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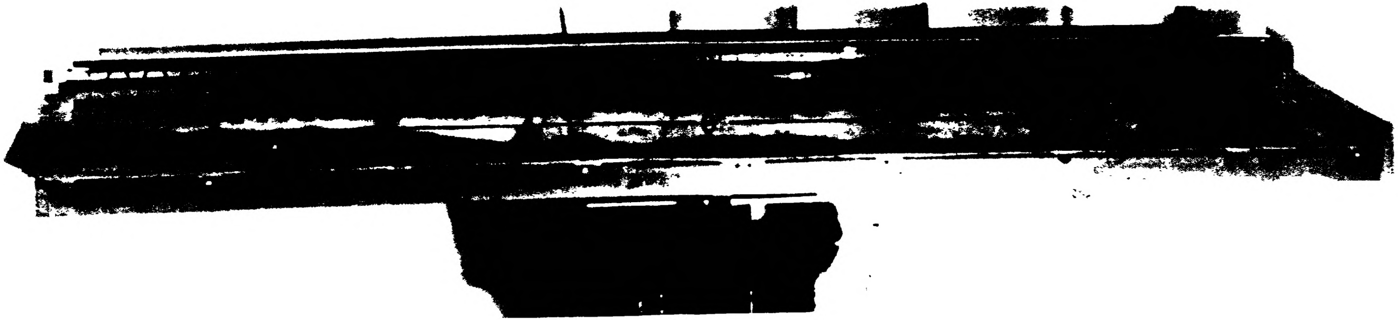
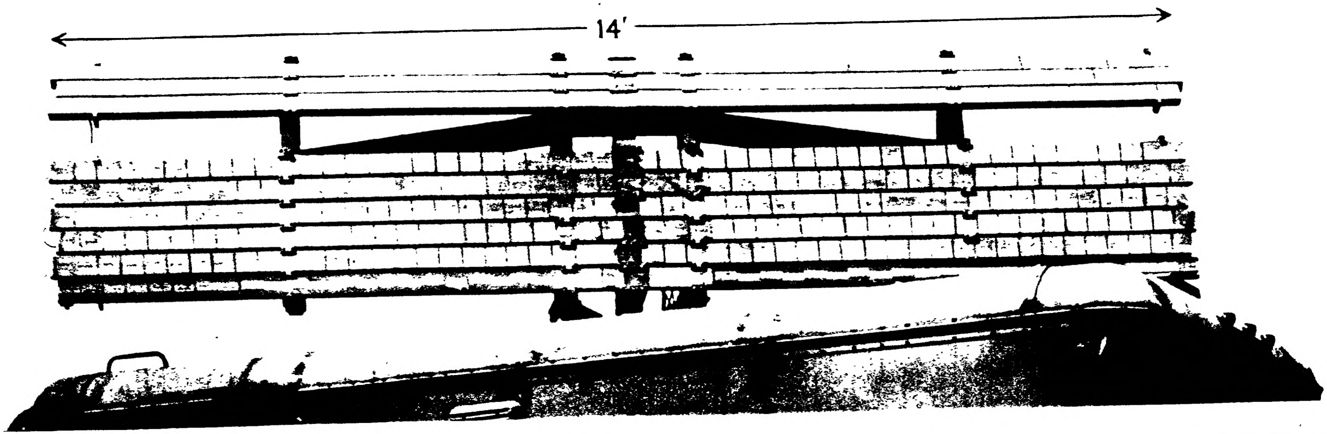


(*a*) Linear array of inclined 'edge' slots feeds rudimentary horn. (Horizontal electric polarization; *X*-band.) (Courtesy of T.R.E.)



(*b*) Two-dimensional array of longitudinal displaced slots. (Vertical electric polarization; *S*-band.) (Courtesy of N.R.C. Canada)

SLOTTED GUIDE ANTENNAS

THE PHYSICAL PRINCIPLES
OF
WAVE GUIDE TRANSMISSION
AND
ANTENNA SYSTEMS

BY

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PREFACE

The propagation of electromagnetic waves of high frequency in wave guides is one of the most fruitful fields for the application of Maxwell's Electromagnetic Theory in the form given by Heaviside. By a wave guide is meant in the simplest sense a hollow tube of metal which contains the energy propagated from one end to the other. For practical reasons the cross-sectional form is usually either rectangular or circular. A dielectric cylinder may also be used to guide electromagnetic waves of sufficiently high frequency and is usually called a dielectric guide. Conventional parallel wire and coaxial transmission lines, of course, also guide waves in that electromagnetic energy is propagated in the vicinity of the conductors. We shall, however, restrict the use of the term 'wave guide' simply to a hollow tube of conductor.

The range of electromagnetic spectrum for which wave guides are used extends over wavelengths from 40 cm. to a fraction of a centimetre and is referred to as the microwave region. To this range Hertz's pioneer work with spark-excited short waves may be said to have extended, but on account of the many years required to develop oscillators producing sufficient microwave power this most interesting region of the spectrum has not been exploited for radio until the last decade. During that period the war intervened and greatly accelerated the growth of knowledge and technological skill. By the operation of security regulations, however, the dissemination of knowledge of this newly acquired technique was restricted to physicists and engineers engaged in research or in the production of instruments of war employing microwaves. This book attempts to make amends in one part of the field.

The fundamental ideas of propagation in wave guides were enumerated by Lord Rayleigh at the end of last century, but his results were not generally well known until the experiments of Southworth† and Barrow‡ and others demonstrated their practical importance. One might almost say that, to judge by the literature of the 1930's, his work had to be rediscovered in America. But perhaps this comment is hardly fair, for the language which the radio engineer uses to-day is very different from that of the theoretical physicist writing before the invention of the thermionic vacuum tubes and before the modern

† Southworth, G. C., *Bell. Syst. Tech. J.* **15**, 284–309 (1936).
‡ Barrow, W. L., *Proc. I.R.E.* **24**, 1298–1328 (1936).

development of telephonic engineering that has contributed much to the elaboration of electric circuit theory.

For the radio engineer, however, the wave guide is fundamentally a novelty in one respect which is a commonplace to the mathematical physicist, namely, multiple propagation. Until the engineer had to deal with radio waves shorter than a metre the transverse dimensions of his ordinary transmission systems were all considerably less than half of the free-space wavelength for the frequencies transmitted over the system. Consequently the propagation was simply treated by the theory which includes the telephone line and submarine cable and uses quasi-static conceptions—inductance, capacity, voltage across the line, and current in it. The waves consist essentially in the propagation of electro- and magneto-static distributions along the line with the velocity of light. This is a great convenience, because for many purposes it is not necessary to refer to the detailed description of the electromagnetic field in space between and around the conductors. As soon as the transverse dimensions of the physical transmission line are sufficiently large compared with the wavelength in free space, this simple method is no longer adequate. The transmission system then behaves like a number of transmission lines propagating at different speeds. Consequently the field distributions characteristic of the individual simple lines, once launched, never recombine in the proper way to reproduce the composite field distribution which initiated the waves. In short, simple periodicity in space, obvious on the Lecher wires with short waves, is secured only by taking account of the possibilities represented in field theory and ensuring that only one characteristic mode of propagation—the dominant wave—is effective in the process of net energy transfer. It is this condition in a wave guide that has been exploited with great success in practice up to the present, and with which most of this book has to deal.

The physical facts that have compelled the use of wave guides to propagate microwaves, rather than the conventional 2- or 3-conductor transmission lines, are the large energy losses occurring essentially in the latter. These losses arise first in the dielectric separating the conductors either continuously or as spaced supports, and secondly on the surface of the conductors themselves. For given power transfer, geometry requires higher current densities in these lines than in wave guides serving the same purpose. Further, no dielectric is necessary in the wave guide.

It has been my aim in this book to emphasize especially those aspects of wave-guide technique the parallels to which are quite unrealizable in practice in connexion with longer waves. There are, of course, many respects in which conventional short-wave radio practice finds its counterpart in the microwave region, but there is a practical limit to the exploitation of this correspondence. This limit is set in part by considerations of mechanical rigidity and in part by the expense of precise fabrication of many very small parts, to be assembled into a permanent electrical whole, and finally by capacity for transmitting large power efficiently. It is obvious that the practical difficulties attending the application of longer wave practice become more serious the higher the microwave frequency to be used.

Among physicists whose limited interests in radio have never given them the opportunity to meet the problem of feeding power to an antenna, it has been a common gesture to dismiss the physical basis of radio practice with very short electromagnetic waves as merely requiring a little physical optics. The existence of this point of view shows how incomplete has been the conventional treatment of optical theory. Electromagnetic optics is greatly simplified by the concept of impedance. Physicists owe the clear appreciation of this fundamental theoretical matter to the development of microwave radio.

The first chapter of this book deals with the wedding of electric circuit ideas to wave conceptions. Physicists will understand that the application of these conceptions need not be restricted only to electromagnetic waves. Whenever we have to deal with plane waves of a physical quantity u satisfying the equation

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=0,$$

the conceptions of impedance and admittance, series and shunt, have a clear application.

It has been customary to introduce ideas of equivalent circuits to represent the loading of waves. This procedure can be applied usefully to wave guides only in a restricted way, and only too often lures one away from the essential simple representation—that waves are incident on an obstruction to electric waves in the guide or the flow of electric current on the wall, and in consequence secondary waves are radiated in both directions from this 'antenna' in the guide. Thus the wave systems on the two sides of the antenna are different. The methods for dealing with such waves are connected with the mathematical theory of bilinear transformations and in this connexion the use of matrices is highly appropriate. Having observed many tedious examples of amorphous elementary algebra, I have endeavoured to encourage the use of the more powerful and, indeed, easier methods. The reader who is not familiar with matrices can find all that he requires in the early pages of one or other of several introductory texts.† These same methods might well find application in other branches of physics.

Accordingly Chapter I contains material of quite general applicability to any characteristic wave; the attention of the reader is directed particularly to what is logically essential about series- and shunt-loading of waves, and the ease with which these conceptions can be generalized. What is important about series loading is that an open-circuit is transformed past a series-load unchanged, and what is important about shunt loading is that a short-circuit is transformed past a shunt-load unchanged. I have called unimodular loading the general case where a pure reactance or susceptance is transformed unchanged past the load, and it will be seen in this book that this type of loading is actually realized in a wave guide when a resonant slot is cut in the wall.

In another respect the treatment of electric waves in this chapter is aimed at inducing versatility in representing them by indicating the use of both current and voltage representations. The terms *current*, *voltage*, *admittance*, *shunt*, and *series* form a system mathematically indistinguishable, although physically distinct, from that of *voltage*, *current*, *impedance*, *series*, and *shunt*. As is well known to electrical engineers, advantage may lie in using one rather than the other in a particular application. The resulting possibility that the circle diagram may be used in two senses is quite clear and, given good notation, should lead to no difficulty.

In Chapter II dominant wave propagation in a rectangular wave guide is presented, and for the purpose of giving in brief compass a general idea of the laboratory work in which theoretical conceptions should be brought into contact with experience Chapter III is devoted to measurements. These chapters, therefore, deal with the essentials of the simple technique of the rectangular wave guide, the conception of multiple propagation being reserved for Chapter IV, in which mathematical detail has been suppressed as far as possible. Chapter V introduces the magnetic half-wave radiator with the electromagnetic version of Babinet's Principle and deals with the ideas for treating obstructions and antennas in rectangular guides.

The aspect of the use of wave guides with which I have been most

† e.g. Aitken, *Determinants and Matrices* (Oliver & Boyd, Edinburgh, 1939); Frazer, Duncan, and Collar, *Elementary Matrices* (Cambridge).

closely associated is the development of slot radiators cut in the wall of the wave guide. On account of their practical advantages and the theoretical interest in their novelty, the study of slot antennas and guide couplings has received perhaps undue attention in this book (Chapters VI, VII, VIII). It seemed, however, that since older aspects of the theory of wave guides have already been treated in other works,† it is unnecessary to repeat them in detail. Instead, the reader will find here the general theory of wave guides presented from a somewhat more physical point of view. This is supplemented by a discussion in the last chapter on Field Representation, treated in sufficient detail to indicate the methods employed, especially the contributions to the mathematical theory by Professors Synge, Infeld, and Stevenson, whose work is best known to me.

On account of its particular relevance in the design of slotted wave-guide antennas, the elegant and successful work of Stevenson on the field theory of resonant slots in wave guides is dealt with at some length. The complete elucidation of guide coupling by means of slots is given in Chapter VII for the first time and a new design procedure for the most useful slot antenna to be used in airborne radar is worked out in Chapter VIII.

In Chapter IX several devices have been discussed with the object of emphasizing the principles relevant to understanding their operation. No attempt has been made to present all the practical details which a radio engineer interested in technique would require and which, no doubt, will be described elsewhere. For this reason it was decided not to treat rotating joints, the 'door-knob' coupling, and a variety of bridge devices. Further, following the decision not to take up dielectric guides, 'polyrod' aerials have been omitted.

This book would not have been written had it not been for the opportunity provided for work in radio during the war at McGill University. After its inception in the basement of the Macdonald Physics Building the work was moved to a more commodious laboratory which with its staff was liberally supported financially by the National Research Council of Canada and received the most helpful consideration by the University. Special acknowledgment is due to Dr. J. S. Foster, F.R.S., who was instrumental during his association with the Radiation Laboratory at the Massachusetts Institute of Technology in securing for the McGill laboratory he had started the use of apparatus

† Stratton, J. A., *Electromagnetic Theory*, McGraw-Hill, 1941; Slater, J. C., *Microwave Transmission*, McGraw-Hill, 1942; Schelkunoff, S. A., *Electromagnetic Waves*, van Nostrand, 1943.

without which the work on slots would have been confined to the 10-cm. band.

To the writings of Dr. H. G. Booker and conversations with him I owe in great measure the stimulation of my thoughts. During the war his visits to this Continent began and ended at Montreal, and in this we were very fortunate.

In the pioneer experiments on slots the measurements were skilfully made by Dr. E. W. Guptill, whose enthusiastic co-operation in invention it is a pleasure to acknowledge. I am likewise indebted to other members of the laboratory at McGill—Messrs. J. W. Dodds, R. H. Johnston, M. Telford, and Dr. F. R. Terroux, on the results of whose work I have drawn freely in the following pages.

Most of this book was already in type before the appearance of the report of the Radiolocation Convention of the Institution of Electrical Engineers and the publication of the Technical Series of the Radiation Laboratory of the Massachusetts Institute of Technology. Had this not been so, more detailed references to work during the war would have been possible. My hearty thanks are due to Sir Edward Appleton, F.R.S., for his kind assistance in managing the references that are given, and for his helpful editing of this monograph. Dr. Guptill has put me further in his debt by reading the proofs and eliminating the errors. Any remaining must be charged to me, for the workmanship of the printer has been meticulous. Lastly, the secretarial staff of the Oxford University Press has at every point made smooth the work of seeing the book through the press. I am very grateful for this efficient and understanding help in avoiding the delays inherent in correspondence at so great a distance.

W. H. W.

SASKATOON,
SASKATCHEWAN, CANADA
1946

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The illustration showing 'Slotted Guide Antennas' is the frontispiece to this book, while Figs. 3 *and* 68 *face respectively pages* 7 *and* 126

S- and *X*-BANDS refer to ranges of wavelength approximating to 10 and 3 cm. and were devised as military codes: the wavelength values of other coded bands of less than 3 cm. are still secret.

NOTE

Reference numbers in the text are used where reference is made to work originally reported in secret documents. These references are collected on p. 205.

I

PLANE ELECTROMAGNETIC WAVES

WE shall be concerned at the outset to treat the elements of wave propagation using the idea of impedance. Our object is to bridge the gap which exists between the treatment of transmission lines by means of circuit theory and the theory of more general forms of wave propagation based on Maxwell's equations in which, until quite recently, the impedance concept has not been used. Impedance has been introduced into the treatment of wave propagation by Schelkunoff.†

In the present chapter we shall deal only with plane waves, and as a starting-point we consider a simple transmission line which lends itself very simply to the conventional transmission-line treatment and also to field representation.

1.1. The Strip Transmission Line and Circle Diagram

Imagine two parallel infinite conducting planes distant b metres apart, in the empty space between which is propagated a uniform plane

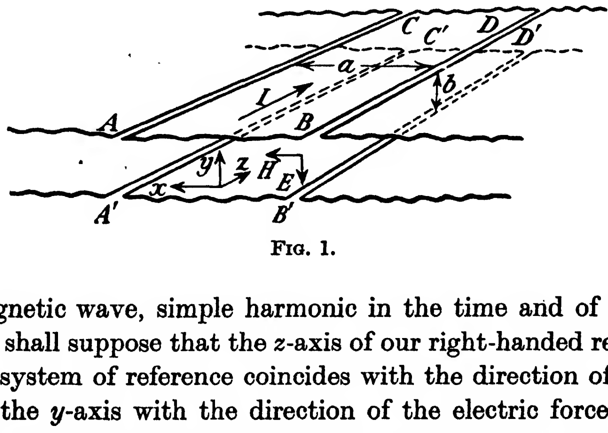


FIG. 1.

electromagnetic wave, simple harmonic in the time and of frequency $\omega/2\pi$. We shall suppose that the z-axis of our right-handed rectangular Cartesian system of reference coincides with the direction of propagation, and the y-axis with the direction of the electric force E in the wave, the magnitude of which we shall suppose not to depend on x and y. The conducting planes between which the waves travel are parallel to the xz-plane. Imagine both conductors cut parallel to z so as to isolate two strips of width a metres opposite each other. We shall think of these two strips ($ABCD$ and $A'B'C'D'$ in Fig. 1) as constituting the transmission line, the remainder of the

† Schelkunoff, S. A., *Bell Syst. Tech. J.* **17**, 17 (1938).

planes being retained as guard plates to permit the simplest possible geometry.†

Since we have assumed a uniform plane wave with E parallel to y,

$$E_x = E_z = 0, \qquad \frac{\partial E_y}{\partial y} = \frac{\partial E_y}{\partial x} = 0.$$

It follows from Maxwell's equations (see § 10.2) that when the field components depend on the time through $e^{j\omega t}$ $(j^2 = -1)$

$$\left.\begin{aligned} \frac{\partial E_y}{\partial z} &= \mu_0 \frac{\partial H_x}{\partial t} = j\omega\mu_0 H_x, \\ \frac{\partial H_x}{\partial z} &= \kappa_0 \frac{\partial E_y}{\partial t} = j\omega\kappa_0 E_y, \\ H_y &= H_z = 0. \end{aligned}\right\} \tag{11.1}$$

Now in order to sustain the uniform magnetic field H_x between the plates, there must be a longitudinal current I parallel to z on the faces of the strips bounding the field between them, namely,

$$I = aH_x \tag{11.2}$$

on the upper strip, there being an equal and opposite current on the lower one. The voltage between the strips is

$$V = -bE_y. \tag{11.3}$$

Accordingly the equations (11.1) may be rewritten

$$\frac{\partial V}{\partial z} = -\frac{j\omega\mu_0 b}{a} I, \qquad \frac{\partial I}{\partial z} = -\frac{j\omega\kappa_0 a}{b} V. \tag{11.4}$$

These are just the equations governing the propagation of current and voltage on an ideal transmission line for which the inductance and capacitance per unit length are respectively

$$L_1 = \mu_0 b/a \quad \text{and} \quad C_1 = \kappa_0 a/b;$$

indeed these relations suggest a picturesque method for defining the constants

$$\mu_0 \; (= 4\pi \times 10^{-7} \text{ henry/metre}) \quad \text{and} \quad \kappa_0 \left(= \frac{1}{36\pi} 10^{-9} \text{ farad/metre}\right).$$

(11.4) shows that both I and V and both E_y and H_x satisfy the equation

$$\frac{\partial^2 u}{\partial z^2} = -\omega^2 \mu_0 \kappa_0 u, \tag{11.5}$$

† If one considers a coaxial line, this device is unnecessary, but the complexity of cylindrical polar coordinates is introduced. It is a good exercise to work out the calculation in this case.

so that, given the harmonic time factor, u has the general form

$$u = Ae^{j\omega(t-z/c)}+Be^{j\omega(t+z/c)}, \tag{11.6}$$

where $c = \dfrac{1}{\sqrt{(\mu_0 \kappa_0)}} = 3 \cdot 10^8$ m./sec. is the velocity of propagation. When u stands for V we call this a voltage representation, and correspondingly for I or E_y or H_x a current or electric force or magnetic force representation. The variables V and E_y are essentially electric; I and H_x are magnetic variables.

In (11.6) A is the complex amplitude of the wave travelling in the direction of increasing z: that is, $|A|$ is the physical amplitude and $\arg A$ the phase of the wave at $z = ct$. B has the like meaning for the wave travelling in the direction z-decreasing.

Since all electromagnetic variables on the line are proportional to $e^{j\omega t}$ we shall follow the practice of electrical engineers and suppress this factor. Physical magnitudes are the real parts of the corresponding complex numbers. Any of the variables V, I, E, H is completely specified once we know the corresponding pair of complex numbers A and B. Now it should be noted that for a given wave system the arguments of A and B depend on the choice of the origin of z. To emphasize this fact we shall write A and B as functions of the reference z-position. For instance,

$$A(0) = A(z_0)e^{j\omega z_0/c}, \qquad B(0) = B(z_0)e^{-j\omega z_0/c}, \tag{11.7}$$

corresponding to (11.6) written in the form

$$u(z',t) = A(z_0)e^{j\omega(t-z'/c)}+B(z_0)e^{j\omega(t+z'/c)}, \tag{11.6$'$}$$

where $z' = z-z_0$.

The equations (11.7) constitute a transformation of the complex number pair (A,B) which we shall treat mathematically as a column 2-vector and, using matrix notation, write

$$\begin{pmatrix} A(0) \\ B(0) \end{pmatrix} = P(z_0)\begin{pmatrix} A(z_0) \\ B(z_0) \end{pmatrix}, \tag{11.8}$$

where $P(z_0)$ is the matrix

$$\begin{pmatrix} \omega(z_0) & 0 \\ 0 & \omega^{-1}(z_0) \end{pmatrix}$$

and

$$\omega(z_0) = e^{j\omega z_0/c}. \tag{11.9}$$

The wavelength on the line is $\lambda = 2\pi c/\omega$, and it is evident that displacement by an integral number of half-waves corresponds to the unit matrix E as follows:

$$P\left(\frac{m\lambda}{2}\right) = (-1)^m E, \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where m is an integer.

Consequently the ratio $w(z_0) = B/A$ regarded as a function of z_0

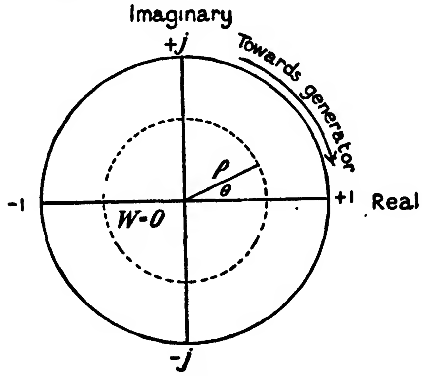


FIG. 2.

repeats its behaviour every half-wavelength. For any displacement z_0, $\arg w$ changes by $(2z_0/\lambda)2\pi$. We speak of $\frac{z_0}{\lambda} \times \begin{Bmatrix} 360 \\ 2\pi \end{Bmatrix}$ as the electrical distance $\begin{Bmatrix} \text{degrees} \\ \text{radians} \end{Bmatrix}$ corresponding to z_0. On displacement in a given wave system on an unloaded line, $\arg w$ therefore changes at twice the rate of the electrical distance, and on account of (11.7) $\operatorname{mod} w = |w|$ is unaltered by displacement. Let us use the complex w-plane as in Fig. 2 to represent wave systems.

The point $w = 0$ denotes a pure travelling wave in the direction z-increasing. The circle $|w| = 1$ represents the standing wave produced by two waves of equal amplitude travelling in opposite directions; each point of the circle corresponds to a point in the wave system. The point $w = -1$ corresponds to the position on the line where u vanishes, and $w = +1$ denotes the position where u is a maximum. If u denotes V or E_y, $w = -1$ is a short-circuit point in the wave; $w = +1$ is an open-circuit point. On the other hand, if u denotes I or H_x, the electrical roles of the places where $w = -1$ and $w = +1$ are interchanged. It should be evident that short-circuit and open-circuit points in a standing-wave system occur alternately a quarter-wavelength apart.

It is convenient to think of z increasing to the right and to regard the source of waves on the left. By this convention it is possible always to represent any wave system on the line within the circle $|w| = 1$. Any other circle $|w| = \rho < 1$ represents a wave system in which there is imperfect reflection. Where the circle cuts the real axis u is a maximum on the positive side of the origin and a minimum on the negative side. The ratio

$$r = \frac{u_{\max}}{u_{\min}} = \frac{1+\rho}{1-\rho} \tag{11.10}$$

is called the standing-wave ratio. Places of maximum u and minimum u occur alternately $\frac{1}{4}\lambda$ apart. Hence if r is measured and we know, for example, the position of $u_{\min}$, the circle diagram may be employed to obtain w at any other point of the line. The angle coordinate $\theta = \arg w$, measured counter-clockwise from the real axis of w, is twice the electrical distance from the position represented on the positive real axis and measured to the right, i.e. away from the generator.

So far we have considered the transformation of A and B on displacement through the wave system. Let us now consider transformations which involve physical changes of the wave system on the transmission line. Suppose that at the point $z = 0$ there is a localized obstacle to propagation; say, a thin sheet of dielectric or imperfect conductor. For simplicity, assume that the sheet is at right angles to z and is continued indefinitely between the guard plates. The obstacle will then reflect the waves travelling in both directions; consequently there will be discontinuities in A and B at $z = 0$. Let A' and B' denote the values of A and B to the left of $z = 0$, and let unprimed letters denote values to the right of the origin. Suppose the wave of complex amplitude A' incident from the left. The obstructing sheet acts as an antenna sending a plane wave of complex amplitude $-fA'$ to the right and $-fA'$ to the left, where f is a complex number whose modulus is less than 1, and all waves are treated according to the same convention as to the positive direction of u in the wave. Similarly the wave B incident from the right excites the antenna. We now apply the *principle of superposition* which holds for waves obeying linear laws; we equate the outgoing wave amplitudes to the sum of the ingoing and scattered amplitudes. Since we are dealing with complex numbers, the sums in question take phase differences into account automatically. We obtain

$$B' = -fA'+(1-f)B,$$

$$A = (1-f)A'-fB,$$

or, solving for A' and B',

$$A' = \frac{1}{1-f}A+\frac{f}{1-f}B,$$

$$B' = -\frac{f}{1-f}A+\frac{1-2f}{1-f}B. \tag{11.11}$$

Now let

$$\frac{2f}{1-f} = \alpha, \tag{11.12}$$

then

$$A' = \left(1+\frac{\alpha}{2}\right)A+\frac{\alpha}{2}B,$$

$$B' = -\frac{\alpha}{2}A+\left(1-\frac{\alpha}{2}\right)B. \tag{11.13}$$

The matrix transforming from the right to the left of the point of loading is therefore

$$\begin{pmatrix} 1+\frac{1}{2}\alpha & \frac{1}{2}\alpha \\ -\frac{1}{2}\alpha & 1-\frac{1}{2}\alpha \end{pmatrix}. \tag{11.14}$$

We shall call f the radiation coefficient of the load, and the meaning of α will appear later.

It is clear that $$A'+B' = A+B.$$

Hence if u denotes the voltage, the load is a *shunt* load since the voltage at $z = 0$ suffers no discontinuity at the point of loading. On the other hand, if u denotes I, the load is a *series* load since the current at the point of loading suffers no discontinuity, but, of course, this does not apply to the example just given.

1.2. Impedance and Admittance

Consider a travelling wave to the right. From (11.4)

$$\frac{V}{I} = \sqrt{\frac{L_1}{C_1}} = \sqrt{\frac{\mu_0}{\kappa_0}}.\frac{b}{a} = Z_c \quad \text{(definition).} \tag{12.1}$$

Z_c is the characteristic impedance of the line and the ratio is independent of position on the line. In general, when waves travel in both directions in the wave system, the ratio V/I is a function of z and is the impedance in the wave at the place where the ratio is evaluated; for a travelling wave to the left this impedance is $-Z_c$. If u denotes V in the (A, B) representation, then at $z = 0$ the impedance

$$Z = \frac{V}{I} = Z_c\frac{A+B}{A-B} = Z_c\frac{1+w}{1-w}, \tag{12.2}$$

whereas the admittance

$$Y = \frac{I}{V} = Y_c\frac{A-B}{A+B} = Y_c\frac{1-w}{1+w}. \tag{12.3}$$

Y_c is the reciprocal of Z_c. Unless they are required explicitly we eliminate Z_c and Y_c from (12.2) and (12.3) by understanding that Z and Y are measured with Z_c and Y_c respectively as unit.

It is noted that impedance and admittance are mutually interchangeable merely by reversing the sign of w, and that the complex number Z is derived by the conformal transformation of the w-plane

$$Z = \frac{1+w}{1-w}, \tag{12.4}$$

where the real part of Z is R, the resistance, and the imaginary part X is the reactance; hence

$$2R = \frac{1+w}{1-w}+\frac{1+w^*}{1-w^*}, \qquad 2jX = \frac{1+w}{1-w}-\frac{1+w^*}{1-w^*}, \tag{12.5}$$

where w^* denotes the conjugate complex of w. It is easy to show that the loci of constant R are circles centred on the real axis of w, passing through $w = +1$ and of radius $1/(R+1)$: whereas the loci of constant X are circles centred on the tangent to the unit circle at $w = +1$ and passing through that point.

It is therefore possible to superpose on the simple circle diagram of Fig. 2 a grid of two orthogonal sets of circles all passing through $w = +1$, so that (R, X) coordinates may be read off directly. This is the circle diagram which is fundamental in all transmission-line work. An example is shown in Fig. 3. The same grid system may be used admittance-wise, conductance and susceptance being expressed as fractions of the characteristic admittance of the line. The common limit of the two circle systems, being the open-circuit point on the impedance diagram, becomes the short-circuit point in the admittance diagram, and in accordance with the rule given in (12.4) the sign of j must be changed, so that numbers on the side of the real axis that formerly referred to positive reactance now refer to negative susceptance. Further, it is sometimes convenient to adopt the inverse coordination in which (R, X) are used as rectangular coordinates while $|w|$ and $\arg w$ are given by the auxiliary grid of circles.

The simple equations (11.2) and (11.3) allow us to transfer very easily all the ideas which have just been expressed in the familiar context of the transmission line, in order to set up the idea of field impedance, which in the case of the plane wave considered above is $-E_y/H_x$. In a travelling wave this ratio is $\sqrt{(\mu_0/\kappa_0)} = 376{\cdot}6$ ohms (120π). In the case of a medium between the plates in place of vacuum, let μ and κ be its magnetic susceptibility and dielectric constant. The velocity of propagation is now $c/\sqrt{(\mu\kappa)}$, and the intrinsic impedance of the medium is $120\pi\sqrt{(\mu/\kappa)}$ ohms, corresponding to the characteristic impedance of the transmission line.

The great advantage obtained by introducing the concept field impedance is that problems involving uniform plane waves are very simply managed by the (A, B) representation already discussed. At the boundary between two dielectrics the tangential components of E and H are continuous, consequently, in the electric force representation, $(A+B)$ and $(A-B)/Z_c$ are continuous. Thus the matrix governing transformation from the medium 1 to the medium 2 at the interface is as follows:

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \frac{1}{Z_2}\begin{pmatrix} \frac{1}{2}(Z_2+Z_1) & \frac{1}{2}(Z_2-Z_1) \\ \frac{1}{2}(Z_2-Z_1) & \frac{1}{2}(Z_2+Z_1) \end{pmatrix}\begin{pmatrix} A_2 \\ B_2 \end{pmatrix}, \tag{12.6}$$

where $Z_1 = \sqrt{(\mu_1/\kappa_1)}120\pi$, $Z_2 = \sqrt{(\mu_2/\kappa_2)}120\pi$.

If we put $B_2 = 0$, we readily obtain from (12.6) the relations between the electric force in the reflected, transmitted, and incident wave travelling towards the interface in medium 1, when medium 2 is unbounded on the right, viz.

$$\frac{A_1}{Z_1+Z_2} = \frac{B_1}{Z_2-Z_1} = \frac{A_2}{2Z_2}, \tag{12.7}$$

or, in terms of admittance,

$$\frac{A_1}{Y_1+Y_2} = \frac{B_1}{Y_1-Y_2} = \frac{A_2}{2Y_1}. \tag{12.8}$$

1.3. Energy Flux and Phase in the Wave System

In a voltage representation the mean rate of energy transfer† to the right is (the * denoting conjugate complex)

$$\tfrac{1}{2}\,\mathrm{Re}(VI^*) = \frac{1}{2Z_c}\{|A|^2-|B|^2\}, \tag{13.1}$$

which is proportional to $u_{\max}\times u_{\min}$, and the root-mean-square of u is proportional to the square root of this product of the maximum and minimum values of u in the wave system. In the electric field representation the above expression yields, when Z_c has been replaced by the intrinsic impedance of the medium, the mean energy transfer rate per unit area perpendicular to the direction of propagation. It is

† Stratton, *Electromagnetic Theory*, McGraw-Hill, 1941, p. 135.

evident that the more closely $|B|$ tends to $|A|$, the smaller the mean flux: the limiting value 0 is reached for infinite standing-wave ratio (r), because (13.1) can be written

$$\frac{|A|^2}{2Z_c}\frac{4r}{(r+1)^2}.$$

It is important for some purposes to know how phase is distributed

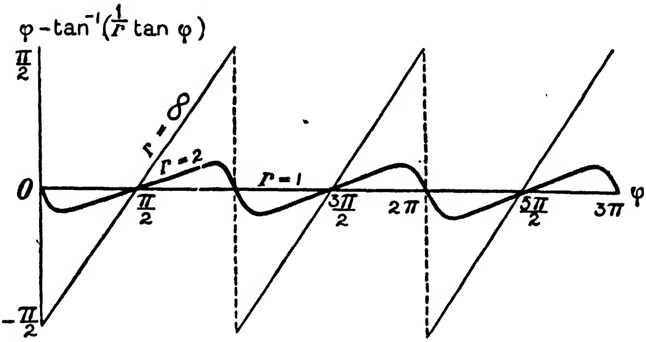


FIG. 4.

along the line in the wave system. Referred to the phase at the place in the line where $w = \rho$, $\arg\rho = 0$ as origin of z,

$$\begin{aligned}\arg u &= \arg(e^{-j\phi}+we^{j\phi}), \quad \text{where } \phi = \frac{2\pi z}{\lambda},\\ &= \tan^{-1}\{(1/r)\tan\phi\}. \qquad (13.2)\end{aligned}$$

Only in a pure travelling wave to the right ($r = 1$) is $\arg u = \phi$. A graph of $\phi - \tan^{-1}\{(1/r)\tan\phi\}$ for different values of the standing-wave ratio r is shown in Fig. 4. Note that the phase of u changes rapidly in the vicinity of the positions where $|u|$ is a minimum. In a perfect standing wave the phase changes discontinuously by π at the minima (zero). A fixed position on a line does not mean a position of constant phase if the terminating load on the line and consequently r is changed. If r is large, however, the phase at a maximum does not change rapidly with r and may in practice be used for a reference phase.

1.4. The Fundamental Matrices

We have still to give an electrical meaning to the coefficient α introduced in equation (11.12). Let us rewrite (11.13) in the form

$$A'+B' = A+B,$$

$$A'-B' = A-B+\alpha(A+B).$$

If (A, B) is a voltage (or electric field) representation these equations are equivalent to

$$V' = V,$$
$$I' = I+\alpha V.$$

Hence, by Kirchhoff's laws, α is the admittance of the load.

In order to obtain a series load, *with the same meaning for* (A, B), we have to make the load radiate in opposite phases on its two sides. Thus, when a wave of complex amplitude A' is incident from the left, the scattered wave is $-fA'$ to the right and $+fA'$ to the left. Apply the principle of superposition to obtain

$$A = (1-f)A'+fB,$$
$$B' = (1-f)B+fA'.$$

Hence

$$A' = \frac{1}{1-f}A-\frac{f}{1-f}B,$$
$$B' = \frac{f}{1-f}A+\left(1-\frac{f}{1-f}\right)B. \tag{14.1}$$

Let $\dfrac{2f}{1-f} = \gamma$ and the transformation matrix is

$$\begin{pmatrix} 1+\frac{1}{2}\gamma & -\frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 1-\frac{1}{2}\gamma \end{pmatrix}. \tag{14.2}$$

We show the series character of the load by combining the transformation equations (14.1) in the form

$$A'-B' = A-B,$$
$$A'+B' = A+B+\gamma(A-B),$$

which are Kirchhoff's laws for a series element when we have regard to the fact that in a voltage representation $(A+B)$ is the voltage and $(A-B)$ is the current at $z = 0$. The quantity γ is therefore the impedance of the load.

Thus the series and shunt loading matrices transforming right to left in a voltage representation are respectively

$$E+\gamma U_2 \text{ (series)} \quad \text{and} \quad E+\alpha U_1 \text{ (shunt)}, \tag{14.3}$$

where

$$U_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \qquad U_2 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}. \tag{14.4}$$

Any combination of series and shunt loads may be represented by multiplying in order the corresponding matrices and noting that

$$U_1^2 = U_2^2 = 0, \qquad U_1 U_2 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad U_2 U_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$
$$U_1 U_2+U_2 U_1 = E. \tag{14.5}$$

If the loads are separated we must introduce the corresponding displacement matrices P in order to take account of propagation.

Since the choice of a voltage (electric, impedance) representation or a current (magnetic, admittance) representation is dependent on the types of loading, both representations may be used. It is convenient to know the matrix which transforms from one to the other. In an obvious notation

$$\begin{pmatrix} A_V \\ B_V \end{pmatrix} = Z_c \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A_I \\ B_I \end{pmatrix} \tag{14.6}$$

and a loading matrix M in current representation becomes for voltage representation

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{14.7}$$

This shows that U_1 is appropriate to series and U_2 to shunt in an admittance representation on the w-plane.

1.5. Standing-wave Representations

For certain purposes it is convenient to represent the wave system in terms of standing waves in space quadrature instead of running waves. Thus, suppressing the time factor, (11.6) may be written

$$u = (A+B)\cos kz + j(B-A)\sin kz. \tag{15.1}$$

The transformation from travelling to standing waves is therefore given by the matrix

$$\begin{pmatrix} 1 & 1 \\ -j & j \end{pmatrix}, \tag{15.2}$$

and the inverse transformation is

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2}j \\ \frac{1}{2} & -\frac{1}{2}j \end{pmatrix}. \tag{15.3}$$

In this system of representation impedance plays the role of w in respect to the travelling waves.

Displacement in the standing-wave representation is achieved by

$$\begin{pmatrix} \cos kz_0 & -j\sin kz_0 \\ \sin kz_0 & j\cos kz_0 \end{pmatrix} \tag{15.4}$$

in place of (11.9) which applies to travelling waves.

As examples of the construction of loading matrices, we choose shunt and series loads with (A, B) on a voltage basis. For a shunt load of admittance α the transformation is

$$\begin{pmatrix} 1 & 1 \\ -j & j \end{pmatrix} \begin{pmatrix} 1+\frac{1}{2}\alpha & \frac{1}{2}\alpha \\ -\frac{1}{2}\alpha & 1-\frac{1}{2}\alpha \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2}j \\ \frac{1}{2} & -\frac{1}{2}j \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -j\alpha & 1 \end{pmatrix}, \tag{15.5}$$

operating from right to left. Correspondingly for the series load of impedance γ we have

$$\begin{pmatrix} 1 & 1 \\ -j & j \end{pmatrix}\begin{pmatrix} 1+\frac{1}{2}\gamma & -\frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 1-\frac{1}{2}\gamma \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2}j \\ \frac{1}{2} & -\frac{1}{2}j \end{pmatrix} = \begin{pmatrix} 1 & j\gamma \\ 0 & 1 \end{pmatrix}. \tag{15.6}$$

1.6. Transformations on the w-plane

If (A, B) is transformed by the matrix

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \tag{16.1}$$

the corresponding transformation of w is

$$w' = \frac{c_{21}+c_{22}w}{c_{11}+c_{12}w}. \tag{16.2}$$

This bilinear transformation can be found in any practical case, but only the ratios of the elements of the matrix C are involved. Hence in addition to knowledge of the transformation of the w-plane which would be obtained by impedance or admittance measurements only, two quantities (a modulus and phase) must be measured in addition, in order that the corresponding matrix C can be fully determined. It is clear that the most general matrix C represents the most general localized loading of the line or wave. Regarded as a mathematical transformation (16.2) has well-known properties.† There are two self-corresponding points. If these points are coincident, we obtain a restricted class of transformations and correspondingly a restricted class of localized loads. In impedance representations, if the coincident self-corresponding points lie at $w = +1$, the load is a series one, for an open-circuit is transformed unchanged past the series load; if they lie at $w = -1$ the load is a shunt, for a short-circuit is transformed past the shunt load unchanged. Corresponding statements may be made regarding admittance representations.

The general form of the matrix C can be expressed in terms of the radiation coefficients. Let a wave of complex amplitude A' be incident from the left, and let the load radiate $f'A'$ to the left and $-g'A'$ to the right.‡ When the wave of amplitude B is incident from the right, let the corresponding amplitudes of the waves radiated to the right and to the left be fB and $-gB$ respectively. Now apply the principle of superposition to obtain *in current representation*, for example,

$$A = (1-g')A'+fB,$$
$$B' = f'A'+(1-g)B,$$

† Bateman, H., *Partial Differential Equations*, § 4.22.

‡ The negative sign is chosen so that g will be positive when real.

which lead at once to the matrix representation

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} \dfrac{1}{1-g'} & \dfrac{-f}{1-g'} \\ \dfrac{f'}{1-g'} & 1-g-\dfrac{ff'}{1-g'} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}. \tag{16.3}$$

The special case which represents the most general load presented by a lumped circuit load to a wire transmission line is given by $g = g'$. This result is required by reciprocity, for the determinant of the second-order matrix in (16.3) must be unity. It is perhaps worth while to prove this result. Let v' and i' denote the voltage and current on the line on the left of the load and let unprimed letters denote corresponding quantities to the right. On account of the linear circuit laws

$$\begin{aligned} v' &= pv+qi \\ i' &= rv+si \end{aligned} \quad \text{or} \quad \begin{pmatrix} v' \\ i' \end{pmatrix} = Q\begin{pmatrix} v \\ i \end{pmatrix},$$

where p, q, r, and s are constants determined by the circuit. When $v = V$ and $v' = 0$, the value of i' is $-(ps-qr)V/q$. Whereas when $v = 0$ and $v' = V$, the value of i is V/q. Now the reciprocity theorem states that these two currents are equal, hence $|\det Q| = |ps-qr| = 1$. But

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = R\begin{pmatrix} v' \\ i' \end{pmatrix}, \quad \text{where } R = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Thus

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = RQR^{-1}\begin{pmatrix} A \\ B \end{pmatrix},$$

and since $|Q| = 1$, $|RQR^{-1}| = 1$, which is the result stated.

Let us now restrict ourselves to the system in which not merely $g' = g$, but also

$$f = ge^{-j\delta}, \qquad f' = ge^{+j\delta}. \tag{16.4}$$

Then

$$C = \begin{pmatrix} \dfrac{1}{1-g} & \dfrac{-ge^{-j\delta}}{1-g} \\ \dfrac{ge^{+j\delta}}{1-g} & \dfrac{1-2g}{1-g} \end{pmatrix}. \tag{16.5}$$

The corresponding w-transformation may be written

$$w' = \frac{ge^{j\delta}+(1-2g)w}{1-ge^{-j\delta}w}, \tag{16.6}$$

which may be reduced to the form

$$\frac{1}{w'-e^{j\delta}} = \frac{1}{k+e^{j\delta}}+\frac{1}{w-e^{j\delta}}, \tag{16.7}$$

where $k = -e^{j\delta}/g$. This equation shows that $e^{j\delta}$ is the position of the double self-corresponding point of the transformation, and it is evident that the susceptance transformed unchanged past the load is $\tan\frac{1}{2}\delta$, while the transform of the match ($w = 0$) is

$$w' = -\frac{e^{2j\delta}}{k} = ge^{j\delta}. \tag{16.8}$$

With respect to this type of load, the susceptance just mentioned plays the same role as does the short-circuit for a shunt load or the open-circuit for a series load. We thus reach a suggestive generalization of the conceptions shunt and series by giving attention to the pure susceptance which is invariant under the transformation representing the load. It would be convenient to use the name *unimodular loading of argument* δ to describe a load for which the self-corresponding double point lies at $e^{j\delta}$. Series loading would then become *unimodular loading of argument* π, etc. It will always be necessary to indicate whether the w-plane is used on a current or voltage basis so as to render these names unambiguous. Accordingly it may be advantageous to introduce the following refinement in notation: use *s-argument* and *r-argument* respectively to distinguish uses of the w-plane to represent susceptances on the one hand and reactances on the other. Further, we note on writing the matrix C of (16.5) in the form

$$C = E+\frac{2g}{1-g}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2}e^{-j\delta} \\ \frac{1}{2}e^{j\delta} & -\frac{1}{2} \end{pmatrix} = E+\frac{2g}{1-g}U_\delta \tag{16.9}$$

that $U_\delta = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}e^{-j\delta} \\ \frac{1}{2}e^{j\delta} & -\frac{1}{2} \end{pmatrix}$ is the generalization of U_1 and U_2 which in voltage representations become respectively U_π and U_0.

Since the equation (16.7) will be used in practice with actual plots on the circle diagram, it is useful to have the following results relevant to the geometry of the case. Let the self-corresponding points be $e^{j\delta}$ and let the transform of a match be $\rho e^{j\epsilon}$. The unit circle is transformed into the circle with its centre at

$$\frac{2\rho e^{j\delta}}{1-\rho^2}[\cos(\delta-\epsilon)-\rho]$$

and of radius
$$\frac{1-2\rho\cos(\delta-\epsilon)+\rho^2}{1-\rho^2}. \tag{16.10}$$

The fraction of power absorbed by the load when the line is terminated in a match is

$$1-\frac{|1-g|^2}{1-|g|^2}. \tag{16.11}$$

1.7. Oblique Incidence of Plane Waves on a Plane Interface

In § 1.2, it was indicated how to treat, in perfect analogy with the transmission line, the discontinuity in propagation which occurs when unbounded plane waves are incident normally on the plane interface between two dielectrics. We now discuss the oblique incidence of unbounded plane waves on the plane interface. For this purpose we introduce the ideas of propagation-vector and wave-function for plane waves propagated in any direction specified by the set of direction cosines (l, m, n) referred to our coordinate system $Oxyz$. The phase difference between the origin and the point (x,y,z) at the same time is $k(lx+my+nz)$, where $k = 2\pi/\lambda$, λ being the wavelength in the medium supposed loss-free; if κ and μ are the dielectric constant and magnetic permeability of the medium,

$$k = \frac{\omega}{c}\sqrt{(\mu\kappa)} = k_0\sqrt{(\mu\kappa)}.$$

The wave-function for these waves (assumed harmonic) is

$$\exp[j\{\omega t-k(lx+my+nz)\}]$$

and the propagation-vector is (kl, km, kn) radians/metre.

The components of the propagation-vector will be complex in the case of a medium with loss and, as we shall see, this is also possible even when there is no leakage. When this is so, the real parts of the components form a vector which is perpendicular to the planes of constant phase

$$k(lx+my+nz) = \phi,$$

while the imaginary parts form a vector in the direction opposed to that of the exponential attenuation. When these two directions coincide, the infinite plane wave is called simple.

Let the semi-infinite medium 1 be bounded over the xy-plane by a second medium 2 extending to infinity (z positive). Let the constants μ and κ for these media be distinguished by the subscripts 1 and 2. Imagine a simple plane wave travelling in medium 1 towards the interface: let its propagation-vector make θ_1 with the normal to the interface; let the propagation-vectors of the reflected and transmitted waves make $(\pi-\theta_1')$ and θ_2 respectively with the normal which is the positive z-axis. (See Fig. 5.)

Although the law of reflection is known, we shall not for the purpose of this discussion assume it. With an obvious choice of the axes of x and y, the propagation-vectors of the three waves are:

Incident: $\quad k_0\sqrt{(\mu_1\kappa_1)}(0, \sin\theta_1, \cos\theta_1).$

Reflected: $k_0\sqrt{(\mu_1\kappa_1)}(0, \sin\theta_1', -\cos\theta_1')$.

Transmitted: $k_0\sqrt{(\mu_2\kappa_2)}(0, \sin\theta_2, \cos\theta_2)$.

Suppose that in the incident wave the electric vector E_1 is parallel to the interface. Since the wave is transverse we must have E_1 in the x-direction. The component of magnetic force parallel to the interface is $H_y = H_1\cos\theta_1$. Thus the field impedance looking towards

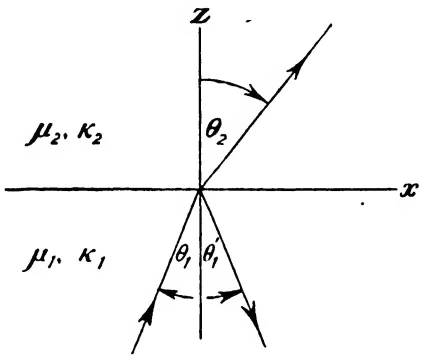


FIG. 5.

the interface in the direction of the positive z-axis is for the incident wave

$$Z_1 = \frac{E_x}{H_y} = \frac{E_1}{H_1}\sec\theta_1 = 120\pi\sqrt{(\mu_1/\kappa_1)}\sec\theta_1.$$

This method can be applied with the following results to each wave, in each of the two fundamental cases from which the behaviour for arbitrary polarization of the incident wave may be deduced:

Case 1. E vibrates parallel to the interface:

Incident: $Z_1 = 120\pi\sqrt{(\mu_1/\kappa_1)}\sec\theta_1$.

Reflected: $Z_1' = -120\pi\sqrt{(\mu_1/\kappa_1)}\sec\theta_1'$. (17.1)

Transmitted: $Z_2 = 120\pi\sqrt{(\mu_2/\kappa_2)}\sec\theta_2$.

Case 2. H vibrates parallel to the interface:

Incident: $Z_1 = 120\pi\sqrt{(\mu_1/\kappa_1)}\cos\theta_1$.

Reflected: $Z_1' = -120\pi\sqrt{(\mu_1/\kappa_1)}\cos\theta_1'$. (17.2)

Transmitted: $Z_2 = 120\pi\sqrt{(\mu_2/\kappa_2)}\cos\theta_2$.

The boundary conditions which must be satisfied at the interface $z = 0$ in each case are the continuity of the tangential components of both E and H. This requires at once that the y-components of the propagation-vectors of the waves shall be identical, i.e.

$$k_0\sqrt{(\mu_1\kappa_1)}\sin\theta_1 = k_0\sqrt{(\mu_1\kappa_1)}\sin\theta_1' = k_0\sqrt{(\mu_2\kappa_2)}\sin\theta_2,$$

which yield at once the laws of reflection and refraction

$$\theta_1' = \theta_1 \quad \text{and} \quad \sqrt{\left(\frac{\mu_2\kappa_2}{\mu_1\kappa_1}\right)} = \frac{\sin\theta_1}{\sin\theta_2}. \tag{17.3}$$

We note that $Z_1' = -Z_1$.

In the second place, we have for *Case* 1

$$\begin{aligned} E_1+E_1' &= E_2, \\ (H_1)_y+(H_1')_y &= (H_2)_y, \\ \text{or} \qquad Y_1E_1-Y_1E_1' &= Y_2E_2, \end{aligned} \tag{17.4}$$

where Y_1 is the admittance corresponding to Z_1, etc.

For *Case* 2

$$\begin{aligned} H_1+H_1' &= H_2, \\ Z_1H_1-Z_1H_1' &= Z_2H_2. \end{aligned} \tag{17.5}$$

Thus

$$(1) \qquad \frac{E_1}{Y_1+Y_2} = \frac{E_1'}{Y_1-Y_2} = \frac{E_2}{2Y_1} \quad (E \text{ parallel to interface});$$

$$(2) \qquad \frac{H_1}{Z_1+Z_2} = \frac{H_1'}{Z_1-Z_2} = \frac{H_2}{2Z_2} \quad (H \text{ parallel to interface}). \tag{17.6}$$

The first of these is equivalent to (12.7) in the case of normal incidence and the second corresponds with what we should obtain in place of (12.7) were we to use the magnetic (A, B) representation.

On the basis of equations (17.6) just derived, we can establish Fresnel's laws, which are well known in electromagnetic optics. Further, these results may be extended to apply to media in which there is appreciable conductivity (σ). The phenomena of reflection and transmission are described by substituting for κ in the above formulae the complex number $(\kappa - j\sigma/\omega\kappa_0)$, so that the intrinsic impedance of the second medium, for example, becomes

$$120\pi\sqrt{\left(\frac{j\omega\mu_2}{\sigma_2/\kappa_0+j\omega\kappa_2}\right)}.$$

The velocity of propagation is also complex, viz. $c/\sqrt{\{\mu_2(\kappa_2-j\sigma_2/\omega\kappa_0)\}}$, which means exponential attenuation of amplitude in the direction of propagation. The results are immediately evident from Maxwell's equations and Ohm's law for

$$\operatorname{curl}\mathbf{H} = j\omega\kappa\kappa_0\mathbf{E}+\mathbf{i} = j\omega\kappa_0\left(\kappa-\frac{j\sigma}{\omega\kappa_0}\right)\mathbf{E},$$

where $\mathbf{i}$ is the current density.

1.8. Evanescent Transverse Electric and Magnetic Waves

The field impedance Z_2 of (17.1) and (17.2) may be expressed directly in terms of the angle of incidence θ_1:

$$(1) \qquad Z_2 = \frac{120\pi\mu_2}{\sqrt{(\mu_2\kappa_2-\mu_1\kappa_1\sin^2\theta_1)}} \quad (E \,||\, \text{interface});$$

$$(2) \qquad Z_2 = -\frac{120\pi}{\kappa_2}\sqrt{(\mu_2\kappa_2-\mu_1\kappa_1\sin^2\theta_1)} \quad (H \,||\, \text{interface}). \qquad (18.1)$$

If $\mu_1\kappa_1 < \mu_2\kappa_2$, Z_2 is real, i.e. purely resistive, for all real values of the angle of incidence. But if $\mu_1\kappa_1 > \mu_2\kappa_2$, Z_2 is purely resistive or purely reactive according as θ_1 is less or greater than $\sin^{-1}\sqrt{(\mu_2\kappa_2/\mu_1\kappa_1)}$, which is called the critical angle of incidence θ_c. When $\theta_1 > \theta_c$, (1) is actually an inductive reactance and (2) is a capacitive reactance. The formulae (17.6) give the relative amplitudes and phases of the three waves whatever the angle of incidence, and when $\theta_1 > \theta_c$, the propagation-vector in medium 2 is complex: its real part is

$$k_0\{0, \sqrt{(\mu_1\kappa_1)}\sin\theta_1, 0\};$$

its imaginary part is

$$k_0\{0, 0, -j\sqrt{(\mu_1\kappa_1\sin^2\theta_1-\mu_2\kappa_2)}\}.$$

Thus the wave in medium 2 is not simple. The propagation of phase takes place in the y-direction with the velocity $\{c/\sqrt{(\mu_1\kappa_1)}\}\operatorname{cosec}\theta_1$, which is less than that for free waves in medium 2. The surfaces of constant phase are perpendicular to y. In the z-direction the amplitude of the disturbance is attenuated exponentially with distance from the interface at the rate

$$20\log_{10}e\,k_0\sqrt{(\mu_1\kappa_1\sin^2\theta_1-\mu_2\kappa_2)} \text{ decibels/metre.} \qquad (18.2)$$

The reactive field impedance Z_2 secures complete reflection. Like a coil or a condenser, it sucks in and ejects energy across the interface in successive quarter cycles; but, of course, these processes happen at different times at different places across the interface, due to oblique incidence. The energy is stored in medium 2 close to the interface, as is shown by the exponential attenuation of field strength in that medium. If E vibrates parallel to the interface, the stored energy is mainly magnetic, and if H vibrates parallel to the interface it is mainly electric. There is no continual flow of energy away from the surface of medium 2 as there is when $\theta_1 < \theta_c$ and the propagation-vector in that medium and the impedance Z_2 are real.

When E vibrates parallel to the interface, the magnetic field in medium 2 is seen, because of the continuity of the tangential magnetic force and normal magnetic induction, to have components

$$H_y = \frac{2H_1\,120\pi\sqrt{(\mu_1/\kappa_1)}}{Z_2(1+Z_1/Z_2)}, \qquad H_z = \frac{2\mu_1 \sin\theta_1 H_1}{\mu_2(1+Z_1/Z_2)}. \tag{18.3}$$

The electric force in medium 2 is parallel to x. Thus we have in that medium, under conditions of total reflection, a wave with the real part of its propagation-vector parallel to y, with the components of E and H at right angles to this direction in phase with each other, while in the y-direction there is a component of magnetic force in phase-quadrature with E_x and H_z and perpendicular to them, for Z_2 (reckoned normal to the interface) is imaginary. This wave is called *transverse electric*. In it the electric force is perpendicular to the direction of phase propagation which is also the direction of the quadrature component of magnetic force.

When H vibrates parallel to the interface, we make use of the continuity of electric induction normal to the interface to obtain the corresponding result, viz. the wave in medium 2 is a *transverse magnetic* wave propagated parallel to y. The longitudinal (y) component of E is in phase-quadrature with the transverse components of E and H.

In both cases these waves are propagated with phase velocity less than that for plane waves with uniform amplitude distribution in the medium. The attenuation of wave amplitude normal to the interface is not attended by loss of energy in the form of heat such as would occur in a medium with finite conductivity. The waves are therefore called *evanescent waves*—since they involve exponential attenuation in a loss-free medium. Evanescent waves are associated with the storage of energy in the neighbourhood of a surface and are closely connected with the idea of lumped shunt reactance in transmission systems whose lateral dimensions are not necessarily small compared with $\lambda/2\pi$.

1.9. Lumped Shunt Admittance and Impedance

Consider the strip transmission line of § 1.1. Let a resistive film whose surface conductance is G mhos be presented normally to the waves between the strips. If we take a square of the film of side 1 metre, the conductance for uniform current flow between two opposite edges of the square will be G mhos. Hence our strip transmission line is shunted by the conductance Ga/b mhos. If Y is the intrinsic admittance of the medium between the strips, the characteristic admittance of the transmission line is Ya/b. For a wave incident from the left, the reflection and transmission on the line shunted by the film and extending indefinitely to the right in the direction of the positive z-axis are calculable by (12.8) if we put Y for Y_1 and $Y+G$ for Y_2. Hence

$$\frac{E_1}{2Y+G} = \frac{E_1'}{-G} = \frac{E_2}{2Y}. \tag{19.1}$$

From the circuit point of view this is exactly what we expect from the admittance Ga/b shunted by Ya/b as the termination of the transmission line. From the wave point of view, at the discontinuity in the propagation, the electric field but not the magnetic field is continuous. The discontinuity in the magnetic field arises from the current through the film constituting the load.

Suppose now that a perfectly conducting plate is situated exactly $\frac{1}{4}\lambda$ beyond the conducting film. The field impedance immediately to the right of the film is now infinite, and in (12.8) we must now put $Y_2 = G$, i.e.

$$\frac{E_1}{Y+G} = \frac{E_1'}{Y-G} = \frac{E_2}{2Y}. \tag{19.2}$$

For complete absorption of the wave in the film we must have $G = Y$: then $E_1' = 0$.

In order to secure complete absorption of the wave energy in the resistive film when the simple infinite plane wave is incident at the angle θ to the normal to the film supposed in a medium (μ, κ), we have to consider field admittances reckoned in the direction of the normal to the film. We have

$$Y = \frac{1}{120\pi}\sqrt{\frac{\kappa}{\mu}}\cos\theta \quad (E \parallel \text{film}),$$

$$Y = \frac{1}{120\pi}\sqrt{\frac{\kappa}{\mu}}\sec\theta \quad (H \parallel \text{film}). \tag{19.3}$$

The complete propagation-vector in the direction of the normal is $k_0\sqrt{(\mu\kappa)}\cos\theta$. There will be no reflection from the film if G is made equal to the appropriate value of Y, and the film is backed by a parallel reflecting plate placed $\frac{1}{4}\lambda\sec\theta$ behind it.

So far we have considered resistive loads on plane waves: it is natural to conceive the possibility of loads having susceptance and to find their realization in a metal grating such as a plane grid of parallel equidistant wires.

Imagine an infinite plane grid of perfectly conducting coplanar strips. Let $2d$ be the distance between the centres of adjacent parallel strips, and let $2b$ be the width of the gap between them. This grid is equivalent to a shunt admittance similar to the resistive film just discussed, but it has susceptance as well as conductance. If $2d < \lambda$, the wavelength in the medium in which the grating is embedded, the surface admittance of the grid is a pure susceptance, whereas if $2d > \lambda$, it has conductance as well. The pure susceptance is of capacitive type if the electric force in the incident wave is perpendicular to the strips, but of inductive type if the electric force is parallel to the strips.

We have already seen (§ 1.8) how, in a field, capacitive and inductive reactance are connected with the storage of energy in the vicinity of the plane where a discontinuity in propagation takes place. Further, the storage of energy is associated with evanescent waves. We shall find the same phenomenon due to the grating. A wave incident normally on the grid from the left with E parallel to the strips induces in them currents which vibrate in phase. The waves reradiated by the strips when $2d < \lambda$ arrive in phase over any plane parallel to the grid and over no other planes. The wave radiated to the left is the reflected wave, while that travelling to the right superposed on the undisturbed incident wave constitutes the transmitted wave. But if $2d > \lambda$ the grid can produce spectra; the waves reradiated by the strips arrive in phase over any one of planes inclined at θ_n to the grid, where

$$\theta_n = \sin^{-1}\left(\frac{n\lambda}{2d}\right) \tag{19.4}$$

and n is an integer less than $2d/\lambda$.

These waves draw energy from the incident wave and give rise to the resistive component in the lumped shunt which represents the grid. Let us consider these side waves when $2d < \lambda$: θ_1 is imaginary and the waves become evanescent just as does the transmitted wave at total reflection. These evanescent waves are concerned only with the storage of energy in the immediate neighbourhood of the grid. With this interpretation of imaginary values of θ_n we see that in general we may regard n as unrestricted, but of course integral. To each value of n corresponds a side-wave: if θ_n is real, the side-wave in question carries away energy continually and hence contributes to the shunt conductance presented by the grid; if θ_n is imaginary, the corresponding side-wave is evanescent and, being associated with the storage of energy in the vicinity of the grid, contributes to the shunt susceptance presented by the grid. Exact expressions for the surface susceptance of such grids have been worked out, both for grids of fine circular wires and for flat strips, and for both the inductive and capacitive cases.

We shall have occasion to return to the discussion of gratings in connexion with obstructions in wave guides.

1.10. Summary

In this initial chapter some fundamental ideas have been presented. With the aid of the simple strip transmission line it has been shown how to transfer the circuit conceptions impedance and admittance to plane waves in space. In this way it is possible to discard the elaborate differential equations conventionally employed even with such simple waves and to use instead simple algebra. The gain in ease of management of quite complicated wave systems is analogous to the corresponding step in the theory of electric circuits where the replacement of systems of ordinary differential equations by algebraic ones involving complex numbers is well known to engineers and physicists. The algebraic equations representing waves either on a transmission line or in space can be easily connected with an elegant geometrical representation in terms of the circle diagram.

Again, algebraic representation immediately suggests that propagation and the loading of waves be regarded as transformations. These are found to belong to a simple type and are conveniently represented by matrices which we have developed in some detail. Appropriate to plane waves, the ideas of energy flux, phase distribution, and the classification of types of load by the mathematical properties of the corresponding transformations were considered.

Finally there was discussed at some length the optical phenomenon of reflection and transmission from one medium into another, and by a grid of conducting strips embedded in a single medium. In both of these cases the idea of impedance has an illuminating role to play. In order to explain the reactive behaviour of grids and of an interface at total reflection, we were led to the consideration of evanescent waves. To anyone accustomed to the representation of plane waves of light as essentially transverse electromagnetic, the appearance of transverse electric and transverse magnetic waves will be a novelty. In part, the justification of the discussion of these waves in this chapter is that we have to deal with them in wave guides in which we shall find that, as the result of a different amplitude distribution in the planes of equal phase, these waves can transfer energy just as do the better-known transverse electromagnetic waves.

II

THE RECTANGULAR WAVE GUIDE. I

2.1. The Dominant Wave

In order to obtain a picture of the simplest or dominant transverse electric wave in a rectangular wave guide, we consider a system of two interfering trains of the plane waves such as we introduced in § 1.1. As before, let the electric vector be parallel to y but let the waves travel between two uncut parallel planes, in such a way that the propagation-vectors each make the angle θ on opposite sides of the z-axis but remain parallel to the xz-plane. The harmonic wave trains are supposed in phase at the origin of coordinates. The propagation-vectors are $k(\pm l, 0, n)$, where $l = \sin\theta$, $n = \cos\theta$, and by the principle of superposition the electric force in the system of the two plane waves of equal amplitude $A/2$ is

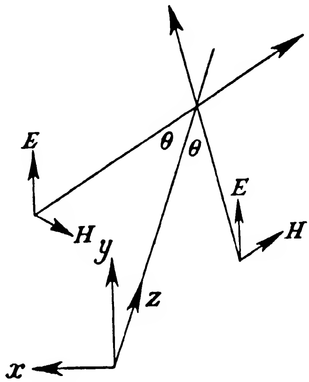


Fig. 6.

$$E_y = \frac{A}{2}\left[e^{j\omega\{t-(nz-lx)/c\}} + e^{j\omega\{t-(nz+lx)/c\}}\right]$$

$$= A\cos\frac{l\omega x}{c}e^{j\omega(t-nz/c)}. \tag{21.1}$$

The magnetic forces of the waves are in different directions, hence they must be combined vectorially (see Fig. 6):

$$H_x = -\frac{An}{Z_0}\cos\frac{l\omega x}{c}e^{j\omega(t-nz/c)}, \tag{21.2}$$

$$H_z = -j\frac{Al}{Z_0}\sin\frac{l\omega x}{c}e^{j\omega(t-nz/c)}, \tag{21.3}$$

where Z_0 is the intrinsic impedance for the medium in which the plane waves are travelling. The other field components vanish.

The wave system therefore comprises a standing-wave pattern in the x-direction and propagation in the z-direction with phase-velocity $c/n = V_g$. The standing-wave pattern is repeated (apart from phase opposition) in the x-interval

$$a = \frac{\pi c}{l\omega}. \tag{21.4}$$

Along the planes $x = \pm a/2$, and on parallel planes distant pa from them (p an integer), the electric force parallel to the planes vanishes. Consequently, we may confine our attention to the part of space contained within the planes $y = 0$, $y = b$, and $x = \pm a/2$, provided that the field is bounded by a perfect conductor. That is, we have arrived at a possible field distribution inside a rectangular tube $a \times b$ in cross-section. For reasons that will be adduced later, we shall assume $b < \frac{1}{2}\lambda < a < \lambda$, where $\lambda = 2\pi c/\omega$. Since the electric force, but not the magnetic force, is transverse, relative to the direction of propagation z, this wave is called a transverse electric (TE) wave.

Let us introduce a into the formulae and displace the origin of coordinates to $(-\frac{1}{2}a, 0, 0)$, then

$$E_y = A \sin\frac{\pi x}{a} e^{j\omega(t-z/V_g)}, \tag{21.5}$$

$$H_x = -\frac{A}{Z_g}\sin\frac{\pi x}{a} e^{j\omega(t-z/V_g)}, \tag{21.6}$$

$$H_z = -\frac{jA}{\omega\mu_0}\frac{\pi}{a}\cos\frac{\pi x}{a} e^{j\omega(t-z/V_g)}, \tag{21.7}$$

where
$$Z_g = \frac{Z_0}{n} = \frac{V_g}{c}\sqrt{\frac{\mu_0}{\kappa_0}} \tag{21.8}$$

and
$$V_g = \frac{c}{\sqrt{\{1-(\lambda/2a)^2\}}}, \quad \text{for } l = \frac{\lambda}{2a}. \tag{21.9}$$

Since $|l| \leqslant 1$, the width a of the tube must not be less than half of the free-space wavelength corresponding to the frequency $\omega/2\pi$. That is, there is a certain minimum frequency

$$f_0 = \frac{c}{2a} \tag{21.10}$$

for which the velocity V_g and the impedance Z_g are real. For frequencies less than f_0, V_g and Z_g become imaginary; the field components, instead of depending harmonically on z, decay exponentially with distance along the guide. At the same time the wave impedance Z_g becomes a pure reactance, which means that although energy can be stored inside the guide close to the place where it is excited, there can be no transfer of energy on the average along a guide with perfectly conducting wall when $\omega < 2\pi f_0$. In fact, the wave is an evanescent one. We speak of f_0 as the cut-off frequency for the TE-wave with this particular amplitude distribution in the xy-plane for the guide cross-section given.

The form of equation (21.7) shows that the longitudinal magnetic field H_z is in phase-quadrature with the transverse electric and magnetic field components, which both vanish on the walls of the guide parallel to the y-axis. Further, over the plane $x = \frac{1}{2}a$, midway between and parallel to these walls, H_z vanishes, but the transverse electric and magnetic field components reach their maximum amplitudes of oscillation.

2.2. Energy Flux

Let us now apply Poynting's theorem in the form giving the mean energy flux density $\mathbf{w}$ in terms of the complex vectors $\mathbf{E}$ and $\mathbf{H}$:

$$\overline{\mathbf{w}} = \tfrac{1}{2}\operatorname{Re}\mathbf{E}\times\mathbf{H}^* = \frac{A^2}{2Z_g}\sin^2\frac{\pi x}{a}\mathbf{z}_1, \tag{22.1}$$

where $\mathbf{z}_1$ is the unit vector in the direction z. The total flux obtained by integrating over the guide section is

$$W = \frac{abA^2}{4Z_g} = \frac{c}{V_g}\frac{abA^2}{4.120\pi}\text{watts.} \tag{22.2}$$

Since H_z is in quadrature with E_y, there is no mean energy transfer in the x-direction. There is, however, oscillation of energy to supply what must be stored most densely at each side of the guide where H_z is maximum, and to remove it again when H_z falls to zero. We call E_y/H_z the transverse impedance in the wave: it is seen to be reactive in accordance with the vanishing mean energy flux in the x-direction.

2.3. The Current System on the Wall

Just as the electric and magnetic fields in the space between the two conductors of an ordinary transmission line require current on the conductors, so the walls of the wave guide are the seat of surface currents which support the field inside the guide. Apply the boundary condition expressed in equation (11.2) and we obtain for the surface-current density ($\mathbf{J}$) measured in amperes per metre perpendicular to the direction of flow:

$$\begin{array}{lll} \text{on } y = 0, & J_x = H_z, & J_z = -H_x; \\ \text{on } y = b, & J_x = -H_z, & J_z = H_x; \\ \text{on } x = 0, & J_y = -H_z, & J_z = 0; \\ \text{on } x = a, & J_y = H_z, & J_z = 0. \end{array} \tag{23.1}$$

Fig. 7 represents the amplitude of oscillation in the system of current flow on a normal cross-section $ABCD$ of the guide, which has been opened out so as to show both longitudinal and transverse currents as functions of distance along the perimeter of the section.

The longitudinal current is zero on BC and DA, and is in opposite directions on AB and CD where its amplitude is proportional to $\sin \pi x/a$. The transverse component of current is in phase-quadrature with the longitudinal current at the same guide section. Its amplitude is constant on BC and DA, but it flows in opposite directions on these

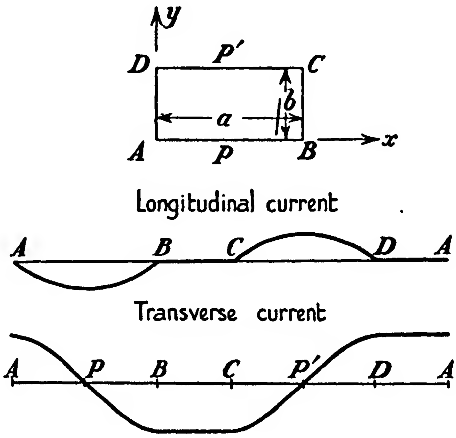


FIG. 7.

narrow faces of the guide wall. On AB and CD the distribution of transverse current is given by $\cos \pi x/a$. If we change our representation so as to make B coincide with C and D with A, the longitudinal is shown by a sine curve and the transverse by a cosine curve on AD. With this device we may therefore say that the two distributions are in time-quadrature and space-quadrature along the perimeter of a cross-section of the guide. At P, the mid-point of AB, the broad face of the guide, and at P', the mid-point of CD, the transverse current is zero. Consequently the current system on the guide wall will, for practical purposes, be undisturbed by a cut or slot in the rectangular metal tube along the centre-line of a broad face. This fact is used in the technique of standing-wave measurements in guides, and in the measurement of relative field strengths, to allow the introduction of an antenna into the guide.

The electric current vector on the walls of the guide executes linear oscillations on the side walls and on the centre-line of the broad faces. Elsewhere it executes elliptical oscillations because it is the resultant of two unequal orthogonal vectors in phase-quadrature; for the particular value of x which makes equal the amplitudes of H_x and H_z, the oscillation is a circular one. These remarks apply only to a pure travelling wave in one direction. If, due to reflection at one end of the guide, a standing-wave system is generated, there will be a stationary pattern of current distribution on the walls, in which the lines of current flow are orthogonal to the loci

$$\sin\frac{\pi x}{a}\sin\frac{\omega z}{V_g} = \text{constant}. \tag{23.2}$$

The distribution is shown in Fig. 8.

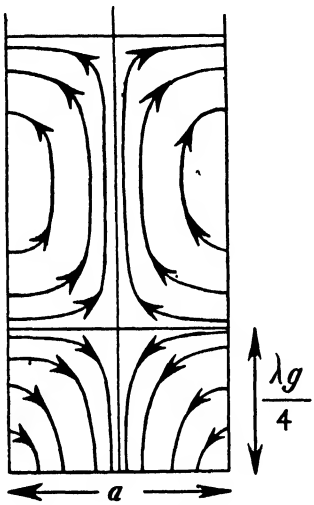


FIG. 8.

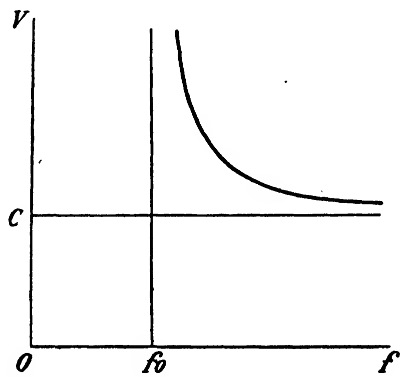


FIG. 9.

2.4. Frequency Dependence of the Propagation

One of the striking characteristics of propagation in a wave guide is dispersion. The velocity of propagation of waves with amplitude (21.5) is a fairly rapidly varying function of frequency (see Fig. 9). Accordingly a group of waves will not retain its group wave form in the course of propagation. For example, if the microwaves are modulated by means of micro-second pulses repeated every milli-second, the shape of the pulse will be appreciably altered in propagation through a guide many wavelengths long. Such modulated microwaves will not permit zero minima in the standing-wave system in a long guide terminated by a perfect reflector. Further, if two trains of coherent modulated waves are sent by different paths to interfere at a detector, cancellation can never be perfect for the modulated wave, even although it may be so for a narrow band of frequencies in the group.

As the frequency of the radiation is increased in a given rectangular wave guide, the relative amplitude of the longitudinal magnetic field, and, consequently, of the transverse surface current on the wall, is decreased. The field impedance Z_g tends more and more closely to the free-space 120π ohms. On the other hand, as the frequency is reduced towards cut-off, the value of Z_g increases indefinitely; the transverse magnetic field and the longitudinal current on the wall tend to zero. At the same time, for given power input to the guide, the electric field strength, the longitudinal magnetic field, and the transverse current all increase.

2.5. Impedance of a Rectangular Wave Guide

So far we have introduced only the field impedance in the TE-wave in the rectangular guide. On account of the simple field distribution in the guide we are easily led, however, to think of the impedance of the guide reckoned as the ratio

$$Z = \frac{2\times\text{total energy flux}}{(\text{total longitudinal current})^2}.$$

It is readily shown that
$$Z = \frac{\pi^2}{8}\frac{b}{a}Z_g. \tag{25.1}$$

If we attempt to define impedance as the voltage/current ratio in a travelling wave, we must choose what we mean by voltage. Some writers use the maximum voltage, others the mean voltage over the guide section. Neither of these gives the same numerical factor as (25.1), but both yield the same dependence on the dimensions a and b of the guide.

The importance of this conception is that it brings out clearly how the depth b of the guide enters to determine the current and maximum voltage to be expected for given power transfer along the guide. A shallow guide requires greater current on the inside of the faces of the guide wall for the same power. Consequently, on account of the finite conductivity of the metal walls on which the current must flow, one expects greater ohmic loss in the transmission of microwave energy in such a shallow guide than in one with larger b.

A wave guide is essentially a high-impedance transmission line. For example, S-band guide ($a = 7{\cdot}2$ cm., $b = 3{\cdot}4$ cm., $Z_g = 1{\cdot}5\times 120\pi$ ohms) gives $Z = 329$ ohms for a typical frequency in the band for which the guide dimensions were chosen. This should be compared with coaxial line impedances of less than 100 ohms. Not merely is the wave guide able to transmit microwave power more efficiently than coaxial line, it has actually a higher power rating than coaxial lines of practically useful impedance which permit the effective propagation of only one mode, namely, the principal wave. In Table I are shown some data on standard wave guides and coaxial lines to illustrate these considerations.

TABLE I

Outer diam.	*Wall*	*Centre rod*	*Char. imped.*	*Support*	*Attenuation*	*Rated*† *power*	
in.	in.	in.	ohms		db/m.	kw.	
Coaxial lines							
$\frac{5}{16}$	0·025	$\frac{1}{8}$	44·4	bead	0·47	5	*X*-band
$\frac{1}{2}$	0·032	$\frac{3}{16}$	50·6	stub	0·49	50	*X*-band
$\frac{5}{8}$	0·035	$\frac{1}{4}$	47·8	bead	0·22	20	*S*-band
$\frac{7}{8}$	0·032	$\frac{3}{8}$	46·4	stub	0·15	200	*S*-band
Wave guides			*Wave type*				
$\frac{1}{2}\times 1$	0·050		H_{10}		0·25	250	*X*-band
$\frac{5}{8}\times 1\frac{1}{4}$	0·064		H_{10}		0·15	450	*X*-band
1‡	0·032		H_{11}			550	*X*-band
$1\frac{1}{4}$‡	0·080		E_0			500	*X*-band
$1\frac{1}{2}\times 3$	0·080		H_0		0·04	2,500	*S*-band
3 I.D.‡	—		H_{11}			3,700	*S*-band
4 I.D.‡	—		E_0			3,700	*S*-band

† The rated power is one-quarter maximum, based on the breakdown field strength in air under normal conditions.

‡ For convenience, data for other wave types in circular guides are included.

2.6. Approximate Theory of Attenuation due to Ohmic Loss

On the assumption that the ohmic losses are small, we may use equations (21.5)–(21.7) as a first approximation to the field in the guide.§ The intrinsic impedance of the metal walls is $Z_w = 120\pi\sqrt{(j\omega\kappa_0/\sigma)}$ very nearly, where the permeability of the metal is taken as 1, and σ is its conductivity in mhos/metre. The flux of energy into the metal to produce heating is determined by the tangential component of electric force which no longer vanishes as in our previous representation of the field. This component is small and may be deduced as follows. The tangential component H_t of magnetic force at the wall and just inside the metal will be very nearly given by (21.6) and (21.7); and the tangential component of E inside the metal is $E_t = Z_w H_t$ directed at right angles to the magnetic vector.

Let R_w be the real part of Z_w, i.e.

$$R_w = 120\pi\sqrt{\left(\frac{\omega\kappa_0}{2\sigma}\right)}. \tag{26.1}$$

§ Cf. Schelkunoff, S. A., *Proc. I.R.E.* **25**, 1457–92 (1937).

Since the transverse and longitudinal currents are in phase-quadrature, we can treat the power dissipated by them separately. The power flowing into the wall of the two broad faces of the guide per unit length due to the longitudinal current is

$$Q_L = 2\int_0^a \tfrac{1}{2}R_w|H_x|_{y=0}^2\,dx. \tag{26.2}$$

The power lost per unit length due to the transverse current is represented in two terms belonging respectively to the broad and narrow faces, as follows:

$$Q_T = 2\left[\int_0^a \tfrac{1}{2}R_w|H_z|_{y=0}^2\,dx + \tfrac{1}{2}bR_w|H_z|_{x=0}^2\right]. \tag{26.3}$$

Since Q_L+Q_T is small compared with the power flux P along the guide, we may obtain the attenuation α reckoned on a power basis, as

$$\alpha = \frac{Q_L+Q_T}{2P}\ \text{nepers/m.}\quad (1\ \text{neper} = 8{\cdot}686\ \text{db.}) \tag{26.4}$$

P is given by (22.2). Apply equations (21.6) and (21.7) to obtain

$$\frac{Q_L}{P} = \frac{2R_w}{Z_g b},\qquad \frac{Q_T}{P} = 2R_w Z_g\left(\frac{\pi}{a\omega\mu_0}\right)^2\left(\frac{1}{b}+\frac{2}{a}\right).$$

Therefore

$$\alpha = \frac{R_w}{bZ_g}\left\{1+\left(\frac{\pi Z_g}{a\omega\mu_0}\right)^2\left(1+\frac{2b}{a}\right)\right\}. \tag{26.5}$$

Since from (21.8) and (21.9)

$$\left(\frac{\pi Z_g}{a\omega\mu_0}\right)^2 = \frac{l^2}{1-l^2},\qquad l = \frac{\lambda}{2a}, \tag{26.6}$$

we have finally

$$\alpha = \sqrt{\left(\frac{1}{120\sigma\lambda}\right)}\frac{1}{b}\frac{1+2l^2b/a}{\sqrt{(1-l^2)}}\ \text{nepers/m.} \tag{26.7}$$

For example, the following approximate values are relevant for S-band guide:

$$\lambda = 0{\cdot}1\ \text{m.},\qquad b = 0{\cdot}035\ \text{m.},\qquad b/a = \tfrac{1}{2},\qquad l = 0{\cdot}7.$$

Assuming $\sigma = 1{\cdot}1\times10^7$ mhos/m. for brass, we find $\alpha = 0{\cdot}045$ db./m., which agrees reasonably well with experience.

It will be noted that the main factors determining α are the depth (b) of the guide, and the 'cut-off' factor $1/\sqrt{(1-l^2)}$. Use of silver in place of brass halves the attenuation, while for a similar guide designed for K-band the attenuation would be three times as large, according to the formula. The ratio b/a may be chosen so that $\partial\alpha/\partial\lambda = 0$.

2.7. The Wave Guide as a Transmission Line

The equations (21.5–7), which present the field components in the dominant wave in a rectangular guide, show explicitly their dependence on a propagation factor common to all and a distribution factor specifying how they vary over the guide cross-section. If the waves are detected by a method which does not vary the aspect of the detecting antenna with respect to the guide cross-section, the only variations in the measured fields arise from propagation. Accordingly the methods of §§ 1.1–1.6 may be applied to such measurements of wave amplitudes, the dominant wave in the guide thus being treated in the same way as the principal wave on a conventional transmission line.

Localized loading of the wave by an antenna or obstruction in the guide causes the radiation of waves, which, at sufficient distance to allow the disappearance of evanescent waves in the vicinity of the load, consist of dominant waves in a special phase and amplitude relation to the incident dominant wave. If the secondary waves of electric force on the two sides of the load are of equal amplitude and in phase at the same distance from it, the load behaves as a shunt, presenting an admittance which can be measured by standing-wave procedure. If the secondary waves of electric force are of equal amplitude and in opposed phase on the two sides at the same distance, the load behaves as a series one, presenting to the wave an impedance which can likewise be measured. Shunt loading involves a discontinuity in the transverse component of magnetic force at the point of loading: series involves a discontinuity in electric force. It should be noted that the phase relations of the secondary waves for shunt and series are interchanged when the dominant wave is represented as a wave of transverse magnetic force. Further, it is clear that types of loading more complicated than simple series or shunt may be treated just as they were in Chapter I for the principal wave on the transmission line.

III

MEASUREMENTS

3.1. Detection

Two methods are used in detecting microwaves. In order that low-frequency amplification may be applied, the microwave generator is usually amplitude-modulated. For this purpose it is desirable to use a square wave form, the fundamental frequency in which is not close to any of the important frequencies present in the a.c. supply used to operate equipment (e.g. 60 and 300 cycles per second). By modulating with a square wave one may avoid the complicating effects of the frequency modulation due to the varying voltage applied to one of the electrodes of the microwave oscillator; the time intervals during which this voltage is changing form an insignificant fraction of the duration of the pulses of oscillation. As detectors, crystals have been successfully stabilized and until recently have far surpassed electronic converters in their freedom from noise. The crystal may be used as a single detector of the modulation which is amplified by a noise-free, high-gain amplifier; the output of the latter is shown on a good vacuum-tube voltmeter. This system must be fully calibrated on a standing wave, the frequency of which is continuously monitored. From time to time this calibration should be checked. The second method is more elaborate but yields linear response by double detection on the superheterodyne principle [1]. A second microwave oscillator, with automatic frequency control and well-regulated excitation, will be required for this purpose.

If sufficient energy is available, as in high-power testing equipment, a bolometer or vacuum thermocouple may be used in place of the crystal. Sensitive detectors of the bolometer type have been used in some laboratories.

Microwave radiation is picked up by an antenna which may be designed to receive in space outside the wave guide or may be of suitable form to introduce into the wave guide through a central slot. In the latter case, it is desirable to locate the detector close to the antenna by mounting it in the carriage which supports the antenna and which can be moved to and fro on the top of the guide. For accurate work, mechanical details require scrupulous attention, and the higher the microwave frequency used, the more exacting the precision of workmanship required to overcome many of the possible sources of experimental trouble both electrical and mechanical.

3.2. Wavemeters

The essential basis of all microwave measurements is the wavemeter and some convenient arrangement by which frequency may be continuously monitored. This is required because of the frequency dependence of propagation in wave guides.

In one form a wavemeter consists of a tunable cavity of sufficiently large Q which will respond only in one mode of oscillation at frequencies in the band over which it is to be used. If the spectrum of the cavity is accurately calculable, as a function of the parameter specifying the change in its configuration, this method can be made absolute. Generally, it is more convenient to use this type as a secondary standard which is calibrated by comparison with a coaxial line type of instrument. Here the cavity consists of a coaxial line which is weakly coupled by means of a probe or loop or through a small hole in its wall to the line from the source to be measured. One end of the wavemeter is short-circuited by a fixed plate, the other by a movable plunger whose position can be accurately ascertained with the aid of vernier or micrometer attachments which form part of the instrument. If the losses in the metal walls are very low, the velocity of propagation of the principal wave in the line is known accurately, and if the cross-section of the coaxial line does not vary, the positions of the movable plunger for maximum response in a detector coupled to the wavemeter are $\frac{1}{2}\lambda$ apart.

As a frequency monitor a secondary standard wavemeter is weakly coupled to the main microwave circuit near the generator. The output of the crystal detector of the wavemeter is amplified and fed to an oscillograph. On the screen will be shown the square wave modulation of the source when the frequency of the latter agrees with the wavemeter setting and when it is constant during the pulse of oscillation. Incipient frequency drift is readily detected by the change in the pattern.

3.3. Microwave Circuits

While the main parts of microwave circuits consist of wave guides and of rigid coaxial lines, for convenience in ordinary laboratory routine flexible coaxial cable is employed. The attenuation in such cable due to loss in dielectric beads or continuous insulation supporting the inner conductor should not be excessive. The bead type of support introduces trouble if the curvature of the cable has to be altered in the course of measurements. If the outside of the cable is exposed to radiation, it is essential to have a satisfactory outer covering. Metallic braids are quite unacceptable for accurate work where it is best to use lead-covered cable with continuous polythene dielectric. Plugs, jacks, and other couplings have been satisfactorily developed, but unless properly designed may introduce mismatch in the circuits of which they form a part.

It seems desirable at this point to distinguish between the weak signals which result on the one hand from absence of matching and on

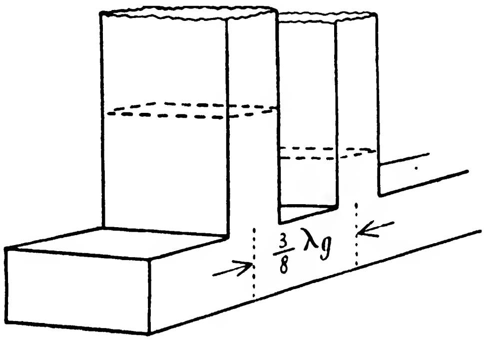


Fig. 10.

the other from intrinsic attenuation in cables or wave guides. The latter is usually desirable in moderate amount—for example, in order to decouple a generator from a reactive load. The former involves strong coupling with the generator as the result of reflection from a discontinuity in the circuit. In order to match a device to which coaxial line is coupled, a double-stub tuner may be employed. On the coaxial line are mounted, $\frac{3}{8}\lambda$ apart, two coaxial line stubs shorted by movable plungers. These stubs shunt the line with variable reactances. The reactance nearer the load must be adjusted to yield a pure resistance at that point of the line. The reactance seen at the position of the second stub is then cancelled by the reactance there. This same principle may be employed in a wave guide. The stubs are now pieces of wave guide abutting the main guide transversely on the broad face as shown in Fig. 10. The reactances introduced are in series with the dominant wave circuit; the stubs are called E-stubs. Shunt reactances could be introduced by H-stubs abutting the narrow face of the main guide. But this arrangement is inconvenient because of the geometrical disposition and the need for guide of smaller width to make the stubs.

Frequency insensitive attenuating devices which present a matched input are common parts of microwave circuits. For non-variable attenuation one may use a piece of guide containing absorbent impregnated with finely powdered graphite, the density of impregnation increasing from a very low value at the input end, and the gradient being sufficiently gradual to prevent reflection. A solid piece of weak conductor in the form of a wedge will also serve. A variable attenuator is very convenient. It consists of a thin segment of plastic which is specially lacquered to make it conducting and is introduced into the guide through a longitudinal slot in the broad face of the guide. The segment is hinged at one vertex, as shown in Fig. 11. It may be

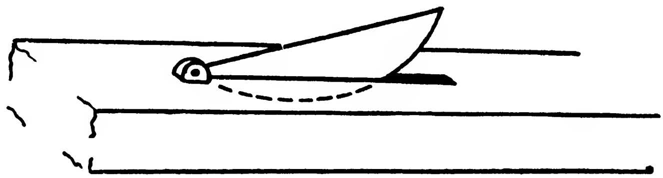


FIG. 11.

necessary to cover the slot to prevent leakage when high attenuation is desired.

The same principles are followed in making a guide termination intended to absorb without reflection all the microwave power incident on it. When high power is used, a wedge of sand and powdered graphite is enclosed by micalex, and the guide containing it is furnished with fins so as to cool it. This device is commonly called a sand-load, which is understood to mean a matched termination capable of dissipating of the order of 100 watts.

As a common part of wave-guide circuits we have to mention choke-couplings. In order to prevent reflection from the junction of two pieces of straight wave guide of nominally the same cross-section it is necessary to connect the guides rigidly and to provide good electrical contact for the flow of longitudinal surface current on the broad face without leakage. While simpler couplers have been used and have been found to function satisfactorily when care is taken, it is found convenient to design flange couplers which are held together by screws after having been soldered to the ends to be joined. By means of them the guides are properly alined, which is otherwise more difficult to achieve the smaller the guide. To prevent the propagation of waves through the gaps which will inevitably exist between the faces of the flanges in juxtaposition, a narrow circular slot or groove $\frac{1}{4}\lambda$ deep is cut in each flange around the junction. The use of such a groove or trough is common to choke microwaves travelling over a metal surface.

The problem of leakage likewise arises at a movable plunger in the guide. It is important not to use a metal handle attached to the plunger for it will form with the guide a 2-conductor transmission line of low impedance making an easy escape path. Various degrees of elaboration have been introduced into the design of plungers; for most purposes well-made plungers will serve if good contact with the guide walls is assured by cleaning.

In wire-transmission work in radio the quarter-wave transformer is a very useful device. In wave guides its principle may be used in matching technique, but it is not generally convenient to introduce a metal block into the guide to alter its depth and thus make a quarter-wave (in the guide) transformer.

3.4. The Standing-wave Detector

After the wavemeter, probably the most important microwave measuring device is the standing-wave detector. In the broad face of the rectangular guide is cut a narrow longitudinal slot accurately in the centre. This slot should be somewhat shorter than one wavelength in the guide. It is important that the cutting of the slot should not deform the guide cross-section and that the outside top surface in which the slot is cut should be quite flat. Through the slot passes a short fine wire surrounded by a larger tube which does not project into the guide. This tube is fixed rigidly in the centre of a key which fits the slot so that lateral displacement of the travelling antenna is reduced to a minimum, and, of course, its distance from the nearest metal, namely the surrounding tube, does not vary. It is desirable to have some easy means of altering the depth of projection of the wire antenna into the guide. If a crystal detector is used, it is more convenient to employ the shunt arrangement of the crystal on the transmission line to the antenna so as to allow the adjustment just mentioned. If the series arrangement is used, it is necessary to bend the antenna back on itself and to solder it to the outer tube so as to complete the low-frequency circuit. The high-frequency circuit is closed by means of the condenser indicated in Fig. 12. In X- and K-band work, a more suitable arrangement of the detector is the following. The wire antenna stretches across a second piece of wave guide forming a tunable cavity; it lies broad-face to broad-face over the central slot in the other guide and can be moved to and fro on it. The probe antenna is coupled to the crystal only through the cavity, and the crystal joins the inner conductor of the low-frequency coaxial output to the opposite inner wall of the cavity, and is therefore placed in the latter. This arrangement permits the easy adjustment of the projection of the probe through the slot.

The keyed block or wave-guide cavity carrying the antenna can be moved to and fro on the top of the guide by means of a rack and pinion and its position determined by means of a pointer and scale with vernier attachment if desired. For the shortest microwaves this

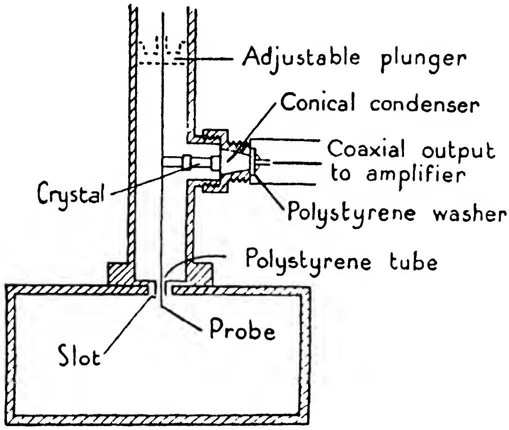


FIG. 12.

arrangement is insufficiently precise. To fix a reference point on the scale, let a short-circuiting (reflecting) plunger be placed at some point in the guide beyond the standing-wave detector (S.W.D.). Move the detector until a minimum is shown by the voltmeter. This position corresponds to the position of the plunger short-circuit (mod $\frac{1}{2}\lambda_g$).† The wavelength in the slotted guide may have to be measured accurately and compared with that in the uncut guide beyond, so that the difference, if any, can be taken into account.

So long as the S.W.D. is used in a nearly pure travelling wave no particular difficulties attend its use. It is when large values of the standing-wave ratio (S.W.R.) occur that skill is required for accuracy. The minimum signal as the detector is moved through the waves has to be discerned in the presence of noise, and if this is not reduced it is not feasible to use a more sensitive vacuum-tube voltmeter. Thus a natural limit is set to the largest values of S.W.R. that can be measured. When the detector is at a position of maximum signal it will tend, in virtue of shunting the line, to underestimate the signal strength just

† $x_1 = x_2 \pmod{a}$ if $x_1 = x_2 \pm na$, where n is an integer.

as does a voltmeter in a high-resistance d.c. circuit. It is therefore desirable to work with very short receiving probes and to depend on efficient amplification of minimum signals.

To measure impedance at a particular point in the wave system a plunger is placed at the position (mod $\frac{1}{2}\lambda_g$) in question, and the position of the S.W.D. for minimum signal noted. If the plunger is a perfect reflector, the minimum should actually reach zero. The plunger is next either removed altogether from the guide or placed somewhere else to open or close the guide circuit (see below). With the load it is desired to measure producing a standing-wave system, the position of the minimum and the values of the maximum and minimum signal strength are noted, due account being taken of the calibration of the detector if necessary. The noise is measured and S.W.R. calculated. This fixes $|w|$ on the circle diagram. Arg w is determined by twice the electrical distance between the minimum in the standing-wave pattern and the position of the load. Impedance or admittance may then be read directly from the circle diagram chart of sufficient size to permit the necessary precision. Two forms of this chart are in common use—one in which the centre of the diagram only is shown, for use with nearly matched systems; in the other, the complete circle diagram is presented.

It has already been pointed out in § 1.6 that standing-wave measurements alone will not suffice in general to indicate how the guide is loaded in any particular case. It is necessary to supplement measurements of impedance transformation from one side to the other of the load with measurements of power and phase transformation.

The power transfer in a standing-wave system is equal to the geometric mean of the powers corresponding to the maximum and minimum signals in the standing wave. When power is to be measured in two different parts of a wave-guide system, care should be taken to see that the probe antenna is coupled to the same extent in the two places; or two probes may be used and interchanged.

3.5. Power

The probe-antenna detector is useful in power measurements only on a relative basis. Too many variables enter to allow its use for absolute measurement of the power transmitted along a guide in watts. For this purpose two devices are employed—the absorption wattmeter and the enthrakometer. The former is a piece of wave guide arranged to absorb, without reflection of the microwaves, all the energy passing down the guide: it acts as a water-flow calorimeter. The latter measures a known fraction of the power passing through the guide and can therefore be used as a wattmeter when the power transmitted is usefully employed.

In the absorption wattmeter [2] the matching device should be insensitive to frequency over the band for which it is intended. Rather than use a quarter-wave transformer of suitable dielectric (usually micalex) it is better to enclose the absorbing matter with a V- or wedge-shaped entry for the microwaves. The guide termination is then cooled by water flowing through a pipe in good thermal contact with the guide.

In the enthrakometer [3] a section of the narrow face of the guide wall is removed and replaced by a metallic film which is insulated electrostatically and thermally from the guide wall without opening the microwave circuit. The film is in the form of a grid of parallel resistive strips perpendicular to the guide axis. These are in series with each other and with a source of d.c. by which change in resistance of the film is measured in a bridge. The change in resistance in the steady state is proportional to the rate of dissipation of energy in the film due to microwaves passing it in the guide. Since the magnitude of the transverse surface current which alone flows on the narrow face is dependent on frequency when the power transfer in the guide is given, this device gives an indication dependent on frequency. It may be made direct reading, but must be calibrated by comparing with the absorption wattmeter.

3.6. Phase

In principle the phasemeter allows the mixing of two microwave signals in a tunable cavity or other mixer (see § 9.8) into which a detecting crystal is coupled (see Fig. 13). One signal is a standard of phase comparison which can be adjusted in amplitude. The other signal is obtained from a probe antenna which receives where the phase of the radiation is to be measured. The line from the probe antenna contains a section which may be varied in length. This section may be telescoping coaxial-line made up of silver tubes or it may be a piece of wave guide which is effectively varied in length by moving the coupling antenna and plug. The latter device may yield better resolution, but it is more difficult to make and it is frequency sensitive. Whatever arrangement is used to secure the phase-shift (see Chapter IX for other phase-shifting devices), the method of measuring phase is to adjust the amplitude of one signal and the phase of the other until the detector of the mixer gives zero or minimum output. The required phase is then equal in magnitude and opposite in sense to that introduced by phase-shift. It is essential to provide sufficient decoupling in the system by means of attenuation, otherwise the method of measurement is quite unreliable. Further, if the probe antenna is to measure phase in free space, one can easily test for radiation from the exploring antenna by shielding it from radiation other than its own which will be reflected from neighbouring pieces of metal, and its presence made

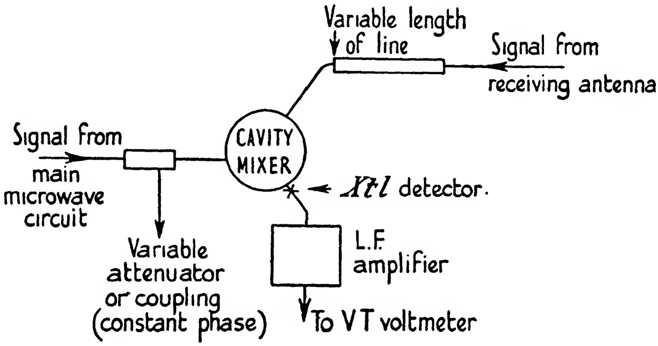


Fig. 13.

evident by the fluctuation of the detector response as the antenna is moved about.

Plane sections of equiphase surfaces in the radiation near an antenna can be plotted directly on a large sheet of squared paper stretched in a plane passing through the antenna. The exploring antenna is moved so as to bring the detector indication in the phasemeter to a minimum: the position of the centre of the antenna is then plotted. In order to reduce to a minimum the distortion of the radiation field being explored, it is desirable to use coaxial line of especially small radius to couple to the exploring antenna.

In the laboratory it is convenient to have a means of reducing the reflection of radiation from objects in it. A large plane screen made of cloth treated to give it a surface resistance of 120π ohms between the opposite sides of a square 1 m. $\times$ 1 m. will act as a perfect absorber of waves incident normally on it when backed by a plane reflector $\frac{1}{4}\lambda$ behind it.

3.7. Directive Pattern and Gain of Antennas

While it is possible to obtain by laboratory measurements a fair idea of the performance to be expected of a microwave antenna, it will, of course, always be necessary ultimately to test a directive antenna by observing its directive pattern when used either as a transmitting or receiving aerial. The latter arrangement is found more convenient in practice, for the antenna is mounted on a turn-table at the receiving station, and indeed the pattern may be recorded by automatic means as the table is turned. It must be possible to investigate reception of both horizontal and vertical polarization.

Since it is the distant pattern in which we are interested, the transmitter and receiver must be set up at a sufficiently great distance apart for the test. If D is the major aperture of the antenna, the distance in question should be four or five times D^2/λ so that the waves from the transmitter are effectively plane over the aperture.

To supplement knowledge of the radiation pattern which is not usually determined by absolute field-strength measurements, when given power is delivered to the antenna, it is usual to estimate the antenna gain. This is specified by using as standard either a hypothetical isotropic radiator or a half-wave dipole. The gain of an antenna is defined as the ratio of the field strength in the maximum of the main-lobe for given power input, to the maximum field strength produced at the same distance by the standard of reference when radiating the same total power. The determination of gain requires either integration of the two-dimensional pattern of the antenna or absolute field measurements. If the average cross-section of the radiator which terminates the transmission line feeding the antenna is known,† the gain of the directive antenna can be found by the method of matched transmission and reception. The transmitted and received power are measured when the antenna to be tested is the receiving one.

With highly directive antennas it is much easier to measure the width of the beam and to infer the gain with sufficient accuracy for practical purposes.

3.8. The Electrical Properties of Dielectrics and Semi-conductors at Microwave Frequencies

The use of wave-guide impedance and attenuation measurements to obtain the electrical properties of fluid and solid substances without having in mind any application to radio is of considerable physical interest. We shall deal first with the measurement of attenuation, which calls for comment only when it is comparable with that due to the finite conductivity of the walls of the guide. A piece of guide about 20 wavelengths long is terminated in a short-circuit by closing its end

† Slater, *Microwave Transmission* (McGraw-Hill, 1942), chapter vi.

with a plate soldered to the guide wall. Near the input coupling is a S.W.D. *On account of the attenuation the S.W.R., instead of being infinite, will have a finite value, for the reflected wave is not equal in* amplitude to the incident. For weak attenuation of electric force according to the law $e^{-\alpha z}$ we have

$$\alpha = \frac{1}{rL}, \tag{38.1}$$

where r is the S.W.R. and L is the length of the guide between the detector and the short-circuit. This method could evidently be applied

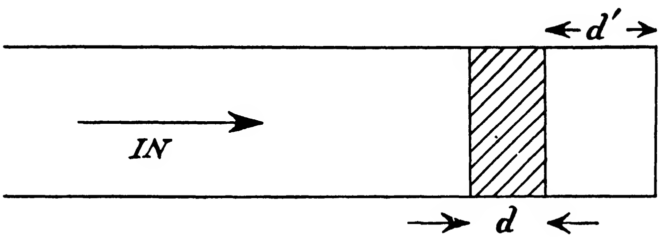


FIG. 14.

to gases and the dielectric constant be determined from the wavelength in the gas-filled guide.

Solid dielectrics and liquids have been investigated by Roberts and von Hippel.† Imagine a section of rectangular wave guide filled with the substance between two parallel planes perpendicular to the axis of the guide. On the side removed from the source of the microwaves the guide is terminated in a short-circuit. The input impedance to the slab is measured (reckoned at the position of the face where the radiation first enters) and knowing the thickness d of the slab and the air gap d' to the reflecting termination (Fig. 14), we may calculate the constants of the material.

Let γ_2 be the propagation constant in the sample in the guide for the dominant wave. Then

$$\gamma_2^2 = -k_1^2+k_0^2(1-\epsilon), \quad \text{where } \epsilon = \kappa+\frac{\sigma}{j\omega\kappa_0}. \tag{38.2}$$

k_1 and k_0 are respectively $2\pi/\lambda$ for the empty guide and for free-space, κ is the dielectric constant, and σ the conductivity of the sample.

We shall apply the matrix method to determine the value of the circle diagram variable w at the face of the slab where the waves are

† Roberts, S., and von Hippel, A., Massachusetts Institute of Technology Publication, *A New Method for Measuring Dielectric Constant and Loss in the Range of Centimeter Waves*, Mar. 1941.

incident from the generator. We require the matrix of (12.5) and its inverse and note that for Z_1/Z_2 we write k_1/γ_2, if the material can be treated as non-magnetic. The matrix M which transforms from the place of the short-circuit termination to the generator side of the slab at the position of its face is

$$M = \frac{1}{\alpha^2-\beta^2}\begin{pmatrix} \alpha & -\beta \\ -\beta & \alpha \end{pmatrix}\begin{pmatrix} \omega_2 & 0 \\ 0 & \omega_2^{-1} \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_1^{-1} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

where

$$\alpha = \frac{1}{2}+\frac{k_1}{2\gamma_2}, \qquad \beta = \frac{1}{2}-\frac{k_1}{2\gamma_2}, \qquad \omega_1 = e^{jk_1 d'}, \qquad \omega_2 = e^{\gamma_2 d}.$$

Since at the short-circuit $w = -1$ (voltage representation), the required value of w' is from (16.2)

$$w' = \frac{M_{21}-M_{22}}{M_{11}-M_{12}} = \frac{-\alpha\beta\omega_1(\omega_2-\omega_2^{-1})-\omega_1^{-1}(\alpha^2\omega_2^{-1}-\beta^2\omega_2)}{\omega_1(\alpha^2\omega_2-\beta^2\omega_2^{-1})-\alpha\beta\omega_1^{-1}(\omega_2-\omega_2^{-1})}.$$

There are two obvious special cases: (I) $d' = 0$, hence $\omega_1^2 = 1$, and (II) $kd' = \frac{1}{2}\pi$, $\omega_1^2 = -1$. The latter would be chosen for a thin slab to obtain maximum sensitivity.

The corresponding values of w' are:

I.

$$-\frac{\frac{\beta}{\alpha}\omega_2^2+1}{\frac{\beta}{\alpha}+\omega_2^2}.$$

The input impedance is

$$\frac{1+w'}{1-w'} = \frac{\alpha-\beta}{\alpha+\beta}\tanh\gamma_2 d = \frac{k_1 d\tanh\gamma_2 d}{\gamma_2 d}.$$

II.

$$\frac{\frac{\beta}{\alpha}\omega_2^2-1}{\frac{\beta}{\alpha}-\omega_2^2}.$$

The input admittance is

$$\frac{1-w'}{1+w'} = \frac{\alpha+\beta}{\alpha-\beta}\tanh\gamma_2 d = \frac{\gamma_2 d}{k_1 d}\tanh\gamma_2 d.$$

The input impedance or admittance having been measured, and since $k_1 d$ is known, we can find $\gamma_2 d$ from charts of the functions $\tanh\theta/\theta$ and $\theta\tanh\theta$ of the complex variable θ respectively. Once γ_2 is known, the dielectric constant κ and the conductivity σ can be calculated from (38.2).

The main point of practical importance in connexion with such measurements is the difficulty of removing all traces of water from the material studied, otherwise the results will have little theoretical value.

3.9. Discontinuities in Propagation

In order to determine how the guide is loaded in any given discontinuity in propagation it is essential in the first place to establish the law of transformation of the circle diagram variable w, or impedance Z, or admittance Y past the discontinuity.† This is accomplished by standing-wave measurements on both sides of the place of discontinuity. The simple series and shunt cases are revealed at once by the fact that an open-circuit and a short-circuit are respectively transformed unchanged past the discontinuity, which, for this purpose, must be regarded as established at a particular definite cross-section in the guide. If the discontinuity can be represented as a lumped circuit quadripole on the equivalent line, the form of the loading matrix is

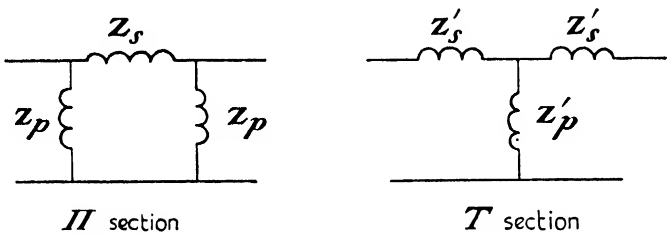


FIG. 15.

given by (16.5). For certain purposes, however, it is found convenient to think of an equivalent circuit. This may be drawn in the form of the symmetrical Π- or T-sections indicated in Fig. 15.

	Π-section	T-section
Let	$\dfrac{Z_0}{Z_p} = \alpha, \quad \dfrac{Z_s}{Z_0} = \beta.$	$\dfrac{Z_s'}{Z_0} = \gamma, \quad \dfrac{Z_0}{Z_p} = \delta.$

The loading matrix transforming in voltage representation from the right to the left of the point of loading is

$$(E+\alpha U_1)(E+\beta U_2)(E+\alpha U_1) \qquad (E+\gamma U_2)(E+\delta U_1)(E+\gamma U_2)$$
$$= (1+\alpha\beta)E+(2\alpha+\alpha^2\beta)U_1+\beta U_2. \qquad = (1+\gamma\delta)E+(2\gamma+\gamma^2\delta)U_2+\delta U_1.$$

In order that the sections be equivalent

$$\alpha\beta = \gamma\delta, \qquad \beta = 2\gamma+\gamma^2\delta, \qquad \delta = 2\alpha+\alpha^2\beta;$$

hence

$$\beta = \frac{2\gamma}{1-\alpha\gamma}, \qquad 1+\alpha\beta = \frac{1+\alpha\gamma}{1-\alpha\gamma}, \qquad 2\alpha+\alpha^2\beta = \frac{2\alpha}{1-\alpha\gamma}.$$

When the guide is terminated by an impedance Z at the position of the section, the input impedance is given by

$$Z_{IN} = \frac{(1+\alpha\beta)Z+\beta}{1+\alpha\beta+Z(2\alpha+\alpha^2\beta)} = \tfrac{1}{2}(\gamma+1/\alpha)-\frac{[1/2\alpha-\frac{1}{2}\gamma]^2}{Z+\frac{1}{2}(\gamma+1/\alpha)}.$$

† This method of studying the loading of a guide by both series and shunt coupled reactances was introduced by Watson and Guptill and reported July 1943.

When Z is a pure reactance, and γ and α are purely reactive, the graph of Z_{IN} against Z is a rectangular hyperbola with centre at $(-\frac{1}{2}(\gamma+1/\alpha),\ \frac{1}{2}(\gamma+1/\alpha))$. The constant of the hyperbola is $[\frac{1}{2}(1/\alpha-\gamma)]^2$. From these facts, $1/\alpha$ and γ and hence β and δ can be deduced. The impedance transformed unchanged past the discontinuity is $+\sqrt{(\alpha\gamma)}$.

If α and γ contain resistive elements, and Z is a pure reactance, Z_{IN} must be represented on the complex plane. Its locus is the circle which is the transform of the imaginary axis on the Z-plane. If the centre and radius of the circle are determined, then $1/\alpha$ and γ follow by simple geometrical considerations.

In general, however, it will be found by experience that the circuit ideas, which are so very attractive in the simple cases, are really inconvenient and should be discarded in favour of the straightforward matrix representation of transformation of the w-plane treated in § 1.6.

The essential difficulties that attend the measurement of weak discontinuities by the method just mentioned arise from the high noise/signal ratio at the minimum in the standing-wave system, where the voltage must be accurately measured, the length of guide required for the measurement, and the junction between the slotted section and test-piece may itself introduce a discontinuity. The cavity resonance method [4] is designed to overcome these difficulties. The discontinuity now occurs in a section of wave guide which is shorted at both ends by carefully designed plungers moved by micrometer screws. The cavity is excited through the input plunger and resonance is detected by a weakly coupled probe through a hole in the broad face. Preliminary tests determine the position of zero electric field in terms of micrometer screw readings which measure the motion of the plungers in the guide.

If the matrix representing the discontinuity is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\omega_2 = e^{j\theta_2}$ and $\omega_1 = e^{j\theta_1}$, where θ_1 and θ_2 are respectively the distances of the input and terminating plungers from the discontinuity, then the matrix transforming from the terminating to the input plunger is

$$\begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_1^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \omega_2 & 0 \\ 0 & \omega_2^{-1} \end{pmatrix} = \begin{pmatrix} a\omega_2\omega_1 & b\omega_2^{-1}\omega_1 \\ c\omega_2\omega_1^{-1} & d\omega_2^{-1}\omega_1^{-1} \end{pmatrix}.$$

Since w has the value -1 at both of these places, we find for resonance the condition

$$a\omega_1\omega_2 - \frac{d}{\omega_1\omega_2} - b\frac{\omega_1}{\omega_2} + c\frac{\omega_2}{\omega_1} = 0,$$

i.e. $$ae^{j(\theta_1+\theta_2)} - de^{-j(\theta_1+\theta_2)} - be^{j(\theta_1-\theta_2)} + ce^{j(\theta_2-\theta_1)} = 0.$$

Special cases:

1. Series: $a = 1+\frac{1}{2}\gamma$, $d = 1-\frac{1}{2}\gamma$, $b = -\frac{1}{2}\gamma$, $c = \frac{1}{2}\gamma$.

$$j\sin(\theta_1+\theta_2)+\gamma[\cos(\theta_1+\theta_2)+\cos(\theta_1-\theta_2)] = 0,$$

γ must therefore be the pure reactance

$$X = \frac{-\sin(\theta_1+\theta_2)}{\cos(\theta_1+\theta_2)+\cos(\theta_1-\theta_2)}.$$

2. Shunt: $a = 1+\frac{1}{2}\alpha$, $d = 1-\frac{1}{2}\alpha$, $b = \frac{1}{2}\alpha$, $c = -\frac{1}{2}\alpha$.

α must be the pure susceptance

$$B = \frac{-\sin(\theta_1+\theta_2)}{\cos(\theta_1+\theta_2)-\cos(\theta_1-\theta_2)}.$$

3. Π- or T-section:

$$j(1+\alpha\gamma)\sin(\theta_1+\theta_2)+(\alpha+\gamma)\cos(\theta_1+\theta_2)+(\gamma-\alpha)\cos(\theta_1-\theta_2) = 0.$$

Let $\alpha = jB$, $\gamma = jX$, then

$$0 = (1-BX)\sin(\theta_1+\theta_2)+(B+X)\cos(\theta_1+\theta_2)+(X-B)\cos(\theta_1-\theta_2).$$

The essence of the method is to operate in the vicinity of $\theta_1 = \theta_2$, which corresponds to a stationary value of $(\theta_1+\theta_2)$, removes the uncertainty in actual position to be assigned to the discontinuity, and permits the use of approximate algebraic equations in place of the transcendental ones.

In the laboratory it is convenient to work with lengths (l) in the guide in place of the electrical distances (θ). The two stationary positions are given by

$$\tan\frac{k(l_1+l_2)}{2} = \frac{1}{X} \quad \text{or} \quad -B.$$

Given one, the other is readily deduced.

It should be recognized that with microwaves of the order of 1 cm. in length, mechanical precision and measuring accuracy of the highest order are required to justify these measurements.

IV

MULTIPLE PROPAGATION

4.1. Propagation in Wave Guides in General

THE wave system described in Chapter II is a particular case which can be realized in practice only if the wave guide and the radio frequency are properly chosen. So long as it is not necessary to probe deeper for practical ends, we are content to treat the wave by the transmission-line method. On the other hand, for a proper understanding of wave-guide propagation one must consider other possibilities.

The propagation of electromagnetic waves in a guide exemplifies an important principle concerning waves. Imagine an unbounded plane wave of which the amplitude is distributed in a pattern across the planes perpendicular to the direction of propagation. Two complementary general consequences follow from the equation of wave propagation

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}: \tag{41.1}$$

(i) the phase velocity of the waves in general differs from c, and (ii) if the amplitude pattern referred to is periodic and preserved in propagation, there exists a cut-off frequency which must be exceeded by the actual frequency of the waves in order to allow the effective propagation of the pattern through space.

If we think of the pattern like an unlimited wall-paper, the wave guide provides a boundary to limit the wave to a single cell of the pattern. On this boundary the tangential component of electric force and the normal component of magnetic force must vanish. Let propagation take place in the z-direction with velocity V, the pattern of amplitude being distributed in planes parallel to the xy-plane. It is easily seen that only those amplitude distributions which satisfy

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \tag{41.2}$$

are propagated with velocity $V = c$. For any bounded portion of the xy-plane there can be only one such distribution, called the principal wave, and it can exist only if the domain in the xy-plane is multiply-connected. Such is the case if there are two or more separate conductors indefinitely extended in the z-direction. There is no principal wave in a hollow tube.

Let $k/2\pi$ and $k_1/2\pi$ denote respectively the wave numbers, of the harmonic waves of frequency $\omega/2\pi$, in free-space and in the guide. Then, when these are different, in place of (41.2) above we have

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = (k_1^2-k^2)u. \tag{41.3}$$

To this equation must be adjoined the boundary condition imposed by the wall of the tube around the curve C which is the form of cross-section of the wave guide. This condition depends on the physical meaning of u. In the problems with which we have to deal it is either u or its gradient normal to C which vanishes on C.

The equation (41.3) is the same as that satisfied by the displacement of a vibrating uniform membrane, viz.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = -\chi^2 u, \tag{41.4}$$

where χ^2 depends on the surface tension and density of the membrane. If the membrane is clamped at its edge, the appropriate condition is $u = 0$ at all points of C.

Now it is known that the frequency of natural vibration of a membrane of given physical properties may take any one of the series of values constituting the spectrum of its possible modes of oscillation. Further, to each frequency there corresponds at least one amplitude distribution characterized by a set of nodal lines on which the displacement vanishes. If the membrane is struck, a combination of notes may be sounded, corresponding to the superposition, in suitable relative proportions, of the amplitude distributions each of which is characteristic of one mode. This physical example has been used to indicate the mathematical result, viz. the differential equation (41.4) with its boundary condition does not in general have a solution. Only for a discrete series of values of χ is a solution possible. We shall name the possible χ-values by means of the pair of integers m, n, χ_{mn}; the meaning of the pair of integers will be made clear later in particular cases. To each m, n there corresponds the characteristic amplitude distribution u_{mn}, and k_1 must now take a different value for each distinct χ_{mn}. In fact

$$\chi_{mn}^2 = k^2-k_{mn}^2. \tag{41.5}$$

This equation shows at once that if $k < \chi_{mn}$, then k_{mn} is imaginary. That is, the corresponding distribution u_{mn} is not propagated, but is choked, its dependence on z being by a real exponential instead of an imaginary one. In fact, the corresponding wave is evanescent; by means of it there is no net transfer of energy on the average. The cut-off frequency is

$$f_{mn} = \frac{c\chi_{mn}}{2\pi}, \tag{41.6}$$

and the velocity of propagation for waves of frequency f in the (m,n)th mode is

$$V_{mn} = \frac{c}{\sqrt{(1-f_{mn}^2/f^2)}}. \tag{41.7}$$

4.2. *TE* and *TM* Characteristic Waves

On account of the association of electric and magnetic vectors in the wave, the system of characteristic distributions in a wave guide is more complicated than for a membrane. The characteristic functions form two groups, in one of which—magnetic type or H-waves—the electric force is transverse only, thus constituting transverse electric (TE) waves: whereas in the other—electric type or E-waves—the magnetic force is transverse only, thus constituting transverse magnetic (TM) waves. The function u by which the characteristic forms are given represents the longitudinal z-component of force in the wave, as follows:

	u represents	*Boundary condition*
TE (*H*-waves)	H_z	$\dfrac{\partial u}{\partial n} = 0$
TM (*E*-waves)	E_z	$u = 0$

The detailed investigation of different forms of wave guide can be carried out on the basis of the information just given. Mathematically, the simplest method is to use the two Hertz-vectors pointing in the direction of propagation—electric type for E-waves, magnetic type for H-waves. The reader is referred to §§ 10.2 and 10.3 for an explanation of this representation.

The characteristic functions u_{mn} form a closed infinite set. They are linearly independent, and by linear superposition of a number of them it is possible to represent any electromagnetic disturbance whatever in the unobstructed wave guide. This form of representation may be regarded as a natural generalization of Fourier's theorem by which a function, given in a finite stretch of a single variable, may under certain conditions be represented as the sum of sine and cosine terms. In the two-dimensional representation, however, the shape of the boundary enters, whereas of course it does not in one dimension. This circumstance causes the system of characteristic functions to alter with the boundary. Thus the rectangular guide has the following forms:

$$TE: \qquad (H_z)_{mn} = \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b},$$

$$TM: \qquad (E_z)_{mn} = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \tag{41.8}$$

where evidently from equation (41.4)

$$\chi^2_{mn} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \tag{41.9}$$

and a and b are respectively the width and depth of the cross-section of the guide. For TE-waves m or n may be zero, whereas for TM neither may be. This explains why the TE-wave discussed in Chapter II is the simplest wave in the rectangular guide and has the lowest cut-off frequency. The inequality $b < \frac{1}{2}\lambda < a < \lambda$ secures that f exceeds f_{10} but is less than f_{01}, f_{20}, and the cut-off frequency for all higher TE modes and for all the TM modes. That is, only the H_{10}-wave can be effectively propagated along the guide, all other modes are evanescent, and the larger m and n the more rapidly are these higher-order characteristic waves attenuated. An antenna or obstruction in the wave guide can excite the higher-order waves, but except in the immediate vicinity of the obstruction or the antenna, only the H_{10}-wave is present, and hence propagation in the guide may be treated by the transmission-line analogy. Since the energy introduced at one part of the cycle into the characteristic waves of higher order does not escape along the guide, but is merely stored and then reflected, it is evident that the reactive behaviour of any load is explained in part by the storage of energy in the higher-order waves. Ignoration of these waves is quite analogous to the procedure in electric practice, where no attempt is made to represent the distribution in space of the fields which account for self-inductance and capacity.

4.3. The Degenerate Case of Square Cross-section

If the guide has square cross-section ($a = b$), there are now two characteristic waves with the same cut-off frequency and propagated with the same velocity. In place of the linearly polarized electric force of the H_{10}- or H_{01}-waves, we now have in general elliptical polarization, with the particular conditions of linear and circular polarization when the proper phase and amplitude relations are fulfilled. To represent this propagation by a pair of characteristic waves—called a degenerate case, since both have the same cut-off frequency and velocity of propagation—it is necessary to introduce two transmission lines which must be thought of as coupled at any place where the propagation is disturbed. Again, it is possible to omit explicit representation of the higher-order waves, but of course the system, being a double transmission line, is a more complicated one (see Chapter VII).

4.4. Circular Wave Guide

When guide of circular cross-section of radius a is to be used, it is convenient to choose polar coordinates (r, θ); the characteristic forms are:

$$TM\text{-waves:}\quad (E_z)_{nm} = J_n(q_{nm} r)\begin{matrix}\cos\\ \sin\end{matrix} n\theta, \qquad \chi_{nm} = q_{nm},$$

where $q_{nm}a$ is the mth non-vanishing root of $J_n(z) = 0$.

$$TE\text{-waves:}\quad (H_z)_{nm} = J_n(q'_{nm} r)\begin{matrix}\cos\\ \sin\end{matrix} n\theta, \qquad \chi_{nm} = q'_{nm},$$

where $q'_{nm}a$ is the mth non-vanishing root of $J'_n(z) = 0$.

The symmetry of the circular guide makes it impossible to obtain the condition analogous to the simple H_{10} propagation found with the rectangular guide. It is true that there is a single E_{01}-wave and a single H_{01}-wave, but on account of the distribution of the roots of the Bessel functions $J_0(z)$, its derivative $J'_0(z)$, and the derivative $J'_1(z)$, these waves have a higher cut-off frequency than the H_{11}-wave.† Since $n = 1$ has two characteristic forms for the same cut-off frequency $c/3{\cdot}412a$, a circular guide involves at least double propagation. This is a great practical inconvenience, because any departure of the cross-section from exact circular form and symmetry may cause coupling of the two modes of propagation, so that there may occur rotation of the plane of polarization, or elliptic polarization may be produced. The same cause would operate to convert some of the energy of an E_{01}-wave into an H-wave. Hence in both cases, any receiving device might be called on to accept waves which it was not designed to receive. The resulting reflection might cause electrical breakdown in the system and would in any case prevent matched transfer of energy.

The method of standing-wave measurements encounters serious difficulties in circular guide, quite apart from the physical difficulty of cutting a slot without distorting the guide wall. It is true that the slot may be cut so as not to disturb propagation in the polarization for

† Stratton, op. cit., p. 537 et seq.

which the slot axis is a line of zero transverse current on the wall. For any other polarization, however, the slot presents a load to the wave and interferes with the propagation.

4.5. Effect of Finite Conductivity of the Walls on Propagation

Since resistance in the walls of the guide requires a flux of energy from the wave in the guide into the metal wall, the tangential component of electric force does not vanish on the boundary of the cross-section of the guide. It is therefore not possible to satisfy the boundary condition by means of the characteristic pure *TE*- and *TM*-waves which we have discussed in § 4.2. It is necessary to superpose waves of both electric and magnetic types whose amplitude distributions do not have a nodal line of tangential electric force on the inner surface of the guide wall. The theory is well known for a tube of circular cross-section.†

4.6. Multiple Propagation on the Outside of the Guide

So far we have had occasion to deal only with waves inside the wave guide. When an aperture is cut in the guide wall, the outside of which will have the same form of cross-section as the inside when the wall is of uniform thickness, waves are launched on the outside of the guide. These waves are reflected from the ends of the finite guide and radiation to space takes place. The discussion of this problem belongs to the field theory of antenna oscillations. In the case of an antenna in the form of a rectangular prism the explicit representation of the waves would be extremely complicated, and little is gained in physical insight by attempting the representation, unless detailed numerical results can be attained. The problem of an antenna of arbitrary cross-section has been discussed by Schelkunoff,‡ who treats the antenna as a transmission line with varying electrical parameters along its length.

If the guide is long and we are interested in the field near the guide and away from its ends, a useful approximation is to regard the guide as infinite in length. The aperture in the guide wall is then coupled to the system of waves possible outside the infinite guide. Propagation is considered parallel to the axis of the guide, and the characteristic waves occur in two groups: *TM*, in which there is no magnetic force component parallel to the guide axis, and *TE*, in which the longitudinal electric force vanishes. The lowest order *TM*-wave is propagated as

† Stratton, op. cit., p. 551.
‡ Schelkunoff, *Proc. I.R.E.* **29**, 493–521 (1941).

the principal wave with the velocity of light and without attenuation when the guide wall is a perfect conductor. The electric force near the guide wall is perpendicular to the wall, there are no nodal lines of electric force, and there is no transverse current on the wall. The absence of transverse current is characteristic of all *TM*-waves, its presence of *TE*-waves. With the exception of the principal wave, unless the guide is sufficiently large, all of these waves are attenuated, because they are of evanescent type. The higher-order modes of both types may be thought of as waves which spiral round the guide—the higher the order, the more rapid the convolution. Two waves of the same type and order and of equal amplitude, when superposed, form a standing-wave pattern of current on each transverse section of the guide wall, nodal lines of electric force being parallel to the axis.

The other approximate form in which the waves outside the guide may be represented is by analogy with the wave-functions of a prolate spheroidal antenna.† This will be more helpful in the treatment of a short antenna. Magnetic and electric types are now classified by the vanishing of the 'radial' components of E and H. The determination of the field is the problem of the diffraction by the spheroid of the radiation from the aperture in the wall. Resonance will occur for each characteristic mode of the spheroid in turn as the frequency of the radiation is increased. Due to radiation damping, no essential infinities occur at resonance.

These methods of picturing the wave system on the outside of the guide are admittedly crude, though the axially symmetric case of the circular wave guide permits exact treatment of the infinitely long guide,‡ and the solution for the finite guide, provided that the terminations are symmetrical about the axis, may be achieved.§ The rectangular guide requires much greater computation because of its geometrical form, even if only a very approximate solution is desired. It is quite possible that an experimental study of the waves launched on the outside of the guide by resonant slots (see Chapter VI) will prove a much more convenient and economical method of achieving a good physical picture of these waves.

† Stratton and Chu, *J. of Applied Phys.* **12,** 241–8 (1941).

‡ Stratton, op. cit., p. 524.

§ See Schelkunoff's recent paper, *Proc. I.R.E.* **33,** 872–8 (Dec. 1945), where references to the published work of Hallen and others on the problem of the cylindrical antenna are given. See also [5].

V

THE RECTANGULAR WAVE GUIDE. II

5.1. Introduction

In our discussion of the rectangular wave guide in Chapter II we confined our attention to the dominant wave and its propagation in the simple guide. We now consider the influence of obstructions in the guide and the behaviour of antennas introduced into it in order to excite or receive the dominant wave. Although at first sight it might appear that an antenna in a wave guide is a more complicated affair than the antenna in free space, with which the radio engineer and physicist are already familiar, this is not the case so far as propagation is concerned, for between transmitter and receiver only a single wave is involved in the transfer of energy in the guide. In space, however, the transfer of energy is achieved by multiple coupling of both the receiving and transmitting antenna to space. Consequently, while the wave guide can be represented by the methods of Chap. I, as a simple transmission line, radiation and reception in space cannot: both require a multiplicity of transmission lines which are coupled together at the antenna.† Nevertheless, the reader who has already become familiar with the radiation problems of ordinary antenna theory will probably find it helpful to think of the wave-guide antenna problem in terms of the other. He will then understand how driven and parasitic antennas and arrays of antennas may be expected to behave in a wave guide.

5.2. The Half-wave Magnetic Radiator

The foundation of classical radiation theory is the electric doublet. It is regarded as the strength of a point singularity in the distribution of the electric Hertz-vector and represents an antenna short compared with the wavelength of the radiation. The equivalent magnetic radiator is the magnetic dipole and in radio practice with longer waves, the only way of realizing it is by means of a loop which is less efficient than the electric one. In microwave radio, however, the half-wave magnetic radiator can be realized by means of a narrow slot about $\frac{1}{2}\lambda$ long cut in a sufficiently large conducting sheet. The current system on the surface of the sheet may be excited by means of a transmission line, the two conductors

† Watson, *Trans. Roy. Soc. Can.*, Sect. III, pp. 33–51, 1945.

of which are connected to the two sides of the slot at its centre, or, as we shall see, by cutting the slot in the conductor forming a wave guide so that the slot is excited by the wave inside the guide. In order to picture the field due to such a resonant slot, imagine that a magnetic current replaces the conductor of a half-wave electric linear antenna with its sinusoidally distributed electric current. The magnetic field due to the magnetic current will now point along meridians on a distant sphere centred on the middle of the radiator which lies along the polar axis of the sphere, as shown in Fig. 16. The electric field will be directed round the circles of latitude and therefore perpendicular to any plane containing the polar axis. Such a plane may therefore be occupied by a conducting sheet which splits the field in two. Since the magnetic force close to the sheet is tangential to it, the conductor will be the seat of currents at right angles to the magnetic field. In order to permit this current system, directed on one side towards and on the other away from that part of the polar axis which is the seat of the magnetic current, it is necessary that the conductor be cut so as to present a narrow slot half a wavelength long. The electric field in the plane of the sheet is continuous through the slot, so the phase of the field in half of Fig. 16 must be reversed to represent the resonant slot cut in the infinite sheet. In the plane of the sheet the magnetic field within the slot is directed parallel to its length and varies sinusoidally along the slot, being zero at its ends. This magnetic distribution may be regarded as the magnetic current from which we started. So long as the slot is narrow and its length approximately equal to half of the free-space wavelength, the distribution of E and H along it will be sinusoidal. Only the longitudinal component of H and the transverse component of E are then physically important, and to a first approximation they may be taken as constant across the width of the slot. The magnetic current from which the field due to the resonant slot may be computed is then equal to the voltage across the slot. The direction of the e.m.f. is related to that of the magnetic current in the sense opposite to the direction of the magnetomotive

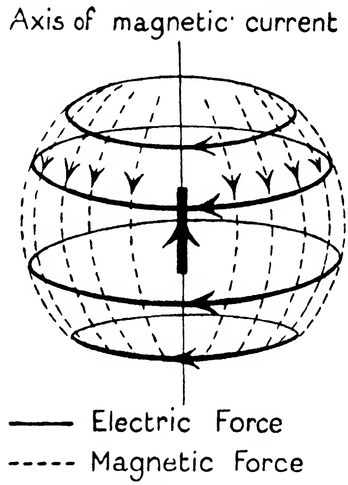


Fig. 16.

force equivalent to the corresponding electric current. This result is immediately evident in Maxwell's equations, and we can derive the field due to the magnetic radiator from that which is well known for the half-wave electric antenna by interchanging the electric and magnetic field components.

The substitution of magnetic for electric quantities and vice versa which we have just mentioned has an intrinsic connexion with the electromagnetic theorem which is the counterpart of Babinet's principle in optics.

5.3. Babinet's Principle

In physical optics Babinet's principle is usually thought of in connexion with a scalar wave and a non-reflecting screen. Suppose the screen is pierced by apertures of any size and shape and that there is incident on the screen an optical disturbance. Let the screen obtained by interchanging the holes and obstructions be called the complementary screen. Then Babinet's principle states that the disturbances produced at any point behind the plane of each of two complementary screens, exposed in turn to the same incident waves, would, if superposed, yield the effect produced at the same point with no screen. In electromagnetism we use reflecting screens, and, of course, vector waves. The principle must therefore be restated. This has been achieved by Booker [6], who first applied it to deduce some of the properties of resonant slots from the known properties of the corresponding electric oscillator.

In the hands of Sommerfeld the mathematical theory of the diffraction of electromagnetic waves by a semi-infinite conducting screen was treated by means of a Riemann space.† We may use this idea in connexion with diffraction by a single aperture. The points of this space do not correspond 1 : 1 with the points of physical space. Instead, a point of physical space corresponds to two points of Riemann space, which is entered and left behind by passing through the reflecting sheet. Let us start in *object space* before the screen, on the same side as the source of waves: on passing through the part of the plane occupied by the metal screen, we enter *image space* which gives a complete representation 1 : 1 of physical space. After passing through the aperture, still in image space, we can again approach the plane of the screen, and on passing through it at a point which corresponds to the metal

† See Baker and Copson, *The Mathematical Theory of Huygens' Principle* (Oxford, 1939).

of the screen, we can reach the object space behind the screen; coming back through the aperture, we can again approach the metal screen in the same way as that in which we started. Thus for a screen with a single aperture there is a dual representation of physical space. One of these is called the object space, in which are presented the waves accessible to physical observation. The other is called the image space; in it are placed the image sources by means of which the reflecting property of the plane of the metal screen is secured. To satisfy the boundary conditions imposed on the field at the screen we have to introduce image sources which look out of image space into object space through the face of the screen.

Since the fields produced behind the two complementary screens are to be added to obtain the field that would be produced with no screen, the image sources associated with the complementary screens must cancel out when superposed. Thus, for Babinet's principle to hold, we require that the images in the complementary screens must vibrate in anti-phase. This necessitates that if one screen is a perfect conductor of electricity the complementary screen must act as a perfect conductor of magnetism. There being no such magnetic conductor, we have to secure the anti-phase relation of the two systems of images in another way. Suppose that in passing to the complementary screen we interchange electric and magnetic quantities everywhere and reverse the sign of one of the field vectors; the complementary screen becomes a perfect conductor of electricity, while the directions of the electric and magnetic vectors are interchanged in the surrounding field. If the source associated with the original metal screen is an electric dipole, that associated with the complementary screen must be a magnetic dipole. Two sources related in this way are called conjugate sources. Two plane waves which differ only in being linearly polarized at right angles to each other are conjugate.

Let S_1 (electric) and S_2 (magnetic) be conjugate sources placed respectively at corresponding points before the mutually complementary screens Σ_1 and Σ_2. Let the corresponding object and image spaces be denoted by O_1, O_2 and I_1, I_2. The source S_1 looks into I_1 and produces there beyond the plane of the screen Σ_1 an *electric* field denoted by $e_2 s_1$, while in O_1 behind the screen the electric field is denoted by $e_1 s_1$, where s_1 is the strength of the field at the point in question due to the source S_1 in the absence of the screen. Since the geometrical relations of the spaces O and I to the source are interchanged when we pass from Σ_1 to Σ_2, the source S_2 produces in O_2 behind Σ_2 the *magnetic* field $e_2 s_2/Z_0$ and in I_2 the *magnetic* field $e_1 s_2/Z_0$, where Z_0 is the intrinsic impedance of the medium, and s_2 denotes the strength of the electric field at the point in question due to the source S_2 in the absence of the screen. Now when the screen Σ_1 is removed, the fields in O_1 and O_2 are superposed to produce the field in the absence of the screen, and since s_2 is equal to s_1 for conjugate sources, we reach the result

$$e_2 s_1 + e_1 s_1 = s_1$$

or

$$e_1 + e_2 = 1. \tag{53.1}$$

Further, due to the images S_1' and S_2' of S_1 and S_2 in Σ_1 and Σ_2 respectively, the contributions to the fields in O_1 and O_2 respectively behind the screen are $e_3 s_1'$ and $e_3 s_2'$. Since S_2' is supposed equal to S_1' and in anti-phase with it, these fields cancel on superposition, and we reach the result which is Babinet's principle for electromagnetism, viz.:

Let e_1 be the ratio of the field behind the screen Σ_1 to the field which would be produced at the same point in its absence, and let e_2 denote the same ratio for the complementary screen Σ_2 due to the conjugate source, then $e_1 + e_2 = 1$.

Further, if f_1 denotes the ratio of the reflected field in front of Σ_1 to the reflected field due to the complete screen without apertures, and if f_2 is the same ratio for the complementary screen Σ_2 with the conjugate source, then $f_1 + f_2 = 1$.

This principle has been studied in detail by Booker and Macfarlane. They have developed methods of thinking about electromagnetic waves which in their physical immediacy remind one of the role of Fresnel's zones in optics. By means of these methods it is possible to avoid recondite mathematical analysis and at the same time keep in view the physically important quantities.

5.4. Applications of Babinet's Principle

Consider two infinite plane complementary gratings. Let Y_1 and Y_2 mhos be their equivalent surface admittances for simple infinite plane waves with their electric vectors vibrating in perpendicular directions and incident normally on the gratings. The ratios of the electric field in the transmitted wave to that in the incident wave for the two gratings are from § 1.9

$$\frac{2Y_0}{2Y_0 + Y_1} \quad \text{and} \quad \frac{2Y_0}{2Y_0 + Y_2}, \tag{54.1}$$

where Y_0 is the intrinsic admittance of the surrounding medium. By Babinet's principle the sum of (54.1) must be unity. That is,

$$Y_1 Y_2 = 4Y_0^2, \tag{54.2}$$

or, in terms of the corresponding impedances,

$$Z_1 Z_2 = \tfrac{1}{4}Z_0^2. \tag{54.3}$$

Thus, for complementary conducting gratings in the field of incident plane waves in which the directions of electric and magnetic vibration are interchanged, the geometric mean of the equivalent surface impedances is half the intrinsic impedance of the surrounding medium.

The same relation can be established between the driving-point impedance of a half-wave strip electric radiator and that of a half-wave

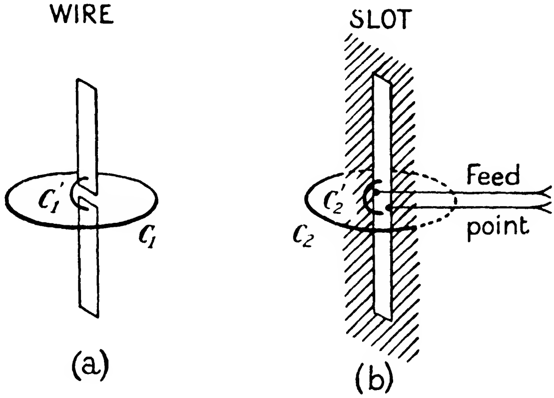


FIG. 17.

resonant slot fed at the same position along its length. In the former a generator of voltage V_1 is applied across a gap small compared with the strip width; in the latter the generator of voltage V_2 is applied across the slot. We shall use the electromagnetic laws in integral form and apply them to the paths indicated in Fig. 17 (*a*) and (*b*). The subscript 1 refers to the electric radiator, the subscript 2 to the magnetic one. The circuit C_1 encircles the wire at its centre, the short path C_1' bridges the gap in the antenna. The circuit C_2 encircles the slot passing through the metal in which it is cut; C_2' is the short path bridging the feed-point in the slot. C_1 is geometrically identical with C_2, and C_1' with C_2' in relation to the two half-wave radiators.

In order that the two fields be transformable into one another in conformity with the conception of conjugate sources, we require

$$\int\limits_{C_1} \mathbf{H}_1 . d\mathbf{s} = \frac{1}{Z_0}\int\limits_{C_2} \mathbf{E}_2 . d\mathbf{s} = \frac{2V_2}{Z_0}. \tag{54.4}$$

Hence, I_1 being the electric current in the metallic strip,

$$V_2 = \tfrac{1}{2}Z_0 I_1. \tag{54.5}$$

Similarly
$$\int_{C_1'} \mathbf{E}_1 . d\mathbf{s} = Z_0 \int_{C_2'} \mathbf{H}_2 . d\mathbf{s}. \tag{54.6}$$

Now from the symmetry of the field on the two sides of the plane of the resonant slot, the integral on the right of (54.6) is equal to one-half of the current I_2 entering and leaving the slot at the feed-point; consequently
$$V_1 = \tfrac{1}{2} Z_0 I_2. \tag{54.7}$$

We now combine (54.5) and (54.7) to obtain the result (54.3), where Z_1 and Z_2 now represent the ratios V_1/I_1 and V_2/I_2.

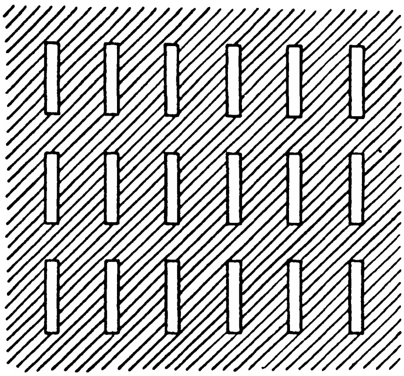


Fig. 18.

This result enables us to calculate the radiation resistance of the centre-fed resonant slot, for $Z_1 = 0{\cdot}194 Z_0$ (73 ohms) and hence $Z_2 = 1{\cdot}29 Z_0$ (485 ohms). Some useful results follow.

If V is the root-mean-square e.m.f. across the centre of the slot in kilovolts, the power (W kilowatts) radiated on both sides of the screen is
$$W = \frac{10^3 V^2}{Z_2} = 2{\cdot}06 V^2. \tag{54.8}$$

Although the foregoing discussion dealt with straight radiators, the argument may be extended to any plane linear conductor and the complementary slot. The driving-point impedances of wire and slot are related by (54.3) at corresponding points. A circular slot about a wavelength in perimeter is equally transparent, no matter what the polarization of the normally incident wave may be. Likewise a pair of straight half-wave slots bisecting each other at right angles is equally transparent to all polarizations.

A grating of resonant slots as shown in Fig. 18 is dealt with like a curtain of resonant electric dipoles. If the spacing is such as to make the latter a perfect reflector of waves polarized with E parallel to the wires, the slot grating is, by Babinet's principle, perfectly transparent to waves polarized with E perpendicular to the length of the slots. It acts in fact as a band-pass filter. The mutual impedance between two slots is related to the mutual impedance of the two complementary strip radiators by the equation (54.3).

We now consider approximately how the selectivity of the slot depends on its width. It is known that the distributions of voltage and current along a half-wave electric radiator are related approximately in terms of the characteristic impedance

$$Z_1^0 = \frac{Z_0}{2\pi}\log_e\left(\frac{2\lambda}{b}\right),$$

where b is the width of the strip, and its selectivity is approximately

$$Q = \frac{\pi Z_1^0}{2Z_1}.$$

Now, regard the slot as approximately half a wavelength of transmission line of characteristic impedance Z_2^0 terminated by the impedance Z_2'. We have

$$Z_2 Z_2' = (Z_2^0)^2$$

and

$$Q = \frac{\pi Z_2^0}{2Z_2'} = \frac{\pi Z_2}{2Z_2^0}.$$

Since by Babinet's principle the two radiators have the same Q,

$$\frac{Z_1^0}{Z_1} = \frac{Z_2}{Z_2^0}, \qquad Z_2^0 = \frac{\pi Z_0}{2\log_e(2\lambda/b)}, \quad \text{and} \quad Q = 1{\cdot}29\log_e\left(\frac{2\lambda}{b}\right). \quad (54.9)$$

The foregoing is quite crude, and the reader is warned to approach the treatment of antennas by the simple transmission-line analogy with great caution.

5.5. Obstructed Propagation in a Rectangular Guide viewed as a Grating Problem

It must be evident that the study of gratings is relevant to understanding the propagation of the dominant wave when irises are present in the rectangular guide, for we may approach the wave guide via the strip transmission line as was done in § 2.1. The strip transmission line permits the propagation of plane waves in longitudinal slabs between its two parallel planes with such boundary conditions at the faces of the slabs that we can imagine an infinity of similar slabs side by side, the conducting sheets withdrawn without disturbing the field, and the resulting field a possible unbounded plane-wave system in the presence of a periodic obstruction. We now offer some preliminary general observations on antennas and obstructions in wave guides.

Consider the limiting form of the linear electric antenna, an electric doublet, placed in a rectangular guide. Because of the multiple reflections from the guide walls, it is equivalent to a two-dimensional infinite array of doublets. Such an array is able to radiate only in certain directions determined by the spacing and phasing of its elements. The excited element D with its adjacent images is shown in Fig. 19. Now in planes parallel to the plane of the array, standing-wave patterns of

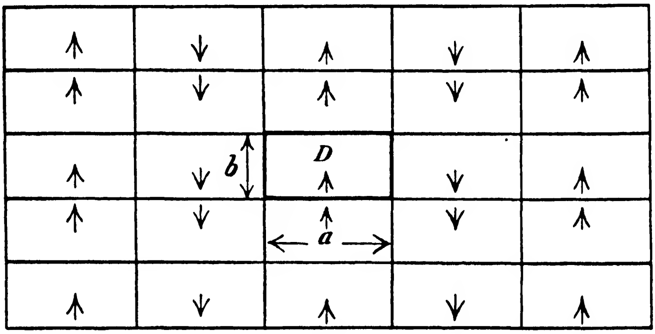


FIG. 19.

field distribution are produced by groups of these directed beams in which the array radiates, all the members of one group being propagated with the same component speed perpendicular to the plane of the array, in a manner analogous to that discussed in § 2.1 for the H_{10}-wave, and, of course, the period of the pattern is the rectangle congruent to the guide cross-section. Each group of waves corresponds to the propagation of one of the characteristic modes in the wave guide. Since the cross-sectional dimensions of the latter determine the spacings of the array, only a limited number of real side-waves is possible from the array, and corresponding thereto only a limited number of non-evanescent waves in the guide. The presence of TE or TM types (or both) depends on the orientation of the doublet as well as on the ratio of the guide dimensions to the free-space wavelength of the radiation. Now the array has associated with it an infinite system of evanescent waves in which the amplitude falls off exponentially with distance from the plane of the array, and phase is propagated parallel to that plane. Waves with the same rate of exponential decay of amplitude may be grouped in TE and TM types to build up standing-wave patterns of amplitude corresponding to evanescent characteristic waves in the guide, just as the non-evanescent ones were.

So far we have thought of the elements of the array as independently excited, but essentially the observations that have just been made are relevant in understanding the effect of any obstruction in the guide when the currents in the obstruction are due to an incident wave in the guide. Each point of the obstruction is now a dipole excited by the incident wave and with its own system of images. We have to think of the periodic repetition of the obstruction as constituting a grating. The characteristic waves in the guide are then regarded as systems of unbounded plane waves incident obliquely on the grating. In this way we can reach a clear understanding of the effects due to wires stretched across the rectangular guide parallel to the electric force in the dominant TE-wave, and of metallic strips introduced in transverse sections thus reducing locally the cross-section of the guide.

5.6. Plane Gratings and Irises [7]

We consider first a plane grating of parallel thin wires of radius $\rho \ll \lambda$, and spaced d apart. On it is incident an unbounded uniform plane wave propagated in a direction making θ with the grating normal, the electric force in the wave being parallel to the wires. The currents excited in the latter are in constant phase relation with the electric force in the incident wave as it reaches them, and the possible directions for distant radiation from the infinite grating are given by

$$\sin\psi_n = \sin\theta \pm \frac{n\lambda}{d} \quad (n = 0,\, 1,\, 2,\ldots), \tag{56.1}$$

where ψ_n is measured in the same sense as θ from the normal to the plane of the grid. The case $n = 0$, $\psi_n = \theta$ or $\pi-\theta$, yields the transmitted and reflected beam directions. For given d/λ and θ there is only a limited number of real values of ψ_n satisfying (56.1): they give the directions of the real side-waves which occur in pairs symmetrical with respect to the plane of the grid. The complex values of ψ_n satisfying (56.1) are

$$\psi_n = \frac{\pi}{2} \pm j\cosh^{-1}\!\left(\frac{n\lambda}{d}+\sin\theta\right) = \frac{\pi}{2} \pm jV_n,$$

$$\psi_n' = \frac{3\pi}{2} \pm j\cosh^{-1}\!\left(\frac{n\lambda}{d}-\sin\theta\right) = \frac{3\pi}{2} \pm jV_n', \tag{56.2}$$

the positive sign to be taken for radiation on one side of the grid, the negative on the other, as can be seen at once by considering the propagation factor

$$\exp\{-j\omega t+jk(z\cos\psi_n+x\sin\psi_n)\}, \tag{56.3}$$

where z is normal to the grid, y parallel to the wires, and x perpendicular to them. Although $\sin\psi_n$ is real, in general $\cos\psi_n$ is not. Indeed we have in such cases for the propagation factor

$$\exp(-kz\sinh V_n).\exp(-j\omega t+jkx\cosh V_n), \tag{56.4}$$

leading to exponential decay of amplitude with distance from the grating. These evanescent waves store energy vibrating 90° out of phase with the incident field in the vicinity of the grid: they do not contribute to the radiation of energy on the average from it.

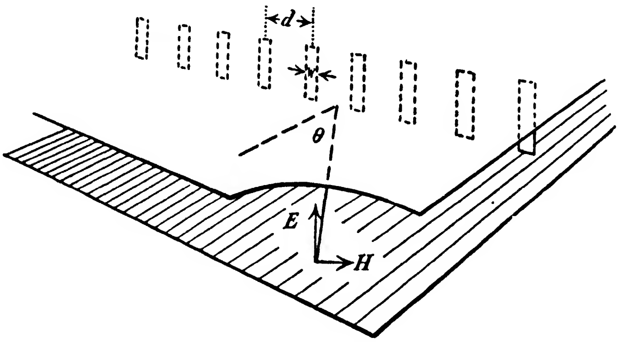


Fig. 20.

Now without altering the field we can bound it by two parallel plane conductors 1 metre apart, perpendicular to the wires, so as to make a parallel strip transmission line of unit width, guarded by the remainder of the parallel planes, and shunted by the grid of wires.

If $2d < \lambda$, there are no real side-waves: all are evanescent, consequently the grid loads the transmission line inductively, the energy stored at reflection being in the magnetic field of the currents in the wires.

The impedance presented by the grid to the transmission line of characteristic impedance $Z_0 \sec\theta$ is found to be

$$Z = \frac{jZ_0 d}{\lambda}\left[\log_e\frac{d}{2\pi\rho}+F\left(\frac{d}{\lambda},\theta\right)\right], \tag{56.5}$$

where

$$F\left(\frac{d}{\lambda},\theta\right) = \tfrac{1}{2}\{f(\theta)+f(-\theta)\}, \tag{56.6}$$

$$f(\theta) = \sum_{n=1}^{\infty}\frac{1}{n}\left\{\frac{n\lambda/d}{\sqrt{(\sin^2\psi_n^+-1)}}-1\right\}, \tag{56.7}$$

and

$$\sin\psi_n^+ = \sin\theta+\frac{n\lambda}{d}. \tag{56.8}$$

$F(d/\lambda,\theta)$ is a correction term taking into account the finite ratio d/λ: when $2d < \lambda$, this correction term is real, but if $2d > \lambda$, some of the terms are imaginary, leading to real conductance which corresponds to the loss of incident energy, for each term in F corresponds to a side-wave.

If the wires are replaced by plane metallic strips of width w, then, without restricting w to be small, we have

$$Z = \frac{jZ_0 d}{\lambda}\left[\log\operatorname{cosec}\frac{\pi w}{2d} + F\left(\frac{d}{\lambda}, \theta\right)\right] \tag{56.9}$$

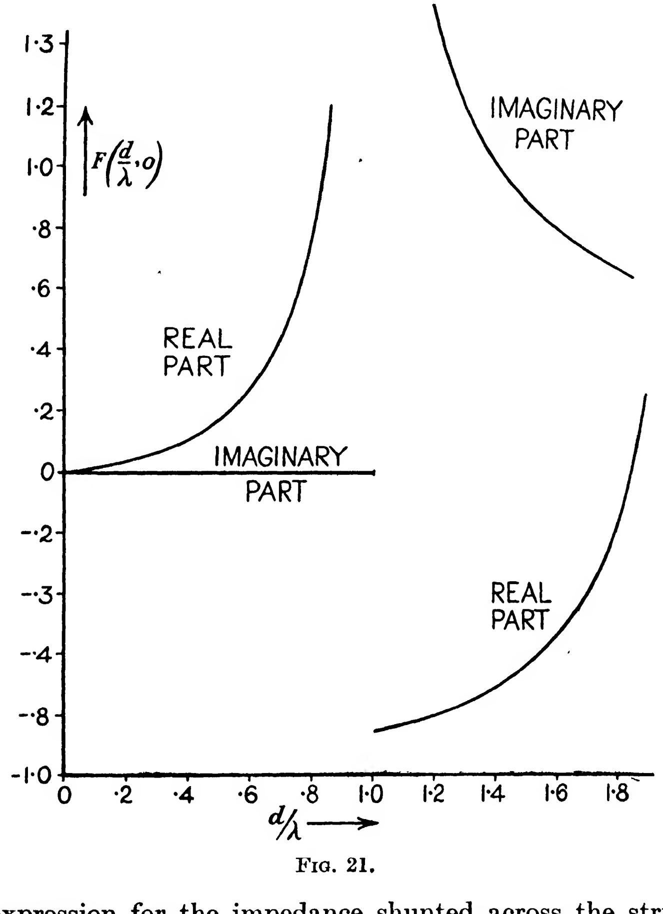


Fig. 21.

as the expression for the impedance shunted across the strip transmission line (Fig. 20). When $d/w \gg 1$, (56.9) reduces to (56.5) with $w = 4\rho$.

Let us restrict ourselves to normal incidence, $\theta = 0$. The self-inductance shunting the strip transmission line is

$$L = \frac{\mu_0 d}{2\pi}\left[\log_e\operatorname{cosec}\frac{\pi w}{2d} + F\left(\frac{d}{\lambda}, 0\right)\right]. \tag{56.10}$$

A graph of $F(d/\lambda, 0)$ is shown in Fig. 21.

By means of Babinet's principle we can find at once the shunt admittance presented by the complementary capacitive iris across the transmission line. Alternatively we could have started with the capacitive iris and inferred the result for the inductive case. From (54.3)

$$\frac{X}{Y'} = \tfrac{1}{4}Z_0^2.$$

Hence
$$jY' = \frac{j4d}{\lambda Z_0}\left[\log_e \operatorname{cosec}\frac{\pi w}{2d} + F\left(\frac{d}{\lambda}, 0\right)\right]. \tag{56.11}$$

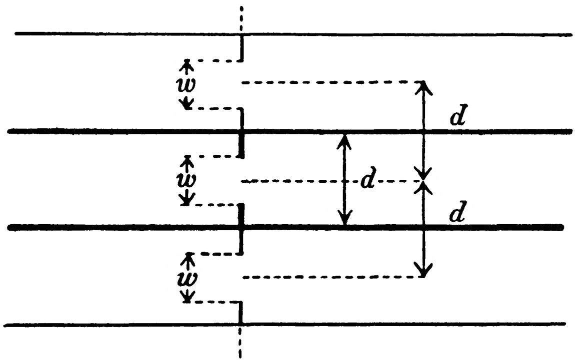


FIG. 22.

Thus the shunt capacity across the transmission line when the waves are incident normally on the grid is given for this symmetrical iris by

$$C = \frac{2d\kappa_0}{\pi}\left[\log_e \operatorname{cosec}\frac{\pi w}{2d} + F\left(\frac{d}{\lambda}, 0\right)\right]. \tag{56.12}$$

w now denotes the width of the gap and d the distance between the centres of adjacent gaps (see Fig. 22).

In order to pass to the dominant wave in a guide of rectangular section, since the strip transmission line and rectangular wave-guide problems for the capacitive iris have essentially the same mathematical form, we have merely to replace λ by λ_g and Y_0 by Y_g, and d by the depth (b) of the guide. Accordingly, the susceptance measured in the standing-wave produced by the iris (Fig. 23) is expected to be

$$B_0 = \frac{4b}{\lambda_g}\left[\log_e \operatorname{cosec}\frac{\pi w}{2b} + F\left(\frac{b}{\lambda}, 0\right)\right]. \tag{56.13}$$

Actually the measured values are very sensitive to the thickness (ϵ) of the metal forming the window. The following theoretical formulae yield the susceptance and conductance referred to the entrance plane of the iris:

$$B = B_0 + \frac{2\pi\epsilon}{\lambda_g}\left(\frac{b}{w} - \frac{w}{b}\right),$$

$$G = \frac{2\pi\epsilon}{\lambda_g} B_0 \frac{w}{b}. \tag{56.14}$$

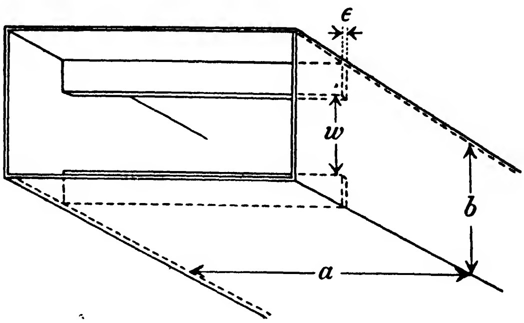


FIG. 23.

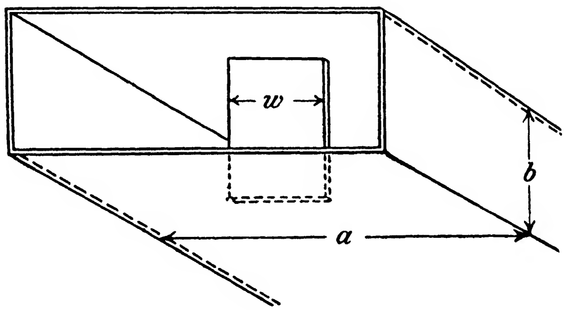


FIG. 24.

For the symmetrical inductive strip (Fig. 24) the case is different, for we have to deal with the images of the strip in the narrow faces; these make up a grating with the currents in adjacent strips in opposite directions. The consequence is that the inductive reactance presented to the transmission line of characteristic impedance $Z_{10} = \frac{Z_0}{\sqrt{[1-(\lambda/2a)^2]}}$ is approximately

$$X = \frac{Z_{10}a}{2\lambda_g}\left[\cosh^{-1}\left(\operatorname{cosec}\frac{\pi w}{4a}\right) - 2\right], \tag{56.15}$$

when w, the width of the strip, is much less than a, the width of the guide.

It will be noticed that the geometric mean of the reactances of the complementary irises in the guide is not equal to one-half of the characteristic impedance Z_{10}. This result is not in conflict with Babinet's principle, for to apply the latter the wave systems in the two complementary cases must be conjugate. That is, if we deal with H_{10} and the capacitive iris, we must treat the E_{10}-wave with the inductive one and vice versa. Actually the error in using (54.3) with Z_{10} substituted for Z_0 depends on the ratio a/b and in the useful practical range will never exceed 25 per cent.

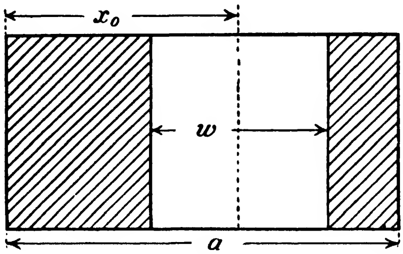


Fig. 25 (a).

Fig. 25 (b) and (c).

An instructive method for deducing the foregoing results for the capacitive iris and inductive strip is given in § 10.5. In practice, irises are used mainly to introduce susceptance for matching and to couple the wave guide to another guide or cavity. The capacitive iris is of little use in high-power circuits because it reduces the maximum power that can be handled without breakdown: as a coupling device to a cavity it yields too strong coupling; for these reasons inductive windows are generally used. It is obviously inconvenient to insert two strips of metal through the guide wall when one will serve; accordingly the results for unsymmetrical inductive irises are of practical interest. These have been worked out by Schwinger's method (§ 10.5), and for the sake of generality we quote the result in the general case. Let x_0 be the distance of the centre of the gap, of width w, from one of the narrow faces of the guide as shown in Fig. 25 (a). The susceptance introduced by the window is

$$B = -\frac{\lambda_g}{a}\cot^2\frac{\pi w}{2a}\left(1+\sec^2\frac{\pi w}{2a}\cot^2\frac{\pi x_0}{a}\right). \tag{56.16}$$

The special cases of the symmetrical window (Fig. 25 (b)) and of the window bounded on one side by the guide wall (Fig. 25 (c)) may be immediately deduced from this:

(i) Symmetrical window, $x_0 = \frac{1}{2}a$,

$$B = -\frac{\lambda_g}{a}\cot^2\frac{\pi w}{2a}. \tag{56.17}$$

(ii) Single-strip window, $x_0 = \frac{1}{2}w$,

$$B_0 = -\frac{\lambda_g}{a}\cot\frac{\pi w}{2a}\left(1+\operatorname{cosec}^2\frac{\pi w}{2a}\right). \tag{56.18}$$

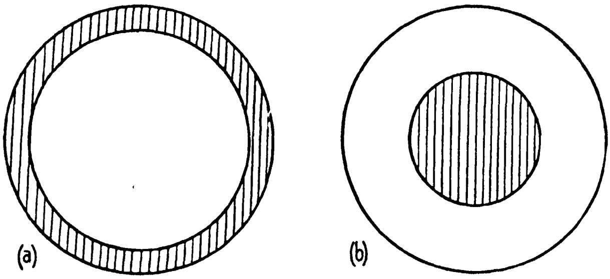


FIG. 26 (a) and (b).

Finally it is of interest to mention the inductive ring (Fig. 26 (a)) and the capacitive disc (Fig. 26 (b)) used in circular wave guides intended for H_{11} propagation.

In practice the band-width of transmission by irises is important. Actually the iris is introduced into the wave guide for the purpose of compensating susceptance in the load; consequently, except for a resonant slot iris, the conception of the Q of an iris itself has no practical relevance. The band-width of the combination is determined by the gradients with respect to frequency of the susceptance both of the iris and the load. Thus, let B_1 be the susceptance introduced by the iris, B_2 that of the load (reckoned at the position of the iris), both of which are functions of λ, then

$$Q = \frac{\lambda_0}{2}\left|\frac{d(B_1+B_2)}{d\lambda}\right|_0, \tag{56.19}$$

c/λ_0 being the resonant frequency of the system. If $dB_2/d\lambda = 0$, then

$$Q_1 = \frac{\lambda_0}{2}\left|\frac{dB_1}{d\lambda}\right|_0.$$

Now in all the cases with which we have dealt we have found either that B_1 is proportional to λ_g (inductive irises) or to λ_g^{-1} (capacitive irises). Hence

$$Q_1 = \frac{\lambda_0}{\lambda_g}\left|B_1\frac{d\lambda_g}{d\lambda}\right|_0 = \left(\frac{\lambda_g}{\lambda}\right)_0^2|B_1|_0 = \frac{|B_1|_0}{1-(\lambda_0/2a)^2}. \tag{56.20}$$

In order to secure band-width, therefore, we require that the gradients of B_1 and B_2 should compensate, or, if this is not possible, B_2 and $dB_2/d\lambda$ must both be small, i.e. only small compensation is necessary.

5.7. Discontinuity at the Junction of Two Guides of Differing Cross-section: Bends, Corners, Twists

Although the junction of two guides of different cross-sections does not commonly arise in practice, it is well worth while to mention it in order to bring out that even when two lines of equal characteristic impedance are joined, a considerable mismatch may occur, unless the dimensions of the cross-sections are all small compared with the wave-length, and unless the geometrical discontinuity itself is small. Since in wave guides the former restriction cannot be met, we must expect reflection from the junction of two different guides on account of the geometrical discontinuity, even when the guides have the same impedance.

When the guide beyond the junction is terminated in a match, reflection can be measured. The S.W.R. to be expected may in certain simple cases be calculated without difficulty [8]. For instance, if the discontinuity consists only in change of the depth (b) of the rectangular guide, then the reflection is approximately what one would expect from a capacitive iris in the deeper guide due to the distortion of the field in that guide only, that is, only half of the capacity of the iris is effective. If a more accurate estimate is desired, the quasi-static method of § 10.5 may be used, or the result be derived by Schwinger's analysis (§ 10.5).

For the numerical treatment of all problems of this type, Motz and Klanfer [9] have outlined a method which, with adequate computational aids, will serve to determine with any desired precision the effects on dominant propagation due to changes in the guide cross-section and due to junctions. The method is based on the representation of a travelling wave as the linear superposition of two standing-waves displaced with respect to each other in space and time phase. Once the nodal lines in the standing-waves are known, the travelling-wave problem is solved.

In order to convert the solution of the boundary-value problem for the partial differential wave equation, into that of a system of difference equations, a portion of the guide containing the discontinuity is imagined divided into cubical cells, and the field strength is found only at the lattice points. The nodal planes can be found either by relaxation methods or by direct solution of the difference equations by determinants and subsequent transformation of the latter, so that in both cases sufficiently refined knowledge of the position of the nodes is obtained by interpolation.

The representation of the standing-waves proceeds differently from what is given in § 1.5, for there the nodal lines are fixed by the choice of the relative space and time phases of the two standing-waves. For a discontinuity symmetrical in planes parallel to the axis of propagation it is convenient to choose as the basis standing-waves symmetrical and anti-symmetrical in z. Thus, let

$$\left.\begin{aligned} &(a) \qquad u = G\sin(kz+\delta)+H\sin(kz+\delta+\epsilon) \qquad (z>0);\\ &(b) \qquad u = -G\sin(kz-\delta)+H\sin(kz-\delta-\epsilon) \quad (z<0), \end{aligned}\right\} \tag{57.1}$$

$z = 0$ being the plane of symmetry of the discontinuity. Since (a) represents the pure travelling wave to the right, we require

$$G = -He^{-j\epsilon},$$

and the travelling wave to the left is

$$u = jH\cos\epsilon\, e^{-j(kz-\delta)}.$$

The reflection coefficient $|w|$ on the left is

$$\left|\frac{e^{j(\epsilon+kz-\delta)}+e^{j(kz-\delta-\epsilon)}}{e^{j(kz-\delta-\epsilon)}+e^{j(kz-\delta-\epsilon)}}\right| = |\cos\epsilon|. \tag{57.2}$$

ϵ is evidently the electrical distance between the two nodal planes, and equation (57.2) leads immediately to the value of the S.W.R. due to the discontinuity when the guide is terminated in a match.

When a wave guide is bent, one expects that only small effects will show on the propagation inside the guide, provided that the radius of curvature of the bend is large compared with the transverse dimensions of the guide. Without seriously deforming the guide, it is difficult to bend rectangular tube in small space: a circular bend of small radius may be electroformed by plating on a suitable base or turned from a solid piece of metal, but it is more convenient to use corners joining straight pieces of guide. For example, one may cut the two pieces of similar straight guide at the same angle to the guide axis, namely, half of the resultant angle between the two straight pieces after joining. When the deviation is not greater than 30°, no appreciable reflection will occur at the join, but for larger angles of turn the corner must be mitred, an example of which is shown in Fig. 27. The proper dimensions for these mitred corners, so as to give matched transmission, have been worked out [10] experimentally for turns in the plane parallel to the broad face of the guide (bend in the H-plane) and for turns in the plane parallel to the narrow face (bend in the E-plane). Such corners have very limited band-width, and have been superseded in practice [11] by mitred corners formed with an intermediate section of guide (see

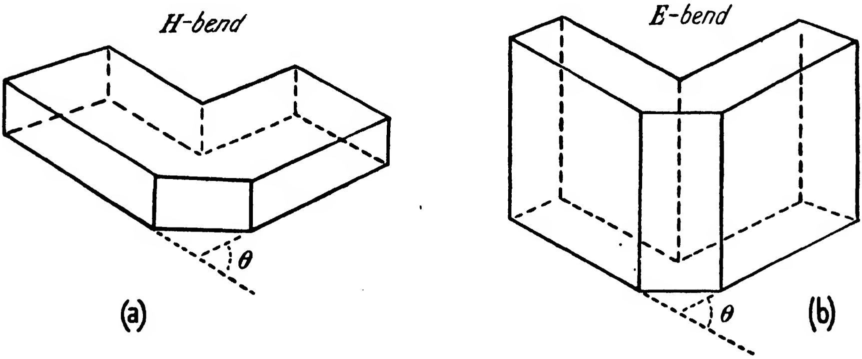


FIG. 27 (*a*) and (*b*).

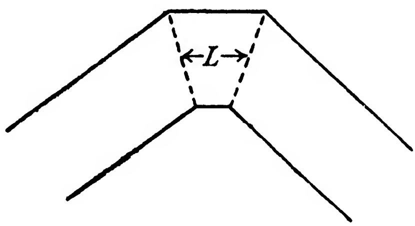


FIG. 28.

Fig. 28). The mean length L of the intermediate section should be close to $\frac{1}{4}\lambda_g$, or an odd integer times this distance, on the theory that the similar reflections from the two discontinuities will then cancel. Actually it is found that the H- and E-type corners differ markedly: the former show a considerable departure from the mean intermediate length $\frac{1}{4}\lambda_g$ for no reflection, and this departure depends on the angle of the corner, whereas the latter show only a small variation. Circular bends and twists introduce reflections in opposed phase arising from entry to and emergence from the bend or twist. In order to cancel them, the mean length of the section introduced as the bending or twisting connector is chosen about $\frac{1}{2}\lambda_g$. The mathematical theory of circular bends has been worked out by Marshak [12].

If, in addition to changing the direction of the guide axis, it is desired to alter the orientation of both the plane of the broad face and that of the narrow face in a single junction, the combined corner and twist can be made in the smallest distance by applying the principles of guide coupling presented in Chapter VII. These same principles have been used in designing a '360° corner' in which the returning guide lies along the outgoing piece.

5.8. The Coupling of Antennas to the H_{10}-Wave in the Rectangular Guide

In order that a linear conductor may be excited by the H_{10}-wave in a rectangular guide, it is necessary that for part of its length, at least, its projection on the direction of the electric force in the wave does not vanish. If the wire is insulated from the wall of the guide and of resonant length, the antenna will effectively short-circuit the wave in the guide and therefore reflect all energy, just as a curtain of resonant dipoles, if sufficiently closely spaced, would reflect a simple plane wave. Since the depth of the guide permitting only H_{10}-propagation is less than $\frac{1}{2}\lambda$, it would, of course, be necessary to bend the wire, but this does not affect its short-circuiting property in the wave guide when resonant.† On the other hand, the curtain of bent wires would have quite a different effect on the plane wave.

A straight-wire antenna may be inserted through a hole in either of the broad faces of the guide, and in virtue of the system to which it is coupled outside the guide may become resonant. For instance, the wire, suitably supported, may project through the wall of the guide and may act as a radiator to space or into another guide or cavity, or it may form the central conductor of a coaxial line, the outer conductor being connected to the wall of the guide as shown in Fig. 29. Experience shows that the last of these is probably the most important case in practice, for by this means energy is commonly introduced into a wave guide from a generator or received from the wave guide by detecting apparatus.

If high power is to be delivered, the problem of electrical breakdown must be surmounted. Accordingly the wire must be replaced by a thick rod with a round end. For 10-cm. waves the diameter of the rod would be of the order of $\frac{1}{4}$ in. in a coaxial line about 1 in. in diameter. The width of the corresponding wave guide is 3 in. With shorter waves this method of coupling a high-power generator is not successful: it is

† Measurements on such bent probes were reported by Watson and Guptill, Jan. 1943.

necessary to build the generator with a wave-guide output, so that in order to use the power one merely couples two wave guides.

Whether a wire of small diameter is used as in low-power measurements, or a rod of considerable diameter as in high-power applications, there are two problems connected with the coupling of an electric antenna to a rectangular wave guide. In order to achieve the efficient use of power, a technique is required for matching; and this must be based on understanding how a wire probe loads the H_{10}-wave in the guide.

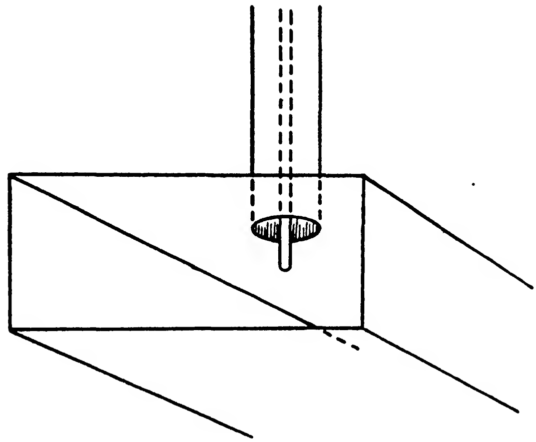


Fig. 29.

On physical grounds it is relatively easy to see that a wire probe will always load the H_{10}-wave as a shunt. Think of the electric field due to the radiating wire as initiating H_{10}-waves along the guide in both directions. These waves must have the same amplitude and the electric force in them must have the same phase at equal distances from the wire. It follows at once from § 1.2 that this type of radiation from the antenna corresponds to shunt loading of the equivalent transmission line representing the dominant wave propagation in the guide. With the actual dependence of the admittance on the position of the antenna in the cross-section of the guide, its length, and the system to which it is connected, we shall deal in the next section. We note, however, that it is not possible to arrange any other type of load on an H_{10}-wave by means of a straight thin wire antenna, for there is only one way in which the field in the guide can excite currents in the wire, namely, by its single electric component. In a guide propagating a TM-wave, however, this restriction would no longer hold. The wire might be coupled to the transverse electric field or to the longitudinal electric field.

Since the magnetic field in an H_{10}-wave possesses two components in quadrature, and corresponding thereto both longitudinal and transverse currents flow on the wall of the guide, a magnetic radiator, realized by a slot cut in the wall, may be coupled in various ways. These will be discussed in Chapter VI. The key to understanding the mode of coupling lies in the answers to the questions, what are the amplitudes and phases of the dominant waves radiated by the antenna on each side, and what is the relation of the antenna oscillation to the incident wave.

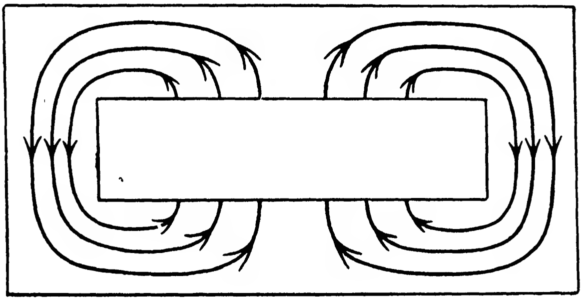


FIG. 30.

When an obstruction is placed in the dominant wave in a rectangular guide, we can understand its effect by regarding it as an antenna which is excited by the wave. We think of the current system flowing in it as made up by the superposition of characteristic distributions, one of which corresponds to the dominant wave and determines the radiation along the guide from the antenna. This radiation superposed on the incident wave yields the transmitted and reflected waves. This principle leads one to a qualitative understanding of the reason why a rectangular aperture, which might be regarded as the combination of symmetrical inductive and capacitive irises, can be expected to transmit the dominant wave without reflection when the aperture is correctly proportioned. In Fig. 30 the arrows indicate the direction of current flow on the metal diaphragm in which the aperture is cut, when the latter is oscillating as a resonant slot. In the centre of the guide cross-section the current flow is opposed to that at the sides. These opposed currents tend to cancel each other's contribution to the dominant wave radiated from the iris.

5.9. The Impedance of a Wire Antenna in the Guide

The study of the relation of impedance to waves in Chapter I makes evident what could not have been brought to light by the conventional circuit treatment of impedance, namely, that we must keep in mind the particular type of wave to which the impedance refers. In transmission lines we have to deal normally only with the principal wave, for the transverse dimensions of the line are small compared with the wavelength of the radiation. When a wire antenna is inserted into a wave guide it may be mounted on the wall of the guide in conducting contact with it, or, as we have already mentioned, it may form the extension of the inner conductor of a coaxial line, the outer conductor of which terminates on the guide wall. If we regard the antenna as fed at the other end of the coaxial line, we see that it presents a certain impedance to the line just as an antenna radiating to space. In order to permit radiation in one direction only along the guide, a reflecting plunger may be placed $\frac{1}{4}\lambda_g$ from the antenna. The load presented to the coaxial line is made up of a radiation resistance and a radiation reactance. Both of these quantities depend on the length of the antenna, the distance of the plunger from the antenna, and the position of the antenna in the guide cross-section. We shall later discuss this dependence in detail. For the present our interest is to distinguish the driving-point impedance presented to the coaxial line at its termination, and the shunt impedance presented to the dominant wave in the guide when travelling towards the antenna. Evidently this shunt impedance will depend on the termination of the coaxial line and its characteristic impedance, just as the radiation impedance will depend on the termination of the wave guide.

Knowledge of driving-point impedance is important in feeding energy to a wave guide from a coaxial line. It is measured by standing-wave technique on the coaxial line. The shunt impedance offered by the receiving antenna to the dominant wave, on the other hand, is measured by standing-wave technique in the wave guide.

Nevertheless these two impedances are related to each other because the antenna really couples two transmission lines—the coaxial line and the dominant wave circuit in the guide. We shall have occasion to treat the subject of coupled lines in Chapter VII: for the present it suffices to remark that the law of impedance transfer from one line to the other at the coupling is

$$Z = \frac{m}{Y} + j\xi, \tag{59.1}$$

where ξ is real and dependent only on the antenna and its position in the guide cross-section, Z is the impedance in the wave system of the coaxial line reckoned at the junction and looking towards it, Y is the total admittance seen from the point of coupling in the wave guide, and m is a real constant like ξ. This is called shunt-shunt coupling.

Consider first the impedance Z_1 presented to the coaxial line when the guide has a matched termination on one side, and on the other the plunger distant z_0 from the antenna. If Y_g is the characteristic admittance of the guide,

$$Y = Y_g\left(1+\frac{e^{jk_1z_0}+e^{-jk_1z_0}}{e^{jk_1z_0}-e^{-jk_1z_0}}\right), \qquad k_1 = \frac{2\pi}{\lambda_g}. \tag{59.2}$$

Hence $$Z_1 = R_1+jX_1 = \frac{m}{2Y_g}(1-e^{-2jk_1z_0})+j\xi. \tag{59.3}$$

The radiation resistance R_1 is $\frac{m}{2Y_g}(1-\cos 2k_1 z_0)$, while the radiation reactance X_1 is $\frac{m}{2Y_g}\sin 2k_1 z_0+\xi$. Now consider the coaxial line terminated by Z', reckoned at the point of coupling, the plunger placed $\frac{1}{4}\lambda_g$ from the antenna, and the input admittance Y' seen in the guide at the antenna. From (59.1) we have

$$-Z' = \frac{m}{-Y'}+j\xi \quad \text{or} \quad Y' = \frac{m}{Z'+j\xi}. \tag{59.4}$$

ξ/m is therefore the reactance shunting the guide when the antenna is short-circuited to the broad face of the guide. If we take $Y_g = 1/Z_g$ (cf. (21.8)), then

$$m = \frac{k^2}{ab}\sin^2\frac{\pi x_0}{a}\tan^2\frac{\pi l}{\lambda}, \tag{59.5}$$

l being the length of the antenna which is supposed thin, x_0 is the distance of the antenna from the narrow face, and a, b, k, λ have their usual meaning. Infeld [13] has deduced the following expression for ξ intended to apply to a thin antenna of radius ρ_0:

$$-\xi = \frac{30\pi\tan^2\pi l/\lambda}{bk}\mathbf{Y}_0(k\rho_0)+$$
$$+\frac{60\pi}{jbk\sin^2 kl}\sum_{n=1}^{\infty}\frac{(\cos kl-\cos n\pi l/b)^2}{1-(n\pi/bk)^2}\mathbf{H}_0^{(1)}\left[k\rho_0\Big/\sqrt{\left\{1-\left(\frac{n\pi}{bk}\right)^2\right\}}\right]+$$
$$+\frac{60\pi}{bk}\tan^2\left(\frac{\pi l}{\lambda}\right)\sum_{m=1}^{\infty}(-1)^m\mathbf{Y}_0(mka), \tag{59.6}$$

where $\mathbf{Y}_0$ is the Bessel function of the second kind and $\mathbf{H}_0^{(1)}$ is the Hankel function of the first kind, both being of order zero.

The first and third terms in (59.6) are of opposite signs, and in the vicinity of $l = \frac{1}{4}\lambda$, ξ vanishes. Thus the guide is short-circuited by a 'quarter-wave' antenna attached to the guide wall. (This result holds also for a bent antenna.) As the length of the antenna is increased, the reactance presented by it to the wave in the guide passes from a capacitive into an inductive load, provided that Z' is real. As the length approaches half-wave, ξ/m tends to infinity as an inductive reactance: of course, an antenna of this length could not be accommodated in the guide without bending, but this fact does not invalidate the argument. If the 'half-wave' antenna is insulated from the wall of the guide, we may think of Z' as a large reactance of capacitive type which will be cancelled by ξ when the antenna has the proper length, so that the guide is again short-circuited. On the basis of Babinet's principle we expect therefore that a half-wave slot, cut perpendicular to the electric force of the H_{10}-wave and in a conducting partition across the guide, loads the guide as an open-circuit—that is, it does not interfere at all with the propagation unless the power is sufficient to cause breakdown in the slot. As we have already mentioned, such a slot may be regarded crudely as the combination of complementary capacitive and inductive irises and behaves therefore as a parallel resonant circuit shunted across the transmission line.

The case of the short probe is of special interest; both ξ and m are proportional to l^2, so that approximately we may write

$$Y' = \frac{k^4 l^2}{4abZ'}\sin^2\frac{\pi x_0}{a}, \qquad |Z'| \gg \xi. \tag{59.7}$$

Since the fraction of power drawn from a travelling wave by the shunt load is Y'/Y_g, it is proportional to the square of the probe length.

It remains for us to discuss the matching problem [14]. We are required to choose z_0, the distance of the plunger, and l, the length of the antenna, so as to make Z_1 in (59.3) equal to Z_c, the characteristic impedance of the coaxial line.

Let $$A(l) = \frac{m}{2Y_g Z_c} \quad \text{and} \quad A'(l) = -\frac{\xi}{Z_c}.$$

From (59.3) we require

$$A(l)\{1-\cos 2k_1 z_0\} = 1 \quad \text{and} \quad A'(l) = A(l)\sin 2k_1 z_0. \tag{59.8}$$

Let l_0 be the value of l which makes these two equations possible, then

$$\tfrac{1}{2}\{1+[A'(l_0)]^2\} = A(l_0). \tag{59.9}$$

Plot the right- and left-hand sides of this equation as functions of l for the given radius ρ_0. The intersection of the two graphs yields l_0: substitution in the first of (59.8) then gives z_0.

The equations (59.3) and (59.9) may be presented in simple geometrical form on an impedance (R, X) diagram. From the former

$$(R_1-A)^2+(X_1-A')^2 = A^2. \tag{59.10}$$

Thus if we alter z_0, keeping the other parameters fixed, the points (R_1, X_1) lie on the impedance circle tangent to the X-axis and with centre at (A, A') which must lie on the parabola (59.9) plotted on the same plane with the axes of A and A' respectively coincident with those of R and X (Fig. 31). The centre of the impedance circle is determined by the length of the antenna; the position on the circle corresponding to the input impedance is determined by z_0.

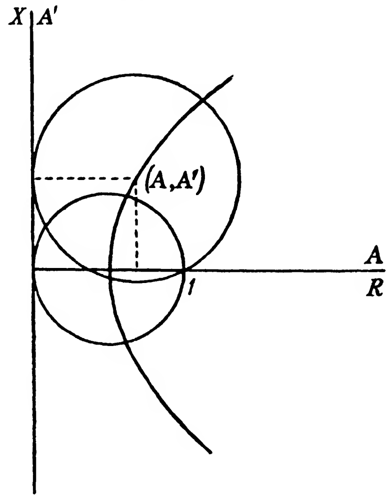


FIG. 31.

The calculation referred to above is based on assumed sinusoidal distribution of current in the antenna. If the latter reaches across the guide so that it touches the opposite face, there is reason to believe that the current amplitude is uniform along it, and we then have the results

$$R = Z_g\frac{b}{a}(1-\cos 2k_1 z_0),$$

$$X = \frac{Z_g b}{a}\sin 2k_1 z_0-\frac{60\pi^2 b}{\lambda}\sum_{m=-\infty}^{\infty}(-1)^m Y_0\{k(\rho_0+ma)\}, \tag{59.11}$$

where m is an integer. It is possible to achieve a match in this case by choosing z_0 and ρ_0, the radius of the antenna.

Useful results for the resistances of antennas of various shapes variously disposed in rectangular and circular wave guides have been worked out by Chu [15] and Chien [16]; the latter corrects results of the former in the case of circular guide.

The principles for treating the excitation of other modes by means of an antenna in the guide are discussed in Chapter X.

VI

THE COUPLING OF A RESONANT SLOT IN A RECTANGULAR WAVE GUIDE

WE shall consider a narrow slot about $\frac{1}{2}\lambda$ long and discuss the different ways in which coupling may take place. Outside the guide the slot may radiate to space or it may radiate into a second guide. Only the first of these is treated in the present chapter; the second is discussed in the following chapter on the coupling of guides by means of slots. The simple rectangular shape is taken as fundamental, for only two parameters are required to specify it; nevertheless it must be pointed out that in actual practice it is desirable to depart from this simple shape in order to facilitate accurate but inexpensive cutting.

6.1. Displaced Longitudinal Shunt Slots

Let a slot be cut in the guide wall with its long axis parallel to the axis of the guide: such a slot presents a shunt load to the dominant

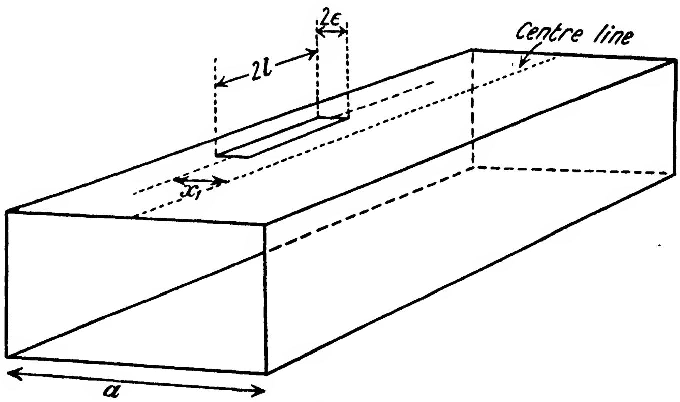


FIG. 32.

wave in the guide. In spite of the length of the slot antenna, this load can be treated as lumped at the position of the centre of the slot.

When, under the chosen conditions outside the guide, the slot has been tuned by cutting to the proper length, which is close to half of the free-space wavelength, the slot presents a pure conductance G to the wave. Let x_1 denote the distance of the centre of the slot from the centre-line of the broad face of the guide (see Fig. 32). Then, expressed as a fraction of the characteristic admittance for the dominant wave in the guide,

$$G = A_1 \sin^2\frac{\pi x_1}{a}. \tag{61.1}$$

A_1 is a constant which is approximated well in Stevenson's [17] formula derived by solving the somewhat idealized field problem in which the width (2ϵ) of the slot is assumed small compared with the length ($2l$), the thickness of the wall of the guide is taken as vanishing, and the diffraction of waves round the outside of the guide is left out of account.

In presenting various results in connexion with slots we shall find it convenient to introduce the angle i (θ of Fig. 6) whose trigonometric ratios connect the free-space, guide, and cut-off wavelengths as follows:

$$\sin i = \frac{\lambda}{2a}, \qquad \cos i = \frac{\lambda}{\lambda_g}, \qquad \tan i = \frac{\lambda_g}{2a}. \tag{61.2}$$

In the theory referred to it is found that

$$A_1 = 2{\cdot}09\frac{a}{b}\,\frac{\cos^2(\frac{1}{2}\pi\cos i)}{\cos i} \qquad (a > \tfrac{1}{2}\lambda > b). \tag{61.3}$$

a and b are, as usual, the internal dimensions of the guide cross-section, λ_g is the dominant wavelength in the guide. Refinement of the theory will affect the numerical factor in (61.3), but it will not seriously affect the dependence on the other parameters of A_1, which is closely related to the slot polar diagram. It will be noted that the nearer the guide is to cut-off for the frequency used ($i \to \frac{1}{2}\pi$) the greater G for given proportionate displacement of the slot from the centre of the broad face. If the guide is wide ($\lambda \to a$), the slot is a less effective shunt.

The conductance can be measured by means of the standing-wave detector, provided that a reflecting plunger is placed $\frac{1}{4}\lambda_g$ from the centre of the slot, at which point in the guide the equivalent line is regarded as loaded. This method is quite unsuited for the measurement of the small conductances ($O(0{\cdot}01)$) which result from placing the slot close to the centre-line. In that case, the difficulties arising from the need to draw energy into the S.W.D. and the great disparity of the field strengths at the maximum and minimum of the S.W. system can be avoided by comparing the radiation fields due to two slots, one of which can be measured well by S.W. technique. On the assumption that the slots radiate with similar directive patterns, the slot conductances are in the ratio of the square of the field strengths measured in space outside the guide at the same position with respect to the

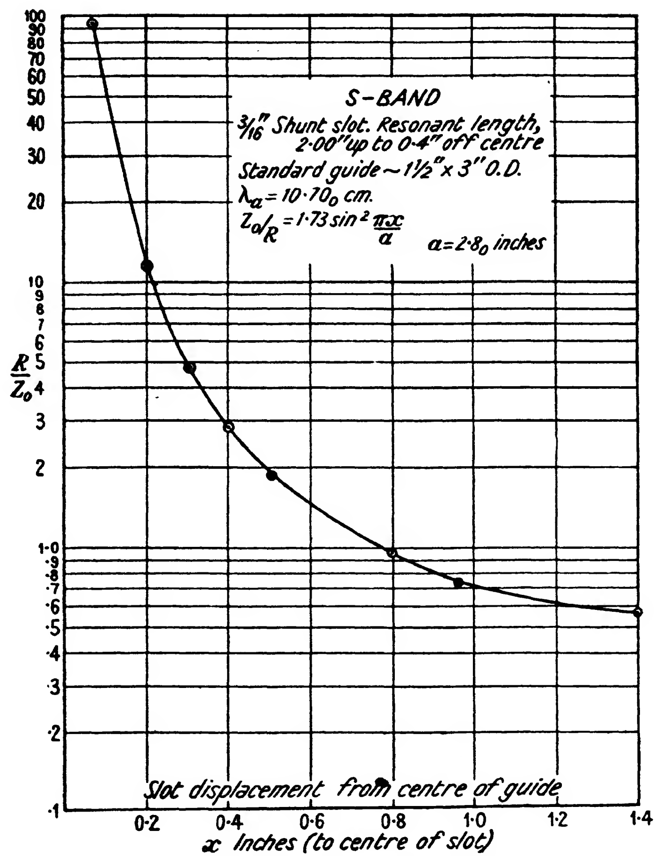


Fig. 33 (a).

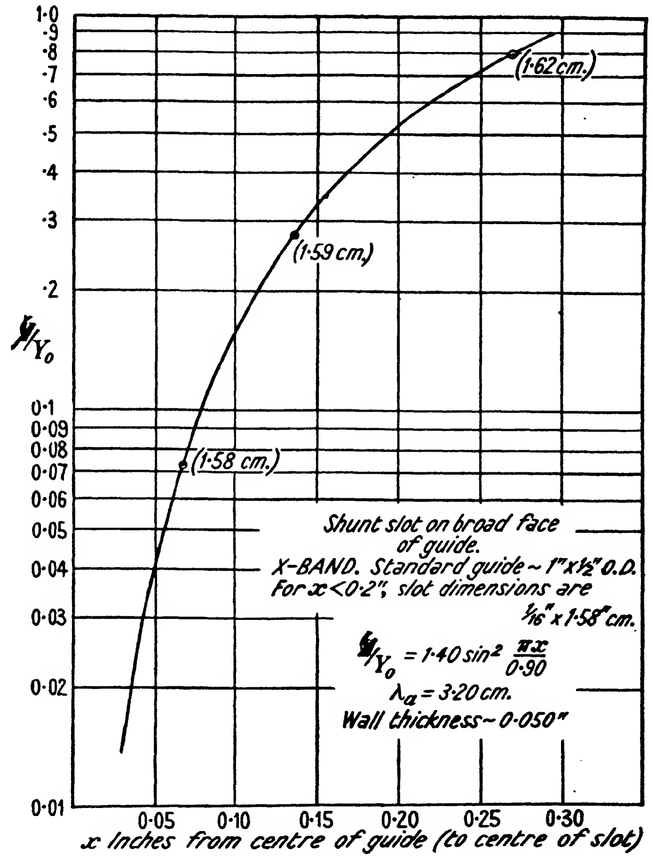


Fig. 33 (b).

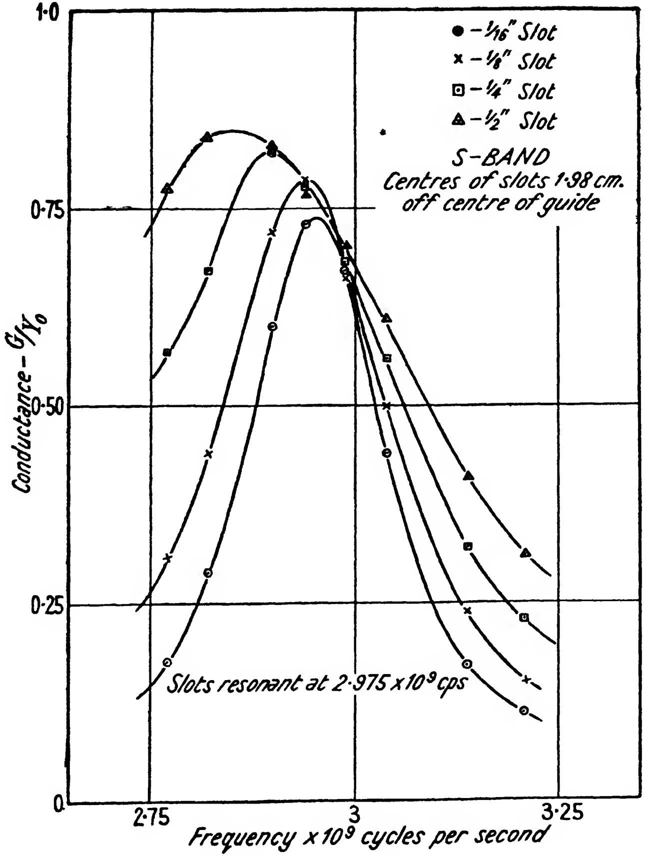


Fig. 34 (*a*).

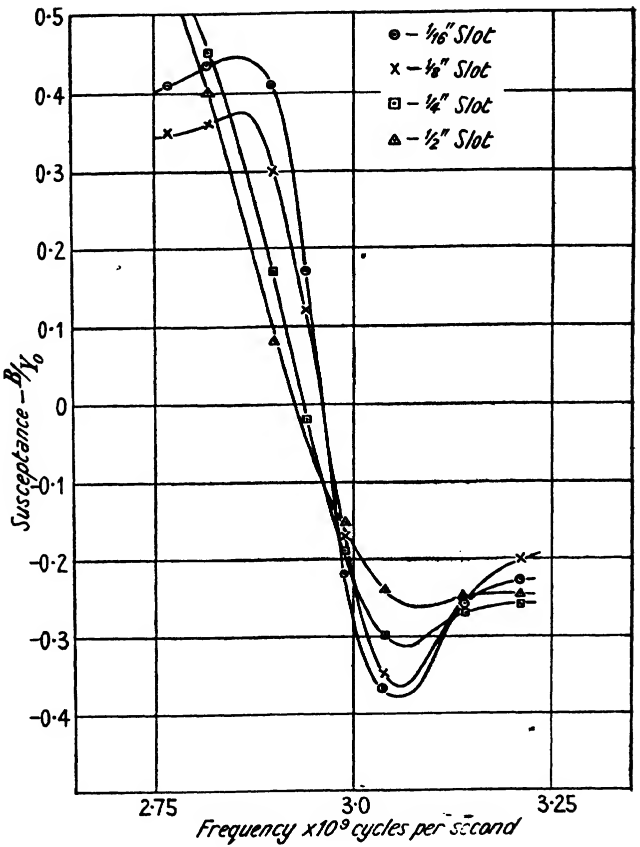


Fig. 34 (*b*).

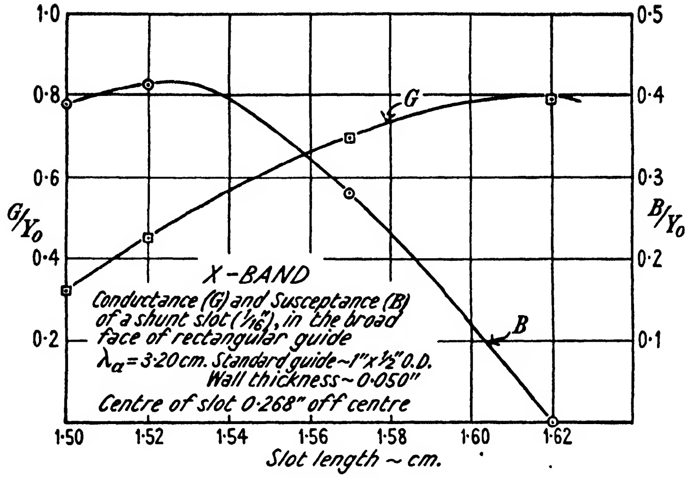


FIG. 34 (c).

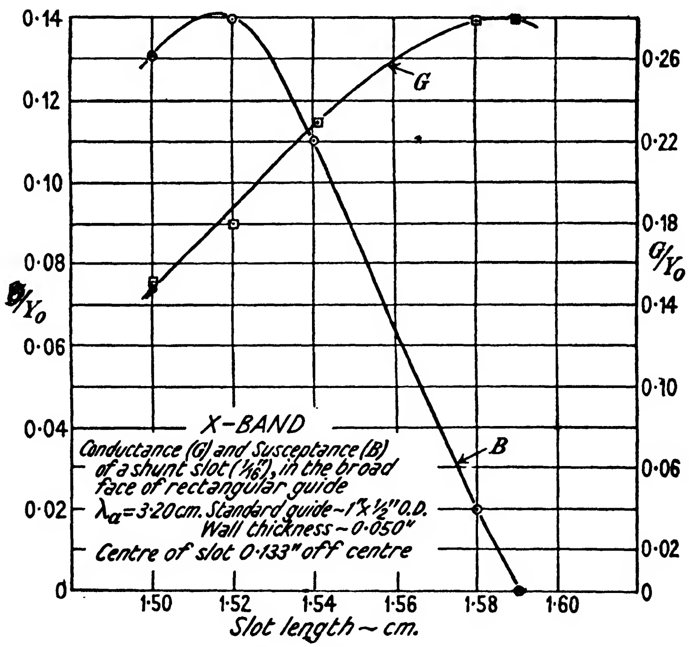


FIG. 34 (d).

slots. Experimental results for S- and X-bands are shown in Figs. 33 (*a*) and (*b*).

The length of slot for resonance, that is, to present a pure conductance, is dependent on the position of the slot on the guide and on its width. Some experimental results are shown in Figs. 34 (*a*)–(*d*).

For a given slot, if the frequency of the radiation is altered in the vicinity of resonance, it is found that for one frequency the conductance attains a maximum: this is not in general the same frequency at which the susceptance vanishes for the slot load (see Fig. 34). It has been

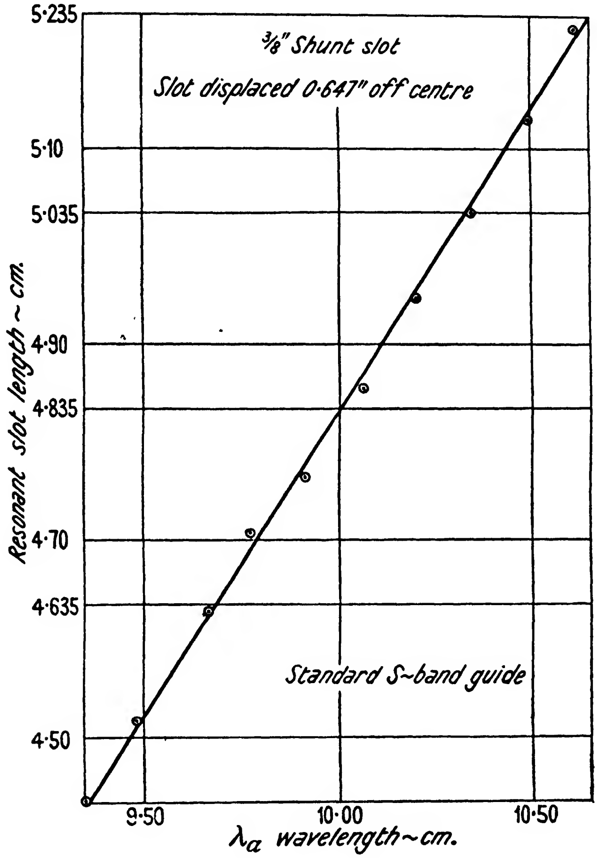


Fig. 35.

stated that the resonant length of the slot for given frequency and guide dimensions is linearly dependent on its displacement from the centre-line, but there is good reason to believe that a parabolic law fits the facts for not too large displacements, and is in accord with theoretical expectations (see § 10.9, Fig. 95). However, for a given displacement on a given guide the resonant length depends linearly on the wavelength, as shown in Fig. 35.

The width of the slot enters in an important way to determine the sensitivity of the slot to change of frequency. It is expressed [18] by the Q-value of the slot: if f_0 is the resonant frequency and λ_0 the corresponding free-space wavelength, B and G the susceptance and conductance of the slot, then

$$Q = \frac{f_0}{2G_0}\left|\frac{dB}{df}\right|_0 = \frac{\lambda_0}{2G_0}\left|\frac{dB}{d\lambda}\right|_0 = \frac{\lambda_0}{8G_0}\left|\frac{dB}{dl}\right|. \tag{61.4}$$

Fig. 36 shows the variation of Q with slot width for an S-band slot

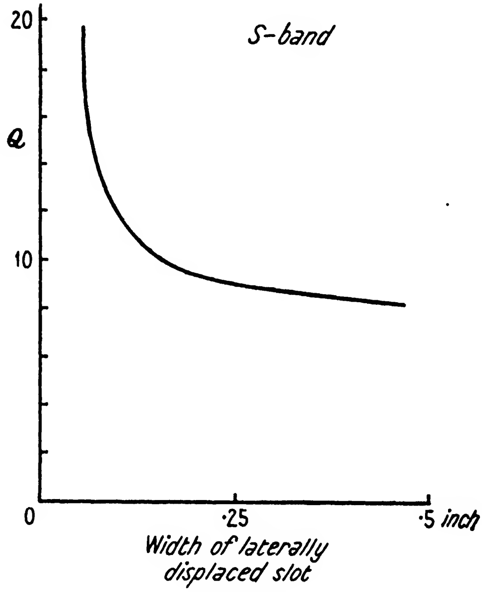


FIG. 36.

displaced 1·98 cm. from the centre-line. Note that the Q is higher than for a slot cut in an infinite plane sheet (eqn. (54.9)).

It follows from (61.1) and it has already been pointed out that when the longitudinal slot is alined with the centre of the broad face, the slot is not coupled to the dominant wave. This is because the transverse component of surface current on the inside wall of the guide vanishes there: this component may be regarded as exciting the slot in a displaced position, and the dependence of G on x_1 can be readily understood on this basis. Further, the transverse surface current reverses phase across the centre-line, consequently the phase in which a resonant slot would be excited would be reversed were the slot placed in the image position with respect to the centre-line of the broad face. In this position the conductance presented to the dominant wave in the guide will, of course, have the same value as before.

6.2. Inclined Transverse Shunt Slots

Since the longitudinal component of surface current vanishes on the narrow walls of the guide, a transverse slot, perpendicular to the axis of the guide, cut in the narrow face, will be parallel to the lines of current flow and hence unexcited by the dominant wave in the guide. In order to achieve resonant length, since $b < \frac{1}{2}\lambda$, it will be necessary to cut the slot partly in the two broad faces, but even so, the slot, if cut symmetrically—its centre on the centre of the narrow face—will

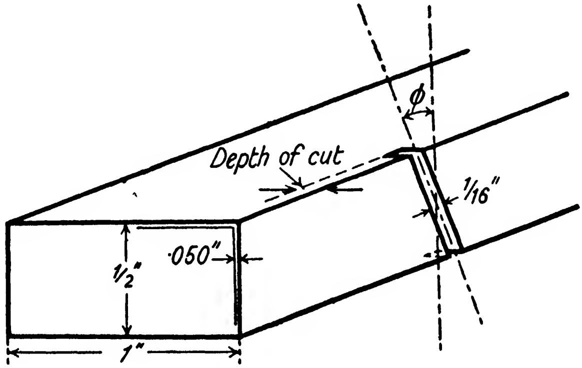


Fig. 37.

still be unexcited by the dominant TE-wave. It may be excited by other types of waves inside or outside the guide (see § 9.3).

Suppose now that the slot is turned about its centre, as shown in Fig. 37. Such a slot could be cut with an end-milling cutter, its axis parallel to the broad face and perpendicular to the axis of the guide, and the cut made in the plane making ϕ with the transverse section of the guide through the centre of the slot. The latter is again excited by the transverse surface current and loads the H_{10} dominant waves as a shunt. At resonance, in Stevenson's theory,

$$G = 2{\cdot}09\frac{a}{b}\frac{\sin^4 i}{\cos i}\left[\frac{\sin\phi\cos(\frac{1}{2}\pi\cos i\sin\phi)}{1-\cos^2 i\sin^2\phi}\right]^2, \tag{62.1}$$

and approximately, when ϕ is less than 30°,

$$G = A_2\sin^2\phi, \tag{62.2}$$

where

$$A_2 = 2{\cdot}09\frac{a\sin^4 i}{b\cos i}. \tag{62.3}$$

The radiation from this slot travels along the outside of the guide: it spreads around the guide and will also be reflected from the ends of the guide or from obstacles on it, for the slot is coupled to the principal wave on the outside of the guide considered as of infinite length. These facts must be considered in measurement.

The phase of the inclined shunt slots may be reversed by changing the sign of ϕ, which, if small, allows in the radiation electric polarization substantially parallel to the guide axis. This is at right angles to the electric force in the radiation from the longitudinal shunt slot.

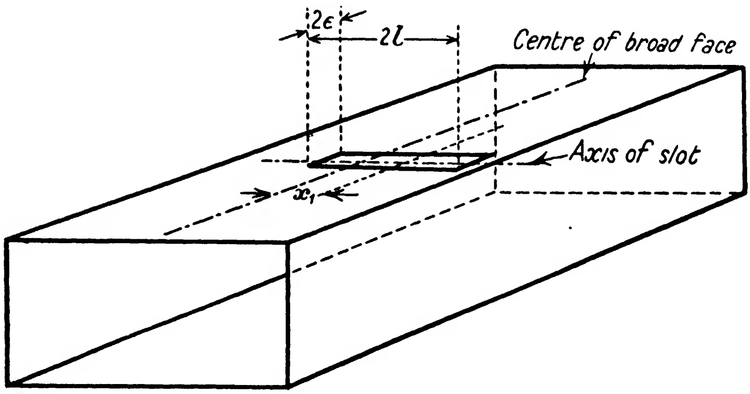


FIG. 38.

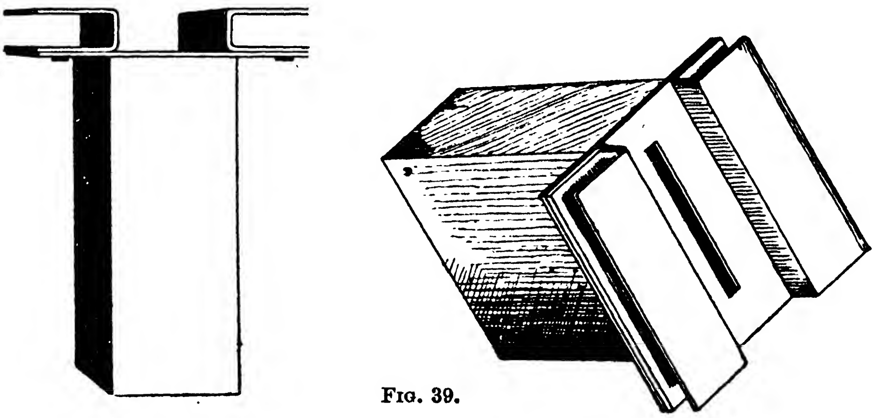


FIG. 39.

6.3. Transverse Series Slots

A transverse slot symmetrically cut in the broad face is excited by the longitudinal component of the surface current in the inside of the guide wall. Such a slot radiates H_{10}-waves of equal amplitude in opposed phase in opposite directions inside the guide, and hence presents a series load to the incident wave. If this slot is displaced parallel to its length on the broad face through the distance x_1, as shown in

Fig. 38, then at resonance its resistance is

$$R = B_1 \cos^2\frac{\pi x_1}{a}, \tag{63.1}$$

where, according to Stevenson's theory,

$$B_1 = 2{\cdot}09\frac{a}{b}\frac{\sin^2 i}{\cos^3 i}\cos^2(\tfrac{1}{2}\pi \sin i) \tag{63.2}$$

The calculation is based on the assumption that the slot lies completely in the broad face. If, in order to secure the resonant length, part of the slot has to be cut in the narrow face (round the corner), the above result will hold fairly well provided that the part off the broad face is not greater than a quarter of the slot length.

This slot is coupled to the principal wave on the outside of the guide. It is interesting to compare the constants A_1 and B_1:

$$\frac{B_1}{A_1} = \tan^2 i\,\frac{\cos^2(\tfrac{1}{2}\pi \sin i)}{\cos^2(\tfrac{1}{2}\pi \cos i)}. \tag{63.3}$$

Since usually i will not differ greatly from $\frac{1}{4}\pi$, it is seen that the above ratio is not far from unity in practice.

A series-coupled slot may be cut in a plate closing the end of the guide, thus presenting a magnetic radiator as the termination of the wave guide. The resistance of the resonant slot when cut symmetrically in the end-plate is the same as for a symmetrically disposed transverse slot in the broad face. Since the radiation from the resonant slot termination will travel back along the outside of the guide, it is necessary to introduce suitable chokes. This may be done in the way indicated in Fig. 39. After adjustment of the geometrical parameters involved, it is possible to secure the radiation pattern shown in Figs. 40 (a) and (b). Further, the termination is a fair approximation to a matched load. For comparison, Fig. 41 shows the transverse pattern for an array of inclined shunt-coupled slots (see Chapter VIII).

6.4. Inclined Series Slots

A transverse slot symmetrically placed on the broad face may be rotated about its centre (Fig. 42) and still remain a pure series load. For small angles of rotation θ from the centre-line of the broad face—that is, from the unexcited longitudinal position—the resistance presented by the slot at resonance is given approximately by

$$R = B_2\theta^2 \quad (\theta \text{ in radians}). \tag{64.1}$$

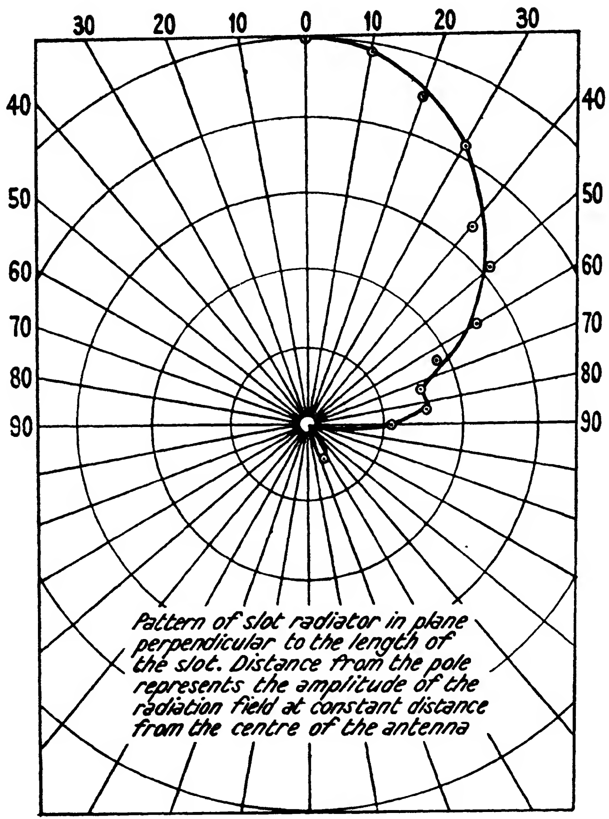


FIG. 40 (*a*).

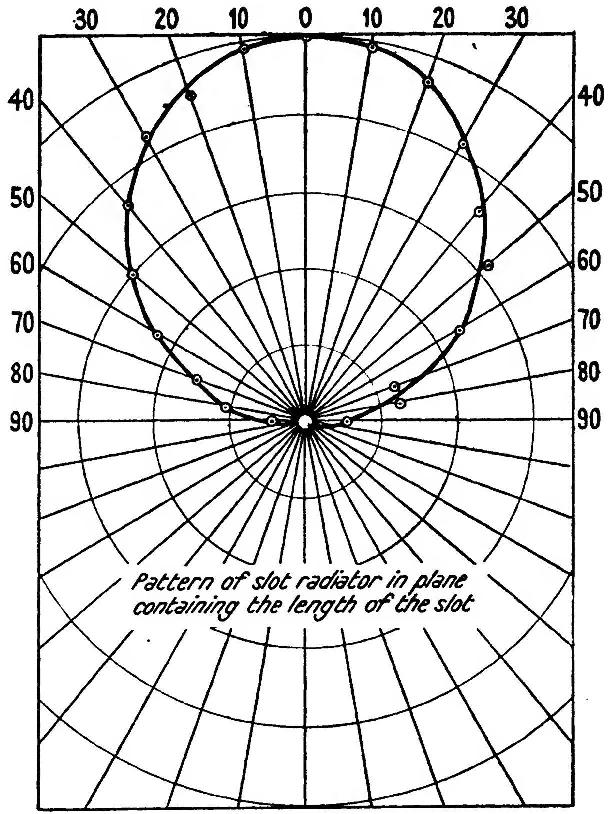


FIG. 40 (*b*).

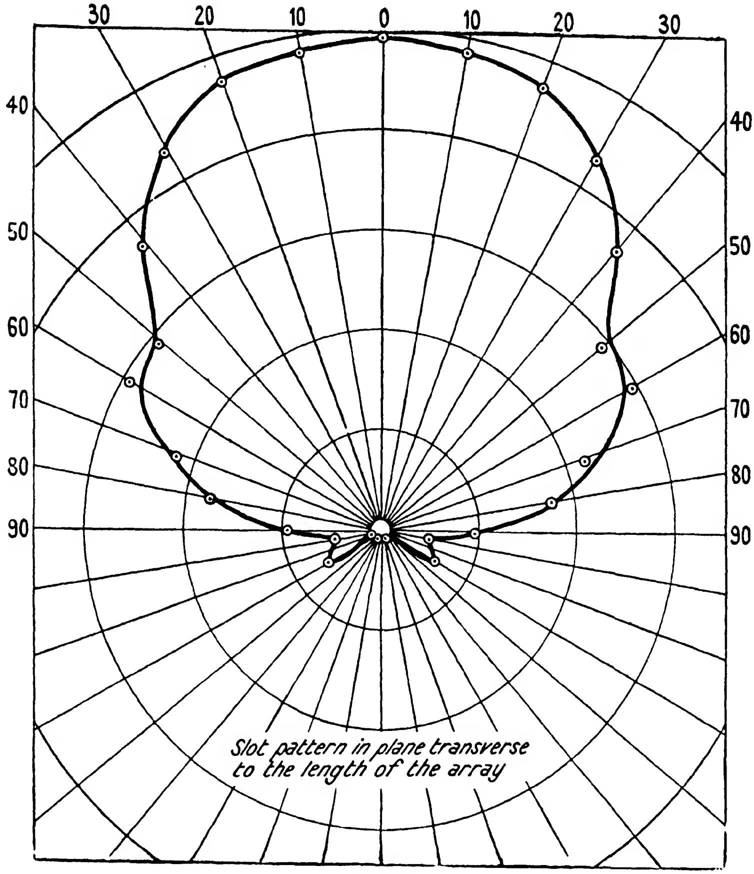

30
20
10
0
10
20
30
40
40
50
50
60
60
70
70
80
80
90
90
Slot pattern in plane transverse to the length of the array

Fig. 41.

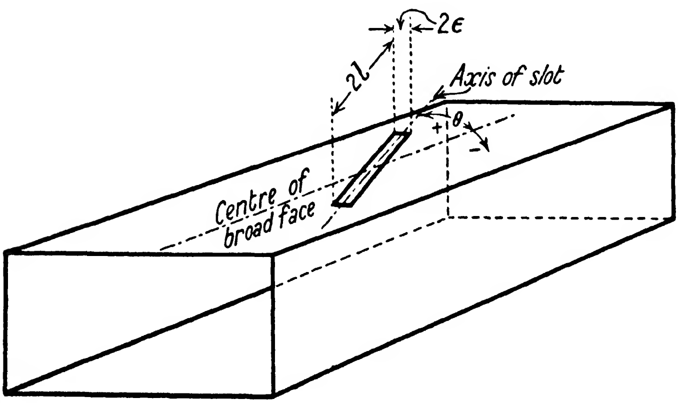

2ε
2l
Axis of slot
θ
+
-
Centre of broad face

Fig. 42.

Experiments easily show the departure from this law as θ increases. Stevenson's theory yields a fairly complicated dependence on θ, of which (64.1) is the limiting form. We now consider the theoretically exact law. Let

$$M(z) = \frac{\cos(\frac{1}{2}\pi\cos z)}{\sin z}, \tag{64.2}$$

then

$$R = 0{\cdot}524\frac{\sin^2 i}{\cos i}\frac{a}{b}[M(i+\theta)-M(i-\theta)]^2. \tag{64.3}$$

The constant in (64.1) is given by

$$B_2 = 2{\cdot}09\frac{\sin^2 i}{\cos i}\frac{a}{b}\left[\frac{d}{dz}M(z)\right]^2_{z=i}. \tag{64.4}$$

These slots are evidently excited by the longitudinal surface current on the broad face, for the transverse current on the guide wall is flowing towards the two halves of the slot in opposite phases. The phase of radiation from the slot is reversed by changing the sign of θ.

6.5. Behaviour off Resonance

It is important to understand that the behaviour of these slots when not resonant is represented by the schemes of the transmission-line

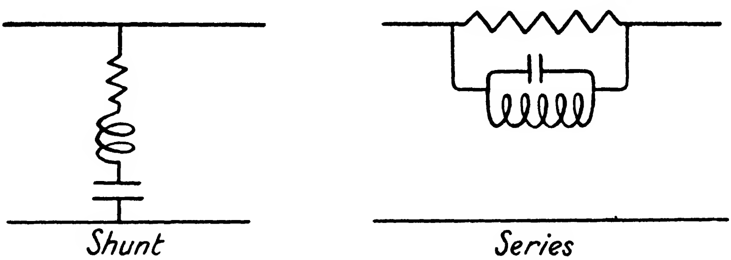


FIG. 43.

analogy shown in Fig. 43. Accordingly, the value of G at resonance does not give the conductance off resonance in the shunt case, and the value of R at resonance in the series case does not yield the resistance off resonance.

The results which have just been presented can be verified experimentally. It has been shown that when there is no mutual interaction, the admittances of shunt loads combine by addition provided that the loads are placed an integral number of guide half-wavelengths apart (i.e. are in the same position (mod $\frac{1}{2}\lambda_g$)) and the line terminated by a short-circuit $\frac{1}{4}\lambda_g$ (mod $\frac{1}{2}\lambda_g$) from this position. The impedances of series loads also combine by addition if the loads are at the same position on the line (mod $\frac{1}{2}\lambda_g$): the line in this case will be terminated by a short-circuit at this position (mod $\frac{1}{2}\lambda_g$). For any other spacing of the loads it is necessary to use the circle diagram or the equivalent matrix method in order to calculate the input impedance or admittance to the combined load. The striking fact revealed by the experiments and supported by Stevenson's field calculations is that the slot may be treated as a lumped load applied at the position of its centre and that the simple types of load may be achieved by the above scheme.

6.6. The General Inclined-Displaced Slot (Broad Face)

We now consider the general case of an inclined laterally displaced slot cut in the broad face of the guide. Let x_0 be the transverse displace-

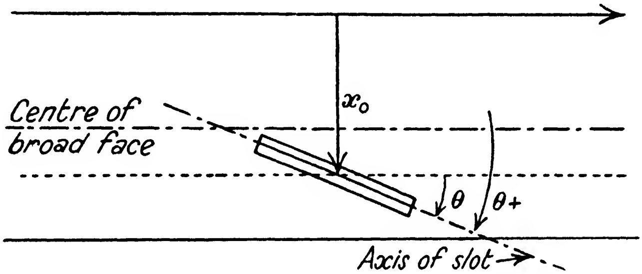


FIG. 44.

ment of the slot-centre from one edge of the broad face, and θ the inclination of the slot-axis to the guide axis, according to the sign conventions of Fig. 44.

If the guide beyond the slot is terminated by a movable reflecting plunger (short-circuit), it is found that there exists a position (mod $\frac{1}{2}\lambda_g$) of the plunger at which the slot is not excited, and the reactance presented by the plunger as seen from the centre of the slot is transformed past the slot unchanged. This reactance plays the part of the short-circuit for the shunt load and the open-circuit for the series load. Thus the resonant slot in the broad face of the guide propagating only the H_{10}-wave realizes in full the possibilities discussed in § 1.6.

Two fundamentally different types of measurement are required to exhibit the properties of such slots. First, by S.W. measurements between the slot and the generator, and between the slot and an arbitrary known termination of the guide beyond the slot—usually a match or a pure reactance—it is possible to find the law of impedance or admittance transformation past the slot, regarded as producing a discontinuity in propagation at the position of its centre. Secondly, measurements are made of the power taken from the guide through the slot, and of the phase change which occurs in propagation past the slot to the matching termination. The S.W. measurement refers to operations on the circle diagram: the latter supplement this information so as to determine the laws of transformation of wave-amplitudes from one side to the other of the slot.

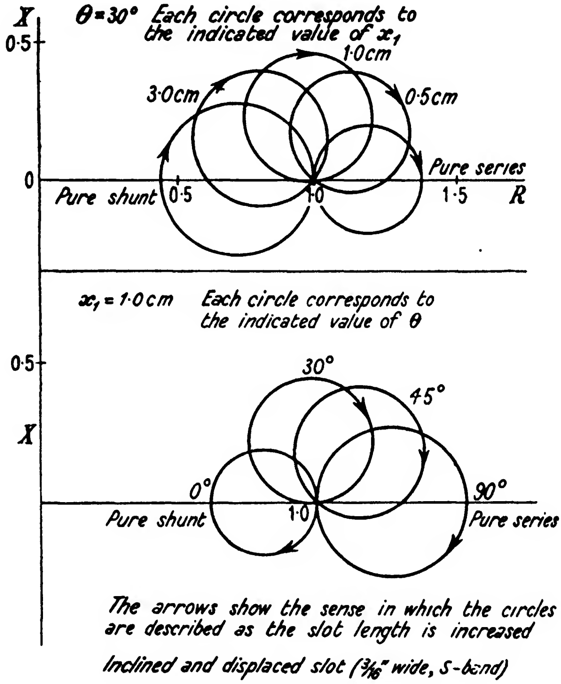


FIG. 45.

Let us consider those properties of the slot made evident by S.W. measurements and represented in terms of the transformation of the plane of the circle diagram variable w, reckoned at the position of the centre of the slot. We shall employ the magnetic representation. Since the self-corresponding points are coincident, and lie on the unit circle on the w-plane, the form of transformation for any such slot can be expressed by equation (16.7), viz.

$$\frac{1}{w'-e^{j\delta}} = \frac{1}{k+e^{j\delta}} + \frac{1}{w-e^{j\delta}}, \tag{16.7}$$

where, as usual, w' refers to the left and w to the right of the slot, the generator being on the left, and k and δ are expected to depend on the displacement, inclination, and length of the slot.

It is found that δ does not depend on the length of the slot whereas k does. Evidently δ can be determined by finding the susceptance Y_1 which is transformed unchanged by the slot, for we know that $Y_1 = \tan \frac{1}{2}\delta$. Further, once we know the transform of a match, k can be calculated from (16.8). Thus by finding first the reactive termination beyond the slot which will put the slot out of action as a radiator, and then measuring the transform of $w = 0$, we can determine the bilinear transformation completely.

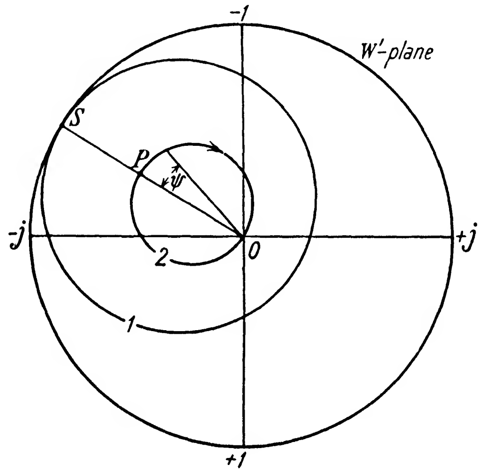


FIG. 46.

The dependence of k on the length of the slot can be explained with the aid of Fig. 45, where the input impedance to an inclined-displaced slot, when the guide is terminated in a match, is shown as a function of the length of the slot in each case. From the circular form of these curves it is evident that w' given by (16.7) must lie on a circle through the origin of the w'-plane as the length $2l$ of the slot is varied $\frac{1}{4}\lambda < 2l < \frac{3}{4}\lambda$. Fig. 46 exhibits the relations involved.

The self-corresponding point S is the point of contact of the unit circle $|w'| = 1$ and the circle 1, which is the transform of the unit circle. The point P is the transform of $w = 0$ when the length of the slot is $2l_0$. As the length of the slot is changed, the displacement x_0 and the inclination θ being kept fixed, $w = 0$ transforms into a point on the circle 2. It will be noticed that the centres of the circles 1 and 2 lie on the same radius OS, making δ with the positive real axis of w'. On the circle 2 the arrow indicates the direction of the displacement of P when the length of the slot is increased; ψ is therefore a function of the length of the slot and it is found also to depend on its width.

Let us now consider how the position of the self-corresponding points of (16.7) depends on the inclination and lateral displacement of the slot. Let α be the conductance of the pure shunt resonant slot with its centre at x_0, and let γ be the resistance of the pure series resonant slot inclined at θ, its centre of course being at $x = \frac{1}{2}a$. Experiment shows that to a first approximation for slots at small inclination with centres near the middle of the guide, $Y_1 = \pm\sqrt{(\alpha/\gamma)}$. The sign to be given to Y_1 depends on the combination of inclination and displacement chosen according to the rules presented in Fig. 47.

The facts just presented are illuminated by the results of field

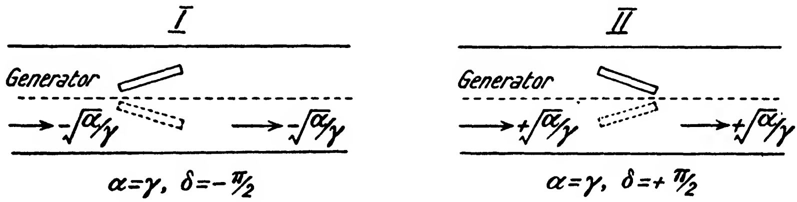


FIG. 47.

representation applied to the problem of the general near-resonant slot in a rectangular wave guide. Without entering fully into the details of the field calculation, we may profit from a brief reference to the principle of it. Suppose that the dominant TE-wave of unit amplitude is incident from the left. Let p' denote the amplitude of the voltage across the centre of the slot when excited by this wave only. The solution of the electromagnetic boundary problem under the assumptions already mentioned at the beginning of this chapter leads to the result

$$p' = \frac{\zeta}{K}, \tag{66.1}$$

where the complex number ζ depends on the displacement and inclination of the slot but not on its length or width. The real part of K, like ζ, can be evaluated without difficulty. The imaginary part of K involves the summation of a doubly-infinite series. When the slot is similarly excited only by a wave of unit amplitude from the right, then the voltage across the slot is

$$p = \frac{\zeta^*}{K}, \tag{66.2}$$

where ζ^* is the conjugate complex of ζ, and

$$\zeta = \frac{\pi}{a}[e^{j\pi x_0/a}M(i-\theta)+e^{-j\pi x_0/a}M(i+\theta)]. \tag{66.3}$$

The function M is defined in (64.2) and the equation (66.3) applies only to slots cut in the broad face. When the slot is cut in the narrow face

$$\zeta = \frac{2\pi \sin i \cos\theta}{a(1-\cos^2 i \cos^2\theta)} \cos(\tfrac{1}{2}\pi \cos i \cos\theta), \tag{66.4}$$

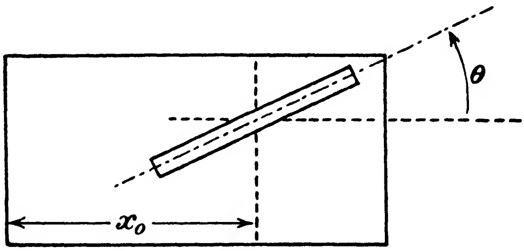


FIG. 48.

while if the slot is cut in the end of a guide as indicated in Fig. 48, which presents the terminating plate seen from the inside of the guide,

$$\zeta = -\frac{2\pi j}{a} \cos i \cos\theta \times$$
$$\times \left[\frac{\sin\dfrac{\pi x_0}{a}\cos\left(\dfrac{\pi}{2}\sin i \cos\theta\right) + \cos\dfrac{\pi x_0}{a}\left\{\sin i \cos\theta \sin\left(\dfrac{\pi}{2}\sin i\cos\theta\right) - 1\right\}}{1-\sin^2 i \cos^2\theta}\right]. \tag{66.5}$$

All of the foregoing formulae are derived on the assumption that the slot is sufficiently narrow to allow approximate treatment.

Now when the slot radiates with unit voltage amplitude at its centre, there is radiated to the left a wave of amplitude $L\zeta$, and to the right a wave of amplitude $L\zeta^*$, where $1/NL$ is the power carried down the guide by a travelling wave of unit amplitude, and N is a numerical factor depending on the particular field vector whose amplitude is to be unity, and of course on the system of units employed. If the magnetic Hertz-vector is chosen, as it is in Chapter X,

$$L = \frac{\mu_0 a}{\pi^2 bkk_g} \quad \text{and} \quad N = 4\sqrt{(\mu_0/\kappa_0)} = 480\pi. \tag{66.6}$$

It follows then that the radiation coefficients of § 1.6 are

$$f' = \frac{L\zeta^2}{K}, \qquad f = \frac{L\zeta^{*2}}{K}, \qquad g = -\frac{L|\zeta|^2}{K}, \tag{66.7}$$

which are exactly of the form (16.4) with

$$\tfrac{1}{2}\delta = \tfrac{1}{2}\pi + \arg\zeta. \tag{66.8}$$

The susceptance Y_1 is therefore given by

$$Y_1 = \tan\frac{\delta}{2} = \frac{M(i+\theta)+M(i-\theta)}{M(i+\theta)-M(i-\theta)}\cot\frac{\pi x_0}{a}, \tag{66.9}$$

which leads in the limit $\theta \to 0$, $x_0 \to \frac{1}{2}a$ to the experimental result for Y_1.

Let $r = L|\zeta|^2$, then

$$f' = \frac{re^{j\delta}}{K}, \qquad f = \frac{re^{-j\delta}}{K}, \qquad g = -\frac{r}{K}. \tag{66.10}$$

Since from (16.8) the transform of a match is $ge^{j\delta}$ on the w'-plane, the angle ψ of Fig. 46 is given by

$$\psi = -\pi+\arg K. \tag{66.11}$$

We shall call the inclined-displaced slot of length such that $\arg K = \pi$ the resonant slot, even although there is no pure resistance or conductance presented to the wave in the guide.

Now we can deduce from energy considerations an important relation between r and K which is the basis for the estimates of resistance and conductance in the cases of the resonant series- and shunt-coupled slots. Let $1/N_0$ be the power radiated by the slot outside the guide when the voltage amplitude at its centre is unity. Then considering a wave of unit amplitude incident from the left inside the guide, we have the following equation of energy:

$$\frac{1}{NL}(1-|f'|^2) = \frac{|1-g|^2}{NL}+\frac{|p'|^2}{N_0}.$$

Hence

$$1-\frac{r^2}{|K|^2} = 1-2\operatorname{Re}g+\frac{r^2}{|K|^2}+\frac{Nr}{N_0|K|^2},$$

$$\operatorname{Re}g = \frac{r}{|K|^2}\left(r+\frac{N}{2N_0}\right),$$

or

$$\operatorname{Re}K = -\left(r+\frac{N}{2N_0}\right). \tag{66.12}$$

If K is real, the slot is resonant, and we obtain for its conductance, if shunt, or its resistance, if series,

$$\frac{2g}{1-g} = \frac{4N_0 r}{N}. \tag{66.13}$$

It is found that

$$\frac{4N_0}{N} = \frac{120\pi}{73} = \frac{\text{intrinsic impedance of space}}{\text{radiation resistance of electric dipole}}. \tag{66.14}$$

The simplifying assumption under which the numerical factor occurring in equation (61.3), and similar equations, was calculated, ignored the diffraction of waves round the outside of the guide. N_0 therefore is taken to have the value corresponding to a half-wave slot cut in an infinite sheet of perfect conductor. Generally speaking, one expects a transverse slot to be better represented in this way than a longitudinal one.

The equation (66.12) shows that according to theory the circle 2 of Fig. 46 is indeed a circle.

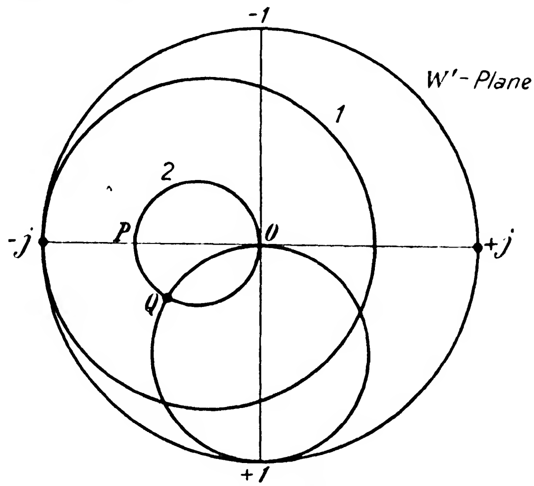


FIG. 49.

There is a very interesting and useful inclined-displaced slot which transforms a terminating match into an admittance whose real part is unity and whose susceptance is negative and therefore inductive, when $\delta = -\frac{1}{2}\pi$. The transform is represented by Q in Fig. 49.

Let us write $$K = -(r+r_0)+jq. \tag{66.15}$$

It is readily seen that in this case

$$\tan\psi = -\frac{r}{r+r_0}. \tag{66.16}$$

Hence from (66.11) $$q = r. \tag{66.17}$$

The matrix transforming from the right to the left of the slot (generator on the left) is

$$\begin{pmatrix} \dfrac{1-j+a}{a-j} & \dfrac{-j}{a-j} \\ \dfrac{-j}{a-j} & \dfrac{a-j-1}{a-j} \end{pmatrix}, \tag{66.18}$$

where $a = r_0/r$, which can easily be determined by measurement since the diameter of circle 2 in Fig. 49 is $1/(a+1)$. Further, from equations (66.13), (66.10), and (66.3) it is seen that a is approximately given by $1/\gamma$, where γ is the resistance of the pure series slot with inclination θ when the latter is small.

The phase of the radiation from the slot when the guide beyond it is terminated in a match is

$$\arg p' = \arg \zeta - \arg K = \frac{\delta}{2} - \psi - \frac{3\pi}{2}. \tag{66.19}$$

6.7. Probe-excited Slots

We have already noted the two cases of resonant slots cut parallel to the stationary lines of current flow on the guide wall and therefore unexcited by the dominant wave. These are the longitudinal slot centred on the middle of the broad face and the transverse slot centred on the middle of the narrow face. Both of these slots will be excited if, by local loading of the guide, we distort the system of current flow on the guide wall. This may be achieved by electric or magnetic means.

In the electric case a probe which has part of its length parallel to the electric vector in the guide is placed so as to produce a field distribution asymmetrical with respect to the vertical plane through the slot-axis when the broad face of the guide is horizontal. For the longitudinal slot the simplest plan is to introduce a thin vertical screw through the broad face in the cross-section containing the slot-centre. When the screw is moved to the image position with respect to the centre-line on the broad face, the phase of excitation of the slot will be reversed. The strength of coupling of course increases as the short length of screw protruding into the guide is extended. This method of coupling does not, however, allow independent adjustment of conductance and susceptance of the resultant load on the dominant wave, which adjustment is desirable in arrays of couplings.

The variable coupling should allow the load presented to the guide to be a variable pure conductance, usually small compared with the guide admittance. Suppose that a bent probe (see Fig. 50) is supported on the narrow face in the vertical plane perpendicular to the guide axis. By itself this probe presents a positive susceptance determined by the total length of the probe. It is possible to choose the length of the slot which is to be coupled so that the probe-slot combination is a pure conductance, the value of which is proportional to $\cos^2\theta_1$, where θ_1 is the angle between the electric vector in the guide and the part of the probe that can be rotated about the fixed horizontal axis through the centre of its support on the narrow face. A $\frac{1}{8}$-in. diameter probe of this type in S-band guide will handle up to 50 kw. without breakdown in the air at normal pressure.

In the magnetic case, a second slot S' cut in the guide wall may be used to excite the first. Since this will produce leakage of energy from the guide, it is necessary to cover the slot S', so that the latter really couples the guide to a cavity. Though more complicated mechanically than the probe, this device gives great flexibility of circuit, since either

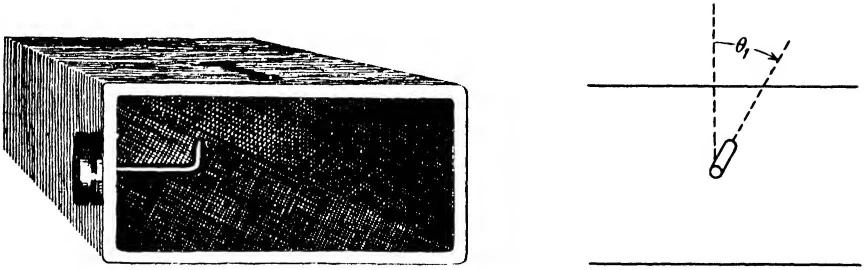


FIG. 50.

series or shunt coupling may be selected. The reactance presented by the cavity and the length of the radiating slot may be chosen to compensate each other so that the resulting load is a pure conductance.

6.8. Probe-compensated Slots

If slots are to be used as radiators in an array, it is advantageous to couple them to the guide so that no reflection of the incident energy takes place. One obvious way of doing this is to insert a probe between the slot and the generator at the point in the wave system before the slot where the conductance is unity and the susceptance negative. If less than $\frac{1}{4}\lambda$ long, the probe presents a capacitive (positive) susceptance which, because of its dependence on the length of the probe, can be adjusted to cancel the negative susceptance in the wave system at the place where the probe is to be introduced. The practical objection to this type of compensation is that it is exact only for the exact frequency which gives the proper electrical distance between probe and slot-centre. In the laboratory, however, where the frequency of the generator may be well regulated, this device is a useful one.

Probe compensation of the special inclined-displaced slot described at the end of § 6.6 does not suffer from this defect, because the susceptance to be cancelled occurs at the position of the slot. If inserted in the cross-section of the guide containing the slot-centre and placed opposite the centre of the slot and perpendicular to the broad face, a probe of the proper length will give perfect compensation at the frequency chosen and a very good approximation to it over a band of frequencies. Thus it is possible to load the guide so as to abstract energy without reflection, provided the guide is terminated in a match. The probe being placed in the broad face opposite to that in which the slot is cut, the direct mutual interaction between the electric and magnetic antennas is made almost negligible.

These considerations bring us quite naturally to discuss the loading produced by a pair of antennas in the same cross-section of the guide. Let the radiation coefficients of one antenna be f_1, f_1', g_1 and of the other f_2, f_2', g_2; then in the absence of direct mutual coupling, the radiation coefficients for the combination are $f_1+f_2, f_1'+f_2', g_1+g_2$, and the loading matrix is

$$E+\frac{1}{1-(g_1+g_2)}(G_1+G_2), \quad \text{where} \quad G = \begin{pmatrix} g & -f \\ f' & -g \end{pmatrix}. \tag{68.1}$$

Suppose that antenna 1 is a pure shunt, then

$$f_1' = f_1 = -g_1.$$

If, moreover, antenna 2 is that represented by (66.18), then

$$g_2 = \frac{1}{a+1-j}, \qquad f_2 = \frac{j}{a+1-j} = -f_2'.$$

In order that the combination be non-reflecting to the left, we require $f_1'+f_2' = 0$, and the susceptance presented by antenna 1 should be

$$X = \frac{2}{a+1}. \tag{68.2}$$

The fraction of power extracted by the radiating antenna 2 is

$$1-|1-g_1-g_2|^2 = \frac{2(a+1)}{1+(a+1)^2}. \tag{68.3}$$

On the other hand, if antenna 2 is pure series,

$$f_2' = f_2 = g_2,$$

and it is easily seen that when $g_2 = g_1$ the pair of antennas will transform a match unchanged in either direction. The transforming matrix (right to left) is then

$$\begin{pmatrix} 1+\dfrac{2g_1}{1-2g_1} & 0 \\ 0 & 1-\dfrac{2g_1}{1-2g_1} \end{pmatrix}. \tag{68.4}$$

The equivalence of g_1 and g_2 requires that the admittance (α) of the shunt antenna shall equal the impedance (γ) of the series antenna. The matrix (68.4) then becomes

$$\begin{pmatrix} 1+\dfrac{\alpha}{1-\frac{1}{2}\alpha} & 0 \\ 0 & 1-\dfrac{\alpha}{1-\frac{1}{2}\alpha} \end{pmatrix}. \tag{68.5}$$

6.9. Pairs of Slots

If two half-wave slots are cut at right angles to each other intersecting at their common centre (Fig. 51), the following facts can be established when the guide is matched beyond the slots:

(*a*) When one of the slots lies along the centre-line on the broad face,

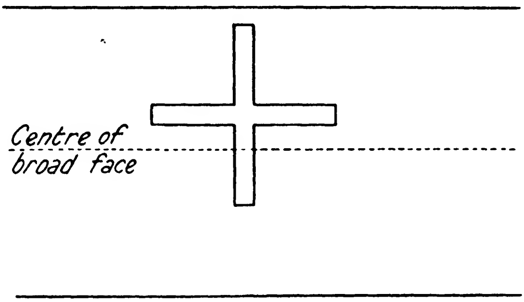


FIG. 51.

only the other slot is effective as a radiator. The radiation is electrically polarized parallel to the length of the guide. This shows that the mutual coupling of these slots is negligibly small.

(*b*) As the cross is turned about its centre both slots are excited and consequently the direction of polarization is not substantially altered. The cross presents a pure series load to the guide.

(*c*) Suppose now that the centre of the cross is displaced from the centre of the broad face, one slot being kept parallel to the guide axis. The transverse slot is a series element and therefore excited in quadrature with the other which is a shunt element. One expects that this device should radiate elliptically polarized waves normal to the guide. This is found to be the case.

In general the radiation pattern of the pair of slots is approximately that of two half-wave magnetic radiators at right angles to each other with their complex amplitudes in the ratio $\zeta_1:\zeta_2$ in accordance with (66.1), the guide termination being matched. For circular polarization we require one arm of the cross parallel to the guide axis and the centre distant x_0 from one edge, where

$$\tan\frac{\pi x_0}{a} = \frac{M(i)}{M(\frac{1}{2}\pi-i)}. \tag{69.1}$$

If the cross is moved to its image with respect to the centre-line on the broad face, the sense of circular polarization is reversed.

The effect of terminating the guide by a movable reflecting plunger can easily be deduced on the basis of the principles already adduced. Radiation from the longitudinal arm of the cross is effectively sup-

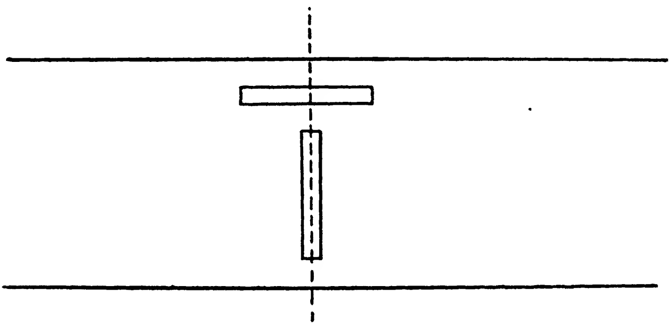


FIG. 52.

pressed when the plunger is $\frac{1}{2}\lambda_g$ from the centre of the cross and from the transverse when the plunger is $\frac{1}{4}\lambda_g$ from the centre.

Another combination of a pair of slots, one longitudinal shunt, the other transverse series, with their centres in the same transverse section of the guide, is shown in Fig. 52. If $\alpha = \gamma = 2$, then the pair of slots presents a match to the generator and all the energy is radiated, none passing the slots in the guide, irrespective of the termination of the latter.

6.10. General Remarks on Apertures in the Guide Wall

The rectangular slot is a particular type of aperture in the wall of the guide. Imagine the slot deformed but still retaining a major and a minor axis of symmetry, the former associated with the length and the latter with the width. So long as the deformation is continuous without changing the connectivity of the aperture, we obtain essentially the same behaviour as with the half-wave slot near its first resonance: that is, its resonance of lowest frequency. Dimensions for the aperture can be found to produce resonance. Such an aperture will load the guide just like a resonant rectangular slot. In practice, on account of the ease of cutting by means of an end-mill, the rectangular slot is replaced by one with semicircular ends. Dumb-bell slots have also been used in the laboratory; the dimensions of such slots for *S*-band are shown in Fig. 53 and the accompanying table. The larger the circles at the ends, the narrower the gap required in the centre. The deformed slot now resembles a conventional resonant circuit in that in the circular ends of the slot the energy is mainly stored in the magnetic field as in inductance, while in the gap the storage is electrical as in capacity.

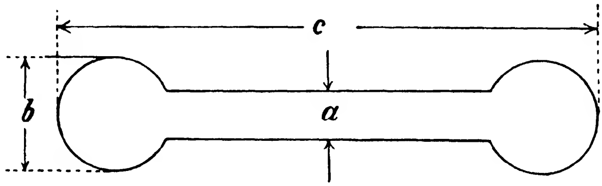


Fig. 53.

Slot	*a*	*b*	*c*
1	$\frac{1}{4}''$	$\frac{3}{8}''$	1·90″
2	$\frac{1}{4}''$	$\frac{1}{2}''$	1·77″
3	$\frac{1}{8}''$	$\frac{3}{8}''$	1·69″
4	$\frac{1}{8}''$	$\frac{1}{2}''$	1·59″
5	$\frac{3}{16}''$	$\frac{1}{4}''$	2·00″

Dimensions of dumb-bell slot, resonant at 10·7 cm., on end of standard *S*-band guide.

Slots may be covered with dielectric so as to close the guide in which they are cut. The main effect of this is to increase the capacity referred to above and thus reduce the length of the slot for resonance.

A slot of length λ, if not driven near its centre, will tend to oscillate in two halves in opposed phase with a node of voltage at its centre, just like the same length of transmission line. If the Q of the oscillation is sufficiently small, this may not be the only oscillation present: there may be superposed on it the distribution without a node in the centre like the half-wave slot. Each of these two different amplitude distributions has its own resonant frequency; hence if the slot is excited by radiation in a guide with frequency close to one of these, that particular form will predominate in the oscillation of the aperture, always supposing that the couplings of the two modes to the waves in the guide are of the same order of magnitude.

More interesting than the long slot in this connexion is an aperture such as a sufficiently large circular or square hole cut in the broad face off-centre. There are now two natural modes with frequencies close to each other, and we may dismiss the somewhat artificial condition introduced above. It is easy to demonstrate experimentally that the radiation from the hole corresponds to different types of oscillation of the current system about it, depending on the mode of excitation of the hole. Thus, suppose a reflecting plunger is used to terminate the guide. Let the plunger be placed $\frac{1}{2}\lambda_g$ beyond the centre of the hole so that the transverse current near the centre of the longitudinal side of

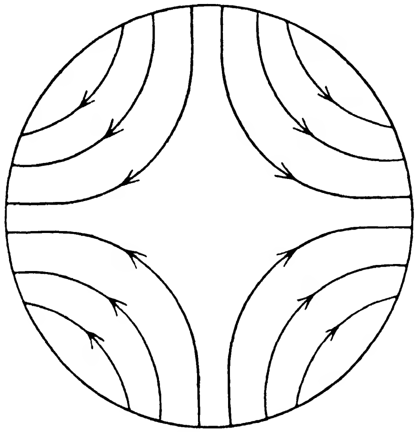


FIG. 54.

the hole would in its absence be reduced to zero. The hole oscillates with electric force parallel to the guide axis like a transverse series slot. If the plunger is moved $\frac{1}{4}\lambda_g$, the other simple mode with transverse electric force predominates, as it does in longitudinal shunt slots. Thus the polarization of the radiation from the hole, and of course also its radiation pattern, depends on the position of the plunger inside the guide. Moreover, these two oscillations have different phases with respect to the waves inside the guide. Corresponding effects can be shown for a circular hole; the two simplest modes correspond to the H_{11}-modes of the transverse field distribution in a guide of circular cross-section. If the hole is sufficiently large, it is possible also to excite the H_{21}-mode of oscillation, the notation again being based on analogy with the names of the circular guide patterns. The electric field distribution in the plane of the hole is shown in Fig. 54.

The foregoing observations are founded on quite rough experiments, so the question should be investigated further.

VII

GUIDE COUPLING BY SLOTS

7.1. The Simple Laws

In microwave practice it is often required to divide wave-guide paths so that energy may be distributed according to a definitely prescribed law, and it frequently occurs that wave guides having different orientations must be joined in the minimum space. In order to understand the practical devices by which these results can be achieved it is necessary to know the laws of the coupling of guides by means of slots or holes cut in the metal sheet which forms part of the wall or termination of both guides.

The slot, once excited, radiates into both guides. Depending on the aspect of the slot in each guide, the type of coupling, and hence the law of impedance or admittance transfer from one guide to the other, will change from one disposition of slot and guides to another. The method used by Stevenson in treating the slot radiating to space can equally well be applied to the coupling of guides. The difficulty of diffraction outside the guide is now removed, but it is replaced by another, due to the finite wall thickness and width of the slot which must be represented if the theory is to yield results of practical value. Accordingly we shall present the method of representation adopted in connexion with the discovery of these laws. In any practical case, where the shape of the slot and of the neighbouring wall may be distorted in order to achieve low energy loss at the junction and high-power transfer through it, it is possible to use the forms presented below and to determine by direct measurement the parameters introduced in the symbolism.

The simplest types of guide coupling by a single slot are classified by the modes of coupling of the corresponding slot radiator cut in the same aspect with respect to each of the coupled guides. Thus, suppose the coupling slot is transverse in the broad face of both guides, which must therefore be parallel; this is called series-series coupling, and an example of it is depicted in Fig. 55. If the slot is longitudinal in the two parallel guides, they are coupled shunt-shunt; an example is shown in Fig. 56. The simplest case of dissimilar aspect is when the slot is transverse to guide (1) and longitudinal in guide (2); the coupling is then series-shunt. In this case the guides must be at right angles to each other, as shown in Fig. 57. It is found that the coupling depends on the length and width of the slot and on the thickness of metal between the adjacent inside surfaces of the two guides. It is likewise determined in part by the disposition of the slot with respect to each of the guides coupled by means of it.

Let guide (2) be terminated on each side of the coupling slot and let guide (1) be fed from an oscillator on one side and be terminated on the other side so as to produce either (*a*) a short-circuit at the centre of the slot, if it is series to guide (1), or (*b*) an open-circuit at the centre

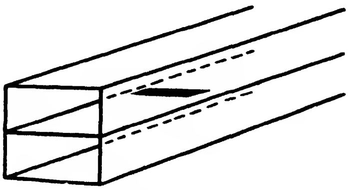


Fig. 55.

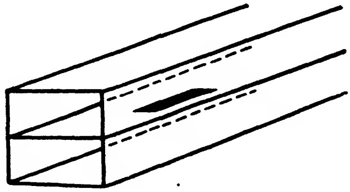


Fig. 56.

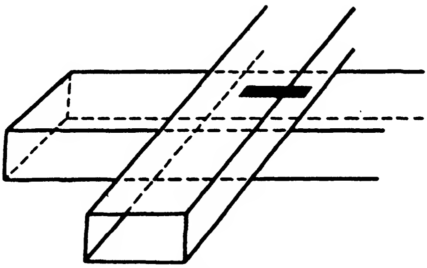


Fig. 57.

of the slot, if it is shunt to guide (1). The two impedances seen from the slot in guide (2) will add by the law of *series* combination if the slot is series-coupled to guide (2), and by the law of *shunt* combination if the slot is shunt-coupled to guide (2).

Let Z', Y' be respectively the total impedance or admittance in guide (2) as seen from the point of coupling which is the centre of the slot, and let Z, Y be the corresponding input impedance or admittance to guide (1) terminated in the way indicated above. The input impedance is reckoned at the point of coupling. Provided that the length of the coupling slot is properly adjusted for the frequency used, for the aspect and width of the slot, and for the dimensions of the guide cross-section, we have the following simple laws of impedance and admittance transfer through the slot at its centre, when the wall thickness is negligible.

I. Series-series: $$Z = \frac{Z'}{n_1}, \tag{71.1}$$

where n_1 is a numerical constant which is equal to 1 for identical guides coupled in similar aspect.

II. Shunt-shunt: $$Y = \frac{Y'}{n_2}, \tag{71.2}$$

where n_2 is a numerical constant which is unity for identical guides coupled in similar aspect.

III. Series-shunt: $$Z = \frac{Y'}{n_3}, \tag{71.3}$$

where n_3 is a numerical constant which may be varied by changing the displacement of the coupling slot from the central line of guide (2).

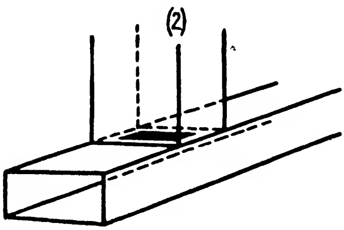


FIG. 58.

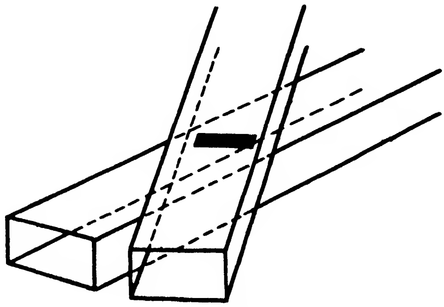


FIG. 59.

Not merely may series coupling be achieved by means of a slot transverse to one of the guides, it may also be achieved for guide (2), for example, when one end of this guide abuts on guide (1), as shown in Fig. 58. In this case the law of impedance transfer (71.1) holds, but Z' now stands for the single terminating impedance in guide (2).

Inclined slots may also be used to achieve these three different types of coupling, but the possibilities are somewhat restricted in practice by the difficulty of accommodating slots of sufficient width to transfer large power. Thus in series-series coupling the slot-centre must lie on the centre-line of the broad faces of the guides which touch one another, as in Fig. 59, or it may lie on the centre-line of one guide and in one of the ends of the other, as in Fig. 60. Shunt-shunt coupling with an inclined slot may be achieved with the narrow faces placed in contact (Fig. 61). Series-shunt coupling with an inclined slot may be accomplished with the narrow face of the shunt-coupled guide in contact with the broad face of the series-coupled one, the centre of the coupling slot lying in the centre of the latter face, as shown in Fig. 62. An important case of shunt-series coupling is shown in Fig. 63. The axes of the guides and also their broad faces are perpendicular to each other.

Guide (1), which carries the inclined coupling slot, fits into the recessed broad faces of guide (2), so that mechanical as well as electrical junction is effected at the coupling. It is unnecessary to place a conducting termination in the series-coupled guide (2) behind the coupling slot. In standard S-band guide, the maximum inclination allowable with a slot $\frac{1}{4}$ in. wide is 40°.

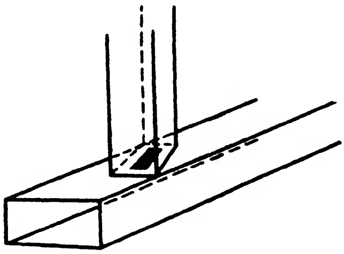


Fig. 60.

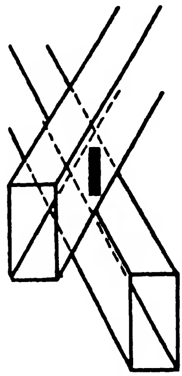


Fig. 61.

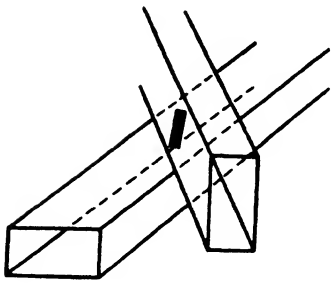


Fig. 62.

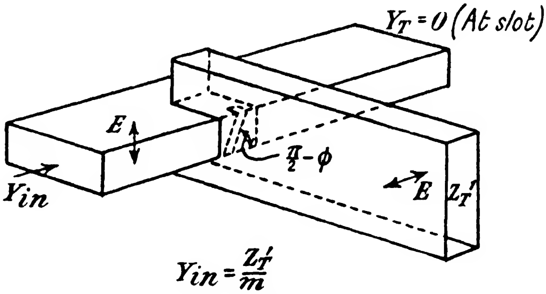


Fig. 63.

7.2. Radiation Coefficients in Guide Coupling

In order to explain the method of radiation coefficients by which the laws of guide coupling may be deduced, we shall consider first the simple case of series-series coupling of similar guides in similar aspect to the coupling slot. When a wave of amplitude A_1' is incident from the left in guide (1), the coupling slot radiates a wave gA_1' to the left and $-gA_1'$ to the right in guide (1) and $-gA_1'$ to the left and gA_1' to the right in guide (2). These are represented in Fig. 64. The senses assumed correspond to the relative voltage due to a series slot cut in both guides. The equations expressing the outgoing wave-amplitudes in terms of the incoming waves are given in (72.1), where subscripts refer to the guides, primed letters to the left of the slot, unprimed to the right, and as usual A denotes a wave travelling to the right, B a wave to the left. Thus

$$\begin{aligned} A_1 &= (1-g)A_1'+gA_2'+gB_1-gB_2, \\ A_2 &= gA_1'+(1-g)A_2'-gB_1+gB_2, \\ B_1' &= gA_1'-gA_2'+(1-g)B_1+gB_2, \\ B_2' &= -gA_1'+gA_2'+gB_1+(1-g)B_2. \end{aligned} \tag{72.1}$$

In order to determine the impedance relations we put $B = Aw$, thus reducing (72.1) to a set of four homogeneous equations in the incoming

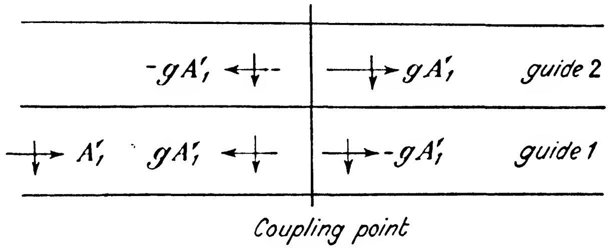


FIG. 64.

wave-amplitudes B_1, B_2, A_1', A_2'. The determinant of the coefficients must therefore vanish; i.e.

$$\begin{vmatrix} 1-g & g & g-1/w_1 & -g \\ g & 1-g & -g & g-1/w_2 \\ g-w_1' & -g & 1-g & g \\ -g & g-w_2' & g & 1-g \end{vmatrix} = 0. \tag{72.2}$$

From this we obtain

$$\frac{1-2g}{g} = \frac{(w_2'-1)(1-w_2)}{w_2'-w_2}+\frac{(w_1'-1)(1-w_1)}{w_1'-w_1}. \tag{72.3}$$

We now substitute $w = \dfrac{Z-1}{Z+1}$, all impedances being reckoned positive looking to the right, and we have

$$\frac{1-2g}{g} = \frac{1}{Z_2-Z_2'}+\frac{1}{Z_1-Z_1'}. \tag{72.4}$$

Since $Z_1 = 0$, and $Z_2-Z_2' = Z'$, we obtain

$$\frac{1}{Z_1'} = \frac{1}{Z'}-\frac{1-2g}{2g}. \tag{72.5}$$

As we shall see, when the slot is properly tuned, $g = \frac{1}{2}$ and we have the special case of the law (71.1).

7.3. The General Case of Single Slot Coupling

Let the slot be excited by a wave of unit amplitude from the left in guide (1), all the other incoming amplitudes being zero, then in place of (66.1) which applies to the slot radiating to space we find for the voltage amplitude at the centre of the slot

$$p_1' = \frac{\zeta_1}{K_{12}}, \tag{73.1}$$

where K_{12} plays the part of K in the previous discussion, and ζ_1 is the value of ζ for guide (1). Similarly for excitation by unit amplitude from the right

$$p_1 = \frac{\zeta_1^*}{K_{12}}.$$

When the incoming wave is of unit amplitude in guide (2), the corresponding voltage amplitudes at the slot are

$$p_2' = \frac{\zeta_2}{K_{12}} \quad \text{and} \quad p_2 = \frac{\zeta_2^*}{K_{12}}. \tag{73.2}$$

Now when the voltage amplitude at the centre of the slot is unity, there are radiated waves of amplitude $L_1\zeta_1$ and $L_2\zeta_2$ to the left in the respective guides and to the right $L_1\zeta_1^*$ and $L_2\zeta_2^*$. Thus if the wave A_1' is incident on the coupling in guide (1) from the left, the wave radiated to the left in guide (2) is $\dfrac{L_2\zeta_2\zeta_1}{K_{12}}A_1'$ and to the right is $\dfrac{L_2\zeta_2^*\zeta_1}{K_{12}}A_1'$, and so on. The equations corresponding to (72.1) are therefore

$$\begin{aligned}
B_1' &= \frac{L_1\zeta_1^2}{K_{12}}A_1'+\left(1+\frac{L_1|\zeta_1|^2}{K_{12}}\right)B_1+\frac{L_1\zeta_1\zeta_2}{K_{12}}A_2'+\frac{L_1\zeta_1\zeta_2^*}{K_{12}}B_2,\\
A_1 &= \left(1+\frac{L_1|\zeta_1|^2}{K_{12}}\right)A_1'+\frac{L_1\zeta_1^{*2}}{K_{12}}B_1+\frac{L_1\zeta_1^*\zeta_2}{K_{12}}A_2'+\frac{L_1\zeta_1^*\zeta_2^*}{K_{12}}B_2,\\
B_2' &= \frac{L_2\zeta_2\zeta_1}{K_{12}}A_1'+\frac{L_2\zeta_2\zeta_1^*}{K_{12}}B_1+\frac{L_2\zeta_2^2}{K_{12}}A_2'+\left(1+\frac{L_2|\zeta_2|^2}{K_{12}}\right)B_2,\\
A_2 &= \frac{L_2\zeta_2^*\zeta_1}{K_{12}}A_1'+\frac{L_2\zeta_2^*\zeta_1^*}{K_{12}}B_1+\left(1+\frac{L_2|\zeta_2|^2}{K_{12}}\right)A_2'+\frac{L_2\zeta_2^{*2}}{K_{12}}B_2.
\end{aligned} \tag{73.3}$$

Let

$$\frac{K_{12}}{L_1\zeta_1} = s_1 \quad \text{and} \quad \frac{K_{12}}{L_1\zeta_1^*} = s_1', \tag{73.4}$$

then the relation between the circle diagram variables w is

$$\begin{vmatrix}
\zeta_1-s_1w_1' & \zeta_1^*+s_1 & \zeta_2 & \zeta_2^*\\
\zeta_1+s_1' & \zeta_1^*-s_1'/w_1 & \zeta_2 & \zeta_2^*\\
\zeta_1 & \zeta_1^* & \zeta_2-s_2w_2' & \zeta_2^*+s_2\\
\zeta_1 & \zeta_1^* & \zeta_2+s_2' & \zeta_2^*-s_2'/w_2
\end{vmatrix} = 0, \tag{73.5}$$

from which on expansion we obtain

$$L_1|\zeta_1|^2\left[\frac{e^{-j\phi_1}w_1 w_1'+2w_1+e^{j\phi_1}}{w_1'-w_1}\right]+L_2|\zeta_2|^2\left[\frac{e^{-j\phi_2}w_2 w_2'+2w_2+e^{j\phi_2}}{w_2'-w_2}\right] = K_{12}, \tag{73.6}$$

where $\phi_1 = 2\arg\zeta_1, \qquad \phi_2 = 2\arg\zeta_2.$

The equation (73.6) may be arranged in more convenient form as follows:

$$L_1|\zeta_1|^2\frac{(w_1 e^{-j\phi_1}+1)(w_1'+e^{j\phi_1})}{w_1'-w_1}+L_2|\zeta_2|^2\frac{(w_2 e^{-j\phi_2}+1)(w_2'+e^{j\phi_2})}{w_2'-w_2}$$
$$= K_{12}+L_1|\zeta_1|^2+L_2|\zeta_2|^2. \tag{73.7}$$

We shall now show that the right-hand side of (73.7) is equal to the imaginary part of K_{12}.

Imagine a wave of unit amplitude incident from the left; the outgoing waves are then given by (73.3) when B_1, A_2', B_2 are put equal to zero and $A_1' = 1$. The energy equation reads

$$\frac{1}{L_1}\left|\frac{L_1\zeta_1^2}{K_{12}}\right|^2+\frac{1}{L_1}\left|1+\frac{L_1|\zeta_1|^2}{K_{12}}\right|^2+\frac{1}{L_2}\left|\frac{L_2\zeta_2\zeta_1}{K_{12}}\right|^2+\frac{1}{L_2}\left|\frac{L_2\zeta_2^*\zeta_1}{K_{12}}\right|^2 = \frac{1}{L_1},$$

i.e.
$$2|\zeta_1|^2\left(\frac{L_1|\zeta_1|^2+L_2|\zeta_2|^2}{|K_{12}|^2}+\operatorname{Re}\frac{1}{K_{12}}\right) = 0,$$

so
$$\operatorname{Re}K_{12} = -(L_1|\zeta_1|^2+L_2|\zeta_2|^2). \tag{73.8}$$

This proves the result.

The tuning of the slot by adjusting its length so as to obtain the simple coupling laws corresponds to making K_{12} real: then the right-hand member of (73.7) vanishes. Further, the assertion made about the value of g in equation (72.5) is seen to be justified. The three cases of simple coupling are:

I. *Series-series*: $\qquad \phi_1 = \phi_2 = \pi.$

$$L_1|\zeta_1|^2\frac{(1-w_1)(w_1'-1)}{w_1'-w_1}+L_2|\zeta_2|^2\frac{(1-w_2)(w_2'-1)}{w_2'-w_2} = 0. \tag{73.9}$$

II. *Shunt-shunt*: $\qquad \phi_1 = \phi_2 = 0.$

$$L_1|\zeta_1|^2\frac{(w_1+1)(w_1'+1)}{w_1'-w_1}+L_2|\zeta_2|^2\frac{(w_2+1)(w_2'+1)}{w_2'-w_2} = 0. \tag{73.10}$$

III. *Shunt-series*: $\qquad \phi_1 = 0, \qquad \phi_2 = \pi.$

$$L_1|\zeta_1|^2\frac{(w_1+1)(w_1'+1)}{w_1'-w_1}+L_2|\zeta_2|^2\frac{(1-w_2)(w_2'-1)}{w_2'-w_2} = 0. \tag{73.11}$$

Since
$$\frac{(1-w_1)(w_1'-1)}{w_1'-w_1} = \frac{2}{Z_1-Z_1'} \tag{73.12}$$

and
$$\frac{(w_1+1)(w_1'+1)}{w_1'-w_1} = \frac{2}{Y_1-Y_1'}, \tag{73.13}$$
it follows that the constants n_1, n_2, n_3 in (71.1) et seq. are all given by
$$n = \frac{L_2|\zeta_2|^2}{L_1|\zeta_1|^2}. \tag{73.14}$$
The values of ζ_1 and ζ_2 can be obtained from the equations given in § 6.6. Thus, for example, when the coupling slot is cut in the broad face of one guide and a somewhat larger registering slot is cut in the

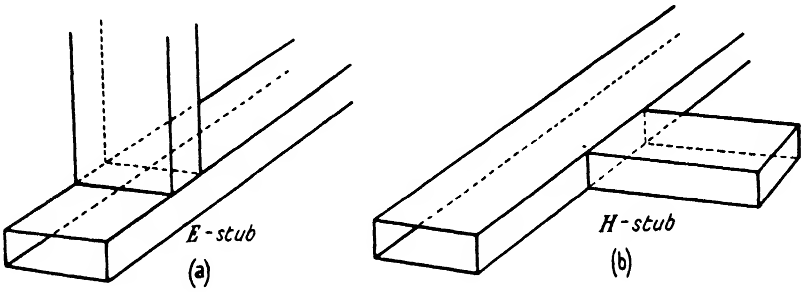


Fig. 65.

broad face of the other guide, one applies equation (66.3). The ratio L_2/L_1 is
$$\frac{L_2}{L_1} = \frac{a_2 b_1 \cos i_1}{a_1 b_2 \cos i_2}. \tag{73.15}$$
When the slot is not of resonant length, the coupling laws are:

I. *Series-series*: $$\frac{1}{Z} = \frac{n_1}{Z'} + j\alpha_1; \tag{73.16}$$

II. *Shunt-shunt*: $$\frac{1}{Y} = \frac{n_2}{Y'} + j\alpha_2; \tag{73.17}$$

III. *Series-shunt*: $$\frac{1}{Z} = \frac{n_3}{Y'} + j\alpha_3, \tag{73.18}$$

where $j\alpha_1$, $j\alpha_2$, $j\alpha_3$ are given by $\dfrac{-\operatorname{Im} K_{12}}{2L_1|\zeta_1|^2}$ and the appropriate values are given to K_{12}, L_1, and ζ_1 for the two similar guides in the case concerned.

It is found experimentally that the form (73.16) applies to the E-type coupling of guides and (73.18) to the H-type coupling. These are shown in Fig. 65.

7.4. Some Simple Consequences of the Laws

The reactance introduced by the non-resonant slot is in shunt with the impedance in a series-coupled guide, and in series with the admittance in a shunt-coupled guide. In the series-shunt case the coupling acts as a transformer.

The resonant slot couplings may be conveniently applied in the design and construction of wave-guide circuits. Since the coupling of similar guides by a resonant slot in the same aspect to both guides allows the transfer of impedance in the series case, and admittance in the shunt case, without change, a means is provided for coupling different loads to a wave guide without requiring at each coupling operation mechanical work on the guide in question. In particular, short-circuits and open-circuits may be introduced without any plunger in the main guide. It should be noted, however, that in the coupling of reactances and susceptances, particularly when transformation takes place with a high ratio, the ohmic losses at the slot may not be negligible due to the very high currents flowing to it.

In practice the shunt-series coupling is of special importance. It has been found that the thickness of guide wall, which is relatively greater in guides intended for higher frequency microwaves, introduces a departure from the simple laws (73.16)–(73.18). The form which fitted the experimental facts with standard S-band guide is

$$\frac{1}{Y} = \frac{m}{Z'+jR} - j\alpha, \tag{74.1}$$

where α may be reduced to zero by tuning the slot-length, R may not: it depends on the thickness of metal between the adjacent inside faces of the guides being coupled by the slot. R may be tuned out by the appropriate series reactance in guide (2). It may be remarked that the form (74.1) was obtained with guide (1) terminated on one side by an open-circuit, and guide (2) terminated on one side by a short-circuit at the position of the slot-centre (mod $\frac{1}{2}\lambda_g$).

7.5. Directive Antenna Coupling of Guides

Suppose that the second guide is coupled by a pair of antennas excited by the first. If one of the antennas is shunt-coupled and the other series-coupled to guide (2) at the same position in that guide, then the radiation from the two antennas will be greater in that direction along guide (2) in which the fields are in phase. When complete cancellation occurs in the direction which was formerly that of weak radiation, all the energy is transmitted in the other. We shall work out the case of coupling by a pair of slots each in the same aspect to the two similar guides, one being shunt-shunt, the other series-series.

Let ζ_1, ζ_2, K_{12} be the parameters already introduced for one slot, and ξ_1, ξ_2, J_{12} be the corresponding parameters for the second. Adding the waves from the pair of slots, we obtain in place of the first of equations (73.3)

$$B_1' = \left(\frac{L_1\zeta_1^2}{K_{12}}+\frac{L_1\xi_1^2}{J_{12}}\right)A_1'+\left(1+\frac{L_1|\zeta_1|^2}{K_{12}}+\frac{L_1|\xi_1|^2}{J_{12}}\right)B_1+$$
$$+\left(\frac{L_1\zeta_1\zeta_2}{K_{12}}+\frac{L_1\xi_1\xi_2}{J_{12}}\right)A_2'+\left(\frac{L_1\zeta_1\zeta_2^*}{K_{12}}+\frac{L_1\xi_1\xi_2^*}{J_{12}}\right)B_2. \quad (75.1)$$

There are four such relations in all.

Now suppose that $\arg\zeta_1 = 0$ (shunt), $\arg\xi_1 = \frac{1}{2}\pi$ (series). Since the guides are in similar aspect to each slot, $\xi_1 = \xi_2$, $\zeta_1 = \zeta_2$, and when the slots are tuned

$$K_{12} = -2L_1|\zeta_1|^2, \qquad J_{12} = -2L_1|\xi_1|^2.$$

On substitution of these values we obtain

$$B_1' = (\tfrac{1}{2}-\tfrac{1}{2})A_1'+(1-\tfrac{1}{2}-\tfrac{1}{2})B_1+(\tfrac{1}{2}-\tfrac{1}{2})A_2'+(-\tfrac{1}{2}-\tfrac{1}{2})B_2 = -B_2. \quad (75.2)$$

Similarly

$$A_1 = -A_2', \qquad B_2' = -B_1, \quad \text{and} \quad A_2 = -A_1'. \quad (75.3)$$

This result shows that a wave entering one guide is switched completely at the slots to emerge from the other, there being no reflection from the junction.

7.6. The Resonant Coupling Paradox

When the slots of the directive antenna pair are not situated with their centres in the same guide cross-section, the problem of computing the coupling is somewhat more difficult than when they are at the same position, for now each antenna scatters the radiation from the other.

Let
$$W = \frac{1}{e^{-j\phi}w+1}, \quad (76.1)$$

then the coupling equation is of the form derived from (73.7)

$$\frac{A_1}{W_1'-W_1''}+\frac{B_1}{W_2'-W_2''} = jC_1, \quad (76.2)$$

where, as before, the primed variables refer to the left of the whole coupling, but the double-primed variables now refer to the right of the left-hand coupling slot. A_1, B_1, and C_1 are constants determined by the position and length of the latter. Let T denote the operation of transferring from the position of the first coupling slot (on the left) to that of the second. The coupling equation for the second slot is

$$\frac{A_2}{T_1 W_1''-W_1}+\frac{B_2}{T_2 W_2''-W_2} = jC_2, \tag{76.3}$$

the unprimed variables referring to the right of the second slot-centre.

Now equation (76.2) relates W_1'' to W_2'' by a bilinear transformation M'' and equation (76.3) relates $T_1 W_1''$ to $T_2 W_2''$ by a second bilinear transformation M. M'' and M can be represented by their appropriate matrices, as can also T_1 and T_2. We therefore require for consistency

$$M = T_2 M'' T_1^{-1}. \tag{76.4}$$

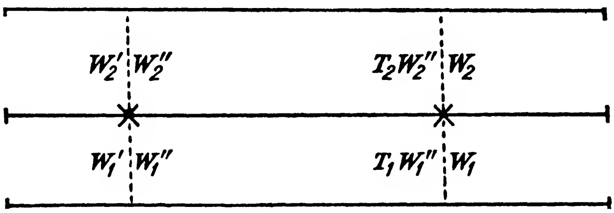


FIG. 66.

In general, this means that the W's on each side of the coupling are related by four equations. In the case of resonant slot couplings these actually impose conditions on the constants A_1, B_1, A_2, and B_2 and the spacing which are supposed given! Indeed they require $A_1 = A_2$, $B_1 = B_2$, and if the velocity of propagation is different in the two guides, the slots must be spaced so that electrical distances in the two guides differ by an integral multiple of 2π. If these conditions are not met, then when energy is fed to the first guide no coupling to the second guide is possible.

The explanation of these apparently paradoxical conclusions lies in this. Suppose we regard equations (73.3) as a set from which the primed wave-amplitudes are to be determined in terms of the unprimed wave-amplitudes. It is easily shown that the determinant of the coefficients is

$$1+\frac{L_1|\zeta_1|^2+L_2|\zeta_2|^2}{K_{12}},$$

which vanishes when the slot is resonant and K_{12} real because of (73.8). It is therefore not possible in this case to set up the fourth-order coupling matrix by which one can transform wave-amplitudes from the right to the left of the coupling slot. The resonant slot has the same effect with respect to the guides as that imposed by a parallel-resonant circuit as a common series element coupling two transmission lines. The latter forces the two lines to carry equal and opposite currents at the point of coupling. Thus the number of independent variables required to describe the coupled line transmission is reduced from four to three.

7.7. Multiple Propagation and Coupling—Matrix Method

In the non-resonant case, let jq be the imaginary part of K_{12}, then the coupling matrix by which we transform the column vector

$$\begin{pmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{pmatrix} \quad \text{into} \quad \begin{pmatrix} A_1' \\ B_1' \\ A_2' \\ B_2' \end{pmatrix}$$

is

$$E-\frac{1}{jq}\begin{pmatrix} L_1|\zeta_1|^2\begin{pmatrix} 1 & e^{-j\phi_1} \\ -e^{j\phi_1} & -1 \end{pmatrix} & L_1|\zeta_1||\zeta_2|\begin{pmatrix} e^{j\frac{1}{2}(\phi_2-\phi_1)} & e^{-j\frac{1}{2}(\phi_2+\phi_1)} \\ -e^{j\frac{1}{2}(\phi_1+\phi_2)} & -e^{j\frac{1}{2}(\phi_1-\phi_2)} \end{pmatrix} \\ L_2|\zeta_1||\zeta_2|\begin{pmatrix} e^{j\frac{1}{2}(\phi_1-\phi_2)} & e^{-j\frac{1}{2}(\phi_1+\phi_2)} \\ -e^{j\frac{1}{2}(\phi_1+\phi_2)} & -e^{j\frac{1}{2}(\phi_2-\phi_1)} \end{pmatrix} & L_2|\zeta_2|^2\begin{pmatrix} 1 & e^{-j\phi_2} \\ -e^{j\phi_2} & -1 \end{pmatrix} \end{pmatrix}. \tag{77.1}$$

The three special cases are

I. *Series-series*:

$$E-\frac{2}{jq}\begin{pmatrix} L_1|\zeta_1|^2U_2 & L_1|\zeta_1||\zeta_2|U_2 \\ L_2|\zeta_1||\zeta_2|U_2 & L_2|\zeta_2|^2U_2 \end{pmatrix}. \tag{77.2}$$

II. *Shunt-shunt*:

$$E-\frac{2}{jq}\begin{pmatrix} L_1|\zeta_1|^2U_1 & L_1|\zeta_1||\zeta_2|U_1 \\ L_2|\zeta_1||\zeta_2|U_1 & L_2|\zeta_2|^2U_1 \end{pmatrix}. \tag{77.3}$$

III. *Shunt-series*:

$$E-\frac{2}{jq}\begin{pmatrix} L_1|\zeta_1|^2U_1 & jL_1|\zeta_1||\zeta_2|U_1\,U_2 \\ -jL_2|\zeta_1||\zeta_2|U_2\,U_1 & L_2|\zeta_2|^2U_2 \end{pmatrix}. \tag{77.4}$$

When q is small these matrices represent the coupling of two transmission lines by a third of electrical length θ which in cases I and II is close to 0 (mod π) and in case III close to $\frac{1}{2}\pi$.

So far as concerns the calculation of impedance transfer these matrices are very useful, for we can pass to the limit $q \to 0$ (resonant case) *after* establishing the general impedance relations. The method may evidently be extended to multiple guide coupling. For instance, for three guides the matrix would be of sixth order. By using the fundamental matrices U_1, U_2 and their products to write this sixth-order matrix formally as of third order, it is possible to calculate quite readily the impedance (or admittance) relations, and the voltages and currents in quite complicated circuits. The same principles may be applied to the coupling of different waves in the same guide. It is necessary to introduce the composite matrix corresponding to (11.9), namelv in the case of three guides

$$\begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{pmatrix},$$

in order to transform from one point to another, there being no discontinuity in propagation between these points.

7.8. Resonant Slot in H_{11}—Circular Guide

The loading of a circular guide by a resonant slot cut in quite general aspect to the wave will be represented in a fourth-order matrix, provided that the frequency of the radiation is between the cut-off frequencies for E_{01}- and H_{11}-waves. Since there are two independent polarizations possible in the latter which constitutes a case of degenerate propagation, there are effectively two independent waves in the guide. If only one is incident, the slot will in general also generate waves of the other polarization in both directions along the guide.

Let ϕ_0 denote the angle between the direction of the electric vector through the centre of the circular section in the linearly polarized H_{11}-wave and the radius (a) to the centre of the slot, which is supposed cut so that it would be straight were the cylindrical guide wall rolled out flat. Let θ denote the inclination of the slot-axis to the cylinder generator through its centre, reckoned positive when the end of the slot farther from the microwave source is rotated from the line $\theta = 0$ in the direction of increasing ϕ. Then with respect to the waves linearly polarized along $\phi = 0$ on the guide axis, it can be shown for this slot that

$$\zeta_1 = k[e^{j(\phi_0-\frac{1}{2}\pi)}F(\theta)+e^{-j(\phi_0-\frac{1}{2}\pi)}F(-\theta)], \tag{78.1}$$

where

$$F(\theta) = \frac{\{H_{11}^2\cos\theta-(k_{11}/a)\sin\theta\}\sin[(\pi/2k)\{k_{11}\cos\theta+(1/a)\sin\theta\}]}{\{k_{11}\cos\theta+(1/a)\sin\theta\}^2-k^2}. \tag{78.2}$$

$H_{11} = 2\pi/\lambda_c$, $k_{11} = 2\pi/\lambda_g$, and λ_c is the cut-off wavelength for the H_{11}-waves in the guide of radius a.

For radiation polarized at right angles to that just treated,

$$\zeta_2 = k[e^{j\phi_0}F(\theta)+e^{-j\phi_0}F(-\theta)]. \tag{78.3}$$

When the incident radiation is circularly polarized in the counter-clockwise sense

$$\zeta_+ = \tfrac{1}{2}(\zeta_1+j\zeta_2) = je^{-j\phi_0}F(-\theta) \tag{78.4}$$

and for circular polarization in the clockwise sense

$$\zeta_{-} = \tfrac{1}{2}(\zeta_1 - j\zeta_2) = -je^{j\phi_0}F(\theta). \tag{78.5}$$

Of course, in general, when radiation in one polarization is incident on the slot, the latter will radiate in both polarizations except in the particular case when $F(\theta)$ or $F(-\theta)$ vanishes. Thus for circular polarization in the counter-clockwise sense, the equation

$$\tan\theta = -\frac{aH_{11}^2}{k_{11}} \quad \text{or} \quad k_{11}a \tag{78.6}$$

gives the inclination of the unexcited slots. The former corresponds to the transverse slot in the rectangular guide, the latter to the longitudinal slot unexcited by the incident TE-wave.

The principles already discussed in § 7.3 together with the results just given enable one to calculate the impedance presented to one wave by the tilted slot, given the terminations for the second wave of either the linearly or the circularly polarized pair; also the rotation of the plane of polarization of the wave on reflection and transmission could be calculated. Many possibilities present themselves in connexion with the loading of a circular guide by reactances coupled to it by means of slots.

7.9. Application to the Optics of Polarized Light

In concluding this chapter it is not inappropriate to return to the brief consideration of plane waves with uniform amplitude distribution in the surfaces of equal phase. The matrix methods we have used in connexion with the coupling of waves in a guide may be employed to represent the changes in polarization produced in propagation. We think of the two component waves linearly polarized at right angles to each other, as propagated on two transmission lines which are in general coupled at a discontinuity.

First we require the matrix

$$T = \begin{pmatrix} \cos\phi\, E & \sin\phi\, E \\ -\sin\phi\, E & \cos\phi\, E \end{pmatrix} \tag{79.1}$$

by which we transform the wave amplitudes to correspond with the rotation of the transverse axes of reference through the angle ϕ. This enables us to find coupling matrices C' corresponding to arbitrary systems of resolving the vibration in the wave, as follows. If C is the coupling matrix corresponding to a particular coupling load on the pair of transmission lines, then

$$C' = TCT^{-1}, \qquad T^{-1} = \begin{pmatrix} \cos\phi\, E & -\sin\phi\, E \\ \sin\phi\, E & \cos\phi\, E \end{pmatrix}. \tag{79.2}$$

In the second place, transformation from a system of linearly polarized waves to a system of circularly polarized waves is accomplished by means of the matrix

$$R = \begin{pmatrix} E & -\Omega \\ E & \Omega \end{pmatrix}, \quad \text{where} \quad \Omega = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}; \tag{79.3}$$

the first row in the transformed matrix will refer to the circularly polarized wave in the counter-clockwise sense, the second row to that in the clockwise sense.

Again, from the coupling matrix C for linearly polarized waves we can derive the coupling matrix C'' for the pair of transmission lines representing the circular polarizations. Thus

$$C'' = RCR^{-1}, \quad \text{where} \quad R^{-1} = \begin{pmatrix} \frac{1}{2}E & \frac{1}{2}E \\ \frac{1}{2}\Omega & -\frac{1}{2}\Omega \end{pmatrix}. \tag{79.4}$$

On this basis it is possible to solve optical problems involving polarization by methods analogous to those used in Chapter I.

VIII

WAVE-GUIDE ARRAYS

8.1. Introduction

One important practical advantage which distinguishes microwaves from longer radio waves is the possibility of constructing radiators of large aperture in order to obtain high directivity. By this means radiation is concentrated in a narrow cone. Accordingly, for given power input to the antenna, a prescribed value of electric intensity in the waves radiated from it can be realized at a greater distance from the radiator than is possible for a less directive one. Secondly, the more directive a radar antenna is, the greater the resolution in most representations made by means of it. The physical considerations entering the design of such an antenna are twofold. In the first place, one must solve an optical diffraction problem to determine the amplitude and phase distribution over the aperture required to secure the desired radiation pattern. There remains then the electrical problem of securing this distribution and at the same time the maximum transfer of power from the generator.

A suitable termination to a wave guide may be used as source of the waves which are concentrated by a metallic reflecting screen in the form of a paraboloid. The effective source of the waves is at the focus of the latter. The primary pattern of this source must be such as to flood the reflector properly, otherwise the advantage of the large aperture will be lost. This primary pattern is usually investigated in the absence of the reflector, equiphase surfaces being traced and the amplitude distributions in both polarizations being measured. To this end the wave guide may feed a dipole or system of driven dipoles and parasites, or it may feed a slot in its end or the guide may be terminated in a horn. The last named has been used to flood the reflector in a 'cheese aerial', which is a shallow section of parabolic cylinder, its depth parallel to the generators of the cylinder and the long axis of the rectangular aperture of the horn. This device provides a fan-shaped beam polarized in the principal plane perpendicular to the focal line of the cylinder.

Whenever great directivity is required, say, in the horizontal plane, as commonly occurs in radar practice, the reflector will be a parabolic cylinder with its greatest dimension parallel to the horizontal generators of the cylinder. To secure satisfactory illumination of the reflector, the power must be fed from a linear radiator lying along the focal line of the cylinder. A linear array of elementary radiators coupled to a wave guide will serve this purpose well. In place of this combination of linear radiator and parabolic reflector, it is sometimes advantageous to use a two-dimensional array consisting of a coplanar parallel system of linear arrays. In the present chapter our main interest will be to discuss the design of linear and two-dimensional wave-guide arrays.

The expression for the distant field from a linear array of similar radiators is well known from works on optics and radio. It is worth while to recall the result. We shall call the straight line on which lie the centres of the elementary radiators the axis of the array. Let the N similar radiators, which we shall suppose similarly oriented, radiate with electric force $E_1(\theta,\psi)/r$ in the direction making θ with planes normal to the axis of the array, and ψ with a fixed plane through the axis. Let the distance between adjacent radiators be d, and let ϕ be the constant phase difference between the excitations of successive radiators along the array taken in the same sense as positive θ. The amplitude of the electric force at a great distance from the array in the direction (θ,ψ) is proportional to

$$\left|E_1(\theta,\psi)\sum_{s=0}^{N-1} e^{-sj(2\pi d\sin\theta/\lambda+\phi)}\right|,$$

i.e. to
$$|E_1(\theta,\psi)|\frac{\sin N(\pi d\sin\theta/\lambda+\frac{1}{2}\phi)}{\sin(\pi d\sin\theta/\lambda+\frac{1}{2}\phi)} = |E_1(\theta,\psi)|A(\theta). \tag{81.1}$$

The array factor $A(\theta)$, regarded as a function of direction, will in general show principal maxima in those directions in which the field from the individual radiators reinforce each other, viz. for θ given by

$$\frac{\pi d\sin\theta}{\lambda}+\frac{\phi}{2} = m\pi \quad (m \text{ integral}). \tag{81.2}$$

The order of the principal maximum is $(m+1)$. Between these directions there will occur subsidiary maxima given by

$$\tan N\left(\frac{\pi d\sin\theta}{\lambda}+\frac{\phi}{2}\right) = N\tan\left(\frac{\pi d\sin\theta}{\lambda}+\frac{\phi}{2}\right), \tag{81.3}$$

due to fluctuations in the degree of destructive interference of the corresponding waves.

The maximum order of principal maximum present is the integer next greater than the larger of

$$\left|\frac{d}{\lambda}+\frac{\phi}{2\pi}\right| \quad\text{and}\quad \left|-\frac{d}{\lambda}+\frac{\phi}{2\pi}\right|.$$

Thus more than one principal maximum will occur if $d > \lambda$. Accordingly, in order to achieve a beam of radiation with only one major lobe or principal maximum in any plane containing the axis of the array, the elements must be spaced somewhat closer than λ or they must themselves be sufficiently directive, so that the maximum in the array factor $A(\theta)$ is rendered ineffective by the weakness of the amplitude of the reinforcing waves which are radiated in the direction corresponding to the principal maximum in question.

For an array of similar co-phased elements the main lobe is normal to the axis and its shape is determined by

$$S(\theta) = \left| \sum_{s=1}^{N} f_s e^{sj(2\pi d\theta/\lambda)} \right|,$$

f_s being the amplitude of excitation of the sth element. For symmetrical arrays we have

$$\begin{aligned} S(\theta) &= f_{\frac{1}{2}(N+1)} + 2f_{\frac{1}{2}(N-1)} \cos\frac{2\pi d\theta}{\lambda} + 2f_{\frac{1}{2}(N-3)} \cos\frac{4\pi d\theta}{\lambda} + \dots \quad (N \text{ odd}) \\ &= 2\left[f_{\frac{1}{2}N} \cos\frac{\pi d\theta}{\lambda} + f_{\frac{1}{2}N-1} \cos\frac{3\pi d\theta}{\lambda} + f_{\frac{1}{2}N-2} \cos\frac{5\pi d\theta}{\lambda} + \dots \right] \quad (N \text{ even}). \end{aligned} \tag{81.4}$$

Since the sharpness of the peak in $S(\theta)$ is dependent on the high-order cosine terms, the sharpness of the lobe is dependent on the excitation of the extreme elements of the array.

8.2. Array Elements

A linear array from which the electric force in the main lobe is at right angles to the axis of the array, we shall call *transversely* polarized. If the electric force is parallel to the plane through the point where it is measured and containing the axis, we shall call it *longitudinally* polarized. The simplest array elements which are practically useful are electric or magnetic half-wave radiators. The former is realized by a wire or rod, the latter by a slot. The electric polarization due to the electric dipole is parallel to its axis and that due to the magnetic dipole transverse. Apart from polarization, E and H being interchanged, these radiators have identical patterns with maximum intensity in the equatorial plane. Consequently in order to suppress end-fire radiation from an array, it is necessary to choose closer spacing in an array of elements with their axes transverse to the array than is required when their axes are parallel to the array. We have already seen how slot radiators may be cut in the wall of the wave guide. Electric dipoles must be mounted on the guide wall, and a suitable coupling device must be introduced to draw energy from the guide and feed it to the dipole without allowing unwanted spurious radiation from the coupling. Usually a probe antenna in the guide forms the extension of the central conductor of a coaxial line which acts as support for the wings of the half-wave radiator. One type of element is shown in Fig. 67. This element is mounted on the centre of the broad face of the feed-guide and the wire probe passes through into the guide. The depth to which the probe extends determines the shunt load imposed on the wave in the guide and hence the fraction of energy drawn from the wave when the rest of the load is given. This method is suitable only when there is a large number of elements so that the energy to be drawn by a single element is not a large fraction of the whole. If this is not so, since the probe constitutes a shunt load on the H_{10}-wave, the reactance in series with the radiation resistance of the dipole is varied rapidly with the probe depth and hence independent control of phase and amplitude is not possible. This difficulty may be avoided by means of the bent-probe coupling described in § 6.7: the dipoles are mounted on the narrow face, and the probe is the bent continuation of the central conductor of the coaxial feed-line terminated in the dipole. The probe is chosen of such overall length as to present a non-reactive load to the guide and the conductance is controlled by rotating the probe, thus altering its inclination to the electric force in the guide. Two limitations should be mentioned: (i) the power which can be radiated by such an element is considerably smaller than that which even a narrow slot can handle without breakdown, and (ii) unless a very large number of such radiators is to be fabricated, the cost of making them is excessive. Nevertheless, electric dipoles mounted on the broad face have been effectively used in the United States [19].

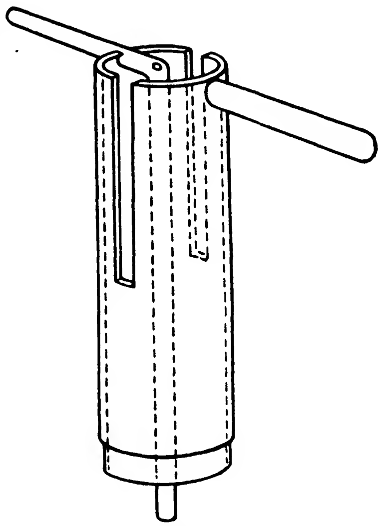


FIG. 67.

Not only do the slot (magnetic) radiators compare favourably with electric dipoles on the basis of power-handling capacity, physical permanence, and cost, but they provide the electrical designer with a much more flexible circuit element, in that both series and shunt couplings are possible. By using the slot coupling described in §§ 6.6 and 6.8, it is also possible to couple the slot radiators so as to maintain in the guide a pure travelling wave between the radiators.

Since we have in mind long arrays with many elements among which the energy is to be divided, each element must be comparatively weakly coupled to the wave in the guide. Accordingly, we can cut slots in the guide in somewhat different ways in order to control the degree of coupling, without at the same time introducing into the radiation pattern serious effects due to the varying aspect or disposition of the slots, either on account of their displacement from the longitudinal central line on the broad face, or on account of their inclination, or in fact, due to a combination of these. If it is absolutely essential to have radiators parallel and in line, then the slots must be probe-excited† or an external structure must be built on the guide at each coupling slot. An array of bent-probe-excited slot radiators built and tested in April 1943 actually preceded arrays of slots cut in the guide-feed. It is shown in Fig. 68.

8.3. Band-width

One of the most important practical requirements to be met in a radar antenna in war-time is to secure adequate band-width, to allow for the spread of the actual frequencies of commercial magnetrons and for operational flexibility. It must be possible for the antenna to present a tolerable impedance over the specified frequency range, and the antenna must retain its directivity or gain without introducing an intolerable side-lobe structure. To maintain a satisfactory radiation pattern as the frequency is altered the condition on the phase distribution is much more stringent than that on the amplitude distribution. The former can always be tested in the laboratory by means of equiphase plots near the array. Such plots for different frequencies through the band will indicate, by the presence of large departures from straightness, that a poor distant pattern is to be expected. In addition, the amplitude distribution can be roughly checked near the array.

Provided that over the band a constant phase-gradient is maintained along the array, a good main lobe is to be expected. The direction of this lobe will in general change with frequency unless the phase-gradient is independent of frequency. If the direction of the main lobe is to remain fixed through the band, it will generally be more difficult to secure band-width than if this requirement is relaxed. Further, it is

† Discussed in [20].

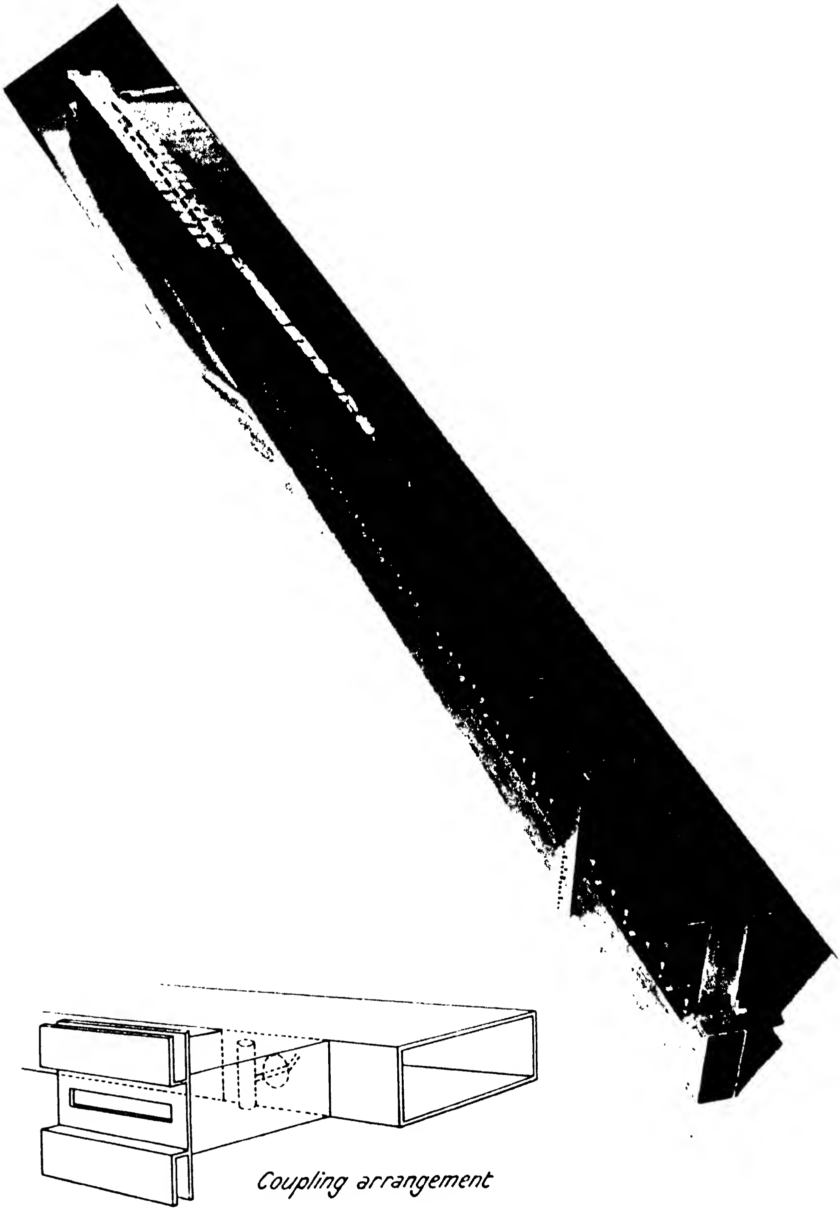


FIG. 68

evident from equation (81.4) that change in frequency affects the shape of the main lobe and hence the width of the beam. We therefore classify linear radiators in two groups:

(*a*) fixed azimuth for main lobe—narrow band-width,

(*b*) variable azimuth for main lobe—wide band-width.

Of course it will be desirable with class (*a*) to secure as great band-width as possible in order to render less critical the operation of the transmitter.

8.4. The Elements of the Design Problem for a Linear Radiator

We shall deal with the straightforward problem of providing a linear radiator with the following properties:

(i) For given length of radiator it shall give the narrowest possible main lobe (total width being reckoned, for instance, at 6 db. below maximum) or inversely, for a given beam-width, the radiator shall have the minimum length, consistent with

(ii) no side lobe should exceed in amplitude 15 per cent. of the main lobe maximum (i.e. the power level in the side lobe should not exceed approximately 2 per cent. of the maximum in the main lobe).

In order to meet these requirements, the phase distribution along the array must be linear, and the amplitude distribution 'gabled' or 'tapered' with its maximum at the centre of the array. However, since the sharpness of the main lobe derives from the rapid onset of destructive interference of the waves from the whole array on each side of the direction of maximum intensity, the radiators at the ends of the array must contribute just as much as condition (ii) will permit. It is possible to achieve very sharp lobes by using only two radiators, one at each end of the array.† In order to produce a single lobe, the intermediate radiators are required to cancel all except one of these, and in the process they widen the remaining lobe. It is therefore necessary to have due regard to optical principles in array design, so that effective use is made of the length of the array. By actual computation, supported by antenna experience, it has been found profitable to choose with care the gabling function by which the amplitude distribution along the array is specified; for example, in an array from which the central element opposite the feed-point was missing, it was chosen to have the form shown in Fig. 69. The amplitude of the end elements

† Cf. Michelson's method for determining stellar diameters.

is about half of that of the centre ones. It should be noted that if no gabling were introduced and the elements therefore equally excited, the maximum side-lobe amplitude would be 24 per cent. (or about 6 per cent. on a power basis) of the main lobe. The uniformly illuminated array gives a main lobe about 10 per cent. narrower than the gabled one referred to.

From these considerations it is clear that, provided the phase

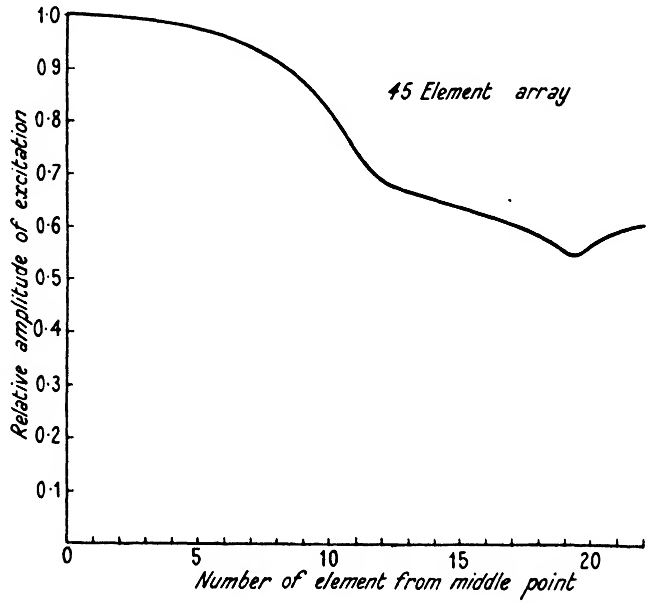


FIG. 69.

distribution along the array is good, the advantage of correct amplitude distribution is that it leads to efficient use of the length of the array. For the purpose of comparing different arrays in this respect, it is desirable to have a term, based on the comparison of the beam-width (α) given by the antenna, supposed to meet condition (ii) above, and that to be expected from a uniformly illuminated array of the same length L for the wavelength λ. We shall define the *figure of merit* for a linear array to be

$$S = \frac{L\alpha}{\lambda}. \tag{84.1}$$

The larger this quantity is the worse the performance, For a good array, S should be less than 1·5.

From the point of view of theory, one should be able to calculate the phase and amplitude of excitation of the elements of an array, each element being coupled to the wave system in the guide, no matter what that system is on account of loading. In fact, such calculations were carried through for comparatively short arrays in the early stages of understanding the process in the feed-guide. In practical arrays, however, there are only two alternative conditions which can be adopted as the basis of design, because of the required stability of the phase distribution. These are:

I. The guide is terminated by a reflecting plunger, and a standing-wave system is established in the guide. Alternate couplings reverse phase and the elements, being spaced $\frac{1}{2}\lambda_g$ apart, radiate in phase. The main beam from the array is normal to the guide. The band-width of this arrangement is proportionally reduced the longer the array, for the limiting condition is that, for N elements, the fractional change $1/N$ in the frequency will put one of the elements in opposed phase. For the fractional change n/N, $(N-n)$ elements will radiate in phase, and n elements in anti-phase, and the beam will split. For this reason it is of the greatest practical importance, in constructing long arrays of this type, to measure the wavelength in the actual guide to be used for the mean frequency of the band, otherwise the array will not function over the desired band. It may, however, function satisfactorily over a displaced band.

II. The alternative condition which allows a good approximation to a constant phase-gradient along the array consists of a travelling wave in the guide. Since the velocity of propagation in the latter is usually about 1·5 times that in free space, the main beam will be radiated at an inconvenient angle to the array normal, when the elements radiate with the phase of the wave in the guide where they are coupled. Accordingly, alternate elements are coupled in reversed phase, and unless the coupling device does not reflect in the guide, it is necessary to space the elements sufficiently differently from $\frac{1}{2}\lambda_g$ in order to present a satisfactory input impedance over a band of frequencies, the guide being terminated by a matched sand-load. Two spacings have been used, viz. 160° and 200° (guide), the latter being preferable [21]. When the spacing of the elements is s with alternately reversed phase coupling, the direction of the main lobe is given by

$$\sin\theta = \left(\frac{\lambda}{\lambda_g}-\frac{\lambda}{2s}\right), \tag{84.2}$$

θ being reckoned in the same sense from the normal as the direction of propagation in the guide.

This type of array was first used with electric dipoles coupled by probes, which therefore act as shunt loads. It is possible to maintain a sufficiently good approximation to a travelling wave in the guide and to obtain the proper amplitude distribution only if all the elements are weakly coupled to the wave in the guide. This requires that a considerable fraction (from 5 to 20 per cent.) of the input energy may have to be dissipated in the sand-load terminating the guide. The waste of energy varies with the frequency of the waves, and is one of the main factors limiting the band-width of arrays of this type. We shall describe later a wide-band antenna which avoids this difficulty.

The fundamental distinction between the standing-wave and the travelling-wave antennas is that in the former the loads are effectively at the same place on the transmission line and compete for power, exactly like the parts of a d.c. circuit, whereas in the latter the exciting wave is attenuated by the absorption of power in the series of loads, hence for the equal excitation of two elements, that farther from the generator must be more strongly coupled than the other. With half-wave spacing, if the elements are all shunt-coupled, the wave of electric force in the guide is unattenuated; the wave of transverse magnetic force is attenuated, but since the elements are voltage-excited, they are unaffected by this attenuation when the frequency corresponds exactly with the spacing. If the elements were series-coupled at the resonant spacing, the current (magnetic force) wave would not be attenuated, whereas the voltage wave would. In order to produce attenuation of both waves, both series and shunt loading would be required at $\frac{1}{2}\lambda_g$ spacing. The matter may be stated somewhat differently, thus: Off resonant spacing, with weak coupling, the standing-wave ratio never departs seriously from unity anywhere along the guide; whereas with resonant spacing, the standing-wave ratio varies from ∞ to 1 if the guide is fed from one end and presents a matched input. It should be noted that a travelling-wave antenna must be end-fed, otherwise a discontinuity in phase-gradient will occur at the feed-point. In a standing-wave antenna the feed-point may be at one end or it may be near the middle of the array, and in the latter case by skilful design it may be possible to secure somewhat greater band-width.

In order to prevent instability of phase when resonant spacing is used, the elements must be coupled at maxima in the standing-wave of mean frequency which excites them, i.e. shunt loads should be placed at voltage maxima, series loads at current maxima. The amount of power drawn from the guide should be determined by presenting to the wave in the guide the proper admittance or impedance as the case may be. Attempts to control power taken from the guide by any other method are likely to lead to instability of the phase distribution.

8.5. Theory of the Wave-guide Feed to a Linear Array

We now discuss the theory of the feeding of a linear array from a wave guide.† We shall assume in the first place that the mutual effects between elements of the array, apart from their coupling to the H-wave in the guide, may be neglected. This method will therefore apply with fair approximation to dipole arrays and to arrays of longitudinal shunt and series slots, but not to transverse slots. For the purpose of the argument, we shall assume shunt loading, but the principle of the method is applicable to any type of loading.

Let α_r be the admittance of the rth shunt load in the array of N elements, the Nth load being most distant from the generator. Let v_r be the voltage in the equivalent transmission line at the position of the rth load. If the loads are spaced equidistant d apart, or θ radians reckoned on the unloaded line, the difference equation satisfied by the voltage is

$$v_{r+1}-2\cos\theta\, v_r+v_{r-1} = -j\alpha_{r-1}\sin\theta\, v_r. \tag{85.1}$$

We note at once that when θ is an integral multiple of π, say $m\pi$,

$$v_{r+1} = (-1)^m v_r, \tag{85.2}$$

showing that the voltage wave is unattenuated. If i_r denotes the current immediately to the right of the rth load,

$$i_r = (-1)^r\Big(i_0-\sum_{s=1}^{r}\alpha_s V\Big), \tag{85.3}$$

where V is the constant voltage, and i_0 is the current $\frac{1}{2}\lambda_g$ in front of the first load. This shows the decreasing current wave in the shunt-loaded line.

It remains now to discuss non-resonant spacing. In a long array the α_r should be small compared with unity for weak coupling in the proper physical sense of negligible local disturbance of the feed, and further, α_r is not a rapidly changing function of r.

Let us treat (85.1) by the analogue of the method of variation of parameters, writing

$$v_r = A_r\omega^r+B_r\omega^{-r}, \qquad \omega = e^{j\theta}, \tag{85.4}$$

† The theory has been discussed in another way by Harvie [22].

with the auxiliary condition

$$(A_{r+1}-A_r)\omega^{r+1}+(B_{r+1}-B_r)\omega^{-(r+1)} = 0, \tag{85.5}$$

so that
$$v_{r+1}-v_r = A_r(\omega-1)\omega^r+B_r\omega^{-r}(\omega^{-1}-1). \tag{85.6}$$

Thus we can replace the second-order difference equation (85.1) in v_r by a pair of first-order equations in A_r and B_r. In consequence of the smallness of α_r, we proceed on the assumption that A_r and B_r do not vary rapidly with r, and hence we may replace the pair of difference equations by differential equations in which the variable x/d replaces the index r. These equations are

$$\omega^{x/d}\frac{dA}{dx}+\omega^{-x/d}\frac{dB}{dx} = 0, \tag{85.7}$$

$$(1-\omega^{-1})\omega^{x/d}\frac{dA}{dx}+(1-\omega)\omega^{-x/d}\frac{dB}{dx} = \frac{-j\sin\theta\alpha(x/d)}{d}(A\omega^{x/d}+B\omega^{-x/d}). \tag{85.8}$$

Comparison of these equations shows that

$$\left.\begin{aligned}(1-\omega^{-1})\frac{dA}{dx}+\frac{A\alpha(\omega-\omega^{-1})}{2d} &= \lambda\frac{dA}{dx},\\ (1-\omega)\frac{dB}{dx}+\frac{B\alpha(\omega-\omega^{-1})}{2d} &= \lambda\frac{dB}{dx},\end{aligned}\right\} \tag{85.9}$$

where λ is a multiplier to be determined.

If
$$f(x) = \int\limits_0^x \alpha\,dx \quad\text{and}\quad f(L) = Q, \tag{85.10}$$

where $L = Nd$, we find

$$\begin{aligned}A &= A_0\exp\left\{-\frac{(\omega-\omega^{-1})}{2d(1-\lambda-\omega^{-1})}f(x)\right\},\\ B &= B_L\exp\left\{\frac{(\omega-\omega^{-1})\{Q-f(x)\}}{2d(1-\omega-\lambda)}\right\}.\end{aligned} \tag{85.11}$$

The value of λ is deduced in terms of the ratio of the input and terminal values of the circle diagram variable $w = B/A$ reckoned at points distant an integral number of half-wavelengths from the origin of x. We find

$$1-\lambda = \cos\theta+j\sqrt{(p+1)}\sin\theta, \quad\text{where}\quad p = \frac{2Q}{d\log_e w_1/w_T}. \tag{85.12}$$

If w_T is near zero (matched termination), p will be small, and therefore $1-\lambda$ tends to ω, so we obtain the following approximate formula:

$$A = A_0\exp\left(\frac{-f(x)}{2d}\right), \tag{85.13}$$

or reverting to the discrete loading,

$$A_r = A_0 \exp\Big[-\sum_{s=1}^{r} \tfrac{1}{2}\alpha_s\Big], \tag{85.14}$$

while

$$B_r = B_L \Big(\frac{w_1}{w_T}\Big)^{\sum_{s=r+1}^{N} \alpha_s \big/ \sum_{1}^{N} \alpha_s}. \tag{85.15}$$

The expression for A_r shows the attenuation of the wave travelling from the generator, that for B_r the growth of the reflected wave. We shall show how to calculate the ratio w_1/w_T in considering input impedance in the succeeding section.

In general α_s is complex, $\alpha_s = g_s - jb_s$. The presence of susceptance b_s modifies the velocity of phase propagation, but near constancy of the phase-gradient along the radiators of the array will be preserved if the gradient of b along the array is small. The attenuation of the wave is determined by the conductances g_s. The fraction of energy reaching the terminating load is

$$\exp\Big[-\sum_{s=1}^{N} g_s\Big]. \tag{85.16}$$

8.6. Amplitude Distribution and Input Impedance

In the case of half-wave spacing the radiators may be co-phased by making them pure conductances g_r, so that the input conductance to the array can be made unity or any desired value G, and the amplitude distribution along the array made to follow the desired law f_r simultaneously, by means of the relation

$$g_r = G \frac{f_r^2}{\sum_{r=1}^{N} f_r^2}. \tag{86.1}$$

To secure the desired amplitude distribution in a travelling-wave antenna, one must take into account the attenuation along the wave guide due to the loads. Let e_r be the fraction of energy to be extracted from the guide by the rth radiator from the generator end, and let f_r be the amplitude of excitation desired for that radiator, then we require

$$\Big(\frac{f_{r+1}}{f_r}\Big)^2 = \frac{e_{r+1}(1-e_r)}{e_r},$$

or

$$e_r = \frac{e_{r+1}}{e_{r+1}+(f_{r+1}/f_r)^2}. \tag{86.2}$$

Since the fraction of energy taken from the guide by a radiator in a travelling wave is equal to the conductance g_r presented by it to the waves, we have

$$g_r = \frac{g_{r+1}}{g_{r+1}+(f_{r+1}/f_r)^2}. \tag{86.3}$$

In order to fix the values of the g_r, we must be given one of them. This will generally be decided by one or other of the following considerations: (a) there is a maximum g that can be presented to the guide consistent with the hypothesis of weak coupling and satisfactory phasing, or in the case of slots, consistent with satisfactory radiating properties with respect to polarization and side lobes from the array; and (b) the amount of energy to be dissipated in the terminating load must be sufficiently small so as not to reduce unduly the efficiency of the array. Except for long arrays, these two considerations usually conflict, and in practice it is necessary to strike a compromise. Computation of the g's is greatly facilitated by noting that if we write

$$g_r = f_r^2/y_r \tag{86.4}$$

we obtain the following simple formula:

$$y_r = y_{r+1}+f_r^2. \tag{86.5}$$

Hence we have to find y_N so that $f_N^2 = y_N g_N$, then

$$y_r = y_N+\sum_{s=r}^{N-1} f_s^2. \tag{86.6}$$

The fraction of energy to be absorbed in the terminating load is then approximately

$$\exp\Big(-\sum_{s=1}^{N} g_s\Big). \tag{86.7}$$

In the foregoing it has been assumed that the wave in the guide is attenuated only by loading due to the radiators. At X-band and K-band, however, appreciable attenuation will take place due to the finite conductivity of the walls of the guide. Let δ be the fraction of energy lost due to this cause between two successive radiators, then equation (86.2) must be replaced by

$$\frac{f_{r+1}^2}{f_r^2} = \frac{e_{r+1}(1-e_r-\delta)}{e_r} \tag{86.8}$$

or

$$g_r = \frac{(1-\delta)}{g_{r+1}+f_{r+1}^2/f_r^2}g_{r+1}. \tag{86.9}$$

It is easily shown that the y's are now given by the rule

$$y_r = \frac{1}{1-\delta}(y_{r+1}+f_r^2). \tag{86.10}$$

We now consider the calculation of input impedance to the guide loaded at equal intervals (θ radians) by admittances α_r which are small compared with the characteristic admittance of the equivalent line.

Let Ω be the displacement matrix $\begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$. The matrix M_n^{n-1} which transforms from the right of the nth point of loading to the right of the $(n-1)$th point of loading, θ radians distant, is

$$M_n^{n-1} = \Omega + \alpha_n \Omega U_1. \tag{86.11}$$

For N loads the resultant transformation from output to input is

$$M = \prod_{n=1}^{N} M_n^{n-1} = \prod_{n=1}^{N} (\Omega + \alpha_n \Omega U_1). \tag{86.12}$$

To a first approximation this may be written

$$\Omega^N + \sum_{s=1}^{N} \Omega^{s-1} \alpha_s \Omega U_1 \Omega^{N-s} = \Omega^N + \sum_{s=1}^{N} \alpha_s \Omega^s U_1 \Omega^{N-s}. \tag{86.13}$$

Now
$$\Omega^s U_1 \Omega^{N-s} = \begin{pmatrix} \frac{1}{2}\omega^N & \frac{1}{2}\omega^{-N+2s} \\ -\frac{1}{2}\omega^{N-2s} & -\frac{1}{2}\omega^{-N} \end{pmatrix}. \tag{86.14}$$

Hence
$$M = \begin{pmatrix} \omega^N(1+\sum \frac{1}{2}\alpha_s) & \omega^{-N} \sum \frac{1}{2}\alpha_s \omega^{2s} \\ -\omega^N \sum \frac{1}{2}\alpha_s \omega^{-2s} & \omega^{-N}(1-\sum \frac{1}{2}\alpha_s) \end{pmatrix}, \tag{86.15}$$

all sums being extended from 1 to N. Thus the transformation from the terminating value of w, w_T, to the input value in the sense of § 1.6, is

$$w_1 = \frac{-\omega^N(\sum \frac{1}{2}\alpha_s \omega^{-2s}) + \omega^{-N}(1-\sum \frac{1}{2}\alpha_s) w_T}{\omega^N(1+\sum \frac{1}{2}\alpha_s) + \omega^{-N}(\sum \frac{1}{2}\alpha_s \omega^{2s}) w_T}. \tag{86.16}$$

When the termination is matched, $w_T = 0$, and

$$w_1 = \frac{-\sum \frac{1}{2}\alpha_s \omega^{-2s}}{1+\sum \frac{1}{2}\alpha_s}. \tag{86.17}$$

As θ is varied, the sum which is the numerator of this fraction undergoes rapid fluctuations, but its modulus remains small compared with $\sum \frac{1}{2}\alpha_s$ except when $\theta = m\pi$ (m integral). Indeed a graph of the modulus of the sum against θ will resemble the diffraction pattern of a grating with principal and subsidiary maxima. Thus when the terminating load is non-reflecting, $|w_1|$, and hence the standing-wave ratio will exhibit fluctuations like the sum in question when the electrical spacing of the elements is altered by varying the velocity of propagation in the guide, either as the result of changing the frequency or changing the width of the guide cross-section. That is, it is not possible to feed the same array when $\theta = m\pi$ and in the vicinity of these values. The fluctuations in S.W.R. to be expected are illustrated in Fig. 70.

On the other hand, if $w_T = 1$ (open-circuit termination) and $\sum_1^N \alpha_s = 1$, then when $\theta = m\pi$, $w_1 = 0$, i.e. the input is matched. Even when $\sum \alpha_s = q < 1$, we can secure a matched input by choosing $w_T = q/(2-q)$, but, of course, if $q > 1$ there is no possible termination of the feed-guide yielding a matched input. From (86.16) it is easily seen

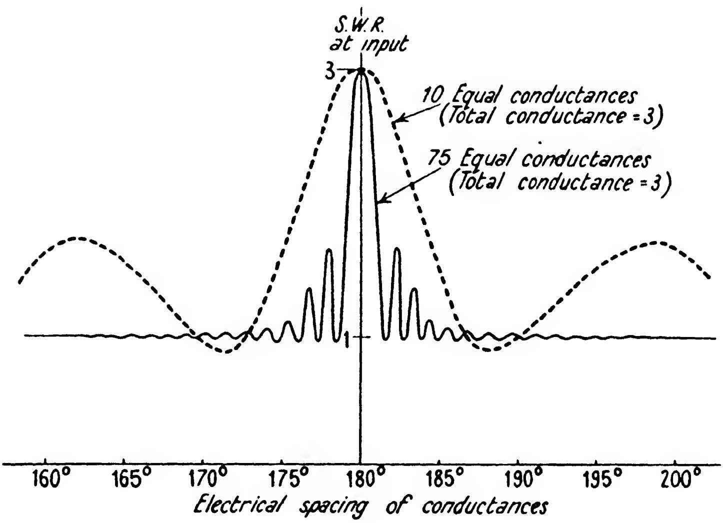


Fig. 70.

that for a system of equal loads intended to match at half-wave spacing, the input admittance is reduced to $\frac{1}{2}$ when θ is changed from π to the nearest zero of $\sin N\theta$, that is the wavelength is changed by the fraction $1/2N$. These considerations explain the sensitivity of the input admittance of a long array to frequency at $\frac{1}{2}\lambda_g$ spacing.

It remains to mention the possible effect of the reflector on the input impedance to the array placed along its focal line. So long as the reflector is wide ($> 10\lambda$), transverse to the axis of the array, and so long as the distance between the face of the guide and the nearest part of the reflector, which is of course parallel to the broad face of the guide opposite it, is not equal to an odd number of quarter-wavelengths, one is justified in expecting no effect on impedance that would be serious in practice. When the array floods a narrower reflector, the energy reflected towards the guide is increased, and this in due turn will partly reach the generator, which thus faces a reactive load which will have to be compensated by means of an iris in the feed-guide.

8.7. Linear Arrays of Slots—Transverse Polarization

The invention of slot arrays at McGill University in 1943 was the natural outcome of the search for radiating elements which can be coupled so as to present a low pure conductance to the wave in the guide. It is quite obvious that dipoles coupled by means of probes perpendicular to the broad face of the guide at its centre are ill adapted, since the coupling is made at the place of strongest electric force in the guide; consequently the probe has to be very short. On the other hand, a longitudinal slot cut close to the centre of the broad face is

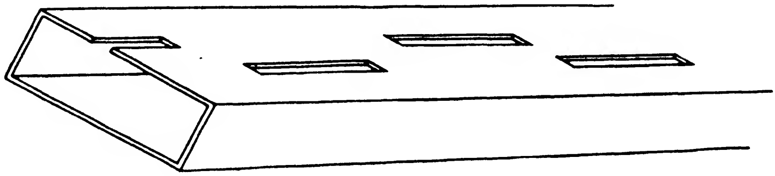


Fig. 71.

weakly coupled. The slot can be tuned to present a pure conductance, whereas the probe-excited dipole cannot, for the essential cause of the weak coupling by a short probe is the high reactance introduced by the latter (see § 5.9).

The first array of slots cut in the guide wall consisted of forty-nine shunt-coupled displaced slots, $\frac{1}{2}\lambda_g$ being the spacing between the centres of successive slots, and the guide terminated by a reflecting plunger $\frac{1}{4}\lambda_g$ from the last slot-centre. The conductance presented to the dominant wave in the guide by each slot was determined by its distance from the centre of the guide in accordance with the law (61.1) presented in Chapter VI, the constant A_1 being determined by measurement. Phase reversal was secured by placing alternate slots on opposite sides of the centre-line as shown in Fig. 71. This particular array was centre-fed: in such a case, it is necessary to take into account the phase relationship of the waves to the right and left of the point of coupling. If the coupling of the generator feed in the guide array is series, the two waves are in anti-phase, if shunt they are in phase. Consequently for a centre-fed array it is necessary to know the method by which the transmitter is coupled to the guide in order to cut the slots correctly.

This array had a sufficiently large number of elements so that even the most strongly coupled slots in the centre of the array did not have to be displaced from the centre-line by a distance great enough to distort sensibly the radiation pattern from the array. The polarization was of course transverse. The following refinements and simplification arise in the practical development of this device:

(i) choice of slot-width sufficient to improve band-width;

(ii) since the slots must be covered with weather-proof dielectric, the length of slot for resonance must be measured for the covered slots;

(iii) the length of slot for resonance varies slightly with displacement of the slot-centre on the broad face of the guide;

(iv) it is not necessary to cut the slots with displacements exactly as calculated, the displacements can be forced to the nearest figure found suitable for the machine work of cutting them (usually with an end-mill).

An array having the same polarization but not as convenient to cut could be made with inclined series slots; the inclination of the slots to the guide axis would be small enough to give a good radiation pattern provided that a sufficient number of elements comprised the array.

8.8. Longitudinally Polarized Slot Arrays

In many practical antennas longitudinal polarization is desired because of the increased contrast possible in the radar pictures resulting

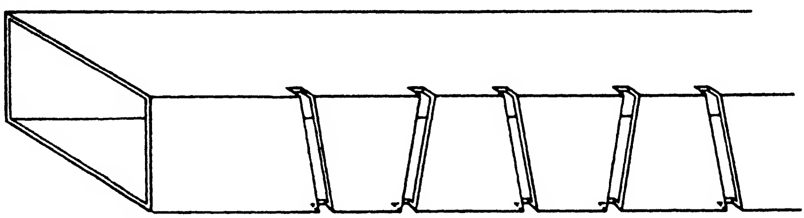


FIG. 72.

from the use of this polarization. It may be provided by means of the inclined shunt-coupled slots cut in the narrow face of the guide. These slots make only a small angle (ϕ) with the plane perpendicular to the guide axis. On account of the mutual interaction between these slots in virtue of waves propagated on the outside of the guide, the exact design problem is quite difficult. In the first place, it is not easy to obtain precise measurements of the parameters which determine the mutual effects; in the second, the conditions governing propagation outside are complicated; and in the third, even if the foregoing information were obtained with the necessary accuracy, special computing methods would be required to give the theory effect in practice.

The presence of mutual interaction can be shown by measurements of the input impedance to an array of similar slots cut at $\frac{1}{2}\lambda_g$ spacing, as the number of slots is increased. If there were no interaction, the graph of susceptance B against conductance G would be a straight line

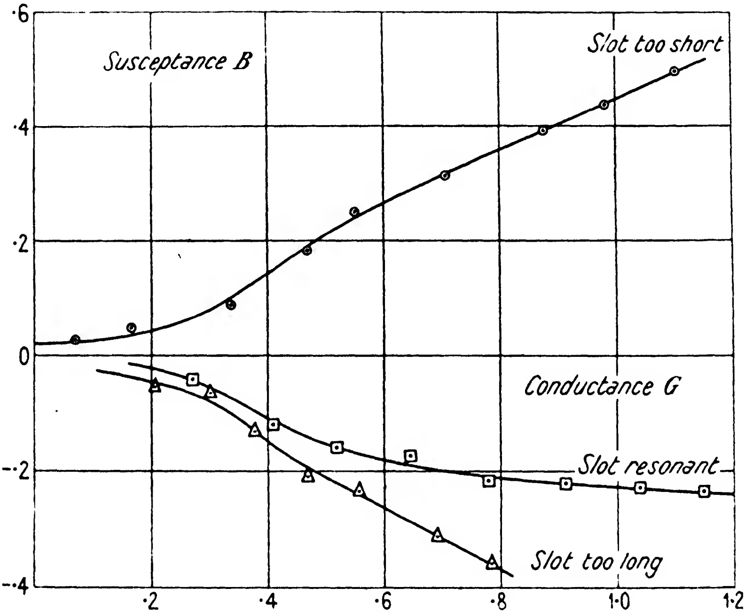


FIG. 73.

through the origin. Actually a somewhat irregular curve is shown for the first six or seven slots; thereafter it becomes smooth, and by proper choice of the common length of the slots, it is found that each of the later slots contributes the same pure conductance to the input admittance of the array. For any other slot-length there is likewise a limiting gradient of B per slot. On account of the weak coupling of slots cut in the narrow face nearly at right angles to the guide axis, it is more convenient to study the mutual interaction of series slots cut transversely in the broad face. In one experiment seven such slots were cut $\frac{1}{2}\lambda_g$ apart. The mutual impedance between the first of these and each of the other six was measured in the following way. The series impedance of each slot by itself was measured, then the impedances of the pairs. Twice the mutual impedance of any pair is the impedance of the slots acting conjointly minus the sum of their separate impedances. On the RX diagram of Fig. 74 are shown the values of the mutual impedances between the pairs 1, 2; 1, 3;..., and 1, 7. The points lie on a spiral. The angle between the radii from the origin to the successive points is 75°, which is the difference between 255° and 180°, the respective electrical spacings of the slots at free-space and guide phase-velocities. The observations quoted indicate that the mutual impedance falls off very approximately as the reciprocal of the

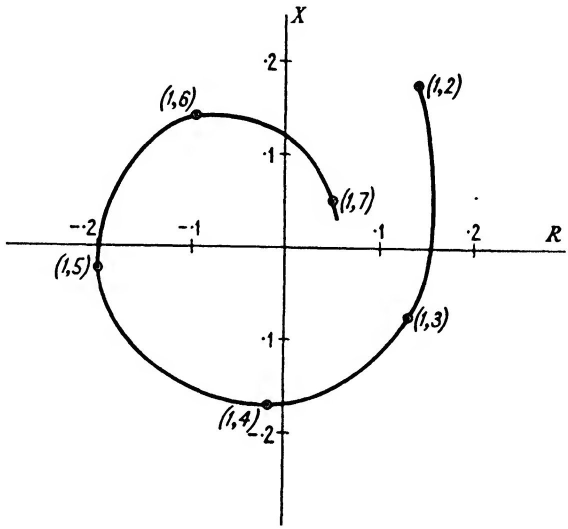


Fig. 74.

distance between the slots. Thus the main factor determining the mutual impedance corresponds to the propagation of a nearly spherical wave from one slot to the other. This wave may be reflected by obstacles mounted on the guide, such as the S.W.D.; care should be taken in measurement to minimize this reflection.

Arrays of inclined shunt slots may be constructed with resonant or non-resonant spacing—usually 200° (guide). The latter affords a somewhat easier crude approximation in design which we now consider. With a matched termination, the attenuation of the wave in the slotted guide due to radiation from 15–20 similar slots cut with alternately positive and negative inclinations is measured as a function of the depth of cut, which, of course, is transverse in the broad face. For a sufficiently great number of slots there is found a particular depth of cut for each angle of inclination (ϕ) of the slot, which makes the power radiated a maximum. For different ϕ ($\leqslant 15°$) the overall length of the slot, measured on the outside of the guide, is found to be the same for maximum radiation: that is, the depth of cut is determined geometrically, once the common length is known. Hence slots should be cut to the same length rather than to the same depth. This attenuation measurement with a large number of similar slots gives a useful estimate of the contribution of the individual slots in a gabled array at the same non-resonant spacing, only provided that the array is sufficiently long, so that the gradient of slot inclination along the array is nowhere rapid. The attenuation constant (κ) per slot so measured is an approximation to the effective conductance G of each slot in the array. For example, for 10° slots in standard S-band guide, the limiting value of κ for a large number of slots is 0·049, whereas the admittance of a single slot of the same length and inclination is $0{\cdot}017+j\,0{\cdot}015$. Thus the effect of mutual interaction is to increase the effective conductance of a single slot, and, of course, to change the susceptance. If the slots had been cut without phase reversal, that is, all parallel, quite a different result would ensue. It is fortunate that at 200° spacing the effect is to increase the power radiated per slot, so that the amount of unwanted polarization to be tolerated for a given radiation per slot is less than would be expected on the basis of an individual slot.

Since arrays of these slots are desired to give a certain measure of broad-band behaviour, it is interesting to compare the effect of varying the frequency with that of varying the slot-length. Since the former changes the relative phases of the contributions to the mutual effect at each slot as well as changing the admittance of the slot itself, it is expected that to produce a given decrease from the maximum value of the power radiated by the array of similar slots the fractional change in frequency is much less than the fractional change in slot-length. This is shown in the following table, where the results are given for ten 10° slots cut at 200° spacing in S-band guide.

Percentage change in radiated power from maximum	*Percentage change in slot-length*	*Percentage change in frequency*
10	3·1	1·5
25	4·9	2·2

The beam leaving a travelling-wave antenna is conical in section; to make it cylindrical, oblique flanges are attached to the slotted guide so that the waves leave the open end of the flanges in phase. The flanges may be flared to obtain the primary distribution of radiation required to flood the mirror. The direction of the beam emerging from the guide is dependent on frequency; to overcome this difficulty, flanges have been so proportioned as to impose a frequency-sensitive phase change to counterbalance. There results a beam whose direction is independent of frequency to the first order.†

The simplest procedure in designing an array of inclined shunt slots, cut in the narrow face of the guide of a longitudinally polarized array, is to make all the slots of the same overall length which corresponds to maximum radiation in a long array of similar slots at the same spacing equally inclined but alternately reversed in phase. The inclination of the individual slots is then chosen so as to approach the desired amplitude distribution and to yield a suitable dissipation of power in the terminating load. The guide is assumed weakly loaded by each slot so that the phase of each radiator is nearly in constant relation to the phase of the wave in the guide at the position of the slot-centre. An array constructed according to this plan will not give the desired amplitude distribution. Whenever the mutual effect increases radiation, one may expect that waves on the outside of the guide travelling towards the generator have larger amplitude than those travelling in the opposite direction. In order to test the foregoing an array of twenty slots of equal overall length and 200° spacing (guide) was cut with inclinations varying from 6·3° at the input end to 8·1° at the other, intended to extract 40 per cent. of the energy in the guide, on the assumption that mutual interaction may be taken into account approximately by assigning to each slot the conductance corresponding to the limiting decrement of energy per slot in a long array of similar slots. It was found that 45 per cent. of the energy was actually abstracted. The amplitude distribution was intended to be uniform; the actual measured distribution is shown in Fig. 75. The hump in the distribution at the input end of the array corresponds with the observation that the wave energy beyond the array on the outside of the guide was twice as much at the input end as at the load end. The actual amplitude distribution is only a fair approximation to the intended one: examination of the equiphase distribution about 20 cm. away from the array showed that the equiphase lines are very nearly straight, as they should be for a good directive pattern from a linear array. They were inclined at 4·2° to the guide: if the radiation were phased according to the travelling wave in the unloaded guide, the angle would be about 3·8°. For many practical purposes, therefore, this method may be adequate. An example of an array of this type used to feed a reflector is shown in the Frontispiece.

† See J. A. Ratcliffe, *Aerials*, Part II, I.E.E. Radiolocation Convention, 1946.

If one wishes to go farther it seems necessary to prescribe the length and inclination of each slot in order to secure the desired amplitude and phase distributions. A design procedure based on approximate theory is discussed in the following paragraphs.

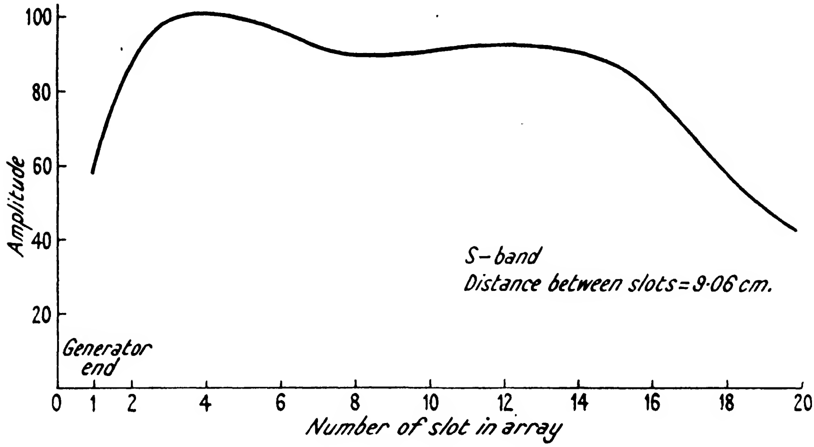


FIG. 75.

8.9. A Design Procedure for a Travelling-wave Antenna with Inclined Shunt Slots cut in the Narrow Face

Let d be the distance between the centres of adjacent slots and $\theta = 2\pi d/\lambda_g$, $\theta' = 2\pi d/\lambda$. If $\eta_r e^{jr\theta}$ is the amplitude of the travelling wave in the guide at the rth slot, the loading of the wave by that slot is represented by

$$\eta_r - \eta_{r+1} = \tfrac{1}{2}\sigma_r \eta_r, \tag{89.1}$$

where σ_r is a parameter which is to be determined. Let $P_r e^{jr(\theta+\pi)}$ be the voltage amplitude across the centre of the rth slot; then taking account of reversed phase and equation (66.7),

$$\tfrac{1}{2}\sigma_r \eta_r = (-1)^r L\zeta_r P_r, \tag{89.2}$$

where ζ_r is determined entirely by the inclination of the slot and the dimensions of the cross-section of the guide, the wall of which is supposed indefinitely thin. $\frac{1}{2}\sigma_r \eta_r$ is the amplitude of the wave scattered by the slot antenna.

Assuming a long guide antenna so that σ_r is small, we have the following approximate expression for the power radiated from the guide by the rth slot:

$$W_r = \mathrm{Re}\, \sigma_r \eta_r^2. \tag{89.3}$$

Since ζ is real for a shunt slot and we wish the P's to be in phase, we have from (89.2)

$$\arg \sigma_r \eta_r = \text{constant along the array}. \tag{89.4}$$

Equations (89.1), (89.3), and (89.4) allow us to determine $\sigma_r\, \eta_r$ and η_r by geometrical construction, provided we are given the power lost in the load. On a straight line choose the origin O, and set off segments $X_1 X_2$, $X_2 X_3, \ldots$, such that

$$OX_1 . X_1 X_2 = \tfrac{1}{2}W_1,$$
$$OX_2 . X_2 X_3 = \tfrac{1}{2}W_2,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$
$$OX_n . X_n X_{n+1} = \tfrac{1}{2}W_n$$

on some suitably chosen scale. Construct the circle shown in Fig. 76;

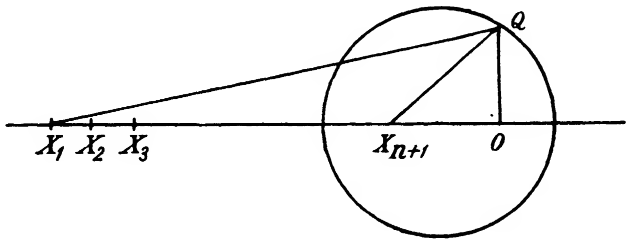


FIG. 76.

it is the locus of points whose distances from X_1 and X_{n+1} are in the ratio of the square roots of the input and output power in the guide. Let this circle meet the normal at O to the line $X_1 O$ in Q. Then $X_1 Q$, $X_2 Q, \ldots$, $X_{n+1} Q$ represent η_1, $\eta_2, \ldots$, η_{n+1} in amplitude and phase. $X_1 X_2$, $X_2 X_3, \ldots$, $X_n X_{n+1}$ represent $\tfrac{1}{2}\sigma_1 \eta_1$, $\tfrac{1}{2}\sigma_2 \eta_2, \ldots$, $\tfrac{1}{2}\sigma_n \eta_n$.

The P's are supposed given by the desired amplitude distribution along the array and a brief conversion of units. Hence ζ_r can be determined from (89.2). This information fixes the inclinations of the slots.

Now field theory methods require us to replace the equation (66.1) for the single slot by

$$\sum_s q_{rs} P_s e^{js(\theta+\pi)} = \zeta_r \eta_r e^{jr\theta}, \tag{89.5}$$

where $q_{rr} = K_r$, the constant already introduced for the single slot, and

$$q_{rs} = \int_{-l}^{+l} F^0_{rs}(\xi) \cos k\xi \, d\xi \quad (r \neq s). \tag{89.6}$$

$F^0_{rs}(\xi)$ is the component of magnetic force along the rth slot due to a sinusoidal voltage of unit amplitude in the sth slot calculated for propagation outside the guide. q_{rs} may be approximated by

$$\frac{\cos\phi_r \cos\phi_s \mu_{rs}}{|r-s|d} e^{j|r-s|\theta'},$$

where μ_{rs} is dependent on the length of both slots to an extent which decreases as the slots are separated from each other. Except in the case of nearest neighbours, μ_{rs} may be taken as constant. We shall also ignore the variation of the cosine factors since the inclinations are supposed less than 10°. That is, we replace $\mu_{rs}\cos\phi_r\cos\phi_s$ by a constant μ. So

$$q_{rs} = \frac{\mu}{|r-s|d} e^{j|r-s|\theta'}.$$

Finally we have

$$K_r = \frac{\zeta_r \eta_r}{P_r} - \sum_s \frac{\mu}{|r-s|d} \frac{P_s}{P_r} e^{j|r-s|\theta' - j(r-s)(\theta+\pi)}. \tag{89.7}$$

Now all the quantities on the right-hand side are known, hence we can determine K_r and the length of each slot. It may be desirable to carry this procedure to a second approximation to improve the representation of the mutual effects of nearest neighbours.

It should be noted that the reflection of waves inside the guide is very weak because the slots are weakly coupled to the inside of the guide. On the outside this is not so: the slots are strongly series-coupled to the principal wave on the outside of the guide. Whether the representation of the propagation on the outside of the guide is sufficiently good in the form adopted here remains to be seen. At the present time the representation of wave-propagation on the outside of a rectangular wave-guide is very crude.

The factor limiting the band-width of arrays of this type is the decrease of effective conductance as the frequency is altered from the value corresponding to maximum radiation from the array, and the consequent increased dissipation in the sand-load.

8.10. Inclined Shunt Slot Arrays at Resonant Spacing

An approximate design for arrays of inclined slots in the narrow face spaced $\frac{1}{2}\lambda_g$ apart with a plunger terminating the guide can be approached through the conception of incremental conductance. Similar slots with equal inclinations alternately reversed in sign to reverse the phase of excitation are cut $\frac{1}{2}\lambda_g$ apart in the narrow face. The overall slot-length is found which will make the input susceptance to the array independent of the number of slots when that number is sufficiently great. The limiting value of the average conductance per slot of this length in a long array is called the incremental conductance, and is used as the effective conductance of the individual slots in any array in which the gradient of inclination is everywhere small. For this purpose the incremental conductance is found as a function of inclination. Arrays are then cut according to the law for shunt slots at resonant spacing without mutual admittance. The terminating plunger must be adjusted by trial. While this procedure yields usable arrays with beams radiated normal to the guide, the procedure is not good for securing the desired input impedance and amplitude distribution.

If we attempt to improve the design by following theory like that used in § 8.9, we find that both the inclinations and the lengths of the slots enter the oscillating sums, leading to very formidable numerical work in the design of an array of over fifty elements. The question is briefly discussed in § 10.10.

It is evident that when polarization is not of prime importance, the inclined shunt slots being the most easy to cut, arrays of them are particularly suitable for X- and K-band work. In the latter the losses in the guide itself should be allowed for in design.

8.11. A Broad-band Array of Slots (Transverse Polarization)

On account of the mutual interaction between the slots of a longitudinally polarized array, and the sensitivity of that interaction to frequency so that the fraction of energy radiated by the slots is fairly strongly frequency sensitive, it is clear that the band-width of these arrays is limited by the loss of energy to the terminating load. Mutual interaction by waves outside the guide operates to reduce band-width. For a broad-band slot array it is necessary to use transverse polarization. We now present a method for achieving this by means of the displaced-inclined slots cut in the broad face and described in § 6.8. The principle involved is that each radiator is so coupled to the guide that it permits a pure travelling wave from the generator to the matched termination of the guide while abstracting a known fraction of the energy from it.

The slots have small displacement (x) and inclination (θ) measured from the longitudinal unexcited position on the broad face, and so chosen that the self-corresponding point on the w-plane transformed by the slot lies at $-j$. The length of the slot is such that a match is transformed into unit conductance with a small negative susceptance. The latter is compensated by the positive susceptance due to the silver probe (of diameter $\frac{1}{8}$ in. for S-band) placed in the same guide cross-section as the centre of the slot and opposite to it in the other broad face. This disposition is required for proper compensation when a considerable fraction of energy is drawn from the guide by the end slots of the array.

Design of the array depends on the following measurements:

(i) The corresponding values of x and θ to yield the proper type of displaced inclined slots; for practical purposes, it is not necessary to treat as variable the slot-length once found for one value of θ, so that it allows perfect compensation by a probe, because only a small range of angles is required.

(ii) The compensating susceptances and the corresponding probe-lengths.

(iii) The fraction of power abstracted by the compensated slot from a travelling wave as a function of θ.

(iv) The phase shift (retardation) produced in the travelling wave as it passes the slot as a function of θ.

The desired connexions between x and θ, and the compensating susceptance are given in § 6.8, and approximate formulae may be derived to represent the data (iii) and (iv) for convenience in interpolation.

At S-band, a slot displaced 0·4 in. and inclined at 25° abstracts 40 per cent. of the energy from a travelling wave in the guide; the corresponding phase-shift is 14·4° in passing the compensated slot. Since most of the radiators in an array with its amplitude distribution tapered symmetrically about its centre will be required to radiate very much smaller fractions of power and will produce phase shift much less than 14°, we may ignore the longitudinal displacement required to compensate the latter when we are concerned with amplitude distribution, and choose as the most convenient variable to characterize a radiator, the fraction of energy to be radiated by the slot. Let e_r be the fraction of energy to be radiated by the rth slot from the generator end of the array, and f_r be the amplitude of the excitation desired for that radiator; then the equation (86.2) serves to connect e_r and f_r, except for the few slots near the end of the array which are sufficiently inclined to the guide axis to make correction necessary. For these we write

$$\frac{f_{r+1}^2}{f_r^2} = \frac{e'_{r+1}(1-e_r)}{e'_r},$$

where e'_r denotes the fraction of power radiated in the desired polarization by the rth radiator. The correction amounts to 5 per cent. in amplitude for slots inclined at 20°.

Given the f_r and having chosen the fraction e_N to be drawn by the last slot of the array of N slots, we can design the array with the aid of graphs in which are plotted against the variable e (i) the displacement

x, (ii) the inclination θ, (iii) the length p of the compensating probe, and (iv) the phase shift converted into equivalent displacements along the guide at mean frequency in the band. The curves are shown in Fig. 77 for standard S-band guide at λ 10·7 cm. The last item (iv) is used to correct the position of the centres of the slots with respect to

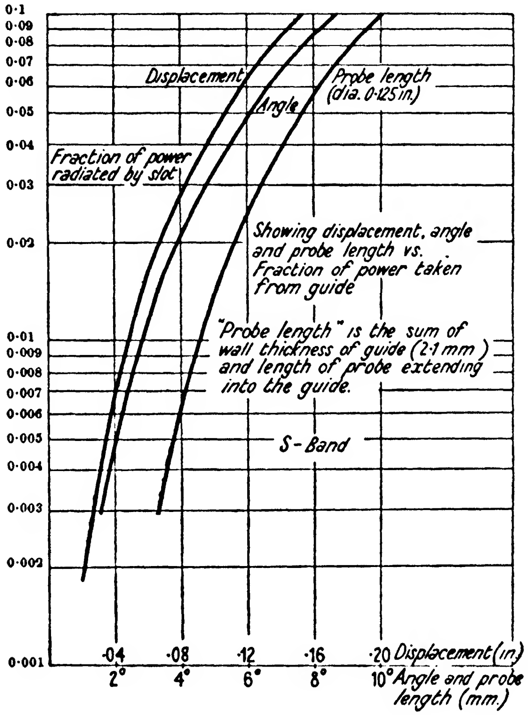


Fig. 77 (a).

the basic spacing $\frac{1}{2}\lambda_g$ calculated for the mean frequency. The correction is cumulative. For a range of frequencies off the centre of the band, the maintenance of a sufficiently good approximation to a travelling wave in the guide is assured even if the individual slot-probe combination fails to transform a match into a perfect match off frequency. Further, provided that the phase shift is taken into account, the beam from the array should be radiated normal to the guide at mean frequency.

In order to secure the alternation of phase necessary to produce the normal beam, the slot-centres will be alternately staggered on each side of the centre of the broad face of the guide. In order that the proper type of slot shall be used, the end of the slot nearer to the

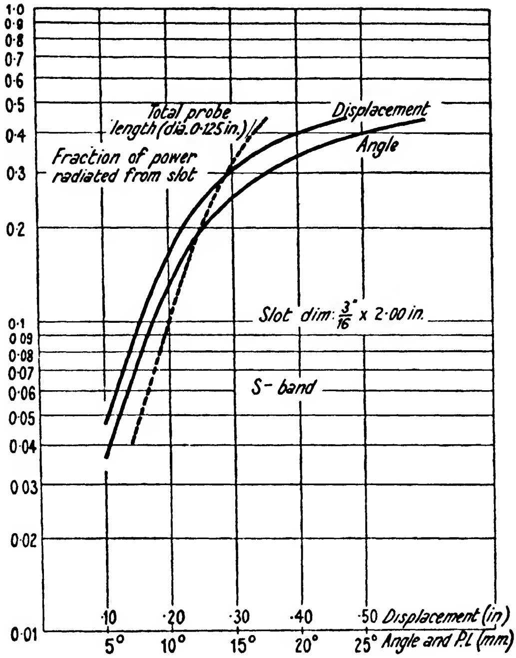


FIG. 77 (b).

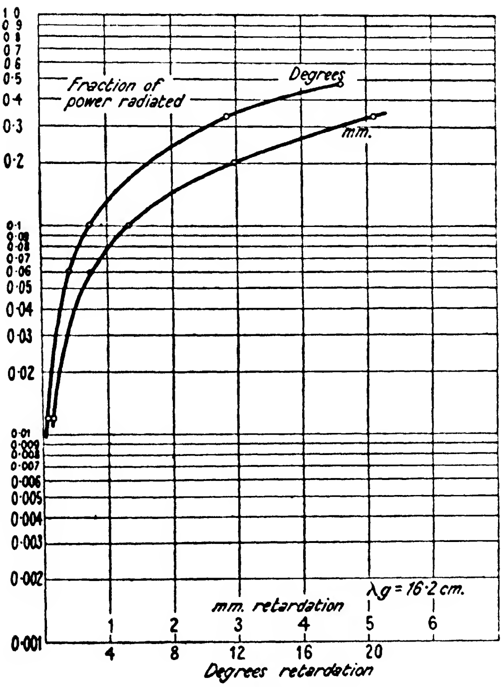


FIG. 77 (c).

generator will be nearer to the centre of the guide in every case. The phase correction is applied by bringing the slots nearer to the generator.

As an example of the application of these principles, a 45-element array was cut for S-band. The slots were $\frac{3}{16}$ in. wide, 2 in. long, and the basic spacing $\frac{1}{2}\lambda_g$ at λ 10·7 cm., and the correction for phase shift was taken into account. The energy reaching the matching load at the

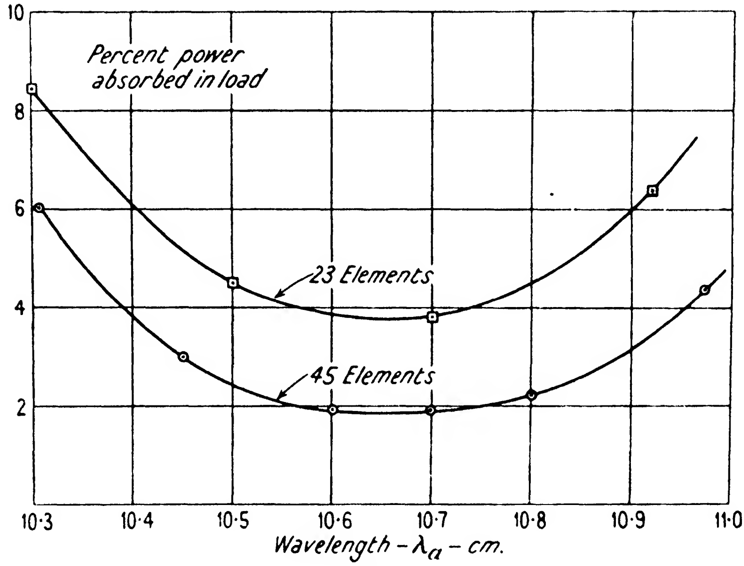


Fig. 78.

end of the guide was measured as a function of frequency, and the result is shown in Fig. 78 both for the whole array and for the last half of it. On the basis that not more than 6 per cent. of the energy shall be dissipated in the terminating load, this array has band-width ± 4 per cent., and it may confidently be predicted that a longer array would have greater band-width for satisfactory power absorption. Throughout the band the S.W.R. (voltage) never exceeded 1·12. In practice, however, it is possible to dispense with the terminating load and to terminate the guide with a plunger $\frac{3}{8}\lambda_g$ from the centre of the nearest slot. This particular reactive termination may be used because the last few slots of the array transform it into a near match, and it is found that the disturbance of phase at the end of the array is not serious. It should be noted that transformation by these slots reduces the radius of a small circle near the origin on the w-plane, thus the matching arrangement is essentially stable.

The results of the field measurements are shown in Fig. 79 (*a*)–(*j*). The average observed beam-width of the main lobe was 2·35°, which is to be compared with the theoretical 2·1° for a uniformly illuminated array of the same length. When the guide was terminated in a match the maximum side-lobe amplitude was 15 per cent. of the maximum in the main lobe, in the range 10·4–11·0 cm. used in the measurements. When the guide was terminated by a reflecting plug, the maximum side-lobe varied from 19 per cent. at 10·5 cm. to 12 per cent. at 10·8 and 16 per cent. at 11·0 cm. The lobe of unwanted polarization, due mainly to the inclination of the slots, was observed at $\pm 45°$ with the normal to the array and did not exceed 11 per cent. in amplitude. The band-width of this array is sufficiently great to allow control of the direction of the beam over 4 or 5 degrees by means of the frequency, variation of which can be tolerated ± 4 per cent. from the mean.

8.12. Two-dimensional Arrays of Slots

The linear arrays which have been described in the foregoing sections can be used as elements in a two-dimensional array of slots, which replaces the combination of a linear radiator with a reflector. The linear arrays may be either standing-wave or travelling-wave radiators. Mutual effect between them will be important only for transverse polarization, but it may be suppressed by introducing a choke between each pair of adjacent guides. This choke is a narrow channel, a quarter-wave deep, which prevents waves from travelling from the surface of one guide to the next. Care should be taken to prevent the transmission line formed by the channel from being resonant and thus becoming the source of unwanted radiation.

In the design of a two-dimensional array, the problem is to arrange that the linear arrays load the transverse feed-guide so as to give the desired transverse amplitude and phase distributions and a suitable input impedance for the array as a whole. To achieve this end, the coupling of guides by means of slots offers a wide variety of circuit arrangement. The spacing of the linear arrays, if transversely polarized, is little open to choice because of the width of the guide, but longitudinally polarized arrays may be spaced more closely. If we are willing to use a feed-guide of specially chosen width, or if the guide may be turned so as to cross the linear radiators at an angle different from right, the spacing of the loads on the transverse feed-guide may be made any desired figure, resonant or non-resonant.

There are two basic types of coupling of shunt-series type that permit

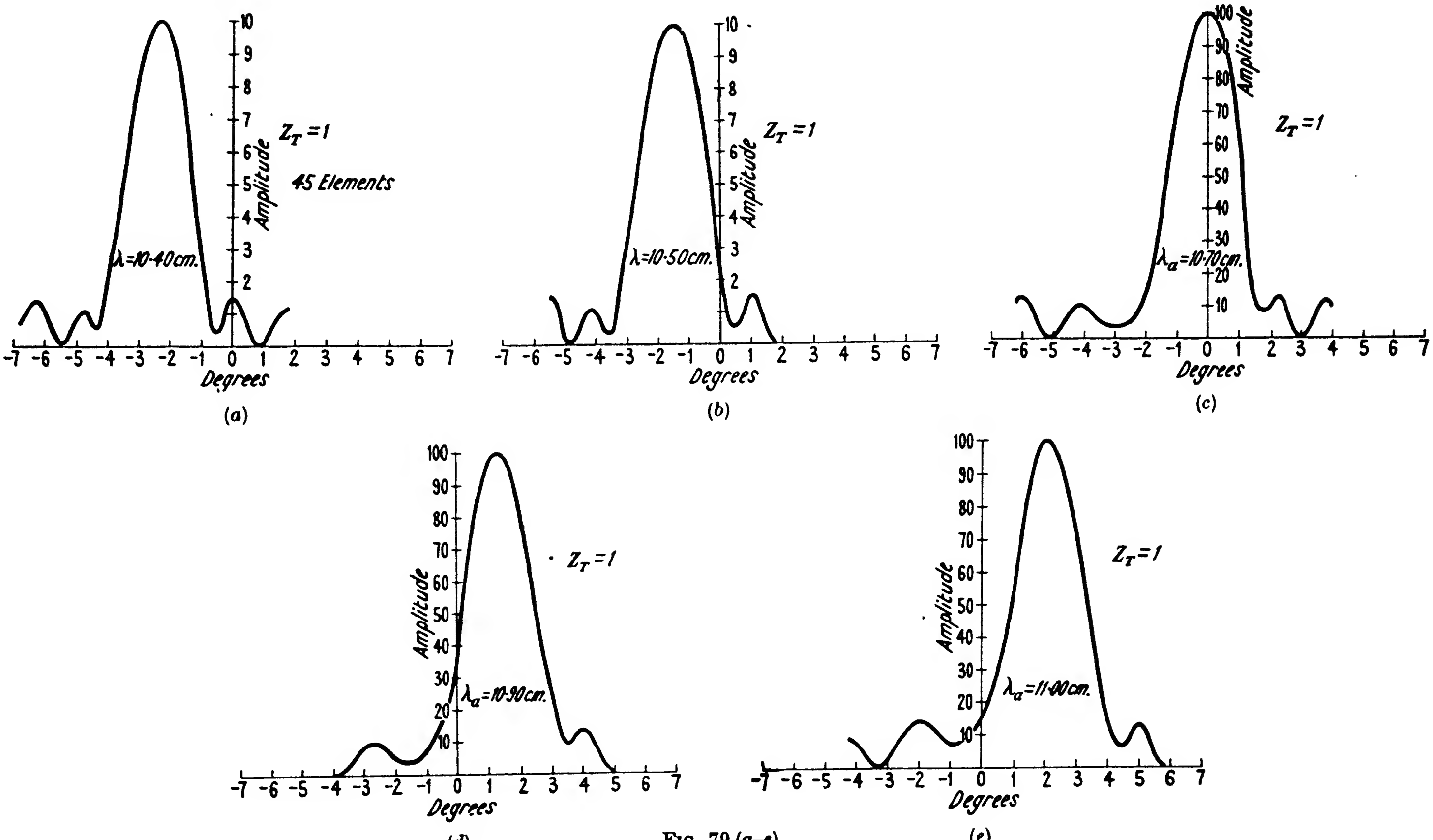
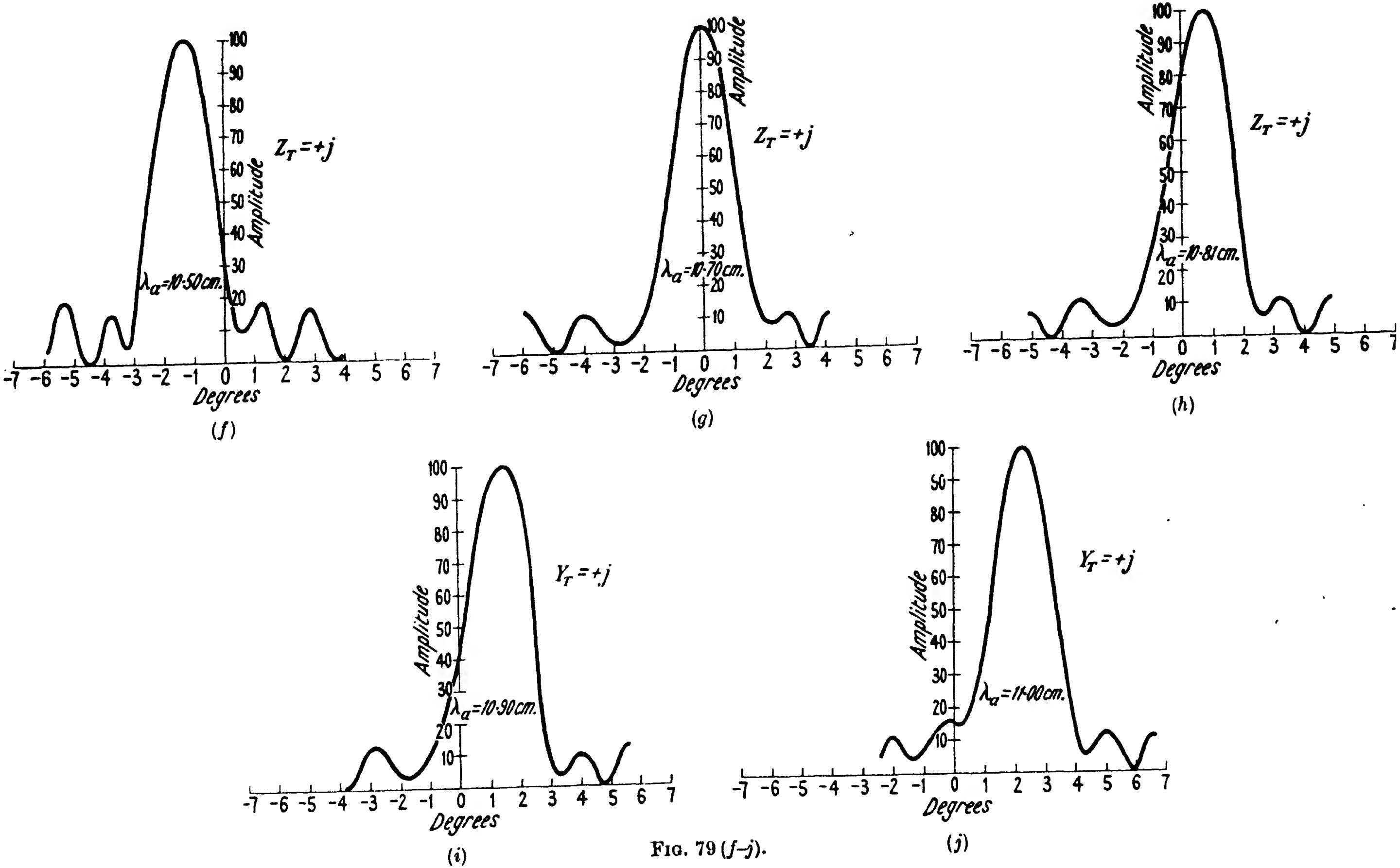

Z_T = 1
45 Elements
λ=10·40cm.
Amplitude
Degrees
(a)
Z_T = 1
λ=10·50cm.
Amplitude
Degrees
(b)
Z_T = 1
λ_a=10·70cm.
Amplitude
Degrees
(c)
Z_T = 1
λ_a=10·90cm.
Amplitude
Degrees
(d)
Z_T = 1
λ_a=11·00cm.
Amplitude
Degrees
(e)

FIG. 79 (a–e).



$Z_T = +j$
$\lambda_a = 10{\cdot}50$ cm.
Amplitude
Degrees
(f)
$Z_T = +j$
$\lambda_a = 10{\cdot}70$ cm.
Amplitude
Degrees
(g)
$Z_T = +j$
$\lambda_a = 10{\cdot}81$ cm.
Amplitude
Degrees
(h)
$Y_T = +j$
$\lambda_a = 10{\cdot}90$ cm.
Amplitude
Degrees
(i)
$Y_T = +j$
$\lambda_a = 11{\cdot}00$ cm.
Amplitude
Degrees
(j)

FIG. 79 (f–j).

choice of impedance transfer. For slots cut in the broad face the couplings have already been explained in Chapter VII, Fig. 57, while for slots cut in the narrow face of the feed-guide the arrangement was depicted in Fig. 63. With both types of coupling the system of wave guides forming the whole antenna can easily be rigidly bound together.

We shall now briefly describe the first two-dimensional array of slots which was built at the Radio Field Station of the National Research Council of Canada. The beam desired was to have 2° width in azimuth and about 15° width in elevation. Six horizontal linear wave-guide radiators were stacked vertically and centre-fed from a transverse feed-guide on which they were spaced $0{\cdot}55\lambda_g$ apart with phase inversion in the alternate couplings. With this arrangement it was possible to feed the transverse guide from either end, and since the main lobe does not point normal to the plane of the two-dimensional array, the elevation of the beam could be altered by means of a switch which caused first one end of the feed and then the other to be connected to the generator.

Since the series-shunt couplings act as transformers, the impedance seen from the centre in the radiating guides, each terminated by a plunger $\frac{1}{4}\lambda_g$ from the nearest slot, is at our disposal, provided that the coupling coefficient is correspondingly adjusted by proper displacement of the coupling slot. The impedance was finally chosen to give best matching band-width for the whole antenna.

The antenna is shown in the Frontispiece mounted on a trailer containing the transmitter. The upper two-dimensional array of three linear radiators is intended to give a less directive beam in elevation. The staggered slots, which are all covered, can be discerned most easily in the lowest guide.

IX

FURTHER MICROWAVE DEVICES

THE plan of this chapter is to deal with a number of loosely related topics connected with the use of wave guides as transmission lines and in antennas. Interest will of course lie mainly in the physical principles involved rather than in the details of particular practical devices.

9.1. Scanning

The highly directive antennas which we have considered in the preceding chapter would obviously be of little operational use unless means are provided to turn them. It is not proposed here to consider the mechanical problems involved. In order to increase the speed of rotation and hence of scanning it is necessary to reduce the weight and size of the antenna and therefore to use the shortest possible microwaves consistent with power or other requirements. For the purpose of securing rapid scan over a limited sector, without turning the antenna as a whole to obtain the scan, several devices have been invented. All of them depend on making the rotation of some part of the device impose a phase gradient across the aperture of the antenna so that the direction of the main lobe is determined by the moving part. For example [23], a feed-guide may be rotated opposite the annular mouth of a thin flat horn which has been rolled into a complete circle at the feed end. Another successful scanning device, due to Foster, introduces the phase gradient into a beam of wide aperture by varying the path across the whole beam. The variable path is the space between two coaxial cones limited by the diametrically opposite input and output slots in the outer cone and the slot through the centre of the inner cone along its length. The inner cone rotates. The main practical problem to be surmounted is to shut off the alternative paths round the inner cone while maintaining a good approximation to matched transfer of energy through the system. It is not our intention to discuss the details of such devices here. We confine ourselves to the problem of scanning with wave-guide arrays.

Steerable-beam antennas depending on the phasing of the elements of an array are well known in short-wave radio work, but to use independent phase changers in a long microwave array of many elements is not practical. From equation (81.2) we see that to produce a 10° change in the azimuth of the main lobe of an array of equispaced elements $0{\cdot}7\lambda$ apart, it is necessary to introduce a phase shift of 48° between each pair of adjacent elements; that is, about 20 per cent. change in the free-space path-length is required to achieve the change in azimuth of the beam. In the wave-guide feed there are only two practical ways of doing this—by changing the frequency of the generator, which is not possible for a high-power magnetron, or by changing the velocity of propagation in the guide through varying its width. The latter alternative has been followed with a travelling-wave antenna. The guide is specially made in two halves which are moved transversely to their length, thus causing the guide to vary periodically in width (a). Since the coupling of any radiator to the guide depends on the width, the main difficulties, when a is varied, concern input impedance and satisfactory absorption of energy by the radiators. The amplitude distribution cannot remain unchanged in the process, consequently a scanning array of this type will be successful only if an adequate compromise can be found; and this is decided by what can be tolerated in practice. The array will scan only on one side of the normal to the guide when the power is fed from one end.

It is easily seen that microwave arrays are not suited for scanning by electrical phasing except over comparatively small angles, because of the difficulty of introducing the large fractional change in path required to turn the beam. In all other cases of scanning antennas, the mechanical device producing the beam swing must have outstanding advantages over that required to turn the antenna as a whole; otherwise its use is not justified.

Two-dimensional (television) scanning is achieved by superposing on the azimuthal displacement of the beam a transverse oscillation which may be produced by rocking the reflector.

9.2. The Cosecant Pattern

A horizontal array designed to give a sharp beam in azimuth and placed along the focal line of a parabolic cylinder of sufficiently great aperture ($> 10\lambda$) produces a narrow beam in elevation, which is satisfactory for height (or depth) coverage at great ranges. At short range, however, effective coverage of vertical height as the beam swings round with the antenna is very much reduced, and so long as the beam is formed in the simple way just mentioned, adequate vertical coverage at short range will be obtained only at the expense of the maximum range obtainable with given power input to the antenna. Indeed, for radar used in aircraft to view the ground, or vice versa, the radiation pattern required in elevation should be asymmetrical about the direction of maximum intensity in the beam, being sharply cut off on one side, and trailing gradually on the other. If the maximum difference in height to be viewed is given, it is obvious that the trailing pattern should yield an energy distribution approximating the cosec$^2\theta$ law [24], θ being the angle measured in the vertical plane from the cut-off side of maximum (see Fig. 80), so that the ground or sky at the chosen height may be illuminated properly. Ideally one wishes to obtain return

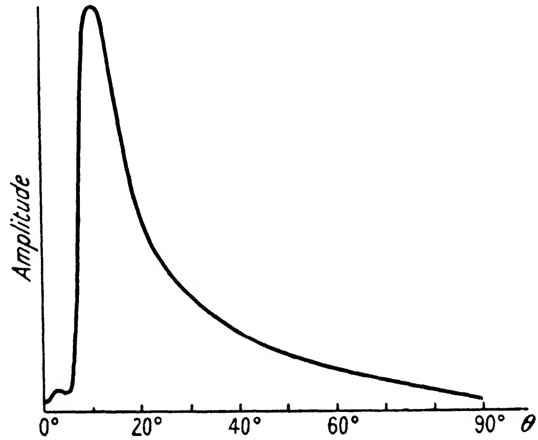


FIG. 80.

signals substantially independent of range up to the maximum range considered practically attainable with the power available, so Lambert's law of diffuse scattering might be taken into account, thus requiring a cosec$^3\theta$ distribution, but experience does not confirm this.

The essential difficulty in producing the beam which we have just described lies in securing with a comparatively narrow aperture the sharp cut-off feature, the strong maximum intensity, and at the same time avoiding the appearance of side-lobe structure in the rest of the pattern. The reflector being illuminated in the normal way, we can obtain the cut-off by splitting the reflector or deforming it so as to obtain radiation normal to the antenna in antiphase from the two halves. In theory this can be done by displacing one half of the reflector $\frac{1}{4}\lambda$ behind the other, but in practice it does not yield good overall results. It is better to replace a segment of the paraboloid by a plane flap or, if the reflector lies only on one side of the axial plane of the paraboloid, one may alter a portion of it near the axis bringing it closer to the focus to get the desired result.

For the rest, in order to obtain a satisfactory beam the illumination of the deformed reflector must be properly adjusted. While it is possible to calculate the required law of flooding, it is quite another matter to secure it from a linear radiator. The problem therefore becomes essentially a practical one of experimental compromise.

If instead of a parabolic reflector we use a two-dimensional array, we can obtain the required cut-off by mounting the array below (or above) a plane sheet of metal projecting about 10λ in front of the array. By using arrays of inclined shunt slots we have some freedom to adjust the 'fill' in the pattern by turning these linear arrays about the guide axes, and arranging the strength of coupling to the feed through inclined coupling slots in the broad face of the latter, and by choosing the transverse spacing between the linear arrays and the reflector so as to improve the pattern. A satisfactory antenna of this type was constructed by Guptill early in 1944.

9.3. Parasitic Radiators: Microwave Yagi

In short-wave practice, end-fire arrays are used as directive antennas, the 'half-wave' wire radiators being transverse to the axis of the array. Only one of these radiators need be driven by the transmitter; the others are parasitic, excited by the radiation from the driven member. One of the parasites is placed about $\frac{1}{4}\lambda$ behind the driven radiator to act as a reflector; one or more others act as directors spaced about $0{\cdot}3\lambda$ along the line of sight. Such a linear array is called a Yagi antenna. The critical parameters are the lengths of the wire radiators and their spacing. When the wavelength is about a metre, the centres of the parasites are attached to a metal rod to which the driving element is joined with metal by folded dipole technique. Successful Yagi antennas for S-band have been made likewise, but for X and K bands the sizes are inconvenient. The Yagi principle may be applied to direct waves over a metal sheet by using 'quarter-wave' wire antennas mounted at right angles to the sheet and in line. This will not be a convenient antenna, and to make it will require fastidious workmanship.

We have seen that a resonant slot cut in the narrow face of a rectangular wave guide at right angles to the axis of the guide is unexcited by the dominant wave in the guide, and that the mutual coupling of inclined slots cut in the narrow face and weakly coupled to the guide takes place by waves on the outside of the guide and plays a large part in determining the admittance presented to the guide by these slots. These two properties make possible the use of slots cut in the narrow face as parasitic radiators and their employment in microwave antennas. Such slots must be cut at right angles to the axis of propagation in the guide.

An example of this type of antenna was constructed as follows.† In each of the narrow faces of a piece of X-band guide was cut an array of slots. Each array consisted of one 20° inclined slot coupled to the guide with a reflecting plunger $\frac{1}{4}\lambda_g$ behind it inside the guide, a single parasitic reflector, and fifteen director parasitic slots at 0·31λ spacing. The two arrays were excited in antiphase and their fields allowed to join in front at the edge of the wedge in which the guide carrying the slots was

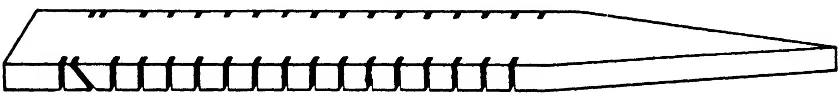


Fig. 81.

terminated in order to minimize reflection. The details are shown in Fig. 81. The radiation is polarized parallel to the broad face of the guide.

The microwave Yagi has all the advantages of the wire Yagis for longer waves; namely, small cross-section presented to the direction of the main lobe which is approximately axially symmetric, extreme portability, and simplicity. The advantage of the slotted Yagi antenna is that it is easily made. Its directivity is limited, however, by the fact that it is an end-fire array.

9.4. Switching Devices

In d.c. and a.c. devices the opening of a circuit to change the path of energy flow is attended by transient phenomena which, for practical purposes, may be suppressed by suitable adjustment of the time-constants of the circuit. Once the circuit is open, no current flows in the steady state. But in a microwave circuit this is not so. The steady state consists in the reflection of waves from the open circuit; this is equivalent to presenting a reactance to the generator. In order to render the circuit quiescent, it is necessary to put the generator out of oscillation. Consequently in microwave work one may have to couple the switching device to the generator to do this. If the switching is repeated, the generator will be pulse-modulated.

There are two main types of switching device: (1) those which modify the circuit by mechanical means, pieces of metal being moved and the rapidity of switching severely limited by this fact, and (2) electronic

† Watson, Guptill, and Terroux, unpublished note, Feb. 1944.

devices which may be triggered by suitable transient waves of voltage applied to auxiliary electrodes. In the former, power-carrying capacity is usually not difficult to achieve, in the latter it is the most important limiting factor in design. When the peak microwave power is not great, a much greater variety of switching devices is possible and consequently greater flexibility of control.

In order to block the passage of microwaves along a wave guide without a metallic partition in the guide, one may couple a second guide to the first. If the coupling in the first guide is series, the reactance presented in the second must be such as to introduce infinite reactance in the first in order to open-circuit it. If the coupling is shunt, the reactance presented in the second guide must be such as to short-circuit the first. As an example of this principle we show a very simple switching device [25] in Fig. 82. It may be mechanically driven for repeated transfer of power alternately into the arms A and B. The narrow faces of the guides are presented in the diagram, so that all couplings are series. When energy is to be fed into A, the plunger a is placed $\frac{1}{2}\lambda_g$ from the junction, so completing the circuit to the guide arm A; meanwhile the plunger b is placed $\frac{1}{4}\lambda_g$ from its coupling, so open-circuiting the guide arm B. If A and B are both matched, it is possible to couple the motions of the plungers so that the input impedance of the whole device does not depart markedly from unity at any time in the cycle of operations.

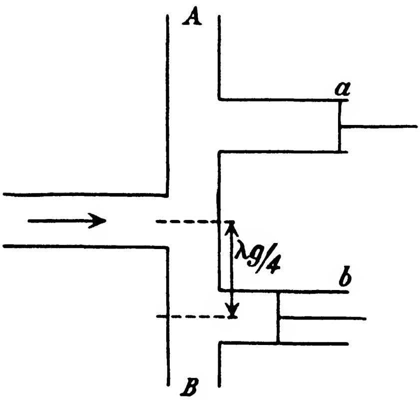


FIG. 82.

The stubs with the moving plungers may be replaced by electronic switches. Let a resonant iris in a guide be enclosed with inert gas at low pressure. When the gas conducts, the iris ceases to transmit; thus in principle we have a switch. A tuned cavity can likewise be detuned and when suitably coupled to the main guide will effectively short- or open-circuit the guide when the arc strikes across the narrow gap surrounded by inert gas inside the cavity or rhumbatron. The arc may be triggered by means of a pulse of voltage applied to an auxiliary electrode in the rhumbatron. The speed of switching is determined by the de-ionization time of the space where the arc occurs.

By far the most important switch is the *TR*-box [26] used for duplexing so that both transmission and reception may be accomplished with a single antenna. In the receiver line, at the proper distance from the junction, is the soft rhumbatron by means of which the receiver is effectively disconnected when the arc passes across the gap between the axial electrodes of the cavity. These are usually conical or cusp-shaped, thus giving the resonant cavity the general shape of a toroid except for the gap. Actually what happens is that when the arc is struck, the cavity is put out of resonance and the admittance presented by the receiver line at the junction is very small. Whereas when the arc is off, the distance of the transmitting generator from the junction being chosen so that a low admittance is presented at the junction by the transmitter stub, the rhumbatron, which is of fairly high Q, effectively couples the receiver to the junction without reflection, even although the two couplings into the rhumbatron are weak. When coaxial line is used, the couplings in question will usually be by means of loops; when wave guide is used, coupling is effected by inductive windows or by holes. It would be out of place in the present work to attempt to treat the immense amount of practical information which has been gathered on the design and perfecting of this device. It is appropriate, however, to mention the names of Sutton of the Clarendon Laboratory, and Samuel and Fisk of the Bell Laboratories, as the leading pioneers in its development. It is fairly obvious that when high power is to be transmitted the device must give adequate protection to the receiver and at the same time it must provide efficient reception. In this connexion the reader is referred to § 9.7 on cavities.

Low-power electronic or other switching devices designed for use in coaxial line may be coupled to an antenna stretched across the wave guide, and by tuning and detuning the antenna, short-circuit or open-circuit the guide. Since in low-power devices one may expect to work with narrow band-width, a high-Q cavity may be used in switching, deformation of the cavity being used as a method of detuning it. A convenient form of cavity for this purpose is the rhumbatron of a klystron oscillator shown in section in Fig. 83. A telephone receiver diaphragm forms the flat circular base, and outside the cavity is mounted opposite the diaphragm the magnet-coil system of the receiver. This device permits switching at telephonic frequency.

As examples of simple mechanical switching devices which do not involve the movement of plungers we may mention the useful ring switch and a rotating switch employing the principles of guide coupling.

In § 5.9 it was pointed out that a wire antenna insulated at both ends will effectively short-circuit the guide when the length of the antenna is close to $\frac{1}{2}\lambda$ and part of it is parallel to the electric force in the dominant wave; it is clear that if two ends of the wire are joined so as to make a ring, which need not be exactly circular, the condition for

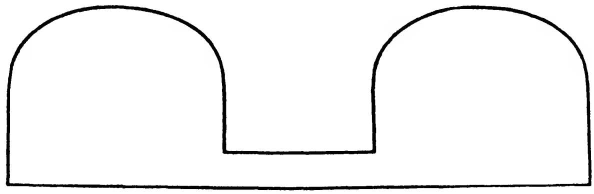


Fig. 83.

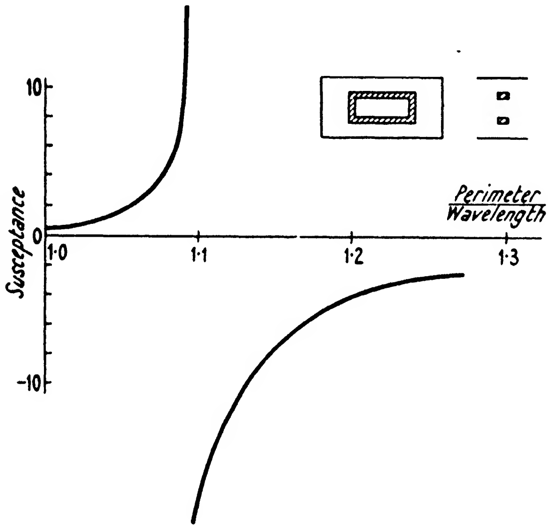


Fig. 84.

resonance and effective short-circuiting of the dominant wave is that the perimeter of the ring should be about a wavelength—actually it should have a particular length somewhat greater than λ. The general shape of the graph of susceptance vs. perimeter of ring in wavelengths is shown in Fig. 84 [27]. With the resonant perimeter, the ring behaves as a perfect reflector. It may be used to switch radiation at a junction, as indicated in Fig. 85. When the proper dimensions have been found, it is possible to have substantially matched transmission in either direction from the junction.

The rotating wave-guide switch† is particularly suited for X-band

† Johnston, Terroux, and Watson, *N.R.C. Report*, June 1944.

or higher frequency radiation. A complete ring of wave guide is made by cutting a groove of rectangular section in a thick metal plate, and closing the top by means of a second plate which can rotate with respect to the first about the axis of circular symmetry. Leakage from the guide is prevented by choke grooves concentric with the ring. Radiation is fed into the guide through the lower plate by a series-series resonant slot coupling, and proceeds only in one direction from the coupling point because on the other side is coupled a series-stub which open-circuits the guide $\frac{1}{4}\lambda_g$ (mod $\frac{1}{2}\lambda_g$) from the main coupling.† Similar output couplings are arranged at intervals on the upper plate, as shown in Fig. 86.

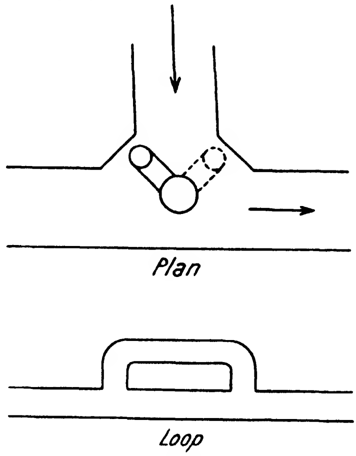


Fig. 85.

Radiation is propagated in the ring wave guide only between the input

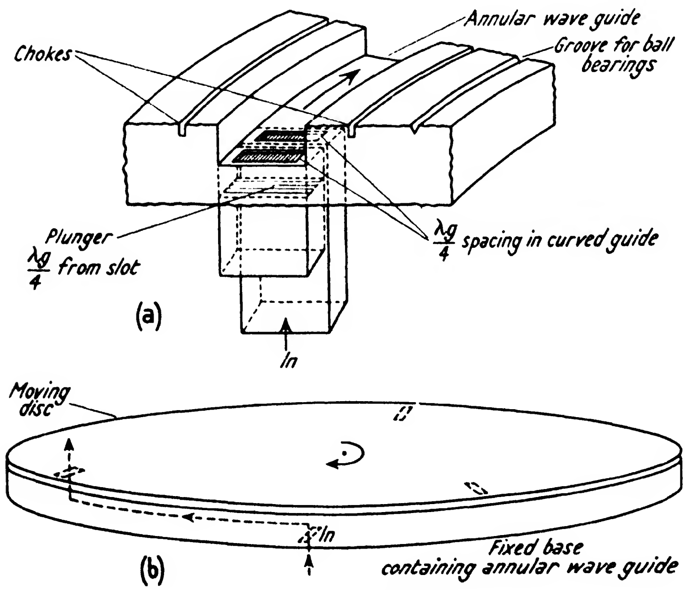


Fig. 86 (*a* and *b*).

coupling and the nearest output coupling in the proper direction. Thus as the upper plate rotates, radiation is fed successively to the wave

† The directive coupling of § 7.5 was found more difficult to make.

guides attached to it. This permits a cycle of switching operations and would also allow a multiple antenna to scan a limited sector of wide angle by continuous rotation. Blanking is required during the period when the switch-over occurs and a coupling point in the upper plate is opposite the input coupling in the lower one.

9.5. Phase Changers

The usefulness of apparatus capable of changing the phase of microwave radiation in a wave guide without altering amplitude is obvious. We have already mentioned in connexion with the measurement of phase the basic method of altering path-length which may be easily accomplished in coaxial line but not in a wave guide; consequently the latter is used only when variable phase change is not required. We shall describe two methods of changing phase which ensure matched transmission through the phase changer.

The first consists in coupling a series and a shunt reactance to the guide by means of resonant slots centred in the same guide cross-section. This arrangement functions as a symmetrical Π- (or T-)section. We have already discussed this way of loading the guide in § 6.8 and found that the susceptance (b) of the shunt must equal the reactance (x) of the series load. The loading matrix (see (68.5)) is

$$\begin{pmatrix} 1+\dfrac{jb}{1-\frac{1}{2}jb} & 0 \\ 0 & 1-\dfrac{jb}{1-\frac{1}{2}jb} \end{pmatrix}.$$

When the guide beyond the load is matched, the input is also matched, for the loading matrix is diagonal. The ratio of the output to the input complex wave amplitude is

$$\frac{1-\frac{1}{2}jb}{1+\frac{1}{2}jb}.$$

The modulus of this ratio is 1 and its argument is $2\tan^{-1}\frac{1}{2}b$. The latter is therefore the delay in phase introduced by the arrangement, when b is positive, i.e. corresponds to capacitive shunt reactance. All that is now required is a convenient method for securing the equality of b and x. Let x be introduced by a series-series coupling to the top broad face of the guide and b by a shunt-shunt coupling to the bottom broad face (see Fig. 87), the two auxiliary guides of the same cross-section as the main guide being closed on one side $\frac{1}{2}\lambda_g$ and $\frac{1}{4}\lambda_g$ respectively from the coupling while on the other side a plunger slides in each. If the plungers are set to correspond—i.e. $\frac{1}{4}\lambda_g$ apart—then they will continue to do so when displaced through the same distance. If this distance (ξ) is reckoned from the position for $b = 0$, the phase shift ϕ is given by

$$\phi = 2\tan^{-1}\tfrac{1}{2}\tan(k_g\xi), \qquad k_g = \frac{2\pi}{\lambda_g}.$$

The second type of phase changer [28] with which we shall deal is

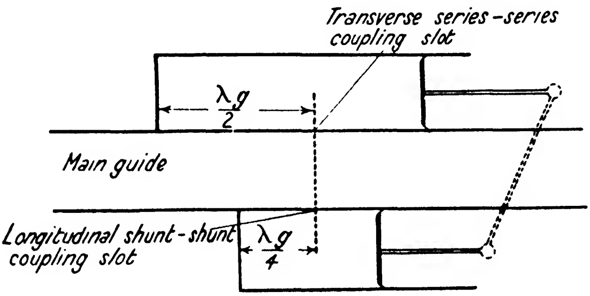


FIG. 87.

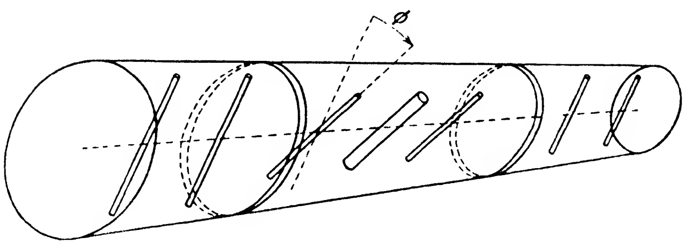


FIG. 88.

more troublesome to make, but once made, is very much more convenient to use since the phase change introduced by means of it is equal to twice the angle through which one of its parts is rotated. The device consists of three pieces of circular guide, two of which serve to convert a linearly polarized H_{11}-wave into circularly polarized and vice versa. These parts are fixed, and the third piece of circular guide coaxial with the others is free to rotate between them about the axis of the system.

The transformation from linear to circular polarization is achieved as follows. The guide is shunted by a pair of parallel wires $\frac{3}{8}\lambda_g$ apart. These extend diametrically across the guide, and their common diameter is chosen such as to make the inductive reactance each presents as a shunt to the H_{11}-wave, polarized with its electric force parallel to the wires in the centre of the guide, equal to one-half of the guide impedance for this wave. That this arrangement will cause a 90° phase lag without reflection is readily checked by the matrix method. The matrix transforming from the immediate right of the second wire met by the waves from the generator, to the immediate left of the other, is

$$M = (E+2jU_1)P(E+2jU_1) = P+2j(U_1P+PU_1)-4U_1PU_1,$$

where P and U_1 are the matrices introduced in Chapter I. M is readily reduced to the form

$$\begin{pmatrix} j\omega & 0 \\ 0 & -j\omega^{-1} \end{pmatrix},$$

which shows the 90° phase change without change of amplitude. Thus, by means of this device, linear polarization is converted into circular polarization if the incident wave is linearly polarized so that the diameter coincident with its electric force makes 45° with the wires; for radiation polarized perpendicular to the wires, which are thin, passes by unchanged. Obviously the process is reversible.

The central section of the phase changer is equivalent to two of the sections just described merged into a single unit, there being three parallel wires spaced $\frac{3}{8}\lambda_g$ apart, the middle one showing twice the admittance of each of the others for waves polarized parallel to the wires. These H_{11}-waves are now retarded 180° in passing through this wave-guide circuit. Thus the direction of oscillation in a circularly polarized wave is derived from that in a similar wave traversing the same length of unloaded guide by reflecting the direction of oscillation of the latter in the plane of the wires. Since the direction of oscillation rotates at a constant rate in propagation, this reflection is equivalent to changing the phase of the circularly polarized wave by 2ϕ, where ϕ is the angle between the plane of the wires and one of the two directions referred to. The sense of phase change is retardation. Thus rotation of this central section of circular guide changes the phase of the linearly polarized H_{11}-wave emerging from the third section.

The main practical difficulties in this apparatus concern coupling from rectangular to circular guide, the suppression of resonance in the system, and the fact that the apparatus is designed for one frequency. So far as power-handling capacity is concerned this device is satisfactory, whereas the simple Π-section in rectangular guide is strictly limited in this respect even when the voltage required to cause sparking at the resonant slot couplings has been raised by using wide slots reinforced by cylindrical bulwarks covering the edges of the slots to reduce the electric field strength (see Fig. 89).

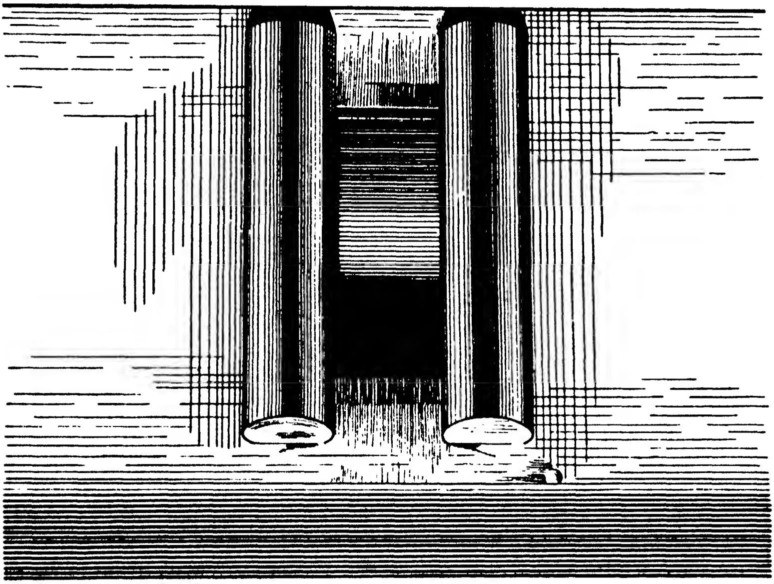


FIG. 89.

9.6. Radiation and Coupling by Small Circular Holes

Small circular holes have sometimes to be drilled in a wave guide. It is well to realize [29] that the radiation by such a hole is not to be calculated by the Kirchhoff-Huygens method of physical optics. The radius (a) of the hole being small compared with λ, the incident field on the hole can be treated without regard to phase retardation across it. Through Babinet's principle, the problem of a hole in an infinite conducting sheet exposed to radiation is identical with the problem of a conducting disc in the conjugate applied field. We may replace the hole by a distribution of magnetic charge and current. The latter arises partly from the tangential component of magnetic force and partly from the normal component of electric force. This distribution may be regarded as radiating on the other side of the screen. It is easily shown that the hole is equivalent to a magnetic doublet of strength $-(2/3\pi)a^3H_0$ amp.-m.^{2} with its axis in the plane of the sheet, and an electric doublet $-(\kappa_0/3\pi)a^3E_0$ coul.-m. perpendicular to the sheet, E_0 being, of course, the electric force (normal to the sheet) and H_0 the tangentially directed magnetic force in the absence of the hole. The doublets are directed in sense opposite to the applied field, and the radiation from them to space or in a guide may be calculated without difficulty.

Two guides may be coupled by such a hole. Now we have already seen that a transverse electric dipole is shunt-coupled to the dominant *TE*-wave in the guide, and likewise the longitudinal magnetic dipole: but a transverse magnetic dipole is series-coupled, hence the hole behaves as a directive antenna; by suitable orientation of the two guides, or if they are parallel to each other, broad face to broad face, by suitable displacement of the hole from the centre, it is possible to arrange that radiation takes place in one direction only in the second guide. Thus, samples of the waves travelling in the two opposed directions in the main guide may be separated to travel in opposite directions in the coupled guide away from the hole. This principle is used in the measurement of S.W.R. Two meters are required, but no motion of a travelling probe is involved. A pair of holes separated $\frac{1}{4}\lambda_g$ serves as a directive coupling [30].

9.7. Cavities

We have had occasion to refer to two important uses of resonant cavities in connexion with wave guides, first in wavemeters and later in the *TR* switch. We now consider cavities with general principles in view. Except when the cavity is coupled to a beam or cloud of electrons from which energy is derived to maintain the electromagnetic oscillation, the coupling of other systems to the cavity will be weak in general, so we turn our attention in the first place to the properties of the cavity by itself. These depend on its shape and size and on the conductivity of the metal of which the containing wall is made, and are summarized physically in the spectrum of resonant frequencies, the field distributions in the corresponding modes of oscillation, and the Q of the oscillation for each resonant frequency. Since generally the best conductor will be used to form the walls, the conductivity of the latter does not enter at all in the representation of the spectrum and field distributions. If it did we should be deprived of the essentially simple notions of pure *TE* and pure *TM* fundamental modes of oscillation just as in wave-guide propagation (§ 4.5).

Accordingly the calculation of the properties of a cavity can be split into two parts. First, the resonant frequencies and corresponding modes of oscillation are determined on the assumption that the wall is perfectly conducting: then one can determine approximately the Q-value for each mode, defined as follows. Let $\omega_s/2\pi$ be the natural frequency of the sth mode, and Q_s the corresponding Q-value, then ω_s/Q_s is the average fractional rate of dissipation of the energy of the cavity oscillation in the sth mode. The corresponding rate of decay of field strength is $\omega_s/2Q_s$. The proportionate band-width $\Delta\omega/\omega$ of cavity response in forced oscillation reckoned at half-amplitude, i.e. 6 db. below peak, is $\sqrt{3}/Q_s$.

Exact analytical expressions for the resonant frequencies of the normal modes of both TE and TM types together with formulae for the field distribution have been deduced for cavities of simple shape—the rectangular prism and cylindrical box (made in each case, for instance, by short-circuiting a piece of wave guide at both ends) and the space between two coaxial cylinders bounded by two planes perpendicular to the axis are all well known.† For practical purposes, the study of the cylindrical cavity with a centre post that may be regarded as the incomplete inner conductor in a shorted coaxial line has been followed in some detail both in theory‡ and experimentally as to spectrum;|| while Bethe, Marshak, and Schwinger [31] have devoted considerable attention to the fundamental mode of cavities of this shape to determine how the natural frequency and Q-value depend on the various geometrical parameters involved. In the theoretical treatment of such a case one must keep in mind that the geometrical form discussed in the theory does not represent the cavities actually used, except topologically. Practical considerations enter to require shapes that are not easy to treat analytically, although they may be managed numerically by methods such as that of Klanfer and Motz (§ 5.7). Generally speaking, however, it is more economical to investigate details by the experimental study of models. The idealized geometrical system is suitable as a good rough guide as to general trends.

With the hollow cylindrical resonator, Stevenson [32] has discussed the best modes to use, the best ways of exciting such modes by a wire-probe or loop in order to obtain maximum Q, taking into account practical restrictions as to the maximum allowable size for the cavity and proximity of neighbouring modes. The procedure for treating the excitation of oscillations is exactly analogous to that used to calculate the field due to a given distribution of currents in a wave

† Schelkunoff, op. cit.
‡ Hansen, W. W., *J. Applied Phys.* **9**, 654–63 (1938); **10**, 38–46 (1939).
|| Barrow, W. L., and Mieher, W. W., *Proc. I.R.E.* **48**, 184 (1940).

guide; it is discussed in § 10.6. It can be shown that the Q_s of the mode s of any cavity can always be expressed in the form

$$\frac{Q_s}{\sqrt{\lambda_s}} = F_s\sqrt{\sigma},$$

where λ_s is the free-space wavelength of the sth mode, σ is the conductivity of the metal walls, and F_s is a dimensionless factor determined by the mode in question and by the ratios of the linear dimensions of the cavity. The subscript s of course stands for the set of integers required to distinguish each mode of the E and H types. If the cavity is symmetrical like a circular cylinder or square prism, degeneracy will occur; that is, each natural frequency may correspond to more than one single mode of oscillation as judged by the field distribution. This fact must be kept in mind when the cavity is loaded, or coupled to another system, for then in general the degeneracy will be removed, and two or three natural frequencies close to each other will appear in the spectrum in place of one.

The response of a cavity to a particular current distribution of frequency $\omega/2\pi$ introduced into it by means of a wire, for example, can be represented when Q_s is large, in terms of coupling coefficients which are proportional to $\left|\omega^2-\omega_s^2+\frac{j\omega_s^2}{Q_s}\right|^{-1}$ and depend linearly on integrals of the type

$$\iiint I_\xi\, E_\xi^{(s)}\, dxdydz$$

over the current distribution. I_ξ is the ξ-component of exciting current density; E_ξ is the ξ-component of electric force in the sth mode in free oscillation. A cavity may, of course, be excited through a circular hole in its wall or through a window in a wave guide abutting the cavity wall. In both of these cases one must distinguish the magnetic excitation of the cavity from the electric in order to understand coupling to the cavity. Again the calculation follows the corresponding one for the excitation of a wave guide through an aperture.

In representing the behaviour of actual cavities we should take into account the effect of coupling arrangements on the natural frequency and Q of the cavity used normally in its lowest-order mode. For example, capacitive or loop coupling increases the natural frequency whereas an inductive window decreases the natural fundamental frequency of a cylindrical cavity with centre post. Also included will be the effect of dielectrics used to close the cavity, e.g. so that it may be evacuated. Usually a cavity will be coupled to at least two systems which it is intended to couple only through oscillations in the chosen mode of the cavity. Actually, coupling may also take place through the excitation of higher-order modes and also directly. The presence of these other modes of coupling may actually vitiate decoupling which is intended by detuning the cavity in respect to its lowest-order mode.

9.8. The 'Magic-Tee' Junction

The arrangement of three coupled guides which was introduced as the Π-section phase changer (§ 9.5) or in the more simple form shown

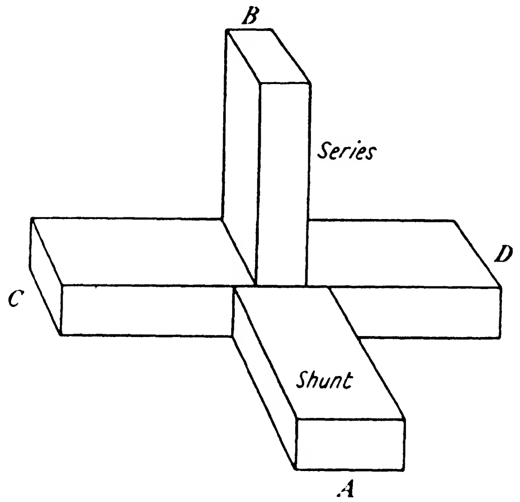


FIG. 90.

in Fig. 90, and known as the Magic-Tee junction [33], can be applied as a balanced mixing-device. Instead of regarding the guides A and B as loading the main guide CD, we now imagine C and D as terminated in matched loads and power-fed into B and A from two sources.

Since the arm B is series-coupled to the guide CD, it will radiate into the latter waves of equal amplitude travelling in opposite directions from the junction and in opposed phase at equal distances. On the other hand, the arm A being shunt-coupled, radiates into CD waves of equal amplitude travelling in opposite directions from the junction and in phase at equal distances therefrom. Further, there is no direct coupling of the arms A and B. It follows, therefore, that if two signals are fed into A and B there will be radiated towards one of C and D the sum of the signals and into the other the difference. If crystals are coupled to the arms C and D and are connected to the opposite ends of a centre-tapped I.F.-transformer, the local oscillator coupled to

B and to A the signal to be detected, there results a very useful balanced mixing device. The main advantages of the system when the arms are all properly matched are that there is no loss of signal into the local oscillator, and that the system rejects local oscillator noise.

The junction may also be used as an R.F. bridge. Power is fed to the shunt branch A and a detector terminates the series branch B. The arm C is terminated by the unknown impedance and D by a calibrated impedance such as a movable plunger behind a variable attenuator. The detector current will be zero when the two impedances are equal, for waves from C and D of equal amplitude and in phase do not excite the series-coupled arm B.

X

FIELD REPRESENTATIONS

10.1. Introduction

For the most part in the foregoing chapters the phenomena concerning wave guides have been treated by representing only propagation and discontinuities in propagation of the individual characteristic waves. In not explicitly exhibiting the actual distribution of electric and magnetic force in space and time, we are in fact abstracting part of the field representation of the waves at such distances from antennas and obstructions in the guide that all evanescent waves may be ignored. We have seen how, for theory, the wave-conception 'radiation coefficient' is a much more useful one in general than the circuit one 'impedance'. The coefficients may be determined by measurement in the laboratory or they may be calculated by applying electromagnetic theory to a sufficiently good geometrical model.

Questions of propagation, including the calculation of the losses due to the presence either of walls with finite conductivity or of imperfect dielectrics in the guide, and the theoretical treatment of junctions, bends, and irises require that the field be completely represented. If only the dominant wave is effectively propagated in the guide, so that the cut-off frequency of all other modes of propagation is very much higher than the frequency of radiation in the guide, the field of the evanescent waves in the vicinity of a localized obstruction may be treated as an electro- and magnetostatic distribution varying harmonically in time just like the fields in low-frequency a.c. theory. This is equivalent to ignoring the frequency dependence of the rates of extinction of the evanescent waves of different order radiated from the obstruction, the wave equation being therefore treated as if the velocity of propagation of free waves were infinite in this approximation.

In this chapter we shall devote our attention to the radiation and reception by antenna systems through which waves are launched in the guide or coupled to other guides or to space. The basis of field representation in this connexion is the determination of expressions for the Hertz-vectors of the field in a wave guide due (1) to an electric dipole, and hence to a current element, and (2) to a magnetic radiator, which is realized by a distribution of tangential electric force on the surface where an aperture replaces part of the guide wall. As a preliminary to this study, for the convenience of the reader, Maxwell's equations are presented in the following section in M.K.S. units, together with the representation of plane waves in terms of Hertz-vectors in the succeeding section.

10.2. Résumé of Field Equations

An electromagnetic field on the molar scale is described by two pairs of vectors:

Electric type: **E** measured in volts/metre,
B ,, webers/square metre.
Magnetic type: **H** ,, amperes/metre,
D ,, coulombs/square metre.

Sources of the field are represented by distributed electric charge of density ρ coulombs/cubic metre and electric current of density **i** amperes/square metre.

Between these there subsist the vector relations

$$\text{(A)}\quad \operatorname{curl}\mathbf{E} = -\dot{\mathbf{B}}, \qquad \text{(B)}\quad \operatorname{curl}\mathbf{H} = \dot{\mathbf{D}}+\mathbf{i},$$
$$\operatorname{div}\mathbf{B} = 0, \qquad \operatorname{div}\mathbf{D} = \rho, \tag{102.1}$$

which express the well-known electromagnetic laws. The dot denotes partial differentiation with respect to the time.

In vacuum $\mathbf{B} = \mu_0\mathbf{H}, \qquad \mathbf{D} = \kappa_0\mathbf{E},$

where $\mu_0 = 4\pi\times 10^{-7}$ henry/m. is the specific inductance of vacuum, and $\kappa_0 = \dfrac{1}{36\pi\times 10^9}$ farad/m. is the specific capacitance of vacuum. In an isotropic medium, μ_0 and κ_0 must be multiplied respectively by the magnetic permeability and dielectric constant of the medium. We shall not have occasion to deal with an anisotropic medium where the magnetic and electric properties are represented by tensors.

The equations (A) are automatically satisfied if we relate **E** and **B** to the magnetic vector potential **A** and electric scalar potential ϕ by means of

$$\mathbf{E} = -\dot{\mathbf{A}}-\operatorname{grad}\phi, \qquad \mathbf{B} = \operatorname{curl}\mathbf{A}. \tag{102.2}$$

The set (B) now become conditions on **A** and ϕ. On the assumption that $\operatorname{div}\mathbf{A}+\mu_0\kappa_0\dot{\phi} = 0$, we find in a rectangular cartesian coordinate system

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = -\mu_0\mathbf{i}, \qquad \left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = -\frac{\rho}{\kappa_0}, \tag{102.3}$$

and, of course, $c^2 = 1/\mu_0\kappa_0$. In the absence of distributed electric charge and current, the components of **A** and ϕ together with the components of the field vectors satisfy the wave equation obtained by annulling the right-hand sides of the above equations.

The sources of the distributions of **A** and ϕ are electric currents and charges; accordingly the conception 'vector potential' has been found very useful in radiation theory. However, current and charge distributions cannot be assigned arbitrarily; one must have regard to the conservation of electricity, and this finds expression in the conception 'electric doublet' (dipole) as a fundamental singularity in the radiation field. The corresponding system of potentials is the electric Hertz-vector, which we define as **Π**, given by

$$\mathbf{A} = \mu_0 \frac{\partial \mathbf{\Pi}}{\partial t}, \qquad \phi = -\frac{1}{\kappa_0} \operatorname{div} \mathbf{\Pi}. \tag{102.4}$$

Hence from (102.2)

$$\left.\begin{aligned} \mathbf{E} &= \frac{1}{\kappa_0}\left(\operatorname{grad}\operatorname{div}\mathbf{\Pi} - \frac{1}{c^2}\frac{\partial^2 \mathbf{\Pi}}{\partial t^2}\right), \\ \mathbf{H} &= \operatorname{curl}\frac{\partial \mathbf{\Pi}}{\partial t}. \end{aligned}\right\} \tag{102.5}$$

Not only may we think of electric dipoles as field singularities, we may also introduce the idea of magnetic dipole; in which case, it is advantageous to use the Hertz-vector of magnetic type **M** given by

$$\mathbf{A} = \operatorname{curl} \mathbf{M}, \qquad \phi = 0. \tag{102.6}$$

Thus

$$\left.\begin{aligned} \mathbf{E} &= -\operatorname{curl}\frac{\partial \mathbf{M}}{\partial t}, \\ \mathbf{H} &= \frac{1}{\mu_0}\operatorname{curl}\operatorname{curl}\mathbf{M}. \end{aligned}\right\} \tag{102.7}$$

In free space, the cartesian components of both **Π** and **M** satisfy the wave equation

$$\Box u \equiv \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)u = 0, \tag{102.8}$$

and hence equations (102.5) and (102.7) are symmetrical as between electric and magnetic quantities, apart from the negative sign in the first of (102.7).

10.3. Plane Waves

Suppose that the equiphase surfaces are parallel to the xy-plane and that the only non-zero Hertz-vector component is Π_z which we shall denote by U. From (102.5) we readily obtain in the case of harmonic waves of frequency $\omega/2\pi$, time dependence being represented by the factor $e^{-j\omega t}$ for convenience in calculation:

$$E_x = \frac{1}{\kappa_0}\frac{\partial^2 U}{\partial x \partial z}, \qquad E_y = \frac{1}{\kappa_0}\frac{\partial^2 U}{\partial y \partial z}, \qquad E_z = \frac{1}{\kappa_0}\left(\frac{\partial^2 U}{\partial z^2} + k^2 U\right),$$

$$H_x = -j\omega\frac{\partial U}{\partial y}, \qquad H_y = j\omega\frac{\partial U}{\partial x}, \qquad H_z = 0, \tag{103.1}$$

where
$$k^2 = \mu_0 \kappa_0 \omega^2 = \frac{\omega^2}{c^2} = \left(\frac{2\pi}{\lambda}\right)^2.$$

This set of field components constitutes a TM-wave propagated parallel to z, and the appropriateness of the name E-wave is evident from the fact that the generating function U is an electric Hertz-vector. On a conducting wall parallel to z, U vanishes.

Now suppose that $V = M_z$ is the only non-vanishing Hertz-vector component. Then, from (102.7), we find

$$E_x = j\omega\frac{\partial V}{\partial y}, \qquad E_y = -j\omega\frac{\partial V}{\partial x}, \qquad E_z = 0,$$

$$H_x = \frac{1}{\mu_0}\frac{\partial^2 V}{\partial x \partial z}, \qquad H_y = \frac{1}{\mu_0}\frac{\partial^2 V}{\partial y \partial z}, \qquad H_z = \frac{1}{\mu_0}\left(\frac{\partial^2 V}{\partial z^2} + k^2 V\right). \tag{103.2}$$

This constitutes a TE-wave with generating function V, the magnetic Hertz-vector in the direction of propagation. On a conducting wall the normal derivative of V vanishes.

It is readily seen that if the transverse components of $\mathbf{\Pi}$ and $\mathbf{M}$ do not vanish we cannot have the simple wave types (103.1) and (103.2), for in the former H_z will differ from zero, and in the latter E_z.

In representing waves in a guide we require the generating functions to satisfy the wave equation subject to the boundary condition on the wall of the guide $U = 0$, $\partial V/\partial \nu = 0$, where ν denotes the normal to the wall.

Any U may be expressed in terms of the normalized eigenfunctions $u_\sigma(x, y)$, where in the region A bounded by the guide cross-section

$$\frac{\partial^2 u_\sigma}{\partial x^2} + \frac{\partial^2 u_\sigma}{\partial y^2} + h_\sigma^2 u_\sigma = 0, \qquad \iint_A u_\sigma^2\, dxdy = 1, \tag{103.3}$$

and h_σ is an eigenvalue of the problem with $u_\sigma = 0$ on the boundary C of A. The subscript σ, of course, stands for the pair of integers by which the set of eigenfunctions and corresponding eigenvalues are distinguished. In unimpeded propagation, U is obtained by superposing the characteristic waves of TM type: thus

$$U(x,y,z,t) = \sum_{\sigma} c_{\sigma} u_{\sigma} e^{-j\omega t+jk_{\sigma}z}$$

if propagation takes place in the direction z-increasing. The complex constants c_{σ} are determined by the mode of exciting the guide, and

$$k_{\sigma} = \sqrt{(k^2-h_{\sigma}^2)}.$$

Similarly, for the waves of magnetic type we have the normalized eigenfunctions $v_{\sigma}(x,y)$ with

$$\frac{\partial^2 v_{\sigma}}{\partial x^2}+\frac{\partial^2 v_{\sigma}}{\partial y^2}+H_{\sigma}^2 v_{\sigma} = 0, \qquad \iint\limits_{A} v_{\sigma}^2\,dxdy = 1, \qquad \frac{\partial v_{\sigma}}{\partial \nu} = 0 \text{ on } C,$$

and

$$K_{\sigma} = \sqrt{(k^2-H_{\sigma}^2)}. \tag{103.4}$$

As the notation states, in the expressions for k_{σ} and K_{σ} the positive square root is taken if it is real, and the positive imaginary square root if pure imaginary.

Consider now the problem of finding a function $f(x,y,z)$ which satisfies

$$(\nabla^2+k^2)f = \phi(x,y,z), \tag{103.5}$$

ϕ being given, together with the boundary conditions that $f = 0$ on the surface S of the guide wall, and that f represents outgoing or damped waves at $z = \pm\infty$. We suppose f expanded in terms of the functions u_{σ}. Let

$$f = \sum_{\sigma} a_{\sigma}(z)u_{\sigma}(x,y). \tag{103.6}$$

Substitute in (103.5) and use the orthogonality relation

$$\iint\limits_{A} u_{\sigma} u_{\tau}\,dxdy = 0 \quad (\tau \neq \sigma) \tag{103.7}$$

and we find

$$\frac{d^2 a_{\sigma}}{dz^2}+k_{\sigma}^2 a_{\sigma} = \iint\limits_{A} \phi(x,y,z)u_{\sigma}(x,y)\,dxdy. \tag{103.8}$$

This equation may now be solved by the method of variation of parameters or by operator methods, using the conditions $a_{\sigma}(z) \sim e^{jk_{\sigma}z}$ as $z \to +\infty$, and $a_{\sigma}(z) \sim e^{-jk_{\sigma}z}$ as $z \to -\infty$, to determine the arbitrary constants.

The foregoing procedure may evidently be applied also to a function whose normal derivative vanishes on S. The method is a general one for calculating the effects of perturbation of the propagation.

10.4. Characteristic Waves on the Strip Transmission Line: Irises

As a step towards the full representation of the waves in a rectangular guide and for the purpose of treating some fundamental ideas in the theory of irises, we shall consider the strip transmission line [34] of Chapter I using the system of rectangular axes displaced so that the zx-plane is half-way between the strips.

For the TM-type waves propagated in the direction z-increasing, the generating function for the nth mode is

$$U_n = \sin\frac{n\pi y}{b}e^{jk_n z}, \quad \text{where} \quad k_n^2 = k^2-\left(\frac{n\pi}{b}\right)^2,$$

and for the TE-type, nth mode,

$$V_n = \cos\frac{n\pi y}{b}e^{jk_n z}.$$

The case $n = 0$ is the principal wave treated in Chapter I. It is included with the others in the corresponding expressions for the field vectors:

	TE	TM
E_x . .	$-j\omega\sin\frac{n\pi y}{b}$	0
E_y . .	0	$\frac{jk_n}{\kappa_0}\frac{n\pi}{b}\cos\frac{n\pi y}{b}$
E_z . .	0	$\frac{n^2\pi^2}{b^2\kappa_0}\sin\frac{n\pi y}{b}$
H_x . .	0	$-j\omega\frac{n\pi}{b}\cos\frac{n\pi y}{b}$
H_y . .	$\frac{-jk_n}{\mu_0}\frac{n\pi}{b}\sin\frac{n\pi y}{b}$	0
H_z . .	$\frac{n^2\pi^2}{b^2\mu_0}\cos\frac{n\pi y}{b}$	0

(104.1)

All expressions are to be multiplied by the propagation factor $e^{jk_n z}$, the time factor $e^{-j\omega t}$ being suppressed as usual. For the principal wave the amplitude factor must be assumed inversely proportional to n, before putting n equal to zero.

Consider now a symmetrical capacitive iris on the line at $z = 0$ (see Fig. 22). There is no physical reason to lead one to expect the generation of electric force E_x parallel to the length of the infinite iris.

Accordingly we may represent the field due to the iris when the principal wave is incident on it by means of

$$E_y(y,z) = a_0 e^{\pm jkz} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi y}{b} \exp\left[\mp\frac{n\pi z}{b}\sqrt{\left(1-\frac{f^2}{f_n^2}\right)}\right] \quad (z \gtrless 0), \tag{104.2}$$

where $f = \dfrac{\omega}{2\pi}$ and $f_n = \dfrac{nc}{2b}$ is the nth cut-off frequency, and the evanescent waves introduced by the iris when $\lambda > 2b$ are all of TM type, because $f < f_n$ (all n). Suppose we put $f = 0$, then

$$E_y(y,z) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi y}{b} \exp\left[\mp\frac{n\pi z}{b}\right] \quad (z \gtrless 0). \tag{104.3}$$

This form disregards the retardation of the field, the sum being just what we should obtain in the electrostatic problem of the two strips with iris charged to different potentials. It is known that this problem is solved by means of the Schwarz-Christoffel transformation

$$\sin(V+iU) = \operatorname{cosec}\left(\frac{\pi w}{2b}\right)\sin\frac{\pi}{b}(y+iz), \tag{104.4}$$

where V is the electric potential taking the values $\pm\frac{1}{2}\pi$ on the strips; U is the stream function. So that

$$E_y = -\frac{\partial V}{\partial y} = \frac{\partial U}{\partial z}. \tag{104.5}$$

Evidently a_0 is the average electric force between the strips at $z = 0$, i.e.

$$a_0 = \frac{\pi}{b}. \tag{104.6}$$

Thus if we represent the propagation only of the principal wave and treat the evanescent waves as of negligible frequency compared with their cut-off frequencies, and this will, of course, be nearer the truth the smaller b compared with λ, we may write

$$E_y = \frac{\pi}{b}(e^{\pm jkz}-1)+\frac{\partial U}{\partial z} \quad (z \gtrless 0). \tag{104.7}$$

Now consider the condenser formed by the surfaces $V = \pm\frac{1}{2}\pi$ of our static problem between $z = -l$ and $z = +l$. The total charge on the surface of the positively charged conductor per unit width in the x-direction is

$$Q = \kappa_0\left[\frac{\pi}{b}\left(\frac{e^{jkl}-e^{-jkl}}{jk} - 2l\right)+U(l)-U(-l)\right], \tag{104.8}$$

and the capacity, referred to the voltage π between the strips at $z = 0$, is

$$C(l) = \frac{\kappa_0}{\pi}\{U(l)-U(-l)\}-\frac{2\kappa_0 l}{b}+\frac{\kappa_0}{b}\frac{e^{jkl}-e^{-jkl}}{jk}. \tag{104.9}$$

The first term is the capacity of the strip with iris in the static problem, the second is the capacity of the same length of strip without iris, and the third is a periodic term which is bounded and, due to the weak attenuation present in any actual guide, represents a capacity which tends to zero as l is increased indefinitely.

Since $$U(l)-U(-l) = 2\cosh^{-1}\left(\operatorname{cosec}\frac{\pi w}{2b}\cosh\frac{\pi l}{b}\right), \tag{104.10}$$

we see that $C(l)$ tends to a limit as $l \to \infty$, viz.

$$C_1 = \lim_{l\to\infty} C(l) = \frac{2\kappa_0}{\pi}\log\operatorname{cosec}\frac{\pi w}{2b} \text{ farads/m.} \tag{104.11}$$

This is the lumped capacity due to the iris, presented to the principal wave on the line. The main part of it arises from the immediate vicinity of the iris. The justification of the limiting process is that the evanescent waves are physically insignificant at distances $> b$ from the iris, and the simplicity of the formula obtained by this approximation. The origin of the correction term in (56.5) can be seen: the frequency is not treated as zero in the evanescent waves.

Macfarlane has discussed the foregoing problem and the complementary one in detail. He has defined the radiation admittance (Y) of the capacitive iris and also of an inductive grid per metre width of strip referred to the voltage V of the given frequency across the iris. Let P be the complex power supplied by the generator of voltage V in order to produce the radiation on both sides of the iris, and to induce the field in its vicinity. Then

$$Y = \frac{2P}{|V|^2}. \tag{104.12}$$

Now the complex power from the iris per metre width is equal to the z-component of the complex Poynting vector $\frac{1}{2}\mathbf{E}\times\mathbf{H}^*$, where * denotes conjugate complex, integrated over two planes parallel to the iris at the small distance Δz from it on each side, thus enclosing the (fictitious) generator. Thus

$$P = -\tfrac{1}{2}\int\limits_{-\frac{1}{2}b}^{\frac{1}{2}b} E_y(y,-\Delta z)H_x^*(y,-\Delta z)\,dy - \tfrac{1}{2}\int\limits_{-\frac{1}{2}b}^{\frac{1}{2}b} E_y(y,\Delta z)H_x^*(y,\Delta z)\,dy. \tag{104.13}$$

These integrals are evaluated by noting that H_x does not depend on y.

Consider an infinite loop C formed by the length dx of the line $y = y_0$ in the gap and by lines extending from its ends to $-\infty$ parallel to z as shown in Fig. 91. Now the magnetomotive force round this circuit is equal to the time rate of change of electric flux $\Phi_e dx$ threading the loop. Since this is the same for each similar loop parallel to C, and since the only contribution to magnetomotive force occurs in the gap, provided that we introduce the weak attenuation required to annul the field at infinity, it follows that H_x is constant across the gap. In fact

$$H_x(y,0) = \frac{d}{dt}[\Phi_e]. \tag{104.14}$$

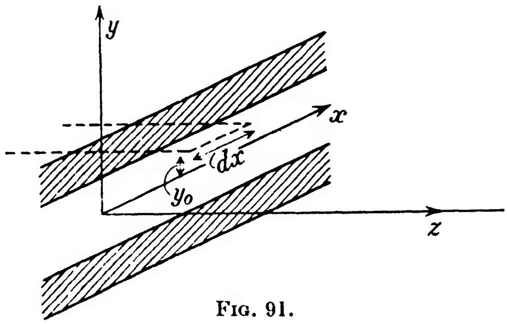


FIG. 91.

Hence
$$P = -2.\tfrac{1}{2}H_x^* \int_{-\frac{1}{2}b}^{\frac{1}{2}b} E_y(y,0)\,dy = \frac{d}{dt}[\Phi_e]V^*, \tag{104.15}$$

and from (104.12) the radiation admittance is

$$Y = -\frac{2j\omega}{V}\Phi_e = G - jS. \tag{104.16}$$

G, of course, turns out to be twice the characteristic admittance of the line.

Similar results can be established for the inductive grating. The problem of a plane grating of parallel equidistant conducting strips in the field of a plane wave incident normally with E parallel to the edges of the strips is exactly the problem of a narrow shorting strip across a uniform strip transmission line mutually coupled to an infinite array of identical transmission lines in parallel. The voltage in the gap of the capacitive iris becomes the current in the metal iris strips of the inductive one; magnetic flux replaces electric flux and is linked to a loop rotated through a right angle with respect to the transmission line and terminated by the iris strip. Current generators act in the strips and are faced by radiation impedance in place of the admittance in the capacitive iris.

10.5. Irises in Rectangular Wave Guide (Dominant Wave)

We now discuss the capacitive iris in the H_{10} wave guide. Referred to rectangular axes bearing the same relation to the wave guide as that adopted above in treating the strip transmission line, the electric force in the incident dominant wave in the plane of the iris is

$$E_y(x,y,0) = E_y(0,y,0)\cos\frac{\pi x}{a}. \tag{105.1}$$

The corresponding transverse magnetic force must be independent of y for the reason already advanced in connexion with the strip transmission line, i.e.

$$H_x(x,y,0) = H_x(0,0,0)\cos\frac{\pi x}{a}. \tag{105.2}$$

The total power that must be fed to the iris in order to maintain the radiation and storage fields is

$$\begin{aligned} P &= -H_x^*(0,0,0)\int\limits_{-\frac{1}{2}a}^{\frac{1}{2}a}\cos^2\frac{\pi x}{a}\,dx\int\limits_{-\frac{1}{2}b}^{\frac{1}{2}b}E_y(0,y,0)\,dy \\ &= \tfrac{1}{2}H_x^*(0,0,0)V, \end{aligned} \tag{105.3}$$

where V is the voltage across the gap at its centre.

In place of (104.2) we have

$$\begin{aligned} &E_y(x,y,z) \\ &\quad = \cos\frac{\pi x}{a}\left[b_0\exp(\pm jk_{10}z)+\sum_{n=1}^{\infty}b_n\cos\frac{n\pi y}{b}\exp\left\{\mp z\sqrt{\left(\frac{n^2\pi^2}{b^2}-k_{10}^2\right)}\right\}\right] \\ &\qquad\qquad (z \gtrless 0), \end{aligned} \tag{105.4}$$

where $k_{10}^2 = k^2-(\pi/a)^2$.

Concentrating our attention on the plane $x = 0$ (cf. V above), we note that apart from the replacement of k^2 of the strip transmission line by k_{10}^2 the problem of the symmetrical capacitive iris in the rectangular wave guide is mathematically equivalent to that for the strip transmission line.

The treatment of the inductive strip across a wave guide is somewhat more complicated, for we have to deal with the images of the strips in the narrow faces so that in the corresponding grating we require that the currents flowing in adjacent strips will be in opposite directions. By an argument in every respect analogous to that given for the capacitive iris we find for the radiation impedance of the iris referred to current I in the strips

$$Z = \frac{2P}{|I|^2} = \frac{j\omega b}{I}[\Phi_m]_0^{-\infty}, \tag{105.5}$$

where $[\Phi_m]_0^{-\infty}$ is the magnetic flux threading through a loop of unit width as shown in Fig. 92. In place of (104.2) for the electric field, we now represent the transverse magnetic field

$$H_x(x,y,z)$$
$$= a_1 \cos\frac{\pi x}{a} e^{\pm jk_{10}z} + \sum_{n=2}^{\infty} a_n \cos\frac{(2n-1)\pi x}{a} \exp\left\{\mp z\sqrt{\left((2n-1)^2\frac{\pi^2}{a^2}-k^2\right)}\right\}$$
$$(z \gtrless 0), \quad (105.6)$$

and pass to the magnetostatic problem in which equal and opposite direct currents flow in adjacent strips of the infinite grating composed

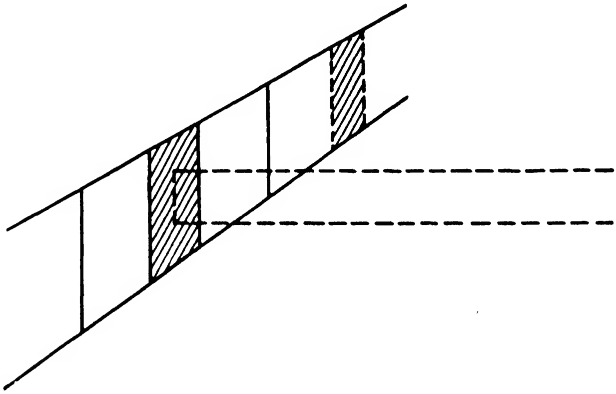


FIG. 92.

of strips of width w spaced a apart between centres. The magnetic force due to these currents can be expanded in the form

$$H_x^0 = \sum_{n=1}^{\infty} a_n \cos(2n-1)\frac{\pi x}{a} \exp\left(\mp(2n-1)\frac{\pi z}{a}\right) \quad (z \gtrless 0), \quad (105.7)$$

which, apart from its first term, is the same as the sum in (105.6) with $k = 0$, i.e. the quasi-static representation of the evanescent waves. Thus we write

$$H_x(x,y,z) = a_1 \cos\frac{\pi x}{a}[e^{\pm jk_{10}z} - e^{\mp \pi z/a}] + H_x^0, \qquad (105.8)$$

and H_x^0 can be determined from the known solution of the magnetostatic problem in terms of the transformation, when $w \ll a$,

$$V+iU = V_1-V_2+i(U_1-U_2),$$

where

$$\sin(V_1+iU_1) = \operatorname{cosec}\frac{\pi w}{4a}\sin\frac{\pi}{2a}(x+iz),$$

$$\sin(V_2+iU_2) = \operatorname{cosec}\frac{\pi w}{4a}\cos\frac{\pi}{2a}(x+iz), \qquad (105.9)$$

and
$$H_x = -\frac{\partial V}{\partial x} = \frac{\partial U}{\partial z}. \tag{105.10}$$

V is the magnetic potential, U the corresponding stream function: the total current in each of the strips is 2π.

The magnetic flux threading the rectangular loop of width 1 metre in the plane $x = 0$ from $z = 0$ to $z = -\infty$ is

$$\begin{aligned}\Phi_m &= \mu_0 \int\limits_0^{-\infty} H_x(0,0,z)\,dz \\ &= \mu_0 a_1\left[\frac{1}{jk_{10}}+\frac{a}{\pi}\right]+\mu_0[U(-\infty)-U(0)],\end{aligned} \tag{105.11}$$

and
$$U(0) = \cosh^{-1}\operatorname{cosec}\frac{\pi w}{4a}, \qquad U(-\infty) = 0. \tag{105.12}$$

The coefficient a_1 is determined as the first in the Fourier series representing H_x when $z \to 0$, i.e.

$$a_1 = \frac{2}{a}\int\limits_{-\frac{1}{2}w}^{\frac{1}{2}w} \frac{\pi}{w}\cos\frac{\pi x}{a}\,dx = \frac{2\pi}{a}. \tag{105.13}$$

Thus the radiation impedance (see (105.5)) is from the last three equations

$$\begin{aligned}Z &= \frac{\omega\mu_0 b}{k_{10} a}+\frac{j\mu_0\omega b}{2\pi}\left\{2-\cosh^{-1}\left(\operatorname{cosec}\frac{\pi w}{4a}\right)\right\} \\ &= \frac{b}{a}Z_{10}-\frac{jbZ_0}{\lambda}\left\{\cosh^{-1}\left(\operatorname{cosec}\frac{\pi w}{4a}\right)-2\right\} \\ &= R-jX.\end{aligned} \tag{105.14}$$

It is instructive to deduce from this the susceptance to be expected in standing-wave measurements in the guide. Since the guide is shunt-coupled to the strip and the two sides of the guide are in parallel with each other to make up the radiation resistance (cf. § 5.9), we write (105.14) in the form

$$Z = 2Z_{10}\frac{b}{a}\left[\frac{1}{2}-\frac{a}{2\lambda_g}\left\{\cosh^{-1}\left(\operatorname{cosec}\frac{\pi w}{4a}\right)-2\right\}\right].$$

Thus the susceptance in question is

$$\frac{2\lambda_g}{a}\left[\cosh^{-1}\left(\operatorname{cosec}\frac{\pi w}{4a}\right)-2\right]^{-1}.$$

For narrow strips we replace the inverse hyperbolic function by $\log 8a/w$.

An analytic method of greater scope than the foregoing is due to Schwinger [35]: the approximation is carried to the same point as before, namely, by quasi-static treatment of the evanescent waves. We shall illustrate it by means of the capacitive iris in the rectangular guide and for the dominant wave.

The electric field in the plane of the iris superposed on the incident wave is represented by

$$E_y(x,y,0) = \cos\frac{\pi x}{a}\left[\tfrac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi y}{b}\right], \qquad (105.15)$$

where $$c_n = \frac{1}{b}\int\limits_{-\frac{1}{2}w}^{\frac{1}{2}w} E_y(0,\eta,0)\cos\frac{n\pi\eta}{b}\,d\eta \quad (n = 0,1,2,...).$$

This field initiates waves travelling on each side of the iris, such that when superposed on the incident dominant wave of amplitude E_0, the transverse magnetic field is continuous across the gap. This requires

$$c_0 + 2jk_{10}\sum_{n=1}^{\infty}\frac{c_n\cos\frac{n\pi y}{b}}{\sqrt{\left\{\left(\frac{n\pi}{b}\right)^2 - k_{10}^2\right\}}} = E_0.$$

We may now substitute for the c's from (105.15) to obtain the following integral equation:

$$\frac{1}{b}\int\limits_{-\frac{1}{2}w}^{\frac{1}{2}w} E(\eta)\,d\eta + \frac{2jk_{10}}{b}\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi y}{b}}{\sqrt{\left\{\left(\frac{n\pi}{b}\right)^2 - k_{10}^2\right\}}}\int\limits_{-\frac{1}{2}w}^{\frac{1}{2}w} E(\eta)\cos\frac{n\pi\eta}{b}\,d\eta = E_0. \qquad (105.16)$$

Now replace $\sqrt{\left\{\left(\frac{n\pi}{b}\right)^2 - k_{10}^2\right\}}$ by $\frac{n\pi}{b}$, interchange integration and summation, use the known value of the sum

$$\sum_{n=1}^{\infty}\frac{1}{n}\cos\frac{n\pi y}{b}\cos\frac{n\pi\eta}{b} = -\tfrac{1}{2}\log 2\left|\cos\frac{\pi y}{b} - \cos\frac{\pi\eta}{b}\right| = K(y,\eta), \qquad (105.17)$$

so that (105.16) becomes

$$\frac{1}{b}\int\limits_{-\frac{1}{2}w}^{\frac{1}{2}w} E(\eta)\,d\eta + \frac{2jk_{10}}{\pi}\int\limits_{-\frac{1}{2}w}^{\frac{1}{2}w} E(\eta)K(y,\eta)\,d\eta = E_0. \qquad (105.18)$$

This equation is solved for $E(y)$ and the radiation coefficient for the iris determined as $c_0/2E_0$.

Solution of the integral equation yields for the electric force parallel to y in the slit

$$E_y = \frac{E_0 \cos\dfrac{\pi y}{b}}{\left(1-\dfrac{jk_{10}b}{\pi}\log\operatorname{cosec}\dfrac{\pi w}{2b}\right)\left(\sin^2\dfrac{\pi w}{2b}-\sin^2\dfrac{\pi y}{b}\right)^{\frac{1}{2}}}. \tag{105.19}$$

The foregoing method has been applied to the capacitive iris centred at a distance y_0 from one broad face. The susceptance for a narrow slit of width w is

$$B_{\text{slit}} = \frac{4b}{\lambda_g}\left[\log\frac{2b}{\pi w\sin(\pi y_0/b)}+2\sum_1^{\infty}\cos^2\frac{n\pi y_0}{b}\left(\frac{1}{\sqrt{\{n^2-(2b/\lambda_g)^2\}}}-\frac{1}{n}\right)\right], \tag{105.20}$$

in which the sum is usually quite unimportant.. If a conducting strip replaces the slitted plate across the guide, then

$$B_{\text{strip}} = \frac{\pi^2 w^2}{2b\lambda_g}. \tag{105.21}$$

When the narrow slit or obstacle is presented at right angles to the foregoing, it is found for these inductive loads,

$$\frac{1}{B_{\text{slit}}} = -\frac{\pi^2 w^2}{4a\lambda_g}\sin^2\frac{\pi x_0}{a}, \tag{105.22}$$

$$\frac{1}{B_{\text{strip}}} = -\frac{a}{2\lambda_g\sin^2(\pi x_0/a)}\left[\log\frac{8a\sin(\pi x_0/a)}{\pi w}-2\sin^2\frac{\pi x_0}{a}+\right.$$
$$\left.+2\sum_{n=2}^{\infty}\sin^2\frac{n\pi x_0}{a}\left(\frac{1}{\sqrt{\{n^2-(2a/\lambda_g)^2\}}}-\frac{1}{n}\right)\right], \tag{105.23}$$

where x_0 is the displacement of the centre of the strip or slit from the narrow face of the guide.

Heins has worked out for a centred strip, whose width is no longer restricted to be small compared with a, the width of the guide,

$$B = \frac{\lambda_g}{a}\left[\frac{E-\sin^2(\pi w/2a)F}{E-\frac{1}{2}\{1+\sin^2(\pi w/2a)\}F}\right], \tag{105.24}$$

where $$F = \int_0^{\frac{1}{2}\pi}\left(1-\cos^2\frac{\pi w}{2a}\sin^2\phi\right)^{-\frac{1}{2}}d\phi$$

and $$E = \int_0^{\frac{1}{2}\pi}\left(1-\cos^2\frac{\pi w}{2a}\sin^2\phi\right)^{\frac{1}{2}}d\phi.$$

10.6. The Hertz-vector of the Radiation induced by an Electric Dipole in the Rectangular Guide

We assume harmonic oscillations of frequency $\omega/2\pi$. From Maxwell's equations

$$(\nabla^2+k^2)\mathbf{E} = -j\omega\mu_0\,\mathbf{i}+\frac{1}{\kappa_0}\operatorname{grad}\rho, \tag{106.1}$$

and the conservation of electricity requires

$$j\omega\rho = \operatorname{div}\mathbf{i}. \tag{106.2}$$

Let us now use the operator method of the Fourier transform, with the vector $j\mathbf{p}$ representing the grad operator. In order to justify this procedure, we must assume that the current $\mathbf{i}$ dies off suitably at infinity. Since we shall be interested in localized current, this is clearly ensured by the physical conditions.

Let $\mathbf{I}(\mathbf{p})$ denote the Fourier transform of the given current distribution $\mathbf{i}(\mathbf{r})$ ($\mathbf{r} = (x,y,z)$). Then the Fourier transform of $\mathbf{E}(\mathbf{r})$

$$\mathbf{E}(\mathbf{p}) = \frac{1}{j\omega\kappa_0}\,\frac{k^2\mathbf{I}-\mathbf{p}(\mathbf{p}.\mathbf{I})}{k^2-p^2}. \tag{106.3}$$

Similarly

$$(\nabla^2+k^2)\mathbf{H} = -\operatorname{curl}\mathbf{i}, \tag{106.4}$$

and hence

$$\mathbf{H}(\mathbf{p}) = \frac{-j(\mathbf{p}\times\mathbf{I})}{k^2-p^2}. \tag{106.5}$$

Now we know that only the characteristic distributions

$$\left.\begin{aligned} u_{mn} &= \sqrt{\left(\frac{2}{ab}\right)}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \\ &\qquad\qquad (m,n \text{ integers}) \\ v_{mn} &= \sqrt{\left(\frac{2}{ab}\right)}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}, \end{aligned}\right\} \tag{106.6}$$

are permitted by the boundary conditions on the guide wall. The former is proportional to Π_z, the z-component of electric Hertz-vector and hence to E_z for the mnth characteristic TM-wave, and the latter, v_{mn}, to the z-component of the magnetic Hertz-vector M_z and hence to H_z in the corresponding TE-wave. From this limitation on H_z and E_z it follows from (106.3) and (106.5) that the components of $\mathbf{i}$ must be expressible as

$$\begin{aligned} i_x = i^{(1)} &= \sum\sum i^{(1)}_{mn}(z)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \\ i_y = i^{(2)} &= \sum\sum i^{(2)}_{mn}(z)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}, \\ i_z = i^{(3)} &= \sum\sum i^{(3)}_{mn}(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \end{aligned} \tag{106.7}$$

These require that the tangential components of $\mathbf{i}$ vanish on the wall.

On account of the discrete spectrum of $\mathbf{i}$ in so far as concerns the dependence of $\mathbf{I}$ on p_x and p_y, which fact is evident in (106.7), we may simplify by using Fourier transforms in connexion with z only, and writing

$$U_{mn}(p_z) = \frac{\kappa_0}{h_{mn}^2}\frac{1}{j\omega\kappa_0}\frac{(k_{mn}^2-p_z^2)I_{mn}^{(3)}(p_z)+p_z\{(m\pi/a)I_{mn}^{(1)}(p_z)+(n\pi/b)I_{mn}^{(2)}(p_z)\}}{k_{mn}^2-p_z^2}$$

$$= \frac{1}{j\omega h_{mn}^2}\left[I_{mn}^{(3)}(p_z)+\frac{p_z\{(m\pi/a)I_{mn}^{(1)}(p_z)+(n\pi/b)I_{mn}^{(2)}(p_z)\}}{k_{mn}^2-p_z^2}\right], \quad (106.8)$$

where $I_{mn}^{(\alpha)}(p_z)$ is the Fourier transform of $i_{mn}^{(\alpha)}(z)$ ($\alpha = 1, 2, 3$); and when $U_{mn}(p_z)$ has been inverted to a function of z it is to be multiplied by $\sin(m\pi x/a)\sin(n\pi y/a)$ to give the mnth TM-wave. Similarly

$$V_{mn}(p_z) = -\frac{j\mu_0}{H_{mn}^2}\frac{[(m\pi/a)I_{mn}^{(2)}(p_z)+(n\pi/b)I_{mn}^{(1)}(p_z)]}{k_{mn}^2-p_z^2}, \quad (106.9)$$

and when $V_{mn}(p_z)$ has been inverted, it is to be multiplied by $\cos(m\pi x/a)\cos(n\pi y/b)$ to yield the mnth TE-wave due to the original current distribution which was supposed given.

These formulae may be applied to find the Hertz-vectors which are the characteristic generating functions for the two types of waves in the guide, excited by a given current distribution. They show clearly that TE-waves are coupled only to transverse electric antennas, whereas TM-waves are coupled in general to any electric antenna, whatever its orientation in the guide.

An isolated electric dipole is a singular current distribution. In order to show the method, we shall deal with the dominant wave ($m = 1$, $n = 0$)

$$V_{10}(p_z) = -\frac{j\mu_0 a}{\pi}\frac{I_{10}^{(2)}(p_z)}{k_{10}^2-p_z^2}. \quad (106.10)$$

This shows that we are concerned only with the component of current flow parallel to y. Now suppose that the current constituting the dipole is confined to the element ds inclined in the direction (L, M, N) at the point (x_1, y_1, z_1) in such a way that

$$\iiint \mathbf{i}\,dxdydz = I_0\,ds. \quad (106.11)$$

Then using the notation of the δ-function, we have

$$i_y = I_0 M\,ds\,\delta(x-x_1)\,\delta(y-y_1)\,\delta(z-z_1) \quad (106.12)$$

and

$$I_{10}^{(2)}(p_z) = \frac{2I_0 M\,ds}{ab}\int\limits_0^a dx\int\limits_0^b dy\int\limits_{-\infty}^{\infty} e^{-jp_z z}\sin\frac{\pi x}{a}\delta(x-x_1)\delta(y-y_1)\delta(z-z_1)\,dz$$

$$= \frac{2I_0 M\,ds}{ab}e^{-jp_z z_1}\sin\frac{\pi x_1}{a}. \qquad (106.13)$$

We require to invert the Fourier transform $\dfrac{I_{10}^{(2)}(p)}{k_{10}^2-p^2}$.

This is given by

$$\frac{2I_0 M\,ds}{ab}\sin\frac{\pi x_1}{a}\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{e^{jp(z-z_1)}}{k_{10}^2-p^2}\,dp. \qquad (106.14)$$

Now the integral represents†

$$\left.\begin{aligned}&\frac{je^{jk_{10}(z-z_1)}}{2k_{10}} \quad \text{if} \quad z > z_1,\\ \text{and} \quad &\frac{je^{-jk_{10}(z-z_1)}}{2k_{10}} \quad \text{if} \quad z < z_1.\end{aligned}\right\} \qquad (106.15)$$

Thus with the aid of (103.2) we see that the electric force in the dominant wave for $z > z_1$ is given by

$$E_y = j\frac{120\pi k}{k_{10}}\frac{I_0 M\,ds}{ab}\sin\frac{\pi x_1}{a}e^{-j\{\omega t-k_{10}(z-z_1)\}}\sin\frac{\pi x}{a}. \qquad (106.16)$$

10.7. Field due to a Linear Antenna: Impedance

For a thin antenna, parallel to y at the position (x_1, z_1), the current distribution will be assumed sinusoidal. If the length (l) is not a resonant one, at least one end of the antenna must be connected to another electrical system, for one end cannot then be the position of a current node. Let us suppose that the antenna is straight and extends between $y = 0$ and $y = l$, the latter being free and therefore a current node, then

$$i_y = I_0 \sin k(l-y)\delta(x-x_1)\delta(z-z_1) \qquad (107.1)$$

and

$$I_{10}^{(2)}(p) = \frac{2I_0}{abk}e^{-jpz_1}\sin\frac{\pi x_1}{a}(1-\cos kl). \qquad (107.2)$$

The electric force in this dominant wave has the amplitude

$$\frac{Z_g I_0}{ab}(1-\cos kl)\sin\frac{\pi x_1}{a}. \qquad (107.3)$$

† See Titchmarsh, *Theory of Fourier Integrals*, Oxford, 1937, pp. 4 and 43 on generalized Fourier integrals.

Thus the power radiated by the antenna in both directions down the guide is

$$W = \frac{Z_g I_0^2}{2abk^2}(1-\cos kl)^2 \sin^2\frac{\pi x_1}{a} \tag{107.4}$$

and the radiation resistance of the antenna at the point where it enters the guide is

$$\frac{W}{\frac{1}{2}I_0^2 \sin^2 kl} = \frac{Z_g}{abk^2}\tan^2 \tfrac{1}{2}kl \sin^2\frac{\pi x_1}{a} \text{ ohms.} \tag{107.5}$$

If a reflecting plunger is placed at $z = 0$, it is easily seen, by introducing the image source in the reflector, that the radiation resistance is

$$\frac{2Z_g}{abk^2}\tan^2\frac{kl}{2}\sin^2\frac{\pi x_1}{a}\sin^2 k_{10} z_1 \text{ ohms.} \tag{107.6}$$

Synge, Infeld, and Stevenson [36] have deduced this result by considering the system of images of the antenna in the guide walls. These images make up a lattice constituted by the superposition of eight basic lattices. The distant field due to the lattice is calculated by summing series. The mathematical difficulty in the calculation lies essentially in the representation of the discontinuous wave-function propagated on the two sides of the antenna in different directions. Copson [37] has worked out in full the Fourier integral representation of the Hertz-vector due to an electric dipole, taking account of the boundary conditions imposed by the guide walls.

While the calculation of the radiation resistance of an antenna in a wave guide does not involve computational difficulties—indeed, as already mentioned, results have been obtained for antennas of different shapes and in different positions both in rectangular and circular guides—the calculation of radiation reactance is protracted. Infeld [38] has treated the radiation impedance of a straight thin antenna in a rectangular guide by assuming sinusoidal distribution of current in the wire and calculating the field due to it. The radiation impedance of the antenna in the guide closed at one end is equivalent to that of a free antenna surrounded by eight lattices of images. We have already seen that the presence of the reflecting termination can be taken account of after the problem of the antenna in the open guide has been treated. Thus only four lattices need be considered. The radiation impedance of the antenna in the guide can therefore be represented as the sum of the self-impedance of the free antenna and of its mutual impedance with each of the elements of all the lattices, due account being taken of the fact that some elements are parallel and some are anti-parallel to it. These impedances are deduced by the e.m.f. method, that is, using exactly the considerations introduced in § 10.4. Let ρ_0 be the radius of the wire, and $E(\rho_0)$ represent the electric force at the surface of the wire, then the self-impedance of the antenna A is

$$Z_{AA} = -\frac{1}{I_0^2}\int\limits_0^l E_y^{(A)} i^{(A)}\,dy = \frac{1}{-I_0 \sin kl}\int\limits_0^l E_y(\rho_0)\sin k(l-y)\,dy, \tag{107.7}$$

I_0 being the driving-point current at $y = 0$. The mutual impedance Z_{AB}, that is, the contribution to the impedance of antenna A from antenna B is

$$Z_{AB} = \frac{1}{[I_0^{(A)}]^2}\int\limits_0^l E_y^{(B)}(\rho_0) i^{(A)}\,dy^{(A)}. \tag{107.8}$$

Infeld set out from Sommerfeld's formula expressing in integral form the Hertz-vector due to a dipole, by summing elementary cylindrical waves, and derived an expression for the electric field parallel to the finite antenna, and so proceeded to evaluate the impedances. The success of the computation lies in the transformation of the double sum over each of the lattices into a rapidly convergent form.

The method of images is applicable only to a few types of wave guide with suitable cross-sectional form. A general method for calculating the impedance of an antenna in a wave guide of arbitrary cross-section has been indicated by Stevenson [39], who has given general formulae for the radiation resistance when only a single E- or H-wave is transmitted by the guide. In the notation of § 10.3 the results (in ohms) are

(i) TM-wave:

$$R = \frac{Z_0 k_\sigma}{|I_0|^2 k h_\sigma^2}\left|\int \left(i_x\frac{\partial u_\sigma}{\partial x} + i_y\frac{\partial u_\sigma}{\partial y}\right)\sin k_\sigma z\,ds - \frac{h_\sigma^2}{k_\sigma}\int i_z u_\sigma \cos k_\sigma z\,ds\right|^2.$$

(ii) TE-wave:

$$R = \frac{Z_0 k}{|I_0|^2 K_\sigma H_\sigma^2}\left|\int \left(i_x\frac{\partial v_\sigma}{\partial y} - i_y\frac{\partial v_\sigma}{\partial x}\right)\sin K_\sigma z\,ds\right|^2. \tag{107.9}$$

I_0 is the current at the feed-point, ds is an element of length along the antenna which is referred to the plane of the reflecting plunger as the xy-plane; i_x, i_y, i_z are current components at the point (x, y, z). The above formulae hold for a guide of any cross-section and for any assumed current distribution along the antenna.

The detailed calculation of impedance for a longitudinal antenna in circular wave guide has been carried through by Infeld [40], who also discussed the matching problem for the E_{01}-wave which the antenna is intended to excite.

10.8. Field in a Wave Guide due to an Assigned Distribution of Tangential Electric Force over an Aperture in the Wall of the Guide

This problem has been solved independently by Bethe [41] and by Stevenson [42]: what follows is based on the latter's method.

Let ν denote the outward normal and s the direction of the tangent to C, the boundary of A, the cross-section of the guide as shown in Fig. 93; the axis of the guide is parallel to z, and rotation from ν to s

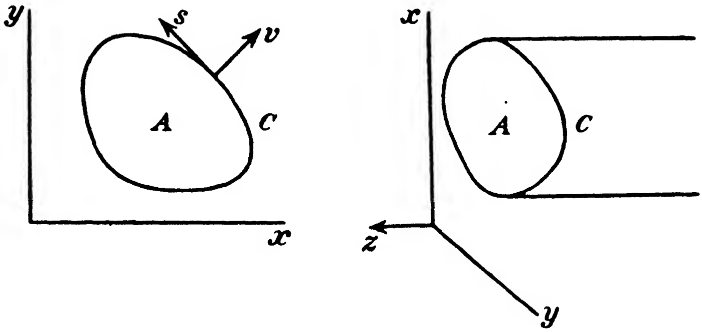


FIG. 93.

is in the same sense as that from x to y. We shall think of E_z and E_s as given functions of position on S, the wall of the guide; of course, they differ from zero only at the aperture. At infinity inside the guide, we shall suppose the field to tend sufficiently rapidly to zero for mathematical purposes; the hypothesis of weak attenuation in the course of propagation is physically acceptable and would serve.

Inside the guide E_z satisfies (102.8). Hence by a well-known result in the theory of Green's functions,†

$$E_z(P) = \int\limits_S \frac{\partial G_1(P,P')}{\partial \nu'} E_z(P')\, dS'. \tag{108.1}$$

P is the point (x,y,z) and P' is (x',y',z') in the domain of integration S. The Green's function $G_1(P,P')$ is defined by the following properties:

(i) G_1 regarded as a function of P satisfies the wave equation everywhere within S except at the point P';

(ii) $G_1 = 0$ when P is on S;

(iii) As $P \to P'$, $G_1 \to \infty$ as $1/4\pi r$, where r is the distance from P to P'.

† Bateman, op. cit., § 2.32.

To obtain the result expressing H_z in terms of E_s and E_z in the aperture, we use Maxwell's equations in the coordinate system ν, s, z. Then

$$-j\mu_0\omega H_z = \frac{\partial E_s}{\partial \nu}-\frac{\partial E_\nu}{\partial s}.$$

Differentiate with respect to ν, use the wave equation, and obtain

$$\frac{\partial H_z}{\partial \nu} = \frac{j}{\omega\mu_0}\left[\left(\frac{\partial^2}{\partial z^2}+k^2\right)E_s-\frac{\partial^2 E_z}{\partial z\partial s}\right]. \tag{108.2}$$

Now introduce the second Green's function $G_2(P,P')$ which satisfies $\partial G_2/\partial\nu = 0$ on S, and tends to infinity like $-1/4\pi r$ as $P \to P'$. Then

$$H_z(P) = \frac{j}{\omega\mu_0}\int\limits_S G_2(P,P')\left[\left(\frac{\partial^2}{\partial z'^2}+k^2\right)E_s(P')-\frac{\partial^2}{\partial z'\partial s'}E_z(P')\right]\,dS'. \tag{108.3}$$

Integrate by parts twice with respect to z' and s', use the condition that E_s and $E_z \to 0$ at $z = \pm\infty$, and we may replace (108.3) by

$$H_z(P) = \frac{j}{\omega\mu_0}\int\limits_S \left[\left(\frac{\partial^2}{\partial z'^2}+k^2\right)G_2(P,P').E_s(P')-\frac{\partial^2 G_2(P,P')}{\partial z'\partial s'}E_z(P')\right]\,dS'. \tag{108.4}$$

Equations (108.1) and (108.4) give E_z and H_z respectively in the guide, if G_1 and G_2 are known. Further, E_s and E_z may have discontinuities on the wall; hence the formulae apply in the case of an aperture.

To find the two Green's functions, we apply equation (103.5) with

$$\phi(x,y,z) = \delta(x-x')\delta(y-y')\delta(z-z'). \tag{108.5}$$

From (103.8)

$$\frac{d^2 a_\sigma}{dz^2}+k_\sigma^2 a_\sigma = u_\sigma(x',y')\delta(z-z'). \tag{108.6}$$

The solution of this equation subject to $a_\sigma(z) \sim e^{jk_\sigma z}$ as $z \to +\infty$, and $a_\sigma(z) \sim e^{-jk_\sigma z}$ as $z \to -\infty$, is

$$a_\sigma(z) = -\frac{1}{2jk_\sigma}u_\sigma(x',y')e^{jk_\sigma|z-z'|}.$$

Hence†

$$G_1(P,P') = -\sum_\sigma \frac{1}{2jk_\sigma}u_\sigma(x,y)u_\sigma(x',y')e^{jk_\sigma|z-z'|}. \tag{108.7}$$

Similarly

$$G_2(P,P') = \sum_\sigma \frac{1}{2jK_\sigma}v_\sigma(x,y)v_\sigma(x',y')e^{jK_\sigma|z-z'|}, \tag{108.8}$$

from which there has been omitted a term proportional to $e^{jk|z-z'|}$ which does not contribute to the field in (108.4).

† Cf. Tamarkin, J. D., and Feller, W., *Partial Differential Equations* (Brown University, 1941), p. 252.

Substituting for the Green's functions in (108.1) and (108.4) and using the contracted notation $u_\sigma(x',y') = u'_\sigma$, etc., we have for the generating functions of the TM- and TE-waves

$$U(P) = -\sum_\sigma \frac{1}{2jk_\sigma h_\sigma^2} u_\sigma \int_S \frac{\partial u'_\sigma}{\partial v'} e^{jk_\sigma|z-z'|} E_z(P')\, dS', \tag{108.9a}$$

$$V(P) = \sum_\sigma \frac{1}{2kK_\sigma} v_\sigma \int_S v'_\sigma e^{jK_\sigma|z-z'|} E_s(P')\, dS' -$$

$$-(\pm)\frac{j}{2kH_\sigma^2} v_\sigma \int_S \frac{\partial v'_\sigma}{\partial s'} e^{jK_\sigma|z-z'|} E_z(P')\, dS', \tag{108.9b}$$

the + or − sign being taken in (108.9b) according as $z > z'$ or $z < z'$.

These formulae show clearly that to excite TM-waves by means of an aperture it is the longitudinal component of the tangential electric force that is effective, whereas to excite TE-waves both longitudinal and transverse electric force in the aperture are effective. Further, for strong coupling E_z should be concentrated where $\partial u'_\sigma/\partial v'$ is maximum for TM-waves, while for TE-waves, E_s should be concentrated at maximum v'_σ, E_z at maximum $\partial v'_\sigma/\partial s'$. In a circular wave guide, for example, these places for TE-wave excitation are 90° apart on the wall.

10.9. The Resonant Slot in the Rectangular Guide

Let us consider first an infinite conducting plane in which is cut a narrow slot and a wave incident on it from one side. Since we are interested in the first resonance of the slot, its length will be about one-half wavelength. From the near-symmetry of Maxwell's equations in E and H we have already seen that this problem is essentially the same as that of an antenna in the form of a narrow conducting strip scattering the incident wave in which the vectors E and H are interchanged. Now, in the electric antenna it is the longitudinal current in the strip and hence the transverse tangential magnetic force at the surface of the strip which is of importance in determining the radiation field due to the antenna. The magnetomotive force between the long edges of the strip is equal to the electric current at the same section and varies very nearly sinusoidally with distance along the strip, the maximum at the centre. Thus, in the slot problem, the important field components are the tangential components of E transverse to the slot and of H along it. For a narrow slot, transverse electric force is associated with the e.m.f. e which is equal to the magnetic current required to produce the radiation field of the slot. The foregoing conclusions may be expected to apply at least approximately to a slot in a wave guide coupled to free space or to a slot in guide-to-guide coupling: near resonance, the voltage is distributed very nearly sinusoidally and vanishes at the ends of the slot. When the slot extends round a corner and is therefore bent sharply, the sinusoidal distribution may be expected to fail.

We have to introduce a convention as to how the position of the slot

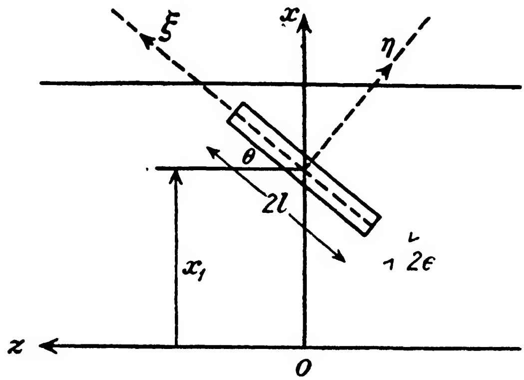


Fig. 94.

is to be specified with respect to the guide. We take the z-axis along one of the edges of the broad face containing the slot, and a right-handed system of axes, so that the x-axis passes through the centre of the slot and the y-axis is parallel to the normal to the guide face drawn into the guide, so that Fig. 94 views the guide face containing the slot from the outside. Let x_1 be the distance of the centre of the slot from the z-axis, and the acute angle θ between the z-axis and the axis of the slot be taken as positive, if in the same sense as the rotation Oz to Ox: so we have always $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$.

To describe the tangential field at the slot, we find it convenient to introduce axes (ξ, η) through the centre of the slot as shown, the ξ-axis making an acute angle with Oz, and a rotation $O\xi$ to $O\eta$ having the same sense as $Oz \to Ox$. The component E_η of $\mathbf{E}$ across the slot in the η-direction, when integrated across the slot yields the voltage $e(\xi)$:

$$e(\xi) = \int\limits_{-\epsilon}^{\epsilon} E(\xi, \eta)\, d\eta, \tag{109.1}$$

2ϵ being the width of the slot.

We combine (103.1) and (103.2) to obtain

$$\left.\begin{aligned} H_z &= \frac{1}{\mu_0}\left(\frac{\partial^2 V}{\partial z^2}+k^2 V\right), \\ H_x &= \frac{1}{\mu_0}\frac{\partial^2 V}{\partial x \partial z} - j\omega\frac{\partial U}{\partial y}. \end{aligned}\right\} \tag{109.2}$$

The functions $U(x,y,z)$ and $V(x,y,z)$ are given by (108.9). Now relying on our conception that only E_η is important at the slot, we have

$$U = U'\sin\theta,$$

$$V = V'\cos\theta + V''\sin\theta, \tag{109.3}$$

where

$$U' = \frac{1}{2j}\sum_\sigma \frac{u_\sigma(x,y)}{k_\sigma h_\sigma^2}\int_S e^{jk_\sigma|z-z'|}\frac{\partial u'_\sigma}{\partial y'}E_\eta(\xi',\eta')\,d\xi' d\eta', \tag{109.4 a}$$

$$V' = \frac{j}{k}\int_S G_2(\xi,\eta;\xi',\eta')E_\eta(\xi',\eta')\,d\xi' d\eta', \tag{109.4 b}$$

$$V'' = \pm\frac{j}{2k}\sum_\sigma \frac{v_\sigma(x,y)}{H_\sigma^2}\int_S e^{jK_\sigma|z-z'|}\frac{\partial v'_\sigma}{\partial x'}E_\eta(\xi',\eta')\,d\xi' d\eta'. \tag{109.4 c}$$

(x',z') or (ξ',η') is a point of the domain of integration, the slot S, and after differentiation we are to put $y' = 0$. The upper or lower sign in the expression for V'' is to be taken according as $z > z'$ or $z < z'$.

We require an expression for H_ξ so as to apply the equation of continuity of longitudinal magnetic force along the slot inside and outside the guide, the wall of which is taken as infinitely thin. Since

$$H_\xi(\xi,\eta) = (H_z\cos\theta + H_x\sin\theta)_{y=0}, \tag{109.5}$$

we have from (109.2) and (109.3)

$$\mu_0 H_\xi(\xi,\eta) = \cos^2\theta\left(\frac{\partial^2 V'}{\partial z^2}+k^2V'\right)+\cos\theta\sin\theta\left(\frac{\partial^2 V''}{\partial z^2}+k^2V''+\frac{\partial^2 V'}{\partial x\partial z}\right)+$$
$$+\sin^2\theta\left(\frac{\partial^2 V''}{\partial x\partial z}-j\omega\mu_0\frac{\partial U'}{\partial y}\right).$$

Now $$\frac{\partial^2}{\partial\xi^2} = \cos^2\theta\frac{\partial^2}{\partial z^2}+2\cos\theta\sin\theta\frac{\partial^2}{\partial x\partial z}+\sin^2\theta\frac{\partial^2}{\partial x^2}.$$

Substitute for $\cos^2\theta(\partial^2/\partial z^2)$ to obtain after rearrangement and use of (109.4)

$$\mu_0 H_\xi(\xi,\eta) = \left(\frac{\partial^2}{\partial\xi^2}+k^2\right)V'+J(\xi,\eta), \tag{109.6}$$

where

$$J(\xi,\eta) = \pm\frac{j}{2k}\sin\theta\cos\theta\sum_\sigma\int\limits_S\left(v_\sigma\frac{\partial v'_\sigma}{\partial x'}+v'_\sigma\frac{\partial v_\sigma}{\partial x}\right)e^{jK_\sigma|z-z'|}E'_\eta\,d\xi'd\eta'+$$
$$+\tfrac{1}{2}\sin^2\theta\left\{\frac{1}{k}\sum_\sigma\int\limits_S\left(\frac{K_\sigma}{H_\sigma^2}\frac{\partial v_\sigma}{\partial x}\frac{\partial v'_\sigma}{\partial x'}-\frac{1}{K_\sigma}\frac{\partial v_\sigma}{\partial x}v'_\sigma\right)e^{jK_\sigma|z-z'|}E'_\eta\,d\xi'd\eta'+\right.$$
$$\left.+k\sum_\sigma\int\limits_S\left(\frac{\sqrt{(\mu_0/\kappa_0)}}{k_\sigma h_\sigma^2}\frac{\partial u_\sigma}{\partial y}\frac{\partial u'_\sigma}{\partial y'}e^{jk_\sigma|z-z'|}-\frac{1}{K_\sigma}v_\sigma v'_\sigma e^{jK_\sigma|z-z'|}\right)E'_\eta\,d\xi'd\eta'\right\}, \qquad (109.7)$$

and $y = y' = 0$ after differentiation.

It has been assumed that term-by-term differentiation of the infinite series is legitimate. On account of the exponential factors, if $z \neq z'$, this is so; but at points of the domain of integration for which $z = z'$ it may not be. In this event, one must exclude a small region from the integrals and add integrals over this small region where differentiations occur outside the sign of summation. With this precaution and excluding the case $\theta = \frac{1}{2}\pi$ we may show that $J(\xi,\eta)$ remains finite as $\epsilon \to 0$. With an error of order ϵ/l we may replace it by

$$J_0(\xi) = \int\limits_{-l}^{l} F(\xi,\xi')e(\xi')\,d\xi', \qquad (109.8)$$

where

$$F(\xi,\xi') = \tfrac{1}{2}\sin^2\theta\left[\frac{1}{k}\sum_\sigma\left(\frac{K_\sigma}{H_\sigma^2}\frac{\partial v_\sigma}{\partial x}\frac{\partial v'_\sigma}{\partial x'}-\frac{1}{K_\sigma}\frac{\partial^2 v_\sigma}{\partial x^2}v'_\sigma\right)e^{jK_\sigma|z-z'|}+\right.$$
$$\left.+k\sum_\sigma\left\{\sqrt{\left(\frac{\mu_0}{\kappa_0}\right)}\frac{1}{k_\sigma h_\sigma^2}\frac{\partial u_\sigma}{\partial y}\frac{\partial u'_\sigma}{\partial y'}e^{jk_\sigma|z-z'|}-\frac{1}{K_\sigma}v_\sigma v'_\sigma e^{jK_\sigma|z-z'|}\right\}\right]\pm$$
$$\pm\frac{j}{2k}\sin\theta\cos\theta\sum_\sigma\left(v_\sigma\frac{\partial v'_\sigma}{\partial x'}+v'_\sigma\frac{\partial v_\sigma}{\partial x}\right)e^{jK_\sigma|z-z'|}, \qquad (109.9)$$

in which after differentiation we put

$$\left.\begin{aligned} x &= x_1+\xi\sin\theta, & z &= \xi\cos\theta, & y &= 0;\\ x' &= x_1+\xi'\sin\theta, & z' &= \xi'\cos\theta, & y' &= 0.\end{aligned}\right\}$$

Thus we have $$\mu_0 H_\xi(\xi,\eta) = \left(\frac{\partial^2}{\partial\xi^2}+k^2\right)V'+J_0(\xi). \qquad (109.10)$$

The magnetic field H_ξ^0 outside the guide is calculated on the assumption that the slot is cut in an infinite conducting plane. It may be inferred in exactly the same way as the electric field due to a thin antenna in the form of a flat strip having a given current distribution, so

$$\mu_0 H^0_\xi(\xi,\eta) = \frac{j}{2\pi k}\left(\frac{\partial^2}{\partial\xi^2}+k^2\right)\int\limits_S \frac{e^{jkr}}{r}E_\eta(\xi,\eta')\,d\xi' d\eta', \tag{109.11}$$

where r is the distance between (ξ,η) and (ξ',η').

Now suppose the incident dominant wave of complex amplitude A (reckoned at the centre of the slot) to travel in the positive z-direction, and let $Af(\xi)/\mu_0$ be the ξ-component of the tangential magnetic force in this wave. Continuity of the tangential magnetic force at the slot requires

$$\mu_0 H_\xi = \mu_0 H^0_\xi - Af(\xi),$$

i.e.

$$\frac{j}{k}\left(\frac{\partial^2}{\partial\xi^2}+k^2\right)\int\limits_S \left[G_2(\xi,\eta;\xi',\eta')-\frac{1}{2\pi}\frac{e^{jkr}}{r}\right]E'_\eta\,d\xi' d\eta' = -J_0(\xi)-Af(\xi). \tag{109.12}$$

This is an integro-differential equation in E_η which would usually be solved by series methods. The principal part of the solution is governed by the same considerations that have already been discussed in the literature of the electric oscillator, and we know physically that the most important term in the distribution of $e(\xi)$ is proportional to $\cos k\xi$. Accordingly let

$$e(\xi) = p'\cos k\xi,$$

and operate on (109.12) in the usual way for finding coefficients in a Fourier series; we then have

$$j\left[\sin k\xi\int\limits_S \left\{G_2(\xi,\eta;\xi',\eta')-\frac{e^{jkr}}{2\pi r}\right\}E'_\eta\,d\xi' d\eta'\right]_{-l}^{l}$$
$$= -A\zeta - p'\iint\limits_{-l}^{l} d\xi d\xi'\,F(\xi,\xi')\cos k\xi\cos k\xi', \tag{109.13}$$

where

$$\zeta = \int\limits_{-l}^{l} f(\xi)\cos k\xi\,d\xi. \tag{109.14}$$

Let us rewrite† the integral on the left as

$$\int\limits_S \left(G_2-\frac{e^{jkr}}{2\pi r}\right)E_\eta(\xi,\eta')\,d\xi' d\eta' + \int\limits_S \left(G_2-\frac{e^{jkr}}{2\pi r}\right)[E_\eta(\xi',\eta')-E_\eta(\xi,\eta')]\,d\xi' d\eta'.$$

Approximate expressions for these two integrals are respectively

$$\frac{1}{2\pi}\log\frac{4l}{\epsilon}e(\xi) \tag{109.15}$$

† This step follows Hallen's method discussed by Schelkunoff, l.c., 1945.

and
$$\int_{-l}^{l} \left\{G_2(\xi,\xi') - \frac{e^{jk|\xi-\xi'|}}{2\pi|\xi-\xi'|}\right\}\{e(\xi')-e(\xi)\}\,d\xi', \tag{109.16}$$

where $G_2(\xi,\xi')$ means $\lim_{\epsilon\to 0} G_2(\xi,\eta;\ \xi',\eta')$. Now substitute in (109.13), put $\sin kl = 1$, and we have

$$\frac{p'}{A} \doteq \frac{\zeta}{K}, \tag{66.1}$$

the result already quoted in Chapter VI, where

$$K = \frac{-j}{\pi}\log\frac{4l}{\epsilon}\cos kl - j\int_{-l}^{l} [G_2(l,\xi')+G_2(-l,\xi')]\cos k\xi'\,d\xi' +$$
$$+\frac{j}{\pi}\int_{-l}^{l} \frac{e^{jk(l-\xi')}}{l-\xi'}\cos k\xi'\,d\xi' - \int\!\!\int_{-l}^{l} F(\xi,\xi')\cos k\xi\cos k\xi'\,d\xi'd\xi. \tag{109.17}$$

The first term is retained because, as Stevenson [43] has shown, the approximate solution† depends on $2\log(4l/\epsilon)$ being much greater than 1; in all other terms we may write $k = \pi/2l$ or $l = \frac{1}{4}\lambda$.

If the incident wave is travelling in the negative z-direction and is of complex amplitude B, the amplitude p of voltage at the centre of the slot is given by

$$\frac{p}{B} = \frac{\zeta^*}{K}. \tag{66.2}$$

To complete the calculation we use the second formula of (108.9) employing only the value of σ for the dominant wave, because all other terms correspond to evanescent waves. We assume a sinusoidal distribution of voltage along the slot, the amplitude at the centre being p'. By referring to (103.2) and (106.6) and (109.5), we readily obtain for the amplitudes of the dominant waves radiated to the left and right respectively $L\zeta p'$ and $L\zeta^* p'$ with $L = a\mu_0/\pi^2 bkk_{10}$ and ζ defined in (109.14). Thus the results on the basis of which § 6.6 was developed are established as the first approximation in the field representation.

For a laterally displaced slot cut in standard S-band guide with $\lambda = 10{\cdot}7$ cm., Pounder [44] has worked out Stevenson's result numerically. The impedance of this shunt slot is $Z = R - jX$, where

$$X = \frac{F(x_1) - C\,\delta l\log(4l/\epsilon)}{\cos^2(\pi x_1/a)}.$$

$\delta l = \frac{1}{4}\lambda - l$, and x_1 is measured from the inside edge of the broad face.

† Cf. Schelkunoff, l.c.

C is the constant $1{\cdot}194$ cm.$^{-1}$ with the appropriate data. In Fig. 95, $F(x_1)$ is shown as a function of x_1/a and the value of δl for resonance ($X = 0$) is shown on the same figure, corresponding to three different widths of slot. From the shape of the $F(x_1)$ graph, we see that the length of a resonant slot increases parabolically with its displacement from the centre of the broad face in which it is cut, provided that the displacement is not too great.

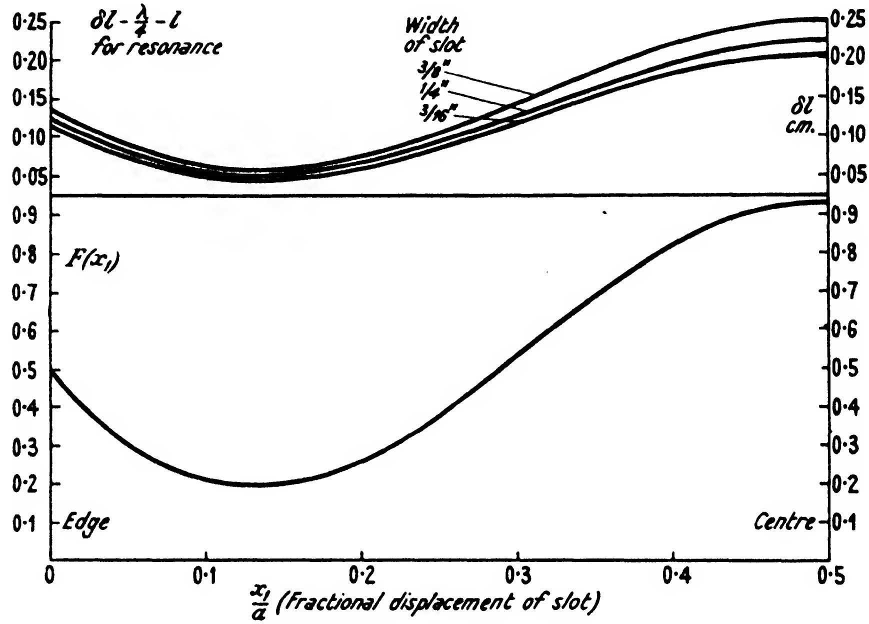


Fig. 95.

If the slot is in the narrow face of the guide, a similar calculation can be carried through, and the results apply in the same way as before, if, in K, a replaces b and vice versa, and the expression for ζ is that given in (66.4). For an end slot, that is, one cut in a plate terminating the guide at right angles to its axis, the same principles apply. We require, however, a new expression in place of (109.10) to represent the longitudinal component of magnetic force in the slot due to the very nearly sinusoidal distribution of electric force across it. The method of images seems best suited to this purpose; the evaluation of K in (66.1) for this case has not yet been carried out.

The coupling of two guides by means of a slot when the wall is infinitely thin can evidently be treated by the foregoing method. We can now dispense with equation (109.11) and the assumption it involves, namely that the slot connects the inside of the guide with space above the infinite plane in which the cut broad face lies. Instead we use the analogue of (109.10) for the second guide. The conventions for specifying the configuration are the following. The position of the slot with respect to the second guide is given by a second pair of parameters (x_1, θ); the ξ-axis, which lies along the length of the slot, is common to both guides, but the direction of the η-axis must be reversed if E_η is to have the same direction in both when positive: in consequence, once the direction of z is chosen in the first guide, that in the second is fixed.

The essence of the argument for the general case of single slot coupling of guides is already given in § 7.3. It remains to specify K_{12} explicitly: it is derived in quite the same way as K above. If we use superscripts to distinguish the functions $G_2(\xi, \xi')$ and $F(\xi, \xi')$, defined respectively in (109.16) and (109.9) for the single guide, then

$$\begin{aligned} K_{12} = \frac{-j}{\pi}\log\frac{4l}{\epsilon}\cos kl - \\ -j\int\limits_{-l}^{l} [G_2^{(1)}(l,\xi')+G_2^{(1)}(-l,\xi')]\cos k\xi'\,d\xi' - \\ -j\int\limits_{-l}^{l} [G_2^{(2)}(l,\xi')+G_2^{(2)}(-l,\xi')]\cos k\xi'\,d\xi' - \\ -\iint\limits_{-l}^{l} F^{(1)}(\xi,\xi')\cos k\xi\cos k\xi'\,d\xi d\xi' - \\ -\iint\limits_{-l}^{l} F^{(2)}(\xi,\xi')\cos k\xi\cos k\xi'\,d\xi d\xi'. \end{aligned} \qquad (109.18)$$

10.10. The Longitudinally Polarized Array of Slots

As the final application of these principles, we shall discuss, to the same degree of approximation as before, the field representation of the mutual interaction between the slots of a longitudinally polarized array. Since the slots are spaced at such distance apart that evanescent waves inside the guide play no part in their mutual interaction, we need only consider the single slot in the array excited by the dominant wave in the guide and under the action of the field due to each of the other slots radiating outside the guide.

Let us number the slots 1, 2, 3,..., N according to their order in one of the narrow faces of the rectangular guide. Since each slot is approximately $\frac{1}{2}\lambda$ long, the voltage distribution along each slot will be approximately sinusoidal. Let p_s denote the complex amplitude of the voltage at the centre of the sth slot. The problem is to establish the connexion between the p_s and the wave incident from the generator inside the guide at one end, in terms of the spacing, inclination, and length of the slots.

In the first place, we have to replace equation (109.11), which served for a single slot, by

$$\mu_0 H^0_{\xi_s}(\xi_s, \eta_s) = \frac{j}{2\pi k}\left(\frac{\partial^2}{\partial \xi_s^2}+k^2\right) \sum_{m=1}^{N} \int\limits_{S_m} \frac{e^{jkr_{ms}}}{r_{ms}} E_{\eta_m}(\xi'_m, \eta'_m)\, d\xi'_m\, d\eta'_m, \tag{110.1}$$

where E_{η_m} is the η-component of electric force in the mth slot, r_{sm} is the distance between (ξ_s, η_s) and (ξ'_m, η'_m), and each integral in the sum is an integral over a slot including the sth one. Accordingly, equation (109.12) must be altered by adding to the expression on the left referring to the sth slot the sum on the right of (110.1) above, from which the sth term of the sum has been omitted. On the right we must write in place of A the complex amplitude A_s which denotes the amplitude of the wave travelling in the direction of z-increasing, reckoned immediately at the sth slot on the incident side, which we take as usual to be the left. Proceeding exactly as with a single slot we find

$$\sum_{m=1}^{N} q_{sm} p_m = A_s \zeta_s, \tag{110.2}$$

where $$q_{ss} = K_s,$$

the value of K calculated in accordance with (109.17) for the sth slot, and

$$q_{sm} = \int\limits_{-l}^{l} F_{sm} \cos k\xi_s\, d\xi_s, \tag{110.3}$$

F_{sm} being the component of H along the sth slot in the waves outside the guide due to a sinusoidal voltage of unit amplitude in the mth slot.

The foregoing is the basis of the discussion in § 8.9 of the array with non-resonant spacing of the weakly coupled slots excited by a travelling wave. When the slots, although still weakly coupled to the guide, are at resonant spacing, we can no longer overlook the scattered waves in the guide. Let B_s denote the complex amplitude of the wave travelling to the left immediately to the right of the sth slot, then

$$\sum_{m=1}^{N} q_{sm} p_m = A_s \zeta_s + B_s \zeta_s^*. \tag{110.4}$$

Now by the principle of superposition, for the dominant wave in the guide,

$$A_s = A_1 e^{jk_1 z_s} + \sum_{m=1}^{s-1} L\zeta_m^* p_m e^{jk_1(z_s - z_m)},$$

$$B_s = B_N e^{-jk_1(z_N - z_s)} + \sum_{m=s+1}^{N} L\zeta_m p_m e^{jk_1(z_s - z_m)}.$$

Let B_0 be the reflected wave amplitude from the whole array reckoned at the position of the first slot, then $B_0/A_1 = w_0$ is the value of the circle-diagram variable corresponding to the input impedance reckoned at the same position.

Thus $$w_0 A_1 = B_N e^{-jk_1 z_N} + \sum_{m=1}^{N} L\zeta_m p_m e^{-jk_1 z_m},$$
and eliminating B_N we find

$$B_s = w_0 A_1 e^{jk_1 z_s} - \sum_{1}^{N} L\zeta_m p_m e^{jk_1(z_s - z_m)} + \sum_{s+1}^{N} L\zeta_m p_m e^{jk_1(z_s - z_m)}$$

and

$$A_s \zeta_s + B_s \zeta_s^* = A_1(\zeta_s e^{jk_1 z_s} + w_0 \zeta_s^* e^{jk_1 z_s}) - \sum_{1}^{N} L\zeta_m \zeta_s^* p_m e^{jk_1(z_s - z_m)} +$$
$$+ \sum_{1}^{s-1} L\zeta_m^* \zeta_s p_m e^{jk_1(z_s - z_m)} + \sum_{s+1}^{N} L\zeta_m \zeta_s^* p_m e^{jk_1(z_s - z_m)}.$$

At $\frac{1}{2}\lambda_g$ spacing, and with weak shunt loading ($\zeta_s^* = \zeta_s$)

$$\sum_m q_{sm} p_m = A_1 \zeta_s (1 + w_0)(-1)^{s-1},$$

so that for matched input we require

$$\sum_m q_{sm} p_m = (-1)^{s-1} A_1 \zeta_s.$$

The design problem for this type of array appears to consist in finding the ζ_s and $\operatorname{Im} K_s$ from which the inclination and length of each slot can be found, when the p_m/A_1 are known, as they will be, from the amplitude distribution and the fact that all the energy is radiated.

The foregoing appears as a straightforward attack on the problem of the antenna design, which it should be possible with sufficient computational aid to carry through by successive approximation. There are, however, reasons which lead one to doubt the correctness of this method of approach. In the writer's experience every array of this type which was made in the laboratory gave a good main lobe normal to the array, even although the amplitude distribution may not have been satisfactory and the input impedance far from expectation. It is well known that the mutual effects are strong and extend over many slots. These facts point to a much more stable phase distribution along the array than the simple theory we have discussed seems capable of explaining. They point strongly to a multiply-coupled system as the mechanism by which the phase is preserved across the array; that is, the array should be thought of as a set of parallel tuned circuits coupled by mutual impedance. In the lowest order mode of this system, all the elements oscillate in phase, and it must be this mode that is excited by the wave in the guide. Furthermore, the possibilities of current flow on the outside of the guide allowing this simple phase distribution are quite complicated, so that slot-length is not a critical factor in determining the phase distribution in the way indicated by the foregoing calculations. Slot-length does enter to affect the amplitude distribution and the input impedance. We may conclude then by remarking that the proper understanding of wave-guide arrays of this type depends on a more adequate representation of waves on the outside of the guide.

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R.C. Radiolocation Convention of the Institution of Electrical Engineers; in connexion with this, papers now in press will appear in the Journal of the Institution.

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