STRUCTURAL THEORY AND DESIGN

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BY

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FOREWORD

THIS book would appear to me to fulfil a useful function, in that it combines the fundamental theory of structures with the practical knowledge of structural design gained by the author over a number of years. In his introduction the author has pointed out that engineering design can be learnt only in practice. While this is perfectly true, the young engineer can always profit by the experience of the practising engineer. This work has, in my opinion, endeavoured to indicate the application of first principles to practical civil engineering problems.

For that reason and also since the author is a former member of my staff, and well known to me, it gives me great pleasure to recommend this work especially to the younger generation of civil and structural engineers.

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C.B.E., M.C., M.I.C.E., M.I.M.E., M.I.E.E.

INTRODUCTION

This book is intended primarily for the student or young engineer, and the theoretical portions should cover the syllabus for the Degree or Associate Membership examinations in the Theory of Structures; for that reason many of the numerical examples are based upon questions set in those examinations, but it is hoped that the book will also prove of service to practising engineers.

Many books on Structural Design are either too academic or ultra-practical, that is, based upon rule-of-thumb approximations. Whilst it is a truism that the art and science of design can be learnt only by experience, the author feels that there is a definite need of, and demand for, a book or books which, whilst dealing with the methods of rigorous analysis, can indicate the application of first principles to practice. It has been stressed in the worked examples that a designer must acquire the skill to be able to modify the theoretical requirements to suit practical design problems.

In presenting the subject the author has assumed a knowledge of elementary mechanics and mathematics; the use of advanced mathematics has been avoided wherever possible, emphasis being placed upon the methods of successive approximation which have been introduced in recent years. The student should find the comparison of methods of analysis in the later chapters of interest, and he should bear in mind that no one method should be applied indiscriminately to every problem. The arrangement of the subject-matter may be unusual, but it has seemed logical to commence with materials and, having dealt with the design of main members, to conclude with the problem of Structural Connexions (a matter which has received scant attention in many text-books). The student should also realize that the engineer who cannot prepare satisfactory details cannot design efficiently, and he should consider connexions in relation to fabrication, erection, and engineering economics.

The book has been divided into two volumes, the first of which covers the more elementary part of the subject. The second volume deals with advanced theory and design suitable for the student specializing in structures and for the practising structural engineer.

The author wishes to thank all who have been kind enough to read the manuscript for their helpful criticism and advice, in particular Mr. R. H. Ray; Mr. A. E. Moxon; Mr. H. P. Smith, B.Sc., M.I.Struct.E.; Mr. A. P. Mainprice, B.Sc., A.M.I.C.E.; Mr. E. E. Markland, M.Sc., A.M.I.C.E., and Mr. L. R. Waddington, B.A., A.M.I.C.E. Part of the text and illustrations in Chapter XIV are reproduced from a previous

publication by kind permission of Mr. G. W. Thomson; the Tables in Appendix A and plates in Chapter XV by permission of the British Steelwork Association; the Tables in Appendix C by permission of H.M. Stationery Office, and those in Appendix D by permission of the Institution of Structural Engineers. The illustrations of timber connexions in Chapter XVI are reproduced from the B.S. Specification by permission of B.S.I. and from photographs kindly supplied by the Timber Engineering Co., Washington, U.S.A.

The author also wishes to express his grateful thanks to Dr. Orr, Glasgow University, for numerical examples, and to J. B. Dwight, M.Sc. for his careful reading of the proofs.

Finally, as it is impossible to avoid errors in presentation and computation, the author will be grateful if readers will notify him of any they may discover, with a view to their elimination.

34, ST. JAMES'S AVENUE
HAMPTON HILL, MIDDX.
June, 1950

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CHAPTER I

STRUCTURAL MATERIALS

THIS book is not intended to deal with the strength of materials at any length. It is, however, necessary for the designer to understand the physical properties of the materials with which he is dealing, and for that reason this chapter has been included as an introduction to structural theory and design.

Steel is the material most universally used in structural work either as structural, i.e. fabricated, steelwork or as reinforcement to concrete in the form of round or square bars, wires, or fabric. Steel manufactured in this country is very uniform in quality and has displaced cast and wrought iron almost entirely. It possesses approximately the same strength in tension and compression and also has a high resistance to the effects of shear and torsion.

The manufacture of steel is beyond the scope of this chapter and the following text is intended to give a brief outline only. Steel used for structural purposes can be classified thus:

1. Mild Steel: Structural Steel for Bridges, etc., and General Building Construction. (B.S.S. 15—1948.)
2. High-tensile Steel: High-tensile Structural Steel for Bridges, etc., and General Building Construction. (B.S.S. 548—1934.)
3. High-tensile (Fusion Welding Quality) Structural Steel for Bridges, etc., and General Building Construction. (B.S.S. 968—1941.)
4. Bars and Wires for Concrete Reinforcement (B.S.S. 785—1938.)

Mild steel is the most common form of steel used for structural purposes. It can be manufactured by the open-hearth process (acid or basic) or the acid Bessemer process and should not contain more than 0·06 per cent. of phosphorus or sulphur. Mild steel may be copper bearing, and the copper content should be either between 0·2 and 0·35 per cent. or between 0·35 and 0·50 per cent.

The *basic open-hearth* process produces nearly 75 per cent. of all the steel made in this country. It was first introduced into this country in 1884 in order that phosphoric pig-iron might be used in place of the more expensive haematite, and it has grown in popularity since it is able to use cheap low-grade steel scrap, cast-iron and wrought-iron scrap which the acid open-hearth or Bessemer processes cannot use. The chief difference between the basic and the acid open hearth lies in the lining of the hearth. In the case of the basic process the hearth is lined with bricks made of crushed dolomite bonded with tar or with dolomite melted to form a covering. In operating a furnace of this type, the first charge consists of steel scrap; afterwards the pig-iron and scrap are added alternately. About 10 per cent. of limestone should be added to help the formation of basic slag. The pig-iron used in this process should be basic, i.e. low silicon content, high manganese phosphoric iron. The phosphorus content is less than that for the acid open-hearth process, since the necessary heat is supplied by producer gas and not by the process of oxidation of the phosphorus. The charge should be melted as quickly as possible, and most of the silicon and some of the manganese is oxidized and absorbed by the slag during this part of the process. When the charge is completely melted the contents of the furnace show about 0·6–0·8 per cent. carbon. Iron ore and mill scrap are then added to reduce the carbon content and fluorspar can be added to make the slag more fluid. As the carbon content is reduced, more lime and ore are added to facilitate the removal of phosphorus, which should be eliminated before the carbon content has fallen below the specified value. Samples of the metal are analysed for carbon, sulphur, and phosphorus. To prevent the return of phosphorus from the slag to the steel it is necessary to deoxidize and recarbonize the steel either in the ladle or in the furnace. The steel is tapped out of the furnace into the ladle and various additions are put into the steel in the ladle to remove impurities, and powdered anthracite to bring the carbon content to the necessary value. Since the steel in the ladle may absorb phosphorus from the residue slag, some manufacturers discard the last ingot cast from the ladle.

In the *acid open-hearth* process the floor of the furnace is lined with silica or similar substances. The charge consists of 30–50 per cent. of haematite pig-iron and selected steel scrap (not cast iron or wrought iron) which must be low in sulphur and phosphorus. The pig-iron is fed in first and the steel scrap follows. The charge should be melted as rapidly as possible (4–7 hours), and during melting a certain proportion of iron and silicon should be oxidized and the oxides form a slag. The iron-oxide content of the slag is raised by adding haematite ore which has the result of oxidizing the rest of the silicon, also the manganese and the carbon. More ore is added to keep the metal boiling and to continue the elimination of carbon, samples being analysed from time to time. When the carbon content is nearly the desired value, no more ore is added, but limestone is added to thin the slag. Just before the end of the process additions are put into the metal to eliminate impurity and to adjust the carbon content, and the metal is then tapped into the ladle and thence into ingot moulds. Acid open-hearth furnaces produce about 18 per cent. of the total output of steel in this country.

The *acid Bessemer process* is carried out in a tilting steel container called a 'converter' which is fitted with a wind box from which air is forced in through openings called 'tuyeres'. Bessemer converters are used in conjunction with blast-furnace plants, and liquid iron from the blast-furnace is tapped into a 'mixer' and thence into the converter. Owing to the action of the air entering through the tuyeres, the molten metal becomes agitated and sparks come from the mouth of the converter. Iron oxide is formed and this reacts with the impurities in the iron, which should contain 1·5 to 2·5 per cent. silicon. As the process continues silicon, manganese, and finally carbon are eliminated, and the metal 'boils', a white flame appears, and the blast should be shut off. Sulphur and phosphorus are not eliminated and therefore the pig-iron must not contain more than a certain amount of these impurities. In order to recarbonize the iron a certain quantity of iron (either solid or liquid) is added to convert the pure iron into steel. It is necessary to remove dissolved oxygen and iron oxide by the addition of alloys such as ferro-silicon, ferro-manganese, and silico-manganese. Haematite pig-iron is used in the acid Bessemer process, which produces about 2 per cent. of the total steel output of this country.

The *basic Bessemer process* uses phosphoric iron. The removal of phosphorus is effected after the elimination of silicon, manganese, and carbon. The lining of the converter is dolomite instead of the silica used in the acid process. The process is known as the Thomas process and is used extensively on the Continent where phosphoric ore deposits occur. It has also been reintroduced into this country to use similar ore found in Northamptonshire.

High-tensile steel (B.S.S. 548 and B.S.S. 968) and also steel to B.S.S. 785 can be manufactured by the open-hearth process (acid or basic) or the acid Bessemer process.

The chemical analysis and physical properties of these steels are given in Table I. It will be noticed that for steel to B.S.S. 15 no yield-point stress is specified.† Tensile tests are carried out on standard test-pieces (see Fig. 1.1), the most common being test-piece A. The specimen is placed in a testing machine (one type is shown diagrammatically in Fig. 1.2) and the load is applied gradually until failure takes place. During this time the extensions on the gauge length are measured by means of an extensometer (various types of these are in use). From the test results a graph can be plotted showing stress (force or load per unit area) as ordinate against strain (extension per unit length) as abscissa. A typical stress-strain graph for mild steel is shown in Fig. 1.3. The characteristic points on the graph are (1) the elastic limit, (2) the limit of proportionality, (3) the yield-point, and (4) the ultimate breaking stress.

† B.S.S. 15—1948 specifies a yield-point for steel.

The elastic limit is usually defined as the point where elasticity ceases to function but actually elasticity is always present in steel to some extent.

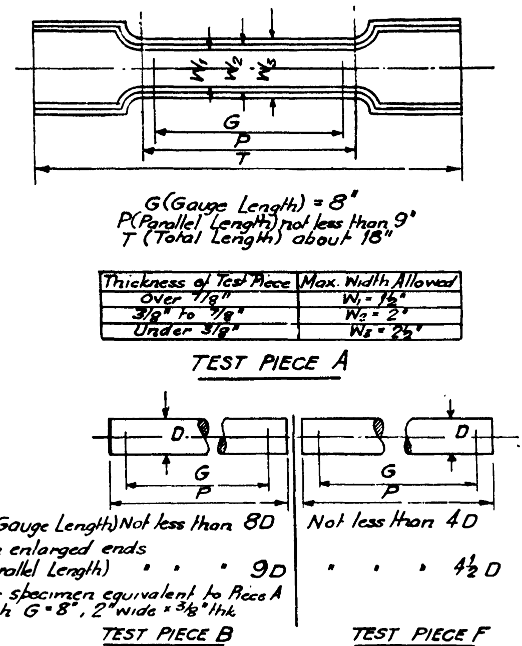

FIG. 1.1

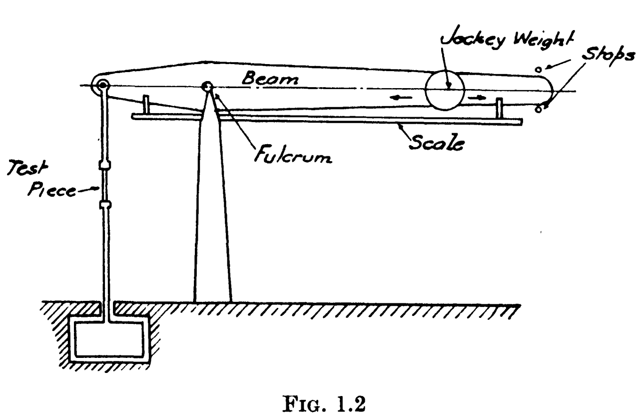

FIG. 1.2

The limit of proportionality is that point on the stress-strain graph where the graph ceases to be straight. In finding this point it is usually correct to neglect any permanent deformation less than 1/100,000. In carrying out a tensile test it will be found that the graph is a straight line from the origin up to a certain point and then deviates slightly from the straight line through the origin. The point where the stress-strain graph leaves the straight line through the origin is the limit of proportionality. Up to that point the specimen obeys Hooke's law, i.e. that stress is proportional to strain. It is necessary to take accurate

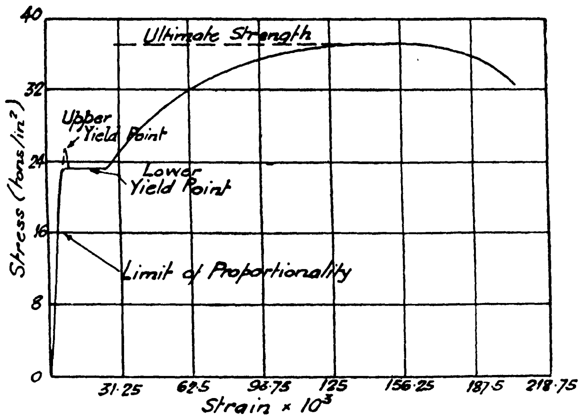

FIG. 1.3

observations with highly sensitive instruments to determine this point with any certainty. From this part of the graph, i.e. up to the limit of proportionality, the value of Young's modulus (E) can be found thus:

$$E = \text{stress} \div \text{strain},$$

$$= \text{slope of stress-strain graph.}$$

It is worth noting that the value of E remains practically constant, regardless of the carbon content of steel or even the method of manufacture or heat treatment.

The yield-point is usually defined as that point at which there is a sharp change in the behaviour of the specimen due to a breakdown of the structure of the material. The point is characterized by an extension much greater than the previous extensions *without increase in the load.* The yield-point is indicated in the testing machine shown in Fig. 1.2 by a marked drop in the lever. If the test is carried out very carefully and the stress at any point is the *actual* stress, the stress-strain will show two yield-points (upper and lower) and the upper value is taken as the yield-point stress. The yield-point stress is most important, and it will be noticed that the specification for high-tensile steel defines the yield-point stress (Table I). In fact, high-tensile steels might be called high yield-point steels, and the present tendency is to base working stresses on the yield-point stress and not on the ultimate stress as was formerly the practice. The yield-point stress appears to be a much more logical basis, since once the material has passed this

TABLE I

Comparison of Structural Steels

B.S.S. no.	Title	Chemical analysis (Max. percentages) S	P	Cu	C	Ni	Si	Mn	Cr	Physical tests For	Sizes	Ultimate stress Tons/in.²	Yield point	Elongation	Gauge length (in.)	Test piece	Radius for bend test
15	Bridges and general building construction	0·06	0·06	..	..	..	..	..	..	Sections, plates, and flats	⅜" and over	28–33	..	20	8	A	1·5 × thickness
		..	(a)	0·2–0·35	†	..	..	..	..		Under ⅜"	28–33	..	16	8	A	
											,, ¼"	Bend test only		..	..	..	
		..	(b)	0·35–0·50	†	..	..	..	..	Squares	⅜" and over	28–33	..	20	8t	B	,,
												,,	..	24	4t	F	
											Under ⅜"	Bend test only		..	..	..	
										Rounds	⅜" and over	28–33	..	20	8d	B	$d > 1''$, $r = 1{\cdot}5d$
												,,	..	24	4d	F	$d = < 1''$, $r = d$
											Under ⅜"	Bend test only		..	..	..	
										Rivet bars	All sizes	25–30	..	26	8d	B	Shanks bent double
												,,	..	30	4d	F	Heads flattened
548	High tensile for bridges and general building construction	..	..	..	0·25 (for rivet bars)	..	..	..	..	Sections, plates, and flats	Under 3/16"	Bend test only		..	..	..	1·5t
											3/16" to less than ⅜"	37–43	23	14	8	A	
											⅜"–1¼"	37–43	23	18	8	A	
											1¼"–1¾"	37–43	22	18	8	A	
		..	..	..	0·30 (for others)	..	..	..	..		1¾"–2¼"	37–43	21	18	8	A	
											2¼"–2¾"	37–43	20	18	8	A	
											Over 2¾"	37–43	19	18	8	A	
		..	..	as agreed (not to exceed 0·6)	..	..	..	..	..	Squares and rounds	Bars under 1¼"φ	Bend test only		..	..	..	1" φ and under $r = d$
		0·05	0·05‡		..	..	..	..	..		1¼"–1½"	37–43	22	18	8d	B	Over 1"φ $r = 1{\cdot}5t$
											1½"–2"	37–43	21	22	4d	F	$r = 1{\cdot}5d$
											2"–2½"	37–43	20				
											Over 2½"	37–43	19				
										Rivet bars	..	30–35	..	22	8d	B	Bars bent double, cold
														27	4d	F	Length 2d compressed to d by hammering hot

† Copper bearing steel only. ‡ May be 0·06 (War Emergency Revision, 1942).

968	High tensile (fusion welding quality) for bridges and general building construction	.. .. ..	.. .. ..	.. .. 0·60 (optional)	0·23	.. .. ..	.. .. ..	.. 1·8§ ..	.. .. ..	Plates	Under 3/8" thick 3/8"–1/2" thick 1/2"–1" thick Over 1" thick	37–43 (Under 3/8" and 3/8"–1/2") 35–41 33–39	23 21 19	14 18 (3/8"–1/2" to over 1")	8 8 8 8t	A A A B	1·5t
		0·06	0·06	.. .. ..	.. .. ..	.. .. 0·5 (optional)	0·35 (optional)	.. .. ..	.. 1·0§ ..	Sections, squares, flats, and rounds	Under 3/8" 3/8"–3/4" Over 3/4" Over 1" (rods and bars)	35–41 (Under 3/4") 33–39 (Over 3/4")	21 19	14 18 18 22	8 or 8t (Under 3/4") 4d	A or B B1	1·5t 1" φ rounds and under r = d
785	Rolled steel bars and hard drawn steel wire for reinforcement	M.S. to B.S.S. 15								Bars	Under 3/8" φ or thickness Up to and including 1" φ or thickness Over 1" φ or thickness	.. 28–33	.. ..	16 20 24	8d 8d 4d	B B F	1" φ and under r = d Over 1" φ r = 1·5d
		Medium high-tensile steel								Bars	Under 3/8" 3/8" up to and including 1" Over 1" up to and including 1½" Over 1½" up to and including 2" Over 2" up to and including 2½" Over 2½" up to and including 3"	.. 33–38 (3/8" to 3")	19·5 (Under 3/8" and up to 1") 18·5 17·5 16·5 (Over 2" to 3")	14 18 22 (Over 1" to 3")	8d (up to 1") 4d	B F	
		High-tensile steel to B.S.S. 548								Bars	Under 3/8" 3/8" up to and including 1" Over 1" up to and including 1½" Over 1½" up to and including 2" Over 2" up to and including 2½" Over 2½" up to and including 3"	.. 37–43 (3/8" to 3")	23 (Under 3/8" and up to 1") 22 21 20 19	14 18 22 (Over 1" to 3")	8d (up to 1") 4d	B F	
										Wire (cold drawn from mild steel)	All sizes	37–42	..	7·5	8d	B	

§ Combined percentage not more than 2·0.

point it becomes plastic and has no recovery. Since the yield-point is important, care should be taken to determine it as accurately as possible, the usual procedure being that laid down in B.S.S. 548. When the specimen is nearing the yield-point the rate of loading should not exceed 0·5 tons per square inch per second. The actual point can be found by the drop of the lever or, if this method is considered unsatisfactory, by means of dividers. The yield-point should not be assumed to have been reached until the increase in the gauge length exceeds 1/200. (The drop of the lever method gives the upper yield-point.) If the rate of loading is excessive, the value of the yield-point obtained is higher than the true value.

Proof stress. During the test, after the yield-point has been passed, if the load is gradually decreased and the lengths between the gauge-points measured, the stress-strain graph is approximately a straight line parallel to the 'loading' stress-strain graph. The intercept of the 'unloading' line on the horizontal axis gives the permanent set. B.S.S. 18 (Tensile Testing of Metals) defines proof stress as the stress just sufficient to produce a permanent elongation equal to a certain percentage of the original gauge length. This can be found either by the unloading test or by drawing a parallel line to the straight part of the 'loading' stress-strain graph to give the required intercept on the horizontal and reading off the corresponding stress on the graph. A proof test can be made by applying the specified proof stress to the specimen for 15 seconds. If, after the removal of the load, the permanent elongation of the specimen does not exceed the percentage elongation specified, then the material shall have passed the proof-stress test satisfactorily. Proof stress is sometimes called the commercial yield-point.

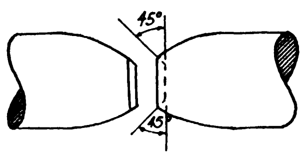

FIG. 1.4

After passing the yield-point, the extension increases rapidly without marked increase of load until the material fails. The nominal stress at failure, i.e. the breaking load divided by the nominal area, is called the *ultimate tensile stress.* Owing to the reduction in area which accompanies the elongation the actual stress is much higher than the nominal stress. The reduction of area is inversely proportional to the extension (measured on the gauge length). On reaching the maximum load a local contraction (or necking) of the specimen takes place and the load falls although the actual stress rises. The elongation on the gauge length and the reduction of area are an indication of the ductility of the specimen. A typical fracture of steel is shown in Fig. 1.4. The ultimate tensile stress is given approximately by the formula

$$U.T.S. = 19+50C+4(Mn)^2,$$

where C = percentage of carbon and Mn = percentage of manganese. The yield-point stress is usually 55 to 60 per cent. of the U.T.S.

Other tests

Bend tests. In this case the specimen shall be able to be bent over until the internal radius is not greater than that given in Table I (last column).

Tests for rivet bars. Rivet bars must be bent as shown in Table I and rivet heads must be able to be flattened, while hot, without cracking at the edges (see Fig. 1.5). The head when flattened must have a diameter at least $2\frac{1}{2}$ times that of the shank.

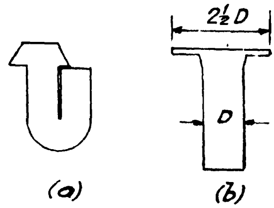

FIG. 1.5

Hardness test. This affords a quick and inexpensive method of finding the approximate U.T.S. The machine most commonly used is the Brinell hardness tester. The hardness is measured by the diameter of the impression made on a smooth, flat surface of the metal by a specially hardened steel ball. The test can also be made by a diamond hardness tester. The Brinell hardness number can be found from tables, then the U.T.S. = hardness number$\times$0·22.

Fatigue stress. This measures the resistance of the specimen to alternating stress, e.g. alternate tension and compression. The 'fatigue range' is defined as the greatest alternating stress which, when applied to the metal for any number of reversals, will not produce failure. The fatigue range is usually 80 to 90 per cent. of the U.T.S. In actual practice, steel structures are often subject to alternating stress due to the effect of rolling loads and may have to be designed accordingly.

Effect of overstrain and cold working

The effect of initial straining at stresses below the yield-point stress is to improve the elastic properties, and the limit of proportionality stress is increased. If the yield-point stress is exceeded, the material is *overstrained.* The effect of overstraining is to raise the yield-point stress and to reduce the ductility. Overstrain produces a condition called strain hardening when repeated.

If the yield-point stress is exceeded and the specimen is reloaded immediately after unloading, the limit of proportionality falls to zero, but the yield-point stress is raised to nearly the maximum stress during the previous loading. The U.T.S. is slightly increased and the elongation slightly decreased. If the yield-point stress is passed, the limit of proportionality stress is restored, if the specimen is allowed to rest. Mild heat treatment, e.g. immersion in boiling water, will completely restore the limit of proportionality stress and the yield-point stress.

The effect of cold working is to increase the yield-point stress and the U.T.S. and to reduce the ductility similar to the effects of overstrain. Normalizing counteracts the effects of cold working and restores the normal yield-point, U.T.S., and ductility.

Special and alloy steels

The demand for high-tensile steels first arose for long-span bridges, where it was important to reduce dead load. Nickel steels were first used in the Queenborough Bridge, the Quebec Bridge, and Delaware River Bridge. Silicon steels have been standardized in the U.S.A. and were used in the Sydney Harbour Bridge. Silicon steel has also been standardized in Germany as well as copper-chrome steels for bridgework. Very high tensile steels are required for the cables of large-span suspension bridges and for the reinforcement in pre-stressed concrete. The steel used for the latter should have an U.T.S. of 100 tons per square inch, 0·2 per cent. proof stress 0·8 of the U.T.S., and an E value of 12,000 to 14,000 tons per square inch. Wire for the same purpose is available up to No. 8 S.W.G. and has an U.T.S. of 150 tons per square inch and 0·2 per cent. proof stress of about 80 to 90 per cent. of the U.T.S. There is no B.S. specification for such steels, but the engineer can specify the U.T.S., etc., for the steel for any particular purpose. For example, for pre-stressed concrete sleepers the reinforcement can be either (*a*) hard drawn steel wire up to $\frac{1}{5}$ in. diameter having an U.T.S. of 80 to 125 tons per square inch and having a permanent set not exceeding 1·0 per cent. after being subject to a stress of 70 per cent. of the U.T.S., or (*b*) alloy steel up to $\frac{3}{8}$ in. diameter with an U.T.S. of 70 tons per square inch.

Concrete

Concrete is a material widely used in structural work. The use of concrete seems likely to increase, especially in view of recent developments in pre-cast and pre-stressed concrete. Concrete is strong in compression and has a moderate strength in shear, but its tensile strength is almost negligible. Plain concrete is suitable for resisting direct compression and where dead weight is the prime consideration, but where tensile forces have to be resisted, some form of steel reinforcement must be used.

Concrete consists of a mixture of cement, fine aggregate (sand), and coarse aggregate (broken stone, gravel, or similar material). The mixture should be as dense as possible to obtain a good concrete. The sand should fill the spaces between the coarse aggregate, and the cement paste should fill the spaces in the sand. The amount of water used has a pronounced influence on the strength of the concrete. The properties of the constituent materials are:

1. *Cement.* This may be of several kinds. The most commonly used cement is *Portland Cement* (B.S. Specn. 12) which is produced by heating a mixture of calcareous (chalky) and argillaceous (clayey) materials to sintering temperature and then grinding to a powder. The requirements as to fineness, chemical composition, setting-time, strength, etc., are laid down in the specification.

Rapid Hardening Portland Cement is very similar to Portland Cement but it has been ground more finely and therefore the setting time is reduced.

Portland Blast-furnace Cement contains a certain amount of finely ground blast-furnace slag.

High Alumina (or aluminous) Cement contains a larger proportion of alumina (aluminium oxide) than Portland Cement. The relatively high percentage of alumina reduces the setting-time and final setting may take place $3\frac{1}{2}$ to $5\frac{1}{2}$ hours after mixing. The B.S. specification for high alumina cement is No. 915 and the L.C.C. Building By-laws lay down certain requirements for fineness, percentage of alumina, ratio of alumina to lime, and setting-times. It possesses rapid hardening properties (note that hardening is *not* the same as setting). The cost of this cement is relatively high and unless there is need for rapidity of construction or fear of corrosion by chemicals it is not economical in use. The 'curing' of concrete made with high alumina cement should be carried out with special care.

2. *Fine aggregate (sand).* Sand used for concreting should be clean, hard, and free from clay or organic materials. The shape of the grains is not of major importance and rounded grains are quite satisfactory. Loam, clay, and similar impurities can be detected by shaking the sand with water in a vessel and noting the amount of material in suspension and the quantity of sediment. Where impurities exist, they should be removed by washing. River and freshwater sand seldom require washing, but pit sand generally must be washed before use. All these sands are suitable for concreting. If sea sand is used, it should not be too fine, and should be taken from *below* high-water mark, otherwise it may contain too much salt. It is as well to make test cubes before using sea sand to any large extent. Screenings from crushed stone can be used, but foundry and silver sands are too fine.

The size of sand varies with the purpose in mind, the usual maximum being $\frac{3}{16}$ in. or $\frac{1}{4}$ in. ($\frac{1}{8}$ in. for rendering). Coarse sands usually give a stronger concrete than fine sands. The grading of the sand plays a greater part in determining the strength of concrete than is generally realized. Some specifications give a $\frac{1}{4}$-in. maximum and state that at least 10 per cent. must be retained on a $\frac{1}{16}$-in. mesh; other specifications state that not more than 10 per cent. shall pass a sieve having 50 meshes per inch, and not more than 3 per cent. shall be under $\frac{1}{600}$ in. L.C.C. By-laws gives $\frac{3}{16}$-in. maximum, with not more than 5 per cent. under $\frac{1}{600}$ in. Some natural sands are well graded, but others may have to be mixed for good results.

3. *Coarse aggregate.* The requirements for this are: hardness, freedom from clay, loam, etc., cubical or spherical shape, without injurious chemicals and, for some work, fire-resisting properties. Coarse aggregate is best obtained from a source of supply which has proved satisfactory over a period of years. Gravel and crushed stone are most suitable for reinforced concrete. For mass concrete, hard brick and tile can also be used. Aggregates which require much washing are not economical. Dust is removed by screening. For reinforced work the aggregate should have a crushing strength of not less than 5,000 lb./in.^{2}, with softer aggregates for mass work. Broken brick aggregates are too porous for reinforced work. The amount of water absorbed in 24 hours should not exceed 10 per cent. of the weight for mass concrete and 5 per cent. for reinforced concrete. Pit and river gravel and Thames ballast are the most commonly used aggregates and are quite suitable after washing. Uncrushed gravel contains fewer voids than crushed stone. Gravel found in excavations should not be used without washing and screening. Beach shingle should be taken from *below* high-water mark. The best stones are granite, whinstone, quartzite, flint, and some of the harder sandstones. Shale or similar materials should not be used, but hard limestones, if free from dust, produce good concrete. Broken brick is banned by the L.C.C. By-laws for R.C. work. Coal residues, clinker, ashes, coke breeze, pan breeze, copper slag, forge breeze, dross, and such must not be used. Gypsum has the effect of causing corrosion of the reinforcement. Good blast-furnace slag makes a good aggregate. Clinker and breeze may be used for mass concrete. Pumice produces a lightweight concrete but should be used with great care.

The size and grading of the coarse aggregate vary with the purpose for which the concrete is intended. For R.C. work the minimum size will be $\frac{3}{16}$ in. or $\frac{1}{4}$ in. and the maximum size about 20 to 25 per cent. of the total thickness, and also be small enough to pass between the reinforcement. For heavy work or mass concrete, the maximum size may be $1\frac{1}{2}$ in. or 2 in., with 3-in. stones in masses. For the usual $\frac{3}{16}$-in. to $\frac{3}{4}$-in. limits, the percentage less than $\frac{3}{16}$ in. should not exceed 10 and the percentage less than $\frac{3}{8}$ in. should not be more than 25. Actually, the size and grading of the coarse aggregate does not play such an important part in determining the strength of concrete as that of the fine aggregate.

Water used for concrete should be fresh and clean, and free from oils, acids, and other deleterious substances. Generally, water suitable for drinking is suitable for concrete. Sea-water produces efflorescence on concrete and lengthens the setting-time. In doubtful cases it is as well to test water before using it.

Water–cement ratio. The ratio weight of water: weight of cement has a definite influence on the strength of concrete. Generally speaking, the lowest value of this ratio consistent with a workable concrete gives maximum strength. Finely ground cements require more water than coarse cements. Concretes whose aggregates contain a relatively large proportion of sand also require more water than those with coarse aggregate, and broken stone aggregates require more water than gravel.

Properties and tests of concrete. The working stress for concrete is based on compression tests (works tests) on a 6-in. cube after 28 days. The working stress (in bending compression 'c') is generally taken as one-third of the cube crushing strength.

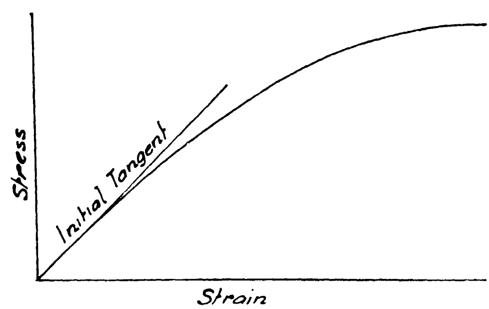

FIG. 1.6

For ordinary Portland cement and high alumina cement concretes the cube strength should be not less than the tabular values (in lb./in.^{2}) after 28 days. For direct compression as in columns the safe stress

	Portland cement concrete and Portland blast furnace cement		*High alumina cement concrete*	
Mix	*Preliminary test*	*Works test*	*Preliminary test*	*Works test*
1:1:2	5,250	4,500	..	..
1:1·5:3	4,375	3,750	..	..
1:2:4	3,500	3,000	6,000	5,000

can be taken as 80 per cent. of c. The shear stress s is generally taken as 10 per cent. of c. The bond or adhesion stress s_b is usually taken as 13 to 14 per cent. of c for ordinary grade concrete, but exhaustive tests have shown that it is as well to be rather more conservative in fixing the value of s_b for high-grade concretes. Punching shear, i.e. shear concentrated over a small area, is usually taken as twice the value of s. The value of E_c, the Young's modulus of concrete, presents some difficulty. Fig. 1.6 shows a typical stress-strain graph for concrete.

It will be noticed that there is no elastic limit or limit of proportionality for concrete, and it is therefore difficult to calculate E_c. A good approximation can be found by taking a value based on the tangent through the origin. It will be noted that E_c decreases as the stress increases. The ratio $m = E_s/E_c$ is an important factor in R.C. design and has been the subject of considerable controversy. Some authorities take a constant value ($m = 15$; $E_s = 30 \times 10^6$ lb./in.^{2}).† This does not seem to be in line with tests, which shows that m decreases as the cube strength increases. Another method is to calculate m by the formula

$$m = \frac{40{,}000}{\text{cube (works) strength}},$$

which appears to be logical. The American Society of Civil Engineers in their 1940 specification give varying values of m, decreasing as the cube strength rises, so that on the whole it appears that m must be a variable quantity and in important work should be found experimentally.

In specifying concrete there are two alternative methods: (1) that in which the proportions of cement, fine aggregate, coarse aggregate, and water-cement ratio are definitely given and the minimum cube strength is also stated; (2) in which the proportions of cement, fine aggregate, coarse aggregate, and water-cement ratio are left to the discretion of the contractor (within certain limits) and the cube strength that must be attained is specified. The difference between ordinary and high-grade concrete lies in the care taken in the grading of the constituents, mixing, placing, and supervision. The values of cube strengths given in the table are rather conservative and the present tendency is to upgrade working stresses in view of the present-day quality of cement and the more stringent supervision and improved workmanship in mixing and placing of concrete.

Timber

Timber is a structural material which differs from steel and concrete inasmuch as, being an organic material, it is not uniform and is highly complex, being subject to variation in grain, moisture content, etc. Before examining the question of physical properties and working stresses it is as well to review briefly the structure of wood and its growth.

Wood structure. Wood is composed essentially of cellulose in minute elongated cells, called fibres, firmly cemented together by lignin. The fibres are tapered at one end, and run vertically in the standing tree, and are about $\frac{1}{8}$ in. long in softwoods and $\frac{1}{24}$ in. in hardwoods. Their

† B.S.S. Code of Practice CP 114 (1948).

central diameter is about length/100. The appearance of different woods varies with the arrangement of the fibres. In addition to the fibres running parallel with the grain there are bands of cells extending radially from the pith, or centre of the tree, across the grain towards the bark. These are called the wood or medullary rays. In most woods these rays are small and inconspicuous. The weight and strength of wood depends on the thickness of the cell-walls. The shape, size, and arrangement of the fibres, the presence of wood rays, and the later effect of springwood and summerwood account for the large differences in properties along and across the grain. The properties across the grain may be a small fraction of the same properties parallel to the grain.

Hardwoods and softwoods. Trees used for structural and building purposes are divided broadly into hardwoods and softwoods. The terms are rather loose, as some 'softwoods' may be harder than 'hardwoods'. The custom has developed in America of calling coniferous trees softwoods and deciduous, or broad-leaved trees, hardwoods. American classification is as follows:

Softwoods	*Hardwoods*	
Cedars and Junipers	Alder	Gums
Cypress	Ashes	Hachberry
Douglas fir	Aspen	Hickories
True firs	Basswood	Locust
Hemlocks	Birches	Magnolia
Larch	Buckeye	Maples
Pines	Butternut	Oaks
Redwood	Cherry	Sycamore
Spruces	Chestnut	Walnut
Tamarack	Cottonwoods	Willow
Yew	Elms	Yellow Poplar

Other hardwoods are karri, jarrah, teak, and greenheart.

The British classification is given in the tables in Appendix C (Vol. II).

Heartwood and sapwood. The end of a timber log shows three distinct zones of wood: (1) the bark on the outside; (2) a light-coloured zone next to it, called sapwood; (3) an inner zone (usually darker), called heartwood. In the centre of the log is the pith or heart centre. A tree grows by forming new layers of wood where the sapwood and bark meet. A young tree is made up entirely of sapwood. As the tree ages, the central portion matures, forming heartwood. Heartwood is more durable in contact with the soil and under conditions conducive to decay. It is therefore better to use heartwood if the material is to be used untreated. On the other hand, sapwood absorbs preservatives better than heartwood and is preferable for treating.

Annual rings. The cross-section of freshly cut trees shows a large number of concentric rings, starting at the centre. Each ring represents the growth the tree makes in one year from spring to autumn. The width of the annual rings varies: narrow rings are formed during a short dry season and wider rings during a more favourable growing season. Each annual ring, in many woods, is made up of two parts: (1) an inner light-coloured part, called springwood; and (2) a darker outer portion known as summerwood. Springwood is made up of relatively large, thin-walled cells formed during the early part of the growing season; summerwood is formed later in the year and is made up of cells with thicker walls and smaller openings in the cells. Summerwood contains more solid wood substance than springwood. The presence of springwood and summerwood occurs in both softwoods and hardwoods; it is less noticeable in hardwoods. The proportion of springwood and summerwood present in softwoods has an important effect on their strength and physical characteristics.

Density and rate of growth. In softwoods the rate of growth has an important effect on their strength properties. An accurate measure of this is provided by the relative width and character of wood in each annual ring. In these woods, pieces having medium to narrow growth-rings have been found to have generally higher strength properties than those having wider rings. In addition, in certain woods, portions where a considerable proportion of each annual ring is made up of summerwood have still higher strength properties. Therefore in grading timbers for structural purposes the rate of growth, i.e. the number of rings per inch radially, and the density, i.e. proportion of summerwood, are considered and are made part of the specification. Timber having a specified minimum number of annual rings per inch is known as 'close-grained' and that having one-third or more of summerwood is known as 'dense'.

Grain and Texture. These terms are used to describe characteristics of wood in various ways. Wood from slow-growing trees in which the annual rings are close together are known as close-grained; that from rapidly growing trees with wide rings as coarse-grained. Straight grain and cross grain describe wood in which the direction of the fibres (not the annual rings) is parallel to or at an angle with the sides of the piece. Cross grain includes spiral grain in which the fibres wind around the trunk of the tree. The 'slope of grain' is employed in the grading of structural timber to describe the extent of cross grain permitted, since this has an important effect on strength properties.

'Grain' and 'texture' refer usually, however, to the physical properties of appearance rather than to properties of strength. For example, fine grain is used to describe woods in which the cells are small and thick-walled, making a compact wood with smooth surface, and coarse grain to denote woods in which the cells are large and open, producing a surface slightly roughened due to the large cells being cut where they intersect the surface.

Knots are portions of a branch or limb which have become incorporated in the body of a tree. Loose knots are a source of weakness.

Wanes or waney edges are parts of the original rounded surface of the tree remaining on the timber which cause a reduction of the effective area.

Shake is a separation along the grain, which usually occurs between the annual rings.

Shakes, loose knots or numbers of knots, or waney edges are definite faults in timber, which impair the strength. The presence of any one or more of these faults has an effect on the grading of structural timbers, and where these faults are present in marked degree, the timber should be rejected or used as second-grade material.

Specific gravity and density. The substance of which wood is composed is actually about 50 per cent. heavier than water. The dry wood of most species floats, however, owing to the large proportion of the volume occupied by air spaces. The specific gravity or density of dry wood is an excellent index of the amount of wood substance and therefore of the strength properties (see Appendix C).

Moisture in wood. Wood in standing trees contains moisture in two forms: (1) as free water held in the cell cavities, and (2) as imbibed hygroscopic moisture held in the cell-walls. The moisture content is expressed as a percentage of the oven-dry weight and can be found by drying a sample at slightly over 212° F. until no further loss of weight occurs.

$$\text{Moisture content} = \frac{\text{loss in weight}}{\text{oven-dry (final) weight}} \times 100.$$

Wood in use over a period of time usually arrives at a moisture content corresponding to the humidity of the surrounding atmosphere. Fig. 1.7 shows the moisture content of wood, when in equilibrium with various relative humidities and at given temperatures. Moisture content has an important effect upon the susceptibility to decay. Wood that is continually wet, as when submerged in water, or wood that is continually dry, will not be attacked by fungi which cause decay. Differences in moisture content above the fibre saturation point have no effect upon the volume or strength of wood. As wood dries below the fibre saturation point, shrinkage occurs and the strength increases.

Shrinkage of wood. Most data on the shrinkage of wood is expressed in terms of the shrinkage which takes place between the green (actually the fibre saturation point) and the oven-dry condition. Actually this shrinkage will never occur in structural timber. The important factor is the change of size that may occur due to changes in the moisture content as wood members absorb or give off moisture due to variation in the atmospheric humidity. Large structural members do not shrink as much proportionately as small members, since drying does not take place simultaneously in the inner and outer fibres of large members.

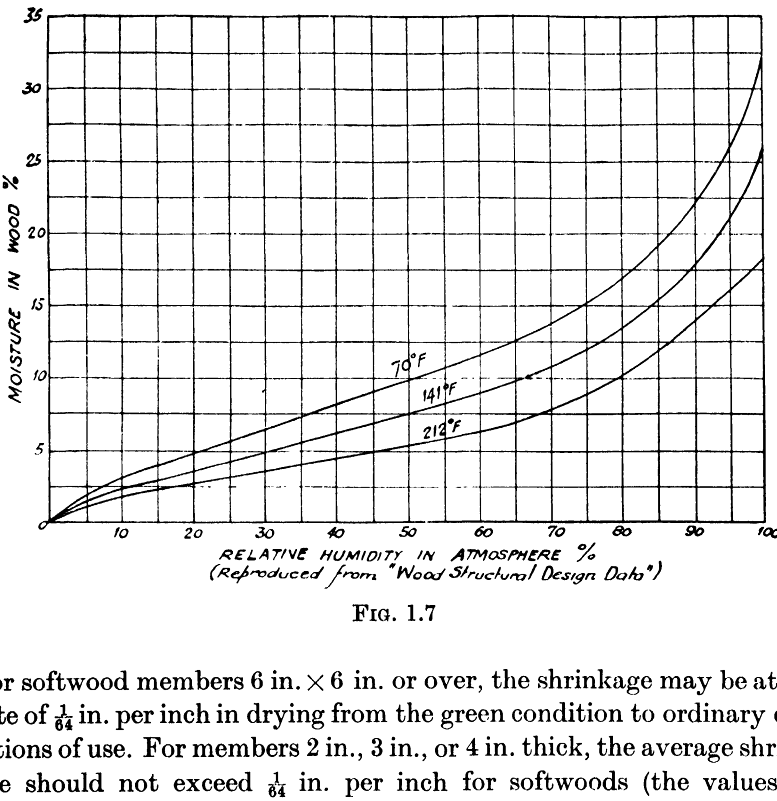

FIG. 1.7

For softwood members 6 in. × 6 in. or over, the shrinkage may be at the rate of $\frac{1}{64}$ in. per inch in drying from the green condition to ordinary conditions of use. For members 2 in., 3 in., or 4 in. thick, the average shrinkage should not exceed $\frac{1}{64}$ in. per inch for softwoods (the values for hardwoods will be slightly more).

Effect of moisture on strength. Increase in strength begins when the cell-walls begin to lose moisture, i.e. after the wood is dried to below the fibre saturation point. From then on, most strength properties increase rapidly as drying progresses. Drying wood down to 5 per cent. moisture content may add anything from $2\frac{1}{2}$ to 20 per cent. to its density. In small members the end crushing strength and bending strength may be doubled or trebled. The effect of seasoning is most marked in the case of small members. Seasoning does not affect all the strength properties in the same proportion. Resistance to shock is not increased and may be decreased. Table II shows the effect of change of moisture content on strength properties.

TABLE II

Values show Average Increase (or Decrease) in Value (%) due to lowering (or raising) Moisture Content 1 per cent. from Fibre Saturation Point

Static bending		*Impact bending*	
Fibre stress at limit of proportionality	5	Fibre stress (limit of proportionality)	3
Cross-breaking strength	4		
Modulus of elasticity	2		
Compression parallel to grain		*Compression at right angles to grain*	
Fibre stress at limit of proportionality	5	Fibre stress (limit of proportionality)	5
Maximum crushing strength	6	Hardness, end grain	4
		,, side grain	2·5
		Shearing strength, parallel to grain	3
		Tension, perpendicular to grain	1·5

Grading of timber and working stresses. It will be seen from the foregoing remarks that it is difficult to lay down definite rules for the grading of structural timbers. The chief factors to be considered are the density, moisture content, closeness of grain, and freedom from defects such as knots, shakes, waney edges, etc. Other factors which affect the grading indirectly are the extent to which exposure may cause decay, the treatment, and the frequency and thoroughness of inspection. L.C.C. By-laws divide timber into two classes, 'graded' and 'non-graded'. Stresses in bending, compression parallel to the grain and at right angles to the grain may be increased for close-grained and dense timbers, but horizontal shearing stress should be increased only in the case of dense timbers. For timbers subject to severe exposure, the working stresses may be reduced at the discretion of the engineer. The basic stress in compression parallel to the grain is given for members whose l/d ratio does not exceed 10 (l = effective length; d = least width). For members where l/d exceeds 10, working stresses should be reduced. Fig. 1.8 shows the comparison between various formulae used in America. L.C.C. By-laws show a reduction in working stress for various values of l/d.

l/d	*Factor*	l/d	*Factor*
0–10	1·00	24–26	0·650
10–12	0·985	26–28	0·600
12–14	0·970	28–30	0·485
14–16	0·950	30–32	0·430
16–18	0·920	32–34	0·380
18–20	0·875	34–36	0·340
20–22	0·820	36–38	0·300
22–24	0·745	38–40	0·275

Uses of timber. While timber has been displaced to a great extent by steel and reinforced concrete, it still has many uses, e.g. piles and temporary work, formwork, roof trusses, and purlins. Seasoned timber has good resistance to attack by smoke and steam. The extent to which timber is used must be governed by the availability of seasoned timber, which at the time of writing is less than normal. The question of the connexion of timber members is dealt with in some detail in Chapter XVI (Vol. II).

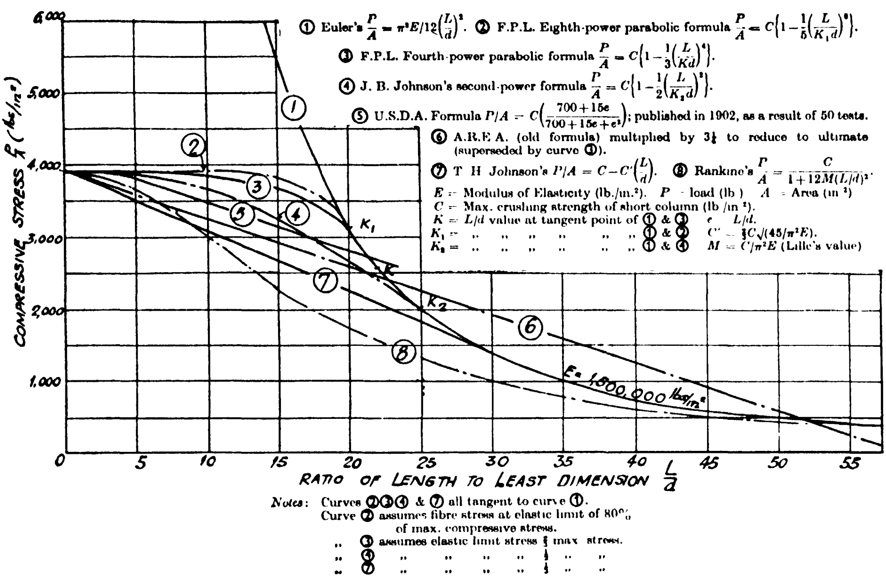

FIG. 1.8

Brickwork

In recent years brickwork has diminished in importance as a structural medium, owing to the introduction of steel-framed work and reinforced concrete. The present tendency is to use brickwork as a facing for brick-filling between steel and R.C. members to keep out the weather, or as permanent shuttering for concrete. At the same time it is probable that brickwork will continue to be used for load-bearing piers, walls, etc., for some time yet, and it is as well to consider its properties.

Bricks consist essentially of an argillaceous or clayey earth, baked or burnt to form an artificial stone, and the properties of the bricks depend on the nature of the earth, its preparation, and the manner in which it is baked or burnt. Brick earths are found in many parts of this country, and the variation in the chemical composition of the earths is responsible for the different kinds of bricks, with different appearance, hardness, porosity, etc. When the brick earth has been dug out it is usually subjected to the following processes: (1) preparation, (2) moulding, (3) drying, and (4) burning. The processes vary somewhat according to the nature of the earth and the purpose for which the brick is intended.

Stocks, i.e. clamp-burnt bricks of yellowish appearance commonly used in London, the south of England, etc., are made by the following processes: (1) preparation of the earth either by spreading layers of the earth on level ground and then layers of breeze and broken chalk and so on to a height of 5 or 6 ft., or alternatively washing the earth and chalk together in a wash-mill and passing the mixture through a grid into settling-pits to remove excess water and spreading a layer of breeze on top. In both cases the mixture is left to weather during the winter and then turned over two or three times. (2) Moulding may be done by hand or machine. Hand-moulding is done by filling a mould about 10 in. × 5 in. × 3 in. (for $8\frac{3}{4}$ in. × $4\frac{1}{4}$ in. × $2\frac{5}{8}$ in. bricks) with the clay and striking off to form a flush surface. With machine-moulding the clay is pressed into a mould containing six bricks. (3) Drying is commenced immediately after moulding. The object of drying is to enable the bricks to be handled and to withstand the pressure due to stacking. It may be carried out in a drying-shed where the bricks are sprinkled with sand to absorb moisture. The process may take 3 to 6 weeks. Where the bricks are moulded by machinery the drying is carried out in long chambers, through which hot air is forced by fans, the period for drying being reduced to 24 hours. (4) Burning may be carried out either in 'clamps' or in kilns. The object of burning is to drive out water and to fuse the constituents into a homogeneous mass with the necessary hardness to withstand pressure and with a vitrified surface to resist the effects of the weather. 'Clamps' are built by raising the site above the surrounding ground and draining it. The surface is paved with bricks; horizontal flues filled with faggots are made; then two layers of brick on edge are laid diagonally, above which is placed a layer of raw bricks; then about 7 in. of breeze, then another layer of raw bricks, above which is placed 4 in. of breeze; then another layer of raw bricks with 2 in. of breeze on top. Above this, bricks are built up in thin unbonded walls to a height of 14 ft. The time of burning varies from 2 to 6 weeks. Bricks produced in clamps are called 'stocks' and are capable of resisting a high degree of compression.

Kilns used in brick-making may be of the Scotch, Hoffmann, or Warren type. Kilns may take 20,000 to 50,000 bricks at a time. The time taken for burning is 2 or 3 days, and the bricks burnt by this process are much more uniform in colour and regular in shape than those burnt in clamps. Bricks produced in kilns are called builders' firsts, seconds, and thirds. *Wire-cut bricks* are forced through orifices

9 in. × $4\frac{1}{2}$ in. and then cut into 3-in. layers by wires arranged in a frame. They are generally burnt in a Hoffmann kiln. *Fletton bricks* are produced from dense bluish-grey shale from the Oxford Clay formation, which is ground in a mill, before going to the pressing machine, where it is pressed into the form of bricks and passed to a kiln, where it remains for about 3 weeks. They absorb about 20 per cent. of their weight of water in 24 hours and vary considerably in properties. *Blue bricks* such as the Staffordshire are used in engineering where it is necessary to have a high compressive strength. They have an ultimate crushing strength (when tested between plywood) of up to 16,000 lb./in.^{2}

Bricks may be classified in various ways. *Hand-moulded* bricks have a frog on one side, are porous, and have no great amount of shape. *Wire-cuts* have no frogs and show the wire-cuts on the beds; they are regular in shape and dense. *Pressed bricks* have smooth faces, regular and sharp arrises, frogs on one or two sides with trade marks in frogs and are very dense. *Clamp-burnt* bricks are not uniform in colour. *Kiln-burnt* bricks may show light and dark stripes owing to the position of the brick in the kiln. Bricks made from clays free from iron burn white. Slight amounts of iron cause a cream colour. Red bricks are made from clay containing slightly more iron with chalk. Clays containing 8 to 10 per cent. of iron produce blue or black bricks.

Characteristics and tests. Desirable qualities for good bricks are regularity of shape, rectangular faces, uniform texture, compactness and freedom from flaws, and the amount of water absorbed should not exceed 15 per cent. of their own weight. Water should be absorbed gradually and given off freely. Bricks should be uniformly burnt, hard, and give a metallic ring when two are struck together. They should be of correct colour for their kind and tough or pasty in texture. Bricks should require repeated blows to break them and should stand carting and handling. From an engineering point of view, the most satisfactory classification of bricks is according to the crushing strength. L.C.C. By-laws grade bricks thus:

Designation of bricks or blocks as regards strength	*Load in lb./in.^{2} of horizontal area*
Special	Over 10,000
First	10,000
Second	7,500
Third	5,000
Fourth	4,000
Fifth	3,000
Sixth	1,500

Where the slenderness ratio of walls and piers does not exceed 6, the permissible pressure for walls, piers, etc., built of bricks or blocks as specified above is given in the following table:

Designation of bricks or blocks	*Proportion of mixture of mortar (by volume)*			*Maximum pressure tons/ft.²*
	Cement	*Lime*	*Sand*	
Special	1	..	2	($\frac{1}{500}$ × strength of brick in lb./in.²) + 10, max. value 40
First	1	..	2½	30
Second	1	..	2½	23
Third	1	..	3	16
Fourth	1	..	3	13½
Fifth	1	..	4	11
,,	1	1	6	10
Sixth	1	..	4	8
,,	1	1	6	7
,,	1	2	9	6
,,	1	3	12	5½
,,	1	4	15	5
,,	1	5	18	4½
,,	..	1	3	4

For a slenderness ratio of 12, 40 per cent. of above values should be taken as the permissible pressure, with corresponding values for intermediate slenderness ratios. B.S.S. 449 (*Use of Structural Steel in Building*) gives a table showing pressures as above but allows certain increases for pressure due to eccentric loading or lateral forces or local loads, e.g. at bearings of girders, stanchion bases, etc. The same specification gives a method for finding the crushing strength of bricks. Twelve bricks are tested and the average value of the ultimate strength is taken. The bricks must be soaked in water at a temperature of between 15° and 20° C. for 24 hours. The frogs are then filled with a mortar composed of one part (by weight) of cement and three parts (by weight) of dry, clean Leighton Buzzard pit sand (cement being either normal or rapid-hardening Portland cement). After the frogs have been filled, the bricks are covered with damp cloths for 24 hours, then immersed in water (15° to 20° C.) for a period of 27 days (for normal Portland cement) or 6 days (for rapid-hardening Portland cement) and then crushed between 3-plywood sheets ⅛ in. thick, loading being applied at the rate of about 2,000 lb./in.² per minute. Note that when rapid-hardening Portland cement is used the period of immersion may be reduced for 2 days, provided that the mean crushing strength of three 3-in. mortar cubes made from the same batch of mortar and stored under the same conditions is 2,000 to 4,000 lb./in.²

L.C.C. By-laws give extensive rules governing brickwork in relation to maximum height, unsupported span, etc., for any particular thickness, and B.S.S. 449 also gives some rules, and the reader is referred to these publications for fuller information. Engineering brickwork should always be built in cement mortar. English bond (alternate courses of headers and stretchers) is to be preferred to Flemish bond (alternate headers and stretchers) for strength, although the latter has a better appearance.

As mentioned earlier, brickwork should possess high crushing strength but has little strength (comparatively speaking) in tension and adhesion. This is due to the effect of the mortar joint. At one time it was customary to ignore the strength of mortar entirely, but actually it can be shown by tests that cement mortar does possess a fair degree of strength in tension or adhesion. When dealing with self-supporting brickwork structures, a distinction should be drawn between cases where the mortar has had time to set and harden before the load is applied and cases where load may be applied when the mortar is green. Brickwork subject to wind loading, e.g. chimney-stacks, shafts, and buttresses, should be treated as if the mortar has not had time to set when the load is applied. Brickwork may be reinforced by placing layers of expanded metal or mesh reinforcement in the mortar joints.

Reinforced brickwork is analogous to reinforced concrete and can be designed in a similar manner. Reinforced brickwork can be built in two ways: (*a*) By building brick skins with a layer of concrete reinforced with steel rods or mesh sandwiched between. The skins are usually $4\frac{1}{2}$ in. thick with headers at intervals bonded into the concrete. This form of construction has been largely used for A.R.P. purposes and compares favourably with reinforced concrete for this, as no shuttering is required, but the work should be carried out carefully to get good results. (*b*) By building reinforcement into the mortar joints which are made thicker than usual or by passing steel rods through holes in specially made bricks. Such bricks are more expensive and difficult to obtain and the arrangement of vertical and diagonal reinforcement presents some practical difficulties.

Common bricks have been standardized throughout England and Wales at $8\frac{3}{4}$ in. long (tolerance $\pm\frac{1}{8}$ in.) by $4\frac{3}{16}$ in. wide (tolerance $\pm\frac{1}{16}$ in.) with a depth of either $2\frac{5}{8}$ in. (tolerance $\pm\frac{1}{16}$ in.) or $2\frac{7}{8}$ in. (tolerance $\pm\frac{1}{16}$ in.). The reason why the depth is varied is to allow for a varying thickness of the mortar bed. (The usual practice is to reckon 4 courses of brickwork to one foot vertically.)

Masonry

Many of the general remarks applying to brickwork apply also to masonry. The use of masonry except as a facing has diminished in engineering work and to-day masonry is used chiefly where mass is desired, e.g. retaining walls, dams, and marine structures, and even in such cases the general tendency is to replace stones by concrete.

Building stones can be divided into two classes: natural and artificial, the latter being usually a concrete with varying kinds of aggregates. Natural stone may still be used where it is found locally and can be quarried economically.

Characteristics of building stones. These are the general structure, fineness of grain, compactness, porosity and absorption, weight, appearance, seasoning, natural bed, and weathering.

General structure. Sandstones consist of grains of sand cemented together by various natural cementing agents. Limestones usually consist of crystallized grains of calcium carbonate joined together by a cement of calcium carbonate (when it can be polished it is called a marble). *Oolitic* limestones consist of calcareous matter formed round a *nucleus*, usually in the form of small shells. Marbles proper consist of calcite of various colours ranging from white to pink and other shades. Dolomites are composed of calcium and magnesium carbonates which should be in nearly equal amounts. All the foregoing stones, except dolomites, belong to the sedimentary class. Granite, which is an igneous rock, is crystalline in structure and is composed of quartz, mica, and felspar.

Fineness of grain is important, as fine-grained stones produce sharp arrises and are better for weathering.

Compactness is necessary for durability. The best building stones are those which have been formed in the lower strata and have been subject to intense pressure. Actually they may be found in some cases near the surface of the earth, owing to disturbance in the strata or to the effects of weathering.

Porosity and absorption are important qualities, as stones with a high degree of porosity are unsuitable for building work, especially in exposed positions. Porous stones absorb much rain-water and may decompose or disintegrate. Where rain contains such acids as sulphuretted, hydrochloric, and sulphurous types (which occur in town and city atmospheres) it may be absorbed into stones and decompose the constituents. Rain in conjunction with frost may cause disintegration of stones.

The weight of stones is important as being a measure of the density. Heavy stones are used for buttresses, gravity walls, marine structures, etc.

The appearance of a stone is a good guide to its durability. Red and brown shades of colour show the presence of oxides of iron, which leads to discoloration and perhaps disintegration. The lighter shades of any particular stone are preferable.

Seasoning of building stones is necessary in order to harden them. Stone when freshly quarried contains a certain amount of moisture known as 'quarry sap'. The dressing of stone should be done before the quarry sap has disappeared. After the stone has been worked to a finished surface, the dressed surface should not be disturbed as is sometimes done in cleaning down the surface of a building. Quarry sap should have disappeared before the stone is positioned, and the time of seasoning varies from 6 to 12 months.

The natural bed of a stone is the surface on which the material was originally deposited. It need not necessarily be horizontal in the quarry. The natural bed of a stone should always be placed at right angles to the direction of pressure, e.g. horizontally in walling or at right angles to the centre line in arches. The question of weathering is also important in considering how the natural bed should be placed in exposed stonework. The natural bed may not be apparent to the naked eye, and in such cases it can be found by pouring some clean water on the stone and observing (through a magnifying-glass) the direction it takes in descending.

Weathering is most apparent on surfaces exposed to the prevailing wind (south-west in England). The best way to determine the weathering properties of a stone is to inspect similar stonework which has been exposed to the weather under similar conditions.

Building stones are classified as igneous, metamorphic, and sedimentary. *Igneous* rocks are volcanic in origin and include granites, syenites, and traps. *Metamorphic* rocks are either igneous or sedimentary rocks which have been subject to enormous pressure or heat or both, and which have altered their structures owing to the action of these agents. They include dolomites. *Sedimentary* rocks are those whose constituent materials have been deposited by air or water. They include most of the limestones and sandstones. *Granite* is found in Aberdeenshire, Cornwall, Guernsey, and other parts of Britain. In addition to being a good building stone it has good wearing properties for use in setts, dock sills, quoins, etc. *Syenite* is imported from Norway. *Limestone* is found in many parts of England. The most commonly used variety is Portland stone. *Dolomites* are found in the midlands and Yorkshire. *Purbeck* is really a marble of oolitic origin found in Dorset. A similar stone is found in Derbyshire. *Sandstones* are found in many parts of England and Scotland. Sandstones may range from greyish-white to red in colour and have good weathering properties.

To ascertain the crushing strength of a building stone (in accordance with B.S.S. 449) three specimens should be tested and the mean value of the ultimate loads taken. The specimens should be in the form of cubes with sides not less than 3 in. and must be soaked in water at a temperature of from 15° to 20° C. for 24 hours. They are then crushed between 3-plywood sheets $\frac{1}{8}$ in. thick, the rate of loading being about 2,000 lb./in.^{2} per minute.

Stone should have a crushing strength of not less than 15,000 lb./in.^{2} and should be free from such defects as cracks and sand-holes.

Stone for engineering work is usually in the form of ashlar, i.e. roughly dressed stone, and should be set in cement mortar. Where it is used as a facing to brick or concrete, it should be properly bonded to the backing. The permissible pressure on masonry walls, piers, etc., can be found in a manner similar to that described in the sections dealing with brickwork.

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CHAPTER II

STRUCTURAL MECHANICS

Before proceeding to consider the theory of bending and shear, it is advisable to state very briefly the conditions for equilibrium of a body acted upon by any system of coplanar forces. These forces may be either (*a*) concurrent or (*b*) non-concurrent.

Case (*a*). Concurrent forces P, Q, and R passing through the same point O (Fig. 2.1). Draw two axes of reference XX and YY at right

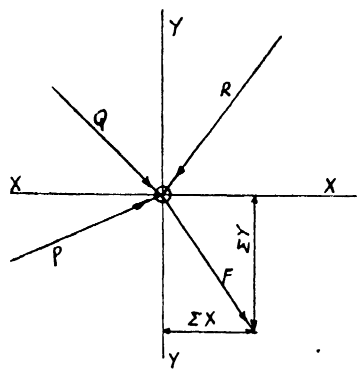

Fig. 2.1

angles to each other. Now the separate forces P, Q, and R can be replaced by a single force F passing through O. Each force can be resolved into two components parallel to the axes X–X and Y–Y and the algebraic sum of such components can be written $\sum X$, $\sum Y$.

For equilibrium F must be zero and therefore

$$F = \sqrt{\{(\sum X)^2+(\sum Y)^2\}} = 0.$$

Hence

$\sum X$ = algebraic sum of components parallel to X–X axis $= 0$,

$\sum Y$ = ,, ,, ,, ,, ,, ,, Y–Y ,, $= 0$.

Case (*b*). Forces P, Q, and R are non-concurrent. The separate forces can be replaced by a single force E which cannot be zero (Fig. 2.2.)

$$F = \sqrt{\{(\sum X)^2+(\sum Y)^2\}}$$

and

$$\tan\phi = \frac{\sum Y}{\sum X}.$$

Now if a perpendicular ON be drawn from O to the line of action, then the turning moment of the force F about the point O is given by

$$M = F\times ON,$$

which must equal

$$\sum \text{(moment of each force about } O\text{)}.$$

For equilibrium a balancing force must be applied to the body equal in magnitude but opposite in direction. The moment of this balancing force must be equal and opposite to that due to F, so that the resultant moment is zero.

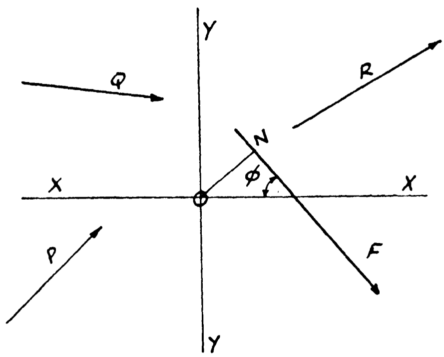

FIG. 2.2

Therefore the third condition of equilibrium is $\sum M = 0$, i.e. the algebraic sum of the moments of all forces acting upon the body is zero.

Hence the conditions of equilibrium are (1) $\sum X = 0$; (2) $\sum Y = 0$; (3) $\sum M = 0$.

First Moment of Area

Any area A of any shape can be divided into a number of elements a (Fig. 2.3). Taking a pair of rectangular axes of references X'–X' and Y'–Y', then each small element of area a has a pair of coordinates x, y with respect to these axes and moments $a \times x$; $a \times y$ about these axes.

The total moments can be written thus:

$$\sum ax \quad : \quad \sum ay.$$

Now if the coordinates of the centre of gravity of area A with respect to the axes are $\bar{x}$, $\bar{y}$, the total moments can be written as $A\bar{x}$, $A\bar{y}$.

$$\therefore \quad \sum ax = A\bar{x} = \text{first moment of area about } Y'\text{–}Y',$$

$$\sum ay = A\bar{y} = \quad ,, \quad ,, \quad ,, \quad ,, \quad X'\text{–}X'.$$

When these quantities become zero, the coordinates $\bar{x}$, $\bar{y}$ must be zero (since A cannot be zero); therefore the axes must pass through the C.G. Hence if two rectangular axes X–X, Y–Y be drawn parallel to the original axes X'–X', Y'–Y' and at distances $\bar{x}$, $\bar{y}$ respectively therefrom, they must pass through the C.G. of the area.

Alternatively we can write

$$\bar{x} = \frac{\sum ax}{A},$$

$$\bar{y} = \frac{\sum ay}{A},$$

and the principle can be used to find the position of the C.G. of any section, as will be seen in later chapters. Note that the first moment of area can be either positive or negative.

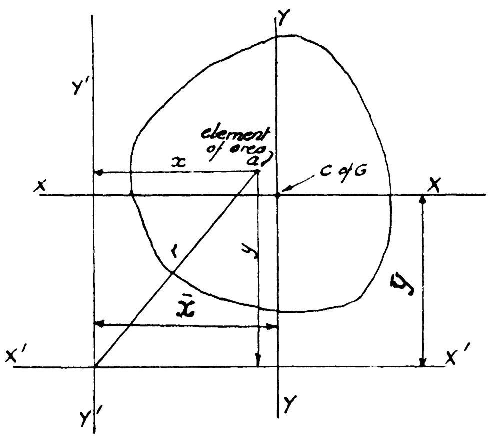

FIG. 2.3

Second Moment of Area or Moment of Inertia

Referring again to Fig. 2.3, the second moments of area a with respect to the axes Y'–Y' X'–X', are ax^2, ay^2 and the second moments of the whole area are

$\sum ax^2$ (with respect to axis Y'–Y')

and $\sum ay^2$ („ „ „ X'–X').

Considering now the second moments of area with respect to the rectangular axes Y–Y and X–X passing through the C.G., these are

$\sum a(x-\bar{x})^2$ and $\sum a(y-\bar{y})^2$ respectively.

But

$$\begin{aligned}\sum a(x-\bar{x})^2 &= \sum a(x^2+\bar{x}^2-2x\bar{x})\\ &= \sum ax^2+\sum a\bar{x}^2-2\bar{x}\sum ax\\ &= \sum ax^2+A\bar{x}^2-2\bar{x}\times A\bar{x}\\ &= \sum ax^2-A\bar{x}^2\\ &= \text{(2nd moment about axis } Y'\text{–}Y')-A\bar{x}^2.\end{aligned}$$

Similarly

$$\sum a(y-\bar{y})^2 = \text{(2nd moment about axis } X'\text{–}X')-A\bar{y}^2.$$

Or

2nd moment of area about Y'–Y'

$$= (\text{2nd moment of area about axis } Y\text{–}Y \text{ through C.G.}) + A\bar{x}^2,$$

and

2nd moment of area about X'–X'

$$= (\text{2nd moment of area about axis } X\text{–}X \text{ through C.G.}) + A\bar{y}^2.$$

Generally this principle can be stated thus: The second moment of area about any axis is equal to the second moment of area about a parallel axis through the C.G. plus a quantity equal to the total area × the square of the perpendicular distance between the two axes. This is often referred to as the *principle of parallel axes* and will be used in succeeding chapters. Note that the second moment of area is always positive since it is equal to $\sum ax^2$ or $\sum ay^2$.

The second moment of area is often referred to as the moment of inertia, which is a misnomer since an area has neither weight nor inertia nor moment of inertia. The term 'moment of inertia' is in general use and the axis of reference is often denoted by a suffix thus:

Moment of inertia about X–X axis $= I_{xx}$

„ „ „ Y–Y „ $= I_{yy}$.

Usually I_{xx}, I_{yy} are the maximum and minimum values of the second moment of area.

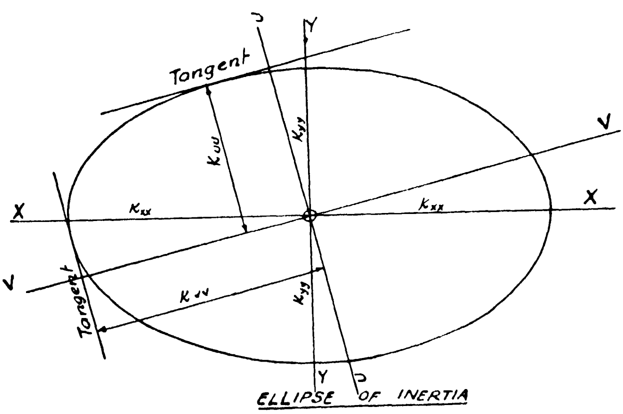

FIG. 2.4

Ellipse of Inertia (Fig. 2.4)

Using the maximum and minimum values of the moment of inertia as the major and minor axes of an ellipse, then the value of the inertia about any other axis passing through the C.G. can be found by drawing a line through the origin parallel to the axis and scaling off the value.

The values I_{xx} and I_{yy} are normally the values of the second moment about the axes of symmetry. (Where there are no axes of symmetry, then the values should represent the maximum and minimum values of the second moment of area.) In cases where the figure has more than two axes of symmetry, e.g. the square or the circle, the ellipse of inertia becomes a circle.

Polar Moment of Inertia

This is often denoted by the letter J and is equal to the quantity $\sum ar^2$, where r is distance from any element of area a to an axis passing through the C.G. and at right angles to the plane in which the area lies. The quantity J is used in calculations relating to torsional problems. It can be used to find the value of I for such figures as circles and annular rings, since by symmetry

$$I_{xx} = I_{yy} = \tfrac{1}{2} \times J.$$

Referring again to Fig. 2.3,

$$J = \sum ar^2,$$

but

$$r^2 = x^2 + y^2.$$

$$\therefore \quad J = \sum ax^2 + \sum ay^2 = I_{xx} + I_{yy}.$$

Hence the polar moment = sum of 'rectangular' moments.

The M.I. of simple shapes such as rectangles, circles, etc., can be

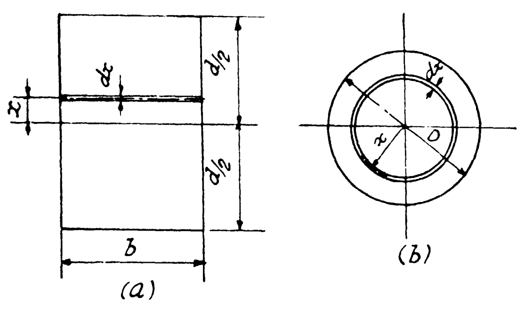

FIG. 2.5

calculated without difficulty. Consider the rectangle shown in Fig. 2.5 (*a*), and find the moments of inertia I_{xx} and I_{yy}.

For an infinitely thin strip of thickness dx at a distance x from the axis X–X the second moment of area of the strip is $b \times dx \times x^2$, and the total moment I_{xx} is given by

$$\int_{-\frac{1}{2}d}^{+\frac{1}{2}d} bx^2\,dx = \left[\frac{bx^3}{3}\right]_{-\frac{1}{2}d}^{\frac{1}{2}d} = \frac{bd^3}{12}.$$

Similarly the moment

$$I_{yy} = \frac{db^3}{12}$$

and $$J = \frac{bd}{12}[b^2+d^2].$$

By using the principle of parallel axes it can be shown that the second moments about the edges are $bd^3/3$ and $db^3/3$ respectively.

To find the values for a circle, find the value of J by considering an annular ring of average radius x and thickness dx (Fig. 2.5 (*b*)).

Then $$J = \int_0^R 2\pi x\, dx\, x^2 = \left[\frac{\pi x^4}{2}\right]_0^R = \frac{\pi D^4}{32}$$

and $$I_{xx} = I_{yy} = \frac{\pi D^4}{64}.$$

For an annular ring $$J = \frac{\pi(D_0^4-D_1^4)}{32}$$

and $$I_{xx} = I_{yy} = \frac{\pi(D_0^4-D_1^4)}{64}.$$

FIGURE	AREA	$\bar{X}$	$\bar{Y}$	I_{xx}	k_{xx}^2	I_{yy}	k_{yy}^2
RECTANGLE	bd	$\frac{d}{2}$	$\frac{b}{2}$	$\frac{bd^3}{12}$	$\frac{d^2}{12}$	$\frac{db^3}{12}$	$\frac{b^2}{12}$
PARALLELOGRAM	bd	$\frac{d}{2}$	-	$\frac{bd^3}{12}$	$\frac{d^2}{12}$	-	-
TRIANGLE	$\frac{bd}{2}$	$\frac{d}{3}$		$\frac{bd^3}{36}$	$\frac{d^2}{18}$	$\frac{db^3}{48}$	$\frac{b^2}{24}$
TRAPEZIUM	$\left(\frac{a+b}{2}\right)d$	$\frac{d}{3}\left(\frac{2a+b}{a+b}\right)$	-	$\frac{bd^3}{36}\left\{\frac{a^2+4ab+b^2}{ab+b^2}\right\}$	-	-	-
SQUARE	b^2	$\frac{b}{2}$	$\frac{b}{2}$	$\frac{b^4}{12}$	$\frac{b^2}{12}$	$\frac{b^4}{12}$	$\frac{b^2}{12}$
CIRCLE	$\frac{\pi}{4}D^2$	$\frac{D}{2}$	$\frac{D}{2}$	$\frac{\pi}{64}D^4$	$\frac{D^2}{16}$	$\frac{\pi}{64}D^4$	$\frac{D^2}{16}$
HOLLOW CIRCLE	$\frac{\pi}{4}(D^2-d^2)$	$\frac{D}{2}$	$\frac{D}{2}$	$\frac{\pi}{64}(D^4-d^4)$	$\frac{D^2+d^2}{16}$	$\frac{\pi}{64}(D^4-d^4)$	$\frac{D^2+d^2}{16}$
ELLIPSE	$\frac{\pi bd}{4}$	$\frac{d}{2}$	$\frac{b}{2}$	$\frac{\pi bd^3}{64}$	$\frac{d^2}{16}$	$\frac{\pi db^3}{64}$	$\frac{b^2}{16}$
PARABOLA	$\frac{2}{3}bd$	$\frac{2}{5}d$	·	$\frac{8}{175}bd^3$	$\frac{12}{175}d^2$		
PARABOLIC SEGMENT	$\frac{1}{3}bd$	$\frac{3}{10}d$	-	-	-		

FIG. 2.6

The values for other shapes can often be found by dividing the figure up into rectangles, circles, triangles etc., and considering the moment of each element about an axis through its own C.G. parallel to the axis of reference and adding an amount equal to the area × square of distance between the axes. The examples in Chapter VI should serve to illustrate the method. Values for various figures are given in Fig. 2.6.

Radius of Gyration

If in Fig. 2.3 the whole area A is taken as being concentrated at a distance k_{xx} from the axis X–X so that

$$A \times (k_{xx})^2 = I_{xx},$$

then k_{xx} is called the 'radius of gyration' with regard to the axis X–X.

Similarly $$A \times (k_{yy})^2 = I_{yy}$$

and $$k_{xx} = \sqrt{\frac{I_{xx}}{A}}; \qquad k_{yy} = \sqrt{\frac{I_{yy}}{A}}.$$

The radius of gyration is used principally in calculating the strength of columns and other struts.

Approximate Moments of Inertia

The M.I. can be found in many cases by dividing the figure up into a number of rectangles.

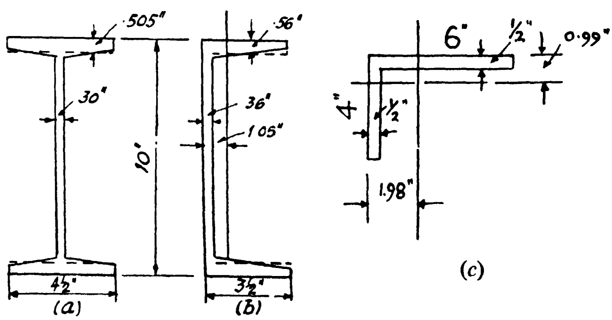

FIG. 2.7

If a 10 in. × 4½ in. joist section be taken as an example (web thickness = 0·30 in.; flange thickness = 0·505 in.) the moments of inertia can be found by taking the gross area and deducting the difference between that and the net area (Fig. 2.7 (*a*)).

Axis X–X

$$I_{xx} \text{ (gross area)} = \frac{4{\cdot}5 \times 10^3}{12} = 375 \text{ in.}^4$$

Difference { breadth $4{\cdot}5 - 0{\cdot}3 = 4{\cdot}2$
depth $10 - 2 \times 0{\cdot}505 = 9$ in. about.

$$I = \frac{4{\cdot}2 \times 9^3}{12} = 255 \text{ in.}^4$$

Net moment = 120 in.^{4}

The value given in the tables (Appendix A) is 122·34 in.^{4}, so that the error is less than 2 per cent.

Axis Y–Y

$$\text{Moment}\begin{cases}\text{flanges} & \dfrac{1{\cdot}01\times4{\cdot}5^3}{12} & = 7{\cdot}6\text{ in.}^4\\ \text{web} & \dfrac{9\times0{\cdot}3^3}{12} & = \text{—}\end{cases}$$

$$7{\cdot}6\text{ in.}^4$$

(Tabulated value = $6{\cdot}49$ in.^{4}).

Applying the same method to a 10 in. × $3\frac{1}{2}$ in. channel section with web and flange thicknesses of 0·36 in. and 0·56 in. respectively, it can be seen that the position of the C.G. must be calculated in relation to the heel of the channel (Fig. 2.7 (*b*)).

$$I_{xx}\text{ (gross area)} = \frac{3{\cdot}5\times10^3}{12} = 292\text{ in.}^4$$

$$\text{Difference}\begin{cases}\text{breadth} & 3{\cdot}5-0{\cdot}36 = 3{\cdot}14\text{ in.}\\ \text{depth} & 10-2\times0{\cdot}56 = 8{\cdot}88\text{ in.}\end{cases}$$

$$I = \frac{3{\cdot}14\times8{\cdot}88^3}{12} = 185\text{ in.}^4$$

$$\text{Net} = 107\text{ in.}^4$$

(Tabulated value = $109{\cdot}5$ in.^{4})

I_{yy}

Part	*Area* A (*in.*^{2})	*Distance from heel* y (*in.*)	*1st moment* Ay (*in.*^{3})	*2nd moment* Ay^2 (*in.*^{4})	I *own axis* (*in.*^{4})	*Total* I (*in.*^{4})
Web	8·88 × 0·36 = 3·21	0·18	0·58	0·10	..	0·10
Flanges	2 × 3·5 × 0·56 = 3·92	1·75	6·88	12·00	4·00	16·00
Totals	7·13	..	7·46	..	..	16·10

Therefore $$\bar{y} = \frac{7{\cdot}46}{7{\cdot}13} = 1{\cdot}05,$$

and by principle of parallel axes

$$I_{yy} = 16{\cdot}10-7{\cdot}13\times(1{\cdot}05)^2 = 8{\cdot}30\text{ in.}^4$$

(Tabulated value = $7{\cdot}40$ in.^{4})

In dealing with such sections as angles and channels, etc., it is often necessary to find the M.I. about axes other than the rectangular axes. This is most conveniently done by means of the principle of the ellipse of inertia. An unequal angle 6 in. × 4 in. × $\frac{1}{2}$ in. will serve to illustrate this (Fig. 2.7 (c)).

First, calculate the properties about the rectangular or 'square' axes.

I_{xx}

Part	*Area* (*in.*²)	*Arm* (*in.*)	*1st moment* (*in.*³)	*2nd moment* (*in.*⁴)	*Own I* (*in.*⁴)	*Total I* (*in.*⁴)
Table	$4\times\frac{1}{2} = 2{\cdot}00$	0·25	0·50	0·12	..	0·12
Stalk	$5\frac{1}{2}\times\frac{1}{2} = 2{\cdot}75$	3·25	8·95	29·00	6·95	35·95
Total	4·75	..	9·45	..	..	36·07

$$\bar{x} = \frac{9{\cdot}45}{4{\cdot}75} \qquad = 1{\cdot}98 \text{ in.}$$

$$I_{xx} = 36{\cdot}07 - 4{\cdot}75 \times 1{\cdot}98^2 = 17{\cdot}42 \text{ in.}^4$$

(Tabulated value $= 17{\cdot}14$ in.^{4})

I_{yy}

Part	*Area* (*in.*²)	*Arm* (*in.*)	*1st moment* (*in.*³)	*2nd moment* (*in.*⁴)	*Own I* (*in.*⁴)	*Total I* (*in.*⁴)
Table	2·00	2·00	4·00	8·00	2·67	10·67
Stalk	2·75	0·25	0·69	0·17	..	0·17
Total	4·75	..	4·69	..	..	10·84

$$\bar{y} = \frac{4{\cdot}69}{4{\cdot}75} \qquad = 0{\cdot}99 \text{ in.}$$

$$I_{yy} = 10{\cdot}84 - 4{\cdot}75 \times 0{\cdot}99^2 = 6{\cdot}19 \text{ in.}^4$$

(Tabulated value $= 6{\cdot}11$ in.^{4})

In order to find the values of the maximum and minimum moments of inertia, it is necessary to calculate the 'product moment' about the axes X–X and Y–Y which is $\sum axy$.

Therefore the product moment is given by

$$\text{Table } 2{\cdot}00 \times (1{\cdot}98 - 0{\cdot}25) \times (2{\cdot}00 - 0{\cdot}99) = 3{\cdot}48 \text{ in.}^4$$

$$\text{Stalk } 2{\cdot}75 \times (3{\cdot}25 - 1{\cdot}98) \times (0{\cdot}99 - 0{\cdot}25) = \underline{2{\cdot}58} \text{ in.}^4$$

$$\text{Total} \quad = 6{\cdot}06 \text{ in.}^4$$

Then angle of inclination between the two sets of axes is given by

$$\tan 2\theta = \frac{2 \times \text{product moment}}{I_{xx} - I_{yy}}$$

$$= \frac{2 \times 6{\cdot}06}{17{\cdot}42 - 6{\cdot}19} = \frac{12{\cdot}12}{11{\cdot}23},$$

$$\therefore \quad 2\theta = 47^\circ\ 11'' \text{ approx.} \qquad \theta = 23^\circ\ 35'\ 30'' \text{ approx.}$$

If I_{vv} and I_{uu} are the maximum and minimum values

$$I_{vv} + I_{uu} = I_{xx} + I_{yy} = 23{\cdot}61 \text{ in.}^4$$

and

$$I_{uu} = 23{\cdot}61 - I_{vv}.$$

Hence, since $I_{xx} = I_{vv}\cos^2\theta + I_{uu}\sin^2\theta = 17{\cdot}42.$

Substituting and solving, we obtain the values

$$I_{vv} = 20{\cdot}1 \text{ in.}^4,$$
$$I_{uu} = 3{\cdot}51 \text{ in.}^4$$

Tabulated values are 19·73 and 3.52 respectively.

More complicated sections such as plate and angle girders can be dealt with in a similar manner. The arithmetical work can be reduced by the intelligent use of a structural section book. (When dealing with net sections, i.e. when rivet holes are to be deducted from the gross area, use the principle: moment of net section = moment of gross section − moment of area deducted.)

Definitions of Shear Force, Bending Moment, etc.

1. The loading diagram shows the intensity of normal loading at any point along the span.

2. The shear force at any point along a loaded span is the algebraic sum of the forces (normal to the span) to the one side or the other of the point in question. The shear-force diagram shows the variation in shear force along the span and is the 'sum curve' of the loading diagram.

3. The bending moment at any point on a loaded span is the algebraic sum of the moments of all (normal) forces, to one side or the other of the point under consideration, about that point. The bending-moment diagram shows the variation of the bending moment along the span.

The bending-moment diagram is the 'sum curve' of the shear-force diagram and conversely the shear-force diagram is the curve showing the rate of change of B.M. along the span. Consider two points A, B on a span and let $AB = dx$.

$$\text{Load on } AB = w \text{ (per unit length)} \times dx$$
$$= w\,dx.$$
$$\text{Moment at } A = M$$
$$\text{,, \quad ,, \quad } B = M + dM.$$
$$\text{Shear at } A = S$$
$$\text{,, \quad ,, \quad } B = S + dS.$$

Then taking moments about A,

$$(M+dM) - M + (S+dS)\,dx - \tfrac{1}{2}w \times (dx)^2 = 0.$$

Neglecting the last two terms as they are the squares of elemental quantities,

$$S = -\frac{dM}{dx},$$

which is the relation between shear and B.M.

Since shear represents rate of change of B.M. it follows that the maximum or minimum value of the B.M. occurs at the point of zero shear, and this fact gives us a convenient method of determining the maximum B.M. (Fig. 2.8).

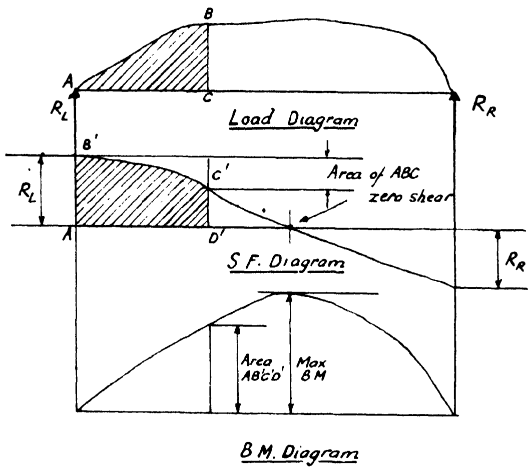

FIG. 2.8

Theory of Bending

The accepted theory of bending is based upon the following assumptions:

1. Sections which are plane before bending remain plane after bending.

2. Stress is proportional to strain, i.e. the material obeys Hooke's law and is elastic; also the material is not stressed beyond its elastic limit.

3. The value of E (Young's modulus of elasticity) is the same for tension and compression.

Referring to Fig. 2.9, AB, CD are two sections which are plane before bending. After bending AB, CD become $A'B'$, $C'D'$ respectively, which when produced meet at O, the centre of curvature of the beam (the curvature has been purposely exaggerated in Fig. 2.9).

Fibres along the neutral axis are unchanged.
,, above ,, ,, shorten.
,, below ,, ,, lengthen.

That is,
the portion of the beam above the N.A. is in compression,
,, ,, ,, below ,, ,, tension.

Considering a fibre at x above the N.A. and having an elemental area a, then

$$\text{strain in } EL = \frac{\text{change in length}}{\text{original length}}$$
$$= \frac{EE'+LL'}{EL}.$$

Since $\quad$ stress $= E \times$ strain, $\quad$ force $= aE \times \dfrac{EE'+LL'}{EL}$.

Similarly strain in $FM = \dfrac{FF'+MM'}{FM}$, $\quad$ force $= aE \times \dfrac{FF'+MM'}{FM}$.

$$\therefore \quad \text{total force in compression} = \sum aE \times \frac{EE'+LL'}{EL},$$

$$\text{,,} \quad \text{,,} \quad \text{tension} = \sum aE \times \frac{FF'+MM'}{FM}.$$

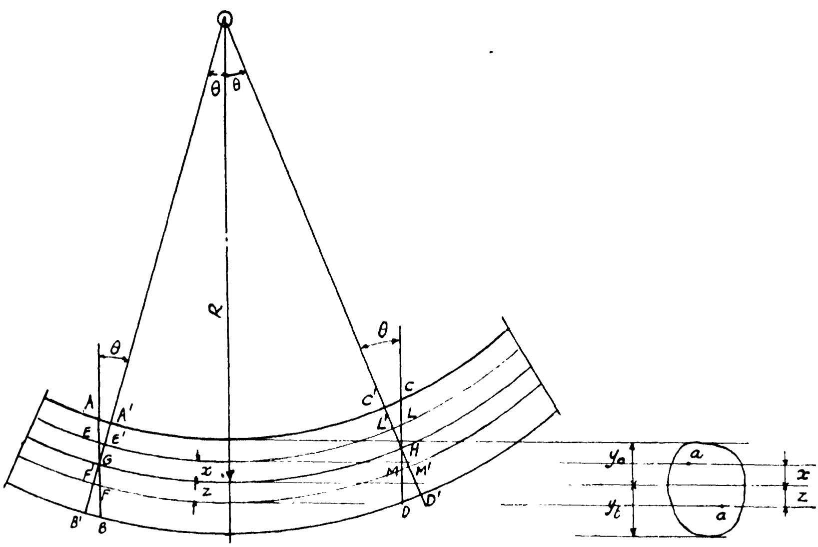

FIG. 2.9

But $EE' = LL'$ and $FF' = MM'$, also $EL = FM = GH$, and by similar triangles,

$$\frac{EE'}{FF'} = \frac{x}{z} = \frac{LL'}{MM'}.$$

$$\frac{\text{Stress in } EL}{\text{Stress in } FM} = \frac{E \times (EE'+LL')/EL}{E \times (FF'+MM')/FM} = \frac{EE'+LL'}{FF'+MM'} = \frac{x}{z}$$

which proves that stress is proportional to the distance from the N.A. or

$$\frac{\text{stress}}{\text{distance from N.A.}} = \text{constant}.$$

Now if f is the stress at the extreme fibre, stress at any other fibre

$$= f \times \frac{\text{distance to that fibre}}{\text{distance to extreme fibre}},$$

and $$\text{stress in } EL = f\times\frac{EG}{AG} = fx/AG$$

$$\text{force in } EL = fxa/AG,$$

and $$\text{total compression} = \sum fxa/AG$$

$$\text{total tension} = \sum fza/AG.$$

For equilibrium these forces must balance.

$$\therefore \quad \sum ax = \sum az,$$

or first moment of area above N.A. = first moment of area below the N.A.

This condition applies to the centre of gravity of the section, therefore the N.A. must coincide with the C.G.

Now taking moments of forces about the N.A.:

$$\text{total moments of forces in compression} = \sum\frac{fx^2a}{AG} = \frac{f}{AG}\sum ax^2,$$

$$\text{,, \quad ,, \quad ,, \quad tension} = \sum\frac{fz^2a}{AG} = \frac{f}{AG}\sum az^2,$$

and sum of these moments

$$= (f/AG)(\textstyle\sum ax^2 + \sum az^2)$$
$$= (f/AG)(\text{second moment of area about C.G.})$$
$$= (fI/AG).$$

Now sum of moments of forces = moment of resistance, which must equal M, the external B.M.

$$\therefore \quad M = I\times\frac{f}{AG},$$

which can be written $$\frac{M}{I} = \frac{f}{AG} = \frac{f}{y}. \tag{1}$$

This formula is one of the basic formulae for use in analysis and design of beams and girders.

Also $$f = E\times\text{strain in } AC$$
$$= E\times\frac{AA'+CC'}{AC}.$$

If angle $AGA' = \theta$, angle CHC' is also θ, and angle $GOH = 2\theta$.

$$AA' = AG\times\theta; \qquad CC' = CH\times\theta = AG\times\theta,$$

and $$AC = R\times 2\theta \text{ approx.}$$

$$\therefore \quad f = E\times\frac{2AG\times\theta}{2R\theta} = E\times\frac{AG}{R},$$

and substituting in equation (1) we find that

$$\frac{M}{I} = \frac{f}{AG} = \frac{E}{R} = \frac{f}{y}. \tag{2}$$

This equation (2) gives the complete relation between B.M., inertia, stress, distance from N.A., and radius of curvature.

From equation (1)

$$M = \frac{fI}{y} = f \times \frac{I}{y} = f \times Z,$$

where Z = section modulus of the section. It should be noted that, as the distance to the extreme fibre in compression is not necessarily the same as for the corresponding fibre in tension, there may be two values of Z for any section, viz.

$Z_c = (I/y_c)$ (section modulus in compression),

$Z_t = (I/y_t)$ (,, ,, tension);

y_c, y_t are distances to the extreme fibres.

If Z be multiplied by the maximum fibre stress the result is the moment of resistance (M.R.) of the section, which must equal M (bending moment).

In many cases in practice the maximum stress in compression is not the same as that in tension. For instance, beams and girders may fail by buckling of the top flange due to strut action or torsion, and for that reason the permissible compressive stress is generally less than the corresponding tensile stress.

Moment of resistance (compression) $= MR_c = Z_c \times f_c$

,, ,, (tension) $= MR_t = Z_t \times f_t$.

Where the two moments of resistance are unequal, the lesser value should be adopted in strength calculations.

I is usually given in in.^{4}, and as y is in inches,

$$Z = I/y = \text{in.}^3$$

As f is usually given in tons per square inch, M will be in in.-tons, whereas bending moment is often calculated in ft.-tons, and care should be taken to convert one or other of these quantities to the same units in which the other is given.

Since $1/R$ can be written as d^2y/dx^2, where y is the deflexion (this will be proved in Chap. IV),

$$\frac{M}{I} = E\frac{d^2y}{dx^2} \quad \text{or} \quad \frac{d^2y}{dx^2} = \frac{M}{EI}. \tag{3}$$

Integrating the R.H.S. of this equation (3) we get, since

$dy/dx = i$ = slope of the elastic line at the point in question,

$$i = \int \frac{M}{EI}\,dx.$$

A second integration gives the value of y, the deflexion at the same point,

$$y(\Delta) = \int \left(\int \frac{M}{EI}\,dx \right) dx.$$

Bending applied to Homogeneous Sections

Steel is a homogeneous material and therefore we can take the 10 in. $\times 4\frac{1}{2}$ in. I section shown in Fig. 2.7 (a) as an example.

$$I_{xx} = 122{\cdot}34 \text{ in.}^4; \qquad I_{yy} = 6{\cdot}49 \text{ in.}^4$$

$$Z_{xx} = \frac{122{\cdot}34}{5} = 24{\cdot}46 \text{ in.}^3$$

$$Z_{yy} = \frac{6{\cdot}49}{2{\cdot}25} = 2{\cdot}88 \text{ in.}^3 \qquad \text{Area} = 7{\cdot}35 \text{ in.}^2$$

$$\text{Radius of gyration } k_{xx} = \sqrt{\frac{24{\cdot}46}{7{\cdot}35}} = 4{\cdot}08 \text{ in.}$$

$$\text{,, \quad ,, \quad ,, } \quad k_{yy} = \sqrt{\frac{6{\cdot}49}{7{\cdot}35}} = 0{\cdot}94 \text{ in.}$$

Assuming $f = 10$ tons per sq. in.,

M.R. $= 10 \times 24{\cdot}46 = 244{\cdot}6$ in.-tons,

$= 20{\cdot}38$ ft.-tons about X–X axis.

Timber is also a homogeneous material (although it is organic and therefore less homogeneous than steel).

For a timber beam b in. wide $\times\, d$ in. deep,

$$Z = bd^2/6 \quad \text{and} \quad \text{M.R.} = fbd^2/6.$$

If $b = 3$ in., $d = 9$ in., $f = 1{,}000$ lb./in.^{2},

M.R. $= 40{,}500$ in.-lb.

$= 3{,}375$ ft.-lb.

Bending applied to Non-homogeneous Sections

Reinforced concrete is an example of a non-homogeneous material, since the moduli of elasticity and working stresses are different for steel and concrete.

The basic theory for R.C. is called the 'straight-line, no-tension' method. It ignores any tensile strength in the concrete, the whole of the tension being taken up by the steel reinforcement. In recent years several theories have been propounded, some of which ignore the modular ratio, and in certain other cases the concrete is treated as a plastic material with a parabolic or hyperbolic distribution of stress (see Bibliography). The inexperienced designer should adhere to the generally accepted theory, since any error therein is upon the side of safety.

Referring to Fig. 2.10, let

E_s = Young's modulus for steel,

E_c = ,, ,, ,, concrete in compression,

$m = E_s/E_c$.

Considering the bending of the section it is obvious that the strain on any fibre is proportional to its distance from the N.A. Also, for equal strains,

stress on steel = $m \times$ stress on concrete at same distance from N.A.

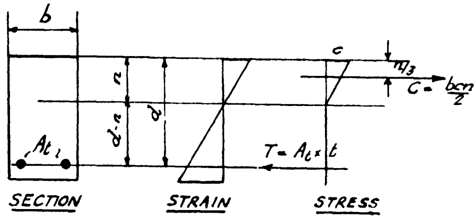

FIG. 2.10

If n = depth from top to N.A.

d = ,, ,, reinforcement,

and b = width of section,

$$\frac{\text{strain on concrete at extreme fibre}}{\text{strain on steel at extreme fibre}} = \frac{n}{d-n}.$$

Hence
$$\frac{\text{stress in concrete at extreme fibre}}{\text{stress in steel at extreme fibre}} = \frac{n}{m(d-n)}.$$

If c = stress in concrete at extreme fibre

and t = ,, steel ,, ,,

$$\frac{c}{t} = \frac{n}{m(d-n)} \quad \text{or} \quad t = \frac{mc(d-n)}{n}. \qquad (4)$$

And if A_t = area of steel reinforcement,

$$\text{total tension } (T) = A_t \times mc \times \{(d-n)/n\}.$$

Now the compressive stress in the concrete varies from c at the top to zero at the N.A.

$$\therefore \quad \text{total compression } (C) = bn \times \tfrac{1}{2}c = \tfrac{1}{2}bcn.$$

But for equilibrium $C = T$.

$$\therefore \quad \tfrac{1}{2}bcn = A_t \times mc \times \frac{d-n}{n}. \qquad (5)$$

The force C must act at $\frac{1}{3}n$ from the top, since the distribution of stress is triangular, and the lever arm $a = d - \frac{1}{3}n = a_1 \times n$.

Moment of resistance (compression) $= \frac{1}{2}bcn \times (d - \frac{1}{3}n)$,

,, ,, (tension) $= A_t \times mc \times \dfrac{d-n}{n} \times (d - \frac{1}{3}n)$.

For any given values of t, c, and m, the depth of the N.A. n can be found in terms of d from equation (4) thus:

$$\frac{t}{c} = r = \frac{m(d-n)}{n},$$

$$\therefore \quad n = \frac{d}{r/m+1} = n_1 d,$$

$$a_1 = 1-\tfrac{1}{3}n_1.$$

MR_c can be written thus:

$$MR_c = \frac{bcn_1 d}{2} \times \left(d - \frac{n_1 d}{3}\right)$$

$$= Qbd^2,$$

where $$Q = \text{constant},$$

$$= \frac{cn_1(1-\tfrac{1}{3}n_1)}{2}.$$

If $t = 20{,}000$ lb./in.², $c = 750$ lb./in.², $m = 15$,

$$r = 26{\cdot}67 \quad \text{and} \quad n_1 = 0{\cdot}36, \quad a_1 = 0{\cdot}88, \quad Q = 119.$$

$$\frac{A_t}{bd} = \text{ratio of steel to concrete}$$

$$= \frac{cn_1}{2t} = 0{\cdot}00675.$$

Similarly for $t = 18{,}000$ lb./in.² and $c = 750$ lb./in.²

$$n_1 = 0{\cdot}385, \quad a_1 = 0{\cdot}872, \quad Q = 125{\cdot}7, \quad \text{and} \quad A_t/bd = 0{\cdot}008.$$

Other values are given in Appendix B. It will be noticed that Q rises with c, i.e. with richer mixes, but that at the same time a higher value of t does not increase Q, although the ratio $Q/A_t/bd$ falls.

Owing to practical considerations in R.C. design it is often necessary to provide compression reinforcement (above the N.A.). The stress in this is m times the stress in the surrounding concrete. The steel can be replaced, for purposes of calculation, by an area equal to $(m-1)A_c$, where A_c = area of compression steel (Fig. 2.11).

If this is placed at d_c below the top, then the additional moment of resistance is given by

$$A_c \times (m-1) \times (1-d_c/n) \times (d-d_c)$$

or $$C_s \times (d-d_c).$$

Then total $$MR_c = Qbd^2 + C_s \times (d-d_c)$$

$$= MR_t.$$

Hence it is necessary to increase the value of A_t to suit.

$$\therefore \quad A_t = \frac{Qbd^2 + C_s \times (d - d_c)}{a_1 dt}.$$

The use of compression steel is not really economical owing to the fact that it is under-stressed (the American Society of Civil Engineers allow a stress in compression steel equal to $2 \times (m-1)c \times (1-d_c/n)$ with an upper limit of 16,000 lb./in.²). Where compression steel is used, stirrups must be provided and spaced so as to prevent any tendency to buckle.

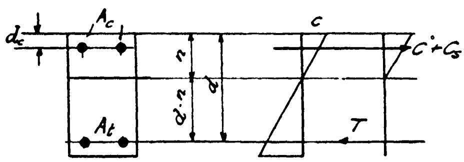

FIG. 2.11

The following examples are useful to illustrate the above principles.

Example 1. A concrete beam is 9 in. wide × 24 in. deep. Using $c = 825$ lb./in.², $t = 20{,}000$ lb./in.², and $m = 14$, find the area of tensile reinforcement and the M.R. of the section.

$$t/c = 24{\cdot}3.$$
$$\therefore \quad n_1 = 0{\cdot}366,$$
$$a_1 = 0{\cdot}878,$$
$$Q = 133,$$
$$\frac{A_t}{bd} = \frac{825 \times 0{\cdot}366}{2 \times 20{,}000} = 0{\cdot}00756.$$
$$d = 24 - 1\tfrac{1}{2} = 22\tfrac{1}{2} \text{ in.}$$

to allow for cover on the underside of the reinforcement.

$$\therefore \quad MR_c = 133 \times 9 \times 22{\cdot}5^2 = 604{,}000 \text{ in.-lb.}$$
$$A_t = 0{\cdot}00756 \times 9 \times 22{\cdot}5 = 1{\cdot}53 \text{ in.}^2$$

Therefore use three $\frac{7}{8}$-in. diameter bars, which gives

$$A_t = 1{\cdot}80 \text{ in.}^2$$

Then
$$MR_t = 1{\cdot}80 \times 20{,}000 \times 0{\cdot}878 \times 22{\cdot}5$$
$$= 711{,}000 \text{ in.-lb.}$$

MR_c is the lower value and will be used as the effective M.R.

Example 2. The beam used in Example 1 is required to resist a B.M. of 800,000 in.-lb. Find the area of compression steel required and adjust A_t to suit.

$$\text{B.M.} - MR_c = 196{,}000 \text{ in.-lb.}$$

Working stress in compression steel

$$= 825 \times \frac{(8{\cdot}2 - 1{\cdot}5)}{8{\cdot}2} \times (14-1)$$

$$= 8{,}750 \text{ lb./in.}^2$$

$$\text{Lever arm} = (0{\cdot}878 \times 22{\cdot}5) - 1\tfrac{1}{2} = 18{\cdot}3 \text{ in.}$$

$$\therefore \quad A_c = \frac{196{,}000}{18{\cdot}3 \times 8{,}750} = 1{\cdot}26 \text{ in.}^2$$

Two $\frac{7}{8}$-in. diameter bars give $A_c = 1{\cdot}20$ in.²

$$A_t = \frac{800{,}000}{19{\cdot}8 \times 20{,}000} = 2{\cdot}00 \text{ in.}^2$$

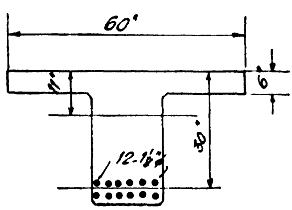

FIG. 2.12

Use two at $\frac{7}{8}$-in. diameter and one at 1-in. diameter, which gives

$$A_c = 2{\cdot}01 \text{ in.}^2$$

Example 3. A T-beam has an effective width of 60 in., an effective depth of 30 in., and the slab is 6 in. thick, Using the same values for t, c, and m as before, find the area of tension steel required.

$n = 0{\cdot}366 \times 30 = 11$ in. approx., i.e. the N.A. lies below the slab.

Stress in concrete at top of slab $= 825$ lb./in.²

" " underside of slab $= 825 \times \dfrac{11-6}{11}$

$$= 375 \text{ lb./in.}^2$$

$$\therefore \quad \text{average stress in slab} = \frac{825+375}{2} = 600 \text{ lb./in.}^2$$

Total compression in slab $= 600 \times 6 \times 60 = 216{,}000$ lb.

In order to find the C.G. of the compressive forces, take moments about the underside of the slab:

Stress	*Area*	*Arm*	
375	$\times (60 \times 6) \times$	$\frac{6}{2}$	$= 405{,}000$
$\dfrac{825-375}{2}$	$\times (60 \times 6) \times$	$\frac{2}{3} \times 6$	$= 324{,}000$
			$729{,}000$

$$\therefore \quad \text{distance} = \frac{729{,}000}{216{,}000} = 3{\cdot}4 \text{ in.}$$

and lever arm $= (30-6) + 3{\cdot}4 = 27{\cdot}4$ in.

$$MR_c = 27{\cdot}4 \times 216{,}000 = 5{,}950{,}000 \text{ in.-lb.}$$

(neglecting compression in the rib).

$$\therefore \quad A_t = \frac{5{,}950{,}000}{20{,}000 \times 27{\cdot}4} = 11 \text{ in.}^2$$

Use twelve $1\frac{1}{8}$-in. diameter bars: $A_t = 12$ in.^{2} (Fig. 2.12).

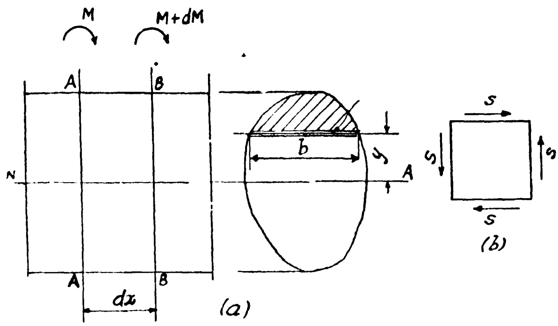

FIG. 2.13

Shear in Homogeneous Members: General Case

It has already been proved that shear represents the rate of change of B.M. Referring to Fig. 2.13,

$$\text{shear} = S = dM/dx.$$

Considering any fibre at distance y from the N.A.

Bending stress at $A\text{–}A = \dfrac{M}{I_1} \times y$ $\quad I_1 =$ M.I. at $A\text{–}A$

" " $B\text{–}B = \dfrac{(M+dM)}{I_2} \times y$ $\quad I_2 =$ " $B\text{–}B$.

Since dx is small, $\quad I_1 = I_2 = I.$

$\therefore$ change in bending stress between $A\text{–}A$ and $B\text{–}B = (dM/I) \times y$.

If the area of the fibre $= a$,

then force in fibre $= a \times \dfrac{dM}{I} \times y$,

and total force in fibres above N.A. $= \sum ay \times dM/I$.

But $dM = S\,dx$,

shear stress $= s$,

breadth of section $= b$,

and shear force must balance force due to change in B.M.

$$\therefore \quad b \times dx \times s = \frac{S\,dx}{I} \times \sum ay,$$

or

$$s = \frac{S}{Ib} \times \sum ay$$

$$= \frac{S}{Ib} \times \text{first moment of area of parts above the N.A.}$$

Therefore s (intensity of shear stress) must be a maximum when the first moment of area is a maximum, i.e. at the N.A. and zero at the extreme fibres. It should be remembered that s refers to the shear stress on a horizontal plane.

The horizontal shear stress acting upon a cube of unit side produces a moment of $s \times 1 \times 1$. For equilibrium this moment must be balanced, and hence it is clear that the intensity of shear stress on a vertical plane must also be equal to s (Fig. 2.13 (b)).

Shear stresses also occur on diagonal planes due to the action of shear on the horizontal and vertical planes. The value of s given by

$$s = \frac{S}{Ib} \times \sum ay$$

is that at any particular distance from the N.A., the average shear stress being given by

$$s_m = S/A \qquad (A = \text{area of cross-section}).$$

Case 1. Rectangular beam.

$$b = \text{breadth},$$

$$d = \text{depth}.$$

$$\text{Intensity of shear } s = \frac{S}{I \times b} \times \sum ay \qquad (I = bd^3/12).$$

$$\sum ay = b\left(\frac{d}{2} - y\right)\left(\frac{d}{4} + \frac{y}{2}\right) = b\left(\frac{d^2}{8} - \frac{y^2}{2}\right).$$

When

$$y = \tfrac{1}{2}d, \quad \sum ay = 0,$$

$$y = 0, \quad \sum ay = \tfrac{1}{8}bd^2.$$

$$\text{Maximum intensity of } s = \frac{3S}{2bd} = \tfrac{3}{2} \times s_m.$$

$$\text{Average intensity } s_m = S/bd.$$

Also, since $\sum ay$ varies as a function of y^2, the graph of shear stress must be parabolic.

Case 2. 10 in. $\times$ $4\frac{1}{2}$ in. joist section dealt with previously.

From the formula

$$s = \frac{S}{I \times b} \times \sum ay$$

it can be seen that s will increase from zero at the extreme fibres up to a certain value at the underside of the flange, the variation of stress being parabolic.

$$\text{Maximum value } s_f = \frac{S \times 4{\cdot}5 \times 0{\cdot}505 \times (5 - 0{\cdot}505/2)}{4{\cdot}5 \times I} = 0{\cdot}0195S.$$

The value of s increases suddenly just below the flange and the web shear is then

$$s_w = \frac{S\times 4{\cdot}5\times 0{\cdot}505\times(5-0{\cdot}505/2)}{I\times 0{\cdot}3} = 0{\cdot}29S.$$

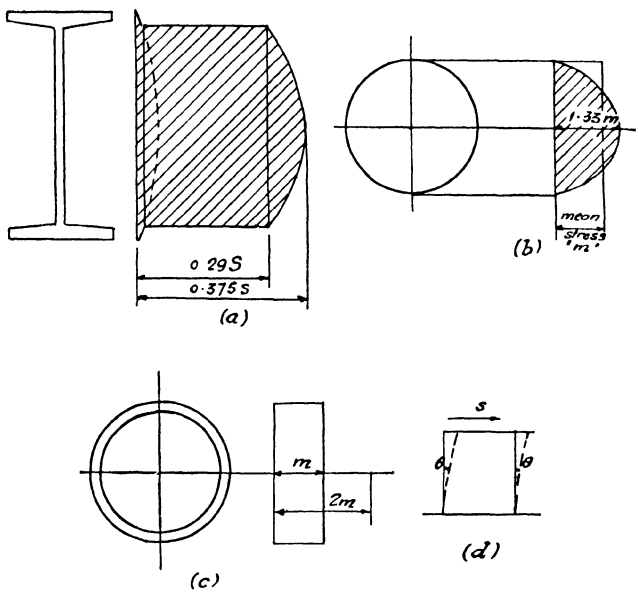

FIG. 2.14

Maximum value of shear at the N.A.

$$s_{\text{max}} = \frac{S\times\{4{\cdot}5\times 0{\cdot}505\times(5-0{\cdot}505/2)+(5-0{\cdot}505)^2\times 0{\cdot}3/2\}}{I\times 0{\cdot}3} = 0{\cdot}375S.$$

Total shear on flanges

$$= 2\times\tfrac{2}{3}\times 0{\cdot}0195S\times 4{\cdot}5\times 0{\cdot}505 \qquad = 0{\cdot}059S.$$

Total shear on web

$$= 0{\cdot}3(10-2\times 0{\cdot}505)\times 0{\cdot}29S \qquad = 0{\cdot}782S$$

$$+\tfrac{2}{3}\times 0{\cdot}3(10-2\times 0{\cdot}505)(0{\cdot}375-0{\cdot}29)S = \underline{0{\cdot}153S}$$

$$0{\cdot}994S$$

(See Fig. 2.14 (*a*).)

Percentage of total shear on web = 94 per cent. (about). The assumption made in practice is that the web carries the whole shear (error is about 10 per cent.).

$$\text{Average shear stress} = \frac{S}{7{\cdot}35} = 0{\cdot}136S.$$

$$\therefore\quad \frac{\text{max. shear stress}}{\text{average shear stress}} = 2{\cdot}76.$$

Case 3. Solid round section (Fig. 2.14 (*b*)).

$$I \text{ for circle} = \frac{\pi D^4}{64} \quad (D = \text{diameter}).$$

$$\text{Average shear stress} = \frac{S}{A} = \frac{4S}{\pi D^2}.$$

Considering shear stress at the N.A.

$$\text{Area above N.A.} = \tfrac{1}{8}\pi D^2.$$

$$\text{Distance from N.A. to C.G. of part above N.A.} = 2D/3\pi.$$

$$\therefore \quad \text{first moment of area} = \frac{\pi D^2}{8} \times \frac{2D}{3\pi} = \frac{D^3}{12}.$$

$$\therefore \quad s_{\max} = \frac{16S}{3\pi D^2},$$

and
$$\frac{\text{max. shear stress}}{\text{average shear stress}} = 1{\cdot}33.$$

Case 4. Thin shell of mean diameter D and thickness t (Fig. 2.14 (*c*)).

$$\text{Average shear stress} = S/\pi Dt.$$

$$\text{Area above N.A.} = \tfrac{1}{2}\pi Dt.$$

$$\text{Distance to C.G. of part above N.A.} = D/\pi.$$

$$\text{First moment of area} = \frac{\pi Dt}{2} \times \frac{D}{\pi} = \frac{D^2 t}{2}.$$

$$I = \text{area} \times (\text{radius of gyration})^2$$
$$= \pi Dt \times \tfrac{1}{8}D^2 = \tfrac{1}{8}\pi D^3 t.$$

$$\text{Max. shear stress} = 2S/\pi Dt = 2 \times \text{average shear stress}.$$

Shear modulus is called the 'modulus of rigidity' (G). For steel

$$G = 0{\cdot}4E \text{ approximately,}$$
$$= 5{,}500 \text{ tons/in.}^2$$

Shear strain is measured in radians (Fig. 2.14 (*d*)). For steel the shear strain is usually very small, and in general shear strain is small in comparison with the strain due to bending.

Shear stress acting in conjunction with stress due to bending can cause diagonal stresses in the webs of plate girders, which should be considered in the design of such members. Shear in webs should not be confused with web buckling due to failure of the web under compression.

In the design of built-up girders, the shear force acting upon the rivets must be calculated. Since

$$s = \frac{S \times \sum ay}{I \times b},$$

$$\text{shear per unit length} = \frac{S \times \sum ay}{I}.$$

In actual practice it is usual to find the shear per unit length thus: since $S = dM/dx$ (Fig. 2.15).

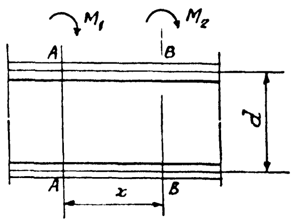

FIG. 2.15

If M_1 = B.M. at A–A

and M_2 = B.M. at B–B,

$$S = \frac{M_2 - M_1}{x} \quad (\text{where } x = AB).$$

Now the flange forces form a couple equal and opposite to the B.M., so that if d = effective depth,

$$\text{flange force at } A\text{–}A = M_1/d,$$

$$\text{,,} \quad \text{,,} \quad B\text{–}B = M_2/d,$$

and increment of flange force $= \dfrac{M_2 - M_1}{d} = \dfrac{Sx}{d}$.

When x is small, shear per unit length $= S/d$. This is usually calculated per linear inch and must be developed by the rivets or welds connecting the flange details, or the flange to the web.

In cases where $\frac{1}{8} \times$ web area has been included in the flange area

$$\text{shear per unit length} = \frac{S}{d} \times \frac{A}{A + \frac{1}{8}W},$$

where A = flange area and W = web area.

Shear in Rivets

When two structural members are connected together by riveting and loaded as shown (Fig. 2.16 (a)), there is a tendency to fail by shear along the line AB. The effective area of the rivet (in single shear) is $\pi d^2/4$ (where d = diameter of the hole), and strength in single shear $= \pi d^2 s/4$ (s = safe shear stress). For rivets in double shear (Fig. 2.16(b)) there is a tendency to shear along two planes and the value in double shear is usually taken as $2 \times \pi d^2 s/4$. Note that a rivet can also fail in bearing by crushing against the edges of the hole and the bearing value = safe bearing stress $\times d \times t$ (t = thickness of metal).

In designing riveted joints the least value of the rivet should be used. For long rivets treble or quadruple shear may occur, but owing to bending which may occur in the rivet and other practical reasons, it is well to reduce the working stresses in such rivets as stated in Chapter XVI (Vol. II). Since riveted joints may fail by tearing of the plate along the dotted lines in Fig. 2.16 (*a*), the distance between the edge of the hole and that of the plate should not be less than the rivet diameter. Values for rivets in shear and bearing are given in the regulations referred to

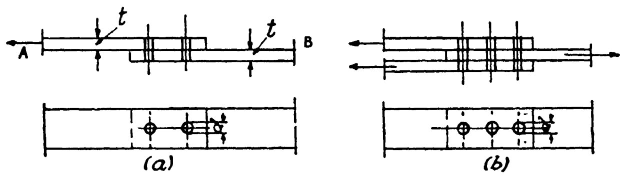

FIG. 2.16

in the Bibliography. Shop-driven rivets are more reliable than those driven at site, and for that reason turned bolts with a tolerance of $\pm 0{\cdot}005$ in. are often used in preference to the latter. (Rivets are not used in tension where this can be avoided. For rivets in tension, use diameter of cold rivet to find the effective area.)

Welding

This method of joining structural members has come into more general use in recent years and the use of welding is likely to increase owing to the tendency to design structures as rigid or semi-rigid frames. Since welded connexions are dealt with in Chapter XVI (Vol. II) it will be sufficient to deal very briefly with the question of welding at this stage.

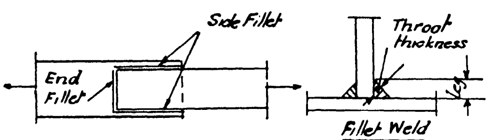

FIG. 2.17

The most common type of welds are (*a*) butt and (*b*) fillet welds. Type (*a*) are more reliable than type (*b*), although they involve more preparation of the parent metal. Fillet welds are rather unreliable under the action of alternating stresses which occur in bridges and similar structures. Welds may be subject to tension, compression, or shear, or to the action of combined stresses.

Fillet welds (Fig. 2.17) may be end, side, or diagonal fillet welds. The size is specified as the length of the shorter leg, but the size used in design calculations is the throat thickness = 0·707 length of leg for symmetrical fillet welds. The effective length = overall length $-2\times$(size of fillet). This should not be less than 2 in. or $6\times$size of fillet for stressed welds. Side fillets are stressed in longitudinal shear. End fillets are stressed in transverse shear.

Effective area = throat thickness $\times$ effective length, and working load = safe stress $\times$ effective area. Values for safe stresses are given in the various regulations already referred to.

Butt welds (Fig. 2.18) may be of various profiles and can be used to resist tension, compression, and shear. The most common types are the square butt, vee, double-vee, J, and U, and their use and working stresses are specified in B.S.S. 538 and other regulations.

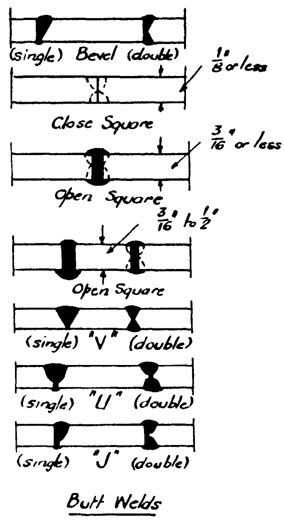

FIG. 2.18

Shear combined with Tension or Compression

It is unusual in the case of simply supported beams to find heavy stresses due to shear occurring at the same point as heavy tensile or compressive bending stress. This case does, however, arise in rigid or semi-rigid frames. Considering the general case (Fig. 2.19) of a direct stress f and a shear stress s acting together. The stress f acts upon the plane AB and s upon the plane AC. Since s acts upon the plane AC there must be a similar stress upon a plane at right angles to AC. If BC be the plane acted upon by the principal stress p, then

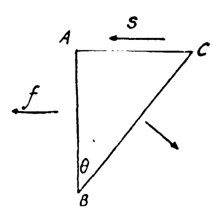

FIG. 2.19

$$AB\times f+AC\times s = BC\times p\cos\theta, \qquad (1)$$

also

$$AB\times s = BC\times p\sin\theta. \qquad (2)$$

From (2)

$$s = p\tan\theta,$$

and from (1)

$$AB\times f+\frac{BCp\sin\theta}{AB}\times AC = BC\times p\cos\theta.$$

Simplifying,

$$f+p\tan^2\theta = p,$$

or

$$f+p\times(s/p)^2 = p.$$

$$\therefore\quad p = \frac{f}{2}\pm\sqrt{\left(\frac{f^2+4s^2}{4}\right)}$$

$$= \frac{f}{2}\left\{1\pm\sqrt{\left(1+\frac{4s^2}{f^2}\right)}\right\}.$$

$$\therefore \quad \text{Max. principal stress} = p_1 = \frac{f}{2}\left\{1+\sqrt{\left(1+\frac{4s^2}{f^2}\right)}\right\}.$$

$$\text{Min. principal stress} = p_2 = \frac{f}{2}\left\{1-\sqrt{\left(1+\frac{4s^2}{f^2}\right)}\right\}.$$

Principal stress can be defined as the normal stress acting upon a plane which is wholly free from tangential stress.

$$\text{Max. shear stress} = \frac{p_1-p_2}{2} = \sqrt{\left(\frac{f^2}{4}+s^2\right)}.$$

The same method can be used for riveted, bolted, or welded connexions subject to direct stress and shear due to eccentric loading.

Shear in Timber Beams

The treatment of this presents no difficulty since timber beams are usually rectangular or square; hence as previously stated the maximum shear stress will be $3S/2bd$, where b, d are the breadth and depth of the beam. Since the shear strength of timber parallel to the grain is less than the corresponding value across the grain, this should be used in checking the strength of the member.

Shear in Non-homogeneous Members; R.C. Beams

In the case of rectangular R.C. beams with tensile steel only, the lever arm $a = d-n/3$.

In Fig. 2.20,

$$M_1 = \text{B.M. at } A\text{–}A,$$
$$M_2 = \text{B.M. at } B\text{–}B,$$
$$AB = l.$$

Then compressive force in concrete = B.M./a = tensile force in steel.

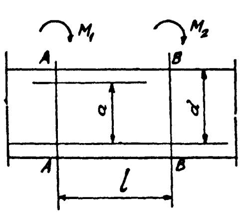

FIG. 2.20

$\therefore$ Force at A–A = M_1/a,

,, ,, B–B = M_2/a.

Increment in tensile force between A–A and B–B = $(M_2-M_1)/a$.

But since $M_2-M_1 = Sl$,

in the limit when l is very small,

increment in tensile force = S/a,

and this must be resisted by the shear stress s acting across the width b.

$$\therefore \quad s = S/ab.$$

The actual shear-stress diagram is a parabola between the top edge (where shear is zero) and the N.A. (where shear = s). The shear stress remains constant from the N.A. to the steel reinforcement.

The allowable shear stress in concrete is usually taken as 10 per cent. of the basic compressive bending stress. If the calculated shear stress does not exceed this value there is no necessity for shear reinforcement, although it is good practice to provide nominal reinforcement. Where the calculated stress exceeds the allowable shear reinforcement must be used (*a*) in the form of stirrups, (*b*) by bending up the main bars as diagonals, (*c*) by a combination of (*a*) and (*b*). Stirrups should be spaced not farther apart than the lever arm *a* or twelve times the diameter of the main bars (whichever is the lesser). The following formula is used to calculate the area of stirrups:

$$S = \frac{t_w \times A_w \times a}{p},$$

where

t_w = allowable tensile stress in stirrups (18,000–20,000 lb./in.^{2} for mild steel),

A_w = area of stirrup = $2 \times \frac{1}{4}\pi d^2$ for double stirrups (d = diameter),

p = pitch of stirrups.

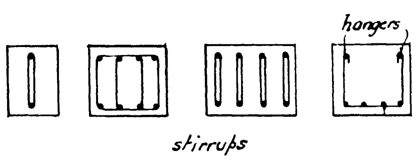

FIG. 2.21

Stirrups are normally $\frac{1}{4}$ in. minimum diameter, but stirrups more than $\frac{1}{2}$ in. diameter are difficult to bend. As the shear is usually a maximum at, or near, the supports, the stirrups should be spaced more closely together at the ends of the beams and may be omitted near mid-span. For doubly reinforced beams stirrups must be provided throughout the length of the beam. Hanger bars should be provided in singly reinforced beams (Fig. 2.21).

Diagonal or bent-up bars. Main bars may be bent up beyond their 'theoretical ends', i.e. where they are no longer required to resist B.M., to act as shear reinforcement. Usually the top layer of bars or the inner bars where there is only one layer, are bent up in this way. The steel acts in tension and the vertical component of the tension resists shear.

Force resisting shear = $At \sin\theta$,

where A = area of bar or bars,

t = tensile force (18,000–20,000 lb./in.^{2} for M.S.),

θ = angle of inclination to horizontal.

For $\theta = 45$,

$$\text{shear value} = 0{\cdot}707At. \quad \text{(Fig. 2.22.)}$$

These bars are analogous to the diagonals in a lattice girder.

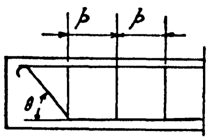

FIG. 2.22

Shear steel should be designed as carefully as the main reinforcement since failures have occurred due to insufficient shear reinforcement. Where the calculated shear stress exceeds the allowable value for concrete, some designers ignore the shear value of the concrete entirely. A more rational method is to reduce the allowable shear stress in the ratio of

$$\frac{\text{allowable shear stress}}{\text{calculated shear stress}}.$$

Then the difference between the shear taken up by the concrete (at the reduced working stress) and the total shear must be taken up by shear reinforcement.

Bond and Anchorage in R.C. Beams

It has been proved that the increment in tensile force between two points along the span of an R.C. beam is given by S/a. Now if no slip occurs between the steel and the surrounding concrete, this must be balanced by the bond or adhesion between the steel and the concrete.

$$\therefore \quad s_b \times O = S/a$$

and

$$s_b = S/(O \times a),$$

where $s_b =$ bond stress,

and $O =$ perimeter of the bar or bars.

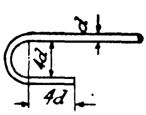

FIG. 2.23

Values for s_b are based upon the basic bending compressive stress C.

Since reinforcing steel is normally used in tension, it is necessary to anchor main bars at their ends by hooking them over other bars or by providing a length of bar to develop the tensile strength in bond.

If

$T =$ tensile force $= \frac{1}{4}\pi d^2 t$ (using the same symbols as before),

then the anchorage length is given by

$$s_b \times O \times l = \tfrac{1}{4}\pi d^2 t,$$

or

$$l = \frac{\pi d^2 t}{4 \times O \times s_b} = \frac{dt}{4 s_b}.$$

For $t = 18{,}000$ lb./in.^{2} and $s_b = 100$ lb./in.^{2},

$$l = 45d.$$

It is usual to provide a hook of inside diameter = $4d$ in addition to the length given by the above formula. In the case of bent-up bars the length assumed to act as anchorage is measured from the N.A. (Fig. 2.23).

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CHAPTER III

STRUCTURAL ANALYSIS: CANTILEVERS AND SIMPLY SUPPORTED BEAMS

Cantilevers are beams or girders which are free at one end and fixed or built-in at the other. The tangent to the elastic line, i.e. the line representing the N.A., must be horizontal at the fixed end.

Case 1. Concentrated load W at free end (Fig. 3.1) of span l. The

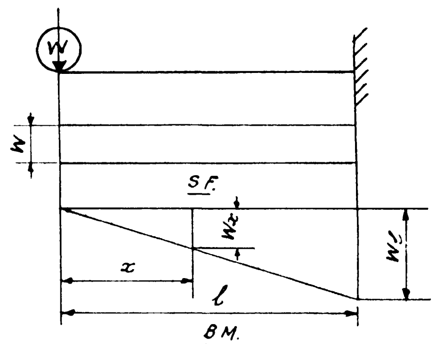

FIG. 3.1

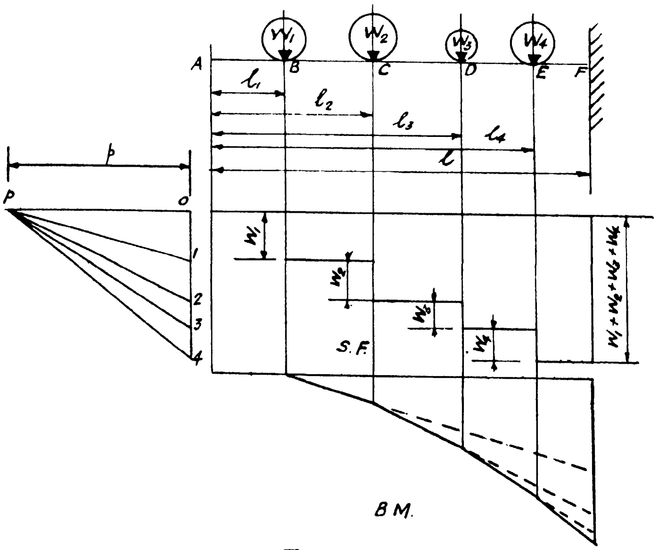

FIG. 3.2

shear at any point is W and the B.M. increases from zero at the free end to a maximum of Wl at the fixed end.

Case 2. Concentrated loads W_1, W_2, W_3, and W_4 at distances l_1, l_2, l_3, and l_4 respectively from the free end (Fig. 3.2).

$$\text{Shear between}\begin{cases} A \text{ and } B = 0, \\ B \;\;,, \;\; C = W_1, \\ C \;\;,, \;\; D = W_1+W_2, \\ D \;\;,, \;\; E = W_1+W_2+W_3, \\ E \;\;,, \;\; F = W_1+W_2+W_3+W_4. \end{cases}$$

$$\text{B.M.}\begin{cases} \text{from } A \text{ to } B = 0, \\ \text{at } C = W_1(l_2-l_1), \\ \text{at } D = W_1(l_3-l_1)+W_2(l_3-l_2), \\ \text{at } E = W_1(l_4-l_1)+W_2(l_4-l_2)+W_3(l_4-l_3), \\ \text{at fixed end} = W_1(l-l_1)+W_2(l-l_2)+W_3(l-l_3)+ \\ \qquad\qquad +W_4(l-l_4). \end{cases}$$

Graphical Method

Set down the span to scale with the loads spaced correctly. On a vertical vector set off the loads to any convenient force scale. Then if the loads are represented by 0–1, 1–2, 2–3, and 3–4, project the loads down vertically to meet horizontal lines from the points 0, 1, 2, 3, and 4 to obtain the S.F. diagram. To draw the B.M. diagram, select a pole P at a horizontal distance p in. from the vector line and draw the rays P–0, P–1, P–2, P–3, and P–4. In space BC draw a line parallel to ray P–1, and from the intersection of this line with the vertical line through C draw a line parallel to ray P–2 and so on, until the diagram is completed. The scale for B.M. = scale of space $\times$ scale of force $\times$ polar distance, e.g. if one inch represents 4 ft., one inch represents 5 tons, and polar distance = 4 in., then one inch on B.M. diagram represents

$$4\times5\times4 = 80 \text{ ft.-tons.}$$

Case 3. Uniformly distributed load w per ft. run and $W = wl$ (Fig. 3.3).

$$\text{Shear at } x \text{ from free end} = wx,$$

$$\text{B.M. at } x \text{ from free end} = \tfrac{1}{2}wx^2.$$

Maximum values when $x = l$, i.e. at fixed end, are:

$$\text{shear} = wl = W; \qquad \text{B.M.} = \tfrac{1}{2}wl^2 = \tfrac{1}{2}Wl.$$

Case 4. Triangular loading increasing in intensity from zero at free end up to a maximum of w at the fixed end (Fig. 3.4). The intensity of loading at distance x from the free end is wx/l and the shear at that point $= wx^2/2l$. The B.M. at x

$$= \frac{wx^2}{2l}\times\frac{x}{3} = \frac{wx^3}{6l}.$$

The maximum values at the fixed end are:

$$\text{shear } \tfrac{1}{2}wl = W \quad \text{and} \quad \text{B.M.} = \tfrac{1}{6}wl^2 = \tfrac{1}{3}Wl.$$

Any combination of the loadings in Cases 1–4 can be dealt with by treating each component loading separately and then adding the values

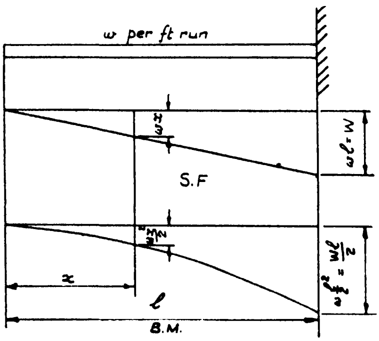

FIG. 3.3

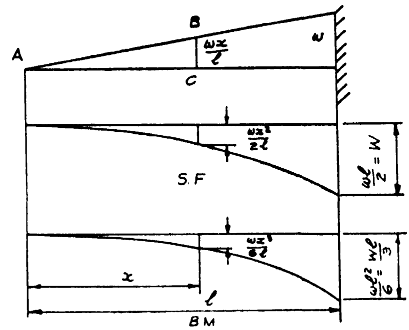

FIG. 3.4

of shear and B.M. at points along the span algebraically to obtain the combined S.F. and B.M. diagrams.

Sign Convention for Shear and B.M.

Shear { Upward shear to the left—positive
,, ,, right—negative.

B.M. { Bending causing tension below N.A.—positive
,, ,, above N.A.—negative.

It will be noticed that the cantilevers dealt with in Cases 1–4 are all subject to negative B.M.

Simply supported beams are freely supported on two supports, generally at each end, and are free from restraining moments. The supporting forces are called reactions, which may have suffixes to denote at which end they occur. By the conditions for equilibrium stated in Chapter II, the sum of the reactions must be equal to the sum of the loads and the B.M. at each support must be zero (except in the case of overhanging ends).

Case 1. Concentrated load W at distance a from R_L (Fig. 3.5 (a)).

Then $$R_L = Wb/l \quad \text{and} \quad R_R = Wa/l$$

and $$R_L + R_R = W.$$

$$\text{Shear from } R_L \text{ to } W = +Wb/l,$$

$$\text{,, ,, } W \text{ to } R_R = -Wa/l.$$

$$\text{B.M. at } x \text{ from } R_L = R_L \times x = Wbx/l.$$

When $x = a$, i.e. at the load, B.M. $= Wab/l$.

When $x = a = \frac{1}{2}l$, i.e. load is at mid-span,

$$R_L = R_R = \tfrac{1}{2}W \quad \text{and} \quad \text{max. B.M.} = \tfrac{1}{4}Wl \quad \text{(Fig. 3.5 }(b)\text{)}.$$

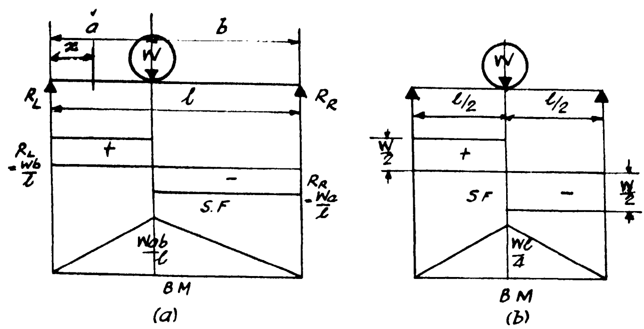

FIG. 3.5

Case 2. Uniform load w per ft. run on span l (Fig. 3.6). Then

$$R_L = R_R = \tfrac{1}{2}wl = \tfrac{1}{2}W$$

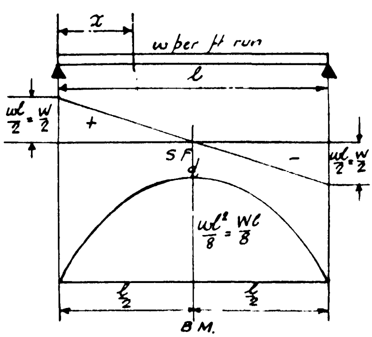

FIG. 3.6

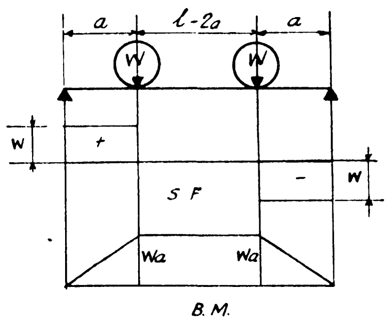

FIG. 3.7

and shear at any point at x from R_L is $R_L - wx$, so that S.F. diagram is a straight line passing through zero at mid-span.

$$\text{B.M. at } x = R_L \times x - \tfrac{1}{2}wx^2 = \tfrac{1}{2}wlx - \tfrac{1}{2}wx^2.$$

Since shear is zero at mid-span, i.e. when $x = \frac{1}{2}l$,

$$\text{max. B.M.} = \tfrac{1}{8}wl^2 = \tfrac{1}{8}Wl$$

and the B.M. diagram is a parabola.

Case 3. Two equal point loads W at equal distances a from the supports (Fig. 3.7).

$$R_L = R_R = W.$$

Shear is constant between the supports and the loads and zero between the loads, so that part of the beam is subject to pure bending. The maximum B.M. is Wa at and between the loads.

Case 4. Loads W_1, W_2, W_3, and W_4 on span l (Fig. 3.8).

Set down the loads to scale on a vertical vector line 0–1–2–3–4. Choose any pole P at a horizontal distance p from the vector line.

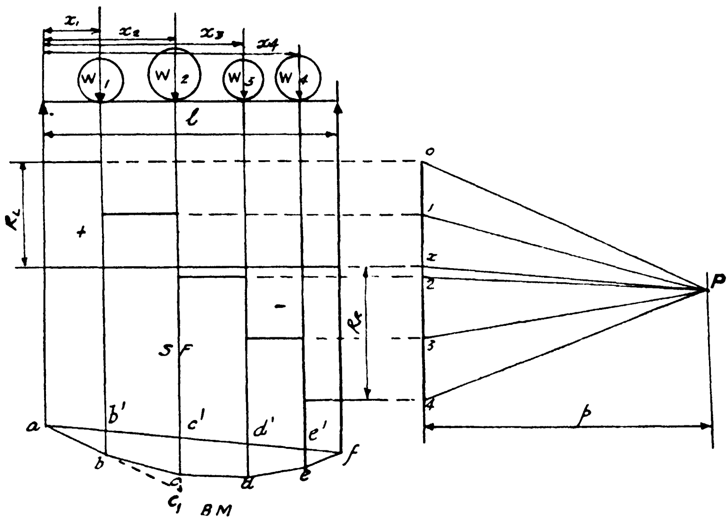

FIG. 3.8

Construct the polygon *abcdef* by drawing lines *ab*, *bc*, etc., respectively parallel to P–0, P–1, etc., and join *af*. Through P draw Px parallel to *af* and project x horizontally, also points 0, 1, 2, 3, and 4 to meet the verticals through W_1, W_2, W_3, and W_4. The resulting diagram is the shear force diagram drawn to the scale chosen for the vector line.

The polygon *abcdef* is the B.M. diagram and the scale for this

$$= p \times \text{space scale} \times \text{scale of vector line}.$$

p should be chosen so as to give a convenient scale. Ox represents R_L and $x4$ represents R_R.

By moments
$$R_R \times l = W_1 x_1 + W_2 x_2 + W_3 x_3 + W_4 x_4,$$
$$R_L = (W_1 + W_2 + W_3 + W_4) - R_R.$$

Shear

From R_L to $W_1 = R_L$,

" W_1 to $W_2 = R_L - W_1$,

" W_2 to $W_3 = R_L - (W_1 + W_2)$,

" W_3 to $W_4 = R_L - (W_1 + W_2 + W_3)$,

" W_4 to $R_R = R_L - (W_1 + W_2 + W_3 + W_4) = R_R$.

B.M.

At $W_1 = R_L x_1$,
,, $W_2 = R_L x_2 - W_1(x_2 - x_1)$,
,, $W_3 = R_L x_3 - \{W_1(x_3 - x_1) + W_2(x_3 - x_2)\}$,
,, $W_4 = R_L x_4 - \{W_1(x_4 - x_1) + W_2(x_4 - x_2) + W_3(x_4 - x_3)\}$,
$= R_R(l - x_4)$.

Proof. By similar triangles

$$\frac{bb'}{x_1} = \frac{Ox}{p}.$$

$$\therefore \quad bb' \times p = Ox \times x_1$$

and $$cc' = c_1 c' - cc_1.$$

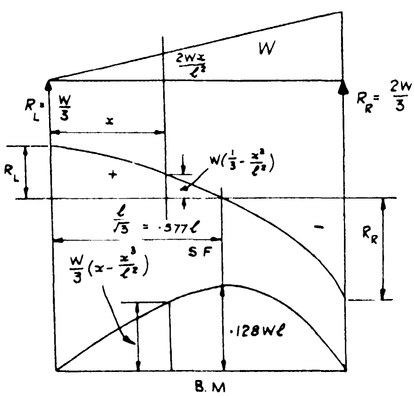

FIG. 3.9

$$\therefore \quad p \times cc' = Ox \times x_2 - W_1(x_2 - x_1)$$

and $$p \times dd' = Ox \times x_3 - \{W_1(x_3 - x_1) + W_2(x_3 - x_2)\}.$$

Also $$p \times ee' = Ox \times x_4 - \{W_1(x_4 - x_1) + W_2(x_4 - x_2) + W_3(x_4 - x_3)\}$$

and $$0 = Ox \times l - \{W_1(l - x_1) + W_2(l - x_2) + W_3(l - x_3) + W_4(l - x_4)\}.$$

$$\therefore \quad Ox \times l = W_1(l - x_1) + W_2(l - x_2) + W_3(l - x_3) + W_4(l - x_4)$$

$$= \text{moments of loads about } R_R.$$

Hence Ox represents R_L and $x4$ represents R_R.

Now $$p \times bb' = Ox \times x_1 = R_L \times x_1 = \text{moment at } W_1$$

and $$p \times ee' = x4(l - x_4) = R_R(l - x_4) = \text{moment at } W_4,$$

and similarly for other points along the span. Therefore the polygon *abcdef* represents the B.M. to scale.

Case 5. Uniformly increasing load, the total load being W (Fig. 3.9). The maximum intensity of loading is $2W/l$ per ft. $R_L = \frac{1}{3}W$ and $R_R = \frac{2}{3}W$.

The intensity of loading at x from $R_L = 2Wx/l^2$

and shear at that point

$$= R_L - \frac{2Wx}{l^2} \times \frac{x}{2} = \frac{W}{3} - \frac{Wx^2}{l^2}.$$

This is zero when $x = l/\sqrt{3}$, which gives the position of the maximum B.M.

$$\text{Max. B.M.} = R_L \times 0{\cdot}577l - \frac{Wl^2}{3l^2} \times \frac{l}{3\sqrt{3}} = 0{\cdot}128Wl.$$

Case 6. Triangular load W, the distance from R_L to the apex of the triangle being a (Fig. 3.10).

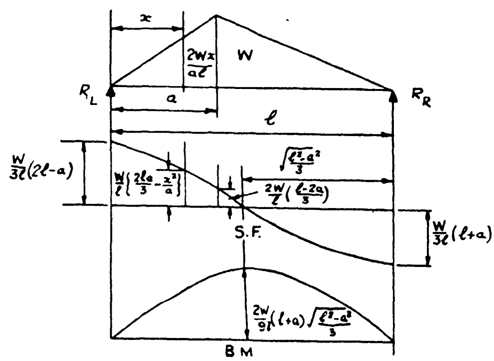

FIG. 3.10

The intensity of loading at the apex $= 2W/l$.

$$R_L \times l = \frac{2W}{l}\left\{(l-a)^2 \times \frac{1}{2} \times \frac{2}{3} + \frac{a}{2}\left(l-a+\frac{a}{3}\right)\right\}$$

$$= \frac{W}{3}(2l-a).$$

$$\therefore \quad R_L = \frac{W}{3l}(2l-a)$$

and $$R_R = W - R_L = \frac{W}{3l}(l+a).$$

The intensity of loading at x from $R_R = \dfrac{2W}{l} \times \dfrac{x}{l-a}$.

For zero shear at x:

$$\frac{2W}{l} \times \frac{x}{l-a} \times \frac{x}{2} = R_R.$$

$$\therefore \quad x = \sqrt{\frac{l^2-a^2}{3}}.$$

$$\text{Max. B.M.} = R_R\sqrt{\left(\frac{l^2-a^2}{3}\right)}-\frac{W}{3l}(l+a)\times\frac{1}{3}\sqrt{\left(\frac{l^2-a^2}{3}\right)}$$
$$= \frac{2W}{9l}(l+a)\sqrt{\left(\frac{l^2-a^2}{3}\right)}.$$

When $a = \frac{1}{2}l$, i.e. apex is at mid-span,

$$R_L = R_R = \tfrac{1}{2}W \quad \text{and} \quad \text{max. B.M.} = \tfrac{1}{6}Wl.$$

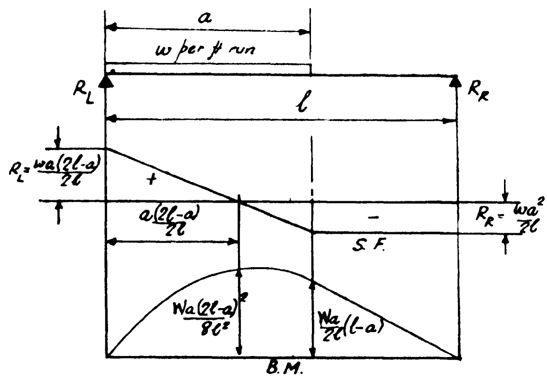

FIG. 3.11

Case 7. Uniform load w per ft. over distance a from R_L (Fig. 3.11).

$$R_R = \frac{wa^2}{2l}; \qquad R_L = \frac{wa(2l-a)}{2l}.$$

$$\text{Distance of point of zero shear from } R_L = \frac{a(2l-a)}{2l}.$$

$$\text{Max. B.M.} = \frac{Wa(2l-a)^2}{8l^2} \qquad (W = wa).$$

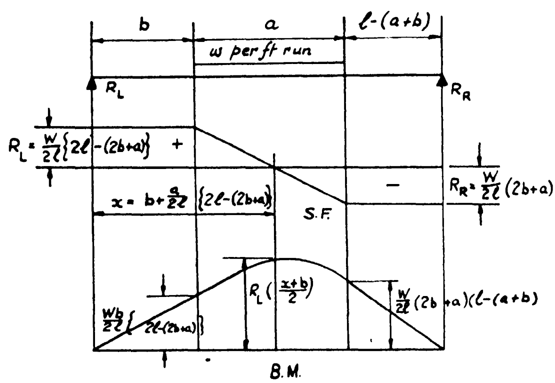

FIG. 3.12

Case 8. Uniform load w per ft. over length a of span l, the distance from R_L to L.H. end of load being b (Fig. 3.12):

$$R_R = \frac{wa}{2l}(2b+a); \qquad R_L = \frac{wa}{2l}(2l-2b-a).$$

Distance of point of zero shear from $R_L = b+\frac{a}{2l}(2l-2b-a)$,

and max. B.M. $= R_L x - R_L\left(\frac{x-b}{2}\right) = \frac{R_L}{2}(x+b)$

$$= \frac{W}{8l^2}(2l-2b-a)(2b+a)(2l-a),$$

where $W = wa$.

Simply Supported Beams with Overhanging Ends

These beams can be treated by dividing them into a cantilever (or cantilevers) and a simple beam, finding the shear and B.M. for each

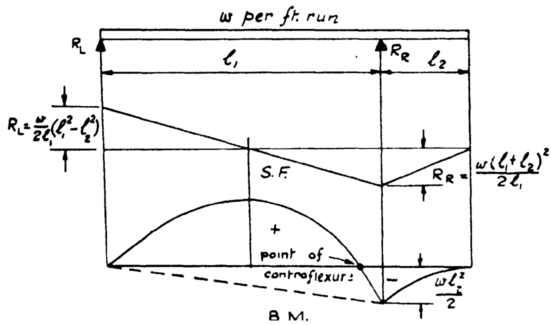

FIG. 3.13

part, and adding the ordinates algebraically. For the case shown in Fig. 3.13, the maximum (negative) B.M. at $R_R = wl_2^2/2$ and the shear is wl_2. For the part between R_L and R_R, the maximum (positive) B.M. at the centre is $wl_1^2/8$ and the shear at R_L and $R_R = wl_1/2$. Combining these values, the S.F. and B.M. diagrams are as shown. It will be noticed that the B.M. diagram crosses the base line near R_R. At this point the B.M. changes sign and this is therefore called a 'point of contraflexure'. Shear at $R_L = wl_1/2$ and shear at $R_R = \frac{wl_1}{2}+wl_2$.

By moments,

$$R_R = \frac{w(l_1+l_2)^2}{2l_1},$$

$$R_L = \frac{w}{2l_1}(l_1^2-l_2^2),$$

and $$R_R+R_L = \text{total load}.$$

The values found by combining the component parts were

$$R_R = \tfrac{1}{2}wl_1+wl_2; \qquad R_L = \tfrac{1}{2}wl_1,$$

the difference in each case being $\pm wl_2^2/2l_1$. Now B.M. at $R_L = 0$ and B.M. at $R_R = -wl_2^2/2$.

$\therefore$ Difference in B.M. $= \pm wl_2^2/2$.

Dividing this by the distance l_1 between R_L and R_R, we get the amount $wl_2^2/2l_1$ and hence the rule for such beams; the reactions must be adjusted by an amount equal to the difference in B.M. divided by

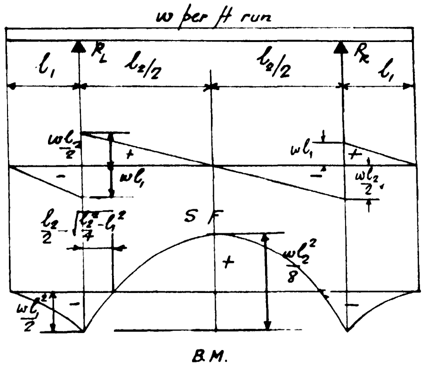

FIG. 3.14

the distance between the reactions. This will be dealt with more fully in Chapter V.

For the case of a beam overhanging both supports by an equal amount l_1 (Fig. 3.14), the central span being l_2, the values for the cantilevers are:

$$\text{shear} = wl_1; \qquad \text{B.M.} = -wl_1^2/2.$$

The values for the central part are:

$$\text{shear} = \frac{wl_2}{2}; \qquad \text{B.M.} = \frac{wl_2^2}{8}.$$

Since there is no difference in B.M. at the supports, no adjustment of the reactions is necessary and the value of these is $wl_1 + wl_2/2$. There are two points of contraflexure equidistant from the ends. The distance from the supports is x, where

$$\frac{(\frac{1}{2}l_2 - x)^2}{(\frac{1}{2}l_2)^2} = \frac{w(\frac{1}{8}l_2^2 - \frac{1}{2}l_1^2)}{w(\frac{1}{8}l_2^2)}.$$

$$\therefore \quad x = \frac{l_2}{2} - \sqrt{\left(\frac{l_2^2}{4} - l_1^2\right)}.$$

Greatest economy is obtained where the net B.M. at the centre is equal numerically to the support moments. Let the central span be l and the cantilevers be nl (Fig. 3.14 (a)).

Then support moment $= -\frac{1}{2}wn^2l^2$;

$$\left.\begin{aligned} \text{Positive moment} &= \tfrac{1}{8}wl^2; \\ \text{Net moment} &= \frac{wl^2}{8} - \frac{wn^2l^2}{2} \end{aligned}\right\} \text{ at centre.}$$

When
$$\frac{wl^2}{8} - \frac{wn^2l^2}{2} = \frac{wn^2l^2}{2} \text{ numerically,}$$

$$n^2 = \tfrac{1}{8} \quad \text{or} \quad n = 0{\cdot}353.$$

$$\text{Total length} = 1{\cdot}706l.$$

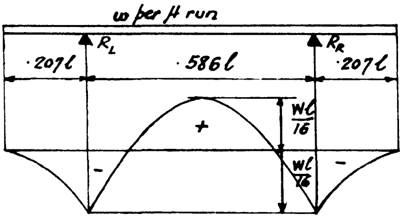

FIG. 3.14(*a*)

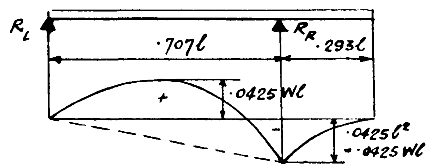

FIG. 3.14(*b*)

Hence we derive the rule that the best method of lifting or slinging is when the supports are at approximately $\frac{1}{5}$ length from the ends.

$$\text{Max. B.M.} = 0{\cdot}0625Wl \quad (W = wl).$$

For a uniformly loaded beam overhanging one end only (Fig. 3.14 (*b*)):

$$\text{Positive B.M.} = \tfrac{1}{8}wl^2; \quad \text{negative B.M.} = -\tfrac{1}{2}wn^2l^2,$$

$$\text{Effective B.M.} = \frac{wl^2}{8} - \frac{1}{2} \times \frac{wn^2l^2}{2} = \frac{wn^2l^2}{2},$$

$$\therefore \quad n = 0{\cdot}41 \text{ approx.}$$

$$\text{Total length} = 1{\cdot}41l, \text{ and ratio } \frac{0{\cdot}41}{1{\cdot}41} = 0{\cdot}29.$$

Graphical Construction for Beams with Overhanging Ends

W_1, W_2, W_3, and W_4 are point loads on a beam overhanging each support (Fig. 3.15). Set down a vector line 0–1–2–3–4 representing the loads to a suitable scale. Project the points 0, 1, 2, etc., across and set off a pole P at a convenient polar distance p in. Draw ab parallel to P–0 in the space between R_L and W_1, then bc parallel to P–1 in the next space, and so on. Join a to the last point f and obtain the polygon $abcdef$ representing the B.M. to scale. Draw Px parallel to af, then P–x represents R_L and x–4 represents R_R. The S.F. diagram can be completed by projecting the x across horizontally.

Example 1. A simply supported girder 120 ft. long is loaded as shown in Fig. 3.16. Draw the S.F. and B.M. diagrams.

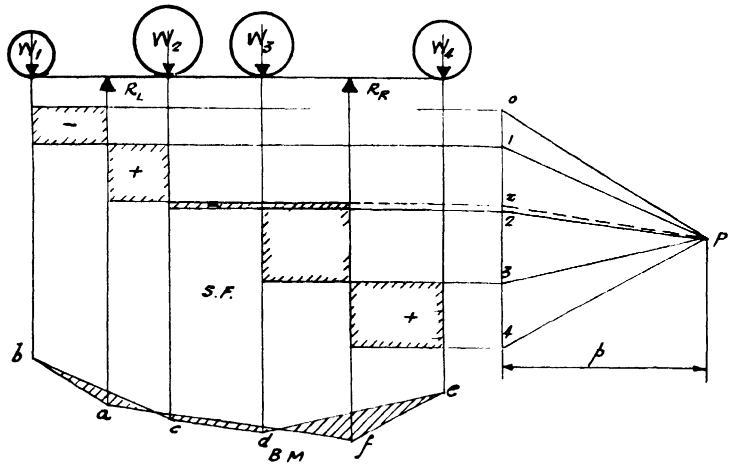

FIG. 3.15

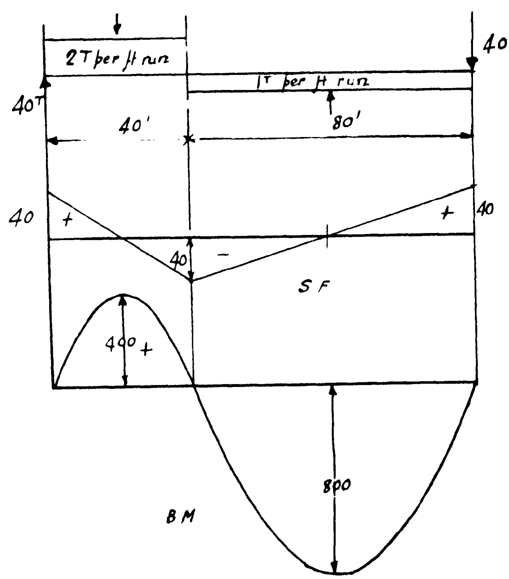

FIG. 3.16

By moments $120 \times R_L = 80 \times 100 - 80 \times 40.$

$\therefore \quad R_L = 40$ tons and $R_R = -40$ tons.

Points of zero shear are at 20 ft. from R_L and 40 ft. from R_R.

Max. positive B.M. $= 40 \times 20 - 40 \times 10 = 400$ ft.-tons.

,, negative B.M. $= 40 \times 40 - 40 \times 20 = 800$,,

and S.F. and B.M. diagrams are as shown.

Example 2. A beam 40 ft. long overhangs each support by 10 ft. and carries a uniformly distributed load of 1 ton per ft. run. Draw S.F. and B.M. diagrams.

$$R_L = R_R = 20 \text{ tons.}$$

Shear at support due to cantilever = 10 tons.

$$\text{Negative B.M. at support} = 1\times\frac{10^2}{2} = 50 \text{ ft.-tons.}$$

$$\text{Max. positive B.M.} = 1\times\frac{20^2}{8} = 50 \quad ,,$$

$\therefore$ Residual B.M. at mid-span is zero.

S.F. and B.M. diagrams are as shown in Fig. 3.17.

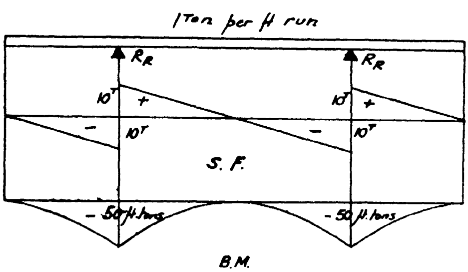

Fig. 3.17

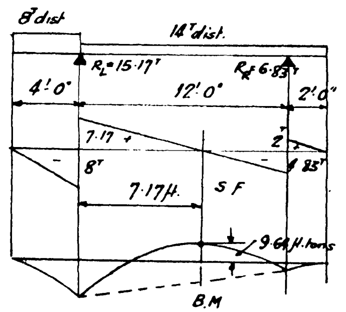

Fig. 3.18

Example 3. A beam 18 ft. long is supported and loaded as shown in Fig. 3.18. Draw S.F. and B.M. diagrams and find the maximum values of the shear and B.M.

$$\text{Total load on beam} = 8+14 = 22 \text{ tons.}$$

$$R_R\times 12 = 14\times 7 - 8\times 2.$$

$$\therefore \quad R_R = 6{\cdot}83 \text{ tons} \quad \text{and} \quad R_L = 15{\cdot}17 \text{ tons.}$$

$$\text{Max. negative B.M. at } R_L = 8\times 2 = 16 \text{ ft.-tons.}$$

$$,, \quad ,, \quad ,, \quad R_R = 2\times 1 = 2 \quad ,,$$

$$\left.\begin{array}{l}\text{Max. positive B.M.} = \dfrac{1\times 12^2}{8} = 18 \text{ ft.-tons} \\ \text{Residual positive B.M.} = 18 - \left(\dfrac{16+2}{2}\right) = 9 \quad ,, \end{array}\right\} \text{at centre of 12-ft. span.}$$

Distance of point of zero shear from $R_L = 15{\cdot}17 - 8 = 7{\cdot}17$ ft.

$$\therefore \quad \text{Max. positive B.M.} = -(8\times 9{\cdot}17) - \left(1\times\frac{7{\cdot}17^2}{2}\right) + 15{\cdot}17\times 7{\cdot}17.$$

$$= 9{\cdot}64 \text{ ft.-tons.}$$

Example 4. A beam is loaded as shown in Fig. 3.18 (*a*) and has a pin at 5 ft. from the L.H. end which is built in. Find the reactions and draw the S.F. and B.M. diagrams.

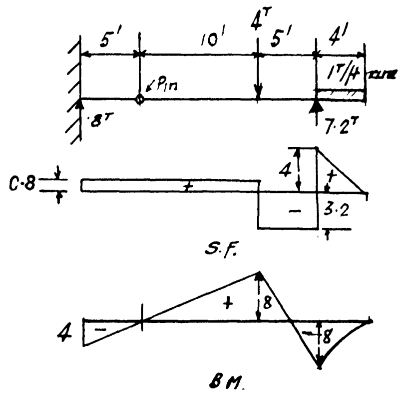

FIG. 3.18 (*a*)

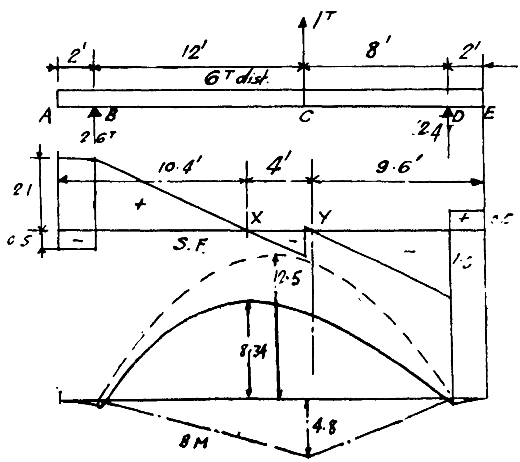

FIG. 3.19

Since B.M. is zero at the pin, by taking moments about this point

$$15 \times R_R = 4 \times 10 + 4 \times 17.$$

$$\therefore \quad R_R = 7{\cdot}2 \text{ tons}; \qquad R_L = 0{\cdot}8 \text{ tons}.$$

B.M. at R_R = −8 ft.-tons,

,, under load = +8 ,,

,, at R_L = −4 ,,

and diagrams are as shown.

Example 5. A beam *ABCDE* is loaded and supported as shown in Fig. 3.19. Draw S.F. and B.M. diagrams and find the maximum B.M.

By moments about D: $R_B \times 20 = 6\times 10 - 1\times 8 = 52.$

,, ,, B: $R_D \times 20 = 6\times 10 - 1\times 12 = 48.$

$R_B = 2{\cdot}6$ tons; $R_D = 2{\cdot}4$ tons.

Shear force. At $B = -0{\cdot}5$ tons and $+2{\cdot}1$ tons.

,, $C = -0{\cdot}9$,, ,, $+0{\cdot}1$,,

,, $D = -1{\cdot}9$,, ,, $+0{\cdot}5$,,

Distance of point of zero shear (X) from $A = \dfrac{2{\cdot}1}{0{\cdot}25}+2 = 10{\cdot}4$ ft.

,, ,, ,, (Y) ,, $E = \dfrac{1{\cdot}9}{0{\cdot}25}+2 = 9{\cdot}6$,,

$$\text{B.M. at } X = 2{\cdot}6\times 8{\cdot}4 - \frac{0{\cdot}25\times 10{\cdot}4^2}{2} = 8{\cdot}34 \text{ ft.-tons (max.)}$$

$$,, \quad Y = 2{\cdot}4\times 7{\cdot}6 - \frac{0{\cdot}25\times 9{\cdot}6^2}{2} = 6{\cdot}75 \quad ,,$$

$$,, \quad C = 2{\cdot}6\times 12 - \frac{0{\cdot}25\times 14^2}{2} = 6{\cdot}70 \quad ,,$$

$$\text{Negative B.M. at } B \text{ and } E = \frac{0{\cdot}25\times 2^2}{2} = 0{\cdot}5 \text{ ft.-tons.}$$

The dotted and chain dotted lines on the diagrams represent the B.M. due to distributed and point loading respectively. The points of contraflexure are X, Y.

$$\frac{(AX)^2\times 0{\cdot}25}{2} = 2{\cdot}6(AX-2). \qquad \therefore \quad AX = 2{\cdot}2 \text{ ft.}$$

$$\frac{(EY)^2\times 0{\cdot}25}{2} = 2{\cdot}4(EY-2). \qquad \therefore \quad EY = 2{\cdot}3 \text{ ft.}$$

In all the preceding examples the loads on the span have been fixed in position. It is now necessary to define dead, superimposed, and live loads and to consider the effects of the latter.

Dead load is the self-weight of a structure, e.g. in the case of a bridge the weight of the girders, flooring, rails or road surfacing, and in the case of a steel-framed or R.C. building, the weight of the structural framework, floors, walls, etc.

Superimposed load (for buildings). In a warehouse this represents the loads due to the materials stored therein. For office blocks the superload is the load due to furniture, fittings, and occupants. The superimposed loads for various classes of buildings are laid down in the Codes of Practice, etc., referred to in the Bibliography. In practice superimposed loads are treated as dead load, but with the exception that they should be applied so as to produce the maximum shear and B.M. It may be found, for instance, that the worst conditions occur when all spans are loaded with dead load and superimposed load is applied to some of the spans (for continuous spans).

Live loads are those which travel across a span, e.g. the vehicular traffic on rail and road bridges and the loads due to end carriages of cranes acting upon gantry girders. The loading on the span varies during the passage of the wheels and the effect of this must be investigated. In addition to the effect of the wheel loads crossing the span, there are certain effects due to impact which will be described later in this chapter.

Influence Lines

Many problems of live or rolling loads can be solved by the use of influence lines. An *influence line* is a curve representing the variation in shear, B.M., or any other result of the loading *at one particular point* as the load crosses the span. The equation to the curve is derived from the value of the function of the loading (i.e. shear, B.M., etc.) due to *unit* loading travelling along the span. It is usually convenient to write the function in terms of x, the distance from one support.

In a bridge, where the load is first applied to the stringers or rail bearers and then transferred through the cross girders to the main girders, the value of any function for the latter must vary at a uniform rate as the load moves from one joint to another, and the influence line will be a straight line between the panel points. It is usually sufficient to calculate the ordinates of the influence line at two or three critical points. Influence lines are useful for showing the effect of the moving load and can be used for finding the forces in the members.

Method of Use. First draw the influence line and find the position of the loads which will produce the worst condition. Although mathematical analysis may be more direct in simple cases, the use of influence line gives a better appreciation of the effect of rolling loads, and is certainly the best method for complex loadings or structures.

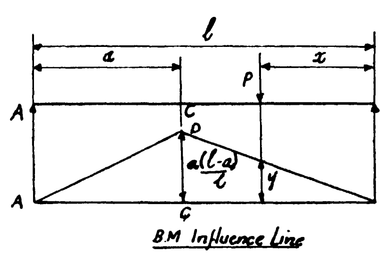

FIG. 3.20

Case 1. Influence line for B.M. at any point due to a single rolling load (Fig. 3.20). C is a point on the span at a from A. Place unit load P at distance x from B. When P is to the right of C,

$$M_c = \frac{Px}{l} \times a = \frac{ax}{l} \times P, \quad \text{but } P = 1.$$

Therefore the equation becomes $M_c = ax/l$, which is the equation to a straight line. The maximum ordinate is at C and equals $a(l-a)/l$.

Case 2. Several concentrated loads on the span. Referring to Fig. 3.20, the moment at C due to any load P at any point is $P \times y$, where y is the ordinate at that point (since y represents the moment due to

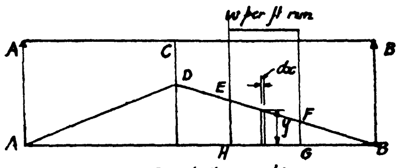

FIG. 3.20 (*a*)

unit load at that point). Therefore total moment $= \sum Py$, or B.M.$_c$ due to *any number of loads* is given by the sum of the *products of the loads multiplied by the respective influence line ordinates.*

In practice a useful method is to draw the influence line and then to set down the spacing of the loads to the same scale used for space on the influence line on tracing-paper. Then by trial and error the maximum value of $\sum Py$ can be found, which gives the maximum B.M. on the span due to the applied loading.

Case 3. Influence line for uniformly distributed load over part of the span (Fig. 3.20 (*a*)). If the intensity of loading $= w$ per ft. run,

$$M_c = \text{area of part of influence line between end ordinates} \times w.$$

Since the moment due to the element $w\,dx$ is given by $w\,dx \times y$,

$$\text{total moment} = \int_{x_1}^{x_2} wy\,dx = w \int_{x_1}^{x_2} y\,dx$$

$$= w \times \text{area of element},$$

and moment due to load $= w \times$ area $EFGH$.

(When the uniform load is longer than the span,

$$M_c = w \times \text{area } ADB.)$$

To find the position of rolling loads for maximum B.M. at any point C.

Case 1. Single concentrated load. M_c due to any load P at $x = Py$. Hence this is a maximum for the load at C.

Case 2. Uniform load. The maximum B.M. at C evidently occurs when the load covers the whole span and is given by

$$M_c = \text{area } ADB \times w$$

$$= \frac{1}{2} \times \frac{a(l-a)}{l} \times l \times w = \frac{wa}{2}(l-a).$$

Case 3. Two equal loads P at distance d apart (Fig. 3.21).

$$M_c = P(y+y'), \quad \text{where } y,\, y' \text{ are ordinates.}$$

Hence M_c is a maximum when $(y+y')$ is a maximum.

$$\text{Max. } M_c = P\left\{\frac{a(l-a)}{l}+\frac{a(l-a)}{l}\times\frac{l-a-d}{l-a}\right\}$$

$$= \frac{P}{l}\{2a(l-a)-ad\}.$$

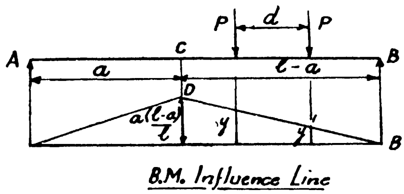

FIG. 3.21

Case 4. Any number of concentrated loads (Fig. 3.22). Let

G_1 = sum of all loads between A and C for any position of the loads,

G_2 = sum of all loads between C and B.

G_1, G_2 are acting at the centroids of the load groups. Since G_1, G_2 have the same effect on the reaction at B as the loads they replace, they

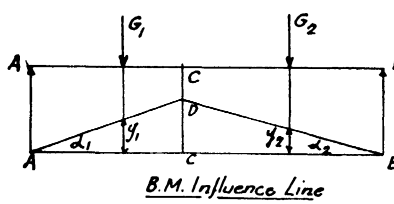

FIG. 3.22

have the same effect on the moment at C. As the loads move to the left across the span, the change in the moment at C is given by

$$dM = G_2\times dy_2 - G_1\times dy_1$$

and
$$\frac{dM}{dx} = G_2\times\frac{dy_2}{dx} - G_1\times\frac{dy_1}{dx},$$

which can be written as

$$\frac{dM}{dx} = G_2\tan\alpha_2 - G_1\tan\alpha_1 \quad \left(\frac{dy_2}{dx} = \tan\alpha_2,\ \frac{dy_1}{dx} = \tan\alpha_1\right)$$

$$= CD\left\{\frac{G_2}{CB}-\frac{G_1}{AC}\right\}.$$

For maximum value of M_c this expression must be zero as the moment changes sign. The values G_1, G_2 can change only as the load passes the point C. A load passing C increases G_1 and diminishes G_2, and therefore the condition for maximum value of M_c is

$$\frac{G_2}{CB} = \frac{G_1}{AC},$$

i.e.

$$\frac{G_1+G_2}{CA+CB} = \frac{G_1}{AC}$$

or

$$\frac{G}{AB} = \frac{G_1}{AC} \qquad (G = G_1+G_2).$$

That is, the maximum B.M. at any point on a loaded span occurs when the *average unit load* to the left of the point in question is equal to the *average unit load* over the *whole span*. The unit of length is the foot for simple girders and the panel for lattice girders. For any given loading there are generally two or three positions which satisfy this condition, and each must be investigated to find the maximum B.M. for the particular point on the span.

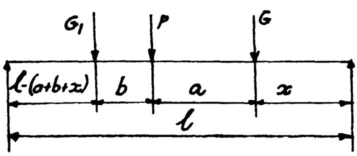

FIG. 3.23

Case 5. Maximum B.M. for a series of point loads (Fig. 3.23). Let P be any load, G the total load on the span, and G_1 the sum of the loads to the left of P. a, b are distances independent of the position of the loads *provided* that *all* the loads are on the span.

$$\begin{aligned}\text{Moment under } P &= R_L(l-x-a)-G_1\times b\\ &= (Gx/l)(l-x-a)-G_1 b\\ &= (G/l)(lx-x^2-ax)-G_1 b.\end{aligned}$$

Now $\quad dM/dx = 0, \quad$ when $\quad l-2x-a = 0,$

or $\quad x = l-x-a.$

Therefore maximum B.M. under a load occurs when the load and the C.G. of the whole load system are equidistant from the ends of the span.

Influence Lines for Shear

Case 1. Shear influence line for any point on a span (Fig. 3.24). The positive shear due to unit load, moving from B to A, increases from zero (for load at B) to a maximum of $(l-a)/l$ at point C. The shear at C due to a load W is Wy. As a load passes C the shear suddenly becomes $-a/l$ and then decreases to zero when the load reaches A.

If there are several loads on the span, a movement to the left increases the positive shear until some load P passes C, when the shear is suddenly decreased by P. For concentrated loads, the shearing force reaches a critical value each time a load reaches C, and the maximum shear occurs when one of the loads near the head of the train is just to the right of C.

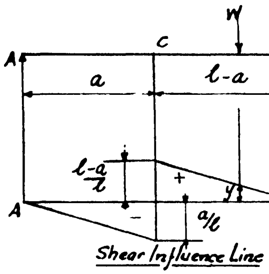

FIG. 3.24

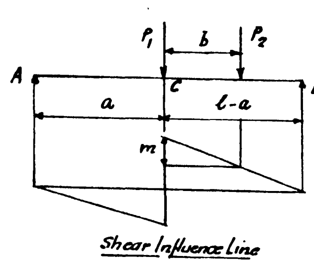

FIG. 3.24 (a)

Case 2. To find which of two loads P_1 and P_2 causes the greater shear when placed just to the right of c (Fig. 3.24 (a)). Let

b = centres of loads P_1 and P_2,

G = total load on span (when P_1 is at c).

Now let loads advance a distance b to the left so that P_2 is at C. The shear at C is suddenly decreased and then gradually increased by $G \times m$. But

$$\frac{m}{b} = \frac{1}{l}.$$

$$\therefore \quad \text{increase} = G \times \frac{b}{l} - P_1.$$

If this is positive, P_2 at C gives the greater shear.

,, ,, negative, P_1 ,, ,, ,, ,,

If $$G \times \frac{b}{l} = P_1 \quad \text{or} \quad \frac{G}{l} = \frac{P_1}{b},$$

we have equal shears with either P_1 or P_2 at C (neglecting the slight increase in shear at C due to additional loads coming on the span at the right).

If G_1 = total load on span for P_2 at C, then increase in shear lies between

$$G\frac{b}{l} - P_1 \quad \text{and} \quad \frac{G_1 b}{l} - P_1.$$

When the first expression is negative and the second is positive, both positions should be tried for maximum shear (this will occur for a short distance, to the left of which both expressions are positive and to the right of which both are negative).

Case 3. Influence line for uniform loading over length shorter than the span (Fig. 3.24 (*b*)). The shear at P is represented by the shaded area $\times$ intensity of loading w. When the head of the load reaches P, the area and the shear will be a maximum. When the head of the load

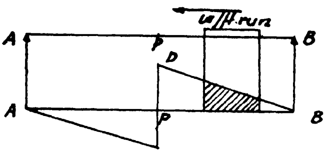

FIG. 3.24 (*b*)

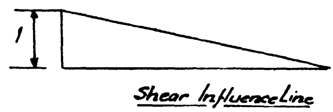

FIG. 3.24 (*c*)

passes P, the shear will decrease, as the portion of the diagram to the left is of opposite sign.

Case 4. Influence line for uniform load longer than the span. The maximum value occurs when the head of the load reaches P as before and is represented by the area $PBD \times w$.

It is common practice to calculate the shear at the quarter points (or panel points for trussed girders).

Case 5. Influence line for shear at the supports (Fig. 3.24 (*c*)). The influence line is a triangle of height unity, and the shear due to a series

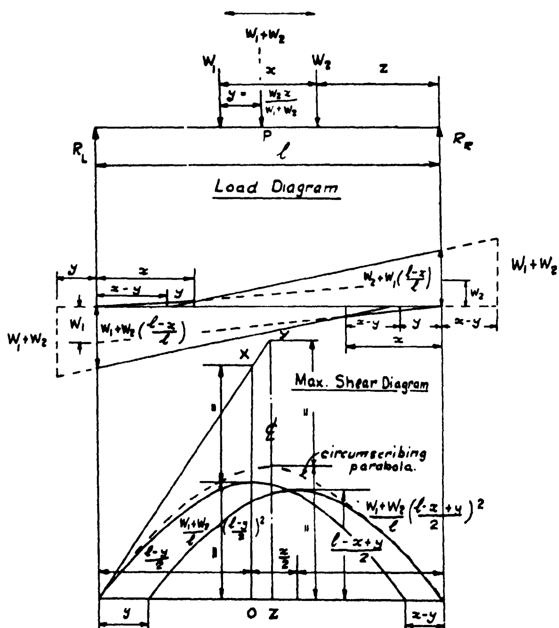

FIG. 3.25

of concentrated loads $= \sum Wy$. For a uniform load, maximum shear when the load covers the span $= wl/2$.

In all the preceding cases, the influence lines have been drawn for a *certain point*. It is often necessary to find the maximum values of shear and B.M. for a number of points along the span.

Case 1. For a single load W, maximum B.M. $= Wl/4$ and maximum shear $= W/2$.

Case 2. Two loads W_1 and W_2 at distance x apart on span l (Fig. 3.25). C.G. of the loads is at P, where

$$y = \frac{W_2 x}{W_1 + W_2}.$$

Distance of W_2 from $R_R = z$.

$$\therefore \quad R_L = \frac{(W_1+W_2)(z+x-y)}{l}$$

and

$$R_R = (W_1+W_2)\left(\frac{l-z-x+y}{l}\right).$$

$$\therefore \quad \text{shear between } R_L \text{ and } W_1 = R_L.$$

This is a maximum when z is a maximum (since $x =$ constant). The maximum value of $z = l-x$ and the maximum shear at L.H. end as load travels from right to left

$$= W_1 + W_2\left(\frac{l-x}{l}\right).$$

For loads travelling from left to right, the maximum value of R_R occurs when $z = 0$.

Then

$$R_L = \frac{W_1 x}{l}$$

and

$$R_R = W_2 + W_1\left(1 - \frac{x}{l}\right).$$

B.M. under $W_1 = R_L \times (l-x-z)$

$$= \left(\frac{W_1+W_2}{l}\right)(zl-2zx-z^2+xl-x^2-yl+xy+zy).$$

For a maximum value, $\dfrac{dM}{dz} = 0 = l-2x-2z+y$,

or

$$z = \frac{l-2x+y}{2}.$$

Distance from R_L to $W_1 = \dfrac{l-y}{2}$.

" " P to $R_R = \dfrac{l-y}{2}$.

Therefore max. B.M. under W_1 occurs when W_1 and P are equidistant from the supports. The same condition holds for max. B.M. under W_2.

$$\text{Max. B.M. under } W_1 = \frac{W_1+W_2}{l}\left(\frac{l-y}{2}\right)^2 = M_1.$$

$$\text{,, \quad ,, } \quad W_2 = \frac{W_1+W_2}{l}\left(\frac{l-x+y}{2}\right)^2 = M_2.$$

Hence M_1 is greater than M_2.

When $W_1 = W_2 = W$,.

$$M_1 = \frac{2W}{l}\left(\frac{l-y}{2}\right)^2$$

$$= \frac{2W}{l}\left(\frac{l-\frac{1}{2}x}{2}\right)^2, \quad \text{since } y = \tfrac{1}{2}x,$$

$$M_2 = M_1.$$

Example. Crane girder 20 ft. span with two 5-ton wheel loads at 10-ft. centres.

$$\text{Max. shear at support} = 5\left(1+\frac{10}{20}\right) = 7\tfrac{1}{2} \text{ tons.}$$

$$\text{Max. B.M.} = \frac{2\times 5}{20}\left(\frac{20-5}{2}\right)^2 = 28{\cdot}125 \text{ ft.-tons.}$$

Case 3. Load w per ft. of length l travelling across a span L.

$$R_L = \frac{wl}{L}(y-x+\tfrac{1}{2}l),$$

$$R_R = \frac{wl}{L}(L-y+x-\tfrac{1}{2}l).$$

$$M = \text{B.M. at point } C = R_R\times y-\frac{wx^2}{2} = \frac{wly}{L}(L-y+x-\tfrac{1}{2}l)-\frac{wx^2}{2}.$$

M is a maximum when $dM/dx = 0$, i.e. when

$$\frac{wly}{L}-wx = 0 \quad \text{or} \quad \frac{x}{l} = \frac{y}{L},$$

or the load is in such a position that the given point divides the load in the same ratio as it divides the span. Hence

$$M = \frac{W}{L}(Ly-y^2)\left(1-\frac{l}{2L}\right) \quad (W = wl),$$

$$\frac{dM}{dy} = 0. \qquad \therefore \quad L-2y = 0 \qquad \left(y = \frac{L}{2}\right),$$

and
$$M = \frac{WL}{4}-\frac{Wl}{8}.$$

Equivalent distributed load W_1 is such that

$$\frac{W_1 L}{8} = \frac{WL}{4} - \frac{Wl}{8}.$$

$$\therefore \quad W_1 = \frac{2W}{L}\left(L - \frac{l}{2}\right).$$

$$\text{Max. shear at supports} = R_L = R_R = W\left(1 - \frac{l}{2L}\right).$$

Equivalent Distributed Load; Circumscribing Parabola

For convenience in practical design it is usual to find the maximum B.M. due to the rolling loads by means of an equivalent distributed

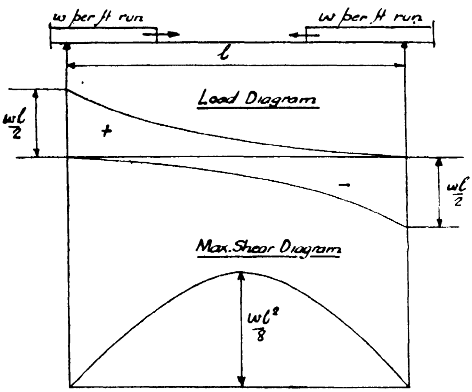

FIG. 3.26

load W so that $Wl/8$ = max. B.M. It is necessary to draw a 'circumscribing parabola' to find the E.D.L. For Case 2 (Fig. 3.25) there are two positions which give maximum B.M. under the two loads. To find the E.D.L. draw a tangent from the L.H. end to the larger parabola by setting off the height $OX = 2 \times$ height of parabola. Produce this tangent to meet the centre line of the span in Y. Draw a parabola with height $= \frac{1}{2} \times YZ$ and base = span. This is the circumscribing parabola or B.M. envelope. If $h = \frac{1}{2} \times YZ$, then

$$\frac{Wl}{8} = h \quad \text{or} \quad W = \frac{8h}{l}.$$

The B.M. envelope must enclose the maximum B.M. diagram for the rolling loads.

For railway bridges B.S.S. 153 gives the values of the E.D.L. and similar values are given for road bridges.

For a uniformly distributed load w per ft. travelling across the span, the maximum shear at any point P is equal to the reaction at the end which the head of the load is approaching (Fig. 3.26).

If

$$x = \text{distance of } P \text{ from the other end,}$$

$$l = \text{span,}$$

then

$$\text{shear at } P = \frac{wx^2}{2l} \quad (\text{when } x = l,\ P = W/2).$$

The maximum B.M. occurs at mid-span and is $WL/8$. For loads shorter than the span an influence line gives the required values.

Combined Dead and Rolling Loads (Fig. 3.26 (*a*))

When the shears due to dead and rolling loads are combined, it may

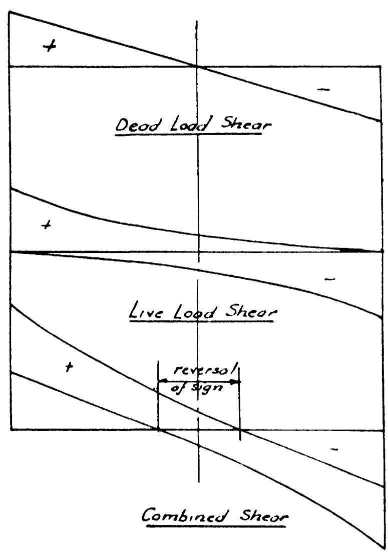

FIG. 3.26 (*a*)

be found that the combined shear changes sign according to the direction of the traffic.

Example 1. A uniform load of $1\frac{1}{2}$ tons per ft. run (longer than the span) rolls across a span of 100 ft., the dead load being $\frac{3}{4}$ ton per ft. run (Fig. 3.26 (*b*)).

Max. live load reaction $= 75$ tons.

,, dead load reaction $= 37{\cdot}5$ tons.

When the head of the load is 10 ft. from the end of the span

$$\text{L.L. reaction} = \frac{1{\cdot}5 \times 90^2}{2 \times 100} = 60{\cdot}75 \text{ tons.}$$

When the head of the load is 20 ft. from the end of the span

L.L. reaction $= 48$ tons, and so on.

From an examination of Fig. 3.26 (*b*) it is found that the dead and live load shears are equal at about 13 ft. from mid-span, so shear force may change sign over the central 26 ft. of the span.

Example 2. A bridge of 100 ft. span carries a dead load of 1 ton per ft. run. The live load can be represented by an E.D.L. of $1\frac{1}{2}$ tons

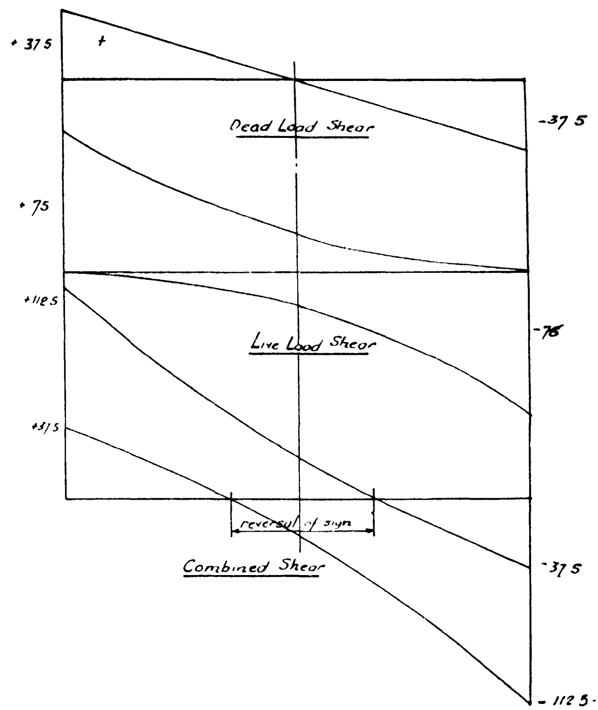

FIG. 3.26 (*b*)

per ft. run. Find the maximum shear on the bridge when the head of the load is 20 ft. from one abutment.

D.L. shear = 50 tons (at abutment).

D.L. shear at 20 ft. from abutment = 50 − 20 = 30 tons.

L.L. ,, ,, ,, $= \dfrac{1{\cdot}5 \times 80^2}{2 \times 100} = 48$,,

Total shear = 78 ,,

Example 3. Draw the influence lines for shear and B.M. for the quarter point of a simply supported span. Hence find the maximum shear and B.M. at the quarter point of a 50-ft. span due to a load of 1 ton per ft. over 20 ft. crossing the span (Fig. 3.27).

Influence line for shear is a triangle of height $\frac{3}{4}$.

$$\therefore \quad \text{shear} = \text{shaded area} \times 1 \text{ ton per ft.}$$

$$= 20 \times \tfrac{1}{2}(0{\cdot}35+0{\cdot}75)\times 1 = 11 \text{ tons.}$$

Influence line for B.M. is a triangle of height $= \dfrac{\frac{1}{4}\times\frac{3}{4}\times 50}{1}$

$$= 50\times\tfrac{3}{16}.$$

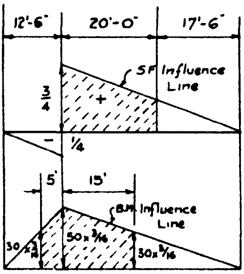

FIG. 3.27

$$\text{B.M.} = \text{shaded area}\times 1 \text{ ton per ft.}$$

$$= 1\times 20\times\tfrac{3}{16}\left(\frac{50+30}{2}\right) = 150 \text{ ft.-tons.}$$

Example 4. A longitudinal girder is supported on columns at 12-ft. centres. Four 6-ton wheel loads at 5-ft. centres roll along the girder. Assuming that the girder is cut so as to be simply supported at the points where it rests on the columns, find the maximum load on a column.

Let the position of the nearest wheel on R.H. span from the column be x, then nearest wheel on L.H. span is $(5-x)$.

$$\text{Then reaction from L.H. span} = \frac{6\{(2+x)+(7+x)\}}{12}$$

$$\text{,, ,, ,, R.H. ,,} = \frac{6\{(7-x)+(12-x)\}}{12}.$$

$$\therefore \quad \text{Total reaction on column} = \frac{6\{9+19\}}{12} = 14 \text{ tons,}$$

and this is independent of the position of the loads so long as these are all on two adjacent spans.

Example 5. A 30-ft. span girder is simply supported. Draw an influence line for B.M. at point A, 10 ft. from L.H. support (Fig. 3.28). Hence find maximum B.M. at A due to a travelling load of 5 tons distributed over a length of 10 ft.

Maximum B.M. at A occurs when the point divides the load in the same ratio as it divides the span.

$$\therefore \quad \text{B.M.} = \text{shaded area}\times\tfrac{1}{2} \text{ ton per ft.}$$

$$= 27{\cdot}78 \text{ ft.-tons.}$$

Example 6. A girder of 16-ft. span is simply supported. Derive from first principles an influence line for shear at A, 4 ft. from L.H. end. If two wheel loads of 8 tons at 6-ft. centres cross the span, find the maximum values of positive and negative shear at A.

Consider unit load at x from R.H. end (Fig. 3.29).

$$R_L = \frac{x}{l}; \text{ positive shear} = R_L; \text{ negative shear} = -\frac{l-x}{l}.$$

Values in this case are $\frac{3}{4}$ and $\frac{1}{4}$ for load at A.

$$\text{Max. positive shear} = (8\times\tfrac{3}{4})+8\times\tfrac{3}{8} = 9 \text{ tons.}$$

$$\text{,, negative ,,} = -\tfrac{1}{4}\times 8 = -2 \text{ tons.}$$

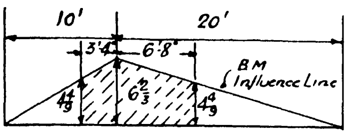

FIG. 3.28

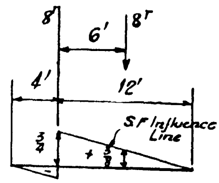

FIG. 3.29

Impact due to Rolling Loads: Dynamic Effects

The dynamic effects of rolling loads have been neglected in the preceding pages and these will now be dealt with. In past years there has been much confusion of ideas on this subject, and the tendency has been generally to overestimate the dynamic effects.

Case 1. Railway underbridges. The effect of a train of loads crossing a bridge is very complex, and the practice of adding a certain percentage to the live load to allow for dynamic effects cannot be justified in the light of modern research. In the early stages of bridge design it was common to double or more than double the live loads, perhaps due to a mistaken analogy with the case of a suddenly applied load. Another type of formula which has been used extensively until comparatively recently was based upon fatigue or range of stress. A typical example is the Launhardt–Weyrauch formula which varies the permissible stress thus

$$f_p = \frac{f}{1\cdot 5}\left\{1+\frac{\text{minimum load}}{2\times\text{maximum load}}\right\},$$

where

f_p = permissible stress for dead+live load,

f = ,, ,, ,, load only.

In actual fact this formula has no connexion with impact, as it is based upon the theory that over a long period a material is less able to withstand the effects of a stress which is applied and then removed very many times than the effects of a uniform stress. It is very doubtful whether this condition occurs in a railway bridge as the dynamic load is applied for short periods and the stress does not reach the elastic limit for the material.

Many empirical formulae have been used to allow for the effects of impact and some of these will be stated very briefly.

(*a*) *Pencoyd formula*

$$I = \frac{300}{300+L}.$$

I = fraction of live load to be added or 'impact factor',

L = span in feet.

This has been widely used and gives very high values for short spans, e.g. $I = 0{\cdot}75$ for $L = 100$.

(*b*) *B.S.S. 153* (1923) *for Girder Bridges*:

$$I = \frac{120}{90+(n+1)/2 \times L} \quad (n = \text{number of tracks}).$$

$$I = 1 \quad \text{for} \quad n = 1 \quad \text{and} \quad L = 30.$$

For rail bearers, etc., I can be greater than unity.

(*c*) *Dr. J. A. L. Waddell*:

$$I = \frac{400}{500+L}.$$

(*d*) *Eyson's formula*:

$$I = \frac{15}{20+L}.$$

(*e*) *Indian Railway Board* (1925):

$$I = \frac{65}{45+L}.$$

(*f*) *A.R.E.A.*:

$$I = \frac{20{,}000}{20{,}000+L^2}.$$

It will be seen that all the foregoing formulae include a function of the span (either linear or quadratic) and that no account is taken of such factors as the peculiarities of the track or vehicles. Therefore such formulae are empirical and not scientific.

The Bridge Stress Committee was constituted to investigate the effects of locomotives and other vehicles on railway bridges and carried out exhaustive tests. Its report, which was published in 1928, gives the results of these tests and the conclusions to be drawn from the analysis thereof. The report covers the work carried out over a period of four years and gives a rational method of finding live loads on any bridge due to the passage of a locomotive or locomotives. The field tests were carried out by a staff of engineers under Mr. Conrad Gribble, O.B.E., M.I.C.E., and the theoretical investigation by Prof. Sir Charles Inglis, M.A., F.R.S., P.P.I.C.E., and his assistants. The report is well worth the study of any bridge engineer and can be summarized as follows.

1. *Effect of hammer blows.* The hammer blow of a locomotive is due to the balancing of the reciprocating parts or the lack of balancing of the revolving parts. This blow varies for different types of locomotive and has no relation to the weight of the locomotive, in fact, it may be much greater for a light than a heavy locomotive. It is less for a three- or four-cylinder locomotive than for a two-cylinder locomotive. In the case of steam locomotives, the report is of opinion that the maximum value of hammer blow should not exceed 5 tons for British railways (when used in conjunction with 20 units of the standard loading) with larger hammer blows where less than 20 units is taken for design. For electric locomotives hammer blow may be entirely absent or be very much less than that for steam locomotives.

The hammer blow is the chief factor causing dynamic effect and is greatly in excess of all other causes for fairly long bridges. For such bridges other causes of impact may be neglected, but for short-span bridges hammer blow should be considered along with other causes of impact.

The hammer blow produces vibrations in a vertical plane due to the periodic impulses as the locomotive travels across the span, and in a long span hammer blow may produce a cumulative effect. This is most marked where the period of the impulses coincides more or less with the free period of vibration of the span. The cumulative effect is limited in practice by the following factors: (1) the number of impulses is limited by the length of the span; (2) the actual vibrations are 'damped' to a certain extent. Damping is due to several causes: (*a*) the span is not perfectly elastic, (*b*) friction at the supports, (*c*) dissipation of energy at the supports, and (*d*) friction in the spring suspension of the locomotive itself. This last factor has the greatest influence in limiting the cumulative effect.

When a locomotive is crossing a span at a slow speed, it acts as if it were not spring-borne, since the vibrations set up in the span are not strong enough to overcome the friction of the spring suspension. At higher speeds, as the vibrations increase, the springs come into operation and then friction helps to damp the vibrations. The effect of damping is most marked when the vibrations are such as to cause violent vibration and in a sense it serves as a factor of safety. (3) The natural or 'free' period of vibration of the span itself varies as the locomotive crosses the span, reaching a maximum when the locomotive reaches mid-span and then decreasing as it approaches the farther abutment. For bridges of moderate spans the variation of the frequency of the span, as the locomotive crosses it, is so marked that it is unlikely that the periods of the hammer blow and of the span will coincide for for any appreciable time and so set up resonance. In addition, the period for large vibrations, when the springs come into operation, is different from those when the load is not spring-borne. The foregoing factors limit the effects of hammer blow, and the conditions for full resonance (i.e. complete agreement of impulse and bridge with regard to period) do not occur in actual service and any synchronism which may occur near mid-span is of a purely transient nature.

For short spans and floor members the free period is shorter than the frequency of the hammer blow which may occur and no cumulative effect is possible. The observed maximum deflexion in such cases is comparable with that due to the axle loads plus a static load equal to the hammer blow.

2. *Effects of irregularities of track and of rail joints.* These effects are most marked in the case of short spans and rail bearers; for long spans they are much less noticeable. The most important causes are: (*a*) Wheels passing over a rail joint drop and then rise and 'batter' the near end of the further rail. This effect varies as the square of the speed. With the present tendency to use long or welded rails this will become much less important. (*b*) Any defect in the permanent way which causes a sudden settlement as the load is applied causes a blow somewhat similar to (*a*). (*c*) Rolling or lurching of the locomotive, which may occur either before or during the time it is crossing the span, produces variation of the load on either rail.

Effects (*a*), (*b*), and (*c*) vary according to the nature of the permanent way and may be 'damped' by the ballast. Effect (*a*) varies according to the product of the axle load and the non-spring part thereof (as well as according to the square of the speed). Lurching causes an addition to, or subtraction from, the load on the rail bearers and main girders. The result can be expressed thus:

$$\text{proportion of load transferred} = \frac{\text{total live load}}{2} \times \frac{160cn}{l+100},$$

where

c = coefficient dependent on the type of spring suspension, weight, and height of rolling stock; also the type of construction and lateral rigidity of the substructure;

n = speed of rolling load (in revolutions per second of the driving wheels of the locomotive);

l = effective span in ft.

For British standard gauge track, $c = \frac{1}{24}$ and $n = 6$ can be used, provided that the structure is adequately stiffened laterally and the maximum value of the proportion of the load should not exceed 0·25. For other railways use $c = \frac{1}{15}$ and a maximum value of 0·40 for the proportion of the load.

Design according to the Bridge Stress Committee Report

In order to simplify the design of railway bridges, certain tables are given in the Report, which embody the test results. To allow for locomotives heavier than those in use at the time of publication of the report, 20 units of the standard loading for railway bridges (B.S.S. 153) are used for main lines and lesser values for secondary and light railways.

Three types of loadings are given, viz.

A:	20 units with	5·0-ton hammer blow	at	5	r.p.s.	($0{\cdot}2n^2$),
B:	16 ,,	12·5 ,,	,,	,,	,,	($0{\cdot}5n^2$),
C:	15 ,,	15·0 ,,	,,	,,	,,	($0{\cdot}6n^2$).

Hammer blow = $(\text{r.p.s.})^2 \times$ factor which varies according to the loading.

Loading A covers the effects of existing three- and four-cylinder locomotives and also provides for future heavier types. The factor is 0·2 owing to the number of cylinders and better balance.

Loading B is intended to cover the effects of somewhat lighter locomotives of the double cylinder type, and the factor is increased to 0·5 to allow for the lesser number of cylinders.

Loading C covers nearly all the remaining types whose hammer blow exceeds 12·5 tons, but such locomotives are comparatively light, and present tendency in design is to reduce hammer blow to a minimum.

Tabulated loads for bending moment and total loads including E.U.D.L. are given in the Appendix to B.S.S. 153. The values are given for speeds of 3, 4·5, and 6 r.p.s., and include *A*, *B*, and *C* loadings. Present practice is to design for *A* or *B*, since the types covered by *C* are obsolescent. Similar tables are given for shear at supports and quarter points. These tables allow for the effect of rail joints but not for lurching, and this must be added by multiplying the 'static' load by coefficients which are:

0·25	for	6	r.p.s.
0·1875	,,	4·5	,,
0·125	,,	3	,,

Cross-girder loadings are also tabulated (these do not allow for effects of lurching which are comparatively small). Rail-bearer loadings include the effects of lurching and rail-joints. The proportion of the live load to be added or subtracted is 25 per cent., i.e. one rail bearer can carry either $\frac{5}{8}$ or $\frac{3}{8}$ of the live load per track.

Alternatively the loadings given in the Appendix to B.S.S. 153 can be taken and increased by such impact allowances as the designer may consider suitable for present and future traffic and the type of structure.

Suggested values for impact allowances are given in Appendix 2, which gives the formulae for general application to cover hammer blow and rail-joint effects, also coefficients depending on the type of bridge. Lurching effect must also be considered.

Impact Effects on Highway Bridges

The effect on such bridges is much less complex owing to the absence of hammer blow and rail joints. There are certain effects due to irregularities in the road surface. Factors are given in B.S.S. 153, viz.

$$I = 0{\cdot}75 - 0{\cdot}002l \text{ (max. } 0{\cdot}60) \text{ for one line of traffic,}$$

and $$I = 0{\cdot}65 - 0{\cdot}002l \text{ (max. } 0{\cdot}50) \text{ for more than one line of traffic.}$$

l = centres of main bearings for main girders, and centres of *adjacent* cross girders for cross girders, *except* in the case of an end cross girder or an intermediate cross girder where there is a break in the continuity of the floor system, in which case l must be taken as the distance between the end of the floor (if overhanging), and the farther cross girder, or for end spans of flooring resting on an abutment or pier, the distance from the cross girder to the centre of the bearing plate. For rail bearers and stringers, l = centres of cross girders. The factor I applies only to live load due to *vehicular* traffic.

Impact Effects on Crane Girders and Gantries

This is a comparatively simple case and can be allowed for by adding a given proportion to the wheel load due to the live load. The Institution of Structural Engineers' Report on *Steelwork for Buildings*, Part I, 9, gives a value of 60 per cent. for electric cranes and 30 per cent. for hand cranes of the lifted load. (The report on Gantry Girders is at present under revision.)

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EXERCISES

1. A simply supported beam has a span of 20 ft. and carries a load of 1 ton per ft. run, also a clockwise moment of 20 ft.-tons applied at mid-span. Find the values of R_L, R_R, and the maximum shear and B.M. [9; 11; 11 tons; 40·5 ft.-tons.]

2. A beam $ABCD$ is freely supported at B and C. The dimensions are $AB = 6$ ft.; $BC = 10$ ft.; $CD = 4$ ft. There is a distributed load of 6 tons on AB, a point load of 10 tons at the centre of span of BC, and an upward force of 3 tons at D. Find the reactions R_B, R_C, and the maximum positive and negative B.M. [14; -1 ton; $+22$ ft.-tons; -18 ft.-tons.]

3. A beam ABC is 12 ft. long. It is fixed in position and direction at A and overhangs B by 2 ft. There is a hinge midway between A and B. If an upward force of 5 tons is applied at C, find the values of the reaction at A and B, also maximum positive and negative B.M. [$+2$; -7 tons; $+10$ ft.-tons; -10 ft.-tons.]

4. The loading of Fig. 3.A crosses a simply supported span of 60 ft. Find (1) maximum shear and B.M. at 20 ft. from L.H. end, (2) maximum B.M., (3) maximum end shear. [(1) 33·6 tons, $-8·9$ tons, 792 ft.-tons; (2) 900 ft.-tons; (3) 59·2 tons.]

5. The loading shown in Fig. 3.B may cross a simple span of 50 ft. in either direction. Find the maximum B.M. at 10 ft. from R.H. support. [204 or 188 ft.-tons.]

6. Fig. 3.C shows the loading crossing a simple span of 50 ft. Find the maximum B.M., also the maximum quarter point B.M., and hence the corresponding equivalent distributed loads per ft. [301; 255 ft.-tons; 0·964 and 1·09 tons/ft.]

7. A girder of 80-ft. span carries the loading shown in Fig. 3.D. Find the maximum quarter-point shear and the equivalent distributed load corresponding to maximum B.M. at quarter point. [25 tons; 72 tons.]

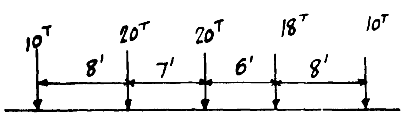

FIG. 3.A

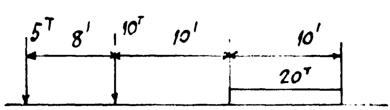

FIG. 3.B

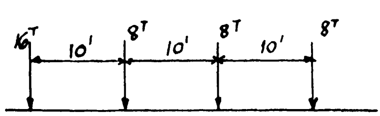

FIG. 3.C

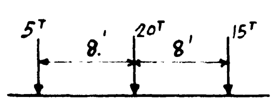

FIG. 3.D

CHAPTER IV

DEFLEXION OF BEAMS, ETC.

CONSIDER any beam bent under the action of an external moment M to a radius of curvature R at a particular point (Fig. 4.1).

ϕ = angle between the tangent to the elastic line at A and the horizontal,

$AB = ds$ which is very small in comparison with R,

$\tan\phi = \dfrac{dy}{dx}$ = slope of elastic line at A,

and $ds = R \times d\phi$.

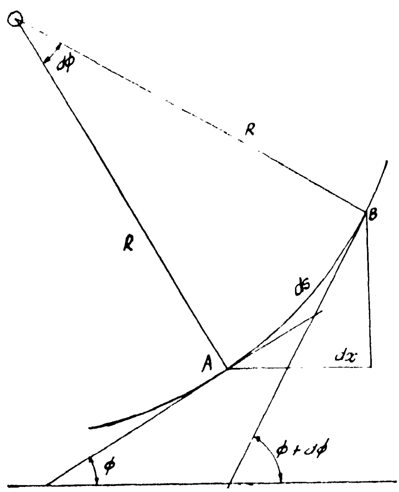

FIG. 4.1

For small lengths $ds = dx$.

$$\therefore \quad \frac{1}{R} = \frac{d\phi}{dx} = \frac{d^2y}{dx^2}.$$

But, from Chapter II, $$\frac{M}{I} = \frac{f}{y} = \frac{E}{R}.$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{M}{EI},$$

and $$\frac{dy}{dx} = \int \frac{M}{EI}\,dx = i = \text{slope of elastic line}, \qquad (1)$$

and $$y = \int\left(\int \frac{M}{EI}\,dx\right)dx = \text{deflexion of elastic line}. \qquad (2)$$

The quantity EI is the flexural rigidity of the beam and is a constant for homogeneous beams of uniform section. From equations (1) and (2) the slope and deflexion of the elastic line at any point along the span can be found either graphically or by calculation.

The Conjugate Beam

This is an imaginary beam of the same span as the beam in question loaded with the B.M. divided by the flexural rigidity. Then the 'shear' at any point gives the value of the slope and the B.M. the value of the deflexion of the elastic line at that point. The 'reactions' give the values of the slopes at the support. The method is very useful in the case of complicated or irregular loading.

Mohr's theorem states that the elastic line of a beam coincides with the shape of a cable loaded with the B.M. diagram and subject to a horizontal pull equal to EI.

Deflexions of Cantilevers

Case 1. Cantilever of length l with point load W at free end.

For any point at x from free end $M = Wx$.

$$\therefore \quad \frac{dy}{dx} = \int_0^x \frac{Wx}{EI}\,dx$$

$$= \frac{Wx^2}{2EI} + C \quad (C = \text{constant}).$$

As $\dfrac{dy}{dx} = 0$ when $x = l$, $\qquad C = -\dfrac{Wl^2}{2EI}$

$= \text{slope at free end.}$

$$\therefore \quad \frac{dy}{dx} = \frac{W}{2EI}(x^2 - l^2)$$

and $\qquad y = \dfrac{W}{2EI}\left(\dfrac{x^3}{3} - l^2x\right) + C_1.$

Since $y = 0$ for $x = l$, $\qquad C_1 = \dfrac{Wl^3}{3EI} =$ deflexion at free end,

and $\qquad y = \dfrac{W}{2EI}\left(\dfrac{x^3}{3} - l^2x + \dfrac{2l^3}{3}\right).$

By conjugate beam method:

$$\text{slope at end} = \text{area of B.M. diagram} \div EI$$

$$= Wl \times \frac{l}{2} \div EI = \frac{Wl^2}{2EI},$$

and $\qquad \text{deflexion} = \dfrac{Wl^2}{2} \times \dfrac{2l}{3} \div EI = \dfrac{Wl^3}{3EI}.$

Case 2. Same cantilever with uniform load w per ft. run.

$$M_x = \frac{wx^2}{2}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{EI}\left(\frac{wx^3}{6}+C\right).$$

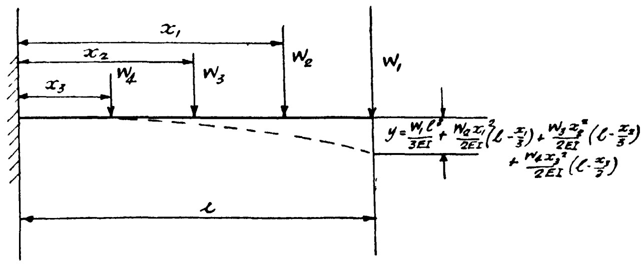

Fig. 4.2

Since $\frac{dy}{dx} = 0$ for $x = l$,

$$C = -\frac{wl^3}{6}$$

and slope at free end $= -\frac{Wl^2}{6EI} \quad (W = wl)$

and $$y = \frac{w}{6EI}\left(\frac{x^4}{4}-l^3x+C_1\right).$$

Since $y = 0$ when $x = l$, $\quad C_1 = \frac{3}{4}l^4.$

$$\therefore \quad y = \frac{w}{6EI}\left(\frac{x^4}{4}-l^3x+\frac{3l^4}{4}\right).$$

When $x = 0$, i.e. at free end,

$$y = \frac{wl^4}{8EI} = \frac{Wl^3}{8EI}.$$

Case 3. Several concentrated loads (Fig. 4.2).

Deflexion at free end due to $W_1 = \frac{W_1 l^3}{3EI}$,

deflexion under $W_2 = \frac{W_2 x_1^3}{3EI}$,

and slope at this point $= \frac{W_2 x_1^2}{2EI}$,

$\therefore$ deflexion at free end due to W_2 = slope $\times$ distance

$$= \frac{W_2 x_1^2}{2EI}(l-x_1).$$

$$\therefore \quad \text{total deflexion at free end due to } W_2 = \frac{W_2 x_1^3}{3EI} + \frac{W_2 x_1}{2EI}(l - x_1)$$

$$= \frac{W_2 x_1^2}{2EI}(l - \tfrac{1}{3}x_1).$$

Similarly

$$\text{deflexion at free end from } W_3 = \frac{W_3 x_2^2}{2EI}(l - \tfrac{1}{3}x_2)$$

$$\text{,, \quad ,, \quad ,, } \quad W_4 = \frac{W_4 x_3^2}{2EI}(l - \tfrac{1}{3}x_3).$$

The same result can be found by taking the moment of the B.M. diagram about the free end and dividing the result by EI. Note that the area of the B.M. diagram should be in in.^{2}-tons and the moment of this in in.^{3}-tons.

Case 4. Triangular load increasing from zero at the free end to a maximum of w per ft. run at the support.

Intensity of load at x from free end $= wx/l$

and
$$M_x = \frac{wx^3}{6l} = EI\frac{d^2y}{dx^2}.$$

$$\therefore \quad EI\frac{dy}{dx} = \frac{w}{6l}\left(\frac{x^4}{4} + C\right).$$

Since $\dfrac{dy}{dx} = 0$ when $x = l$, $\quad C = -\dfrac{l^4}{4}$

and
$$EI\frac{dy}{dx} = \frac{w}{6l}\left(\frac{x^4}{4} - \frac{l^4}{4}\right).$$

$$\therefore \quad \text{slope at free end} = \frac{wl^3}{24EI} = \frac{Wl^2}{12EI},$$

where
$$W = \tfrac{1}{2}wl = \text{total load.}$$

$$EI \times y = \frac{w}{6l}\left(\frac{x^5}{20} - \frac{l^4 x}{4} + C_1\right).$$

Since $y = 0$ when $x = l$, $\quad C_1 = \dfrac{l^5}{5}$

and
$$y = \frac{w}{6EIl}\left(\frac{x^5}{20} - \frac{l^4 x}{4} + \frac{l^5}{5}\right).$$

$$\text{Max. deflexion at free end} = \frac{wl^4}{30EI} = \frac{Wl^3}{15EI}.$$

Any combination of the preceding cases can be dealt with by calculating the separate deflexions and adding the results. For beams of varying inertia it is necessary to draw a corrected B.M. diagram, based on the maximum value of I for convenience. Then the ordinate of the corrected B.M. diagram at any point

$$= \text{B.M.} \times \frac{I(\text{max.})}{\text{inertia at point}}$$

(see Fig. 4.2 (a)).

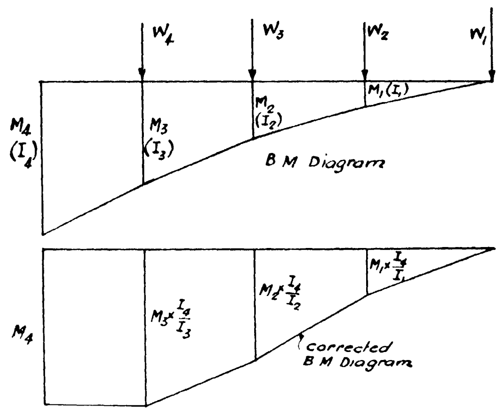

Fig. 4.2 (a)

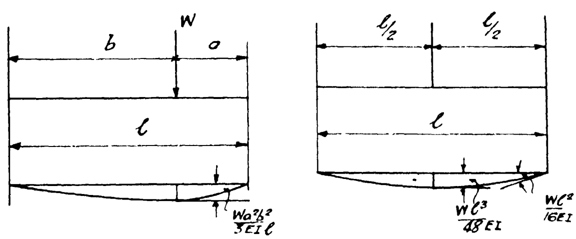

Fig. 4.3

Simply Supported Beams

Case 1. Point load W on span l, b being greater than a (Fig. 4.3).

By conjugate beam method

$$R_L = \frac{Wab}{6EIl}(2b+a) = \text{slope at L.H. end}$$

and $$R_R = \frac{Wab}{6EIl}(2a+b) = \text{slope at R.H. end.}$$

Point of maximum deflexion is where 'shear' is zero.

Let x = distance from R_R.

$$\therefore \quad \frac{Wax^2}{2l} = \frac{Wab}{6l}(2a+b).$$

$$\therefore \quad x = \sqrt{\left\{\frac{b(2a+b)}{3}\right\}}$$

and 'moment' at x

$$= \frac{Wab}{9l}(2a+b)\sqrt{\left\{\frac{b(2a+b)}{3}\right\}} = \text{deflexion} \times EI.$$

Deflexion under load $\times EI$

$$= R_R \times EI \times b - \frac{Wab^2}{2l} \times \frac{b}{3} = \frac{Wa^2b^2}{3l}.$$

When $a = b = l/2$, i.e. at mid-span,

$$\text{deflexion at mid-span} = \frac{Wl^3}{48EI}$$

and slope at end $= \dfrac{Wl^2}{8EI}$.

Case 2. Uniform load w per ft. run over span l.

$$R_R = R_L = wl/2 = W/2 \quad (W = wl).$$

Then B.M. at x from the support

$$= R_L \times x - \frac{wx^2}{2}$$

$$= \frac{wlx}{2} - \frac{wx^2}{2}$$

$$= EI\frac{d^2y}{dx^2}.$$

$$\therefore \quad EI\frac{dy}{dx} = w\left(\frac{lx^2}{4} - \frac{x^3}{6} + C\right).$$

Since $\dfrac{dy}{dx} = 0$ when $x = \dfrac{l}{2}$, $\quad C = -\dfrac{l^3}{24}$

and $\quad$ slope at support $= \dfrac{Wl^2}{24EI}$.

Also $\quad EI \times y = w\left(\dfrac{lx^3}{12} - \dfrac{x^4}{24} - \dfrac{l^3x}{24} + C_1\right).$

Since $y = 0$ when $x = 0$, $\quad C_1 = 0$,

and maximum deflexion at mid-span

$$y = \frac{5Wl^3}{384EI}.$$

Case 3. Two equal point loads W at equal distances a from the ends of a span l. The B.M. diagram is a trapezium, the maximum value being Wa at and between the points of application of the load (Fig. 4.4).

Treating the problem by the conjugate beam method:

$$\text{Area of B.M. diagram} = Wa\left(l-2a+\frac{2\times a}{2}\right)$$
$$= Wa(l-a),$$

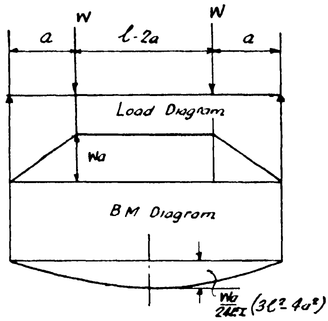

FIG. 4.4

$$\therefore \quad \text{reactions} = \frac{Wa}{2}(l-a)$$

and $$\text{slope at support} = \frac{Wa(l-a)}{2EI}.$$

$$\text{'Moment' at centre} = \frac{Wa}{2}(l-a)\frac{l}{2}-$$
$$-Wa\times\frac{(l-2a)^2}{8}-$$
$$-\frac{Wa^2}{2}\times\left(\frac{l-2a}{2}+\frac{a}{3}\right)$$
$$= \frac{Wa}{24}(3l^2-4a^2),$$

and $$\text{deflexion} = \frac{Wa}{24EI}(3l^2-4a^2).$$

Case 4. Triangular load W on span l with apex of triangle at R_R.

$$R_L = \frac{W}{3}, \qquad R_R = \frac{2W}{3}.$$

Intensity of loading at x from $R_L = \frac{2Wx}{l^2}$.

Hence point of zero shear is at 0·577l from R_L. Considering a point at x from R_L

$$M_x = \frac{W}{3}\left(x-\frac{x^3}{l^2}\right).$$

By the conjugate beam method:

$$\text{area of B.M. diagram} = \int_0^l \frac{W}{3}\left(x - \frac{x^3}{l^2}\right) = \frac{Wl^2}{12},$$

and moment of area of B.M. diagram about R_L

$$= \int_0^l \frac{Wx}{3}\left(x - \frac{x^3}{l^2}\right) = \frac{2Wl^3}{45}.$$

$$\therefore \quad \text{R.H. reaction} = \frac{2Wl^2}{45} \quad \left(\text{slope} = \frac{2Wl^2}{45EI}\right).$$

$$\text{L.H.} \quad \text{,,} \quad = \frac{-7Wl^2}{180} \quad \left(\text{slope} = \frac{-7Wl^2}{180EI}\right).$$

$$EI\frac{d^2y}{dx^2} = \frac{W}{3}\left(x - \frac{x^3}{l^2}\right).$$

$$\therefore \quad EI\frac{dy}{dx} = \frac{W}{3}\left(\frac{x^2}{2} - \frac{x^4}{4l^2} + C\right),$$

but since

$$EI\frac{dy}{dx} = \frac{-7Wl^2}{180} \quad \text{when} \quad x = 0; \qquad C = \frac{-7l^2}{60}.$$

Also deflexion is a maximum when $\frac{dy}{dx} = 0$, i.e. when

$$\frac{x^2}{2} - \frac{x^4}{4l^2} - \frac{7l^2}{60} = 0.$$

$$\therefore \quad x = 0{\cdot}52l.$$

$$EIy = \frac{W}{3}\left[\frac{x^3}{6} - \frac{x^5}{20l^2} - \frac{7l^2x}{60}\right]_0^{0\cdot52l}.$$

$$\therefore \quad \text{max. deflexion} = \frac{0{\cdot}013Wl^3}{EI} \text{ approx.}$$

Case 5. Triangular load W on span l with apex at mid-span.

$$R_L = R_R = \frac{W}{2}.$$

$$\text{B.M. at } x \text{ from support} = W\left(\frac{x}{2} - \frac{2x^3}{3l^2}\right).$$

$$\text{Area of B.M. diagram} = 2\int_0^{\frac{1}{2}l} W\left(\frac{x}{2} - \frac{2x^3}{3l^2}\right)$$

$$= \frac{5Wl^2}{48}.$$

$$\therefore \quad \text{slope at supports} = \frac{5Wl^2}{96EI}$$

and

$$\text{max. deflexion at mid-span} = \frac{Wl^3}{60EI}.$$

Case 6. Uniform load w per ft. run covering L.H. half of span l.

$$R_L = \frac{3}{8}wl, \qquad R_R = \frac{wl}{8}.$$

Using conjugate beam method:

B.M. at x from L.H. end (where x is less than $l/2$)

$$= \frac{3}{8}wlx - \frac{wx^2}{2}.$$

$$\text{B.M. at mid-span} = \frac{wl^2}{16}.$$

$$\text{Area of this half of B.M. diagram} = \int\limits_0^{\frac{1}{2}l} \left(\frac{3wlx}{8} - \frac{wx^2}{2}\right) dx$$

$$= \frac{5wl^3}{192}.$$

$$\text{Area of other half of B.M. diagram} = \frac{wl^3}{64}$$

$$\text{and total area} = \frac{wl^3}{24}.$$

Moment of L.H. half of B.M. diagram about R_L

$$= \int\limits_0^{\frac{1}{2}l} \left(\frac{3wlx^2}{8} - \frac{wx^3}{2}\right) dx = \frac{wl^4}{128}.$$

$$\text{Moment of R.H. half of B.M. diagram about } R_L = \frac{wl^4}{96}.$$

$$\therefore \quad \text{total moment} = \frac{7wl^4}{384}.$$

$$\therefore \quad \text{R.H. 'reaction'} = \frac{7wl^3}{384} \quad \left(\text{slope} = \frac{7wl^3}{384EI}\right).$$

$$\text{L.H.} \quad \text{,,} \quad = \frac{3wl^3}{128} \quad \left(\text{slope} = \frac{3wl^3}{128EI}\right).$$

For any point on L.H. half of span

$$EI\frac{dy}{dx} = \int \left(\frac{3wlx}{8} - \frac{wx^2}{2}\right) dx$$

$$= \frac{3wlx^2}{16} - \frac{wx^3}{6} + C.$$

Since $EI\frac{dy}{dx} = -\frac{3wl^3}{128}$ when $x = 0$; $C = -\frac{3wl^3}{128}$.

Max. deflexion occurs when $\frac{dy}{dx} = 0$, i.e. when

$$\frac{3wlx^2}{16} - \frac{wx^3}{6} - \frac{3wl^3}{128} = 0.$$

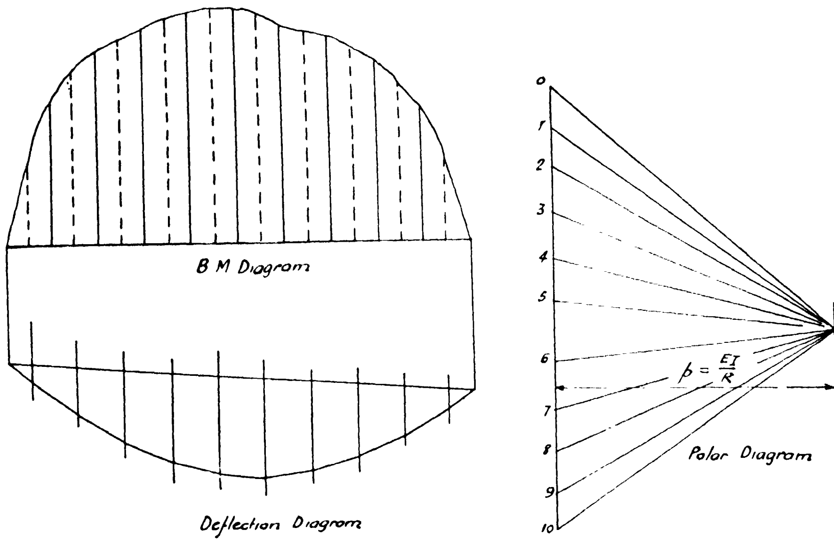

FIG. 4.5

The approximate solution of this equation is given by $x = 0{\cdot}46l$.

$$\therefore \quad EIy = \left[\frac{wlx^3}{16} - \frac{wx^4}{24} - \frac{3wl^3x}{128}\right]_0^{0{\cdot}46l}$$

$$= 0{\cdot}00655wl^4.$$

$$\therefore \quad \text{max. deflexion} = \frac{wl^4}{151EI} \quad \text{approx.}$$

Case 7. Uniform load w per ft. run over length a of span l, the distance from R_L to left of load being b (see Fig. 3.12). This problem can be solved by either the conjugate beam method or the graphical method given in Case 8.

Case 8. Any number of point loads or irregular loads on a span (Fig. 4.5). Using the conjugate beam method, divide the B.M. diagram into a convenient number of strips of equal width. Since the width of the strips is constant the height of the mid-ordinate represents the area of the strip in each case. Set down a vector line, vertically, with the distances 0–1, 1–2, etc., equal to or a fraction of the corresponding mid-ordinates. Set off a polar distance p equal to EI divided by any convenient reduction factor R, so that EI/R is a suitable length. Join P–0, P–1, etc., to form a polar diagram, and draw lines parallel to these lines in the corresponding spaces to form a link polygon. The ordinate of this polygon at any point represents the deflexion at this point to a certain scale.

Scale: If 1 in. on space scale represents S ft.

and 1 in. on B.M. ,, ,, F ft.-tons,

area of any strip = width × mid-ordinate

and 1 in. of mid-ordinate = width × S × F (ft.^{2}-tons).

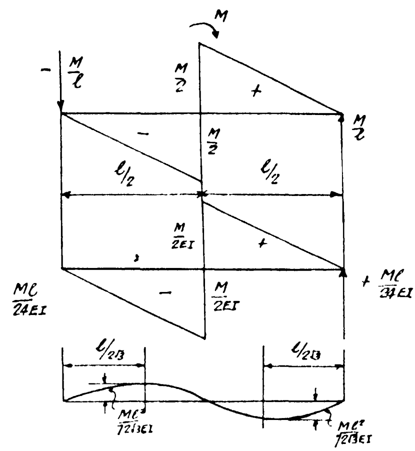

FIG. 4.6

If ordinates are diminished on vector scale by, say, N times, each inch on vector scale represents $N \times S \times F$ (ft.^{2}-tons). Now EI is generally in tons-in.^{2} units. Therefore to convert this to the same units, we must divide by 144 and find the length on the scale: 1 in. to $N \times S \times F$ ft.^{2}-tons. Since this will be large, we divide by R to find p, therefore deflexions are read off to a scale 1 in. to S/R ft. = $12S/R$ in. In choosing R, use a figure so that $12S/R$ gives a convenient scale.

This method is applicable to any loading and is most convenient for complicated or irregular load systems.

Case 9. Span l subject to a moment at mid-span (Fig. 4.6).

$$\text{Reactions} = \pm M/l.$$

$$\text{Max. B.M.} = \pm \tfrac{1}{2}M.$$

Using conjugate beam method:

$$\text{R.H. 'reaction'} = \frac{+Ml}{24}$$

$$\text{and slope} = \frac{+Ml}{24EI}.$$

Point of max. deflexion is where 'shear' is zero.

$$\text{B.M. at } x \text{ from L.H. support} = -\frac{Mx}{l} = EI\frac{d^2y}{dx^2}.$$

$$\text{L.H. 'reaction'} = -\frac{Ml}{24}.$$

Let point of zero 'shear' be at y from R_L. Then

$$= -\frac{My}{l}\times\frac{y}{2} = -\frac{Ml}{24},$$

and so
$$y = \frac{l}{2\sqrt{3}} = 0{\cdot}288l.$$

Then 'B.M.' at $0{\cdot}288l$ from R_L

$$= -\frac{Ml}{24}\times 0{\cdot}288l + \frac{M}{l}\times\frac{(0{\cdot}288l)^3}{6}$$

$$= -0{\cdot}008Ml^2$$

and
$$\text{deflexion} = \pm\frac{0{\cdot}008Ml^2}{EI}.$$

Example 1. A beam of I section, 12 in. deep with $I = 488$ in.^{4} and freely supported at the ends, has to carry a uniformly distributed load of 18 tons. What is the maximum span for a fibre stress of 8 tons per sq. in., and will the beam satisfy the condition that the deflexion must not exceed $\frac{1}{25}$ in. per ft. of span if $E = 13{,}000$ tons per sq. in.?

$$\text{Section modulus} = 81{\cdot}33 \text{ in.}$$

$$\text{Moment of resistance} = \frac{81{\cdot}33\times 8}{12} = 54{\cdot}22 \text{ ft.-tons}$$

$$= \frac{18\times L}{8}.$$

$$\therefore\quad L = 24{\cdot}1 \text{ ft.}$$

$$\text{Deflexion} = \frac{5WL^3}{384EI} = 0{\cdot}86 \text{ in.}$$

$$\text{Permissible deflexion } \frac{24{\cdot}1}{25} = 0{\cdot}96 > 0{\cdot}86 \text{ in.}$$

Example 2. A cantilever 12 ft. long carries a load of 2 tons at 4 ft. and another load of 1 ton at 10 ft. from the fixed end. Find the deflexion at the free end if $I = 60$ in. and $E = 12{,}500$ tons per sq. in.

Due to 2-ton load:

$$\text{deflexion under load} = \frac{WL^3}{3EI} = \frac{2\times 48^3}{3EI},$$

$$\text{slope under load} = \frac{WL^2}{2EI} = \frac{2\times 48^2}{2EI}.$$

Additional deflexion at end = slope × distance $= \dfrac{2\times 48^2\times 96}{2EI}$.

∴ total deflexion at end due to 2-ton load $= \dfrac{2\times 48^3}{EI}\times\dfrac{4}{3}$.

Similarly total deflexion at end due to 1-ton load $= \dfrac{120^2}{EI}\times 52$.

∴ total deflexion at free end = 1·39 in.

Example 3. A simply supported beam of 18-ft. span carries a uniformly distributed load of 3 tons. $I = 36$ in.^{4} and $E = 12{,}000$

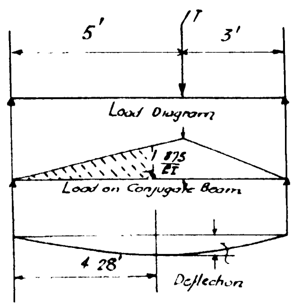

FIG. 4.7

tons per sq. in. Find the maximum deflexion at the centre and what propping force must be applied at the centre if the deflexion is not to exceed span/300.

$$\text{Deflexion due to distributed load} = \frac{5WL^3}{384EI} = 0{\cdot}9125 \text{ in.}$$

$$\frac{\text{Span}}{300} = 0{\cdot}72 \text{ in.}$$

$$\text{Residual deflexion} = 0{\cdot}1925 \text{ in.}$$

$$= \frac{W_1L^3}{48EI}, \quad \text{where } W_1 = \text{propping force.}$$

$$\therefore\ W_1 = 0{\cdot}39 \text{ ton.}$$

Example 4. A simply supported beam of 8-ft. span carries a load of 1 ton at 5 ft. from the L.H. end. Find the maximum deflexion due to the load and the distance. $I = 4{\cdot}25$ in.^{4} and $E = 13{,}000$ tons per sq. in.

B.M. diagram is a triangle (Fig. 4.7).

Using conjugate beam method:

area of B.M. diagram = 7·5 ft.^{2}-tons.

R.H. 'reaction' = 4·0625 ,,

L.H. 'reaction' = 3·4375 ,,

Deflexion is a maximum where the 'shear' is zero.

Let distance from L.H. end be x ft. Then

$$\frac{3}{8}\times\frac{x^2}{2} = 3\cdot4375$$

or

$$x = 4\cdot28,$$

and

$$\left.\begin{aligned}\text{'B.M.'} &= 3\cdot4375\times4\cdot28\\ &\quad -3\cdot4375\times\frac{4\cdot28}{3}\end{aligned}\right\} = 9\cdot75\ \text{ft.}^3\text{-tons.}$$

$$\therefore\quad \Delta = \frac{9\cdot75\times1728}{4\cdot25\times13{,}000} = 0\cdot31\ \text{in.}$$

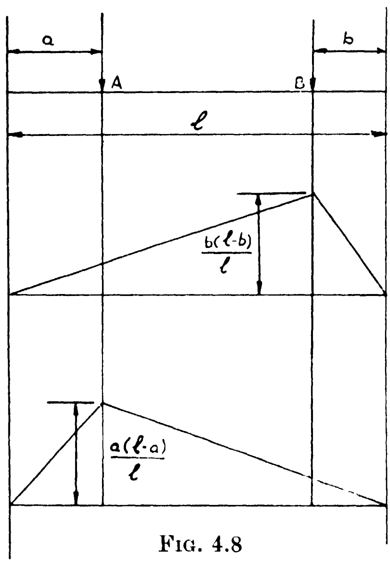

FIG. 4.8

General Cases

Clerk Maxwell's theorem of reciprocal deflexions. In a simply supported span the deflexion at any point A due to a load at any point B is equal to the deflexion at point B due to the load at A.

Proof. Place a unit load on span l at point B distant b from R.H. end (Fig. 4.8). Consider point A at a from L.H. end.

By conjugate beam method:

$$\text{area of B.M. diagram} = \frac{b(l-b)}{2}.$$

$$\text{Distance of C.G. from R.H. end} = \frac{b+l}{3}.$$

$$\text{L.H. 'reaction'} = \frac{b(l^2-b^2)}{6l}.$$

$$\text{Deflexion at } A = \frac{ab(l^2-b^2-a^2)}{6l\times EI}.$$

Now place load at A.

$$\text{Area of B.M. diagram} = \frac{a(l-a)}{2}.$$

$$\text{Distance of C.G. from L.H. end} = \frac{a+l}{3}.$$

$$\text{R.H. 'reaction'} = \frac{a(l^2-a^2)}{6l},$$

and

$$\text{deflexion at } B = \frac{ab(l^2-a^2-b^2)}{6l\times EI}$$

as before, which proves the principle of reciprocity.

Deflexion of a Simply Supported Beam under a series of Concentrated Rolling Loads

The exact solution of this problem becomes rather complicated for a number of loads. For all practical purposes it is sufficiently accurate to take the deflexion at mid-span for a certain position of the loads. There are two positions of the loads which may produce maximum deflexion. One is that position producing maximum B.M. on the span (see Rolling Loads, Chap. III) and the other is that giving the maximum area of the B.M. diagram, which occurs when the C.G. of the loads is at mid-span.

Deflexion of Beams with Overhanging Ends

Such beams can be dealt with by considering (1) the part between the supports, and (2) the cantilever portions.

As an example consider the problem previously dealt with in Chapter III, Fig. 3.13.

At R_R $\qquad$ moment $= -\frac{1}{2}wl_2^2$

and $\qquad$ shear $= R_R$.

Deflexion between R_L and R_R is that due to the distributed load between these points and the negative moment at R_R.

(1) Considering any point at x from R_L,

$$EI\frac{d^2y}{dx^2} = \frac{wl_1 x}{2}-\frac{wx^2}{2},$$

$$EI\frac{dy}{dx} = w\left(\frac{l_1 x^2}{4}-\frac{x^3}{6}-\frac{l_1^3}{24}\right),$$

$$y = -\frac{w}{24EI}(l_1^3 x^3+x^4-2l_1 x^3).$$

(2) Now moment at $R_R = -wl_2^2/2$,

$$,, \quad ,, \quad R_L = 0.$$

$$\text{Area of negative B.M. diagram} = \frac{wl_2^2}{2} \times \frac{l_1}{2}.$$

$$\text{'Reaction' at } R_L = \frac{wl_2^2 l_1}{12},$$

$$,, \quad ,, \quad R_R = \frac{wl_2^2 l_1}{6},$$

and
$$\text{'loading' at } x = \frac{wl_2^2}{2} \times \frac{x}{l_1}.$$

Hence upward deflexion at $x \times EI$ = 'reaction' $\times x$ — moment of area of part of B.M. diagram about x

$$= \frac{wl_2^2 l_1 x}{12}\left(1 - \frac{x^2}{l_1^2}\right).$$

$$\therefore \quad \Delta = \frac{wl_2^2 l_1 x}{12EI}\left(1 - \frac{x^2}{l_1^2}\right).$$

$$\text{Slope at } R_R = \frac{wl_2^2 l_1}{6EI}.$$

Combining this with the slope due to the simply supported portion (Case 2, Chap. III) which is $Wl_1^2/24EI$ we find slope at end

$$= \frac{wl_2^2 l_1}{6EI} - \frac{Wl_1^2}{24EI},$$

which gives the slope of the tangent to the elastic line of the cantilever portion at R_R, from which we can find the deflexion of the overhanging portion.

In the case of a beam overhanging both supports by an equal amount (Fig. 3.14), the problem is very simple. The deflexion at any point between R_R and R_L is that due to a uniform load on a simply supported span minus that due to a constant negative B.M. of $wl_2^2/2$. The slopes and deflexions of the cantilever portions are found as in the preceding case.

Deflexions due to Unsymmetrical Loading

It is common in practice for a beam or girder to be subject to loading in a plane which does not coincide with one of the principal axes, e.g. a purlin under vertical load. In such cases it is necessary to calculate the B.M. and deflexion with respect to each of the two principal axes.

General Case. Moment M applied at an angle α to the principal axis OY (Fig. 4.9).

Then components of M are

$$M \sin\alpha \text{ about } OY$$

and

$$M \cos\alpha \text{ about } OX.$$

At any point Q whose coordinates are x, y

$$\text{stress due to bending} = \frac{M\cos\alpha\, y}{I_{xx}} - \frac{M\sin\alpha\, x}{I_{yy}}$$

(x, y may be either positive or negative).

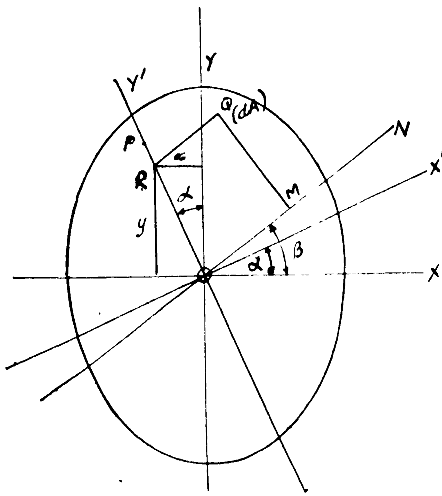

FIG. 4.9

For points on the neutral axis, the stress is zero.

$$\therefore \quad \frac{y\cos\alpha}{I_{xx}} = \frac{x\sin\alpha}{I_{yy}},$$

or

$$y = \frac{I_{xx}}{I_{yy}} \times x\tan\alpha,$$

which is the equation to a straight line ON inclined at an angle β where $y = x\tan\beta$, or

$$\tan\beta = \frac{I_{xx}}{I_{yy}}\tan\alpha$$

(I_{xx} and I_{yy} being constant for any given section).

$$\tan\beta = (I_{xx}/I_{yy})\tan\alpha$$

is the relationship between the slopes of the conjugate axes of an ellipse. Therefore the neutral axis is conjugate to the plane of bending.

This can also be proved thus: at Q the stress is proportional to the distance from the N.A.

$$f_Q \propto QM = C \times QM.$$

Therefore moment of any elemental area δA about OY' (the plane of loading)

$$= \delta A \times C \times QM \times QR.$$

$$\text{Total moment} = C \sum \delta A \times QM \times QR,$$

but total moment must be zero and $\sum \delta A \times QM \times QR$ is a product moment for axes OY' and ON; and if the product moment is zero, the axes must be conjugate. In general $\tan\beta$ is not equal to $\tan\alpha$ and therefore the N.A. is not normal to the plane of loading. The plane of the deflexion curve is normal to the N.A. when $\alpha = 0$, the loading plane is normal to N.A. when $I_{xx} = I_{yy}$, the ellipse of inertia is a circle and any two perpendicular axes may become the principal axes of the section. When I_{xx} is large compared with I_{yy}, $\tan\beta$ is large compared with $\tan\alpha$. When $\tan\alpha$ is small, β will approach 90° and the N.A. will approach the vertical axis. The deflexion will be greatest in the plane of maximum flexibility.

For most sections used in structural engineering, the shapes are symmetrical about one or two axes and the properties can be found from section books.

Approximate method. For simply supported beams of span L

(*a*) with central load W: $\Delta = \dfrac{WL^3}{48EI} = \dfrac{ML^2}{12EI}$,

(*b*) with uniform load W: $\Delta = \dfrac{5WL^3}{384EI} = \dfrac{5ML^2}{48EI}$

$$(M = \text{max. B.M.}).$$

In all cases Δ is proportional to M and for a constant span

$$\Delta = \frac{CM}{I},$$

where C is a constant involving L^2 and a number depending on the nature of the loading.

Then component deflexions are

$$\Delta_y = \frac{CM\cos\alpha}{I_{xx}},$$

$$\Delta_x = \frac{CM\sin\alpha}{I_{yy}}.$$

Resultant deflexion

$$= \sqrt{(\Delta_y^2+\Delta_x^2)} = CM\sqrt{\left(\frac{\cos^2\alpha}{I_{xx}^2}+\frac{\sin^2\alpha}{I_{yy}^2}\right)},$$

and the direction is given by $\tan^{-1}\Delta_y/\Delta_x$.

Example. A 9 in. × 3 in. timber purlin carries a vertical load of 250 lb. over a simply supported span of 10 ft. The slope of the rafter is 1 in 2. Find the maximum fibre stress and the central deflexion if

$$E = 1{,}500{,}000 \text{ lb./in.}^2$$

Properties of section

$$I_{xx} = 182\cdot25 \text{ in.}^4, \qquad z_{xx} = 40\cdot5 \text{ in.}^3$$

$$I_{yy} = 20\cdot25 \text{ in.}^4, \qquad z_{yy} = 13\cdot5 \text{ in.}^3$$

$$\tan\alpha = 0\cdot5; \qquad \sin\alpha = 0\cdot447; \qquad \cos\alpha = 0\cdot894.$$

$$\text{Max. vertical B.M.} = 37{,}500 \text{ in.-lb.}$$

$$\text{Component B.M.s}\begin{cases} M_{xx} = 37{,}500\times0\cdot894 = 33{,}500 \text{ in.-lb.} \\ M_{yy} = 37{,}500\times0\cdot447 = 16{,}750 \quad ,, \end{cases}$$

$$\text{Max. fibre stress} = \frac{33{,}500}{40\cdot5}+\frac{16{,}750}{13\cdot5} = 952 \text{ lb./in.}^2$$

$$\text{Central deflexions}\begin{cases} \Delta_y = \dfrac{5}{384}\times\dfrac{2{,}500\times120^3\times0\cdot894}{182\cdot25\times1{,}500{,}000} = 0\cdot183 \text{ in.} \\ \Delta_x = \dfrac{5}{384}\times\dfrac{2{,}500\times120^3\times0\cdot447}{20\cdot25\times1{,}500{,}000} = 0\cdot823 \text{ in.} \end{cases}$$

$$\text{Vertical deflexion} = \Delta_y\cos\alpha+\Delta_x\sin\alpha$$

$$= 0\cdot532 \text{ in.}$$

$$\text{Resultant deflexion} = \sqrt{(\Delta_y^2+\Delta_x^2)} = 0\cdot84 \text{ in.}$$

in the direction $\tan^{-1}\dfrac{183}{823}$ from the principal axis.

Stiffness of Beams and Girders

In all cases for beams and cantilevers $\Delta = K_1\times WL^3/EI$ and maximum bending moment $K_2\times WL$, where K_1, K_2 are coefficients depending on the loading and the position of the supports.

Δ can be written as

$$\frac{K_1}{K_2}\times\frac{ML^2}{EI}, \quad \text{and} \quad \frac{M}{I} = \frac{f}{y} = \frac{f}{d/2}$$

or

$$\frac{\Delta}{M} = \frac{K_1}{K_2}\times\frac{L^2}{EI} = K\frac{L^2}{EI} \qquad \left(K = \frac{K_1}{K_2}\right).$$

Also, since $M = f\times Z$ and if $f = 10$ tons/in.² (for steel) with $E = 13{,}000$ tons/in.²

$$M = 10\times\frac{I}{d/2} = \frac{20I}{d} = K_2\,WL.$$

$$\therefore \quad \Delta = K \times \frac{L^2}{650d} \qquad (L = \text{span in inches}).$$

For other values of f, $\Delta = f/10 \times \dfrac{KL^2}{650d}$.

Type of span	*Loading*	K_1	K_2	K	$\Delta\begin{cases}= 10 \text{ tons/in.}^2\\ = 13{,}000 \text{ ,,}\end{cases}$
Cantilever . .	W at end	$\frac{1}{3}$	1	$\frac{1}{3}$	$0{\cdot}0738 \times L^2/d$
,, . .	W dist.	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{4}$	$0{\cdot}0554 \times$,,
Simply supported.	W at centre	$\frac{1}{48}$	$\frac{1}{4}$	$\frac{1}{12}$	$0{\cdot}0185 \times$,,
,, ,, .	W dist.	$\frac{5}{384}$	$\frac{1}{8}$	$\frac{5}{48}$	$0{\cdot}0231 \times$,,

In calculating the deflexion of R.C. beams, it is usual to find the 'equivalent' inertia by replacing the tensile reinforcement by $(m-1)$ times its area of concrete (m = modular ratio). Values of m are given in the various codes.

In dealing with timber beams, the inertia is $bd^3/12$ for rectangular beams and the value of E varies between 1,200,000 and 1,600,000 lb./in.^{2}

Work done in Bending (Strain Energy)

In any beam bent under an external moment, work done

$$= \text{average B.M.} \times \text{angle } d\phi \quad (\text{Fig. 4.1})$$

$$= M\,d\phi/2.$$

$$\therefore \quad \text{total work} = \frac{1}{2}\int M\,d\phi \qquad \left(d\phi = \frac{M}{EI}dx\right)$$

$$= \frac{1}{2}\int_0^L \frac{M^2}{EI}dx = \frac{M^2L}{2EI} = U \quad [M \text{ constant}].$$

But
$$M^2 = \frac{f^2I^2}{y^2}.$$

$$\therefore \quad U = \frac{f^2I^2L}{2EIy^2}.$$

For a rectangular section $b \times d$: $I = bd^3/12$, $y = d/2$

$$U = \frac{f^2}{6E} \times V \qquad (V = bdL).$$

$$\text{Resilience per unit volume} = \frac{f^2}{6E}.$$

Case 1. Cantilever span l with load W at free end.

For any point at x from W,

$$M_x = Wx$$

$$U = \int_0^l \frac{M^2}{2EI}dx = \frac{W^2l^3}{6EI}$$

$$\text{max. B.M.} = M = Wl.$$

$$\therefore \quad U = \frac{M^2l}{6EI}.$$

For rectangular section:

$$U = \frac{f^2}{18E} \times V.$$

$$\text{Resilience per unit volume} = \frac{f^2}{18E}.$$

If $\Delta =$ deflexion at free end,

$$U = \Delta \times \frac{W}{2} = \frac{W^2l^3}{6EI}.$$

$$\therefore \quad \Delta = \frac{Wl^3}{3EI}.$$

Case 2. Same cantilever with load w per ft. run.

$$M_x = \frac{wx^2}{2}.$$

$$U = \int_0^l \frac{w^2x^4}{8EI}dx = \frac{W^2l^3}{40EI} = \frac{M^2l}{10EI} \quad \left(M = \frac{Wl}{2}\right).$$

For rectangular section:

$$U = \frac{f^2}{30E} \times V.$$

$$\text{Resilience per unit volume} = \frac{f^2}{30E}.$$

Case 3. Simply supported span l with central load W.

Moment at x from support $= M_x = \dfrac{Wx}{2}$.

For half the beam:

$$\frac{U}{2} = \int_0^{\frac{1}{2}l} \frac{M^2\,dx}{EI} = \frac{W^2l^3}{192EI}.$$

$$\therefore \quad U = \frac{W^2l^3}{96EI} = \Delta \times \frac{W}{2}$$

$$\Delta = \frac{Wl^3}{48EI}.$$

For rectangular section:

$$U = \frac{f^2}{18E} \times V.$$

$$\text{Resilience per unit volume} = \frac{f^2}{18E}.$$

Case 4. Same span with load w per ft. run.

$$M_x = \frac{wlx}{2} - \frac{wx^2}{2},$$

$$\frac{U}{2} = \int_0^{\frac{1}{2}l} \frac{M^2}{2EI} dx = \frac{w^2 l^5}{480EI}.$$

$$\therefore \quad U = \frac{W^2 l^3}{240EI}$$

$$= \frac{4M^2 l}{15EI} \quad \left(M = \frac{Wl}{8}\right).$$

For rectangular section:

$$U = \frac{4f^2}{45E} \times V.$$

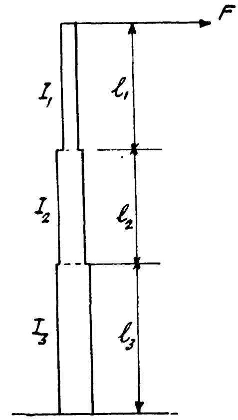

FIG. 4.10

$$\text{Resilience per unit volume} = \frac{4f^2}{45E}.$$

Applications of Strain Energy Method

1. To find the deflexion of a pole at the free end due to a force F applied at that point (Fig. 4.10).

	Length	*Inertia*
Top section	l_1	I_1
Middle section . . .	l_2	I_2
Bottom section . . .	l_3	I_3

Work done by $F = \frac{1}{2}F \times \Delta$, where Δ = deflexion,

$$= \sum \frac{M^2}{2EI} dx.$$

$$\Delta = \frac{2}{F} \sum \frac{M^2}{2EI} dx \qquad (M = Fx)$$

$$= \frac{2}{F} \times F^2 \sum \frac{x^2\,dx}{2EI} = \frac{F}{E} \sum \frac{x^2\,dx}{I}$$

$$= \frac{F}{E}\left\{ \int_0^{l_1} \frac{x^2\,dx}{I_1} + \int_{l_1}^{l_1+l_2} \frac{x^2\,dx}{I_2} + \int_{l_1+l_2}^{l_1+l_2+l_3} \frac{x^2\,dx}{I_3} \right\}$$

$$= \frac{F}{3E}\left\{ l_1^3\left(\frac{1}{I_1} - \frac{1}{I_2}\right) + (l_1+l_2)^3\left(\frac{1}{I_2} - \frac{1}{I_3}\right) + (l_1+l_2+l_3)^3 \frac{1}{I_3} \right\}.$$

2. To find the deflexion of a simply supported beam at the ends, due to a load W falling through a height h, at centre of span l.

Let deflexion during impact $= \Delta$,
equivalent load producing $\Delta = P$

$$\frac{Pl^3}{48EI} = \Delta \quad \text{or} \quad P = \frac{48EI\Delta}{l^3}.$$

$$\text{Work done by } P = \frac{P\Delta}{2} = \text{energy stored in beam}$$

$$= \frac{24EI\Delta^2}{l^3}.$$

Energy of weight W falling through $(h+\Delta)$

$$= W(h+\Delta).$$

$$\therefore \quad W(h+\Delta) = \Delta^2 \times \frac{24EI}{l^3}.$$

For example, let a 5-cwt. load fall 2 in. on to the centre of a 12-in. I-beam of 20-ft. span ($I = 220$ in.^{4}).

Then $5 \times 112(2+\Delta) = \frac{1}{2}P\Delta$,

$$\Delta = \frac{Pl^3}{48EI}.$$

$$\therefore \quad P = 1{,}160 \text{ lb.},$$

from which Δ and the stress f can be found.

$$M = 69{,}600 \text{ in.-lb.}, \quad f = 1{,}900 \text{ lb./in.}^2, \quad \Delta = 0{\cdot}05 \text{ in.}$$

Deflexion of a Continuous Web Girder at any point by Method of Virtual Work

Imagine a load P placed at the point at which the deflexion is required and let the actual loading be gradually applied.

Let u = deflexion due to actual loading. Work done by P acting through distance $u = P \times u$. Actual loading gradually applied does work $= \sum \frac{1}{2}W\delta$, where

W = any part of actual load,

and δ = deflexion of C.G. of W.

$$\therefore \quad \text{total external work} = Pu + \sum \tfrac{1}{2}W\delta.$$

Let M' = B.M. at any point due to P,
f' = stress intensity on a horizontal strip of breadth b at y from N.A.

$\therefore$ force on elemental strip due to $P = f'b\,dy$ (Fig. 4.11).

If f = stress intensity on same strip due to the actual loading

$$\text{strain on } dx = \frac{f}{E} \text{ and extension} = \frac{f}{E} \times dx.$$

$$\text{Load on strip} = fb\,dy.$$

$$\text{Total load on strip} = f'b\,dy + fb\,dy$$
$$= b\,dy(f'+f).$$

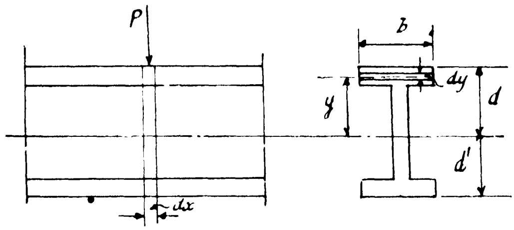

FIG. 4.11

Work on elemental strip when actual loading is gradually applied = mean stress × extension due to actual load. Since

$$\text{mean stress} = \frac{\text{initial stress} + \text{final stress}}{2},$$

$$\text{work done on strip} = \frac{f'+(f+f')}{2} b\,dy \times \frac{f}{E} dx$$
$$= \frac{(f'+\frac{1}{2}f)fb\,dy}{E} dx.$$

Let d = distance of extreme fibre from N.A. and

f_1' = stress intensity at d due to P,

f_1 = ,, ,, ,, actual load.

$$\therefore \quad f' = f_1' \times \frac{y}{d} \quad \text{and} \quad f = f_1 \times \frac{y}{d}.$$

$$\therefore \quad \text{work done} = (f_1'+\tfrac{1}{2}f_1)\frac{y}{d} \times f_1\frac{y}{d} \times \frac{b}{E}\,dydx.$$

Integrating this expression over the total length and depth of the girder:

$$\text{total work} = \int_0^l \frac{(f_1'+\frac{1}{2}f_1)}{d^2E} f_1\,dx \times \int_{d'}^{d} by^2\,dy.$$

$$\int_{d'}^{d} by^2\,dy = \text{M.I. of section about N.A.}$$

If M = B.M. due to actual load,

$$\frac{f_1'}{d} = \frac{M'}{I} \quad \text{and} \quad \frac{f_1}{d} = \frac{M}{I},$$

so that the expression for the total work becomes

$$\int_0^l \frac{(M'+\frac{1}{2}M)M}{I^2E} \times I\,dx,$$

$$\text{or total work in girder} = \int_0^l \frac{M'M}{EI}\,dx + \int_0^l \frac{M^2}{2EI}\,dx$$

$$= \int_0^l \frac{M'M}{EI}\,dx + \sum \frac{W\delta}{2}$$

$$= \int_0^l \frac{M'M}{EI}\,dx + \text{work done by actual loading.}$$

$$\therefore \quad Pu = \int_0^l \frac{M'M\,dx}{EI},$$

or

$$u = \frac{1}{P}\int_0^l \frac{M'M\,dx}{EI}.$$

P can be taken as unit load (which may not necessarily be vertical).

Then

$$u = \int_0^l \frac{M'M\,dx}{EI}.$$

Example. Cantilever length l with load W at free end.

$$\Delta = \int_0^l \frac{Wx^2\,dx}{EI} = \frac{Wl^3}{3EI}.$$

If the cantilever carries in addition a uniformly distributed load w per ft. run,

$$\Delta = \int_0 \frac{x(Wx+wx^2/2)}{EI}\,dx$$

$$= \frac{Wl^3}{3EI} + \frac{wl^4}{8EI} = \frac{Wl^3}{3EI} + \frac{W_1\,l^3}{8EI}.$$

Deflexion due to Shear

Case 1. Rectangular beam with central load W on span l (Fig. 4.12). Divide the beam into four equal sections as shown. Shear $= W/2$ throughout. Therefore the same energy is stored in each section.

$$\text{Shear stress } s_y = \frac{S}{I} \times \frac{b(h/2-y)\times\frac{1}{2}(h/2+y)}{b}$$

$$= \frac{W}{4I}\left(\frac{h^2}{4}-y^2\right).$$

Energy stored in each section due to shearing stress

$$= \int_0^{\frac{1}{2}h} \frac{s_y}{2} \times \frac{s_y}{G} \times b\, dy \times \frac{l}{2} \quad (G = \text{modulus of rigidity})$$

$$= \frac{1}{2G} \int_0^{\frac{1}{2}h} \frac{W^2}{16I^2} \left(\frac{h^2}{4} - y^2\right)^2 \frac{bl}{2}\, dy$$

$$= \frac{144W^2bl}{16b^2h^6 \times 4G} \int_0^{\frac{1}{2}h} \left(\frac{h^2}{4} - y^2\right)^2 dy$$

$$= \frac{3}{80} \times \frac{W^2l}{bhG}.$$

Total energy due to shear $= \frac{3}{20} \times \frac{W^2l}{bhG}$.

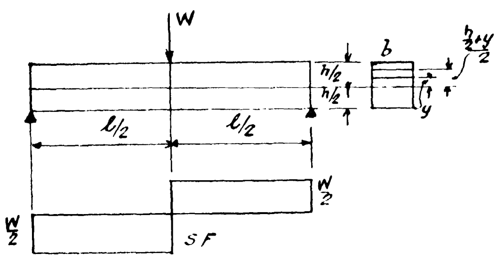

FIG. 4.12

Total energy in beam

$$= \tfrac{1}{2}W(\Delta_1 + \Delta_2) \quad \begin{cases} \Delta_1 = \text{deflexion due to bending,} \\ \Delta_2 = \quad ,, \quad ,, \quad \text{shear.} \end{cases}$$

$$\therefore \quad \Delta_2 = \frac{3Wl}{10bhG},$$

$$\Delta_1 = \frac{Wl^3}{4bh^3E}.$$

$$\therefore \quad \frac{\Delta_2}{\Delta_1} = \frac{6h^2E}{5l^2G}.$$

Case 2. Same beam with distributed load w per ft. run (Fig. 4.12 (*a*)).

Energy stored is proportional to s_y^2 and shear force diagram is a straight line.

Area of S.F. diagram varies according to (distance)^2.

Ordinate of curve of energy is proportional to (distance)^2.

Therefore average energy stored for distributed load $= \frac{1}{3}$(average energy for point load).

Slope of curve varies as the distance from the centre, i.e. curve of shear deflexion is a parabola.

Let Δ_1 = shear deflexion for point load,

Δ_2 = ,, ,, distributed load,

U_1 = energy for point load.

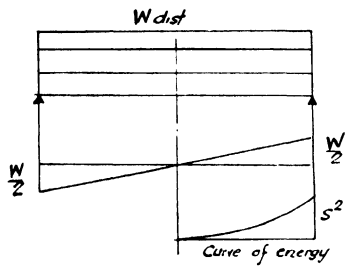

FIG. 4.12(*a*)

Then
$$U_1 = \frac{W\Delta_1}{2}$$

and
$$\frac{U_1}{3} = \frac{W}{2}\times\frac{2\Delta_2}{3} \quad \text{(average load} \times \text{mean height).}$$

$$\therefore \quad \Delta_2 = \frac{\Delta_1}{2}$$

which is general for uniform sections.

Case 3. I-beam: assuming flanges take all the B.M. and the web all the shear.

Let d = depth,

A_f, A_w = area of flange and web respectively.

$$\therefore \quad I = \tfrac{1}{2}A_f\, d^2 \text{ approx.}$$

(The error is about $2\frac{1}{2}$ per cent. for most cases.)

For point load

$$\text{deflexion due to bending} = \frac{Wl^3}{48EI} = \Delta_1.$$

Assume shear stress uniform over web

$$s_y = \frac{W}{2A_w}.$$

$$\text{Deflexion due to shear} = \Delta_2 = \frac{Wl}{4A_w G}.$$

$$\therefore \quad \frac{\Delta_2}{\Delta_1} = \frac{6EA_f d^2}{GA_w l^2}.$$

The ratio Δ_2/Δ_1 varies as $(d/l)^2$.
For steel $G = 0{\cdot}4E$.
For short beams Δ_2 may be comparable with Δ_1.
For distributed loads:

$$\Delta_2 = \tfrac{1}{2}\times\text{deflexion due to point load} = \frac{Wl}{8A_w G},$$

$$\Delta_1 = \frac{5Wl^3}{384EI}.$$

$$\therefore\quad \frac{\Delta_2}{\Delta_1} = 4{\cdot}8\times\frac{E}{G}\times\frac{A_f}{A_w}\left(\frac{d}{l}\right)^2.$$

In practice it is common to neglect shear deflexion. This can be done with safety except for short beams or where the shear is great relative to B.M.

Example 1. A rectangular cantilever has the dimensions b, d, and l. A point load W is applied at the free end. Show that the shear deflexion is $6Wl/5Gbd$.

Shear force at any point $= W$.

Shear stress (s_y) at y from N.A.

$$= \frac{S}{I}\times\frac{b(\frac{1}{2}d+y)(\frac{1}{2}d-y)}{2b}$$

$$= \frac{6W}{bd^3}\left(\frac{d^2}{4}-y^2\right).$$

$$\text{Resilience} = \frac{s_y^2}{2G} = \frac{36W^2}{2Gb^2d^6}\left(\frac{d^2}{4}-y^2\right)^2.$$

Total work done by shear

$$= \frac{18W^2}{Gb^2d^6}\int\limits_{-\frac{1}{2}d}^{+\frac{1}{2}d}\left(\frac{d^2}{4}-y^2\right)^2 dy\times bl$$

$$= \frac{18W^2l}{30Gbd}$$

$$= \text{average force}\times\text{distance}$$

$$= \tfrac{1}{2}W\times\Delta.$$

$$\therefore\quad \Delta = \frac{6Wl}{5Gbd}.$$

Example 2. A beam 50-ft. span, freely supported at the ends, deflects 2 in. under a uniformly distributed load of 50 tons. If the beam is propped at the centre so that the deflexion there is reduced to 0·75 in., calculate the propping force. Sketch the B.M. and S.F. diagrams, indicating the maximum B.M.

Shear at end = 25 tons (from distributed load).

Max. B.M. at centre = 312·5 ft.-tons.

$$\Delta = \frac{5Wl^3}{384EI} = 2 \text{ in.}$$

$$\therefore \quad EI = \frac{250}{768} \times l^3.$$

If residual deflexion = 0·75 in.,

deflexion due to prop = 1·25 in.

If propping force = P tons,

$$\frac{Pl^3}{48EI} = 1{\cdot}25,$$

and

$$P = \frac{60EI}{l^3} = 19{\cdot}8 \text{ tons.}$$

Shear at end = 25—9·9 = 15·1 tons.

Negative B.M. = 247·5 ft.-tons.

Effective B.M. at centre = 65 ft.-tons.

Max. B.M. occurs when shear is zero, i.e. where shear from distributed load = 9·9 tons, which is at 10 ft. (approx.) from the centre.

$$\text{Positive B.M.} = 312{\cdot}5\{1-(\tfrac{10}{25})^2\} = 262{\cdot}5 \text{ ft.-tons.}$$

$$\text{Negative B.M.} = 9{\cdot}9 \times 15 \qquad = 148{\cdot}5 \quad ,,$$

$$\therefore \quad \text{Max. B.M.} = \overline{114{\cdot}0} \quad ,,$$

Example 3. It is estimated that a certain beam will deflect 0·2 in. under a central load of 8 tons and that the resulting maximum bending stress will be 6 tons per sq. in. If a load of 1 ton is allowed to fall 1 in. on to the centre of the span, what deflexion and maximum stress will be produced? Assume that impact stresses are negligible and state any other assumptions which are necessary.

Let P = equivalent static load which will produce the same deflexion as the 1-ton load falling on the beam.

$$\text{Deflexion for } P \text{ tons} = \frac{P}{8} \times 0{\cdot}2 = \frac{P}{40} = \Delta.$$

$$\text{Work done} = \frac{P}{2} \times \Delta = 20\Delta^2$$

= energy of 1 ton falling $(1+\Delta)$ in.

$= 1+\Delta$.

$$\therefore \quad \Delta = 0{\cdot}25 \text{ in.}$$
$$P = 10 \text{ tons.}$$

If stress is 6 tons per sq. in. for a load of 8 tons,
then ,, 7·5 ,, ,, 10 tons.

It is unnecessary to make any assumption other than that no energy is lost in the impact and that there is no rebound.

Torsion

Torsion occurs in beams curved in plan and in beams subject to unsymmetrical loading, etc. Torsion is also important when considering the stability of certain struts and of beams which are not adequately supported laterally.

The solution of the problem of torsion is complicated except in the case of circular sections, where the formula is

$$\frac{T}{J} = \frac{q}{r} = G\frac{\theta}{l},$$

where T = torque,
J = polar moment of inertia,
θ = angle of twist per length l.

For other shapes the analysis becomes very complicated. Investigations of such problems have been carried out by various experimenters. Dr. Orr has made a very useful comparison between the results of actual tests and the empirical formula propounded by Dr. Griffith. The results of his investigation are given in his paper. A number of formulae are given which show remarkably close agreement with the experimental values.

For structural sections

$$T = G\frac{\theta}{l} \times C \tag{3}$$

and

$$q = G\frac{\theta}{l} \times R, \tag{4}$$

where C = rigidity constant for the section
and R = stress factor.

$C = K(Dt_w^3 + nBt_f^3)$,
D = depth; t_w = web thickness,
B = breadth; t_f = flange thickness,
K = 0·42 for British standard beams,
= 0·37 ,, ,, channels,
= 0·34 ,, ,, tees and angles,
n = 2 for beams and channels,
= 1 ,, tees and angles.

From (4) $$G\frac{\theta}{l} = \frac{q}{R}.$$

Substituting in (3) $$T = \frac{q}{R}\times C,$$

or $$q = \frac{RT}{C}.$$

t is taken as the greatest thickness of metal, and in the case of beams and channels, this occurs at the junction of the flange and the web. This agrees with the experimental result that the maximum shear stress is at this point. The stress factor R can be taken as t_f for beams and channels (error is about 10 per cent.) and the thickness for angles and tees.

In order to find the value of T, it is necessary to calculate C. For $q = 5$ tons per sq. in., $T = 5C/R$ with proportionate values for other shear stresses. In the case of built-up sections (compound and plated girders), it is recommended that the value of C be calculated for each component and $\sum C$ evaluated for the whole section.

For rectangles such as web and flange plates

$$C = \frac{bt^3}{3} - 0{\cdot}21t^4,$$

which is approximately correct. By using this method, the effect of the rivets or welds in preventing slipping is neglected. Any error caused thereby is on the side of safety. It may be possible to increase the value of q for built-up sections for this reason.

American practice is to divide all structural sections into a number of separate rectangles, then $C = \sum (bt^3/3)$ which is a good enough approximation for design purposes.

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EXERCISES

1. A cantilever of span l is fixed at one end and carries a distributed load W and a point load F at the centre. Find the maximum deflexion.

$$[l^3/48EI \times (5F+6W).]$$

2. A 12 in. × 6 in. joist ($I = 316{\cdot}8$ in.^{4}) carries a distributed load over a 30-ft. span and is strengthened by a 6 in. × ½ in. plate top and bottom over the central 10 ft. Find the distributed load if the maximum stress is 9 tons per sq. in., and the maximum deflexion. [16·9 tons; 1·82 in.]

3. Two 10 in. × 6 in. joists ($I = 211{\cdot}4$ in.^{4}; $A = 12{\cdot}4$ in.^{2}) are placed (*a*) side by side, (*b*) one above the other but not connected thereto, (*c*) as (*b*) but riveted on. If the maximum bending stress is the same in each case, find (1) the ratio of the loads carried and (2) the ratio of the deflexions. [1 : 1 : 1·23; 1 : 1 : 0·5.]

4. A steel pole is composed of an upper portion of outside diameter 5 in. and inside diameter 4 in.; the lower portion is 6 in. outside and 5 in. inside diameter. The pole is fixed at the base and the overall height is 16 ft. Find the length of the upper portion (1) if the maximum stress due to a load of 600 lb. applied at the top is the same for both parts, and (2) if the maximum deflexion is 1½ in. [10·5 ft.; 6·6 ft.]

5. A beam carries a distributed load of 15 tons over a span of 20 ft. Find a suitable section for a maximum stress of 8 tons per sq. in. If a load of 2½ cwt. is allowed to fall 2 in. on to the centre of the span, find the additional stress and the deflexion. [14 in. × 6 in. × 46 B.S.B.; $I = 443$ in.^{4}; 3 tons per sq. in.; 0·16 in.]

CHAPTER V

FIXED AND CONTINUOUS BEAMS

Fixed Beams

WHERE one or both ends of a beam are fixed or 'encastred', the supporting forces cannot be calculated by the methods previously used for simply supported beams. The system is statically indeterminate owing to the restraining moments which must be applied so that the tangent to the elastic line is horizontal at the ends, i.e. $dy/dx = 0$.

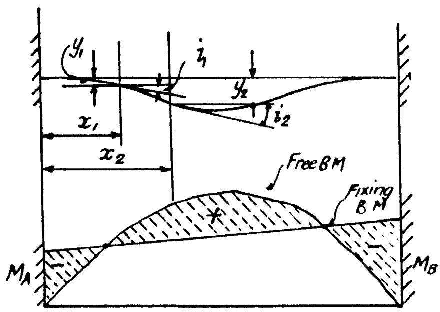

FIG. 5.1

General case (Fig. 5.1). Consider the points at x_1 and x_2 from L.H. end

$$\int_{x_1}^{x_2} \frac{d^2y}{dx^2}\,dx = \int_{x_1}^{x_2} \frac{M}{EI}\,dx.$$

If $\quad i_1 =$ slope of tangent to the elastic line at x_1

and $\quad i_2 =$,, ,, ,, ,, x_2,

$$i_2 - i_1 = \int_{x_1}^{x_2} \frac{M}{EI}\,dx = \text{area of B.M. diagram between } x_1 \text{ and } x_2 \text{ divided by } EI. \tag{1}$$

Also

$$\int_{x_1}^{x_2} x\frac{d^2y}{dx^2}\,dx = \int_{x_1}^{x_2} \frac{Mx}{EI}\,dx.$$

$$\therefore \quad \left[x\frac{dy}{dx} - y\right]_{x_1}^{x_2} = \frac{1}{EI}\int_{x_1}^{x_2} Mx\,dx, \tag{2}$$

or $\quad x_2 i_2 - x_1 i_1 - [y_2 - y_1] =$ moment of area of B.M. diagram between x_1 and x_2 about L.H. end divided by EI.

Now if $x_1 = 0$ and $x_2 = l$, then since the tangent to the elastic line is horizontal at the supports, $i_2 = i_1 = 0$.

Therefore equation (1) becomes: area of B.M. diagram $= 0$, i.e. the area of the 'fixing' moment diagram is equal and opposite to that of the free B.M. diagram. Also, since $y_1 = y_2 = 0$, equation (2) becomes: moment of area of B.M. diagram $= 0$, i.e. moment of 'fixing' moment diagram is equal and opposite to that of the 'free' B.M. diagram. Since the areas are equal numerically, the distance to the centroid must be the same in each case, i.e. they must lie on the same vertical.

Case 1. Span $AB = l$, with central point load W, built in at ends.

$$\text{Free B.M.} = \frac{Wl}{4}.$$

$$\text{Area of free B.M. diagram} = \frac{Wl^2}{8}.$$

By symmetry,

$$M_A = M_B = \frac{Wl^2}{8} \div l = \frac{Wl}{8} = \text{max. negative moment.}$$

$$\text{Max. positive B.M. at centre} = \frac{Wl}{4} - \frac{Wl}{8} = \frac{Wl}{8}.$$

$$\text{Positive deflexion} = \frac{Wl^3}{48EI}.$$

Negative deflexion due to fixing moment

$$= \frac{Wl}{8} \times \frac{l}{2}\left(\frac{l}{2} - \frac{l}{4}\right) \div EI = \frac{Wl^3}{64EI}.$$

$$\therefore \quad \text{effective deflexion} = \frac{Wl^3}{48EI} - \frac{Wl^3}{64EI} = \frac{Wl^3}{192EI}$$

$= \frac{1}{4}$(deflexion for same beam simply supported).

Case 2. Same beam AB with W uniformly distributed.

$$\text{Free B.M.} = \frac{Wl}{8}.$$

As B.M. diagram is a parabola, area

$$= \frac{Wl}{8} \times \frac{2l}{3} = \frac{Wl^2}{12}.$$

$$\therefore \quad M_A = M_B = \frac{Wl}{12},$$

and effective B.M. at centre

$$= \frac{Wl}{8} - \frac{Wl}{12} = \frac{Wl}{24}.$$

Distance of point of contraflexure (where moment changes sign) from the supports $= 0{\cdot}211l$.

Positive deflexion for simply supported span

$$= \frac{5Wl^3}{384EI}.$$

Upward deflexion due to fixing moment

$$= \frac{Wl}{12} \times \frac{l}{2} \times \frac{l}{4} \div EI$$

$$= \frac{Wl^3}{96EI}.$$

$\therefore$ effective deflexion

$$= \frac{Wl^3}{384EI} = \frac{1}{5}(\text{deflexion for same beam simply supported}).$$

Case 3. Triangular load W on span l (apex of triangle at mid-span) with fixed ends.

For freely supported beam, B.M. at x

$$= \frac{Wx}{2} - \frac{2Wx^3}{3l^2} = \frac{W}{2}\left(x - \frac{4x^3}{3l^2}\right).$$

$$\therefore \quad \text{area} = 2 \times \frac{W}{2} \int_0^{l/2} \left(x - \frac{4x^3}{3l^2}\right) dx = \frac{5Wl^2}{48}.$$

$$\therefore \quad M_A = M_B = \frac{5Wl}{48}.$$

For simply supported beam, deflexion $= \dfrac{Wl^3}{60EI}$.

Upward deflexion due to fixing moment

$$= \frac{5Wl}{48} \times \frac{l}{2} \times \frac{l}{4} \div EI$$

$$= \frac{5Wl^3}{384EI}.$$

$\therefore$ effective deflexion

$$= \frac{7Wl^3}{1920EI} = \frac{7}{32}(\text{deflexion for same beam simply supported}).$$

Case 4. Two point loads W at a from ends of span $AB = l$, with fixed ends.

$$\text{Max. free B.M.} = Wa$$

and

$$\text{deflexion} = \frac{Wa(3l^2 - 4a^2)}{24EI}.$$

Where $a = l/3$, i.e. loads are at third points,

$$\text{B.M.} = \frac{Wl}{3} \quad \text{and} \quad \text{deflexion} = \frac{23Wl^3}{648EI}.$$

For general case, area of 'free' B.M. diagram $= Wa(l-a)$.

$$\therefore \quad M_A = M_B = \frac{Wa(l-a)}{l},$$

and

$$\text{negative deflexion} = \frac{Wa(l-a)l^2}{8l \times EI}.$$

$$\therefore \quad \text{effective deflexion} = \frac{Wa^2(3l-4a)}{24EI}.$$

For $a = l/3$: effective deflexion

$$= \frac{5Wl^3}{648EI} = \frac{5}{23}(\text{deflexion for same beam simply supported}).$$

Case 5. Point load W at a from L.H. support of span $AB = l$, with ends fixed.

$$\text{Free B.M.} = \frac{Wa(l-a)}{l}$$

and area of B.M. diagram

$$= \frac{Wa(l-a)}{2} = \frac{M_A+M_B}{2} \times l.$$

$$\therefore \quad M_A+M_B = \frac{Wa(l-a)}{l}.$$

Also, since moment of 'fixing' moment diagram = moment of free B.M. diagram

$$\frac{Wa(l-a)}{l}\left\{\frac{a}{2} \times \frac{2a}{3} + \left(\frac{l-a}{2}\right)\left(a + \frac{l-a}{3}\right)\right\} = \frac{l^2}{6}(M_A+2M_B).$$

$$\therefore \quad M_B = \frac{Wa^2(l-a)}{l^2},$$

$$M_A = \frac{Wa(l-a)^2}{l^2},$$

and deflexions can be found by the conjugate beam method.

Now since M_A and M_B are not equal in this case, the reactions R_A and R_B must be adjusted.

For the simply supported beam

$$R_A = \frac{W(l-a)}{l},$$

$$R_B = \frac{Wa}{l}.$$

Shear force at any point = rate of change of B.M.

At R_A shear due to fixing moment

$$= \frac{M_A - M_B}{l}$$

$$= \frac{Wa(l-2a)(l-a)}{l^3},$$

which may be + or − in sign.

$$\text{Total reaction } R_A = \frac{W(l-a)^2(l+2a)}{l^3},$$

$$\text{,, \quad ,, \quad } R_B = \frac{Wa^2(3l-2a)}{l^3}.$$

Case 6. Triangular load W with apex at R.H. end.

Free B.M. at x from L.H. end $= \frac{W}{3}\left(x - \frac{x^3}{l^2}\right)$.

$$\text{Total area} = \frac{W}{3}\int_0^l \left(x - \frac{x^3}{l^2}\right)dx = \frac{Wl^2}{12} = \frac{M_A + M_B}{2} \times l.$$

$$\therefore \quad M_A + M_B = \frac{Wl}{6}.$$

Moment of B.M. diagram (about L.H. end)

$$= \frac{W}{3}\int_0^l \left(x^2 - \frac{x^4}{l^2}\right)dx$$

$$= \frac{2Wl^3}{45} = \frac{l^2}{6}(M_A + 2M_B).$$

$$\therefore \quad M_B = \frac{Wl}{10} \quad \text{and} \quad M_A = \frac{Wl}{15}.$$

Effect of Haunching on Fixed Beams

If the depth (and therefore the moment of inertia) of a fixed beam varies along the span, the end fixing moments must be adjusted accordingly. The end moments can be expressed thus:

$$M_F = KWl,$$

where K is a coefficient depending on two factors:

(1) $K_1 = \sqrt[3]{\frac{I_S}{I_C}}$, where $\begin{cases} I_S = \text{M.I. at support,} \\ I_C = \text{ ,, \quad centre,} \end{cases}$

and (2) $K_2 = \frac{\text{length of haunch}}{\text{span } l}$.

For a uniformly distributed load, the normal fixing moment

$$= \frac{Wl}{12} = 0{\cdot}0833Wl.$$

If $K_1 = 2$ and $K_2 = 0{\cdot}2$,

$$M_F = 0{\cdot}10Wl \text{ approx.}$$

and max. positive moment $= Wl(0{\cdot}125-0{\cdot}1) = 0{\cdot}025Wl$.

Where $I_C = 0$ and $K_2 = 0{\cdot}5$, i.e. where depth increases from zero at the centre to a maximum at the supports (virtual hinge at the centre),

$$M_F = 0{\cdot}125Wl$$

and max. positive moment = zero (approx.).

Therefore it can be seen that the fixing moment varies from a maximum of $0{\cdot}125Wl$ to a minimum of $0{\cdot}0833Wl$. The centre (positive) moment varies from zero to a maximum of $0{\cdot}0416Wl$.

Treatment of Fixed Beams by the Column Analogy

This method, first introduced by Professor Hardy Cross, has not perhaps received the attention that it deserves in this country. Since it has useful applications for rigid frames, etc., it will be outlined in this chapter.

For any elastic beam, the angle between the tangents at two points

$$= i_x - i_{x_0} = \frac{1}{EI}\int_{x_0}^{x} M\,ds,$$

and the relative vertical deflexion

$$= y_x - y_{x_0} = \frac{1}{EI}\int_{x_0}^{x} Mx\,ds,$$

and the relative horizontal deflexion

$$= x_y - x_{y_0} = \frac{1}{EI}\int_{y_0}^{y} My\,ds.$$

Now, as will be seen when dealing with columns, the equations relating to columns subject to loading eccentric about the two axes X–X and Y–Y are:

$$\text{total load on column} = W = \int w\,dA.$$

$$\text{B.M. about } X\text{–}X \text{ axis} = M_{xx} = \int wy\,dA,$$

$$\text{,,} \quad Y\text{–}Y \quad \text{,,} \quad = M_{yy} = \int wx\,dA.$$

Comparing the two sets of equations, we find that

$$\frac{1}{EI}\int ds \quad \text{for beams corresponds to} \int dA \text{ for columns}$$

$$\frac{1}{EI}\int x\,ds \quad ,, \quad ,, \quad ,, \quad \int x\,dA \quad ,,$$

$$\frac{1}{EI}\int y\,ds \quad ,, \quad ,, \quad ,, \quad \int y\,dA \quad ,,$$

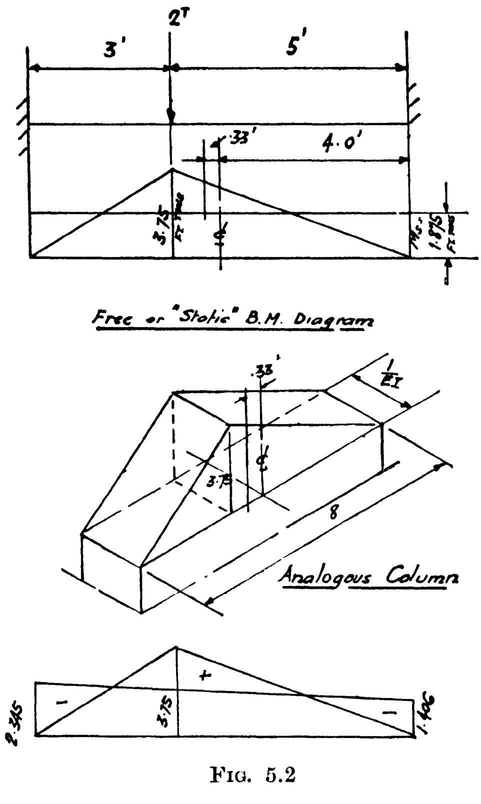

FIG. 5.2

and these relations form the column analogy. Therefore the rule can be laid down that the indeterminate (restraint) moments in a fixed beam correspond to the stresses in a short column with the following dimensions, viz.

length of cross-sectional area = length of beam,

width ,, ,, $= \dfrac{1}{EI}$.

Since the tangent to the elastic line is horizontal at both ends of a fixed beam, the fixing moment must be such that $i_x - i_{x0} = 0$, when $x = l$ = length of the beam. The loading on the analogous column is the 'free' B.M. at the corresponding point on the beam.

Convention for signs:

Positive load—downwards.

,, stress—compressive.

,, B.M.—produces tension on underside.

,, coordinates—measured upwards and to the right of the origin.

The method is best illustrated by an example (Fig. 5.2). Draw the B.M. diagram for a simply supported beam.

$$\text{Max. B.M.} = \frac{Wab}{l} = 3{\cdot}75 \text{ ft.-tons.}$$

Area of B.M. diagram $= 3{\cdot}75 \times \frac{8}{2} = 15$ ft.^{2}-tons.

Average B.M. $= 1{\cdot}875$ ft.-tons.

Distance of C.G. from $R_R = 4{\cdot}33$ ft.

Eccentricity $= 0{\cdot}33$ ft.

$$\text{Load} = W = \frac{15}{EI}.$$

$$\text{M.I. of section} = \frac{(1/EI) \times 8^3}{12} = \frac{42{\cdot}67}{EI} = I_1.$$

$$\text{Stress at ends} = \frac{W}{A} \pm \frac{W \times l \times \frac{1}{2}l}{I_1}$$

$$= 1{\cdot}875 \pm \frac{20}{42{\cdot}67} = \begin{cases} 2{\cdot}345 \text{ ft.-tons} \\ 1{\cdot}406 \quad ,, \end{cases}$$

which will be found to correspond with the values given by Wa^2b/l^2 and Wab^2/l^2. The method is particularly useful for beams of varying moment of inertia (Fig. 5.3). In this case the analogous column section must be of varying width.

$$\text{Max. B.M. for freely supported beam} = \frac{Wl}{4} = 5 \text{ ft.-tons.}$$

Average free B.M. $= 2{\cdot}5$ ft.-tons.

$$\text{Width of L.H. half of column section} = \frac{1}{2EI}$$

$$\text{,, R.H. ,, ,,} = \frac{1}{EI}.$$

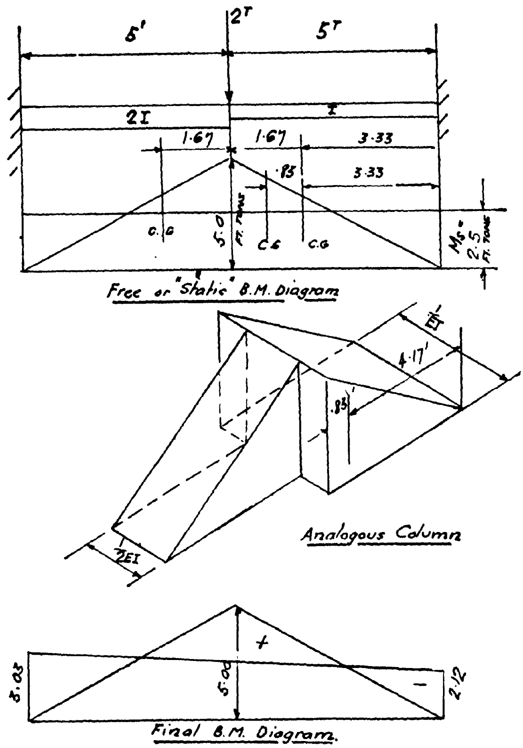

FIG. 5.3

Multiplying throughout by EI and taking moments about R.H. support, the calculation is set out in the table below:

Part	*Area (A)*	*Distance (x)*	*Ax*	*Ax²*	*Own I*	*Total I*
L.H.	2·5	7·5	18·75	140·625	5·208	145·833
R.H.	5·0	2·5	12·5	31·25	10·417	41·667
Total	7·5	. .	31·25	. .	. .	187·5

$$\text{Distance of C.G. from R.H. end} = \frac{31\cdot25}{7\cdot5} = 4\cdot17 \text{ ft.}$$

$$\therefore \quad \text{eccentricity} = 0\cdot83 \text{ ft.}$$

$$\text{Net } I \times EI = 187\cdot5 - 7\cdot5 \times 4\cdot17^2 = 57\cdot282.$$

Total Load on Analogous Column and Moments

Part	*Area × EI*	*Load × EI*	*Distance*	*Moment × EI*
L.H.	2·5	6·25	2·5	15·625
R.H.	5·0	12·5	0·83	−10·417
Total	7·5	18·75	. .	5·208

End moments:

$$\text{L.H.} = \frac{18{\cdot}75}{7{\cdot}5} + \frac{5{\cdot}208 \times 5{\cdot}83}{57{\cdot}282} = 3{\cdot}03 \text{ ft.-tons.}$$

$$\text{R.H.} = \frac{18{\cdot}75}{7{\cdot}5} - \frac{5{\cdot}208 \times 4{\cdot}17}{57{\cdot}282} = 2{\cdot}12 \quad ,,$$

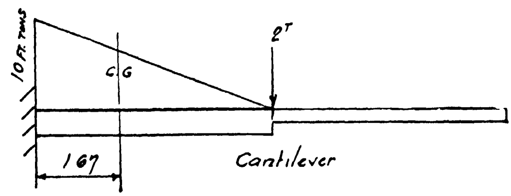

FIG. 5.3 (*a*)

Checking as cantilever from L.H. support (Fig. 5.3 (*a*)):
Determinate moment at L.H. = 10 ft.-tons, and
'Static' moment = 5 ft.-tons.
Load on L.H. of analogous column $\times EI = 12{\cdot}5$.
Moments from above:

$$\text{L.H.} = \frac{12{\cdot}5}{7{\cdot}5} + \frac{12{\cdot}5 \times 4{\cdot}17 \times 5{\cdot}83}{57{\cdot}282} = +6{\cdot}97 \text{ ft.-tons.}$$

$$\text{R.H.} = \frac{12{\cdot}5}{7{\cdot}5} - \frac{12{\cdot}5 \times 4{\cdot}17^2}{57{\cdot}282} = -2{\cdot}12 \text{ ft.-tons.}$$

Subtracting above moments from the 'cantilever' moments, final moments then become

$$\left.\begin{array}{l}\text{L.H.} = 10 - 6{\cdot}97 = 3{\cdot}03 \text{ ft.-tons} \\ \text{R.H.} = 0 - (-2{\cdot}12) = 2{\cdot}12 \quad ,,\end{array}\right\}\text{as before.}$$

Effect of Settlement of one Support

All the preceding calculations have been based upon the assumption that the supports are level. If, for some reason, settlement occurs at one of the supports relative to the other, then the moments and shears must be adjusted to suit. In the case shown in Fig. 5.4, the R.H. support sinks a distance Δ relative to the L.H. support. Now, since the tangent to the elastic line is horizontal at each support, it follows that there must be a point of contraflexure at mid-span. Assume an imaginary W at this point and divide the beam into two cantilevers each loaded with W at its end. The deflexion of each cantilever

$$= W\left(\frac{l}{2}\right)^2 \div 3EI = \frac{Wl^3}{24EI} = \frac{\Delta}{2}.$$

$$\therefore \quad W = \frac{12EI\Delta}{l^3},$$

and end moment $$= \frac{Wl}{2} = \frac{6EI\Delta}{l^2} = \frac{6EK\Delta}{l},$$

where $$K = \frac{I}{l}.$$

Note that the end moments are of opposite sign and therefore the reactions will have to be altered by an amount equal to $12EK\Delta/l^2$. Also that the C.G. of the fixing moment diagram will not be on the same vertical as the C.G. of the free B.M. diagram.

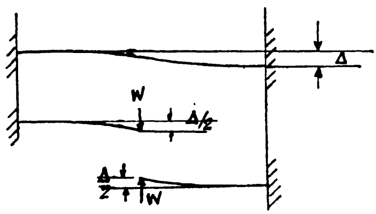

FIG. 5.4

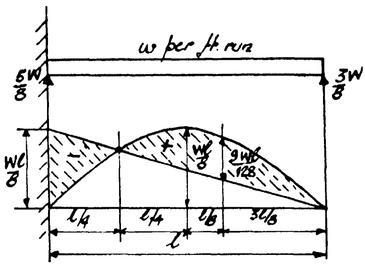

FIG. 5.5

Beams fixed One End only

Such beams are best treated as 'propped cantilevers', i.e. by finding the deflexion at the free end when the beam acts as a cantilever and then finding the force necessary to cancel this deflexion.

Case 1. Span l, uniform load w per ft. run (Fig. 5.5).

$$\text{Deflexion as cantilever} = \frac{Wl^3}{8EI} = \Delta.$$

If propping force $= F$,

$$\frac{Fl^3}{3EI} = \Delta = \frac{Wl^3}{8EI}.$$

$$\therefore \quad F = \frac{3W}{8},$$

and fixing moment $$= \frac{Wl}{2} - \frac{3Wl}{8} = \frac{Wl}{8},$$

and B.M. diagram is as shown with a point of contraflexure at $l/4$ from the fixed end.

Max. positive B.M. at $\frac{5}{8}l$ from fixed end $= 9Wl/128$.

The maximum deflexion occurs where the tangent to the elastic line is horizontal. By the conjugate beam method, the reactions are

$$+\frac{Wl^2}{24} \text{ at each end from 'free' B.M. diagram,}$$

and

$$\left.\begin{array}{l} -\dfrac{Wl^2}{24} \text{ at fixed end} \\ -\dfrac{Wl^2}{48} \text{ at free end} \end{array}\right\} \text{from fixing moment.}$$

Therefore net 'reaction' at fixed end is zero.

Let x = distance from fixed end to point of zero 'shear', then

$$\int \left(\frac{wlx}{2} - \frac{wx^2}{2}\right) dx = \frac{Wl^2}{16}\left\{1 - \left(\frac{l-x}{l}\right)^2\right\}.$$

$$\therefore \quad x = 0{\cdot}579l.$$

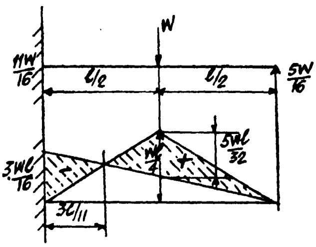

FIG. 5.6

Taking moments about fixed end,

$$EI\Delta = \int \left[\frac{wlx^2}{2} - \frac{wx^3}{2} - \left(\frac{wl^2}{8} \times \frac{(0{\cdot}58l)^2}{2} - \frac{wl}{8} \times \frac{0{\cdot}58l}{2} \times \frac{2}{3}(0{\cdot}58)^2\right)\right] dx$$

$$= 0{\cdot}0054wl^4.$$

$$\therefore \quad \Delta = \frac{0{\cdot}0054\,Wl^3}{EI} = \frac{Wl^3}{185EI} \text{ approx.}$$

Case 2. Same beam with load W at mid-span (Fig. 5.6). Treating as cantilever.

$$\text{Deflexion under load} = \frac{W(\tfrac{1}{2}l)^3}{3EI} = \frac{Wl^3}{24EI},$$

$$\text{Slope} \quad ,, \quad = \frac{W(\tfrac{1}{2}l)^2}{2EI} = \frac{Wl^2}{8EI}.$$

$$\therefore \text{ additional deflexion} = \frac{Wl^2}{8EI} \times \frac{l}{2} = \frac{Wl^3}{16EI},$$

$$\therefore \text{ total} \quad ,, \quad = \frac{5Wl^3}{48EI} = \frac{Fl^3}{3EI}.$$

$$F = \frac{5}{16}W \text{ and reaction at fixed end} = \frac{11W}{16}.$$

$$\text{Fixing moment} = \frac{Wl}{2} - \frac{5Wl}{16} = \frac{3Wl}{16}.$$

Distance from fixed end to point of contraflexure $= x$.

Then

$$\frac{3}{16}\left(\frac{l-x}{l}\right) = \frac{1}{4}\left(\frac{x}{\tfrac{1}{2}l}\right).$$

$$\therefore \quad x = \frac{3l}{11}.$$

Positive moment under load $= \frac{5Wl}{32}$.

Deflexion under load (freely supported) $= \frac{Wl^3}{48EI}$.

Upward deflexion due to fixing moment $= \frac{Wl^3}{64EI}$.

$\therefore$ Net deflexion under load $= \frac{Wl^3}{192EI}$.

Reactions from B.M. diagram:

	Fixed	*Free*
'Free' moment	$+\frac{Wl^2}{16}$	$+\frac{Wl^2}{16}$
Fixing moment	$-\frac{Wl^2}{16}$	$-\frac{Wl^2}{32}$
Total moment	0	$\frac{Wl^2}{32}$

Proceeding as before, as 'reaction' at fixed end is zero, area of B.M. diagram between the free end and point of zero shear must be $Wl/32$.

$$\text{Area of effective B.M. diagram in length } \frac{l}{2} = \frac{5Wl^2}{128}.$$

$$\frac{\text{Distance to point of zero shear}}{\frac{1}{2}l} = \sqrt{\frac{1/32}{5/128}} = \frac{2}{\sqrt{5}}.$$

$$\therefore \text{ distance} = \frac{l}{\sqrt{5}}.$$

$\therefore$ $EI\Delta =$ area of B.M. diagram $\times$ distance of C.G. from end

$$= \frac{Wl^3}{48\sqrt{5}}.$$

$$\therefore \Delta = \frac{Wl^3}{48\sqrt{5}EI} = \frac{Wl^3}{107EI} \text{ approx.}$$

Case 3. Point load W at a from fixed end (Fig. 5.7).

Deflexion under load (as cantilever) $= \frac{Wa^3}{3EI}$.

Slope ,, ,, $= \frac{Wa^2}{2EI}$.

Additional deflexion under load $= \frac{Wa^2(l-a)}{2EI}$.

$\therefore$ Total deflexion under load $= \frac{Wa^2(3l-a)}{6EI} = \frac{Fl^3}{3EI}$.

$$\therefore \quad F = \frac{Wa^2(3l-a)}{2l^3},$$

and $\qquad$ reaction at fixed end $= W\left(\frac{(l-a)(2l^2+2al-a^2)}{2l^3}\right),$

moment ,, ,, $= \frac{Wa(2l-a)(l-a)}{2l^2}.$

B.M. is as shown and deflexions can be found as for Cases 1 and 2.

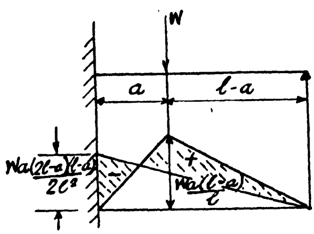

FIG. 5.7

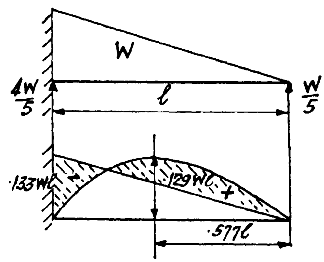

FIG. 5.8

Case 4. Triangular load W with apex at fixed end (Fig. 5.8).

From Chapter IV deflexion at free end

$$= \frac{15EI}{Wl^3} = \frac{Fl^3}{3EI}.$$

$$F = \frac{W}{5} \text{ and reaction at fixed end} = \frac{4W}{5}.$$

$$\text{Fixing moment} = \frac{Wl}{3} - \frac{Wl}{5} = \frac{2Wl}{15}$$

and B.M. diagram is as shown.

The distance from the freely supported end to the point of maximum B.M. is $0{\cdot}447l$. ($0{\cdot}577l$ for simply supported beam.)

Deflexions can be found as before.

'Propped cantilevers' can also be treated as one half of a beam continuous over two equal spans, as will be seen later in this chapter. The moment at the fixed end can be found by treating the beam as fixed at both ends and then transferring half the moment from the far end to the fixed end (this can be checked by comparing Cases 1–4 with the corresponding cases of fixed beams).

Example. A uniform beam of length L is rigidly built into a wall at each end and rests on a prop at mid-span. It carries a uniformly distributed load (including its own weight) of w per unit length. If the prop sinks a distance d below the level of the ends of the beam, show that the end B.M.

$$= \frac{WL^2}{48} + \frac{24EId}{L^2}$$

and sketch B.M.

$$\text{Deflexion due to uniform load} = \frac{WL^3}{384EI}.$$

$$\text{Residual deflexion} = d.$$

$$\text{Upward deflexion due to prop} = \frac{WL^3}{384EI} - d$$

$$= \frac{FL^3}{192EI}.$$

$$\therefore \quad F = \frac{W}{2} - \frac{192EId}{L^3}.$$

$$\text{End moment} = \frac{WL}{12} - \left(\frac{W}{2} - \frac{192EId}{L^3}\right)\frac{L}{8}$$

$$= \frac{WL}{48} + \frac{24EId}{L^2}.$$

B.M. diagram is shown in Fig. 5.9.

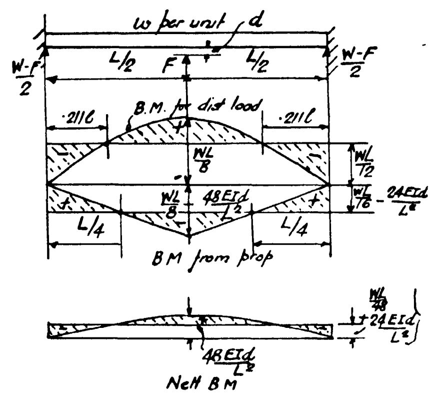

FIG. 5.9

Continuous Beams

Where a beam or girder is continuous over more than two supports, the system is statically indeterminate. The number of redundancies (the support moments) is equal to the number of intermediate supports.

Methods of analysis

1. Clapeyron's Theorem of Three Moments and its variations.
2. Method of 'Fixed' or 'Characteristic' Points.
3. Slope Deflexion method.
4. Hardy Cross or Distribution method.
5. Experimental method.

1. Clapeyron's theorem of three moments can be called the 'classic' method of analysis and the proof can be stated thus:

Consider any portion of a continuous beam ABC resting on three supports A, B, and C at the same level and assume that no relative settlement takes place (Fig. 5.10).

Let M_A, M_B, M_C be the support moments at A, B, and C respectively.

A_1 = area of free B.M. diagram for AB,

A_2 = ,, ,, ,, BC,

$AB = l_1$,

$BC = l_2$,

x_1 = distance of C.G. of free B.M. on AB to A,

x_2 = ,, ,, ,, BC to C.

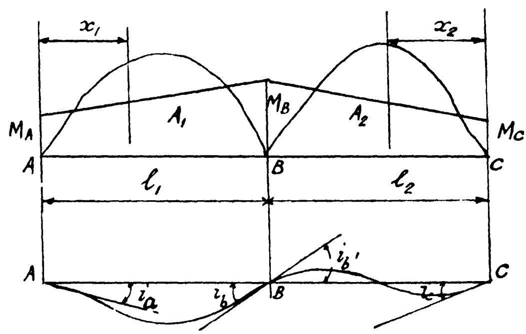

FIG. 5.10

Using the same reasoning as for the general case of fixed beams at the beginning of this chapter, and considering span AB with support A as the origin,

$$l_1 \times i_B - y_B - (0 \times i_A - y_A) = \frac{1}{EI_1}\int_0^{l_1} Mx\,dx.$$

i_B, i_A are the slopes of the tangent to the elastic line at B and A respectively and y_B, y_A are the corresponding relative deflexions; but since the latter are zero,

$$l_1 \times i_B = \frac{1}{EI_1}\int_0^{l_1} Mx\,dx = \frac{1}{EI_1}\times[\text{moment of B.M. diagram about } A]$$

$$= \frac{1}{EI_1}[A_1 x_1 - \tfrac{1}{6}(M_A l_1^2 + 2M_B l_1^2)].$$

Similarly, considering span BC with C as origin,

$$l_2 \times i_{B'} = \frac{1}{EI_2}[A_2 x_2 - \tfrac{1}{6}(M_C l_2^2 + 2M_B l_2^2)].$$

Since the elastic lines for AB and BC have a common tangent at B,

$$-i_{B'} = i_B,$$

and

$$i_B = \frac{1}{EI_1 l_1}[A_1 x_1 - \tfrac{1}{6}(M_A l_1^2 + 2M_B l_1^2)]$$

$$i_{B'} = \frac{1}{EI_2 l_2}[A_2 x_2 - \tfrac{1}{6}(M_C l_2^2 + 2M_B l_2^2)],$$

$$\therefore \quad \frac{M_A l_1 + 2M_B l_1}{I_1} + \frac{M_C l_2 + 2M_B l_2}{I_2} = 6\left(\frac{A_1 x_1}{l_1 I_1} + \frac{A_2 x_2}{l_2 I_2}\right), \qquad (3)$$

which is the general form of Clapeyron's theorem.

For constant moment of inertia $I_1 = I_2$ and the equation can be written as

$$M_A l_1 + 2M_B(l_1 + l_2) + M_C l_2 = 6\left(\frac{A_1 x_1}{l_1} + \frac{A_2 x_2}{l_2}\right).$$

By treating any two adjacent spans in this way, a number of simultaneous equations are derived, which can be solved algebraically to find the support moments, the number of equations being equal to that of the intermediate supports. For the case where the beam is *hinged* at the end supports, the end moments are zero. In the case where the beams are *fixed* at the ends, then the number of redundancies is more than the number of equations. A solution can be found by adding, at each end, another span which is symmetrical with the adjacent span, about the end support, and solving as before. For beams overhanging an end support, the moment at that support is equal to that produced by the cantilever portion.

A modification of the Clapeyron method was devised by the late Mr. T. R. Sturgeon. When used in conjunction with the graph in Fig. 5.11, this method saves a certain amount of time.

The equation (3) can be written thus:

$$\frac{l_1}{I_1}(M_A + 2M_B + K_1) + \frac{l_2}{I_2}(M_C + 2M_B + K_2) = 0, \qquad (4)$$

where $\quad K_1 = \dfrac{6A_1 x_1}{l_1^2} \quad$ and $\quad K_2 = \dfrac{6A_2 x_2}{l_2^2}$ numerically.

A number of corresponding equations can be derived. Where an end, such as A, is fixed or built in, then an additional equation such as

$$2M_A + M_B + K_1\left(\frac{l_1}{x_1} - 1\right) = 0 \qquad (5)$$

is required.

When the loading on the span is symmetrical, then

$$x = \frac{l}{2} \quad \text{and} \quad K = \frac{3A}{l}.$$

For a uniformly distributed load W on a span l,

$$\text{area of B.M. diagram} = \frac{Wl^2}{12}$$

and
$$K = \frac{Wl}{4}.$$

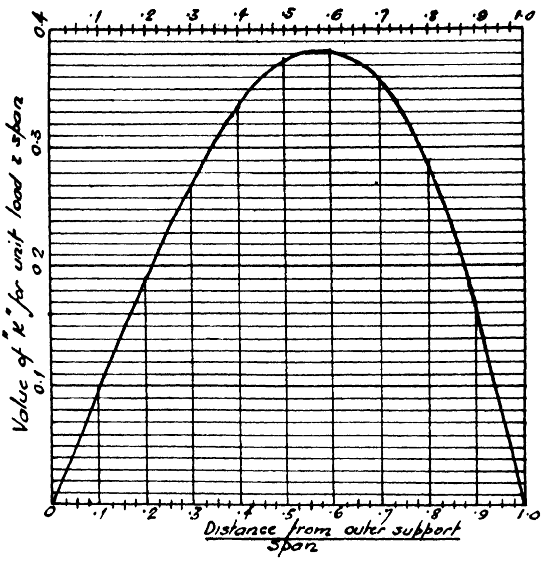

FIG. 5.11

For a number of point loads on a span AB, the B.M. due to a load w at x from A

$$= \frac{wx(l-x)}{l}.$$

$$\text{Area of B.M. diagram} = \frac{wx(l-x)}{2},$$

$$\therefore \quad \text{total area} = \sum \frac{wx(l-x)}{2}.$$

The moment of B.M. diagram about A

$$= \frac{wx(l-x)}{2} \times \frac{l+x}{3},$$

$$\therefore \quad \text{total moment} = \sum \frac{wx(l-x)(l+x)}{6},$$

and distance of C.G. from A

$$= \sum \frac{w(l+x)}{3\sum w}$$

Referring to Fig. 5.11, K can be found by the expression

$$\sum \text{load} \times \text{ordinate on graph} \times \text{span}.$$

For each load divide the distance from the outer support by the span length to find a decimal. Set off this decimal along the horizontal axis and move vertically to meet the curve and then horizontally to read off the value on the vertical scale. Note that since the graph is drawn for *unit* span and load, the value on the vertical scale must be multiplied by the actual values of the span and the load.

For an unsymmetrical system of loading

$$x_1 = \sum \frac{w(l+x)}{3\sum w},$$

$$K_1 = \frac{6A_1 x_1}{l_1^2} = \sum wx\left(\frac{l_1-x}{l_1}\right)\left(\frac{l_1+x}{l_1}\right),$$

from which K_1 can be found by slide rule. Some cases will now be worked out for the sake of comparison between Clapeyron's method and the above modifications.

Case 1. Two equal spans AB, BC with any loading which is symmetrical about B and constant I

$$x_1 = x_2; \qquad M_A = M_C = 0.$$

By Clapeyron's theorem,

$$2M_B \times 2l = 6 \times 2 \times \frac{Ax_1}{l}.$$

$$\therefore \quad M_B = \frac{3Ax_1}{l^2}.$$

For uniformly distributed load W

$$\text{B.M.} = \frac{Wl}{8}; \qquad A = \frac{Wl^2}{12}.$$

$$\therefore \quad M_B = -\frac{Wl}{8}.$$

From the modified method:

$$2M_B + K = 0; \qquad K = \frac{Wl}{4}.$$

$$\therefore \quad M_B = -\frac{Wl}{8}.$$

Addition to R_B due to moment

$$= 2 \times \frac{Wl}{8 \times l} = \frac{W}{4}.$$

$$\therefore \quad R_B = \frac{5W}{4} \quad \text{and} \quad R_A = R_C = \frac{3W}{8}.$$

(Note that half the beam gives the same result for moment and reactions as Case 1 for 'propped cantilevers'.)

Case 2. Two unequal spans $AB = l_1$ and $BC = l_2$. Moments of inertia are I_1 and I_2.

By Clapeyron's theorem

$$\frac{M_A l_1+2M_B l_1}{I_1}+\frac{M_C l_2+2M_B l_2}{I_2} = 6\left(\frac{A_1 x_1}{l_1 I_1}+\frac{A_2 x_2}{l_2 I_2}\right).$$

If $M_A = M_C = 0$,

$$\frac{M_B l_1}{I_1}+\frac{M_B l_2}{I_2} = 3\left(\frac{A_1 x_1}{l_1 I_1}+\frac{A_2 x_2}{l_2 I_2}\right).$$

By using modified method

$$\frac{l_1}{I_1}(2M_B+K_1)+\frac{l_2}{I_2}(2M_B+K_2) = 0.$$

For uniformly distributed load and constant inertia

$$K = \frac{Wl}{4} \quad \text{and} \quad l_1\left(2M_B+\frac{W_1 l_1}{4}\right)+l_2\left(2M_B+\frac{W_2 l_2}{4}\right) = 0.$$

$$\therefore \quad M_B = -\frac{1}{8}\left(\frac{W_1 l_1^2+W_2 l_2^2}{l_1+l_2}\right).$$

$$R_B = \frac{W_1+W_2}{2}+\frac{W_1 l_1^2+W_2 l_2^2}{(l_1+l_2)l_1}+\frac{W_1 l_1^2+W_2 l_2^2}{(l_1+l_2)l_2}.$$

$$R_A = \frac{W_1}{2}-\frac{W_1 l_1^2+W_2 l_2^2}{(l_1+l_2)l_1}; \qquad R_C = \frac{W_2}{2}-\frac{W_1 l_1^2+W_2 l_2^2}{(l_1+l_2)l_2}.$$

Case 3. Three unequal spans AB, BC, and CD, with moments of inertia I_1, I_2, and I_3 respectively, and loads on spans W_1, W_2, and W_3.

$$M_A = M_D = 0.$$

By Clapeyron,

$$2M_B(l_1+l_2)+M_C l_2 = 6\left(\frac{A_1 x_1}{l_1}+\frac{A_2 x_2}{l_2}\right),$$

$$2M_C(l_2+l_3)+M_B l_2 = 6\left(\frac{A_2(l_2-x_2)}{l_2}+\frac{A_3 x_3}{l_3}\right).$$

By the modified method,

$$\frac{l_1}{I_1}(2M_B+K_1)+\frac{l_2}{I_2}(2M_B+M_C+K_2) = 0,$$

$$\frac{l_3}{I_3}(2M_C+K_3)+\frac{l_2}{I_2}(2M_C+M_B+K_4) = 0.$$

For uniformly distributed load and constant inertia

$$l_1\left(2M_B+\frac{W_1 l_1}{4}\right)+l_2\left(2M_B+M_C+\frac{W_2 l_2}{4}\right) = 0,$$

$$l_3\left(2M_C+\frac{W_3 l_3}{4}\right)+l_2\left(2M_C+M_B+\frac{W_2 l_2}{4}\right) = 0,$$

from which M_B and M_C can be found and the support reactions calculated.

Case 4. Three spans AB, BC, and CD with loads W_1, W_2, and W_3 and inertias I_1, I_2, and I_3 respectively.

Ends A and D are fixed horizontally.

Then applying Clapeyron's theorem,

$$\frac{M_A l_1 + 2M_B l_1}{I_1} + \frac{M_C l_2 + 2M_B l_2}{I_2} = 6\left(\frac{A_1 x_1}{l_1 I_1} + \frac{A_2 x_2}{l_2 I_2}\right),$$

$$\frac{M_B l_2 + 2M_C l_3}{I_2} + \frac{M_D l_3 + 2M_C l_3}{I_3} = 6\left(\frac{A_2(l_2 - x_2)}{l_2 I_2} + \frac{A_3 x_3}{l_3 I_3}\right).$$

This cannot be solved directly as there are four unknowns and two equations and the only method possible is to 'reflect' AB and CD about A and D respectively.

Using the revised method,

$$\frac{l_1}{I_1}(M_A + 2M_B + K_2') + \frac{l_2}{I_2}(2M_B + M_C + K_2) = 0,$$

$$\frac{l_3}{I_3}(M_D + 2M_C + K_3) + \frac{l_2}{I_2}(2M_C + M_B + K_3') = 0,$$

$$2M_A + M_B + K_1' = 0,$$

$$2M_D + M_C + K_3' = 0.$$

If end D is freely supported, the last equation can be omitted.

For constant inertia and uniform load

$$l_1\left(M_A + 2M_B + \frac{W_1 l_1}{4}\right) + l_2\left(2M_B + M_C + \frac{W_2 l_2}{4}\right) = 0,$$

$$l_3\left(M_D + 2M_C + \frac{W_3 l_3}{4}\right) + l_2\left(2M_C + M_B + \frac{W_2 l_2}{4}\right) = 0,$$

$$2M_A + M_B + \frac{W_1 l_1}{4} = 0,$$

$$2M_D + M_C + \frac{W_3 l_3}{4} = 0,$$

from which the moments and reactions can be found.

2. *Method of fixed or characteristic points.* The following construction can be used to find the position of the 'fixed points', i.e. where the negative B.M. diagram crosses the base line. In Fig. 5.12, set off the third points in the spans AB, BC, and CD of lengths l_1, l_2, l_3, etc., and draw verticals through these points. Working from A, set off the fixed point F_1.

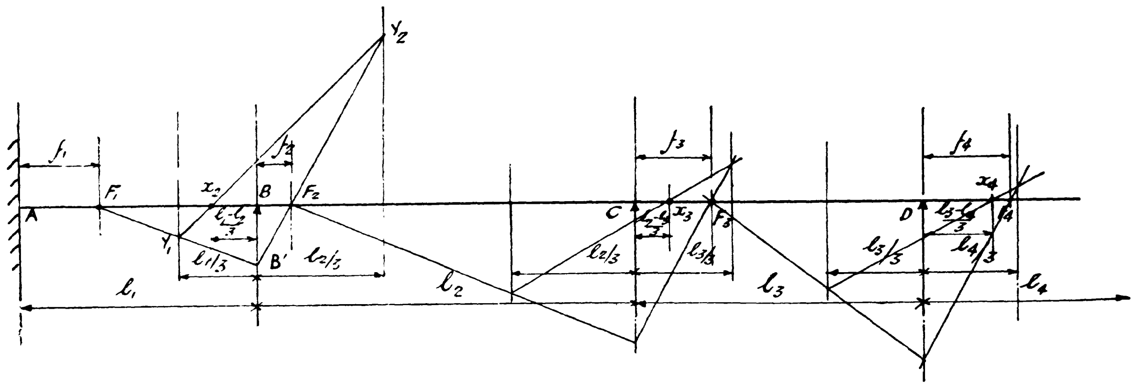

Fig. 5.12

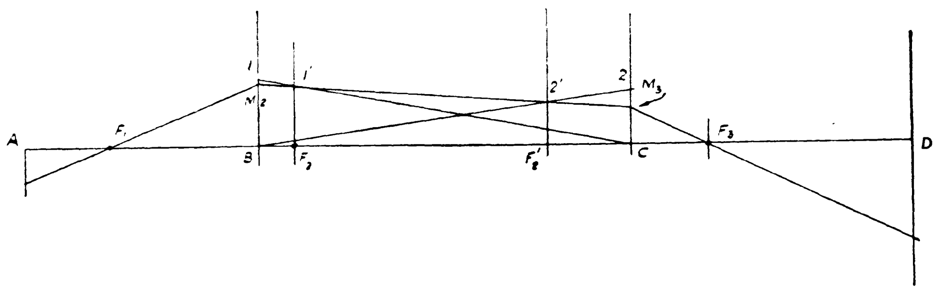

Fig. 5.12 (a)

L

When A is simply supported, distance of F_1 from $A = 0$.

When A is fixed, distance of F_1 from $A = l_1/3$.

From B set off $Bx_2 = \frac{1}{3}(l_1-l_2)$,

,, C ,, $Cx_3 = \frac{1}{3}(l_2-l_3)$,

,, D ,, $Dx_4 = \frac{1}{3}(l_3-l_4)$, and so on.

Draw any line through x_2 to meet the third point vertical in Y_1 and produce it to meet the third point vertical at Y_2. Join $F_1 Y_1$ and produce it to meet the vertical through B at B'. Join $B'Y_2$ and find the fixed point F_2. F_3, F_4 can be found in the same way, and working from the other end, the corresponding points F'_1, F'_2, F'_3, F'_4 can be found. For equal spans, the values of the fixed points are:

Distance of fixed point . .	AF_1	BF_2	CF_3	DF_4
End A fixed	$l/3$	$2l/9$	$0{\cdot}2121l$	$0{\cdot}2113l$
End A freely supported .	0	$l/5$	$0{\cdot}2105l$	$0{\cdot}2113l$

Having found the fixed points, the free B.M. diagram can be drawn. Considering span BC (Fig. 5.12 (a)), set up

$$\begin{cases} B1 = \dfrac{6A_2 x_2}{l_2^2}, \\ C2 = \dfrac{6A_2(l_2-x_2)}{l_2^2}. \end{cases}$$

$$\frac{6A_2 x_2}{l_2^2} = \begin{cases} \dfrac{Wl}{4} \text{ for distributed load,} \\ \dfrac{3Wl}{8} \text{ for central point load.} \end{cases}$$

Join $C1$ and $B2$ to intersect the verticals through F_2 and F'_2 in points $1'$ and $2'$. Join $1'2'$ and produce this line to intersect $B1$, $C2$ at M_2 and M_3. Join $M_2 F_1$ and produce the line to meet the vertical through A. Also join $M_3 F_3$ and produce this line to meet the vertical through D. Treat the other spans similarly and combine B.M. values to obtain the complete B.M. diagram.

It will be noticed that the values $6Ax/l^2$ must be found as for the Clapeyron method, and up to this point the work involved is the same. The difference lies in the fact that this method solves the problem graphically, whereas in the previous method a number of simultaneous equations had to be solved. For the method of fixed points, the graph of Fig. 5.11 can be used to find the values Ax, etc., and reduce the amount of work necessary.

3. *Slope deflexion methods.* This method was introduced by Mohr and developed by Wilson and Maney and extended in scope by Professor Ostenfeld. It uses the deflexions and rotations as the redundants. It should be noticed that relative settlements at the supports are taken

into account automatically, whereas in the two previous methods the effects of settlements are ignored and must be calculated separately.

The general form of the basic equation is

$$M_{AB} = 2EK\left(2\theta_A+\theta_B-\frac{3\Delta}{l}\right),$$

$$M_{BA} = 2EK\left(2\theta_B+\theta_A-\frac{3\Delta}{l}\right),$$

where M_{AB}, M_{BA} are end moments, θ_A, θ_B are the angular rotations,

$$K = \frac{I}{l} \quad \text{and} \quad \Delta = \text{relative settlement in span } l.$$

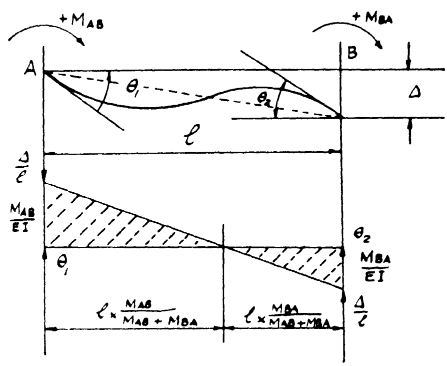

FIG. 5.13

It should also be noted that this method is based upon a beam fixed at the ends as the angular rotations are measured from the horizontals at A and B. In order to find the solution of the problem it is necessary to know the fixing moments M_{FAB} and M_{FBA} and to add the moments M_{AB}, M_{BA} algebraically to find the actual moments at A and B.

Proof of slope deflexion equation. Consider the case shown in Fig. 5.13 with moments M_{AB}, M_{BA} and deflexion at $B = \Delta$.

The angular rotations at A and B are θ_1, θ_2 respectively. Using the conjugate beam method, load the beam with M_{AB}/EI and M_{BA}/EI and 'reactions' θ_1, θ_2. There is an equivalent 'reaction' at $A = \Delta/l$ and a balancing 'reaction' at $B = \Delta/l$. The total 'loading' must balance the 'reactions'.

$$\therefore \quad \frac{1}{EI}\left[\frac{(M_{AB})^2 l}{2(M_{AB}+M_{BA})}-\frac{(M_{BA})^2 l}{2(M_{AB}+M_{BA})}\right] = \theta_1-\theta_2+\left(\frac{\Delta}{l}-\frac{\Delta}{l}\right).$$

$$\left(K = \frac{I}{l}\right) \qquad \therefore \quad M_{AB}-M_{BA} = 2EK(\theta_1-\theta_2). \tag{6}$$

Also, by taking moments about R.H. support,

$$\left(\theta_1-\frac{\Delta}{l}\right)l = \frac{1}{EI}\left[\frac{(M_{BA})^2 l}{2(M_{AB}+M_{BA})}\times\frac{M_{BA}\,l}{3(M_{AB}+M_{BA})}\right].$$

$$\therefore \quad 2M_{AB}-M_{BA} = 6EK\left(\theta_1-\frac{\Delta}{l}\right). \tag{7}$$

Combining equations (6) and (7),

$$M_{AB} = 2EK\left(2\theta_1+\theta_2-\frac{3\Delta}{l}\right),$$

$$M_{BA} = 2EK\left(2\theta_2+\theta_1-\frac{3\Delta}{l}\right),$$

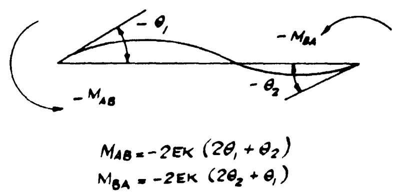

FIG. 5.13 (*a*)

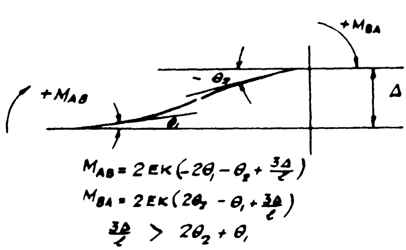

FIG. 5.13 (*b*)

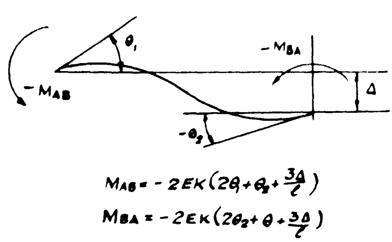

FIG. 5.13 (*c*)

which are the relations for Fig. 5.13. If there is no deflexion the term $3\Delta/l$ disappears. These relations hold good only if

$$2\theta_1+\theta_2 > \frac{3\Delta}{l}.$$

For other cases the values of M_{AB}, M_{BA} are as shown in Figs. 5.13 (*a*), (*b*), and (*c*). The sign convention rather detracts from the usefulness of this method and the fact that the fixed-end moments must be calculated must be remembered. One or two examples will be worked to illustrate the method.

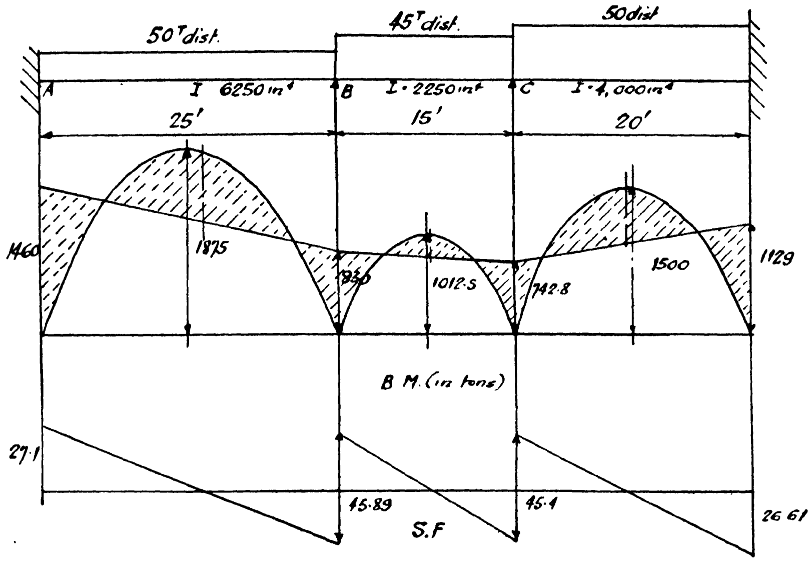

FIG. 5.14

Example 1. A beam $ABCD$ is as shown in Fig. 5.14. Ends A and D are fixed ($\theta_A = \theta_D = 0$).

$$M_{FAB} = -\frac{Wl}{12} = -1{,}250 \text{ in.-tons.} \qquad M_{FBA} = +1{,}250 \text{ in.-tons,}$$

$$M_{FBC} = -675 \text{ in.-tons} \qquad M_{FCB} = +675 \quad ,,$$

$$M_{FDC} = -1{,}000 \quad ,, \qquad M_{FDC} = +1{,}000 \quad ,,$$

$$M_{AB} = 500E\theta_B - 1{,}250,$$

$$M_{BA} = 1{,}000E\theta_B + 1{,}250,$$

$$M_{BC} = 600E\theta_B + 300E\theta_C - 675,$$

$$M_{CB} = 600E\theta_C + 300E\theta_B + 675,$$

$$M_{CD} = 800E\theta_C - 1{,}000,$$

$$M_{DC} = 400E\theta_C + 1{,}000.$$

For equilibrium

$$M_{BA} = -M_{BC} \quad \text{and} \quad M_{CB} = -M_{CD}.$$

$$\therefore \quad E\theta_B = -0{\cdot}420 \quad \text{and} \quad E\theta_C = +0{\cdot}323.$$

Hence

$$M_{AB} = -1{,}460 \text{ in.-tons;} \qquad M_{BA} = +830 \text{ in.-tons} = -M_{BC}.$$

$$M_{CB} = +743 \text{ in.-tons} = -M_{CD}; \qquad M_{DC} = +1{,}129 \text{ in.-tons.}$$

Reactions	*A*	*B*	*C*	*D*
Freely supported	25	47·5	47·5	25·0
From moments	+2·1	−2·1 +0·49	−0·49 −1·61	+1·61
Totals	27·1	45·89	45·4	26·61

Total reactions = 145 tons = total load.

Check by theorem of three moments. In order to obtain sufficient equations, it is necessary to add spans $A'A$ and DD' symmetrical with AB and CD respectively. Alternatively, using modified form, four equations are found to solve for four unknowns, viz.

$$\frac{1}{250}\left(M_A+2M_B+\frac{1250}{4}\right)+\frac{1}{150}\left(M_C+2M_B+\frac{675}{4}\right)=0,$$

$$\frac{1}{150}\left(M_B+2M_C+\frac{675}{4}\right)+\frac{1}{200}\left(M_D+2M_C+\frac{1000}{4}\right)=0,$$

$$2M_A+M_B+\frac{1250}{4}=0,$$

$$2M_D+M_C+\frac{1000}{4}=0.$$

From above equations

$$\left.\begin{array}{ll} M_A=-1{,}457; & M_B=-839 \\ M_C=-740; & M_D=-1{,}129 \end{array}\right\}\text{in.-tons.}$$

The above values, obtained by slide rule, show close agreement with the slope deflexion results.

Example 2. Spans $ABCD$ with point loading as shown in Fig. 5.15. Ends A, D are hinged ($M_{AB}=M_{DC}=0$).

Fixed end moments

$$M_{FAB}=-48 \text{ ft.-tons};\quad M_{FBA}=+72 \text{ ft.-tons},$$
$$M_{FBC}=-M_{CB}=-50 \text{ ft.-tons},$$
$$M_{FCD}=-22{\cdot}22 \text{ ft.-tons};\quad M_{FDC}=+44{\cdot}44 \text{ ft.-tons}.$$

Relative values of K are taken for simplicity in equations

$$M_{AB}=2E(2\theta_A+\theta_B)-48=0,$$
$$M_{BA}=2E(2\theta_B+\theta_A)+72=-M_{BC}=-[2{\cdot}5E(2\theta_B+\theta_C)-50],$$
$$M_{CB}=2{\cdot}5E(2\theta_C+\theta_B)+50=-M_{CD}=-[3E(2\theta_C+\theta_D)-22{\cdot}22],$$
$$M_{DC}=3E(2\theta_D+\theta_C)+44{\cdot}44=0.$$

Solving, the following values are obtained:

$$E\theta_A=+15{\cdot}035;\quad E\theta_B=-6{\cdot}07;\quad E\theta_C=+1{\cdot}011;\quad E\theta_D=-7{\cdot}912.$$

$$\therefore\ M_{BA}=+77{\cdot}79;\ M_{BC}=-77{\cdot}81;\ M_{CB}=+39{\cdot}88;\ M_{CD}=-38{\cdot}89.$$

(All above moments are in ft.-tons.)

Check by Clapeyron's theorem of three moments:

Span AB: Free B.M. $=$ 120 ft.-tons; Area $=$ 1,500 ft.^{2}-tons
,, BC: ,, $=$ 100 ,, ; ,, $=$ 1,000 ,,
,, CD: ,, $=$ 66·67 ft.-tons; ,, $=$ 500 ,,

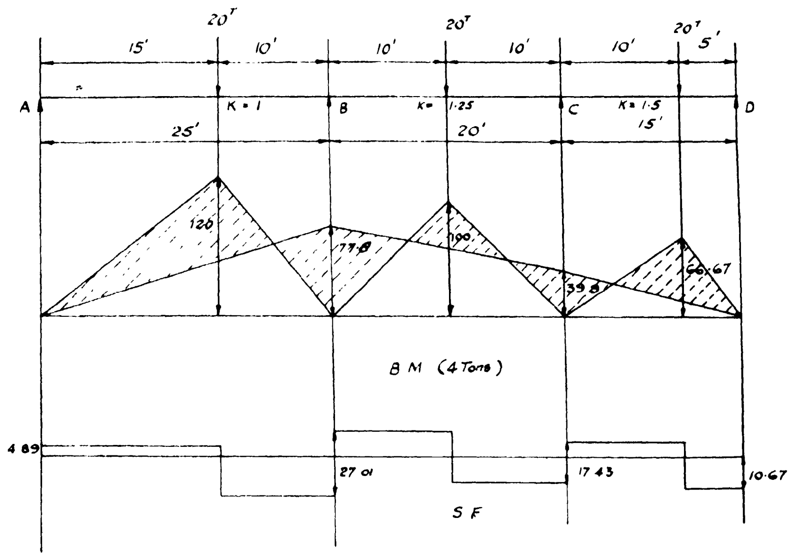

FIG. 5.15

Distances from C.G. of free B.M. diagram to supports are:

span AB:—13·33 ft.; span CD:—6·67 ft.

$$\therefore \quad 1\left(2M_B + \frac{6\times 1500\times 13{\cdot}33}{25^2}\right) + \frac{1}{1{\cdot}25}\left(2M_B + M_C + \frac{3000}{20}\right) = 0,$$

$$\frac{1}{1{\cdot}25}(M_B + 2M_C + 150) + \frac{1}{1{\cdot}5}\left(2M_C + \frac{6\times 500\times 6{\cdot}67}{15^2}\right) = 0.$$

$$\therefore \quad M_B = -77{\cdot}8 \text{ ft.-tons}; \qquad M_C = -39{\cdot}9 \text{ ft.-tons}.$$

	A	B	C	D
Freely supported reactions (tons)	8	22	16·67	13·33
Reactions from moments (tons)	$-3{\cdot}11$	$+(3{\cdot}11+1{\cdot}9)$	$-1{\cdot}9+2{\cdot}66$	$-2{\cdot}66$
Total reactions (tons)	4·89	27·01	17·43	10·67

Total reactions $=$ 60 tons $=$ total loads.

Note that in these two examples no settlement of the supports has been allowed for. In cases where this occurs, the induced moment is $6EK\Delta/l$.

4. *Hardy Cross*† *or moment distribution method.* This method has been widely used since it was first suggested by Professor Hardy Cross, not only for continuous beams but for other statically indeterminate structures. Like the slope deflexion method it involves the use of fixed-end moments, but on the other hand it is a method of successive approximations.

The beam is first considered to be fixed in direction at each joint before loading, and therefore when the loading is applied the continuous beam consists of a series of fixed-end beams. Owing to the fact that the fixed-end moments do not, as a rule, balance, the joints are 'out of balance' by a certain amount. Now, if any one joint is released, it will tend to rotate into a position of equilibrium and the unbalanced moment will distribute itself among the members meeting at that joint. In order to prevent rotation, it is necessary to apply a moment equal and opposite to the algebraic sum of the fixed-end moments. Now if the joint is released, this moment will be distributed among the members in the ratio $= \dfrac{\text{stiffness of member}}{\text{total stiffnesses of members}}$. At the same time there will be a 'carry-over' of moments to the remote ends of the members equal to half the amount distributed to the near ends. This is obvious from the slope deflexion equations

$$M_{AB} = 2EK(2\theta_A+\theta_B); \quad M_{BA} = 2EK(2\theta_B+\theta_A).$$

If $\theta_B = 0$, i.e. far end remains fixed,

$$M_{AB} = 4EK\theta_A; \qquad M_{BA} = 2EK\theta_A = \tfrac{1}{2}M_{AB}.$$

The carry-over is added to the algebraic sum of the balancing moments. The other joints are released in turn and the process repeated until the out-of-balance moments are negligible. The method can be summarized thus:

1. Write down the stiffnesses and the fixed-end moments due to the loading.
2. At each joint, evaluate the 'balancing' moment and the distribution factors $= \dfrac{\text{stiffness}}{\sum \text{stiffness}}$.
3. Release the first joint and distribute the balancing moment.
4. Make the carry-over to the adjacent joints.
5. Release the other joints and continue the process until all joints are balanced.

Special cases.

1. Beams freely supported at one end.

† H. Cross, 'Analysis of Continuous Frames by Distributing Fixed-end Moments', *Trans. Amer. Soc. C.E.*, **96** (1932), p. 1 et seq.

By slope deflexion,

$$M_{AB} = 2EK(2\theta_A+\theta_B); \qquad M_{BA} = 2EK(2\theta_B+\theta_A) = 0.$$

$$\therefore \quad \theta_B = -\tfrac{1}{2}\theta_A.$$

$$\therefore \quad M_{AB} = 2EK(\tfrac{3}{2}\theta_A) = 3EK\theta_A$$

$$= \tfrac{3}{4}\times M_{AB} \text{ for beam fixed at } B.$$

Hence this case can be dealt with by assuming an equivalent stiffness $= \frac{3}{4}\times$nominal stiffness I/l.

2. Overhanging ends can be treated by considering the last support to be hinged as (1) and then adding the cantilever moment to the support moment (which of course is zero).

Since this method is arithmetical, it is best illustrated by its application to the two preceding examples.

Example 1. Fixed-end moments are $AB = 1{,}250$, $BC = 675$, $CD = 1{,}000$ in.-tons.

The ratios of the stiffnesses are $AB:BC:CD = 5:3:4$.

The distribution factors are: at B, $AB = \frac{5}{8}$, $BC = \frac{3}{8}$; at C, $BC = \frac{3}{7}$, $CD = \frac{4}{7}$.

The work is best done in tabular form as below. Note that the beam cannot be released at A and D as it is fixed at these points. The balancing moments are shown thus (−575).

Rel. K	*A* 5	*B*	3	*C*	4	*D*
F.E.M.	−1,250	+1,250	−675	+675	−1,000	+1,000
Bal.			(−575)			
Dist.		*−359·37*	*−215·63*			
C.O.	−179·69			−107·82		
				(+432·82)		
Dist.				*+185·50*	*+247·32*	
C.O.		−(92·75)	+92·75			+123·66
Dist.		*−57·97*	*−34·78*			
C.O.	−28·99			−17·39	(+17·39)	
Dist.				*+7·60*	*+9·79*	
C.O.		(−3·80)	+3·80			+4·90
Dist.		*−2·37*	*−1·43*			
C.O.	−1·18			−0·72	(+0·72)	
Dist.				*+0·41*	*+0·31*	
C.O.		(−0·20)	+0·20			+0·16
Dist.		*−0·12*	*−0·08*			
C.O.	−0·06			−0·04	(+0·04)	
Dist.				*+0·02*	*+0·02*	
C.O.			+0·01			+0·01
Total moments	−1,459·92	+830·17	−830·16	+742·56	−742·56	+1,128·73

It will be seen that the moments balance at B and C and that the values agree with those previously found.

Example 2. In this case ends A and D are hinged so that

$$M_{AB} = M_{DC} = 0.$$

In order to have zero moments at these points, it is necessary, after each carry-over to A and D, to apply an equal and opposite balancing moment. The balancing moment is then carried over and the process is repeated so far as is necessary. As an exercise the student is recommended to work this example using the equivalent stiffnesses for AB and CD, i.e. $\frac{3}{4} \times I/l$, and it should be noted that the fixed-end moments should be altered to suit the condition 'one end free'. Note that there is no carry-over to the free ends. The amount of arithmetic is reduced by using this alternative method.

$$\text{Distribution factors } B\begin{cases} AB = \frac{4}{9} \\ BC = \frac{5}{9}. \end{cases}$$

$$C\begin{cases} BC = \frac{5}{11} \\ CD = \frac{6}{11}. \end{cases}$$

Rel. K	*A*	4	*B*		5	*C*		6	*D*
F.E.M.	−48		+72	−50		+50	−22·22		+44·44
				(−22)					
Bal. and C.O.	−4·89	←	*−9·79*	*−12·21*	→	−6·10	(−21·68)		
,, ,,	*+52·89*	→	+26·45	−4·92	←	*−9·85*	*−11·83*	→	−5·92
				(−21·53)					
,, ,,	−4·78	←	*−9·57*	*−11·96*	→	−5·98	−19·26	←	*−38·52*
						(+25·24)			
,, ,,	*+4·78*	→	+2·39	+5·73	←	*+11·47*	*+13·77*	→	+6·88
				(−8·12)					
,, ,,	−1·80	←	*−3·61*	*−4·51*	→	−2·25	−3·44	←	*−6·88*
						(+5·69)			
,, ,,	*+1·80*	→	+0·90	+1·30	←	*+2·59*	*+3·10*	→	+1·55
				(−2·20)					
,, ,,	−0·49	←	*−0·98*	*−1·22*	→	−0·61	−0·78	←	*−1·55*
							(+1·39)		
,, ,,	*+0·49*	→	+0·25	+0·31	←	*+0·63*	*+0·76*	→	+0·38
				(−0·56)					
,, ,,	−0·12	←	*−0·25*	*−0·31*	→	−0·16	−0·19	←	*−0·38*
							(+0·35)		
,, ,,	*+0·12*	→	+0·06	+0·08	←	*+0·16*	*+0·19*	→	+0·10
				(−0·14)					
,, ,,	−0·03	←	−0·06	−0·08	→	−0·04	−0·05	←	*−0·10*
							(+0·09)		
,, ,,	*+0·03*	→	+0·01	+0·02	←	*+0·04*	*+0·05*	→	+0·02
				(−0·03)					
			−0·01	−0·02	→	−0·01	−0·01	←	*−0·02*
							(+0·02)		
						+0·01	*+0·01*		
Total moments	0		+77·78	−77·79		+39·90	−39·90		0

If settlement of the supports occurs, the moments due to this ($\pm 6EK\Delta/l$) are written down and distributed until balance is obtained. The resulting moments are then added to those due to the loading to find the final moments.

5. *Experimental methods.* Such methods have not been used extensively in this country up to the present (apart from research carried out by scientific bodies), but the use of these for actual design purposes is now being widely adopted. These methods can be classified under two main headings: (*a*) direct, and (*b*) indirect.

(*a*) The direct method consists of applying to a model the actual forces, reduced, of course, in a convenient ratio. This method is applicable only to members subject to a symmetrical B.M. diagram.

From the slope deflexion equations

$$M_{AB} = 2EK\left(2\theta_A+\theta_B-\frac{3\Delta}{l}\right)-M_{FAB},$$

$$M_{BA} = 2EK\left(2\theta_B+\theta_A-\frac{3\Delta}{l}\right)+M_{FBA}.$$

Now for a symmetrical B.M. diagram, $M_{FAB} = M_{FBA}$ numerically.

Also they are equal to A/l, where A = area of free B.M. diagram. Hence the equations can be written in the form

$$M_{AB} = 2EK\left(2\theta_A+\theta_B-\frac{3\Delta}{l}\right)-\frac{A}{l},$$

$$M_{BA} = 2EK\left(2\theta_B+\theta_A-\frac{3\Delta}{l}\right)+\frac{A}{l}.$$

Therefore, knowing A and K, the values of θ_A, θ_B, and Δ can be measured and the moments calculated. Usually the model is mounted upon rollers on a drawing-board and the loads are applied by weights and pulleys or by spring balances. Light pointers attached to each end of the members give the end movements. Care should be taken to prevent buckling of the model under load (this can be done by placing weights upon the model). A full description of the application of this method to the analysis of building frames has been given by Prof. J. F. Baker (see Bibliography).

(*b*) The indirect method consists usually of supporting an unloaded model in a manner similar to the beam under consideration. The method was first described by Prof. G. E. Beggs in the *Proceedings of the American Concrete Institute*, and has been developed in this country by Prof. A. J. S. Pippard and Dr. S. R. Sparkes. It has much to recommend it as it can be used with the simplest of apparatus by any careful investigator after a little practice in its use. The principle is based upon a theorem enunciated by Prof. Müller-Breslau. If A, B, and C are the supports of a continuous beam (see Fig. 5.16) it is required to find the influence line for the reaction at C. This support can be removed and replaced by a unit vertical force. Then the deflected shape of ABC is shown by the curved line, which represents the deflexion at all points on ABC, and is therefore the influence line for deflexion at C. Now if a unit load be placed at any point P, the deflexion produced at C will be δ, the ordinate under that load. The unit force

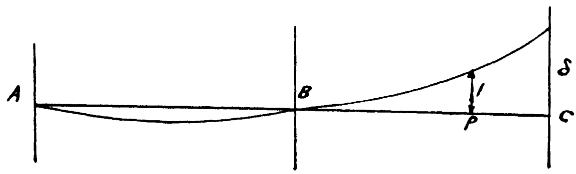

FIG. 5.16

acting at C produces a deflexion Δ at that point. Therefore in order to bring the point C back to its original position, a force of $1 \times \delta/\Delta$ must be applied. Hence it follows that if the ordinates of the deflected shape are divided by Δ, the resulting figure is the influence line for reaction at C, and this principle is used to find the reactions at the supports.

Considering the case of a continuous beam $ABCD$, the model, which is made of thin celluloid, xylonite, or even a thin spline to a suitable scale for length, is pinned at A, B, C, and D to a sheet of paper on a drawing-board. The edge of the model is traced on the paper. To find the reaction at any support, take away the pin at that point and displace the point vertically upwards and trace the line of the edge again. Then, by taking a number of points along the span and dividing their ordinates (between the pencil lines) by the displacement at the support, the influence line for reaction at that support is obtained. The same process is repeated for the other supports. The displacements given should be large enough for the ordinates to be measured directly on a finely divided scale (the section of the beam is assumed to be uniform). The reaction for any particular load is given by the expression

$$\sum \text{load} \times \frac{\text{ordinate}}{\text{displacement}}.$$

For distributed loading this expression becomes

$$\sum \text{area of influence line} \times \text{load intensity}.$$

These expressions must also be multiplied by a number depending upon the linear scale, e.g. for $\frac{1}{2}$ in. to 1 ft., multiply by 2. Further reference will be made to this method when dealing with rolling loads, and fuller information can be found in the papers mentioned in the Bibliography.

Comparison of Methods used in the Analysis of Continuous Beams

1. Clapeyron's theorem of three moments is useful for a small number of spans or for a large number of equal spans of uniform sections. For a larger number of spans or irregular loading the solution of the simultaneous equations involved becomes laborious.

2. The graphical methods are also convenient in fairly straightforward problems, but for unequal spans or loadings, the work involved in constructing the diagrams, after having found the 'fixed points', is considerable, and the possibility of errors is correspondingly increased. Like Method 1, it does not allow for moments caused by settlement at the supports.

3. Slope deflexion is applicable to any case as it allows for settlement at the supports. While the solution of the equations may prove laborious for large numbers of spans, the method is worthy of close study as it can be applied to all types of rigid frames.

4. The Hardy Cross method is one of the most useful tools in the hands of the designer. Its simplicity will appeal to many, as once the end moments and distribution factors are evaluated, the process is purely arithmetical. As in method 3, there is a certain check on the accuracy of the working owing to the fact that the moments at any joint must be in equilibrium. This method can also be applied to a wide range of rigid frames.

5. *Experimental methods.* The great virtue of such methods lies in the fact that problems may be solved by experiment in a matter of hours which would take days or even weeks to analyse by calculation. The agreement between observed and calculated results is remarkable, and any error is unlikely to exceed that due to faulty initial estimates of the loading. It is certainly extremely useful as a check on the results of mathematical analysis.

6. *Other methods.* These include the method of least work combined with moment area; moment balance and four-moment theorem, but space precludes any attempt to deal with these.

Effect of Settlement of Supports

Where settlement occurs it is probably spread over an appreciable time and the *relative* settlement between adjacent supports is probably almost negligible, so that the resulting moments and shears will have comparatively slight effects on the structure. Large relative settlements are unlikely to occur unless the foundations have been badly designed or constructed. For R.C. beams it is probable that plastic flow will relieve the effect of settlements very considerably.

Deflexions of Continuous Beams

Once the support moments have been found, the effective B.M. diagram can be drawn and the deflexion at any point found by the conjugate beam method. (The slope deflexion method gives the slopes of the elastic line at each support.)

Rolling Loads on Continuous Beams

When dealing with this subject we must consider two cases: (1) rolling loads on continuous beam bridges (which is the commoner case), and (2) rolling loads on other continuous structures. The problem is simplified when the rolling load can be expressed as an equivalent uniformly distributed load. Some authorities take the case of alternate spans loaded which, theoretically, gives the maximum values of B.M. at the mid-spans and supports, but this condition is unlikely to occur in practice. The more probable condition is that two trains of vehicles, with a short space between them, are crossing the span. The influence line method is particularly useful in the analysis of such cases. The principles stated under 'Experimental methods' can be applied. In every case a *unit* force or *unit* moment is applied at the point under consideration. Then by finding the ordinates to the deflected shape and dividing by the displacement (unity), the influence line ordinates are obtained. The method for finding an influence line for reaction has already been explained. To find an influence line for *shear* at any point, a vertical unit displacement must be applied at that point (without relative rotation at the two ends). Finally, to find the influence line for B.M. a unit rotation (i.e. one radian) or unit couple must be applied at the point. Typical influence lines for shear and B.M. are shown in Fig. 5.17.

For design purposes the structural engineer must obtain the curves of maximum B.M. and shear for the spans so that he can design for the worst cases. Reference has been made under 'Graphical methods' to fixed or characteristic points. Generally speaking, these lie at or near the third points for continuous beams (values are given on p. 146). From a study of Fig. 5.17 it will be noticed that for any central portion of any span between the fixed points, the maximum B.M. occurs when the spans are fully loaded. For the parts of the spans lying between the fixed points and the supports, maximum B.M. occurs when the spans are partially loaded (these can be approximated when the B.M.s at the fixed points have been found). Maximum B.M. at the supports occurs when the spans are fully loaded. For positive B.M. between the fixed points, find the sum of the moments produced by live load on that span and on each alternate span (plus, of course, the moment due to dead load). For negative B.M. find the sum of the moments due to live load on the *other* spans and the dead load. Then the maximum B.M. curve for the parts of the spans between the fixed points can be drawn. For the support moments, the maximum (negative) values occur when both the spans supported and each alternate span are loaded, the positive values being obtained by reversing this condition

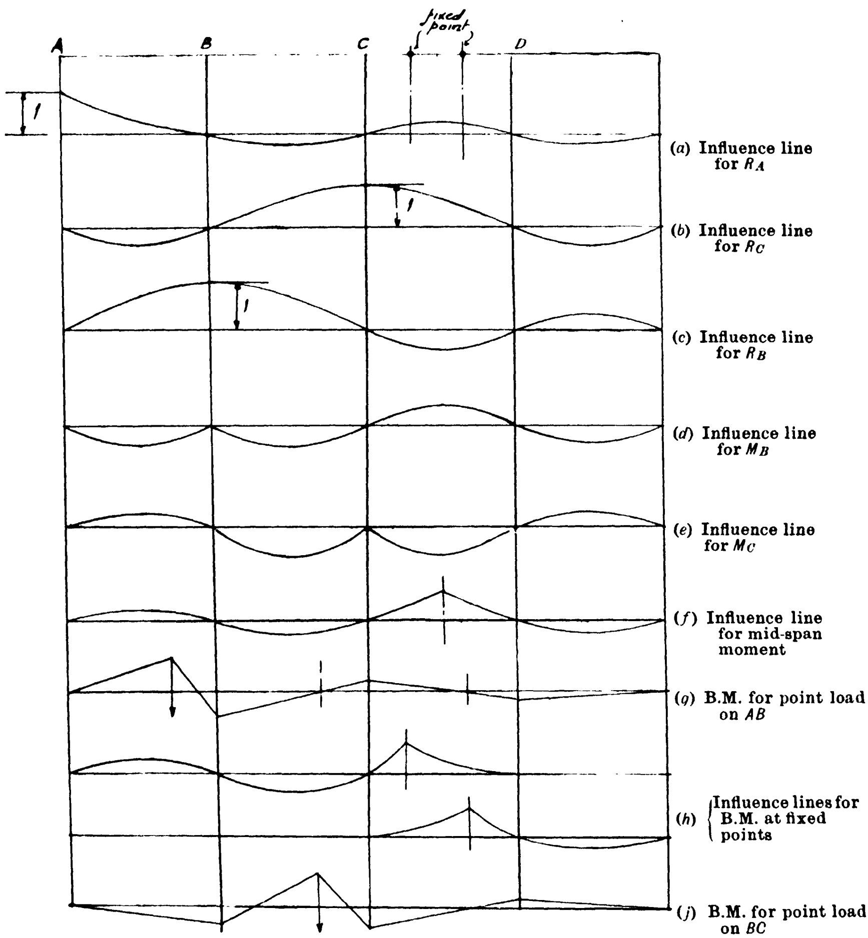

FIG. 5.17

(in each case dead load moments must be added). The remainder of the B.M. curve can be sketched in (Fig. 5.18).

For maximum shear near a support use the same loading condition as that for maximum negative B.M. at that support. The shear at or near the support is the value used in the design of beams of constant inertia. For haunched beams it may be necessary to find the shear at mid-span.

Continuous spans other than bridges. The most common example of this is the crane gantry. It is unlikely that more than two or three cranes will be carried on any one gantry. Therefore, if the influence lines for B.M. and shear have been drawn and the wheel loads and

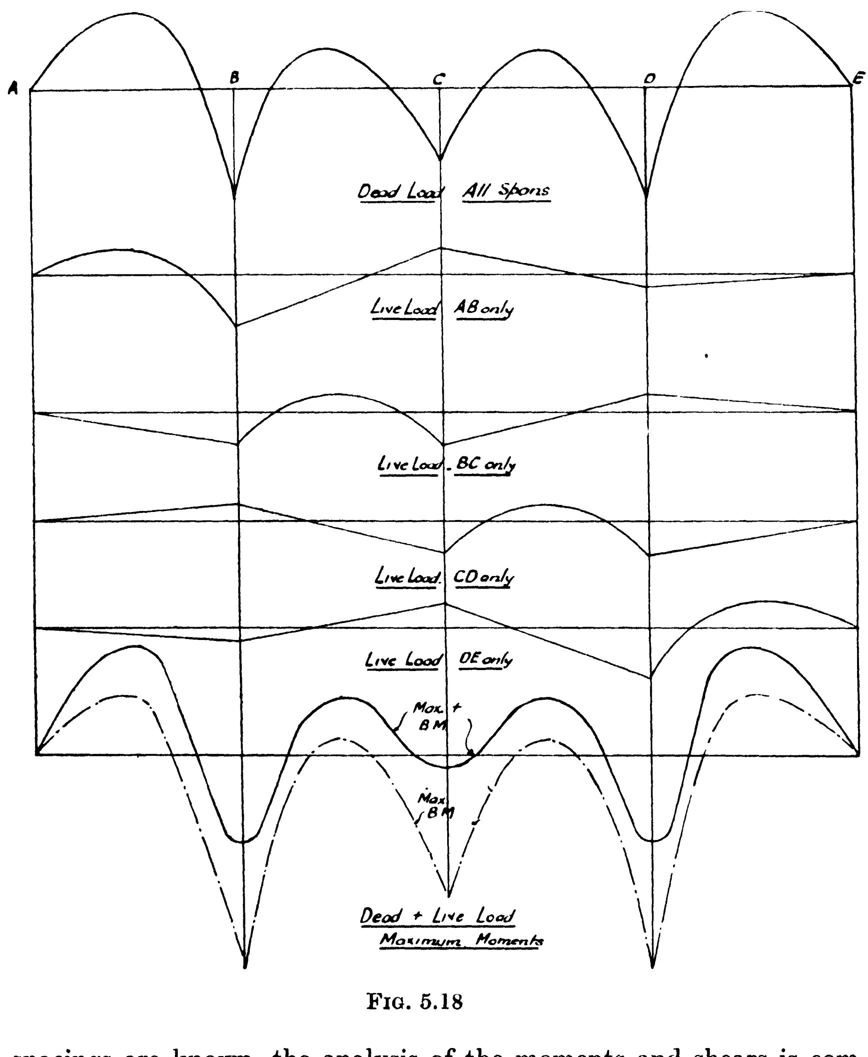

FIG. 5.18

spacings are known, the analysis of the moments and shears is comparatively simple.

Example. A uniform continuous beam of length l rests upon supports at each end and at a point distant $l/3$ from the L.H. end. The beam carries a uniformly distributed load, including its own weight of w per unit length. Assuming that the supports are on the same level, prove that the support moment is $wl^2/24$, and find the reactions.

$$\text{L.H. span:}\quad \text{span} = \frac{l}{3}.$$

$$\text{Max. B.M.} = \frac{wl^2}{72}.$$

$$A = \frac{wl^3}{324};\quad \text{distance } x_1 = \frac{l}{6}.$$

$$\text{R.H. span:}\quad \text{span} = \frac{2l}{3}.$$

$$\text{Max. B.M.} = \frac{wl^2}{18}.$$

$$A = \frac{2wl^3}{81};\quad \text{distance } x_2 = \frac{l}{3}.$$

$$\text{Support moment} \times 2l = 6\left(\frac{wl^3}{324} \times \frac{1}{2} + \frac{2wl^3}{81} \times \frac{1}{2}\right)$$

$$= \frac{wl^3}{12},$$

$$\therefore\quad \text{moment} = \frac{wl^2}{24}.$$

$$\text{Centre reaction} = \frac{11wl}{16},$$

$$\text{L.H.}\quad ,, \quad = \frac{wl}{24},$$

$$\text{R.H.}\quad ,, \quad = \frac{13wl}{48}.$$

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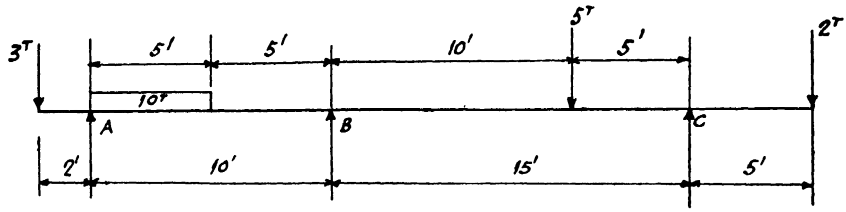


FIG. 5.A

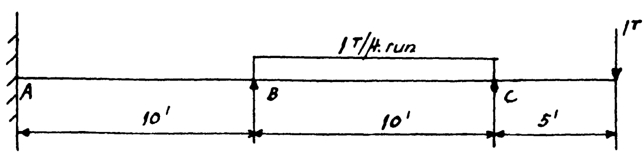


FIG. 5.B

EXERCISES

1. A beam is loaded as shown in Fig. 5.A. Find the moment at B and the reactions. [15·24 ft.-tons; 9·58, 5·42, and 5 tons.]

2. A beam is as shown in Fig. 5.B. A is fixed. Find the moments at A, B, and C, and draw the B.M. and shear diagrams. [4·28; 8·56; 5 ft.-tons.]

3. A continuous beam ABC rests on columns at A, B, and C. $AB = BC = 10$ ft. The load is 1 ton per ft. run. The columns are 100 in. high and their areas are $A = C = 6$ in.^{2}; $B = 5$ in.^{2} If the beam has an inertia of 144 in.^{4}, find the reactions at A and C. [3·796 tons.]

4. A beam $ABCD$ is supported at A, B, C, and D and carries a distributed load of w per ft. run. $AB = CD = 3$ ft. 9 in.; $BC = 27$ ft. 6 in. Find the moment at B and the reaction at A. [57·92w; $-13·59w$.]

5. A beam ABC is 30 ft. long and carries a load of 1 ton per ft. run. It is built in at A and supported at B ($AB = 20$ ft.) and at C. Find the maximum positive and negative moments. [$M_A = 37·5$; $M_B = 25$.]

6. A beam $ABCD$ is 42 ft. long. It is fixed at A, supported at B and C, and free at D. $AB = 18$ ft., $BC = CD = 12$ ft. There is a point load of 20 tons midway between A and B, a load of 2 tons per ft. on BC, and a point load of 10 tons at free end D. Find moments at A, B, C, also deflexion at D if $I = 2{,}000$ in.^{4} and $E = 12{,}000$ tons/in.^{2} [$M_A = 61·23$; $M_B = 12·53$; $M_C = 120$ ft.-tons; $\Delta = 0·417$ in.]

7. A beam ABC is continuous over two equal spans l and carries a point load W at the mid-span points. Find the settlement at B so that the reactions are all equal. [$17Wl^3/144EI$.]

CHAPTER VI

DESIGN OF BEAMS AND GIRDERS; DESIGN IDEALS AND WORKING STRESSES; STEEL, REINFORCED CONCRETE, AND TIMBER BEAMS AND GIRDERS

THE analysis of beams and girders under dead and live loading has been dealt with in Chapters III to V. In fact, given the loading conditions and particulars of any beam or girder, the investigation of stresses and deflexion, either graphically or analytically, is perfectly straightforward. On the other hand, the problem facing the designer of structures is complex, in view of the large number of factors to be taken into consideration. Firstly there are general considerations, such as choice of materials, economic life of the structure, etc. The choice of material depends on the purpose for which the structure is to be designed, local conditions, relative costs, and maintenance. The question of maintenance is important as it affects the overall cost of the structure, as distinct from the initial cost, but unfortunately it is frequently overlooked by designers. The deterioration of many metal structures has been due to the fact that the design has not provided for periodic inspection and painting. Maintenance must be studied in relation to the economic or expected life of the structure. The choice of material must be governed by several factors, for instance, in certain cases it may be economical to use mass concrete in preference to reinforced concrete especially where steel or skilled labour is in short supply. In cases where floor space is valuable, it may be advisable when designing R.C. columns to use a concrete mix richer than the usual. Where noxious fumes or liquids may come in contact with a structure or any part thereof, the choice of material becomes doubly important and the results of inspection of structures under similar conditions should be studied closely. In general, it can be said that material is usually cheaper than workmanship. Therefore design should aim at simplicity in fabrication and erection and it is often good policy to sacrifice economy of material to secure simplicity in the works or field. Over-refinement in design calculations should be avoided. Since the assumptions made in design as to dead and live loading may be in error to the extent of several per cent., it is unnecessary to spend too much time in calculating stresses or sections. It should also be remembered that errors in fabrication or erection, or settlement of the foundations, may upset the assumed loading conditions and it is advisable to revise 'theoretical' design in the light of practical experience, without adopting 'rule-of-thumb' methods. It can be said that skill in design is a combination of experience and 'engineering sense'.

At present, the design of structures is in a transitional or fluid stage. Up to quite recently, practically all beams and girders (other than continuous) were designed as 'simply supported'. In such cases, the necessary sections can be found directly. The present tendency in design, both for economic and aesthetic reasons, is towards monolithic frames in welded steel and R.C. Recently, the superimposed loadings have been reduced and at the same time working stresses have been increased somewhat, in view of improved materials and workmanship. There is a good deal to be said for the policy of allowing the skilled designer to design for working stresses somewhat higher than those in common use. Working stresses for various materials and types of structures are set out in the specifications and by-laws mentioned in the bibliography at the end of this chapter.

Every structure should be treated as a separate problem and it is impossible to generalize. For that reason, it is best to deal with the behaviour of the various materials under load and to give design examples from actual practice, so far as the available space will allow.

Steel

The properties of steel have been described in Chapter I. Steel may be used in the form of simple I or [beams, as compound girders formed of I or [beams with flange plates, as plated girders, or as built-up sections formed by welding plates together. Steel beams and girders may fail by flexure, by elastic instability, by buckling of the compression flange together with distortion of the web, by shear, web buckling, or by diagonal compression. The buckling of the compression flange is the most common cause of failure. In short spans shear or web buckling may have to be provided against. Web buckling must not be confused with shear. Buckling of the web is due to the failure of the web as a short strut. It is liable to occur at points of concentrated load, especially in grillage beams supporting heavy stanchion loads. Where there is heavy B.M. and shear at the same point, diagonal stress may be important.

Buckling of the compression flange is due to the safe stress in compression being exceeded. In order to prevent this, the compression flange must be efficiently supported in a lateral direction by subsidiary members, by concrete casing, or by being embedded in a concrete slab. There is a certain amount of divergence of opinion as to the permissible stresses in compression flanges of beams or girders. The Institution of Structural Engineers gives values of 10 tons/in.^{2} for mild steel and 14 tons/in.^{2} for high-tensile steel where l/k does not exceed 100 and 75 respectively. Where these values of l/k are exceeded, the safe stresses are: $1000k/l$ for mild steel and $1050k/l$ for high-tensile steel (see

Report on Steelwork for Buildings, Addendum, 1942). In other specifications the safe stress is based on l/b, where b is the least width.

B.S.S. 449 (*Use of Structural Steel in Building*), War Emergency Revision, 1940,† gives a maximum value of 10 tons/in.^{2} for mild steel up to $L/b = 20$. Where L/b exceeds 20, the safe stress is given by $14{\cdot}4 - 0{\cdot}22 \times L/b$ tons/in.^{2}

B.S.S. 153—1937 for Girder Bridges gives $9(1 - 0{\cdot}0075L/b)$ tons/in.^{2} for stiffened edges and $9(1 - 0{\cdot}01L/b)$ tons/in.^{2} for unstiffened edges.

L.C.C. By-laws give values similar to those of B.S.S. 449 before War Emergency Revision, viz. 8 tons/in.^{2} for L/b less than 20 and $11{\cdot}0 - 0{\cdot}15 \times L/b$ for L/b greater than 20. It should be noted that the above values apply to uncased beams only.

Where beams or girders are cased in concrete, it is usual to increase the permissible stresses in virtue of the additional lateral support. The Institution of Structural Engineers Report allows an increase on permissible stresses of up to 10 per cent. B.S.S. 449 allows the 'b' value to be increased by up to 4 in. where beams are encased in concrete and there is a similar provision in the L.C.C. By-laws. For grillage beams embedded in a solid block of concrete not leaner than 1:6 mix, the stresses in single and top tiers may be increased by 25 per cent. or for other tiers by 50 per cent.

For beams (other than grillage beams) embedded in a solid block of concrete, such as filler beams, it is usual to calculate on the basis of the combined moment of inertia of steel and the surrounding concrete as for R.C. The extreme fibre stress in steel should not exceed 11 tons/in.^{2} for mild and 15 tons/in.^{2} for high-tensile steel (Institution of Structural Engineers Report) and similar provisions are made in B.S.S. 449 and the L.C.C. By-laws, also in Draft Code of Practice.

Web shear is generally taken as 5 to 6 tons/in.^{2} for mild steel and 7·5 to 9 tons/in.^{2} for high-tensile steel. The same proviso holds good for grillage beams encased in a solid block of concrete, viz. that the stress can be increased by 25 per cent. to 50 per cent. for mild steel. Shear stress should be investigated at the bearings or under heavy concentrated loads.

Web buckling is often neglected and is not directly referred to in the specifications mentioned, except in B.S.S. 449, although some steel handbooks give values of safe loads in tons per linear inch. The value of the permissible stress is given by some authorities as

$$5{\cdot}0 - (0{\cdot}04 \times d/t) \text{ tons/in.}^2,$$

where d is the effective depth of the web and t is the web thickness. Where the safe stress is exceeded, stiffeners fitted into the flanges should be riveted or welded to the web.

† To be superseded by B.S. Code of Practice which gives different values.

Where heavy bending and shearing stress occur at the same point on a span, diagonal stresses are set up in the web which, when in compression, may exceed the safe stress. This can be dealt with by providing stiffeners sufficiently close together. This is specially important in deep girders such as plated girders and some specifications lay down rules for spacing of stiffeners.

The economical depth of beams or girders is difficult to define. Generally, the deeper the beam the more economical, provided such points as web buckling are not overlooked. L.C.C. By-laws give a maximum span: depth ratio of 24, unless the calculated deflexion is less than span/325. For filler beams the span: depth ratio should not exceed 32, the depth being measured from the underside of the beam to the top of the concrete. The Institution of Structural Engineers Report gives the same values for filler beams but does not give any rule for other beams except for deflexion. B.S.S. 449 gives the same values as L.C.C. By-laws. In actual practice, the depth may be limited by such factors as the available construction depth, architectural features, etc., and the permissible deflexion must be governed by the purpose for which the girder or floor is to be used. In some cases it may be necessary to use a section which is not really economical. Provided the maximum B.M. is known, the required section modulus is found by dividing by the permissible bending stress and then a suitable section can be selected from the steel handbook.

The design of compound beams is on similar lines. The rivets or welds connecting the flange plates to the I or [beams must be strong enough to transmit the shear force. If V is the shear force at any point and d the effective depth, then shear per inch is V/d and the shear per foot $12V/d$ and therefore the rivets or welds must be designed to suit. In the case of rivets the minimum strength (either in shear or in bearing) must be used to calculate the number required.

Plate girders are used where the strength required is more than that of compound girder sections. The economical depth of plate girders is about $\frac{1}{12}$ to $\frac{1}{10}$ of the span, although such considerations as construction depth may cause a greater span: depth ratio to be used. The web of the girder should be thick enough to resist the maximum shearing force on the girder. One-eighth of the web area can be taken as equivalent flange area, provided that the web joints are designed to transmit horizontal as well as vertical stress. In designing the flanges it should be noted that the angles should form as large a part of the flange area as practicable. Two methods of designing can be used: (1) by calculating the moment of inertia and section moduli of the section of the girder about its neutral axis; (2) by dividing the maximum B.M. by the distance between the centres of gravity of the flanges, in order to find the 'flange load'. In commercial design, the depth over the flange angles is often taken as equal to the distance between the centres of gravity. While this method may be sufficiently accurate where the span:depth ratio does not exceed 12, it is not exact as it neglects the flexural strength of the web. Where the span:depth ratio is more than 12, the moment of inertia should be calculated as in the design example. As the B.M. decreases towards the supports, the outer flange plates are stopped off as they become unnecessary.

✓ Web plates should never be less in thickness than $\frac{1}{180}$ of the clear distance between the flange angles (in practice it is as well to use $\frac{3}{8}$ in. minimum on account of corrosion). B.S.S. 153 gives shear in M.S. web plates as 5 tons/in.² The Report of the Institution of Structural Engineers on Steelwork for Buildings gives the same figure for the shear on the gross area where ratio unsupported length:thickness does not exceed 80 and $9{\cdot}44 - \dfrac{\text{length}}{18 \times \text{thickness}}$ otherwise, provided that shear stress on the net area does not exceed 6 tons/in.²

The web must be supported by stiffeners riveted or welded thereto, at the bearings and all points of concentrated loading. At bearings and under concentrated loads, the stiffeners should be sufficiently strong to carry the whole shear, when designed as struts having an effective depth of three-quarters of the depth of the girder. A certain amount of the web can be included as acting together with the stiffeners for this purpose. Intermediate stiffeners must be provided throughout at centres not exceeding the depth of the girder, with a maximum value of 6 ft. They can be designed according to the formula

$$s = \frac{Sp}{4D},$$

where s = load per pair of stiffeners,

S = max. vertical shear at that point,

p = distance between stiffeners along the girder,

D = overall depth of the girder.

Where $\frac{1}{8}$ of the web area has been taken as forming part of the flange area, the rivets or welds fastening the flange angles to the web plate should be designed by

$$F = \frac{S}{D} \times \frac{A}{A+(W/8)},$$

where A = flange area, W = web area,

and F = shear force per unit length.

Where it is necessary to make a joint in the flange angles, flange plates, or web plate, then the joint should be designed to develop the full strength of the member, even if that member is not stressed to its maximum permissible stress at the point in question.

Design example. To design 45-ft. span crane girders to carry 25-ton and 10-ton cranes, the cross centres of crane rails being 43 ft. $11\frac{1}{8}$ in. (to suit existing girders).

Data: (1) 25-ton crane.

Weight of crab = $5\frac{3}{4}$ tons

Weight of crane bridge and end carriages = $25\frac{1}{4}$,,

Wheel centres, 11 ft. $3\frac{1}{2}$ in.; distance over buffers, 13 ft. 9 in.; minimum hook approach to centre of girder, 4 ft.

∴ *wheel loads*

(*a*) from bridge and end carriages $= \dfrac{25\cdot25}{4} = 6\cdot31$ tons

(*b*) from crab $\frac{1}{2}\times5\cdot75\times\dfrac{39\cdot93}{43\cdot93} = 2\cdot62$,,

(*c*) from load $\frac{1}{2}\times25\times\dfrac{39\cdot93}{43\cdot93} = 11\cdot4$,,

Add impact (60 per cent. of load) = 6·84 ,,

Max. wheel load = 27·17 ,,

(2) 10-ton crane.

Weight of crab = 2 tons

Weight of crane bridge and end carriages = 10 ,,

Wheel centres, 10 ft. $1\frac{1}{2}$ in.; distance over buffers, 12 ft. $4\frac{1}{2}$ in.; minimum hook approach to centre of girder, 3 ft. 6 in.

∴ *wheel loads*

(*a*) from bridge and end carriages = 2·5 tons

(*b*) from crab $\frac{1}{2}\times2\times\dfrac{40\cdot43}{43\cdot93} = 0\cdot93$,,

(*c*) from load $\frac{1}{2}\times10\times\dfrac{40\cdot43}{43\cdot93} = 4\cdot65$,,

Add impact $\dfrac{60}{100}\times4\cdot65 = 2\cdot79$,,

Max. wheel load = 10·87 ,,

To find max. B.M. due to wheel loads. Find the position of centre of gravity of the loads (Fig. 6.1).

By moments $27\cdot17\times11\cdot29 = 306\cdot75$

$$\left.\begin{matrix}10\cdot87\times13\cdot646\\10\cdot87\times23\cdot77\end{matrix}\right\} = 406\cdot71$$

713·46

$$\bar{X} = \frac{713{\cdot}46}{\text{total load}} = \frac{713{\cdot}46}{76{\cdot}08} = 9{\cdot}36 \text{ ft.}$$

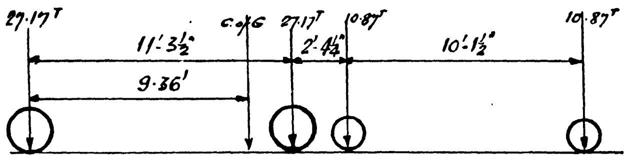


FIG. 6.1

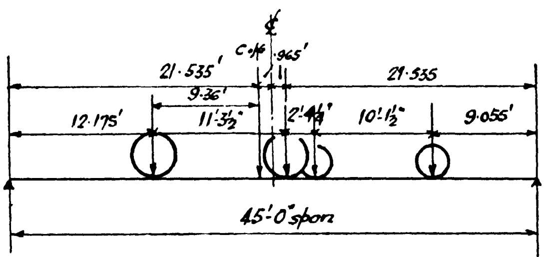


FIG. 6.1 (*a*)

For max. B.M. under one wheel load, that wheel and the C.G. of the loads must be equidistant from the ends of the span (Fig. 6.1 (*a*)):

$$R_L = \frac{76{\cdot}08 \times 23{\cdot}465}{45} = 39{\cdot}6 \text{ tons.}$$

$$\begin{aligned} \text{Max. B.M.} &= 39{\cdot}6 \times 23{\cdot}465 = 929{\cdot}16 \text{ ft.-tons} \\ &\quad -27{\cdot}17 \times 11{\cdot}29 = \underline{306{\cdot}75} \quad ,, \\ &\qquad\qquad\qquad\qquad 622{\cdot}41 \quad ,, \end{aligned}$$

Assuming weight of girder at 7 tons,

$$\text{Max. D.L. B.M.} = \frac{7 \times 45}{8} = \underline{39{\cdot}5} \text{ ft.-tons}$$

$$\text{Total max. B.M.} = 661{\cdot}89 \quad ,,$$

B.M. at quarter-point of span. Taking L.L. values from influence line (Fig. 6.2)

$$\text{L.L. B.M.} = 463 \text{ ft.-tons}$$

$$\left.\begin{aligned} \text{D.L. B.M. } & 3{\cdot}5 \times 11{\cdot}25 \\ & -1{\cdot}75 \times 5{\cdot}625 \end{aligned}\right\} = \underline{29{\cdot}58} \quad ,,$$

$$\text{Total B.M.} = 492{\cdot}58 \quad ,,$$

In addition to the vertical loads from the wheels, the horizontal forces transverse to the rails must be allowed for in the design. The Institution of Structural Engineers Report on Steelwork for Buildings (Addendum) gives a value of 20 per cent. of the load for electric cranes (to be divided equally between the two girders). The lateral B.M. is calculated for the same position of the wheel load as was taken for the vertical B.M.

Loads per girder: 25-ton crane, 1·25 tons per wheel,

10-ton ,, 0·50 ,, ,, (Fig. 6.3).

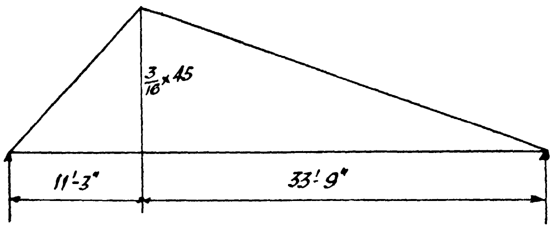


FIG. 6.2

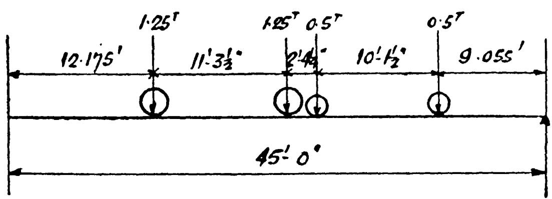


FIG. 6.3

By moments: $R_L \times 45 = 0{\cdot}5(9{\cdot}035+19{\cdot}16) = 14{\cdot}1$ ft.-tons

$1{\cdot}25(21{\cdot}51+32{\cdot}82) = 67{\cdot}9$,,

$82{\cdot}0$,,

$$\therefore \quad R_L = \frac{82}{45} = 1{\cdot}82 \text{ tons.}$$

$\therefore$ B.M. $= 1{\cdot}82 \times 23{\cdot}475 = 42{\cdot}7$ ft.-tons

$-1{\cdot}25 \times 11{\cdot}29 = 14{\cdot}15$,,

Max. lateral B.M. $= 28{\cdot}55$ ft.-tons.

Lateral B.M. at quarter-point (Fig. 6.2):

$R_L \times 45 = 0{\cdot}5(9{\cdot}975+20{\cdot}1) = 15{\cdot}04$ ft.-tons

$1{\cdot}25(22{\cdot}45+33{\cdot}75) = 70{\cdot}25$,,

$85{\cdot}29$,,

$$\therefore \quad R_L = \frac{85{\cdot}29}{45} = 1{\cdot}90 \text{ tons.}$$

$\therefore$ Lateral B.M. $= 1{\cdot}9 \times 11{\cdot}25 = 20{\cdot}45$ ft.-tons.

Having found the B.M. near mid-span and at the quarter-point, the maximum shear at end of girder must be found. Maximum shear occurs when one wheel of 25-ton crane is over the end (Fig. 6.4).

$$\text{Then shear} = 27{\cdot}17\left(1+\frac{33{\cdot}71}{45}\right) = 47{\cdot}25 \text{ tons}$$

$$+21{\cdot}74\left(\frac{26{\cdot}27}{45}\right) = 12{\cdot}75 \quad ,,$$

$$\text{Max. L.L. shear} = 60{\cdot}00 \quad ,,$$

$$\text{D.L. shear} = 3{\cdot}5 \quad ,,$$

$$\text{Max. shear} = 63{\cdot}5 \quad ,,$$

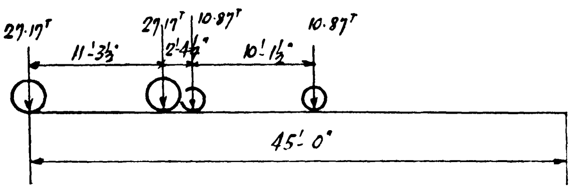


FIG. 6.4

Taking web shear at 5 tons/in.², web area required is

$$\frac{63{\cdot}5}{5} = 12{\cdot}7 \text{ in.}^2$$

In this case depth over the flange angles is 3 ft. 8 in. to suit existing girders,

$$\therefore \quad \text{thickness required} = \frac{12{\cdot}7}{44} = 0{\cdot}29 \text{ in.}$$

Use ⅜ in. web plate throughout.

Riveted design. Use section shown in Fig. 6.5 as 'basic' section. The calculation for the moment of inertia is best done in tabular form, thus:

Part	*Area* (*in.*²)	*Arm* (*in.*)	*1st moment* (*in.*³)	*2nd moment* (*in.*⁴)	*Own I* (*in.*⁴)	*Total I*	I_{yy}
Web . .	16·5	. .	. .	. .	2,662	2,662	. .
Flange angles .	11·44	20·88	. .	4,986	17	5,013	38
Channel . .	13·04	+21·56	+282 }	6,061	15	6,076	520
Top plate . .	8·00	+22·73	+182 } 539·5	4,125	. .	4,125	170·67
Top 6½" × ½" pkg.	3·25	+23·23	+ 75·5 }	1,750	. .	1,750	11·4
Bottom plate .	8·00	−22·25	−178	3,950	. .	3,950	170·67
Gross . .	60·23	. .	+361·5	. .	. .	23,576	910·74
−2 holes $\frac{15}{16}$" × ⅞"	1·65	−22·06	+36·4	. .	. .	−800	. .
Net . . .	58·58	. .	397·9	. .	. .	22,776	. .

$$\bar{x} = \frac{397{\cdot}9}{58{\cdot}58} = 6{\cdot}8 \text{ in.}$$

$$I_{xx} = 22{,}776$$

$$-58{\cdot}58 \times 6{\cdot}8^2 = \underline{2{,}700}$$

$$20{,}076 \text{ in.}^4$$

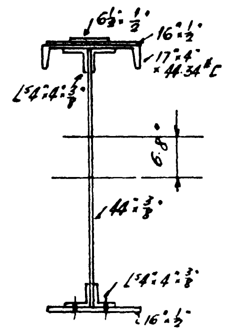


FIG. 6.5

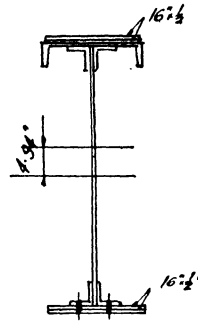


FIG. 6.5 (a)

$$k_{yy} = \sqrt{\left(\frac{910{\cdot}74}{58{\cdot}58}\right)} = 3{\cdot}95 \text{ in.} \qquad l/k_{yy} = 137.$$

$$Z_{yy} = \frac{910{\cdot}74}{8{\cdot}0}$$

$$= 113{\cdot}83 \text{ in.}^3$$

$$\text{Safe stress} = \frac{1{,}000k}{l}$$

$$= \frac{1{,}000}{137} = 7{\cdot}3 \text{ tons/in.}^2$$

For quarter-point

$$f_v = \frac{492{\cdot}58 \times 12 \times 16{\cdot}18}{20{,}076} = 4{\cdot}77 \text{ tons/in.}^2$$

$$f_h = \frac{20{\cdot}45 \times 12}{113{\cdot}83} = 2{\cdot}16 \quad ,,$$

$$6{\cdot}93 \quad ,,$$

Less than $7{\cdot}3$ tons/in.^{2}

This section is therefore suitable at quarter-points.

Full section as shown in Fig. 6.5 (*a*).

$$\bar{x} = \frac{347{\cdot}44}{70{\cdot}4} = 4{\cdot}94.$$

$$\text{Total } I = 28{,}990$$

$$70{\cdot}4 \times 4{\cdot}94^2 = \underline{1{,}720}$$

$$\text{Net } I_{xx} = 27{,}270 \text{ in.}^4$$

Part	Area (in.²)	Arm (in.)	1st moment (in.³)	2nd moment (in.⁴)	Own I (in.⁴)	Total I	I_{yy}
Web . .	16·5	..	..	..	2,662	2,662	..
Angles . .	11·44	20·88	..	4,986	17	5,013	38
Channel . .	13·04	+21·56	+282 }	6,061	15	6,076	520
Top plate .	8·00	+22·73	+182 } 649·84	4,125	..	4,125	170·67
16″×½″ . .	8·00	+23·23	+185·84 }	4,325	..	4,325	170·67
Bottom plate .	8·00	−22·25	−178 } 360	..	..	3,950	170·67
16″×½″ . .	8·00	−22·75	−182 }			4,125	170·67
Gross . .	72·98	..	289·84	..	..	30,276	1,240·67
−2 holes $\frac{15}{16}$″×1⅜″ .	2·58	..	57·6	..	..	1,286	..
Net . .	70·40	..	347·44	..	..	28,990	..

$$k_{yy} = \sqrt{\left(\frac{1,240\cdot67}{70\cdot4}\right)} = 4\cdot18 \text{ in.} \qquad l/k_{yy} = 129.$$

$$Z_{yy} = 155\cdot05 \text{ in.}^3$$

$$\text{Safe stress} = \frac{1,000}{129} = 7\cdot75 \text{ tons/in.}^2$$

$$Z_c = \frac{27,270}{18\cdot54} = 1,470 \text{ in.}^3$$

$$Z_t = \frac{27,270}{27\cdot94} = 975 \text{ in.}^3$$

For max. B.M.

$$f_c = \frac{661\cdot89\times12}{1,470} = 5\cdot41$$

$$f_h = \frac{28\cdot85\times12}{155\cdot05} = 2\cdot20$$

$$7\cdot61 \text{ tons/in.}^2 < 7\cdot75.$$

For riveting at ends:

$$V = 63\cdot5 \text{ tons}; \qquad d = 44 \text{ in.}$$

$$F = \frac{V}{d}\times\frac{\text{Area of flange}}{\text{Area of flange}+\frac{1}{8}\times\text{web area}}.$$

Area of flange	8·00
	5·72
	13·72
$\frac{1}{8}$ web	2·06
	15·78

$$= \frac{63\cdot5}{44}\times\frac{13\cdot72}{15\cdot78}$$

$$= 1\cdot255 \text{ tons/in.} = 15\cdot06 \text{ tons/ft.}$$

Value of $\frac{15}{16}$ in. rivet bearing in $\frac{3}{8}$ in. plate $= \frac{15}{16} \times \frac{3}{8} \times 12 = 4{\cdot}23$ tons.

$$\text{No. required per foot} = \frac{15{\cdot}06}{4{\cdot}23} = 3{\cdot}56,$$

i.e. 3·37 in. pitch.

End stiffeners. Max. load = 63·5 tons.

Effective length $= \frac{3}{4} \times 44$ in. $= 33$ in.

Section two angles $3\frac{1}{2}$ in. $\times 3\frac{1}{2}$ in. $\times \frac{3}{8}$ in.	Area =	4·97 in.^{2}
	End plate 16 in. $\times \frac{1}{2}$ in. =	8·00 ,,
	web $3\frac{1}{2}$ in. $\times \frac{3}{8}$ in. =	1·31 ,,
		14·28 ,,

$$\text{Actual stress} = \frac{63{\cdot}5}{14{\cdot}28} = 4{\cdot}45 \text{ tons/in.}^2$$

$$\frac{l}{k} = \frac{33}{1{\cdot}65} = 20,$$

$\therefore$ allowable stress = 6·8 tons/in.^{2}

For intermediate stiffeners use two angles $3\frac{1}{2}$ in. $\times 3\frac{1}{2}$ in. $\times \frac{3}{8}$ in.

The 'basic' section will be used from the ends to the quarter-points and the 'full' section for the centre portion. The outer 16 in. $\times \frac{1}{2}$ in. flange plates will be extended beyond their 'theoretical' ends to cover the joints in the inner 16 in. $\times \frac{1}{2}$ in. plates. The $6\frac{1}{2}$ in. $\times \frac{1}{2}$ in. packing plate on top is intended to take up the thickness of the outer 16 in. $\times \frac{1}{2}$ in. plate and to form an even seating for the crane rail. The latter is not taken as forming part of the girder section as it is bolted thereto at about $10\frac{1}{2}$ in. reeled pitch.

Alternative welded section

1. At quarter-point (Fig. 6.6).

Part	*Area (in.^{2})*	*Arm (in.)*	*1st moment (in.^{3})*	*2nd moment (in.^{4})*	*Own I (in.^{4})*	*Total I*	I_{yy}
Web	16·5	..	..	..	2,662	2,662	..
Channel	13·04	+21·56	+282 }	6,061	15	6,076	520
$14'' \times \frac{3}{4}''$	10·5	+22·85	+240 } +637	5,490	..	5,490	171·5
$6\frac{1}{2}'' \times \frac{3}{4}''$	4·88	+23·60	+115 }	2,710	..	2,710	17·1
$14'' \times \frac{3}{4}''$	10·5	−22·375	−235	5,300	..	5,300	171·5
	55·42	..	402	..	..	22,238	880·1

$$\bar{x} = \frac{402}{55{\cdot}42} = 7{\cdot}4.$$

$$\text{Total } I = 22{,}238$$

$$55{\cdot}42 \times 7{\cdot}4^2 = 2{,}980$$

$$\text{Net } I_{xx} = 19{,}258 \text{ in.}^4$$

$$k_{yy} = \sqrt{\left(\frac{880 \cdot 1}{55 \cdot 42}\right)} = 4 \text{ in.} \qquad \frac{l}{k} = 135.$$

Safe stress = 7·4 tons/in.²

Max. compressive stresses in channel

$$f_v = \frac{492 \cdot 58 \times 12 \times 15 \cdot 08}{19{,}258} = 4 \cdot 63 \text{ tons/in.}^2$$

$$f_h = \frac{20 \cdot 45 \times 12 \times 8 \cdot 5}{880} = 2 \cdot 37 \quad ,,$$

$$\text{Total} = \overline{7 \cdot 00} \quad ,,$$

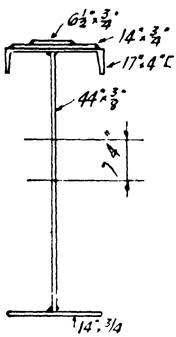


FIG. 6.6

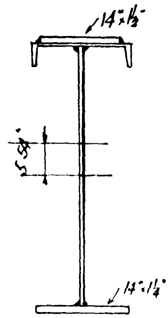


FIG. 6.6(*a*)

and plate

$$f_v = \frac{492 \cdot 58 \times 12 \times 15 \cdot 83}{19{,}258} = 4 \cdot 86 \text{ tons/in.}^2$$

$$f_h = \frac{20 \cdot 45 \times 12 \times 7}{880} = 1 \cdot 96 \quad ,,$$

$$\text{Total} = \overline{6 \cdot 82} \quad ,,$$

Less than safe stress; therefore section is adequate.

Full section (Fig. 6.6(*a*)):

Part	*Area (in.²)*	*Arm (in.)*	*1st moment (in.³)*	*2nd moment (in.³)*	*Own I (in.⁴)*	*Total I*	I_{yy}
Web . .	16·5	..	..	..	2,662	2,662	..
Channel . .	13·04	+21·56	+282 } +770	6,061	15	6,076	520
Top plate . .	21·00	+23·23	+488 }	11,300	..	11,300	343
Bottom plate .	17·5	−22·625	−396	9,000	..	9,000	285·8
	68·04	..	374	..	..	29,038	1,148·8

$$\bar{x} = \frac{374}{68 \cdot 04} = 5 \cdot 54 \text{ in.}$$

$$\text{Total } I = 29{,}038$$

$$68 \cdot 04 \times 5 \cdot 54^2 = \underline{2{,}070}$$

$$\text{Net } I_{xx} = 26{,}968 \text{ in.}^4$$

$$k_{yy} = \sqrt{\left(\frac{1{,}148 \cdot 8}{68 \cdot 04}\right)} = 4 \cdot 1 \text{ in.} \qquad \frac{l}{k} = 132.$$

Safe stress $= 7\cdot7$ tons/in.^{2}

Max. compressive stress in channel

$$f_v = \frac{661 \cdot 89 \times 12 \times 16 \cdot 94}{26{,}968} = 4 \cdot 98 \text{ tons/in.}^2$$

$$f_h = \frac{28 \cdot 55 \times 12 \times 8 \cdot 5}{1{,}148 \cdot 8} = 2 \cdot 54 \quad ,,$$

$$\text{Total} = 7 \cdot 52$$

and plate

$$f_v = \frac{661 \cdot 89 \times 12 \times 18 \cdot 44}{26{,}968} = 5 \cdot 43 \text{ tons/in.}^2$$

$$f_h = \frac{28 \cdot 55 \times 12 \times 7}{1{,}148 \cdot 8} = 2 \cdot 09 \quad ,,$$

$$\text{Total} = 7 \cdot 52 \quad ,,$$

Max. shear at ends $= 63\cdot5$ tons.
Use double $6\frac{3}{4}$ in. $\times \frac{1}{2}$ in. stiffeners at ends. Area $= 13\cdot5$ in.^{2}
Value of two $\frac{1}{4}$-in. fillet weld $= 2\cdot1$ tons/in.
Length required $= 30$ in. Make full depth.
Intermediate stiffeners $6\frac{3}{4}$ in. $\times \frac{3}{8}$ in. with intermittent welds.
Welding of channel to web:

$$s = S \times \frac{\sum \text{area above section} \times \text{distance}}{I}$$

$$= 63 \cdot 5 \times \frac{(13 \cdot 04 \times 16 \cdot 02) + (21 \cdot 00 \times 17 \cdot 69)}{26{,}968}$$

$$= 63 \cdot 5 \times \frac{580 \cdot 4}{26{,}968} = 1 \cdot 27 \text{ tons/in.}$$

Use two $\frac{1}{4}$-in. fillet welds. Value $= 2\cdot1$ tons/in.

It will be seen that the welded section is lighter and therefore more economical as well as being better from a maintenance point of view, but at present, tonnage rates for welded steel are 10 to $12\frac{1}{2}$ per cent. higher than for riveted steel, so first costs are about equal.

Reinforced Concrete

The fundamental assumptions made in R.C. design have been dealt with in Chapter II. In applying these principles to design in practice, it must be remembered that the designer is dealing with a non-homogeneous material and that some of the assumptions made are convenient, but not strictly true. Whilst steel is a perfectly elastic material (within certain limits), concrete is a material with no true elastic modulus. The value of E_c and therefore that of the modular ratio in E_s/E_c has been a subject of considerable controversy. Up to quite recently it was standard practice in this country to adopt a constant value of 15. At present most concrete design in Great Britain is in accordance with the Code of Practice for Reinforced Concrete which is incorporated by reference in the Model Building By-laws.† The Code gives values of m varying according to the concrete mix and is based on the cube crushing strength at 28 days ($3x$). The basic bending compressive stress is taken as x and the modular ratio at $40{,}000/3x$. In considering the modular ratio it must be borne in mind that the value of E_c varies with the time during which the concrete is loaded. For an instantaneous load (which is seldom likely to occur in practice) there is no permanent deformation, i.e. concrete will behave as an elastic material. For sustained loading, on the other hand, concrete has an elastic and also a plastic deformation known as creep. The creep will increase with time until it reaches a value which may be four or five times that of the elastic deformation. The effect of creep is to redistribute the stresses in a R.C. beam between the steel and the concrete. In the earlier stage of loading the concrete stress may be somewhat higher than that calculated according to the accepted theory and the steel stress correspondingly lower. After the section has been under load some time, the concrete stress will diminish and the steel stress increase. Creep is roughly proportional to the stress applied, and for that reason the value of m given by $40{,}000/3x$, while higher than that obtained by tests on specimens, takes into account the redistribution of stress and is therefore safe for design purposes. It is interesting to compare the values for m given by the American Society of Civil Engineers Report for Concrete and Reinforced Concrete. This gives values as follow, viz.:

m	*cube crushing strength (lb./in.²)*
15	2,000–2,400
12	2,500–2,900
10	3,000–3,200
8	4,000–4,900
6	over 5,000

which show fairly close agreement with the Code values. Other American research workers give a value for m of $5+10{,}000/u$ (u = crushing strength of 12 in. × 6 in. cylinder) for gravel and broken stone concretes.‡ The straight-line no-tension theory of R.C. outlined in Chapter II is generally accepted, although other theories based on different stress distribution have been put forward by various investigators from time

† CP 114 (1948) gives value of m as 15.

‡ *Ultimate Strength of Reinforced Concrete Beams as Related to the Plasticity Ratio of Concrete*, University of Illinois Bulletin No. 345.

to time. An interesting comparison of these is made by Dr. R. H. Evans in a paper presented to the Institution of Civil Engineers.† It can be said that the accepted theory errs on the side of safety, if at all, and failures in R.C. structures have been due generally to faulty materials or workmanship rather than to errors in design.

Another phenomenon peculiar to concrete is that of shrinkage. The setting of concrete involves a certain amount of heat, the effect being most marked with quick-setting and other special cements. As the heat is dissipated into the surrounding atmosphere a certain amount of shrinkage must take place, causing tension in the concrete and compression in the steel. The amount of shrinkage which takes place in any particular case depends on several factors, chiefly the care with which 'curing' is done (particularly with special cements). Again, the shrinkage stresses may be affected by a certain amount of slip between the concrete and steel. Usually shrinkage stresses can be neglected (except for indeterminate structures) in design and the usual effect is the formation of hair cracks in the concrete.

In considering the question of failure of R.C. sections due to bending, it is found that, provided the tensile steel is securely anchored, the most common causes of failure are yielding of the steel or failure of the concrete along a diagonal. The tensile reinforcement is designed for a safe stress of 18,000 lb./in.^{2} for mild and 25,000 for high tensile steel and the yield-point is approximately twice the safe stress. It is a matter of regret that no specification lays down a value for the yield-point‡ as this is a most reliable guide to the safe stress and certain formulae, notably that for steel stanchions in B.S.S. 449, are based thereon. The amount or percentage of tensile steel is constant for any given ratio of stresses and modular ratio, and provided this is placed in the tension side of the beam, failure by yielding is unlikely to take place. The design of shear reinforcement either in the form of stirrups or bent-up bars (or both) is most important, particularly near the supports or under concentrated loads, and the detailing of this steel should receive special care. The failure of R.C. beams in compression is not so common as for steel, but buckling of compression steel may take place if the distance between stirrups exceeds the safe value as laid down in the specification.

Rectangular beams. The effective depth should be

$$\frac{\text{span} \times \text{tensile stress}}{320,000}.$$

† 'The Plastic Theories of the Ultimate Strength of Reinforced Concrete Beams', *Journal I.C.E.*, Dec. 1943.

‡ This point is brought out in B.S. Code of Practice, *The Structural Use of Normal R.C. Buildings.*

For $t = 18{,}000$, depth = span/18; for $t = 20{,}000$, depth = span/16, and for $t = 25{,}000$, depth = span/13, with other values in proportion. A rough rule is that overall depth (inches) = clear span (feet). However, such factors as headroom and construction depth must govern the design. The breadth/depth ratio may vary from $\frac{1}{3}$ to 1, with an average value of $\frac{1}{2}$. The breadth is governed by the amount of steel

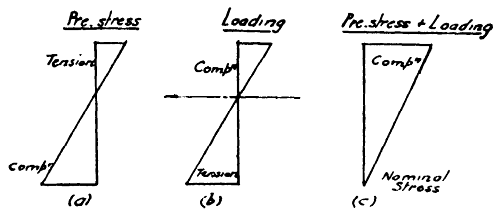


FIG. 6.7

and its cover and the area required in shear. The breadth should, if possible, be fixed to suit commercial sizes of timber for shuttering (4 in., 7 in., 8 in., 11 in., etc.) or steel shutters. For rectangular beams the ratio, unsupported length : width of compression flange, should not exceed

$$20\left(3-2\times\frac{\text{actual compressive stress}}{\text{permissible compressive stress}}\right).$$

In recent years the principle of pre-stressing has been introduced for simply supported beams. The process consists essentially of stretching the tensile steel by jacks or other devices while the concrete is formed round it.† The steel used should be high or medium yield-point steel. When the tension is released the effect is to introduce an initial compression in the concrete in the tension zone (Fig. 6.7 (*a*)). This, combined with the normal stress distribution (Fig. 6.7 (*b*)), produces a final effect as shown in Fig. 6.7 (*c*). The object is to obtain a smaller section and the principle is particularly useful for pre-cast units. For large spans the pre-stressing of the shear steel presents practical difficulties, and where there is reversal of B.M. the method is not of much use.

T- *and -beams.* The design of flanged beams is similar to that of rectangular beams, with the exception that the slab, being monolithic with the rib, forms the compression flange for positive B.M. The effective width of **T**-beam flanges is

	Code of Practice	*American Society of Civil Engineers*
least of	$\frac{\text{effective span}}{3}$	$\frac{\text{span length}}{4}$
	centres of ribs	centres of ribs
	breadth of rib + 12 × slab thickness	breadth of rib + 16 × slab thickness

† See also Vol. II, Chap. XV.

Corresponding values for L-beams:

least of	$\frac{\text{effective span}}{6}$	breadth of rib $+ \frac{\text{span length}}{12}$
	breadth of rib $+ \frac{1}{2} \times$ clear distance between ribs	breadth of rib $+ \frac{1}{2} \times$ clear distance between ribs
	breadth of rib $+ 4 \times$ slab thickness	breadth of rib $+ 6 \times$ slab thickness

The maximum positive B.M. will occur at mid-span and the maximum negative B.M. at the supports. In dealing with negative B.M., the slab comes in the tension zone and must be neglected. At the same time, additional strength can be obtained by 'haunching' the underside of the beam at the supports. In the design of buildings, the following conditions must be investigated: (*a*) alternate spans loaded, (*b*) adjacent spans loaded and all other spans unloaded. (For bridge floor members the influence line method outlined in Chapter V can be used.)

Secondary beams generally frame into main beams at each end and are restrained to a certain extent by the continuity of members and the torsional rigidity of the main beams. For exact analysis, the effect of the latter can be taken into account by assuming its stiffness equal to that of a column having a stiffness half that of the beam. Beams may be taken as continuous over supports about which they are free to rotate. This assumption ignores the torsional rigidity and can hardly represent the conditions existing in practice. For a preliminary design for uniformly distributed load and approximately equal spans, the maximum B.M.s can be taken as

Near middle of end span	*Penultimate support*	*At middle of interior span*	*Other interior supports*
$+\frac{Wl}{10}$	$-\frac{Wl}{10}$	$+\frac{Wl}{12}$	$-\frac{Wl}{12}$

Where necessary, the final analysis can be done by taking the relative stiffnesses into account, but this is seldom necessary for secondary beams.

Main beams are generally monolithic with columns and should be treated as part of an elastic frame and analysed by use of Hardy Cross, slope deflexion, or method of virtual work. In order to carry out an exact analysis it is necessary to know the stiffnesses or relative stiffnesses of the members. The stiffnesses of members can be calculated on the area of concrete only, the error involved being small. To obtain preliminary sections the coefficients stated for secondary beams can be used. For beams framing into exterior columns the value of $-Wl/24$ for end B.M. can be taken into preliminary calculations. In order to reduce the negative B.M. at supports and consequently the amount of reinforcement at column connexions, it is permissible to make a reduction of up to 15 per cent., *provided* that the positive mid-span is increased by that percentage of the negative B.M. For haunched beams the shear is altered by the vertical component of the inclined stress. Usually shear in beams need only be investigated at supports or at a distance from column faces equal to the effective depth. The B.M. at a column face can be taken as

$$\text{B.M. at centre of column} - \text{shear} \times \frac{\text{breadth of column}}{3}.$$

Slabs can span in one or in two directions at right angles. It should be clearly understood that these remarks do not apply to slabs in 'mushroom' floors which are designed according to rules derived as a result of experience in this country and the U.S.A. (see Report of Institution of Structural Engineers on Flat Slab Floors). The design of one-way slabs follows that of rectangular beams with the proviso that distribution steel must be provided at right angles to main steel equal to 20 per cent. of such and at spacing not exceeding 4 times the effective depth of slab. The slab thickness (effective) should be not less than $\frac{1}{20}$ of the span. Shear in slabs does not usually require investigation.

The load in two-way slabs is distributed in the two directions according to the ratio long span/short span. The distribution coefficients have been derived by various methods, usually by equating deflexions of strips at right angles at their crossing-points. If

$$K = \frac{\text{long span}}{\text{short span}},$$

values of distribution coefficients are:

	Grashof and Rankine		*French Government*	
K	*Short span*	*Long span*	*Short span*	*Long span*
1·00	0·50	0·50	0·33	0·33
1·25	0·71	0·29	0·55	0·17
1·50	0·83	0·17	0·71	0·09
1·75	0·90	0·10	0·83	0·05
2·00	0·94	0·06	0·89	0·03

The Code of Practice gives other values based on Marcus's method, which allows for corner restraint and torsion, and values are also given by the Report of the Institution of Structural Engineers and by the American Society of Civil Engineers. The effective depth should be not less than $\frac{1}{30}$ span, or thickness can be calculated by

$$t = \frac{1}{72}\left[L+l-\frac{P}{10}\right]\sqrt[3]{\left(\frac{2{,}500}{3x}\right)},$$

where

L, l are lengths of sides in inches,

P = perimeter in continuity with adjacent slabs (in.),

x = permissible bending stress.

In general it can be said that slabs less than 4 in. thick are not economical and thicker slabs form a more effective flange for T- and L-beams. Point loads on slabs can be treated according to the Report of the Institution of Structural Engineers. In calculating the load taken by the supporting edges, the method shown in Fig. 6.8 should be used, the shaded area being that supported on the shorter edges.

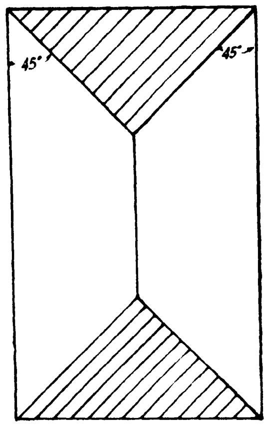


FIG. 6.8

Design example

Warehouse floor: superimposed load 2 cwt. per square foot; columns 20-ft. centres both ways; secondary beams 6 ft. 8 in. centres.

Concrete 1:2:4 mix; steel stress 18,000 lb./in.² (See Fig. 6.9.)

$$\text{Thickness of slab} = \frac{1}{72}\left[240+80-\frac{640}{10}\right]\sqrt[3]{\left(\frac{2{,}500}{2{,}250}\right)}$$
$$= 3{\cdot}65 \text{ in.},$$

or
$$\frac{80}{20} = 4 \text{ in.}$$

Use 4-in. slab:

then dead load per square foot is

$$\begin{aligned} \text{Slab} &= 48 \text{ lb.} \\ \text{Finish} &= \underline{12} \text{ ,,} \\ & 60 \text{ ,,} \end{aligned}$$

$$c = 750 \text{ lb./in.}^2; \quad t = 18{,}000 \text{ lb./in}^2; \quad m = 18.$$
$$\therefore \quad n_1 = 0{\cdot}425; \quad a_1 = 0{\cdot}858; \quad Q = 136{\cdot}9.$$

Slab design. Since ratio

$$\frac{\text{long span}}{\text{short span}} = 3,$$

slabs will be designed to span in one direction.

	Positive B.M.		*Negative B.M.*	
	Dead	*Super*	*Dead*	*Super*
Interior spans	$wl^2/24$	$wl^2/12$	$wl^2/12$	$wl^2/12$
End spans	$wl^2/12$	$wl^2/10$	$wl^2/10$	$wl^2/10$

Interior spans:

Positive B.M. $\begin{cases} \text{Dead} = \dfrac{60 \times 6{\cdot}67^2 \times 12}{24} = 1{,}333 \text{ in.-lb.} \\ \text{Super} = \dfrac{224 \times 6{\cdot}67^2 \times 12}{12} = 9{,}967 \text{ ,,} \end{cases}$

$$11{,}300 \text{ ,,}$$

Negative B.M. $\begin{cases} \text{Dead} = \dfrac{60 \times 6{\cdot}67^2 \times 12}{12} = 2{,}667 \text{ ,,} \\ \text{Super (as Positive)} = 9{,}967 \text{ ,,} \end{cases}$

$$12{,}634 \text{ ,,}$$

M.R. of 4-in. slab ($d = 3{\cdot}25$ in.)

$$= 12 \times 136{\cdot}9 \times 3{\cdot}25^2 = 17{,}500 \text{ in.-lb.}$$

Make reinforcement same throughout, then

$$A_t = \frac{12{,}633}{0{\cdot}858 \times 3{\cdot}25 \times 18{,}000} = 0{\cdot}25 \text{ in.}^2/\text{ft.}$$

Area of $\frac{3}{8}$-in. ϕ bars at 5-in. centres = 0·265 in.^{2}/ft.
For distribution steel use $\frac{5}{16}$-in. ϕ bars at 12-in. centres.

Area given = 0·077 in.^{2}/ft.

End spans:

Positive B.M. $\begin{cases} \text{Dead} = \dfrac{60 \times 6{\cdot}67^2 \times 12}{12} = 2{,}667 \text{ in.-lb.} \\ \text{Super} = \dfrac{224 \times 6{\cdot}67^2 \times 12}{10} = 11{,}960 \text{ ,,} \end{cases}$

$$14{,}627 \text{ ,,}$$

Negative B.M. $\begin{cases} \text{Dead} = \dfrac{60 \times 6{\cdot}67^2 \times 12}{10} = 3{,}200 \text{ ,,} \\ \text{Super (as Positive)} = 11{,}960 \text{ ,,} \end{cases}$

$$15{,}160 \text{ ,,}$$

M.R. of 4-in. slab = 17,500 in.-lb.

Use same reinforcement throughout.

$$A_t = \frac{15{,}160}{0{\cdot}858 \times 3{\cdot}25 \times 18{,}000} = 0{\cdot}303 \text{ in.}^2$$

Use $\frac{3}{8}$-in. ϕ bars at 4-in. centres: area $= 0{\cdot}331$ in.^{2}, and distribution bars $\frac{5}{16}$-in. ϕ at 12-in. centres as before.

Secondary beams:

$$\text{Effective span} = 20 \text{ ft.}$$

$$\text{Centres} = 6 \text{ ft. } 8 \text{ in.}$$

Area supported by secondary beams

$$= 20\times 6{\cdot}67 - 6{\cdot}67\times 3{\cdot}33 = 111{\cdot}11 \text{ sq. ft.}$$

$$\text{Dead load} = 60\times 111{\cdot}11 = 6{,}667 \text{ lb.}$$

$$\text{O.W.} = \tfrac{3}{4}\times\tfrac{7}{6}\times 19\times 144 = \underline{2{,}432} \text{ ,,}$$

$$9{,}099 \text{ ,, say } 9{,}100.$$

$$\text{Super load} = 224\times 111{\cdot}11 = 25{,}000 \text{ lb.}$$

Interior spans:

$$\text{Positive B.M.}\begin{cases}\text{Dead} = \dfrac{9{,}100\times 20\times 12}{24} = 91{,}000 \text{ in.-lb.}\\[2ex] \text{Super} = \dfrac{25{,}000\times 20\times 12}{12} = \underline{500{,}000} \text{ ,,}\end{cases}$$

$$591{,}000 \text{ ,,}$$

Effective width

Span/3	= 6 ft. 8 in.
Centres of ribs	= 6 ft. 8 in.
Rib + 12 × slab = 8 in. + 12 × 4 in.	= 4 ft. 8 in.

Take least value = 4 ft. 8 in. = 56 in.
Take effective depth = 18 in.; then

$$n = 18\times 0{\cdot}425 = 7{\cdot}65 \text{ in.}$$

$$\text{Max. compressive stress in slab} = 750 \text{ lb./in.}^2$$

$$\text{Min. ,, ,, ,,} = 750\times\frac{3{\cdot}65}{7{\cdot}65} \text{ ,,}$$

$$= 358 \text{ lb./in.}^2$$

Moment of resistance of sections:

$$\text{Rib } 136{\cdot}9\times 8\times 18^2 = 355{,}000 \text{ in.-lb.}$$

$$\text{Slab}\begin{cases}(56-8)4\times 358 = 68{,}550\times 16 = 1{,}095{,}000 \text{ ,,}\\[2ex] (56-8)4\times\dfrac{750-358}{2} = 38{,}200\times 16{\cdot}67 = \underline{642{,}000} \text{ ,,}\end{cases}$$

$$\text{Total} = 2{,}092{,}000 \text{ ,,}$$

$$\therefore\ \text{M.R. is greater than B.M.}$$

$$A_t = \frac{590{,}667}{18{,}000\times 0{\cdot}858\times 18} = 2{\cdot}15 \text{ in.}^2$$

Use $\begin{cases} 2 \text{ at } 1\text{-in. } \phi = 1{\cdot}57 \text{ in.}^2 \\ 1 \text{ at } \frac{7}{8}\text{-in. } \phi = 0{\cdot}60 \text{ ,,} \end{cases}$

$$2{\cdot}17 \text{ ,,}$$

Negative B.M.:

$$\text{Dead} = \frac{9{,}067 \times 20 \times 12}{12} = 181{,}330 \text{ in.-lb.}$$

$$\text{Super (as Positive)} = 500{,}000 \text{ ,,}$$

$$681{,}330 \text{ ,,}$$

M.R. of 8 in. × 18 in. beam $= 8 \times 136{\cdot}9 \times 18^2 = 355{,}000$,,

∴ moment to be taken by compression steel $= 326{,}330$,,

$$n = 0{\cdot}425 \times 18 = 7{\cdot}65 \text{ in.}$$

Permissible stress in compression steel

$$= \frac{6{\cdot}15}{7{\cdot}65} \times 750 \times 17 = 10{,}300 \text{ lb./in.}^2$$

$$\therefore \quad A_c = \frac{326{,}330}{10{,}300 \times 15{\cdot}45} = 2{\cdot}05 \text{ in.}^2$$

Use $\begin{cases} 1 \text{ at } \frac{7}{8}\text{-in. } \phi = 0{\cdot}60 \text{ in.}^2 \\ 2 \text{ at } 1\text{-in. } \phi = 1{\cdot}57 \text{ ,,} \end{cases}$

$$2{\cdot}17 \text{ ,,}$$

$$A_t = \frac{681{,}330}{18{,}000 \times 15{\cdot}45} = 2{\cdot}46 \text{ in.}^2$$

Use $\begin{cases} 2 \text{ at } 1\text{-in. } \phi = 1{\cdot}57 \text{ in.}^2 \\ 1 \text{ at } 1\frac{1}{8}\text{-in. } \phi = 0{\cdot}99 \text{ ,,} \end{cases}$

$$2{\cdot}56 \text{ ,,}$$

$$\text{End shear} = 17{,}000 \text{ lb.}$$

Shear taken by two 1-in. ϕ bars at 45°

$$= \frac{1{\cdot}57}{\sqrt{2}} \times 18{,}000 = 20{,}000 \text{ lb.}$$

Shear beyond end of bent-up bars $= 17{,}000 \times \frac{6{\cdot}5}{9{\cdot}5} = 11{,}650$ lb.

Shear taken by concrete $= 8 \times 0{\cdot}858 \times 18 \times 75 = 9{,}250$,,

Shear to be taken by stirrups $= 2{,}400$,,

Using $\frac{5}{16}$-in. ϕ stirrups

$$A_w = 0{\cdot}153 \text{ in.}^2$$

$$\therefore \quad p = \frac{18{,}000 \times 0{\cdot}153 \times 0{\cdot}858 \times 18}{2{,}400}.$$

$$\therefore \quad p = 18 \text{ in.}$$

Use 9-in. spacing at ends and 12-in. in middle.

End spans:

$$\text{Positive B.M.}\begin{cases}\text{Dead} = \dfrac{9{,}067\times 20\times 12}{12} = 181{,}340 \text{ in.-lb.}\\ \text{Super} = \dfrac{25{,}000\times 20\times 12}{10} = 600{,}000 \text{ ,,}\end{cases}$$

781,340 ,,

This is less than M.R. of beam.

$$A_t = \frac{781{,}340}{18{,}000\times 0{\cdot}858\times 18} = 2{\cdot}80 \text{ in.}^2$$

Use three $1\frac{1}{8}$-in. ϕ.

$$\text{Negative B.M.}\begin{cases}\text{Dead} = \dfrac{9{,}067\times 20\times 12}{10} = 217{,}000 \text{ in.-lb.}\\ \text{Super (as Positive)} = 600{,}000 \text{ ,,}\end{cases}$$

817,000 ,,

M.R. of 18 in. × 8 in. = 355,000 ,,

B.M. to be taken by compression steel = 462,000 ,,

$$\therefore \quad A_c = \frac{462{,}000}{10{,}300\times 15{\cdot}45} = 2{\cdot}88 \text{ in.}^2$$

$$A_t = \frac{817{,}000}{18{,}000\times 15{\cdot}45} = 2{\cdot}93 \text{ ,,}$$

Use three $1\frac{1}{8}$-in. ϕ = 2·98 in.²

Shear as for interior spans.

Main beams:

Effective span = 20 ft.

Floor area supported = 22·22 sq. ft.

Distributed dead load = 22·22 × 60 = 1,333 lb.

O.W. 19 × 1 × 1·5 × 144 = 4,100 ,,

∴ Total distributed dead load = 5,433 ,,

Distributed super load = 22·22 × 224 = 5,000 ,,

Loads from secondary beams at third points:

Dead = 9,067 lb.

Super = 25,000 lb.

Interior spans:

Positive B.M.

$$\text{Dead load}\begin{cases}\text{dist. } \dfrac{5{,}433\times 20\times 12}{24} = 54{,}330 \text{ in.-lb.}\\ \text{conc. } \dfrac{9{,}069\times 20\times 12}{8\cdot 25} = 266{,}000 \text{ ,,}\end{cases}$$

$$\text{Super load}\begin{cases}\text{dist. } \dfrac{5{,}000\times 20\times 12}{12} = 100{,}000 \text{ ,,}\\ \text{conc. } \dfrac{25{,}000\times 20\times 12}{4\cdot 5} = 1{,}333{,}333 \text{ ,,}\end{cases}$$

$$\therefore \text{ Total} = 1{,}753{,}663 \text{ ,,}$$

$$\text{Effective width}\begin{cases}\text{span}/3 = 6 \text{ ft. } 8 \text{ in.}\\ 12\times\text{slab}+\text{rib} = 4 \text{ ft.}+1 \text{ ft.} = 5 \text{ ft.}\end{cases}$$

Adopt 5 ft. as lesser value.

Take overall depth as 21 in. and allow for 1 layer of bars. Then effective depth $d = 19\cdot 25$ in. and $n = 0\cdot 425\times 19\cdot 25 = 8\cdot 18$ in.

Stress at underside of slab

$$= \frac{4\cdot 18}{8\cdot 18}\times 750 = 385 \text{ lb./in.}^2$$

Moment of resistance (compression)

$$\text{Rib} \quad 136\cdot 9\times 12\times 19\cdot 25^2 = 600{,}000 \text{ in.-lb.}$$

$$\text{Slab}\begin{cases}4(60-12)385 = 74{,}000\times 17\cdot 25 = 1{,}280{,}000 \text{ ,,}\\ 4(60-12)\left(\dfrac{750-385}{2}\right) = 35{,}000\times 17\cdot 92 = 625{,}000 \text{ ,,}\end{cases}$$

$$2{,}505{,}000 \text{ ,,}$$

which is greater than the B.M.

$$A_t = \frac{1{,}753{,}663}{18{,}000\times 0\cdot 858\times 19\cdot 25} = 5\cdot 85 \text{ in.}^2$$

Use four $1\frac{3}{8}$-in. ϕ = $5\cdot 93$ in.^{2}

Negative B.M.

$$\text{Dead load}\begin{cases}\dfrac{5{,}433\times 20\times 12}{12} = 108{,}660 \text{ in.-lb.}\\ \dfrac{9{,}067\times 20\times 12}{4\cdot 75} = 460{,}000 \text{ ,,}\end{cases}$$

$$\text{Super load}\begin{cases}\dfrac{5{,}000\times 20\times 12}{12} = 100{,}000 \text{ ,,}\\ \dfrac{25{,}000\times 20\times 12}{3\cdot 5} = 1{,}710{,}000 \text{ ,,}\end{cases}$$

$$\therefore \text{ Total} = 2{,}378{,}660 \text{ ,,}$$

Allowing for 9-in. haunch at columns, effective depth = 28·25 in.

$$\text{M.R.} = 136{\cdot}9 \times 12 \times 28{\cdot}25^2 = 1{,}310{,}000 \text{ in.-lb.},$$

$\therefore$ moment to be taken by compression steel = 1,068,660 in.-lb.

$$n = 0{\cdot}425 \times 28{\cdot}25 = 12{\cdot}00 \text{ in.}$$

Stress in compression steel

$$= \frac{10{\cdot}5}{12} \times 17 \times 750 = 11{,}150 \text{ lb./in.}^2$$

$$\therefore \quad A_c = \frac{1{,}068{,}660}{11{,}150 \times 26{\cdot}75} = 3{\cdot}58 \text{ in.}^2$$

Use $\begin{cases} \text{two } 1\tfrac{1}{8}\text{-in. } \phi = 1{\cdot}97 \text{ in.}^2 \\ \text{two 1-in. } \phi = 1{\cdot}57 \text{ ,,} \end{cases}$

$$\underline{\qquad 3{\cdot}54 \text{ ,,}}$$

$$A_t = \frac{2{,}378{,}660}{18{,}000 \times 24{\cdot}25} = 5{\cdot}45 \text{ in.}^2$$

Use $\begin{cases} \text{four } 1\tfrac{1}{8}\text{-in. } \phi = 3{\cdot}94 \text{ in.} \\ \text{two 1-in. } \phi = 1{\cdot}57 \text{ ,,} \end{cases}$

$$\underline{\qquad 5{\cdot}51 \text{ ,,}}$$

Shear at column faces = $\begin{cases} \text{dist. dead load} = 2{,}716 \text{ lb.} \\ \text{conc. dead load} = 9{,}067 \text{ ,,} \\ \text{conc. super load} = 25{,}000 \text{ ,,} \end{cases}$

$$\underline{\qquad 36{,}783 \text{ ,,}}$$

B.M. at column faces = 2,378,660.

$$\therefore \quad \text{Effective shear} = 36{,}783 - \frac{2{,}378{,}660}{3 \times 28{\cdot}25}$$

$$= 36{,}783 - 28{,}000$$

$$= 8{,}783 \text{ lb.}$$

$$\therefore \quad \text{Shear stress} = \frac{8{,}783}{12 \times 0{\cdot}858 \times 28{\cdot}25} = 30 \text{ lb./in.}^2$$

$$\text{Shear at end of haunch} = \frac{7{\cdot}25}{9{\cdot}5} \times 36{,}783$$

$$= 28{,}200 \text{ lb.}$$

$$\text{Shear taken by two 1-in. } \phi \text{ bars bent} = \frac{1{\cdot}57}{\sqrt{2}} \times 18{,}000$$

$$= 20{,}000 \text{ lb.}$$

Shear on concrete = 8,200 ,,

$$\text{Stress ,, ,,} = \frac{8{,}200}{12 \times 0{\cdot}858 \times 19{\cdot}25} = 41 \text{ lb./in.}^2$$

End spans:

Positive B.M.

$$\text{Dead load}\begin{cases}\dfrac{5{,}433\times 20\times 12}{12} = 108{,}660 \text{ in.-lb.}\\[2ex] \dfrac{9{,}067\times 20\times 12}{4} = 544{,}000 \quad ,,\end{cases}$$

$$\text{Super load}\begin{cases}\dfrac{5{,}000\times 20\times 12}{10} = 120{,}000 \quad ,,\\[2ex] \dfrac{25{,}000\times 20\times 12}{3\cdot5} = 1{,}715{,}000 \quad ,,\end{cases}$$

$$\text{Total} = 2{,}487{,}660 \quad ,,$$

This is less than M.R. in compression.

$$A_t = \frac{2{,}487{,}660}{18{,}000\times 0\cdot858\times 19\cdot25} = 8\cdot4 \text{ in.}^2$$

$$\text{Use}\quad \begin{cases}\text{four } 1\tfrac{3}{8}\text{-in. } \phi = 5\cdot94 \text{ in.}^2\\ \text{two } 1\tfrac{1}{4}\text{-in. } \phi = 2\cdot45 \quad ,,\end{cases}$$
$$8\cdot39 \quad ,,$$

Negative B.M.

$$\text{Dead load}\begin{cases}\dfrac{5{,}433\times 20\times 12}{10} = 130{,}000 \text{ in.-lb.}\\[2ex] \dfrac{9{,}067\times 20\times 12}{3\cdot5} = 622{,}000 \quad ,,\end{cases}$$

$$\text{Super load}\begin{cases}\dfrac{5{,}000\times 20\times 12}{10} = 120{,}000 \quad ,,\\[2ex] \dfrac{25{,}000\times 20\times 12}{3\cdot25} = 1{,}840{,}000 \quad ,,\end{cases}$$

$$\text{Total} = 2{,}712{,}000 \quad ,,$$
$$\text{M.R.} = 1{,}310{,}000 \quad ,,$$

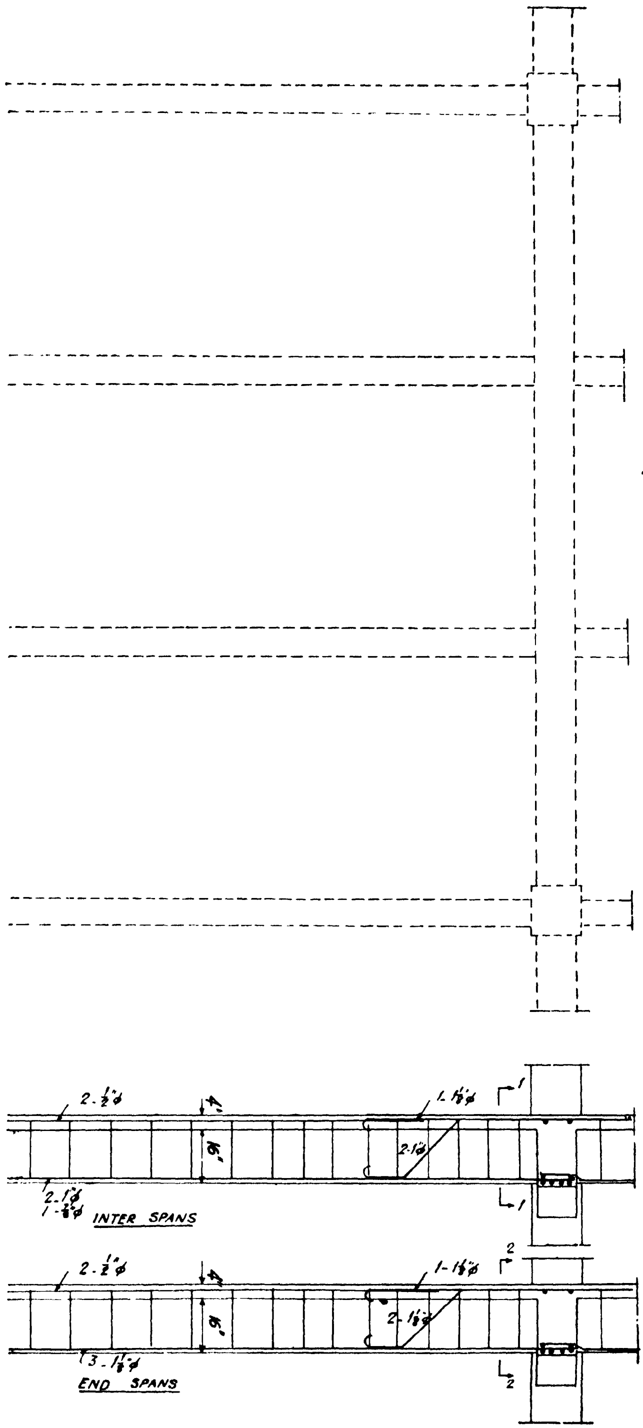
B.M. to be taken by compression steel $= 1{,}400{,}000$,,

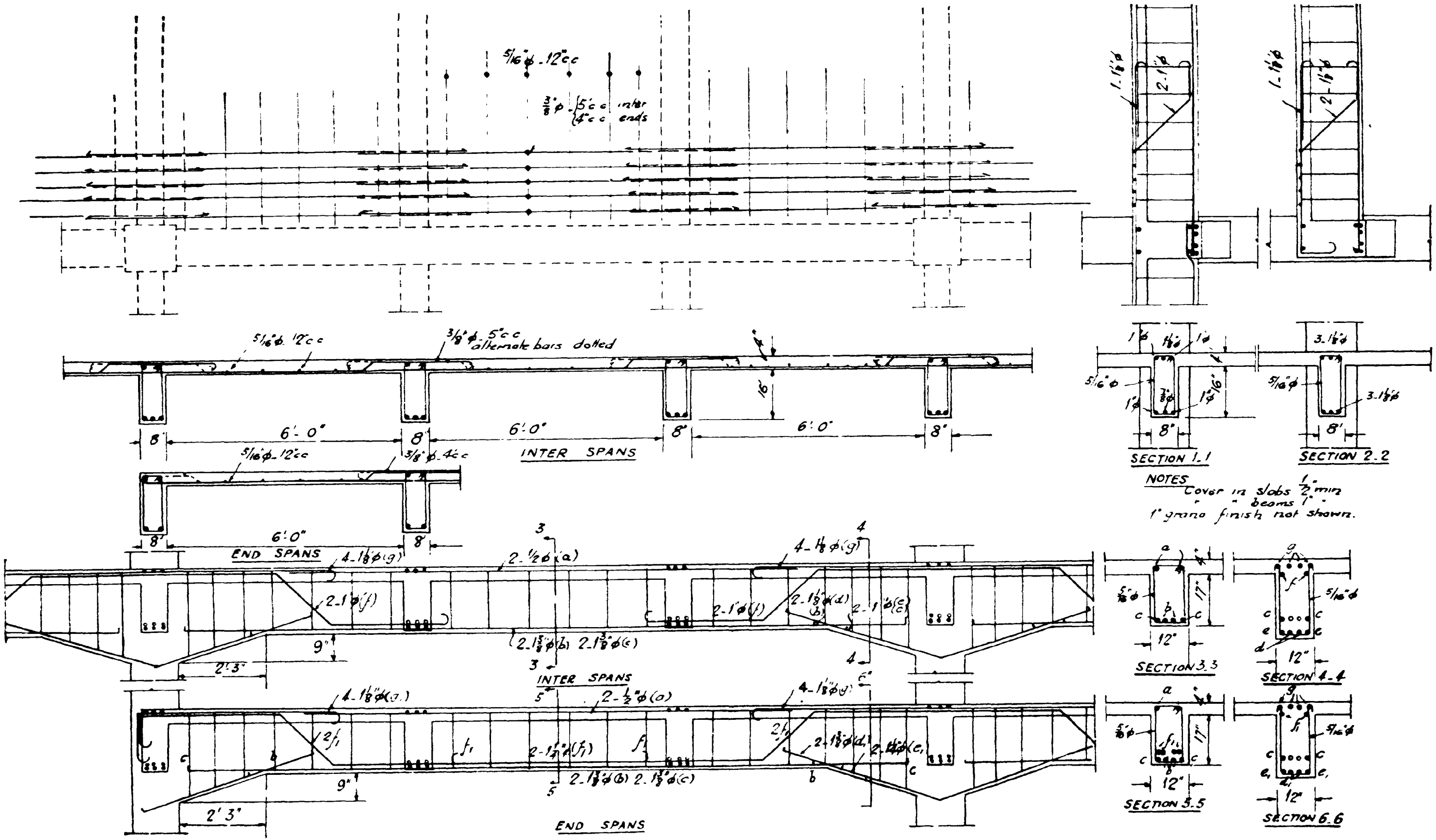
$$A_c = \frac{1{,}400{,}000}{11{,}150\times 26\cdot75} = 4\cdot73 \text{ in.}^2$$

$$\text{Use}\quad \begin{cases}\text{two } 1\tfrac{3}{8}\text{-in. } \phi = 2\cdot97 \text{ in.}^2\\ \text{two } 1\tfrac{1}{4}\text{-in. } \phi = 2\cdot45 \quad ,,\end{cases}$$
$$5\cdot42$$

$$A_t = \frac{2{,}712{,}000}{18{,}000\times 0\cdot858\times 28\cdot25} = 6\cdot2 \text{ in.}^2$$

$$\text{Use}\quad \begin{cases}\text{two } 1\tfrac{1}{4}\text{-in. } \phi = 2\cdot45 \text{ in.}^2\\ \text{four } 1\tfrac{1}{8}\text{-in. } \phi = 3\cdot98 \quad ,,\end{cases}$$
$$6\cdot43 \quad ,,$$


INTER SPANS
END SPANS
16"
4"



5/16"φ. 12" c c
3/8"φ. 5" c c inter, 4" c c ends
3/8"φ. 5" c c alternate bars dotted
5/16"φ. 12" c c
3/8"φ. 4" c c
INTER SPANS
END SPANS
6'-0"
8"
16"
9"
2' 3"
4-1 1/8"φ(g)
2-1/2"φ(a)
2-1"φ(f)
2-1 1/8"φ(b) 2-1 1/8"φ(c)
SECTION 1.1
SECTION 2.2
SECTION 3.3
SECTION 4.4
SECTION 5.5
SECTION 6.6
NOTES
Cover in slabs 1/2" min
" " beams 1" "
1" grano finish not shown.

FIG. 6.9

Shear as for interior spans.

For arrangement and details see Fig. 6.9. (All B.M. coefficients taken from Institution of Structural Engineers Report on Reinforced Concrete for Buildings and Structures, Part I (Loads).)

Composite Beams

When steel beams are used in conjunction with concrete slabs, it is usual to encase the beams in concrete in order to protect them against the effects of fire or weather. The weight of the concrete increases the dead load to be carried by the beams, but, at the same time, the steel is strengthened by the encasement and many authorities advocate higher working stresses in such cases.

The strength of composite sections has been investigated by a number of tests carried out in this country and in America, notably by Prof. Batho, Dr. Lash, and Mr. Kirkham; cf. 'The Properties of Composite Beams, consisting of Steel Joists encased in Concrete, under Direct and Sustained Loading' (*Journal Inst. C.E.* **11**, 1938–9, pp. 61–114). Other data are to be found in 'Composite Beams of Concrete and Structural Steel', R. A. Caughey (*Proc. Iowa Engineering Soc.* 1929); 'Tests of Steel Floor Framing encased in Concrete' (*Proc. Western Society of Engineers*, June 1930); Bulletin 75, Iowa Engineering Experiment Station (A. H. Fuller and R. A. Caughey), and in Arrol's *R.C. Handbook*.

Tests have shown that the composite section of steel and concrete behaves as a unit, provided that the bond stress between the steel and concrete does not exceed the permissible stress. Professor Caughey in his book *Reinforced Concrete* gives a value of 60 lb./in.^{2} for bond and 240 lb./in.^{2} for horizontal shear on the concrete. He also suggests that for preliminary design purposes the safe working stress in steel joist be increased by $33\frac{1}{3}$ per cent., and that the strength of the composite section be checked according to the rules for R.C. Tee-beams. In designing such sections the unknown factors are: (1) The effective width of concrete to be taken. Mr. E. A. Scott (Arrol's *R.C. Handbook*) gives the rule that the least of following quantities should be used. (*a*) $8D+b_1$, where D = depth from top of concrete to underside of joist and b_1 = width of rib measured on concrete. (*b*) joist span/3, and (*c*) $\frac{3}{4}\times$ slab span. (2) Depth of neutral axis. In calculating this the full section of the steel joist should be used, although part of it may lie within the compression zone.

Method of design (see Fig. 6.10). If b = effective width of slab (in.), then

$$bn\times\frac{n}{2} = mA_s(y_s-n),$$

where A_s = area of steel, y_s = depth to steel C.G., m = modular ratio;

or
$$bn^2 - 2mA_s y_s + 2mA_s n = 0.$$

$$\therefore \quad n = -\frac{mA_s}{b} + \sqrt{\left\{\left(\frac{mA_s}{b}\right)^2 + \frac{2mA_s y_s}{b}\right\}},$$

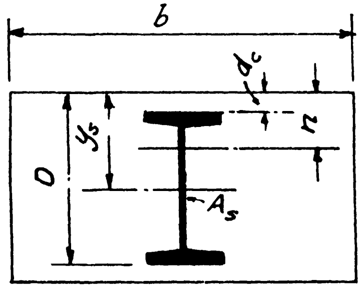


FIG. 6.10

from which n is found. Having found n, the equivalent moment of inertia about N.A. can be calculated:

$$I_e = \frac{bn^3}{3m} + I_s + A_s(y_s - n)^2$$

in steel units, where I_s = moment of inertia of steel about its C.G.

Hence if M is applied moment

$$f_s = \frac{M(D-n)}{I_e} = \text{fibre stress in steel at underside of joist,}$$

$$f_c = \frac{Mn}{mI_e} = \text{fibre stress in concrete at top,}$$

$$f_d = \frac{M(y_s - n)}{I_e} = \text{direct stress in steel,}$$

$$f_b = \frac{M(D-y_s)}{I_e} = \text{max. bending stress in steel.}$$

Also
$$s_b = \frac{Sd_c(n-(d_c/2))}{mI_e} = \text{bond stress at top of joist,}$$

where S = total shear and d_c = concrete cover at top.

The joist in a composite beam is, in general, subject to axial and flexural stresses and the maximum fibre stress is the sum of these. The flexural strength of the joist is much greater than that of ordinary reinforcement and may form a considerable proportion of the total strength. The strength can be increased by the provision of anchorage plates or angles on the top flange of the joist. Where composite sections are subject to reverse B.M., the strength can be increased by the addition of steel rods in the compression zone. The underside of the concrete should be not less than 1 in. below the N.A. and the cover to the lower flange of the joist not less than 2 in. Some authorities recommend that concrete at the sides of the ribs should be sloped as shown in Fig. 6.10 (*a*), and it is advisable to have the lower flange wrapped as shown.

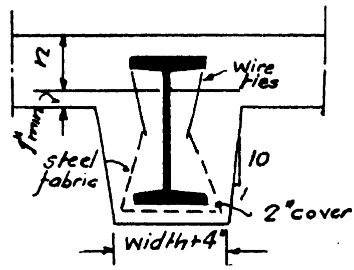


FIG. 6.10 (*a*)

Example. 7 in. × 4 in. × 16 in. B.S.B., spaced at 2 ft. centres. $d_c = 2$ in.; $t = 18{,}000$ lb./in.²; $c = 750$ lb./in.²; $m = 15$; $A_s = 4{\cdot}75$ in.²; $I_s = 39{\cdot}5$ in.⁴; $y_s = 5{\cdot}5$ in.

Take

$$b = \tfrac{3}{4}\times 24 = 18 \text{ in.}$$

$$n = -\frac{15\times 4{\cdot}75}{18} + \sqrt{\left\{\left(\frac{15\times 4{\cdot}75}{18}\right)^2 + \frac{2\times 15\times 4{\cdot}75\times 5{\cdot}5}{18}\right\}} = 3{\cdot}72 \text{ in.}$$

$$I_e = \frac{18\times 3{\cdot}72^3}{3\times 15} + 39{\cdot}5 + 4{\cdot}75(1{\cdot}78)^2$$

$$= 75 \text{ in.}^4 \text{ approx.}$$

$$M_{\text{steel}} = \frac{75\times 18{,}000}{9-3{\cdot}72} = 232{,}000 \text{ in.-lb.}$$

$$M_{\text{conc}} = \frac{750\times 15\times 75}{3{\cdot}72} = 226{,}000 \quad ,,$$

$$M_{\text{joist}} = \frac{39{\cdot}5}{3{\cdot}5}\times 18{,}000 = 203{,}000 \quad ,,$$

Increase in moment = 23,000 in.-lb. = 11·3 per cent.
Max. shear S for $s_b = 60$ lb./in.²

$$S = \frac{15\times 75\times 60}{2\times 2{\cdot}72} = 12{,}400 \text{ lb.}$$

For $b = 24$ in.; $n = 3{\cdot}46$ in.; $I_e = 81{\cdot}4$ in.⁴; $M = 265{,}000$ in.-lb.
Increase in moment = 62,000 in.-lb. = 30·5 per cent.
The figure 30·5 per cent. is roughly in accordance with Prof. Caughey's rule for approximate design.

Design of Timber Beams and Joists

The physical properties of timber have been dealt with in Chapter I and in the Appendix. The moisture content has a considerable effect on the strength of timber and the latter may vary accordingly. In this country timber is divided into graded and non-graded, and the L.C.C. Regulations give the following values for stresses in lb. per sq. in.:

Nature of stress	*Graded*	*Non-graded*
Extreme fibre stress in bending .	1,200	800
Shear stress parallel to grain .	100	90
Compression perpendicular to grain	325	165
Tension parallel to grain . .	1,200	800
Modulus of elasticity . . .	1,600,000	1,200,000

The maximum deflexion should not exceed span/360.

In calculating the strength of timber beams the span should be taken as centre to centre of bearings. In timber floors the distribution of concentrated loads can be taken as (for B.M.)

Thickness of floor	*Load on beam nearest load*
2 in.	Joist centres ÷ 4·0
4 in.	,, ÷ 4·5
6 in.	,, ÷ 5·0

Where the joist spacing (in feet) exceeds the value of the coefficient given above, then the floor between the joists can be treated as a simple beam and reactions calculated accordingly. The above values to be used in calculating bending moment. To calculate the shear on floor joists due to concentrated load, when the distribution of moment at mid-span is known, the distribution to adjacent beams, loaded at or near the quarter-points, can be obtained from the following table:

Distribution in Terms of Proportion of Total Load

Load at mid-span		*Load at quarter-point*	
Centre beam	*Proportion on side beams*	*Centre beam*	*Proportion on side beams*
1·00	0	1·00	0
0·90	0·1	0·94	0·06
0·80	0·2	0·87	0·13
0·70	0·3	0·79	0·21
0·60	0·4	0·69	0·31
0·50	0·5	0·58	0·42
0·40	0·6	0·44	0·56
0·33	0·67	0·33	0·67

The design of timber beams and joists is governed by three factors: (1) bending stress, (2) shear stress parallel to the grain, (3) deflexion.

(A rough rule for the depth of timber floor joists is span/24+2 in.) The safe B.M. on a timber joist of breadth b in. and depth d in. is $fZ = f \times bd^2/6$ and $f = 1{,}200$ (graded) and 800 (non-graded). The shear stress is a maximum at the N.A. and is $\frac{3}{2} \times$ average shear stress $= \frac{3}{2} \times S/bd$, where S is total shear. Therefore maximum shear $S = \frac{2}{3}sbd$ and $s = 100$ (graded) and 90 (non-graded). In many cases shear will be found to be the limiting factor. Deflexion can be calculated according to the formulae given in Chapter IV.

For rectangular joists as before $I = bd^3/12$

and the deflexion at mid-span $= \dfrac{5WL^3}{384EI} = 5WL^3/32Ebd^3$ for distributed load,

and $= WL^3/4Ebd^3$ for central load.

$E = 1{,}600{,}000$ lb.-in.² (graded) and 1,200,000 lb./in.² (non-graded). Now if deflexion is limited to span/360, the maximum span can be calculated in each case in terms of d.

For graded timbers max. span $= d \times \frac{40}{27}$.

For non-graded timbers max. span $= d \times \frac{5}{3}$.

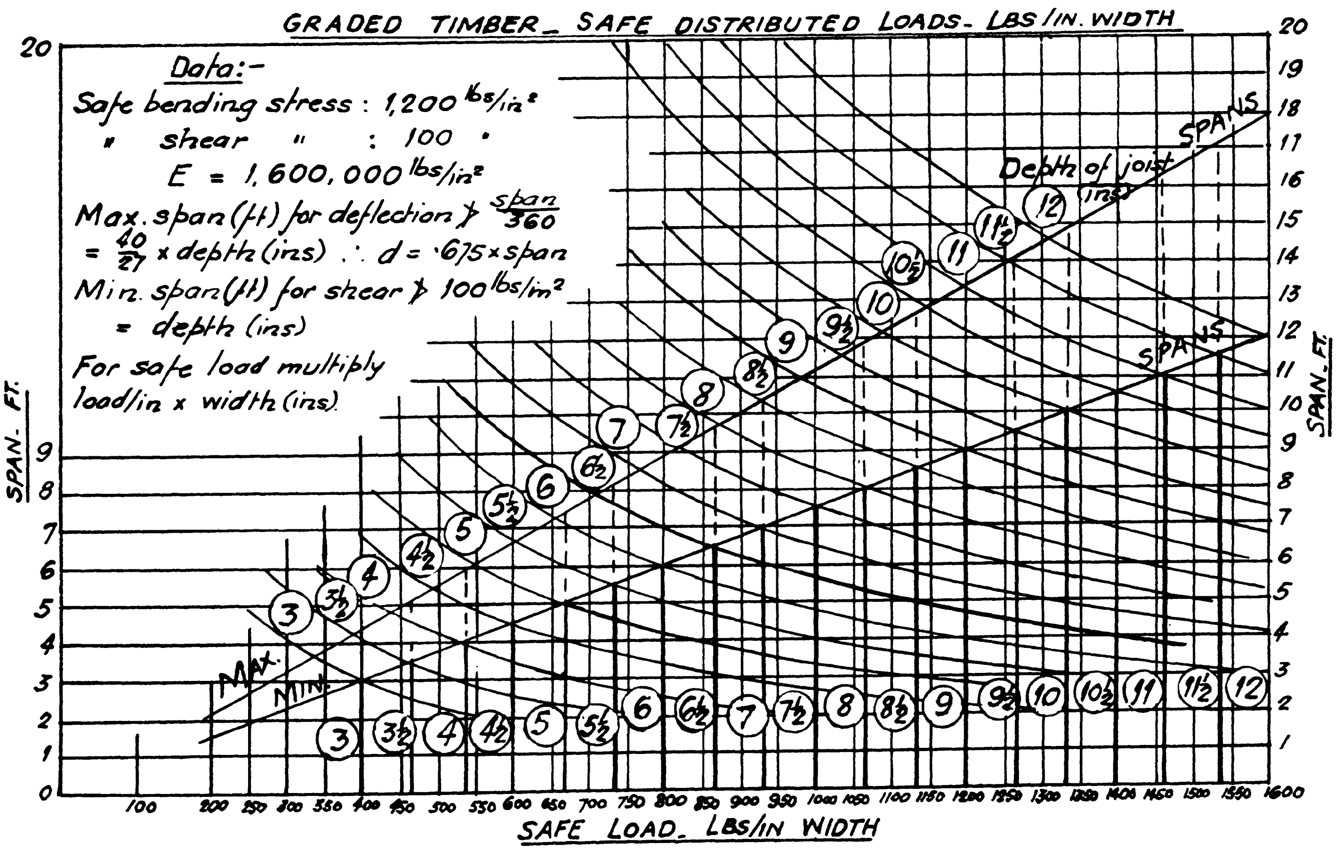
Similarly the minimum span can be calculated from the shear stress in terms of d.

For graded timbers min. span $= d$.

For non-graded timbers min. span $= d \times 0{\cdot}731$.

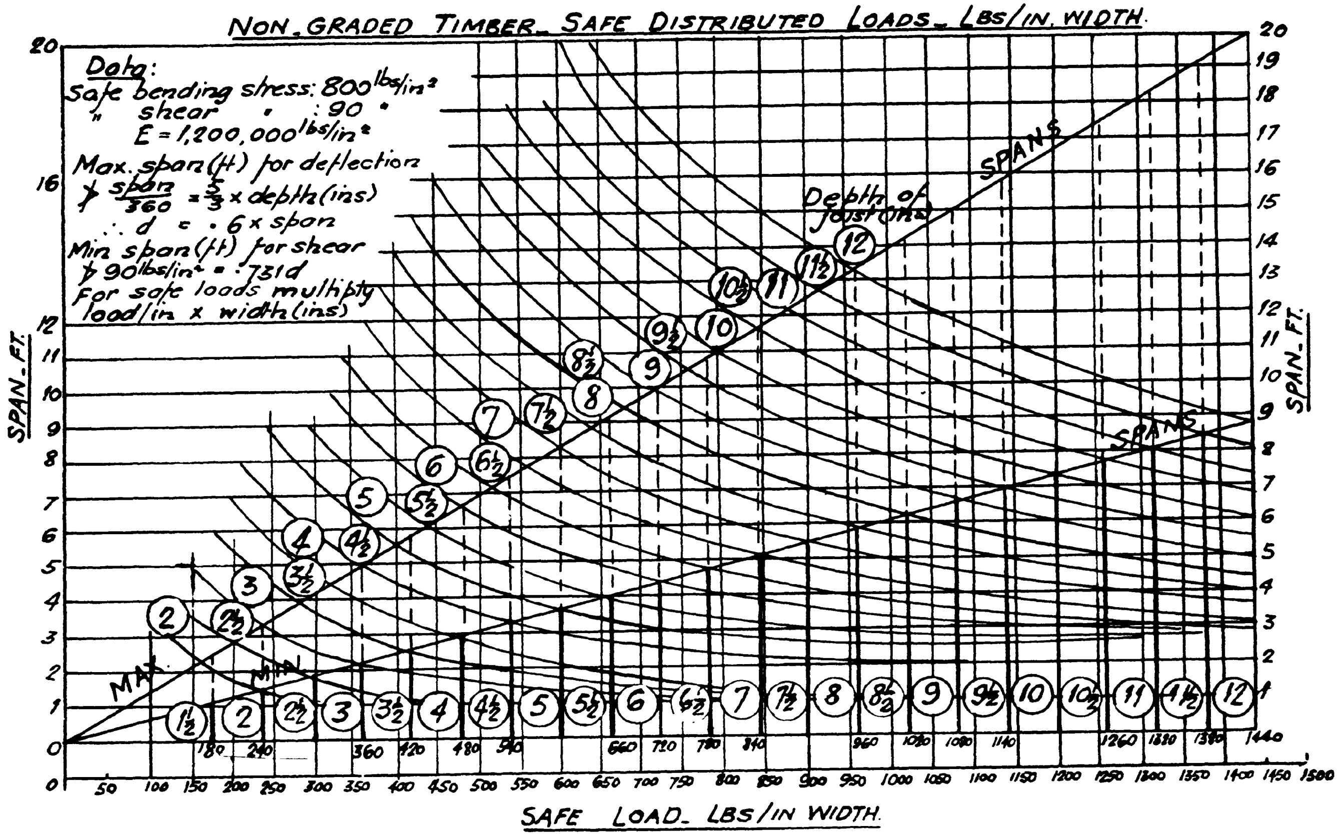
Figs. 6.11 and 6.12 give the total safe distributed loads in pounds for joists up to 12 in. deep. The actual values are found by multiplying the figures given per unit width by the width of the joist. Take as an example a 12 in. × 6 in. (graded) timber joist on 16-ft. span. Then using Fig. 6.11 safe load = 1,200 × 6 = 7,200 lb. Note that for 12-in. depth maximum span (for deflexion) is 17·8 ft. and minimum span (for shear) = 12 ft. These values are read off by following the vertical line marked 12 in. until it meets the sloping lines marked max. spans and min. spans.

For 9 in. × 3 in. timber (non-graded) use Fig. 6.12. It is required to find maximum and minimum spans, also the corresponding loads. Move up line marked 9 in. vertically until it meets line of maximum spans at 15 ft., then horizontally to intersect curve of 9 in. depth, then by moving vertically downwards the value of 480 lb. is read on base line. Therefore safe load = 480 × 3 = 1,440 lb. Minimum span = 6 ft. 8 in. and corresponding load = 1,080 × 3 = 3,240 lb. Other values can



GRADED TIMBER_ SAFE DISTRIBUTED LOADS- LBS/IN. WIDTH
Data:-
Safe bending stress : 1,200 lbs/in²
" shear " : 100 "
E = 1,600,000 lbs/in²
Max. span (ft) for deflection ≯ span/360
= 40/27 x depth (ins) ∴ d = ·675 x span
Min. span (ft) for shear ≯ 100 lbs/in²
= depth (ins)
For safe load multiply
load/in x width (ins)
Depth of joist (ins)
SPANS
MAX.
MIN.
SPAN. FT.
SAFE LOAD. LBS/IN WIDTH

Fig. 6.11



NON. GRADED TIMBER. SAFE DISTRIBUTED LOADS. LBS/IN. WIDTH.
Data:
Safe bending stress: 800 lbs/in²
" shear " : 90 "
E = 1,200,000 lbs/in²
Max. span (ft) for deflection
≯ span/360 = 5/3 × depth (ins)
∴ d = ·6 × span
Min span (ft) for shear
≯ 90 lbs/in² = ·731d
For safe loads multiply
load/in × width (ins)
Depth of joist (ins)
SPANS
MAX.
MIN.
SPAN. FT.
SAFE LOAD. LBS/IN WIDTH.

Fig. 6.12

be found in the same way. Circular sections can be taken as equivalent to a square section of equal cross-sectional area.

In calculating flexural strength of beams notched at or near mid-span, the net depth should be used. For beams notched at or near bearings, the net depth should be used in calculating the shear.

Lateral Restraint for Timber Joists

Ratio depth/breadth	*Restraint required*
Not exceeding 2	None.
3	At ends.
4	Joist to be held in line.
5	One edge to be held in line.
6	To be bridged at 6 times depth.
7	Both edges to be held in line.

For joists subject to bending plus compression parallel to the grain, the ratio depth/breadth can be 5 if one edge is held in line. In general, joists should be bridged at not more than 8 ft. centres or not more than 8 ft. from bearing to nearest bridge. If these values are exceeded, safe stresses should be reduced accordingly.

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CHAPTER VII

SIMPLE FRAMED STRUCTURES

Definition. A framed structure consists theoretically of a number of straight bars, pin-jointed together at their ends. Such structures can be either two-dimensional or three-dimensional.

The following assumptions are made unless otherwise stated:

(1) that the intersection of two or more members forms a perfectly frictionless joint;

(2) that the external forces are applied at the joints only (where the load is applied between two joints, it should be replaced by two equivalent loads at the nearest joints and the local B.M. in the member calculated separately);

(3) that the forces in the members are axial tension or compression;

(4) that the members are rigid and that they do not alter appreciably in length under the applied loading.

Types of Framed Structures

1. *Firm or completely triangulated frames* which have just sufficient members to prevent distortion and in which any alteration to the length of a member will not induce a stress in any other member. To fulfil this condition the following equation must apply:

$$n = 2N-3,$$

where n = number of bars,

N = ,, joints

(Fig. 7.1 (*a*)). (This equation applies to two-dimensional frames.)

2. *Deficient frames* do not have sufficient members to prevent distortion and to satisfy the above equation. They are prevented from

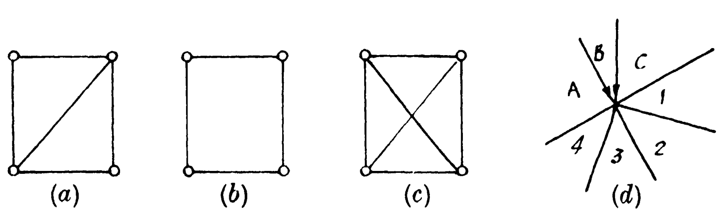


FIG. 7.1

collapse by the stiffness of the joints and will be dealt with in the chapters dealing with portals and rigid frames. (Fig. 7.1 (*b*).)

3. *Redundant frames* have more members than are necessary for complete triangulation or to satisfy the equation referred to. (Fig.

7.1 (*c*).) Any error in the length of a member or change in temperature may induce forces in the members. Such frames are treated in Chapter VIII.

The analysis of the forces in completely triangulated frames can be made by graphics or by calculation.

Graphic Statics

In order to specify a force completely, the following data must be known: (1) magnitude and direction of force, (2) point of application. For equilibrium there must be no translation and the resultant of the reactions must be equal and opposite to that of the external forces or loads. In drawing the force diagram (erroneously called the 'stress' diagram) it is necessary to know (1) all the data regarding all the forces at any one point except one force about which nothing is known, or (2) all the data regarding all the forces except for two forces, of which either the magnitudes or the directions are unknown.

Bow's notation is used in lettering the forces, and each external force must have a letter and one only on each side of it (see Fig. 7.1 (*d*) and examples). The usual practice is to go round the frame in a clockwise direction when lettering the forces, and the internal spaces between the members may be lettered or numbered. The rule for finding the 'sense' of the force in any member, i.e. whether it is in tension or compression, is as follows: take the letters or numbers on each side of the member in the space diagram in the same direction as that in which the external forces were lettered, with respect to the joint at one end. Then the direction of the force in the force diagram is from the first to the second letter or number. If this is *towards* the joint used for reference, the member is in *compression*; if this is *from* the joint, the member is in *tension*. The usual convention (apart from the method of tension coefficients) is for compression to be shown by a plus (+) and tension to be shown by a minus (−). The *magnitude* of the forces is found from the force diagram, i.e.

$$\text{force} = \text{length of line on diagram} \times \text{force scale adopted.}$$

Examples of Firm Frames

1. (Fig. 7.2.) R.H. reaction is vertical since the R.H. end is on rollers and therefore incapable of resisting horizontal forces. Set down load line *bcdef* to a scale of 1 in. to 4 tons. Draw *cg*, *dg* parallel to *CG*, *DG* respectively to intersect in *g*. Draw *gh*, *eh* parallel to *GH*, *EH* respectively to meet in *h*, and so on. The values of the forces in the members and the reactions can be scaled from the force diagram.

2. (Fig. 7.3.) Truss subject to vertical dead and normal wind load. R.H. reaction vertical. Analysis as for (1).

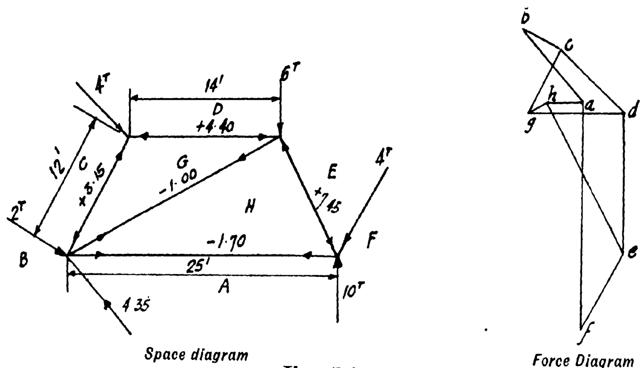


FIG. 7.2

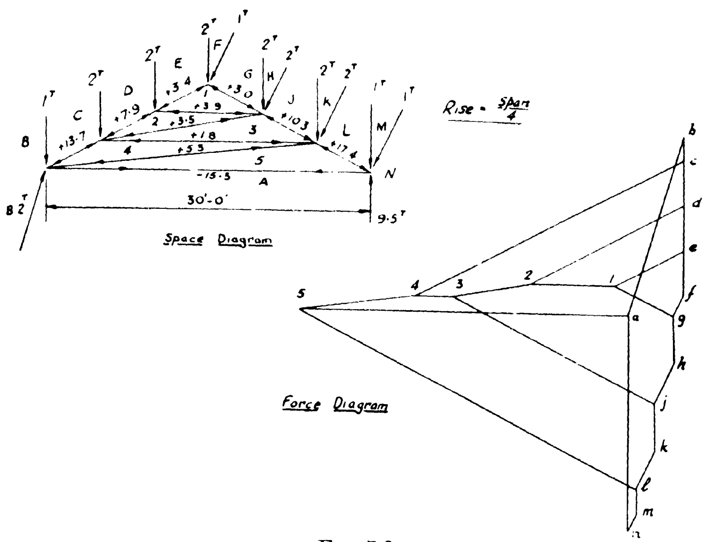


FIG. 7.3

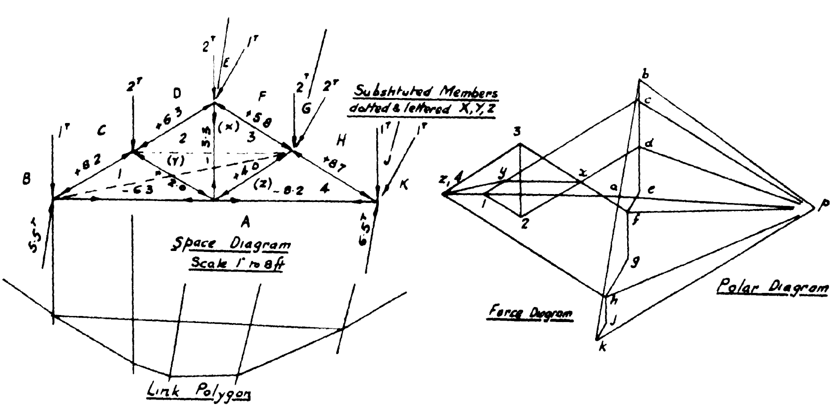


FIG. 7.4

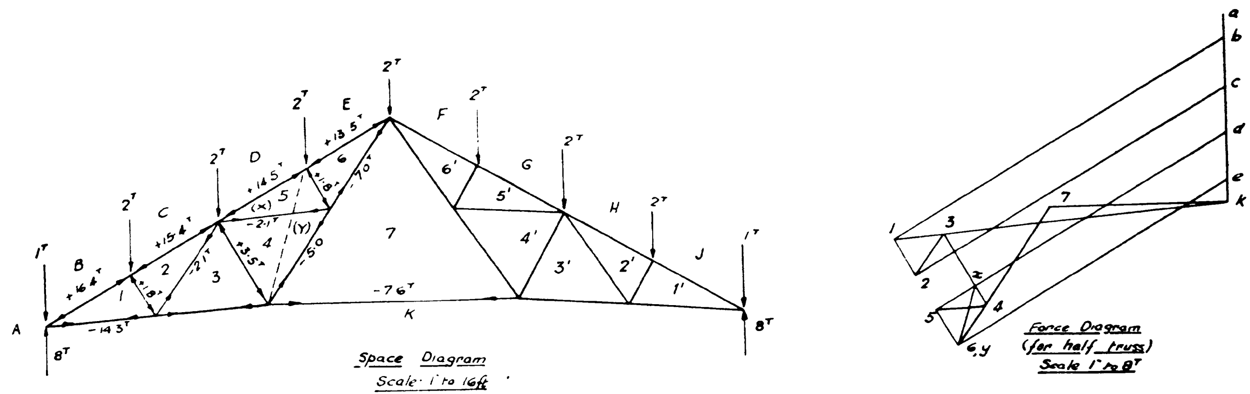

Space Diagram
Scale: 1" to 16ft
Force Diagram
(for half truss)
Scale 1" to 8T

FIG. 7.5

3. (Fig. 7.4.) King-post truss subject to vertical dead and normal wind load. In this case both ends are fixed and the reactions must be parallel if the supports are equally rigid. The reactions can be found by means of a 'substituted frame'.

In some cases it is found that, in drawing the force diagram, a point is reached where the number of unknown forces is more than the usual two. In such cases the 'substituted frame' can be used. Since the magnitudes of the reactions are independent of the internal framing, they can be found by using this method, i.e. a 'substituted frame' carrying the same loads. Having found the reactions, the forces in the members can be found without difficulty.

The reactions can also be found by using a polar diagram and link polygon. Referring to Fig. 7.4, a suitable pole P can be taken and rays Pb, Pc, Pd, Pf, Ph, and Pk drawn. Then, by drawing a line Pa parallel to the closing line of the link polygon, the values of the reactions KA and AB are obtained and will be found to agree with those obtained by means of the substituted frame.

4. (Fig. 7.5.) French truss subject to dead load only. In drawing the force diagram, it is found that at the joint C–D, there are three members meeting at one point. The problem can be solved by substituting the single member shown dotted for the two members 4–5, 5–6, and finding E, and afterwards working back to find the loads in the actual members. Alternatively, it can be assumed that the equal loads BC and DE set up equal forces in the diagonal members and parallel lines are drawn in the force diagram.

Method of Moments or Sections (Ritter's Method)

This method is commonly used for the analysis of framed structures with parallel chords. In such cases it is as simple as the graphical method. Where the chords are not parallel, e.g. roof trusses, it is more laborious than the graphical method, but it may be used as a check on the graphical method. In any framed structure, if a section line X–X is drawn cutting several members, it is obvious that if the members were actually severed, the structure would collapse. Therefore, the moments of the forces in the members about any point must balance the moments due to the external loads. For purposes of analysis, it is convenient to choose a point through which one or more members (or members produced) pass, so that the forces in these members have no moment about that point. Some numerical examples will illustrate this method.

Case 1 (Fig. 7.6). 200-ft. span Pratt truss girder, divided into eight panels of 25 ft. and with dead load on top and bottom panels.

Booms

$L_0 L_1$, moments about U_1:

$$\text{Force} = \frac{10{\cdot}5 \times 25}{31} = 8{\cdot}48 \text{ tons,}$$

and $L_1 L_2 = L_0 L_1$ since members are continuous.

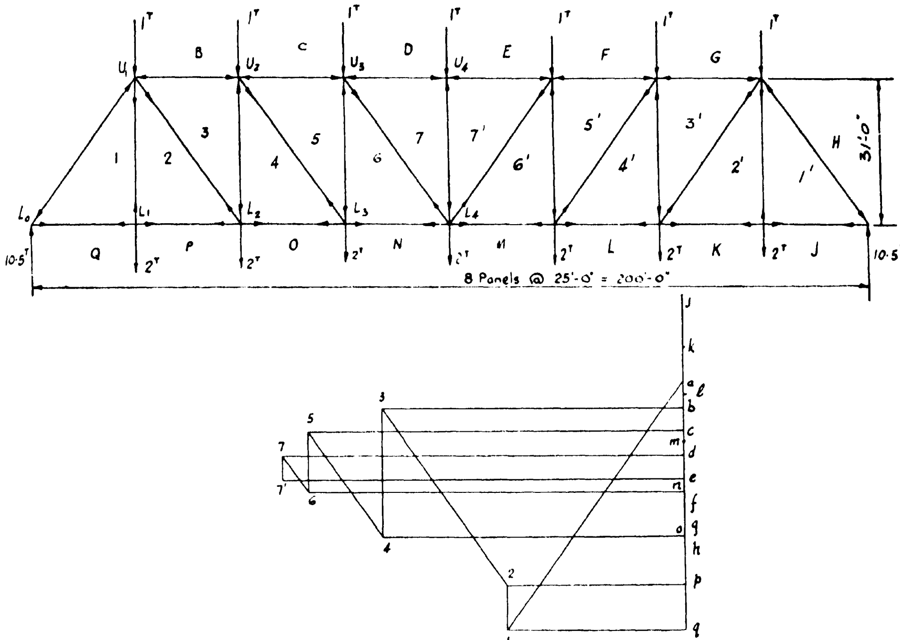


FIG. 7.6

$L_2 L_3$, moments about U_2:

$$\text{Force} = \frac{(10{\cdot}5 \times 25 \times 2) - (3 \times 25)}{31} = 14{\cdot}5 \text{ tons.}$$

$L_3 L_4$, moments about U_3:

$$\text{Force} = \frac{(10{\cdot}5 \times 25 \times 3) - (3 \times 75)}{31} = 18{\cdot}1 \text{ tons.}$$

Member $U_1 U_2$: take moments about L_2. Since moment about L_2 = moment about U_2, $U_1 U_2 = L_2 L_3$.

Similarly $U_2 U_3 = L_3 L_4$.

Member $U_3 U_4$ (moments about L_4):

$$\text{Force} = \frac{(10{\cdot}5 \times 25 \times 4) - (3 \times 25 \times 6)}{31} = 19{\cdot}4 \text{ tons.}$$

To find the force in the diagonals, first find the net force to one side of the section line and multiply by the secant of angle of inclination to the vertical (in this case $\sec\theta = 1{\cdot}285$).

Member	*Shear (tons)*	*Force (tons)*
$L_0 U_1$	10·5	13·6
$U_1 L_2$	(10·5 − 3)	9·65
$U_2 L_3$	(10·5 − 6)	5·78
$U_3 L_4$	(10·5 − 9)	1·93

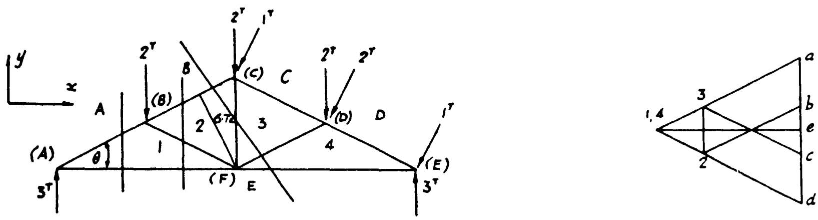


FIG. 7.7

To find the forces in the verticals, use inclined section lines as shown, then force = net load to left.

Force in hanger $U_1 L_1$ = load at panel point L_1 = 2 tons.

,, ,, $U_2 L_2 = 10{\cdot}5 - 5$ = 5·5 tons

,, ,, $U_3 L_3 = 10{\cdot}5 - 8$ = 2·5 ,,

,, ,, $U_4 L_4$ = load at U_4 = 1 ton.

Case 2. King-post truss in Fig. 7.4 (with dead load only for sake of simplicity). (See Fig. 7.7.)

$$\text{Span} = 30 \text{ ft.}; \quad \text{rise} = \tan^{-1}\tfrac{1}{2}; \quad \theta = 26^\circ\ 34';$$

$$\sin\theta = 0{\cdot}447; \quad \cos\theta = 0{\cdot}894.$$

Use section line cutting rafter $A1$ and main tie $E1$.

For $E1$; moment of external force about mid-point of rafter

$$= 3 \text{ tons} \times 7{\cdot}5 \text{ ft.} = 22{\cdot}5 \text{ ft.-tons}.$$

$$\text{Perpendicular distance} = 3{\cdot}75 \text{ ft.},$$

$$\therefore\ \text{force} = 6 \text{ tons}.$$

To find the force in $A1$, take moments about the mid-point of the main tie.

$$\text{Moment of external force} = 3 \times 15 = 45 \text{ ft.-tons}.$$

$$\text{Perpendicular distance} = 15\sin\theta = 6{\cdot}72 \text{ ft.},$$

$$\therefore\ \text{force} = \frac{45}{6{\cdot}72} = 6{\cdot}7 \text{ tons}.$$

Use second section line for members $B2$ and 12.

$B2$. Taking moments about mid-point of main tie.

$$\text{Force} = \frac{3\times 15-2\times 7{\cdot}5}{6{\cdot}72} = 4{\cdot}47 \text{ tons.}$$

12. Taking moments about L.H. support.

$$\text{Force}\times 15\sin\theta = 2\times 7{\cdot}5,$$

$$\therefore \quad \text{force} = 2{\cdot}23 \text{ tons.}$$

To find the force in 2–3, use an inclined section line cutting $B2$, 2–3, 3–4 and the main tie, and calculate the moments about L.H. support so that $B2$ and the main tie have no effect. Then

$$\text{moment of external force} = 2\times 7{\cdot}5 = 15 \text{ ft.-tons.}$$

$$= \text{moment of 3–4} + \text{moment of 2–3}$$

$$= -15+(\text{force in 2–3})15$$

$$\therefore \quad \text{force in 2–3} = 2 \text{ tons.}$$

(The signs of the forces must be taken into account.)

Method of Resolution of Forces

Applying the principles stated in Chapter II to the last example, and considering the forces in the members acting at the mid-point of the main tie, two of these (diagonals) are in compression and the post is in tension. Now the forces acting at the point must balance, i.e. $\sum H = 0$ and $\sum V = 0$. Knowing the forces in the diagonals as 2·23 tons (acting towards the point), it is obvious from symmetry that their horizontal components must balance.

Vertical component $= 2{\cdot}23\sin\theta = 1$ ton (acting towards the point).

Therefore force in post $= 2\times 1$ ton $= 2$ tons (acting away from the point), i.e. post is in tension. The same result could be obtained by considering the forces at the apex.

Method of Tension Coefficients

This method, first introduced into this country by Professor Sir R. V. Southwell, has come into greater use in recent years and is particularly useful in the analysis of space frames.

If AB is any bar in a space frame

$$L_{AB} = \text{length of } AB,$$

$$t_{AB} = \text{tension coefficient of } AB.$$

$$\text{Tension in } AB = L_{AB}\times t_{AB} = T_{AB}$$

(if t_{AB} is negative, AB is in compression).

Referring to Fig. 7.8, if A is taken as the origin of three rectangular axes Ax, Ay, Az, the coordinates of the ends of AB are x_A, y_A, z_A and x_B, y_B, z_B. Resolving the force T_{AB}, its component in the direction of the x-axis is given by

$$T_{AB}\cos BAx = T_{AB}\times\frac{x_B-x_A}{L_{AB}}$$
$$= t_{AB}\times(x_B-x_A).$$

Similarly the other components are

$$t_{AB}\times(y_B-y_A); \qquad t_{AB}\times(z_B-z_A).$$

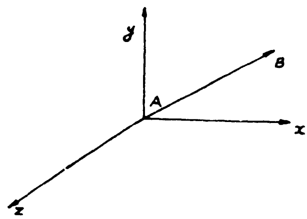


FIG. 7.8

Considering the other end of the bar B, the components may be written as

$$t_{AB}(x_A-x_B); \qquad t_{AB}(y_A-y_B); \qquad t_{AB}(z_A-z_B).$$

Now, if there are a number of bars AB, AC, AD,..., AN all meeting at the point A, the external forces acting at A can be written as $\sum X_A$, $\sum Y_A$, and $\sum Z_A$. By first principles, the algebraic sum of the forces at any one point must be zero. Hence we can write:

$$t_{AB}(x_B-x_A)+t_{AC}(x_C-x_A)+ +t_{AD}(x_D-x_A)+...+t_{AN}(x_N-x_A)+\sum X_A = 0, \quad (1)$$

$$t_{AB}(y_B-y_A)+t_{AC}(y_C-y_A)+ +t_{AD}(y_D-y_A)+...+t_{AN}(y_N-y_A)+\sum Y_A = 0, \quad (2)$$

$$t_{AB}(z_B-z_A)+t_{AC}(z_C-z_A)+ +t_{AD}(z_D-z_A)+...+t_{AN}(z_N-z_A)+\sum Z_A = 0, \quad (3)$$

and all other joints can be treated in the same way. Note that the quantities (x_B-x_A), (y_B-y_A), (z_B-z_A) are the projections of the length AB on the three axes. Also, if all such equations are written out, it will be found that for every expression such as $t_{AB}(x_B-x_A)$ there is an equal and opposite expression $t_{AB}(x_A-x_B)$.

If, therefore, we add these equations for all joints we obtain:

$$\left.\begin{array}{l}\sum X_A+\sum X_B+\sum X \;+...+\sum X_N = 0\\ \sum Y_A+\sum Y_B+\sum Y_C+...+\sum Y_N = 0\\ \sum Z_A+\sum Z_B+\sum Z_C+...+\sum Z_N = 0\end{array}\right\}. \quad (4)$$

That is, for equilibrium the algebraic sum of the external forces in any direction must be zero. Also, multiplying equations (1) and (2) by y_A and x_A respectively, we obtain:

$$t_{AB}(x_B y_A - x_A y_A) + t_{AC}(x_C y_A - x_A y_A) + t_{AD}(x_D y_A - x_A y_A) + \ldots + \\ + t_{AN}(x_N y_A - x_A y_A) + y_A \sum X_A = 0. \quad (5)$$

$$t_{AB}(x_A y_B - x_A y_A) + t_{AC}(x_A y_C - x_A y_A) + t_{AD}(x_A y_D - x_A y_A) + \ldots + \\ + t_{AN}(x_A y_N - x_A y_A) + x_A \sum Y_A = 0. \quad (6)$$

Subtracting equation (6) from (5) we find

$$t_{AB}(x_B y_A - x_A y_B) + t_{AC}(x_C y_A - x_A y_C) + t_{AD}(x_D y_A - x_A y_D) + \ldots + \\ + t_{AN}(x_N y_A - x_A y_N) + y_A \sum X_A - x_A \sum Y_A = 0,$$

and similar results can be obtained for other joints. Again it will be found that for every expression such as $t_{AB}(x_B y_A - x_A y_B)$ there is an equal and opposite expression. Hence by adding all such equations we find

$$\left.\begin{aligned} (y_A \sum X_A - x_A \sum Y_A) + (y_B \sum X_B - x_B \sum Y_B) + \ldots &= 0 \\ (x_A \sum Z_A - z_A \sum X_A) + (x_B \sum Z_B - z_B \sum X_B) + \ldots &= 0 \\ (z_A \sum Y_A - y_A \sum Z_A) + (z_B \sum Y_B - y_B \sum Z_B) + \ldots &= 0 \end{aligned}\right\}. \quad (7)$$

These equations mean that the algebraic sum of the components of all the forces about the three axes is zero in every case.

Now for a space frame there must be six support forces or reactions. The equations (4) and (7) enable the reactions to be found. Having found the reactions, the equations (1), (2), and (3) are used to find the tension coefficients and hence the forces in each member. For a space frame the value of

$$T_{AB} = t_{AB}\sqrt{\{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2\}}.$$

For a plane frame the last term disappears. One or more examples will be worked to show the use of this method.

Example 1. Take king-post truss with vertical load only (Fig. 7.7), as solved by Ritter's method. Length of members AF, EF = 15 ft.; FC = 7 ft. 6 in.; others 8·385 ft.

Joint	*Axis*	*Equation*	*Member*	t	L	T
A	x	$7{\cdot}5t_{AB} + 15t_{AF} = 0$	AB	$-0{\cdot}8$	8·385	$-6{\cdot}71$
	y	$3{\cdot}75t_{AB} + 3 = 0$	AF	$+0{\cdot}4$	15	$+6{\cdot}00$
B	x	$-7{\cdot}5t_{AB} + 7{\cdot}5t_{BC} + 7{\cdot}5t_{BF} = 0$	BC	$-0{\cdot}533$	8·385	$-4{\cdot}47$
	y	$-3{\cdot}75(t_{AB} + t_{BF}) + 3{\cdot}75t_{BC} - 2 = 0$	BF	$-0{\cdot}267$		$-2{\cdot}24$
C	x	$-7{\cdot}5t_{BC} + 7{\cdot}5t_{CD} = 0$	CF	$+0{\cdot}267$	7·5	$+2{\cdot}00$
	y	$-3{\cdot}75(t_{BC} + t_{CD}) - 7{\cdot}5t_{CF} - 2 = 0$	..	..	..	..
F	x	$-15(t_{AF} - t_{FE}) - 7{\cdot}5(t_{BF} - t_{DF}) = 0$	..	..	..	..
	y	$3{\cdot}75(t_{BF} + t_{DF} + 2t_{CF}) = 0$	..	..	..	..

Example 2. Same truss with positive wind loading. To calculate the reactions, resolve the wind loads. Then total horizontal force

$$= H_A = 1{\cdot}788 \text{ tons} \quad (E \text{ is free to move}).$$

$$V_B = (\tfrac{3}{4}\times 3{\cdot}596) - \frac{1{\cdot}788}{8} = 2{\cdot}46 \text{ tons}; \quad V_A = 3{\cdot}58 - 2{\cdot}46 = 1{\cdot}12 \text{ tons}.$$

Joint	*Axis*	*Equation*	*Member*	*t*	*L*	*T*
A	*x*	$7{\cdot}5t_{AB}+15t_{AF}+1{\cdot}788 = 0$	*AB*	−0·298	8·385	−2·49
	y	$3{\cdot}75t_{AB}+1{\cdot}117 = 0$	*AF*	+0·03	15	+0·45
B	*x*	$7{\cdot}5(t_{BC}+t_{BF}-t_{AB}) = 0$	*BC*	−0·298	8·385	
	y	$3{\cdot}75(t_{BC}-t_{BF}-t_{AB}) = 0$	*BF*	..	..	..
C	*x*	$7{\cdot}5(t_{CD}-t_{BC})-0{\cdot}447 = 0$	*CD*	−0·239	8·385	−2·00
	y	$-3{\cdot}75(t_{CD}+t_{BC}+2t_{CF})-0{\cdot}894 = 0$	*CF*	+0·150	7·5	+1·12
D	*x*	$7{\cdot}5(t_{DE}-t_{CD}-t_{DF})-0{\cdot}894 = 0$	*DE*	−0·417	8·385	−3·50
	y	$3{\cdot}75(t_{CD}-t_{DE}-t_{DF})-1{\cdot}788 = 0$	*DF*	−0·298		−2·49
E	*x*	$-7{\cdot}5(t_{DE}+2t_{EF})-0{\cdot}447 = 0$	*EF*	+0·179	15	+2·62
	y	$3{\cdot}75t_{DE}-0{\cdot}894+2{\cdot}46 = 0$				

Effects of Live Loads

The effects of live or rolling loads on framed structures (such as bridges) can be analysed by the use of influence lines. In truss bridges

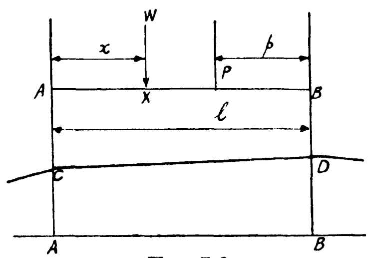


FIG. 7.9

the loads are usually carried on bearers or stringers and then transferred by the cross girders to the chord points of the main girders (top chords for deck spans and bottom chords for through spans). Therefore the stringer or bearer acts as a simple beam spanning between the panel points. If then a load W is at X on a span AB (Fig. 7.9) and AC, BD are influence-line ordinates at A and B, then

$$\begin{cases} \text{reaction at } B = \dfrac{Wx}{l} \\ \quad ,, \qquad A = \dfrac{W(l-x)}{l}. \end{cases}$$

For any point P,
$$\begin{cases} \text{shear} = \dfrac{Wx}{l} \\ \text{B.M.} = \dfrac{Wxp}{l}. \end{cases}$$

Both of these latter expressions are linear functions of x and therefore the influence lines for shear and B.M. are straight between A and B. Hence if the value of the influence line ordinates at the panel points are known, the influence line can be drawn by joining up these points.

Influence Lines for Trussed Girders with Parallel Chords

This category includes Warren, Pratt, and N-girders. As an example, find the force in the bottom chord member AB of a Warren type

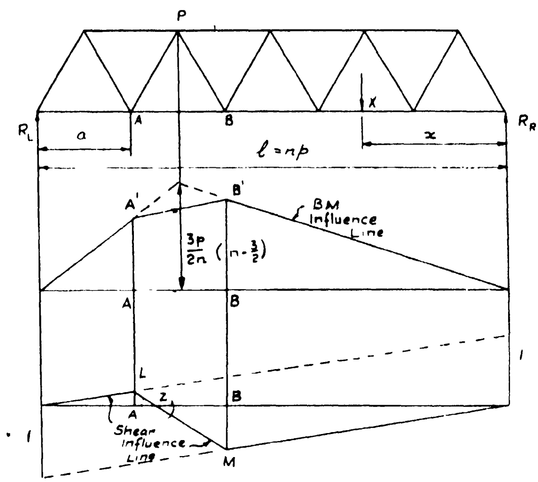


FIG. 7.10

through span (Fig. 7.10). The B.M. at P on the top chord is formed by an influence line and hence the force in AB. For unit load placed at X distant x from R.H. end, $R_L = x/l$ and $R_R = (l-x)/l$,

$$l = np,$$

where n = number of panels, p = panel length.

$$\therefore \quad R_L = \frac{x}{np} \quad \text{and} \quad M_P = \frac{x}{np} \times \frac{3p}{2} = \frac{3x}{2n}.$$

Now
$$M_A = \frac{px}{pn} = \frac{x}{n}$$

and
$$\frac{M_A}{M_P} = \frac{2}{3} = \frac{\text{distance of } A \text{ from L.H.}}{\text{distance of } P \text{ from L.H.}}.$$

Similarly it can be shown that

$$\frac{M_B}{M_P} = \frac{\text{distance of } B \text{ from R.H.}}{\text{distance of } P \text{ from R.H.}}.$$

But
$$M_P = \frac{3x}{2n} \quad \text{and} \quad \text{max. value of } x = l - \frac{3p}{2}$$
$$= p(n - \tfrac{3}{2}).$$

$$\therefore \quad M_P = \frac{3p(l-3p/2)}{2l},$$

which is the same as the ordinate for the B.M. influence line for a simple beam span. Therefore the B.M. influence line can be drawn by setting up the ordinate at P and joining the ends of the base line and then joining $A'B'$ to complete the influence line.

Similarly, to obtain the influence line for shear in panel AB, set off the influence line as for a simple beam and join points L and M.

Position of loads for maximum B.M. For a single load, maximum B.M. will occur when the load reaches B. For a uniformly distributed load, maximum value of B.M. will occur when the whole span is covered by the load.

Position of loads for maximum shear. For single loads, maximum shear will occur with the load at A or B. For uniformly distributed

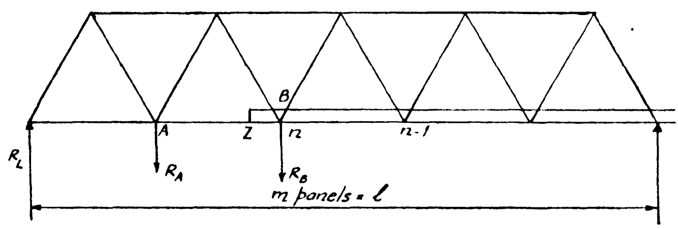


FIG. 7.11

loading the maximum shear will occur when the head of the load reaches Z (as once the head of the load passes that point, shear changes sign). The position of Z relative to A and B is dependent on the number of panels. In the general case (Fig. 7.11), let

n = number of panel points to the right of panel in question (counting from the right),
p = panel length,
m = total number of panels.

$$R_L = \frac{(np+x)^2}{2pm}; \qquad R_A = \frac{x^2}{2p} \qquad (x = BZ).$$

$$\text{Shear in panel} = R_L - R_A = S.$$

$$\text{For maximum } \frac{dS}{dx} = 0.$$

$$\therefore \quad x = \frac{np}{m-1}.$$

B.M. Influence Lines for Lower Chord Panel Point

Let A be the point distant a from L.H. support.

For unit load at x from R.H. support,

$$R_L = \frac{x}{l} \quad \text{and} \quad M_A = \frac{ax}{l}$$

and max. value of $x = l-a$.

$$\therefore \quad \text{influence line ordinate} = \frac{a(l-a)}{l}.$$

For point loads, the maximum value of B.M. can be found as for simple beam spans.

For uniformly distributed loads the B.M. is given by [(area of influence line) × intensity of load]. Maximum value occurs when the span is fully loaded.

For deck spans the reverse procedure is adopted in calculating the forces in the chord members.

For girders with vertical posts, such as the Pratt and **N**-types, the influence lines for B.M. will be as for the panel points on the lower chord of the Warren girder. The shear in the panels is obtained as for the Warren girder.

Influence Lines for Trussed Girders with Non-parallel Chords

Consider the Warren girder shown in Fig. 7.12 and the same panel as before, and a section line cutting the chords and diagonal. To find the force in the upper chord, a B.M. influence line must be drawn for point A on the lower chord. Then force = B.M ÷ perpendicular distance from A.

To find the force in the lower chord AB, a B.M. influence line must be drawn for P. These lines are drawn in the same manner as for the Warren girder with parallel chords. To find the force in the diagonal, produce the inclined top chord member PQ to meet the lower chord produced in X. Then force in diagonal = M_x ÷ perpendicular distance XR.

To draw the influence line for M_x, consider a unit load placed at y from R.H. support. Then

$$R_L = \frac{y}{l} \quad \text{and} \quad M_x = -\frac{y}{l} \times \text{distance } z \quad \text{(anti-clockwise moment).}$$

For any value of y between R.H. support and B,

$$M_x = -\frac{y}{l} \times \text{distance } z \quad (\text{varying as } y).$$

$$\text{Max. value of } y = BS \text{ and max. value of } M_x = -\frac{BS}{l} \times z.$$

For any distance r, so that the load is between L.H. support and A,

$$R_L = \frac{l-r}{l} \quad \text{and} \quad M_x = r\left(1+\frac{z}{l}\right)$$

which is a linear function of r and M_x is a maximum when r is a maximum (equal to OA).

The ordinates of the B.M. influence line at A and B can be drawn and the points joined up to complete the influence line.

For maximum force in the diagonal, maximum B.M. occurs (for a single load) when the load is at A. For uniformly distributed load the maximum B.M. at X is obviously produced when the head of the load reaches the point where the influence line cuts the base line.

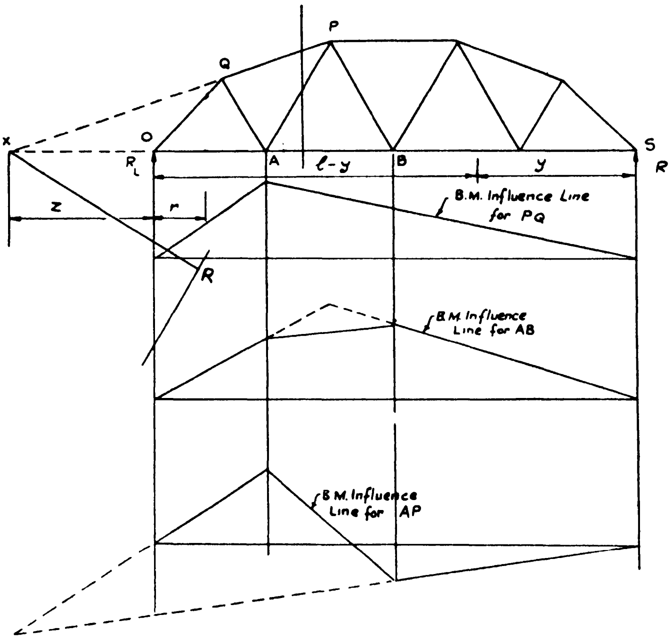


FIG. 7.12

As in the case of the Warren girder, the position of this point relative to the panel points depends upon the number of panels and the position of the panel point immediately behind the head of the load.

For maximum shear in any panel of a truss, the load in the panel must be equal to the total load on the truss divided by the number of panels (Fig. 7.13).

$$S_C = \text{shear at } C = R_L - \left(W_1 + \frac{W_2 \times FD}{p}\right) \quad (p = \text{panel length})$$

$$= W\left(\frac{l-x-d_1}{l}\right) - W_1 - W_2 \times \frac{FD}{p} \quad (W = \text{total live load})$$

$$= W\left(\frac{l-x-d_1}{l}\right) - W_1 - W_2\left(\frac{3p-x-d_1}{p}\right).$$

$$\text{For max. shear } \frac{dS_C}{dx} = 0 = -\frac{W}{l} + \frac{W_2}{p}.$$

$$\therefore \quad \frac{W_2}{p} = \frac{W}{l} \quad \text{or} \quad W_2 = \frac{Wp}{l} = \frac{W}{n},$$

where n equals number of panels.

Some numerical examples will be worked to illustrate the application of these principles.

Example 1. A single-track deck bridge is shown in Fig. 7.14. The members are each 20 ft. long and the cross girders are connected to the top chord panel points. Draw an influence line for the force in the

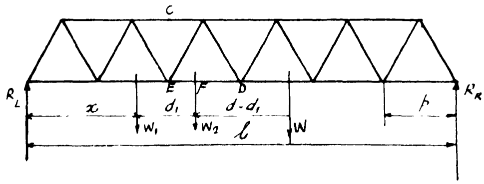


FIG. 7.13

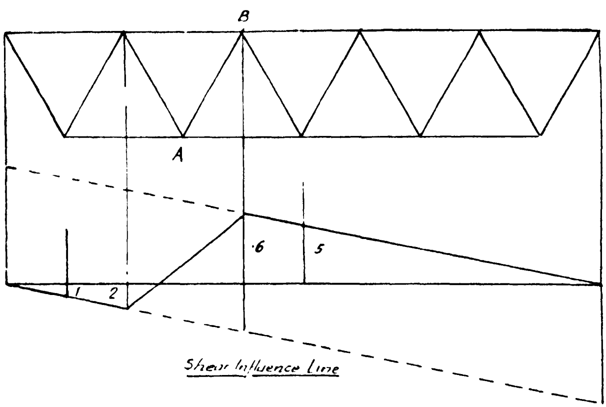


FIG. 7.14

diagonal *AB*. Hence find the maximum compression and tension in *AB* due to the two 8-ton loads coupled together at 10-ft. centres.

The influence line is as shown. Multiplying the maximum ordinates by the values of the loads,

$$\text{Shear} = \tfrac{1}{2}(8\times 0{\cdot}2+8\times 0{\cdot}1) = 1{\cdot}2 \text{ tons,}$$

$$\text{,,} \quad = \tfrac{1}{2}(8\times 0{\cdot}6+8\times 0{\cdot}5) = 4{\cdot}4 \quad \text{,,}$$

$$\sec\theta = 2/\sqrt{3}.$$

$$\text{Tension} = 1{\cdot}2\times 2/\sqrt{3} = 1{\cdot}39 \text{ tons.}$$

$$\text{Compression} = 4{\cdot}4\times 2\sqrt{3} = 5{\cdot}08 \quad \text{,,}$$

Example 2. A bridge has main girders as shown in Fig. 7.15. The width of each panel is 25 ft., and the cross-girders are connected to the lower chord panel points. Draw an influence line for the force in vertical *AB*. Hence find the maximum tension and compression in the member due to a uniformly distributed load of 2 tons per foot run per girder, longer than the span.

The influence line is as shown in Fig. 7.15. In order to find the force produce the top chord to meet the lower chord produced in Z.

Then distance $z = \left(30 \div \frac{8}{25}\right) - 50 = 43{\cdot}75$ ft.

For unit load between R_R and B, maximum distance y from $R_R = 75$ ft.

$$\therefore \quad \text{ordinate} = -\frac{75}{150} \times 43{\cdot}75 \div 93{\cdot}75 = -0{\cdot}234.$$

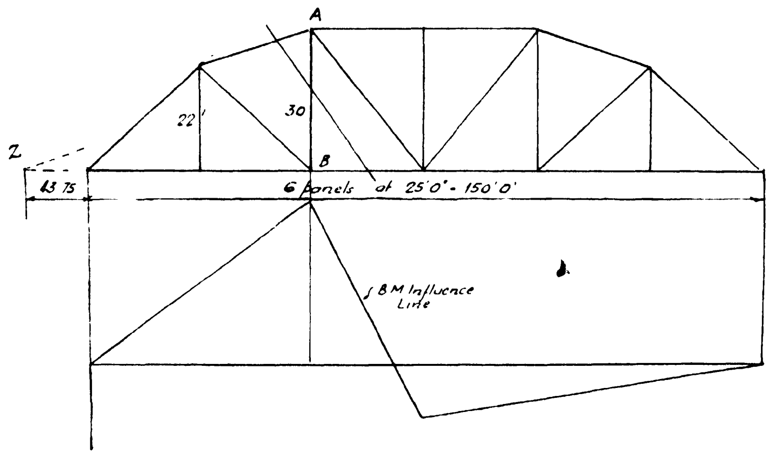


FIG. 7.15

For unit load between R_L and A, maximum distance $r = 50$ ft.

$$\therefore \quad \text{ordinate} = 50\left(1 + \frac{43{\cdot}75}{150}\right) \div 93{\cdot}75 = +0{\cdot}69.$$

Max. tension value $= \frac{1}{2} \times 2 \times 0{\cdot}69 \times 68{\cdot}75 = 47{\cdot}5$ tons.

Max. compression value $= \frac{1}{2} \times 2 \times 0{\cdot}23 \times 81{\cdot}25 = 18{\cdot}7$,,

Example 3. A single-track bridge has two main girders as shown in Fig. 7.16. The panels are each 25 ft. long and the cross girders are connected to the lower chord panel points. Draw an influence line for the force in member AB and hence find the maximum force in the member due to the train of moving loads.

Draw an influence line for B.M. at P as before.

$$\text{Maximum ordinate} = 36{\cdot}46.$$

To convert this into an influence line for force in AB, divide by the distance from AB to $P = 25 \times \sqrt{3}/2$.

$\therefore$ ordinate of influence line for force in AB

$$= 36{\cdot}46 \div 25\sqrt{3}/2 = 1{\cdot}685.$$

By trial and error, maximum force in AB occurs with one 5-ton load over B and another 5-ton load at 8 ft. to the right of B.

Max. force $= (5\times 1{\cdot}44)+(4\times 1{\cdot}38)+(11\times 1{\cdot}42)+(5\times 1{\cdot}29) = 34{\cdot}8$ tons.

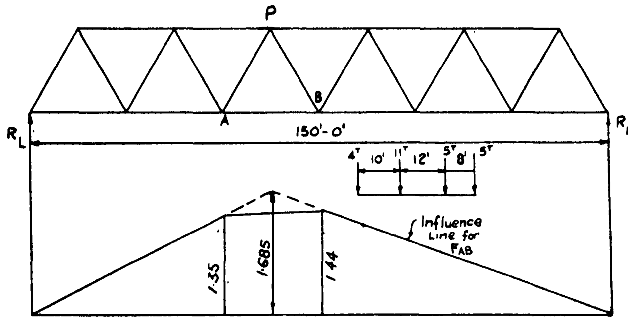


FIG. 7.16

Deflexions of Framed Structures

The deflexion of a framed structure at any point and in any direction can be found either graphically or by calculation. The latter method used is based on the principle of work or strain energy. The general law of work for framed structures can be enunciated thus: 'For any framed structure at constant temperature acted upon by loads which are gradually applied, the actual work produced during the deformation of the structure is independent of the manner in which the loads are applied and is always half as great as the work which would be produced by the same system of loads retaining their full values during the whole process of deformation.'

If W = any external load or reaction,

δ = displacement of the point of application of W in its line of action,

F = force in any member due to load W,

Δl = change in length of any members due to W,

then total external work $= \frac{1}{2}\sum W\delta$,

,, internal ,, $= \frac{1}{2}\sum F\times \Delta l$

$$= \frac{1}{2}\sum F\times\frac{Fl}{AE} = \frac{1}{2}\sum\frac{F^2l}{AE},$$

where l = length of member,

A = cross-sectional area,

E = Young's modulus.

By the principle of conservation of energy,

$$\tfrac{1}{2}\sum W\delta = \frac{1}{2}\sum\frac{F^2l}{AE} = U.$$

If a system of forces and loads acting at any joint of a framed structure is in equilibrium, then the algebraic sum of their components parallel to any axis of reference is zero, e.g. $\sum X = 0$.

If θ be the angle made by any load or force (F) with the axis of reference, then

$$\sum F\cos\theta = 0.$$

Now, if the joint in question is displaced by an amount Δ parallel to the axis and if the joint is still in equilibrium

$$\sum (F\cos\theta)\Delta = 0.$$

But $\Delta\cos\theta =$ displacement Δ projected on the line of action of the force in question,

$$= \delta.$$

$$\therefore \quad \sum F'\delta = 0,$$

where F' represents a force independent of δ. This relation is called the law of *virtual work*.

If the forces are in equilibrium, the displacements δ may be any possible displacements and it can be stated that for any point, frame, or body acted upon by loads F' in an established state of equilibrium, the total work done by those forces in moving through any given possible displacements is always zero. This rule can be developed to give other equations. Take for example a framed structure subject to loads W'_1, W'_2, W'_3, etc.,..., which, in conjunction with temperature effects, cause forces

$$F'_1,\ F'_2,\ F'_3,\ldots, \text{etc.}$$

in the members of the frame, and form a system in equilibrium.

Now all joints are subject to small displacements Δ_1, Δ_2, Δ_3,..., etc., which may be due to another system of loads (the values of Δ are variable).

For any joint in the structure, the algebraic sum of the components parallel to any axis is always zero.

$$\therefore \quad W'_1\cos\theta_1 + \sum F'_1\cos\alpha_1 = 0,$$

where $\theta_1 =$ angle of inclination for W'_1,

$\alpha_1 =$,, ,, F'_1.

Also $\Delta_1 W'_1\cos\theta_1 + \Delta_1 \sum F'_1\cos\alpha_1 = 0,$

but $\Delta_1\cos\theta_1 = \delta_1$

$=$ displacement in the direction of W'_1.

$$\therefore \quad W'_1\delta_1 + \Delta_1 \sum F'_1\cos\alpha_1 = 0.$$

Similarly for all joints in the structure, and the summation of these equations will give the virtual work equations for the structure.

The expression $W'_1\delta_1$ will occur once only but the term $\Delta_1 F'_1\cos\alpha_1$ will occur twice, i.e. at each end of the member in question.

Also

$$\Delta_1 \cos \alpha_1 = \text{displacement of the joint in the direction of } F'_1.$$

$$\therefore \quad \sum W'_1 \delta_1 + \sum F'_1 \Delta l = 0,$$

where Δl = change in length of the member.

But internal work is always opposite in sign to external work.

$$\therefore \quad \sum W'_1 \delta_1 - \sum F'_1 \Delta l = 0.$$

$$\therefore \quad \sum W'_1 \delta_1 = \sum F'_1 \Delta l.$$

If the supports are subject to movement, the equation becomes

$$\sum W'_1 \delta + \sum R' \Delta_r = \sum F'_1 \Delta l \quad (R, R', \text{ etc., are reactions}),$$

where δ, Δ_r, and Δl are caused by the loads W_1, W_2, W_3.

If W'_1, W'_2, W'_3,...; F'_1, F'_2, F'_3,..., and R' are identical with W_1, W_2, W_3,...; F_1, F_2, F_3,..., and R respectively, then

$$\sum W_1 \delta + \sum R \Delta_r = \sum F_1 \Delta l = \frac{1}{2} \sum \frac{F_1^2 l}{AE}.$$

If there are no movements of the supports,

$$\sum W\delta = \sum F \Delta l.$$

If $\sum W = 1, \qquad 1 \times \delta = \sum \bar{F} \times \Delta l,$

where $\bar{F}$ are the forces due to a unit loading. Unit loading may be a single force, a force pair, or a moment of unity, in which case δ = rotation measured by an arc.

The work done by unit loading may be denoted by $\bar{U}$.

If then, it is required to find the relative displacement δ of two points of a framed structure when changes of length Δl are given to each member (neglecting the effects of temperature, and assuming the supports immovable), assume that two unit forces are applied at the points in question along the line joining the two points. The direction of these forces should be such as to make $\bar{U}$ positive.

Then $$1 \times \delta = \sum \bar{F} \Delta l = \sum \frac{\bar{F} \times Fl}{AE}.$$

The forces F are produced by the actual loading.

For the case of deflexion at any point use a single load at the point at which the deflexion is required.

Then $$\delta = \sum \frac{KFl}{AE},$$

where K = force in any member due to unit load at point where deflexion is to be found.

In cases where there are changes of temperature and movements of the supports,

$$\delta = \sum \bar{F}\left(\frac{Fl}{AE} + \alpha t l\right) - \sum R \Delta_r,$$

where α = coefficient of linear expansion,

t = change of temperature.

(Compare these formulae with the corresponding equation for beams in bending, viz.

$$\delta = \sum \frac{\bar{M} \times M\,dx}{EI}.\Big)$$

Applying this principle to the case of the lattice girder shown in Fig. 7.17 and assuming that the stresses in tension and compression members are 8 and 6·8 tons/in.² respectively, find the deflexion of the

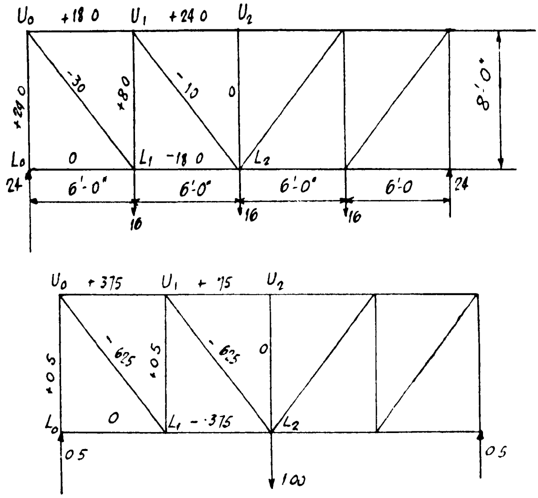


FIG. 7.17

mid-point of the lower chord. The value of E can be taken as 13,000 tons/in.² It is necessary to find the value of K, i.e. the load in the member due to a unit load applied at the point in question. As the girder and loading are symmetrical, calculate the deflexion for half the truss and multiply by 2. The calculation is best done in tabular form, thus:

Member	*Force (tons)*	*Stress (tons/in.²)*	*L (in.)*	$\frac{FL}{AE}$	*K*	$\frac{KFL}{AE}$
U_0U_1	+18·0	6·8	72	+0·0376	+0·375	0·0141
U_1U_2	+24·0	6·8	72	+0·0376	+0·75	0·0282
L_0L_1	0	0	..	..	..	..
L_1L_2	−18·0	8·0	72	−0·0443	−0·375	0·0166
U_0L_0	+24·0	6·8	96	+0·0502	+0·5	0·0251
U_1L_1	+8·0	6·8	96	+0·0502	+0·5	0·0251
U_2L_2	0	0	..	..	..	..
U_0L_1	−30	8	120	−0·0738	−0·625	0·0461
U_1L_2	−10	8	120	−0·0738	−0·625	0·0461

$$\sum \frac{KFL}{AE} = 0{\cdot}2013$$

Total deflexion = $2 \times 0{\cdot}2013 = 0{\cdot}4026$ in.

Graphical Treatment of Deflexions

This can be done by means of a Williot diagram. In order to understand the principle, take the case of the simple frame shown in Fig. 7.18 carrying a load W. The member AB is in tension and extends by the small amount BB_1, while the member CB is in compression and shortens by the small amount BB_2. To find the deflected position, swing arcs from points A, C with radii AB, CB_2 respectively to intersect in B'. Since the extensions are small and the arcs are nearly

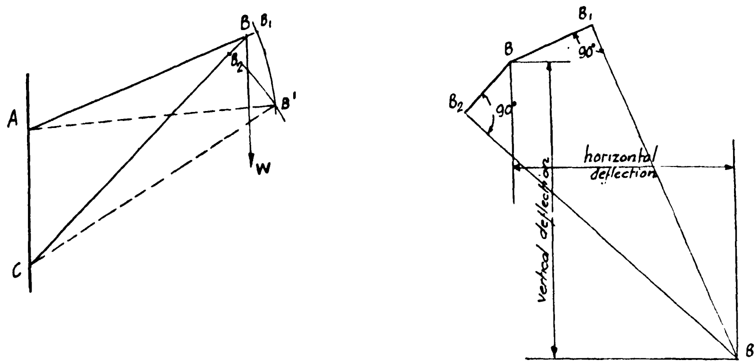


FIG. 7.18

perpendicular to the members, the same result can be found by setting off the extensions in the correct directions and drawing perpendiculars from the ends to obtain the deflexion of B from its original position.

Now, applying this method to the lattice girder, we take L_2 as the reference point and $U_2 L_2$ as the reference direction. The force in $U_2 L_2$ is zero. The extensions are generally set off to an enlarged scale. Set off from L_2 the shortening of $U_1 U_2$ horizontally to the right and the lengthening of $U_1 L_2$ upwards to the left, parallel to the slope of the respective member. Then by drawing perpendiculars the deflected position of U_1 is found relative to L_2. Also set off from L_2 the extension of $L_1 L_2$ horizontally to the left and the extension of $U_1 L_1$ vertically upwards and draw perpendiculars to find the new position of L_1. Then set off the extension of $U_0 L_1$ upwards to the left and the shortening of $U_0 U_1$ horizontally to the right and draw perpendiculars to find the new position of U_0. Finally from that point set off the shortening of $U_0 L_0$ vertically upwards and draw perpendicular to meet perpendicular from the point L_1 (since there is no force in $L_0 L_1$) to find the position of L_0. The horizontal and vertical deflexions are as shown in Fig. 7.19. If in drawing Williot diagrams there is any doubt as to the direction of any line, it should be remembered, that, if the member is in tension, the other end moves away from the end which has already been located in the line of the member and vice versa for members in compression.

(These movements are all relative to the reference line chosen at the beginning.) In this case the total vertical deflexion obtained from the Williot diagram is $\frac{4}{10} = 0{\cdot}4$ in., which agrees with the calculated value of 0·4026 in. The advantage of the graphical compared with the mathematical method is that the former gives the deflexion of *all* joints in *both* directions, whereas the latter gives the deflexion at *one* joint in one direction *only*. The Williot diagram should be carefully drawn to an enlarged scale in order to obtain accurate results and the mathematical method may be used as a check on the accuracy of the drawing.

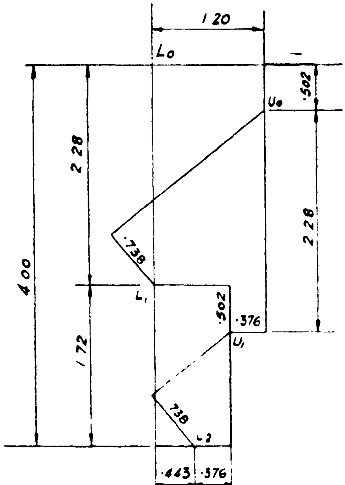


FIG. 7.19. Williot diagram. Displacement × 10

In the case of unsymmetrically loaded framed structures, the deflexion at any joint can be calculated exactly as before. In drawing the Williot diagram, however, it must be remembered that this gives the displacements relative to one point and on the assumption that one bar is fixed in direction, e.g. in previous example with reference to L_2, and assuming that the direction of $U_2 L_2$ remained unchanged. Now, if the bar in question is subject to a change of direction, it is obvious that the actual displacement of any joint is made up of a relative displacement, as obtained from the Williot diagram, combined with a displacement due to a rotation of the whole frame about an instantaneous centre of rotation. Taking the same girder as dealt with in previous examples (p. 221) but with an unsymmetrical load (Fig. 7.19 (*a*)), the Williot diagram can be drawn. Fig. 7.20 shows the diagram, the joint L_0 being taken as the point of reference and $U_0 L_0$ being assumed as fixed in direction. The displacement of L_4 with respect to L_0 is represented by $L_0 L_4$, which is almost completely vertical. It is obvious that the displacement of L_4 with respect to L_0 cannot have a vertical component and therefore a correction must be applied to obtain absolute displacements. This is most conveniently done by means of a Mohr diagram.

If in Fig. 7.19 (*a*), P is the instantaneous centre of rotation and θ the angle of rotation, then any joint will move in a direction at right angles to the line joining the joint in question to the pole P and the amount of movement will be distance $(L) \times \theta$.

If a pole O be taken and rays be drawn parallel to the displacements of the joints and representing them to any convenient scale, then, by joining the ends of these rays together the Mohr diagram will be obtained, and since the angles in each triangle are respectively equal to those of the girder, the Mohr diagram is similar to the frame diagram and the rotation of the Mohr diagram relative to the frame diagram is 90°.

The Mohr diagram must be superimposed on the Williot diagram in

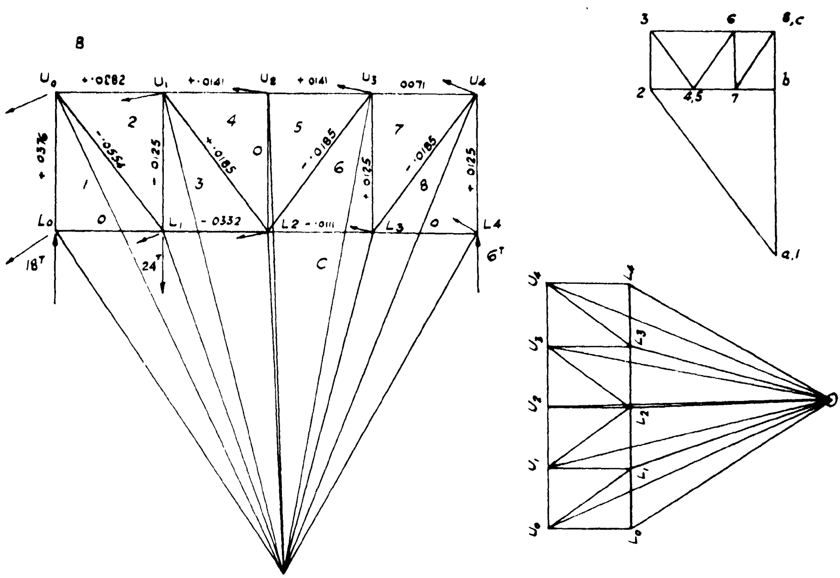


FIG. 7.19 (a)

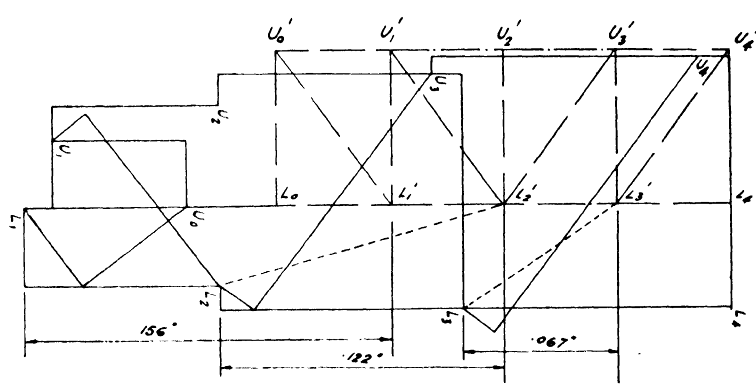


FIG. 7.20. Williot–Mohr diagram. Displacement × 25

order to obtain absolute displacement, and in order to do so, two points on the Mohr diagram must be located on the Williot diagram. Now, the points L_0 must obviously coincide and also the point L_4 can have no vertical displacement. The point L_0 is therefore chosen as the instantaneous centre of rotation. From L_0 draw a vertical line to meet the horizontal line from L_4 in L_4'. Then draw with $L_0 L_4'$ as base the outline of the girder to form the Mohr diagram and letter it accordingly L_1', L_2', etc. (shown chain dotted). The absolute displacement of any point is found by joining that point on the Mohr diagram to the corresponding point on the Williot diagram (dotted lines) and can be resolved into a vertical and a horizontal displacement. For example the vertical displacements of L_1, L_2, and L_3 are found to be 0·122 in., 0·156 in., and 0·067 in., and these values will be calculated as a check on the accuracy of the drawing. It will be seen that the amount of time involved in drawing the Williot–Mohr diagram is much less than that necessary for the calculation of all the displacements.

The calculation of deflexions at these points is as under:

Member	Load (tons)	*FL/AE* (by proportion) (in.)	Δ at L_2		Δ at L_1		Δ at L_3	
			K	*KFL/AE* (in.)	*K*	*KFL/AE* (in.)	*K*	*KFL/AE* (in.)
U_0U_1	+13·5	$\frac{3}{4}$ × 0·0376 = +0·0282	+0·375	+0·0106	+0·5625	+0·0159	+0·1875	+0·0053
U_1U_2	+9·0	$\frac{9}{24}$ × 0·0376 = +0·0141	+0·75	+0·0106	+0·375	+0·0053	+0·375	+0·0053
U_2U_3	+9·0	+0·0141	+0·75	+0·0106	+0·375	+0·0053	+0·375	+0·0053
U_3U_4	+4·5	+0·0071	+0·375	+0·0026	+0·1875	+0·0013	+0·5625	+0·0039
L_1L_2	−13·5	$\frac{3}{4}$ × −0·0443 = −0·0332	−0·375	+0·0124	−0·5625	+0·0187	−0·1875	+0·0062
L_2L_3	−4·5	−0·0111	−0·375	+0·0041	−0·1875	+0·0021	−0·5625	+0·0062
U_0L_0	+18·0	$\frac{3}{4}$ × 0·502 = +0·0376	+0·5	+0·0188	+0·75	+0·0282	+0·25	+0·0094
U_1L_1	−6·0	−0·0125	+0·5	} ..	−0·25	+0·0031	+0·25	−0·0031
U_3L_3	+6·0	+0·0125	+0·5		+0·25	+0·0031	−0·25	−0·0031
U_4L_4	+6·0	+0·0125	+0·5	+0·0062	+0·25	+0·0031	+0·75	+0·0093
U_0L_1	−22·5	$\frac{3}{4}$ × −0·0738 = −0·0554	−0·625	+0·0346	−0·9375	+0·0520	−0·3125	+0·0173
U_1L_2	+7·5	+0·0185	−0·625	} ..	+0·3125	+0·0058	−0·3125	−0·0058
U_3L_2	−7·5	−0·0185	−0·625		−0·3125	+0·0058	+0·3125	−0·0058
U_4L_3	−7·5	−0·0185	−0·625	+0·0115	−0·3125	+0·0058	−0·9375	+0·0173
Totals	..	..	..	+0·1220	..	+0·1555	..	+0·0677

The calculated values will be seen to agree very closely with those obtained from the Williot–Mohr diagram. In practice, the actual deflexion is generally found to be considerably less than the calculated value. (For bolted members the deflexions may be modified owing to the 'slip' of the bolts in the clearance holes.)

Secondary Stresses

In estimating deflexions of framed structures in the preceding pages, the stiffnesses of the gussets have been neglected and the deflexions have been assumed to be as shown in dotted lines in Fig. 7.21 (deflexions have been purposely exaggerated). The members have been taken as free to rotate about pin connexions whereas, in actual fact, they are restrained by the rivets, bolts, or welds connecting them to the gussets to take up shapes as indicated by the full lines. Therefore there must be restraining moments at each end of the members and these cause secondary bending stresses in addition to the primary stresses due to the axial loads in the members. The computation of these bending stresses may be rather involved and for that reason many engineers have decided to neglect them and allow for their effect by reducing the permissible working stresses. The problem can be solved by an application of the slope-deflexion method previously mentioned in Chapter V, e.g. referring to Fig. 7.21,

$$M_{AB} = \frac{2EI_{AB}}{l_1}\left(2\theta_A + \theta_B - \frac{3\Delta_{AB}}{l_1}\right),$$

$$M_{AC} = \frac{2EI_{AC}}{l_2}\left(2\theta_A + \theta_C - \frac{3\Delta_{AC}}{l_2}\right), \quad \text{etc.,}$$

and the equations are then solved for θ_A, θ_B, θ_C, etc. Where there are a large number of members the solution of the simultaneous slope-deflexion equations becomes involved and may almost be impossible.

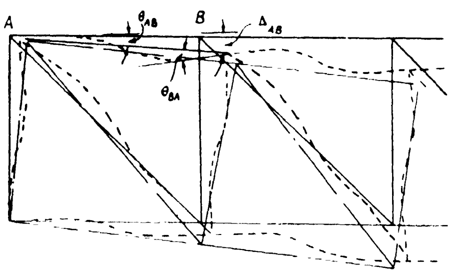


FIG. 7.21

The problem is most easily solved by an application of the Hardy Cross method of moment distribution previously mentioned in Chapter V.† This application has been dealt with by Prof. J. F. Baker.‡ The deflexions having been found by means of a Williot diagram, then if Δ is the deflexion of one end of a member relative to the other, measured at right angles to the axis, the end B.M. is given by $6EI\Delta/l^2$. Having found the moments at the ends of all the members, these moments can then be distributed until balance is obtained. The method is best illustrated by an example and the symmetrically loaded girder already dealt with (Fig. 7.17) can be taken. The horizontal and vertical displacements are shown in Fig. 7.19 and the moments at the ends of the posts are obtained as shown. For convenience in calculation I is assumed 1,000 in.^{4} throughout. Note that for the diagonals the relative displacement of the ends must be found by combining the horizontal and vertical displacements and then resolving the resultant displacement at right angles to the member, also that there is no moment in $U_2 L_2$.

It will be seen, as all members, except $U_2 L_2$, tend to rotate in clock-

† *Trans. American Society Civil Engineers*, **96**, 1932, p. 108 et seq.

‡ 'Modern Methods of Structural Design', *Journal Inst. C.E.* **8**, 1936–7, p. 297 et seq.

Moments in Members

Member	*Relative displacement* Δ (*in.*)	*Length* (*in.*)	*Moment* $= \dfrac{6\times 13{,}000\times 1{,}000\Delta}{length^2}$ (*in tons*)
U_0U_1	0·228	72	3,440
U_1U_2	0·122	72	1,840
L_0L_1	0·228	72	3,440
L_1L_2	0·172	72	2,590
U_0L_0	0·120	96	1,015
U_1L_1	0·0819	96	692
U_0L_1	0·200	120	1,083
U_1L_2	0·108	120	585

wise direction, 'fixing' moments required to constrain them in their original directions at their ends must be anti-clockwise and therefore that in order to balance the joint a clockwise moment equal to the sum of the anti-clockwise fixing moments must be applied to the joint. Now this balancing moment must be divided among the members in proportion to their stiffnesses $K = I/l$ (in this case in the inverse ratio of their lengths). The calculation is best done in tabular form (see p. 227). The relative stiffnesses of the members at each joint are written down and immediately below them the fixing moments from the above table. Then the distributed balancing moment is written in the next line and process completed for each joint. Then the joints are 'released', i.e. half the moment at one end of each member is carried over to the other end of the same member. Due to the carry-over moments the joints are again out of balance and balancing moments will then be written down. These again must be carried over and balanced. The process must be continued until the 'out-of-balance' moments are so small as to be comparatively negligible and then the moments for each member at each joint added up to find final moments, which should balance at each joint for equilibrium. A study of the table should show the method, which is purely arithmetical and should be self-checking (see p. 227).

(*Note.* In practice critical B.M. will occur at the edge of a gusset.)

Influence Lines for Deflexion

When necessary these may be obtained from the Williot or Williot–Mohr diagram by means of Clerk Maxwell's theorem of reciprocal deflexions. Referring to the method of virtual work dealt with on pp. 217–20, take the case of a framed structure loaded with W_1, W_2, W_3, etc., at the panel points and let the supports be immovable. Now, let the loaded points be subject to displacements Δ_1', Δ_2', Δ_3', etc., in the lines of action of the loads. By the principle of virtual work $W_1\Delta_1'+W_2\Delta_2'+W_3\Delta_3'+...+\sum F_1\Delta l = 0$ where F_1 etc., are forces in

	U_0				U_1					U_2			
Member	U_0L_0	U_0L_1	U_0U_1	*Totals*	U_0U_1	U_1L_1	U_1U_2	U_1L_2	*Totals*	U_1U_2	U_2L_2	U_2U_3	*Totals*
Relative K	1·25	1	1·33	3·58	1·33	1·25	1·33	1	4·91	1·33	1·25	1·33	3·91
Moment	-1,015	-1,083	-3,440	-5,538	-3,440	-692	-1,840	-585	-6,557	-1,840	0	+1,840	0
Balance	+1,933	+1,547	+2,058	+5,538	+1,778	+1,667	+1,778	+1,334	+6,557	. .	. .	. .	. .
Carry-over	+1,079	+795	+889	+2,763	+1,029	+993	0	0	+2,022	+789	0	-789	0
Balance	-964	-772	-1,027	-2,763	-548	-514	-548	-412	-2,022	. .	. .	. .	. .
Carry-over	-490	-280	-274	-1,044	-514	-351	0	0	-865	-274	0	+274	0
Balance	+365	+292	+387	+1,044	+234	+220	+234	+176	+864	. .	. .	. .	. .
Carry-over	+207	+119	+117	+443	+193	+148	0	0	+341	+117	0	-117	0
Balance	-154	-124	-165	-443	-93	-87	-93	-69	-342	. .	. .	. .	. .
Carry-over	-82	-48	-46	-176	-82	-60	0	0	-142	-46	0	+46	0
Balance	+61	+49	+66	+176	-39	-36	-39	-29	+143	. .	. .	. .	. .
Carry-over	+35	+20	+19	+74	+33	+24	0	0	+57	+20	0	-20	0
Balance	-26	-21	-27	-74	-15	-15	-15	-12	-57	. .	. .	. .	. .
Carry-over	-12	-8	-7	-27	-13	-10	0	0	-23	-7	0	+7	0
Balance	+9	+8	+10	+27	-6	-6	-6	+5	+23	. .	. .	. .	. .
Totals	+3,689 -2,733	+2,830 -2,336	+3,536 -4,986	. .	+3,312 -4,707	+3,094 -1,729	-2,057 -2,496	+1,544 -1,078	-1	+926 -2,167	0	+2,167 -926	. .
Net moment	+956	+494	-1,450	0	-1,395	+1,365	-439	+466	-3	-1,241	. .	+1,241	0

	L_0			L_1					L_2					
Member	U_0L_0	L_0L_1	*Totals*	L_0L_1	U_0L_1	U_1L_1	L_1L_2	*Totals*	U_1L_2	L_1L_2	U_2L_2	L_2L_3	U_3L_2	*Totals*
Relative K	1·25	1·33	2·58	1·33	1	1·25	1·33	4·91	1	1·33	1·25	1·33	1	5·91
Moment	-1,015	-3,440	-4,455	-3,440	-1,083	-692	-2,590	-7,805	-585	-2,590	0	+2,590	+585	0
Balance	+2,158	+2,297	+4,455	+2,114	+1,590	+1,987	+2,114	+7,805	. .	. .	. .	. .	. .	. .
Carry-over	+967	+1,057	+2,024	+1,148	-773	+833	0	+2,754	+667	+1,057	0	-1,057	-667	0
Balance	-981	-1,043	-2,024	-746	-561	-701	-746	-2,754	. .	. .	. .	. .	. .	. .
Carry-over	-482	-373	-855	-522	-386	-257	0	-1,165	-206	-373	0	+373	+206	0
Balance	+414	+441	+855	+316	+237	+297	+316	+1,165	. .	. .	. .	. .	. .	. .
Carry-over	+182	+158	+340	+221	+146	+110	0	+477	+88	+158	0	-158	-88	0
Balance	-165	-175	-340	-130	-97	-121	-130	-478	. .	. .	. .	. .	. .	. .
Carry-over	-77	-65	-142	-88	-62	-43	0	-193	-35	-65	. .	+65	+35	0
Balance	+69	+73	+142	+52	-40	+49	+52	+193	. .	. .	. .	. .	. .	. .
Carry-over	+30	+26	+56	+36	+24	+18	0	+78	+15	+26	0	-26	-15	0
Balance	-25	-31	-56	-21	-16	-20	-21	-78	. .	. .	. .	. .	. .	. .
Carry-over	-13	-10	-23	-16	-10	-8	0	-34	-6	-10	0	+10	+6	0
Balance	+11	+12	+23	-9	+7	+9	+9	+34	. .	. .	. .	. .	. .	. .
Totals	+3,831 -2,758	+4,064 -5,137	. .	-3,896 -4,963	+2,817 -2,215	+3,303 -1,842	+2,491 -3,487	0	+770 -832	+1,241 -3,038	0	+3,038 -1,241	+832 -770	0
Net moment	+1,073	-1,073	0	-1,067	+602	+1,461	-996	0	-62	-1,797	. .	+1,797	+62	0

members due to loads W_1, etc., and Δl = change in length of member due to another system of loads W'_1, W'_2, W'_3, etc., or

$$\sum W_1 \Delta'_1 = \sum F_1 \Delta l = \sum F_1 \times \frac{F'_1 l}{AE} = \sum \frac{F_1 F'_1 l}{AE}.$$

Similarly, by taking the same frame loaded with W'_1, W'_2, W'_3, etc., and displacing the loaded points Δ_1, Δ_2, Δ_3, etc., in direction of loading

$$\sum W'_1 \Delta_1 = \sum F'_1 \Delta l' = \sum F'_1 \times \frac{F_1 l}{AE} = \sum \frac{F_1 F'_1 l}{AE}.$$

$$\therefore \quad \sum W_1 \Delta'_1 = \sum W'_1 \Delta_1.$$

Now if Δ'_1, Δ'_2, Δ'_3 are deflexions at panel points due to load system W'_1, W'_2, W'_3, etc., and if Δ_1, Δ_2, Δ_3 are corresponding deflexions for W_1, W_2, W_3, etc., we can write:

$$W_1 \Delta'_1 + W_2 \Delta'_2 + W_3 \Delta'_3 + \ldots + W_n \Delta'_n = W'_1 \Delta_1 + W'_2 \Delta_2 + W'_3 \Delta_3 + \ldots + W'_n \Delta_n.$$

That is, if the supports are immovable, the product of one system of loads and the deflexions due to another load system is equal to the other load system multiplied by the deflexions due to the first system. This is the general case of Clerk Maxwell's law, which can be applied to beams and elastic frames as well as to framed structures. The law is often stated in the form that 'the deflexion at point A when a load is placed at point B = deflexion at B when the same load is placed at A'. Let load be W and points A and B; then if deflexions for W at A be denoted by Δ_{A1}, Δ_{A2}, Δ_{A3}, etc., with corresponding values Δ_{B1}, Δ_{B2}, Δ_{B3} for W at B, the equation becomes

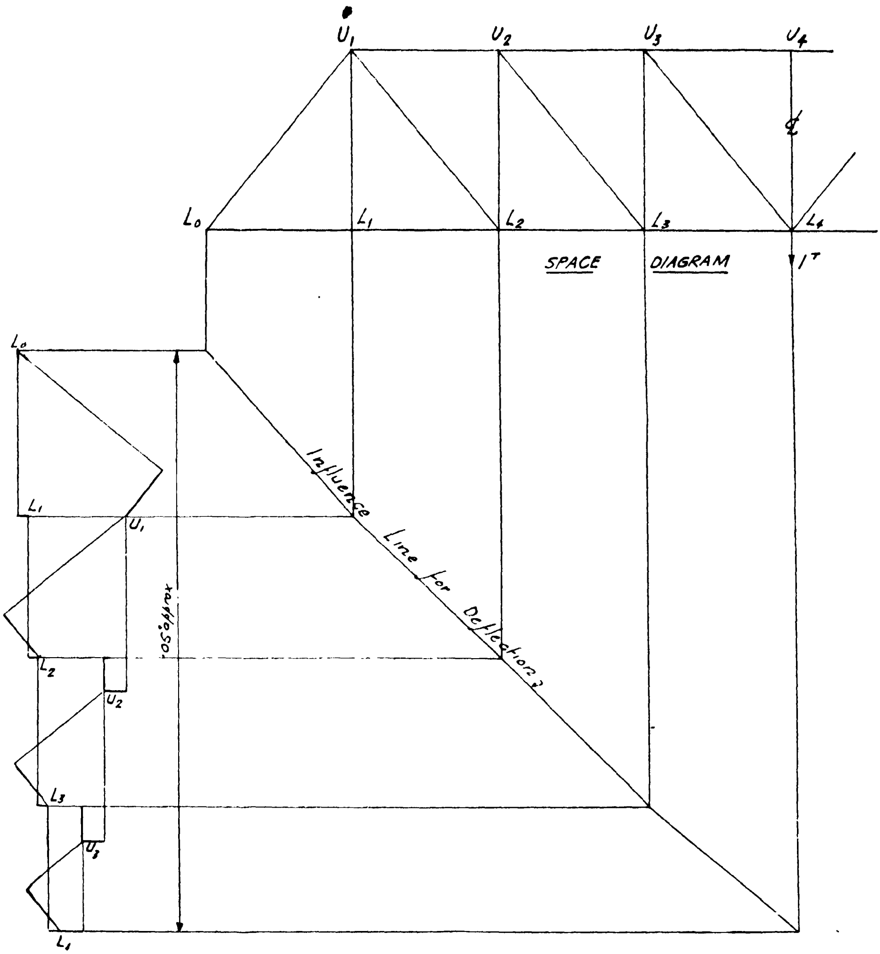
$$W \times \Delta_{BA} = W \times \Delta_{AB},$$

since A, B are the only points loaded and the other terms in the equation become $0 \times \Delta = 0$, i.e.

$$\Delta_{BA} = \Delta_{AB},$$

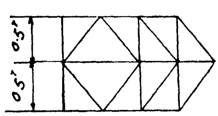
or deflexion at B due to load at A = deflexion at A due to load at B.

If we take the case of the lattice girder in Fig. 7.6, suppose that it is required to draw an influence line for the deflexion at L_4, the mid-point of the lower chord. Since the deflexion at any other panel point due to a load at the centre equals deflexion at the centre due to that load at the panel point, let the load be unity and find the influence line thus: set off the panel points on a horizontal line and project them vertically downwards to meet the corresponding points on the Williot diagram projected horizontally across. By joining the points of intersection a deflexion polygon or influence line for deflexion at the centre of the lower chord is obtained (Fig. 7.22). Note that the Williot diagram must be drawn for displacements due to a unit load at the point for which the influence line has to be drawn. These displacements can be



Williot diagram. Displacement × 125

Influence Line for Deflexion at L_4



Force diagram

Member	Force (tons)	Area (in.)	Length (in.)	Fl/AE (in.)
$U_1 U_2$	+0·81	10	300	0·00187
$U_2 U_3$	+1·21	15	300	0·00187
$U_3 U_4$	+1·62	20	300	0·00187
$L_0 L_1$	−0·40	10	300	0·00093
$L_1 L_2$	−0·40	10	300	0·00093
$L_2 L_3$	−0·81	15	300	0·00125
$L_3 L_4$	−1·21	20	300	0·00093
$U_2 L_2$	+0·5	5	372	0·00286
$U_3 L_3$	+0·5	5	372	0·00286
$U_1 L_0$	+0·64	5	478	0·00473
$U_1 L_2$	−0·64	5	478	0·00473
$U_2 L_3$	−0·64	5	478	0·00473
$U_3 L_4$	−0·64	5	478	0·00473

FIG. 7.22

derived by proportion from the changes in length due to the forces (p. 227) and the values of K, i.e. force due to unit load already tabulated on page 224.

Space or Three-dimensional Frames

In dealing with two-dimensional frames it was shown that the relation between joints and bars for a simply firm frame is given by

$$n = 2N-3 \begin{cases} (n = \text{number of bars}); \\ (N = \text{number of joints}). \end{cases}$$

If then it is required to build up a simply firm frame in space, three bars meeting at three joints and forming a triangle can be taken as a basis. Now, if a fourth joint be taken and connected to the three joints by three bars, the result is a simply firm frame in space (Fig. 7.23). The number of joints is 4 and the number of bars 6. The process of building up the frame in space can be continued, and for a firm frame the relation will always be $n = 3N-6$, where n = number of bars, and N = number of joints.

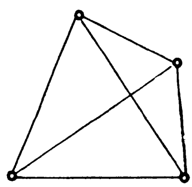


FIG. 7.23

If the number of bars is less than that given by the equation, the frame will be deficient; if more, then certain of the bars will be redundant and it will be necessary to treat the structure as a simply firm frame with one or more additional redundant bars. It is obvious that for stability a space frame must be supported at not less than three points. Referring to the method of tension coefficients dealt with on pp. 207–9 we have three equations of the form

$$\sum t_{AB}(x_B-x_A)+\sum X_A = 0$$

at each joint and therefore we have $3N$ equations for N joints, which are used in finding the forces in the members. Also we have three equations of the form

$$\sum X_A+\sum X_B+\sum X_C+...+\sum X_N = 0$$

and three equations of the form

$$y_A \sum X_A - x_A \sum Y_A + y_B \sum X_B - x_B \sum Y_B + ... = 0.$$

The total number of equations is therefore $(3N+6)$.

In order to find the forces in the members we can use the equations

$$\sum t_{AB}(x_B-x_A)+\sum X_A = 0$$

and the remaining six equations can be used to find the external forces. Since there are six linear equations, there should be six unknown forces to be determined and therefore it can be said that the essential number of supporting forces for a space frame is 6. The forces may be distributed among the supports in several ways; for example, if the case of three points of support is taken, then there may be three reactions X_1, Y_1, and Z_1 at the first point, two forces Y_2 and Z_2 at the second point, and only one force Z_3 at the remaining point.

As an example of the use of tension coefficients, take the case of the top cross-arms of a transmission tower carrying vertical loads (due to the weight of the cables, etc.) at the ends of the cross-arm (Fig. 7.24). The work is set out in tabular form and should be self-explanatory. In this case it is obvious that the vertical force in each leg must be a quarter of the total load.

As a second example the case of a simple tower, which is triangular in plan and subject to loads in each direction, can be dealt with. The equations are again set out in tabular form and as a check the values of the supporting forces have been calculated and found to be equal and opposite to the loads. It will be noticed that at each support there are three reactions X, Y, and Z, but it must be remembered that any two of these can be combined to form a single force in one plane. The equations become rather cumbersome where there are more than three or four members meeting at a joint, and care must be taken to include all the factors in the equations and due attention paid to the signs (Fig. 7.25).

(See Fig. 7.24)

Joint	*Axis*	*Equation*	*Bar*	t	L (*ft.*)	T (*tons*)
A	x	$6t_{AB}+6t_{AC}+8t_{AG} = 0$	AB	−0·1667	6·3246	−1·0541
	y	$4t_{AG}-1 = 0$	AG	+0·25	8·9443	+2·2361
	z	$2t_{AB}-2t_{AC} = 0$	AC	−0·1667	6·3246	−1·0541
F	x	$-6t_{FE}-6t_{FD}-8t_{FG} = 0$	FE	−0·1667	6·3246	−1·0541
	y	$4t_{FG}-1 = 0$	FG	+0·25	8·9443	+2·2361
	z	$2t_{FE}-2t_{FD} = 0$	FD	−0·1667	6·3246	−1·0541
B	x	$4t_{BE}+4t_{BD}+2t_{BG}-6t_{BA} = 0$	BE	−0·1875	4	−0·75
	y	$4t_{BG}+Y_B = 0$	BG	−0·125	4·8990	−0·6124
	z	$-4t_{BC}-4t_{BD}-2t_{BG}-2t_{BA} = 0$	BD	0	..	..
C	x	$4t_{CD}+2t_{CG}-6t_{CA} = 0$	CD	−0·1875	4	−0·75
	y	$4t_{CG}+Y_C = 0$	CG	−0·125	4·8990	−0·6124
	z	$4t_{CB}+2t_{CG}+2t_{CA} = 0$	BC	+0·14583	4	+0·5833
D	x	$6t_{DF}-4t_{DC}-2t_{DG} = 0$	DF	−0·1667	6·3246	−1·0541
	y	$4t_{DG}+Y_D = 0$	DG	−0·125	4·8990	−0·6124
	z	$2t_{DF}+2t_{DG}+4t_{DE} = 0$	DE	+0·14583	4	+0·5833
E	x	$6t_{EF}-2t_{EG}-4t_{EB} = 0$				
	y	$4t_{EG}+Y_E = 0$	EG	−0·125	4·8990	−0·6124
	z	$-2t_{EF}-4t_{ED}-2t_{EG} = 0$				
G	x	$8t_{GF}-8t_{GA}+2t_{GE}+2t_{GD}-2t_{GB}-2t_{GC} = 0$				
	y	$-4(t_{GE}+t_{GB}+t_{GC}+t_{GD}+t_{GA}+t_{GF}) = 0$				
	z	$2t_{GB}+2t_{GE}-2t_{GC}-2t_{GD} = 0$				

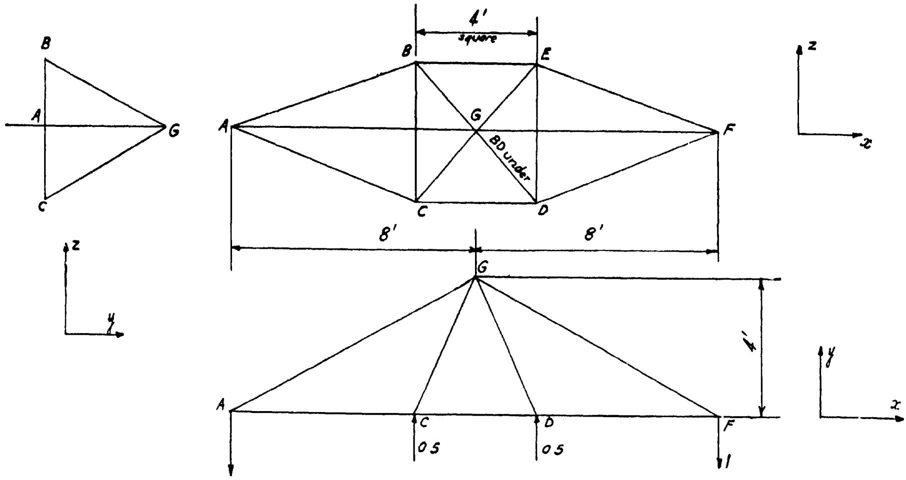


FIG. 7.24

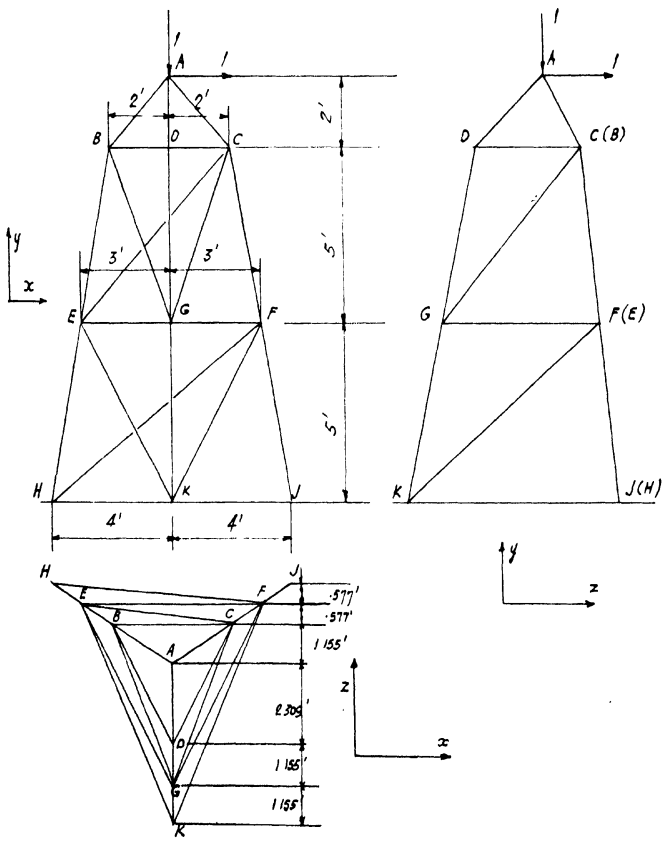


FIG. 7.25

(See Fig. 7.25)

Joint	*Axis*	*Equation*	*Bar*	*t*	*L (ft.)*	*T (tons)*
A	*x*	$2t_{AC}-2t_{AB}+1 = 0$	*AB*	−0·0608	3·0551	−0·186
	y	$-2(t_{AC}+t_{AB}+t_{AD})-1 = 0$	*AC*	−0·5608	3·0551	−1·712
	z	$+1{\cdot}155(t_{AB}+t_{AC})-2{\cdot}309t_{AD}+1 = 0$	*AD*	+0·1217	3·6519	+0·445
B	*x*	$4t_{BC}+2t_{BA}-t_{BE}+2t_{BG}+2t_{BD} = 0$	*BC*	+0·0162	4	+0·0648
	y	$2t_{BA}-5(t_{BE}+t_{BG}) = 0$	*BE*	−0·0568	5·1316	−0·2915
	z	$-1{\cdot}155t_{BA}+0{\cdot}577t_{BE}-$	*BD*	−0·0325	4	−0·130
		$-3{\cdot}464t_{BD}-4{\cdot}619t_{BG} = 0$	*BG*	+0·0325	7·0953	+0·230
C	*x*	$t_{CF}-4t_{CB}-5t_{CE}-2t_{CA}-2t_{CG}-2t_{CD} = 0$	*CE*	+0·0889	7·0947	+0·630
	y	$2t_{CA}-5(t_{CF}+t_{CE}+t_{CG}) = 0$	*CF*	−0·4345	5·1316	−2·225
	z	$+0{\cdot}577(t_{CF}+t_{CE})-1{\cdot}155t_{CA}-$				
		$-4{\cdot}619t_{CG}-3{\cdot}464t_{CD} = 0$	*CG*	+0·1214	7·0953	+0·861
D	*x*	$2(t_{DC}-t_{DB}) = 0$	*DC*	−0·0325	4	−0·130
	y	$2t_{DA}-5t_{DG} = 0$	*DB*	−0·0325	4	−0·130
	z	$2{\cdot}309t_{DA}+3{\cdot}464(t_{DB}+t_{DC})-1{\cdot}155t_{DG} = 0$	*DG*	+0·0487	5·1317	+0·2445
E	*x*	$6t_{EF}+5t_{EC}+t_{EB}-t_{EH}+3t_{EG}+3t_{EK} = 0$	*EF*	−0·0593	6	−0·3558
	y	$5(t_{EB}+t_{EC})-5(t_{EH}+t_{EK}) = 0$	*EG*	−0·0216	6	−0·1296
	z	$-0{\cdot}577(t_{EB}+t_{EC})-5{\cdot}196t_{EG}+$	*EH*	+0·0159	5·1316	+0·0819
		$+0{\cdot}577t_{EH}-6{\cdot}351t_{EK} = 0$	*EK*	+0·0162	8·6216	+0·1493
F	*x*	$-6t_{FE}-t_{FC}-3t_{FG}-7t_{FH}+t_{FJ}-3t_{EK} = 0$	*FH*	+0·0445	8·6216	+0·3840
	y	$5t_{FC}-5(t_{FJ}+t_{FK}+t_{FH}) = 0$	*FJ*	−0·5397	5·1316	−2·765
	z	$+0{\cdot}577(t_{FJ}+t_{FH}-t_{FC})-$				
		$-5{\cdot}196t_{FG}-6{\cdot}351t_{FK} = 0$	*FK*	+0·06075	8·6216	+0·524
G	*x*	$+2t_{GC}-2t_{GB}+3t_{GF}-3t_{GE} = 0$	*GF*	−0·0810	6	−0·486
	y	$5t_{GD}-5t_{GK}+5t_{GC}+5t_{GB} = 0$	*GK*	+0·2026	5·1316	+1·075
	z	$1{\cdot}155t_{GD}-1{\cdot}155t_{GK}+4{\cdot}619(t_{GC}+t_{GB})+$				
		$+5{\cdot}196(t_{GE}+t_{GF}) = 0$				
H	*x*	$7t_{HF}+t_{HE}+X_H = 0$				
	y	$5(t_{HF}+t_{HE})+Y_H = 0$				
	z	$-0{\cdot}577(t_{HF}+t_{HE})+Z_H = 0$				
J	*x*	$-t_{JF}+X_J = 0$				
	y	$5t_{JF}+Y_J = 0$				
	z	$-0{\cdot}577t_{JF}+Z_J = 0$				
K	*x*	$3(t_{KF}-t_{KE})+X_K = 0$				
	y	$5(t_{KF}+t_{KE}+t_{KG})+Y_K = 0$				
	z	$6{\cdot}351(t_{KF}+t_{KE})+1{\cdot}155t_{KG}+Z_K = 0$				

As a check on the tabular analysis the reactions at each support in each direction can be calculated from the coefficients.

$$X_H = -(7t_{HF}+t_{HE}) = -0{\cdot}3274$$
$$X_J = t_{JF} = -0{\cdot}5397$$
$$X_K = -3(t_{KF}-t_{KE}) = -0{\cdot}1337$$
$$\text{Total} = -1{\cdot}0008 \quad (\text{error} -0{\cdot}0008)$$

$$Y_H = -5(t_{HF}+t_{HE}) = -0{\cdot}3020$$
$$Y_J = -5t_{JF} = +2{\cdot}6985$$
$$Y_K = -5(t_{KF}+t_{KE}+t_{KG}) = -1{\cdot}3977$$
$$\text{Total} = +0{\cdot}9988 \quad (\text{error} -0{\cdot}0012)$$

By moments:

	Y_H	Y_J	Y_K
From unit load in Y-axis	$+0{\cdot}3333$	$+0{\cdot}3333$	$+0{\cdot}3333$
,, ,, X-axis			
$\frac{1\times 12}{8}$	$-1{\cdot}5$	$+1{\cdot}5$	..
From unit load in Z-axis			
$\frac{12}{6{\cdot}9284}$	..	..	$-1{\cdot}7321$
$\frac{12}{2\times 6{\cdot}9284}$	$+0{\cdot}8660$	$+0{\cdot}8660$	..
	$-0{\cdot}3007$ tons	$+2{\cdot}6993$ tons	$-1{\cdot}3988$ tons

$$Z_H = 0{\cdot}577(t_{HF}+t_{HE}) = +0{\cdot}0348 \text{ tons}$$
$$Z_J = 0{\cdot}577(t_{JF}) = -0{\cdot}3114 \text{ ,,}$$
$$Z_K = -6{\cdot}351(t_{KF}+t_{KE})-1{\cdot}155t_{KG} = -0{\cdot}7227 \text{ ,,}$$
$$\text{Total} = -0{\cdot}9993 \text{ (error} +0{\cdot}0007)$$

In practice, it is quite usual in the commercial design of towers, etc., to assume that all vertical loads are carried by the legs and that the horizontal loads are carried by the face or faces in whose plane they occur. The separate forces due to these conditions are then added algebraically to find the total force in each member. It is also quite common practice to neglect redundants, e.g. if a panel is counter-braced then it is assumed that the diagonals are capable of resisting tensile forces only, and that one or the other diagonal becomes redundant according to the direction of the horizontal load. It can be seen from the foregoing analysis that such assumptions are justifiable.

EXERCISES

1. The frame shown in Fig. 7.A is loaded as shown. Find the forces in the members and the reactions. [$AB = BC = 0{\cdot}2^{TC}$; $CD = 1{\cdot}8^T$; $AE = FD = 1{\cdot}28^{TC}$; $CF = 5^{TC}$; $CE = 5{\cdot}38^T$; $GH = 1{\cdot}2^{TC}$; $EG = 4{\cdot}3^T$; $GJ = 4{\cdot}85^T$; $FH = 6{\cdot}5^{TC}$; $HK = 6{\cdot}67^{TC}$; $FG = 64^T$; $GH = 0{\cdot}2^{TC}$; $HJ = 26^T$; $R_L = +6{\cdot}67$ (vert.); $R_R = -4{\cdot}67$ (vert.) and 2^T (hor.).]

2. The frame shown in Fig. 7.B carries three loads. Find the deflexion under the centre load if the area of all members is 4 sq. in. and $E = 12{,}000$ tons per sq. in. [0·058 in.]

3. In the frame shown in Fig. 7.C, a rope passes over C and is fixed to B. It carries a load of 5 tons at its free end. The areas are: $AC = AD = DE = 3$ sq. in.; $BD = 4$ sq. in.; $CE = 2$ sq. in. Find vertical deflexion at D. [0·074 in.]

4. Fig. 7.D shows a simply supported bridge girder consisting of 8 panels of 10 ft. each. Draw influence lines for forces in the members AD and AB. If vertical DB is removed, compare new influence line for AC with previous influence line for AB. For a live load of 1 ton per ft. run longer than the span, find the maximum compression in AD and compare the maximum forces in AB and AC. [23·9 tons; 37·6 tons (tension); 34·2 tons (tension).]

5. For frame in Fig. 7.E find the vertical deflexion at B if l/A is 10 (inch units) for all members and $E = 12{,}000$ tons per sq. in. [0·0167 in.]

6. A swing bridge (Fig. 7.F) is supported at A, B, and C. When the outer

supports are removed and unit vertical loads are applied at A and C, the deflexion curve for the bottom boom is as shown. Find the forces in the members X and Y due to a loading of $\frac{1}{2}$ ton per ft. run over the whole span, the loading being applied at the panel points. Find also the reactions. [$R_A = R_C = 5{\cdot}5$ tons; $R_B = 19$ tons; $X = 6{\cdot}93$ tons; $Y = 2{\cdot}31$ tons.]

7. Fig. 7.G represents the end of a Warren girder, each member being 200 in. long and having a second moment of 1,000 in.^{4} All joints are rigid and joint A is prevented from rotating. The rotations of AB, AC, and BC by Williot diagram are respectively $+30/E$, $+40/E$, $+10/E$ ($+$ = clockwise) and the bending moments in members BD, CD, CE at joints B and C are $M_{BD} = 300$, $M_{CD} = 500$, and $M_{CE} = 300$ in. tons as shown. Find the moments in members at A and sketch distorted shape for AC.

[$M_{AB} = -707$; $M_{AC} = -1{,}072$; $M_{CA} = 944$ in. tons.]

8. Find the forces in members of the space frame of Fig. 7.H due to the point load of 10 tons at G. [$FB = FE = FG = 0$; $DG = 8{\cdot}29$ compr.; $CG = 8{\cdot}29$ tens.; $EG = 11{\cdot}18$ tens.; $ED = 18{\cdot}2$ compr.; $AE = 22{\cdot}8$ tens.; $BE = 8{\cdot}4$ compr.]

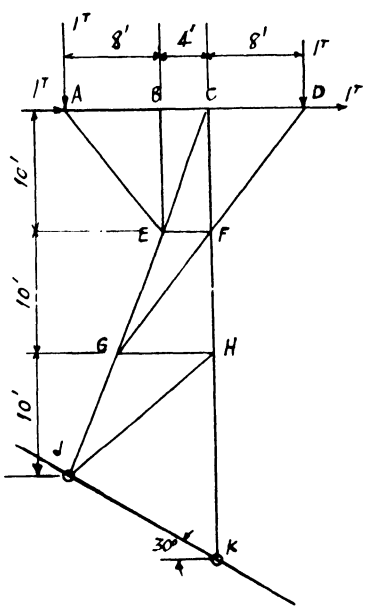


FIG. 7.A

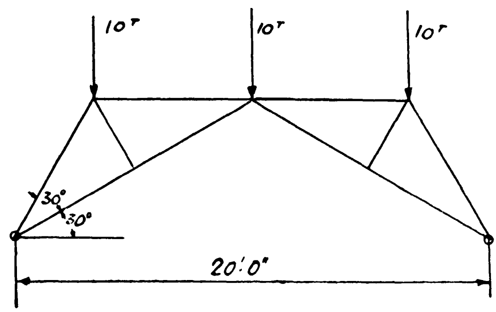


FIG. 7.B

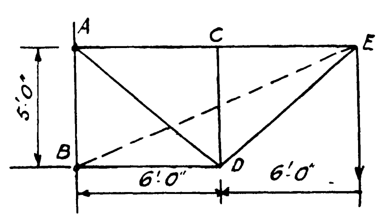


FIG. 7.C

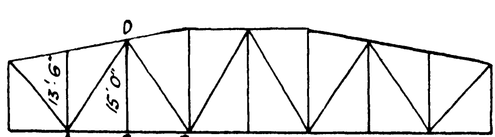


FIG. 7.D

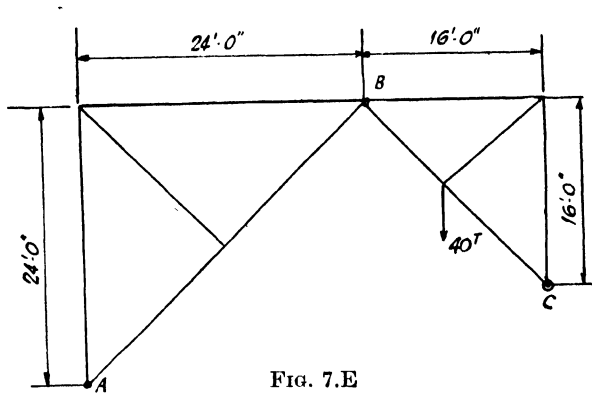


Fig. 7.E

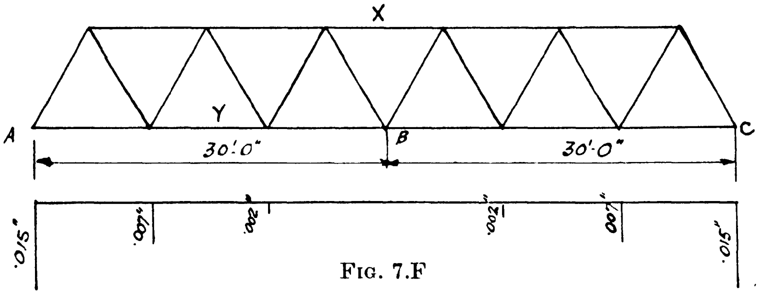


Fig. 7.F

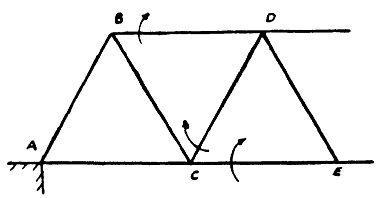


Fig. 7.G

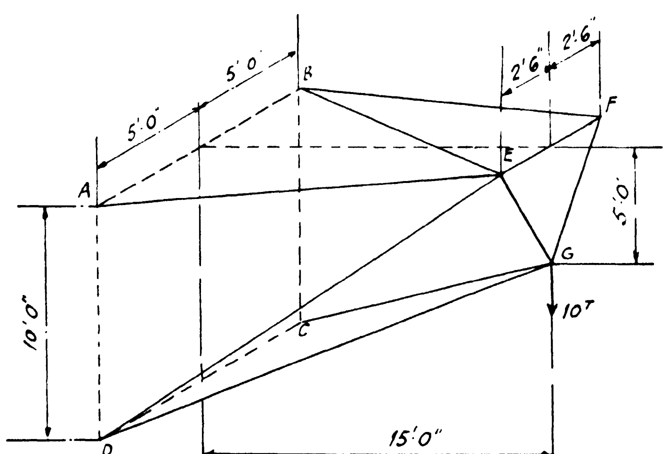


Fig. 7.H

CHAPTER VIII

COLUMNS AND STRUTS

A SHORT member subject to compressive stress will fail by direct crushing or shear. A long slender member, however, is more liable to fail by buckling. The deformation or deflexion of the member caused by buckling will induce bending stresses in addition to the direct compressive stress.

Bending stress which may be induced by buckling, by eccentricity of loading, or by lateral loads, is in general, more important than direct stress in column design, as will be seen in the text and worked examples.

It is impossible to do more in this chapter than to deal with some of the many column formulae which have been propounded since Euler first enunciated his theory in 1759. Since this may be regarded as the 'classical' theory, it may be worth expounding now, remembering that this is a purely mathematical formula and that Euler made certain assumptions, viz. (*a*) that direct stress due to compression can be neglected, (*b*) that the load is perfectly concentric, (*c*) that the member is perfectly straight initially and homogeneous.

Case 1. Considering now the strut free to rotate at both ends although the ends are fixed in position (see Fig. 8.1 (*a*)).

Then the moment at x from one end $= -Wy = M_x$.

Since
$$M_x = EI\frac{d^2y}{dx^2},$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{Wy}{EI} = -n^2y,$$

where
$$n^2 = \frac{W}{EI}.$$

The solution to this differential equation is given by

$$y = A\sin nx + B\cos nx$$

and
$$\frac{dy}{dx} = nA\cos nx - nB\sin nx.$$

$$\therefore \quad \frac{d^2y}{dx} = -n^2A\sin nx - n^2B\cos nx = -n^2y.$$

In this case, when $x = 0$, then $y = 0$.

$$\therefore \quad B = 0 \quad \text{and} \quad y = A\sin nx.$$

Also, when $x = L$, then also $y = 0$.

$$\therefore \quad A\sin nL = 0.$$

Since A cannot be zero, $\sin nL = 0$.

The values of nL satisfying this condition are 0, π, 2π, 3π, etc.

Taking $$nL = \pi, \quad \text{then} \quad n^2 = \frac{\pi^2}{L^2}.$$

$$\therefore \quad \frac{W}{EI} = \frac{\pi^2}{L^2} \quad \text{and} \quad W = \frac{\pi^2 EI}{L^2}.$$

W = critical load or load which when applied concentrically to an initially straight column will cause failure by buckling.

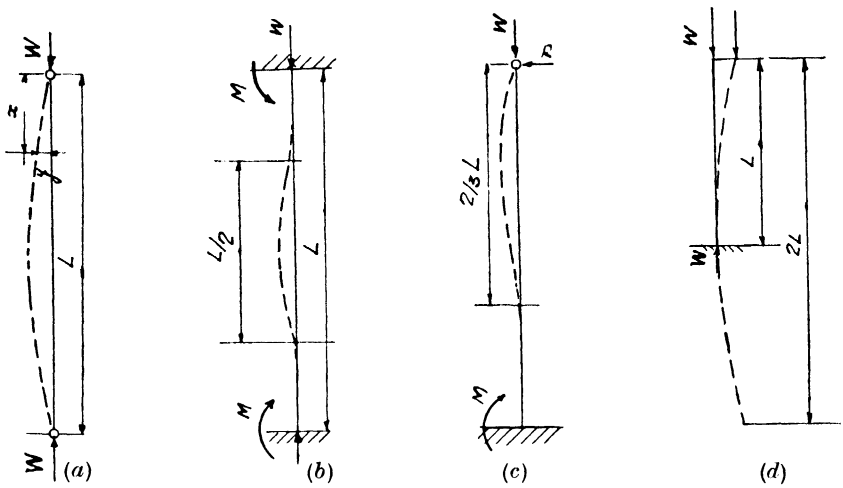


FIG. 8.1

Case 2. Ends fixed in position and direction (Fig. 8.1 (*b*)).
Let fixing moment at each end be M.

Then $$M_x = M - Wy = EI\frac{d^2y}{dx^2}.$$

$$\frac{d^2y}{dx^2} = -\frac{1}{EI}(Wy - M)$$
$$= -\frac{W}{EI}\left(y - \frac{M}{W}\right)$$
$$= -\frac{W}{EI}(y-a), \quad \text{where} \quad a = \frac{M}{W},$$
$$= -n^2(y-a), \quad (n \text{ being as in Case 1}).$$

Now let $y-a = z$; then
$$\frac{dz}{dx} = \frac{dy}{dx},$$
$$\frac{d^2z}{dx^2} = \frac{d^2y}{dx^2}$$
$$= -n^2z.$$
$$\therefore \quad z = A\sin nx + B\cos nx,$$

or
$$y-a = A\sin nx+B\cos nx.$$
$$\frac{dy}{dx} = nA\cos nx-nB\sin nx.$$

When $x = 0, \frac{dy}{dx} = 0.\qquad \therefore\quad A = 0.$
$$\therefore\quad y-a = B\cos nx.$$
When $x = 0$, $y = 0.\qquad \therefore\quad B = -a.$
$$y-a = -a\cos nx.$$
$$y = a(1-\cos nx).$$
Also, when $x = L$, $y = 0$.
$$\therefore\quad \cos nL = 1.$$
Hence nL must be 0, 2π, 4π, etc.

Taking
$$nL = 2\pi,$$
$$n^2 = \frac{4\pi^2}{L^2} = \frac{W}{EI}.$$
$$\therefore\quad W = \frac{4\pi^2EI}{L^2} = 4W \text{ for free ends.}$$
Let 'equivalent' length of free-ended strut $= L_1$.

Then
$$\frac{\pi^2EI}{L_1^2} = \frac{4\pi^2EI}{L^2}.$$
$$\therefore\quad L_1 = L/2,$$
$\therefore$ 'equivalent' length for fixed ends $= \frac{1}{2}L$.

Case 3. One end fixed in position and direction, the other end fixed in position only (Fig. 8.1 (*c*)).

Let fixed end moment $= M$, and horizontal reaction at other end $= R$.

Then
$$M_x = R(L-x)-Wy = EI\frac{d^2y}{dx^2}.$$
$$\frac{d^2y}{dx^2} = -\frac{W}{EI}\left\{y-\frac{R}{W}(L-x)\right\}$$
$$= -n^2\left\{y-\frac{R}{W}(L-x)\right\} \quad (n \text{ being as in Case 1}).$$
Let
$$z = y-\frac{R}{W}(L-x);$$
$$\frac{dz}{dx} = \frac{dy}{dx}+\frac{R}{W}$$

and $$\frac{d^2z}{dx^2} = \frac{d^2y}{dx^2} = -n^2z.$$

$$\therefore \quad z = A\sin nx + B\cos nx$$

$$= y - \frac{R}{W}(L-x).$$

For $x = 0$, $y = 0$. $\quad \therefore \quad B = -\frac{RL}{W}.$

Also $$\frac{dy}{dx} + \frac{R}{W} = nA\cos nx - nB\sin nx$$

and $\frac{dy}{dx} = 0$ when $x = 0$.

$$\therefore \quad nA = \frac{R}{W} \quad \text{and} \quad A = \frac{R}{nW}.$$

$$y - \frac{R}{W}(L-x) = \frac{R}{nW}\sin nx - \frac{RL}{W}\cos nx$$

$$= \frac{R}{W}\left(\frac{\sin nx}{n} - L\cos nx\right).$$

But $y = 0$ when $x = L$.

$$\therefore \quad 0 = \frac{R}{W}\left(\frac{\sin nL}{n} - L\cos nL\right)$$

or $$\sin nL = nL\cos nL.$$

$$\therefore \quad \tan nL = nL.$$

By solving graphically, $nL = 4{\cdot}493$ radians,

$$n^2L^2 = 2{\cdot}047\pi^2,$$

$$n^2 = \frac{2{\cdot}047\pi^2}{L^2}$$

$$= \frac{W}{EI}.$$

$$\therefore \quad W = \frac{2{\cdot}047\pi^2 EI}{L^2}.$$

To find equivalent length

$$\frac{\pi^2 EI}{L_1^2} = \frac{2{\cdot}047\pi^2 EI}{L^2}.$$

$$\therefore \quad L_1 = L\sqrt{\left(\frac{1}{2{\cdot}047}\right)} = \frac{L}{1{\cdot}43} = 0{\cdot}7L \text{ approx.}$$

Case 4. One end fixed in position and direction, the other free in both position and direction (Fig. 8.1 (*d*)). By inspection it is obvious that the member can be reflected and that the equivalent length $= 2\times$actual length,

$$\therefore \quad \text{load} = \tfrac{1}{4}\times\text{load for Case 1}.$$

In dealing with Euler's formula it must be remembered that it applies only to ideal conditions unlikely to be found in practice, also that even then its limits of application are not the same as those for actual column design.

The value W is known as the 'crippling' load, generally taken as the limiting load under which the column will buckle with progressive deformation and collapse.

Taking Case 1, then $W = \dfrac{\pi^2 EI}{L^2}$

$$\text{and compressive stress} \quad = \frac{W}{A} = \frac{\pi^2 EI}{AL^2} = \frac{\pi^2 EAk^2}{AL^2}$$

$$= \pi^2 E/(L/k)^2.$$

$$\therefore \quad W = \frac{A\pi^2 E}{(L/k)^2}.$$

The value L/k is generally called the 'slenderness ratio', and k is taken as the *least* radius of gyration.

The Eulerian theory aroused much controversy and many efforts have been made to modify it, and it is possible to deal with only a few of such formulae now. Perhaps the earliest formula in general use was that of Gordon, who adapted the Eulerian formula to allow for the effect of direct compression. Gordon's formula can be written

$$W = \frac{f_c A}{1+c(L/d)^2},$$

where d = least diameter or breadth, c = a constant.

Gordon's formula was further modified by Professor J. W. M. Rankine to read

$$W = \frac{f_c A}{1+c_2(L/k)^2},$$

where f_c = compressive stress for very short members and c_2 = a constant.

This formula was obtained by interpolation between the formula for a short strut, which will fail by direct compression, and that for a long, slender strut which will behave according to the Eulerian theory, i.e. fail by buckling.

Let $W_1 = f_c A$ = value for short strut, and

$$W_2 = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 EA}{(L/k)^2} = \text{Eulerian value}.$$

Then according to Rankine,

$$\frac{1}{W} = \frac{1}{W_1} + \frac{1}{W_2},$$

since for very short struts $1/W_2$ is negligible and $W = W_1$, and for very long struts $1/W_1$ is negligible in comparison with $1/W_2$ and $W = W_2$. Also, since the change in W by increasing l must be continuous, it is reasonable to accept the value of W, viz.

$$W = \frac{W_1 W_2}{W_1 + W_2}$$

$$= \frac{f_c A \times \pi^2 EA(k/L)^2}{f_c A + \pi^2 EA(k/L)^2}$$

$$= \frac{f_c A}{1 + \frac{f_c}{\pi^2 E(k/L)^2}} = \frac{f_c A}{1 + C(L/k)^2},$$

where $$C = \frac{f_c}{\pi^2 E} = \text{constant}$$

and stress $$= \frac{W}{A} = \frac{f_c}{1 + C(L/k)^2}.$$

Johnson's Parabolic Formula

This is an empirical formula propounded by Prof. J. B. Johnson and can be expressed thus:

$$f = f_y - b(L/k)^2,$$

where f = permissible stress,

f_y = stress at yield-point,

and b = a constant determined so as to make the parabola giving the value of f tangential to the Eulerian curve.

Claxton Fidler. This formula, given by the author in his *Bridge Construction*, can be expressed thus:

$$\text{stress} = \frac{f_E + f_c - \sqrt{\{(f_E + f_c)^2 - 2{\cdot}4 f_E f_c\}}}{1{\cdot}2},$$

where f_E = Eulerian stress,

and f_c = ultimate strength in compression.

Fidler's formula is based on the assumption that the value of E varies from one side of the cross-section to the other, causing differential contraction and hence a deflexion which increases with the bending moments.

It has been pointed out that the Eulerian formula neglects two factors which are likely to arise in actual struts: (*a*) accidental eccentricity of loading, and (*b*) original curvature of the strut (as well as the direct compressive stress).

The formulae under heading (*a*) may be termed 'eccentricity' formulae and those under heading (*b*) as 'curvature' formulae. It should be understood that 'eccentricity' means 'equivalent' eccentricity, i.e. including effect of curvature, and similarly that 'curvature' means a curvature representing the combined effect of curvature and eccentricity.

The Moncrieff formula is typical of type (*a*) and embodies the result of much research work by the late J. Mitchell Moncrieff described in the *Proceedings of the American Society of Civil Engineers* (1901). Moncrieff assumed that the column was originally straight, the loading accidentally eccentric, the deflected form is parabolic, and the central deflexion is given by

$$d = \frac{We_0 L^2}{8EI-(5WL^2/6)},$$

where e_0 = accidental eccentricity,

W = load on the column,

and I = least moment of inertia.

If the denominator becomes zero, then d becomes infinite, i.e. when

$$8EI = 5WL^2/6$$

or

$$W = \frac{48EI}{5L^2},$$

and

$$\frac{W}{A} = \frac{9 \cdot 6E}{(L/k)^2},$$

which is approximately the Eulerian value,

viz.

$$\frac{W}{A} = \frac{\pi^2 E}{(L/k)^2} = \frac{9 \cdot 81E}{(L/k)^2}.$$

Since the total deflexion = e_0+d,

$$\text{max. stress} = p\left\{1+\frac{(e_0+d)n}{k^2}\right\},$$

where n = distance to extreme fibre.

In order to find the L/k value corresponding to p, Moncrieff gives the following formula:

$$\left(\frac{L}{k}\right)^2 = \frac{48E\left(\frac{f_c}{p}-1-\frac{e_0 n}{k^2}\right)}{p\left(\frac{e_0 n}{k^2}-5\right)+5f_c},$$

which assumes that e_0 is known.

In order to obtain a formula for columns intended to be loaded centrally, Moncrieff plotted the results of hundreds of tests and came to the conclusion that $e_0 = 0{\cdot}6k^2/n$. Also, in order that the working stress should not exceed one-third of the Eulerian value, Moncrieff took the value of f_c as 10·7 tons/in.²

When $L/k = 0$ and $e_0 = 0{\cdot}6k^2/n$,

$$f_c = 1{\cdot}6p \quad \text{or} \quad p = 6{\cdot}67 \text{ tons/in.}^2$$

and the formula can be written as

$$\frac{L}{k} = 100\sqrt{\left\{\left(\frac{21{\cdot}4}{53{\cdot}5-4{\cdot}4p}\right)\left(\frac{10{\cdot}7}{p}-1{\cdot}6\right)\right\}}.$$

Considering *curvature* formulae (type *b*) the Perry formula may be taken as typical and can be written as:

$$p = \frac{p_1+(n+1)p_E}{2} - \sqrt{\left\{\left(\frac{p_1+(n+1)p_E}{2}\right)^2 - p_1p_E\right\}}$$

for stress p_1 on concave side of column and

$$p = \frac{(1-n)p_E-p_1}{2} - \sqrt{\left\{\left(\frac{(1-n)p_E-p_1}{2}\right)^2 + p_1p_E\right\}}$$

for stress p, on convex side of column, where p_E = Eulerian value and n = measure of the original curvature.

Prof. A. Robertson in his paper *The Strength of Struts* (Inst. C.E., Selected Paper No. 28) took the value of p_1 as the yield-point in compression of the material. The lower limit of the tests could be taken as represented by $n = 0{\cdot}003l/k$. Prof. Robertson also took the yield-stress as 18 tons/in.² and E as 13,000 tons/in.² (for mild steel).

Then
$$p_E = \frac{9{\cdot}81\times 13000}{(l/k)^2}$$

and
$$p = \frac{18+1{\cdot}003p_E}{2} - \sqrt{\left\{\left(\frac{18+1{\cdot}003p_E}{2}\right)^2 - 18p_E\right\}}$$

or
$$p = \frac{0{\cdot}997p_E-18}{2} - \sqrt{\left\{\left(\frac{0{\cdot}997p_E-18}{2}\right)^2 + 18p_E\right\}}.$$

In the First Report of the Steel Structures Research Committee Prof. Robertson makes the statement that, for struts having a definite eccentricity e_0, he is of opinion that the term C_0 from the curvature formula should be added to e_0 and that for working loads these can be represented by a factor of 2·36. C_0 can be taken as the measure of the curvature (see Fig. 8.2).

The formula used in B.S.S. No. 449 for the use of structural steel in building is of this type, viz.:†

$$Ap = \frac{p_y+(n+1)p_E}{2} - \sqrt{\left\{\left(\frac{p_y+(n+1)p_E}{2}\right)^2 - p_y p_E\right\}},$$

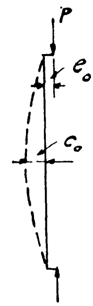


FIG. 8.2

where p = working stress for mild steel,

$A = 2{\cdot}36$,

p_y = yield stress = 18 tons/in.^{2},

p_E = Eulerian stress,

$n = 0{\cdot}003l/k$.

It must be realized that for many columns in practice the stress due to direct loading, including the effects of additional eccentricity and curvature, may not be the determining factor in the design. In fact many columns in buildings may carry B.M.s so large in comparison to the direct load that it is a moot point whether they should not be designed as vertical beams. To allow for the effect of bending the B.S.S. formula is modified thus. Suppose that the value p is determined from the formula given above and let $f_c = W/A$, where W = direct load and A = area. Then if $f_c < p$, the safe working stress can be increased thus:

$$F_2 = f_c + 7{\cdot}5\left(1-\frac{f_c}{p}\right)\left(1-0{\cdot}002\frac{L}{k}\right)$$

for mild steel (corresponding values are given for high-tensile steel).

The formula given in B.S.S. 449 is widely adopted, as it is incorporated by reference in the Ministry of Health Model Building By-laws and the L.C.C. By-laws. It is worthy of note that where columns are subject to the effects of wind pressure, the allowable stresses may be further increased. B.S.S. 449 applies to all building work.

B.S.S. No. 153 applies to girder bridges. Parts 3, 4, and 5 give the following values for struts in truss and lattice girders:

$9(1-0{\cdot}0038l/k)$ for riveted connexions

and $9(1-0{\cdot}0054l/k)$ for pin connexions,

† Draft revision of B.S.S. 449 gives this amended formula (1) for values of l/k greater than 80:

$$2p = \frac{p_y+(n+1)p_E}{2} - \sqrt{\left\{\left(\frac{p_y+(n+1)p_E}{2}\right)^2 - p_v + p_E\right\}}$$

and (2) for values of l/k less than 80; then p can be found by joining that point on the curve for l/k to the value when $l/k = 0$ [when $p = 0{\cdot}59p_y$ (9·00 tons/in.^{2})].

The Draft Code of Practice for the structural use of steel in buildings gives: (1) for values of l/k greater than 80, formula as in B.S.S. 449/1937, but with constant A reduced from 2·36 to 2; (2) for values less than 80, by interpolation between that for $l/k = 80$ and that for $l/k = 0$, i.e. $0{\cdot}63p_y$ (9·00 tons/in.^{2}).

values being in tons/in.^{2} for axial stress, with an upper limit of 7·65 tons/in.^{2} for mild steel.

The values for high-tensile steel are generally taken as 50 per cent. higher than the above values. The '153' formulae are of the straight-line or Pencoyd type. Though this is of an empirical nature, the value of the working stress shows a fairly close agreement with those derived from mathematical or experimental studies.

Another formula used in this country is that of the Institution of Structural Engineers given in the *Report of Steelwork for Buildings*. Although the formula itself is not stated explicitly in the report, the values for L/k ratios of 10 to 240 are given for mild and high-tensile steels, and correspond with those given in B.S.S. 449. The effect of combined stress, viz. compression and bending, is also dealt with by allowing a working stress

$$f_1 = f_a + f_b(1 - f_a/f_p),$$

where $f_a = W/A$ (corresponding to f_c in B.S.S. 449),

f_b = permissible compressive stress for laterally unsupported beams, viz.

$$f_b = 1{\cdot}0 + \frac{750}{L/k} \text{ (M.S.)} \quad \text{or} \quad 0{\cdot}5 + \frac{900}{L/k} \text{ (H.T.S.)}$$

and f_p = strut value for ratio L/k given in table (as B.S.S. 449).

A further proviso is that, for a distance not exceeding one-fifth of the actual length above or below a restrained end, the working stress may be increased to $4 + \frac{1}{2}f_1$ (mild steel) or $6 + \frac{1}{2}f_1$ (high-tensile steel). Values are also given for eccentricity of loads on columns. It should be understood that in using this formula or the B.S.S. 449 formula (*a*) that the stress W/A must be less than the 'primary' allowable stress f_1, and (*b*) that the algebraic sum of stresses W/A and that due to bending (in cases where there is bending about both axes, then the bending stress must be calculated about both axes and added algebraically) must be less than the maximum stress, f_2, allowable under the formulae when W/A is less than the 'primary' working stress, f_1.

Other column formulae are in use in the U.S.A. and other countries, but space precludes any attempt to deal with them. It must be understood that the foregoing remarks apply to steel and that columns of other materials will be dealt with later in this chapter. In Fig. 8.3 working stresses according to various formulae have been plotted for the purpose of comparison. The student should bear in mind the fact that while some of the formulae may be complicated, the values of the working stresses for varying L/k values can be obtained by reference to tables or graphs.

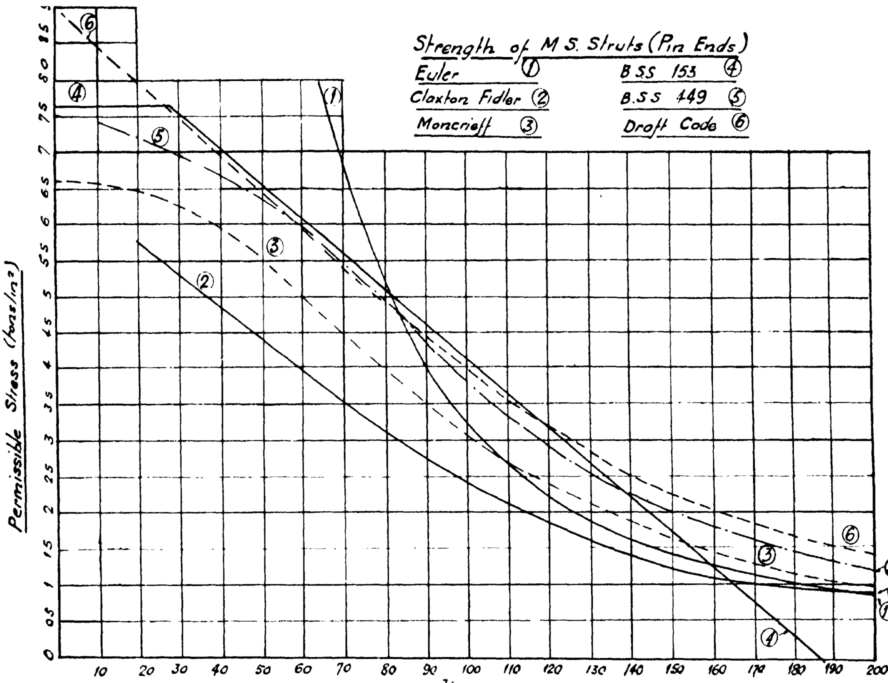


FIG. 8.3

Design of Steel Columns

It must be recognized that all column design must be by trial and error; in other words, a section must be selected and the safe stress found from the column formula and the actual stress calculated to ensure that it does not exceed the safe stress. Only experience can minimize the time to select a satisfactory section. In cases where there is no B.M. to allow for, the ideal strut would be one where the M.I. of the cross-section is the same about both axes, e.g. a hollow circle. In steelwork tables, joists whose flange widths are broad relative to the overall depths are suitable, and of course a suitable section can be made by adding flange plates to a single or double joist or channel.

Effective length of columns. One point which must be decided in column design is the *effective length* or length of the equivalent column with pin ends. It is best to treat the problem in two parts: (*a*) single-story columns; (*b*) continuous columns such as are found in multi-story buildings.

Under heading (*a*) it will be remembered that according to Euler the equivalent length ratios are (1) both ends fixed, 0·5; (2) one end fixed and the other end hinged, 0·7; (3) one end fixed and the other end free, 2·0. Claxton Fidler was more conservative and took the equivalent length for both ends fixed as 0·6. Present-day practice is inclined to be still more conservative in assessing equivalent lengths, probably on account of the difficulty in estimating the degree of fixity at the ends.

The following values are given in the report of the Institution of Structural Engineers and may be taken as representative of good practice:

End conditions	*Ratio* $\frac{\text{equivalent length}}{\text{actual length}}$
Both ends fixed	0·7
One end fixed / ,, ,, hinged	0·85
Both ends hinged	1·00
One end fixed, other end fixed wholly or partly in direction but not in position	1·0 to 1·5†
One end fixed, other end not held in position and either partly or not restrained, not held in direction	1·5 to 2·0‡

† Given as 1·5 in draft revision of B.S.S. 449.
‡ Given as 2·0 in draft revision of B.S.S. 449.

The equivalent length under heading (*b*) is more open to controversy. The table below gives values which may be accepted:

End conditions	*Ratio* $\frac{\text{equivalent length}}{\text{actual length}}$
Both ends fixed	0·75
Both ends fixed in position; one or both ends imperfectly fixed in direction	0·75 to 1·00
One end fixed in position and direction; other end imperfectly fixed in position and direction . .	1·00 to 2·00

The amount of restraint depends on the relative stiffnesses of the column and the beams connected thereto, also the connexions between the column and other members.

For columns forming part of a frame with rigid joints the following values have been suggested for equivalent lengths:

Ratio $\frac{\text{stiffness of beam}}{\text{stiffness of column length}}$	0·25	0·50	1·00	1·5 and over
Ratio $\frac{\text{effective length}}{\text{actual column length}}$	0·90	0·75	0·65	0·60

Flat ends. Moncrieff dealt with flat-ended columns in his investigations already referred to. These behave as columns fixed at the ends until the compressive stress at one point on the end becomes zero, when the end rotates and the column tends to behave as a hinged-end column with initial curvature. Flat-ended columns may be taken as fixed for the values less than 120 and as partially fixed for L/k more than 120.

Equivalent concentric loads. Where a load or a system of loads has a definite eccentricity about one or both axes, it can be reduced to an equivalent concentric load thus:

Let load $= W$; eccentricity about X–X axis $= e_{xx}$.

Then B.M. about X–X axis $= W \times e_{xx}$.

Stress due to bending $= \dfrac{W \times e_{xx} \times d/2}{I_{xx}}$.

Direct load to produce same stress $= \dfrac{W \times e_{xx} \times d/2 \times A}{I_{xx}}$

$= \dfrac{W \times e_{xx} \times d/2}{k_{xx}^2}$.

Similarly for Y–Y axis: Load $= \dfrac{W \times e_{yy} \times b/2}{k_{yy}^2}$,

$\therefore$ total or equivalent concentric load

$$= W\left(1 + \frac{e_{xx} \times d}{2k_{xx}^2} + \frac{e_{yy} \times b}{2k_{yy}^2}\right),$$

which is a function of the column and its properties.

Example 1. A column is 10 ft. high and hinged at each end. It carries a load of 40 tons which has eccentricities of 6 in. and $1\frac{1}{2}$ in. about the X–X and Y–Y axes respectively. Design a section to comply with B.S.S. No. 449.

Use section composed of two channels 8 in. $\times$ $3\frac{1}{2}$ in. $\times$ 20·21 lb. and two plates 10 in. $\times$ $\frac{5}{8}$ in. Properties: Area $= 24{\cdot}38$ in.^{2}; $k_{yy} = 2{\cdot}62$ in.; $Z_{xx} = 76{\cdot}5$ in.^{3}; $Z_{yy} = 33{\cdot}4$ in.^{3};

$$L/k = \frac{120}{2{\cdot}62} = 46; \qquad p = 6{\cdot}44 \text{ tons/in.}^2$$

Direct stress (f_c) $= \dfrac{40}{24{\cdot}38} = 1{\cdot}64$ tons/in.^{2} $\quad \dfrac{f_c}{p} = 0{\cdot}255.$

Bending stress X–X $= \dfrac{240}{76{\cdot}5} = 3{\cdot}14$,, $\quad \therefore \; F_2 = 6{\cdot}70 > 6{\cdot}61$ (and section is adequate).

,, ,, Y–Y $= \dfrac{60}{33{\cdot}4} = 1{\cdot}80$,,

Total $= 6{\cdot}58$ tons/in.^{2} $> 6{\cdot}44$.

Check by finding equivalent concentric load:

$$W_E = 40(1 + 6 \times 0{\cdot}32 + 1{\cdot}5 \times 0{\cdot}73) = 160{\cdot}6 \text{ tons}$$

and total stress $= \dfrac{160{\cdot}6}{24{\cdot}38} = 6{\cdot}61$ tons/in.^{2} as before.

Example 2. A column has an effective length of 12 ft. and carries loads as shown in Fig. 8.4. Assume loads are applied on brackets so that there is an eccentricity from the face of flanges or of web. Design suitable section to B.S.S. 449.

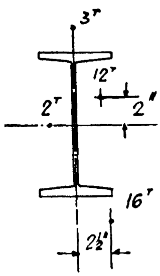


FIG. 8.4

Try 12 in. × 8 in. × 65 lb. B.S.H.B.

Properties: Area $= 19{\cdot}12$ in.^{2}; $k_{yy} = 1{\cdot}85$ in.;

$Z_{xx} = 81{\cdot}3$ in.^{3}; $Z_{yy} = 16{\cdot}3$ in.^{3};

$$L/k = \frac{144}{1{\cdot}85} = 78; \qquad p = 4{\cdot}99 \text{ tons/in.}^2$$

$$\text{Direct stress } (f_c) = \frac{33}{19{\cdot}12} = 1{\cdot}73 \text{ tons/in.}^2$$

$$f_c/p = 0{\cdot}35.$$

Bending moment X–X axis $= (16-3)(6+2)-12\times 2 = 80$ in.-tons.

" " Y–Y " $= 16\times 2{\cdot}5+(12-2)2{\cdot}22 = 62{\cdot}2$ "

$$\text{Bending stress } X\text{–}X = \frac{80}{81{\cdot}3} = 0{\cdot}98 \qquad F_2 = 5{\cdot}8 \text{ tons/in.}^2 < 6{\cdot}53$$

(not strong enough).

$$\text{" " } Y\text{–}Y = \frac{62{\cdot}2}{16{\cdot}3} = 3{\cdot}82$$

$$f_c = 1{\cdot}73$$

$$\text{Total} = 6{\cdot}53 > 4{\cdot}99.$$

Try 8 in. × 5 in. × 28 lb. B.S.B. with two 10 in. × ½ in. flange plates.

Properties: Area $= 18{\cdot}28$ in.^{2}; $k_{yy} = 2{\cdot}26$ in.;

$Z_{xx} = 60{\cdot}1$ in.^{3}; $Z_{yy} = 18{\cdot}7$ in.^{3};

$$L/k = \frac{144}{2{\cdot}26} = 64; \qquad p = 5{\cdot}71 \text{ tons/in.}^2$$

$$f_c = \frac{33}{18{\cdot}28} = 1{\cdot}80 \text{ tons/in.}^2 \qquad f_c/p = 0{\cdot}32.$$

Bending moment X–X axis $= (16-3)6{\cdot}5-12\times 2 = 60{\cdot}5$ in.-tons.

Y–Y " $= 16\times 2{\cdot}5+(12-2)2{\cdot}18 = 61{\cdot}8$ "

$$\text{Bending stress } X\text{–}X = \frac{60{\cdot}5}{60{\cdot}1} = 1{\cdot}01 \text{ tons/in.}^2$$

$$Y\text{–}Y = \frac{62{\cdot}2}{18{\cdot}7} = 3{\cdot}32 \text{ "}$$

$$f_c = 1{\cdot}80 \text{ "}$$

$$6{\cdot}13 > 5{\cdot}71.$$

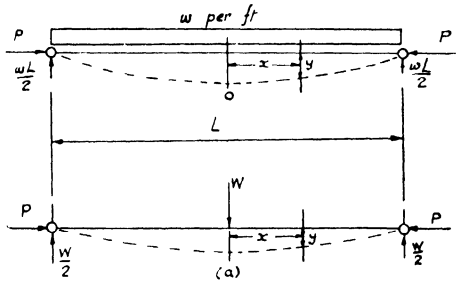
$F_2 = 6{\cdot}25 > 6{\cdot}12$, so section is suitable.

Equivalent concentric load	=	33 tons
$+60{\cdot}5 \times 0{\cdot}30$	=	18 ,,
$+62{\cdot}2 \times 0{\cdot}98$	=	61 ,,
		112 ,,

$$\text{Stress} = \frac{112}{18{\cdot}28} = 6{\cdot}12 \text{ tons/in.}^2 \text{ as above.}$$

Columns with Lateral Loads

Case 1. Hinged ends, concentric load P, lateral load w per unit length. (Fig. 8.5.)



FIGS. 8.5 and 8.5(*a*)

Taking the origin at O on the centre line, then the B.M. at x from origin

$$M_x = -\frac{w}{2}\left(\frac{L^2}{4} - x^2\right) - Py = EI\frac{d^2y}{dx^2}.$$

Perry's approximation to above is

$$EI\frac{d^2y}{dx^2} = -\frac{wL^2}{8}\cos\frac{x}{L}\pi - Py$$

or

$$EI\frac{d^2y}{dx^2} + Py = -\frac{wL^2}{8}\cos\frac{x}{L}\pi.$$

The solution to this differential equation is

$$y = \frac{wL^2}{8}\,\frac{\cos \pi x/L}{P_e - P},$$

where P_e = Eulerian value.

When $x = 0$ at centre line,

$$y = y_0 = \frac{wL^2}{8(P_e - P)},$$

and

$$M_0 = -\frac{wL^2}{8} \times \frac{P_e}{P_e - P} = \text{max. value},$$

from which the bending stress can be calculated. It will be noticed that

$$M_0 = -\frac{wL^2}{8} \times \frac{1}{1-P/P_e}$$

and the first part of the expression represents the B.M. due to lateral load only.

Case 2. Lateral point load W at centre (Fig. 8.5 (*a*)).

Then
$$-M = Py + \frac{W}{2}\left(\frac{L}{2} - x\right),$$

or
$$Py + EI\frac{d^2y}{dx^2} = -\frac{W}{2}\left(\frac{L}{2} - x\right).$$

Hence when $x = 0$,

$$y_0 = \frac{W}{2P}\sqrt{\left(\frac{EI}{P}\right)}\tan\left\{\frac{L}{2}\sqrt{\left(\frac{P}{EI}\right)}\right\} - \frac{WL}{4P}.$$

$$\therefore \quad -M_0 = \frac{W}{2}\sqrt{\left(\frac{EI}{P}\right)}\tan\left\{\frac{L}{2}\sqrt{\frac{P}{EI}}\right\}.$$

Putting
$$\frac{L}{2}\sqrt{\frac{P}{EI}} = \theta$$

and using the expansion

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \frac{17\theta^7}{315} + \ldots,$$

$$-M_0 = \frac{WL}{4}\left(1 + \frac{\theta^2}{3} + \frac{2\theta^4}{15} + \frac{17\theta^6}{315} + \ldots\right),$$

and since

$$\theta^2 = \frac{L^2}{4} \times \frac{P}{EI} = \frac{\pi^2 P}{4P_e} \quad (P_e = \text{Eulerian value}),$$

$$-M_0 = \frac{WL}{4}\left\{1 + \frac{\pi^2}{12}\frac{P}{P_e} + \frac{\pi^4}{120}\left(\frac{P}{P_e}\right)^2 + \frac{17\pi^6}{20{,}160}\left(\frac{P}{P_e}\right)^3 + \ldots\right\}.$$

Since the expression in brackets is a geometric progression approximately,

$$-M_0 = \frac{WL}{4}\left\{1 + \frac{P}{P_e} + \left(\frac{P}{P_e}\right)^2 + \left(\frac{P}{P_e}\right)^3 + \ldots\right\}$$

$$= \frac{WL}{4}\left(\frac{1}{1-(P/P_e)}\right), \quad \text{since } P/P_e \text{ is less than 1},$$

$$= \frac{WL}{4}\left(\frac{1}{1-(PL^2/10EI)}\right), \quad \text{since } P_e = \frac{10EI}{L^2}.$$

The expression $WL/4$ is, of course, the B.M. due to point load W

only. Therefore it is apparent that to find the B.M. it is necessary to multiply the B.M. due to lateral load only by

$$\frac{1}{1-(PL^2/10EI)} = \frac{1}{1-(P/P_e)}$$

(note that this also holds good for Case 1).

Columns acted on by End Moments

In building frames, column lengths are often restrained at the ends by moments due to the beams framing into them. The usual effect of such moments is to reduce the bending due to initial curvature.

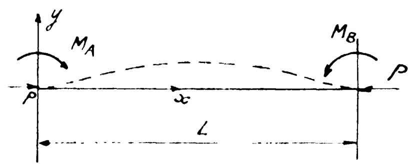


FIG. 8.6

If then we take the moments at A, B as M_A, M_B respectively and direct load as P, and if the assumption is made that the equation to the original shape is given by

$$y = \frac{\epsilon \sin \pi x}{l} \quad \text{(as in B.S.S. 449 formula)},$$

where $\epsilon = 0{\cdot}003\,Lk/a$,

a = distance to extreme fibre.

Then the equation to the curve under M_A, M_B becomes:

$$y = \frac{M_A}{P}\left[\frac{L-x}{L} - \frac{\sin\alpha(L-x)}{\sin\alpha L}\right] + \frac{M_B}{P}\left[\frac{x}{L} - \frac{\sin\alpha x}{\sin\alpha L}\right] + \frac{\epsilon\pi^2}{(\pi^2-\alpha^2L^2)}\frac{\sin\pi x}{L}$$

where $\alpha = \sqrt{(P/EI)}$, and the moment at any point x is given by

$$M_x = -M_A\left[\frac{\sin\alpha(L-x)}{\sin\alpha L}\right] - M_B\left[\frac{\sin\alpha x}{\sin\alpha L}\right] + \frac{P\epsilon\pi^2}{(\pi^2-\alpha^2L^2)}\frac{\sin\pi x}{L}.$$

When $M_A = M_B = M$,

then moment at centre ($= M_c$)

$$= -M\sec\frac{\alpha L}{2} + \frac{P\epsilon\pi^2}{\pi^2-\alpha^2L^2}$$

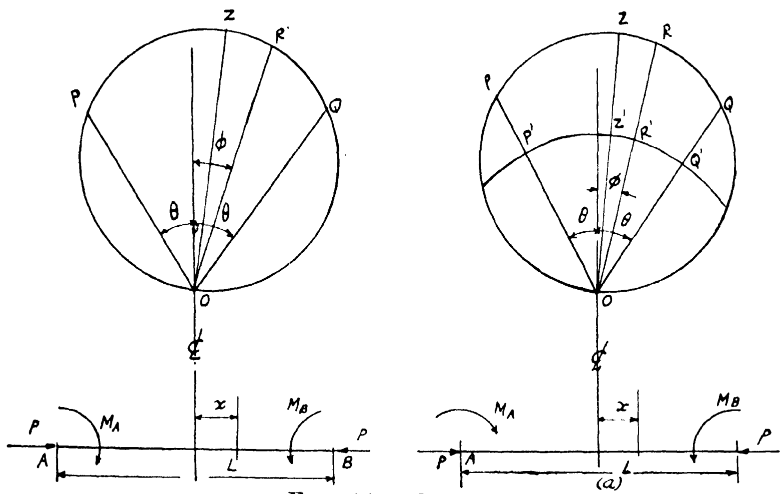
$$= -M\sec\frac{\alpha L}{2} + \frac{0{\cdot}003\pi^2PLk}{a(\pi^2-\alpha^2L^2)}.$$

The equivalent length l for a hinged end strut restrained by end moments M is given by:

$$-M\sec\frac{\alpha L}{2}+\frac{0{\cdot}003\pi^2PLk}{a(\pi^2-\alpha^2L^2)}=\frac{0{\cdot}003\pi^2Plk}{a(\pi^2-\alpha^2l^2)}\quad\text{(for } M_c \text{ greater than } M\text{)}.$$

If M is greater than M_c, then to find equivalent length

$$M=\frac{0{\cdot}003\pi^2Plk}{a(\pi^2-\alpha^2l^2)}\quad (k = \text{least radius of gyration}).$$



FIGS. 8.7 and 8.7 (a)

Howard Circle Diagrams

The moment at any point on a column subject to end moments as well as to direct load can be found graphically by means of a modification to the method proposed by H. B. Howard for the analysis of beams subject to end moments and direct compression. Referring to Fig. 8.7, from the point O on centre line of the beam set off two rays $OP = M_A$ and $OQ = M_B$, making an angle θ with OY so that

$$\theta=\frac{L}{2}\sqrt{\frac{P}{EI}}.$$

Describe a circle to pass through O, P, and Q. To find M_x draw a line OR at angle of $\phi = x\sqrt{(P/EI)}$ to centre line, then the moment is represented by OR to the scale used for OP, OQ. The maximum B.M. is represented by the diameter OZ.

Certain modifications must be introduced in order to allow for the effect of initial curvature. In the formula on page 253 the equation to the centre line of the column has been assumed to be

$$y = 0{\cdot}003\frac{Lk}{a}\sin\frac{\pi x}{L},$$

representing the initial curvature. The ordinates representing this are approximately equal to those for an initially straight beam under a constant B.M. of $M = 0{\cdot}024KEI/aL$ and the error involved in substituting the latter is very small. Therefore the diagram should be modified thus: when the rays OP, OQ are set off from O make $OP = M+M_A$ and $OQ = M+M_B$, then the circle drawn through O, P, and Q represents the moment at any point on a column acted on by end moments $M+M_A : M+M_B$. In order to find M_x set off OR at $\phi = x\sqrt{(P/EI)}$ then $M_x = OR-M$ and maximum B.M. $= OZ-M$ (where OZ is diameter). The points P', Q', R', Z' lie on the line giving the values of the net moments (see Fig. 8.7 (*a*)).

Design of Columns for Single-story Workshop Buildings

The design of such columns depends on the method of attachment at the cap and base. For spans up to about 30 ft. columns can be taken as fixed at the base and pinned at the cap. The B.M. due to the lateral wind load = sum of moments about the base. The column and also the foundation must be designed to resist this moment.

For larger spans, it may be economical to insert a knee-brace connecting the column to a panel point on the main tie of the truss. In this case the B.M. is often assumed to be zero at the cap and at the base and a maximum at the foot of the knee-brace.

Another type of column is that designed to span as a vertical beam between the base and a wind girder at main-tie level or in the plane of the rafter. The moment is zero at the base and the cap. The reactions at the column caps are carried by the wind girder to the end frames. This design can be economical for high buildings provided that the building is not too long.

It should be understood that the foregoing remarks apply only to simple columns and single spans. For more complicated columns and for more than one span, effective heights are given in the Code of Practice for the *Use of Structural Steel in Buildings* and the draft revision of B.S.S. 449.

Reinforced Concrete Columns

In dealing with R.C. columns it must be remembered that this is a composite or non-homogeneous material. Under direct compression the stress in the steel will be m times the stress in the surrounding concrete. In addition to the main longitudinal reinforcement R.C. columns must have secondary reinforcement to prevent buckling of the main bars and bursting of the concrete outwards. Columns without this are 'rodded' columns. Secondary reinforcement is in tension and may be

(1) in the form of hoops;
(2) ,, ,, a spiral or helix;
(3) ,, ,, (1) and (2) combined.

The effective area of a R.C. column can be taken as either (1) gross area $D \times D$ or (2) net area $d \times d$ (see Fig. 8.8).

The minimum concrete cover to steel should be 1 in. (measured from outside of transverse reinforcement). The effective area of a R.C. column is equal to concrete area$+(m-1)$ area of main steel or

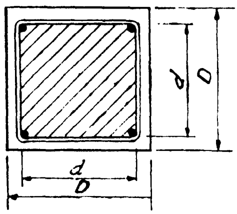


Fig. 8.8

$$A = D \times D + (m-1)A_c.$$

Then $A \times f_c =$ safe load on short column, where $f_c =$ allowable compressive stress.

Main or longitudinal reinforcement should be between 0·8 per cent. and 8 per cent. of gross column area. Where helical transverse steel is used, the main steel must be in the form of not less than 6 bars, but with hoops 4 bars can be used.

Transverse reinforcement. The volume of this should not be less than 0·4 per cent. of the gross volume of the column. For hoops the spacing should be 6 to 12 in. For spirals the pitch should be not more than 3 in. or $\frac{1}{6} \times$ core diameter (whichever is the lesser) and not less than 1 in. or $3 \times$ diameter of main steel (whichever is the greater).

Allowable Stress in Concrete in Direct Compression (f_c)

Mix	*Stress lb./in.*^{2}
1:1:2	780
1:1·2:2·4	740
1:1·5:3	680
1:2:4	600

The allowable compressive stress in the main steel can be taken at 13,500 lb./in.^{2} and the same value can be taken for tension in the spiral reinforcement. These values are for mild steel only; where steel with a yield-point of not less than 44,000 lb./in.^{2} is used, these values can be increased to 15,000 lb./in.^{2}

Loads in Columns (modern practice)

(*a*) *Short columns.* (1) With hoops:

$$\text{Safe load} = f_c \times \text{gross area of concrete} + 13{,}500A_c.$$

(2) With spiral reinforcement:

$$\text{Safe load} = f_c \times \text{core area of concrete} + 13{,}500A_c + $$
$$+ 2 \times 13{,}500 \times \text{equivalent area of spiral reinforcement}$$

(equivalent area of spiral reinforcement = volume of spiral divided by length of column).

The reason why spirally reinforced columns can carry a higher load is that tests have shown conclusively that these can develop a higher stress in the concrete core before failure takes place. These figures apply only to columns under direct load.

(*b*) *Long columns.* The safe load in this case is obtained by multiplying the safe load for a short column by a coefficient depending on the ratio $\frac{\text{effective length}}{\text{least lateral dimension}}$. In case (1) the least lateral dimension is the least *overall* dimension, and in (2) the least *core* dimension is taken. These coefficients are given:

Effective length of column / *Least lateral dimension*	*Coefficient*
15	1·0
18	0·9
21	0·8
24	0·7
27	0·6
30	0·5
33	0·4
36	0·3
39	0·2
42	0·1
45	0

L.C.C. By-laws are based on L/k values thus:

L/k	*Coefficient*
50	1·0
60	0·9
70	0·8
80	0·7
90	0·6
100	0·5
110	0·4
120	0·3

$k = \sqrt{(I/A)}$, where I = equivalent moment of inertia found by taking equivalent area of steel $= (m-1)A_c$.

These coefficients allow for the effects of buckling. The effective length can be defined as the equivalent length of a hinged end column. This depends, of course, on the conditions of the ends. The following values are typical:

Single-story Columns and Top Lengths

End conditions	*Effective length* / *Actual length*	*Connexions*	*Effective length* / *Actual length*
Both ends fixed	0·75	4- and 3-way	0·875
,, ,, hinged	1·00	2-way	1·0
One end fixed, ,, ,, hinged	1·00–2·00	1-way	1·25

Columns of Two or More Stories

End conditions	*Effective length* / *Actual length*	*Connexions*	*Effective length* / *Actual length*
Both ends fixed	0·75	4- and 3-way	0·75
,, ,, hinged	0·75–1·00	2-way	0·875
One end fixed, ,, ,, free	1·00–2·00	1-way	1·00

Example 1. A concrete column 10 in. square (1:2:4 mix) is 15 ft. high and fixed at both ends. The main steel is four 1-in. diameter bars with binders. Find safe load.

$$\text{Safe load for short column} = (600\times 10\times 10)+(13{,}500\times 3{\cdot}14)$$
$$= 102{,}500 \text{ lb.}$$
$$\text{Actual length} = 180 \text{ in.}$$
$$\text{Effective length} = 135 \text{ ,,}$$
$$\frac{\text{Effective length}}{\text{Least lateral dimension}} = \frac{135}{10} = 13{\cdot}5.$$
$$\text{Coefficient} = 1{\cdot}0,$$
$$\therefore \text{ safe load} = 102{,}500 \text{ lb.}$$

Check by L.C.C.:

$$A_E = 100+3{\cdot}14\times 14 = 143{\cdot}96 \text{ in.}^2$$
$$I_E = \frac{10^4}{12}+(m-1)A_c\times \text{distance}^2$$
$$= \frac{10^4}{12}+14\times 3{\cdot}14\times (5-1{\cdot}75)^2$$
$$= 1{,}298.$$
$$K = \sqrt{\left(\frac{I_E}{A_E}\right)} = 3{\cdot}02 \text{ in.}$$
$$l/K = \frac{135}{3{\cdot}02} = 45. \quad \text{Coefficient} = 1{\cdot}0.$$

Example 2. A concrete column 12 in. diameter is 20 ft. high and hinged at each end. The mix is 1:1·5:3, and the main steel consists of six $1\frac{1}{8}$-in. diameter bars. Find what transverse reinforcement is required and the safe load.

$$\text{Pitch of spiral reinforcement} = 2 \text{ in.}$$
$$\text{Min. diameter of spiral reinforcement} = \tfrac{5}{16} \text{ in.}$$

Volume of spiral reinforcement per foot length

$$= 6\times \pi\times 10\times 0{\cdot}076 = 14{\cdot}3 \text{ in.}^3$$
$$\text{Equivalent area} = 1{\cdot}19 \text{ in.}^2$$
$$\text{Core area} = \frac{\pi}{4}\{12-2\times 1\tfrac{1}{2}\}^2 = 64 \text{ in.}^2$$
$$A_c = 6{\cdot}00 \text{ in.}^2$$

$\therefore$ safe load for short column

$$= (64\times 680)+(6\times 13{,}500)+(2\times 1{\cdot}19\times 13{,}500)$$
$$= 156{,}600 \text{ lb.}$$

$$\text{Effective length} = 240 \text{ in.}$$

$$\text{Least lateral dimension} = \frac{240}{9} = 26{\cdot}67.$$

$$\text{Coefficient} = 0{\cdot}6,$$

$$\therefore \quad \text{safe load} = 0{\cdot}6 \times 156{,}600 = 93{,}960 \text{ lb.}$$

Check by L.C.C.:

$$A_E = \frac{\pi}{4} \times 12^2 + 14 \times 6$$

$$= 113{\cdot}6 + 84 = 197{\cdot}6 \text{ in.}^2;$$

$$I_E = \frac{\pi}{64} D^4 + 4 \times 14 \times 1 \left(4{\cdot}5 \times \frac{\sqrt{3}}{2}\right)^2$$

$$= 1{,}870 \text{ in.}^4$$

$$k = \sqrt{\left(\frac{1870}{197{\cdot}6}\right)} = 3{\cdot}07.$$

$$L/k = \frac{240}{3{\cdot}07} = 78. \quad \text{Coefficient} = 0{\cdot}72.$$

This gives safe load $= 0{\cdot}72 \times 156{,}600 = 112{,}752$ lb.

R.C. Columns subject to Bending Moments

In practice R.C. columns are often subject to B.M. as well as direct compression, e.g. columns in building frames are acted on by bending moment due to unequal spans, loadings, wind pressure, etc. As this matter will be dealt with more fully in another chapter, the following coefficients can be used for preliminary design:

	Single-bay frames	*Multiple-bay frames*
Moment at base of upper column length	$\left\{\frac{K_U}{K_U+K_L+\frac{1}{2}K_B}\right\}M_1$	$\left\{\frac{K_U}{K_U+K_L+n_1K_{B_1}+n_2K_{B_2}}\right\}\times(M_1-M_2)$
Moment at top of lower column length	$\left\{\frac{K_L}{K_U+K_L+\frac{1}{2}K_B}\right\}M_1$	$\left\{\frac{K_L}{K_U+K_L+n_1K_{B_1}+n_2K_{B_2}}\right\}\times(M_1-M_2)$

where K_U = stiffness of upper column length,
K_L = ,, lower column length,
K_B = ,, beam,
K_{B_1} = ,, ,, B_1,
K_{B_2} = ,, ,, B_2,

n = 1 for beam continuous beyond next support,
= $\frac{1}{2}$,, non-continuous beyond next support,
suffixes for beams 1, 2, etc.

M_1, M_2 = fixed-end moments for beams B_1, B_2.

Where the moment M is known as well as the direct load W, then it is usual to calculate the equivalent eccentricity $e = M/W$. The design of R.C. columns subject to B.M. is somewhat complicated, as the position of the neutral axis varies and the formulae are somewhat unwieldy. In order to solve the equations Mörsch's curves can be used (see Fig. 8.9). The graph shows the ratio n_1 knowing the value of M/Nd (d = overall depth) and the percentage of reinforcement on the tension side. The method of calculation is apparent from the figure.†

The graphs given in the Institution of Structural Engineers' Report on *Formulae for the Computation of Stresses in R.C.* (1946) can also be used.

In cases where the eccentricity is less than half of the column width and the ratio of steel : concrete $\times m$ is more than 0·3, the following formula for fibre stress can be used:

$$\text{Stress} = \frac{W}{Ag} \times \frac{1+K}{1+(m-1)(As/Ag)} \qquad [Ag = \text{gross area},\ As = \text{area of steel}],$$

$$K = 6e/d \text{ for rectangular columns},$$
$$= 8e/d \text{ for round columns}.$$

In some specifications it is permissible to use somewhat higher stresses in cases of combined bending and axial load thus:

$$\text{allowable stress} = f_c \times \frac{1+K}{1+CK},$$

where $C = f_c/\text{allowable stress in bending},$
$= f_c/c.$

The critical stresses are the maximum fibre stress in concrete and, for large eccentricities, the stress in the steel on the tension.

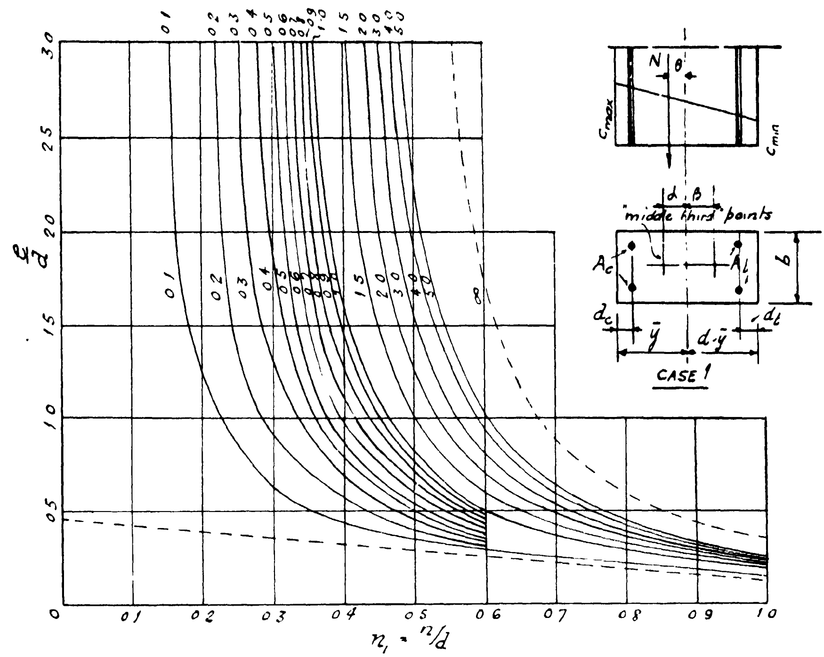
Composite columns. Where steel columns are cased with concrete properly reinforced with main and spiral reinforcement, the safe load can be taken as

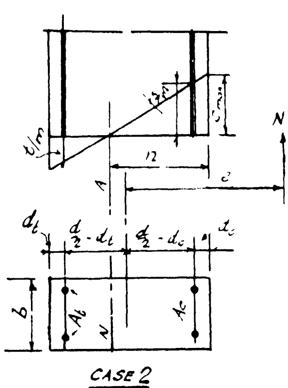
$$P = f_c A_n + t \times A_c + \text{area of steel col.} \times \text{working stress},$$
$$A_n = \text{net concrete area}.$$

Timber Struts

The strength of timber has been already dealt with in Chapter I and various American formulae have been shown for comparative purposes in Fig. 1.8. It must be remembered that timber is an organic material and that the strength of any particular timber must depend on the tree from which it is obtained and the method and effect of seasoning.

† For preliminary calculation divide moment by inertia and multiply by half depth to find bending stress.





MÖRSCH CURVES

TO FIND 'n' IN DOUBLY AND SYMMETRICALLY REINFORCED RECTANGULAR SECTIONS

d =	overall depth.	A_t =	tensile steel.
b =	,, width.	A_c =	comp. ,,
		A_E =	equivalent area.
N =	load.	e =	eccentricity.
p_t =	$\frac{A_t}{bd} \times 100.$	$\bar{y}$ =	depth of C.G. (Case 1).
p_c =	$\frac{A_c}{bd} \times 100.$	n =	,, N.A. (Case 2).

I_c = 2nd moment about C.G. (Case 1).

$\therefore \quad M = N \times e.$

To find 'middle third' points

$$\alpha = I_c/A_E \times d - \bar{y}, \qquad \beta = I_c/A_E\,\bar{y}.$$

CASE 1. N acting *inside* 'middle third' points ($e < \alpha, \beta$)

$$c_{\max} = \frac{N}{A_E}\left(1 + \frac{e}{\beta}\right), \qquad c_{\min} = \frac{N}{A_E}\left(1 - \frac{e}{\alpha}\right).$$

CASE 2. N acting *outside* 'middle third' points ($e > \alpha, \beta$) then, neglecting tension in concrete,

$$c_{\max} = \frac{N}{(bn/2 + mA_c/n)(n - d_c) - (mA_t/n)(d - d_t - n)}$$

$$= \frac{Ne}{bn/2(d/2 - n/3) + \{mA_c(d/2 - d_c)\}/n(n - d_c) + \{mA_t(d/2 - d_t)\}/n(d - d_t - n)}.$$

n_1 is found from graph by finding intersection of e/d line with curve showing p_t for section.

In usual case when $A_c = A_t$ and $d_c = d_t$,

$$c_{\max} = \frac{N}{bn/2 + (mA_t/n)(2n - d)},$$

$$t = \frac{mc_{\max}}{n}(d - d_t - n), \qquad f_s \text{ (comp. in steel)} = \frac{mc_{\max}}{n}(n - d_c).$$

FIG. 8.9

For British practice the following values can be taken: (1) for slenderness ratios of not more than 10, strength in compression parallel to the grain 1,000 lb./in.^{2} (graded timber) and 800 lb./in.^{2} (non-graded timber); (2) for other slenderness ratios:

Slenderness ratio	*Non-graded (lb./in.²)*	*Graded (lb./in.²)*
10–12	785	985
12–14	775	970
14–16	755	950
16–18	725	920
18–20	690	875
20–22	635	820
22–24	565	745
24–26	485	650
26–28	420	565
28–30	365	485
30–32	320	430
32–34	285	380
34–36	255	340
36–38	225	300
38–40	205	275

The curve of stress-slenderness is rather similar to B.S.S. 449 curve. The slenderness ratio $\dfrac{\text{effective length}}{\text{least lateral dimension}}$ should not exceed 40. Fig. 8.10 shows comparative values for British and American formulae for timber struts.

American practice with regard to timber columns is rather different as they are divided into three categories: (1) Short columns, viz. l/d equal to 11 or less. In this case, P/A = allowable stress in compression parallel to the grain (c). (2) Intermediate columns, i.e. l/d greater than 11 but less than the value given by

$$\frac{\pi}{2}\sqrt{\left(\frac{E}{5c}\right)} = 0{\cdot}702\sqrt{\left(\frac{E}{c}\right)},$$

the safe stress is reduced to†

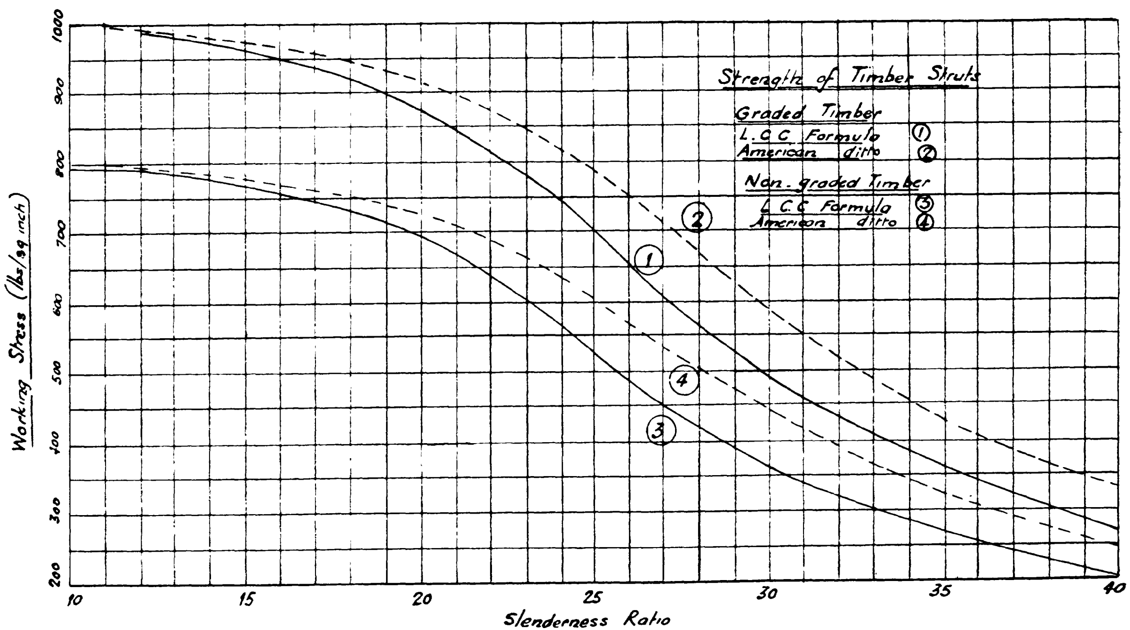
$$\frac{P}{A} = c\left[1-\frac{1}{3}\left(\frac{l}{Kd}\right)^4\right] \quad (d = \text{least dimension}).$$

(3) Long columns, l/d equal to or greater than $0{\cdot}702\sqrt{(E/c)}$, then safe stress $= 0{\cdot}329E/(l/d)^2$. This is the Eulerian value, and $l/d = 0{\cdot}702\sqrt{(E/c)}$ is the point where the Eulerian curve is tangential to the curve for intermediate columns.

The limiting value of l/d for simple solid columns is 50, and for component parts of spaced columns l/d should not exceed 80 and the

† K is l/d ratio at a point on the curve where intermediate (parabolic) and long (Eulerian) column formulae are tangent (stress $= \frac{2}{3}c$ at this point)

$$K = 0{\cdot}702\sqrt{(E/c)}.$$



Strength of Timber Struts
Graded Timber
L.C.C. Formula ①
American ditto ②
Non-graded Timber
L.C.C. Formula ③
American ditto ④
1
2
3
4
1000
900
800
700
600
500
400
300
200
Working Stress (lbs/sq inch)
10
15
20
25
30
35
40
Slenderness Ratio

FIG. 8.10

distance between spacing blocks should not exceed $40d$. The design of spaced columns is governed by factors such as position and spacing of connectors, etc.

For round columns the load should be calculated on the basis of a square column of the same area (Fig. 8.10).

The treatment of timber columns subject to lateral, eccentric, and bracket loads is somewhat more complicated and cannot be dealt with here, and the reader is referred to the bibliography at the end of the chapter for more detailed information.

Columns subject to Bracket Loads

Case 1. Ends hinged; inertia uniform throughout (Fig. 8.11). Obviously $R = \pm M/n$ and B.M. diagram is as shown. If depth of bracket is taken as d, then B.M. diagram is modified as shown in Fig. 8.11 (a).

Max. B.M. $= Mn$ or $M(1-n)$ or

$$\left.\begin{aligned} &= M(1-n-d/2n) \\ \text{or} \quad & M(n-d/2n) \end{aligned}\right\} \text{for bracket.}$$

Case 2. Ends fixed; inertia uniform throughout (Fig. 8.12). Let end-fixing moments be M_A, M_B, and R = reaction.

If M_x = B.M. at x from A, H = overall height, h = height of point of application of M,

$$M_x = M_A - Rx + M$$

(the last term applies when $x > h$). Since ends are fixed,

$$\int_0^H \frac{M\,dx}{EI} = 0 \quad \text{and} \quad \int_0^H \frac{Mx\,dx}{EI} = 0.$$

$$\int_0^H M\,dx = \left[M_A x - \frac{Rx^2}{2}\right]_0^H + [Mx]_h^H$$

$$= M_A H - \frac{RH^2}{2} + M(H-h) = 0.$$

$$\int_0^H Mx\,dx = M_A\frac{H^2}{2} - \frac{RH^3}{3} + \frac{M}{2}(H^2-h^2) = 0.$$

$$\therefore \quad R = \frac{6Mh}{H^3}(H-h) = \frac{6Mn(1-n)}{H},$$

where

$$n = \frac{h}{H}.$$

$$M_A H = \frac{RH^2}{2} - M(H-h).$$

$$\therefore \quad M_A = \frac{RH}{2} - M(1-n)$$

$$= 3Mn(1-n) - M(1-n)$$

$$= -M(1-4n+3n^2),$$

and

$$M_B = M - 6Mn(1-n) - M(1-n)(1-3n)$$

$$= -M(2n-3n^2).$$

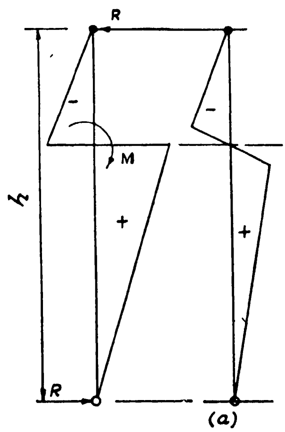


FIG. 8.11

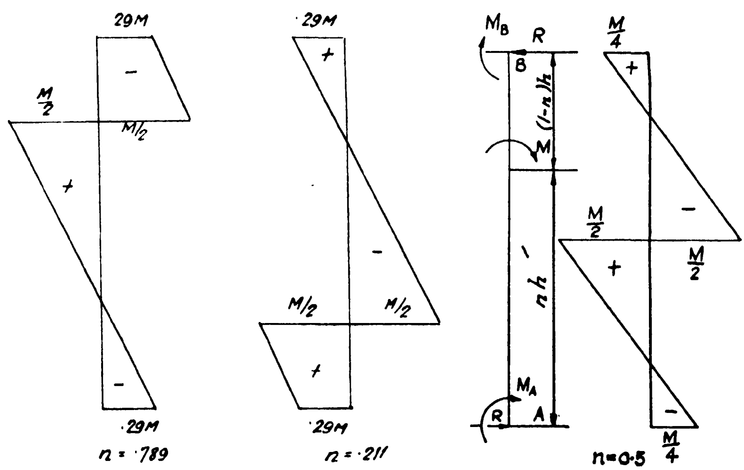


FIG. 8.12

For B.M. just below bracket

$$M_x = -M(1-n)(1-3n) - 6Mn^2(1-n)$$

$$= -M(1+3n^2-4n+6n^2-6n^3).$$

When $$M_x = \frac{M}{2} \quad \text{(max. value)}$$

$$n = 0{\cdot}5,\ 0{\cdot}7887,\ \text{or}\ 0{\cdot}2113.$$

When $n = 0{\cdot}5$, $M_A = M/4$; $M_B = M/4$.

$n = 0{\cdot}7887$, $M_A = 0{\cdot}289M$; $M_B = 0{\cdot}289M$.

$n = 0{\cdot}2113$, $M_A = 0{\cdot}289M$; $M_B = 0{\cdot}289M$.

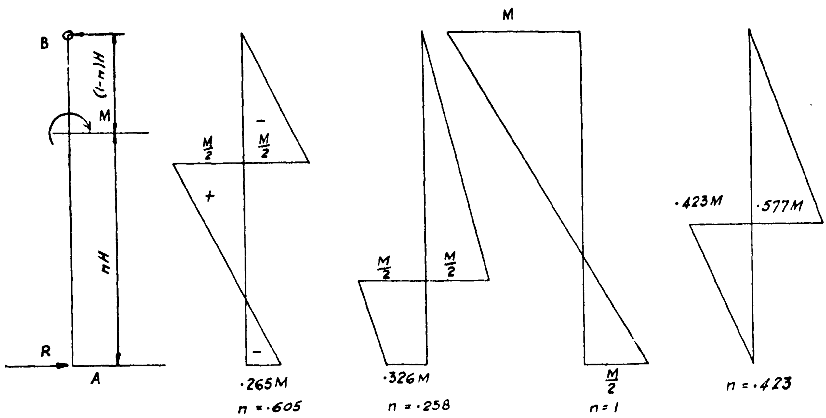


FIG. 8.13

In the limit when $n = 1$, $R = M/H$

and $$M_A = 0, \quad M_B = M.$$

For $n = \frac{1}{3}$, $$R = \frac{4M}{3H}$$

and $$M_A = 0.$$

$$M_B = \frac{M}{3}.$$

Case 3. One end hinged; one end fixed. Inertia constant throughout (Fig. 8.13).

$$M_B = 0.$$

$$M_A = RH - M.$$

Also $$\int_0^H M_x x\,dx = 0$$

and $$M_x = Rx - M$$

(second term applies when $x > H - h$).

$$\therefore \int_0^H Rx^2\,dx - \int_{H-h}^H Mx\,dx = 0.$$

$$\frac{RH^3}{3} - \frac{M}{2}[H^2 - (H^2 + h^2 - 2Hh)] = 0$$

or
$$\frac{RH^3}{3} = \frac{M}{2}[2Hh-h^2] = \frac{Mh}{2}[2H-h].$$

$$\therefore \quad R = \frac{3Mn}{2H}[2-n].$$

$$M_A = \frac{3Mn}{2}[2-n]-M$$

$$= \frac{M}{2}[6n-3n^2-2].$$

B.M. at bracket $= R(H-h)-M$

$$= \frac{3Mn}{2}[2-n][1-n]-M.$$

For max. value $= -\frac{M}{2}$

$$\frac{3Mn}{2}[2-n][1-n] = \frac{M}{2}$$

$$6n+3n^3-9n^2-1 = 0.$$

$$n = 0{\cdot}605 \quad \text{or} \quad 0{\cdot}258.$$

For $n = 0{\cdot}605$, $R = \frac{1{\cdot}28M}{H}$; $M_A = 0{\cdot}265M$.

,, $n = 0{\cdot}258$, $R = \frac{0{\cdot}675M}{H}$; $M_A = 0{\cdot}326M$.

,, $n = 1$, $R = \frac{3M}{2H}$; $M_A = 0{\cdot}5M$.

For $M_A = 0$,

$$n = 0{\cdot}423.$$

Then $R = M/H$ and moments at bracket $= \begin{cases} 0{\cdot}423M, \\ 0{\cdot}577M. \end{cases}$

Case 4. Fixed at base; hinged at top; inertia constant; eccentric load at nh from base (Fig. 8.13).

This is really the same as Case 3, with $n = 1$, i.e. $M_A = M/2$ and $R = 3M/2H$.

Other cases, such as eccentric load at top and bracket load in addition, can be solved by analysing the effect of each load separately as Cases 1, 2, or 3 and adding moments and reactions algebraically.

When the column is of varying moment of inertia, as often occurs in practice, the analysis is more difficult. In numerical cases distribution methods can be used with advantage. To illustrate the use of this method take a simple case, viz. a column 30 ft. high with fixed ends and a moment due to bracket loading of 60 ft.-tons at two-thirds of height. Referring to Case (2), $n = 0{\cdot}667$,

$$M_A = 60(\tfrac{1}{3}) = 20 \text{ in.-tons.}$$

$$M_B = 0.$$

In order to apply the distribution method apply a propping force at the point of application of the moment (Fig. 8.14). As $K_{BC}/K_{AC} = 2$, the balancing moments are $M_{AC} = 20$ ft.-tons, $M_{BC} = 40$ ft.-tons, and carrying over in the usual way $M_{AC} = 10$ ft.-tons and $M_{BC} = 20$ ft.-tons. It is now necessary to find the magnitude of the imaginary propping force.

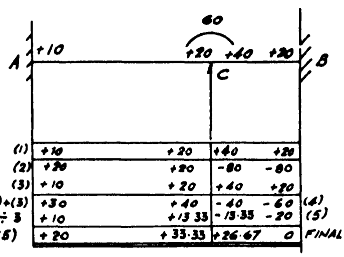


FIG. 8.14

The propping force due to moments

$$= \frac{40+20}{10} - \frac{10+20}{20} = 6 - 1{\cdot}5 = 4{\cdot}5.$$

If a force is applied at C the moments at ends of AC, BC, are proportional to I/l^2 values. Distributing these moments values are as Fig. 8.14 and corresponding force is

$$\frac{60+40}{10} + \frac{40+30}{20} = 13{\cdot}5.$$

Therefore ratio for reducing moments is

$$\frac{4{\cdot}5}{13{\cdot}5} = \frac{1}{3}$$

and final moments are

	AC	CA	CB	BC
Non-sway .	+10	+20	+40	+20
Sway . .	+10	+13·33	−13·33	−20
Final . .	+20	+33·33	+26·67	0

and $$R = 2{\cdot}667 \text{ tons.}$$

Check: Moment below bracket $= 60 \times x = 33{\cdot}33$.

Case 5. Column hinged at top and fixed at base with roof and bracket loads; varying moment of inertia (see Fig. 8.15).

Let moment of inertia of lower length $= I_1$,

,, ,, upper ,, $= I_2$,

and $$\frac{H}{H-h} = m,$$

also $$\frac{I_1 - I_2}{mI_2} = K.$$

From B to C: $\quad M_x = -Rx, \quad \frac{dM_x}{dR} = -x$

and $\quad U = \int \frac{M_x^2\,dx}{2EI}. \quad \therefore \quad EI\frac{dU}{dR} = \int \frac{dU}{dM_x}\frac{dM_x}{dR}\,dx$

$$= Rx^2\,dx.$$

$$\frac{dU}{dR} = \int\limits_0^{H-h} \frac{Rx^2}{EI_2}\,dx = \left[\frac{Rx^3}{3EI_2}\right]_0^{H-h} = \frac{R}{3EI_2}\times\frac{H^3}{m^3}.$$

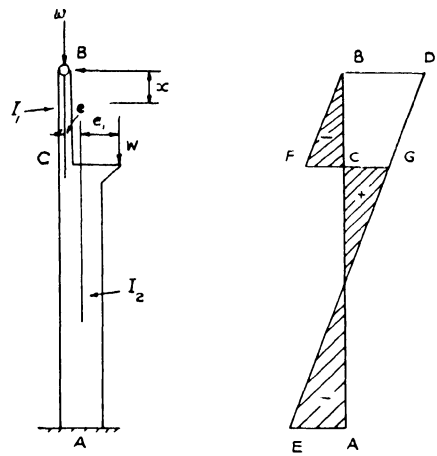


FIG. 8.15

From C to A:

$$M_x = M - Rx. \quad \therefore \quad \frac{dM_x}{dR} = -x.$$

$$\frac{dU}{dR} = \int\limits_{H-h}^{H} -\frac{Mx - Rx^2}{EI_1}\,dx$$

$$= -\frac{MH^2}{2EI_1}\left(\frac{m^2-1}{m^2}\right) + \frac{RH^3}{3EI_1}\left(\frac{m^3-1}{m^3}\right).$$

Since $$\sum \frac{dU}{dR} = 0,$$

$$R = \frac{M}{H}\times\frac{3(m^2-1)}{2(m^2+K)}$$

and $$M_A = M\times\frac{m^2-3-2K}{2(m^2+K)}.$$

Graphical construction for above case. Set off

$$BD = M$$

and $$AE = M\times\frac{m^2-3-2K}{2(m^2+K)}.$$

Join ED and draw BF parallel to ED, and complete B.M. diagram by drawing FCG horizontally.

Example. Let

$$H = 30 \text{ ft.} \quad \text{and} \quad H-h = 10 \text{ ft.}$$

Also $$I_1 = 2{,}000 \text{ in.}^4, \qquad I_2 = 1{,}000 \text{ in.}^4,$$

and $$M = 70 \text{ ft.-tons.}$$

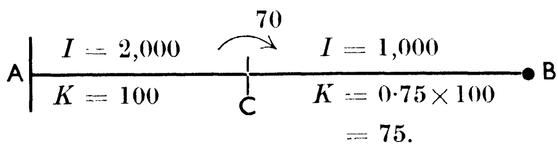
Then $$m = 3, \qquad K = \tfrac{1}{3}.$$

$$R = \frac{70}{30} \times \frac{24}{2\times 9\frac{1}{3}} = 3 \text{ tons.}$$

$$M_A = 70 \times \frac{5\frac{1}{3}}{2\times 9\frac{1}{3}} = 20 \text{ ft.-tons.}$$

$$\text{Moments at brackets} = \left.\begin{matrix} -30 \text{ ft.-tons} \\ +40 \quad ,, \end{matrix}\right\}.$$

Check by distribution method:



$$\text{Distribution of moment} \begin{cases} CA = 40 \text{ ft.-tons.} \\ CB = 30 \quad ,, \end{cases}$$

Then $$AC = 20 \quad ,,$$

Propping force at C

$$= \frac{40+20}{20} - \frac{30}{10}$$

$$= 3-3 = 0.$$

Therefore above moments are final and agree with those already calculated.

Column details

Lattice bracing to columns. Where columns are composed of two or more members braced together by diagonal bars, care must be taken to ensure that such bars are capable of resisting the lateral forces. These consist of any known lateral forces plus a percentage of the direct load on the column. This percentage is rather difficult to assess, and some specifications call for $2\frac{1}{2}$ per cent. of the direct load to be taken up by the bracings at any point. A more exact method is to take the difference in the working stresses p and F_2 given in B.S.S. 449, and multiply this value by the appropriate section modulus of the column to find the safe B.M. of the column

$$M = (F_2 - p)Z.$$

Now $$\frac{dM}{dx} = \text{rate of change of B.M.}$$

$$= \text{shear.}$$

Therefore if the analogy of a beam is taken, to find the lateral force, divide the B.M. found above by half the length of the equivalent pin-jointed strut. This force must be divided by 2 for single bracings on each face and by 4 for double bracings on each face and multiplied by the ratio $\frac{\text{diagonal length}}{\text{horizontal length}}$. In addition, diagonal bracings are subject to small secondary stresses due to the elastic shortening of the main column. It is well to be conservative in the design of column bracings, especially for comparatively short and heavy columns. Where diagonal bracings consist of single angles riveted to the main members by one leg only, it must be borne in mind that the effective area of such members is less than the full area. The usual rule is to take half the area of the free leg (this is a generous allowance according to research on such members) in addition to that of the riveted leg. In addition there is usually eccentric bending at the ends of such members. These remarks apply to all asymmetrical struts such as single or double angles connected by one leg only. It is worthy of note that reverse cleats are of little value in developing the strength of the member.†

Column bases. In recent years it has been common practice to use 'bloom' plates for column bases, and these must be thick enough to resist the bending moment due to the pressure on the concrete foundation.

If, then, B = dimension of base plate parallel to column flanges,

D = dimension of base plate parallel to web,

$\left.\begin{matrix} b \\ d \end{matrix}\right\}$ = corresponding dimensions of column,

f = working stress, usually 9 tons/in.^{2},

t = thickness of base plate,

W = load on column.

Then negative B.M. about Y–Y axis

$$= \frac{W}{2} \times \frac{b}{4} = \frac{Wb}{8} \quad \text{(from column).}$$

B.M. due to pressure under base

$$= \frac{W}{2} \times \frac{B}{4} = \frac{WB}{8}.$$

Net moment $$= \frac{WB}{8} - \frac{Wb}{8} = \frac{W}{8}(B-b).$$

† See B.S.S. 449.

The section modulus of base plates about Y–Y axis

$$= Dt^2/6,$$

and resisting moment $$= \frac{Dt^2}{6} \times f.$$

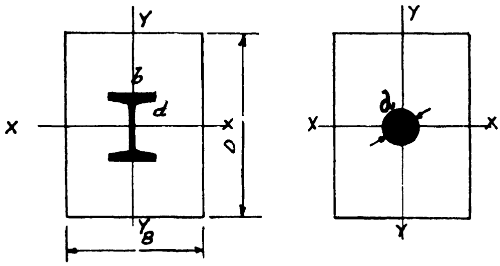


FIG. 8.16

$$\therefore \quad t^2 = \frac{3}{4} \times \frac{W}{f} \frac{(B-b)}{D}$$

$$t = \sqrt{\left\{\frac{3W(B-b)}{4fD}\right\}}.$$

Similarly $$t' = \sqrt{\left\{\frac{3W(D-d)}{4fB}\right\}},$$

and the greater value of t is taken.

For solid round columns:

let d = diameter of column,

then $$t = \sqrt{\left\{\frac{3W}{f} \frac{D}{(D-d)}\right\}} \text{ approx.}$$

For $D = B$, $$t = \sqrt{\left\{\frac{3W}{f} \frac{B}{(B-d)}\right\}} \text{ approx.}$$

(See Fig. 8.16.) The length or diameter should be not less than

$$\tfrac{3}{2}(d+3).\dagger$$

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† These formulae are as in B.S.S. 449. L.C.C. By-laws give rather more conservative results.

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EXERCISES

1. A compression member of a bridge truss consists of two web plates 18 in. $\times \frac{1}{2}$ in. with four 4 in. $\times$ 4 in. $\times \frac{3}{8}$ in. angles and a top flange plate 14 in. $\times \frac{1}{2}$ in., the overall width being 15 in. (the angles are inside the web plates). The effective length is 15 ft. Given that the area of one angle $= 2{\cdot}86$ in.^{2}, $I_{xx} = 4{\cdot}26$ in.^{4}, and distance from heel to C.G. $= 1{\cdot}12$ in., find safe load if $f = 9(1-0{\cdot}0038l/K)$. [290 tons approx.]

2. A strut is 25 ft. effective length and is composed of four angles $3\frac{1}{2}$ in. $\times 3\frac{1}{2}$ in. $\times \frac{3}{8}$ in. at the corners of a square of 18-in. side. The angles are laced on all sides. If the area of one angle $= 2{\cdot}49$ in.^{2} and $I_{xx} = 2{\cdot}80$ in.^{4}, $I_{min} = 1{\cdot}15$ in.^{4}; $I_{max} = 4{\cdot}45$ in.^{4} and distance to C.G. $= 1$ in., find the safe load by B.S.S. 449 formula and the theoretical spacing of the lacing bars. [67 tons; 25·5 in.]

3. (*a*) A column is hinged at the top and fixed at the base and is 25 ft. high. Find the horizontal reaction and the moment at the base and draw the B.M. diagram if a clockwise moment of 25 ft.-tons is applied at 5 ft. from the top.

(*b*) Also find reaction and moment if a moment of 25 ft.-tons is applied by means of a bracket 2 ft. 6 in. deep.

4. A column carries a load of 100 tons and is 14 in. square. The column base consists of a bloom resting on a R.C. foundation. If the permissible pressure on the concrete is 20 tons per sq. ft., find size and thickness of the base. [2 ft. 3 in. sq. $\times$ 2 in.]

5. A concrete column carries a load of 150 tons and the story height is 15 ft. The ends may be considered as fixed. Using a safe stress of 680 lb./in.^{2} in the concrete, find the effective length, the size of column required, and the main and secondary reinforcement. [135 in.; 20 in. sq.; four $1\frac{1}{2}$-in. diam.; $\frac{1}{2}$-in. stirrups at 9-in. centres.]

6. A R.C. column is 18 in. square and is reinforced by four $1\frac{1}{8}$-in. diam. bars. The cover to the bars is $1\frac{1}{2}$ in. The column carries a B.M. in addition to the direct load. If the stress in the concrete must not exceed 800 lb./in.^{2}, find the safe load (if $e = 3$ in. and $m = 15$), and indicate the amount of secondary steel required. [152,000 lb.; 1·3 in.^{2} per ft.]

7. A timber strut consists of a 10 in. $\times$ 10 in. post 15 ft. high and fixed at each end. Find the safe load according to the formula

$$f = C[1-\tfrac{1}{3}(l/Kd)^4], \qquad C = 1{,}000 \text{ lb./in.}^2$$

[98,000 lb.]

8. A C.I. column is 10 in. external and 8 in. internal diameter, but the core is $\frac{1}{2}$ in. eccentric. The column is 10 ft. effective height. Find the radius of gyration and the crippling load according to Euler formula ($E = 6{,}000$ tons/in.^{2}). [3·00 in.; 1,040 tons.]

APPENDIX A

STEELWORK

Table I. *British Standard Joists: Dimensions and Properties*

Size ($D \times B$)	*Weight* (*lb./ft.*)	*Area* (*in.*^{2})	*Standard thicknesses* *Web* (*in.*)	*Flange* (*in.*)	*Moments of inertia* X–X	Y–Y	*Moduli of section* X–X	Y–Y	*Radii of gyration (in.)* X–X	Y–Y
24″ × 7½″	95	27·94	0·57	1·011	2533·04	62·54	211·09	16·68	9·52	1·50
22″ × 7″	75	22·06	0·50	0·834	1676·80	41·07	152·44	11·73	8·72	1·36
20″ × 7½″	89	26·19	0·60	1·010	1672·85	62·54	167·29	16·68	7·99	1·55
20″ × 6½″	65	19·12	0·45	0·820	1226·17	32·56	122·62	10·02	8·01	1·31
18″ × 8″	80	23·53	0·50	0·950	1292·07	69·43	143·56	17·36	7·41	1·72
18″ × 7″	75	22·09	0·55	0·928	1151·18	46·56	127·91	13·30	7·22	1·45
18″ × 6″	55	16·18	0·42	0·757	841·76	23·64	93·53	7·88	7·21	1·21
16″ × 8″	75	22·06	0·48	0·938	973·91	68·30	121·74	17·08	6·64	1·76
16″ × 6″	62	18·21	0·55	0·847	725·05	27·14	90·63	9·05	6·31	1·22
16″ × 6″	50	14·71	0·40	0·726	618·09	22·47	77·26	7·49	6·48	1·24
15″ × 6″	45	13·24	0·38	0·655	491·91	19·87	65·59	6·62	6·10	1·23
15″ × 5″	42	12·36	0·42	0·647	428·49	11·81	57·13	4·72	5·89	0·98
14″ × 8″	70	20·59	0·46	0·920	705·58	66·67	100·80	16·67	5·85	1·80
14″ × 6″	57	16·78	0·50	0·873	533·34	27·94	76·19	9·31	5·64	1·29
14″ × 6″	46	13·59	0·40	0·698	442·57	21·45	63·22	7·15	5·71	1·26
13″ × 5″	35	10·30	0·35	0·604	283·51	10·82	43·62	4·33	5·25	1·03
12″ × 8″	65	19·12	0·43	0·904	487·77	65·18	81·30	16·30	5·05	1·85
12″ × 6″	54	15·89	0·50	0·883	375·77	28·28	62·63	9·43	4·86	1·33
12″ × 6″	44	13·00	0·40	0·717	316·76	22·12	52·79	7·37	4·94	1·30
12″ × 5″	32	9·45	0·35	0·550	221·07	9·69	36·84	3·88	4·84	1·01
10″ × 8″	55	16·18	0·40	0·783	288·69	54·74	57·74	13·69	4·22	1·84
10″ × 6″	40	11·77	0·36	0·709	204·80	21·76	40·96	7·25	4·17	1·36
10″ × 5″	30	8·85	0·36	0·552	146·23	9·73	29·25	3·89	4·06	1·05
10″ × 4½″	25	7·35	0·30	0·505	122·34	6·49	24·47	2·88	4·08	0·94
9″ × 7″	50	14·71	0·40	0·825	208·13	40·17	46·25	11·48	3·76	1·65
9″ × 4″	21	6·18	0·30	0·457	81·13	4·15	18·03	2·07	3·62	0·82
8″ × 6″	35	10·30	0·35	0·648	115·06	19·54	28·76	6·51	3·34	1·38
8″ × 5″	28	8·28	0·35	0·575	89·69	10·19	22·42	4·08	3·29	1·11
8″ × 4″	18	5·30	0·28	0·398	55·63	3·51	13·91	1·75	3·24	0·81
7″ × 4″	16	4·75	0·25	0·387	39·51	3·37	11·29	1·69	2·89	0·84
6″ × 5″	25	7·37	0·41	0·520	43·69	9·10	14·56	3·64	2·44	1·11
6″ × 4½″	20	5·89	0·37	0·431	34·71	5·40	11·57	2·40	2·43	0·96
6″ × 3″	12	3·53	0·23	0·377	20·99	1·46	7·00	0·97	2·44	0·64
5″ × 4½″	20	5·88	0·29	0·513	25·03	6·59	10·01	2·93	2·06	1·06
5″ × 3″	11	3·26	0·22	0·376	13·68	1·45	5·47	0·97	2·05	0·67
4¾″ × 1¾″	6·5	1·91	0·18	0·325	6·73	0·26	2·83	0·30	1·88	0·37
4″ × 3″	10	2·94	0·24	0·347	7·79	1·33	3·89	0·88	1·63	0·67
4″ × 1¾″	5	1·47	0·17	0·239	3·66	0·19	1·83	0·21	1·58	0·36
3″ × 3″	8·5	2·52	0·20	0·332	3·81	1·25	2·54	0·83	1·23	0·70
3″ × 1½″	4	1·18	0·16	0·249	1·66	0·13	1·11	0·17	1·19	0·33

Table II. *British Standard Channels: Dimensions and Properties*

Size	*Weight*	*Area*	*Distance of neutral axis from toe*	*Standard thicknesses*		*Moments of inertia*		*Moduli of section*		*Radii of gyration (in.)*	
(D × B)	*(lb./ft.)*	*(in.²)*	*(in.)*	*Web (in.)*	*Flange (in.)*	*X–X*	*Y–Y*	*X–X*	*Y–Y*	*X–X*	*Y–Y*
17″ × 4″	51·28	15·08	3·21	0·60	0·68	569·31	16·96	66·98	5·28	6·14	1·06
17″ × 4″	44·34	13·04	3·08	0·48	0·68	520·18	15·26	61·20	4·96	6·32	1·08
15″ × 4″	42·49	12·50	3·18	0·53	0·62	382·85	14·97	51·05	4·71	5·54	1·09
15″ × 4″	36·37	10·70	3·03	0·41	0·62	349·10	13·34	46·55	4·40	5·71	1·12
13″ × 4″	38·92	11·45	3·12	0·53	0·62	270·66	14·51	41·64	4·64	4·86	1·13
13″ × 4″	33·18	9·76	2·96	0·40	0·62	246·86	12·76	37·98	4·31	5·03	1·14
12″ × 4″	36·63	10·77	3·11	0·53	0·60	218·81	13·80	36·47	4·44	4·51	1·13
12″ × 4″	31·33	9·21	2·94	0·40	0·60	200·09	12·12	33·35	4·12	4·66	1·15
12″ × 3½″	30·45	8·96	2·79	0·48	0·50	174·13	7·96	29·02	2·86	4·41	0·94
12″ × 3½″	26·37	7·76	2·67	0·38	0·50	159·73	7·15	26·62	2·68	4·54	0·96
11″ × 3½″	30·52	8·98	2·69	0·48	0·58	152·96	8·86	27·81	3·30	4·13	0·99
11″ × 3½″	26·78	7·88	2·57	0·38	0·58	141·87	7·93	25·80	3·09	4·24	1·00
10″ × 3½″	28·54	8·39	2·68	0·48	0·56	119·52	8·50	23·90	3·17	3·77	1·01
10″ × 3½″	24·46	7·19	2·53	0·36	0·56	109·52	7·42	21·90	2·93	3·90	1·02
10″ × 3″	21·33	6·27	2·33	0·38	0·45	87·66	4·31	17·53	1·85	3·74	0·83
10″ × 3″	19·28	5·67	2·26	0·32	0·45	82·66	3·98	16·53	1·76	3·82	0·84
9″ × 3½″	25·63	7·54	2·64	0·45	0·54	89·30	7·86	19·84	2·98	3·44	1·02
9″ × 3½″	23·49	6·91	2·55	0·38	0·54	85·05	7·26	18·90	2·85	3·51	1·03
9″ × 3½″	22·27	6·55	2·50	0·34	0·54	82·62	6·90	18·36	2·76	3·55	1·03
9″ × 3″	19·91	5·86	2·32	0·38	0·44	67·38	4·18	14·97	1·80	3·39	0·85
9″ × 3″	17·46	5·14	2·22	0·30	0·44	62·52	3·75	13·89	1·69	3·49	0·86
8″ × 3½″	23·20	6·82	2·60	0·43	0·52	65·27	7·30	16·32	2·81	3·09	1·03
8″ × 3½″	20·21	5·94	2·45	0·32	0·52	60·57	6·37	15·14	2·60	3·19	1·04
8″ × 3″	18·68	5·49	2·29	0·38	0·44	50·99	4·11	12·75	1·79	3·05	0·87
8″ × 3″	15·96	4·69	2·17	0·28	0·44	46·72	3·58	11·68	1·65	3·16	0·87
7″ × 3½″	20·18	5·94	2·51	0·38	0·50	45·12	6·48	12·89	2·58	2·76	1·05
7″ × 3½″	18·28	5·38	2·41	0·30	0·50	42·83	5·83	12·24	2·42	2·82	1·04
7″ × 3″	17·07	5·02	2·28	0·38	0·42	36·18	3·87	10·34	1·70	2·68	0·88
7″ × 3″	14·22	4·18	2·12	0·26	0·42	32·75	3·26	9·36	1·53	2·80	0·88
6″ × 3½″	18·52	5·45	2·49	0·38	0·48	30·68	6·05	10·23	2·43	2·37	1·05
6″ × 3½″	16·48	4·85	2·36	0·28	0·48	28·88	5·29	9·63	2·25	2·44	1·05
6″ × 3″	17·53	5·16	2·15	0·43	0·48	27·18	3·95	9·06	1·84	2·30	0·88
6″ × 3″	16·51	4·86	2·09	0·38	0·48	26·28	3·70	8·76	1·77	2·33	0·87
6″ × 3″	13·64	4·01	2·19	0·31	0·38	22·35	3·10	7·45	1·42	2·36	0·88
6″ × 3″	12·41	3·65	2·11	0·25	0·38	21·27	2·83	7·09	1·34	2·41	0·88
5″ × 2½″	11·24	3·31	1·80	0·31	0·38	12·50	1·82	5·00	1·01	1·94	0·74
5″ × 2½″	10·22	3·01	1·73	0·25	0·38	11·87	1·64	4·75	0·95	1·99	0·74
4″ × 2″	7·91	2·33	1·47	0·30	0·31	5·38	0·79	2·69	0·94	1·52	0·58
4″ × 2″	7·09	2·09	1·40	0·24	0·31	5·06	0·70	2·53	0·50	1·56	0·58
3″ × 1½″	5·11	1·50	1·07	0·25	0·28	1·94	0·30	1·29	0·28	1·14	0·44
3″ × 1½″	4·60	1·35	1·02	0·20	0·28	1·82	0·26	1·22	0·26	1·16	0·44

Table III. *British Standard Equal Angles: Dimensions and Properties*

Size $D \times B \times t$ (*in.*)	*Weight* (*lb./ft.*)	*Area* (*in.*^{2})	*Distance of N.A. from toe* (*in.*)	*Radii of gyration* (*in.*) *Axis X–X Axis Y–Y*	*Max.*	*Min.*
$8 \times 8 \times 1$	51·01	15·00	5·65	2·42	3·05	1·56
,, $\frac{7}{8}$	45·00	13·24	5·70	2·43	3·07	1·56
,, $\frac{3}{4}$	38·89	11·44	5·75	2·45	3·09	1·57
,, $\frac{5}{8}$	32·68	9·61	5·80	2·46	3·11	1·57
$7 \times 7 \times \frac{7}{8}$	39·05	11·48	4·94	2·12	2·67	1·36
,, $\frac{3}{4}$	33·79	9·94	4·99	2·13	2·69	1·37
,, $\frac{5}{8}$	28·42	8·36	5·04	2·14	2·71	1·37
,, $\frac{1}{2}$	22·95	6·75	5·09	2·16	2·72	1·38
$6 \times 6 \times \frac{7}{8}$	33·10	9·73	4·19	1·80	2·26	1·16
,, $\frac{3}{4}$	28·69	8·44	4·24	1·81	2·28	1·17
,, $\frac{5}{8}$	24·17	7·11	4·29	1·83	2·30	1·17
,, $\frac{1}{2}$	19·55	5·75	4·34	1·84	2·32	1·18
,, $\frac{3}{8}$	14·82	4·36	4·39	1·85	2·34	1·18
$5 \times 5 \times \frac{3}{4}$	23·59	6·94	3·49	1·50	1·88	0·97
,, $\frac{5}{8}$	19·93	5·86	3·53	1·51	1·90	0·97
,, $\frac{1}{2}$	16·16	4·75	3·58	1·52	1·92	0·98
,, $\frac{3}{8}$	12·28	3·61	3·63	1·54	1·94	0·98
$4\frac{1}{2} \times 4\frac{1}{2} \times \frac{3}{4}$	21·04	6·19	3·11	1·34	1·68	0·87
,, $\frac{5}{8}$	17·80	5·24	3·16	1·35	1·70	0·87
,, $\frac{1}{2}$	14·45	4·25	3·21	1·37	1·72	0·88
,, $\frac{3}{8}$	11·00	3·24	3·26	1·38	1·74	0·88
$4 \times 4 \times \frac{3}{4}$	18·49	5·44	2·74	1·18	1·48	0·77
,, $\frac{5}{8}$	15·68	4·61	2·78	1·19	1·50	0·77
,, $\frac{1}{2}$	12·75	3·75	2·83	1·21	1·52	0·78
,, $\frac{3}{8}$	9·73	2·86	2·88	1·22	1·54	0·78
$3\frac{1}{2} \times 3\frac{1}{2} \times \frac{5}{8}$	13·55	3·99	2·41	1·03	1·30	0·68
,, $\frac{1}{2}$	11·05	3·25	2·45	1·05	1·32	0·68
,, $\frac{3}{8}$	8·45	2·49	2·50	1·06	1·34	0·68
,, $\frac{5}{16}$	7·11	2·09	2·53	1·07	1·35	0·68
$3 \times 3 \times \frac{1}{2}$	9·35	2·75	2·41	1·03	1·12	0·58
,, $\frac{3}{8}$	7·17	2·11	2·45	1·05	1·14	0·58
,, $\frac{5}{16}$	6·04	1·78	2·50	1·06	1·15	0·58
,, $\frac{1}{4}$	4·89	1·44	2·53	1·07	1·15	0·59
$2\frac{1}{2} \times 2\frac{1}{2} \times \frac{3}{8}$	11·8	3·47	1·75	0·74	0·94	0·48
,, $\frac{5}{16}$	10·0	2·93	1·77	0·75	0·95	0·48
,, $\frac{1}{4}$	8·1	2·38	1·80	0·76	0·95	0·49
$2\frac{1}{4} \times 2\frac{1}{4} \times \frac{3}{8}$	10·5	3·09	1·56	0·67	0·84	0·43
,, $\frac{5}{16}$	8·9	2·62	1·58	0·67	0·85	0·43
,, $\frac{1}{4}$	7·2	2·13	1·61	0·68	0·85	0·44
,, $\frac{3}{16}$	5·5	1·62	1·63	0·68	0·86	0·44
$2 \times 2 \times \frac{5}{16}$	7·8	2·30	1·39	0·59	0·75	0·38
,, $\frac{1}{4}$	6·4	1·87	1·42	0·60	0·75	0·39
,, $\frac{3}{16}$	4·9	1·43	1·44	0·60	0·76	0·39

TABLE IV. *British Standard Unequal Angles: Dimensions and Properties*

Size D × B × t (in.)	*Weight (lb./ft.)*	*Area (in.²)*	*Distances of N.A. from toe (in.)*		*Radii of gyration (in.)*			
			Axis X–X	*Axis Y–Y*	*Axis X–X*	*Axis Y–Y*	*Max.*	*Min.*
9 × 4 × 7/8	36·07	10·61	5·57	3·05	2·85	1·00	2·90	0·83
,, 3/4	31·24	9·19	5·62	3·10	2·87	1·01	2·92	0·83
,, 5/8	26·30	7·73	5·67	3·15	2·88	1·02	2·94	0·83
,, 1/2	21·25	6·25	5·73	3·20	2·90	1·03	2·96	0·84
8 × 6 × 7/8	39·05	11·48	5·41	4·40	2·49	1·73	2·75	1·27
,, 3/4	33·79	9·94	5·46	4·45	2·51	1·74	2·77	1·28
,, 5/8	28·42	8·36	5·51	4·50	2·52	1·75	2·79	1·28
,, 1/2	22·95	6·75	5·56	4·55	2·54	1·77	2·81	1·29
8 × 4 × 3/4	28·69	8·44	5·07	3·06	2·54	1·04	2·61	0·84
,, 5/8	24·17	7·11	5·12	3·10	2·55	1·05	2·63	0·85
,, 1/2	19·55	5·75	5·17	3·15	2·57	1·06	2·65	0·85
8 × 3½ × 5/8	23·11	6·80	5·00	2·73	2·55	0·88	2·60	0·72
,, 1/2	18·70	5·50	5·05	2·78	2·57	0·89	2·62	0·73
7 × 4 × 3/4	26·14	7·69	4·51	3·00	2·21	1·07	2·30	0·85
,, 5/8	22·05	6·48	4·56	3·05	2·22	1·09	2·32	0·86
,, 1/2	17·85	5·25	4·61	3·10	2·24	1·10	2·34	0·86
7 × 3½ × 5/8	20·99	6·17	4·45	2·69	2·22	0·91	2·29	0·74
,, 1/2	17·00	5·00	4·50	2·74	2·24	0·92	2·31	0·74
,, 3/8	12·91	3·80	4·56	2·79	2·25	0·93	2·32	0·75
6 × 4 × 3/4	23·59	6·94	3·94	2·93	1·87	1·11	2·00	0·85
,, 5/8	19·93	5·86	3·98	2·98	1·88	1·12	2·02	0·86
,, 1/2	16·16	4·75	4·03	3·03	1·90	1·13	2·04	0·86
6 × 3½ × 5/8	18·86	5·55	3·89	2·63	1·89	0·95	1·98	0·75
,, 1/2	15·30	4·50	3·94	2·68	1·91	0·96	2·00	0·75
,, 3/8	11·63	3·42	3·99	2·73	1·92	0·97	2·01	0·76
6 × 3 × 5/8	17·80	5·24	3·78	2·27	1·89	0·77	1·95	0·63
,, 1/2	14·45	4·25	3·83	2·32	1·91	0·78	1·97	0·63
,, 3/8	11·00	3·24	3·88	2·37	1·93	0·80	1·98	0·64
5 × 4 × 5/8	17·80	5·24	3·39	2·89	1·54	1·16	1·74	0·84
,, 1/2	14·45	4·25	3·44	2·94	1·56	1·17	1·76	0·84
,, 3/8	11·00	3·24	3·49	2·99	1·57	1·18	1·77	0·85
5 × 3½ × 5/8	16·74	4·92	3·31	2·56	1·55	0·98	1·68	0·74
,, 1/2	13·61	4·00	3·36	2·60	1·57	0·99	1·70	0·75
,, 3/8	10·37	3·05	3·41	2·65	1·58	1·01	1·72	0·75
5 × 3 × 1/2	12·75	3·75	3·37	2·26	1·58	0·82	1·66	0·64
,, 3/8	9·73	2·86	3·32	2·31	1·59	0·83	1·68	0·65
,, 5/16	8·17	2·40	3·34	2·33	1·60	0·84	1·68	0·65
4½ × 3 × 1/2	23·80	7·00	2·98	2·22	1·41	0·84	1·51	0·64
,, 3/8	18·20	5·35	3·03	2·27	1·42	0·85	1·53	0·64
,, 5/16	15·30	4·50	3·06	2·30	1·43	0·85	1·53	0·65
4 × 3½ × 5/8	29·20	8·60	2·71	2·46	1·21	1·02	1·41	0·71
,, 1/2	23·80	7·00	2·76	2·51	1·22	1·03	1·43	0·72
,, 3/8	18·20	5·35	2·81	2·56	1·24	1·04	1·45	0·72
,, 5/16	15·30	4·50	2·84	2·58	1·24	1·05	1·46	0·72
4 × 3 × 1/2	22·10	6·50	2·68	2·18	1·24	0·85	1·36	0·63
,, 3/8	16·90	4·97	2·73	2·23	1·25	0·87	1·38	0·64
,, 5/16	14·20	4·18	2·76	2·25	1·26	0·87	1·39	0·64
4 × 2½ × 3/8	15·60	4·60	2·64	1·89	1·26	0·69	1·34	0·53
,, 5/16	13·20	3·87	2·67	1·91	1·27	0·70	1·34	0·54
,, 1/4	10·60	3·13	2·70	1·94	1·27	0·70	1·35	0·54
3½ × 3 × 1/2	20·40	6·00	2·38	1·63	1·06	0·87	1·23	0·61
,, 3/8	15·60	4·60	2·43	1·68	1·08	0·88	1·25	0·62
,, 5/16	13·20	3·87	2·46	1·71	1·08	0·89	1·26	0·62
3½ × 2½ × 3/8	14·30	4·22	2·35	1·85	1·09	0·71	1·19	0·53
,, 5/16	12·10	3·56	2·38	1·87	1·10	0·71	1·20	0·53
,, 1/4	9·80	2·88	2·41	1·90	1·10	0·72	1·20	0·54
3 × 2½ × 3/8	13·10	3·85	2·06	1·80	0·92	0·73	1·05	0·52
,, 5/16	11·00	3·24	2·08	1·83	0·93	0·73	1·06	0·52
,, 1/4	8·90	2·63	2·11	1·85	0·93	0·74	1·07	0·52
3 × 2 × 3/8	11·80	3·47	1·97	1·47	0·93	0·55	1·00	0·42
,, 5/16	10·00	2·93	2·00	1·49	0·94	0·56	1·00	0·43
,, 1/4	8·10	2·38	2·02	1·52	0·94	0·56	1·01	0·43
2½ × 2 × 3/8	10·50	3·09	1·68	1·43	0·76	0·57	0·85	0·41
,, 5/16	8·90	2·62	1·70	1·45	0·77	0·57	0·86	0·42
,, 1/4	7·20	2·13	1·73	1·47	0·77	0·58	0·87	0·42
,, 3/16	5·50	1·62	1·75	1·50	0·78	0·58	0·88	0·42
2½ × 1½ × 1/4	6·40	1·87	1·64	1·13	0·78	0·40	0·82	0·32
,, 3/16	4·90	1·43	1·67	1·16	0·79	0·41	0·83	0·32
2 × 1½ × 1/4	5·50	1·63	1·35	1·09	0·61	0·42	0·68	0·31
,, 3/16	4·20	1·24	1·67	1·12	0·62	0·43	0·68	0·32

TABLE V. *British Standard Tees: Dimensions and Properties*

Size $B \times D \times t$ (in.)	Weight (lb./ft.)	Area (in.²)	Dist. from heel to N.A. (in.)	Moments of inertia		Radii of gyration	
				I_{xx} (in.⁴)	I_{yy} (in.⁴)	Axis X–X (in.)	Axis Y–Y (in.)
1½×1½×¼	2·36	0·69	0·46	0·14	0·07	0·44	0·31
2×2×¼	3·21	0·94	0·58	0·34	0·16	0·60	0·41
2½×2½×¼	4·07	1·20	0·70	0·68	0·30	0·75	0·50
,, ⅜	5·92	1·74	0·75	0·96	0·47	0·74	0·52
3×3×⅜	7·20	2·12	0·87	1·71	0·81	0·90	0·62
4×3×⅜	8·49	2·50	0·77	1·86	1·91	0·86	0·87
,, ½	11·09	3·26	0·82	2·37	2·60	0·85	0·89
4×4×⅜	9·77	2·87	1·10	4·19	1·90	1·21	0·81
,, ½	12·79	3·76	1·16	5·40	2·59	1·20	0·83
5×3×⅜	9·79	2·88	0·69	1·97	3·72	0·83	1·14
,, ½	12·80	3·77	0·74	2·51	5·04	0·82	1·16
5×4×⅜	11·06	3·25	1·00	4·47	3·70	1·17	1·07
,, ½	14·50	4·27	1·05	5·77	5·02	1·16	1·09
6×3×⅜	11·08	3·26	0·63	2·06	6·40	0·80	1·40
,, ½	14·52	4·27	0·68	2·63	8·67	0·78	1·42
6×4×½	16·22	4·77	0·97	6·07	8·64	1·13	1·35
,, ⅝	19·99	5·88	1·02	7·33	10·93	1·12	1·36
6×6×½	19·62	5·77	1·63	19·04	8·56	1·82	1·22
,, ⅝	24·23	7·13	1·69	23·31	10·87	1·81	1·23

APPENDIX B

REINFORCED CONCRETE

TABLE I. *Ordinary Concrete* (*m* = 15 *throughout*)

Concrete mix	Working stresses (lb./in.²)					Design factors				
	Steel tension (*t*)	Concrete								
		Bending (*c*)	Shear (*s*)	Bond	Comp.	*t*/*c*	n_1	a_1	*Q*	*r*
1:2:4	18,000	750	75	100	600	24·00	0·385	0·872	125·7	0·008
	20,000					26·67	0·360	0·880	119·0	0·00675
	25,000					33 30	0·312	0·896	105·0	0·00467
	27,000					36·00	0·294	0·902	99·4	0·00408
	18,000	1,000	100	B_1 120, B_2 180	760	18	0·455	0·848	193	0·0126
	20,000					20	0·428	0·857	183·7	0·0107
	25,000					25	0·375	0·875	164	0·0075
	27,000					27	0·357	0·881	157	0·0066
1.1½:3	18,000	850	85	110	680	21·20	0·414	0·862	151·5	0·00975
	20,000					23·55	0·390	0·870	143·7	0·00827
	25,000					29·40	0·338	0·887	127·5	0·00575
	27,000					31·70	0·321	0·893	121·8	0·00505
	18,000	1,250	115	B_1 135, B_2 200	950	14·4	0·511	0·830	264·4	0·0177
	20,000					16	0·483	0·839	253·2	0·0150
	25,000					20	0·428	0·857	229·9	0·0107
	27,000					21·6	0·411	0·863	221·7	0·0095
1:1:2	18,000	975	98	123	780	18·50	0·447	0·851	185·7	0·0121
	20,000					20·50	0·423	0·859	177·0	0·0103
	25,000					25·70	0·369	0·877	158·0	0·0072
	27,000					27·70	0·351	0·883	151·5	0·00633
	18,000	1,500†	130	B_1 150, B_2 200	1,140	12	0·556	0·815	339·6	0·0232
	20,000					13·33	0·529	0·824	328·7	0·0198
	25,000					16·67	0·473	0·842	299·6	0·0142
	27,000					18	0·455	0·848	289·2	0·0126

† Same values for High Alumina Cement Concrete, 1:2:4 mix.

For other values of *t*, *m*, and *c*

$$\frac{t}{mc} = \frac{1}{n_1} - 1; \qquad a_1 = 1 - \frac{n_1}{3}; \qquad Q = \frac{c}{2} \times n_1 a_1; \qquad r = \frac{c}{2} \times \frac{n_1}{t}.$$

TABLE II. *Vibrated Concrete* (*m* = 15 *throughout*)

Concrete mix	Working stresses (lb./in.²)					Design factors				
	Steel tension (*t*)	Concrete								
		Bending (*c*)	Shear (*s*)	Bond	Comp.	*t*/*c*	n_1	a_1	*Q*	*r*
1:2:4	18,000	1,100	110	B_1 132, B_2 198	836	16·36	0·479	0·840	221·4	0·0146
	20,000					18·18	0·453	0·849	211·3	0·0124
	25,000					22·73	0·398	0·867	190	0·0088
	27,000					24·55	0·379	0·874	182·8	0·0077
1:1½:3	18,000	1,375	127	B_1 149, B_2 220	1,045	13·07	0·535	0·822	303	0·0204
	20,000					14·54	0·508	0·831	290	0·0175
	25,000					18·18	0·453	0·849	264·1	0·0124
	27,000					19·61	0·433	0·856	254·8	0·0110
1:1:2	18,000	1,650	143	B_1 165, B_2 220	1,254	10·91	0·578	0·807	385	0·0264
	20,000					12·11	0·553	0·816	372	0·0228
	25,000					15·15	0·498	0·834	342·4	0·0165
	27,000					16·35	0·478	0·841	332·0	0·0146

TABLE III. *Concrete: Variable Values of m*

	Concrete mix	Modular ratio m	Working stresses (lb./in.²) Steel tension (t)	Concrete: Bending (c)	Concrete: Shear (s)	Concrete: Bond	Concrete: Comp.	Design factors: t/c	n_1	a_1	Q	r
ORDINARY CONCRETE	1:2:4	18	18,000	750	75	100	600	24·00	0·43	0·86	138	0·009
			20,000					26·67	0·405	0·865	131·3	0·0076
			25,000					33·30	0·351	0·883	116	0·0053
			27,000					36·00	0·333	0·889	111	0·0046
	1:1½:3	16	18,000	850	85	110	680	21·15	0·43	0·86	156	0·0101
			20,000					23·50	0·405	0·865	148·5	0·0086
			25,000					29·40	0·353	0·882	132·5	0·0060
			27,000					31·75	0·334	0·889	125·8	0·0052
	1:1:2	14	18,000	975	98	123	780	18·48	0·43	0·86	180	0·0116
			20,000					20·55	0·404	0·865	170	0·0098
			25,000					25·65	0·354	0·882	152	0·0069
			27,000					27·70	0·336	0·889	145	0·0061
HIGH-GRADE CONCRETE	1:2:4	14	18,000	950	95	120	760	18·97	0·425	0·858	173	0·0112
			20,000					21·05	0·400	0·867	165	0·0095
			25,000					26·35	0·347	0·884	145·5	0·0066
			27,000					28·45	0·333	0·889	140	0·0059
	1:1½:3	12	18,000	1,100	110	135	880	16·37	0·423	0·859	200	0·0129
			20,000					18·17	0·398	0·867	189·5	0·0110
			25,000					22·70	0·346	0·885	168·5	0·0076
			27,000					24·55	0·333	0·889	162·8	0·0068
	1:1:2	11	18,000	1,250	125	150	1,000	14·40	0·433	0·856	231·5	0·0015
			20,000					16·0	0·408	0·864	220	0·0013
			25,000					20·0	0·356	0·881	196	0·0009
			27,000					21·60	0·338	0·887	187·5	0·0008

TABLE IV. *Areas of Round Bars (in.²)*

Bar Diam. (in.)	Number of bars: 1	2	3	4	5	6	7	8	9	10	Bar Diam. (in.)
1/4	0·049	0·098	0·147	0·196	0·245	0·294	0·343	0·392	0·441	0·491	1/4
5/16	0·076	0·153	0·230	0·306	0·385	0·460	0·536	0·613	0·690	0·767	5/16
3/8	0·110	0·220	0·331	0·441	0·552	0·662	0·772	0·883	0·993	1·104	3/8
7/16	0·150	0·300	0·450	0·601	0·751	0·901	1·052	1·202	1·352	1·503	7/16
1/2	0·196	0·392	0·588	0·785	0·981	1·177	1·374	1·570	1·766	1·963	1/2
9/16	0·248	0·497	0·745	0·994	1·242	1·491	1·739	1·988	2·236	2·485	9/16
5/8	0·306	0·613	0·920	1·227	1·534	1·840	2·147	2·454	2·761	3·068	5/8
11/16	0·371	0·742	1·113	1·484	1·856	2·227	2·598	2·969	3·340	3·712	11/16
3/4	0·441	0·883	1·325	1·767	2·209	2·650	3·092	3·534	3·976	4·418	3/4
13/16	0·518	1·037	1·555	2·074	2·592	3·111	3·629	4·148	4·665	5·185	13/16
7/8	0·601	1·202	1·803	2·405	3·006	3·607	4·209	4·816	5·411	6·013	7/8
15/16	0·690	1·380	2·070	2·761	3·451	4·141	4·832	5·522	6·212	6·903	15/16
1	0·785	1·570	2·356	3·142	3·927	4·712	5·497	6·285	7·068	7·854	1
1 1/8	0·994	1·988	2·982	3·976	4·970	5·964	6·958	7·952	8·946	9·940	1 1/8
1 1/4	1·227	2·454	3·681	4·908	6·136	7·363	8·590	9·817	11·044	12·272	1 1/4
1 3/8	1·484	2·969	4·454	5·939	7·424	8·909	10·394	11·879	13·364	14·849	1 3/8
1 1/2	1·767	3·534	5·301	7·068	8·835	10·602	12·369	14·136	15·903	17·671	1 1/2
1 5/8	2·073	4·147	6·221	8·295	10·369	12·443	14·517	16·591	18·665	20·739	1 5/8
1 3/4	2·405	4·810	7·215	9·621	12·026	14·431	16·837	19·242	21·647	24·053	1 3/4
1 7/8	2·761	5·522	8·283	11·044	13·806	16·567	19·328	22·089	24·850	27·612	1 7/8
2	3·142	6·283	9·424	12·566	15·708	18·849	21·991	25·132	28·274	31·416	2

Table V. *Areas of Round Bars (in in.² per ft. width)*

Bar Diam. (in.)	3	3½	4	4½	5	5½	6	6½	7	7½	8	8½	9	10	11	12
3/16	0·110	0·095	0·083	0·074	0·066	0·06	0·055	0·051	0·047	0·044	0·041	0·039	0·037	0·033	0·030	0·028
1/4	0·196	0·168	0·147	0·13	0·118	0·107	0·098	0·091	0·084	0·079	0·074	0·069	0·065	0·059	0·054	0·049
5/16	0·307	0·263	0·230	0·204	0·184	0·167	0·153	0·142	0·132	0·123	0·115	0·108	0·102	0·092	0·084	0·077
3/8	0·442	0·379	0·331	0·294	0·265	0·241	0·221	0·204	0·190	0·177	0·166	0·156	0·147	0·133	0·121	0·110
1/2	0·785	0·672	0·589	0·524	0·471	0·428	0·393	0·364	0·337	0·314	0·295	0·277	0·262	0·236	0·214	0·196
5/8	1·23	1·05	0·92	0·818	0·736	0·669	0·614	0·569	0·526	0·491	0·460	0·433	0·409	0·368	0·335	0·307
3/4	1·77	1·52	1·325	1·18	1·06	0·964	0·884	0·819	0·757	0·707	0·663	0·624	0·589	0·53	0·482	0·442
7/8	2·41	2·06	1·8	1·60	1·44	1·31	1·20	1·11	1·03	0·962	0·902	0·849	0·801	0·722	0·656	0·601
1	3·14	2·69	2·36	2·09	1·89	1·71	1·57	1·45	1·35	1·26	1·18	1·11	1·05	0·943	0·859	0·783
1⅛	3·98	3·41	2·98	2·65	2·39	2·17	1·99	1·84	1·70	1·59	1·49	1·40	1·33	1·19	1·08	0·994

TABLE VI. *Shear Value of Single Binders (Two Arms)*

Diameter (in.)	Area (in.²) (for one arm)	Stress (lb./in.²) t_w	Values in lb. per unit lever at varying pitches															
			2″	3″	4″	4½″	5″	6″	7″	7½″	8″	9″	10″	11″	12″	15″	18″	24″	Pitch
3/16	0·028	18,000	497	331	248	221	199	166	142	132	124	110	99	90	83	..	..	..	..
		20,000	552	368	276	245	221	184	158	147	138	123	110	100	92	..	..	..	..
1/4	0·049	18,000	883	589	442	393	353	294	252	236	221	196	177	161	147	..	..	..	..
		20,000	982	654	491	436	393	327	281	262	245	218	196	178	164	..	..	..	..
5/16	0·077	18,000	1,381	920	690	613	552	460	395	368	345	307	276	251	230	..	..	..	..
		20,000	1,534	1,023	767	682	613	511	438	409	384	341	307	279	255	..	..	..	..
3/8	0·110	18,000	1,988	1,325	994	884	795	663	568	530	497	442	398	361	331	265	221	166	..
		20,000	2,209	1,473	1,105	982	883	736	631	589	552	491	442	401	368	294	245	184	..
7/16	0·150	18,000	2,706	1,804	1,353	1,203	1,082	902	775	722	676	601	541	492	451	361	301	225	..
		20,000	3,007	2,004	1,503	1,336	1,203	1,002	859	802	752	668	601	547	501	401	334	250	..
1/2	0·196	18,000	3,534	2,356	1,767	1,571	1,414	1,180	1,010	942	883	785	707	642	589	471	393	294	..
		20,000	3,929	2,618	1,963	1,745	1,571	1,309	1122	1,047	982	873	785	714	654	524	436	327	..

$$S \text{ per unit lever arm} = \frac{\text{area of two arms} \times t_w}{\text{pitch}} = s_1, \quad \text{where } t_w = \text{permissible stress.}$$

$$\text{Effective shear value } S = s_1 \times a = \text{shear strength}, \quad \text{where } a = \text{lever arm.}$$

TABLE VII. *Value of Bent-up Bars in Shear (lb.)*

Diameter (in.)	Area (in.²)	Stress t_w (lb./in.²)	Inclination and angle: 1 in 2, 26° 34′	30°	1 in 1½, 33° 41′	1 in 1, 45°	60°
½	0·196	18,000	1,580	1,767	1,960	2,498	3,060
		20,000	1,756	1,963	2,177	2,776	3,400
⅝	0·307	18,000	2,470	2,761	3,063	3,905	4,783
		20,000	2,744	3,068	3,403	4,339	5,314
¾	0·442	18,000	3,556	3,976	4,411	5,623	6,886
		20,000	3,952	4,418	4,900	6,248	7,653
⅞	0·601	18,000	4,840	5,411	6,003	7,654	9,374
		20,000	5,378	6,013	6,670	8,504	10,415
1	0·785	18,000	6,323	7,067	7,840	9,996	12,243
		20,000	7,025	7,854	8,712	11,107	13,603
1⅛	0·994	18,000	8,002	8,946	9,923	12,651	15,495
		20,000	8,892	9,940	11,026	14,057	17,228
1¼	1·227	18,000	9,876	11,043	12,246	15,618	19,128
		20,000	10,975	12,270	13,608	17,352	21,252
1⅜	1·485	18,000	11,956	13,365	14,827	18,902	23,150
		20,000	13,283	14,850	16,475	21,002	25,722
1½	1·767	18,000	14,226	15,903	17,640	22,492	27,590
		20,000	15,805	17,670	19,602	24,992	30,556
1⅝	2·074	18,000	16,695	18,666	20,706	26,397	32,329
		20,000	18,550	20,740	23,004	29,330	35,921
1¾	2·405	18,000	19,360	21,645	24,010	30,611	37,490
		20,000	21,512	24,050	26,680	34,012	41,656
1⅞	2·761	18,000	22,226	24,849	27,567	35,142	43,038
		20,000	24,695	27,610	30,630	39,047	47,820
2	3·142	18,000	25,292	28,278	31,370	39,990	48,975
		20,000	28,102	31,420	34,858	44,435	54,420

$S = A_w \times t_w \sin\theta$, A_w = area of bar, t_w = working stress, θ = angle.

SUBJECT INDEX

AUTHOR INDEX

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