text{ ft.} \qquad T = N\cos\theta = -1.79\, L \text{ lb.}_{\text{w}} \text{ ft.} \tag{9-29}$$

The minus sign appearing in the magnitude of T indicates that the direction of T is opposite to that of F.

By properly placing F, the moment S may be supplied by F. The force F, as given by Eq. (9-27), is to be moved parallel to itself in a plane perpendicular to S and must be situated so that the perpendicular distance from the origin to the line of F is S/F or 1.54 L ft.

The other component T is to be supplied by a couple whose magnitude is $-1.79\, L$ lb.$_w$ ft. The plane containing this couple is perpendicular to F. The negative sign indicates the direction of the couple.

An alternative method for determining whether N is perpendicular to F or not and of evaluating the vectors S and T is included for purposes of comparison and at the same time to provide a check on the results obtained above. This method employs the scalar product of two vectors.

If two vectors are perpendicular to each other, their scalar product is zero. The scalar product of F and N is

$$F \cdot N = (2 \times 9 - 5 \times 2 - 4 \times 5)L = -12L \tag{9-30}$$

Since this product is not zero, N is not perpendicular to F.

In determining the angle between the two given vectors we may again use the scalar product. Since

$$F \cdot N = F\, N \cos\theta$$

it follows that $\cos\theta = -0.1707$.

To find the component (T) of N which is parallel to F we must determine the magnitude of T. In the scalar product of F and N we are at liberty to regard N as being projected into the line of F and then being multiplied by F; hence it follows that

$$T = \frac{(F \cdot N)}{F}$$

$$= \frac{(18\,L - 10\,L - 20\,L)}{\sqrt{45}} = -1.79\,L \qquad (9\text{-}31)$$

Since the vector $-T$ is parallel to F, the ratios of the coefficients of the unit vectors must be equal to the ratio of the magnitudes of the two vectors. If we let a, b, and c represent the coefficients, respectively, of the unit vectors of $-T$, then

$$\frac{a}{2} = -\frac{b}{5} = \frac{c}{4} = \frac{1.79\,L}{\sqrt{45}}$$

from which, $a = 0.534\ L$, $b = -1.333\ L$, and $c = 1.067\ L$. Hence

$$T = (-0.534\,i + 1.333\,j - 1.067\,k)L \qquad (9\text{-}32)$$

The other component (S) of N may be found by the following vector equation:

$$S = N - T$$
$$= (9.534\,i + 0.667\,j - 3.933\,k)L \qquad (9\text{-}33)$$

This vector (S) lies in the plane determined by N and F and is perpendicular to F as may readily be shown by expanding the scalar product of $S \cdot F$.

The position of the force F necessary to supply the moment S may now be found by the method used in the preceding illustration [Eq. (9-26)]. If we let x, y, and z be the coordinates of the point of application of F and let r be the radius vector to that point, then the moment S must be equal to $r \times F$. Putting $r = xi + yj + zk$, expanding the vector product, and putting the coefficients of the unit vectors equal to the corresponding magnitudes of the components of S gives

$$\begin{aligned} S_x &= 4\,y + 5\,z = 9.534\,L \\ S_y &= 2\,z - 4\,x = 0.667\,L \\ S_z &= -5\,x - 2\,y = -3.933\,L \end{aligned} \qquad (9\text{-}34)$$

From these three equations the ratios of the coordinates of the point of application of F may be found.

Problem.—Find the force and couple needed to produce equilibrium on a cube upon which the following forces are applied (Fig. 109):

At A, parallel to the X-axis, $-5.$ lb.
At C, parallel to the Y-axis, $+6.$ lb.
At F, parallel to the Z-axis, $-3.$ lb.

e. Truss.—A simple horizontal bridge truss is made up of three equilateral triangles, all being of the same size (Fig. 110). If the truss carries a load of 1,000 lb. at its central point, find the stress in each member. The weights of the members of the truss are to be neglected and all joints are to be regarded as freely hinged.

The solution of a problem of this sort is obtained by applying the two conditions of equilibrium to each joint of the truss. In order to simplify the designation of the forces at the various joints, we shall use the letters of the terminal points of the member as subscripts to the symbol F and let the sequence of the subscripts indicate the direction of the force at the point under consideration.

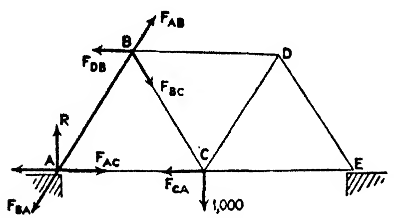

Fig. 110.

For example, the force at A due to the stress in the member AB is parallel to AB and is directed from B toward A and will be designated as F_{BA}. The force at B along the same member is to be written F_{AB}.

We shall first find the forces acting at the joint A. Since the load (1,000 lb.) is at the center of the truss and the weights of the members are to be neglected, the reaction (R) of the pier at A is directed vertically upward and is equal to 500 lb. If desired, this value could readily be established by writing the moments of R and the 1,000-lb. weight about an axis through E. The two other forces acting at A are F_{BA} and F_{AC}. Since F_{AC} is horizontal, the vertical component of F_{BA} must neutralize R, therefore

$$F_{BA} \cos 30° = 500 \text{ lb.} \qquad \text{and} \qquad F_{BA} = 577 \text{ lb.}$$

The horizontal component of F_{BA} must be equal and opposite to F_{AC}; hence

$$F_{AC} = F_{BA} \cos 60° = 288.5 \text{ lb.} \tag{9-35}$$

Of the three forces acting at the point B, the force F_{AB} (equal and opposite to F_{BA}) is completely known and the directions of the two others (F_{DB} and F_{BC}) are known. There are therefore only two unknowns. Analytically the simplest procedure is to put the sums of the horizontal and vertical components separately equal to zero. These two equations are

$$\begin{aligned} -F_{DB} + F_{AB} \cos 60° + F_{BC} \cos 60° &= 0 \\ F_{AB} \sin 60° - F_{BC} \sin 60° &= 0 \end{aligned} \qquad (9\text{-}36)$$

From which we find the magnitudes to be

$$F_{DB} = F_{AB} = F_{BC} = 577 \text{ lb.}$$

This result is confirmed by observing that the three forces make equal angles with each other.

We shall consider next the forces at the point C. There are five forces acting at this point. The directions of all five forces and the magnitudes of three are known. There remain as unknown quantities the magnitudes of F_{CD} and F_{CE}. Since the arrangement is one of symmetry, it is readily seen that

$$F_{CE} = F_{CA} \qquad \text{and} \qquad F_{CB} = F_{CD}$$

In a similar manner the forces at the points D and E may be found. Again by symmetry we may readily write the values for the forces at D and E from those already found.

Problem.—Using the diagram of Fig. 110, assume that the angles BAC, ACB, DCE, and CED are 45 deg. each and that the load (1,000 lb.) is midway between A and C. Find the stresses in the various members.

9-13. Principle of Virtual Work.—The principle of virtual work played an important role in the development of the subject of statics. At the present time it is used but little in this branch of mechanics because the vector relations expressed in the two conditions of equilibrium are easier to understand and to apply in the solutions of problems. The student of mechanics should, however, be familiar with this principle. For a more detailed presentation he should consult the literature on this subject, for example, Appell, "Mécanique rationnelle"; Lagrange, "Mécanique analytique," Crew and Smith, "Mechanics for Students of Physics and Engineering," and others of similar nature.

The credit for formulating the principle of virtual velocities is given to Stevinus (1548–1620). Galileo also used this principle in a theoretical consideration of some simple machines, but it was not until more than a century later (1717) that Jean Bernoulli was able to announce the general principle and its usefulness in problems of statics. The earlier ideas about virtual velocities and their uses in statics were somewhat nebulous before Bernoulli's efforts in this field. In a letter to Varignon he clearly explains the principle and gives illustrations of the application of it.

a. Virtual Displacement.—Suppose that a particle which is in equilibrium has a force ($\boldsymbol{F}$) applied to it and suppose that the force $\boldsymbol{F}$ causes the particle to have a very small displacement which we may call Δs. The displacement is not to be a real displacement but a purely imaginary one. Such a displacement is called a *virtual displacement* to distinguish it from any actual displacement which the particle might have.

b. Virtual Work.—The work ΔW produced by the force $\boldsymbol{F}$ in giving the particle the virtual displacement $\Delta \boldsymbol{s}$ is expressed by the equation

$$\Delta W = \boldsymbol{F} \cdot \Delta \boldsymbol{s} \tag{9-37}$$

It may readily be shown that the virtual work of the resultant of the several forces which may be applied to the particle is equal to the sum of the virtual works of the component forces.

c. Virtual Velocity.—If we let Δt be the element of time in which the virtual displacement $\Delta \boldsymbol{s}$ takes place, then the virtual velocity is $\Delta \boldsymbol{s}/\Delta t$.

d. The Principle.—The ideas expressed by the virtual displacement and work of a particle may readily be extended to a rigid body. We must remember that we are dealing with a body in equilibrium and, hence, there must be at least two forces acting on the body. A virtual displacement is to be associated with each force and is to express a very small displacement of the point of the body to which that force is applied. From this it may be seen that some, but not all, of the virtual displacements in a particular case may not be selected arbitrarily. The virtual work, in the case of a rigid body, done by any force is determined by that force and its associated virtual displacement by taking the scalar product of those two quantities.

Let us consider a rigid body which is in equilibrium under the action of several forces applied to it and then imagine that one or more of the points of application of the forces are given virtual displacements. The principle of virtual work, or of virtual velocities as it was formerly called, as stated by Appell is as follows:

"The necessary and sufficient condition needed for a system to be in equilibrium is that for any virtual displacement of the system, compatible with the liaisons, the sum of the virtual works of the applied forces shall be zero."

In order to understand just what is meant by the phrase "compatible with the liaisons," one must remember that, if the body is rigid and two or more forces are applied to it, the virtual displacements to be associated with the forces are not independent of each other but depend upon the positions of the forces relative to the body and to each other and also may depend upon any constraints to which the body is subject. For example, let us consider a simple lever *AB* (Fig. 111) which is in equilibrium under the influence of the two forces *W* and *F* and which is constrained to rotational motion by having a fixed axis at *C*. If we imagine that one end (*B*) of the lever has a virtual displacement Δs, then the other end (*A*) cannot have any arbitrary virtual displacement if the lever is regarded as rigid. The virtual displacement of *A* is determined by the structural arrangement—the liaisons.

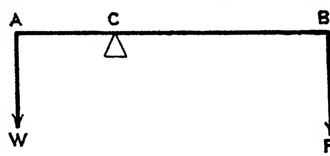

Fig. 111.

To make a further use of the illustration, let us apply the principle of virtual work in order to obtain a relation between the forces *F* and *W*. Let the virtual displacement of *A* be Δd. The sign of a virtual displacement is regarded as positive if it is supposed to take place in the positive direction of the corresponding force; otherwise it is negative. If we assume that Δs is positive, then the sum of the virtual works is

$$F \cdot \Delta s - W \cdot \Delta d = 0 \tag{9-38}$$

A second relation may be obtained from a knowledge of the positions of the two forces with respect to the fixed axis. This relation is one between the relative magnitudes of the virtual displacements and may be written as follows:

$$\frac{\Delta d}{\Delta s} = \frac{AC}{CB} \tag{9-39}$$

From these two equations the ratio of the magnitudes of the two forces may be found, provided the directions of the forces are known.

e. Illustration.—A single illustration will be given to show how the principle of virtual work may be used in a problem of statics.

The radii of the two pulleys of a differential pulley are r_1 and r_2. It is required to find the ratio of the applied force to the weight of the object which may be lifted.

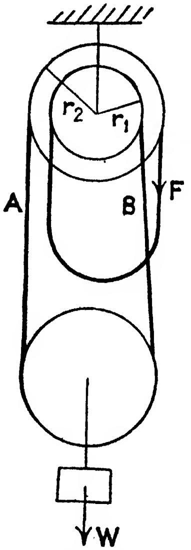

FIG. 112.

Let $\boldsymbol{W}$ be the weight lifted, $\boldsymbol{F}$ the applied force with $\Delta\boldsymbol{d}$ and $\Delta\boldsymbol{s}$, respectively, the virtual displacements (Fig. 112). Applying the principle of virtual work gives the following equation:

$$\boldsymbol{F} \cdot \Delta\boldsymbol{s} - \boldsymbol{W} \cdot \Delta\boldsymbol{d} = 0 \tag{9-40}$$

To find the relation between the two virtual displacements, we may imagine that the virtual displacement of the fixed pulley is $\Delta\theta$. Corresponding to this virtual displacement, the chain A is pulled upward a distance $r_2\Delta\theta$. The chain B on the other side is lowered a distance $r_1\Delta\theta$. Hence the virtual displacement of the weight W is

$$-\Delta d = \tfrac{1}{2}(r_2 - r_1)\Delta\theta$$

The virtual displacement Δs is equal to $r_2\Delta\theta$.

If we substitute the values of $\Delta\boldsymbol{d}$ and $\Delta\boldsymbol{s}$ in Eq. (9-40) and evaluate the scalar products, since the virtual displacements are parallel to the corresponding forces, we obtain the following desired relation:

$$\frac{F}{W} = \frac{(r_2 - r_1)}{2r_2} \tag{9-41}$$

In the ordinary differential pulley, $r_2 - r_1$ is small in comparison with r_2 and hence, neglecting friction, a small force may be used to lift a comparatively large weight.

f. The Work Principle.—The method described above for solving problems in statics by means of the principle of virtual work is very similar to the principle frequently described as the work principle. It will be recalled that the work principle is used in the same type of problems and differs from the principle of virtual work only to the extent of assuming a real and finite displacement in place of the imaginary and very small displacement. The terms *input* and *output* as used in the work principle might very well be employed in the principle of virtual work. The indicated similarity will make an understanding of the principle of virtual work easy to acquire.

9-14. Flexible Cables.—In this section we shall study the characteristics of equilibrium of a heavy flexible cable or rope when hanging between two fixed points. The diameter of the cable is considered to be small in comparison with its length. It is to be regarded as being perfectly flexible; in other words, it offers no elastic resistance to forces tending to bend it. The nature of the curve which the cable will assume when under the influence of gravity alone and the tension at any point of the cable are the two characteristics of chief interest.

If the cable is uniform, its linear density is constant and its weight is therefore distributed evenly along its length. A uniform cable suspended by its extremities between two fixed points and carrying a load, distributed with linear uniformity, will assume a definite curve which is called the catenary. This curve is not a simple curve. Its derivation is given below. If we may assume that the load carried, including the weight of the cable, is uniform horizontally, the curve taken on by the cable is a parabola. Because the latter assumption yields the simpler curve, it will be considered first.

a. Uniform Horizontal Distribution of the Load.—We are to determine the curve assumed by the cable under the limitation that the load is distributed uniformly along a horizontal line.

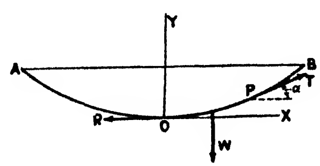

FIG. 113.

Let the cable be suspended from the two points A and B which are in the same horizontal line (Fig. 113). The reference system $X\ O\ Y$ is placed in the plane of the curve with the origin at the lowest point of the curve and with the X-axis horizontal.

Let $P\ (x\ y)$ be any point on the curve. We may consider the portion $O\ P$ of the chain. It is in equilibrium under the influence of three forces, the horizontal force R which is acting at O, the tension T which is tangent to the curve at P, and its weight W which is directed vertically downward at the center of the portion. Since the sums of the horizontal and vertical components of these three forces must be separately equal to zero, we have

$$R - T \cos \alpha = 0 \qquad \text{and} \qquad W - T \sin \alpha = 0 \quad (9\text{-}42)$$

where α is the angle which T makes with the X-axis.

Since the horizontal distribution of the weight is assumed to be uniform, we may put $W = nx$ where n is the weight per unit of horizontal length of the cable. Substituting this value for W in Eqs. (9-42) and eliminating T gives

$$\tan \alpha = \frac{nx}{R} \tag{9-43}$$

But $\tan \alpha = dy/dx$; hence on integrating and putting $x = y = 0$ for initial conditions we find that

$$y = \frac{nx^2}{2R} \tag{9-44}$$

Since P is any point on the curve and n and R are constants, Eq. (9-44) is a parabola with origin at the lowest point of the curve and with the axis directed vertically upward. It follows, therefore, that under the imposed weight distribution the curve assumed by the cable is a parabola.

The tension R in the cable at the point O will obviously depend upon the vertical distance of O below the points A and B and also the horizontal distance from A to B. If the former distance is called d and the distance between A and B is identified as $2a$, then we may evaluate R by using Eq. (9-44), and the coordinates a and d of the point B. Hence we find that

$$R = \frac{n\,a^2}{2d} \tag{9-45}$$

The value of the tension T' at B (or at A) is next to be found. The direction of T' may be found from Eq. (9-43) which gives if we write α' for the particular value of d,

$$\tan \alpha' = \frac{n\,a}{R} = \frac{2\,d}{a} \tag{9-46}$$

The magnitude of T' may be determined by using Eqs. (9-42). Squaring both equations, writing T' for T, and introducing the values for R and W gives

$$T' = \frac{n\,a}{2\,d}(4\,d^2 + a^2)^{\frac{1}{2}} \tag{9-47}$$

The tension T at any point in the cable may be determined in a similar manner.

b. Uniform Linear Distribution of the Load.—We shall next consider the conditions of equilibrium when the load is distributed uniformly along the cable. If we use the symbol m to represent the weight of a unit length of the cable and s to express the length of a segment of the cable such as OP (Fig. 113), the differential equation for the curve becomes

$$\tan \alpha = \frac{dy}{dx} = \frac{m\, s}{R}$$

For the sake of convenience we may put $R/m = c$, a constant, which gives

$$\frac{dy}{dx} = \frac{s}{c} \tag{9-48}$$

This equation may be recognized as belonging to the catenary. The origin of the curve is to be taken at a distance c below O in the diagram. The constant c has the dimension of a length. The equation may be transformed into one which contains only two variables s and x by the use of identity

$$\tan^2 \alpha = \frac{1}{\cos^2 \alpha} - 1$$

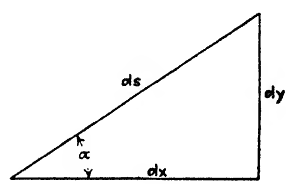

FIG. 114.

and by using the relation shown in Fig. 114. Since $\cos \alpha = dx/ds$, it follows that

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{ds}{dx}\right)^2 - 1 \tag{9-49}$$

Eliminating dy/dx from Eqs. (9-48) and (9-49), we obtain the desired differential equation

$$dx = ds\left(1 + \frac{s^2}{c^2}\right)^{-\frac{1}{2}} \tag{9-50}$$

Integrating this equation and writing $\log D$ for the constant of integration gives

$$\frac{x}{c} = \log\left[\sqrt{1 + \frac{s^2}{c^2}} + \frac{s}{c}\right] + \log D$$

By selecting $x = 0$ at $s = 0$ for the initial conditions, we find that $\log D = 0$; hence we may write

$$e^{c} = \sqrt{1 + \frac{s^2}{c^2}} + \frac{s}{c} \tag{9-51}$$

This equation may be manipulated so that it is explicit for s/c. By taking the reciprocal of both members of the equation and then simplifying, we obtain

$$e^{-\frac{x}{c}} = \sqrt{1 + \frac{s^2}{c^2}} - \frac{s}{c} \qquad (9\text{-}52)$$

Subtracting this equation from Eq. (9-51) gives

$$e^{\frac{x}{c}} - e^{-\frac{x}{c}} = 2\,\frac{s}{c}$$

Hence

$$\frac{s}{c} = \sinh \frac{x}{c} \qquad (9\text{-}53)$$

If this value of s/c is substituted in Eq. (9-48), a differential equation in x and y is obtained. This is

$$\frac{dy}{dx} = \sinh \frac{x}{c}$$

Integration yields the following equation:

$$\frac{y}{c} = \cosh \frac{x}{c} \qquad (9\text{-}54)$$

The constant of integration is equal to zero.

A useful relation between x and y may be obtained from the fact that

$$\cosh^2 \frac{x}{c} - \sinh^2 \frac{x}{c} = 1$$

Applying this relation to Eqs. (9-53) and (9-54) gives

$$y^2 - s^2 = c^2 \qquad (9\text{-}55)$$

The expression of Eq. (9-54) may be simplified if the constant c is large, or physically if the horizontal tension is large in comparison with the linear weight of the cable. By expanding the exponential terms of cosh x/c, it may be shown that the series

$$e^{\frac{x}{c}} = 1 + \frac{x}{c} + \frac{x^2}{2c^2} + \frac{x^3}{6c^3} + \cdots$$

$$e^{-\frac{x}{c}} = 1 - \frac{x}{c} + \frac{x^2}{2c^2} - \frac{x^3}{6c^3} + \cdots$$

converge rapidly if c is large in comparison with x.

The approximate expression then becomes

$$\frac{y}{c} = 2 + \frac{x^2}{c^2} \tag{9-56}$$

This approximate expression is a parabola. We may conclude, therefore, that when the horizontal tension is large the curve assumed by the cable is approximately a parabola.

Problems.—1. A weight of 500 lb. is suspended by two massless cords which make angles of 30 and 50 deg. with the vertical. Find the tension in the cords.

2. A uniform bar 10 ft. long and weighing 250 lb. is supported in a horizontal position by two vertical ropes which are placed 1 and 2 ft. from the ends of the bar. Find the tension of the ropes.

3. A uniform bar 8 ft. long and weighing 25 lb. carries three weights, 50, 75, and 125 lb., which are 1, 3, and 6 ft., respectively, from one end of the bar. Find the center of gravity.

4. A uniform horizontal bar of mass m g. and of length $2L$ cm. is suspended by two parallel massless threads. The threads are $2L$ cm. apart and are D cm. long. Find the couple needed to rotate the bar through an angle θ. (This arrangement is called the bifilar pendulum.)

5. A bug is placed in a spherical-shaped bowl. If the coefficient of friction between the bug and the walls of the bowl is 0.4, how far up the side can the bug crawl before he slips back?

6. Find the ratio of the applied force to the force exerted by a wedge if the angle of the wedge is 2α, and the force exerted is perpendicular to the line of the applied force. Use the principle of virtual work.

7. A ladder of length L and weight W is placed on a horizontal floor and leans against a vertical wall. If the coefficients of friction are similar and both are equal to μ, find the smallest angle which the ladder can make with the floor and not slip.

CHAPTER X

FORCES OF ATTRACTION AND POTENTIAL

10-1. The Law of Gravitation.—The single contribution which made the greatest advance in the study of celestial mechanics was the law of gravitation, formulated by Newton in 1666. It is of interest to observe that, in stating the universal law of gravitation, Newton brought into coalescence many important ideas which were current prior to his time. The works of such men as Galileo, Copernicus, Huygens, Mersenne, and Kepler were milestones to the greater achievements of Newton.

The law of gravitation was not the product of a few hours' work. In order to achieve the final result, much time was required and new mathematical tools had to be invented. He published a description of this work in his "Mathematical Principles of Natural Philosophy"— "a work which as an exhibition of individual intellectual effort is unsurpassed in the history of the human race."

Newton's first great step was in guessing that the earth's gravitation extended to the moon. Using some of Kepler's ideas, he deduced that the forces which keep the planets in their orbits must be inversely proportional to the squares of their distances from the sun, and then he compared the centripetal force needed to hold the moon in its orbit with the force of gravity at the surface of the earth and "found them answer pretty well."

Although Newton found that his gravitational force did "answer pretty well" with the necessary centripetal force, the difference was, in his opinion, sufficiently large to warrant delay in publication. So he waited until 1672, at which time he was able to prove that for certain purposes the mass of the body could be regarded as being concentrated at its geometrical center. This fact together with new and more accurate data of the size of the earth enabled him to make a satisfactory substantiation of his theory. He published the results in his now famous "Principia."

An interpretation of the law, as formulated by Newton, leads us to the view that, when we are concerned with the gravitational attraction between two bodies, we are in reality dealing with resultants of groups of differential forces. Considering two differential masses dm and dm', the force dF exerted by dm on dm' is expressed by the relation

$$dF = -K\,dm\frac{dm'}{r^2} \tag{10-1}$$

in which r is the distance from dm to dm' and K is a constant which depends upon the system of units employed. The minus sign indicates that the force is one of attraction.

The force dF acts mutually upon the two masses. It may be regarded as a tension between them, as though a stretched rubber band was attached to the two particles and was pulling them together. The direction of the force is along the line joining the particles. The magnitude of the force decreases as r^2 increases and *vice versa*. As r approaches zero, the force approaches infinity. Obviously r cannot be made equal to zero.

If we wish to determine the attraction between any two bodies, of mass m and m', we shall have to integrate the differential forces over both masses. Every differential particle of the one body will exert a differential force upon each differential particle of the other body. A double mass integration is therefore necessary to determine the resultant gravitational force between the two masses.

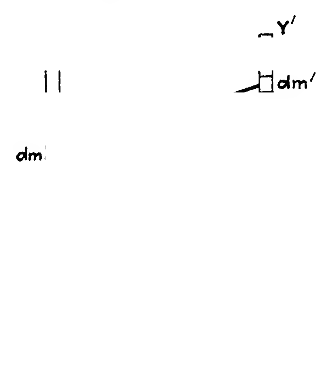

Fig. 115.

Illustration.—Given two thin homogeneous rods, of lengths $2L$ and $2L'$, and of masses M and M', respectively. Let the rods be so placed that they are parallel to each other and that the line joining their centers of mass is perpendicular to each of the rods. Let D be the distance between the centers of mass. Find an expression for the gravitational attraction of M for M'.

Let dm and dm', the differential masses, be situated at the distances y and y' from the centers of mass of M and M', respectively, with the reference system as shown in Fig. 115. If A is the common area of cross section and if ρ and ρ' are the densities of the rods, then the differential force between dm and dm' is

$$dF = -KA^2\rho\rho'\, dy\frac{dy'}{r^2} \tag{10-2}$$

This force is directed along the line r which connects dm with dm'. We cannot integrate this expression over the two rods and obtain a correct expression for the resultant force because the directions of the differential forces in the integration will not all be the same. We may, however, project these differential forces into lines parallel to OX and to OY and then, by separate integrations, determine the components of the resultant force which are parallel to OX and to OY.

The component parallel to OY is found by the following expression:

$$F_y = -KA^2\rho\rho'\int_{-L}^{L}\int_{-L'}^{L'}\frac{\cos\alpha\, dy\, dy'}{r^2}$$

where α is the angle between r and the Y-axis. Since

$$\cos\alpha = \frac{y' - y}{r} \qquad \text{and} \qquad r = [(y' - y)^2 + D^2]^{\frac{1}{2}}$$

then it follows that

$$F_y = -KA^2\rho\rho'\int_{-L}^{L}\int_{-L'}^{L'}\frac{(y' - y)\, dy\, dy'}{[(y' - y)^2 + D^2]^{\frac{3}{2}}} = 0 \tag{10-3}$$

The other component F_x is found by integrating the following expression:

$$\begin{aligned} F_x &= -KA^2\rho\rho'\int_{-L}^{L}\int_{-L'}^{L'}\frac{D\, dy\, dy'}{[(y' - y)^2 + D^2]^{\frac{3}{2}}} \\ &= -\frac{KA^2\rho\rho'}{LL'D}\left[\sqrt{(L' + L)^2 + D^2} - \sqrt{(L' - L)^2 + D^2}\right] \\ &= -\frac{KMM'}{2\, LL'D}\left[\sqrt{(L' + L)^2 + D^2} - \sqrt{(L' - L)^2 + D^2}\right] \end{aligned} \tag{10-4}$$

Since the component F_y is zero, the resultant force is parallel to the axis OX. This result does not prove that the resultant force passes through the centers of mass of the two bodies. In this case, however, because of the symmetry of the arrangement it is not difficult to see that such is the case. In general, it is not correct to say that the resultant force between two bodies does pass through the centers of mass. As an illustration of a case of an unsymmetrical arrangement in which the resultant attraction does not pass through the centers of mass, the following problem is given for the student to work out.

Problem.—Find the direction and magnitude of the attraction between a thin uniform rod, of length L, linear density ρ, and mass M, for a small dimensionless mass m situated at any point on a line which passes through one end of the rod and perpendicular to it. Prove that the direction of the force does not pass through the center of mass of the rod but bisects the angle formed at m by the two lines drawn to the extremities of the rod.

To locate the position of the resultant force in either rod of Fig. 115, say M, it is necessary to determine the moment of the X component of the force on M about some axis perpendicular to the diagram and passing through some point P of the rod. If the moment so determined is zero, then from that fact we should know that the resultant attraction passes through the point P. If the resultant moment were not zero, then dividing the moment by the resultant force would give the lever arm of the force necessary to produce that moment. The point of the body through which the resultant force passes will be at a distance equal to the lever arm from the point selected as axis of moments. The sign of the moment will indicate on which side of the selected axis the line of the force passes.

Let us select the point O in M (Fig. 115) as the axis of moments. The X component of the differential force (dF_x) due to the attraction of the rod M' on any differential mass dm (O, y) of M is found as follows:

$$dF_x = KA\rho' dm \int_{-L'}^{L'} \frac{D\,dy'}{[(y'-y)^2 + D^2]^{\frac{3}{2}}}$$

$$= \frac{KA\rho' dm}{D}\left[\frac{L'-y}{\sqrt{a - 2L'y + y^2}} + \frac{L'+y}{\sqrt{a + 2L'y + y^2}}\right] \quad (10\text{-}5)$$

in which $a = L'^2 + D^2$. The sign is taken positive because the force is in the direction of a positive displacement along OX.

According to our convention of signs, a positive moment about an axis through O would produce a counterclockwise rotation. The moment of the force dF_x about O is therefore $-y\,dF_x$. If we multiply the foregoing expression for dF_x by $-y$, replace dm by $A\rho dy$, and then integrate from $-L$ to L, the resultant moment M is found. Replacing the values of a, the expression for the moment may then be written as follows:

$$M = -\frac{KA^2\rho\rho'}{D}\left[\int_{-L}^{L} \frac{y(L'-y)dy}{\sqrt{D^2 + (L'-y)^2}} + \int_{-L}^{L} \frac{y(L'+y)dy}{\sqrt{D^2 + (L'+y)^2}}\right] \quad (10\text{-}6)$$

The integration of this expression is more easily obtained if we make the following changes:

First Integral		Second Integral
Let		Let
$u = L' - y$		$v = L' + y$
$du = -dy$		$dv = dy$
$L' + L$	Upper limits	$L' - L$
$L' - L$	Lower limits	$L' + L$

Making these substitutions and in the second integral reversing the limits and changing its sign gives

$$M = -\frac{KA^2\rho\rho'}{D}\left[\int_{L'-L}^{L'+L}\frac{u(L' - u)du}{\sqrt{D^2 + u^2}} - \int_{L'-L}^{L'+L}\frac{v(L' - v)dv}{\sqrt{D^2 + v^2}}\right] \quad (10\text{-}7)$$

It is not even necessary to evaluate the integrals because it may be seen that, since the limits are alike and the forms of expression identical, the final value must be equal to zero. We may therefore draw the conclusion that the resultant gravitation attraction of M' for M passes through the centers of mass of both bodies.

10-2. The Gravitational Constant.—After Newton's discovery of the law of gravitation, the attention of many scientists was directed to determining the value of the constant K. This problem was not an easy one, for, it must be remembered, the mass neither of the earth nor of any other "heavenly" body was known. The size of the earth had been determined; hence there were, in reality, two quantities, K and the density δ of the earth, in the equation which had to be determined.

The experiments for finding K and δ are of two general types. In one class some natural mass is selected, usually a mountain whose slope and size were easy to measure. In some of these experiments measurements were made on the deflection of a plumb bob placed near the side of the mountain, and in the others the periods of a pendulum, placed on top and below the mountain, were determined. These experiments utilized the effect of a known mass (the mountain) upon the plumb bob or pendulum.

The other type of experiment was designated to be done in the laboratory. A large and a small mass were selected, the masses of course, being known. Usually the small mass was suspended by a very fine quartz fiber and then the large mass was placed first on one side of the small mass and then on the other. The displacement produced by the attraction was measured. The actual gravitation force between these two masses was compared with the weight of one of the masses.

Our space here is too limited to describe all or any of these interesting experiments. The student should consult the literature on this subject.[1] Some of the milestones in the solution of this problem are given here.

First Type of Experiment. 1. *Bouger*, 1740.—Bouger was the first to make any experiment intended to evaluate K. An account of his efforts on the slopes of Chimborazo in Peru is filled with many thrilling incidents. His results were not good numerically but did indicate "that the attraction of the mountain existed and that the earth, as a whole, is denser than the surface strata." As he remarks, "the experiments at any rate proved that the earth was not merely a hollow shell, as some had till then held; nor was it a globe full of water, as others had maintained."

2. *Maskelyne*, 1774.—Maskelyne made measurements on Schiehallion, a hill (3,547 ft. high) in Scotland. He used a plumb-bob method. His results indicated that δ was 4.5 g. per cubic centimeter.

3. *Carlini*, 1821.—Carlini used a pendulum at the Hospice on Mt. Cenis, about 6,000 ft. above sea level and found $\delta = 4.5$.

4. *Airy*, 1826 *and* 1854.—In these experiments the period of a pendulum was measured at the earth's surface and also at the bottom of a coal mine. He found $\delta = 6.5$ g. per cubic centimeter.

Second Type of Experiment. 1. *Mitchell.*—Credit should be given to Rev. John Mitchell for designing and completing laboratory apparatus for this purpose. Unfortunately, he died before doing the experiment.

2. *Cavendish*, 1797–1798.—This famous experiment is probably well-known by all students of physics. An average of 29 results gave a mean value of $\delta = 5.448 \mp 0.033$.

3. *Reich*, 1837, found values of 5.49 and 5.58 for δ.

4. *Bailey*, 1841, results gave $\delta = 5.674$.

5. *Boys*, 1895, used very fine apparatus and probably obtained the best results. His final values are $G = 6.6576 \times 10^{-8}$ and $\delta = 5.5270$.

The numerical value of K may be found from a knowledge of the weight of a body on the earth's surface and the mass of the earth. This is left for the student to do. Assume that the

[1] Poynting and Thomson, "Properties of Matter."

average density of the earth is 5.53 g. per cubic centimeter, that the radius is 6.37×10^8 cm., and that the weight of 1 g. mass is 980 dynes.

K is not merely a numerical constant used to make the magnitudes agree on both sides of the equation. It has dimensions. These are easily found if we write Eq. (10-1) so that it is explicit for K.

$$K = -\frac{r^2\,dF}{dm\,dm'} \tag{10-8}$$

10-3. Gravitational-field Intensity.—It is customary in physics to speak of magnetic-field intensity at a point in a magnetic field as the magnetic force exerted by the magnetic field upon a unit (N) magnetic pole placed at that point. Similarly the electric-field intensity is defined as the electric force exerted by the electric field upon a unit positive electric charge placed at the point in question. In both of these cases we think of the magnetic or electric field as a region of such a nature that there would be a force exerted upon any magnetic pole or electric charge if brought into the field. In a like manner we may regard the gravitational field as a region of gravitational influence in the space surrounding some mass or masses. Because of this similarity, gravitational-field intensity is defined in a manner analogous to that used in defining the magnetic- or electric-field intensities. *Gravitational-field intensity* is the gravitational force exerted by the field upon a unit mass placed at the particular point at which the field intensity is to be expressed. It is the gravitational force per unit mass. If the force on a differential particle dm at a given point is dF, then dF/dm is the field intensity at that point. It is obvious that field intensity is of the same dimensions as acceleration.

To determine a component of the field intensity G, at any point in the field of any given mass, we may replace dm in Eq. (10-1) by unit mass and, assuming the unit mass to be a dimensionless particle situated at the given point, evaluate the integral

$$G_x = -K\int_M \frac{dm}{r^2}\cos\alpha \tag{10-9}$$

where α is the angle between r and OX.

This equation gives only one component of the field intensity. The resultant field may be found by determining the three components referred to a selected reference system. The position of the force in unsymmetrical bodies is determined by using the moments of the forces as shown in Sec. **10-1.** A number of illustrations are given in the following sections to show the method of determining the field intensity.

10-4. The Gravitational-field Intensity of a Homogeneous Sphere.—The problem of finding the field intensity due to a homogeneous spherical mass at some point outside the mass is an important one in astronomical calculations. There are several ways in which the general expression for the field intensity may be determined. The method shown here involves three steps. Each step will, however, be complete in itself. The first step will be to find the field intensity at some point on the axis of a thin ring, the second will involve a determination of the intensity on the axis of a thin circular disk, and the last step will show the process of finding the intensity at some point outside the solid homogeneous sphere. In the second and third steps the results of the preceding step will be used. By this method the normal process of a triple integration over the volume of the solid is replaced by steps which reveal a physical meaning of the volume integral.

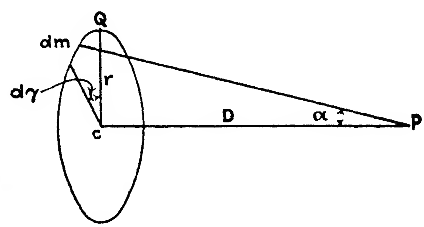

Fig. 116.

a. Thin Ring.—To find the field intensity at some point on the axis of a thin ring.

Let the radius of the ring be r (Fig. 116), its area of cross section A, and its density ρ. We are to find the field intensity at some point P, which is at a distance D from C, the center of the ring. If radii are drawn to the ends of a differential mass at Q and $d\gamma$ is the angle between them, then

$$dm = A\,\rho\,r\,d\gamma$$

The common distance of all such differential masses from the point P is $\sqrt{D^2 + r^2}$. The field intensity at P, due to dm, is in the line PQ. We must therefore project this differential force into the line of PC by multiplying it by the cos α where α is the angle between PC and PQ.

The general expression for the field intensity at P is

$$G = -K \int_M \frac{A\,\rho\,D\,r\,d\gamma}{(D^2 + r^2)^{\frac{3}{2}}} \tag{10-10}$$

Since γ is the only variable, the limits of integration, which correspond to taking the integration over the entire mass, now are from zero to 2π. Hence

$$G = -\frac{K\,A\,\rho\,r\,D}{(D^2+r^2)^{\frac{3}{2}}}\int_0^{2\pi} d\gamma$$

$$= -\frac{KA\,\rho\,rD\,2\pi}{(D^2+r^2)^{\frac{3}{2}}}$$

$$= -\frac{K\,m\,D}{(D^2+r^2)^{\frac{3}{2}}} \qquad (10\text{-}11)$$

in which m, the mass of the ring, is used to replace $A\rho 2\pi r$.

b. Thin Disk.—To find the intensity at a point on the axis of a thin disk.

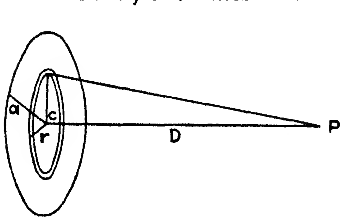

Fig. 117.

Let a be the radius of the disk (Fig. 117), C its center, t the thickness, and ρ the density. The field intensity is to be found at the point P which is on the axis of the disk and situated at a distance D from C, the center of the disk.

We may use a thin ring, of radius r, and width dr in the plane of the disk and concentric with C, as the differential mass. This selection will permit us to use the results obtained above. It is to be noticed that in this case the field intensity of the differential ring is in the line PC which must contain the resultant field intensity of the disk, and therefore no projection factor is necessary. The differential mass will be

$$dm = 2\pi\,r\,t\,\rho\,dr$$

The limits of integration will be from 0 to a, since r is the variable. The equation for the field intensity will therefore be

$$G = -K2\pi t\rho\int_0^a \frac{r\,D\,dr}{(D^2+r^2)^{\frac{3}{2}}} \qquad (10\text{-}12)$$

$$= -K2\pi t\rho\left[1 - \frac{D}{\sqrt{D^2+a^2}}\right]$$

$$= -\frac{K2m'}{a^2}\left[1 - \frac{D}{\sqrt{D^2+a^2}}\right] \qquad (10\text{-}13)$$

in which m', the mass of the disk, is equal to $\pi a^2\rho t$.

c. Solid Homogeneous Sphere.—To find the gravitational-field intensity at any point outside a solid homogeneous sphere.

Let the center of the sphere be C (Fig. 118), the density ρ, and the radius R. We are to find the field intensity at P which is at a distance S from C.

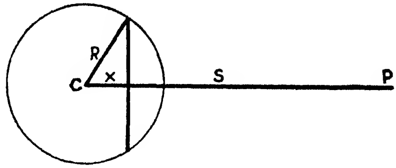

Fig. 118.

We may utilize here the results just obtained for the intensity at a point on the axis of a disk, provided that we select the thin disk as the differential element and place it as shown in the figure, perpendicular to PC. Let the disk be situated at a distance x from C as origin. The thickness of the disk is then dx. This selection will make x the variable and the integration limits will be from $x = R$ to $x = -R$.

Using the results given in Eq. (10-13) and replacing D by its equivalent $S - x$, and putting $a^2 = R^2 - x^2$, the expression for the field intensity will be

$$G = -2\pi K\rho \int_{-R}^{R} \left[1 - \frac{x - S}{\sqrt{R^2 + S^2 - 2Sx}} \right] dx \qquad (10\text{-}14)$$

$$= -\frac{4\pi K R^3 \rho}{3S^2}$$

$$= -\frac{KM}{S^2} \qquad (10\text{-}15)$$

This result expresses the field intensity at P in terms of M the mass of the sphere and S the distance from P to the center of mass of the sphere. As long as the sphere is homogeneous, we may therefore regard the intensity at P as though it were due to the entire mass concentrated at the center of mass. This result is also true for spheres in which the density varies, provided that the density may be expressed in terms of the radius only.

10-5. The Gravitational-field Intensity at a Point inside a Hollow Sphere.—The problem is to determine the field intensity at a point P which is inside of a hollow sphere. Let the sphere be of uniform density with R and B the outer and inner radii, respectively.

It is convenient here to use a differential mass in the shape of a thin ring so situated that its plane is perpendicular to the line PO (Fig. 119) which joins the point P with the center of the sphere O. To establish further the boundaries of the differential ring, let us select a spherical shell of radius r (shown in section in the diagram) and of thickness dr. If now we draw to this shell two radii which make the angles γ and $\gamma + d\gamma$ with the line PO, and then imagine that the radii so drawn are rotated about PO, keeping the angle γ constant, they will cut from the spherical shell the desired thin ring. The dimensions of the ring will be length, $2\pi r \sin\gamma$, width $r\,d\gamma$, and thickness dr. If ρ is the density, then the differential mass of the ring will be

$$dm = 2\pi\rho r^2\,dr \sin\gamma\,d\gamma \qquad (10\text{-}16)$$

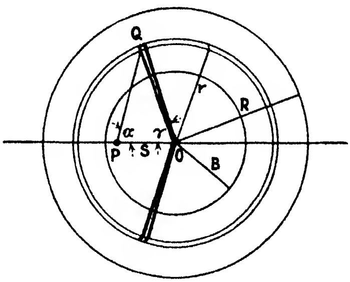

FIG. 119.

We have shown in the preceding section that the contribution of each differential length of the ring to the field intensity at a point P on the axis of the ring must be projected into the axis PO. This is done by introducing the $\cos\alpha$, in which α is the angle between PO and PQ, the line joining P to the differential element at Q.

To obtain the final expression a double integration appears to be necessary. First we may integrate over the spherical shell by letting the angle α vary from 0 to π and then over the solid portion of the sphere by letting the radius of the shell vary from B to R. With these conditions in mind we may now write the expression for the resultant field intensity at P, which is

$$G = -2\pi K\rho \int_B^R r^2 dr \int_0^\pi \frac{\sin\gamma\,d\gamma}{x^2}\cos\alpha \qquad (10\text{-}17)$$

in which x is written for PQ.

In order to evaluate the first integral we must either express x in terms of γ or γ in terms of x. In this case we choose to make the latter substitution. For this purpose we may utilize the following equations which are obtained from geometrical relations. If s is written for the distance PO, then

$$\cos\alpha = \frac{s - r\cos\gamma}{x}$$
$$x^2 = r^2 + s^2 - 2rs\cos\gamma \qquad (10\text{-}18)$$

Eliminating γ from these two equations gives

$$\cos\alpha = \frac{x^2 - r^2 + s^2}{2xs}$$

Differentiating Eq. (10-18) gives

$$\sin \gamma \, d\gamma = \frac{x}{rs} \, dx$$

The limits of integration for the new variable x will be from $r - s$ to $r + s$. Substituting these equivalent expressions in Eq. (10-17) gives

$$\begin{aligned} G &= -\frac{K\pi\rho}{s} \int_B^R r^2 dr \int_{r-s}^{r+s} \frac{(x^2 - r^2 + s^2)}{x^2} dx \\ &= -\frac{K\pi\rho}{s} \int_B^R r^2 dr \times 0 \\ &= 0 \end{aligned} \qquad (10\text{-}19)$$

In carrying out the details of integration for the first integral, the result is found to be zero. The conclusion to be drawn from this result is that the field intensity at any point inside a spherical shell is zero. It is true for all spherical shells of which the hollow sphere is composed, and therefore the intensity at any point inside the hollow of the sphere is zero.

It also follows from the foregoing conclusion that, if the point P were located at any point inside a solid sphere, the field intensity at P would be due only to that portion of the sphere which lies within the spherical surface passing through P which has its center concentric with the given sphere. This result may be obtained by integration or it may be seen to be true from the following consideration. Suppose P is within the hollow of a hollow sphere, and concentric with this sphere there is another sphere with a radius less than P's distance to the center. The field at P is then due only to the inner sphere. By increasing the inner sphere so that its surface approaches P and by decreasing the size of the hollow in the outer sphere, the space between the two may be reduced to zero. This result is true only for spheres the densities of which are uniform throughout or are functions of the radii.

From these results it also follows that the force between two homogeneous spheres, of mass m and m', is

$$F = -\frac{K\,mm'}{r^2} \qquad (10\text{-}20)$$

if r is the distance between their centers.

Problems.—1. Determine the gravitational-field intensity at a point outside a homogeneous spherical shell.

2. Using the results of the preceding problem, determine the field intensity at a point outside a homogeneous sphere.

3. Find the gravitational attraction between a homogeneous sphere and a thin homogeneous rod if the rod is placed so that the line connecting the centers of mass is perpendicular to it.

4. Find the field intensity at a point outside a homogeneous cube if the point is situated on a line which passes through the center of mass of the cube and which is perpendicular to any two parallel faces.

10-6. Conservative Forces.—Suppose we have a force field, such as a region surrounding a gravitational mass, and in this field we move a body from a point A to some other point B. In doing so, in general, a certain amount of work W would have to be done by the forces acting on the body. If the force is that of the field, the work is positive. If, however, work is done against the field, the work is negative. Now if the body is moved back from B to A, and the work done is exactly the same in magnitude as that done in moving the body from A to B, then the field is spoken of as a conservative field. In moving the body from A to B the work might have been done by the field forces. Suppose that this is the case. Then, in moving the body back to A, the work would have to be done against the field forces. In *conservative* fields there is no net work required to move a body from any point over any path of the field back to the same point, regardless of the path used in the displacement. There are fields, however, where forces of resistance are operative, in which there is a net work resulting from moving a body from any point over some path back to the starting point. Such a field is a *nonconservative field.* Fluid resistance is a typical nonconservative force. Work done against such forces is converted into heat energy manifested as kinetic energy of the molecules of the medium and of the body.

10-7. The Force Function.—In a conservative-force field, such as a gravitational field in a vacuum, the force due to the field acting upon a particle placed in the field will be a function of the coordinates of the position of the particle. Suppose that the position of the particle is given by the vector $\mathbf{r}$. For a very small displacement of the particle, such as $d\mathbf{r}$, the element of work dW done by the force $\mathbf{F}$ will be given by the equation

$$dW = \mathbf{F} \cdot d\mathbf{r} \tag{10-21}$$

If we write the vectors $\mathbf{F}$ and $d\mathbf{r}$ in terms of their components in the reference system,

$$F = i F_x + jF_y + kF_z \qquad dr = i\,dx + j dy + k dz$$

then the total work done, expressed in terms of the components of the forces, is

$$W = \int F_x\,dx + \int F_y\,dy + \int F_z\,dz \tag{10-22}$$

where the integration is to be taken over the entire path along which the particle is moved.

The components of the force in a conservative system are single-valued functions of the coordinates. Now suppose that there exists a function $U(xyz)$ such that

$$F_x = \frac{\partial U}{\partial x} \qquad F_y = \frac{\partial U}{\partial y} \qquad F_z = \frac{\partial U}{\partial z} \tag{10-23}$$

then the function U is called the force function because from it the force may be obtained.

One may also derive the force from another function which is called the potential and is usually designated by the symbol V. This is not to be confused with velocity, for which the same symbol is used. The context will indicate which is meant. The potential function V is closely related to the force function U. The relation is expressed by the equation $V = -U$. If there is a potential function at a given point in a field, the potential energy of a particle placed at the given point in the field may be found by simply multiplying the potential by the mass of the particle. In other words, the potential is the potential energy per unit mass.

It is obvious that the potential may be used for determining the force exerted by the field upon a unit mass placed at the point at which V is given. This is expressed symbolically as follows:

$$F_x = -\frac{\partial V}{\partial x} \qquad F_y = -\frac{\partial V}{\partial y} \qquad F_z = -\frac{\partial V}{\partial z}$$

If there is such a function V from which the components of F may be found, then the element of work dW done by the field forces in producing a differential displacement of a unit mass is

$$\begin{aligned} dW &= -\frac{\partial V}{\partial x}\,dx - \frac{\partial V}{\partial y}\,dy - \frac{\partial V}{\partial z}\,dz \\ &= -dV \end{aligned} \tag{10-24}$$

If we assume that V is a single-valued function, then, since V is an exact differential under the given conditions, we may integrate this expression from any point at which the value of V is V_2 to any other point at which the value of V is V_1 which would give

$$W = -(V_2 - V_1) \tag{10-25}$$

This result shows that the work done is independent of the path as it should be in a conservative field.

The single condition which establishes the existence of a force function in a conservative system is that

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}, \qquad \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}, \qquad \frac{\partial F_z}{\partial x} = \frac{\partial F_x}{\partial z} \tag{10-26}$$

If the force function were not a single-valued function of the coordinates in a given conservative system, then it would be possible to do work upon the system in moving a particle from some given point over an arbitrary path back to the starting point, for in so doing the force function might attain a second value. Now, because the forces are conservative, if the particle were then moved in the reverse direction around the selected path, work would be done by the field forces and hence the body would acquire energy. This could be repeated indefinitely and any amount of energy acquired. Such a procedure is contrary to the principle of the conservation of energy. Hence the force function must be a single-valued function.

10-8. Potential Energy and Potential.—There are two kinds of energy: kinetic and potential. *Kinetic energy* is energy of motion and, as we have seen above, may be expressed in terms of the mass of the body and the speed of its center of mass in translational motion or in terms of the moment of inertia of the body and its angular velocity in rotational motion. *Potential energy* is energy of position or of a strained condition. If a body is moved against the forces of some force field, it gains potential energy. If a body is stressed by forces, such that an elastic deformation occurs, as in the case of the stretching of a spring, it gains potential energy. In either case, whether a body gains kinetic energy or potential energy, work must be done upon the body in order that the body may obtain this energy. Work may be used as the measure of the energy, or, in other words, energy is the capacity for doing work.

The total amount of potential energy which a body possesses by virtue of its position is entirely a relative matter. In case the gravitating body is the earth, we might say that the potential energy which a mass possesses relative to the earth is equal to the work that must be done against the gravitational-field forces of the earth in moving the body from the center of the earth to the particular point at which the potential energy is to be expressed. This definition is, however, unsatisfactory.

In electrostatics the forces are repulsive when the charges are similar; hence the potential energy which one charged body has when placed at some point in the field of another charged body is equal to the work done in moving the body from infinity to the point at which the potential energy is to be expressed. A similar definition may be given for the potential energy of magnetic poles in a magnetic field.

The term potential is used in gravitational fields to express the potential energy of a unit mass. Potential in electric or magnetic fields has a similar meaning. In electric or magnetic fields, potential is frequently defined as the work done against the field forces in moving a unit charge or pole from infinity to the point at which the potential is expressed. The work done under these circumstances gives the potential energy; hence these two aspects are consistent with each other. In gravitational fields, however, the forces are attractive and not repulsive as they are in the electric and magnetic fields (between like charges or poles); hence we do not desire to define potential in gravitational fields in terms of the work done against the field in bringing a unit mass from infinity to the point, nor do we wish to define it in terms of the work done in moving the unit mass from the center of mass of the attracting body to the point at which the potential is to be expressed. The difference of potential which exists between any two points in a gravitational field may be measured by the work done in moving a unit mass from one point to another.

Now there is a certain convenience to be gained by having a single definition of potential which is applicable to all three fields. It is not easy to do this and use the conception of the work done against the field forces as a basis for the definition. A common basis for the definition is supplied, however, in the force function. As used in the preceding section, the force function was used to obtain the force on an indefinite mass particle. If the particle has a mass of unity, the forces then become field intensities. Let us therefore define potential as that force function from which the components of the field intensity may be obtained by taking the negative of the partial derivatives with respect to the three coordinates of some reference system. If, then, G is the field intensity, the components of G will be

$$G_x = -\frac{\partial V}{\partial x} \qquad G_y = -\frac{\partial V}{\partial y} \qquad G_z = -\frac{\partial V}{\partial z} \tag{10-27}$$

and V is the potential.

Even by this definition the potential still retains the characteristic of being potential energy for a unit mass. The only significant difference is that in the gravitational case the place at which the potential energy is zero is not given. Since potential energy is a relative quantity anyway, this difference is of no importance.

The method for finding the potential V at any point in a gravitational field due to a mass m is given by the following expression:

$$V = -K \int_m \frac{dm}{r} \tag{10-28}$$

in which r is the distance from the differential mass dm to the point at which the potential is to be expressed. The integration extends over the mass or masses involved.

The validity of this expression may be established by a direct application of Eqs. (10-27) to (10-28) after replacing r by its equivalent expression $(x^2 + y^2 + z^2)^{\frac{1}{2}}$.

10-9. Problems.—1. Find the potential at a point on the axis of a thin ring. Determine also the field intensity from the potential.

If we use the symbols of problem 1 in Sec. **10-4** and the diagram of Fig. 116, we may conveniently check our first result with that given for the field intensity in that problem. Since potential is a scalar quantity, no direction is involved; hence the potential is obtained in the following manner:

$$\begin{aligned} V &= -K \int_m \frac{dm}{r} \\ &= -\frac{KA\, r}{\sqrt{D^2 + r^2}} \int_0^{2\pi} d\gamma \qquad (10\text{-}29) \\ &= -\frac{KA\rho\, 2\pi r}{\sqrt{D^2 + r^2}} \\ &= -\frac{K\, m}{\sqrt{D^2 + r^2}} \qquad (10\text{-}30) \end{aligned}$$

We may now determine the field intensity by taking the negative partial derivative of this expression with respect to D, since the resultant field intensity is in the line of D. Hence

$$G = -\frac{\partial V}{\partial D}$$

$$= -\frac{Km\,D}{(D^2 + r^2)^{\frac{3}{2}}} \qquad (10\text{-}31)$$

This result is identical with that found in problem a of Sec. **10-4.**

2. Find the potential and field intensity at a point on the axis of a thin disk.

We may again use the symbols and the diagram of Sec. **10-4** and Fig. 117. The potential is found by evaluating the integral.

$$V' = -K2\pi t\rho \int_0^a \frac{r\,dr}{\sqrt{D^2 + r^2}}$$

$$= -K\,2\pi t\rho\,[\sqrt{D^2 + a^2} - D]$$

$$= -\frac{K\,2m'}{a^2}\,[\sqrt{D^2 + a^2} - D] \qquad (10\text{-}32)$$

To determine the field intensity, we may differentiate this expression with respect to D and change the sign. This operation gives

$$G = -\frac{\partial V'}{\partial D}$$

$$= -\frac{2\,K\,m'}{a^2}\left[1 - \frac{D}{\sqrt{D^2 + a^2}}\right] \qquad (10\text{-}33)$$

The result agrees with the one giving the field intensity for the thin disk which is given in problem b of Sec. **10-4.**

3. Find the potential and field intensity at any point outside a solid homogeneous sphere.

We may use here the result obtained above for the thin disk and integrate over the entire mass. If we call dx the thickness of the disk and replace the radius of the disk by the expression $\sqrt{R^2 - x^2}$, the potential for the thin disk becomes a differential quantity with x the variable. To obtain the potential for the sphere for the point P (Fig. 118), we must integrate this differential expression over the entire mass. This is expressed as follows if we replace D by $S - x$:

$$V = -K\,2\pi\rho \int_{-R}^{R} [\sqrt{S^2 + R^2 - 2Sx} - (S - x)]dx$$

$$= -\frac{KM}{S} \qquad (10\text{-}34)$$

This simple result shows that the potential at any point outside a sphere is proportional to the mass of the sphere and inversely proportional to the distance from the center of the sphere to the point at which the potential is to be expressed.

The field intensity may be readily found by taking the negative differential of the potential with respect to S, which yields the results given above in problem c of Sec. **10-4.**

4. Another illustration is introduced to emphasize the difference in working with the differential *mass* in potential integrals as compared with handling the differential *forces* in field-intensity integrals. The potential integral is a summation of scalar quantities, while the field-intensity integral represents a summation of vector elements, each of which must be projected into the direction of the resultant field intensity at the point under consideration, if this direction is known.

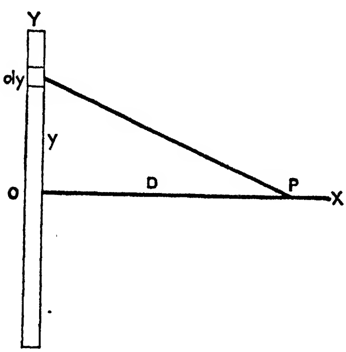

FIG. 120.

Given a thin rod of length $2L$ and linear mass ρ (Fig. 120). To find the potential and field intensity at a point P which is in a line perpendicular to the rod and at a distance D from it. Let the differential mass be $\rho\, dy$ with the coordinate y measured from the center of the rod. We have here a purely scalar summation of elements $K\rho dy/r$. The potential is therefore

$$\begin{aligned} V &= -K\rho \int_{-L}^{L} \frac{dy}{\sqrt{D^2 + y^2}} \\ &= -K\rho \log \frac{\sqrt{D^2 + L^2} + L}{\sqrt{D^2 + L^2} - L} \end{aligned} \qquad (10\text{-}35)$$

The field intensity at P is found by summing vector elements whose magnitudes are represented by $K\rho dy/r^2$, each one having a direction different from the others. It is easily seen that because of the symmetry the resultant field intensity at P is in the line of OX; hence each element of force must be projected into this line by multiplying by the proper cosine factor. The integral is therefore:

$$G = -K\rho \int_{-L}^{L} \frac{D\,dy}{(D^2 + y^2)^{\frac{3}{2}}}$$
$$= -\frac{2L\rho K}{D\sqrt{D^2 + L^2}} \tag{10-36}$$

Problems.—1. Find the potential and field intensity at a point on the axis of a cylinder.

2. Find the potential of a thin uniform rod, of length L, at any point P.

3. Find the gravitational field at any point P near a uniform rod of length L.

4. Find the gravitational potential and field intensity at any point on the axis of a homogeneous right circular cone (the point to be on either side of the cone). The density of the cone is 5 g. per cubic centimeter, the radius of its base is 3 cm., and its altitude is 10 cm.

5. Two homogeneous spheres, whose masses are 100 and 200 g., respectively, are situated with their centers of mass 30 cm. apart. What is the locus of a point so situated in the two gravitational fields that the resultant field intensity is zero?

6. A straight hole is drilled through a large homogeneous sphere of mass M and passes through the center of the sphere. A particle of mass m is dropped from rest at the surface of the sphere into the hole. Under the mutual gravitational attraction between the two masses, the particle will oscillate in the hole. If friction is to be neglected, what will be the motion of the particle? Find the general expression for the acceleration of the particle. How long will it take the particle to pass once through the larger mass?

7. Three homogeneous spheres (masses 100, 200, and 300 g.) are placed with their centers of mass, respectively, coinciding with the vertices of an equilateral triangle, each side of which is 10 cm. long. Find the point of zero gravitational-field intensity.

8. Two homogeneous spheres (masses 1,000 and 2,000 g., respectively) are placed with centers 1 m. apart and are initially at rest. Assuming that the gravitational constant (K) in the metric system is unity and that the radii of the spheres are 3 and 5 cm., respectively, find the position of the collision and the velocities just before impact. How long will it take the spheres to come together. What is the potential energy of the system in its initial position? Does all of this potential energy reappear as kinetic energy at the instant just before impact?

9. Find the attraction between a small mass m at any point on the axis of a thin uniform rod of length L and linear density ρ and situated at a distance s from the nearer end of the rod. Find also the attraction for the case in which the rod extends to infinity in the direction away from m.

Ans. $Km \dfrac{L\rho}{S(S + L)}$, $Km\dfrac{\rho}{s}$

10. Find the attraction between two thin rods, lengths L and L', linear densities ρ and ρ', lying in the same straight line with their nearer ends separated by the distance s.

Ans. $K\rho\,\rho' \log \dfrac{(L + S)(L' + S)}{(L + L' + S)S}$.

11. Find the attraction between a particle of unit mass and a thin circular disk of radius R, mass M, and surface density σ when the particle is situated on the axis of the disk and at a distance s from the center of the disk.

Ans. $\frac{2\,KM}{R^2}\left(1 - \frac{S}{\sqrt{S^2 + R^2}}\right).$

What is the value of the attraction when the radius of the disk is infinite?

12. Plot the field intensity for any point on a line passing through the center of a homogeneous sphere (mass M, radius R).

13. Find the field intensity at any point inside a spherical cavity which is within a homogeneous sphere. Assume that the density of the sphere is ρ and the distance between the center of the sphere and center of the cavity is s.

Ans. $\frac{4}{3}\pi K\rho s.$

CHAPTER XI

CENTRAL FORCES

11-1. Central Forces.—In a consideration of the motions of particles under the influence of central forces we are concerned with a certain type of motion which finds an application in astronomical fields and also in atomic physics. In these two fields we find two extreme cases as far as mass is involved. The motion of a planet about the sun and that of an electron about the nucleus of a hydrogen atom may be described equally well by the same set of equations.

The system under consideration consists of only two bodies, one of which has a large mass in comparison with the other. If the difference in the masses of the two bodies is very large, then the larger body may be considered to be stationary. In such cases we are concerned with the motion of the smaller body. If the difference in mass is not large, the motions of both bodies are to be studied and, in this case, the motions are usually referred to the center of mass of the system.

A *central* force is a force the direction of which always passes through a point fixed in a selected reference system. The fixed point is called the center of the motion and is usually the center of mass of the larger body. The force may be a force of attraction, as in the gravitational cases, or either attraction or repulsion, as found in the deflection of an alpha particle or an electron by the nucleus of a heavy atom.

The central force is always expressible in terms of the distance between the two particles. It may be directly proportional to the first power of the distance between the particles as in those cases where the force is a harmonic force. In gravitational and electrostatic cases the central force is inversely proportional to the square of the distance between the particles. There are still other cases in which the force is a function of the time or some other variable as well as being a function of the distance between the particles. For our purposes we shall be concerned only with those motions in which the central force depends upon the distance between the particles.

The use of particles in this consideration is made in order to exclude rotational motions of the bodies about axes through their centers. Rotational motions of this sort would introduce complications in the expressions, which are undesirable for an introductory study. It should be pointed out that a particle need not necessarily be a small body. Any homogeneous spherical-shaped body, however large, may be considered as a particle (Sec. **10-4c**). In such a case the mass of the particle is considered to be concentrated at the center of mass of the particle.

It is to be noticed that, in some of the considerations which follow, two systems of coordinates are used. This plan is adopted for purposes of illustrating the relative merits of the two systems.

11-2. The General Equations of Motion.—Given two particles of mass M and m, with r the distance between their centers of mass. Let us select a rectangular reference system with origin at the center of M. We are to consider the force, whatever its origin or nature, to act upon m. The force is always a mutual force, acting upon both particles. Let F be the force which acts upon m. The force F may be either attractive or repulsive. If attractive, the sign of F will be negative; if repulsive, it will be positive. Let the coordinates, which define the position of m, be x, y, and z.

The components of F along the reference axes are found by multiplying F by the cosines of the angles between r and OX, OY, and OZ, respectively. Hence

$$F_x = F\frac{x}{r}, \qquad F_y = F\frac{y}{r}, \qquad \text{and} \qquad F_z = F\frac{z}{r}$$

since F is directed along r, the radius vector giving the position of m.

The equations of motion parallel to the reference axes are therefore

$$\frac{Fx}{r} = m\frac{d^2x}{dt^2}, \qquad \frac{Fy}{r} = m\frac{d^2y}{dt^2}, \qquad \text{and} \qquad \frac{Fz}{r} = m\frac{d^2z}{dt^2} \qquad (11\text{-}1)$$

It will be shown below that motion due to central forces is limited to a single plane. If we select the XY plane of the reference system to be the particular plane in which the motion is confined, then the Z component of F would be permanently equal to zero and the third of the foregoing equations may therefore be disregarded.

If we select a plane polar-coordinate system with origin at the center of M, then the two components of F which are to be used for expressing the motion will be parallel and perpendicular to the radius vector r, respectively. The equations of motion in this system are therefore

$$F = m\left(\frac{d^2r}{dt^2} - r\omega^2\right) \qquad 0 = m\left(2\omega V_r + r\frac{d\omega}{dt}\right) \tag{11-2}$$

in which r and γ are the coordinates of m, $\omega = d\gamma/dt$, and $V_r = dr/dt$.

11-3. Constancy of Rotational Momentum.—The rotational momentum of m, referred to an axis through the center of the motion (*i.e.*, through the center of M) and perpendicular to the plane of the motion, is measured by the product of the moment of inertia of m with respect to the selected axis, and the angular velocity of m about the axis, which is the angular velocity of r. In polar coordinates the moment of inertia of m is mr^2 and the angular velocity is ω. Attention should be directed to the fact that in the general case, which we are considering, the resultant velocity of m is not in the line of r. If the resultant velocity of m were in the line of r, then ω would be zero. In the general case, ω is not zero nor is it constant.

To find an expression for the rotational momentum, we may use the second of Eqs. (11-2). The equation may be simplified by writing it in the following manner:

$$m\left(2\omega V_r + r\frac{d\omega}{dt}\right) = \frac{m}{r}\frac{d}{dt}(r^2\omega) = 0 \tag{11-3}$$

Hence

$$m\frac{d}{dt}(r^2\omega) = 0$$

Integration of this expression gives

$$m\, r^2\, \omega = D \text{ (a constant)} \tag{11-4}$$

The resulting equation expresses the fact that the rotational momentum is constant.

The equations of motion expressed in rectangular coordinates may be used to obtain the same result, although the procedure is not so simple. Eliminating the quantity F/r from the three equations of Eqs. (11-1) gives

$$m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right) = 0 \qquad m\left(z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2}\right) = 0$$

$$m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) = 0 \tag{11-5}$$

These equations may be integrated with the following results:

$$m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) = A \qquad m\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right) = B$$

$$m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = C \tag{11-6}$$

where A, B, and C are the constants of integration.

An inspection of these three expressions will show that each represents a component of the rotational momentum. If we use the expression moment of momentum in place of rotational momentum, the interpretation may become obvious. The first equation expresses the component of the rotational momentum parallel to the X-axis, the second expresses the component parallel to the Y-axis, the third gives the component parallel to the Z-axis.

11-4. Motion Confined to a Plane.—In order to prove that the motion of the particle m is confined to one plane, use may be made of the general equations [Eqs. (11-6)] which express the components of the rotational momentum. All that is necessary to do is to eliminate the components of the velocity from these equations and obtain thereby an equation which contains the coordinates x, y, and z, of m. The resulting expression will indicate the character of the surface which must always contain m.

If we multiply the first of Eqs. (11-6) by x, the second by y, and the third by z, and then add the three equations so obtained, the result is

$$Ax + By + Cz = 0 \tag{11-7}$$

This equation is that of a plane which passes through the origin. The particular plane in any given case may be determined by the value of the constants of integration (A, B, and C). Since A, B, and C are constants of integration, they, in turn, depend upon the selected initial conditions, which must include the coordinates of m when at the initial position as well as the initial velocity of m. We may therefore conclude that the plane of motion is determined by the initial velocity and the central force in the initial position. In other words, the plane of motion is determined by the "center" of the motion and the initial velocity.

The planarity of motion produced by central forces may be proved in a simpler manner. Since the acceleration of the satellite m is in the line of the force, *i.e.*, in the line of r, there can be no change of the velocity along a line which is perpendicular to the plane determined by the initial velocity and r. The motion must therefore be confined to a single plane.

11-5. Conservation of Energy.—Since no resistance is included in the fundamental equations of motion, the forces which are responsible for the motion are conservative forces. In view of this fact we should expect the initial stock of energy to remain in the system as long as no external forces are introduced. Under the conditions of the present consideration the total energy must be made up partly of kinetic energy of translation and partly of potential energy of position.

In order to obtain an expression for the energy of m, the general equations [Eqs. (11-1)] may serve as a starting point. If we multiply the three equations by dx/dt, dy/dt, and dz/dt, respectively, in the order written, and add the resulting expressions, we obtain

$$m\left(\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{d^2z}{dt^2}\right) - \frac{F}{r}\left(x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}\right) = 0 \tag{11-8}$$

The coefficient of m in this equation is an exact differential, because

$$\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{d^2z}{dt^2} = \frac{1}{2}\frac{d}{dt}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]$$

Since

$$r^2 = x^2 + y^2 + z^2$$

and

$$r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}$$

we may therefore write

$$\frac{1}{2}m\frac{d}{dt}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] - F\frac{dr}{dt} = 0 \tag{11-9}$$

Now if F is a function of r only, the quantity $F\,dr$ is also an exact differential. When F is a function of r only, there exists a potential function P from which the field intensity G may be obtained by taking the first partial derivative of P with respect to r, as expressed in the following symbols [see Eq. (10-27)]:

$$G = -\frac{\partial P}{\partial r}$$

Multiplying the field intensity by the mass of the particle m gives the force F when one is dealing with gravitational fields. Hence

$$F = -m\frac{\partial P}{\partial r} \tag{11-10}$$

Writing this value for F in Eq. (11-9), integrating, and writing E for the constant of integration gives

$$\frac{1}{2}m\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] + mP = E \tag{11-11}$$

It will be readily seen that the kinetic energy of the particle m is given by the first part of this expression. The quantity mP expresses the potential energy of m, since the potential P is the potential energy of a unit mass in the position of m. Hence Eq. (11-11) indicates that the sum of the kinetic and potential energies is a constant. The student must bear in mind that, in order for this generalization to be true, it is necessary for F to be expressible as a function of r alone. If the magnitude of F depended upon some other variable, the total energy of the particle would not, in general, be constant.

We may also establish the validity of the conservation of energy in this restricted case by the use of the expressions for the force equations [Eqs. (11-2)] in polar coordinates. If we multiply the first of Eqs. (11-2) by dr/dt and the second by ωr and then add the resulting expressions, we obtain

$$m\left(\frac{dr}{dt}\frac{d^2r}{dt^2} - r\frac{dr}{dt}\omega^2 + 2r\frac{dr}{dt}\omega^2 + r^2\omega\frac{d\omega}{dt}\right) - F\frac{dr}{dt} = 0 \tag{11-12}$$

The first part of this equation is an exact differential and may be written as follows:

$$\frac{1}{2}m\frac{d}{dt}\left[\left(\frac{dr}{dt}\right)^2 + r^2\omega^2\right]$$

Substituting this expression for its equivalent in Eq. (11-12), replacing F (dr/dt) by its value as given by Eq. (11-10), and then integrating gives the following equation for the total energy of m:

$$\frac{1}{2}m\left[\left(\frac{dr}{dt}\right)^2 + r^2\,\omega^2\right] + m\,P = E \qquad (11\text{-}13)$$

The first part of this equation gives the kinetic energy of m and the second part expresses the potential energy.

11-6. The General Equation of the Orbit.—The general equation of the orbit of the particle is not easily obtained when rectangular coordinates are used but is readily expressed in polar coordinates. To find the path in polar coordinates, we may use Eq. (11-4) and the first of Eqs. (11-2). The procedure is one which has for its purpose the elimination of the time and the velocity component. The angular velocity ω and d^2r/dt^2 must be expressed in terms of r and γ or the derivative of r with respect to γ.

If we put

$$R = \frac{D}{m} \qquad \text{and} \qquad s = \frac{1}{r}$$

then from Eq. (11-4) we find that

$$\omega = Rs^2$$

If we differentiate the expression $r = 1/s$ with respect to the coordinate γ, we obtain

$$\frac{dr}{d\gamma} = -\frac{1}{s^2}\frac{ds}{d\gamma}$$

To express dr/dt in terms of $ds/d\gamma$, we may take the following steps:

$$\frac{dr}{dt} = \frac{dr}{d\gamma}\frac{d\gamma}{dt} = \frac{dr}{d\gamma}\,\omega = -\frac{\omega}{s^2}\frac{ds}{d\gamma} = -R\frac{ds}{d\gamma}$$

The second derivative of r with respect to the time is therefore

$$\frac{d^2r}{dt^2} = -R\frac{d^2s}{d\gamma^2}\,\omega = -R^2s^2\frac{d^2s}{d\gamma^2}$$

Substituting the expressions for d^2r/dt^2 and ω in the first equation of Eqs. (11-2) gives

$$\frac{F}{m} = -R^2s^2\left(\frac{d^2s}{d\gamma^2} + s\right) \qquad (11\text{-}14)$$

The right-hand member may be integrated directly if both sides of the equation are multiplied by the expression $(2/s^2)(ds/d\gamma)$.

The left-hand member may also be integrated if F is a function of r only. The final expression may be written

$$\frac{2}{m}\int F\,dr = \frac{R^2}{r^4}\left[\left(\frac{dr}{d\gamma}\right)^2 + r^2\right] + L \qquad (11\text{-}15)$$

where L is a constant of integration, the value of which depends upon initial conditions. The equation as written is valid for any central force. In case F is expressible as some function of r only so that the $\int F\,dr$ could be evaluated, the resulting expression would then give a relation between r and $dr/d\gamma$. To obtain the equation of the orbit, still another integration would be necessary. The final equation should express r as some function of γ.

Problems.—1. Making use of Eq. (11-14), obtain an expression for the force F when it is known that the particle m moves along an elliptical path. The equation of the ellipse in polar coordinates with origin at the center is

$$r^2 = \frac{b^2}{(1 - e^2 \cos \gamma)}$$

in which b is the minor axis and e the eccentricity.

2. Using rectangular coordinates, prove that the particle m moves in an elliptical path when the force $F = -k^2mr$.

3. Using polar coordinates, prove that the particle m moves in an elliptical path when $F = -k^2mr$.

11-7. Planetary Motion.—One of the most important cases of central motion is that of planetary motion. We have already shown that the force in gravitational attraction between homogeneous spherical bodies is inversely proportional to the square of the distance between the centers of the masses. It is true that the densities of the sun and planets are not uniform throughout each body, but they are probably proportional to some power of the radius in each case and hence the inverse square law is applicable. We may therefore write the value of F, the attracting force, as follows:

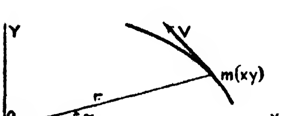

Fig. 121.

$$F = -\frac{mk}{r^2}$$

where k, in this case, includes the mass of the attracting or central body.

In planetary motion we are particularly interested in the nature of the path of the satellite. Starting with the components of the acceleration taken along the X- and Y-axes of a rectangular reference system (Fig. 121), we have the expressions

$$\frac{d^2x}{dt^2} = -\frac{k}{r^2}\cos\gamma, \qquad \frac{d^2y}{dt^2} = -\frac{k}{r^2}\sin\gamma \tag{11-16}$$

where γ is the angle between r, the radius vector drawn to the particle m, and the X-axis.

Using the value of r^2 as may be found from Eq. (11-4), which expresses the constancy of the rotational momentum, we may eliminate r^2 from each of the above equations. Hence if we put $H = km/D$, then

$$\frac{d^2x}{dt^2} = -H\,\omega\cos\gamma, \qquad \frac{d^2y}{dt^2} = -H\,\omega\sin\gamma$$

Since $\omega = d\gamma/dt$, the integration of each of these equations may be effected. If, at the position $\gamma = 0$, the components of the initial velocity U are U_x and U_y, the results of the integration are

$$\frac{dx}{dt} - U_x = -H\sin\gamma, \qquad \frac{dy}{dt} - U_y = H(\cos\gamma - 1) \tag{11-17}$$

In order to eliminate the time from this equation, we may make use of the value of the rotational momentum as expressed in rectangular components. The third equation of Eqs. (11-6) suggests the procedure. Since the motion, in the present case, is confined to the XY plane, the constant C of Eqs. (11-6) becomes equal to the total rotational momentum which we have designated by the symbol D. If we multiply the first equation of Eqs. (11-17) by y and the second by x, subtract the first from the second, and then introduce D/m for its equivalent expression, we obtain

$$x(H - U_y) + U_x y + \frac{D}{m} = H(x\cos\gamma + y\sin\gamma)$$
$$= H\sqrt{x^2 + y^2}$$

This equation may be transformed into one which more readily indicates that it represents a conic section. If we divide both sides of the equation by the constant $\sqrt{(H - U_y)^2 + U_x^2}$, we obtain

$$\frac{x(H - U_y) + U_x y + \frac{D}{m}}{\sqrt{(H - U_y)^2 + U_x^2}} = \frac{H\sqrt{x^2 + y^2}}{\sqrt{(H - U_y)^2 + U_x^2}} \tag{11-18}$$

The left-hand member of this equation represents a straight line. It also represents the perpendicular distance of a point whose coordinates are x and y, *i.e.*, m, from the same straight line. The quantity $\sqrt{x^2 + y^2}$ expresses the distance of m from the origin. Therefore Eq. (11-18) represents a conic section because it expresses the fact that the given point m must move so that its distance from a given straight line is always proportional to the distance of m from a fixed point, a focus. The straight line is the directrix of the conic section.

Whether the path of m will be an ellipse, a parabola, or a hyperbola depends upon the relative values of the constants. Mathematically, this condition may be expressed in terms of the absolute value of the ratio of the distance of m from the origin to the distance of m from the directrix. Physically, the character of the path depends upon the magnitude of the initial velocity.

The ratio of the distances referred to above may be expressed as follows:

$$\frac{1}{H^2}[(H - U_y)^2 + U_x^2]$$

The conic will be an ellipse, a parabola, or a hyperbola according as the magnitude of this ratio is less than unity, equal to unity, or greater than unity, respectively.

From the expression for the foregoing ratio of distances we may derive a criterion for the physical condition which determines the character of the orbit. Using the mathematical requirement, we may write

$$(H - U_y)^2 + U_x^2 \lesseqgtr H^2$$

Hence

$$U_x^2 + U_y^2 \lesseqgtr 2HU_y \tag{11-19}$$

In order to simplify this expression, we have the following relations which are valid at the initial position:

$$H = \frac{K}{r_0^2\,\omega}, \qquad r_0\,\omega = U_y$$

where r_0 is the particular value of r, the radius vector drawn to m, in the initial position. If we now multiply both sides of Eq. (11-19) by $\frac{1}{2}m$ and introduce the given value of H, we obtain the final expression for the desired physical criterion.

$$\tfrac{1}{2}mU^2 \lesseqgtr \frac{Km}{r_0}$$

The left-hand member expresses the original kinetic energy of the particle. The right-hand member gives a value for the potential energy of the particle in the initial position. The path of the particle is therefore an ellipse, a parabola, or a hyperbola according as the kinetic energy of the particle is less than, equal to, or greater than the potential energy of the particle in the initial position.

11-8. Historical Development of Central Forces.—Every student of mechanics should be familiar with the historical development of this subject, particularly with the contributions of Copernicus, Brahe, Kepler, and Newton. A very interesting and lucid account of the efforts of these men to explain planetary motions is given in Buckley's "A Short History of Physics." A brief summary of the contributions of these pioneers is all that may be needed for present considerations.

The first to suggest the heliocentric theory was probably Aristarchus nearly two thousand years before Kepler's time. The idea remained dormant until the first part of the sixteenth century, at which time Copernicus in his "De Revolutionibus Orbium Coelestium" (published in 1543) made the hypothesis that the apparent daily motion of the sun and stars from east to west was in reality due to a rotation of the earth about an axis of its own and that the earth and the other planets revolved about the sun. While this theory made possible a simplification of the description of planetary motion and received some support, objections were raised. Lack of sufficiently accurate astronomical apparatus needed to make conclusive observations was largely responsible for the nonacceptance of the heliocentric theory. Other factors, such as the attitude of the Church and reverence for the "authorities," made men hesitate to accept the new ideas.

Tycho Brahe, though hostile to the Copernican theory, was sufficiently open minded to realize that more data were needed to settle the question. For many years Brahe made careful measurements on the positions of the planets and stars. Considering the quality of the instruments at his disposal, Brahe made remarkably accurate observations.

Kepler, a student of Brahe's, favored the Copernican theory and, using the results of Brahe's work, was able to arrive at those important conclusions which have since borne his name. The laws may be stated as follows:

1. *The orbits of the planets are ellipses with the sun at one of the foci.*

2. *A straight line drawn from the sun to a planet describes equal areas in equal intervals of time.*

3. *The squares of the periods of any two planets are proportional to the cubes of their mean distances from the sun.*

With Kepler's work as a basis for further progress, Newton was able to arrive at still more fundamental conclusions. Kepler's first law in Newton's hands contributed the fact that the attractive force for each planet varies inversely as the square of the distance from the sun to the planet. From Kepler's second law, Newton discovered that the force which holds a planet in its orbit is a central force with the sun at the center. The universal law of gravitation was the result obtained by a coalescence of all three of Kepler's laws.

11-9. Kepler's Second Law by Vector Methods.—It is of interest to develop two of Kepler's laws by vector methods and to see, particularly for the second law, that this method is an easier one. Attention should be directed to the details of the procedure.

In the particular problem it is not necessary to assume that the central force varies inversely as the square of the distance (r) between the two bodies (treated as particles). This fact is not obvious but may be proved to be true. It is necessary, however, to assume that the force is a central force and is expressible as some function of r. Since the masses of the two bodies are assumed to be constant, the acceleration ($d^2\mathbf{r}/dt^2$) of the planet in a reference system fixed to the sun may be expressed as follows:

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{r}_1 f(r) \qquad (11\text{-}20)$$

in which $\mathbf{r}$ is the vector which gives the position of the planet with origin at the center of the sun, $\mathbf{r}_1$ the unit vector in the positive direction of $\mathbf{r}$, and $f(r)$ is some scalar function of r. Note the omission of the minus sign which would have to be included if it were desired to specify a force of attraction. Multiplying both sides of the equation by $\mathbf{r}\times$ (thus obtaining a vector product on each side) gives

$$\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = \mathbf{r} \times \mathbf{r}_1 f(r) \qquad (11\text{-}21)$$

Since $\mathbf{r}_1$ is parallel to $\mathbf{r}$, the right-hand member is zero; hence

$$\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = 0 \qquad (11\text{-}22)$$

This result may be more readily interpreted if we write it in the following form:

$$\frac{d}{dt}\left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right) = 0 \qquad (11\text{-}23)$$

By carrying out the indicated differentiation, the equivalence of this expression to that in Eq. (11-22) may be shown. Since the time derivative of the vector product in Eq. (11-23) is equal to zero, the vector product must be equal to a *constant vector* which we shall designate by $\mathbf{A}$; hence

$$\mathbf{r} \times \frac{d\mathbf{r}}{dt} = \mathbf{A} \qquad (11\text{-}24)$$

The vector $\mathbf{A}$ is perpendicular to both $\mathbf{r}$ and $d\mathbf{r}/dt$ and therefore perpendicular to the plane of motion of the planet, since $d\mathbf{r}/dt$ is velocity of the planet. The velocity of the planet is always tangent to the orbit. Since the vector product of the two vectors is equal to twice the area of the triangle formed upon the two vectors as sides, and since $\mathbf{A}$ is constant, we may conclude that the area swept out in unit time by the radius vector, drawn from the sun to the planet, is constant. This conclusion is Kepler's second law.

It is of interest to observe that Kepler's second law permits a wider application than in the more restricted planetary motion. In Eq. (11-20) the limitation imposed upon $f(r)$ includes all cases in which the central force is any function of r only. Attractive forces which vary inversely as the square of r represent only one type of force to which Kepler's second law is applicable. We may include any force, attractive or repulsive, which is expressible as a function of r.

Still another conclusion may be drawn from Eq. (11-24). Since the vector $\mathbf{A}$ is constant and is always perpendicular to $d\mathbf{r}/dt$, the velocity of the planet, it follows that the motion of the planet must be confined to a single plane. This conclusion was obtained from analytical considerations above, in Sec. **11-4.**

11-10. Kepler's First Law by Vector Methods.—In order to establish Kepler's first law by vector methods, it will be necessary to utilize those vector expressions which are characteristic of planetary motion, *viz.*, Eqs. (11-20) and (11-24), and from them derive an expression for the orbit of the satellite. We may start with the following identity:

$$\boldsymbol{r}_1 \times \frac{d\boldsymbol{r}_1}{dt} = \frac{1}{r^2}\left(\boldsymbol{r} \times \frac{d\boldsymbol{r}}{dt}\right) \tag{11-25}$$

which is true regardless of the meaning to be assigned to the vector $\boldsymbol{r}$. If we choose to let $\boldsymbol{r}$ represent a radius vector as was done in the preceding section, then we may use Eq. (11-24) to replace the vector product in the right-hand member of Eq. (11-25), which gives:

$$\boldsymbol{r}_1 \times \frac{d\boldsymbol{r}_1}{dt} = \frac{\boldsymbol{A}}{r^2} \tag{11-26}$$

It would be possible to obtain an expression for r from this equation, but the result so obtained would be too general for our present purpose. It will be remembered that Eq. (11-24) was derived under the assumption that the central force could be either attractive or repulsive and could be any function of r. In this problem we must impose a further restriction, *viz.*, that of the inverse square law which in vector form may be written

$$\frac{d^2\boldsymbol{r}}{dt^2} = -\frac{k\,\boldsymbol{r}_1}{r^2} \tag{11-27}$$

A little manipulation of Eq. (11-26) is necessary in order to make possible a substitution of this condition [Eq. (11-27)]. Multiplying both sides of Eq. (11-26) by $k\boldsymbol{r}_1 \times$ gives

$$k\boldsymbol{r}_1 \times \left(\boldsymbol{r}_1 \times \frac{d\boldsymbol{r}_1}{dt}\right) = \frac{k\,\boldsymbol{r}_1 \times \boldsymbol{A}}{r^2} \tag{11-28}$$

On expanding the triple vector product, the left-hand member is found to be equal to $-k\,d\boldsymbol{r}_1/dt$. Making this substitution and at the same time introducing in the right-hand member the acceleration as given by Eq. (11-27) leads to the expression

$$k\frac{d\boldsymbol{r}_1}{dt} = \frac{d^2\boldsymbol{r}}{dt^2} \times \boldsymbol{A} \tag{11-29}$$

Integration of this equation yields the following result:

$$\frac{d\boldsymbol{r}}{dt} \times \boldsymbol{A} = k\,\boldsymbol{r}_1 + \boldsymbol{B} \tag{11-30}$$

in which $\boldsymbol{B}$ is a constant vector.

Multiplying both sides by $A^{-1}(A^{-1} = 1/A)$ and expanding the triple vector product gives

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{A}^{-1} \times (k\, \boldsymbol{r}_1 + \boldsymbol{B}) \tag{11-31}$$

This equation is an important one. It expresses the velocity of the planet in terms of the vector sum of two vectors. A discussion of this relation will be given in Sec. **11-12.**

To find the equation for the orbit, we may first multiply Eq. (11-31) by $\boldsymbol{r}\times$. Remembering Eq. (11-24), we then have

$$\begin{aligned} \boldsymbol{A} &= k\, \boldsymbol{r} \times (\boldsymbol{A}^{-1} \times \boldsymbol{r}_1) + \boldsymbol{r} \times (\boldsymbol{A}^{-1} \times \boldsymbol{B}) \\ &= k\, \boldsymbol{A}^{-1} r + \boldsymbol{A}^{-1}\, \boldsymbol{r} \cdot \boldsymbol{B} \end{aligned} \tag{11-32}$$

Multiplying by $\boldsymbol{A}\,\cdot$ gives

$$A^2 = kr + \boldsymbol{r} \cdot \boldsymbol{B}$$

By solving this scalar equation for r, the desired expression for the orbit is obtained.

$$r = \frac{\dfrac{A^2}{k}}{1 + \left(\dfrac{B}{k}\right) \cos \gamma} \tag{11-33}$$

In this equation, γ is the angle between $\boldsymbol{r}$ and $\boldsymbol{B}$ measured from $\boldsymbol{B}$. This equation, in the polar coordinates r and γ, represents a conic, the origin being at one of the foci, and the reference line is along the major axis, parallel to $\boldsymbol{B}$. The eccentricity of the conic is B/k. If the eccentricity is less than unity, the conic is an ellipse. Further data would be necessary before one could decide in the present problem whether or not the eccentricity is less than unity. The constant k is positive. Since the orbits of the planets are closed paths, we may conclude that the particular conic is an ellipse. The fact that the orbit is an ellipse and that the gravitating body is at the origin substantiates Kepler's first law.

11-11. Kepler's Third Law.—It is required to prove that the square of the period of planetary motion is proportional to the cube of the major axis of the elliptical path. Now it may be shown that A^2/k [Eq. (11-33)] is equal to the semi *latus rectum*. If a and b represent the major and minor axes, respectively, and e is the eccentricity of the ellipse, then we may write

$$A^2 = k\, a\, (1 - e^2) \tag{11-34}$$

With reference to Eq. (11-24), it will be seen that the scalar quantity $2A$ is the time rate at which the radius vector sweeps over area during the motion of the planet. If we put $A = dS/dt$ where S is the area, then

$$2\frac{dS}{dt} = [k\ a(1 - e^2)]^{\frac{1}{2}} \tag{11-35}$$

Integrating this equation and assuming that, at the time $t = 0$, $S = 0$, gives

$$2S = [k\ a\ (1 - e^2)]^{\frac{1}{2}}t \tag{11-36}$$

The period of the motion (P) is the time required for the planet to make a complete cycle of its motion or for the radius vector to describe an area equal to that of the ellipse, *viz.*, $\pi\ ab$. It follows therefore that

$$2\ \pi\ ab = [k\ a\ (1 - e^2)]^{\frac{1}{2}}\ P \tag{11-37}$$

Since

$$b = a\ (1 - e^2)^{\frac{1}{2}}$$

an expression for the period P in terms of the major axis alone may be obtained by eliminating b from Eq. (11-37); hence

$$P^2 = \frac{4\ \pi^2\ a^3}{k}$$

It is interesting to observe the simplicity of the final result.

11-12. The Hodograph for Planetary Motion.—In Sec. **11-10** a vector equation [Eq. (11-31)] was derived which expresses the velocity of the planet in terms of two constant vectors and the unit vector $\mathbf{r}_1$. The equation referred to is rewritten here for the sake of convenience.

$$\frac{d\mathbf{r}}{dt} = \mathbf{A}^{-1} \times \mathbf{B} + k\ \mathbf{A}^{-1} \times \mathbf{r}_1 \tag{11-38}$$

For simplicity in writing we shall designate $d\mathbf{r}/dt$ by $\mathbf{V}$, and the vectors represented by the first and second terms of the right-hand member by the symbols $\mathbf{C}$ and $\mathbf{D}$, respectively. It is instructive to identify each vector of Eq. (11-38) for a definite position of the satellite.

Let the orbit be in the plane of the paper as shown in Fig. 122, with the planet at P and the sun at F. The velocity $\mathbf{V}$ will be tangent to the orbit at P. The vector $\mathbf{A}$ (not shown in the diagram) is perpendicular to the diagram and directed toward the reader and may be localized as passing through F. The vector $\mathbf{B}$

is drawn along the major axis as shown. Since the focus is taken at the origin, r is drawn from F to P.

The first term of the right-hand member of Eq. (11-38) is $\boldsymbol{C}$, the vector product of $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}$. Since the direction of $\boldsymbol{A}^{-1}$ is the same as that of $\boldsymbol{A}$, the vector $\boldsymbol{C}$, which is perpendicular to both $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}$, will lie in the plane of the diagram and will be directed along the *latus rectum* as shown. $\boldsymbol{C}$ is constant in both direction and magnitude.

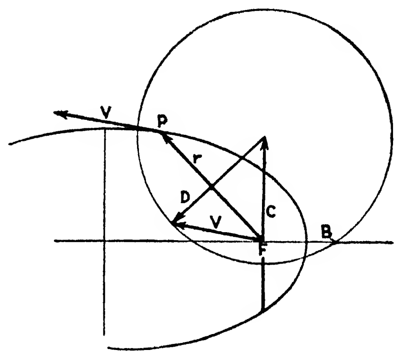

FIG. 122.

The last term of the equation is represented by the vector $\boldsymbol{D}$ in the diagram. It also is in the plane of the orbit because it is perpendicular to $\boldsymbol{A}^{-1}$ and r_1. The magnitude of $\boldsymbol{D}$ is constant, although its direction is changing as r_1 changes. If the vector $\boldsymbol{D}$ is drawn from the terminal point of $\boldsymbol{C}$, a constant vector, then the locus of the terminal point of $\boldsymbol{D}$ is a circle.

With this analysis of the separate vectors in mind, a more general view of Eq. (11-38) shows that the velocity of the planet is the vector sum of the two vectors $\boldsymbol{C}$ and $\boldsymbol{D}$ and may be found for any position of the planet by drawing $\boldsymbol{D}$ from the center of the circle and at the same time perpendicular to r_1 and terminating in the circle. The circle is therefore the hodograph of the planetary motion. It is of interest to study the variation of the velocity as the planet passes along its orbit.

11-13. Deflection of an Alpha Particle by a Stationary Nucleus. One of the problems in modern physics is a determination of how close to the nucleus of a heavy atom an alpha particle may be made to go. Experimentally the procedure is to bombard a piece of thin metal foil with alpha particles and to observe the change in direction of the velocity of the alpha particle as it passes through the metal foil. The source of the alpha particles may be some radioactive matter. The initial speed of the particle is considered to be known. Accounts of these experiments and others of similar nature are intensely interesting. Every student should read some of the original articles or at least summaries of some of them. Mention may be made here of the method which

C. T. R. Wilson devised to make visible the tracks of alpha particles as they pass through a gas. It is indeed very remarkable that we may see, or photograph in perspective, the track of a single particle or of an electron. Some of these photographs reveal collisions of the particle with the nucleus of an atom. Sir William Crookes designed a simple little instrument, the spintharoscope, which makes visible the bombardment of a fluorescent screen by a particle. Another piece of apparatus is available, which is called the alpha-ray track apparatus. This ingenious and fascinating device shows the tracks of alpha particles in a gas according to Wilson's methods.

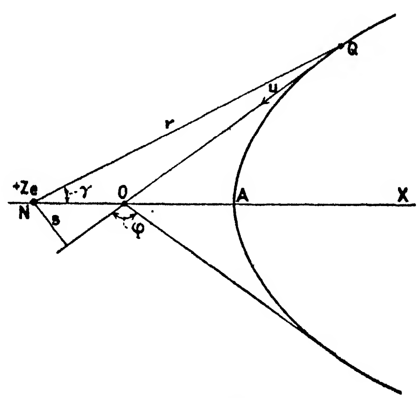

FIG. 123.

It is proposed to develop equations from which the distance of nearest approach of the alpha particle to the nucleus may be calculated from the known factors together with the measured angle of deflection of the alpha particle. The problem may be solved by using some of the equations already developed in the study of central forces.

The mass of the nucleus of the deflecting atom is supposed to be large in comparison with that of the alpha particle. The electric charge on the nucleus is $+Ze$, where Z is the atomic number of the atom and e is the unit of electrostatic charge. The presence of the normal electrons about the nucleus is neglected—an assumption which could not be made unless the alpha particle goes very close to the nucleus. As a first approximation we shall assume that the nucleus is stationary. The charge on the alpha particle is $+2e$. The mass of the alpha particle will be called m.

Let N, the nucleus (Fig. 123), be the origin of a system of polar coordinates r and γ, with NX the reference line. The plane containing the orbit of the alpha particle is the plane of the diagram. Initially, when the alpha particle is at a remote distance, the velocity of the alpha particle is U. The initial position of the alpha particle is at Q. The velocity U is in the line QO.

The force acting upon the particle is a central force, always directed along the line drawn through the alpha particle and the nucleus N. The force F may be expressed as follows:

$$F = \frac{2Ze^2}{r^2} \tag{11-39}$$

The force is repulsive and is therefore positive.

Since the acceleration is always in the line of r, the component of the acceleration in the line of r, *viz.*, J_r, is the resultant acceleration and J_γ, the component of the acceleration perpendicular to r, is always zero. The two differential equations of motion are therefore

$$F = \frac{2Ze^2}{r^2} = m\left(\frac{dV_r}{dt} - r\omega^2\right) \tag{11-40}$$

$$J_\gamma = 2\omega V_r + r\frac{d\omega}{dt} = 0 \tag{11-41}$$

where ω and $d\omega/dt$ are the first and second derivatives, respectively, of γ with respect to the time.

The second of these equations [Eq. (11-41)] has already been integrated [see Eq. (11-4)] and gives an expression for the constancy of the rotational momentum.

The first of these equations has also been integrated [see Eq. (11-13)]. From this integration an expression for the total energy is obtained. In the present case the form of the expression for the potential energy must be changed, because the acting force is an electric force and not gravitational as was assumed in the derivation of Eq. (11-13). If, however, we insert the value of F as given by Eq. (11-39) in Eq. (11-12) and integrate the resulting equation, the following expression for the total energy is obtained:

$$\frac{2Ze^2}{r} + \frac{1}{2}m\left[\left(\frac{dr}{dt}\right)^2 + r^2\,\omega^2\right] = E \tag{11-42}$$

The first term represents the potential energy and the remainder of the left side of the equation expresses the kinetic energy. The total energy remains constant during the motion.

We shall use the energy equation [Eq. (11-42)] for determining the path of the particle, instead of the general equation of the path developed above [Eq. (11-15)]. The time factor may be eliminated from Eqs. (11-42) and (11-4) if we first substitute for dr/dt its equivalent $(d\gamma/dt)(dr/d\gamma)$ and then use the value of $(d\gamma/dt)(\omega)$, as given by Eq. (11-4). Rewriting Eq. (11-42) with the indicated substitution gives

$$\frac{2Ze^2}{r} + \frac{1}{2}m\, r^2 \left(\frac{d\gamma}{dt}\right)^2 \left[\frac{1}{r^2}\left(\frac{dr}{d\gamma}\right)^2 + 1\right] = E$$

Since

$$\frac{d\gamma}{dt} = \frac{D}{mr^2}$$

$$\frac{2Ze^2}{r} + \frac{D^2}{2mr^2}\left[\frac{1}{r^2}\left(\frac{dr}{d\gamma}\right)^2 + 1\right] = E \tag{11-43}$$

Solving this expression for $(dr/d\gamma)^2$ gives

$$\left(\frac{dr}{d\gamma}\right)^2 = \frac{2mE}{D^2}\, r^4 - \frac{4Ze^2m}{D^2}\, r^3 - r^2 \tag{11-44}$$

This equation represents a hyperbola with origin at the center of the acting force. The particular branch of the hyperbola which represents the orbit of the alpha particle is that branch which turns its convex side toward the attracting center. The nucleus is located at a focus of hyperbola. The equation contains two constants of integration, D and E, which may be evaluated by using the initial conditions. When the alpha particle is at a great distance from the nucleus, its potential energy will be zero and hence all of its energy is kinetic; hence

$$E = \tfrac{1}{2}\, m\, U^2 \tag{11-45}$$

Since the rotational momentum may be expressed as the moment of the linear momentum, we may write

$$D = m\, Us$$

where s is the perpendicular distance from the nucleus N to the line of U. By substituting the values of E and D in Eq. (11-44), the path is fully determined.

In the experimental case, however, we do not know the value of s. An expression from which s may be determined from the change of the direction of the velocity of the alpha particle is next to be found. In order to express φ, the angle of deflection of the alpha particle, in terms of s, it will be convenient to use the standard form of expression for the hyperbola. This is usually written as follows:

$$\left(\frac{dr}{d\gamma}\right)^2 = \frac{r^4}{a^2(\epsilon^2 - 1)} - \frac{2\,r^3}{a(\epsilon^2 - 1)} - r^2 \tag{11-46}$$

where a is the distance OA in Fig. 123 and $a\epsilon$ is equal to NO.

The angle φ may be expressed in terms of ϵ as follows:

$$\cot^2 \tfrac{1}{2}\,\varphi = \epsilon^2 - 1$$

By comparing the coefficients of r^4 and r^3 in Eqs. (11-46) and (11-44), it follows that

$$a^2(\epsilon^2 - 1) = \frac{D^2}{2\,m\,E} \quad \text{and} \quad a(\epsilon^2 - 1) = \frac{D^2}{2Ze^2m} \tag{11-47}$$

Hence we find that

$$a = \frac{Ze^2}{E} \qquad \epsilon^2 - 1 = \frac{D^2E}{2Z^2e^4m} \tag{11-48}$$

$$\cot^2 \frac{\varphi}{2} = \frac{D^2E}{2Z^2e^4m} \qquad \cot \frac{\varphi}{2} = \frac{m\,U^2s}{2Ze^2} \tag{11-49}$$

From the last equation the value of s may be found from the experimentally measured value of φ.

The distance of nearest approach of the alpha particle to the nucleus is of interest in the physical case. This distance NA is equal to $a(\epsilon + 1)$. From the foregoing equations it may be shown that

$$NA = \frac{s}{\cot \frac{1}{2}\,\varphi}\left(1 + \sqrt{1 + \cot^2 \tfrac{1}{2}\,\varphi}\right) \tag{11-50}$$

Since s and φ may be found, the distance of nearest approach is readily calculated.

11-14. Deflection of an Alpha Particle by a Movable Nucleus. In the preceding section it was assumed that the nucleus was stationary. We shall now consider the case of a movable nucleus and see to what extent the path of the alpha particle and its distance of nearest approach to the nucleus are affected by introducing this condition. Experimentally, the condition may be realized by bombarding a gas with alpha particles.

In the particular problem under consideration the center of mass of the two bodies will be used as origin for the reference system. The central force acting between the two particles is an internal force as far as the system of two particles is concerned and therefore cannot affect the motion of the center of mass of the two particles. Hence we may consider the center of mass to be stationary.

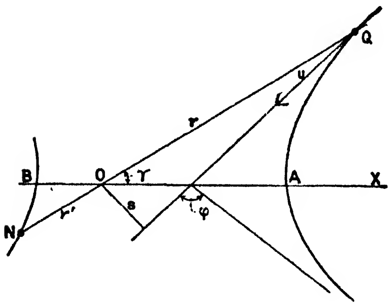

FIG. 124.

Let N and Q (Fig. 124) be the nucleus and alpha particle, respectively. We may take O, the center of mass, as origin of a polar reference system with OX the reference line. Let r, γ be the coordinates of Q and r', $\pi + \gamma$ the coordinates of N.

To find the path of the alpha particle or of the nucleus, we may use the expressions for the rotational momentum and total energy of the system. The procedure is very similar to that employed in the preceding section.

The total rotational momentum is constant (D), since no external force is acting; hence

$$D = m\,r^2\frac{d\gamma}{dt} + M\,r'^2\frac{d\gamma}{dt} \tag{11-51}$$

Since O is the center of mass,

$$mr = Mr'$$

Eliminating r' from these two expressions gives

$$D = m\,r^2\frac{d\gamma}{dt}\left(1 + \frac{m}{M}\right) \tag{11-52}$$

The total kinetic energy (KE) of the system is next to be found. Omitting the details, the final expression may be written as follows:

$$KE = \frac{D^2}{2mr^2\left(1 + \dfrac{m}{M}\right)}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\gamma}\right)^2\right] \tag{11-53}$$

The potential energy (PE) of the system is the potential energy of m with respect to M, which is

$$PE = \frac{2Ze^2}{r\left(1 + \dfrac{m}{M}\right)} \tag{11-54}$$

The total energy (E) of the system is the sum of the kinetic and potential energies. If we add the right-hand members of Eqs. (11-53) and (11-54), put the sum equal to E, and then solve for $(dr/d\gamma)^2$, we obtain

$$\left(\frac{dr}{d\gamma}\right)^2 = \frac{2mE}{D^2}\left(1 + \frac{m}{M}\right) r^4 - \frac{4Ze^2m}{D^2} r^3 - r^2 \qquad (11\text{-}55)$$

This equation is the desired expression for r in terms of γ and therefore the path of m in the selected reference system.

It may be readily seen that Eq. (11-55) is similar to Eq. (11-44) of the preceding section and could be reduced to Eq. (11-44) if we put m/M and r' equal to zero.

Proceeding along the lines of development used above, the angle of deflection φ and the distance of nearest approach (OA) of the alpha particle to O, the center of mass, may be found. The results are

$$\cot \frac{1}{2}\varphi = \frac{mU^2s}{2Ze^2}\left(1 + \frac{m}{M}\right)^{\frac{1}{2}} \qquad (11\text{-}56)$$

$$OA = \frac{s\left(1 + \frac{m}{M}\right)^{-\frac{1}{2}}}{\cot \frac{1}{2}\varphi}\left(1 + \sqrt{1 + \cot^2 \frac{1}{2}\varphi}\right) \qquad (11\text{-}57)$$

It will be noticed that s in the preceding equation is the perpendicular distance from O, the center of mass, to the line of U, the initial velocity. In order to find the distance of nearest approach of the alpha particle to the nucleus, we must add to OA the distance from the nucleus to O. Since O is the center of mass, then, at the particular instant at which the alpha particle is at A, the nucleus will be at B (Fig. 124). But

$$BO = OA\left(\frac{m}{M}\right)$$

Hence the distance from the nucleus to the alpha particle at the instant of nearest approach is

$$BA = OA\left(1 + \frac{m}{M}\right)$$

From this it follows that

$$BA = \frac{s\left(1 + \frac{m}{M}\right)^{\frac{1}{2}}}{\cot \frac{1}{2}\varphi}\left(1 + \sqrt{1 + \cot^2 \frac{1}{2}\varphi}\right) \qquad (11\text{-}58)$$

Some difficulty may be experienced by the student in realizing that the nucleus also moves toward the center of mass during the first half of the encounter in spite of the fact that the force is repulsive. A consideration of the situation, however, reveals the fact that we are dealing with relative motion. We assume that the nucleus is initially at rest in the selected reference system. If we had selected some other reference system which could reveal the motion of the center of mass, then we would have seen that center of mass moves with a constant velocity. The concept of the conservation of momentum is of help in this connection. Initially all of the momentum is possessed by the alpha particle. Since the total momentum is conserved, then, at any later time, if $\boldsymbol{V}$ is the velocity of the center of mass, we have the following relation:

$$(m + M)\,\boldsymbol{V} = m\,\boldsymbol{U}$$

The direction of $\boldsymbol{V}$ must be parallel to that of $\boldsymbol{U}$ and the magnitude of $\boldsymbol{V}$ will remain constant.

As the alpha particle approaches the nucleus, its velocity decreases and at the same time its direction changes. During this period of approach the magnitude of the velocity of the nucleus increases and the direction also changes. The vector sum of the momenta of the two particles remains equal to the constant momentum of the system *i.e.*, $m\,\boldsymbol{U}$.

While the alpha particle is approaching the nucleus, the center of mass also approaches the nucleus. Hence when the motion of the nucleus is expressed in a reference system fixed to the center of mass, the nucleus will approach the center of mass.

While the alpha particle is receding from the nucleus, the speed of the alpha particle is increasing as is also the speed of the nucleus. When the alpha particle has receded to a great distance, its speed approaches a limiting constant value, but the magnitude of this speed is not so large as its initial speed. A part of the initial momentum of the alpha particle has been imparted to the nucleus by the encounter.

Problems.—1. Find the hodograph for planetary motion by analytical methods.

2. Prove Kepler's third law for a circular path from Newton's law and the equation for centripetal force.

3. Assume that a satellite is attracted by a large mass with a force which is proportional to $-k^2r$. Find the orbit of the satellite.

4. If the central force exerted by the larger body is proportional to $+k^2r$, find the orbit.

5. A particle is attracted by two fixed masses with forces which are proportional to the distances of the particle from the attracting centers. Show that the motion of the particle is the same as it would be if there were only one center of force (located at the center of mass of the two attracting bodies) which attracted the particle with a force proportional to the distance of the particle from the mean center.

6. What must be the initial velocity of a particle which is attracted by a fixed mass in order that the orbit may be a circle? Assume that the central force is proportional to $-k^2/r^2$.

7. A satellite is attracted by a body with a force which is proportional to $-k^2r$. Show that the areal velocity (time rate at which the radius vector, drawn from the center to the particle, sweeps over the surface bounded by the orbit) is equal to kab, where a and b are the semi-axes of the orbit.

8. Two equal masses are fixed with their centers separated by a distance L. One attracts a satellite and the other repels it with forces which are proportional to $\mp k^2r$ where r is the distance from either fixed mass to the satellite. Find the orbit.

9. Show that the sum of the kinetic and potential energies of a particle of mass m in any position of its orbit under the influence of the central force of magnitude $-mk^2r$ is $\frac{1}{2}mk^2(a^2 + b^2)$, where a and b are the semi-axes of the orbit.

CHAPTER XII

MOTION OF A PARTICLE IN FLUIDS WITH RESISTANCE

12-1. Resistance of Fluids.—Various attempts have been made to derive a theoretical expression for the resistance offered by a fluid to the motion of a particle as it passes through the fluid. It is not difficult to point out what some of the contributory factors are that affect the magnitude of the resistance, but it is difficult to express these contributions in an accurately quantitative manner. Some of the factors which influence the resistance are the shape, size, and speed of the particle, the viscosity, density, pressure, and temperature of the fluid.

As a particle moves through a fluid, some of the medium must be displaced. From the energy standpoint, energy is used to effect the displacement of the fluid. The necessary energy comes from the kinetic energy of the particle if the fluid resistance is the only external force. If, in addition to the fluid resistance, some other externally applied force is acting, then the energy may be supplied wholly or in part by the work done by the applied force. In either case, as a consequence of the energy acquired by the medium during the passage of the body through it, a force must be exerted by the fluid upon the moving body. Although the resistance definitely depends upon the speed of the particle, no simple, and at the same time accurate, expression for such a relation has been found, except, perhaps, in a few isolated cases.

For any given body and fluid conditions the resistance is probably approximately proportional to some definite power of the speed for a given speed of the body. In many cases it is customary arbitrarily to assume that the friction is proportional to the square of the speed. Experimental data show that the relation between friction and speed is by no means so simple and also that the resistance is not proportional to the square of the speeds except within narrow limits of the speed. The range of speeds over which such a relation is even approximately valid is dependent upon the nature of the body and character of the fluid. Because of the complications involved, theoretical considerations are exceedingly difficult. A considerable amount of experimental work, particularly in the field of ballistics, has been done in order to determine the effect of the speed of rotating projectiles upon the resistance. The following table is given to indicate the nature of the results obtained:

Speed, m./sec.	50	240	295	375	419	550	800	1,000
Resistance	av^2	bv^3	cv^5	dv^3	ev^2	$fv^{1.7}$	$gv^{1.55}$	

The table is written so that the force of resistance is expressed in terms of a constant (a, b, c, etc.) times some power of the speed for the range of speeds indicated above it. For example, for the range of speeds varying from 240 to 295 m. per second the resistance may be expressed with approximate accuracy in terms of bv^3.

12-2. Resistance Proportional to the First Power of the Speed—Pure Translation.—Let us first assume that the resistance of a particle moving through a fluid is proportional to the first power of the speed and determine the three equations which describe the motion. In order to simplify the expressions, we shall assume that, if the particle has an initial velocity, that velocity is parallel to the applied force, in order that pure translational motion may result. In the section immediately following we shall consider the case of motion with fluid resistance in which the initial velocity makes an angle, not equal to zero, with the applied force.

If a particle of constant mass m is subject to a constant force F and if the fluid through which the particle moves offers a constant force of resistance (kV), which is proportional to the first power of the speed V, then the force equation may be written as follows:

$$F - kV = m\frac{dV}{dt} \qquad (12\text{-}1)$$

Let s be the coordinate which measures the displacement of the body from a fixed reference point O. For the initial conditions we may select $s = 0$, $V = U$ at the time $t = 0$.

To find an expression for the velocity in terms of the time, we may first separate the variables in Eq. (12-1) and then integrate as shown:

$$\frac{m\,dV}{kV - F} = -dt$$

$$\frac{m}{k}\log(kV - F) = -t + C$$

where C is the constant of integration. Multiplying through by k/m, putting $kC/m = \log A$, and then writing in the exponential form gives

$$kV - F = A\, e^{-\frac{kt}{m}} \tag{12-2}$$

By utilizing the initial conditions, we find that the constant $A = kU - F$.

Hence

$$V = \frac{F}{k} + \left(U - \frac{F}{k}\right)e^{-\frac{kt}{m}} \tag{12-3}$$

which is one of the desired equations.

To express the displacement s in terms of the time, we may put $V = ds/dt$ in Eq. (12-3); hence

$$\frac{ds}{dt} = \frac{F}{k} + \left(U - \frac{F}{k}\right)e^{-\frac{kt}{m}}$$

Separating the variables and integrating gives

$$s = \frac{Ft}{k} - \frac{m}{k}\left(U - \frac{F}{k}\right)e^{-\frac{kt}{m}} + B$$

Since $s = 0$ at the time $t = 0$, we find that

$$B = \frac{m}{k}\left(U - \frac{F}{k}\right)$$

Hence the second desired equation becomes

$$s = \frac{Ft}{k} - \frac{m}{k}\left(U - \frac{F}{k}\right)\left(e^{-\frac{kt}{m}} - 1\right) \tag{12-4}$$

The third desired relation expresses the velocity in terms of the displacement. To obtain this equation, dV/dt in Eq. (12-1) may be replaced by its alternative form VdV/ds, which gives

$$F - kV = m\,\frac{V\,dV}{ds} \tag{12-5}$$

Separating the variables and integrating gives

$$F - kV - F\log(F - kV) = \frac{k^2 s}{m} + D$$

where D is the constant of integration. Putting $V = U$ at the position where $s = 0$ gives the value of D.

$$D = F - kU - F \log (F - kU)$$

Hence the relation between V and s is

$$k(U - V) + F \log \frac{F - kU}{F - kV} = \frac{k^2 s}{m} \qquad (12\text{-}6)$$

Problems.—1. In what units must k be expressed when using the c.g.s. absolute system?

2. If a 100-g. spherical mass falls from rest in air which is of such a density that $k = 5$, c.g.s. absolute units, find the velocity of the mass at the end of 10 sec. How far would it go in that time? Compare these results with the time and distance obtained under similar conditions but with the friction equal to zero.

3. If a 5-lb. spherical mass is thrown vertically upward in air with a speed of 100 ft. per second and if $k = 0.1$, English absolute units, how long will it take for it to reach its highest point and for it to return to the starting point? What will be the velocity on arriving at the starting point?

12-3. Initial Velocity Not Parallel to the Applied Force.—When the initial velocity is not parallel to the applied force, the moving particle describes a curvilinear path. If the shape of the particle is such that the resistance of the medium ($-kV$) is always parallel to the resultant velocity, then the body will remain in the plane which contains the applied force and the initial velocity. If the applied force is the force due to gravity, the case under consideration is that of the motion of a projectile in a resisting medium.

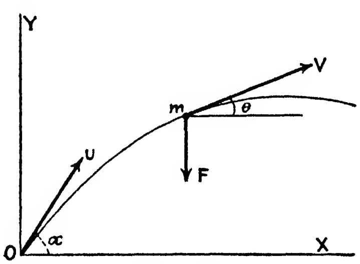

Fig. 125.

Let the moving particle, mass m, have an initial velocity U which makes an angle α with the X-axis (Fig. 125). Also let the constant force F be always parallel to the Y-axis. For any position of the particle, subsequent to its initial position, let V represent the velocity and θ the angle which V makes with OX.

Since the resistance offered by the medium is to be taken proportional to the instantaneous value of the velocity V, we may write $-kV$ for this force. The direction of the resistance is to be always parallel but opposite to V.

The desired equations may be obtained by solving the differential force equations which express the motions parallel and perpendicular to OX. We may therefore write

$$-F - kV_y = m\frac{dV_y}{dt} \text{ (parallel to } OY) \tag{12-7}$$

$$-kV_x = m\frac{dV_x}{dt} \text{ (parallel to } OX) \tag{12-8}$$

in which V_y and V_x are the components of V parallel to OY and OX, respectively.

Since Eq. (12-7) is similar to Eq. (12-1), we may write the solution of Eq. (12-7) immediately, if we replace F by $-F$ in the solution; hence

$$\frac{m}{k}\log(kV_y + F) = -t + C \tag{12-9}$$

If we put

$$C = \frac{m}{k}\log B$$

then, after rearranging, Eq. (12-9) becomes

$$kV_y + F = B\,e^{-\frac{kt}{m}} \tag{12-10}$$

By the use of initial conditions—which may be selected as follows: $t = 0$, and $V_y = U\sin\alpha$—the value of B is found to be

$$B = kU\sin\alpha + F$$

Hence the equation for V_y becomes

$$V_y = -\frac{F}{k} + \left(U\sin\alpha + \frac{F}{k}\right)e^{-\frac{kt}{m}} \tag{12-11}$$

By putting $V_y = dy/dt$, integrating, and evaluating the constant of integration (putting $y = 0$ at $t = 0$), we obtain

$$y = -\frac{Ft}{k} - \frac{m}{k}\left(U\sin\alpha + \frac{F}{k}\right)\left(e^{-\frac{kt}{m}} - 1\right) \tag{12-12}$$

By integrating Eq. (12-8), it may be readily shown that

$$V_x = (U\cos\alpha)\,e^{-\frac{kt}{m}} \tag{12-13}$$

Replacing V_x by dx/dt, integrating, and evaluating the constant of integration by putting $x = 0$ at $t = 0$, we obtain an expression for x in terms of t:

$$x = -\frac{m}{k}(U\cos\alpha)\left(e^{-\frac{kt}{m}} - 1\right) \tag{12-14}$$

The magnitude of the resultant velocity V may now be found for any value of t from Eqs. (12-11) and (12-13) by extracting the square root of the sum of the squares of the right-hand members. The direction of V may be found by putting $\tan \theta = V_y/V_x$.

Equations (12-12) and (12-14) give the position of the body at any time t.

The third desired equation of motion, *viz.*, that which expresses the velocity components in terms of the coordinates may be obtained by replacing the component accelerations dV_x/dt and dV_y/dt by the alternative forms $V_x\, dV_x/dx$ and $V_y\, dV_y/dy$ in Eqs. (12-7) and (12-8), respectively, and integrating the resulting expressions as shown in the preceding section. The details of this process will be left for the student.

Problem.—A spherical mass ($m = 100$ g.) is projected with an initial velocity, of 400 m. per second, which makes an angle of 40 deg. with the horizontal line. The resistance $(-kV)$ varies with the first power of the speed and k is equal to 0.001 g. per second. Find the greatest height to which the projectile ascends, its range (horizontal distance on striking the horizontal line passing through the starting point), and velocity on striking the ground. Take $g = 980$ cm. per second squared.

12-4. Resistance Proportional to the Square of the Speed.—The equations which express the motion of a particle, as it moves through a fluid with resistance proportional to the square of the speed under the influence of a constant force, are to be derived in this section. The particle is to have an initial velocity which is parallel to the applied force, although the initial velocity may be directed oppositely to the applied force. Under this limitation, translational motion will be obtained. The mass of the particle is to remain constant.

If F is the constant force, m the mass of the particle, V the speed of the particle, and k a constant, the force equation may be written as follows:

$$F - kV^2 = m\frac{dV}{dt} \tag{12-15}$$

Separating the variables and integrating gives

$$\frac{m}{2\sqrt{Fk}} \log \frac{\sqrt{Fk} + kV}{\sqrt{Fk} - kV} = t - C \tag{12-16}$$

For the sake of brevity we may put

$$b = \frac{2\sqrt{Fk}}{m}, \qquad h = \frac{\sqrt{Fk}}{k}, \qquad \text{and} \qquad \log A = \frac{2\sqrt{Fk}}{m}C$$

Substituting these values in Eq. (12-16), inverting, and writing in exponential form gives

$$\frac{h - V}{h + V} = A\,e^{-bt} \qquad (12\text{-}17)$$

The constant A may be evaluated by the use of the initial conditions $t = 0$, $V = U$, which gives

$$A = \frac{h - U}{h + U}$$

The desired equation, which expresses the velocity V in terms of the time, is therefore Eq. (12-17). We shall leave it as written in order to simplify the following expressions.

In order to find the second equation of motion, *viz.*, an expression for the distance s in terms of the time, the velocity may be written in its differential form ds/dt. If we solve Eq. (12-17) for V and write ds/dt for V, we obtain

$$\frac{ds}{dt} = \frac{h\,(1 - Ae^{-bt})}{1 + Ae^{-bt}} \qquad (12\text{-}18)$$

Multiplying both sides of this equation by dt and integrating gives

$$s = \frac{h}{b} \log \frac{(1 + Ae^{-bt})^2}{e^{-bt}} + D \qquad (12\text{-}19)$$

The constant of integration (D) may be determined by using the initial conditions $s = 0$ at $t = 0$ and is found to be

$$D = -\frac{h}{b} \log\,(1 + A)^2$$

The third equation of motion may be obtained, in the usual manner, by putting $dV/dt = V\,dV/ds$ in Eq. (12-15). Making this substitution, separating the variables, and integrating gives

$$s = -\frac{m}{2k} \log \left(V^2 - \frac{F}{k}\right) + E$$

where E is the integration constant. Since $s = 0$ when $V = U$, we find that

$$E = \frac{m}{2k} \log \left(U^2 - \frac{F}{k}\right)$$

Hence

$$s = \frac{m}{2k} \log \frac{kU^2 - F}{kV^2 - F} \qquad (12\text{-}20)$$

12-5. Terminal Velocity in Motions with Resistance.—In all cases of the motion of a particle, where a resistance is present and is proportional to some power of the velocity of the particle, there is *terminal velocity*, *i.e.*, a velocity of such a magnitude that the force of resistance becomes numerically equal to the applied constant force. The magnitude of the terminal velocity is dependent upon the resistance factor (k in the preceding sections) and the magnitude of the applied force. When the force of resistance becomes equal to the applied force, the resultant force upon the particle is equal to zero. When this condition is obtained, the acceleration of the particle is equal to zero. Hence, to obtain the terminal velocity, we may put the acceleration, in the force equations, equal to zero and solve for the value of the velocity.

For this purpose we may put the acceleration equal to zero in Eq. (12-5) and solve for the velocity, which gives

$$V = \frac{F}{k}$$

for those cases where the resistance is proportional to the first power of the speed. In a similar manner the terminal velocity in motions in which the resistance is proportional to the square of the speed is

$$V = \sqrt{\frac{F}{k}} \tag{12-21}$$

With these results before us it is of interest to examine the three equations of motion developed in Sec. **12-2.**

The first of these equations [Eq. (12-3)] contains t in the quantity $e^{-kt/m}$. As t increases, the exponential quantity decreases and becomes zero when $t = \infty$. At this value of the time, the velocity attains the terminal velocity F/k, although practically the terminal is attained in much shorter time.

The second equation [Eq. (12-4)] contains t in two terms of the right-hand member. The first of these terms shows that a part of the displacement s increases uniformly with the time, while the second term contains t in an exponential quantity which changes more slowly with larger values of the time. The physical meaning of this relation is that, after the velocity becomes nearly equal to the terminal velocity, the displacement is nearly proportional to the time, as it should be in motions with no acceleration.

The third equation [Eq. (12-6)] expresses the relation between V and s. As the velocity approaches the terminal velocity (F/k), the denominator of the fraction in the last term of the left-hand member approaches zero and hence the logarithm of the fraction approaches infinity. The distance increases indefinitely as the velocity approaches its limiting value.

In a similar manner, interpretations may be obtained for Eqs. (12-17), (12-19), and (12-20) which describe the motion in which the resistance is proportional to the square of the speed. This matter will be left for the student.

12-6. An Experimental Illustration.—In this section we shall present some accurate experimental data giving the time-distance relations of three balls, of different masses but of nearly equal sizes, falling from rest in air. It is proposed to apply the equations developed above to show the adequacy or inadequacy of their use in the description of the actual motion.

The apparatus was set up in a stair well so that vertical distances up to about 40 ft. were available. There were no detectable air currents. A ping-pong ball, a wooden ball, and a golf ball were used. The times required to fall from rest over measured distances were determined by a special electrical clock which was started by the closing of an electric circuit and stopped by the opening of that circuit. The circuit was closed at the instant of releasing the ball and opened when the ball struck the bottom of the desired distance. The error in the measured time intervals was ∓ 0.01 sec. The following data were recorded:

	Ping-pong ball	Wooden ball	Golf ball	Distance, cm.
Mass, g	2.902	18.15	45.15	
Diameter, cm	3.76	3.90	4.08	
Time, sec	0.43	0.41	0.395	66.6
	0.60	0.57	0.57	126.0
	0.745	0.72	0.70	198.0
	1.01	0.95	0.94	354.0
	1.20	1.085	1.075	469.0
	1.51	1.34	1.32	695.0
	1.65	1.44	1.42	805.0
	1.98	1.66	1.64	1,065.0
	2.105	1.74	1.72	1,163.0

Regarding the three sets of records from a general point of view, one readily notices that there is very little difference between the records for the wooden ball and the golf ball, in spite of the fact that the golf ball is more than twice as heavy as the wooden ball. It is also evident that for the ping-pong ball the velocity does not increase so rapidly as in the other cases. Hence the resultant force must decrease more rapidly. In other words the resisting force, in this case, approaches the weight of the ball more closely. Whether the resultant force is reduced to zero at, or near, the end of the greatest height is not apparent. A more detailed study is needed to decide this question.

The student should plot the time-distance records for the ping-pong ball and golf ball. There is hardly sufficient difference between the records of the wooden and golf balls to make an inclusion of the graph of the former of any value. The two plots should be made in a single reference system. It is instructive to include a third graph, *i.e.*, that of a body falling in a vacuum. A second set of graphs should also be made. This set should express the variation of speed with time for all three cases, ping-pong ball, golf ball, and that of a freely falling body. An interpretation of the results should be made.

One observation which was made from the mathematical development of the equations of motion was the existence of the terminal velocity. It is interesting to see whether or not this appears in the limited range of distances used in the experimental cases. That a terminal velocity was reached by the ping-pong ball is obvious from the graphs. This means that the resultant force on the ping-pong ball became zero (or very nearly zero) at the maximum speed. Using the terminal speed, we may easily determine k of the equations. Data are thereby provided for checking the accuracy with which the equations describe the motion. Such tests may be made for the cases in which the resistance was assumed to be proportional to the first and also the second power of the speed. There are various ways in which a test may be carried out. One simple way is to compare the calculated distances, through which the ball fell in several time intervals, with the actual measured distances. The student is asked to carry out the details of the process.

Another way of checking the equations is to calculate k by using several pairs of values of the speed (or distance) and the corresponding times. The criterion, by this method, is whether or not k is constant for any single record. This method has one advantage, in that it is applicable to those cases in which the terminal velocity is not known.

The results of such tests show, for example, that the values of s obtained from Eq. (12-4), when using data for the ping-pong ball may differ from the measured values by nearly 10 per cent. The same data, applied to Eq. (12-19), show even larger differences. While the representation of the motion of the particular experimental case by the equations developed in the preceding sections is not all that could be desired, it is far better than a complete omission of the resistance. In the next section we shall present another method of representing an experimentally determined distance-time curve by which the accuracy may be materially increased.

12-7. Representation by a Polynomial.—Let us consider a continuous single-valued relation such as the distance-time curve given for the ping-pong ball in the preceding section. Now it can be shown that such a relation may be expressed by a polynomial of the following form:

$$s = a + bt + ct^2 + dt^3 + et^4 + \cdots \qquad (12\text{-}22)$$

in which the letters a, b, c, d, etc., are constants. The number of terms is to be finite. In general, the degree of accuracy with which this equation represents the particular relation to which it is applied will depend upon the number of terms used. By increasing the number of terms the accuracy of representation is increased.

We shall not present here a rigid mathematical proof for the statements just made. A consideration of the case under discussion will, however, show that such an equation might be expected to serve our purposes. If the acceleration were constant we know that the first three terms of the right-hand member would suffice to give an accurate expression of the relation. Suppose, however, that the acceleration were changing and that its time rate were constant. In this case, four terms would suffice. If the second time rate of acceleration were constant, five terms would be necessary and so on. The number of terms to be used depends upon which derivative, if it exists, is constant. If d^ns/dt^n is constant, or is small enough to neglect, then $n + 1$ terms will be needed.

It is interesting to apply the equation, using five terms, to the case of the ping-pong ball. To do this we shall need to evaluate the constants by selecting any five pairs of values of s and t. These have been selected from the graph (not shown here) as follows:

$$s = 0, \quad 90, \quad 346, \quad 690, \quad 1{,}085$$
$$t = 0, \quad 0.5, \quad 1.0, \quad 1.5, \quad 2.0$$

Using these values, we may write five simultaneous equations, from which we find

$$a = 0 \qquad b = -59.1 \qquad c = 563.2 \qquad d = -185 \qquad e = 26.9$$

Inserting these values in Eq. (12-22), we obtain the desired equation.

Now to check the validity of the equation, we may determine s for various values of t and compare the results with the observed values. Two such results are as follows:

t	s (calculated)	s (experimental)
0.8	229	227
1.75	884	885

The differences in these selected cases are certainly within the limits of experimental error.

The student should find an equation which will represent the corresponding curve for the golf ball and test its accuracy.

Problems.—1. Derive the three equations of motion which describe the motion of a particle moving in a fluid offering resistance proportional to the square of the speed. Take the initial velocity along a line which makes an angle of $\frac{\pi}{2} + \alpha$ with the direction of the force. The procedure is similar to that given in Sec. **12-3.**

2. If two spherical bodies of equal sizes but of unequal masses are dropped simultaneously from equal distances above the ground and fall under the influence of their weights and resistances due to the air, which body will arrive at the ground first?

3. Develop the equations of motion for a particle which is projected with an initial velocity U in a medium offering a resistance kV. No force, except the resistance, is to be included. How long would it take the particle to go a distance equal to U/k?

CHAPTER XIII

DAMPED HARMONIC MOTION

13-1. Damped Motions.—The subject of simple harmonic motion was discussed in Chap. V. In that chapter the only force included was the restoring force, *i.e.*, that force which is proportional to the displacement of the body and which acts in a direction that is always toward the neutral or rest position of the body. The motion was considered to take place without friction. Probably no such conditions actually exist in nature, because there is present a force of friction which will eventually bring the oscillating body to rest, provided no other external force is introduced to balance or neutralize the friction.

Because work must be done against the friction, the energy of the moving body is dissipated into the surrounding fluid in the form of heat. This constant drain upon the original stock of energy, which was placed in the body previous to the beginning of the motion, must eventually reduce the supply of energy to zero if the body be left to itself. In this chapter we are to study harmonic motions with resistance. Such motions are called *damped* harmonic motions.

13-2. Friction in Fluids.—The harmonic motions of bodies in nature, at least as far as ordinary motions are concerned, take place in some fluid, usually air, although in some cases the surrounding fluid may be much more dense, as in the case of water.

Resistance to motions in water has been intensively studied, particularly because of the immediate applications in navigation and hydraulics. The results of this work may be briefly described in the few following general statements. Friction in liquids is subject to considerable range of variation. When the velocity is large, friction may be proportional to the second, third, or even higher power of the speed. It is approximately proportional to the first power of the speed when the body is moving slowly and approaches zero as a limit value as the speed approaches zero. Pressure in a liquid does not appear to have any effect upon friction. In contrast with the friction between two solids, friction in liquids is dependent upon the area in contact with the fluid. The nature of the surface, as long as it is smooth, has very little effect upon the friction offered by liquids to bodies moving in them.

13-3. Three Cases of Damped Harmonic Motion.—It is now proposed to develop the differential equations of motion for three typical cases in which there is damped harmonic motion. It will then be shown that each of these equations may be readily converted into what might be called a standard or general form. The solution of the general equation will next be obtained and then interpretations of the solution will be given.

The first case involves pure translational motion. For a concrete case we may consider a mass which is suspended by a spring from a rigid support. When the hanging mass is displaced vertically in either direction from its rest position, the resulting motion will be subject to three forces, the weight of the body (and of the spring), the elastic force of the spring, and the resistance. The force of resistance produces a damping of the motion; *i.e.*, the amplitudes of the motion as measured from the rest position become successively smaller and approach zero as a limit. The mass of the spring enters into the inertia of the moving system. It is not the entire mass of spring, however, which is to be included, for the various parts of it have velocities which differ from that of the suspended body. In the following section the effective mass of the spring in the illustration selected is determined.

The second case consists of a weight pendulum. As shown above, the motion of the pendulum is not strictly harmonic. If the angular displacement is small, we may consider the motion to be approximately harmonic.

The third illustration involves a magnetic pendulum. While any rotational or torsional pendulum would have served equally well, the selection of the magnetic pendulum is made because in addition to the magnetic couple there is an additional restoring moment, *i.e.*, the moment of the suspending fiber.

13-4. Effective Mass of the Spring.—In a case such as the first one under consideration, it is not correct to disregard the mass of the spring unless it is very small in comparison with the mass of the supported body. It is perhaps obvious that the entire mass of the spring cannot be added to that of the body when considering the equations of motion. In order to determine the inertia contribution of the spring, the kinetic energy of the system as a whole is to be expressed for some definite position. It is immaterial which position is chosen as long as there is velocity in that position.

The question might come up as to the reason for using a kinetic-energy expression for this purpose rather than one involving momentum, since the momentum also depends upon the mass and velocity. The answer to this is to be found in the fact that we are concerned here with the effect of the forces and the distances through which they act and not the forces and the corresponding time factors. The former combination gives the quantity work; the latter, impulse. Work, not momentum, is associated with the changes in energy. Hence in this situation the kinetic energy, alone, can be used for determining the effective mass of the spring.

We take, therefore, an instantaneous position of the system and express the kinetic energy of the system for this selected position. Suppose for this position that V is the speed of the supported mass. To find the kinetic energy of the spring, we may first express the kinetic energy of a differential element of the spring in terms of V and then integrate over the entire spring. If ρ is the linear density of the spring, and dx the differential length, the mass of the differential element will be $\rho\, dx$. The speed of this element will be proportional to its distance from the point of support, so that at a distance x from the fixed point the speed of the element of the spring will be Vx/s if s is the total length of the spring in the selected position. The kinetic energy of the differential element will therefore be

$$\frac{1}{2}\rho\, dx\left(\frac{Vx}{s}\right)^2$$

The kinetic energy of the entire spring will be found by integrating the expression

$$\begin{aligned} KE &= \frac{1}{2}\rho\frac{V^2}{s^2}\int_0^S x^2dx \\ &= \frac{1}{2}\left(\frac{\rho s}{3}\right)V^2 \qquad (13\text{-}1) \end{aligned}$$

But ρs is the mass of the spring; hence one-third of the mass of the spring is effective in the motion of the system. We must therefore add one-third of the mass of the spring to that of the supported body in order to express accurately the entire effective mass of the system.

Problem.—What would be the effective mass of the spring if the problem were such that we were concerned with the momentum of the system? (Since the directions of the momenta of all differential particles of the spring are the same, the integration may be effected.)

13-5. The Equation for Translational Damped Harmonic Motion.

—The force equation may be used as the starting point for determining the motion of the mass suspended by a spring as shown in Fig. 126. The motion is to be restricted to linear motion by assuming a small initial displacement in a vertical line. With this limitation the acceleration of the mass will be expressed by d^2x/dt^2 if we let x be the coordinate which measures the displacement of the mass from the rest position. Let the upward vertical direction be selected as the positive direction for all vector quantities.

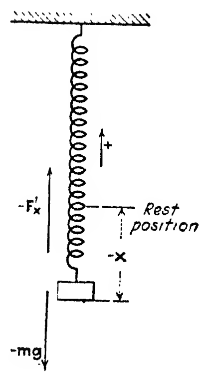

FIG. 126.

The resultant force system will be made up of three separate forces, the weight of the mass (and one-third the weight of the spring), the upward pull of the spring, and friction. If m represents the total effective mass, then $-mg$ is the effective weight and is constant for all positions. Since the pull of the spring in the rest position (from which x is measured) is equal to the weight of the effective mass and since the spring is assumed to follow Hooke's law, the sum of the two forces weight and spring tension is proportional to the displacement and may be written as $-F'x$. One must remember that it is legitimate to consider the system as consisting of a massless spring and a body which has a mass equal to the mass of the suspended object plus the effective mass of the spring.

The force due to resistance is taken proportional to the velocity dx/dt because the speed is assumed to be small. If we let R' be the proportionality constant, then the force of resistance may be written $-R'\ dx/dt$. The minus sign is used because the resistance is always acting in a direction opposite to that of the velocity. If the body is moving upward, the velocity is positive and the resistance negative, and *vice versa*.

With these specifications the force equation becomes

$$-F'x - R'\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

or, as it usually is written,

$$m\frac{d^2x}{dt^2} + R'\frac{dx}{dt} + F'x = 0 \qquad (13\text{-}2)$$

If now we divide through by m and change the coefficients of x and its derivatives by writing $2R = R'/m$ and $F = F'/m$, the equation becomes

$$\frac{d^2x}{dt^2} + 2R\frac{dx}{dt} + Fx = 0 \qquad (13\text{-}3)$$

which is the desired expression.

13-6. The Equation for Rotational Damped Harmonic Motion.—In pure rotational motion an equation similar to Eq. (13-3) may be obtained. To illustrate this type of motion, the weight pendulum has been selected. This may consist simply of a spherical bob suspended by a wire from a rigid support or it may be a long rod mounted to oscillate about a fixed axis near one end. In either case suppose the axis is at O (Fig. 127) and C the center of mass, with OC equal to r. Let I be the moment of inertia of the system with respect to the axis through O and perpendicular to the plane of motion. Also let m be the mass of the pendulum and γ the angular displacement measured from the rest position with positive values in the standard counter-clockwise direction.

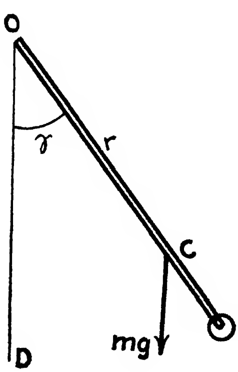

Fig. 127.

If we assume that the moment of the force of resistance is proportional to the first power of the angular velocity $d\gamma/dt$ and let R' be a proportionality constant, then the moment of the resistance will be $-R'\ d\gamma/dt$.

The moment of the weight will be $-mgr \sin \gamma$. The student should verify the necessity for using the minus sign.

The fundamental equation is the moment of force equation, since we are concerned with pure rotational motion. Combining the two force moments, the equation is then

$$-mgr \sin \gamma - R' \frac{d\gamma}{dt} = I \frac{d^2\gamma}{dt^2} \tag{13-4}$$

If we now divide each term of this equation by I and replace $\sin \gamma$ by the angle γ, which we may do if we restrict the motion to small angles, the equation becomes

$$\frac{d^2\gamma}{dt^2} + 2R \frac{d\gamma}{dt} + F\gamma = 0 \tag{13-5}$$

where $2R = R'/I$ and $F = mgr/I$.

Upon comparing this expression with Eq. (13-3), it will be observed that the two are similar and if γ were to be replaced by x they would be identical.

13-7. The Equation for the Magnetic Pendulum.—If a magnet is suspended by a fiber so that it may rotate about a vertical axis, it will be subject to a magnetic moment due to the horizontal component (H) of the earth's magnetic field. If M is magnetic moment of the magnet, then the magnetic restoring couple due to H, for a positive angular displacement γ from the rest position, will be $-MH \sin \gamma$.

In addition to the magnetic couple there will be another force moment caused by a twist of the suspending fiber. If the fiber has zero twist when the magnet is in the rest position, the restoring moment due to the fiber may be expressed as $-T\gamma$, where T depends upon the elasticity of the material of which the fiber is made and also upon its dimensions. The quantity T is the torque per radian displacement of the magnet.

If we designate the resistance proportionality factor by the letter B and assume that the resisting moment is proportional to the angular velocity and let I be the moment of inertia of the system about the axis of rotation, the force-moment equation may be written

$$-MH \sin \gamma - T\gamma - B \frac{d\gamma}{dt} = I \frac{d^2\gamma}{dt^2} \tag{13-6}$$

If we assume, as above, that the angular displacement is small, so that $\sin \gamma$ may be replaced by γ, and divide through the equation by I, the expression reduces to the form

$$\frac{d^2\gamma}{dt^2} + 2R \frac{d\gamma}{dt} + F\gamma = 0 \tag{13-7}$$

in which $2R = B/I$ and $F = (MH + T)/I$. This equation also takes the standard form for damped harmonic motion.

Since the particular equation for each of the illustrations discussed above reduces to one general form, it will be readily seen that the general solution of this differential equation will apply equally well to all three cases.

13-8. Solution of the General Equation.—We may select Eq. (13-3) as being typical of damped harmonic motion in which the resistance is proportional to the first power of the speed. There are several ways in which the general solution of this equation may be obtained. Two methods will be given. Attention should be directed to the fact that the general equation is a second-order linear differential equation (see any textbook on differential equations). Such equations possess two distinct particular solutions. The general solution of a second-order differential equation is a linear combination of the two separate solutions, each having a constant coefficient. For example, if s_1 and s_2 are the particular solutions, s the general solution, and A and B are constant, then

$$s = As_1 + Bs_2$$

is the general solution. The constants A and B are in reality integration constants and are to be evaluated in the usual manner.

a. In the first method of solution we shall first reduce the equation to one which does not have the term containing the velocity (dx/dt). This may be done by a change of variables. If we introduce a new variable y and put

$$x = e^{-Rt}\, y \tag{13-8}$$

and then evaluate the first and second derivatives of x with respect to the time and substitute these expressions in Eq. (13-3), we obtain the following differential equation:

$$\frac{d^2y}{dt^2} + y\,(F - R^2) = 0 \tag{13-9}$$

It will be observed that by this change of variables the term containing the velocity has been suppressed. In order to afford an abbreviation in writing, let us put $\omega^2 = F - R^2$. The symbol ω is selected because each term of the quantity $F - R^2$ is dimensionally that of the square of an angular velocity. That this is the case may be readily verified by referring to the original equation [Eq. (13-3)], in which each term has the dimensions of a linear acceleration. It is to be remembered that the dimension of angular velocity is simply sec^{-1}. The quantity ω is here a constant. With this change the equation becomes

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \tag{13-10}$$

As solutions for this equation we may try the pair of imaginary roots $e^{i\omega t}$ and $e^{-i\omega t}$, in which i is written $\sqrt{-1}$. By taking the second derivative of each expression and substituting in the equation, we find that the equation is satisfied. Introducing the two constants A and B and combining the particular solutions into a single solution gives the following expression:

$$y = A\, e^{i\omega t} + B\, e^{-i\omega t} \tag{13-11}$$

By substituting this value of y in Eq. (13-8) we obtain

$$x = e^{-Rt}\,[Ae^{i\omega t} + Be^{-i\omega t}] \tag{13-12}$$

If the constants A and B are conjugate imaginary quantities, then x is real. This equation may be put into a form which is more convenient for present purposes by replacing the exponential quantities by their trigonometric equivalents. The general solution may then be written as follows:

$$x = e^{-Rt}[(A + B)\cos \omega t + i(A - B)\sin \omega t] \tag{13-13}$$

b. Another method for obtaining the general solution [Eq. (13-3)] is to assume that

$$x = e^{kt} \tag{13-14}$$

and then take the first and second time derivatives of x and substitute the resulting expressions in Eq. (13-3). This procedure gives

$$e^{kt}(k^2 + 2Rk + F) = 0$$

Hence

$$k^2 + 2Rk + F = 0 \tag{13-15}$$

By this substitution we may reduce the solution to one which is an ordinary quadratic in k. The values of k are readily found to be

$$k = -R \pm \sqrt{R^2 - F}$$

It will be observed that the quantity under the radical sign is the negative of the quantity which was arbitrarily defined as ω^2. In order to reduce the solution to one which may be readily compared with that obtained above [Eq. (13-12)], we may first take from the radical the factor i. By introducing ω, as defined above, the expression for k becomes

$$k = -R \pm i\omega \tag{13-16}$$

This gives two values for k. By substituting these values in the expression of Eq. (13-14), two solutions are obtained. The general solution is now obtained by multiplying the solution which contains the plus sign by the constant A and the other by B, and by putting x equal to the sum of the resulting expressions. This equation is identical with Eq. (13-12) obtained above.

An inspection of Eq. (13-13) will be useful here to afford a clue as to the physical meaning of the general solution. A critical factor in the physical solution is to be found in the relative magnitudes of F and R. This fact is more readily observed in the mathematical expression if we replace ω in Eq. (13-13) by its value in terms of F and R, which gives

$$x = e^{-Rt}[(A + B) \cos (\sqrt{F - R^2}\, t) + i(A - B) \sin (\sqrt{F - R^2}\, t) \tag{13-17}$$

There are, obviously, three cases to be considered, one in which F is greater than R^2, one in which F is equal to R^2, and the other in which F is less than R^2. These cases are sufficiently important to justify separate treatments. The three cases will be identified as follows:

Small damping:

$$F > R^2$$

Critical damping:

$$F = R^2$$

Large damping:

$$F < R^2$$

It will be shown below that vibratory motion exists in only the first case. The period of the motion in this case will be discussed next.

13-9. The Period in the Case of Small Damping.—Although the constants A and B of Eq. (13-17) have not as yet, in our development of the subject, been expressed in terms of initial conditions, we may use this equation for an evaluation of the period of the motion. Neglecting the exponential factor for the present, we readily see that, if $F > R^2$, periodic variation in x must be due to the periodic variation of the trigonometric functions. The period of this variation is called the period of the damped harmonic motion. The period depends only upon the coefficient of t in the cosine or sine factors.

As we have shown above in connection with simple harmonic motion, the period may therefore be expressed as follows:

$$T = \frac{2\pi}{\sqrt{F - R^2}} \tag{13-18}$$

From this result we can see that the period increases as R increases, if F remains constant.

13-10. Small Damping.—An interpretation of the solution of the general equation as given in Eq. (13-13) is made easier if the equation is put in a different form. This may be done by introducing two new constants (C and α) which are defined in terms of A and B by the equations

$$A = \tfrac{1}{2}\, Ce^{i\alpha} \qquad B = \tfrac{1}{2}\, Ce^{-i\alpha} \tag{13-19}$$

With these defining relations A and B may be eliminated from Eq. (13-13). If follows from Eqs. (13-19) that

$$A + B = C \cos \alpha \qquad i\,(A - B) = -C \sin \alpha$$

Substituting these values in Eq. (13-13) gives

$$\begin{aligned} x &= Ce^{-Rt}\,(\cos \alpha \cos \omega t - \sin \alpha \sin \omega\, t) \\ &= Ce^{-Rt} \cos\,(\omega t + \alpha) \end{aligned} \tag{13-20}$$

This form has still two constants, one of which, *viz.*, α, is obviously an epoch angle.

In order to evaluate C, initial conditions for the motion are to be selected. Let these be $x = a$ at $t = 0$ and $dx/dt = 0$. Putting $t = 0$ and $x = a$ in the foregoing equation gives

$$C = \frac{a}{\cos \alpha}$$

To eliminate α from this equation, it will be necessary to find another equation containing C and α. Such an equation may be found by differentiating Eq. (13-20) with respect to the time. We have therefore an equation for the speed:

$$\frac{dx}{dt} = -CRe^{-Rt} \cos\,(\omega t + \alpha) - C\omega e^{-Rt} \sin\,(\omega t + \alpha)$$

Putting both t and dx/dt equal to zero gives

$$\tan \alpha = \frac{-R}{\omega}$$

Introducing the value of ω as expressed in terms of F and R, we may find the value for $\cos \alpha$, which is

$$\cos \alpha = \frac{\omega}{\sqrt{F}}$$

Hence

$$C = a \frac{\sqrt{F}}{\omega}$$

Substituting these values for C in Eq. (13-20) gives

$$x = \frac{a\sqrt{F}}{\omega} e^{-Rt} \cos (\omega t + \alpha) \qquad (13\text{-}21)$$

With this expression the value of the displacement may be found for any value of t in a given case where the constants are known. It is to be remembered that, while x is usually taken to mean a coordinate which measures a linear displacement, here it may also represent an angular coordinate, for this solution is valid for those physical cases which involve rotational motion as well as translational.

An expression for the velocity is found by differentiating Eq. (13-21) with respect to the time. This gives

$$\frac{dx}{dt} = -\frac{a\sqrt{F}}{\omega} e^{-Rt}[R \cos (\omega t + \alpha) + \omega \sin (\omega t + \alpha)] \qquad (13\text{-}22)$$

The character of the variation of the displacement as a function of the time may be determined by an examination of Eq. (13-21). The right-hand member is a product of three factors: a constant ($a \sqrt{F}/\omega$), an exponential, and a cosine factor.

If the effect of the exponential factor may be disregarded for a first consideration, then the rest of the equation expresses the time variation of the amplitude in simple harmonic motion. The amplitude of the simple harmonic motion is the constant factor $a \sqrt{F}/\omega$ and the period of the motion is $2\pi/\omega$.

The exponential factor (e^{-Rt}) has a negative exponent; hence, as the time increases from zero, the factor decreases logarithmically from the value 1. When t is infinitely large, this factor becomes equal to zero. It does not have any negative value for positive (or negative) values of the time.

The complete expression for the displacement in damped harmonic motion may therefore be regarded as harmonic motion with an amplitude which decreases logarithmically with time.

A graphical representation of the variation of the amplitude with time is shown in Fig. 128 (curve B). The curve was drawn

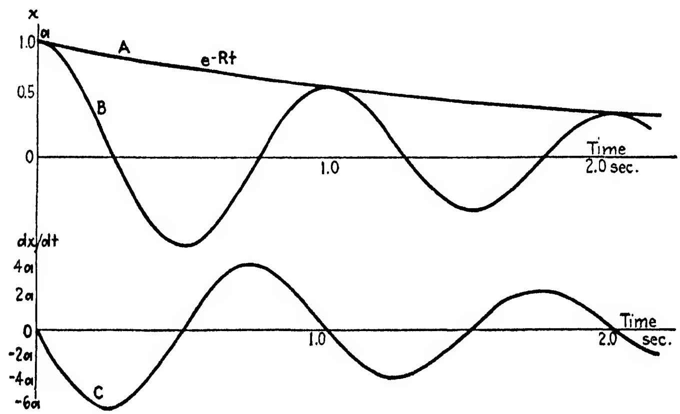

FIG. 128.

from data obtained by assuming certain values for the constants. These values are tabulated as follows:

$$m = 100 \text{ g.}$$
$$F' = 4{,}000 \text{ g. per second squared}$$
$$R' = 100 \text{ g. per second}$$
$$F = \frac{F'}{m} = 40 \text{ sec.}^{-2}$$
$$R = \frac{R'}{2m} = 0.5 \text{ sec.}^{-1}$$
$$\omega = \sqrt{F - R^2} = 6.3 \text{ sec.}^{-1}$$
$$T = \frac{2\pi}{\omega} = 0.997 \text{ sec.}$$
$$\tan \alpha = -\frac{R}{\omega} = -0.0794$$
$$\alpha = -4^\circ\, 32'$$
$$\frac{\sqrt{F}}{\omega} = 1 \qquad a = 1$$

The exponential factor e^{-Rt} is represented by curve A in the same figure. It is to be noticed that, with the value of the constant a put equal to unity, the exponential curve is tangent to the displacement. This condition of tangency would not occur if the value of a were not unity. An inspection of Eq. (13-21) will show that with $a = 1$, whenever $\omega t = 2\pi$ or any integral multiple of 2π, the displacement x will equal e^{-Rt}, since $\cos\alpha = \omega/\sqrt{F}$. These points of tangency will occur at the particular values of the time at which the speed of the particle is zero. That this is true may be seen by putting $\omega t = 2\pi n$, where n is any integer, in Eq. (13-22) which makes the quantity within the bracket become equal to zero, or

$$R\cos\alpha + \omega\sin\alpha = 0$$

since $$\cos\alpha = \frac{\omega}{\sqrt{F}} \qquad \text{and} \qquad \sin\alpha = \frac{-R}{\sqrt{F}}$$

because α is a negative angle.

The variation of the speed with time [Eq. (13-22)] is represented by C of Fig. 128. An analysis of this curve may be made in a manner similar to that used above in examining the displacement equation. It is instructive to compare the simultaneous behavior of the displacement and speed and to picture mentally the physical relations.

Problems.—1. Show that the time interval between successive transits of the body through the rest position is equal to one-half the period.

2. Show that the time interval between any two successive positions of zero velocity is equal to one-half the period.

3. Prove for an initial positive displacement of the body that the velocity is negative for the first half cycle. Assume that the body starts from rest.

4. Prove that the positions of maximum speed do not occur at the instants of zero displacement.

5. For the case of damped harmonic motion with small damping, obtain the third equation of motion, *viz.*, an expression containing x and dx/dt but without t.

13-11. Critical Damping.—The imposed relation between the constants of the original differential equation is that $F = R^2$. To find an expression for the displacement and the velocity in terms of the time, it is necessary to put $F = R^2$ in Eq. (13-9). This substitution gives

$$\frac{d^2y}{dt^2} = 0 \qquad (13\text{-}23)$$

Integration of this equation gives

$$y = Dt + E$$

in which D and E are integration constants. Substituting this value for y in Eq. (13-8) gives

$$x = e^{-Rt}\,(Dt + E) \tag{13-24}$$

The expression for the speed is readily determined from this relation by differentiation and is

$$\frac{dx}{dt} = -e^{-Rt}\,[R\,(Dt + E) - D] \tag{13-25}$$

These two equations, giving displacement and speed in terms of the time, contain two constants. To evaluate these constants the initial conditions

$$x = a \qquad t = 0 \qquad \frac{dx}{dt} = 0$$

which are the same as those given above in the case of small damping, are to be used. By substituting these values for the

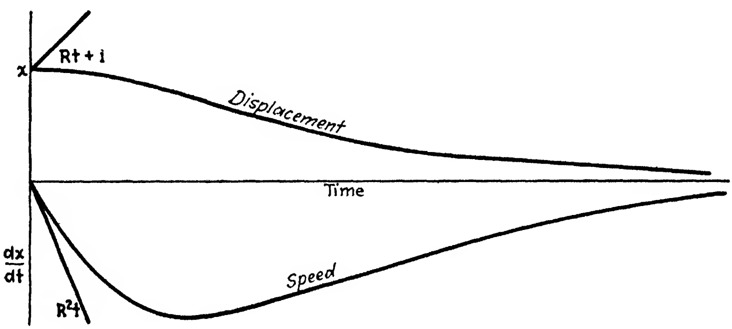

FIG. 129.

variables in Eqs. (13-24) and (13-25), the constants are found to have the following values:

$$D = Ra \qquad E = a \tag{13-26}$$

Hence the desired equations may be written as follows:

$$x = ae^{-Rt}\,(Rt + 1) \tag{13-27}$$

$$\frac{dx}{dt} = -ae^{-Rt}\,(R^2t) \tag{13-28}$$

It is of interest to plot both displacement and speed in terms of the time in order to reveal the character of their variation. The result is shown in Fig. 129 in which both quantities are plotted to the same time axis.

The displacement curve may be regarded, mathematically, as resulting from the product of a straight line, $x = a\,(Rt + 1)$, by an exponential factor e^{-Rt}. The effect of the exponential factor upon the displacement is clearly shown by the plotted curve. The speed curve may be similarly regarded. The speed reaches a maximum negative value at the instant when the displacement curve shows a point of inflection.

The value of the time (t') at which the speed is a maximum is obtained by putting the acceleration equal to zero and solving for t'. Differentiating Eq. (13-28) with respect to the time gives the acceleration.

$$\frac{d^2x}{dt^2} = aR^3t'e^{-Rt'} - aR^2e^{-Rt'} = 0 \qquad (13\text{-}29)$$

Hence

$$t' = \frac{1}{R}$$

Problems.—1. Make a plot which shows the variation of the acceleration with the time in the case of critical damping.

2. Eliminate t from Eqs. (13-27) and (13-28) to obtain the third equation of motion for the case of critical damping.

13-12. Large Damping.—In order to derive a simplified equation for the displacement in the case of large damping, it is convenient to replace ω of Eq. (13-12) by its value $\sqrt{F - R^2}$. To make this radical real, since R^2 is greater than F in the case of large damping, the factor -1 is taken from the quantities under the radical. This factor becomes i when outside the square-root sign; hence

$$\sqrt{F - R^2} = i\sqrt{R^2 - F}$$

If we designate the second radical by the letter s and then substitute is for ω in Eq. (13-12), the equation becomes

$$x = e^{-Rt}\,[Ae^{-st} + Be^{st}] \qquad (13\text{-}30)$$

By differentiating this equation, an expression for the speed is obtained. This is

$$\frac{dx}{dt} = -e^{-Rt}\,[Ae^{-st}\,(R + s) + Be^{st}\,(R - s)] \qquad (13\text{-}31)$$

With the same initial conditions as given in the section above, the constants A and B are found to be

$$A = \frac{a\,(s - R)}{2s} \qquad B = \frac{a\,(R + s)}{2s}$$

Introducing these values in the foregoing equations and combining the exponential factors gives the desired expressions for the displacement and speed in terms of the time.

$$x = \frac{a}{2s}\,[(s - R)e^{-(R+s)t} + (R + s)e^{-(R-s)t}] \tag{13-32}$$

$$\frac{dx}{dt} = -\frac{a}{2s}\,[(s^2 - R^2)e^{-(R+s)t} + (R^2 - s^2)e^{-(R-s)t}] \tag{13-33}$$

In order to interpret these expressions, it is necessary to know the algebraic sign of each term. Since the primary condition required R^2 to be greater than F, it follows that R is greater than

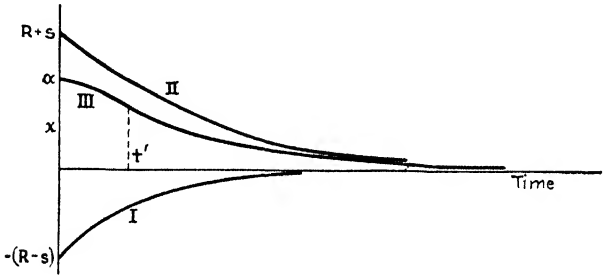

FIG. 130.

s. We may regard each of the quantities $s - R$, $R + s$, $s^2 - R^2$, and $R^2 - s^2$ as single factors and the right-hand members of the equations for x and dx/dt as consisting of only two terms. In the expression for x, since s is positive, the first term is negative and the second one is positive. In the speed equation the reverse is true.

All of the exponential quantities have negative exponents for all values of the time (from 0 to ∞); hence these quantities will vary from $+1$ to zero as the time increases from zero to infinity.

By considering first the displacement equation, it is obvious that the displacement at any instant is made up of the sum of two quantities, one of which is negative and the other positive. Plotting the two terms separately gives the curves, identified as I and II in Fig. 130, for the first and second terms, respectively.

Because of the fact that the coefficient of the time factor in the first term is $R + s$ and that in the second term is $R - s$, the negative curve (I) approaches the time axis more rapidly. The curve marked III is the resultant displacement. This curve has a point of inflection at the time marked t'. At this instant the speed has its maximum value.

The speed curve may be determined in a similar manner. The results are plotted in Fig. 131. Curve I shows the contribution of the first term, curve II that from the second term, and III is the resultant speed.

As in the case of critical damping, the speed curve shows a maximum negative value at some time which we may call t'. The

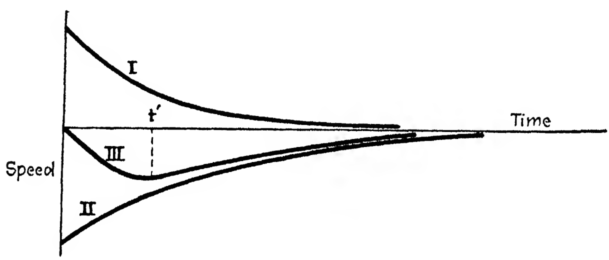

FIG. 131.

expression for t' is found in a manner described above and is as follows:

$$t' = \frac{1}{2s} \log \frac{R + s}{R - s} \qquad (13\text{-}34)$$

13-13. The Logarithmic Decrement.—In this section a more detailed inspection of the decreasing amplitude of the displacement in the case of small damping is to be made. For this purpose Eq. (13-20) will be used. It is rewritten here for convenience.

$$x = Ce^{-Rt} \cos (\omega t + \alpha) \qquad (13\text{-}20)$$

Since the cosine factor can vary only from $+1$ to -1, the curve expressing the displacement x must always lie between the two logarithmic curves $+Ce^{-Rt}$ and $-Ce^{-Rt}$. This is shown in the Fig. 132.

The values of the amplitudes at two successive positive maxima are next to be determined. At these positions the speed is zero. If, then, the values of the times, at which these maxima occur, are determined and substituted in the equation for the displacement, the corresponding displacements may be found.

The equation for the speed is found by differentiating Eq. (13-20); hence

$$\frac{dx}{dt} = -Ce^{-Rt}\,[R\cos(\omega\,t + \alpha) + \omega\sin(\omega\,t + \alpha)] \qquad (13\text{-}35)$$

Putting the speed equal to zero and writing t' for the particular value of the time gives

$$\tan(\omega\,t' + \alpha) = -\frac{R}{\omega} \qquad (13\text{-}36)$$

This relation remains true when the angle $\omega\,t' + \alpha$ is increased by $2\,\pi\,n$ or when the time is increased by nT (where T is the period).
Hence

$$t' + nT = \frac{1}{\omega}\left[-\alpha + \tan^{-1}\left(-\frac{R}{\omega}\right)\right] \qquad (13\text{-}37)$$

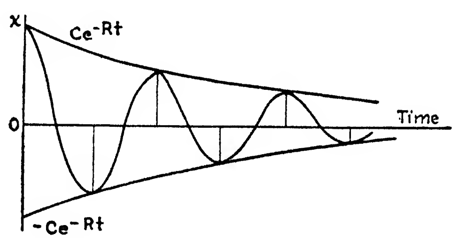

Fig. 132.

The times at which two particular successive maxima (x_1 and x_2) occur may be taken as $t' + T$ and $t' + 2T$. Substituting these values separately in Eq. (13-20), equating the resulting expressions to x_1 and x_2, respectively, and then dividing x_1 by x_2 gives

$$RT = \log\frac{x_1}{x_2} \qquad (13\text{-}38)$$

The quantity RT is called the logarithmic decrement. This equation supplies the means for evaluating R, the damping factor, from experimental observations. All that is necessary is a determination of the period and the ratio of any two successive maximum displacements taken in the same direction.

In *undamped* harmonic motion the values of the time at which the maximum displacements occur are midway between the values of the time at which two consecutive displacements are zero. In damped harmonic motion this is not the case. To prove the truth of this statement, we may determine the values of the time at which the displacements are zero and then, by referring to the values of the time at which the displacement maxima occur, the relative values of these times may be determined. Letting t_1 represent the value of the time when $x = 0$ and putting $x = 0$ in Eq. (13-20) gives

$$0 = Ce^{-Rt_1} \cos (\omega t_1 + \alpha)$$

Since neither C nor e^{-Rt_1} is zero, $\cos (\omega t_1 + \alpha) = 0$ and hence

$$t_1 = \frac{1}{\omega}\left[(2n + 1)\frac{\pi}{2} - \alpha\right] \qquad (13\text{-}39)$$

in which n is any integer.

If one-quarter of the period $(\pi/2\,\omega)$ is added to the value of t_1, the resulting expression is not equal to t' [Eq. (13-37)], the value of the time at which the maximum displacements occur.

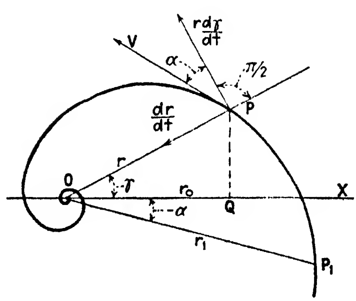

FIG. 133.

13-14. Use of the Logarithmic Spiral to Express Oscillations in Damped Harmonic Motion.—In the chapter dealing with simple harmonic motion it was shown that simple harmonic motion could be obtained by projecting the uniform circular motion of a particle on to the diameter of the circle. It is to be shown here that linear damped harmonic motion, in which the damping is small, may be described in a somewhat similar manner. In place of the circular path used in simple harmonic motion, we shall show that a logarithmic spiral may serve a similar function.

A characteristic of the logarithmic spiral is that the angle between the radius vector drawn to the curve at any point and the tangent to the curve at that point is constant. Let $\frac{\pi}{2} + \alpha$ be this angle (Fig. 133).

Given the point P on the spiral with O the center of the spiral, OX the reference line, and r the radius vector drawn to P. A particle at P is to move with a variable speed along the curve in such a manner that the angular velocity $d\gamma/dt$ of r shall be constant.

It is to be shown that Q, the projection of P upon the reference line OX, moves with damped harmonic motion.

The component of the velocity of P parallel to r is $-dr/dt$ and the component perpendicular to r is $r\, d\gamma/dt$. Since the angle (α) between the resultant velocity V and $r\, d\gamma/dt$ is constant, it follows that

$$\tan \alpha = -\frac{\dfrac{dr}{dt}}{r\left(\dfrac{d\gamma}{dt}\right)} = \frac{1}{r}\frac{dr}{d\gamma} = -A \text{ (a constant)} \qquad (13\text{-}40)$$

Integrating this equation gives

$$\log r = -A\,\gamma + C$$

with C the integration constant.

By selecting the following initial conditions, $r = r_0$, $t = 0$, and $\gamma = 0$, the constant C is found to be equal to $\log r_0$. Hence

$$r = r_0\, e^{-A\gamma} \qquad (13\text{-}41)$$

is the equation which expresses the length of the radius vector in terms of γ and the constants.

If now we let x measure the displacement of Q from the center O along the fixed line OX, the equation for the acceleration of Q may be expressed in terms of x and its derivatives.

It is the form of the equation expressing the acceleration of Q in which we are particularly interested. The starting point is the expression for x, which is

$$x = r \cos \gamma$$

Differentiating this equation and putting $d\gamma/dt = \omega$ (constant) gives

$$\frac{dx}{dt} = \frac{dr}{dt}\cos\gamma - r\,\omega \sin\gamma$$

The value of dr/dt is found by differentiating Eq. (13-41) and is

$$\frac{dr}{dt} = -r\,\omega\,A$$

Substituting this value in the expression for dx/dt gives

$$\begin{aligned}\frac{dx}{dt} &= -r\,\omega\,A\cos\gamma - r\,\omega\sin\gamma \\ &= -x\,\omega\,A - r\,\omega\sin\gamma \qquad (13\text{-}42)\end{aligned}$$

The second derivative of x is therefore

$$\begin{aligned}\frac{d^2x}{dt^2} &= -\frac{dx}{dt}\,\omega\,A - \frac{dr}{dt}\,\omega\sin\gamma - r\,\omega^2\cos\gamma \\ &= -\frac{dx}{dt}\,\omega\,A - \frac{dr}{dt}\,\omega\sin\gamma - x\,\omega^2 \qquad (13\text{-}43)\end{aligned}$$

If now we multiply both sides of Eq. (13-42) by $\omega\,A$, we obtain

$$\frac{dx}{dt}\,\omega\,A = -x\omega^2A^2 - rA\,\omega^2\sin\gamma$$

A substitution of this value for the first term of the right-hand member of Eq. (13-43) gives

$$\begin{aligned}\frac{d^2x}{dt^2} &= x\omega^2\,(A^2 - 1) + rA\omega^2\sin\gamma - \frac{dr}{dt}\,\omega\sin\gamma \\ &= x\omega^2\,(A^2 - 1) + 2rA\omega^2\sin\gamma \\ &= -x\omega^2\,(1 + A^2) + 2x\omega^2A^2 + 2rA\omega^2\sin\gamma \\ &= -x\omega^2\,(1 + A^2) - 2A\omega\frac{dx}{dt}\end{aligned}$$

A rearrangement of the terms of this result yields the final expression for the acceleration of Q, which is

$$\frac{d^2x}{dt^2} + 2A\omega\,\frac{dx}{dt} + \omega^2\,(1 + A^2)x = 0 \qquad (13\text{-}44)$$

The coefficients of dx/dt and x are both constant. If we write $2R$ and F, respectively, for these coefficients, the equation takes the form of the general equation for damped harmonic motion. The projection of the motion of P in the logarithmic spiral, subject to the given limitations, may therefore be used to describe damped harmonic motion.

It is profitable to interpret the equation for the displacement [Eq. (13-21)] and for the velocity [Eq. (13-22)] in terms of the auxiliary motion of P in the logarithmic spiral.

Let us first examine the angular velocity of P in the spiral. The period of P's motion is $2\pi/\omega$, which is also the period of the displacement and of the velocity. Since ω is constant, it follows that the quantity ω, as used in the equations of the damped harmonic motion, may be represented by the angular velocity of P in the spiral.

The angle α in the equation for the damped harmonic motion is an epoch angle. This angle is represented in the spiral by the value which γ has at the time $t = 0$. To show that this is the case, we may determine the position of the radius vector when $t = 0$. Since, at this instant, the speed is zero, the position of r must be such that the velocity of P is perpendicular to the reference line OX. The resultant velocity of P is always in the line of the tangent to the curve at the point P and makes an angle α with the perpendicular to r; hence at the time $t = 0$ the radius vector will lie below OX and make an angle $-\alpha$ with OX. A verification of this statement is to be obtained by comparing the value of α as found in the equations of the motion in the spiral with the value of α determined from the equations of the damped harmonic motion. In the case of the motion in the spiral, α may be found from the coefficient of $2(dx/dt)$ of Eq. (13-44) together with Eq. (13-40), which gives

$$R = +A\,\omega$$

From which

$$-A = \tan\alpha = -\frac{R}{\omega}$$

A similar value for α has already been found.

It is also of interest to observe that the spiral shows that the maximum displacement does not occur at the instant when P crosses the reference line.

Problems.—1. Devise a method for determining the value of the damping factor from a displacement curve in the case of critical damping or of large damping.

2. Show how the damping factor might be determined by using values of the speed corresponding to the times when the displacement is zero.

13-15. Forced Vibrations.—The harmonic vibrations of a particle, as described in the preceding sections, are known as "free" vibrations. In such motions, the elastic restoring force and fluid resistance are the only forces included. It is now proposed to study the effect of adding another force to the other two. The nature of the additional force is to be restricted to a harmonic character; *i.e.*, the force is to be expressible in terms of the time by a cosine or sine term, such as $b \cos \Omega t$. When such a force is added to elastic restoring force and fluid resistance, the resulting vibrations of the particle are somewhat complex for a time. After an initial transient period, the length of which depends upon the constants (inertia, elastic force, and resistance), the vibrations settle down to a steady state and remain so as long as the applied harmonic force does not change.

The equations derived in this section have been found useful in describing several important phenomena which occur in various fields of physics, particularly in sound, electricity, and light. One reference may be made to point out the fact that the fundamental equation may be successfully used to explain the behavior of the very smallest of particles, *i.e.*, the electron. In the classical theory of light, use of these equations has been made to obtain expressions for the dispersion of light. It is interesting to observe that the same fundamental assumptions have to be made for the elastic forces which hold the bound electrons to the atoms of the dielectric and for the resistance offered to the motions of the electrons as have been made here in describing the forced vibrations of a particle. The experimental measurements validate the assumptions.

The addition of an impressed harmonic force to the elastic force and fluid resistance gives the following differential equation:

$$\frac{d^2x}{dt^2} + 2R\frac{dx}{dt} + Fx = b \cos \Omega t \tag{13-45}$$

where b is the harmonic force per unit mass and $2\pi/\Omega$ is the period of variation of this force.

To solve this equation, we may try

$$x = D \cos (\Omega t - \beta)$$

If this value for x is a solution of Eq. (13-45), then, upon substituting the first and second time derivatives of x, the given differential equation should be satisfied.

$$\frac{dx}{dt} = -D\Omega \sin (\Omega t - \beta)$$

$$\frac{d^2x}{dt^2} = -D\Omega^2 \cos (\Omega t - \beta)$$

Substituting in Eq. (13-45), expressing the sine and cosine functions of the sum of two angles in terms of the corresponding functions of the single angles, and rearranging terms gives

$$[b - D\,(F - \Omega^2)\cos\beta - 2DR\,\Omega\sin\beta]\cos\Omega\,t \\ - [D\,(F - \Omega^2)\sin\beta - 2DR\,\Omega\cos\beta]\sin\Omega\,t = 0 \quad (13\text{-}46)$$

In order that this equation should be equal to zero, it is necessary for the coefficients of sin $\Omega\,t$ and cos $\Omega\,t$ to be separately equal to zero. If we put the coefficient of sin $\Omega\,t$ equal to zero, we find that

$$\tan\beta = \frac{2R\,\Omega}{F - \Omega^2} \quad (13\text{-}47)$$

From this relation we may write

$$\sin\beta = \frac{2R\,\Omega}{\sqrt{(F - \Omega^2)^2 + 4R^2\Omega^2}}$$

$$\cos\beta = \frac{F - \Omega^2}{\sqrt{(F - \Omega^2)^2 + 4R^2\Omega^2}}$$

Equating the coefficient of cos $\Omega\,t$ to zero gives

$$D = \frac{b}{(F - \Omega^2)\cos\beta + 2R\Omega\sin\beta}$$

Substituting the values of cos β and sin β as written above and simplifying yields the following value for D:

$$D = \frac{b}{\sqrt{(F - \Omega^2)^2 + 4R^2\Omega^2}} \quad (13\text{-}48)$$

Using this value of D, we may now write

$$x = \frac{b}{\sqrt{(F - \Omega^2)^2 + 4R^2\Omega^2}}\cos(\Omega\,t - \beta) \quad (13\text{-}49)$$

which is a solution of Eq. (13-45). This solution, however, is a particular solution and not a complete solution, for it contains no arbitrary constants.

The complete solution is obtained by adding to Eq. (13-49) the solution of Eq. (13-3), *viz.*, Eq. (13-20), since the sum of the two expressions contains two arbitrary constants and satisfies the general differential equations. The complete solution is therefore

$$x = C\,e^{-Rt}\cos(\omega t + \alpha) + \frac{b\cos(\Omega t - \beta)}{\sqrt{(F - \Omega^2)^2 + 4R^2\Omega^2}} \quad (13\text{-}50)$$

The constants may be evaluated by the use of initial conditions.

The displacement at any instant may be seen to be a sum of two displacements. The first term gives the "free" displacement and the second the "forced" displacement. We have seen above that the amplitude due to the first component approaches a zero value because of the damping factor. The second component contains no exponential factor and therefore its amplitude does not decrease. After a sufficient length of time the free vibrations may become small enough to neglect in comparison with the forced vibrations.

In the present consideration we are interested only in the amplitude of the forced vibration. We shall therefore neglect the free vibration.

The period of the forced vibration is readily seen to be that of the impressed harmonic force, *viz.*, $2\pi/\Omega$. The vibration, however, lags behind the impressed force by an angle β.

The amplitude of the forced vibration is the coefficient of the cosine factor.

It is of interest to see how the amplitude of the vibration varies with the frequency Ω of the impressed force. The conditions for obtaining a maximum amplitude may be found by differentiating the equation for D [Eq. (13-48)] with respect to Ω and putting the resulting expression equal to zero. Performing the indicated operation gives the particular value of the frequency Ω':

$$\Omega' = \sqrt{F - 2R^2} \qquad (13\text{-}51)$$

This value of the frequency is known as the resonance frequency. In order for Ω' to have a real value, it is necessary for F to be greater than $2R^2$. If F is less than $2R^2$, the value of Ω' is imaginary. The effect of increasing the resistance of the medium is to decrease the resonance frequency.

Problems.—1. For given values of F, R, and b [Eq. (13-51)] find the character of the curve which expresses the dependence of D upon Ω.

2. Show that the ratio of the amplitude of the motion, in the case of small damping, at any time, to the initial amplitude is proportional to the time.

3. Derive the differential equation of motion of a particle, which executes simple harmonic vibrations, from the principle of the conservation of energy.

CHAPTER XIV

VECTOR FIELDS

14-1. Nature of a Vector Field.—In Chap. X it was shown that in conservative fields there exists a scalar function, the potential, which is everywhere single valued and finite, and that from this scalar function the field intensity may be determined. At every point in the scalar field there exists a definite value of the potential and also a definite value of the field intensity. Field intensity, however, is a vector quantity. Hence from the scalar field a vector field may be established. A vector field is therefore a region to every point of which there corresponds some value of the vector quantity.

The vector field may include all portions of space but it may be necessary to use two or more analytical expressions to define the vector throughout all portions of the space. For example, if the field is that of gravitational-field intensity due to a single continuous mass, one expression will be necessary for all points outside the boundaries of the gravitating mass and another expression for those points within the boundaries of the mass. If the field is due to three separate masses, then four expressions may be necessary.

14-2. Gradient of Potential and Field Intensity.—Let us consider any scalar field, such as the potential field which exists in the region surrounding a gravitating mass. We may select a reference system XYZ with P any point in the field. The value of the scalar function at P is dependent upon the coordinates of P. The function will, in general, change as we go a distance dx from P along a line which is parallel to the X-axis. If the change in the scalar function is dV, then the rate of change in the direction of dx will be $\partial V/\partial x$. This quantity expresses the rate of change of the scalar function in a definite direction. We may write $\mathbf{A}_x$ for the vector which combines the direction with the magnitude; hence

$$\mathbf{A}_x = \mathbf{i}\frac{\partial V}{\partial x} \tag{14-1}$$

in which i has its usual vector significance—that of the unit vector in the positive direction along the OX-axis.

Similar expressions may be written for the components parallel to the two other reference axes.

To find the maximum rate of change in the scalar function (for the particular point) and the direction of this maximum rate of change, we may determine the magnitude and direction of the vector A which represents the vector sum of the x, y, and z components. Since

$$A = i\left(\frac{\partial V}{\partial x}\right) + j\left(\frac{\partial V}{\partial y}\right) + k\left(\frac{\partial V}{\partial z}\right) \tag{14-2}$$

the magnitude of A is equal to the square root of the sum of the squares of the coefficients of the unit vectors. The direction of A may be readily expressed in terms of its direction cosines l, m, and n. It is readily seen that

$$l = \left(\frac{\partial V}{\partial x}\right)\left[\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right]^{-\frac{1}{2}}$$

Similar expressions may be written for m and n.

The vector A is called the *gradient* of the scalar function V. The *gradient* of a scalar function expresses the direction and the magnitude of the greatest change of that function per unit distance.

Because of the common use of the operation of finding the rate of change of scalar and also vector quantities, as indicated for the scalar function V in Eq. (14-2), it has been found convenient to abbreviate the expression by introducing the symbol ∇ (read del). It is expressed as follows:

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \tag{14-3}$$

and may be treated as a vector quantity. ∇ is frequently called an operator.

In the particular case we may write

$$A = \nabla V \tag{14-4}$$

and regard the right-hand member as a product of a vector and a scalar quantity, which, if expanded, would give the right-hand member of Eq. (14-2).

If V is to represent the potential in the region of a gravitating mass, the vector A as defined by Eq. (14-4) gives the negative of the field intensity. It follows, from Eq. (10-27) if we write G for the vector field intensity, that

$$G = -\nabla V \tag{14-5}$$

This result may be generalized. If there exists a scalar field in which there is a potential which is single valued, finite, and continuous, the negative gradient of the potential will give the field intensity. Applications of this process are to be found in gravitational, electrostatic, and magnetic fields.

Illustration.—Find the field intensity at any point in the region near a thin rod, first by determining the general expression for the potential and then by applying the operator ∇ to the potential as indicated in Eq. (14-5).

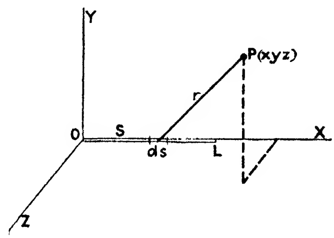

FIG. 134.

Let the rod be of length L and of linear density ρ and, for convenience, let it be placed on the axis OX of the reference system with one end at O. We are first to find the potential at any point P $(x\ y\ z)$ as shown in Fig. 134. If ds is a differential length of the rod which is at a distance r from P, and at a distance s from O, then the general expression for the potential at P is

$$\begin{aligned} V &= -\int_0^L K\rho\left(\frac{ds}{r}\right) \\ &= -K\rho\int_0^L [(x - s)^2 + y^2 + z^2]^{-\frac{1}{2}}ds \\ &= -K\rho \log \frac{\sqrt{x^2 + y^2 + z^2 - 2xL + L^2} + L - x}{\sqrt{x^2 + y^2 + z^2} - x} \end{aligned} \tag{14-6}$$

This result holds for any point P $(x\ y\ z)$ in the field. To obtain a numerical value of the potential for any particular point, all that is necessary to do is to substitute the values of the coordinates of that point in the given expression.

We may now find the general expression for the field intensity at any point P $(x\ y\ z)$ by finding the negative of the gradient as is expressed by Eq. (14-5). The process is indicated as follows:

$$G = -\left[i\frac{\partial V}{\partial x} + j\frac{\partial V}{\partial y} + k\frac{\partial V}{\partial z}\right]$$

The results of the indicated differentiations are more readily obtained if we use the following abbreviations:

$$A = \sqrt{x^2 + y^2 + z^2 - 2xL + L^2} + L - x$$
$$B = \sqrt{x^2 + y^2 + z^2 - 2xL + L^2}$$
$$C = \sqrt{x^2 + y^2 + z^2} - x$$
$$D = \sqrt{x^2 + y^2 + z^2}$$

The final result may be then written:

$$G = iK\rho\frac{x(AB - CD) + D\,(CL + BC + AB)}{ABCD} + jK\rho\frac{y\,(AB - CD)}{ABCD} + kK\rho\frac{z\,(AB - CD)}{ABCD} \quad (14\text{-}7)$$

This expression gives the field intensity in terms of the vector sum of the three components along the reference axes. If the intensity is desired at some definite point, it may be found by substituting the given coordinates of that point in Eq. (14-7). The square root of the sum of the squares of the coefficients of the unit vectors gives the magnitude of the intensity.

14-3. The Divergence of a Vector.—We have seen how the operator ∇ is applied to a scalar quantity in the determination of the gradient, and that a vector quantity, which expresses the direction and magnitude of the rate of greatest increase of the scalar, is obtained. In this section the same operator is to be applied to a vector quantity by forming the scalar product of ∇ and the given vector. The result obtained from this operation is to be examined and its significance interpreted.

If we let F represent the given vector, then the scalar product of ∇ and F may be expanded as follows:

$$\nabla \cdot F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot (i\,F_x + j\,F_y + k\,F_z)$$
$$= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \quad (14\text{-}8)$$

Each term of the right-hand member is a scalar quantity. The quantity $\nabla \cdot F$ is then a scalar quantity, as it should be, for we have formed a scalar product of two vectors.

The significance of this result is to be determined. Before we do this, a digression is made to explain the term *flux of field intensity*. The meaning of the term flux of field intensity, as used here, is similar to its meaning in electric or magnetic fields. If we imagine a uniform field (gravitational, electric, or magnetic) in which the field intensity has a value of unity, *i.e.*, 1 dyne per unit mass, positive charge or north pole, in the c.g.s. system, then this field intensity may be represented conventionally by assigning a single line of force to each unit area (square centimeter) of surface perpendicular to the field. If the field intensity is of strength n, then n lines would be assigned to each unit surface which is perpendicular to the field. This assignment is a purely conventional matter. As an extension of this convention, $4\pi K$ lines of force are assigned to each unit mass and 4π lines to each unit charge or pole. The reason for this assignment is easy to see if we take the surface integral of the field intensity over the surface of a sphere of unit radius which encloses a unit spherical mass particle placed concentric with the spherical surface. With this arrangement the field intensity on the spherical surface is everywhere perpendicular to the spherical surface and has a constant value of $-K$ over the entire surface. The total number of lines of force passing through the surface is called the flux and is found as follows:

$$\begin{aligned}\text{Flux} &= \iint_s -K\,ds \\ &= -4\pi K \qquad (14\text{-}9)\end{aligned}$$

in which ds is the differential element of surface. Because the result of this integration yields $4\pi K$, it has been convenient to adopt the convention of assigning $4\pi K$ lines to each unit mass. Since field intensity has a direction, the convention goes a little further and states that in electric and magnetic fields the lines come out of positive charges and north poles and go into negative charges and south poles. In the gravitational case we cannot make this distinction, because there are only attractive forces. The minus sign in the preceding result is to be understood as meaning that the flux is entering the surface. A positive result in electric and magnetic fields would indicate that the flux is leaving the surface.

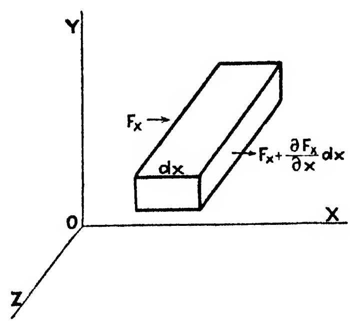

FIG. 135.

Let us determine the excess of flux which comes out of a differential element of volume $dx\,dy\,dz$ (as shown in Fig. 135) over that which enters when placed in a gravitational field.

The flux entering the face parallel to YZ nearer the origin is $F_x\,dy\,dz$, where F_x is the flux intensity or number of lines per square centimeter of surface perpendicular to F_x.

The flux leaving the opposite face would be

$$\left[F_x + \left(\frac{\partial F_x}{\partial x}\right) dx\right] dy\,dz$$

The excess of flux parallel to X would therefore be

$$\left[F_x + \left(\frac{\partial F_x}{\partial x}\right) dx\right] dy\,dz - F_x\,dy\,dz = \left(\frac{\partial F_x}{\partial x}\right) dx\,dy\,dz$$

In a similar manner it may be shown that the excess of flux emanating from the two other pairs of surfaces would be

$$\left(\frac{\partial F_y}{\partial y}\right) dx\,dy\,dz \qquad \text{and} \qquad \left(\frac{\partial F_z}{\partial z}\right) dx\,dy\,dz$$

Since these are all scalar quantities, the total excess of flux leaving the elemental volume over that entering is

$$\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dx\,dy\,dz$$

If we divide this expression by the volume of the element, *viz.*, $dx\,dy\,dz$, we obtain the excess of flux which would emanate from a unit volume at the point where the differential volume is situated.

This quantity has been defined as the *divergence* of $\boldsymbol{F}$ (div. $\boldsymbol{F}$) and may be written as follows:

$$\nabla \cdot \boldsymbol{F} = \text{div. } \boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \qquad (14\text{-}10)$$

The divergence of the field intensity at a given point gives the excess amount of flux, or number of lines of force leaving the surface of a unit volume enclosing the given point over that entering.

The process of finding the amount of mass in a given region in which the field intensity is known involves a determination of the divergence at any point in the region, multiplying this result by the differential volume surrounding the point, then integrating over the entire region, and dividing the result by $-4\pi K$.

In the derivation just given for obtaining the divergence of a vector, a reference system was selected with axes perpendicular and parallel to the edges of the differential parallelopiped (Fig. 135). The result obtained is not dependent upon the particular reference system selected, as may be shown. The divergence of a vector is invariant to a change of the reference system. The proof for this statement may be established by a transformation of axes, but the details of the process are long and uninteresting.

14-4. Applications of the Operator ∇.—For some of the work which follows, it will be convenient to evaluate some of the more commonly encountered expressions which contain ∇. The illustrations given will show the procedures to be followed in such expansions as well as to provide useful relations. We shall first evaluate some expressions in which ∇ is to differentiate a scalar quantity and then expand expressions in which ∇ is applied to a vector.

We have shown in Sec. **14-2** the use of ∇ in determining the gradient of a scalar quantity, such as potential, and have seen that a vector quantity, which gives the direction and magnitude of the maximum space rate of change of the scalar as a function of the coordinates, is thereby obtained. In some of the present illustrations we shall use the scalar quantity r which measures the distance of any point from the origin of a selected reference system. The scalar quantity r may be expressed in terms of the coordinates $(x\ y\ z)$ of the particular point as follows: $r^2 = x^2 + y^2 + z^2$. The vector quantity $\mathbf{r}$ is $r\mathbf{r}_1$ where $\mathbf{r}_1$ is the unit vector in the direction of $\mathbf{r}$ and, in terms of the unit vector $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, may be written

$$\mathbf{r} = r\mathbf{r}_1 = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$

The expansions in the following cases are written in detail in order that the procedure may be clear.

(*a*) ∇r:

$$\begin{aligned} \nabla r &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{\frac{1}{2}} \\ &= \frac{\mathbf{i}\,x + \mathbf{j}\,y + \mathbf{k}\,z}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} \\ &= \frac{\mathbf{r}}{r} \\ &= \mathbf{r}_1 \end{aligned} \tag{14-11}$$

(b) ∇r^n:

$$\nabla r^n = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{\frac{n}{2}}$$
$$= n\,(i\,x + j\,y + k\,z)(x^2 + y^2 + z^2)^{\frac{n}{2}-1}$$
$$= \frac{n}{r}(i\,x + j\,y + k\,z)(x^2 + y^2 + z^2)^{\frac{n}{2}-\frac{1}{2}}$$
$$= n\,r_1\,r^{n-1} \qquad \text{or} \qquad n r\,r^{n-2} \tag{14-12}$$

(c) $\nabla\frac{1}{r}$:

$$\nabla\frac{1}{r} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{-\frac{1}{2}}$$
$$= -(i\,x + j\,y + k\,z)(x^2 + y^2 + z^2)^{-\frac{3}{2}}$$
$$= -\frac{r}{r^3} \qquad \text{or} \qquad -\frac{r_1}{r^2} \tag{14-13}$$

This result may also be obtained from (b) by putting $n = -1$.

(d) $\nabla\,(A \cdot B)$.—Notice that A and B are vectors, but their product $A \cdot B$ is a scalar quantity. Differentiation by ∇ in this case follows the rules of ordinary differentiation, in that ∇ is to operate upon each vector separately while the other remains constant. The procedure is indicated symbolically as follows:

$$\nabla\,(A \cdot B) = \nabla\,(A \cdot B)_B + \nabla\,(A \cdot B)_A$$
$$= (B \cdot \nabla)A + (A \cdot \nabla)B$$

In the first expression the subscripts to the right of the parentheses indicate which vector is to remain constant. The vectors A and B may be expressed in terms of their components A_x, A_y, A_z and B_x, B_y, B_z, respectively, as follows:

$$A = iA_x + jA_y + kA_z \qquad B = iB_x + jB_y + kB_z$$

Introducing these values in the expression for $\nabla(A \cdot B)$ gives

$$\nabla(A \cdot B) = \left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)(iA_x + jA_y + kA_z)$$
$$+ \left(A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right)(iB_x + jB_y + kB_z) \tag{14-14}$$

A further reduction of this expression is not possible except in special cases. If A_x, A_y, A_z (or B_x, B_y, B_z) are functions only of x, y and z respectively, then a simplification may be made.

(e) $\nabla \cdot r$:

$$\nabla \cdot r = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot (i\,x + j\,y + k\,z)$$
$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$
$$= 3 \tag{14-15}$$

(f) $\nabla \cdot (rA)$, in which r is a scalar function and A is a vector function of the coordinates. In this expression ∇ is to differentiate each factor.

$$\nabla \cdot (rA) = (\nabla r) \cdot A + r\,(\nabla \cdot A)$$
$$= r_1 \cdot A + r\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) \tag{14-16}$$

(g) $\nabla \cdot r_1$

$$\nabla \cdot r_1 = \nabla \cdot \left(\frac{r}{r}\right)$$
$$= \left(\nabla\frac{1}{r}\right) \cdot r + \frac{1}{r}\nabla \cdot r$$
$$= -\frac{r_1 \cdot r}{r^2} + \frac{3}{r}$$
$$= -\frac{1}{r} + \frac{3}{r} = \frac{2}{r} \tag{14-17}$$

(h) Other formulas which may be of use are written below without including the detailed proofs by expansion. In these expressions, a and b are scalar functions and A and B are vector functions.

$$\nabla(a + b) = \nabla a + \nabla b \tag{14-18}$$
$$\nabla \cdot (A + B) = \nabla \cdot A + \nabla \cdot B \tag{14-19}$$
$$\nabla(ab) = a\,(\nabla b) + b(\nabla a) \tag{14-20}$$

14-5. Illustration of a Use of Divergence.—In this section we shall make use of the divergence of G, to determine the mass of a body; in particular, the mass of a sphere. For this purpose we must assume that the field intensity at any point within the sphere is known.

Suppose that the radius of the sphere is R and that the field intensity G at any point *within* the sphere is expressed by the relation

$$G = -K\,\mu\,\pi\,r^2 r_1$$

where K and μ are constants and r is distance of the point at which G is given from the center of the sphere.

The procedure to be followed in a determination of the mass of the sphere consists of three steps: first, finding the divergence of G; second, integrating the divergence over the entire volume of the sphere; and, third, since the result of the volume integration of the divergence gives the total flux, dividing the flux by $-4\pi K$.

The divergence of G is found by the following manner:

$$\begin{aligned}\text{Div. } G &= \nabla \cdot (-K\mu\pi\, r^2 \mathbf{r}_1) \\ &= -K\mu\pi\, \nabla \cdot (r^2 \mathbf{r}_1) \\ &= -K\mu\pi[(\nabla r^2) \cdot \mathbf{r}_1 + r^2\, \nabla \cdot \mathbf{r}_1] \\ &= -4K\mu\pi\, r \qquad (14\text{-}21)\end{aligned}$$

This result gives the flux or number of lines per unit volume which enter a differential mass at the point at which G is given. Expressed differently, the divergence of G gives the flux entering a unit volume enclosing the point if div. G were constant throughout the unit volume. But div. G is not constant over the region and we must multiply the result obtained above by dv, a differential volume surrounding the point and integrate over the entire region.

Since r is the only variable in the expression for div. G, we may use for dv a spherical shell which is of thickness dr and of radius r and whose center coincides with the center of the sphere. Hence $dv = 4\pi r^2 dr$. The total flux entering the sphere is found as follows:

$$\begin{aligned}\text{Flux} &= \iiint_{\text{volume}} \text{div. } G\, dv \\ &= -16\, K\mu\pi^2 \int_0^R r^3 dr \\ &= -4\, K\mu\pi^2 R^4 \qquad (14\text{-}22)\end{aligned}$$

The mass of the sphere may now be readily determined by dividing the total flux by $-4\pi K$, since $-4\pi K$ lines have been assigned to a unit mass. If M is the mass of the sphere,

$$\begin{aligned}M &= \frac{\text{flux}}{-4\pi K} \\ &= \mu\pi R^4 \qquad (14\text{-}23)\end{aligned}$$

This is the mass of a sphere of radius R in which the density is not uniform but is equal to μr, as may be readily verified.

Problem.—Find the mass of a sphere in which the field intensity has the value $-\frac{4}{3}K\rho\pi \boldsymbol{r}$ where K and ρ are constants and $\boldsymbol{r}$ is the radius vector from the center of the sphere to the point at which $\boldsymbol{G}$ is given.

14-6. The Gauss Integral.—There are two ways of determining the excess of flux which may enter or leave a given closed surface in a vector field and therefore, in the gravitational case, of determining the amount of mass which lies within the closed surface. One method makes use of the volume integral of the divergence over the entire region, as is expressed by Eq. (14-22) and the other depends upon a surface integral of the normal component of flux intensity over the entire surface. It is not necessary to go to the extent of introducing a rigid mathematical proof to establish the validity of equating these two integrals. For present purposes it is sufficient to say that it is certainly reasonable that the excess flux leaving or entering a closed surface completely surrounding a given region must be equal to the algebraic sum of the excesses of flux leaving or entering all of the infinitesimal elements of volume which completely fill the boundary surface.

This relation is expressed by the following equation:

$$\iint_{\text{surface}} \boldsymbol{n}_1 \cdot \boldsymbol{F}\, ds = \iiint_{\text{volume}} \text{div. } \boldsymbol{F}\, dv \tag{14-24}$$

where $\boldsymbol{n}_1$ is a unit vector perpendicular to the surface element ds, $\boldsymbol{F}$ is the field intensity at that point, and dv is the differential volume. This equation is known as the Gauss theorem.

As a simple illustration of the equality of the two integrals given in Eq. (14-24), let us put the vector $\boldsymbol{F}$ equal to the vector $\boldsymbol{r}$ which expresses the position of any point with reference to some fixed point as origin, and let the region be a sphere of radius R with its center at the origin. We shall evaluate each integral separately and by comparing the results so obtained test the validity of the equation.

Since $\boldsymbol{r} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z$, we may write

$$\begin{aligned}\text{Div. } \boldsymbol{r} &= \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right) \cdot (\boldsymbol{i}\,x + \boldsymbol{j}\,y + \boldsymbol{k}\,z) \\ &= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \\ &= 3\end{aligned} \tag{14-25}$$

Substituting this result in the volume integral gives

$$\iiint_{\text{volume}} \text{div. } \boldsymbol{r}\, dv = 3 \iiint_{\text{volume}} dv$$
$$= 4\pi R^3 \qquad (14\text{-}26)$$

If the surface integral is taken over the surface of the sphere, then $\boldsymbol{r}$ is everywhere perpendicular to the surface and of uniform magnitude all over the surface. Hence

$$\iint \boldsymbol{n}_1 \cdot \boldsymbol{r}\, ds = r \iint_{\text{surface}} ds$$
$$= 4\pi R^3 \qquad (14\text{-}27)$$

which is equal to the result obtained for the volume integral.

Problem.—Test the validity of the Gauss theorem by evaluating each integral of Eq. (14-24), in the case where the field intensity at any point in the interior or on the surface of a sphere of radius R is given by the relation

$$\boldsymbol{G} = -\tfrac{4}{3}K\mu\pi\, r^3 \boldsymbol{r}_1$$

14-7. Poisson's and Laplace's Equations.—The operator ∇ has been applied to potential for the determination of the field intensity and has also been applied to field intensity to find the divergence. It follows therefore, since $\boldsymbol{G} = -\nabla V$, that

$$\begin{aligned} \text{Div. } \boldsymbol{G} &= \nabla \cdot \boldsymbol{G} \\ &= -\nabla \cdot (\nabla V) \\ &= -\nabla^2 V \\ &= -\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) \end{aligned} \qquad (14\text{-}28)$$

In the gravitational case the divergence gives the number of lines or the flux per unit volume entering a differential volume at the point at which the divergence is expressed. We may connect this result with the mass in the unit volume and hence the density by writing $-4\pi K\rho$ for the number of lines coming to ρ g. of matter per unit volume. Hence

$$\text{Div. } \boldsymbol{G} = -4\pi K\rho$$

and

$$\nabla^2 V = +4\pi K\rho$$

or

$$\nabla^2 V - 4\pi K\rho = 0 \qquad (14\text{-}29)$$

which is *Poisson's* equation for gravitational fields.

In particular, if $\rho = 0$, we have

$$\nabla^2 V = 0 \tag{14-30}$$

which is *Laplace's* equation.

It is to be noticed that in the gravitational case the lines of flux *enter* the mass, an assumed characteristic of mass; consequently the sign of the result obtained by applying the divergence operation to gravitational fields will be negative. In the electrical or magnetic cases, however, the sign may be either positive or negative according to the sign of the charge or pole within the region to which the operation is applied. Hence in electric and magnetic fields Poisson's equation is written

$$\nabla^2 V + 4\pi\rho = 0 \tag{14-31}$$

In hydrodynamics or thermodynamics the flux may be a flow of fluid or heat out of the region or into it.

If it is found that flux is leaving a given region, the region is said to contain a "source." If the flux enters a given region, then a "sink" is present in that region.

Problems.—1. In free space outside a homogeneous sphere of mass M, the potential at any point is $-(KM/r)$ where r is the distance from the center of the sphere to the point. Show that $\nabla^2 V = 0$.

2. Inside a particular sphere the potential may be expressed by the relation $V = \frac{1}{3}K\mu\pi r^3$ where K and μ are constants and r is the distance from the center of the sphere to the point at which the potential is given. Find the expression for the density of the sphere by the use of Poisson's equation.

14-8. The Curl of a Vector.—The *curl* of a vector is a function of the vector obtained also by applying the operator ∇ to that vector. Unlike divergence, which is obtained by taking the scalar product of ∇ and the given vector, curl is found from a vector product of the two vectors, one of which is ∇ and the other is the vector to which the operator is applied. This operation gives a vector quantity, as is characteristic of the vector product of two vectors. The following equation may be used as a definition of the curl of a vector:

$$\text{Curl } \boldsymbol{F} = \nabla \times \boldsymbol{F}$$

Writing in terms of the components of the vectors gives

$$\text{Curl } F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times (i F_x + j F_y + k F_z)$$
$$= i\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + j\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + k\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$
$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \qquad (14\text{-}32)$$

This equation expresses the curl F in terms of the three components of a vector, the components being parallel to the axes of a selected reference system. The magnitudes of the components are given by the coefficients of the unit vectors in the foregoing equation. While the magnitudes of the components are individually affected by the particular reference system selected, the resultant vector is independent of this selection.

An illustration is introduced to show something about the meaning of the curl of a vector. For this purpose let us determine the curl of the velocity vector of any point of a rigid body which is moving with any general uniplanar motion, *i.e.*, a combination of translational with rotational motion. To be explicit, let the body be a right circular cylinder and let it be rolling down an inclined plane. Now to every point of the body there is a definite assignable velocity. We may therefore regard the velocities of all points of the body collectively as building up a vector field. If the curl is applied to the velocity of any point in this field, *i.e.*, to a point of the body, the result will be some function of V.

Let the reference system be selected so that the origin coincides, instantaneously, with the center of mass of the cylinder. The velocity V of any point P of the body may be expressed in terms of the vector sum of its velocity relative to the center of mass and the velocity of the center of mass of the cylinder in the fixed reference system [see Eq. 1-19)]. If the point P is at a distance r from the center of the cylinder and if ω is the instantaneous value of the angular velocity of the cylinder, then

$$V = U + \omega \times r \qquad (14\text{-}33)$$

If the curl is taken of this equation, we may express the operation in terms of ∇ as follows:

$$\text{Curl } V = \nabla \times U + \nabla \times (\omega \times r) \qquad (14\text{-}34)$$

Each term of the right-hand member is to be evaluated separately. The first term $\nabla \times U$ is equal to zero. In order to see that this is true, we may expand this vector product in terms of the components of the two vectors as indicated in Eq. (14-32) and thereby obtain

$$\nabla \times U = i\left(\frac{\partial U_z}{\partial y} - \frac{\partial U_y}{\partial z}\right) + j\left(\frac{\partial U_x}{\partial z} - \frac{\partial U_z}{\partial x}\right) + k\left(\frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y}\right) \qquad (14\text{-}35)$$

Since each velocity component of U is not a function of either coordinate which is measured perpendicular to it, then each term of the foregoing expression must be equal to zero. For example, U_x is not a function of either y or z; hence $\partial U_x/\partial y$ and $\partial U_x/\partial z$ must be zero.

The conclusion that $\nabla \times U$ is zero may be reached from a different viewpoint. We must remember that the application of the operator ∇ to a vector involves an operation which determines, in a peculiar way, just how that vector varies with respect to the coordinates of a particular point of the body at which the curl is to be found. In the case under consideration the vector U is one component of the velocity of each point of the body and is common to all points of the body. Therefore, since U does not change from point to point in the body, $\nabla \times U$ is zero.

The second term of the right-hand member of Eq. (14-34) involves a triple vector product. Since ∇ is one of the factors, the triple product may be expanded according to the general rule for such a product, provided we arrange for ∇ to operate upon both of the vectors in each term of the expanded expression. Hence

$$\begin{aligned} \nabla \times (\omega \times r) &= \omega(\nabla \cdot r) - r(\nabla \cdot \omega) \\ &= \omega(\nabla_r \cdot r) + (r \cdot \nabla_\omega)\omega \\ &\quad - r(\nabla_\omega \cdot \omega) - (\omega \cdot \nabla_r)r \end{aligned} \qquad (14\text{-}36)$$

In the last expression, subscripts are written to indicate upon which vector ∇ is to operate. In such expressions as appear in this equation in which ∇ is to operate upon only one of the vectors, the other vector is not to be differentiated by ∇ but is not necessarily constant. For example, in the expression $\omega(\nabla_r \cdot r)$, ∇ differentiates r only but the vector ω need not be a constant vector. The validity of this expression may be established by expressing each vector in terms of its components and carrying out the details of the multiplication and subsequent simplification in the usual manner.

In order to evaluate the expression, we shall examine each term and expand the indicated products where necessary. The first term may be written as follows:

$$\begin{aligned}\omega(\nabla_r \cdot r) &= \omega\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) \\ &= 3\omega\end{aligned}$$

The second and third terms are both equal to zero because ∇_ω is to operate only upon ω, and ω is not a function of the coordinates. The remaining term may be expanded in the following manner:

$$\begin{aligned}(\omega \cdot \nabla_r)r &= \left(\omega_x\frac{\partial}{\partial x} + \omega_y\frac{\partial}{\partial y} + \omega_z\frac{\partial}{\partial z}\right)(ix + jy + kz) \\ &= i\,\omega_x\frac{\partial x}{\partial x} + j\,\omega_y\frac{\partial y}{\partial y} + k\,\omega_z\frac{\partial z}{\partial z} \\ &= \omega\end{aligned}$$

Hence the final expression for the curl V may be written as follows:

$$\text{Curl } V = 2\,\omega \tag{14-37}$$

This result indicates that the curl V is simply a function of V, which, in this particular case, has the value 2ω.

Let us determine the curl of the velocity of any point of the cylinder rolling down an inclined plane by expressing the motion in a different way. It has been shown that the velocity of every point of the cylinder, which is rolling down an inclined plane, may be expressed in terms of an angular velocity and the distance of the point from the instantaneous axis of rotation. The instantaneous axis, in this particular case, is the line of contact of the cylinder with the inclined plane. Let the origin of the reference system be on the instantaneous axis. If ω be the angular velocity of the cylinder and r be the radius vector drawn from the origin to the particular point of the cylinder, then the instantaneous linear velocity V of the point is $\omega \times r$.

The curl V may be found by proceeding as indicated in Eq. (14-36). The result is $2\,\omega$ as before, for the curl must obviously be independent of the manner in which the motion is described. Aside from the absence of U in the second method of describing the velocity of a point of the cylinder, the two cases differ only in the selection of the origin from which the vector r is to be measured. That this difference is immaterial in the particular illustration is indicated by the equivalence of the results.

The vector field just described is typical of those vector fields which have a curl the value of which is not zero.

In order to present a more complete view of the application of the operator $\nabla\times$ to a vector, it will be helpful to discuss two aspects of the illustrations elected. We shall first make some comments regarding the selection of the reference system to be used in any given problem and then, by a special selection of the reference system, show analytically what the curl of a vector really expresses.

The selection of a reference system is frequently as important in dealing with vectors which involve the operator $\nabla\times$ as it is in writing analytical expressions. Although the curl of a vector is independent of the particular reference system selected, convenience may be gained by a judicious selection, often clearly indicated by the peculiarities of the problem. In the case of the cylinder rolling down the inclined plane it was not necessary to select a reference system with the origin on the axis of the cylinder, but it was more convenient to do so. The reason for this selection is that the expression for the velocity of any point of the body includes a term $(\omega \times r)$ in which the factor r is to be measured from some point in the axis of the cylinder. If r is to be expressed as $ix + jy + kz$, as is necessary when applying $\nabla\times$ to it, then the XYZ system in which these coordinates are to be measured must have its origin at the same point which is origin for r. We could not, in general, express $r = ix + jy + kz$ in one reference system and then select another reference system in which the differentiations expressed by ∇ are to be made. If two such reference systems are fixed relatively to each other, the results obtained by applying $\nabla\times$ to r would be the same but the procedure is fundamentally incorrect.

As an illustration of the changes in procedure made necessary when using another reference system in a determination of curl V, we shall determine the curl of the velocity of any point of the cylinder by using a reference system so selected that the position of the point is defined by the vector $s = q + r$ where q gives the position of a point fixed in the axis of the cylinder and $\boldsymbol{r}$ is the position of the point with respect to a moving system whose axes are always parallel to the fixed system from which $\boldsymbol{s}$ is measured and whose origin is the terminal point of $\boldsymbol{q}$. Equation (14-34) may be used provided that we substitute for $\boldsymbol{r}$ its value $\boldsymbol{s} - \boldsymbol{q}$. Since $\nabla \times \boldsymbol{U}$ is zero as before, we have only to consider Eq. (14-36) in which the new value is written for $\boldsymbol{r}$. Wherever ∇ is to operate upon $\boldsymbol{r}$, it must now operate separately upon $\boldsymbol{s}$ and $\boldsymbol{q}$; hence the first term of the right member of Eq. (14-36) may be written

$$\begin{aligned} \omega[\nabla \cdot (\boldsymbol{s} - \boldsymbol{q})] &= \omega[\nabla_s \cdot (\boldsymbol{s} - \boldsymbol{q}) + \nabla_q \cdot (\boldsymbol{s} - \boldsymbol{q})] \\ &= 3\omega \end{aligned} \tag{14-38}$$

The last term of Eq. (14-36) may be handed in a similar manner with the final result for the curl $\boldsymbol{V}$ equal to 2ω as before.

An analytical discussion of the meaning of curl $\boldsymbol{V}$ may be given by using a special selection of the reference system. Suppose the reference system be placed so that its origin is in the axis of rotation of the cylinder and with the Z-axis parallel to the axis of rotation. Let us select a point P in the XY plane and examine curl $\boldsymbol{V}$ for this point.

We may write curl $\boldsymbol{V}$ as follows:

$$\nabla \times \boldsymbol{V} = \boldsymbol{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) \tag{14-39}$$

Since $\nabla \times \boldsymbol{V}$ is known to be 2ω, then it follows that the $\boldsymbol{i}$ and $\boldsymbol{j}$ components are both zero and we need examine only the $\boldsymbol{k}$ component.

The velocity $\boldsymbol{V}$ has only the two components V_x and V_y, both of which may be expressed in terms of the angular velocity ω and the coordinates of the point P as follows:

$$V_y = \omega\, x \qquad V_x = -\omega\, y$$

We shall first examine the meaning of $\partial V_y/\partial x$. If we let V_y' represent the y component of the velocity of a point Q which has the same y coordinate as P and is situated at a distance dx from P, then we may write

$$\frac{\partial V_y}{\partial x} = \frac{V_y' - V_y}{dx}$$

This equation defines the partial differential quotient $\partial V_y/\partial x$ as the space rate of change in the y component of the velocity of P at P, taken along the X-axis. The other differential quotient may be described in an analogous manner.

Since $V_y = \omega x$ and $V_x = -\omega y$ and ω is not a function of either x or y, then

$$\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = 2\omega$$

Hence the curl $\boldsymbol{V}$ may be written

$$\nabla \times \boldsymbol{V} = 2\boldsymbol{k}\omega = 2\boldsymbol{\omega} \tag{14-40}$$

If the axis of rotation had not been selected parallel to the Z-axis, then the velocity components in the XY plane would have to be written

$$V_y = \omega_z x \qquad V_x = -\omega_z y$$

and then the $\boldsymbol{k}$ component of the curl $\boldsymbol{V}$ would have been $2\boldsymbol{k}\omega_z$. In this case the two other components of curl $\boldsymbol{V}$ would not be zero but would be $2\boldsymbol{i}\omega_x$ and $2\boldsymbol{j}\omega_y$. Hence

$$\begin{aligned}\nabla \times \boldsymbol{V} &= 2\boldsymbol{i}\omega_x + 2\boldsymbol{j}\omega_y + 2\boldsymbol{k}\omega_z \\ &= 2\boldsymbol{\omega}\end{aligned}$$

14-9. Illustration—Curl of Moment of Momentum of a Rigid Body.—As a further illustration of applying the curl operation to dynamical quantities we shall find the curl of the moment of momentum of a homogeneous rigid body which is in a state of general uniplanar motion. Let the body be in rotation with angular velocity $\boldsymbol{\omega}$ about some axis which is fixed in the body. Let $\boldsymbol{r}$ be a vector which expresses the position of any differential mass dm in a reference system which has its origin on the selected axis of rotation and $\boldsymbol{V}$ be the velocity of dm. The moment of momentum $\boldsymbol{R}$ is given by the following equation:

$$\boldsymbol{R} = \int_m dm\,(\boldsymbol{r} \times \boldsymbol{V}) \tag{14-41}$$

in which the integration is to be extended over the entire mass.

It is to be observed that we have used the term moment of momentum to describe the quantity expressed by the preceding expression. Ordinarily this term is used to describe the moment of momentum with reference to a line—the axis of rotation. In this case the quantity is written with reference to a point, which is selected as origin. If the moment were referred to a line, the integral would be equal to $I\omega$. The quantity selected, however, is more suitable for the present situation.

We shall determine curl $\boldsymbol{R}$ first by expanding the vectors and then by the shorter method of direct vector analysis.

a. By Expansion of the Vectors.—If we write $\boldsymbol{r}$ and $\boldsymbol{V}$ in terms of their components,

$$\boldsymbol{r} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z$$
$$\boldsymbol{V} = \boldsymbol{i}V_x + \boldsymbol{j}V_y + \boldsymbol{k}V_z$$

and then expand the vector product, the magnitudes of the components of rotational momentum, taken along the reference axes, may be expressed as follows:

$$R_x = \int_m dm\ (yV_z - zV_y)$$
$$R_y = \int_m dm\ (zV_x - xV_z)$$
$$R_z = \int_m dm\ (xV_y - yV_x) \qquad (14\text{-}42)$$

Each component of the linear velocity of dm may be expressed in terms of the components of the angular velocity $\boldsymbol{\omega}$ and of $\boldsymbol{r}$. In general,

$$\boldsymbol{V} = \boldsymbol{\omega} \times \boldsymbol{r} = (\boldsymbol{i}\omega_x + \boldsymbol{j}\omega_y + \boldsymbol{k}\omega_z) \times (\boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z)$$

hence the magnitudes of the component velocities are

$$V_x = \omega_y z - \omega_z y$$
$$V_y = \omega_z x - \omega_x z$$
$$V_z = \omega_x y - \omega_y x$$

If we substitute these expressions in Eqs. (14-42), the components of the moment of momentum may be obtained. The x component only is written here because the two others are easily written by a cyclic interchange of the variables.

$$R_x = \int_m dm\ [y\ (\omega_x y - \omega_y x) - z\ (\omega_z x - \omega_x z)]$$

We may now determine the curl of the moment of momentum. This is expressed in general terms as follows:

$$\text{Curl } \boldsymbol{R} = \nabla \times \boldsymbol{R}$$
$$= \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right) \times (\boldsymbol{i}R_x + \boldsymbol{j}R_y + \boldsymbol{k}R_z)$$
$$= \boldsymbol{i}\left(\frac{\partial R_z}{\partial y} - \frac{\partial R_y}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial R_x}{\partial z} - \frac{\partial R_z}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial R_y}{\partial x} - \frac{\partial R_x}{\partial y}\right) \qquad (14\text{-}43)$$

The x component of the curl may be written $\text{curl}_x \boldsymbol{R}$. It is necessary to determine the value of only one component, for the two others may then be written by cyclic permutation. If we carry out the indicated partial differentiations by differentiating expressions for R_z and R_y under the integral signs and remember that the components of the angular velocity are constant and that dm is not a function of the coordinates, since the body is homogeneous, we may write

$$\begin{aligned}
\text{Curl}_x \boldsymbol{R} &= \boldsymbol{i}\left(\frac{\partial R_z}{\partial y} - \frac{\partial R_y}{\partial z}\right) \\
&= \boldsymbol{i}\int_m dm[(2\omega_z y - \omega_y z) - (2\omega_y z - \omega_z y)] \\
&= \boldsymbol{i}\int_m 3dm(\omega_z y - \omega_y z) \\
&= \boldsymbol{i}\int_m 3dm(-V_x)
\end{aligned} \tag{14-44}$$

One of the limitations imposed in this problem was an assumption of rigidity of the body. It should be observed that the effect of this limitation is to make the components of $\boldsymbol{\omega}$ independent of the coordinates. The result just obtained is not true for a nonrigid body.

Similarly, the values for the two other components of the curl are

$$\text{Curl}_y \boldsymbol{R} = \boldsymbol{j}\int_m 3\, dm\, (-V_y)$$
$$\text{Curl}_z \boldsymbol{R} = \boldsymbol{k}\int_m 3\, dm\, (-V_z)$$

These three components may now be put together to give the final result, which is

$$\begin{aligned}
\text{Curl}\ \boldsymbol{R} &= \boldsymbol{i}\int_m 3\, dm\, (-V_x) + \boldsymbol{j}\int_m 3dm\, (-V_y) + \boldsymbol{k}\int_m 3dm\, (-V_z) \\
&= -3\int_m dm\, \boldsymbol{V} \quad \text{or} \quad -3m\boldsymbol{U}
\end{aligned} \tag{14-45}$$

where m is the mass of the body and $\boldsymbol{U}$ is the velocity of its center of mass.

This result indicates that the curl of the rotational momentum of the body may be expressed in terms of the linear momentum of the body.

b. By Direct Vector Methods.—In order to determine the curl of the moment of momentum by direct vector methods, it will be necessary to introduce two additional formulas which involve the operator ∇. If a be a scalar function of the coordinates and $\boldsymbol{C}$ and $\boldsymbol{D}$ be any two vectors which are also functions of the coordinates, then it may be shown that

$$\nabla \times (a\boldsymbol{C}) = (\nabla a) \times \boldsymbol{C} + a \nabla \times \boldsymbol{C} \tag{14-46}$$

$$\nabla(\boldsymbol{C} \cdot \boldsymbol{D}) = \nabla_C(\boldsymbol{C} \cdot \boldsymbol{D}) + \nabla_D(\boldsymbol{C} \cdot \boldsymbol{D}) \tag{14-47}$$

The curl of $\boldsymbol{R}$, the moment of momentum of the body, may be written symbolically as follows:

$$\text{Curl } \boldsymbol{R} = \int_m dm \; \nabla \times [\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})]$$

Normally the operator $\nabla \times$ should be placed outside the integration sign but in this particular case selected for illustration the body was assumed to be homogeneous and hence dm is not a function of the coordinates. In this case the introduction of ∇ within the integration sign is valid.

The triple vector product within the brackets may be expanded by the ordinary formula; hence

$$\text{Curl } \boldsymbol{R} = \int_m dm \; \nabla \times [\boldsymbol{\omega}\, r^2 - \boldsymbol{r}\,(\boldsymbol{\omega} \cdot \boldsymbol{r})]$$

By the use of Eq. (14-46) the first term may be evaluated as follows:

$$\begin{aligned} \nabla \times (\boldsymbol{\omega}\, r^2) &= (\nabla r^2) \times \boldsymbol{\omega} + r^2(\nabla \times \boldsymbol{\omega}) \\ &= [\nabla(x^2 + y^2 + z^2)] \times \boldsymbol{\omega} + 0 \\ &= 2\,\boldsymbol{r} \times \boldsymbol{\omega} \qquad \text{or} \qquad -2\,\boldsymbol{\omega} \times \boldsymbol{r} \end{aligned}$$

In applying $\nabla \times$ to the second term, we regard $\boldsymbol{\omega} \cdot \boldsymbol{r}$ as a scalar and hence, using Eqs. (14-46) and (14-47), we obtain

$$\begin{aligned} \nabla \times [\boldsymbol{r}(\boldsymbol{\omega} \cdot \boldsymbol{r})] &= [\nabla(\boldsymbol{\omega} \cdot \boldsymbol{r})] \times \boldsymbol{r} + \boldsymbol{\omega} \cdot \boldsymbol{r}(\nabla \times \boldsymbol{r}) \\ &= \nabla_\omega(\boldsymbol{\omega} \cdot \boldsymbol{r}) \times \boldsymbol{r} + \nabla_r(\boldsymbol{\omega} \cdot \boldsymbol{r}) \times \boldsymbol{r} + 0 \\ &= \left[\left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right)\left(x\omega_x + y\omega_y + z\omega_z\right)\right] \times \boldsymbol{r} \\ &= \boldsymbol{\omega} \times \boldsymbol{r} \end{aligned}$$

Hence

$$\text{Curl } \boldsymbol{R} = -\int_m 3dm\, \boldsymbol{\omega} \times \boldsymbol{r} = -\int_m 3dm\boldsymbol{V} \tag{14-48}$$

The integral in the right-hand member of this equation represents the total linear momentum of the body. As a general conclusion we may, therefore, say that the curl of the moment of momentum of a rigid homogeneous body in a state of uniplanar motion is equal to the negative of three times the linear momentum of the body.

Problem.—If the body described above were rotating about a fixed axis, containing the center of mass, curl $\boldsymbol{R}$ would be zero. Which expression in the analytical or vector evaluation of curl $\boldsymbol{R}$ would indicate this fact?

14-10. Stokes's Theorem.—This theorem which involves the curl of a vector is analogous to the Gauss theorem which involves the divergence of a vector. Stokes's theorem may be stated in the following manner: the line integral of a vector, say $\boldsymbol{V}$, taken around any closed path is equal to the surface integral of the normal component of the curl of that vector over the surface which is bounded by that closed path. Symbolically it is expressed by the equation

$$\int_L \boldsymbol{V} \cdot d\boldsymbol{s} = \int_A\int \boldsymbol{n}_1 \cdot \text{curl } \boldsymbol{V}\, da \qquad (14\text{-}49)$$

In the expression for the line integral, the scalar product of the vector and the element of length $d\boldsymbol{s}$ of the path is to be integrated over the entire closed path. In the surface integral the normal component of the curl of $\boldsymbol{V}$ is indicated by writing the scalar product of $\boldsymbol{n}_1$, the unit vector drawn to the element of surface da, and curl $\boldsymbol{V}$. In evaluating the surface integral the normal component of the curl of $\boldsymbol{V}$ is to be found. Then, by writing $dx\, dy$ or an equivalent expression for da, the integral may be determined.

One use of the theorem is to prove whether or not the curl of the vector is zero. Those fields in which the curl is zero are spoken of as curl-free fields. The value of the curl of a vector at a given point may also be determined by the use of the line integral. This is done by enclosing the point at which the curl is to be found by a very small path which includes a differential surface over which the curl may be regarded as being constant. If then the line integral of the vector is determined and its value is divided by the area of the differential surface, the component of the curl perpendicular to that surface is obtained. By using three such differential surface elements mutually perpendicular to each other, three rectangular components of the curl are found from which the resultant curl is easily obtained. In many cases it is easy to determine by inspection in which direction the resultant curl lies and hence a single line integral may suffice.

We shall illustrate the use of Stokes's theorem as a means for finding the value of the curl of some vector at a given point. Let us select a disk which is rotating about an axis through its center but not necessarily perpendicular to the plane of the disk and determine the curl of the velocity of any point of the disk.

We shall determine the curl of the velocity at the point P (Fig. 136). In order to make the determination of the line integral easy, we shall let the differential surface $EBCD$ be bounded by two circular arcs ED and BC drawn with their centers coincident with the center of the disk and also bounded by portions of two radii, *viz.*, EB and DC. Let the mean length of the two arcs be s and the length of the longer arc be $s + \delta_1$, and that of the shorter arc be $s - \delta_2$. Let the length of the radius to the mean arc be r. Also let the radii to the two other arcs BC and DE be $r + \epsilon$ and $r - \epsilon$, respectively.

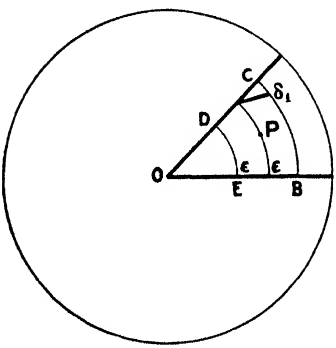

FIG. 136.

Let the origin of a reference system be placed at the center of the disk and let ω be the angular velocity of the disk. If Q is any point in arc BC and $(r + \epsilon)\mathbf{r}_1$ is the radius vector drawn to Q, then $\omega \times (r + \epsilon)\mathbf{r}_1$ is the velocity of Q. Similarly the velocity of any point in the arc ED is $\omega \times (r - \epsilon)\mathbf{r}_1$.

The area of the differential surface, which we may call A, is

$$A = \tfrac{1}{2}(s + \delta_1)(r + \epsilon) - \tfrac{1}{2}(s - \delta_2)(r - \epsilon) \qquad (14\text{-}50)$$

The integral of the velocity taken around the selected closed path may be divided into four parts, from E to B, from B to C, from C to D, and from D to E. In the line integral the component of the vector parallel to the element of the path is multiplied by the element of the path ds; hence the line integrals of the velocity along the lines EB and CD are equal to zero because the velocity of every point in these segments is perpendicular to corresponding elements of the path. There remain, then, but two integrals to be evaluated. If we let $\mathbf{ds}$ be the vector element of the path and $\mathbf{n}_1$ be the unit vector drawn normal to the disk, the normal component of the curl of V may be expressed as follows:

$$n_1 \cdot \text{curl } V = \frac{1}{A}\int_B^C [\omega \times (r + \epsilon)r_1] \cdot ds + \frac{1}{A}\int_D^E [\omega \times (r - \epsilon)r_1] \cdot ds$$
$$= \frac{r+\epsilon}{A}\int_B^C (\omega \times r_1) \cdot ds + \frac{r-\epsilon}{A}\int_D^E (\omega \times r_1) \cdot ds \qquad (14\text{-}51)$$

The triple vector product within each integral may be written $\omega \cdot (r_1 \times ds)$ [see Eq. (2-18)].

In the line integral taken from B to C, ds is positive and hence $r_1 \times ds = n_1 ds$; but in the integral taken from D to E, ds is negative and therefore according to the convention of signs for a vector product $r_1 \times ds = -n_1 ds$. The vectors ω and n_1 are common to all elements of the paths in the two integrals and hence $\omega \cdot n_1$ may be placed outside the integration signs. The expression for $n_1 \cdot$ curl V may then be evaluated as follows:

$$n_1 \cdot \text{curl } V = \frac{r+\epsilon}{A}\, \omega \cdot n_1 \int_B^C ds - \frac{r-\epsilon}{A}\, \omega \cdot n_1 \int_D^E ds$$
$$= \frac{\omega \cdot n_1}{A}[(r+\epsilon)(s+\delta_1) - (r-\epsilon)(s-\delta_2)]$$
$$= 2\,\omega \cdot n_1 \qquad (14\text{-}52)$$

This expression gives the component of the curl V which is perpendicular to the selected differential area. Ordinarily it is necessary to determine the two other components of curl V. In the particular case the second and third components of curl V are both zero and hence we obtain curl $V = 2\omega$.

Problems.—1. Show that Laplace's equation holds in the case of the gravitational potential at any point in the field of a sphere when the point at which the operation is applied lies outside the boundaries of the mass.

2. At any point in a gravitational field at which the potential is given by the expression $V = -KM/r$, find the x, y, and z components of the field intensity and show that their vector sum is equal to $-KMr_1/r^2$, where as usual r_1 is the unit vector in the line of r.

3. Prove that the curl of the field intensity at any point in space outside the boundaries of any gravitating mass is zero.

4. Prove that the curl of the gravitational field intensity at any point outside a homogeneous sphere is zero.

5. Find the divergence of the rotational momentum of a rigid homogeneous body at any point in the body. Formula:

$$\nabla \cdot (aC) = \nabla a \cdot C + a\nabla \cdot C$$

6. What is the gradient of $V = f(r)$ where $f(r)$ is any function of r?

CHAPTER XV

PROBLEMS ILLUSTRATING THE FUNDAMENTAL PRINCIPLES

15-1. Introduction.—In this chapter we have given a number of typical problems and their solutions. The primary purpose of presenting the material below is to provide the student with concrete subject matter which may be used in verifying the results of his own efforts. It is highly important for the student who desires to be analytical in his work to learn to scrutinize his procedure. He must carefully examine the tools which are available and he must discriminate in his selection of the tools which are to be used. Too frequently the student, when confronted by a problem, abandons all thought of a logical or methodical procedure and attempts to find the required "answer" by using the first suggestion which pops into his head. He may succeed and the average student will frequently succeed by this procedure as long as the problem contains no element which is essentially new to him. But such a student will, in all probability, be hopelessly lost in a mathematically uncharted region.

The writer has expressed himself upon this subject several times before in preceding chapters. It is only because he feels that it is of utmost importance for the student to regard the material here presented as being given primarily for the purpose of guiding him in the formation of habits of mind which are intellectually progressive, that he has insisted upon keeping this viewpoint in the foreground.

After reading the statement of a particular problem the student should give the text no further attention until he either has completed his solution or has found that, after a reasonable effort, he can make no headway. In the former case he should check his results with those given in the text. In the latter case he should read as much of the development given as may seem desirable before trying again. To facilitate a check of final results it is suggested that important physical or mathematical quantities be identified by those symbols used in the text.

15-2. The Rolling Cylinder.—A uniform right circular cylinder is permitted to roll from rest, without slipping, down an inclined plane. The friction may be taken as constant and sufficient to prevent slipping. Find an expression for the linear velocity in terms of the linear displacement of the cylinder from its initial rest position.

Let m and r be the mass and radius, respectively, of the cylinder and I the moment of inertia about the geometrical axis. Let α be the angle of inclination of the plane as shown in Fig. 137.

The first step in obtaining a solution is to identify the known and unknown quantities. This step is facilitated by making a diagram of the arrangement and including all of the forces which act upon the cylinder. In this particular problem there are three forces, the weight of the cylinder (mg) acting vertically downward, the reaction (R) of the plane upon the cylinder, a

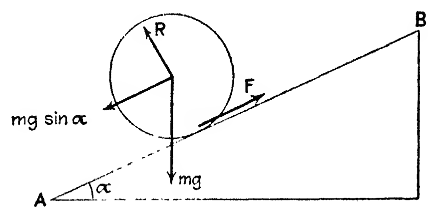

FIG. 137.

force which is perpendicular to the inclined plane, and the friction (F) which is parallel to the plane. It is apparent that the cylinder must move so that its center of mass will remain in a line which is parallel to the inclined plane. Since this is the case, we need concern ourselves only with those vector quantities which are parallel to the inclined plane. There are only two forces which may be regarded as being responsible for the motion, the component of weight ($mg \sin \alpha$) and friction (F). Since the resultant of these forces is parallel to the inclined plane, the acceleration of the center of mass is parallel to the inclined plane.

There are two unknown quantities, the acceleration of the center of mass of the cylinder and the force F. We must, therefore, write at least two fundamental equations. It is immaterial whether the quantity F is eliminated before or after integration of the fundamental equations.

We may use the following tools: the force equation, the work equation, and the force-moment equation. For purposes of illustration three solutions of the problem are given. Each solution is identified by the equation which is most prominent.

a. Force Equation.—Since the friction is assumed to be constant and just sufficient to prevent slipping, the force equation may then be written as follows:

$$mg \sin \alpha - F = m\frac{dV}{dt} \tag{15-1}$$

in which dV/dt is the acceleration of the center of mass.

This equation may be integrated directly. We may assume for initial conditions that, when $t = 0$, both V and s are zero, where V is the velocity and s is the linear displacement of the center of mass of the cylinder. Positive quantities being measured down the incline, the first two equations of motion are found to be

$$V = \left(g \sin \alpha - \frac{F}{m}\right)t \tag{15-2}$$

and

$$s = \frac{1}{2}\left(g \sin \alpha - \frac{F}{m}\right)t^2 \tag{15-3}$$

Eliminating the time factor gives the third equation; *i.e.*,

$$V^2 = 2\left(g \sin \alpha - \frac{F}{m}\right)s \tag{15-4}$$

With the three equations of motion containing the unknown quantity F, it is impossible to find numerical values of V corresponding to selected values of s. It will be necessary, therefore, to evaluate F in terms of the other constants which are assumed to be known or measurable. The effect of F upon the motion of the cylinder is to be examined. If R were equal to zero, the cylinder would slide down the inclined plane without rolling. If F is sufficient to prevent slipping, then F is responsible for the angular acceleration of the cylinder. The other force, $mg \sin \alpha$ acts through the center of mass and hence cannot contribute to the angular acceleration. Using the force-moment equation, we may write, for the effect of F, the following expression:

$$rF = I\,\omega\frac{d\omega}{d\gamma}$$

in which γ is the angle through which the cylinder rotates and ω is the angular velocity. Integration gives

$$rF\gamma = \tfrac{1}{2} I \omega^2$$

Since $r\gamma = s$ and $V = \omega r$, the value of F is found to be

$$F = \frac{IV^2}{2sr^2} \tag{15-5}$$

Substituting this value of F in Eq. (15-4) gives

$$V^2 = 2s\left(g \sin \alpha - \frac{IV^2}{2msr^2}\right) \tag{15-6}$$

which is the desired expression for the velocity of the center of mass of the cylinder after moving a distance from s.

If the cylinder is of uniform density, then its moment of inertia is $\frac{1}{2}mr^2$. Introducing this value for I in Eq. (15-6) gives

$$V^2 = \tfrac{4}{3}\, gs \sin \alpha \tag{15-7}$$

b. Work Equation.—To find an expression for V in terms of s and the known constants by means of the work equation, we must remember that, since the cylinder does not slide, the force F does no work. All of the work is done by the component of the weight of the cylinder which acts parallel to the displacement of the center of mass. Since the cylinder gains both translational and rotational kinetic energy, we may write

$$mgs \sin \alpha = \tfrac{1}{2}mV^2 + \tfrac{1}{2}I\omega^2 \tag{15-8}$$

assuming that the initial conditions are those given above.

If we again put $I = \frac{1}{2}mr^2$ and replace ω^2 by its equivalent expression V^2/r^2, this equation readily reduces to that written above [Eq. (15-7)]. From Eq. (15-8) one may readily find expressions for the velocity and the displacement in terms of the time, if these equations be desired.

c. Pure Rotation.—The motion of the cylinder may be regarded as one of pure rotation and from this standpoint the three equations of motion may be found. In order to write an equation expressing the motion as one of pure rotation, we must regard the cylinder as rotating about the instantaneous axis and then equate the moment of the force or forces about that axis to the product of the moment of inertia and the angular acceleration, both of the latter quantities being expressed about the instantaneous axis.

The instantaneous axis of rotation is the line of contact between the cylinder and the inclined plane. In the present problem the force of friction acts through the line of contact and hence contributes nothing to the total force moment. The force moment is due to the weight of the cylinder, is equal to $mg\ r\ \sin\alpha$, and remains constant during the motion. The moment of inertia of the cylinder about the instantaneous axis of rotation is $I + mr^2$. If $\omega\ d\omega/d\gamma$ is the angular acceleration, then the force-moment equation may be written as follows:

$$mg\ r\sin\alpha = (I + mr^2)\ \omega\frac{d\omega}{d\gamma} \tag{15-9}$$

Integration of this equation gives

$$mg\ r\ \gamma\ \sin\alpha = (I + mr^2)\tfrac{1}{2}\ \omega^2 \tag{15-10}$$

taking the initial condition as given above. This equation may be translated into the form given in Eq. (15-7) by substituting s and V for their equivalent rotational expressions.

Problem.—If a cylinder and a sphere, both homogeneous, are released simultaneously from rest upon an inclined plane, which will arrive at the bottom first if there is no slipping?

Is it necessary to assign relative values to the masses or radii or the two bodies?

15-3. The Falling Rod.—One end of a thin uniform rod is attached by a hinge to a fixed point. It is initially at rest in a vertical position with the free end above the fixed end. It is allowed to fall through a vertical plane. Find an expression for the angular velocity in any position and also the reaction exerted by the support upon the hinged end.

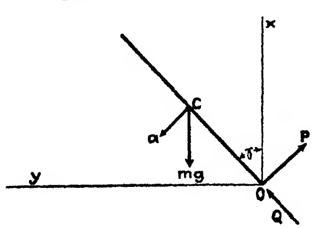

FIG. 138.

Let the plane of motion be that of the diagram of Fig. 138. We shall select the reference system XOY with OX taken vertically and with the origin O at the hinged end of the rod. Let the length of the rod be $2r$, the center of mass of the rod at C, and let γ measure the angular displacement from OX.

The forces acting upon the rod are to be identified. The weight mg acts vertically downward through C. We may consider the reaction of the support upon the rod to be made up of two components (P and Q) which are perpendicular and parallel, respectively, to the rod and with directions as shown in the diagram.

The acceleration of the point C may be advantageously described in terms of the radial and normal components. The motion of C may then be expressed by two equations, one of which expresses the acceleration of C parallel to the rod and the other contains those factors which are concerned with the component acceleration of C which is perpendicular to the rod.

We shall consider first the normal component of acceleration. Since r is of constant length, one term—$2\omega(dr/dt)$—of the general expression for the normal component of acceleration is zero and hence the force equation for the part of the motion under immediate consideration is

$$mg \sin \gamma - P = mr\frac{d^2\gamma}{dt^2} \qquad (15\text{-}11)$$

Another equation containing P is needed before this equation may be solved because P is evidently a function of γ. Such an equation is obtained by writing a force-moment equation. Selecting an axis passing through C and perpendicular to the plane of motion and remembering that the moment of inertia of the rod in this case is $\frac{1}{3}\, mr^2$, the desired equation becomes

$$Pr = \frac{1}{3}mr^2\frac{d^2\gamma}{dt^2} \qquad (15\text{-}12)$$

Dividing through by r and then substituting the resulting value of P in Eq. (15-11) gives

$$mg \sin \gamma = \frac{4}{3}mr\frac{d^2\gamma}{dt^2} \qquad (15\text{-}13)$$

If we now replace the angular acceleration by its alternative expression $\omega\ d\omega/d\gamma$ and then integrate the altered equation between definite limits, *i.e.*, from 0 to γ and from 0 to ω, the following result is obtained:

$$mg(1 - \cos \gamma) = \tfrac{2}{3}\, mr\omega^2 \qquad (15\text{-}14)$$

This result expresses the desired relation between the angular velocity and the position of the rod.

It is interesting to observe that the work equation might have been used in place of the force equation as a means of evaluating the angular velocity. The force equation was selected because by its use attention is directed to the necessity for including the force P in the force equation [Eq. (15-11)] even though that part of the rod, upon which P is applied, does not move. An application of the work equation would not include P because P does no work even though it contributes to the acceleration. It is suggested that the student derive Eq. (15-14) by using the work equation. In the derivation it is immaterial whether the motion be regarded as a combination of translational and rotational motion or as pure rotational. Both treatments are suggested as being instructive.

An evaluation of P, one component of the reaction exerted by the support, is next to be obtained. The most direct procedure for this purpose is by an elimination of the angular acceleration $d^2\gamma/dt^2$ from Eqs. (15-11) and (15-12). The result is

$$P = \tfrac{1}{4}mg \sin \gamma \tag{15-15}$$

The other component (Q) of the reaction exerted by the support upon the hinged end of the rod may be found by writing the force equation for those forces which are parallel to the rod. This equation is

$$-mg \cos \gamma + Q = -mr\omega^2 \tag{15-16}$$

If we eliminate ω^2 from this equation by the use of Eq. (15-14), the desired evaluation of Q is obtained:

$$Q = \tfrac{1}{2}mg(5 \cos \gamma - 3) \tag{15-17}$$

The resultant reaction, which we may call T, may now be found by combining vectorially P and Q. The absolute magnitude of T is

$$T = \tfrac{1}{2}mg[\tfrac{1}{4} \sin^2 \gamma + (5 \cos \gamma - 3)^2]^{\frac{1}{2}} \tag{15-18}$$

The direction of T makes an angle (α) with the rod, the value of which is given by the equation

$$\tan \alpha = \tfrac{1}{2} \sin \gamma (5 \cos \gamma - 3)^{-1} \tag{15-19}$$

in which α is measured in a clockwise direction from the rod.

Problem.—Find expression for the angular velocity and for the angular displacement of the rod in terms of the time.

It is of interest to solve the problem of the falling rod by vector methods. Using the terminology adopted above for the various quantities and including $\boldsymbol{T}$, the resultant reaction, in place of the two components P and Q, we may write the following two equations for the motion:

$$-mg\boldsymbol{i} + \boldsymbol{T} = m\frac{d^2\boldsymbol{r}}{dt^2} \tag{15-20}$$

$$\boldsymbol{r} \times \boldsymbol{T} = -I\frac{d^2\gamma}{dt^2}\boldsymbol{k} \tag{15-21}$$

in which $\boldsymbol{i}$ is the unit vector parallel to the X-axis and $\boldsymbol{k}$ is the unit vector perpendicular to the plane of motion. The first of these equations is the force equation and the second equation expresses the effect of the force moment about an axis through the center of mass and perpendicular to the plane of the diagram.

As in the analytic method, so here we must eliminate $\boldsymbol{T}$ from the two equations. This may be done by a direct substitution of the value of $\boldsymbol{T}$, found by solving Eq. (15-20) for $\boldsymbol{T}$, in Eq. (15-21), or one may multiply Eq. (15-20) by $\boldsymbol{r} \times$ and then add the resulting equation to Eq. (15-21). In either case the following expression is obtained:

$$-mg\, \boldsymbol{r} \times \boldsymbol{i} = m\boldsymbol{r} \times \frac{d^2\boldsymbol{r}}{dt^2} + I\frac{d^2\gamma}{dt^2}\boldsymbol{k} \tag{15-22}$$

To simplify this equation, use may be made of the following relations, which the student should verify:

$$\boldsymbol{r} \times \boldsymbol{i} = -(r \sin \gamma)\boldsymbol{k}$$

$$\boldsymbol{r} \times \frac{d^2\boldsymbol{r}}{dt^2} = \frac{d}{dt}\left(\boldsymbol{r} \times \frac{d\boldsymbol{r}}{dt}\right) = \boldsymbol{k}\frac{d}{dt}(r^2\omega)$$

Introducing the right-hand members of these two relations for their equivalent expressions in Eq. (15-22) and remembering the particular value of the moment of inertia leads to an equation which is identical with Eq. (15-13). From this point on, the solution is obtained by a single integration as shown above.

We shall next evaluate the reaction $\boldsymbol{T}$ by vector methods. For this purpose Eqs. (15-20) and (15-21) will serve as a starting point. It will be convenient to eliminate the linear and angular acceleration factors in order to obtain an equation containing $\boldsymbol{T}$ without differential factors of the second order. The necessary relation between $d^2\boldsymbol{r}/dt^2$ and $d^2\gamma/dt^2$ for this transformation may be obtained by writing $d^2\boldsymbol{r}/dt^2$ in terms of the acceleration components which are parallel and perpendicular to $\boldsymbol{r}$. Using $\boldsymbol{r}_1$

and γ_1 to represent the unit vectors which are parallel and perpendicular, respectively, to $\boldsymbol{r}$, it follows that

$$\frac{d^2\boldsymbol{r}}{dt^2} = r\frac{d^2\gamma}{dt^2}\,\boldsymbol{\gamma}_1 - r\,\omega^2\boldsymbol{r}_1 \tag{15-23}$$

Substituting this expression for the acceleration in Eq. (15-20) gives

$$-mg\,\boldsymbol{i} + \boldsymbol{T} = mr\frac{d^2\gamma}{dt^2}\,\boldsymbol{\gamma}_1 - mr\omega^2\boldsymbol{r}_1 \tag{15-24}$$

Before solving Eq. (15-21) for $d^2\gamma/dt^2$, it is desirable to change the direction of the vectors in that equation from $\boldsymbol{k}$ to $\boldsymbol{\gamma}_1$. This change may be effected by multiplying both sides of Eq. (15-21) by $\boldsymbol{r}_1 \times$ which gives (after interchanging the two members)

$$\begin{aligned} I\frac{d^2\gamma}{dt^2}\boldsymbol{\gamma}_1 &= \boldsymbol{r}_1 \times (\boldsymbol{r} \times \boldsymbol{T}) \\ &= \boldsymbol{r}\,(\boldsymbol{r}_1 \cdot \boldsymbol{T}) - \boldsymbol{T}r \end{aligned} \tag{15-25}$$

Replacing $(d^2\gamma/dt^2)\boldsymbol{\gamma}_1$ in Eq. (15-24) by its value as found from Eq. (15-25) and at the same time eliminating ω^2 as expressed by Eq. (15-14), and introducing the value of I, gives the following equation:

$$4\boldsymbol{T} = mg\boldsymbol{i} + 3\boldsymbol{r}_1(\boldsymbol{r}_1 \cdot \boldsymbol{T}) - [\tfrac{3}{2}mg(1 - \cos\gamma)]\boldsymbol{r}_1$$

Multiplying by $\boldsymbol{r}_1 \cdot$ to convert this expression into a scalar equation gives

$$\boldsymbol{r}_1 \cdot \boldsymbol{T} = \tfrac{1}{2}\,mg(5\cos\gamma - 3) \tag{15-26}$$

Hence

$$T\cos\alpha = \tfrac{1}{2}\,mg(5\cos\gamma - 3)$$

where α is the angle between $\boldsymbol{r}$ and $\boldsymbol{T}$. By using the relation between α and γ [Eq. (15-19)] the final expression for T, as given in Eq. (15-18), may be readily found.

Problem.—A uniform rod of length l and mass m stands vertically upright on a horizontal frictionless plane. If the rod, initially at rest, falls from its vertical position, find an expression for its angular velocity in terms of the angle through which it has rotated. Find also an expression for the reaction exerted by the floor upon the rod in terms of the angular displacement. Obtain solutions by analytical and vector methods.

15-4. The Cylinder and the Falling Weight.—The apparatus is to be arranged as shown in Fig. 139. A weight of mass m is suspended by a light massless cord which passes over a fixed frictionless pulley and is wrapped around a cylinder. The cylinder is supported by a horizontal plane and is placed with its axis perpendicular to the plane of the diagram. The cord is wrapped around the cylinder in such a manner that, as the weight falls, the cylinder is pulled toward the pulley. There is assumed to be sufficient friction between the cylinder and the horizontal plane to prevent slipping. It is required to find the equations of motion of the three bodies and the tensions in the cord.

Let the mass of the cylinder be M, its moment of inertia I, and its radius b. Also let the moment of inertia of the pulley be P and its radius r. We shall designate the tension of the cord between the hanging weight and the pulley by T and between the cylinder and the pulley by t, as shown in the diagram.

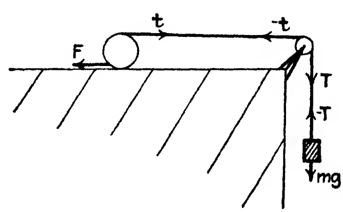

FIG. 139.

Assume that the system is initially at rest. Let the positive direction of all vector quantities be taken parallel to the direction of the motion.

By the indicated arrangement it is obvious that both the linear velocity and linear displacement of the cylinder will be instantaneously one-half of the corresponding velocity and displacement of the hanging weight.

There are two methods of procedure which may be followed in obtaining the desired relations. We may begin by writing the force equations for the three bodies or we may use the work equation. The work equation presents the simpler mode of expressing the velocities in terms of the displacement but by writing the force equations a more complete analysis is immediately obtained. If it were not desired to determine the tensions of the cord, the work equation would perhaps be the better to use for expressing the velocities in terms of the displacements.

The forces acting on the hanging mass are mg and $-T$, the force moments applied to the pulley are rT and $-rt$, and the forces responsible for the motion of the cylinder are t and the friction $-F$. The force equations for the hanging mass and the cylinder are

$$mg - T = ma \qquad t - F = M\frac{a}{2} \tag{15-27}$$

where a is acceleration of the hanging weight. The two force-moment equations may be written as follows:

$$r(T - t) = P\frac{d^2\alpha}{dt^2} \qquad b(t + F) = I\frac{d^2\beta}{dt^2} \tag{15-28}$$

where α and β are the angular displacements of the pulley and cylinder, respectively.

Since all of the forces and force moments are constant, the accelerations, translational and rotational, are also constant. Both of the angular accelerations may be expressed in terms of the linear acceleration (a) of the hanging weight. These relations are

$$\frac{d^2\alpha}{dt^2} = \frac{a}{r} \qquad \text{and} \qquad \frac{d^2\beta}{dt^2} = \frac{a}{2b}$$

The values of the angular accelerations given above may be substituted in Eqs. (15-28). We then have four equations with four unknowns: a, T, t, and F. From these four equations a single equation may be obtained which expresses the acceleration a in terms of the masses, the moments of inertia, and g. This equation may be written as follows:

$$m\,g = a\left(m + \frac{M}{4} + \frac{I}{4b^2} + \frac{P}{r^2}\right) \tag{15-29}$$

If the masses (m and M) and the moments of inertia are known, then the acceleration of the hanging weight may be determined. This result leads readily to determinations of the linear acceleration of the cylinder and the angular accelerations of the pulley and cylinder. When the accelerations of the three bodies are known, the equations of motion may be written directly from the fundamental equations for uniformly accelerated bodies starting from a rest position.

The tensions in the cord may be determined by substituting the known value of the acceleration in the force equations [Eqs. (15-27)].

Problem.—Derive Eq. (15-29) by using the work equation.

15-5. The Sliding Mass on a Smooth Rigid Rod.—A rigid rod is arranged to rotate at a constant rate about a fixed axis which passes through one end of the rod. A spherical mass with a cylindrical hole cut through its center is guided by the rod and may slide along the rod without friction. It is required to find the equations of motion and the path described by the sliding mass.

This problem has been selected because it represents a type of motion in which the path described by the mass is determined by using an equation in which the acceleration is zero. Because of the absence of friction there is no force parallel to the rod. Even though the mass were initially at rest relatively to the rod, the mass moves parallel to the rod. It must be remembered that, in order to maintain a body in motion along a circular path, an inwardly directed radial force is required. Absence of the radial force permits motion along the radius as will be seen in the following treatment of the problem.

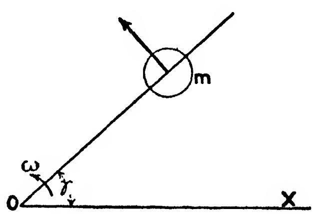

FIG. 140.

Let the reference line be OX (Fig. 140) with O the origin of a system of polar coordinates. The position of the sliding mass upon the rod is indicated by the coordinate r and γ gives the angular displacement of the rod. The rod rotates at the constant angular speed ω.

The components of the acceleration, perpendicular and parallel (J_r and J_γ) to the rod, are in general

$$J_r = \frac{d^2r}{dt^2} - r\omega^2 \qquad J_\gamma = 2\omega V_r + r\frac{d\omega}{dt}$$

But ω is constant; hence if we let N represent the force exerted by the rod upon the sliding mass (of mass m), the force equation for motion perpendicular to the rod is

$$N = 2\, m\, \omega\, V_r \tag{15-30}$$

Since there is no force acting upon m which is parallel to the rod, the acceleration (J_r) of the mass in this direction is permanently equal to zero; hence

$$\frac{d^2r}{dt^2} = r\omega^2 \tag{15-31}$$

The path described by the sliding mass may be found from this equation. If we replace d^2r/dt^2 by its alternative expression

$V_r dV_r/dr$, the equation may be integrated. For initial conditions we shall put

$$V_r = U_r \qquad r = s \qquad \gamma = 0 \qquad t = 0$$

With these initial values of the variables, integration of Eq. (15-31) gives the following result:

$$V_r^2 - U_r^2 = \omega^2 (r^2 - s^2)$$

Hence

$$\frac{dr}{dt} = [U_r^2 + \omega^2 (r^2 - s^2)]^{\frac{1}{2}}$$

Integration of this equation gives

$$\gamma = \omega t = \log \frac{[U_r^2 + \omega^2(r^2 - s^2)]^{\frac{1}{2}} + \omega r}{U_r + \omega s} \qquad (15\text{-}32)$$

which is the desired equation for the path of the sliding mass and may also be used for expressing the coordinates in terms of the time.

The equation which expresses the velocity (V) of the mass in terms of the coordinates may be found by combining V_r and $V_\gamma = \omega r$, the velocity perpendicular to r.

The magnitude of the force N may be found for any position since V_r may be determined for any position. It is interesting to observe that the function of N is to accelerate the mass along a line perpendicular to r to that extent which is necessary for m to keep up with the rod.

Problem.—A somewhat similar problem is left for the student to work out. The mass m is to slide without friction upon a second rod which is in the same plane as OX and r but is fixed rigidly and at right angles to r at some distance d from O. Find the path of the sliding mass and the force N.

15-6. Motion of the Center of Mass of Two Attracting Particles.—Two particles in free space are in a state of mutual attraction. No external forces act upon the system. Prove that the center of mass has no acceleration.

Let the two masses be m_1 and m_2 with coordinates x_1y_1 and x_2y_2, respectively, in a fixed reference system XOY. The plane of the reference system is to be the plane of motion of the two particles. Let α be the angle which the line joining the centers of the two bodies makes with the X-axis.

The mutual attraction between the two particles may be expressed by a force F acting on m_1 and $-F$ upon m_2, since the two forces are equal but oppositely directed. Both forces are parallel to the line joining the centers of mass of the two particles and therefore each makes an angle α with the X-axis. Parenthetically it may be stated that the nature of the mutual force is immaterial to the solution of the problem under consideration. For that matter the forces need not be attractive. The force equations for the two masses, written for the components of F parallel to the X- and Y-axes, respectively, are

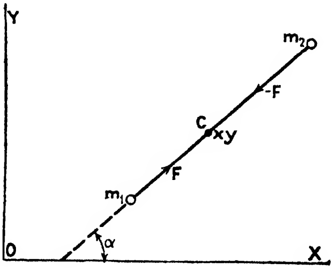

FIG. 141.

$$F\cos\alpha = m_1\frac{d^2x_1}{dt^2} \qquad F\sin\alpha = m_1\frac{d^2y_1}{dt^2}$$

$$-F\cos\alpha = m_2\frac{d^2x_2}{dt^2} \qquad -F\sin\alpha = m_2\frac{d^2y_2}{dt^2}$$

Eliminating F from each pair of equations gives

$$m_1\frac{d^2x_1}{dt^2} + m_2\frac{d^2x_2}{dt^2} = 0 \qquad m_1\frac{d^2y_1}{dt^2} + m_2\frac{d^2y_2}{dt^2} = 0 \qquad (15\text{-}33)$$

When regarded as vectors and combined into a single equation, Eqs. (15-33) indicate that the sum of the products of each mass by its acceleration is zero.

Let the coordinates of the center of mass of the system by $\bar{x}$ and $\bar{y}$. If M is the mass of the system, then

$$M\bar{x} = m_1x_1 + m_2x_2 \qquad M\bar{y} = m_1y_1 + m_2y_2$$

If we differentiate each of these two equations twice with respect to the time, we obtain

$$M\frac{d^2\bar{x}}{dt^2} = m_1\frac{d^2x_1}{dt^2} + m_2\frac{d^2x_2}{dt^2} = 0$$

$$M\frac{d^2\bar{y}}{dt^2} = m_1\frac{d^2y_1}{dt^2} + m_2\frac{d^2y_2}{dt^2} = 0$$

Since the right-hand members of these equations are zero [Eq. (15-33)] it may be concluded that the acceleration of the center of mass of the system is zero.

The results of this analysis may be extended to a system consisting of any number of particles which are under the influence of mutually attractive (or repulsive) forces, each particle attracting (or repelling) each of the other particles. It is not necessary for the particles (if more than three) to be in the same plane. The details of the proof will be left for the student.

15-7. Effects of a Change of Mass in a Planetary System of Two Particles.—Given a planetary system consisting of two particles moving under the influence of their mutual gravitational attraction. The mass of the primary particle is supposed to be increasing at a constant rate due to acquisition of meteoric material. It is to be assumed that the orbit of the satellite is circular. Find the effect of the alteration of the mass of the primary upon the period of the satellite and also the change in the orbit.

If the masses of the primary and satellite are M and m, respectively, and r is the distance between their centers of mass, then the gravitational force F exerted by M upon m is

$$-F = k\frac{Mm}{r^2}$$

The component of the acceleration of m in the line of r is $-r\omega^2$; hence

$$\frac{kM}{r^2} = r\omega^2 \qquad \text{or} \qquad kM = r^3\omega^2 \tag{15-34}$$

The other component of the acceleration of m, *i.e.*, that perpendicular to r, is zero. From this fact it follows [compare Eq. (13-4)] that

$$r^2\omega = h \text{ (a constant)} \tag{15-35}$$

If we raise the exponents of both members of this equation to the three-half power and multiply both sides by $\omega^{\frac{1}{2}}$, we obtain

$$r^3\,\omega^2 = h^{\frac{3}{2}}\omega^{\frac{1}{2}} \tag{15-36}$$

Eliminating $r^3\omega^2$ from this equation by the use of Eq. (15-34) gives

$$kM = h^{\frac{3}{2}}\omega^{\frac{1}{2}}$$

Taking the logarithm of both members and differentiating with respect to the time gives the following result:

$$\frac{1}{M}\frac{dM}{dt} = \frac{1}{2\omega}\frac{d\omega}{dt}$$

This equation indicates the effect of the change of mass of the primary particle upon the angular velocity of the satellite and hence upon the period of the satellite. Since ω may be regarded as a positive quantity, it is readily seen that an increase in M will produce an increase in the angular velocity of the satellite and hence shorten its period.

To determine the effect of an increase of M upon the orbit, it will be necessary to obtain an equation in which the only variables are M and r. Such an equation is obtained by eliminating ω from Eqs. (15-34) and (15-35), which results in the following:

$$k\,M\,r = h^2$$

Taking the logarithm of this equation and differentiating the resulting expression with respect to the time gives

$$\frac{1}{M}\frac{dM}{dt} = -\frac{1}{r}\frac{dr}{dt}$$

This equation shows that an increase in the mass of the primary gives the satellite a velocity component which is inward along the radius vector and hence decreases the size of the orbit.

Problem.—The angular acceleration of the moon is thought to be 12 sec. per century. If this change is due to an increase in the mass of the earth, what is the rate at which the earth accumulates mass? Assume that the density of the earth is 5.52 g. per cubic centimeter, its radius 6.37×10^8 cm., and the distance from the earth to the moon 3.84×10^{10} cm.

15-8. Motion of a Mass Subject to a Double Constraint.—A vertical shaft is made to rotate at a constant rate. A horizontal tube is fixed rigidly to the shaft and carries a mass (n) which may slide along the axis of the tube without friction. An inelastic cord is fastened to the mass n and passes over a massless pulley at the axis of rotation to a second mass (m) which is constrained to slide without friction vertically upon the rotating shaft. Subject to this arrangement the two masses are kept at a constant distance apart, the distance being the length of the cord. It is required to find the equations of motion for the two bodies.

Let the apparatus be arranged as indicated in Fig. 142. It is convenient in this problem to use a rotating reference system, with OX the axis of rotation and OY measured along the axis of the rotating tube.

The mass n is subject to a double constraint. Its velocity along the Y-axis must be equal to the velocity of m along the

X-axis, although opposite in sign. Furthermore the mass n is constrained to move with the rotating tube; hence its velocity perpendicular to OY must always be equal to product of the angular velocity (ω) of the tube by the distance (y) of n from the axis of rotation. The mass m is subject to a single constraint.

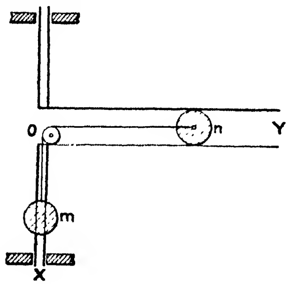

FIG. 142.

Let the positions of m and n on the reference axes be designated by x and y, respectively. Let T be the tension of the cord.

The force equation for the motion of m is

$$mg - T = m\frac{d^2x}{dt^2} \qquad (15\text{-}37)$$

Two force equations will be needed to express the motion of n. If F is the force exerted by the rotating tube upon n (and is therefore perpendicular to the tube), we may write

$$-T = n\left(\frac{d^2y}{dt^2} - y\omega^2\right) \qquad F = n\,2\omega V_y \qquad (15\text{-}38)$$

It is to be noticed that the accelerations of m and n are not equal though the cord is attached to both. The effect of the cord is to make

$$\frac{d^2x}{dt^2} = -\frac{d^2y}{dt^2}$$

The velocity of m and hence the radial velocity of n may be found from Eq. (15-37) and the first equation of Eqs. (15-38). Eliminating T from these equations and substituting suitable alternative forms for the second derivatives of the coordinates gives

$$mg = -(m + n)\frac{V_y dV_y}{dy} + ny\omega^2$$

For initial conditions we may put $x = x_0$, $y = y_0$,

$$V_y = -V_x = V_0$$

at the time $t = 0$. Multiplying the right-hand member of the foregoing equation by $-dy$ and the left member by dx, for $dx = -dy$, and integrating gives

$$mg(x - x_0) = \tfrac{1}{2}(m + n)(V_y^2 - V_0^2) - \tfrac{1}{2}n\omega^2(y_0^2 - y^2) \quad (15\text{-}39)$$

In obtaining this equation some trouble may be experienced in establishing the sign of the last term as given. A realization of the physical relations may make this process easier. A positive displacement of m produces a negative displacement of n; hence $x - x_0 = y_0 - y$. The limits of integration of the term $mgdx$ are taken from x_0 to x. Corresponding to these limits we must integrate the other terms with limits from y to y_0.

An interpretation of the work equation [Eq. (15-39)] is to be made. The left member gives the work done by the force mg for an assumed positive displacement. The first quantity of the right-hand member expresses the change in kinetic energy of m and the kinetic energy change of n for velocities along the y-axis. The last quantity shows that n has lost some of its kinetic energy, that which is due to its velocity perpendicular to the y-axis.

Equation (15-39) is one of the desired equations of motion, since it expresses the velocity of m and one component of the velocity of n in terms of the displacements. This equation may be made explicit for V_y and then, by replacing V_y by dy/dt and integrating, an expression for y (or x) in terms of the time could be obtained. This detail will be omitted. The third equation of motion, expressing the velocity of m or V_y for n, in terms of either coordinate x or y, could then be obtained if desired. The other component (V_γ) of velocity for the mass n may readily be obtained for any position or time, since $V_\gamma = \omega y$.

We shall now turn our attention to the force F which acts upon n along a line which is always perpendicular to the Y-axis. This force is required to supply n with the necessary acceleration which is needed to change the direction of the radial velocity of n and to change the magnitude of V_γ. The force equation expressing this requirement is

$$F = n\, 2\omega\frac{dy}{dt}$$

The energy changes produced by F may be expressed by multiplying both sides of the force equation by the differential displacement in the direction of F and then integrating. This differential displacement is $y\, d\gamma = y\omega dt$ where $d\gamma$ is the differential angular displacement. The differential work equation is

$$\begin{aligned} Fy\omega dt &= 2n\omega y\frac{dy}{dt}d\gamma \\ &= 2n\omega^2 y dy \end{aligned}$$

or

$$\int_0^t F\, y\, \omega\, dt = n\, \omega^2(y^2 - y_0^2) \qquad (15\text{-}40)$$

The left-hand member cannot be integrated unless F and y are expressed in terms of the time.

The total work done by the forces on the system may now be found by adding Eqs. (15-39) and (15-40). For present purposes, since it is not convenient to evaluate the integral of Eq. (15-40), we shall add the right-hand member of Eq. (15-40) to both sides of Eq. (15-39) which results in the expression

$$mg(x - x_0) + n\omega^2(y^2 - y_0^2) = \tfrac{1}{2}(m + n)(V_y^2 - V_0^2) + \tfrac{1}{2}n\omega^2(y^2 - y_0^2)$$

It is interesting to examine the distribution of the work done by F when the mass m is removed from the system. Putting $m = 0$ in the foregoing equation gives

$$n\,(V_y^2 - V_0^2) = n\,\omega^2\,(y^2 - y_0^2)$$

The left-hand member gives the energy change for velocities parallel to y and the right-hand member expresses the energy changes for velocities perpendicular to y. The fact that the two expressions of energy are equal indicates that the work done by F, in the particular case of m being zero, is equally divided by this selection of method of expression.

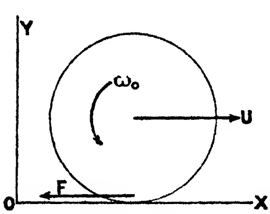

FIG. 143.

15-9. The Sliding Sphere.—A solid homogeneous sphere of radius r is given an initial linear velocity U along a rough horizontal plane. At the same instant the sphere has an initial angular velocity ω_0 about a horizontal axis which is perpendicular to the direction of the linear velocity. The direction of the angular velocity is opposite to that which it would have if it were rolling, without slipping, in the direction of U. Find the character of the motion.

Let the reference system XOY (Fig. 143) be fixed to the horizontal plane with OX parallel to U and Y-axis perpendicular to the horizontal plane. For present consideration it is unnecessary to assign the initial position of the sphere in the reference system.

The only force acting on the sphere is friction (F), the point of application of which is the point of contact between the sphere and the horizontal plane. The direction of F is negative.

If m is the mass of the sphere and V is the velocity of the center of mass, the force equation will be

$$-F = m\frac{dV}{dt} \tag{15-41}$$

If the moment of inertia of the sphere about its axis of rotation is represented by I and the angular velocity by ω, the force-moment equation may be written as follows:

$$-r\,F = I\frac{d\omega}{dt} \tag{15-42}$$

From these two equations it is readily seen that the linear and angular accelerations are both negative.

It is instructive to see to what extent these equations are applicable to the motion. There is always the danger of setting up equations which are adequate accurately to describe the motion of the body concerned for one phase of the motion and then, owing to some change which takes place at a later stage in the motion, to overlook the possibility that the equations may no longer be applicable. For example, in writing the force equation [Eq. (15-41)] for this problem, the friction is written with a negative sign. Since m is positive, the acceleration dV/dt, is negative. In this particular problem, as will be shown later, owing to certain relative initial values of V and ω, it is possible for V to reverse its direction. In such a case F would also reverse its sign at the instant at which V reverses. The force equation [Eq. (15-41)] would then no longer be correct. If applied as written to the motion after a reversal of V, we might be led to believe that the velocity would increase indefinitely in the negative direction. Physically such an increase would be impossible when friction is the only external force. The change in sign of V in this problem presents a sort of physical discontinuity at the instant or position where V is zero, beyond which a new relation is required.

The force-moment equation [Eq. (15-42)] also has its limitations in this problem. The sign of $d\omega/dt$ is negative, as the equation indicates, which is correct certainly for a first stage of the motion. Under certain initial conditions, which will be discussed later, a change will occur during the progress of the motion which makes a change in the force-moment equation necessary. One must be careful, therefore, to inspect peculiarities which may occur during the progress of the motion with the view of ascertaining a need for modification in the equations already written.

Returning to a development of the equations needed to describe the motion under consideration, we proceed to combine Eqs. (15-41) and (15-42) by eliminating the unknown quantity F and at the same time we may introduce the particular value of I. We obtain thereby

$$\frac{dV}{dt} = \frac{2}{5} r \frac{d\omega}{dt} \tag{15-43}$$

a kinematical equation which shows that the motion is independent of the mass of the sphere.

This equation may be integrated and yields the following relation:

$$V - U = \tfrac{2}{5} r (\omega - \omega_0)$$

The relation expressed by the last equation is more readily interpreted if we rearrange the terms as follows:

$$V - \tfrac{2}{5} r\omega = U - \tfrac{2}{5} r\omega_0 \tag{15-44}$$

The right-hand member of the equation as now written is a constant; hence the variations of V and ω must be such as will be consistent with the expressed relation. There are two possibilities to be considered, one in which the right-hand member is positive and the other in which this quantity is negative. These two cases will be discussed separately.

We shall consider first the case in which the left-hand member of Eq. (15-44) is positive. Since V, r, and ω are initially positive quantities and the quantities V and $\frac{2}{5} r\omega$ decrease at the same rate [Eq. (15-43)], it follows that ω will become zero before V does. At the instant at which ω is zero, the sphere slides without rotation. After this instant, ω will continue to decrease positively by increasing negatively. This phase of the motion will continue until $V = -r\omega$, at which instant the motion reaches a critical point and the sphere will roll from then on without slipping.

Beyond this critical point in the motion the validity of the equations is to be considered. An examination of the force equation does not reveal any necessity for making a change in it. The angular acceleration in the force-moment equation will evidently serve as a clue for recasting the force-moment equation. Two alternatives would seem to be possible: either the angular acceleration becomes a positive quantity or it becomes zero. The quantities r and I can be only positive. The force F cannot be positive so long as the linear velocity V be positive and the angular velocity ω be negative and less in absolute magnitude than V/r. These facts show that the sign of $d\omega/dt$ cannot be positive under the assigned limitations. The other alternative remains as the only possibility. We may therefore draw the conclusion that $d\omega/dt$ is zero for the motion of the sphere, after the angular velocity becomes equal to $-V/r$. This conclusion requires that the force F must become zero at the critical point under consideration. From this it follows that neither V nor ω will change after the critical point has been reached and the sphere will roll on with constant linear and angular velocities indefinitely.

The conclusions obtained above give us some light upon the behavior of the force F during the entire motion. Under the conditions of the problem the magnitude of F is dependent upon the relative values of V and ω. So long as the sphere slides upon the horizontal plane, F is not zero and, when the sphere rolls without sliding, F is zero.

The magnitude of F, for any selected value of V or ω, cannot be determined in this particular problem without additional information.

There remains for consideration, however, the case referred to above in which the right-hand member of Eq. (15-44) is negative. The analysis of this case will be left to the student.

Problem.—If $F = k\ (V + r)$, where k is a constant, derive the equations which express V as a function of the time and also as a function of the displacement of the center of mass of the sphere.

15-10. The Skyrocket.—A skyrocket is to be projected vertically upward in a vacuum by means of the force exerted by gas issuing from an orifice centrally located in the lower end of the rocket. The gas is generated by burning powder within the rocket. The rate of generation of gas is assumed to be constant and the gas is to issue from the orifice at a constant velocity relative to the rocket. It is required to find the height to which the rocket will ascend.

We shall assume that the rocket is initially at rest and shall take the initial position of the rocket as origin for a reference system with positive quantities being measured vertically upward. Let m be the mass at any instant t and m_0 the initial mass of the rocket. Let n be the mass of gas issuing per second and $-w$ the constant velocity of the gas as it leaves the rocket relative to the orifice.

There are two forces acting upon the rocket, the weight of the rocket, which is $-(m_0 - nt)g$, and the upward force due to the reaction of the gas as it leaves the rocket. Both forces are parallel to the vertical line of motion. The force caused by the escaping gas is equal to the rate of change of momentum of the gas. Since the velocity of the gas relative to the rocket is $-w$ and n is the rate of generation of gas, this force is $+nw$. Since $m_0 - nt$ is the mass of the rocket at the time t, the force equation may be written as follows:

$$n\,w - (m_0 - nt)g = (m_0 - nt)\frac{dV}{dt} \qquad (15\text{-}45)$$

in which V is the velocity of the rocket.

Since this equation contains only the two variables V and t, we may separate the variables and integrate. If V is zero at the time $t = 0$, the result of integration may be written

$$V = -gt + w \log \frac{m_0}{m_0 - nt} \qquad (15\text{-}46)$$

This equation expresses the velocity of the rocket as a function of the time for any value of the time from zero up to the instant at which the powder is exhausted.

If we let s be the coordinate which measures the upward displacement of the rocket from the initial position, then, since $V = ds/dt$, we may replace V by ds/dt, separate the variables, and integrate the resulting expression to obtain an equation for the displacement. If $s = 0$ at the time $t = 0$, the expression for the displacement is

$$s = -\frac{1}{2}gt^2 + wt \log m_0 + \frac{wC}{n}(\log C - 1) + \frac{wm_0}{n}(1 - \log m_0) \qquad (15\text{-}47)$$

where C is written for $m_0 - nt$. This equation gives the height to which the rocket will ascend in any time t, provided that t is not greater than the time required for the powder to burn up. The particular time t_1 at which the powder will be exhausted will be equal to the mass of the powder divided by n, the rate at which the powder is being used. If the total mass of the powder is known, the value of s which corresponds to t_1 may be found.

At the time t_1 the upward force, due to the reaction of the gas, becomes zero but the rocket will go still higher because of the kinetic energy which it possesses at that instant. The additional height, which we may call h, may be found by applying the work equation. If m_1 is the residual mass of the rocket and V_1 is the velocity at the time t_1, then

$$-m_1gh = -\tfrac{1}{2}\, m_1V_1^2$$

The total height to which the rocket will ascend may now be readily found.

15-11. The Water Stream and Bucket.—Apparatus is arranged whereby a horizontal stream of water, flowing at a constant rate, is caught in a bucket. The bucket, initially at rest and empty, may move along a smooth horizontal plane. Find the equations of motion and the total change of kinetic energy of the system for a given mass of water.

We shall use the following assignment of symbols:

w = rate of flow of the water in terms of mass per second.
u = velocity of the water.
P = mass of the empty bucket.
V = velocity of the bucket and of the water which it may contain.
M = mass of water in the bucket at any time t.

This problem is somewhat similar to the preceding problem (Sec. **15-10**) in that the force which accelerates the bucket and its contents has its origin in the reaction between the moving stream and the bucket. The procedure is to express the force and the mass of the bucket with its contents in terms of the given quantities and then, by using the force equation, to find the acceleration of the bucket and the water in it.

If the stream of water were directed against some stationary object, the force exerted by the stream upon the stationary object would be constant. In the present problem the bucket is not stationary but moves with an increasing velocity in the direction of the moving stream; hence the force accelerating the bucket will decrease. The velocity of the bucket will approach the velocity of the stream as a limit. The mass of the bucket together with the water in it increases, although the rate of increase of mass will not be constant. From the fact that the accelerating force is decreasing and the total mass of the bucket is increasing, one may say definitely that the acceleration of the bucket will decrease. This preliminary analysis of the problem serves as a qualitative check upon the equations which may be written to describe the motion quantitatively.

The mass of water per second flowing by any stationary point in pipe carrying the water is $w = \rho A u$, if ρ is the density of the water and A the area of cross section of the stream. The mass of water w' entering the bucket per second is $\rho A (u - V)$. Hence we may write

$$\frac{w'}{w} = \frac{\rho A (u - V)}{\rho A u}$$

or

$$w' = \frac{w (u - V)}{u}$$

The force F exerted by the stream upon the bucket is equal to the rate of change of the momentum of the stream. Since $u - V$ is the velocity of the stream relative to the moving bucket, we may express the force F as follows:

$$F = \frac{w (u - V)^2}{u} \tag{15-48}$$

The quantity of water in the bucket at any time t may be expressed as follows:

$$M = \int_0^t \frac{w (u - V)}{u} dt$$

if the bucket is initially empty. We cannot at present evaluate this integral because we do not know the way in which V varies with the time. It is possible, however, to express the quantity of water in the bucket in terms of the mass of the bucket and the velocities u and V by using the principle of conservation of momentum. The momentum (Mu) of any given quantity of water (M) before it enters the bucket must be equal to the momentum of the bucket and the water at the instant this quantity of water is all in the bucket; hence

$$Mu = (M + P)V$$

or

$$M = P\frac{V}{u - V} \tag{15-49}$$

We may now write the force equation, which is

$$F = \frac{d}{dt}[(M + P)V]$$

with the values given by Eqs. (15-48) and (15-49) this becomes

$$w\frac{(u - V)^2}{u} = P\frac{u^2}{(u - V)^2}\frac{dV}{dt}$$

Rewriting this equation so that it is explicit for the acceleration gives

$$\frac{dV}{dt} = \frac{w}{P}\frac{(u - V)^4}{u^3} \tag{15-50}$$

Since this equation contains only two variables, it may be readily integrated. It will be remembered that the initial conditions give $V = 0$ at $t = 0$; hence separating the variables and integrating yields one of the equations of motion, which is

$$\frac{Pu^3}{3(u - V)^3} = wt + \frac{P}{3} \tag{15-51}$$

The second equation of motion, *i.e.*, that which expresses the displacement in terms of the time, may now be found. If the displacement of the bucket is measured from its initial position by the coordinate s, then, since $V = ds/dt$, we have

$$\frac{ds}{dt} = u - u\sqrt[3]{\frac{P}{3wt + P}}$$

Integration of this equation gives

$$s = ut - \frac{uP^{\frac{1}{3}}}{2w}[(3wt + P)^{\frac{2}{3}} - P^{\frac{2}{3}}] \tag{15-52}$$

The third equation of motion may be found by eliminating t from Eqs. (15-51) and (15-52), or, in the usual way, by substituting VdV/ds for dV/dt in Eq. (15-50). In either case the result is

$$\frac{u}{3(u-V)^3} - \frac{1}{2(u-V)^2} + \frac{1}{6u^2} = \frac{ws}{Pu^3}$$

There remains to be obtained an expression which will give the total change of kinetic energy of the system for a given mass of water. If we let M represent the given mass of water under consideration, the kinetic energy of this mass before entering the bucket will be $\frac{1}{2}Mu^2$. The velocity of the bucket together with the water in it is V; hence the change in the kinetic energy of the system ($\Delta\, KE$) is

$$\Delta\, KE = \tfrac{1}{2}\, Mu^2 - \tfrac{1}{2}\,(M + P)V^2$$

Using the relation given in Eq. (15-49), we may express the change in the energy in terms of the velocities and the mass of the pail. Making the indicated substitution gives

$$\Delta\, KE = \tfrac{1}{2}\, PuV$$

Problems.—1. Using any of the equations written above, show that V cannot be greater than u.

2. Find the mass of the water which enters the bucket in terms of the time.

3. Show that the acceleration of the bucket approaches zero as a limit.

15-12. The Two Rolling Cylinders.—A homogeneous right circular cylinder is arranged to roll upon a horizontal plane. A second similar cylinder is placed on top of the first cylinder with its axis parallel to and almost vertically above the axis of the lower cylinder. The surfaces of the two cylinders and of the plane are sufficiently rough to prevent any slipping. It is required to write the equations from which the equations of motion may be derived.

We shall use capital letters to identify the quantities which are to be associated with the lower cylinder and small letters for the corresponding quantities of the upper cylinder. Hence we may use the symbols M and m for the masses, R and r for the radii, and Ω and ω for the angular velocities.

Let the reference system be XOY (Fig. 144) with its plane parallel to plane of motion of the two cylinders and with OX horizontal. Let the line of contact of the lower cylinder with the horizontal plane be as shown at D. The centers of the XY

sections of the cylinders are A and C for the lower and upper cylinders, respectively, and B is their point of contact. We shall let γ be the angle which the line of centers AC makes with the Y-axis.

The forces which act upon the system and their points of application are to be identified. The weights of the cylinders are $-Mg$ and $-mg$ and act through A and C, respectively. The reaction (Q) of the horizontal plane upon the lower cylinder is at D and is directed vertically upward. Friction ($-E$) between the plane and the lower cylinder is horizontal and is directed along the negative direction of the X-axis if we assume that the lower cylinder rolls so that the linear velocity of its center (V) is positive along the X-axis. The friction which prevents slipping of the upper cylinder upon the lower gives rise to a pair of equal and oppositely directed forces (F) which lie in the common tangent plane and are therefore perpendicular to the line AC. There is also another pair of equal and oppositely directed forces (P) between the two cylinders which are parallel to AC. There are therefore four unknown forces which must be either evaluated or eliminated in order to find expressions for the accelerations.

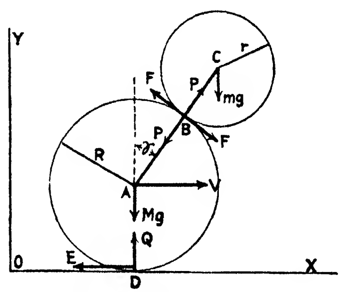

FIG. 144.

The simplest procedure analytically is to write force equations for two mutually perpendicular directions for each of the two cylinders, making four force equations, and also write two force-moment equations, one for each cylinder. Each force-moment equation is to be written for the geometric axis of the particular cylinder. For the lower cylinder we shall write force equations for horizontal and vertical motion, and for the upper cylinder it is more convenient to use directions which are parallel and perpendicular to the line AC.

a. Force Equation for the Lower Cylinder.

Horizontal motion:

$$-E - P \sin \gamma + F \cos \gamma = M\frac{dV}{dt} \qquad (15\text{-}53)$$

Vertical motion:

$$Q - Mg - P\cos\gamma - F\sin\gamma = 0 \tag{15-54}$$

b. Force Equation for the Upper Cylinder.
Perpendicular to AC:

$$mg\sin\gamma - F = m\left[(R + r)\frac{d^2\gamma}{dt^2} + \frac{dV}{dt}\cos\gamma\right] \tag{15-55}$$

Parallel to AC:

$$mg\cos\gamma - P = m\left[(R + r)\left(\frac{d\gamma}{dt}\right)^2 - \frac{dV}{dt}\sin\gamma\right] \tag{15-56}$$

The sign of each term in the equations written above should be carefully examined. The student should remember that it is immaterial to the accuracy of the equation which direction along the selected line is to be considered as positive but it is necessary to be consistent.

An explanation for the validity of the terms written in the right-hand members of the last two equations may be helpful. The quantities within the brackets in both equations represent the linear acceleration of m with respect to the fixed reference system along the direction indicated. These are written with the help of the theorem for the change of origin at A and with axes always parallel to the fixed X- and Y-axes.

The force-moment equation for the two cylinders (for axes through A and C) are

c. For the Lower Cylinder.

$$-rE - rF = \frac{1}{2}MR^2\frac{d\Omega}{dt} \tag{15-57}$$

d. For the Upper Cylinder.

$$-rF = \frac{1}{2}mr^2\frac{d\omega}{dt} \tag{15-58}$$

in which the particular values of the moments of inertia have been introduced.

It is also of importance to verify the signs of the quantities written in these equations. Perhaps the best procedure for this purpose is to determine the sign of each force moment by the use of the convention of signs for rotational quantities and always write the sign of the angular acceleration positive. Whether the sign of the angular acceleration is positive or negative will be determined by the resultant force moment. A common mistake made by students is to include a negative sign in the acceleration term when the force moment is negative and is expressed as a negative quantity.

In the six equations written above there are eight unknown quantities: E, F, P, Q, $d\Omega/dt$, $d\omega/dt$, dV/dt, and $d^2\gamma/dt^2$; hence two more equations are needed in order to obtain the equations of motion of either cylinder. These may be found from a consideration of the kinematical relations which must exist between the linear and angular velocities.

One of these relations may be found by expressing the linear velocity of a point on the axis of the lower cylinder in terms of Ω. If we consider the linear velocity V of A to be due to rotation about the instantaneous velocity center, *i.e.*, the point D, then it follows that $V = -R\Omega$ and hence

$$\frac{dV}{dt} = -R\frac{d\Omega}{dt} \tag{15-59}$$

The other kinematical relation may be found by expressing the x component of the linear velocity $V_x(B)$ of a point (B) in the line of contact between the two cylinders in terms of the linear velocity of that cylinder. These relations are

e. For the Lower Cylinder.

$$V_x(B) = V + R\Omega\cos\gamma$$

f. For the Upper Cylinder.

$$V_x(B) = V + (R + r)\frac{d\gamma}{dt}\cos\gamma - r\omega\cos\gamma$$

Since these two expressions must be equal, we have

$$R\Omega = (R + r)\frac{d\gamma}{dt} - r\omega \tag{15-60}$$

It is to be noted that in this section $d\gamma/dt$ is not equal to ω.

With the eight equations we have written it is possible to evaluate any one of the accelerations and hence obtain the equations of motion of either cylinder.

Problem.—With the arrangement as given above, but with friction considered to be zero, write the relations from which the equations of motion may be obtained.

15-13. The Swinging Bar.—A uniform bar is hung from a rigid support by a massless cord which is attached to one end of the bar. Initially the bar is at rest in a horizontal position with the cord vertical.

Find the acceleration of the center of mass of the rod (*a*) by using the coordinate relations, and (*b*) by the use of the standard forms of components of acceleration J_r and J_γ. (*c*) By means of the work equation write an equation which expresses the angular velocity of the rod in terms of the angular displacement. (*d*) Using the force and force-moment equations, express the angular acceleration of the rod in terms of the x and y components of acceleration of the center of mass and the angular displacements.

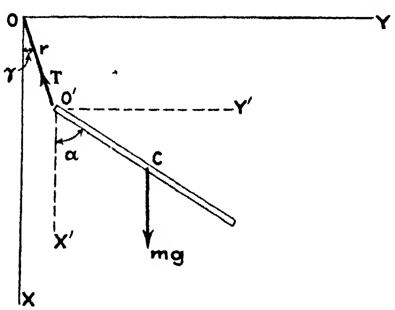

FIG. 145.

Let the reference system (Fig. 145) be placed so that the XY plane is the vertical plane of motion, with the X-axis vertical, the Y-axis horizontal, and the origin O at that point of the rigid support to which the cord is attached. Also let r be the length of the cord, $2l$ the length of the bar, and γ and α the angles which the X-axis makes with the cord and bar, respectively.

It will be convenient to employ a moving coordinate system $X'O'Y'$ fixed to the bar, with axes parallel to the reference axes and with origin O' at one end of the bar.

There are two forces acting upon the bar. The weight of the bar mg (where m is the mass of the bar) acts vertically downward through C, the center of mass of the bar. The cord exerts a force, which we may call T, upon one end (O') of the bar. The direction of T will always be toward O, the point of support.

a. The Coordinate Relations.—If the coordinates of the center of mass of the rod are x and y in the reference system, we may write the following relations for any position of the bar:

$$x = r\cos\gamma + l\cos\alpha \qquad y = r\sin\gamma + l\sin\alpha$$

Differentiating these equations with respect to the time and writing ω for $d\gamma/dt$ and Ω for $d\alpha/dt$ gives

$$\frac{dx}{dt} = -r\omega \sin \gamma - l\Omega \sin \alpha \quad \frac{dy}{dt} = r\omega \cos \gamma + l\Omega \cos \alpha \qquad (15\text{-}61)$$

Differentiating again with respect to the time gives expressions for the components J_x and J_y of the acceleration of C:

$$J_x = -r\omega^2 \cos \gamma - r\frac{d\omega}{dt} \sin \gamma - l\Omega^2 \cos \alpha - l\frac{d\Omega}{dt} \sin \alpha$$

$$J_y = -r\omega^2 \sin \gamma + r\frac{d\omega}{dt} \cos \gamma - l\Omega^2 \sin \alpha + l\frac{d\Omega}{dt} \cos \alpha \qquad (15\text{-}62)$$

b. The r and γ Components.—The acceleration of C may be expressed by applying the theorem of the change of origin for accelerations. We must express the acceleration of C with reference to the moving coordinate system $X'O'Y'$ and the acceleration of the moving system in the fixed reference system. The vector sum of these two accelerations gives the resultant acceleration of C. In place of writing the accelerations as vectors it is desired here to write them in terms of their components parallel to the two reference axes. The components of the acceleration of C in the moving coordinate system, parallel to the X'- and Y'-axes, respectively, are

$$-l\Omega^2 \cos \alpha - l\frac{d\Omega}{dt} \sin \alpha, \qquad -l\Omega^2 \sin \alpha + l\frac{d\Omega}{dt} \cos \alpha$$

Similarly the components of the acceleration of O' in the fixed reference system are

$$-r\omega^2 \cos \gamma - r\frac{d\omega}{dt} \sin \gamma, \qquad -r\omega^2 \sin \gamma + r\frac{d\omega}{dt} \cos \gamma$$

The components J_x and J_y of the acceleration of C in the reference system may now be written by combining the foregoing expressions. The results of the combinations have already been written in Eqs. (15-62).

c. The Work Equation.—The work done by the external forces in producing a displacement of the rod may be expressed in terms of the loss of potential energy of the rod. The change in the potential energy (Δ P.E.) is equal to the weight of the rod multiplied by the vertical displacement. With the initial position of the rod at $\gamma = 0$ and $\alpha = 90°$, the loss of potential energy or the work done upon the rod is

$$\Delta \text{ P.E.} = mg\,(l \cos \alpha + r \cos \gamma - r)$$

Assuming the presence of conservative forces only, we may equate the loss of potential energy to the gain in kinetic energy, translational and rotational. Putting the moment of inertia of the rod about an axis through the center of mass equal to $\frac{1}{3}ml^2$ and using the expressions given in Eqs. (15-61) for the components of the linear velocity of the center of mass, we may write the complete work equation as follows:

$$mg(l \cos \alpha + r \cos \gamma - r) = \tfrac{1}{2}m[(-r\omega \sin \gamma - l\Omega \sin \alpha)^2 + (r\omega \cos \gamma + l\Omega \cos \alpha)^2] + \tfrac{1}{6} ml^2\Omega^2 \quad (15\text{-}63)$$

d. The Force Equations.—The force equations for the motion of the rod may be written for forces parallel to the reference axes. These equations are

$$\begin{aligned} mg - T \cos \gamma &= m J_x \\ -T \sin \gamma &= m J_y \end{aligned} \quad (15\text{-}64)$$

The force-moment equation for rotational motion about an axis through the center of mass is

$$-T\, l \sin (\alpha - \gamma) = \tfrac{1}{3}\, m\, l^2 \frac{d\Omega}{dt} \quad (15\text{-}65)$$

Solving this equation for T and substituting in the force equations gives the desired expressions

$$\begin{aligned} mg + \frac{ml \cos \gamma}{3 \sin (\alpha - \gamma)} \frac{d\Omega}{dt} &= m J_x \\ \frac{ml \sin \gamma}{3 \sin (\alpha - \gamma)} \frac{d\Omega}{dt} &= m J_y \end{aligned} \quad (15\text{-}66)$$

Problems.—1. Differentiate Eq. (15-63) with respect to the time and show that the resulting expression may be obtained from Eq. (15-66).

2. An hourglass, with all of the sand in the lower glass, is accurately counterpoised on a balance. Determine whether or not a state of balance will be maintained when the glass is inverted and the sand is running. Write the necessary equation to prove your answer.

3. One end of a massless thread, which is wrapped around a spool, is fastened to a rigid support. Initially, the spool is at rest with its center of mass in a vertical line which passes through the point of support and a length L of the thread, which is unwound, is stretched tight. If, after releasing the spool, the axis remains horizontal, find the acceleration of the spool and the tension of the thread in any position.

4. A dumb-bell with its axis horizontal is placed on an inclined plane which makes an angle α with a horizontal line. A thread is wrapped around the handle of the dumb-bell and passes up and parallel to the plane over a

pulley at the upper end of the plane. A weight hangs from the free end of the thread. If the system is initially at rest and the moment of inertia of the pulley is to be considered, find the acceleration of the dumb-bell and the hanging weight and also find the tensions in the thread.

5. A circular groove is cut in the middle section of a cylinder so that the plane of the groove is perpendicular to the axis of the cylinder. The depth of the groove is one-fourth of the diameter of the cylinder. A thread is wound around the bottom of the groove. The cylinder is placed upon a horizontal plane and a constant force is applied to the free end of the thread. If the coefficient of friction between the cylinder and the plane is 0.1 and the angle (θ) which the thread makes with the plane is kept constant, in which direction will the cylinder roll? What effect will different values of θ have upon direction of motion? Consider values of θ which vary from 0 to π.

6. A body of mass M has a cylindrical hole cut through it so that it may slide without friction upon a horizontal rod. One end of a massless thread of length L is attached to the lower side of body and the other end is fastened to a particle of mass N. With both masses initially at rest and with the thread stretched tight and making an angle α with the rod, find the distance the body will slide along the rod while the thread is moving into a vertical position, upon releasing the two masses. (Use the principles of conservation of momentum and energy.) Also find expressions for the angular velocity of the thread and for the tension of the thread.

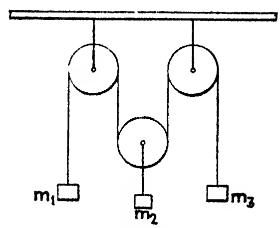

Fig. 146.

7. A system of pulleys, two fixed and one movable, are arranged as shown in Fig. 146. A massless thread is placed as indicated, with the masses m_1 and m_3 attached to the free ends. The four portions of the thread, which are between the pulleys or masses and pulleys, are vertical. The moments of inertia and radii of the three pulleys are equal. The mass of the movable pulley is m_2. If the system is initially at rest and is then set free to move, find expressions for the accelerations of the three masses and the tensions in the four vertical portions of the thread.

8. An empty bucket of mass M and a weight of equal mass are hung by a long string over a fixed pulley whose moment of inertia is I. An arrangement is provided whereby a stream of water falling vertically may be caught by the bucket as it falls. Assume that the water leaves the supply tank at a speed of q cm. per second and at the rate of n g. per second. The bucket is of a cylindrical shape and has an internal diameter of D cm. If the bucket and its counterpoise are initially at rest and the bottom of the bucket is at a distance L cm. below the orifice in the supply tank, find an expression for the acceleration of the bucket.

9. Two masses, m and M, are suspended by a string over a pulley whose moment of inerta is I and whose radius is r. The system is released from rest, the masses move a distance L, and then the larger mass (M) strikes an inelastic object. Assuming that the string does not slip on the pulley, find the velocity with which M is jerked from the inelastic object. How far will the masses then move before coming to rest?

CHAPTER XVI

GENERAL MOTION OF A RIGID BODY

16-1. Introduction.—In this chapter we shall present some of the fundamental equations which describe the motions of a system of particles and of a rigid body, using a different viewpoint from that used in the preceding chapters. The difference is largely in a more complete use of the vector form of expression. The student should have, at this stage of his mastery of the subject, a sufficiently secure command of the fundamental vector processes to appreciate the value of their application to such descriptions as are given, and at the same time to learn those methods which are commonly employed in an advanced treatise. We shall also derive Euler's equations and show how they may be used to describe the motion of a rigid body with one fixed point.

16-2. Motion of a System of Particles.—We start with a system of particles whose masses are $m_1, m_2, \ldots, m_i, \ldots$, etc., and whose position vectors, referred to the origin of a fixed reference system, are, respectively, $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_i, \ldots$, etc. In general, there may be two systems of forces acting upon the particles, one system consists of the external forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_i, \ldots$, etc., and the other includes the internal forces, due to mutual interactions of the particles, $\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_i, \ldots$, etc. The force equation for the ith particle is

$$\mathbf{F}_i + \mathbf{R}_i = m_i \frac{d^2\mathbf{r}_i}{dt^2} \tag{16-1}$$

and for the entire system is

$$\sum_n \mathbf{F}_i + \sum_n \mathbf{R}_i = \sum_n m_i \frac{d^2\mathbf{r}_i}{dt^2} \tag{16-2}$$

Since the $\mathbf{R}$ forces are due to mutual interactions, they will occur in pairs of equal and oppositely directed forces, and hence the sum of such pairs over the entire system will be zero. This term, therefore, vanishes.

We are particularly interested in the motion of the system as a whole and therefore find it necessary to introduce the center of mass of the system if a simplification of Eq. (16-2) is to be obtained. The center of mass has been defined by Eq. (6-4). Here, however, we shall use a vector expression in place of that used above. If $\bar{r}$ gives the position of the center of mass and m is the mass of the system, it is readily seen that

$$\bar{r} = \frac{1}{m}\sum_n m_i r_i \tag{16-3}$$

If we multiply both members by m and differentiate twice with respect to the time, we obtain

$$m\frac{d^2\bar{r}}{dt^2} = \sum_n m_i \frac{d^2 r_i}{dt^2}$$

A substitution of this equivalence in Eq. (16-2) gives

$$\sum_n F_i = m\frac{d^2\bar{r}}{dt^2} \tag{16-4}$$

This equation expresses the fact that the sum of the external forces, which is the resultant force upon the system, is equal to the mass of the system multiplied by the acceleration of the center of mass. Here, again, we see the significance of the center of mass in translational motion. A word of explanation should be included here regarding the present use of the Σ sign. As here employed, it indicates the *vector* sum of the quantities expressed.

Problem.—Find the center of mass of the four particles whose masses are 2, 3, 4, and 5 g., and whose positions are given respectively by the vectors $i + 2j + k$, $2i + j - k$, $i - 2j - k$, and $i - 2j + 2k$.

16-3. Translational Momentum of a System of Particles.—We may obtain the impulse equation for the system of particles by first multiplying each term of Eq. (16-2) by dt, which gives

$$\sum_n F_i\, dt + \sum_n R_i dt = \sum_n m_i \frac{d^2 r_i}{dt^2}\, dt$$

and then integrating from 0 to t yields the following result:

$$\sum_n \int_0^t F_i\, dt = \sum_n m_i V_i - \sum_n m_i U_i \tag{16-5}$$

in which V_i and U_i are the final and initial velocities of the ith particle.

The first term of the right-hand member is the translational momentum of the system at the time t and the last term expresses the value of the same quantity at the time $t = 0$. The right-hand member represents the change in the translational momentum which occurs in the time interval t. This quantity may be represented by a single vector quantity. If we put

$$G = \sum_n m_i V_i, \qquad G_0 = \sum_n m_i U_i, \qquad \text{and} \qquad F = \sum_n F_i \tag{16-6}$$

Eq. (16-5) may be written as follows:

$$\int^t F\, dt = G - G_0 \tag{16-7}$$

The left-hand member is the impulse of the resultant force acting on the system. This equation is the impulse equation for a system of particles.

The quantity G (or G_0), as defined by Eq. (16-6), is the vector quantity obtained by adding the momenta of the particles. G may also be expressed in terms of the velocity of the center of mass of the system. Again we return to the center of mass equation [Eq. (16-3)], which we may differentiate once with respect to the time with the result that

$$m\frac{d\bar{r}}{dt} = \sum_n m_i \frac{dr_i}{dt} = G \tag{16-8}$$

Hence we may conclude [Eq. (16-7)] that the impulse of the resultant force acting upon a system of particles for a given time interval is equal to the change of momentum of the system during that time interval, and that the momentum may be considered as equivalent to the product of the entire mass of the system by the velocity of the center of mass.

16-4. Rotational Momentum of a System of Particles.—In order to obtain the force-moment equation for the system of particles, we may conveniently use the force equation [Eq. (16-1)] as a starting point. The forces, which are acting upon the ith particle, may be converted to force moments by multiplying them by $r_i \times$. Let us, therefore, multiply each term of Eq. (16-1) by $r_i \times$, which gives

$$\boldsymbol{r}_i \times \boldsymbol{F}_i + \boldsymbol{r}_i \times \boldsymbol{R}_i = m_i\, \boldsymbol{r}_i \times \frac{d^2\boldsymbol{r}_i}{dt^2} \tag{16-9}$$

The two force moments are written with respect to the origin (to which $\boldsymbol{r}_i$ is referred). The right-hand member is to be converted into an expression for moment of momentum, which means that we must introduce a first time derivative of $\boldsymbol{r}_i$ and at the same time remove its second derivative. This is readily done by using the following relation:

$$\begin{aligned}\frac{d}{dt}\left(\boldsymbol{r}_i \times \frac{d\boldsymbol{r}_i}{dt}\right) &= \boldsymbol{r}_i \times \frac{d^2\boldsymbol{r}_i}{dt^2} + \frac{d\boldsymbol{r}_i}{dt} \times \frac{d\boldsymbol{r}_i}{dt} \\ &= \boldsymbol{r}_i \times \frac{d^2\boldsymbol{r}_i}{dt^2}\end{aligned} \tag{16-10}$$

The last term in the first expression drops out because the vector product of a vector by itself is zero. Introducing this equivalence in Eq. (16-9) gives

$$\boldsymbol{r}_i \times \boldsymbol{F}_i + \boldsymbol{r}_i \times \boldsymbol{R}_i = m_i \frac{d}{dt}\left(\boldsymbol{r}_i \times \frac{d\boldsymbol{r}_i}{dt}\right) \tag{16-11}$$

This equation is the force-moment equation for a single particle. The corresponding equation for the system of particles may be obtained by taking the sum for the n particles. It may readily be seen that the sum of the quantities represented by the second term is zero, provided that the $\boldsymbol{R}$'s occur in pairs of equal and oppositely directed forces. There remains the following expression:

$$\sum_n \boldsymbol{r}_i \times \boldsymbol{F}_i = \sum_n m_i \frac{d}{dt}\left(\boldsymbol{r}_i \times \frac{d\boldsymbol{r}_i}{dt}\right) \tag{16-12}$$

which is the force-moment equation for the system.

The equation, which gives the impulse of the force moment for the system, is readily written by multiplying both members of Eq. (16-12) by dt and integrating from 0 to t. Carrying out this procedure and introducing the symbol $\boldsymbol{L}_i$ for $\boldsymbol{r}_i \times \boldsymbol{F}_i$ gives

$$\sum_n \int_0^t \boldsymbol{L}_i\, dt = \sum_n m_i\, (\boldsymbol{r}_i \times \boldsymbol{V}_i)_t - \sum_n m_i\, (\boldsymbol{r}_i \times \boldsymbol{V}_i)_0 \tag{16-13}$$

It is to be noticed that the subscripts 0 and t, as used in the right member, are to identify the moments of momentum of the ith particle at the beginning and end of the time interval over which the integration is extended. During this interval, in general, both r_i and V_i change. It would be incorrect to write $\sum_n m_i (r_i \times U_i)$ for the last term.

The moment of momentum or rotational momentum for the system may be designated by the symbol $\boldsymbol{H}$, using the subscripts t and 0 (*i.e.*, H_t and H_0) to designate the final and initial values. The left member of Eq. (16-13) expresses the sum of the impulses of the force moments. We may call this sum the resultant rotational impulse and identify it by the symbol J. With these abbreviations the rotational impulse equation becomes

$$J = H_t - H_0 \tag{16-14}$$

This equation is valid for any system of particles and therefore applies to a rigid body, since the latter may be regarded as a system of particles rigidly connected together. The rotational momentum (H) is not equal to $I\omega$ (moment of inertia times angular velocity) because the r's, which are used in the definition of $\boldsymbol{H}$, are measured from a point, while in moment of inertia the scalar r's are measured from a line, the axis to which the moment of inertia is referred. A similar observation may be made for $\boldsymbol{L}$ to distinguish it from the moment of force (M) as used in some of the preceding chapters.

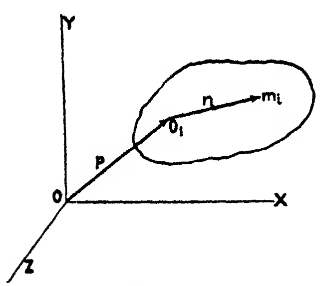

FIG. 147.

16-5. Force-moment Equation for a Rigid Body.—In addition to developing, in this section, the force-moment equation for a rigid body, we shall prove a very important theorem which extends the range of application of the fundamental equations. The theorem shows that the same general equation is valid whether the rotational quantities force moment and rate of change of rotational momentum be referred to the center of mass of the rigid body or to any point in the reference system. The development also shows that we may include those cases of motion of a rigid body which have one point fixed, provided the fixed point be selected as origin for the vector quantities. Furthermore, the fixed point, in the latter case, need not be the center of mass.

We shall select a reference system XYZ with origin at O. Let the center of mass of the rigid body be at O_1, the origin of a moving system fixed in the body. The position of O_1 in the fixed system is given by the vector $\boldsymbol{p}$ (Fig. 147), and the position of m_i, any particle of the body, in the moving system is given by $\boldsymbol{r}_i$. If we let $\boldsymbol{V}_i$ be the velocity of the particle m_i, referred to the fixed origin, then

$$\boldsymbol{V}_i = \dot{\boldsymbol{p}} + \dot{\boldsymbol{r}}_i \tag{16-15}$$

in which $\dot{\boldsymbol{p}}$ is the velocity of the center of mass, and $\dot{\boldsymbol{r}}_i$ is the velocity of m_i with respect to the center of mass.

The rotational momentum of the system (referred to the fixed system) may be identified by the symbol $\boldsymbol{H}'$ and may be evaluated in terms of $\boldsymbol{p}$ and $\boldsymbol{r}_i$ in the following manner:

$$\begin{aligned} \boldsymbol{H}' &= \sum_n m_i \, (\boldsymbol{p} + \boldsymbol{r}_i) \times \boldsymbol{V}_i \\ &= \sum_n m_i \, (\boldsymbol{p} + \boldsymbol{r}_i) \times (\dot{\boldsymbol{p}} + \dot{\boldsymbol{r}}_i) \\ &= \sum_n m_i \boldsymbol{p} \times \dot{\boldsymbol{p}} + \sum_n m_i \boldsymbol{p} \times \dot{\boldsymbol{r}}_i + \sum_n m_i \boldsymbol{r}_i \times \dot{\boldsymbol{p}} + \sum_n m_i \boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i \end{aligned} \tag{16-16}$$

Since $\boldsymbol{p}$ and $\dot{\boldsymbol{p}}$ are common to all particles, the first term of the right member may be simplified by writing m for $\sum_n m_i$. In the third term $\dot{\boldsymbol{p}}$ may be put outside the summation sign. This change leaves $\sum_n m_i \boldsymbol{r}_i$ which is zero, since $\boldsymbol{r}_i$ is measured from the center of mass. The second term is also zero as the student may readily see. The fourth term is the rotational momentum of the body with respect to the center of mass. We may use $\boldsymbol{H}$ to represent this quantity. By using these values Eq. (16-16) becomes

$$\boldsymbol{H}' = m \, \boldsymbol{p} \times \dot{\boldsymbol{p}} + \boldsymbol{H} \tag{16-17}$$

If we let $\boldsymbol{L}'$ be the resultant force moment (referred to the fixed system) and $\boldsymbol{F}_i$, as above, be the force acting on the ith particle, then

$$\boldsymbol{L}' = \sum (\boldsymbol{p} + \boldsymbol{r}_i) \times \boldsymbol{F}_i \tag{16-18}$$

$$= \sum_n p \times F_i + \sum_n r_i \times F_i$$

$$= p \times \sum_n F_i + L \tag{16-19}$$

We have introduced the symbol L in the last term to express the sum of the force moments with respect to the center of mass. It may be readily shown by using Eq. (16-12) that

$$L' = \frac{dH'}{dt} \tag{16-20}$$

We may now eliminate H' and L' from Eqs. (16-17) and (16-19) by using Eq. (16-20) which gives

$$p \times \sum_n F_i + L = \frac{d}{dt} m\, p \times \dot{p} + \frac{dH}{dt} \tag{16-21}$$

The first terms of the two sides of the equation are equal. This is not obvious but can be shown by the following considerations. First we may put

$$\sum_n F_i = m \frac{d^2 p}{dt^2}$$

by referring to Eq. (16-4) and remembering that $\bar{r}$ of Eq. (16-4) is equivalent to p of the present equation. Hence

$$p \times \sum_n F_i = p \times m \frac{d^2 p}{dt^2}$$

The first term of the right member of Eq. (16-21) may be altered by carrying out the indicated differentiation, or

$$\frac{d}{dt} m\, p \times \dot{p} = m\, p \times \frac{d^2 p}{dt^2} + m\, \dot{p} \times \dot{p}$$

$$= m\, p \times \frac{d^2 p}{dt^2}$$

since the last term is zero. Hence we see that the two terms are equal and may therefore be canceled from Eq. (16-21). The final result is

$$L = \frac{dH}{dt} \tag{16-22}$$

This simplified equation, obtained from equations in which the rotational quantities were referred to a fixed point as origin, show that, as far as the rotational effects are concerned, we may express the vectors in a reference system which has the origin at the center of mass. The body must be rigid. The motion of the body may be of any character. No point of the body need be fixed. In case one point of a body is fixed, the equation is applicable, provided we select the fixed point as origin for the reference system. If no point of the body is fixed, the center of mass must be the origin to which the quantities are referred.

16-6. The Inertial Constants and the General Equation.—For purposes of expression the single vector $\boldsymbol{H}$ serves admirably; but when one has a concrete problem to solve, an expansion of $\boldsymbol{H}$ in terms of the inertial constants and components of angular velocity is necessary. In order to evaluate $\boldsymbol{H}$ we shall use either a fixed point of the body or the center of mass as origin of the reference system, which is attached to the body. The expression for $\boldsymbol{H}$ may be manipulated as follows:

$$\boldsymbol{H} = \sum_n m_i \boldsymbol{r}_i \times \boldsymbol{V}_i \tag{16-23}$$

$$= \sum_n m_i \boldsymbol{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{r}_i)$$

We have put $\boldsymbol{V}_i = \boldsymbol{\omega} \times \boldsymbol{r}_i$ in which $\boldsymbol{\omega}$ is the angular velocity of the rotating body measured by a nonrotating set of axes whose origin may be selected as instantaneously coincident with the origin of the set of axes fixed in the body. The vector $\boldsymbol{V}_i$ expresses the velocity of the ith particle relative to this nonrotating system. If we expand the triple vector product, the result is

$$\boldsymbol{H} = \sum_n m_i[(\boldsymbol{r}_i \cdot \boldsymbol{r}_i)\boldsymbol{\omega} - (\boldsymbol{r}_i \cdot \boldsymbol{\omega})\boldsymbol{r}_i] \tag{16-24}$$

If we let the coordinates of m_i be x, y, and z in the coordinate system fixed in the body, the rotational momentum may be expanded further by using the vector relations

$$\boldsymbol{r}_i = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z \qquad \text{and} \qquad \boldsymbol{\omega} = \boldsymbol{i}\omega_x + \boldsymbol{j}\omega_y + \boldsymbol{k}\omega_z$$

We may substitute these values and carry out the details of the indicated scalar products. It is perhaps more convenient at this point to replace the summation by an integration and, of course, to write dm for m_i. The details of the process of expansion are somewhat lengthy but offer no difficulty and may therefore be left for the student to carry out. We may abbreviate the final result by introducing the six constants of inertia A, B, C, D, E, and F which are defined by Eqs. (8-39). The final result is

$$\boldsymbol{H} = \boldsymbol{i}(A\omega_x - F\omega_y - E\omega_z) + \boldsymbol{j}(B\omega_y - D\omega_z - F\omega_x) + \boldsymbol{k}(C\omega_z - D\omega_y - E\omega_x) \quad (16\text{-}25)$$

Our next step is to substitute this value for $\boldsymbol{H}$ in the force-moment equation [Eq. (16-22)]. If we differentiate $\boldsymbol{H}$ with respect to the time, we must remember that, since the unit vectors are parallel to moving axes, $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are not constant. The time derivatives of the unit vectors are the rates of change of direction of the moving axes and consequently may be expressed in terms of the components of $\boldsymbol{\omega}$. The following equations express the desired relations

$$\frac{d\boldsymbol{i}}{dt} = \omega_z\boldsymbol{j} - \omega_y\boldsymbol{k}, \qquad \frac{d\boldsymbol{j}}{dt} = \omega_x\boldsymbol{k} - \omega_z\boldsymbol{i}, \qquad \text{and} \qquad \frac{d\boldsymbol{k}}{dt} = \omega_y\boldsymbol{i} - \omega_x\boldsymbol{j} \quad (16\text{-}26)$$

The details of working out these relations may be left for the student. He should remember that $d\boldsymbol{i}/dt$ must be perpendicular to $\boldsymbol{i}$ and therefore may be expressed in the form $a\boldsymbol{j} + b\boldsymbol{k}$ where a and b are to be evaluated.

If we put

$$\boldsymbol{H} = H_x\boldsymbol{i} + H_y\boldsymbol{j} + H_z\boldsymbol{k}$$

then

$$\frac{d\boldsymbol{H}}{dt} = \boldsymbol{i}\frac{dH_x}{dt} + H_x\frac{d\boldsymbol{i}}{dt} + \boldsymbol{j}\frac{dH_y}{dt} + H_y\frac{d\boldsymbol{j}}{dt} + \boldsymbol{k}\frac{dH_z}{dt} + H_z\frac{d\boldsymbol{k}}{dt} \quad (16\text{-}27)$$

The components of $\boldsymbol{H}$ are given by the coefficients of the unit vectors of Eq. (16-25). The inertial constants are not functions of the time but the components of $\boldsymbol{\omega}$ are; hence, to select one component as an illustration, we have

$$\frac{dH_x}{dt} = A\dot{\omega}_x - F\dot{\omega}_y - E\dot{\omega}_z \quad (16\text{-}28)$$

Corresponding equations may be written for the two other components. We may now write the general equation for the force-moment equation by using Eqs. (16-22), (16-27), and (16-28). It is more convenient to abbreviate the expression by writing H_x, H_y, and H_z in place of their detailed values. The result is

$$L = i(\dot{H}_x - \omega_z H_y + \omega_y H_z) + j(\dot{H}_y + \omega_z H_x - \omega_x H_z) + k(\dot{H}_z + \omega_x H_y - \omega_y H_x) \quad (16\text{-}29)$$

This is an important general equation. In applying it to any particular problem one must remember the limitations used in its derivation. When applied to a given situation, simplifications may be introduced by a judicious selection of the position of the moving axes in the body. If the body possesses symmetry, the axes may be placed so that one or more of the products of inertia will be zero. In case there is a fixed axis, by placing the axes so that one of them will coincide with the fixed axis, two of the components of the angular velocity will be zero. One should look for possible simplifications in any application of the general equation.

Problems.—1. Write the simplified expressions for H and L when the rigid body is rotating about a fixed axis. Put the X-axis coincident with the rotation axis.

2. Write the equations for H and L when there is a fixed axis and when the body is symmetrical with respect to the rotation axis.

16-7. Euler's Equation.—If the rigid body has only one fixed point, we shall take that point as the origin for our moving axes. The axes may usually be so oriented in the body that the products of inertia will be zero. This possibility depends upon the existence of sufficient symmetry. There are many cases to be studied in which these simplifications may be introduced. For such cases, the general equation reduces to the form given below.

If the products of inertia (D, E, and F) are zero, then H becomes

$$H = A\omega_x i + B\omega_y j + C\omega_z k \quad (16\text{-}30)$$

To evaluate L, we may differentiate this equation with respect to the time, but we must remember that the directions of the unit vectors are changing; hence the derivatives of i, j, and k are not zero. By using the values of the derivatives of the unit vectors as given by Eqs. (16-26) the expression for L is readily obtained.

$$L = i[A\dot{\omega}_x + (C - B)\omega_y\omega_z] + j[B\dot{\omega}_y + (A - C)\omega_z\omega_x] + k[C\dot{\omega}_z + (B - A)\omega_x\omega_y] \quad (16\text{-}31)$$

This is Euler's equation.

The components of L may be equated to the corresponding components of the right member. One such equation is

$$\begin{aligned} L_x &= A\dot{\omega}_x + (C - B)\omega_y\omega_z \\ &= \frac{d(A\omega_x)}{dt} + C\omega_y\omega_z - B\omega_y\omega_z \end{aligned} \tag{16-32}$$

It is to be noticed that the first term of the right member is an expression for a time rate of change of a magnitude $(A\omega_x)$. The second and third terms express the rates of change of the directions of angular momenta. In these two types we recognize rotational quantities which correspond to magnitudinal and directional time rates of linear momentum. A more detailed study of the directional changes of angular momentum will be made in a following section.

The chief advantage of Euler's equation is to be found in the fact that, because the moving axes are fixed in the body, the moments and products of inertia are constant. Hence in the equation $L = dH/dt$ the derivatives of these quantities are zero. The derivative of the unit vectors, however, are not constant.

It is well to point out the fact that the components of the angular velocity and angular acceleration along the moving axes are instantaneously equal to the corresponding components taken along axes fixed in space, provided that the two sets of axes are coincident for that particular instant (see Routh, "Dynamics").

16-8. Types of Motion.—The complete analogy which exists between translational and rotational quantities provides a tool

Quantity	Translation, changes in		Rotation, changes in	
	Magnitude	Direction	Magnitude	Direction
Velocity.........	$dV/dt\ \mathbf{V}_1$	$V\ d\mathbf{V}_1/dt$	$d\omega/dt\ \boldsymbol{\omega}_1$	$\omega\ d\boldsymbol{\omega}_1/dt$
Momentum......	$d(mV)/dt\ \mathbf{V}_1$	$mV\ d\mathbf{V}_1/dt$	$d(I\omega)/dt\ \boldsymbol{\omega}_1$	$(I\omega)\ d\boldsymbol{\omega}_1/dt$
	$\rightarrow \mathbf{F}$	$\rightarrow \mathbf{F}$	$\rightarrow \mathbf{L}$	$\rightarrow \mathbf{L}$
	$\rightarrow \mathbf{V}$	$\downarrow \mathbf{V}$	$\rightarrow \boldsymbol{\omega}$	$\downarrow \boldsymbol{\omega}$

which makes it easier to understand the less familiar quantities. We have studied the magnitudinal and directional changes of linear velocity and momentum. We have seen that pure translational motion occurs when there is no change of the direction of the velocity, and that pure rotational motion takes place when the velocity has only directional change. Corresponding to these types of motion in translation there are two types in rotational motion: uniform rotational motion and precessional motion. The table on page 380 is arranged to show these similarities.

16-9. Precessional Motion.—In the study of linear acceleration we observed that the two types of changes of velocity could be obtained by differentiating the velocity V with respect to the time.

$$\frac{dV}{dt} = \frac{d(VV_1)}{dt} = \frac{dV}{dt} V_1 + V \frac{dV_1}{dt}$$

The magnitude of the quantity given by the last term of this equation has been expressed, in terms of the speed V and an angular velocity ω, in the form $V\omega$. The direction of this change is perpendicular to V. The form $V\omega$ is typical of the linear acceleration which expresses the directional time rate of change of the velocity.

In a similar manner we may differentiate the angular velocity $\boldsymbol{\omega}$ and obtain the resulting expression

$$\frac{d\boldsymbol{\omega}}{dt} = \frac{d(\omega\boldsymbol{\omega}_1)}{dt} = \frac{d\omega}{dt} \boldsymbol{\omega}_1 + \omega \frac{d\boldsymbol{\omega}_1}{dt} \qquad (16\text{-}33)$$

in which $\boldsymbol{\omega}_1$ is a unit vector in the line of $\boldsymbol{\omega}$. The first term of the right member expresses the magnitude change of the angular velocity. The last term may be changed by putting $\omega\, d\boldsymbol{\omega}_1/dt = \omega\Omega\, \mathbf{v}_1$ where Ω is an angular velocity and $\mathbf{v}_1$ is a unit vector perpendicular to $\boldsymbol{\omega}_1$. The quantity $\omega\Omega$ is typical of an angular acceleration which expresses the directional rate of change of an angular velocity. The angular velocity Ω is the time rate at which the direction of the angular velocity is changing.

Motions in which there is a directional change of the angular velocity are called *precessional* motions, and Ω is the rate of precession.

The relative directions of the angular velocity $\boldsymbol{\omega}$, the angular acceleration which we may designate by the symbol $\boldsymbol{\alpha}$, and the precessional angular velocity $\boldsymbol{\Omega}$ may be shown advantageously by the use of a diagram (Fig. 148). If $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are both positive and are represented along the X- and Y-axes, then $\boldsymbol{\Omega}$ will be in the positive direction along the Z-axis. The vector equation

$$\boldsymbol{\alpha} = \boldsymbol{\Omega} \times \boldsymbol{\omega}$$

expresses the relation correctly, for this special case, as to both magnitude and direction.

With this analysis of the kinematical relations in mind it is instructive to examine the dynamical relation. For this purpose we may use Euler's equation. To avoid unnecessary complication, we may select the special case in which the force moment $\boldsymbol{L}$ is perpendicular to the angular velocity $\boldsymbol{\omega}$. Let the moving system be placed so that $\boldsymbol{\omega}$ is parallel to the X-axis and $\boldsymbol{L}$ parallel

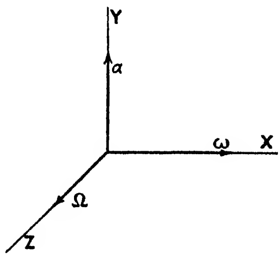

FIG. 148.

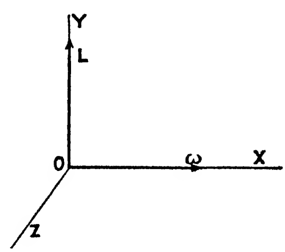

FIG. 149.

to Y (Fig. 149). With this limitation Euler's equation becomes

$$L = [A\dot{\omega}_x + (C - B)\omega_z\omega_x]\boldsymbol{j}$$

For the particular position selected, the $\boldsymbol{i}$ and $\boldsymbol{k}$ components of $\boldsymbol{L}$ are zero. Since $\boldsymbol{L}$ is perpendicular to $\boldsymbol{\omega}$ (or ω_x), there can be no change in the magnitude of $\boldsymbol{\omega}$ and consequently the term $A\dot{\omega}_x$ is zero. The angular velocity ω_z becomes the precessional rate and may be written Ω. Hence the equation becomes

$$L = [(C - B)\ \Omega\omega]\boldsymbol{j} \tag{16-34}$$

The quantities $C\Omega\omega$ and $B\Omega\omega$ are the rates of change of the directions of the rotational momenta about the Z- and Y-axes, respectively. There is, of course, no change in the magnitude of the rotational momentum about the Y-axis, because $\boldsymbol{\omega}$ has no component along that line.

The equations written above express an instantaneous relation. If $\boldsymbol{L}$ and $\boldsymbol{\omega}$ remain perpendicular and $\boldsymbol{L}$ is constant in magnitude, then the motion is called *steady* precessional motion. The magnitudes of $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ will be constant under this limitation.

16-10. Euler's Angles.—The quantities contained in Euler's equations are referred to a set of axes fixed in the body and therefore moving with the body. In order to observe the progress of the motion, it is desirable to express the position of the moving coordinate system by means of three angular coordinates measured with respect to a fixed reference system. These angular coordinates are usually selected according to the plan used by Euler and therefore are called Euler's angles.

Let $X_0Y_0Z_0$ be the fixed system and XYZ the moving system both with origin at O. To afford a means of expressing angular displacements, let us imagine a sphere of unit radius to be constructed about O as center. In the diagram (Fig. 150) the use of the arcs of the great circles in which the reference planes intersect the spherical surface greatly assists us in visualizing the relations. Euler's angles are usually designated by ψ, θ, and φ. Imagine the two systems initially coincident. The moving system is first rotated about OZ_0 through an angle ψ which brings OX up to ON. (This line is called the nodal line.) From this position it is turned about ON through θ, bringing OZ into its final position. The third angular displacement φ is about the OZ-axis. As shown in the diagram, all displacements are taken in the positive sense.

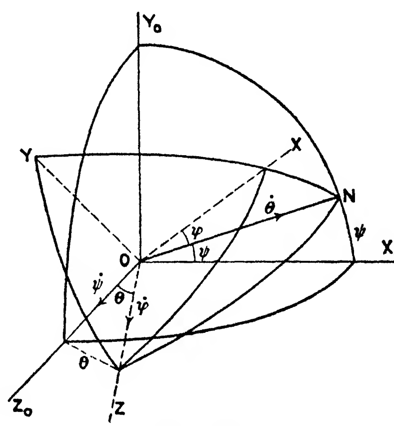

FIG. 150.

We shall next proceed to write equations which express the relations between the two sets of angular velocity components. The components of the resultant angular velocity ω may be written $\dot\psi$, $\dot\theta$, and $\dot\varphi$ and are taken parallel to the axes about which the angular displacements are taken, *viz.*, OZ_0, ON, and OZ. We may now write ω_x, ω_y, and ω_z in terms of $\dot\psi$, $\dot\theta$, and $\dot\varphi$ by projecting the latter set into the lines of the moving axes. Hence

$$\begin{aligned}\omega_x &= \dot\theta \cos\varphi + \dot\psi \sin\theta \sin\varphi \\ \omega_y &= -\dot\theta \sin\varphi + \dot\psi \sin\theta \cos\varphi \\ \omega_z &= \dot\varphi + \dot\psi \cos\theta \end{aligned} \qquad (16\text{-}35)$$

These equations are useful in connection with Euler's equation to determine the motion of the moving system. In any given case we may substitute the values given here for ω_x, ω_y, and ω_z in Euler's equation. The resulting vector equation may then be written as three scalar equations by equating the coefficients of similar unit vectors. This procedure gives three differential equations of the second order in θ, φ, and ψ. The solution of these equations supplies the desired information.

Problem.—Solve Eqs. (16-35) for ψ, $\dot{\theta}$, and $\dot{\varphi}$.

16-11. Precessional Motion of a Heavy Top.—Let us consider the motion of a symmetrical top in the fixed reference system $X_0Y_0Z_0$ (Fig. 151) in which the OZ_0-axis is vertical and the two others are in the horizontal plane. The moving axis OZ is taken along the axis of symmetry and makes the angle θ with OZ_0. The positions of the two other moving axes are to be indicated by values of the Euler angles φ and ψ. The fixed point of the top is at the origin. The only two forces to be considered are the weight of the top and the reaction of the support upon the apex of the top. These two forces form a couple whose magnitude is $mgr \sin \theta$, where m is the mass of the top and r is the distance from the origin to the center of mass. We shall assume that the top is spinning at a constant rate ω; hence we may put $\omega_z = \omega$. Since the moments of inertia (A and B) are equal, we may now write the three following scalar equations from Euler's vector equation.

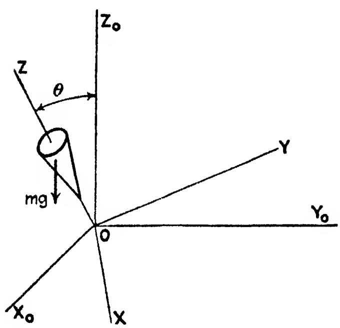

FIG. 151.

$$\begin{aligned} mgr \sin \theta \cos \varphi &= A\dot{\omega}_x + (C - A)\,\omega_y\omega \\ -mgr \sin \theta \sin \varphi &= A\dot{\omega}_y + (A - C)\,\omega\omega_x \\ 0 &= C\dot{\omega} \end{aligned} \qquad (16\text{-}36)$$

An expression for the kinetic energy may be obtained by multiplying the first equation by ω_x, the second by ω_y, and the third by ω, adding the three altered equations and then integrating. The final result is

$$-mgr \cos \theta = \tfrac{1}{2}A(\omega_x^2 + \omega_y^2) + \tfrac{1}{2}C\omega^2 + D \qquad (16\text{-}37)$$

where D is the constant of integration. If $\theta = \theta_0$ at the time when the kinetic energy of the system is T_0, then

$$mgr(\cos \theta_0 - \cos \theta) = \tfrac{1}{2}A(\omega_x^2 + \omega_y^2) + \tfrac{1}{2}C\omega^2 - T_0 \qquad (16\text{-}38)$$

The equation expresses the equivalence between the work done by the couple and the change in kinetic energy.

Since the only couple acting on the top is always perpendicular to the Z_0-axis, the angular momentum about this axis is a constant which we may call h. We may write an expression for h by projecting the angular momentum about the X-, Y-, and Z-axes in the line OZ_0, and equating the sum of these projections to h. This gives

$$h = A\omega_x \sin\theta \sin\varphi + A\omega_y \sin\theta \cos\varphi + C\omega \cos\theta \quad (16\text{-}39)$$

By using the energy and momentum equations [Eqs. (16-38) and (16-39)], expressions containing only $\dot{\theta}$ and $\dot{\psi}$ may be obtained. A substitution of the values of ω_x and ω_y, as given by Eq. (16-35), in Eq. (16-38) gives

$$mgr(\cos\theta_0 - \cos\theta) + \tfrac{1}{2}C\omega^2 - T_0 = \tfrac{1}{2}A(\dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta) \quad (16\text{-}40)$$

Similarly, by putting the values of ω_x and ω_y in Eq. (16-39), we obtain

$$h - C\,\omega\cos\theta = A\,\dot{\psi}\sin^2\theta \quad (16\text{-}41)$$

This equation may be used to express $\dot{\psi}$ in terms of constants and θ. We may, therefore, eliminate $\dot{\psi}$ from Eqs. (16-40) and (16-41) and obtain thereby an equation containing only $\dot{\theta}$ and θ, from which θ could be found. The details of evaluating θ are difficult and may be left for special cases in which the constants are known. If, in any particular problem, θ has been found, then $\dot{\psi}$ may be determined from Eq. (16-41). This value with the known values of ω and θ could be substituted in the third of Eqs. (16-35) for a determination of $\dot{\varphi}$. This information suffices to describe the movement of the XYZ system and therefore that of the top.

16-12. Axes Moving But Not Fixed in the Body.—The problem of the top, which was taken up in the preceding section, may be solved by the use of a set of axes which are moving, but are not fixed in the body. The primary advantage of using axes which are fixed in the body, as was done in Sec. **16-11**, lies in the fact that the inertial constants about those axes cannot be functions of the time. There are cases, such as that of the symmetrical top which we are now considering, in which the axes may not be fixed in the body and still there may be no change of the moment of inertia about these axes.

To use a different procedure in this illustration, we shall assume that the motion is steady precession and determine the necessary force moment for its maintenance.

Let the $X_0Y_0Z_0$ system be fixed as in Sec. **16-11**, with the apex of the top at O. We may let the moving axis OZ always be coincident with the axis of the spinning top, OY is to move in the X_0Y_0 plane, and X is perpendicular to the OY and OZ. The top is to spin about OZ with a constant angular velocity n. The moving axes will rotate about OZ_0 with the precessional angular velocity Ω which is not equal to n. With this arrangement the moments of inertia about OX and OY remain constant, since the top is symmetrical about OZ.

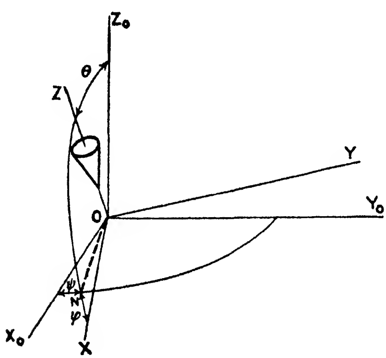

FIG. 152.

Now if ω_x, ω_y, and ω_z are the angular velocities of the top about the X-, Y-, and Z-axes, respectively, and if ω_1, ω_2 and ω_3 are the angular velocities of the X-, Y-, and Z-axes about their instantaneous position, we may write

$$\begin{aligned} \omega_x &= -\Omega \sin\theta & \omega_1 &= -\Omega\sin\theta \\ \omega_y &= 0 & \omega_2 &= 0 \\ \omega_z &= n & \omega_3 &= \Omega\cos\theta \end{aligned} \tag{16-42}$$

It will be observed that these components are written by projecting $\mathbf{\Omega}$, which is along OZ_0, and $\boldsymbol{n}$ (along OZ) into the three moving axes. The significance of each component, however, must be kept in mind.

If H_x, H_y, and H_z be the components of the angular moments about the X-, Y-, and Z-axes, the general values and also the particular values for the present problem are

$$\begin{aligned} H_x &= A\omega_x - F\omega_y - E\omega_z = -A\Omega\sin\theta \\ H_y &= B\omega_y - F\omega_x - D\omega_z = 0 \\ H_z &= C\omega_z - D\omega_y - E\omega_x = Cn \end{aligned}$$

The general and particular values for the components of the force moment, since θ is constant if the motion is steady, are

$$\begin{aligned} L_x &= \dot{H}_x - H_y\omega_3 + H_z\omega_2 = 0 \\ L_y &= \dot{H}_y - H_z\omega_1 + H_x\omega_3 = Cn\Omega\sin\theta - A\Omega^2\sin\theta\cos\theta \\ L_z &= \dot{H}_z - H_x\omega_2 + H_y\omega_1 = 0 \end{aligned} \tag{16-43}$$

If we assume that L_y is caused by the weight moment of the top and is equal to $mgr \sin \theta$, then the force-moment equation for the assigned motion is

$$mgr \sin \theta = Cn\Omega \sin \theta - A\Omega^2 \sin \theta \cos \theta \qquad (16\text{-}44)$$

It is instructive to examine the precessional velocity Ω. If we solve the equation for Ω, the result is

$$\Omega = \frac{1}{2A \cos \theta}(Cn \pm \sqrt{C^2n^2 - 4Amgr \cos \theta}) \qquad (16\text{-}45)$$

There are, evidently, two values of Ω which may be obtained from the action of a given force moment. These values will both be real if the quantity under the radical is positive. There will be only one real value if

$$C^2n^2 = 4Amgr \cos \theta \qquad (16\text{-}46)$$

If the quantity under the radical is negative, Ω will be imaginary. In other words, there will be no precessional motion if n is less than that value which satisfies Eq. (16-46) for a given set of constants.

One other important conclusion may be drawn from the results here obtained. If the force moment in Eq. (16-44) is put equal to zero, we may still have precessional motion. With this assignment the right member may be written

$$Cn\Omega = A\Omega^2 \cos \theta$$
$$\Omega = 0$$

or

$$\Omega = \frac{Cn}{(A \cos \theta)}$$

Experimentally this condition may be realized by providing a suitable weight on the side of the axis of the top opposite the fixed point. If the top spins with a given value n and the frame holding it is given an angular velocity Ω, the precessional motion will continue even though no force moment is applied. This fact has been demonstrated experimentally.

16-13. Precession of the Earth.—The subject of precessional motion is only the introduction to a rather difficult study. The student should consult the literature to learn more about this interesting part of mechanics. We can hardly bring this short introduction to a close without mentioning one rather important application of the development given above.

The earth may be regarded as a huge top spinning about its axis. Owing to the fact that it is not uniformly spherical in shape, we find, in its motion about the sun, the presence of precession. In fact, the term "precession" was first applied to the progressively changing position of the equinoxes or "precession of the equinoxes." Because of the rapid spin about its axis, the balance between gravitational and centrifugal forces upon the more or less fluid-like matter of the earth has resulted in the production of a shape which may be called an oblate spheroid. For mathematical purposes, we could regard the shape of the earth to be that of an approximately spherical mass with an equatorial belt or girdle.

The gravitational attraction of the sun upon the spherical portion (so considered) of the earth is to be regarded as acting through its center of mass and can therefore have no effect upon the rotational part of the motion. It could change only the velocity of the earth, regarded as a particle. The sun's attraction upon the equatorial girdle produces a force moment because of the differences in the distances from the sun to the various portions of the girdle. When the earth is at perihelion or aphelion, because the attraction on the nearer half of the girdle is greater than that on the farther half, there is a resultant force moment which would tend to rotate the plane of the earth's equator into coincidence with the plane of the ecliptic, provided there was no spin about the axis. The presence of this force moment produces a precessional motion of the earth in much the same way as the weight moment causes a precession of the top in the illustration given above. The combination of the spin about the axis and the force moment caused by the differential gravitational attractions does not tend to make the earth's axis become perpendicular to the plane of the orbit but does cause it slowly to describe a cone about the line passing through the center and perpendicular to the plane of the orbit.

The magnitude of the precessional motion of the earth has been calculated from a knowledge of the constants. The result obtained indicates that the length of time required for the axis completely to describe the cone is about twenty-six thousand years. Along with the conical motion the line of equinoxes rotates in the plane of the orbit with the same period.

In the preceding discussion of the precessional motion of the earth we have tried to simplify the presentation by omitting the part the moon plays in this phenomenon. As a matter of fact, the moon's contribution is 2.4 greater than that of the sun. The reason for this difference is because the moon, though very much smaller, is considerably nearer to the earth. A consideration of the entire phenomenon is rather complicated. In the first place, the plane of the moon's orbit is not parallel to the ecliptic nor does it make a constant angle with the ecliptic. One must also take account of the fact that the force moments due to the sun and the moon are not constant but both have periodic variations with different periods. One result which accrues from the variation of the force moment is to produce variation in the precession which is called nutation (or nodding), that is to say, the angle of the precessional cone is not constant but increases and decreases from a mean value. Another effect of the irregularity of the combined force moments is to produce a movement of the axis of rotation in the earth itself. The whole subject is fascinating and should appeal to the student of mechanics for further study.

Problems.—1. Consider a rigid body which is rotating about a fixed point O. How would you locate the instantaneous axis of rotation in the moving system as well as in the fixed? Derive the equations which describe its position.

2. Consider a case of steady precessional motion. Find the locus of the instantaneous axis in the fixed and moving systems.

3. Find an expression for the kinetic energy of a body for axes fixed in the body by using Euler's equations.

4. Express the time rate of change of the kinetic energy of a body referred to axes fixed in the body in terms of the components of the resultant force moment about the moving axes and the components of angular velocity.

5. Express the resultant angular momentum in terms of the constants referred to a set of axes fixed in the body.

6. If the resultant external force moment is zero, the resultant angular momentum remains constant in magnitude and direction. Is the direction of the resultant angular velocity constant? Does the magnitude of the resultant angular velocity change?

7. Find an expression for the cosine of the angle between the instantaneous axis and the axis of the resultant angular momentum, when the external force moment is zero.

8. If a body is rotating with one point fixed, show that the angular acceleration about an axis, the direction cosines of which are l, m, and n with respect to the moving axes, is $l\dot{\omega}_x + m\dot{\omega}_y + n\dot{\omega}_z$ (Gray).

CHAPTER XVII

OTHER GENERAL PRINCIPLES

17-1. D'Alembert's Principle.—Before the publication of D'Alembert's famous principle in his *Traité de dynamique* (1743), mathematicians of the time had solved numerous dynamical problems. The nature of the problems varied but generally they involved the interactions of several bodies which were connected together by various mechanical contraptions, levers, pulleys, etc. As a rule, the method of solution used was peculiar to the individual problem. Some of the solutions were ingenious but were usually very elaborate or, from our point of view today, were anything but direct. The reason for this procedure was simply because there were so few general principles which were known at that time. It is interesting to observe how the discovery of the general principles, many of which are in common use today, came about as a natural result of this interest in mechanical and other problems. Contributions to the advancement of mathematical physics were made, in this way, by a large number of men such as Bernoulli, Euler, and Huyghens. D'Alembert's principle was one of the most useful at that time. Lagrange thought it sufficiently important to use it as a basis for his development of the entire subject of dynamics.

D'Alembert's principle is readily derived from the fundamental force and force-moment equations. Its uniqueness lies not so much in a novelty of form or in the introduction of new quantities as in a point of view. We shall first explain the principle as it is usually used and illustrate its use by a particular problem. In Sec. **17-6** (below) we shall discuss its application to problems involving constraints and show how the unknown reactions of the constraints may be eliminated. It is in this feature that D'Alembert's principle is of special interest.

Let us consider a system of n particles, upon any one of which there may be two kinds of external forces, those which have their origin in things that are external to the system and those which are caused by mutual interactions between the particles of the system. We may designate, as before, the *external* force on the ith particle by $\boldsymbol{F}_i$ and the *internal* force on the same particle by $\boldsymbol{R}_i$. If m_i is the mass of the particle and $\boldsymbol{r}_i$ is the position vector of the particle referred to some arbitrarily selected reference system, then the force equation for this particle may be written as follows:

$$\boldsymbol{F}_i + \boldsymbol{R}_i = m_i \frac{d^2\boldsymbol{r}_i}{dt^2} \qquad (17\text{-}1)$$

Now the point of view which D'Alembert expressed was with regard to the right-hand member of this equation. Instead of the usual dynamical conception of $m_i\ (d^2\boldsymbol{r}_i/dt^2)$ as a measure or result of the applied forces $\boldsymbol{F}_i$ and $\boldsymbol{R}_i$, he chose to consider the right member as a force—a reaction to the agency supplying the resultant external force $(\boldsymbol{F}_i + \boldsymbol{R}_i)$.

The quantity $m_i\ (d^2\boldsymbol{r}_i/dt^2)$ is, of course, dimensionally equivalent to a force. In fact, from this point of view, the quantity expressed by the product of the mass by the acceleration, or the time rate of change of the momentum, may be treated as a force. This quantity has been called the *force of inertia* or the *effective force*. We shall use the former term.

D'Alembert observed that, if the force of inertia be reversed in direction and then combined with the ordinary applied forces, the vector sum of the entire system was zero. This conception is expressed by the following equation:

$$\boldsymbol{F}_i + \boldsymbol{R}_i - m_i \frac{d^2\boldsymbol{r}_i}{dt^2} = 0 \qquad (17\text{-}2)$$

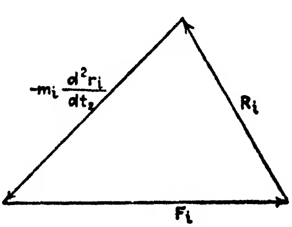

FIG. 153.

and graphically by the diagram of Fig. 153. Mathematically all that has been done was to transpose the right-hand member of Eq. (17-1) to the left side and change its sign.

The next step made by D'Alembert was a more important one. The form in which the equation appeared probably suggested to him the possibility of using the principles of statics to solve dynamical equations similar to the one written. Whether or not the conception was developed in this manner we cannot say, but the important thing is that he found that the idea was feasible and that problems expressed by such equations could be solved by the procedure used in statics.

A necessary and sufficient condition for equilibrium to exist between the forces acting upon the particle and the force of inertia is that the virtual work of these forces for any arbitrary virtual displacement, compatible with the liaisons, must be zero. If $\delta \mathbf{r}_i$ is the virtual displacement for the ith particle, the virtual work for that particle is

$$\left(\mathbf{F}_i + \mathbf{R}_i - m_i \frac{d^2\mathbf{r}_i}{dt^2}\right) \cdot \delta \mathbf{r}_i = 0 \qquad (17\text{-}3)$$

If we sum up such equations, from 0 to n, for the entire system of particles, since the sum of the $\mathbf{R}_i$ forces is zero, the result is

$$\sum_n \left(\mathbf{F}_i - m_i \frac{d^2\mathbf{r}_i}{dt^2}\right) \cdot \delta \mathbf{r}_i = 0 \qquad (17\text{-}4)$$

It is instructive to write a similar equation for a single free particle of mass m upon which the components of the external resultant force are F_x, F_y, and F_z. If the virtual displacements parallel to the reference axes are δx, δy, and δz, the equation is

$$\left(F_x - m\frac{d^2x}{dt^2}\right)\delta x + \left(F_y - m\frac{d^2y}{dt^2}\right)\delta y + \left(F_z - m\frac{d^2z}{dt^2}\right)\delta z = 0 \qquad (17\text{-}5)$$

Since the virtual displacements must be independent of each other, the coefficients of these quantities must be separately equal to zero which gives the necessary three equations for determining the motion of the free particle.

17-2. D'Alembert's Principle for Rigid Bodies.—We shall consider the system of rigid bodies to be made up of a system of particles. We may then write the equations for any particle and express the terms by a sum over the entire system. The introduction of rigid bodies into the consideration necessitates a use of the effects of the force moments in producing rotational motions.

Let the mass of the ith particle be m_i and let its coordinates be x, y, and z in the selected reference system. If F_x, F_y, and F_z be the components of the impressed accelerating forces upon m_i, then, by D'Alembert's principle, these forces together with the forces of inertia upon this particle will be in equilibrium; hence, by the first condition for equilibrium in statics,

$$F_x - m_i\frac{d^2x}{dt^2} = 0, \quad F_y - m_i\frac{d^2y}{dt^2} = 0, \quad \text{and} \quad F_z - m_i\frac{d^2z}{dt^2} = 0$$

For rigid bodies, the second condition for equilibrium requires that the sum of the moments of the forces about each of the three axes must be zero. To apply this condition we must consider the moments of the inertia forces as well as the moments of the impressed forces; hence for the moments about the X-axis we may write

$$yF_z - zF_y - m_i\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right) = 0 \qquad (17\text{-}6)$$

Similar equations may be written for the moments about the two other axes.

The equations for the system may be written by taking the sum over the entire system. For the force equations it is perhaps preferable to write the time rate of the momentum in place of the mass acceleration; *e.g.*,

$$m_i\frac{d^2x}{dt^2} = \frac{d}{dt}\left(m_i\frac{dx}{dt}\right)$$

Similarly for the moment of mass acceleration we may write the time rate of angular momentum; *e.g.*,

$$m_i\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right) = \frac{d}{dt}\left[m_i\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right)\right]$$

Introducing these changes the six equations, which are necessary completely to describe the motion of the system, are

$$\sum_n F_x - \frac{d}{dt}\sum_n m_i\frac{dx}{dt} = 0$$

$$\sum_n F_y - \frac{d}{dt}\sum_n m_i\frac{dy}{dt} = 0$$

$$\sum_n F_z - \frac{d}{dt}\sum_n m_i\frac{dz}{dt} = 0 \qquad (17\text{-}7)$$

$$\sum_n (yF_z - zF_y) - \frac{d}{dt}\sum_n m_i\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) = 0$$

$$\sum_n (zF_x - xF_z) - \frac{d}{dt}\sum_n m_i\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right) = 0$$

$$\sum_n (xF_y - yF_x) - \frac{d}{dt}\sum_n m_i\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = 0 \qquad (17\text{-}8)$$

It will be left to the student to write these equations in the vector form of expression.

In the following section we shall give illustrations of the use of the principle in solving problems.

17-3. Illustrations of the Use of D'Alembert's Principle.—In this section we shall solve three simple problems by using D'Alembert's principle. The problems involve translational motion, rotational motion, and a combination of the two types of motion.

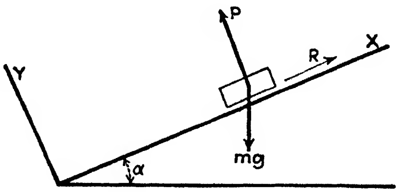

Fig. 154.

a. Translational Type.—A body slides down an inclined plane. Friction is assumed to be constant. Find the equation of motion.

Let mg be the weight of the body, R the force of resistance, and P the normal reaction to the plane. If the reference system is selected as shown in the diagram (Fig. 154), then the equations, written by the use of D'Alembert's principle, which express equilibrium between the forces along the X- and Y-axes, are

$$mg \sin \alpha - R - m\frac{d^2x}{dt^2} = 0$$

$$mg \cos \alpha - P = 0 \quad (17\text{-}9)$$

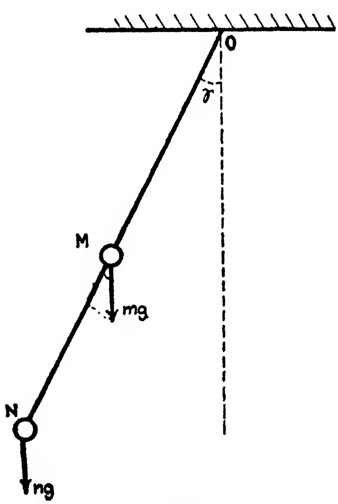

Fig. 155.

These two equations, together with initial conditions, completely determine the motion.

b. Rotational Type.—A massless rod is hung from a rigid support by means of a frictionless hinge so placed that the rod may move only in a vertical plane. Attached to the rod are two small bodies M and N whose masses are to be designated by m and n. The distances from M and N to the center of oscillation O (Fig. 155) we shall call r and s, respectively. It is required to find the differential equation of motion by using D'Alembert's principle.

Since the motion is limited to pure rotation by the nature of the constraints, we may use Eqs. (17-8) for describing the motion. The second condition for equilibrium requires that the sum of the force moments about any axis be zero. A selection of O for the axis of moments eliminates from consideration the reaction of the support at O as well as forces upon M and N which are parallel to the rod.

In place of using rectangular coordinates, it is more convenient to use polar coordinates. The forces of inertia may, in this case, be expressed in terms of the component accelerations, which are parallel and perpendicular to the rod. However, only the inertia forces which are perpendicular to the rod contribute to the moment about O. The moments of the inertia forces may readily be written by the use of the expression for the component of acceleration $J\gamma$ [Eq. (4-15)] if it is remembered that r and s are constant. The moment equation by D'Alembert's principle is therefore

$$mgr \sin \gamma + ngs \sin \gamma - mr^2 \frac{d\omega}{dt} - ns^2 \frac{d\omega}{dt} = 0 \quad (17\text{-}10)$$

or

$$mgr \sin \gamma + ngs \sin \gamma - \frac{d}{dt}(mr^2\omega + ns^2\omega) = 0$$

This equation is the desired differential equation from which, together with the initial conditions, the motion may be determined. The student should verify the signs used in this equation. It is interesting to show that the equation may be converted into one the form of which is similar to that which describes the motion of a single particle.

Problem.—Suppose that the rod carrying the two masses M and N, of the preceding illustration, is made to rotate about the vertical axis through O and makes a constant angle with the vertical axis. Find an expression for the angular velocity of the rod.

c. Translational and Rotational Motion.—For this case we shall consider the motion of a cylinder rolling down an inclined plane.

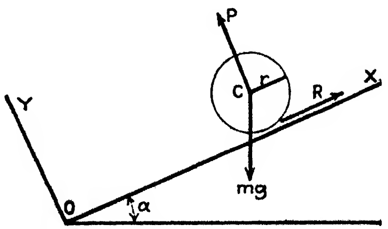

FIG. 156.

Let the mass of the cylinder be m, its radius r, and the angle of inclination of the plane α. The forces upon the cylinder are *mg*, the resistance *R*, and the reaction of the plane *P*. The reference system may be selected as shown in Fig. 156. Equilibrium among the forces parallel and perpendicular to the plane (*OX*) is given by the equations

$$mg \sin \alpha - R - m\frac{d^2x}{dt^2} = 0$$

$$mg \cos \alpha - P = 0 \quad (17\text{-}11)$$

By taking moments about the axis of the cylinder, the balance between the force moments is given by the equation

$$Rr - \frac{1}{2}mr^2\frac{d\omega}{dt} = 0 \quad (17\text{-}12)$$

Eliminating *R* from the first of Eqs. (17-11) and (17-12) gives

$$mg \sin \alpha - \frac{1}{2}mr\frac{d\omega}{dt} - m\frac{d^2x}{dt^2} = 0$$

or

$$mg \sin \alpha - \frac{3}{2}m\frac{d^2x}{dt^2} = 0 \quad (17\text{-}13)$$

A solution of this equation may be readily obtained.

Problem.– Find the differential equation of motion of a ladder which falls with its ends in contact with a rough horizontal floor and a smooth vertical wall, assuming that the friction is not sufficient to prevent slipping.

17-4. D'Alembert's Principle and Conservation of Energy.— We shall consider a system of *n* free particles and show how D'Alembert's principle may be used to derive the principle of the conservation of energy. The system is to be regarded as *free* if there are no conditions imposed upon the way in which the coordinates of the particles may vary, *e.g.*, such as the requirement that the x coordinate of all or any of the particles must remain constant, which would restrict the motion to a single plane. We shall designate the *external* forces upon the ith particle by F_x, F_y, and F_z and the *internal* forces by R_x, R_y, and R_z.

The virtual work of the forces, by D'Alembert's principle, for any arbitrary displacement δs of the ith particle is zero and may be written as follows:

$$\sum_n\left(F_x + R_x - m_i\frac{d^2x}{dt^2}\right)\delta x + \sum_n\left(F_y + R_y - m_i\frac{d^2y}{dt^2}\right)\delta y + \sum_n\left(F_z + R_z - m_i\frac{d^2z}{dt^2}\right)\delta z = 0 \quad (17\text{-}14)$$

where δx, δy, and δz are the components of δs in the selected reference system. Since the sums of the internal forces, for the entire system, must be zero, the preceding equation may be written in the form

$$\sum_n m_i\left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z\right) = \sum_n (F_x\delta x + F_y\delta y + F_z\delta z) \quad (17\text{-}15)$$

We may replace the virtual displacements in this equation, since they are arbitrary and the particles are free, by the actual displacements dx, dy, and dz. With this change the equation may be integrated. If we write v_0 and v for the initial and final velocities of m_i for the particular interval over which the integration is to be extended, and let s_0 and s be coordinates which designate the corresponding terminal positions of m_i, then the equation reduces to

$$\sum_n \tfrac{1}{2}m_i(v^2 - v_0^2) = \sum_n \int_{s_0}^{s} (F_x\,dx + F_y\,dy + F_z\,dz) \quad (17\text{-}16)$$

The right-hand member may be expressed in terms of the potential energy V if the system is a conservative system. In such a case, there exists a force function U from which the forces may be found [see Eq. (10-23)]. Since $V = -U$,

$$(F_x\,dx + F_y\,dy + F_z\,dz) = -dV$$

Hence the integration of the right-hand member of Eq. (17-16) may be carried out. If V_0 and V represent the values of the potential energy in the initial and final configurations, and T_0 and T stand for the kinetic energy in the corresponding positions, Eq. (17-16) may be written in the following abbreviated form

$$T + V = T_0 + V_0 \quad (17\text{-}17)$$

This equation expresses the important fact that in conservative systems there is no change in the energy of the system. This principle is called the *principle of the conservation of energy.*

17-5. System with Constraints.—Constraints are mechanisms or arrangements which influence or restrict the motions of particles or bodies in some definite manner. For example, a particle which is attached by an inelastic string of finite length to some fixed point is constrained to move within the sphere whose radius is equal to the length of the string. A door, which is fastened by hinges to some rigid support, can have only rotational motion about a definite axis. Its motion is therefore subject to a certain constraint. The forces of constraint are those forces which are produced by the constraining mechanism and, in acting upon the body under consideration, modify the motion accordingly.

In the case of a body or particle upon which there are constraining forces, it is possible to regard the body or particle as free, provided we can include the forces of constraint along with the other existing forces. In the case of the simple pendulum the constraining force is the tension of the cord. If this tension could be expressed in suitable terms, we could write the differential equation of motion of the simple pendulum by introducing, in the general equations for a free particle, the resultant force formed by adding the tension of the cord to the weight of the bob. Such a procedure depends upon being able to express the constraining forces.

If the constraining forces were unknown, the equations of motion might still be obtained by using D'Alembert's principle. It is primarily in problems of this nature, where the forces of constraint are unknown, that D'Alembert's principle is particularly advantageous. The key to the procedure lies in being able to select the virtual displacements in equations such as Eq. (17-5) so that the virtual work of the unknown forces is zero, and hence such terms as may contain the constraining forces may be omitted. For example, suppose the tension in the cord of a simple pendulum were unknown. We could use spherical coordinates in place of the ordinary rectangular coordinates and, with the origin at the fixed point, the virtual displacement parallel to the cord would be zero. By a proper orientation of the reference lines a second virtual displacement may be made equal to zero, if the motion is still further restricted to a vertical plane. After zero is put for these two virtual displacements, the remaining terms in the general equation would suffice completely to describe the motion, after the selection of initial conditions. It is a useful exercise for the student to follow the procedure indicated in the case of the simple pendulum and derive the differential equation of the motion.

17-6. Constraints and the Work Equation.—We shall discuss, in this section, a general method by which the constraining forces may be eliminated from the general equations of motion for a system of n particles. Our treatment of the problem will be limited, however, to a system of particles in which the constraints are expressible by means of equations containing the coordinates of the particles and the time, but not containing any time derivatives of the coordinates. Such a system is called a *holonomic* system.

In the system each particle may be subject to one or more constraints. Such a constraint may be expressed by an equation of the form

$$\varphi(x, y, z, t) = 0 \tag{17-18}$$

which in reality represents a surface, the nature of which is to be expressed by a particular equation. If, for example, the particle were constrained to move on the surface of a fixed sphere of radius r, then the equation

$$\varphi(x, y, z, t) = x^2 + y^2 + z^2 - r^2 = 0$$

could be used. The constraint might be such as would require a moving spherical surface. In such a case, terms containing the time would have to be included.

If a single particle were subject to two constraints, then two equations, similar to that written in Eq. (17-18), would be needed. The intersection of the two fixed surfaces is, in general, a curved line upon which the particle would move. In our general case, however, the constraints include the time; hence for two constraints the motion of the particle is restricted to a curve, the shape and position of which are changing. With the imposition of a third constraint, the position of the particle is determined by the instantaneous intersection of three surfaces.

Suppose in the system of n particles that there are k constraints for each particle, then there will be, in general, nk constraints altogether. Since three equations are required completely to describe the motion of a free particle, there will be $3n - k$ independent equations of motion for the system with k constraints for each particle.

Procedure for a Single Particle.—Before applying D'Alembert's principle to the system of particles, it is better to simplify the procedure by considering how the forces of constraint are eliminated when we are dealing with a single particle and only one constraint. Suppose the single particle is given a virtual displacement δs along the surface of the constraint $\varphi(x, y, z, t) = 0$.

By such a selection the work of the constraining force, which is normal to the surface and therefore to δs, is zero. Now if we can introduce, in the general equation, a limitation upon δs so that it must remain in the surface of constraint at the time t, the forces of constraint will be eliminated. If the components of δs along the reference axes are δx, δy, and δz, and if δx, δy, and δz satisfy the equation

$$\frac{\partial \varphi}{\partial x}\delta x + \frac{\partial \varphi}{\partial y}\delta y + \frac{\partial \varphi}{\partial z}\delta z = 0 \tag{17-19}$$

then δs will be in the surface at the particular time t. It will be noticed that the partial derivatives are written with the assumption that t remains constant.

Equations (17-19) and (17-5) together give the equations of motion for the particle on the constraining surface φ. One of the component virtual displacements may be eliminated from the two equations. The single equation resulting thereby would contain the two other virtual displacements. Since these components are purely arbitrary, the coefficients of these two components may be equated to zero, which [with Eq. (17-18)] give the three necessary equations of motion.

Lagrange devised a rather clever way of eliminating one component of the virtual displacement and at the same time provided a form for writing the general equation which is instructive. The device has been called the *method of indeterminate multipliers*. For the simple case of the single particle with one constraint, all we have to do is to multiply Eq. (17-19) by an arbitrary multiplier, which may be called λ, and then adding the result to Eq. (17-5) we obtain

$$\left(F_x + \lambda\frac{\partial \varphi}{\partial x} - m\frac{d^2x}{dt^2}\right)\delta x + \left(F_y + \lambda\frac{\partial \varphi}{\partial y} - m\frac{d^2y}{dt^2}\right)\delta y + \left(F_z + \lambda\frac{\partial \varphi}{\partial z} - m\frac{d^2z}{dt^2}\right)\delta z = 0 \tag{17-20}$$

Now λ is arbitrary; hence we may assign a value to λ such that the coefficient of δx, say, would be equal to zero, or

$$\lambda = \frac{m\dfrac{d^2x}{dt^2} - F_x}{\dfrac{\partial \varphi}{\partial x}} \tag{17-21}$$

The coefficients of δy and δz would also be zero; hence we obtain two equations from Eq. (17-20) which with the equation of the constraint is sufficient to determine the motion.

Before returning to the general case of a system of n particles with several constraints, it is worth while to examine the terms in Eq. (17-20) which contain λ. Each of these terms is a component of the force exerted by the constraint upon the particle. If we let $\boldsymbol{R}$ represent this force, or reaction as it is generally called, then

$$\boldsymbol{R} = \lambda\frac{\partial\varphi}{\partial x}\boldsymbol{i} + \lambda\frac{\partial\varphi}{\partial y}\boldsymbol{j} + \lambda\frac{\partial\varphi}{\partial z}\boldsymbol{k}$$

and

$$R = \lambda\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right]^{\frac{1}{2}} \tag{17-22}$$

If the surface is fixed, the work done by R is zero, as may be readily seen by letting the virtual displacement at the time t be equal to the actual differential displacement. If the actual displacement is $\boldsymbol{ds} = dx\,\boldsymbol{i} + dy\,\boldsymbol{j} + dz\,\boldsymbol{k}$, then the work is found to be zero [see Eq. (17-19)].

$$\boldsymbol{R}\cdot\boldsymbol{ds} = \lambda\left(\frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz\right) \tag{17-23}$$

If, however, the surface is not fixed, then in place of Eq. (17-19) we should have

$$\frac{\partial\varphi}{\partial x}\delta x + \frac{\partial\varphi}{\partial y}\delta y + \frac{\partial\varphi}{\partial z}\delta z + \frac{\partial\varphi}{\partial t}\delta t = 0 \tag{17-24}$$

and the right-hand member of Eq. (17-23) would not be zero. In the general case of a moving surface the work equation formed from Eq. (17-20) would contain terms expressing the work done by the forces exerted by the moving surface upon the particle.

Case of a System of Particles.—For the holonomic system of n particles let the constraints for the ith particle be

$$\varphi(x_i\, y_i\, z_i\, t) = 0 \qquad \psi(x_i\, y_i\, z_i\, t) = 0 \tag{17-25}$$

and others of similar form if they be present. As in the case of a single particle, the displacement of the ith particle will lie in the constraining surfaces at the time t if

$$\begin{aligned} \frac{\partial \varphi}{\partial x_i}\delta x_i + \frac{\partial \varphi}{\partial y_i}\delta y_i + \frac{\partial \varphi}{\partial z_i}\delta z_i &= 0 \\ \frac{\partial \psi}{\partial x_i}\delta x_i + \frac{\partial \psi}{\partial y_i}\delta y_i + \frac{\partial \psi}{\partial z_i}\delta z_i &= 0 \end{aligned} \tag{17-26}$$

If there are k constraints for each particle, then there will be k equations similar to those written. By means of these equations, k of the virtual displacements for each particle may be eliminated from D'Alembert's general equation for virtual work. Making use of Lagrange's method of indeterminate multipliers, we may multiply each of Eqs. (17-26) by λ_i, μ_i, etc., and then add the resulting expressions to the general equation for the system. The results are more conveniently written, however, if we equate each of the coefficients of the virtual displacements to zero and, for the sake of simplicity, omit the subscript i from the coordinates, indeterminate multipliers, and mass. Hence

$$\begin{aligned} \sum_n \left(F_x + \lambda\frac{\partial \varphi}{\partial x} + \mu\frac{\partial \psi}{\partial x} + \cdots \right) &= \sum_n m\frac{d^2x}{dt^2} \\ \sum_n \left(F_y + \lambda\frac{\partial \varphi}{\partial y} + \mu\frac{\partial \psi}{\partial y} + \cdots \right) &= \sum_n m\frac{d^2y}{dt^2} \\ \sum_n \left(F_z + \lambda\frac{\partial \varphi}{\partial z} + \mu\frac{\partial \psi}{\partial z} + \cdots \right) &= \sum_n m\frac{d^2z}{dt^2} \end{aligned} \tag{17-27}$$

in which values for λ, μ, . . . may be arbitrarily selected for each particle so that a corresponding number of virtual displacements will vanish. The resulting equations, together with those for the constraints [Eq. (17-26)], are sufficient completely to determine the motion for the system of particles.

Illustration.—We shall select for illustration a case in which a particle is constrained to move in a circular path under no applied forces. The only forces acting are to be those of the constraint which we shall assume to be unknown. It is to be noticed that these unknown forces will be eliminated by the procedure described above.

Let the particle of mass m move in a circle of radius r. We shall place the reference system so that the plane of the circle is in the XY reference plane with the center of the circle at the origin. There are therefore two constraints, one which keeps the particle in the XY plane and the other requires that it move in a circular path. The equations for these two constraints are

$$\varphi = x^2 + y^2 - r^2 = 0 \qquad \text{and} \qquad \psi = z = 0 \tag{17-28}$$

We may next find the partial derivatives of each of these functions with respect to the coordinates and then, by using equations of the type shown in Eq. (17-19), write in the requirement that the virtual displacement shall be in the two surfaces. These equations are therefore

$$x\,\delta x + y\,\delta y + z\,\delta z = 0 \qquad \text{and} \qquad \delta z = 0 \qquad (17\text{-}29)$$

After multiplying the first of these equations by the indeterminate multiplier λ and the second by μ, we may write the D'Alembert equation for virtual work [see Eq. (17-20)]. Since F_x, F_y, and F_z are zero, the equation is

$$\left(\lambda x - m\frac{d^2x}{dt^2}\right)\delta x + \left(\lambda y - m\frac{d^2y}{dt^2}\right)\delta y + (\lambda z + \mu)\,\delta z = 0 \qquad (17\text{-}30)$$

We may select values for λ and μ so that the coefficients of two of the components (say δx and δz) of the virtual displacements are zero. Hence

$$\lambda = \frac{m}{x}\frac{d^2x}{dt^2} \qquad \text{and} \qquad \mu = -\lambda z$$

The coefficient of δy must also be zero; hence upon substitution of the value for λ in this coefficient we find that

$$\frac{my}{x}\frac{d^2x}{dt^2} - m\frac{d^2y}{dt^2} = 0 \qquad (17\text{-}31)$$

The form of this equation may be altered, after multiplying through by x, so that it may be more easily integrated. The revised form is

$$\frac{d}{dt}\left(my\frac{dx}{dt} - mx\frac{dy}{dt}\right) = 0$$

Hence

$$my\frac{dx}{dt} - mx\frac{dy}{dt} = C \qquad (17\text{-}32)$$

where C is a constant. The physical meaning of the equation is that the moment of the momentum, or the angular momentum, is constant. This equation, together with those for the constraints, is sufficient for a determination of the motion. Since Eq. (17-32) does not contain the coordinate z, only the first equation of the constraints is needed.

The details of integration and of evaluating the constants of integration, of which there are two, may be omitted.

Problems.—1. What are the forces exerted by the constraint in the illustration given above?

2. Find the equations of motion for the particle of the preceding illustration (Sec. **17-6**) by using the coefficients of δy and δz for evaluating λ and μ.

3. A particle of mass m slides down a smooth wire which is bent in the form of an ellipse. The major axis of the ellipse is vertical. The only applied force to consider is the weight of the particle. The reaction of the constraint is to be regarded as unknown. Find the equations of motion.

17-7. Degrees of Freedom.—The phrase *degrees of freedom* is commonly used in connection with constrained motion of particles or rigid bodies to indicate the number of independent coordinates necessary completely to specify the position of the particle or rigid body. A free particle has three degrees of freedom because three independent coordinates are needed to give its position in a selected reference system. If, however, the particle is constrained to move on some fixed surface, it has two degrees of freedom. In the case where two constraints are present, the particle must move on the line of intersection of the two surfaces and has but one degree of freedom.

A free rigid body has six degrees of freedom, for three coordinates are needed to specify the position of some point of the body, such as the center of mass, and three others are required to give its orientation with respect to a moving system, whose origin is at the selected point and whose axes are always parallel, say, to those of the fixed reference system. A rigid body with one fixed point has three constraints and three degrees of freedom. If the rigid body has a fixed axis, there is but one degree of freedom.

17-8. Generalized Coordinates.—In many of the problems encountered in dynamics, particularly when constraints are introduced, it is often convenient to introduce certain parameters which may be useful in specifying the positions of a particle or of a rigid body in place of the ordinary coordinates. The number of such parameters is usually made equal to the number of degrees of freedom in order that all of the parameters may be independent of each other. For example, if a particle is constrained to move upon a cylindrical surface, there are two degrees of freedom, and two parameters would be necessary. One parameter would be required to specify the position of the particle along any element of the surface, and the other would be needed to give the angular position of that element with respect to some reference plane.

The term *generalized coordinates* has been introduced to describe these parameters. These are usually designated by the letter q with appropriate subscripts. If we wished to specify the position of a particle which has one constraint and two degrees of freedom, we should use the two generalized coordinates q_1 and q_2. These two coordinates would be sufficient to locate the position of the particle upon the surface of the constraint.

In general, it would be necessary, or at least convenient, to define the parameters in terms of the ordinary rectangular coordinates by what might be called transformation equations. Such equations may be written in the following general forms:

$$x = f_1(q_1\, q_2) \qquad y = f_2(q_1\, q_2) \qquad z = f_3(q_1\, q_2) \tag{17-33}$$

Suppose that the generalized coordinates q_1, q_2 of a particle which has one constraint be given, and it is desired to find the equations of the constraining surface. It would be only necessary to eliminate the two parameters from these three equations to obtain the desired equation.

Now the principal advantage of using generalized coordinates is to provide a means for removing the necessity of including the constraints. In this respect the use of generalized coordinates is similar to D'Alembert's principle. The differences in the two procedures will appear in the consideration given below. To show how the equations for the constraints are removed by a suitable selection of the generalized coordinates, let us suppose we are concerned with a particle which is constrained to move on the surface of a sphere of radius r. The equation of the constraint is

$$\varphi = x^2 + y^2 + z^2 - r^2 = 0$$

Since there are two degrees of freedom, we shall need two generalized coordinates which we shall call q_1 and q_2. If we put

$$x = r \cos q_1 \qquad y = r \sin q_1 \cos q_2 \qquad z = r \sin q_1 \sin q_2 \tag{17-34}$$

and substitute these values for the rectangular coordinates in the equation for the constraint, we find that it is identically satisfied. Hence the values of q_1 and q_2, as defined by Eq. (17-34), are such that a use of them removes the necessity for including the constraint in the problem.

It will be instructive to write expressions for some of the familiar quantities, using generalized coordinates. We shall again make use of the simple case of a particle with one constraint and two degrees of freedom. In the general sense dq_1 and dq_2 are to be regarded as differential displacements. We can write directly relations between the differential displacements in the two systems of coordinates as follows:

$$dx = \frac{\partial x}{\partial q_1}dq_1 + \frac{\partial x}{\partial q_2}dq_2 \qquad dy = \frac{\partial y}{\partial q_1}dq_1 + \frac{\partial y}{\partial q_2}dq_2$$

$$dz = \frac{\partial z}{\partial q_1}dq_1 + \frac{\partial z}{\partial q_2}dq_2 \qquad (17\text{-}35)$$

in which the coefficients of dq_1 and dq_2 are to be regarded as cosine factors. For example, $\partial x/\partial q_1$ may be regarded as the cosine of the angle between dx and dq_1 and projects dq_1 into the line of dx. The relations for the component velocities in the two systems may be similarly written. These equations may be obtained directly from Eq. (17-35) by dividing each term of the equations by dt. To abbreviate the expressions, we shall use $\dot{x}$ for dx/dt, etc. The equations are

$$\dot{x} = \frac{\partial x}{\partial q_1}\dot{q}_1 + \frac{\partial x}{\partial q_2}\dot{q}_2 \quad \dot{y} = \frac{\partial y}{\partial q_1}\dot{q}_1 + \frac{\partial y}{\partial q_2}\dot{q}_2 \quad \dot{z} = \frac{\partial z}{\partial q_1}\dot{q}_1 + \frac{\partial z}{\partial q_2}\dot{q}_2 \qquad (17\text{-}36)$$

The magnitude of the generalized velocity could be found from these three equations by taking the square root of the sum of the squares of the right-hand members. We shall omit the expression. The components of the momentum may be written by multiplying each term of the three equations by m, the mass of the particle.

The kinetic energy T of the particle may be expressed in the generalized coordinates and their time derivatives by squaring each equation of Eqs. (17-35), dividing throughout by $(dt)^2$, adding the three resulting equations, and introducing $\frac{1}{2}m$ on both sides. The final result is written below with abbreviations for the coefficients of the squares of the generalized velocity components.

$$T = \tfrac{1}{2}m(A\dot{q}_1^2 + 2B\dot{q}_1\dot{q}_2 + C\dot{q}_2^2)$$

in which

$$
\begin{aligned}
A &= \left(\frac{\partial x}{\partial q_1}\right)^2 + \left(\frac{\partial y}{\partial q_1}\right)^2 + \left(\frac{\partial z}{\partial q_1}\right)^2 \\
B &= \frac{\partial x}{\partial q_1}\frac{\partial x}{\partial q_2} + \frac{\partial y}{\partial q_1}\frac{\partial y}{\partial q_2} + \frac{\partial z}{\partial q_1}\frac{\partial z}{\partial q_2} \\
C &= \left(\frac{\partial x}{\partial q_2}\right)^2 + \left(\frac{\partial y}{\partial q_2}\right)^2 + \left(\frac{\partial z}{\partial q_2}\right)^2 \qquad (17\text{-}37)
\end{aligned}
$$

It will be observed that the kinetic energy in generalized coordinates is expressed as a homogeneous quadratic function of the generalized velocity components. The coefficients A, B, and C are functions of the generalized coordinates alone.

17-9. Lagrange's Equations.—In deriving Lagrange's generalized equations, it is immaterial to the form of the final result whether we use the equations for a single particle or for a system of particles. The difference in the final equations is to be found only in an interpretation of the meaning of the symbols used. For the sake of simplicity in writing the details of the development we shall consider only the case of a single particle. In fact, since we may select the generalized coordinates so as to eliminate the constraints, we may consider the particle to be free.

We start with the ordinary equations of motion of the free particle which are

$$F_x = m\,\ddot{x}, \qquad F_y = m\,\ddot{y}, \qquad \text{and} \qquad F_z = m\,\ddot{z} \quad (17\text{-}38)$$

Each of these equations is to be projected into the line of the q's by multiplying it by $\partial x/\partial q_1$, $\partial y/\partial q_1$, $\partial z/\partial q_1$, and $\partial x/\partial q_2$, $\partial y/\partial q_2$, $\partial z/\partial q_2$, respectively, for each of the q coordinates. Again we may simplify the procedure by carrying out some of the details for only one coordinate, say q_1. Adding the three equations after multiplying each by the proper factor gives

$$m\left(\ddot{x}\frac{\partial x}{\partial q_1} + \ddot{y}\frac{\partial y}{\partial q_1} + \ddot{z}\frac{\partial z}{\partial q_1}\right) = F_x\frac{\partial x}{\partial q_1} + F_y\frac{\partial y}{\partial q_1} + F_z\frac{\partial z}{\partial q_1} \quad (17\text{-}39)$$

The right-hand member may be called the generalized force and is usually identified by the symbol Q_1.

The left-hand member is to be expressed in terms of the generalized kinetic energy T. To do this requires some manipulation. The first term may be written in the following manner:

$$m\ddot{x}\frac{\partial x}{\partial q_1} = m\left[\frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial q_1}\right) - \dot{x}\frac{\partial \dot{x}}{\partial q_1}\right] \qquad (17\text{-}40)$$

With the same type of transformation applied to the two other terms, Eq. (17-39) takes the form

$$m\frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial q_1} + \dot{y}\frac{\partial y}{\partial q_1} + \dot{z}\frac{\partial z}{\partial q_1}\right) - m\left(\dot{x}\frac{\partial \dot{x}}{\partial q_1} + \dot{y}\frac{\partial \dot{y}}{\partial q_1} + \dot{z}\frac{\partial \dot{z}}{\partial q_1}\right) = Q_1 \tag{17-41}$$

Similar equations could be written for the other generalized coordinates, q_2, q_3, etc.

Since the kinetic energy of the particle is

$$T = \tfrac{1}{2}m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

the second part of the left-hand member is evidently the partial derivative of the kinetic energy with respect to q_1. We may leave this part of the analysis and examine the first part of the expression.

The rectangular coordinates are all functions of the q's and the time t; hence [Eq. (17-36)]

$$\dot{x} = \frac{\partial x}{\partial q_1}\dot{q}_1 + \frac{\partial x}{\partial q_2}\dot{q}_2 + \frac{\partial x}{\partial q_3}\dot{q}_3$$

with similar equations for $\dot{y}$ and $\dot{z}$. Now, in general, $\dot{x}$ will be some function of the q's, $\dot{q}$'s, and t; hence we differentiate the equation for $\dot{x}$ with respect to $\dot{q}_1$, $\dot{q}_2$, and $\dot{q}_3$ with the following results:

$$\frac{\partial \dot{x}}{\partial \dot{q}_1} = \frac{\partial x}{\partial q_1} \qquad \frac{\partial \dot{x}}{\partial \dot{q}_2} = \frac{\partial x}{\partial q_2} \qquad \frac{\partial \dot{x}}{\partial \dot{q}_3} = \frac{\partial x}{\partial q_3} \tag{17-42}$$

Similar equations may be written for $\dot{y}$ and $\dot{z}$. With these relations the first part of Eq. (17-41) may be transformed, thus:

$$\begin{aligned} m\frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial q_1} + \dot{y}\frac{\partial y}{\partial q_1} + \dot{z}\frac{\partial z}{\partial q_1}\right) &= m\frac{d}{dt}\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_1} + \dot{y}\frac{\partial \dot{y}}{\partial \dot{q}_1} + \dot{z}\frac{\partial \dot{z}}{\partial \dot{q}_1}\right) \\ &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) \end{aligned} \tag{17-43}$$

And again we must remember that terms (not written) containing derivatives with respect to q_2 and q_3 could be similarly transformed. Combining the results of these changes with the generalized force equations for Q_1, Q_2, and Q_3, we may write

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} = Q_1$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} = Q_2$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_3}\right) - \frac{\partial T}{\partial q_3} = Q_3 \qquad (17\text{-}44)$$

These are *Lagrange's equations of motion* for the free particle. In these equations Q_1, Q_2, and Q_3 are the generalized force components corresponding to the coordinates q_1, q_2, and q_3, respectively. The quantity T is the kinetic energy of the particle. Without going into the details of the derivation we could obtain equations similar to those written for a system of particles in which, however, T refers to the kinetic energy of the system of particles and Q_1, Q_2, and Q_3 are the components of the resultant generalized force.

17-10. Lagrange Equations for a Conservative System.—If the forces are conservative forces, then there is a force function U from which the components of the force may be found, and

$$F_x = \frac{\partial U}{\partial x} \qquad F_y = \frac{\partial U}{\partial y} \qquad F_z = \frac{\partial U}{\partial z} \qquad (17\text{-}45)$$

The generalized force components may then be expressed by the following equation:

$$Q_1 = \frac{\partial U}{\partial x}\frac{\partial x}{\partial q_1} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial q_1} + \frac{\partial U}{\partial z}\frac{\partial z}{\partial q_1} = \frac{\partial U}{\partial q_1} \qquad (17\text{-}46)$$

and, similarly,

$$Q_2 = \frac{\partial U}{\partial q_2} \qquad Q_3 = \frac{\partial U}{\partial q_3}$$

If now we write for Q_1 its value in the first of Eqs. (17-44), we obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial (T + U)}{\partial q_1} = 0 \qquad (17\text{-}47)$$

The force function U is a function of position and therefore a function of the coordinates alone. It may therefore be included in the first term of Eq. (17-47), gaining thereby a simpler form of expression. In place of $-U$ we may substitute V, which we have used in Chap. X, to represent the potential. Here, however, we use V to represent the potential energy of the mass whose kinetic energy is T. This substitution gives the quantity $T - V$ which is designated by the single symbol L and has been called the *kinetic potential.* With this change the equation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = 0 \tag{17-48}$$

which is one of the equations of motion for a conservative system. The others may be readily written.

17-11. Constraints and Lagrange's Equation.—We have seen above that for a particle the number of degrees of freedom plus the number of constraints must be equal to three, and for a rigid body this sum must be six. In using Lagrange's equations only for the solution of a problem we need as many generalized coordinates as there are degrees of freedom. The Lagrangian function L must be expressed in terms of the independent generalized coordinates and their time derivatives, if we are to use only Lagrange's equations. If we are considering the motion of a particle with two degrees of freedom and one constraint, we shall need two independent generalized coordinates, say q_1 and q_2, and must be able to express L in terms of the q_1q_2 and $\dot{q}_1\dot{q}_2$.

In some problems it is difficult to reduce the number of coordinates to that of the independent coordinates because of complicated geometrical relations. In such cases we may have L expressed in terms of more than the number of independent coordinates and use expressions giving the geometrical relations in conjunction with the Lagrangian equations.

To show how to set up the equations, let us consider the case of a particle (in a conservative system) with two constraints. Ordinarily we would try to express L in terms of only one independent coordinate. Suppose that it is not easy to do so but that we can express L in terms of the three coordinates q_1, q_2, and q_3 and their time derivatives. Suppose that the geometrical relations are expressed by the equations

$$f_1\,(q_1\,q_2\,q_3) = 0 \qquad f_2\,(q_1\,q_2\,q_3) = 0 \tag{17-49}$$

Now applying the principle of virtual work to Lagrange's equations, we may write

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1}\right)\delta q_1 + \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2}\right)\delta q_2 + \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_3} - \frac{\partial L}{\partial q_3}\right)\delta q_3 = 0 \tag{17-50}$$

Two other equations which are derived from Eq. (17-49) are

$$\frac{\partial f_1}{\partial q_1}\delta q_1 + \frac{\partial f_1}{\partial q_2}\delta q_2 + \frac{\partial f_1}{\partial q_3}\delta q_3 = 0$$
$$\frac{\partial f_2}{\partial q_1}\delta q_1 + \frac{\partial f_2}{\partial q_2}\delta q_2 + \frac{\partial f_2}{\partial q_3}\delta q_3 = 0 \qquad (17\text{-}51)$$

If the method of Lagrange's indeterminate multipliers is used, these two equations are to be multiplied by λ and μ, respectively. The resulting expressions may be combined with Eq. (17-50) by adding the coefficients of the similar virtual displacements δq_1, δq_2, and δq_3 and equating these sums separately to zero. The results are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} + \lambda\frac{\partial f_1}{\partial q_1} + \mu\frac{\partial f_2}{\partial q_1} = 0$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} + \lambda\frac{\partial f_1}{\partial q_2} + \mu\frac{\partial f_2}{\partial q_2} = 0$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_3} - \frac{\partial L}{\partial q_3} + \lambda\frac{\partial f_1}{\partial q_3} + \mu\frac{\partial f_2}{\partial q_3} = 0 \qquad (17\text{-}52)$$

These three equations together with the two geometrical equations [Eqs. (17-49)] are sufficient to determine the three coordinates and the indeterminate multipliers.

The method used above is a general one and may be used in cases involving more coordinates and one or more constraints.

17-12. Illustrations of the Use of Lagrange's Equations. *a. Falling Particle Subject to Two Constraints.*—A particle of mass m falls under the influence of its weight. It is constrained to move in a vertical circular path of radius r. Find the differential equation of motion.

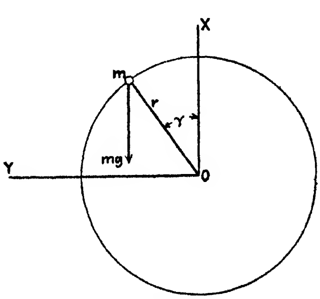

FIG. 157.

We shall select the reference system with the origin at the center of the constraining circle and with the X-axis vertical, as shown in Fig. 157. Let γ be the angle which the radius, drawn to m, makes with the X-axis. Since there are two constraints, a single generalized coordinate only is necessary. We shall choose γ to be this coordinate, since the kinetic potential is readily expressed in terms of γ.

The kinetic energy of m is $\frac{1}{2}mV^2$ or $\frac{1}{2}mr^2\omega^2$, since $V = \omega\, r$. The potential energy, referred to the Y-axis as a reference line, is $mgr \cos \gamma$. Hence the kinetic potential is

$$L = T - V = \tfrac{1}{2}mr^2\omega^2 - mgr \cos \gamma \tag{17-53}$$

from which

$$\frac{d}{dt}\frac{\partial L}{\partial \omega} = mr^2\frac{d\omega}{dt} \qquad \text{and} \qquad \frac{\partial L}{\partial \gamma} = mgr \sin \gamma$$

Therefore the equation of motion becomes

$$mr^2\frac{d\omega}{dt} - mgr \sin \gamma = 0 \tag{17-54}$$

The validity of this equation may be checked by observing that it could be written from an application of the force-moment equation. An inspection of the equation shows that mr^2 is the moment of inertia about the Z-axis and $mgr \sin \gamma$ is the moment of the weight about the same axis. The moment of the force exerted by the constraint is zero.

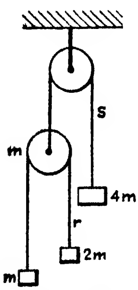

FIG. 158.

The solution of Eq. (17-54) involves an elliptical integral unless one restricts the motion to a small angle so that $\sin \gamma$ may be replaced by the angle γ. If it is desired to obtain expressions for x and y in terms of the time, the equations of the constraints ($\varphi = x^2 + y^2 + z^2 - r^2 = 0$ and $\psi = z = 0$) may be used.

b. System of Pulleys.—Two masses (m and $2m$) are suspended over a movable pulley of mass m by a string of length p. The movable pulley is connected to another mass ($4m$) by a string of length l which passes over a fixed pulley. The system is to be regarded as conservative and the moments of inertia of the pulleys are to be neglected (Fig. 158). Find the acceleration of the various parts of the system.

This problem is of interest because the application of Lagrange's equations makes it possible to avoid the introduction of the unknown tensions which one would have to include in a solution by the ordinary dynamical methods. It also illustrates the use of two generalized coordinates.

We shall let s and r represent the distances of the masses $4m$ and $2m$ below the fixed and the movable pulleys, respectively.

Then it follows that the movable pulley is $l - s$ below the fixed pulley and the mass m is $p - r$ below the movable pulley.

The kinetic energy of the system may be expressed in terms of $\dot{s}$ and $\dot{r}$ as follows:

$$T = \tfrac{1}{2}\,4m\dot{s}^2 + \tfrac{1}{2}\,m\,\dot{s}^2 + \tfrac{1}{2}\,2m\,(\dot{s} - \dot{r})^2 + \tfrac{1}{2}\,m\,(\dot{s} + \dot{r})^2 \qquad (17\text{-}55)$$

The potential energy V is

$$V = -4mgs - mg(l - s) - 2mg\,(l - s + r) - mg\,(l - s + p - r) \qquad (17\text{-}56)$$

These two expressions contain the two coordinates s and r and we shall therefore need two equations to determine the motion.

The two general equations may be written thus:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = 0 \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 \qquad (17\text{-}57)$$

Substituting the values of T and V in the equation $L = T - V$ and taking the derivatives as indicated gives

$$\frac{\partial L}{\partial \dot{s}} = 8m\dot{s} - m\dot{r} \qquad \frac{\partial L}{\partial \dot{r}} = 3m\dot{r} - m\dot{s}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = 8m\ddot{s} - m\ddot{r} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 3m\ddot{r} - m\ddot{s}$$

$$\frac{\partial L}{\partial s} = 0 \qquad \frac{\partial L}{\partial r} = mg \qquad (17\text{-}58)$$

Putting these results in the general equations of Eqs. (17-57), we may write

$$8m\ddot{s} - m\ddot{r} = 0 \qquad 3m\ddot{r} - m\ddot{s} - mg = 0$$

$$\ddot{r} = 8\ddot{s} \qquad 3\ddot{r} - \ddot{s} = g \qquad (17\text{-}59)$$

From which we find

$$\ddot{s} = \frac{g}{23} \qquad \text{and} \qquad \ddot{r} = \frac{8g}{23}$$

If the initial conditions were specified, the velocities and displacements of the several masses could be expressed as functions of the time by integrating these equations.

c. Derivation of Euler's Equations.—It is required to derive Euler's equations by an application of Lagrange's equations.

We shall place the reference axes coincident with the principal axes of the body. The kinetic energy of the rigid body is expressed by the equation

$$T = \tfrac{1}{2}A\omega_x^2 + \tfrac{1}{2}B\omega_y^2 + \tfrac{1}{2}C\omega_z^2 \qquad (17\text{-}60)$$

There are three degrees of freedom; consequently three equations will be needed. The selection of proper generalized coordinates is first to be considered. We cannot use ω_x, ω_y, and ω_z for this purpose because it is impossible to express the position of every point of the body in terms of these quantities without introducing differential coefficients. We may, however, use the Euler angles θ, φ, and ψ for the generalized coordinates.

The relations between the ω's and the generalized velocity components were written above [Eq. 16-35)] but are reproduced here for convenience.

$$\begin{aligned} \omega_x &= \dot\theta \cos \varphi + \dot\psi \sin \theta \sin \varphi \\ \omega_y &= -\dot\theta \sin \varphi + \dot\psi \sin \theta \cos \varphi \\ \omega_z &= \dot\varphi + \dot\psi \cos \theta \end{aligned} \qquad (17\text{-}61)$$

We shall derive but one of the three equations, *viz.*, the one in which φ is the coordinate. The partial derivative of T with respect to $\dot\varphi$ is

$$\frac{\partial T}{\partial \dot\varphi} = C\omega_z \frac{\partial \omega_z}{\partial \dot\varphi} = C\omega_z \qquad \text{and} \qquad \frac{d}{dt} C\omega_z = C\dot\omega_z$$

The partial derivative of ω_z with respect to $\dot\varphi$ is unity, as may be found by differentiating the equation for ω_z given above. In a similar manner we may find $\partial T/\partial\varphi$. The steps are as follows:

$$\begin{aligned} \frac{\partial T}{\partial \varphi} &= A\omega_x \frac{\partial \omega_x}{\partial \varphi} + B\omega_y \frac{\partial \omega_y}{\partial \varphi} \\ &= A\omega_x(-\dot\theta \sin \varphi + \dot\psi \sin \theta \cos \varphi) + \\ &\qquad\qquad B\omega_y(-\dot\theta \cos \varphi - \dot\psi \sin \theta \sin \varphi) \\ &= A\omega_x\omega_y - B\omega_y\omega_x \end{aligned} \qquad (17\text{-}62)$$

We have left to evaluate $-\partial V/\partial\varphi$ or $\partial U/\partial\varphi$. We know, however, that this quantity must yield the force moment about the Z-axis which we may call M_z. Hence, putting the parts of the equation together, we have

$$C\dot\omega_z + (B - A)\omega_x\omega_y = M_z \qquad (17\text{-}63)$$

The two other equations may be found by a similar procedure. The details will be left for the student.

Problems.—1. Two masses, $3m$ and $4m$, are hung by a string over a massless pulley. If the forces are conservative and the moment of inertia of the pulley is to be neglected, find the acceleration of the system by the use of Lagrange's equations.

2. Using Lagrange's equations, derive the equations of motion for a particle without constraints, expressed in spherical coordinates.

3. A ladder of length $2L$ and mass m is supported initially in a nearly vertical position by a smooth vertical wall and a smooth horizontal floor. It falls in a vertical plane. Find the differential equations of motion by using Lagrange's method.

17-13. Impulsive Forces.—In this section we shall show how the form of Lagrange's equations is to be modified in order to express the effects of impulsive forces. We are to consider a particle, system of particles, or a rigid body which may be at rest or may have any motion at the instant at which the impulsive forces are applied. For simplicity in deriving the equations we shall consider the case of a single particle but the results may be readily applied to a system of particles or to a rigid body. The validity of this extension may be easily established.

The impulsive forces are to act during a short time interval, say from $t = t_1$ to $t = t_2$. For each of the independent coordinates we shall have an equation of the form [Eq. (17-44)]

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} = Q_1$$

This equation (and the others if they be present) is to be multiplied by dt and integrated from t_1 to t_2. This step is expressed as follows:

$$\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) dt - \int_{t_1}^{t_2} \frac{\partial T}{\partial q_1} dt = \int_{t_1}^{t_2} Q_1 dt \qquad (17\text{-}64)$$

The first term is readily integrated, with the result given by the expression

$$\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) dt = \left(\frac{\partial T}{\partial \dot{q}_1}\right)_{t_2} - \left(\frac{\partial T}{\partial \dot{q}_1}\right)_{t_1} \qquad (17\text{-}65)$$

in which the subscripts t_1 and t_2 are used to indicate that the particular value of the function corresponding to the times t_1 and t_2 are to be used. The difference of the two quantities [right member, Eq. (17-65)] measures the change which is produced by the impulses in the small time interval $t_2 - t_1$.

The second term of Eq. (17-64) is equal to zero if the time interval is very small, because $\partial T/\partial q_1$ is finite. The final form of Eq. (17-64) is

$$\left(\frac{\partial T}{\partial \dot{q}_1}\right)_{t_2} - \left(\frac{\partial T}{\partial \dot{q}_1}\right)_{t_1} = \int_{t_1}^{t_2} Q_1 dt \qquad (17\text{-}66)$$

The right-hand member may be called the *generalized impulse* because of its similarity to the impulse of an ordinary force. The left-hand member represents a quantity which may be called the change in the *generalized momentum.*

Illustration.—A uniform rod is rotating in a horizontal plane about a fixed axis through its center C with an angular velocity ω. The axis at C is suddenly removed and at the same instant one end of the rod is fixed. Find the new value of the angular velocity and the impulse.

In order to apply Eq. (17-66), it will be necessary to determine T just before and after the impulse is applied. To find such expressions, we shall need to obtain a relation between the angular velocities ω_1 and ω_2 before and after the application of the impulse. Such a relation may be found by using the principle of the conservation of the rotational momentum about an axis through the point (say O) in space at which the end of the rod becomes fixed. Since the only force moment introduced is applied at O, it cannot influence the value of the rotational momentum about an axis through that point.

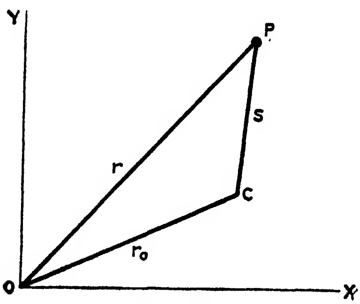

FIG. 159.

We digress for a moment to develop an expression for the rotational momentum about an axis through O. The procedure will be simplified if we select O as origin of the reference system and limit the motion to the XY reference plane. Let $\boldsymbol{r}$ be the position vector of any point P of the body (Fig. 159), $\boldsymbol{r}_0$ the position vector of the center of mass C, and $\boldsymbol{s}$ the vector from C to P. The velocity ($\boldsymbol{V}$) of P may be expressed in terms of the velocity ($\boldsymbol{V}_0$) of C and the velocity $\boldsymbol{\omega} \times \boldsymbol{s}$ (if the body is rigid) of P relative to C by the equation

$$\boldsymbol{V} = \boldsymbol{V}_0 + \boldsymbol{\omega} \times \boldsymbol{s} \tag{17-67}$$

The rotational momentum ($\boldsymbol{M}$) of the body about an axis through O and perpendicular to the XY plane is

$$\begin{aligned} \boldsymbol{M} &= \int_m \boldsymbol{r} \times \boldsymbol{V}\, dm \\ &= \int_m (\boldsymbol{r}_0 + \boldsymbol{s}) \times (\boldsymbol{V}_0 + \boldsymbol{\omega} \times \boldsymbol{s}) dm \end{aligned}$$

$$= \int_m [r_0 \times V_0 + r_0 \times (\omega \times s) + s \times V_0 + s \times (\omega \times s)]dm$$
$$= m\, r_0 \times V_0 + \int_m dm\, s^2\, \omega \tag{17-68}$$

The steps used in obtaining the final expression will be left for the student to work out. The result indicates that the rotational momentum consists of two parts: the first gives the rotational momentum in terms of the velocity of the center of mass and the second part gives the rotational momentum relative to the center of mass and is equal to the moment of inertia about an axis through the center of mass multiplied by the angular velocity.

Returning now to the particular problem, we may select the reference system with origin at O, the point in space at which the end of the rotating rod is to be fixed, and the X-axis drawn through the center of mass of the rod as shown in Fig. 160. The first thing to do is to express the rotational momentum about O for the conditions immediately before and after the point O becomes fixed. If $2l$ is the length of the rod and m its mass, the moment of inertia about an axis through C is $\frac{1}{3}ml^2$. The equation expressing the constancy of rotational momentum about O is

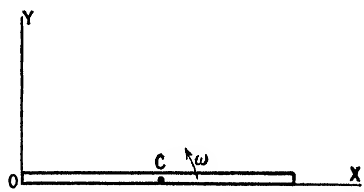

FIG. 160.

$$ml^2\omega_2 + \tfrac{1}{3}ml^2\omega_2 = \tfrac{1}{3}ml^2\omega_1$$

from which

$$\omega_2 = \tfrac{1}{4}\omega_1 \tag{17-69}$$

We may now express the kinetic energy for the two positions which may be called T_2 and T_1.

$$T_2 = \tfrac{1}{2}\left(\tfrac{4}{3}\, ml^2\right)\omega_2{}^2 \qquad\qquad T_1 = \tfrac{1}{2}\left(\tfrac{1}{3}ml^2\right)\omega_1{}^2$$
$$= \tfrac{1}{24}\, ml^2\omega_1{}^2$$
$$\frac{d}{dt}\left(\frac{\partial T_2}{\partial \omega_1}\right) = \frac{1}{12}ml^2\dot{\omega}_1 \qquad \frac{d}{dt}\left(\frac{\partial T_1}{\partial \omega_1}\right) = \frac{1}{3}ml^2\dot{\omega}_1$$

Substituting these values in Eq. (17-66) gives

$$\tfrac{1}{12}ml^2\dot{\omega}_1 - \tfrac{1}{3}ml^2\dot{\omega}_1 = \int_1^{t_2} Q\, dt$$

or

$$-\tfrac{1}{4}ml^2\dot{\omega}_1 = \int_1^{t_2} Q\, dt \tag{17-70}$$

The left-hand member gives the value of the impulse required to effect the prescribed change in the motion. The presence of the minus sign indicates that the direction of the impulse of Q is in the negative sense.

Problems.—1. A thin circular lamina is rotating in its own plane about a fixed point on its circumference. This point is suddenly released and another point on the circumference, which is at a distance equal to a quarter of the circumference from the first point, is fixed. Show that the angular velocity after the application of the impulse is one-third of that before.

2. A square lamina is rotating about a fixed axis coinciding with one of the diagonals of the square when suddenly one of the moving vertices of the square is fixed, while at the same time the axis is released. Find the changed angular velocity and the impulse.

17-14. Hamilton's Principle.—From a consideration of the use of virtual displacements in D'Alembert's principle and their assistance in revealing the relations which exist between forces in problems of equilibrium, the idea was conceived to apply a similar device to a study of motions. This purely imaginative process in motions would suggest a comparison between the actual motion and one which is fictitious but infinitesimally close to it and which satisfies the prescribed conditions. With every point in the actual path is to be associated an adjacent point located by a very small variation of the coordinates of the given point. One assumption introduced is that the time shall be unaffected by the variation.

This principle may be derived in several different ways. The details of the derivation are perhaps minimized by starting with either D'Alembert's principle or Lagrange's equations. One advantage of using Lagrange's equations for a starting point is that it may be more obvious to the student that generalized coordinates may be employed in using Hamilton's principle.

Consider a situation in which there may be several degrees of freedom, for each of which a Lagrangian equation may be written. Each of these equations [Eq. (17-44)] is to be multiplied by a very small displacement δq and the element of time dt and then integrated between the time limits t and t_0. All such equations may be combined by expressing the sum symbolically as follows:

$$\int_{t_0}^{t} \sum \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} - Q \right) \delta q \, dt = 0 \qquad (17\text{-}71)$$

There is a so-called *dynamical* path which each particle of the system will take under the influence of the forces and constraints. Since δq is arbitrary, the effect of introducing it into the equations is to take the particle from the dynamical path into an adjacent *varied* path. The dynamical path is consistent with the equations of motion but the varied path is not necessarily the same. We shall later impose a limitation upon varied paths to the effect that the terminal points of it must be the same as those of the dynamical path.

The equation written above is to be manipulated into a form suitable for the present derivation. In order to understand the change that is to be made in the first term, let us consider the following relation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\delta q\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right)\delta q + \frac{\partial T}{\partial \dot{q}}\frac{d(\delta q)}{dt} \tag{17-72}$$

However, since

$$\frac{d(\delta q)}{dt} = \delta \dot{q}$$

we may introduce the equivalence in the last term and then, after multiplying each term of Eq. (17-72) by dt and rearranging, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right)\delta q\, dt = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\delta q\right)dt - \frac{\partial T}{\partial \dot{q}}\delta \dot{q}\, dt \tag{17-73}$$

The left-hand member of this equation is the first term of Eq. (17-71).

Before Eq. (17-71) is written with this alteration, it will be more convenient to show that the integral of the first term of Eq. (17-73) is zero. The integration of this term may be effected and is

$$\int_{t_0}^{t}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\delta q\right)dt = \frac{\partial T}{\partial \dot{q}}\delta q\Bigg|_{t_0}^{t}$$

If the restriction be made that the terminal points of the varied path, over which δq is taken, coincide with the end points of the dynamical path, then δq vanishes at the limits and the particular integral is zero. Hence we may write Eq. (17-71) in the following form:

$$\int_{t_0}^{t}\sum\left(\frac{\partial T}{\partial \dot{q}}\delta \dot{q} + \frac{\partial T}{\partial q}\delta q + Q\,\delta q\right)dt = 0 \tag{17-74}$$

Since T is some function of the q's and $\dot{q}$'s, then

$$\delta T = \frac{\partial T}{\partial \dot{q}} \delta \dot{q} + \frac{\partial T}{\partial q} \delta q$$

and the first two terms may be replaced by δT. The third term $\Sigma Q \, \delta q$ represents the work of the Q forces and may be replaced by δW, in the general case, or δU, if the forces are conservative. The equation, in the general case, reduces to

$$\delta \int_{t_0}^{t} (T + W) \, dt = 0 \tag{17-75}$$

in which δ is placed outside the integration sign because the variation is to be independent of the time. In a conservative system, the equation becomes

$$\delta \int_{t_0}^{t} (T + U) \, dt = 0 \qquad \text{or} \qquad \delta \int_{t_0}^{t} L \, dt = 0 \tag{17-76}$$

These equations express Hamilton's principle in symbolic form. We may interpret these equations to mean that the time integral of the kinetic potential in a conservative system has a *stationary* value for the dynamical path of a particular system when compared with the time integral over varied paths which have the same termini, provided also that the varied paths are described in the same time as the dynamical path. The term stationary, as used here, means that the time integral for the dynamical path does not vary for closely adjacent paths. In some cases the value is a minimum and in others a maximum.

Hamilton's principle is a very broad principle. From it all of the principles of dynamics may be derived. It is frequently used as a starting point for the derivation of Lagrange's equations, the principle of least action, the conservation of energy, and many others.

17-15. Principle of Least Action.—The principle of least action was first proposed by Maupertius. He, however, did not make use of mathematics to establish this principle but advanced theological arguments in favor of it. He believed that one of the fundamental laws of nature was that phenomena in nature take place with the least action. Perhaps the easiest way to derive the principle of least action is from Hamilton's principle.

In Hamilton's principle the fictitious varied path, very close to the dynamical path, is arbitrary except that it satisfies the constraints. To obtain the principle of least action, it is necessary to prescribe the additional condition that the motion must take place in such a way that the total energy shall remain constant; *i.e.*,

$$T + V = E \qquad \text{or} \qquad \delta T = -\delta V \tag{17-77}$$

where T and V are the kinetic and potential energies and E is the total energy. On the varied paths, if T' and V' are the kinetic and potential energies at any instant, we should have $T' + V' = E$.

The Lagrangian functions L and L' in the dynamical and varied paths, respectively, may be expressed as follows:

$$L = 2T - E \qquad L' = 2T' - E$$

The time integral of the Lagrangian function L, from $t = t_1$ to $t = t_2$, is therefore

$$\begin{aligned}\int_{t_1}^{t_2} L\,dt &= \int_{t_1}^{t_2} (2T - E)\,dt \\ &= \int_{t_1}^{t_2} 2T\,dt - E(t_2 - t_1)\end{aligned} \tag{17-78}$$

since E is a constant. The last term is a constant. By Hamilton's principle the time integral of L must be a maximum or a minimum; therefore the integral of $2T$ must also be a maximum or a minimum. It is, however, usually a minimum and it is from this fact that the term least action has been applied. The integral

$$\int_{t_1}^{t_2} 2T\,dt$$

is called action. Hence the principle is called the principle of least action. This principle means that, of all the possible varied paths which satisfy the given restrictions, that one which is the dynamical path is the one in which the value of the time integral of twice the kinetic energy is a maximum or a minimum.

Problems.—1. A particle of mass m is executing simple harmonic motion in a horizontal path in which the displacement may be expressed by the equation $x = r \cos \omega t$. (*a*) Express the restoring force, the kinetic energy, and the potential energy. (*b*) Find the time integral of the kinetic potential in Hamilton's equation [Eq. (17-76)] from $t = 0$ to $t = \pi/\omega$. (*c*) Using a varied path, which is expressed by the equation $x = r(\cos \omega t + k \cos 3\omega t)$ and which has the same termini as the path used in (*b*), find the value of the integral for the same limits and show that, if k is very small, the variation of the time integral is zero.

2. A simple pendulum of varying length r is hung from a fixed point and oscillates in a vertical plane. If γ is the angle which the string makes with a vertical line through the point of the support, find the equations of motion by the use of Lagrange's equations.

3. A particle of mass m is guided by a massless rod which is rotated at a constant angular velocity ω about a fixed vertical axis. Considering the motion to be without friction, find the equations of motion by Lagrange's equations.

4. A particle moves without friction on the surface of a circular cylinder the axis of which is vertical. The radius of the cylinder is to increase uniformly with the time. Assuming that the particle has an initial angular velocity ω_0 about the axis and that the vertical acceleration is constant, find the equations of motion by Lagrange's equations.

5. A uniform cylinder of radius r and of mass m has an inextensible string wrapped around its central section. One end of the string is fastened to the cylinder and the other to a fixed point. Initially the cylinder is at rest with its axis horizontal and a small portion of the string taut and in a vertical line. The cylinder is then released. Find the resulting motion.

6. A particle is describing an orbit under a central force. The velocity of the particle varies inversely with the square of the length of the radius vector drawn from the center to the particle. Using the principle of least action, find the orbit and an expression for the force.

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