Introduction
to
Electric Power Systems

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By

J. G. TARBOUX, Ph.D.

Professor of Electrical Engineering
Assistant Director, School of Electrical Engineering,
Cornell University

REVISED EDITION

THE INTERNATIONAL TEXTBOOK COMPANY
Scranton, Pennsylvania

PREFACE

The material of this textbook has been accumulated by the author over a period of several years and used in mimeograph form as the basis of undergraduate courses in the field of transmission line networks. It includes a rather wide variety of subjects, the thought being that an undergraduate student should be given a broad point of view of the general field and that more detailed and exact studies should be left for graduate work. For this reason some subjects are very briefly covered, only enough of a discussion being given to whet the appetite in the hope that the good student will thirst for more knowledge.

This volume covers two general fields, namely, the theory and operation of power transmission systems under balanced steady-state conditions, and system characteristics under unbalanced or faulty operation. This choice of content was made on the basis that most electrical engineering graduates who enter the power field come into contact with lines already built and in operation. They are not called upon to enter the design rooms, but rather drift into the operating departments. A rather broad introduction to the major operating characteristics of systems is therefore of benefit. With this in mind, the text covers the following general topics:

I. A hurried review of line characteristics as they apply to the detail problems at hand. Basic material of this type should already be available through a fundamental course.

II. The derivation of the long-line equations, the generalized four-terminal network, and circle diagrams as applied to steady-state performance.

III. An introduction to the very important problem of system stability.

IV. Derivation of fundamental laws of symmetrical components and simple application to networks, without going too deeply into the theory of machine impedances.

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V. An introduction to the very important operating problems of alternator transients and the effect of line surges. These problems are treated more from the qualitative point of view rather than quantitatively.

The introductory material on inductance and capacity presupposes that the student is well grounded in fundamental electromagnetics and electrostatics. This material is as short as is consistent with the complete plan of the text.

In order to limit the size of the volume, it was necessary to omit such subjects as physical properties of conductor materials, skin effect, economic aspects, choice of size of conductor, choice of voltage, choice of span, conductor corona, insulators, underground cables, lightning protection, mechanical design, conductor and ground-cable installation, operation and maintenance, and many other allied topics. A great deal of this material is of a statistical nature and can be obtained from many sources, such as Section 13 of *The Standard Handbook for Electrical Engineers*, Eighth Edition. The topics which appear in the text form a generally connected development of the most important analytical or theoretical phases of electric transmission systems. Enough theory covering the operation and details of transformers is included in order that the subject of system faults may be reasonably complete.

The text is designed for a full year's work. However, for shorter periods, certain sections may be omitted without any great difficulty. The author has attempted to cover only those analytical theoretical phases which are quite permanent and are not subject to change with time. All physical and construction details which may change with the development of the art have been omitted. It is suggested that students consult other up-to-date sources of information on physical details as a means of reinforcing the theoretical content of the course.

Whenever good material was available elsewhere in the literature, it has been used. Suitable acknowledgment has been given in all cases where a definite source was known. Inspiration and material were obtained from Wagner and Evans' fine book on Symmetrical Components. Also Lyon's

volume on the Application of Symmetrical Components has been a help. These two volumes have continually been used by the author as reference volumes in classes on power transmission, and are recommended to all who wish to study in this field.

Very special thanks are due to Miss Edith Clarke of the General Electric Company for suggestions concerning zero sequence capacity of long lines. To the Westinghouse Electric and Manufacturing Company the author is thankful for the contribution of data on their short circuit analyzer.

The author is also deeply indebted to those persons who were kind enough to review the manuscript and contribute suggestions for its improvement.

J. G. TARBOUX

The University of Tennessee
October 1943

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CHAPTER 1

INTRODUCTION

1. Historical Data.—The commercial electrical systems of the United States had their beginning in 1882 with the steam plant known as the Pearl Street Station of the Edison Electric Illuminating Company of New York City and a hydro station in Appleton, Wisconsin. From those small beginnings the electrical industry has developed to undreamed proportions, both as to the physical results accomplished and as to the financial investment involved. Instead of small isolated systems serving only restricted areas, the country is now spanned by high voltage lines connecting together large steam and hydro stations, with the load centers tied into a vast high-tension network.

The original developments were made with direct current, but in 1885 the alternating-current system received a tremendous impetus with the development of the transformer.

Alternating-current transmission originated in 1886 with a line 17 miles long supplying Rome, Italy. In 1887 the capacity of the plant supplying this line was 2,700 hp. The first three-phase line was put into operation in 1891 between Lauffen and Frankfort, a distance of 112 miles, at a voltage of 12,000 volts. Alternating-current transmission in the United States started in 1889 at Oregon City. Power was generated by two 300-hp Victor wheels, belted to 4,000-volt single-phase generators, and was transmitted a distance of 13 miles to Portland. Following this development came the Telluride Power Company in Colorado. In this case two 150-kw single-phase generators, direct connected to Pelton water wheels operating under a head of 500 feet, supplied power at 3,000 volts to the city of Telluride over a line 5 miles long. The first three-phase line in the United States was installed in 1893 in California by the Redlands Electric Light and Power Company (now the Southern California Edison Company).

Up to about 1890, the most common frequency was 133 cycles per second. In 1891, the advisability of a standard

frequency was realized, and 60 cycles per second was established as one standard; and, in 1893, 25 cycles was introduced. Today, these two frequencies are considered as the standards for this country. It is rather unfortunate that the final frequencies of 60 and 25 cycles per second were the ones chosen. There are some definite advantages of a dual-frequency standard, but it would have been much better if 60 and 30, or 50 and 25, cycles per second had been chosen. With a ratio of 2 to 1, the interchange of power between systems of these two frequencies would be more economical and practical than that obtained with the 60- and 25-cycle systems. As a matter of fact, 50 cycles per second has been introduced to a limited extent in this country along with 25 cycles, thus obtaining the advantages of the 2:1 ratio.

In 1895, the first 5,000-hp generators, which were then the largest ever built, were installed at Niagara Falls. From that date on, the growth of the electrical industry has steadily progressed. In 1896, 25,000-volt transmission was used by the Pioneer Electric Power Company of Utah; while, in 1903, 60,000 volts were used by the Guanajuato Power and Electric Company of Mexico. The first company to use 110,000-volt transmission was the Au Sable Electric Company of Grand Rapids, Michigan. In 1913, the Pacific Light and Power Company installed their first 150,000-volt transmission line, which has since been raised to 220,000 volts.

Fig. 1 gives a graphical picture of the growth in transmission line voltages from 1895 to 1940.

2. Boulder Dam-Los Angeles Transmission Line.—An unusual step in power transmission in the United States came in 1936 with the construction of the 266-mile Boulder Dam-Los Angeles line which operates at 287,000 volts at the Boulder Dam end and at 275,000 volts at the Los Angeles end.

A brief specification of the Boulder Dam-Los Angeles line is as follows:

Number of three-phase circuits.   2
Length of single-circuit tower section, in miles   225.3
Length of double-circuit tower section, in miles   40.8

Total length of double-circuit line, in miles. .   266.1
Conductor diameter, in inches. 1.4
Type. . . . . .Segmental hollow copper
Cross-sectional area, in circ. mils. 512,000
Flat spacing of conductors (single-circuit
   towers), in feet. 32.5
Number of segments in conductor.   10
Spiral pitch of conductor, in inches.   28
Weight of conductor per foot, in pounds. 1.57
Resistance per 1,000 feet, at 25° C and 60
   cycles, in ohms. 0.0214
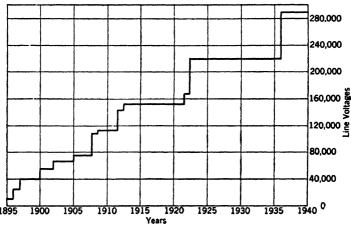
Fig. 1.   Growth in Transmission Line Voltages.

Inductive reactance per phase per 1,000 feet at
   60 cycles, in ohms. 0.1517
Shunt capacitive reactance to neutral per 1,000
   feet at 60 cycles, in megohms. 1.025

The inductive and capacitive reactances are based upon the equivalent transposed delta spacing.

Details of the single-circuit and double-circuit tower sections of the Boulder Dam-Los Angeles line are shown in Fig. 2. A detailed description of this line is given in the issues of *Electrical Engineering* for April and May, 1935.
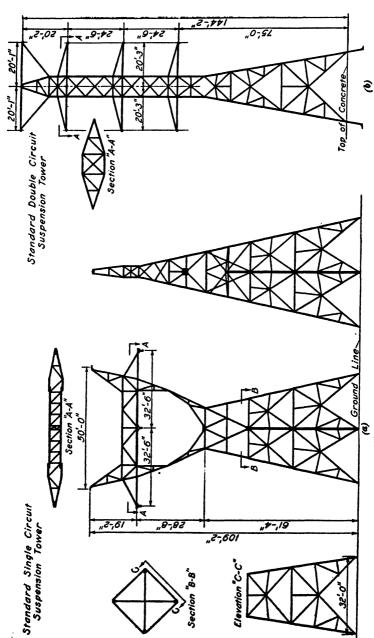
Fig. 2. Boulder Dam–Los Angeles Line Towers.

Standard Double Circuit
Suspension Tower

Section "A-A"

144'-2"
20'-2"
24'-6"
24'-6"
75'-0"
20'-1"
20'-1"
20'-3"
20'-3"

Top of Concrete

(b)

Standard Single Circuit
Suspension Tower

Section "A-A"
50'-0"

32'-6"
32'-6"

Ground Line

(a)

109'-2"
19'-2"
28'-6"
61'-4"

Section "B-B"

Elevation "C-C"
32'-0"

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The most recent step in transmission-line research was taken in 1946 by the construction of a very short experimental 500,000-volt three-phase line. This circuit is energized from the Tidd Substation of the Ohio Power Company at Brilliant, Ohio.

A series of articles describing this line will be found in *AIEE Transactions*, 1947.

3. Problems of Transmission Engineer.—With the rapid increase in the capacity and length of transmission systems, there have developed a good many important problems of operation which do not exist in small isolated systems. The subject of transmission lines cannot any longer be divorced from the characteristics of the terminal equipment. Besides the problem of the steady-state solution of long lines, there are such topics as the stability behavior of systems, the operation under all types of faults, and also the operation under the effect of lightning. Other problems are also present, and the field of the transmission engineer is a highly specialized one, requiring nevertheless a broad understanding of electrical equipment.

CHAPTER 2

RESISTANCE, INDUCTANCE, AND CAPACITY

4. General Nature of the Electrical Circuit.—Let a potential E_s be applied to the sending end of a single-phase transmission line composed of two parallel conductors, as shown in Fig. 3. Analysis of this circuit will reveal three distinctly different phenomena, as follows:
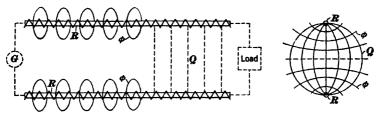
Fig. 3. Single-Phase Transmission Line.

(*a*) Associated with the potential applied between the two conductors is a dielectric flux Q terminating on the two wires. This dielectric field represents in effect a condenser, the capacitance of which is uniformly distributed from the end near the generator G to the load end of the line.

(*b*) The charging current required by this condenser, together with the load current, sets up a magnetic flux ϕ around each conductor, which in turn is responsible for the inductance of the circuit.

(*c*) The conductors themselves offer electrical friction or resistance R to the flow of current.

It will be noticed that these three phenomena occur in distinctly different manners and affect the circuit as a whole in different ways. These effects are independently classified as follows:

(*a*) *Dielectric circuit*, as that part of the circuit involving the dielectric field or condenser action.

(*b*) *Magnetic circuit*, as that part of the circuit involving the magnetic field set up within and around the conductors.

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(c) *Electric circuit*, as that part of the phenomena dealing with the internal conductor resistance.

Thus we may represent Fig. 3 by the equivalent conventional ladder circuit of Fig. 4, in which R represents the conductor friction in the electric circuit, L represents the inductance parameter of the magnetic circuit, and C stands for the capacitance parameter of the dielectric circuit.

5. Mechanical Equivalents.—The three parameters just described are in reality nothing abnormal. The same fundamental quantities are present in mechanics, being designated
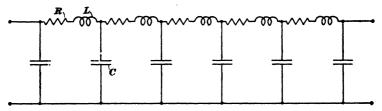
Fig. 4.   Ladder Circuit for Single-Phase Transmission Line.

by the actions of friction, inertia, and elasticity. In mechanics all three parameters affect the general motion of the device; similarly, in electrical circuits the parameters R, L, and C affect the resultant flow of current.

6. Fundamental Laws.—In a mechanical device possessing only friction, we may define ohm's law as follows:

$$f = vR \tag{1}$$

where f is the pressure or force, v is the velocity, and R is the coefficient of friction, expressed in some suitable units.

Similarly, ohm's law for the electric circuit is

$$e = iR \tag{2}$$

where e is the pressure, in volts; i is the flow of current, in amperes or in coulombs per second; and R is the resistance or friction, in ohms.

In other words, ohm's law applies to any circuit involving force, motion, and friction. Equations similar to equation 2

can be written in other fields, such as in hydraulics, heat flow, etc. The similarity of units is very significant. Thus, f and e are the pressure or force in mechanics and in electric circuits, respectively; and v and i can be defined as feet per second and coulombs per second, both representing rates of motion. In actual practice the equations of mechanics are seldom as simple as equation 1, since the units chosen generally require the introduction of constants of proportionality.

Passing on to the magnetic circuit, we define the voltage of self induction as follows:

$$e_{s.i.} = -L\frac{di}{dt} \tag{3}$$

Equation 3 states that, if the current (coulombs per second) is changing with respect to time, there will be developed within the circuit a force of reaction which is opposite to that change. If we write equation 3 in mechanical units, we have the following:

$$f_{s.i.} = -M\frac{dv}{dt} \tag{4}$$

Equation 4 is, of course, the fundamental law of inertia of mechanical bodies, in which $f_{s.i.}$ = the reaction force of inertia, while $\frac{dv}{dt}$ = acceleration and M = mass.

Obviously the voltage or force necessary to overcome the emf of self-induction or the force of inertia is:

$$\left.\begin{array}{l} e_a = L\dfrac{di}{dt} \\[2mm] f_a = M\dfrac{dv}{dt} \end{array}\right\} \tag{5}$$
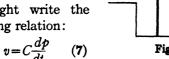
Comparing these two equations, one sees a similarity between L and M, from which we conclude that L has some of the properties of mass.

The equation expressing the fundamental law of the dielectric circuit is

$$i = C\frac{de}{dt} \tag{6}$$

This law may be stated as follows: In a circuit of elastance, the displacement is directly proportional to the rate of change of pressure or force. Imagine that the hydraulic system of Fig. 5 is completely filled with liquid, the chamber A being equipped with an elastic diaphragm. As long as the pressure on the two sides is the same, there will be no motion of the diaphragm; but, let the pressure on one side suddenly increase, and there will then be a movement of the diaphragm and also of the liquid. In other words, the velocity of the liquid will be directly proportional to the rate of change in pressure. Furthermore, the size and physical properties (or capacity C) of the diaphragm will also influence the velocity of liquid motion. Thus, in mechanical units, we might write the following relation:
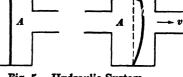
Fig. 5. Hydraulic System.

$$v = C\frac{dp}{dt} \qquad (7)$$

where v = velocity, C = capacity, and $\frac{dp}{dt}$ = rate of change of pressure.

7. **Energy Relations.**—The energy content, in watt-seconds, in any electrical circuit for a period of T seconds may be given as follows:

$$\text{Energy} = \int_0^T ei\,dt \qquad (8)$$

This fundamental law may be applied to each of the three circuits of Art. 4.

(a) *Electric Circuit*: In this case,

$$\text{Energy} = \int_0^T ei\,dt$$

In the case of a constant voltage E and a constant current I, this equation becomes:

$$\text{Energy} = EIT \text{ watt-seconds} \qquad (9)$$

The same general relationship exists in mechanics for a body moving at a velocity v under force f; thus, the energy in foot-pounds is:

$$\text{Energy} = \int_0^T fv\, dt$$

For a constant force F and constant velocity V, this equation becomes:

$$\text{Energy} = FVT \text{ foot-pounds} \tag{10}$$

(b) *Magnetic Circuit*: Applying equation 8 to the magnetic circuit, we have:

$$\text{Energy} = \int_0^T ei\, dt = \int_0^T \left(L\frac{di}{dt} \right)i\, dt$$

Consider a circuit in which the current i is changed from zero, at time $t=0$, to the maximum value I_m at time $t=T$.   Then,

$$\text{Energy} = \int_0^T \left(L\frac{di}{dt} \right)i\, dt = \int_0^{I_m} Li\, di = \tfrac{1}{2}LI_m^2 \text{ watt-seconds} \tag{11}$$

Remembering that a current I_m (coulombs per second) corresponds to velocity V_m (feet per second) and that inductance L corresponds to mass M, we can write the equivalent equation for mechanics.   Thus,

$$\text{Energy} = \tfrac{1}{2}\, M\, V_m^2 \tag{12}$$

Equation 12 is recognized as the formula for kinetic energy of a moving mass.   Hence, equation 11 is often called the "electro-kinetic" energy equation of a magnetic circuit.

Energy is required in accelerating an electric current as well as in accelerating a mechanical body.   From the point of view of the magnetic circuit, an amount of energy equal to that given by equation 11 is stored in the magnetic field surrounding the current-carrying conductors.   Thinking of the inductance L as electrical mass, we may say that the electro-kinetic energy is stored in the mass L in the same manner as kinetic energy is stored in the mechanical mass M.

(c) *Dielectric Circuit*: The energy content of the dielectric circuit may be obtained in the same general manner.   Thus,

$$\text{Energy} = \int_0^T ei \, dt = \int_0^T e\left(C\frac{de}{dt}\right)dt$$

If the potential is zero at time $t = 0$ and is a maximum E_m at time $t = T$, then,

$$\text{Energy} = \int_0^T e\left(C\frac{de}{dt}\right)dt = \int_0^{E_m} Ce \, de = \tfrac{1}{2}CE_m^2 \text{ watt-seconds} \quad (13)$$

Equation 13 is known as the equation for "electro-potential" energy of an electric circuit, or the energy necessary to charge a condenser to a final voltage E_m.   In mechanics it represents the potential energy required in stressing an elastic body, such as a spring.

8. Energy Relations in Circuit Following Simple Harmonic Variation.

—Consider a sine wave of current flowing through a series circuit of resistance, inductance, and capacity.   The relations of current and voltages are given in Fig. 6 (a), where i = current, e_r = voltage across the resistance, e_l = voltage across the inductance, and e_c = voltage across the capacity.   The energy associated with the resistance, the inductance, or the capacity can be obtained by direct application of equation 8. The power curves are shown in Fig. 6 (b).

For the resistance, for one-half of a cycle,

$$\text{Energy} = \int_{t=0}^{t=\frac{1}{2f}} E_{rm}I_m \sin^2 \omega t \, dt = \frac{E_{rm}I_m}{4f}$$

in which f is the frequency in cycles per second.

Substituting the effective values of voltage and current and noting that $\frac{1}{2f}$ is the time for one-half of a cycle, we find that

$$\text{Energy in a resistance} = E_r I T \text{ watt-seconds} \quad (14)$$

Similarly, the energy stored in the magnetic field during one-fourth of a cycle is:

$$\text{Energy} = \int_{t=0}^{t=\frac{1}{4f}} E_{lm} I_m \cos \omega t \sin \omega t \, dt = \int_{t=0}^{t=\frac{1}{4f}} \frac{E_{lm} I_m}{2} \sin 2\omega t \, dt$$
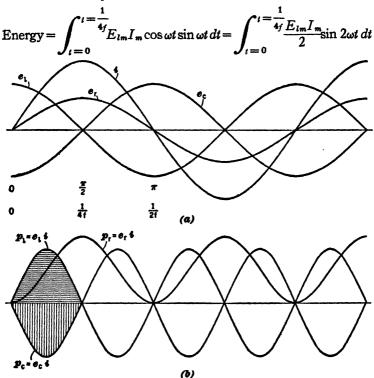
Fig. 6.   Current, Voltage, and Power Curves for Series Circuit.

Carrying out the integration, we obtain:

$$\text{Energy} = \frac{E_{lm} I_m}{4\pi f} \text{ watt-seconds}$$

Since $E_{lm} = 2\pi f L I_m$,

$$\text{Energy} = \tfrac{1}{2} L I_m^2 \text{ watt-seconds} \qquad (15)$$

This result is seen to check with equation 11.

Similarly, the energy involved in charging a condenser is:

$$\text{Energy} = \int_{t=0}^{t=\frac{1}{4f}} -E_{cm} I_m \cos \omega t \sin \omega t = -\frac{E_{cm} I_m}{4\pi f} \text{ watt-seconds}$$

Since $I_m = 2\pi f C E_{cm}$,

$$\text{Energy} = -\tfrac{1}{2}C E_{cm}^2 \qquad (16)$$

The energy of the magnetic circuit is indicated in Fig. 6 (*b*) by the area with horizontal shading; and the energy of the dielectric circuit, by the area with vertical shading.   It will be noticed that these values of energy are alternately positive and negative, suggesting that energy is alternately stored in and discharged from these circuits.   Furthermore, when energy is being stored in the magnetic field. it is being discharged from the dielectric field.

TABLE 1
SUMMARY OF BASIC INFORMATION

	Electric	Magnetic	Dielectric
Motion	I = amperes	ϕ = magnetic flux	Q = dielectric flux
	I = coulombs/second	ϕ = lines	Q = coulombs
Force	E = volts	F = ampere-turns	E = volts
Impedance	R = resistance R = ohms	R = reluctance	S = elastance
Ohm's Law	$E = IR$	$F = \phi R$	$E = QS$
Admittance	$g = \dfrac{1}{R}$ = conductance	$P = \dfrac{1}{R}$ = permeance	$C = \dfrac{1}{S}$ = capacitance
Impedance Law	L = length; A = area $R = \rho\dfrac{L}{A}$ ρ = constant	$R = K\dfrac{L}{A}$ K = constant	$S = K\dfrac{L}{A}$ K = constant
Ohm's Law	$E = I\rho\dfrac{L}{A}$	$F = \phi K\dfrac{L}{A}$	$E = QK\dfrac{L}{A}$
Gradients	$G = \dfrac{E}{L}$	$H = \dfrac{F}{L}$	$G = \dfrac{E}{L}$
Densities	$D = \dfrac{I}{A}$	$\beta = \dfrac{\phi}{A}$	$D = \dfrac{Q}{A}$
Unit Ohm's Law	$G = \rho D$	$H = K\beta$	$G = KD$
Energy	EIt	$\tfrac{1}{2}LI_m^2$	$\tfrac{1}{2}CE_m^2$

9. Summary of Fundamental Relations.—A few of the basic laws of electric, magnetic, and dielectric phenomena have been reviewed in Arts. 4 to 8. These laws, together with other pertinent basic information, are summarized in Table 1. It would be quite worth while to expand this table to cover other fields of engineering, such as mechanics and hydraulics. It is very enlightening to see how these basic laws are applicable to many fields of engineering.

In some major fields (such as mechanics), friction, inertia, and elasticity are also fundamental parameters. They may all be present in a certain mechanical device; or one—or even two—of them may be absent or dormant, that is, its effect is not observable from the over-all action of the device.

10. Systems of Units.—Various systems of units have been used in electrical engineering. Until recently, the generally accepted system was the CGS (Centimeter-Gram-Second) system. This system has the distinct disadvantage of not being consistent in the electromagnetic and electrostatic fields. As a result of this inconsistency, the MKS (Meter-Kilogram-Second), or "practical," system has been generally accepted as the most basic and desirable system. The MKS is definitely desirable whenever theoretical developments are carried out.

The system of units used throughout this text, in the fields of electromagnetic and electrostatic phenomena, is neither one of the foregoing systems, but is rather a mixed system which is a modified CGS system. There is no criticism of the MKS and CGS systems of units; on the contrary, the student should have had a fundamental training in basic electromagnetic and electrostatic theory, in which the value of the MKS system has been definitely demonstrated, before attempting the work of this volume.

To compare these three systems as they apply to electromagnetic theory, Table 2 is given.

Notice that the units of length, time, magnetic flux, and magnetic flux density are the same in the CGS system and the

modified system.   For all systems of units, the following funda-
mental law applies:

$$\beta = \mu H$$

TABLE 2
COMPARISON OF SYSTEMS OF UNITS

Quantity	MKS	CGS	Modified
Length	Meter	Centimeter	Centimeter
Time	Second	Second	Second
Current	Ampere	Al ampere	Ampere
Potential	Volt	Abvolt	Volt
Magnetic flux	Weber	Maxwell or Lines	Lines
Magnetic flux density	Weber per square meter	Gauss = Maxwell per square centimeter	Gauss = Lines per square centimeter
Magnetomotive force	Pragilbert	Gilbert	Ampere-turn
Magnetic intensity	Praoersted	Oersted = Gilbert per centimeter	Ampere-turn per centimeter
Permeability, for non-magnetic materials	1.257×10^{-6}	1	$\frac{4\pi}{10} = 1.257$

In the CGS system, with $\mu = 1$, the magnetic intensity
becomes:

$$H_c = \beta = \text{gilberts per cm}$$

In the modified system:

$$H_m = \frac{\beta}{1.257} = \text{ampere-turns per cm}$$

Furthermore, the total mmf in the CGS system is defined as
follows:

$$F = \frac{4\pi NI}{10} = 1.257 \, NI \text{ gilberts}$$

where $NI = \text{ampere-turns}$.

Notice the presence of the constant 1.257 as a proportionality
constant between gilberts and ampere-turns.   Thus,

$$H_c = 1.257 \, H_m$$

CHAPTER 3

TRANSMISSION LINE INDUCTANCE

11. Definition.—The coefficient of inductance of a circuit of one single turn may be defined with the aid of the fundamental equation of induced voltage. Thus,

$$e_{s.i.} = -L\frac{di}{dt} = -\frac{d\phi}{dt} \times 10^{-8} \tag{17}$$

This can be rearranged, as follows:

$$L = \frac{d\phi}{dt} \times \frac{dt}{di} \times 10^{-8} = \frac{d\phi}{di} \times 10^{-8} \text{ henries} \tag{18}*$$

Equation 18 will give the inductance in henries if the magnetic flux linkage ϕ is measured in maxwells or lines and the current i is in amperes. Furthermore, if ϕ involves all the magnetic flux linkages around a particular conductor as produced by all possible sources, and the current i is the current of the conductor in question, then equation 18 will give the total conductor inductance.

For any magnetic medium having a straight-line saturation curve, the conductor inductance becomes:

$$L = \frac{\phi}{I} \times 10^{-8} \text{ henries} \tag{19}$$

If ϕ in equation 19 is the total flux linkage per centimeter length of conductor, then the total inductance per centimeter length becomes:

$$L = \frac{\phi}{I} \times 10^{-8} \text{ henries/cm} \tag{20}$$

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*If the CGS system of units were used in equation 17, $e_{s.i.}$ would be expressed in abvolts; L, in abhenries; and i, in abamperes. The main question involved is whether 10^{-8} should be introduced in the definition of inductance, and at what point in the general development the constant 1.257 should be introduced.

The author has no objection to other points of view. In fact, he recognizes their historic and basic importance. However, he feels that, in the type of work involved in the present volume, there is some advantage in using the modified system of units because such a system materially simplifies the immediate problem.

The inductive reactance, in ohms per centimeter, is:

$$X = 2\pi f L = 2\pi f \times \frac{\phi}{I} \times 10^{-8} \text{ ohms/cm} \tag{21}$$

Also, the inductive reactance voltage drop is:

$$E_l = IX = 2\pi f\phi \times 10^{-8} \text{ volts/cm} \tag{22}$$

12. Inductance of Single Round Solid Conductor.—In Fig. 7 is represented a solid conductor of circular area with radius a, carrying current which is assumed uniformly distributed over its cross-section. The assumption of uniform current density,
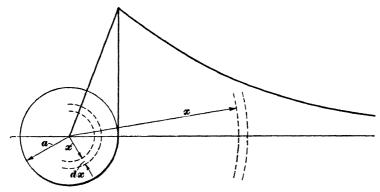
Fig. 7. Flux Linkages.

although not absolutely correct, is satisfactory for most overhead power circuits. Such a conductor will be subjected to magnetic fields which are external to its area and also to magnetic fields which are within the area of the conductor. The external fields will link with all of the current flowing in the conductor, and such fields are therefore known as *total flux linkages* or *external flux linkages*. The flux within the conductor does not link with all of the current, and its linkages are known as *partial flux linkages* or *internal flux linkages*.

To apply equation 20, it is necessary to determine the external flux linkages and then an *equivalent value* of the partial

flux linkages which may be considered as linking with the entire current.

13. Partial Flux Linkages.—As an explanation of the meaning of partial flux linkages, consider the simple coil illustrated in Fig. 8, which is composed of six turns. The current passes through all six turns, and we will assume that in this case the conductor is quite small and no magnetic flux exists within the cross-sectional area of the conductor.

The magnetic flux is illustrated by lines linking with the individual turns of the coil; thus, line 1 links with only two turns, line 2 links with four turns, and line 3 links with all six turns. In other words, the linkages of the flux represented by line 3 are total flux linkage; while the flux represented by line 1 and the flux represented by line 2 link only with some of the turns, and these linkages are partial flux linkage.

Fig. 8.   Partial Flux Linkages.

To compute the total inductance of the entire coil, it is necessary to take due account of the difference in linkages. Thus, applying equation 19 to this coil, we have:
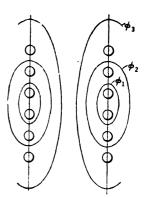
$$L = \frac{6\phi_3 + 4\phi_2 + 2\phi_1}{I} \times 10^{-8}$$

The last two terms in the numerator of the fraction need further study. If only internal flux linkages are considered,

$$L_{\text{internal}} = \frac{4\phi_2 + 2\phi_1}{I} \times 10^{-8}$$

This equation can be rearranged as follows:

$$L_{\text{internal}} = \frac{6}{I}\left(\frac{4\phi_2}{6} + \frac{2\phi_1}{6}\right) 10^{-8}$$

The total inductance is, therefore,

$$L = \frac{N}{I}(\phi_3 + \frac{4}{6}\phi_2 + \frac{2}{6}\phi_1) 10^{-8}$$

where $N = 6$ turns.

From the last equation, we might say that the flux ϕ_2 is $\frac{4}{6}$ effective, and that the flux ϕ_1 is $\frac{2}{6}$ effective, in producing the total inductance.   In general, whenever there is some magnetic flux which does not link with all turns, or—in the case of a large-cross-sectional area—does not link with all the current, then such partial flux linkages will not be fully effective in setting up the total inductance and must be suitably corrected in the ratio of the current linked to the total current.

Consider now a tube of infinitesimal thickness dx and radius x within the cylindrical conductor represented in Fig. 7.   The area contained inside this tube is πx^2.   If the flux density at this tube is β_x, then:

$$\beta_x = \mu H_x$$

where H_x = mmf per centimeter of magnetic path available to maintain the flux density β_x.   Also, if I is the total conductor current, in amperes, then the current I_x in area πx^2, assuming uniform current density, is:

$$I_x = I\frac{x^2}{a^2}$$

Therefore, the mmf per centimeter of the flux path is:

$$H_r = \frac{I_x}{2\pi x} = \frac{Ix}{2\pi a^2} \text{ ampere-turns/cm}$$

Since $\beta_x = \mu H_x$,

$$\beta_x = \mu\frac{Ix}{2\pi a^2} \text{ gausses}$$

The flux within the elementary tube and a unit length of the conductor is:

$$d\phi_x = \beta_x\,dx = \mu\frac{Ix}{2\pi a^2}dx$$

Since this elementary flux $d\phi_x$ links with only a portion of the total conductor area, the equivalent partial flux linkage is:

$$d\phi_{ex} = \mu\frac{Ix}{2\pi a^2}\times\frac{x^2}{a^2}\,dx = \mu\frac{Ix^3}{2\pi a^4}\,dx$$

Hence, the total equivalent partial flux linkages are:

$$\phi_e = \int_0^a \mu \frac{Ix^3}{2\pi a^4}\, dx = \left[\frac{\mu Ix^4}{8\pi a^4}\right]_0^a = \frac{\mu I}{8\pi} \tag{23}$$

14. External Flux Linkages.—For an elementary tube of flux outside of the conductor in Fig. 7, the flux density is

$$\beta_x = \mu H_x = \frac{\mu I}{2\pi x}$$

and the total magnetic flux outside the conductor up to a distance D from the center of the conductor is:

$$\phi = \int_a^D \mu \frac{I}{2\pi x}\, dx = \frac{\mu I}{2\pi} \operatorname{Ln} \frac{D}{a} \tag{24}$$

Therefore, the total flux linkage is:

$$\phi_T = \phi_e + \phi = \frac{\mu I}{8\pi} + \frac{\mu I}{2\pi} \operatorname{Ln} \frac{D}{a} \text{ lines/cm} \tag{25}$$

By equation 20, inductance per centimeter of conductor is:

$$L = \left(\frac{\mu}{8\pi} + \frac{\mu}{2\pi} \operatorname{Ln} \frac{D}{a}\right) 10^{-8} \text{ henries/cm of conductor} \tag{26}$$

15. Inductance in Terms of Conductor GMR.—Equation 26 has two terms, the first representing the inductance due to partial flux linkages and the second representing the inductance due to external flux linkages. This equation may be converted into a simpler form by replacing the solid conductor by a hollow tube of such small thickness that there are no internal flux linkages.* The radius r of such a tube is known as the *geometric mean radius* (GMR) of the conductor. Thus, equation 26 can be rewritten as follows:

$$L = \frac{\mu}{2\pi}\left(0.25 + \operatorname{Ln} \frac{D}{a}\right) 10^{-8}$$

$$= \frac{\mu}{2\pi}\left(\operatorname{Ln} \epsilon^{0.25} + \operatorname{Ln} \frac{D}{a}\right) 10^{-8} = \frac{\mu}{2\pi}\left(\operatorname{Ln} \frac{D}{a\epsilon^{-0.25}}\right) 10^{-8}$$

---

*This can be done if the permeability of the conductor and that of the surrounding air are the same.

Putting $r = a\epsilon^{-0.25}$ = the geometric mean radius of a solid circular conductor, then:

$$L = \frac{\mu}{2\pi}\left(\text{Ln}\frac{D}{r}\right)10^{-8} \text{ henries/cm of conductor} \qquad (27)$$

Equation 27 is important because, assuming uniform current density, it can be used for any type of wire—such as a stranded
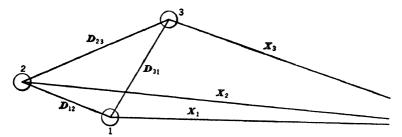
Fig. 9.   Flux Linkages Due to Several Conductors.

conductor—merely by using the proper value of the GMR. For the solid circular conductor:

$$r = a\epsilon^{-0.25} = 0.779a \qquad (28)$$

The GMR of a circular area can be obtained by solving a problem in geometry, as shown in Appendix II.

16. Flux Linkages Due to Any Number of Conductors.
Consider three conductors, such as those marked 1, 2, and 3 in Fig. 9, which have a return circuit at distances X_1, X_2, and X_3. Let it be required to develop an equation for the total flux linkage about conductor 1.

From equation 25, the flux linkage about conductor 1, due to its own current, is:

$$\phi_{11} = \frac{\mu I_1}{8\pi} + \frac{\mu I_1}{2\pi} \text{ Ln } \frac{X_1}{a} \text{ lines/cm of conductor}$$

By the aid of the GMR (see equation 27), this relation can be written thus:

$$\phi_{11} = \frac{\mu I_1}{2\pi} \text{ Ln } \frac{X_1}{r} \qquad (29)$$

The flux linkage about conductor 1 due to the current in conductor 2 may be written from the fundamental relation of equation 24 by using the proper limits. Thus:

$$\phi_{12} = \int_{D_{12}}^{X_2} \mu \frac{I_2}{2\pi x} \, dx = \frac{\mu I_2}{2\pi} \operatorname{Ln} \frac{X_2}{D_{12}} \qquad (30)$$

Notice that the lower limit of the preceding integral is taken as D_{12}. This process assumes that the diameters of the conductors are quite small compared to the spacing between conductors, and ignores the partial flux linkage effect of ϕ_{12} with conductor 1.

Similarly, the flux set up by conductor 3 and linking with conductor 1 is:

$$\phi_{13} = \int_{D_{31}}^{X_3} \mu \frac{I_3}{2\pi x} \, dx = \frac{\mu I_3}{2\pi} \operatorname{Ln} \frac{X_3}{D_{31}} \qquad (31)$$

The flux linkages due to additional conductors could be obtained by similar expressions. Thus, the total flux linkage about conductor 1 due to all sources is:

$$\phi_T = \frac{\mu}{2\pi} \left(I_1 \operatorname{Ln} \frac{X_1}{r} + I_2 \operatorname{Ln} \frac{X_2}{D_{21}} + I_3 \operatorname{Ln} \frac{X_3}{D_{31}} + \ \ldots \ I_n \operatorname{Ln} \frac{X_n}{D_{n1}} \right)$$

If all conductors of a polyphase system are considered, then, by Kirchhoff's law:

$$I_n = -I_1 - I_2 - I_3 - I_4 \ \ldots \ldots \ldots -I_{n-1}$$

Therefore,

$$\phi_T = \frac{\mu}{2\pi} \left[I_1 \left(\operatorname{Ln} \frac{X_1}{r} - \operatorname{Ln} \frac{X_n}{D_{n1}} \right) + I_2 \left(\operatorname{Ln} \frac{X_2}{D_{21}} - \operatorname{Ln} \frac{X_n}{D_{n1}} \right) + \right.$$

$$\left. I_3 \left(\operatorname{Ln} \frac{X_3}{D_{31}} - \operatorname{Ln} \frac{X_n}{D_{n1}} \right) + \ldots \ldots + I_{n-1} \left(\operatorname{Ln} \frac{X_{n-1}}{D_{(n-1)1}} - \operatorname{Ln} \frac{X_n}{D_{n1}} \right) \right]$$

or

$$\phi_T = \frac{\mu}{2\pi} \left[I_1 \left(\operatorname{Ln} \frac{D_{n1}}{r} \times \frac{X_1}{X_n} \right) + I_2 \left(\operatorname{Ln} \frac{D_{n1}}{D_{21}} \times \frac{X_2}{X_n} \right) + \right.$$

$$\left. I_3 \left(\operatorname{Ln} \frac{D_{n1}}{D_{31}} \times \frac{X_3}{X_n} \right) + \ldots \ldots + I_{n-1} \left(\operatorname{Ln} \frac{D_{n1}}{D_{(n-1)1}} \times \frac{X_{n-1}}{X_n} \right) \right]$$

Now, let $X_1 = X_2 = X_3 = \ldots \ldots = X_n \to \infty$; that is, let each of these distances approach infinity.   Then:
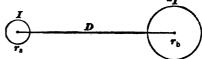
$$\phi_T = \frac{\mu}{2\pi}\left(I_1 \operatorname{Ln}\frac{D_{n1}}{r} + I_2 \operatorname{Ln}\frac{D_{n1}}{D_{21}} + I_3 \operatorname{Ln}\frac{D_{n1}}{D_{31}} + \ldots + I_{n-1} \operatorname{Ln}\frac{D_{n1}}{D_{(n-1)\,1}} \right)$$

$$= \frac{\mu}{2\pi}\left(I_1 \operatorname{Ln}\frac{1}{r} - I_1 \operatorname{Ln}\frac{1}{D_{n1}} + I_2 \operatorname{Ln}\frac{1}{D_{21}} - I_2 \operatorname{Ln}\frac{1}{D_{n1}} + I_3 \operatorname{Ln}\frac{1}{D_{31}} - \right.$$

$$\left. I_3 \operatorname{Ln}\frac{1}{D_{n1}} + \text{etc.} \right)$$

Keeping in mind that $I_1 + I_2 + I_3 + \ldots \ldots I_n = 0$, then:

$$\phi_T = \frac{\mu}{2\pi}\left(I_1 \operatorname{Ln}\frac{1}{r} + I_2 \operatorname{Ln}\frac{1}{D_{21}} + I_3 \operatorname{Ln}\frac{1}{D_{31}} + \ldots + I_n \operatorname{Ln}\frac{1}{D_{n1}} \right) \quad (32)$$

Equation 32 should be carefully examined, as it forms the basis from which it is possible to evaluate the total inductance of any system of conductors.

Fig. 10.   Inductance of Single-Phase Line.

17. Inductance of Single-Phase Line.—Consider the two conductors represented in Fig. 10; and assume that their geometric mean radii are r_a and r_b and that they carry currents I and $-I$, respectively.

According to equation 32, the effective flux linkages about conductors a and b are:

$$\phi_a = \frac{\mu}{2\pi}\left(I \operatorname{Ln}\frac{1}{r_a} - I \operatorname{Ln}\frac{1}{D} \right)$$

and

$$\phi_b = \frac{\mu}{2\pi}\left(-I \operatorname{Ln}\frac{1}{r_b} + I \operatorname{Ln}\frac{1}{D} \right)$$

Therefore, the inductance per centimeter of each conductor is:

$$\left. \begin{aligned} L_a &= \frac{\mu}{2\pi}\left(\operatorname{Ln}\frac{D}{r_a} \right)10^{-8} \text{ henries/cm of conductor} \\ L_b &= \frac{\mu}{2\pi}\left(\operatorname{Ln}\frac{D}{r_b} \right)10^{-8} \text{ henries/cm of conductor} \end{aligned} \right\} \quad (33)$$

The inductance of the entire line per centimeter of line is obtained as the sum of L_a and L_b. Thus:

$$L_1 = L_a + L_b = \frac{\mu}{2\pi}\left(\text{Ln}\,\frac{D}{r_a} + \text{Ln}\,\frac{D}{r_b}\right)10^{-8}$$

or

$$L_1 = L_a + L_b = \frac{\mu}{2\pi}\left(\text{Ln}\,\frac{D^2}{r_a r_b}\right)10^{-8} \text{ henries/cm of line} \quad (34)$$

In case the two conductors have the same GMR, then:

$$L_1 = \frac{\mu}{2\pi}\left(\text{Ln}\,\frac{D^2}{r^2}\right)10^{-8}$$

or

$$L_1 = \frac{\mu}{\pi}\left(\text{Ln}\,\frac{D}{r}\right)10^{-8} \text{ henries/cm of line} \quad (35)$$

18. Inductance of Three-Phase Lines.—Let a three-phase line be composed of three identical conductors such as 1, 2, and 3 in Fig. 9. According to equation 32, the flux linkages about these conductors will be:

$$\left. \begin{aligned}
\phi_1 &= \frac{\mu}{2\pi}\left(I_1\,\text{Ln}\,\frac{1}{r} + I_2\,\text{Ln}\,\frac{1}{D_{12}} + I_3\,\text{Ln}\,\frac{1}{D_{31}}\right) \\
\phi_2 &= \frac{\mu}{2\pi}\left(I_2\,\text{Ln}\,\frac{1}{r} + I_1\,\text{Ln}\,\frac{1}{D_{12}} + I_3\,\text{Ln}\,\frac{1}{D_{23}}\right) \\
\phi_3 &= \frac{\mu}{2\pi}\left(I_3\,\text{Ln}\,\frac{1}{r} + I_2\,\text{Ln}\,\frac{1}{D_{23}} + I_1\,\text{Ln}\,\frac{1}{D_{31}}\right)
\end{aligned} \right\} \quad (36)$$

For lines which are not transposed, the quantities in equation 36 may be evaluated only if the complex values of all three currents are known. Then, by dividing each complex flux linkage by the particular complex current, it is possible to obtain a complex expression for inductance for each conductor. This complex inductance is, of course, a combination of self inductance and mutual inductance. The complex inductance, when multiplied by $j(2\pi f I)$, will give the inductive reactance voltage drop in its proper phase relation; or the inductive reactance voltage drop is, according to equation 22,

$$E_l = j(2\pi f \phi)10^{-8} \text{ volts/cm}$$

19. Inductance of Transposed Three-Phase Line.—In this case the effective flux linking each conductor can be obtained as the average of the flux linkages of each conductor as it is placed in each of the three tower positions. Considering conductor number 1 in position number 1, we may write:

$$\phi_{11} = \frac{\mu}{2\pi}\left(I_1 \operatorname{Ln} \frac{1}{r} + I_2 \operatorname{Ln} \frac{1}{D_{12}} + I_3 \operatorname{Ln} \frac{1}{D_{31}} \right)$$

With conductor 1 in position number 2 and the orher conductors rotated in position accordingly:

$$\phi_{12} = \frac{\mu}{2\pi}\left(I_1 \operatorname{Ln} \frac{1}{r} + I_2 \operatorname{Ln} \frac{1}{D_{23}} + I_3 \operatorname{Ln} \frac{1}{D_{12}} \right)$$

Sinilarly, for conductor 1 in position 3:

$$\phi_{13} = \frac{\mu}{2\pi}\left(I_1 \operatorname{Ln} \frac{1}{r} + I_2 \operatorname{Ln} \frac{1}{D_{31}} + I_3 \operatorname{Ln} \frac{1}{D_{23}} \right)$$

The average of these three equations will be:

$$\phi_1 = \frac{\mu}{6\pi}\left(3I_1 \operatorname{Ln} \frac{1}{r} + I_2 \operatorname{Ln} \frac{1}{D_{12}D_{23}D_{31}} + I_3 \operatorname{Ln} \frac{1}{D_{12}D_{23}D_{31}} \right) \quad (37)$$

Since $I_3 = -I_1 - I_2$,

$$\phi_1 = \frac{\mu}{6\pi}\left(3I_1 \operatorname{Ln} \frac{1}{r} - I_1 \operatorname{Ln} \frac{1}{D_{12}D_{23}D_{31}} \right)$$

and

$$\phi_1 = \frac{\mu}{2\pi}\left(I_1 \operatorname{Ln} \frac{\sqrt[3]{D_{12}D_{23}D_{31}}}{r} \right) \quad (38)$$

Therefore, the inductance per phase is:

$$L = \frac{\mu}{2\pi}\left(\operatorname{Ln} \frac{\sqrt[3]{D_{12}D_{23}D_{31}}}{r} \right) 10^{-8} \text{ henries/cm of conductor} \quad (39)$$

Equation 39 was derived for conductor number 1. It is obvious, however, that the same equation will apply to the other two conductors. The term $\sqrt[3]{D_{12}D_{23}D_{31}}$ may be interpreted as an effective spacing D_e.

20. Inductive Reactance in Ohms.—The inductive reactance, in ohms per mile, for single-phase lines and three-phase

transposed lines may now be obtained from equations 35 and 39, since $X_l = 2\pi f L$. For a single-phase circuit, considering both conductors:

$$X_1 = 2\mu f\left(\text{Ln}\,\frac{D}{r}\right)10^{-8} \text{ ohms/cm of line}$$

For three-phase lines, considering only one phase:

$$X_3 = \mu f\left(\text{Ln}\,\frac{D_e}{r}\right)10^{-8} \text{ ohms/cm of conductor}$$

Reducing to ohms per mile, introducing $\mu = \frac{4\pi}{10} = 1.257$, which is the correct value for non-magnetic materials, and also changing to common logarithms, we have:

$$X_1 = \frac{2 \times 1.257 f \times 2.54 \times 12 \times 5{,}280 \times 2.3026}{10^8} \log\frac{D}{r}$$

or

$$X_1 = 2 \times 4.657 \times 10^{-3} f \log\frac{D}{r} \text{ ohms/mile of line} \qquad (40)$$

Similarly:

$$X_3 = 4.657 \times 10^{-3} f \log\frac{D_e}{r} \text{ ohms/mile/phase} \qquad (41)$$

Thus, at 60 cycles,

$$X_3 = 0.2794 \log\frac{D_e}{r} \text{ ohms/mile/phase}$$

CHAPTER 4

INDUCTANCE BY GEOMETRIC MEAN DISTANCES

21. Fundamental Derivation.—In Art. 15 the term geometric mean radius was introduced and was given as equal to $a\epsilon^{-0.25} = 0.779a$ for a solid round wire of radius a. This concept was found to have certain useful properties, particularly in simplifying the general equations of inductance.

It was also suggested in Art. 15 that the method of geometric mean radii was particularly useful in the analysis of inductance for stranded and other odd cross-sectional conductor shapes. Furthermore, the method is useful in handling problems of
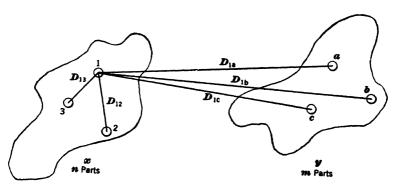
Fig. 11. General Conditions for Determination of Inductance.

inductance when more than one conductor is used in parallel per phase. This last feature is encountered in the solution of zero sequence inductance, which involves the condition of all three power conductors of a three-phase system being treated as parallel wires of a single-phase circuit. Therefore, in this chapter, we will be concerned in establishing a new basis for the calculation of inductance, which will be adaptable to certain special commercial situations. A few of these situations are considered in later chapters of this volume.

27

Consider two conductors, represented in Fig. 11 by the irregular cross-sections marked x and y. The current in conductor y is taken as the negative of the current in x. Let conductor x be divided into n equal parts, and conductor y divided into m equal parts. Assuming uniform current distribution, the current in each element of conductor x is $\dfrac{I}{n}$, and the current in each element of conductor y is $\dfrac{I}{m}$. From equation 32 we may now write the total flux linkage of element 1 as follows:

$$\phi_{T1}=\frac{\mu}{2\pi}\left[\frac{I}{n}\left(\mathrm{Ln}\,\frac{1}{r_1}+\mathrm{Ln}\,\frac{1}{D_{21}}+\mathrm{Ln}\,\frac{1}{D_{31}}+\ \ldots\ +\mathrm{Ln}\,\frac{1}{D_{n1}}\right)\right.$$
$$\left.-\frac{I}{m}\left(\mathrm{Ln}\,\frac{1}{D_{a1}}+\mathrm{Ln}\,\frac{1}{D_{b1}}+\mathrm{Ln}\,\frac{1}{D_{c1}}+\ \ldots\ +\mathrm{Ln}\,\frac{1}{D_{m1}}\right)\right]$$

The first set of terms in parenthesis includes the contribution of all n elements of conductor x, while the second set of terms in parenthesis includes the contribution of all m elements of the conductor y. This equation can be simplified to the following form:

$$\phi_{T1}=\frac{\mu I}{2\pi}\,\mathrm{Ln}\,\frac{\sqrt[m]{D_{a1}D_{b1}D_{c1}\ldots\ldots}}{\sqrt[n]{r_1 D_{21}D_{31}\ldots\ldots}} \tag{42}$$

Similarly, the flux linkage of element 2 is:

$$\phi_{T2}=\frac{\mu I}{2\pi}\,\mathrm{Ln}\,\frac{\sqrt[m]{D_{a2}D_{b2}D_{c2}\ldots\ldots}}{\sqrt[n]{r_2 D_{12}D_{32}\ldots\ldots}} \tag{43}$$

And for element n of conductor x,

$$\phi_{Tn}=\frac{\mu I}{2\pi}\,\mathrm{Ln}\,\frac{\sqrt[m]{D_{an}D_{bn}D_{cn}\ldots\ldots}}{\sqrt[n]{r_n D_{1n}D_{2n}\ldots\ldots}} \tag{44}$$

The average of the flux linkages around any one element in the conductor x can be obtained by adding the results of equations like 42, 43, and 44 and dividing that sum by the number of elements. Thus,

$$\phi_{T \; avg} = \frac{u}{2\pi n}\left(Ln \; \frac{\sqrt[m]{D_{a1}D_{b1}D_{c1} \cdots \cdots}}{\sqrt[n]{r_1 D_{21} D_{31} \cdots \cdots}} \right.$$

$$+ Ln \; \frac{\sqrt[m]{D_{a2}D_{b2}D_{c2} \cdots \cdots}}{\sqrt[n]{r_2 \; D_{12} D_{32} \cdots \cdots}} + \cdots \cdots$$

$$\left. + Ln \; \frac{\sqrt[m]{D_{an}D_{bn}D_{cn} \cdots \cdots}}{\sqrt[n]{r_n D_{1n} D_{2n} \cdots \cdots}} \right)$$

which can be reduced to

$$\phi_{T \; avg} = \frac{\mu I}{2\pi} \; Ln \; \frac{\sqrt[nm]{D_{a1}D_{b1} \cdots \cdots D_{a2}D_{b2} \cdots \cdots D_{an}D_{bn} \cdots \cdots}}{\sqrt[n^2]{r_1 r_2 \cdots r_n \cdots D_{21} D_{31} \cdots \cdots D_{12}D_{32} \cdots \cdots D_{1n}D_{2n} \cdots \cdots}}$$

This expression may now be written in the following simple form:

$$\phi_{T \; avg} = \frac{\mu I}{2\pi} \; Ln \; \frac{D_m}{r_m} \qquad (45)$$

where D_m = geometric mean distance between conductors;
$\quad r_m$ = geometric mean radius of single conductor or group of parallel conductors.

Therefore, from equation 20:

$$L = \frac{\mu}{2\pi}\left(Ln \; \frac{D_m}{r_m} \right)10^{-8} \; \text{henries/cm} \qquad (46)$$

This equation is similar to equation 27, but D_m appears instead of D, and r_m instead of r. For solid round conductors, $D_m = D$ and $r_m = r$; and both equations are identical.

Equation 27 or equation 46 will give the inductance per conductor or per group of conductors which may be in parallel, but the equation does not take into account the effect of the ground* when used as the return circuit.

22. Single-Phase Lines.—For single-phase lines with identical conductors, the inductance per unit length of line is twice that given by equation 27 or 46. Thus,

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*For the effect of ground return, see the treatment under zero sequence impedance.

$$L_1 = \frac{\mu}{\pi}\left(\operatorname{Ln}\frac{D_m}{r_m}\right)10^{-8}\text{ henries/cm of line}\qquad(47)$$

To illustrate the application of equations 34 and 47, consider first the case of a single-phase line made up of three conductors for each side of the circuit as indicated in Fig. 12. Since conductors 1, 2, and 3 are in parallel, they must be replaced by a single conductor (see Art. 21) whose radius is equal to the geometric mean radius of the configuration 1, 2, and 3. Thus,

$$r_x = \sqrt[9]{r_1 r_2 r_3 d_{12}^2 d_{23}^2 d_{31}^2}$$

Similarly,

$$r_y = \sqrt[9]{r_a r_b r_c d_{ab}^2 d_{bc}^2 d_{ca}^2}$$

Fig. 12. Single-Phase Line with Identical Conductors.

And the equivalent distance between these two equivalent conductors is:

$$D_m = \sqrt[9]{D_{1a}D_{1b}D_{1c}D_{2a}D_{2b}D_{2c}D_{3a}D_{3b}D_{3c}}$$

If the value of r_x is not the same as that of r_y, then the reactance must be considered in two parts, as follows (see equation 34):

$$X_1 = 4.657 \times 10^{-8} f\left(\log\frac{D_m}{r_x} + \log\frac{D_m}{r_y}\right)$$

$$= 4.657 \times 10^{-8} f \log\frac{D_m^2}{r_x r_y}$$

or
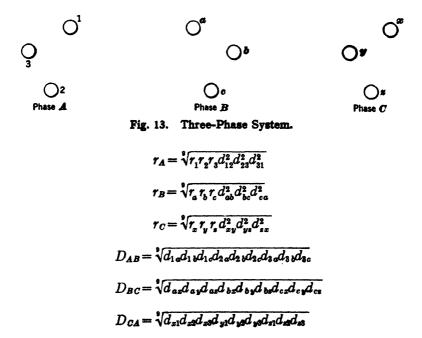
$$X_1 = 2 \times 4.567 \times 10^{-8} f \log\frac{D_m}{\sqrt{r_x r_y}}\text{ ohms/mile}\qquad(48)$$

In case the conductors of group y were reduced to only one conductor, the geometric mean radius of that conductor would be $r_y = r_b$, and, the value of r_z remaining the same as before,

$$D_m = \sqrt[3]{D_{1b}D_{2b}D_{3b}}$$

23. Three-Phase Lines.—The general application of the foregoing method to a three-phase circuit of several conductor elements per phase is similar to that for a single-phase line. To illustrate, assume a three-phase system, each phase composed of three conductors in parallel having the configurations shown in Fig. 13. In this case,

Fig. 13.   Three-Phase System.

$$r_A = \sqrt[9]{r_1 r_2 r_3 d_{12}^2 d_{23}^2 d_{31}^2}$$

$$r_B = \sqrt[9]{r_a r_b r_c d_{ab}^2 d_{bc}^2 d_{ca}^2}$$

$$r_C = \sqrt[9]{r_x r_y r_z d_{xy}^2 d_{yz}^2 d_{zx}^2}$$

$$D_{AB} = \sqrt[9]{d_{1a}d_{1b}d_{1c}d_{2a}d_{2b}d_{2c}d_{3a}d_{3b}d_{3c}}$$

$$D_{BC} = \sqrt[9]{d_{ax}d_{ay}d_{az}d_{bx}d_{by}d_{bz}d_{cx}d_{cy}d_{cz}}$$

$$D_{CA} = \sqrt[9]{d_{x1}d_{x2}d_{x3}d_{y1}d_{y2}d_{y3}d_{z1}d_{z2}d_{z3}}$$
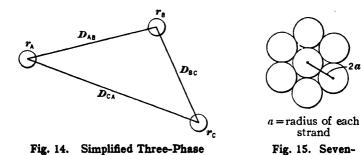
From the foregoing relations, the original system is converted into the arrangement shown in Fig. 14. If the system is fully transposed and if $r_A = r_B = r_C$, then the reactance per phase is obtained from equation 41 by using the following value:

$$D_e = \sqrt[3]{D_{AB}D_{BC}D_{CA}}$$

In using the method of Arts. 22 and 23, it must be remembered that uniform current density is assumed in all parallel elements.   In case current density is not uniform, the method will introduce errors.

24. Inductance of Stranded Conductors.—Most modern transmission lines are built with stranded conductors instead of solid wires. For this reason it is necessary to develop proper methods of handling the calculation of inductance of such conductors. This is done by replacing the actual conductor

Fig. 14. Simplified Three-Phase System.

a = radius of each strand

Fig. 15. Seven-Strand Conductor.

by an equivalent cylindrical wire of equal geometric mean radius. A stranded conductor can be considered as made up of a number of parallel wires as was demonstrated in Art. 21. Then it is only necessary to obtain the proper geometric mean radius, as follows·

$$r_m = \sqrt[n^2]{(r_1 r_2 \ldots) (D_{21} D_{31} \ldots) \ldots (D_{1n} D_{2n} \ldots)} \quad (49)$$

In this equation, r_1, r_2, etc. refer to the geometric mean radii of the several strands, each of which has been shown to be equal to $a\epsilon^{-0.25} = 0.779a$, where a = radius of the strand; the distances D refer to the distances between strands; and n is the number of strands.

The procedure is illustrated for a seven-strand conductor shown in Fig. 15. First, we proceed to list all the terms involved in equation 49, as follows:

$$r = 0.779a \ldots \ldots \ldots \ldots \quad 7 \text{ terms}$$
$$2a \ldots \ldots \ldots \ldots \ldots \ldots 24 \text{ terms}$$
$$2\sqrt{3}\,a \ldots \ldots \ldots \ldots \ldots 12 \text{ terms}$$
$$4a \ldots \ldots \ldots \ldots \ldots \ldots \quad 6 \text{ terms}$$
$$\text{Total} \ldots \ldots \ldots \ldots 49 \text{ terms}$$

Therefore,

$$r_7 = \sqrt[49]{(0.779a)^7(2a)^{24}(2\sqrt{3}\,a)^{12}(4a)^6} \tag{50}$$

In this case, $n = 7$ and $n^2 = 49$.

Ordinarily, it is best to express the geometric mean radius in terms of the outside radius b of the conductor. For the seven-strand cable, $b = 3a$ and equation 50 will reduce to:

$$r_7 = 0.726b \tag{51}$$

The foregoing method can be used to obtain the geometric mean radius of any non-magnetic conductor in which uniform
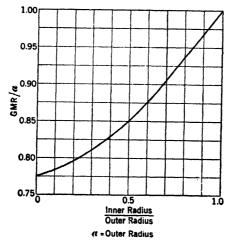
Fig. 16.   GMR of Circular Tubes.

current density is probable. For steel-cored cables it is however possible to measure the inductance and then to obtain an effective value for the geometric mean radius which will satisfy the equations of inductance. In Table 3 are listed common conductors with their corresponding geometric mean radii.

TABLE 3*

CONDUCTOR GMR

Solid Round Conductor....................................	$0.779b$
Full stranding:	
7 strands...	$0.726b$
19 strands..	$0.758b$
37 strands..	$0.768b$
61 strands..	$0.772b$
91 strands..	$0.774b$
127 strands.......................................	$0.776b$
Hollow stranded conductors and ACSR (neglecting steel strands)	
30 (two layer).....................................	$0.826b$
26 (two layer).....................................	$0.809b$
54 (three layer)....................................	$0.810b$
Single layer ACSR................................	$0.55b$ to $0.70b$
Rectangular section of sides α and β....................	$0.2235 (\alpha + \beta)$
Circular tubes.......................................	See Fig. 16

* From Wagner and Evans, "Symmetrical Components," p. 138.

CHAPTER 5

CONDUCTORS WITH EARTH RETURN

25. Inductance of Single Conductor with Earth Return.
Consider a conductor a with a geometric mean radius r_a located a distance h above the surface of the ground, as indicated in Fig. 17. The return side of the circuit is completed through the earth. Such a circuit corresponds to the overhead ground wires commonly used on high-tension lines, and also is the simplified equivalent of the power conductors in so far as zero sequence behavior is considered.

Any analysis of such a circuit must include a certain amount of empirical or test data, even though the fundamental basis be obtained from theoretical considerations. The treatment generally used for such circuits is based on the work of Carson.* His results check experimental data quite well for lines of such lengths that the end effects are negligible.
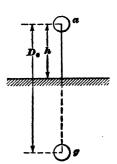
Fig. 17. Conductor with Earth Return.

The return currents will obviously follow rather irregular paths through varying cross-sectional areas. However, the basic idea involved in Carson's work involves the use of the general equation for the inductance of a two-conductor line, with certain empirical modifications. Thus, it is assumed that the return current can be concentrated in a fictitious earth conductor g with a geometric mean radius r_g of 1 foot. A rather extensive experimental study was carried out, and from it a final equivalent separation D_e between the overhead conductor and the fictitious earth conductor was obtained. A number of factors are involved in obtaining the proper value of D_e, the most important being earth con-

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*John R. Carson, "Wave Propagation in Overhead Wires with Ground Return," *The Bell Technical Journal*, Vol. V, p. 529-554, October, 1926.

ductivity, frequency, topography, and presence of bodies of water.

An expression for the inductance of a single-phase line with conductor separation D and conductor geometric mean radii r_a and r_b was derived in Art. 17 and given in equation 34. This expression is:

$$L_1 = \frac{\mu}{2\pi}\left(\mathrm{Ln}\,\frac{D^2}{r_a r_b}\right)10^{-8} \text{ henries/cm of line} \qquad (52)$$

To adapt this equation to the case of a grounded return circuit, it is only necessary to substitute the fictitious conductor separation D_e for D and $r_g = 1$ foot for r_b. Thus:

$$L_1 = \frac{\mu}{2\pi}\left(\mathrm{Ln}\,\frac{D_e{}^2}{r_a}\right)10^{-8} \text{ henries/cm of line} \qquad (53)$$

The line reactance will then become:

$$X_1 = 4.657 \times 10^{-3} f \log \frac{D_e{}^2}{r_a} \text{ ohms/mile of line} \qquad (54)$$

From the work of Carson, the value of the equivalent conductor separation is given by the following empirical equation:

$$D_e^2 = \frac{6.84 \times 10^{-3*}}{\sqrt{\lambda f}} \qquad (55)$$

where λ = absolute conductivity in mhos per cm cube, and f = frequency.

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* Some authors write equation 54 as follows:

$$X_1 = 4.657 \times 10^{-3} f \log \frac{D_e}{r_a}$$

Then, the following relation is obtained instead of equation 56:

$$D_e = 2,160 \sqrt{\frac{\rho}{f}}$$

The author prefers to use equation 56, since it conforms more definitely to the fundamental equation of a single-phase line (see equation 34 or 52), and the depth D_e has a much smaller numerical value, which seems more logical.

Equation 55 may be expressed in terms of ρ, the resistivity in ohms per meter cube, as follows:

$$D_e^2 = 2,160\sqrt{\frac{\rho}{f}} \qquad (56)$$

Typical values of D_e, as obtained by using equation 56 and taking f as 60, are given in Table 4.

TABLE 4
VALUES OF D_e FOR $f=60$

ρ	D_e^2	D_e Feet	Medium
1	280	16.7	Sea water
100	2,800	52.9	Damp earth
1,000	8,840	94	Dry earth

26. **Mutual Inductance.**—The inductance of a transmission-line conductor, in a magnetic medium of constant permeability, has been defined by equation 20 as follows:

$$L = \frac{\phi}{I}10^{-8} \text{ henries}$$

This equation may have a good many different interpretations, depending on the source of the flux linkages and on the particular value of the current used. As far as the work of this text is concerned, there are three interpretations which are important; namely, self inductances, total inductances, and mutual inductances.

(a) By self inductance the foregoing equation would involve only the flux linkages (internal and external) set up by the current flowing in the conductor for which the inductance is desired. From this self inductance, the voltage of self inductance is:

$$z_{s.i.} = -L\frac{di}{dt}$$

in which L = self inductance;

   i = current in the conductor which sets up its own flux linkages.

(b) Equation 32 gives an expression for the total flux linkages about conductor number 1, as set up by any number of conductors including conductor number 1, for the special case for which the vector sum of all the currents considered shall be zero. This expression is:

$$L = \frac{\phi_T}{I_1} 10^{-8}$$

The value of L cannot be called the self inductance, but must be denoted as the total inductance of conductor 1, since the flux linkages ϕ_T include the flux set up by all currents including the self flux linkages.

(c) Now, consider two overhead conductors a and b with earth returns, as in Fig. 18. There may be considerable room for speculation as to exactly what path and locality the two earth return currents will take. Since the area covered by the return current is quite large, there is probably good logic in assuming that the conductors a and b have a common earth return path located at an average distance D_e from a and b. With this assumption, the mutual flux set up by a and its
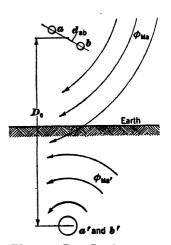
Fig. 18.   Two Conductors with Earth Return.

return a' and linking with the circuit containing b and b' can be calculated along the same lines as given in equation 24, Art. 14.

The mutual flux due to the current in conductor a is:

$$\phi_{Ma} = \frac{\mu}{2\pi} I_a \operatorname{Ln} \frac{D_e}{d_{ab}}$$

The mutual flux due to the current in the earth return with GMR equal to unity is:

$$\phi_{Ma'} = \frac{\mu}{2\pi} I_a \operatorname{Ln} D_e$$

Thus, the total mutual flux threading the circuit b and b' is:

$$\phi_M = \frac{\mu}{2\pi} I_a \operatorname{Ln} \frac{D_e^2}{d_{ab}}$$

Also, the mutual inductance of circuit a upon b is:

$$M_{ab} = \frac{\phi_M}{I_a} 10^{-8}$$

or

$$M_{ab} = \frac{\mu}{2\pi}\left(\operatorname{Ln}\frac{D_e^2}{d_{ab}}\right)10^{-8} \text{ henries/cm of conductor}$$

Similarly, the total flux set up by the currents in conductors b and b' and linking with the circuit composed of conductors a and a' will be:

$$\phi_M = \frac{\mu}{2\pi} I_b \operatorname{Ln} \frac{D_e^2}{d_{ab}}$$

Hence, the mutual inductance of circuit b upon circuit a is:

$$M_{ba} = \frac{\phi_M}{I_b} 10^{-8}$$

or

$$M_{ba} = \frac{\mu}{2\pi}\left(\operatorname{Ln}\frac{D_e^2}{d_{ab}}\right)10^{-8} \text{ henries/cm of conductor}$$

The foregoing results conform to the generally known relation among the mutual inductances, which may be expressed as follows:

$$M_{ab} = M_{ba} = M^* \tag{57}$$

Therefore,

$$M = \frac{\mu}{2\pi}\left(\operatorname{Ln}\frac{D_e^2}{d_{ab}}\right)10^{-8} \text{ henries/cm of conductor} \tag{58}$$

Finally, the mutual reactance per mile of line is:

$$X_M = 4.657 \times 10^{-3} f \log \frac{D_e^2}{d_{ab}} \text{ ohms/mile of line} \tag{59}$$

*For proof of this statement, see Timbie and Bush, *Principles of Electrical Engineering*, 3rd Edition, p. 402.

27. Impedance of Circuits with Earth Returns.—The resistance of a circuit with earth return must take into account the earth resistance as well as the conductor resistance. This resistance cannot be computed very easily, but Carson has introduced a factor in the impedance equation which has been found to be satisfactory. Thus, the total self-impedance of a circuit with earth return is:

$$Z = R_c + 1.588 \times 10^{-3} f + j\, 4.657 \times 10^{-3} f \log \frac{D_e^2}{r_a} \text{ ohms/mile} \quad (60)$$

where R_c = resistance of overhead conductor in ohms per mile, and $1.588 \times 10^{-3} f$ = earth resistance
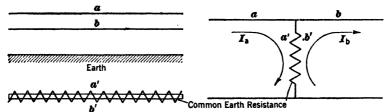
Fig. 19. Schematic Diagram of Coupled Circuit for Two Conductors with Earth Return.

The total mutual impedance will also include a resistive element, namely, that of the earth return which is common to the conductors a' and b' in Fig. 18. This feature is more clearly shown in Fig. 19, which gives a schematic representation of the coupled circuit.

Thus, the mutual impedance is:

$$Z_M = 1.588 \times 10^{-3} f + j\, 4.657 \times 10^{-3} f \log \frac{D_e^2}{d_{ab}} \text{ ohms/mile} \quad (61)$$

If conductor a is made up of three parallel strands called 1, 2, and 3, the value of d_{ab} to be used in equation 61 becomes:

$$d_{ab} = \sqrt[3]{d_{1b} d_{2b} d_{3b}}$$

If, in addition, the conductor b is composed of two parallel elements called x and y, then:

$$d_{ab} = \sqrt[6]{d_{1x} d_{1y} d_{2x} d_{2y} d_{3x} d_{3y}}$$

Other combinations of parallel elements per conductor can be handled by the same general method, as outlined in Arts. 22 and 23.

CHAPTER 6

TRANSMISSION LINE CAPACITY

28. Dielectric Flux Density Around Circular Wire.*—In Fig. 20, let the charge, or the dielectric flux, radiating from a circular wire be denoted by Q coulombs per cm of wire. Assuming an isolated wire, so that the flux may be taken as *uniformly distributed* over the conductor surface, we may then express the dielectric density at any radius as follows:

$$D = \frac{Q}{2\pi x} \text{ per sq cm} \quad (62)$$

Then, the potential across the element dx is:

$$G\,dx = \frac{1}{K}D\,dx = \frac{1}{K}\frac{Q}{2\pi x}dx \text{ volts} \quad (63)$$
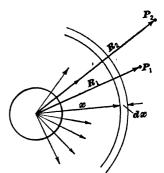
Fig. 20. **Charge from Isolated Circular Wire.**

Also, the potential between two points P_1 and P_2, at distances R_1 and R_2 from the center of the conductor, is:

$$E = \frac{Q}{2\pi K} \operatorname{Ln} \frac{R_2}{R_1} \text{ volts}$$

29. Single-Phase Transmission Line.—Before developing an expression for the capacity of a single-phase line, it is quite

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*In the solution of capacity problems as well as in the solution of inductance problems, modified CGS units are used in this text. The classical electrostatic system of units is based on the concept of a unit spherical point charge. In such a system the dielectric constant of air is defined as unity; and, because of the spherical nature of the unit charge and the introduction of constants of proportionality, the term $\dfrac{10 \times 10^{-6}}{36\pi} =$ 0.08842 × 10⁻⁶ mfds per cm cube is evolved.

In modified CGS units, the dielectric constant of air is defined as 0.08842 × 10⁻⁶, and one may look upon such a value purely as the equivalent of a test coefficient, very much as one interprets the test value for the resistivity of copper. For further details on the electrostatic system, consult "Fundamentals of Electrical Engineering," by Reed, or "Principles of Electrical Engineering," by Timbie and Bush.

important that we examine the dielectric field pattern produced by two long circular conductors supported in a homogeneous medium. In Fig. 21 are shown two small conductors a and b with respective charges, per cm of length, of $-Q$ and $+Q$ coulombs. The origin of coordinates is taken halfway between the points of concentration of the charges. Since the dielectric medium involved can be assumed to have a linear characteristic, we may obtain the potential of point P by considering the effects of the two charges as if they were acting independently.
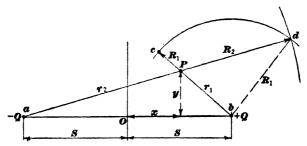
Fig. 21.   Potential of Point Due to Charges on Two Conductors.

Considering only the charge upon conductor a, the potential difference between point P and some other remote second point at a distance R_2 from conductor a is:

$$E_P = -\frac{Q}{2\pi K} \operatorname{Ln} \frac{R_2}{r_2}$$

Similarly, the charge on conductor b will contribute a potential difference between point P and a second point at a distance R_1 from that conductor. This potential difference is:

$$E_P = \frac{Q}{2\pi K} \operatorname{Ln} \frac{R_1}{r_1}$$

Notice that these potentials are from any points on the circles of radii R_1 and R_2 to point P. With respect to each individual charge, any circle represents a contour of constant potential level. Thus, with respect to charge $+Q$ upon conductor b, the potential difference between c and P is the same as the potential difference between d and P. Therefore, the

total potential difference between points d and P due to *both* charges is:

$$E_P = \frac{Q}{2\pi K}\left(\text{Ln } \frac{R_1}{r_1} - \text{Ln } \frac{R_2}{r_2}\right)$$

or

$$E_P = \frac{Q}{2\pi K}\text{ Ln }\frac{R_1 r_2}{R_2 r_1}$$

Now, let point d be removed to a very remote position, so that the distances to it from conductors a and b approach infinity. Then the ratio $\frac{R_2}{R_1}$ will approach unity, and the final form of the expression for the potential of point P becomes:

$$E_P = \frac{Q}{2\pi K}\text{ Ln }\frac{r_2}{r_1}$$

Notice particularly that, from the basic process followed in the foregoing derivation, the expression for the potential of point P really refers to its potential with respect to infinity where the dielectric field intensity becomes equal to zero.

It is now of interest to investigate the possibility of the presence of resultant contours of constant potential. Such contours can be obtained if the point P is allowed to move in space and at the same time maintains a constant potential level. To satisfy this demand, it follows that:

$$E_P = \frac{Q}{2\pi K}\text{ Ln }\frac{r_2}{r_1} = \text{constant}$$

Since the conductor charges are constant, the foregoing constant value of potential level can be obtained only if the ratio $\frac{r_2}{r_1}$ be kept equal to a constant as P occupies different positions. Thus, for constant potential level,

$$r_2 = k r_1$$

From the geometry of Fig. 21, we may express the values of r_2 and r_1 in terms of rectangular coordinates. Then,

$$\sqrt{(S+x)^2+y^2} = k\sqrt{(S-x)^2+y^2}$$

Squaring both sides and expanding, we obtain:

$$S^2 + 2Sx + x^2 + y^2 = k^2(S^2 - 2Sx + x^2 + y^2)$$

Rearranging and collecting terms, the result is:

$$x^2(1-k^2) + 2x(S + k^2 S) + y^2(1-k^2) = S^2(k^2 - 1)$$

Dividing throughout by $1-k^2$ and completing the quadratic equation, we have:

$$x^2 + 2xS\frac{1+k^2}{1-k^2} + S^2\frac{(1+k^2)^2}{(1-k^2)^2} + y^2 = S^2\frac{(1+k^2)^2}{(1-k^2)^2} - S^2$$

or

$$\left(x + S\frac{1+k^2}{1-k^2}\right)^2 + y^2 = \frac{4k^2 S^2}{(1-k^2)^2}$$

This is the equation of a circle. The coordinates of its center are:

$$x_0 = -S\frac{1+k^2}{1-k^2} \text{ and } y_0 = 0$$

and its radius is:

$$R = \frac{2kS}{1-k^2}$$

Thus, the contours of constant potential are indicated by circles with centers located on the horizontal axis, as shown in Fig. 22. For the particular case of $k = \frac{r_2}{r_1} = 1$, then $x_0 = -\infty$ and $R = \infty$. This circle will degenerate to the vertical straight line through the origin of coordinates. It should be noticed that all constant potential circles are not concentric.

In the foregoing development, it was assumed that the charges were concentrated at points a and b which are $2S$ cm apart; while in reality the charges will be carried by circular conductors of radii r which are D cm apart. Since the conductor forms an equipotential surface, we may now determine the correct conductor spacing by introducing the following limiting conditions:

$$R = \frac{2kS}{1-k^2} = r$$

and

$$2x_0 = 2S\frac{1+k^2}{1-k^2} = D$$

Hence,

$$2S = \frac{(1-k^2)r}{k}$$

and

$$D = \frac{(1-k^2)}{k} \times \frac{(1+k^2)}{(1-k^2)} r = \frac{(1+k^2)}{k} r.$$

Solving for k, we have:

$$k = \frac{D}{2r} \pm \sqrt{\frac{D^2}{4r^2} - 1}$$
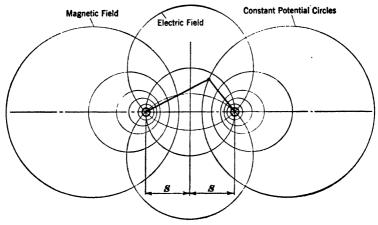
Fig. 22.   Contours of Constant Potential.

There are two possible values of $k = \dfrac{r_2}{r_1}$ which satisfy this last equation.   They are:

$$k_b = \frac{D}{2r} + \sqrt{\frac{D^2}{4r^2} - 1} \text{ and } k_a = \frac{D}{2r} - \sqrt{\frac{D^2}{4r^2} - 1}$$

The value of k_b will apply when obtaining the potential level of the surface of conductor b.   Thus:

$$E_b = \frac{Q}{2\pi K} \text{ Ln} \left(\frac{D}{2r} + \sqrt{\frac{D^2}{4r^2} - 1} \right)$$

Similarly, the value of k_a will apply to the surface of conductor a.   Thus:

$$E_a = \frac{Q}{2\pi K} \text{ Ln} \left(\frac{D}{2r} - \sqrt{\frac{D^2}{4r^2} - 1} \right)$$

Finally, the voltage drop between conductors a and b is:

$$E_{ab} = E_b - E_a$$

$$E_{ab} = \frac{Q}{2\pi K} \, \mathrm{Ln} \frac{\dfrac{D}{2r} + \sqrt{\dfrac{D^2}{4r^2} - 1}}{\dfrac{D}{2r} - \sqrt{\dfrac{D^2}{4r^2} - 1}}$$

This can be arranged in the following form:

$$E_{ab} = \frac{Q}{\pi K} \, \mathrm{Ln} \left(\frac{D}{2r} + \sqrt{\frac{D^2}{4r^2} - 1} \right) \tag{64}$$

For most overhead lines, the term $\left(\dfrac{D^2}{4r^2} - 1 \right)$ is very nearly

equal to $\dfrac{D^2}{4r^2}$. Therefore, an approximate equation for the voltage between two circular conductors is:

$$E_{ab} = \frac{Q}{\pi K} \, \mathrm{Ln} \frac{D}{r} \tag{65}$$

Finally, the capacity of a single-phase line, according to the result of equation 64, is:

$$C_{ab} = \frac{Q}{E_{ab}} = \frac{\pi K}{\mathrm{Ln} \left(\dfrac{D}{2r} + \sqrt{\dfrac{D^2}{4r^2} - 1} \right)} \tag{66}$$

An approximate value, which is quite correct for most cases, is:

$$C_{ab} = \frac{Q}{E_{ab}} = \frac{\pi K}{\mathrm{Ln} \dfrac{D}{r}} \tag{67}$$

Equations 65 and 67 can be obtained by a somewhat simpler method, based upon the assumption of uniform dielectric flux density around the surface of the two conductors. This is practically true when the spacing of conductors is large compared to the radii of the wires. The total voltage drop along all lines of dielectric flux is the same; and, for the purpose of this solution, it is convenient to take the line of flux along the horizontal line between the two conductors. The left-hand

conductor is assumed to have a charge of Q_a coulombs, and the right-hand conductor a charge of Q_b coulombs.

The voltage gradient at a point x cm from the center of the left-hand conductor, due to the charge Q_a, is:

$$G = \frac{1}{K}D = \frac{1}{K}\frac{Q_a}{A} = \frac{1}{K}\frac{Q_a}{2\pi x}$$

Also, the voltage across the element dx is:

$$G\,dx = \frac{1}{K}\frac{Q_a}{2\pi x}dx$$

Therefore, the voltage between conductors a and b, due to the charge Q_a, is:

$$E_{ab} = \int_r^{D-r}\frac{1}{K}\frac{Q_a}{2\pi x}dx = \left[\frac{1}{K}\frac{Q_a}{2\pi}\operatorname{Ln}x\right]_r^{D-r}$$

or

$$E_{ab} = \frac{1}{K}\frac{Q_a}{2\pi}\operatorname{Ln}\frac{D-r}{r}$$

In a similar manner, the voltage across dx, due to the charge Q_b, is: /

$$G\,dx = \frac{1}{K}\frac{Q_b}{2\pi(D-x)}dx$$

and the voltage between a and b, due to Q_b, is:

$$E_{ab} = -E_{ba} = \int_r^{D-r} -\frac{1}{K}\frac{Q_b}{2\pi(D-x)}dx = \left[\frac{1}{K}\frac{Q_b}{2\pi}\operatorname{Ln}(D-x)\right]_r^{D-r}$$

$$= \frac{1}{K}\frac{Q_b}{2\pi}\operatorname{Ln}\frac{r}{D-r} = -\frac{1}{K}\frac{Q_b}{2\pi}\operatorname{Ln}\frac{D-r}{r}$$

Therefore, due to Q_a and Q_b, the voltage is:

$$E_{ab} = \frac{Q_a}{2\pi K}\operatorname{Ln}\frac{D-r}{r} - \frac{Q_b}{2\pi K}\operatorname{Ln}\frac{D-r}{r}$$

But, since $Q_b = -Q_a$,

$$E_{ab} = \frac{Q_a}{\pi K}\operatorname{Ln}\frac{D-r}{r}$$

and

$$C_{ab} = \frac{Q_a}{E_{ab}} = \frac{\pi K}{\operatorname{Ln}\dfrac{D-r}{r}} \tag{68}$$

Using $K = \dfrac{10 \times 10^{-6}}{36\pi} = 0.08842 \times 10^{-6}$ mfds/cm cube, which is the proper value for air, the capacity is:

$$C_{ab} = \frac{0.278 \times 10^{-6}}{\operatorname{Ln} \dfrac{D-r}{r}} \text{ mfds/cm of line} \tag{69}$$

For overhead lines, $(D-r)$ is practically equal to D, and equation 69 can be simplified as follows:

$$C_{ab} = \frac{0.278 \times 10^{-6}}{\operatorname{Ln} \dfrac{D}{r}} \text{ mfds/cm of line}$$

or

$$C_{ab} = \frac{0.01941}{\log \dfrac{D}{r}} \text{ mfds/mile of line} \tag{70}$$

This simplification might have been introduced earlier in the foregoing procedure. Thus, due to each charge acting independently, the voltage is:

$$\left.\begin{aligned}
E_{ab} &= \frac{Q_a}{2\pi K} \operatorname{Ln} \frac{D}{r} \\[2mm]
E_{ab} &= -\frac{Q_b}{2\pi K} \operatorname{Ln} \frac{D}{r}
\end{aligned}\right\} \tag{71}$$

30. Voltage Between Two Conductors Due to Charge on Third Conductor.—The conditions in this case are represented in Fig. 23, where there are three conductors, a, b, and c; and it is required to determine the potential E_{ab} due to the charge of c. Applying Kirchhoff's law of voltages, in which all voltages must be taken in vector form, the following relation may be written:

$$E_{ab} = E_{ac} + E_{cb}$$

As normally applied, Kirchhoff's law refers to the voltages existing when all charges are taken into account. However, it is also permissible to apply Kirchhoff's law for each charge acting independently, as long as the circuit involved has linear characteristics. This procedure corresponds to the use of a

form of superposition, by which the final behavior of the circuit is the result of several effects taken individually.

From equation 71 the following relations are obtained:

$$E_{ac} = -\frac{Q_c}{2\pi K} \operatorname{Ln} \frac{D_{ac}}{r} \text{ and } E_{cb} = \frac{Q_c}{2\pi K} \operatorname{Ln} \frac{D_{cb}}{r}$$

Therefore, due to the charge Q_c, the required voltage between conductors a and b is:

$$E_{ab} = -\frac{Q_c}{2\pi K} \operatorname{Ln} \frac{D_{ac}}{r} + \frac{Q_c}{2\pi K} \operatorname{Ln} \frac{D_{cb}}{r}$$

or

$$E_{ab} = \frac{Q_c}{2\pi K} \operatorname{Ln} \frac{D_{cb}}{D_{ac}} \qquad (72)$$
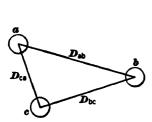
Fig. 23. Conditions for
Equation 72.
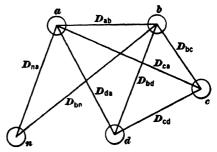
Fig. 24. Conditions for Equation 73.

31. Voltage Between Two Conductors Due to n Adjacent Conductors.—In Fig. 24 is represented a system of n conductors. It is required to determine the voltage between conductors a and b due to the charges on all the conductors.   From equations 71 and 72, we may write the general expression of the voltage between conductors a and b, due to all n charges, as follows:

$$E_{ab} = \frac{1}{2\pi K} \Bigg[(Q_a - Q_b) \operatorname{Ln} \frac{D_{ab}}{r} + Q_c \operatorname{Ln} \frac{D_{cb}}{D_{ca}}$$

$$+ Q_d \operatorname{Ln} \frac{D_{db}}{D_{da}} + \ldots + Q_n \operatorname{Ln} \frac{D_{nb}}{D_{na}} \Bigg] \qquad (73)$$

32. Capacity of Three-Phase Equilaterally Spaced Line. The simplest three-phase example is that involving balanced

voltages and equilateral conductor spacing, as indicated in Fig. 25.

The so-called capacity per phase of a polyphase line is an equivalent quantity which is conceived in order to facilitate calculations. In the case of a single-phase line (see Art. 29), the two conductors form the plates or terminals of the condenser, and the dielectric flux between these conductors is represented by the symmetrical lines from conductor to conductor in Fig 22.

If we consider the equilateral arrangement of Fig. 25, the actual capacity effect is somewhat altered. Assuming that these three conductors are located in a uniform constant dielectric medium, then we should represent the condenser
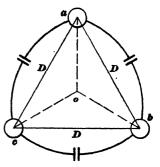
Fig. 25.   Equilateral
Conductor Spacing.

of such an arrangement of conductors by three lumped capacitors connected in delta between the three conductors, as in Fig. 25. However, from fundamental circuit work, it is known that any delta set of impedances can be represented by an equivalent set of Y-connected impedances. Thus, it is more convenient to think of the system of Fig. 25 as composed of three condensers connected in Y, with phase voltage applied between the three Y-connected conductors and the neutral point.

For known charges on the conductors, we now calculate the capacity per phase as follows:

$$C_{ao}{}^* = \frac{Q_a}{E_{ao}} \tag{74}$$

in which E_{ao} is the voltage of conductor a to neutral. For a grounded system, the neutral may be taken at ground potential.

To evaluate this expression, it is obvious that the first step is to obtain the value of charge Q_a. This step may be carried out as follows. Applying equation 73 to this case, we obtain:

*C_{ao} will turn out to be a "real" number for a physically balanced three-phase system of balanced line to neutral voltages (see equation 78).

$$E_{ab} = \frac{1}{2\pi K}\left[(Q_a - Q_b)\,\mathrm{Ln}\,\frac{D}{r} + Q_c\,\mathrm{Ln}\,\frac{D}{D}\right]$$

$$E_{ab} = \frac{1}{2\pi K}(Q_a - Q_b)\,\mathrm{Ln}\,\frac{D}{r}$$

Similarly:

$$E_{ac} = \frac{1}{2\pi K}(Q_a - Q_c)\,\mathrm{Ln}\,\frac{D}{r}$$

It will be noticed that dashes have been used under all values of Q and also under the voltage symbols, indicating that these quantities are considered as vectors. This, of course, is a necessary procedure in all polyphase capacity problems, just as it was found necessary in Art. 16 in the solution of inductance of polyphase systems.

The foregoing equations for E_{ab} and E_{ac} contain three unknowns, namely, Q_a, Q_b, and Q_c. Obviously, to solve for Q_a, another equation is necessary. Assuming an ungrounded three-phase system, then:

$$Q_a + Q_b + Q_c = 0$$

Substituting for Q_c in the equation for E_{ac} gives:

$$E_{ac} = \frac{1}{2\pi K}(2Q_a + Q_b)\,\mathrm{Ln}\,\frac{D}{r}$$

Therefore,

$$E_{ab} + E_{ac} = \frac{3}{2\pi K}Q_a\,\mathrm{Ln}\,\frac{D}{r}$$

and

$$Q_a = \frac{2\pi K(E_{ab} + E_{ac})}{3\,\mathrm{Ln}\dfrac{D}{r}} \tag{75}$$

From equation 74,

$$C_{ao} = \frac{2\pi K(E_{ab} + E_{ac})}{3E_{ao}\,\mathrm{Ln}\dfrac{D}{r}}\text{mfds/cm} \tag{76}$$

In the case of a balanced system, equation 76 may be simplified somewhat. From the vector diagram of Fig. 26, it is seen that:

$$E_{ab} + E_{ac} = 3E_{ao}$$

Thus, equation 76 becomes:

$$C_{ao} = \frac{2\pi K}{\operatorname{Ln}\dfrac{D}{r}}$$

If the value of K given in Art. 29 is introduced in the foregoing equation, then:

$$C_{ao} = \frac{0.556 \times 10^{-6}}{\operatorname{Ln}\dfrac{D}{r}}\text{mfds/cm} \tag{77}$$
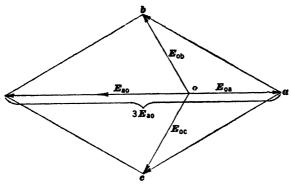
Fig. 26.   Balanced Three-Phase Line.

Equation 77 will, obviously, apply to the other two phases. Reducing this equation to the scale of miles, we have:

$$C_{ao} = \frac{0.03882}{\log\dfrac{D}{r}}\text{mfds/mile} \tag{78}$$

It will be of interest to compare this result with equation 70 for the single-phase line.

33. Capacity of Unsymmetrical Three-Phase Lines.—Let the three conductors of a three-phase line be spaced as indicated in Fig. 27.   Assuming that the line is not grounded, then:

$$Q_a + Q_b + Q_c = 0 \tag{79}$$

From Fig. 27 and equation 73, we may write:

$$E_{ab} = \frac{1}{2\pi K}\left[(Q_a - Q_b)\,\mathrm{Ln}\,\frac{D_{12}}{r} + Q_c\,\mathrm{Ln}\,\frac{D_{23}}{D_{31}} \right]$$

$$E_{ac} = \frac{1}{2\pi K}\left[(Q_a - Q_c)\,\mathrm{Ln}\,\frac{D_{31}}{r} + Q_b\,\mathrm{Ln}\,\frac{D_{23}}{D_{12}} \right] \quad (80)$$

A simple algebraic solution of equations 79 and 80 is not possible. If, however, the values of line voltages, conductor spacings, and conductor radii are known, then it is possible to solve these equations for the values of Q_a, Q_b, and Q_c; and from these results the respective values of capacities can be obtained by the application of equation 74. It will be noticed that the capacities thus obtained will be indicated by complex numbers, and the absolute value of capacity is therefore obtained as the modulus of the complex number.

Fig. 27. Unsymmetrical Three-Phase Line.

The capacities computed by this method are obviously equivalent star-connected condensers. For an unbalanced three-phase system, the line voltages will be different and the question may arise as to what is the correct phase voltage, or voltage to neutral. Such a question should not be given a general answer, but should be clearly stated for each particular problem. If the three-phase system is supplied from a Y-connected transformer bank, then the neutral potential may be considered as the potential of the transformer neutral. If the system is grounded, then phase voltages might be taken as the voltages to ground. Unless a definite clear statement is made, there is liable to be considerable confusion in such calculations.

34. Capacity of Transposed Unsymmetrical Three-Phase Lines.—The phase capacities, in an unsymmetrical system, can be equalized by a system of conductor transpositions very much as equalization of inductance was obtained in Chapter 3. Rotating the positions of the three wires in Fig. 27, we have the second conductor arrangement as shown in Fig. 28 (a).

Thus, equations 80 become:

$$E_{ab} = \frac{1}{2\pi K}\left[(Q_a - Q_b)\ \mathrm{Ln}\ \frac{D_{23}}{r} + Q_c\ \mathrm{Ln}\ \frac{D_{31}}{D_{12}} \right]$$
$$E_{ac} = \frac{1}{2\pi K}\left[(Q_a - Q_c)\ \mathrm{Ln}\ \frac{D_{12}}{r} + Q_b\ \mathrm{Ln}\ \frac{D_{31}}{D_{23}} \right]$$

(81)

Finally, rotating the positions of the conductors a third time, so as to obtain the arrangement shown in Fig. 28 (b), the following equations may be written:

$$E_{ab} = \frac{1}{2\pi K}\left[(Q_a - Q_b)\ \mathrm{Ln}\ \frac{D_{31}}{r} + Q_c\ \mathrm{Ln}\ \frac{D_{12}}{D_{23}} \right]$$
$$E_{ac} = \frac{1}{2\pi K}\left[(Q_a - Q_c)\ \mathrm{Ln}\ \frac{D_{23}}{r} + Q_b\ \mathrm{Ln}\ \frac{D_{12}}{D_{31}} \right]$$

(82)

Fig. 28.   Transposed Three-Phase Line.

Assuming that each conductor is in each possible position for a distance of one-third of the length of the line, and taking the average of equations 80, 81, and 82, we now have:

$$E_{ab} = \frac{1}{6\pi K}(Q_a - Q_b)\ \mathrm{Ln}\ \frac{D_{12}D_{23}D_{31}}{r^3}$$
$$E_{ac} = \frac{1}{6\pi K}(Q_a - Q_c)\ \mathrm{Ln}\ \frac{D_{12}D_{23}D_{31}}{r^3}$$

If we substitute an equivalent separation $D_e^3 = (D_{12}D_{23}D_{31})$ these equations become:

$$E_{ab} = \frac{1}{2\pi K}(Q_a - Q_b)\ \mathrm{Ln}\ \frac{D_e}{r}$$
$$E_{ac} = \frac{1}{2\pi K}(Q_a - Q_c)\ \mathrm{Ln}\ \frac{D_e}{r}$$

(83)

where $D_e = \sqrt[3]{D_{12}D_{23}D_{31}}$.

Assuming balanced line voltages, we can solve equations 79 and 83 for the three values of charges and then for the capacity per phase according to equation 74; we thus obtain the following result:

$$C_{ae} = \frac{0.03882}{\log \dfrac{D_e}{r}} \text{mfds/mile} \quad (84)$$

Fig. 29. Earth-Return Single-Phase Circuit.

It will be noticed that this result is of the same form as that given in equation 78 for an equilateral three-phase line, with the introduction of the equivalent conductor spacing D_e.

35. Capacity of Single-Phase Line with Earth Return. The capacity of any electrical circuit depends on the distribution of the charge or dielectric flux between positive and negative potential elements of the circuit. Fig. 22 illustrates the dielectric flux distribution between two circular conductors located in a uniform dielectric. Consider now a system like that shown in Fig. 29, which consists of a circular conductor and a flat surface, as obtained in the case of an earth-return single-phase circuit.

Since the flat surface can be treated as an equipotential plane, all lines of dielectric flux enter this plane normal to the equipotential surface. Comparing Figs. 29 and 22, it is obvious that the previous solution of Art. 29 can be applied to the present case by introducing a fictitious image conductor, as shown in Fig. 30.

This assumption takes for granted that such a flat equipotential surface can be located. There are limitations to this assumption, as caused by the irregular topography and variations in types of earth surfaces. Furthermore, the concept of an image conductor located at the same distance below

the reference plane as the conductor is located above this reference plane holds true only for a homogeneous earth having exactly the same dielectric properties as air.

Obviously, all these assumptions cannot possibly be true; but the fact still remains that we have systems connected to earth, and the earth potential or charge must not be neglected. The foregoing assumption, even though it has its faults and limitations, has been found worthy of application. Thus, adapting equation 70 to this case, we have the following result:

Fig. 30.   Earth-Return Circuit with Fictitious Image Conductor.

$$C_{ao} = \frac{0.03882}{\log \dfrac{2h}{r}} \text{ mfds/mile of line} \qquad (85)$$

36.   Effect of Earth on Three-Phase Lines.—In Fig. 31 let three conductors a, b, and c be charged to certain potentials E_{ao}, E_{bo}, and E_{co} above ground.   The general distribution of dielectric flux density between these conductors and also between the conductors and the ground is quite complex; and, as before, the method of solution is to obtain an equivalent Y-connected set of condensers, which will account for all of the charging current of the three conductors.

From the discussion of Art. 35, we may include in Fig. 31 the image conductors with their dielectric flux distribution. We thus have the complex mapping of flux in a six-conductor system, having charges Q_a, Q_b, and Q_c in the active conductors and charges $-Q_a$, $-Q_b$, and $-Q_c$ in the image conductors. The voltage between any two conductors is the sum of six separate components, each of which is due to one of the six charges. Applying equation 73 to Fig. 31, we may write:

$$E_{ab} = \frac{1}{2\pi K}\left[(Q_a - Q_b) \operatorname{Ln} \frac{D_{ab}}{r} + Q_c \operatorname{Ln} \frac{D_{cb}}{D_{ca}} \right.$$

$$\left. -Q_a \operatorname{Ln} \frac{H_{ab}}{H_a} - Q_b \operatorname{Ln} \frac{H_b}{H_{ab}} - Q_c \operatorname{Ln} \frac{H_{bc}}{H_{ac}} \right] \tag{86}$$

Similarly,

$$E_{ac} = \frac{1}{2\pi K}\left[(Q_a - Q_c) \operatorname{Ln} \frac{D_{ac}}{r} + Q_b \operatorname{Ln} \frac{D_{bc}}{D_{ab}} \right.$$

$$\left. -Q_a \operatorname{Ln} \frac{H_{ac}}{H_a} - Q_b \operatorname{Ln} \frac{H_{bc}}{H_{ab}} - Q_c \operatorname{Ln} \frac{H_c}{H_{ac}} \right] \tag{87}$$

Furthermore, we will assume the following relation:

$$Q_a + Q_b + Q_c = 0 \tag{88}$$

It is then possible to obtain from equations 86, 87 and 88 solutions for Q_a, Q_b, and Q_c in terms of the dimensions and the line voltages E_{ab} and E_{ac}. Finally, for known line to neutral voltages, E_{ao}, E_{bo}, and E_{co}, the equivalent phase capacities are of the following form:

$$C_{ao} = \frac{Q_a}{E_{ao}}$$

Fig. 31.   Earth-Return Three-Phase Line.

37. Capacity of Other Multi-Conductor Lines, with Effect of Earth.—It is common practice to operate two three-phase circuits on common towers, or in some cases to have lines of different voltages on the same towers. It should be noticed from the preceding articles that all conductors, regardless of their currents or voltages, will affect the capacity of any one conductor. Furthermore, if parallel lines on separate pole structures are close enough to each other, it will be necessary to take into account all con-

ductors of both pole structures when obtaining the capacity of any one conductor. Also, if the earth is introduced, the problem involves a system having twice as many conductors.

CHAPTER 7

THE LONG TRANSMISSION LINE

38. Derivation of Fundamental Steady-State Equations.
In low-voltage lines the shunt current between conductors is generally small enough to be neglected, unless the line is particularly long. As the line voltage is increased, this shunt current becomes more and more important. It is not possible to distinguish definitely between so-called short lines and long lines; but certainly a line is long if the shunt current amounts to a definite percentage of the full-load current.
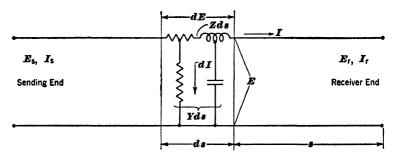
Fig. 32. One Phase of Three-Phase Line.

As pointed out at the outset of this volume, a transmission line is distinguished by the fact that the series resistance and inductance and the shunt capacity and conductance are fully distributed along the entire length of the line. Therefore, no method of circuit analysis dealing with lumped constants can be used accurately to describe the general line behavior.

In Fig. 32 is illustrated one phase of a three-phase line. Considering a balanced system, the return or neutral conductor is omitted. Since the constants of the line are distributed over the·entire length óf the line, the fundamental equations must be developed on the basis of an infinitesimal length of line ds. On this basis we may proceed to define the basic quantities involved.

60

Let $Z = R + jX$ = series impedance per mile;

$Y = G + jB$ = shunt admittance per mile.

Then, $Z\,ds$ = series impedance of length ds;

$Y\,ds$ = shunt admittance of length ds.

If the current flowing through the element ds is I, then the voltage drop across ds is:

$$dE = IZ\,ds \qquad (89)$$

If the voltage across the line at the element ds is E, then the current flowing through the shunt admittance is:

$$dI = EY\,ds \qquad (90)$$

Equations 89 and 90 form the foundation of the long-line formulas; however, the results must be obtained in a more usable form. The basic requirement is to obtain equations of voltage and current at any point along the line in terms of the terminal voltage, the terminal current, and the fundamental line constants. Since the voltage E and the current I of equations 89 and 90 are both unknown quantities, the first step in the solution is to solve these two equations simultaneously for the two unknowns. Thus, equations 89 and 90 become, by differentiation:

$$\frac{d^2E}{ds^2} = Z\frac{dI}{ds} \qquad (91)$$

and

$$\frac{d^2I}{ds^2} = Y\frac{dE}{ds} \qquad (92)$$

Substituting the value from equation 90 in equation 91 and the value from equation 89 in equation 92 will give:

$$\frac{d^2E}{ds^2} = ZYE = m^2E \qquad (93)$$

and

$$\frac{d^2I}{ds^2} = ZYI = m^2I \qquad (94)$$

where $m^2 = ZY$.

These two results are recognizable as second-order differential equations, and anyone familiar with differential equations will immediately write down the answer by inspection.

However, these equations are so important and of such a unique form that a more detailed examination will be submitted. Both of these equations are of the form:

$$\frac{d^2X}{ds^2}=m^2X$$

It should be kept in mind that X represents the unknown voltage or current and must be some function of the terminal voltage and current, of the impedance and admittance, and also of the distance s from the receiver end of the line.

The unique form of this equation is in the fact that the second derivative of X must be equal to m^2X. With this fact in mind it becomes obvious that X must be defined by an infinite-series equation of increasing exponential powers, and must also be of factorial type; otherwise, the exponential powers of the second derivative of X would be less and the coefficients greater than those found in the initial equation for X. The following expansion seems to fulfil these requirements:

$$X=1+ms+\frac{m^2s^2}{2!}+\frac{m^3s^3}{3!}+\frac{m^4s^4}{4!}+ \text{ etc.}$$

This may be tested by taking the second derivative, which is:

$$\frac{d^2X}{ds^2}=m^2\left(1+ms+\frac{m^2s^2}{2!}+\frac{m^3s^3}{3!}+\frac{m^4s^4}{4!}+ \text{ etc.}\right)$$

The foregoing expansion for X can be written in two parts, as follows:

$$X=\left(1+\frac{m^2s^2}{2!}+\frac{m^4s^4}{4!}+\ldots\right)+\left(ms+\frac{m^3s^3}{3!}+\frac{m^5s^5}{5!}+\ldots\right)$$

Applying this expansion to voltage, we have:

$$E=K_1\left(1+\frac{m^2s^2}{2!}+\frac{m^4s^4}{4!}+\ldots\right)+K_2\left(ms+\frac{m^3s^3}{3!}+\frac{m^5s^5}{5!}+\ldots\right) \quad (95)$$

The coefficients K_1 and K_2 are merely constants of proportionality which are necessary in order to balance the equation as to units.

But,

$$\cosh \textit{ms} = \frac{\epsilon^{ms} + \epsilon^{-ms}}{2} = 1 + \frac{m^2 s^2}{2!} + \frac{m^4 s^4}{4!} + \ldots \qquad (96)$$

and

$$\sinh \textit{ms} = \frac{\epsilon^{ms} - \epsilon^{-ms}}{2} = ms + \frac{m^3 s^3}{3!} + \frac{m^5 s^5}{5!} + \ldots \qquad (97)$$

Therefore,

$$E = K_1 \cosh \textit{ms} + K_2 \sinh \textit{ms} \qquad (98)$$

Similarly,

$$I = K_3 \cosh \textit{ms} + K_4 \sinh \textit{ms} \qquad (99)$$

To determine the value of the four constants of proportionality, proceed as follows: If $s = 0$, then $E = E_r$, $I = I_r$, where E_r and I_r are the receiver voltage and current. Also, $\cosh \textit{ms} = 1$ and $\sinh \textit{ms} = 0$ (see equations 96 and 97). Therefore, for $s = 0$:

$$K_1 = E_r \text{ and } K_3 = I_r$$

Substituting these results in equations 98 and 99 gives:

$$E = E_r \cosh \textit{ms} + K_2 \sinh \textit{ms} \qquad (100)$$

$$I = I_r \cosh \textit{ms} + K_4 \sinh \textit{ms} \qquad (101)$$

The last two constants, K_2 and K_4, may be determined as follows: Taking the first derivative of equation 100 gives:

$$\frac{dE}{ds} = E_r m \sinh \textit{ms} + m K_2 \cosh \textit{ms} \qquad (102)$$

But, according to equation 89, $\frac{dE}{ds} = IZ$; or, using the value of I from equation 101:

$$\frac{dE}{ds} = I_r Z \cosh \textit{ms} + K_4 Z \sinh \textit{ms} \qquad (103)$$

Equations 102 and 103 are seen to represent identities. Hence,

$$K_4 Z = E_r m \text{ and } m K_2 = I_r Z$$

from which:

$$K_2 = I_r \frac{Z}{m} = \frac{Z}{\sqrt{ZY}} I_r = \sqrt{\frac{Z}{Y}} I_r = I_r Z_0 \qquad (104)$$

and
$$K_4 = E_r \frac{m}{Z} = \frac{E_r}{Z_0} = E_r Y_0 \qquad (105)$$

where $Z_0 = \sqrt{\dfrac{Z}{Y}} = \dfrac{1}{Y_0} =$ characteristic impedance of the line.

Therefore:
$$\left. \begin{array}{l} E = E_r \cosh ms + I_r Z_0 \sinh ms \\ I = I_r \cosh ms + E_r Y_0 \sinh ms \end{array} \right\} \qquad (106)$$

Also:
$$\left. \begin{array}{l} E = A E_r + B I_r \\ I = C E_r + D I_r \end{array} \right\} \qquad (107)$$

where:
$$\left. \begin{array}{l} A = \cosh ms \\ B = Z_0 \sinh ms \\ C = Y_0 \sinh ms \\ D = A \end{array} \right\} \qquad (108)$$

39. Vector Significance of cosh ms and sinh ms.—By definition:
$$m = \sqrt{ZY} = \sqrt{(R+jX)(G+jB)}$$

If $Z = Z\underline{/\phi_1}$ and $Y = Y\underline{/\phi_2}$, then:
$$m = \sqrt{ZY\underline{/\phi_1 + \phi_2}} = \sqrt{ZY} \; \underline{\bigg/ \frac{\phi_1 + \phi_2}{2}} = m\underline{/\theta_p} \qquad (109)$$

Also, if $ms =$ complex number $= u + jv$, then:
$$\cosh ms = \cosh (u+jv)$$
$$\sinh ms = \sinh (u+jv)$$

But hyperbolic functions can be expressed in the following exponential forms:
$$\left. \begin{array}{l} \cosh ms = \dfrac{\epsilon^u \epsilon^{jv} + \epsilon^{-u} \epsilon^{-jv}}{2} \\[2mm] \sinh ms = \dfrac{\epsilon^u \epsilon^{jv} - \epsilon^{-u} \epsilon^{-jv}}{2} \end{array} \right\} \qquad (110)$$

There are two types of operators in equation 110, namely, $\epsilon^{\pm u}$ and $\epsilon^{\pm jv}$. The first of these may be called a magnitude

operator, while the second is the rotational operator as used in the exponential vector notation. The component parts of cosh ms are shown diagrammatically in Fig. 33. In other words, $\dfrac{\epsilon^u \epsilon^{jv}}{2}$ represents a positively revolving vector of increasing magnitude, while $\dfrac{\epsilon^{-u}\epsilon^{-jv}}{2}$ represents a negatively revolving vector of decreasing magnitude. Generally, therefore, hyperbolic functions combine the properties of vector rotation and change in vector magnitude.

The following relations are general:

$$\epsilon^{jv} = \cos v + j \sin v \text{ and } \epsilon^{-jv} = \cos v - j \sin v$$
$$\epsilon^u = \cosh u + \sinh u \text{ and } \epsilon^{-u} = \cosh u - \sinh u$$

Also:

$$\left.\begin{array}{l} \cosh ms = \cosh u \cos v + j \sinh u \sin v \\ \sinh ms = \sinh u \cos v + j \cosh u \sin v \end{array}\right\} \quad \textbf{(111)}$$

40. Modified Transmission-Line Equations.—The transmission-line equations 106 and 107 are in the general form most convenient for power network computations. Equations 107 are in the so-called *generalized four-terminal network* form, which will be used as the basis of the circle diagram analysis in Chapter 9. However, in order to get a better understanding of the reactions involved within a long line, it is convenient to obtain other forms of these equations. The result desired either can be obtained
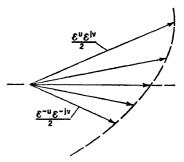
Fig. 33. Component Parts of cosh ms.

directly from the original differential equations 93 and 94 or can be evaluated from the results of equations 106. The second method will be chosen.

Substituting the values in equations 110 in the general forms of equations 106 will give:

$$E = \frac{E_r}{2}(\epsilon^u \epsilon^{jv} + \epsilon^{-u} \epsilon^{-jv}) + \frac{I_r Z_0}{2}(\epsilon^u \epsilon^{jv} - \epsilon^{-u} \epsilon^{-jv})$$

$$I = \frac{I_r}{2}(\epsilon^u \epsilon^{jv} + \epsilon^{-u} \epsilon^{-jv}) + \frac{E_r Y_0}{2}(\epsilon^u \epsilon^{jv} - \epsilon^{-u} \epsilon^{-jv})$$

With a slight rearrangement of terms, we have:

$$E = \left(\frac{E_r + I_r Z_0}{2}\right)\epsilon^u \epsilon^{jv} + \left(\frac{E_r - I_r Z_0}{2}\right)\epsilon^{-u} \epsilon^{-jv}$$

$$I = \left(\frac{I_r + E_r Y_0}{2}\right)\epsilon^u \epsilon^{jv} + \left(\frac{I_r - E_r Y_0}{2}\right)\epsilon^{-u} \epsilon^{-jv}$$

These two expressions are of the same form. Hence, we will consider only the first one in detail, the qualitative results being applicable to the current equation also. Thus, the voltage equation may be written as follows:

$$E = E_1 \epsilon^u \epsilon^{jv} + E_2 \epsilon^{-u} \epsilon^{-jv}$$

For given load conditions of voltage E_r, the current I_r and terms E_1 and E_2 are constants. The terms u and v are related in the following manner:

$$u + jv = ms$$

where $s =$ miles of line measured from the receiver end. Thus, ϵ^u will have its highest value greater than unity at the sending end of the line and will drop to a value of unity at the receiver end of the line. On the other hand, ϵ^{-u} will be equal to unity at the receiver end of the line and will drop off toward zero as the sending end is approached. The rotating operators ϵ^{jv} and ϵ^{-jv} indicate unit vectors which are in phase at the receiver end, so that $\epsilon^{\pm j0} = 1\underline{/0^\circ}$. Also, as the sending end is approached, ϵ^{jv} indicates positive or counter-clockwise rotation, while ϵ^{-jv} indicates negative or clockwise rotation.

With these basic ideas in mind, we notice that the term $E_1 \epsilon^u \epsilon^{jv}$ indicates forward wave motion from the sending end to the receiver end of the line. As the receiver is approached, this term decreases in magnitude and rotates clockwise, becoming $E_1 \epsilon^{j0}$ at the receiver end of the line. Similarly, looking at

the line from the receiver end, the term $E_2\epsilon^{-u}\epsilon^{-jv}$ represents a traveling wave moving toward the sending end. Starting from the receiver end with a value of $E_2\epsilon^{-j0}$, it gradually decreases in value and rotates clockwise as the wave travels toward the sending end.

From the point of view of wave motion there are a few special cases which it is worth while to investigate. These follow:

(a) *Open-circuited line*: For this case the receiver current I_r will be equal to zero. Therefore:

$$E = \frac{E_r}{2}\epsilon^{u}\epsilon^{jv} + \frac{E_r}{2}\epsilon^{-u}\epsilon^{-jv}$$

$$I = \frac{E_r Y_0}{2}\epsilon^{u}\epsilon^{jv} - \frac{E_r Y_0}{2}\epsilon^{-u}\epsilon^{-jv}$$

The forward and reflected waves are seen to be equal in magnitude at the receiver end. The actual receiver voltage is equal to the sum of the two waves; while the receiver current is seen to be equal to the difference between the two current waves.

(b) *Short-Circuited Line*: In this case the receiver voltage E_r will be equal to zero, so that the two basic equations become:

$$E = \frac{I_r Z_0}{2}\epsilon^{u}\epsilon^{jv} - \frac{I_r Z_0}{2}\epsilon^{-u}\epsilon^{-jv}$$

$$I = \frac{I_r}{2}\epsilon^{u}\epsilon^{jv} + \frac{I_r}{2}\epsilon^{-u}\epsilon^{-jv}$$

As before, the forward and reflected waves are equal at the receiver end of the line; but in this case the two component voltage waves neutralize each other, while the two component current waves are additive.

(c) *Line of no reflection*: A very important special case in communication networks is the line of zero reflection. Inspection of the general equations will reveal the fact that the reflected term can only vanish under the following conditions:

$$E_r = I_r Z_0 \text{ or } I_r = E_r Y_0$$

But the receiver voltage and current are obviously related through the load impedance Z_r; thus, $E_r = I_r Z_r$. Therefore,

in order to eliminate reflection, the load impedance Z_r must be equal to the characteristic impedance of the line Z_0. In this case the general equations become:

$$E = E_r \epsilon^u \epsilon^{jv}$$
$$I = I_r \epsilon^u \epsilon^{jv}$$

41. Equivalent π Circuit.—The two fundamental equations 107 are not in a suitable form for use in most symmetrical-components problems. Either π or T equivalents are used. The equivalent π circuit of Fig. 34 is derived as follows:

$$E_s = E_r + (I_r + E_r Y_\pi) Z_\pi = (1 + Y_\pi Z_\pi) E_r + Z_\pi I_r \qquad (112)$$
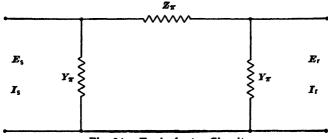
Fig. 34.   Equivalent π Circuit.

$$\left. \begin{array}{l} I_s = I_r + E_r Y_\pi + (E_r + I_r Z_\pi + E_r Y_\pi Z_\pi) Y_\pi \\ \text{or} \qquad I_s = (2Y_\pi + Y_\pi^2 Z_\pi) E_r + (1 + Y_\pi Z_\pi) I_r \end{array} \right\} \qquad (113)$$

Comparing equations 112 and 113 with the long-line equations 107, the following relations must hold if Fig. 34 is to be the equivalent of the long line:

$$1 + Y_\pi Z_\pi = A = \cosh ms$$
$$Z_\pi = B = Z_0 \sinh ms \qquad (114)$$

Therefore:

$$Y_\pi = \frac{-1 + \cosh ms}{Z_0 \sinh ms} \qquad (115)$$

42. Equivalent T Circuit.—The T circuit is shown in Fig. 35, from which the following relations are obtained:

$$E_\bullet = E_r + I_r Z_T + [I_r + (E_r + I_r Z_T) Y_T] Z_T \Bigg\}$$

$$\text{or} \qquad E_\bullet = (1 + Y_T Z_T) E_r + (2 Z_T + Y_T Z^2_T) I_r \qquad (116)$$

$$I_\bullet = I_r + (E_r + I_r Z_T) Y_T = Y_T E_r + (1 + Y_T Z_T) I_r \qquad (117)$$

Comparing equations 116 and 117 with equations 107, we find that:

$$Y_T = C = Y_0 \sinh \emph{m}s \qquad (118)$$

and $\qquad 1 + Y_T Z_T = A = \cosh \emph{m}s$

from which:

$$Z_T = \frac{-1 + \cosh \emph{m}s}{Y_0 \sinh \emph{m}s} \qquad (119)$$

43. Advantages of Equivalent Circuits.—By means of the expedients of Arts. 41 and 42, a transmission line can be accurately represented by an equivalent π or T circuit. In so far as the terminal conditions E_r, I_r, E_s, and I_s are involved, the solution of the line by either the long-line equations or the π and T circuit equivalents will of course be identical.

The equivalent circuits have a distinct advantage in that they may be more conveniently used in connection with network
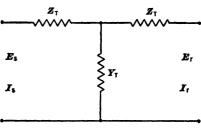
Fig. 35. Equivalent T Circuit.

problems than is possible with the actual distributed line. In case long lines are to be introduced into the general circuit layout of modern short-circuit analyzers (Art. 147), equivalent circuits must be adopted, since the design of the circuits of such a device can only be accomplished with lumped constants.

CHAPTER 8

GENERALIZED CIRCUIT EQUATIONS

44. General Statement.—In the preceding chapters the author has presented a discussion of the parameters belonging to transmission lines and also a development of the basic equations defining long lines. Transmission lines constitute only one small part of power networks. It is worth while to investigate the possibility of generalizing certain features of the operation of transformers, generators, loads, and transmission lines into one single fundamental group of equations. There will obviously be certain restrictions in the use of any mathematical expression which is applied to both machinery and circuits. The most important of these restrictions follow.

(*a*) First in importance is the assumption that a machine can accurately be represented by an equivalent circuit. It is not within the scope of this text to enter into a lengthy discussion of electrical machinery, and it will suffice to say that within certain limitations most equipment—such as transformers and generators—can be represented by some sort of equivalent static circuit, the particular type chosen being dictated by the accuracy required.

(*b*) The circuit equations commonly used are vector equations; that is, the equations involve complex numbers. This implies that all operation is of single-frequency type.

(*c*) The constants involved in these equations imply that the *R*, *L*, and *C* parameters of the equipment considered are truly constants. In actual practice such may not be the case, and the final results may be somewhat in error.

Because of the complexity of networks, such a generalization is quite worth while; and experience will determine how far such an approach can be used and where and to what extent errors may creep into the general results. The foregoing requirements may be summarized by stating that such an equivalent generalized treatment is possible if the machine or circuit possesses qualities of linearity and is also bilateral.

These statements imply that the defining parameters are actually constants and that the circuit will operate in a reversed direction in the same general manner as in its normal direction. For example, a transformer is bilateral in that it transmits power of the same frequency equally well in either direction, that is, from primary to secondary or from secondary to primary. A rectifier, on the other hand, is not bilateral. With this background of explanation, we should be ready to proceed with the development of such a generalized theory.

All electrical circuits receive power at one end and deliver power at the other. In Fig. 36 is shown such a power circuit, the input voltage and current being denoted by E_s and I_s, while the receiver voltage and current are indicated by E_r and I_r. In conformity with the general plan and as will be shown by several examples which follow, it is possible to express E_s and I_s in terms of E_r and I_r as follows:

$$\begin{aligned} E_s &= A\,E_r + B I_r \\ I_s &= C E_r + D I_r \end{aligned} \right\} \quad (120)$$

Fig. 36.   Generalized Circuit.

where A, B, C, and D are known as the generalized circuit constants. It is of interest to note that A and D are numerics, B has the dimension of ohms, while C has the dimension of mhos. These equations imply that the direction of power flow occurs from the s terminals to the r terminals.

In some cases it is advantageous to express E_r in terms of E_s and I_s. This can be accomplished by solving equations 120 as follows:

$$I_r = \frac{I_s - C E_r}{D}$$

Therefore:

$$E_s = A E_r + \frac{B I_s - B C E_r}{D}$$

from which:

$$E_r = H E_s - K I_s \qquad\qquad (121)$$

where

$$H = \frac{D}{AD - BC} \text{ and } K = \frac{B}{AD - BC}$$

For simplicity, let $N = AD - BC$. Then:

$$E_r = \frac{D}{N}E_s - \frac{B}{N}I_s \tag{122}$$

Substituting this value of E_r in the expression for I_r, we have:

$$I_r = \frac{I_s}{D} - \frac{C}{N}E_s + \frac{BC}{DN}I_s = -\frac{C}{N}E_s + \frac{N+BC}{DN}I_s$$

or

$$I_r = -\frac{C}{N}E_s + \frac{A}{N}I_s \tag{123}$$

In the case of four-terminal networks which are linear and bilateral, the term $N = AD - BC$ will always be found to be equal to unity. This statement can be proved with the aid of the "reciprocity theorem," in conjunction with two simple tests.

Test a: Apply voltage E to the sending end of Fig. 36, the receiver end being loaded with an impedance Z_L. Then:

$$E = AE_r + BI_r$$

The transfer impedance Z_{ts} from the sending end to the receiver end is defined as the ratio $\dfrac{E}{I_r}$. Thus:

$$Z_{ts} = \frac{E}{I_r} = A\frac{E_r}{I_r} + B$$

Since $Z_L = \dfrac{E_r}{I_r}$,

$$Z_{ts} = AZ_L + B \tag{124}$$

Test b: Insert the same voltage E at the load end of the network in series with Z_L and short-circuit the sending end. This is illustrated in Fig. 37. Since power is now flowing from the receiver end to the sending end of this line, equations 122 and 123 must be slightly modified by reversing the signs of all current vectors, as follows:

$$E_r = \frac{D}{N}E_s + \frac{B}{N}I_s$$

$$I_r = \frac{C}{N}E_s + \frac{A}{N}I_s$$

But, in this case, $E_s = 0$ and $E_r = E - I_r Z_L$. Therefore:

$$E - I_r Z_L = \frac{B}{N} I_s$$

and

$$I_r = \frac{A}{N} I_s$$

Substituting this last expression for I_r in the preceding expression, we have:

$$E = \frac{A}{N} Z_L I_s + \frac{B}{N} I_s = \frac{A Z_L + B}{N} I_s$$

The transfer impedance between the receiver end and the sending end is defined as follows:

$$Z_{tr} = \frac{E}{I_s} = \frac{A Z_L + B}{N} \tag{125}$$
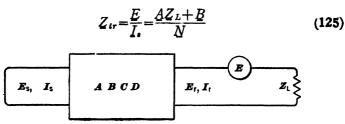
Fig. 37. Generalized Circuit.

The *reciprocity theorem* states that in a linear, bilateral four-terminal network $Z_{ts} = Z_{tr}$. This means that

$$A Z_L + B = \frac{A Z_L + B}{N}$$

or

$$N = A D - B C = 1 \tag{126}$$

Thus, for linear, bilateral networks, equations 122 and 123 become:

$$\left. \begin{array}{l} E_r = D E_s - B I_s \\ I_r = -C E_s + A I_s \end{array} \right\} \tag{127}$$

The generalized constants are generally complex in form, and may be represented as follows:
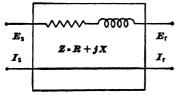
$$A = A \underline{/a} = a_1 + j a_2$$
$$B = B \underline{/\beta} = b_1 + j b_2$$
$$D = D \underline{/\Delta} = d_1 + j d_2$$

As an example of the application of the generalized equations, a few of the circuits most commonly encountered in network problems are given in Arts. 45 to 51. These examples are sufficient to indicate the generality of the method.

45. Simple Series Circuit.—In Fig. 38 is shown a circuit of resistance and inductance in series. This circuit may
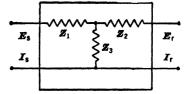
Fig. 38. Simple Series Circuit.

Fig. 39. Simple T Circuit.

represent a short transmission line (capacity neglected) or the simplified circuit of a transformer. In the case of a transformer, the total impedance is obviously based on a single voltage. By inspection, it is evident that to satisfy equations 120, the following relations must be true:

$$\left.\begin{aligned} A &= 1 \\ B &= Z = R + jX \\ C &= 0 \\ D &= 1 \end{aligned}\right\} \tag{128}$$

46. Simple T Circuit.—A T circuit like that shown in Fig. 39 may represent a transmission line or a transformer. The generalized constants for this circuit are:

$$\left.\begin{aligned} A &= 1 + \frac{Z_1}{Z_3} \\ B &= Z_1 + Z_2 + \frac{Z_1 Z_2}{Z_3} \\ C &= \frac{1}{Z_3} \\ D &= 1 + \frac{Z_2}{Z_3} \end{aligned}\right\} \tag{129}$$

The proof of these results is left to the student. The T circuit is generally used for the transformer or the induction motor, but it may be used also for a transmission line.

47. Simple π Circuit. A transmission line may sometimes be represented by a π circuit in which the capacity susceptance is divided equally between Y_s and Y_r. The generalized constants for the π circuit of Fig. 40 are:
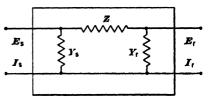
Fig. 40. Simple π Circuit.

$$\left.\begin{array}{l} A = 1 + ZY_r \\ B = Z \\ C = Y_s + Y_r + ZY_sY_r \\ D = 1 + ZY_s \end{array}\right\} \quad \textbf{(130)}$$
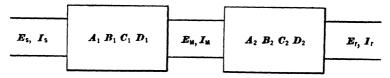
Fig. 41. Series Network.

48. Long Transmission Line.—The fundamental equations of the long line were developed in Chapter 7 (see equations 107 and 108). According to those results, the generalized constants are:

$$\left.\begin{array}{l} A = \cosh ms \\ B = Z_0 \sinh ms \\ C = Y_0 \sinh ms \\ D = \cosh ms \end{array}\right\} \quad \textbf{(131)}$$

49. Series Network.—Consider two systems in series as shown by Fig. 41. Such a combination may represent a transformer connected to a transmission line or two different types of lines connected in series. In this case it is convenient to convert the transformer constant to the voltage scale of the

transmission line; thus, E_s and I_s will be the transformer input voltage and current referred to the transmission line base.

The foregoing two systems can be combined into a single equivalent system, for which the resultant constants are:

$$\left.\begin{aligned} A_0 &= A_1 A_2 + B_1 C_2 \\ B_0 &= A_1 B_2 + B_1 D_2 \\ C_0 &= C_1 A_2 + D_1 C_2 \\ D_0 &= C_1 B_2 + D_1 D_2 \end{aligned}\right\} \tag{132}$$

The results in equations 132 can be verified by applying equations 120 to each section and then eliminating the voltage and the current at the junction of the two sections.
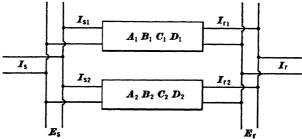
Fig. 42.   Parallel Network.

If a third network is connected in series with those of Fig. 41, a resultant may again be obtained by a second application of equations 132.   The same relative order of network notation of Fig. 41 must be maintained.   This proposition is very valuable because it makes possible a single solution for a power system which might include lines of different designs as well as transforming equipment.

50.   Parallel Network.—Parallel transmission systems may be combined so as to obtain a set of generalized constants, as in the case of a series circuit.   For example, consider the circuit of Fig. 42, in which the two parallel circuits are designated by constants A_1, B_1, C_1, D_1 and A_2, B_2, C_2, D_2.   The generalized equations for this circuit follow:

$$
\begin{aligned}
E_s &= A_1 E_r + B_1 I_{r1} & (a) \\
I_{s1} &= C_1 E_r + D_1 I_{r1} & (b) \\
E_s &= A_2 E_r + B_2 I_{r2} & (c) \\
I_{s2} &= C_2 E_r + D_2 I_{r2} & (d)
\end{aligned}
\qquad (133)
$$

Furthermore, the total current is equal to the sum of the currents to networks 1 and 2. Therefore:

$$
\begin{aligned}
I_s &= I_{s1} + I_{s2} & (a) \\
I_r &= I_{r1} + I_{r2} & (b)
\end{aligned}
\qquad (134)
$$

Substituting the value of I_{r1} from equation 134 (b) in equation 133 (a) gives:

$$
E_s = A_1 E_r + B_1 I_r - B_1 I_{r2} \qquad (135)
$$

But, from equation 133 (c),

$$
I_{r2} = \frac{E_s}{B_2} - \frac{A_2 E_r}{B_2} \qquad (136)
$$

Then, substituting equation 136 into equation 135, we have:

$$
E_s = A_1 E_r + B_1 I_r - \frac{B_1}{B_2} E_s + \frac{A_2 B_1}{B_2} E_r
$$

from which:

$$
E_s = \frac{(A_1 B_2 + A_2 B_1)}{B_1 + B_2} E_r + \frac{B_1 B_2}{B_1 + B_2} I_r \qquad (137)
$$

The sum of equations 133 (b) and (d) is:

$$
I_s = (C_1 + C_2) E_r + D_1 I_{r1} + D_2 I_{r2} \qquad (138)
$$

Eliminating I_{r1} from equation 138, the result is:

$$
I_s = (C_1 + C_2) E_r + D_1 I_r - (D_1 - D_2) I_{r2}
$$

Finally, introducing the value of I_{r2} as given by equation 136, we have:

$$
I_s = (C_1 + C_2) E_r + D_1 I_r - \frac{(D_1 - D_2)}{B_2} E_s + \frac{A_2 (D_1 - D_2)}{B_2} E_r
$$

Introducing the value of E_s from equation 137 and rearranging, we obtain the following result:

$$I_s = \left[C_1 + C_2 + \frac{(A_1 - A_2)(D_2 - D_1)}{B_1 + B_2} \right] E_r + \frac{B_1 D_2 + D_1 B_2}{B_1 + B_2} I_r \quad (139)$$

Thus, for this case, the generalized constants are:

$$A_0 = \frac{A_1 B_2 + A_2 B_1}{B_1 + B_2} \qquad (a)$$

$$B_0 = \frac{B_1 B_2}{B_1 + B_2} \qquad (b)$$

$$C_0 = C_1 + C_2 + \frac{(A_1 - A_2)(D_2 - D_1)}{B_1 + B_2} \qquad (c)$$

$$D_0 = \frac{B_1 D_2 + D_1 B_2}{B_1 + B_2} \qquad (d)$$

$$(140)$$
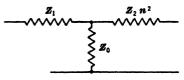
51. Special Transformers.—*A: Tap Changing.*—According to Art. 49, it is very often convenient to combine a transmission line with sending-end and receiver-end transformers and to express the relations for the entire circuit in terms of generalized constants A, B, C, and D. It has already been indicated that a transformer can be represented by a T circuit; and that, with some approximation, a simple series circuit is sufficiently accurate. The T circuit, in which all secondary impedances have been referred to the scale of the primary, is shown in Fig. 43. In this circuit,

Fig. 43.   Tap-Changing Transformer.

$Z_1 =$ primary leakage impedance;

$Z_0 =$ shunt or exciting impedance;

$Z_2 =$ secondary leakage impedance;

$n = \dfrac{N_1}{N_2} =$ ratio of primary to secondary turns.

Substituting these data in the formulas for A, B, C, and D of equations 129 gives:

$$A = 1 + \frac{Z_1}{Z_0} \qquad (a)$$

$$B = Z_1 + Z_2 n^2 + \frac{Z_1 Z_2 n^2}{Z_0} \qquad (b)$$

$$C = \frac{1}{Z_0} \qquad (c)$$

$$D = 1 + \frac{Z_2 n^2}{Z_0} \qquad (d)$$

$$(141)$$

As will be seen from the next chapter, it is often convenient to develop circle diagrams for constant sending and receiver
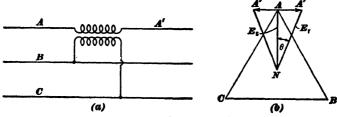
Fig. 44.   Phase-Shifting Transformer.

voltages, but for different transformer taps.   Such a procedure can be carried out quite successfully by the aid of the results of equations 141.

   B: Phase Shifting.—It is not the purpose of this text to go into the details of equipment connection or operation; but, in order to understand the fundamental idea of a phase shifter, the elementary circuit must be given, as illustrated in Fig. 44.

   The method used to obtain a shift in the phase voltage of phase A is illustrated   A transformer secondary is connected in series with line A, and this secondary is energized from a primary coil connected across the line terminals B and C. Thus, a small voltage can be added to or subtracted from the phase voltage V_{NA}, Fig. 44 (b); or the phase voltage is shifted by a small angle θ in either a positive or a negative direction, depending on the relative connections made.   Taps may also be included in the shifting transformer, so that different amounts of shifting are possible.

Neglecting the losses incident to making such a transformation, the relation between the sending and receiver voltages is:

$$E_s = (E_r \cos \theta) \underline{/\pm \theta} \tag{142}$$

In equation 142 the plus sign is used for counter-clockwise angular shifts, while the negative angle is used for clockwise angular shifts.

Obviously, if the phase-shifting transformer impedances and losses are neglected, the load power factor will be transmitted through the shifter without any modification. Thus:

$$I_s = \frac{I_r}{\cos \theta} \underline{/\pm \theta} \tag{143}$$

From equations 142 and 143, the generalized constants of a phase shifter are:

$$\left. \begin{array}{l} A = (\cos \theta) \underline{/\pm \theta} \\ B = 0 \text{ and } C = 0 \\ D = \dfrac{1}{\cos \theta} \underline{/\pm \theta} \end{array} \right\} \tag{144}$$

Phase-shifting transformers are often used in transmission networks in somewhat the same general manner as tap-changing transformers to control the relative distribution of power and reactive volt-ampere distribution as well as to make certain inter-connections possible. In some cases the steady-state solution of a problem of constant terminal voltages but with variable phase-shifter angles may be desired. By means of the results of equations 144, it is possible to set up circle diagrams for constant terminal voltages but variable angular shift, and thus obtain a measure of the power and reactive volt-ampere distribution as a function of phase shifting.

52. Power and Reactive Volt Amperes of Four-Terminal Networks.—The voltage equation for a four-terminal network has been defined as follows:

$$E_s = A E_r + B I_r$$

Solving this equation for I_r, the receiver current, we have:

$$I_r = \frac{E_s - A E_r}{B}$$

Notice that, if $I_r = I_r\underline{/\theta_a}$ and $E_r = E_r\underline{/\theta_e}$, and the conjugate of the receiver voltage $\bar{E}_r = E_r\underline{/\theta_e}$, then:

$$\begin{aligned} P_r + jQ_r^* &= \bar{E}_r I_r = E_r\underline{/-\theta_e}\, I_r\underline{/\theta_a} \\ &= E_r I_r \cos\ (\theta_a - \theta_e) + j E_r I_r \sin\ (\theta_a - \theta_e) \end{aligned}$$

The real part of this product is equal to the power, while the imaginary part equals the reactive volt-amperes of the circuit.

Applying this principle to the receiver end of a four-terminal network, we obtain for the power and reactive volt-amperes:

$$P_r + jQ_r = \bar{E}_r I_r = \frac{E_s \bar{E}_r - A E_r \bar{E}_r}{B}$$

Let:

$$E_r = E_r\underline{/0}$$

$$E_s = E_s\underline{/\phi}$$

$$A = A\underline{/a}$$

$$B = B\underline{/\beta}$$

where $\phi = $ the angle between E_s and E_r.

Substituting these data in the last expression, we have:

$$\begin{aligned} P_r + jQ_r = &\left[\frac{E_s E_r}{B} \cos\ (\phi - \beta) - \frac{A}{B} E_r^2 \cos\ (\beta - a) \right] \\ &+ j\left[\frac{E_s E_r}{B} \sin\ (\phi - \beta) + \frac{A}{B} E_r^2 \sin\ (\beta - a) \right] \end{aligned} \quad (145)$$

In equation 145:

$$\left. \begin{aligned} P_r &= \frac{E_s E_r}{B} \cos\ (\phi - \beta) - \frac{A}{B} E_r^2 \cos\ (\beta - a) \\ Q_r &= \frac{E_s E_r}{B} \sin\ (\phi - \beta) + \frac{A}{B} E_r^2 \sin\ (\beta - a) \end{aligned} \right\} \quad (146)$$

*Capacitive reactive power is taken as positive, while inductive reactive power is taken as negative. See Appendix III for a discussion of reactive power.

These equations give the variation of receiver power and receiver reactive volt-amperes as a function of the system torque-angle ϕ, and also the sending and receiver voltages which are generally held constant.

In a similar manner (see equation 127):

$$E_r = DE_s - BI_s$$

Solving for I_s, we have:

$$I_s = \frac{DE_s - E_r}{B}$$

Also, the expression for the sending-end power and reactive volt-amperes is:

$$P_s + jQ_s = \bar{E}_s I_s = \frac{DE_s \bar{E}_s - E_r \bar{E}_s}{B}$$

Substituting the respective polar values, we have:

$$P_s + jQ_s = \frac{DE_s^2 \underline{/\Delta} - E_s E_r \underline{/\phi}}{B \underline{/\beta}}$$

$$= \frac{DE_s^2}{B} \underline{/\Delta - \beta} - \frac{E_s E_r}{B} \underline{/\phi + \beta}$$

$$P_s + jQ_s = \left[-\frac{E_s E_r}{B} \cos(\phi + \beta) + \frac{DE_s^2}{B} \cos(\beta - \Delta) \right]$$

$$+ j\left[\frac{E_s E_r}{B} \sin(\phi + \beta) - \frac{DE_s^2}{B} \sin(\beta - \Delta) \right] (147)$$

In equation 147:

$$\left.\begin{array}{l} P_s = -\dfrac{E_s E_r}{B} \cos(\phi + \beta) + \dfrac{DE_s^2}{B} \cos(\beta - \Delta) \\[4mm] Q_s = \dfrac{E_s E_r}{B} \sin(\phi + \beta) - \dfrac{DE_s^2}{B} \sin(\beta - \Delta) \end{array}\right\} (148)$$

Equations 148 give the power and reactive volt-amperes at the sending end as functions of the torque angle ϕ, and sending and receiver voltages.

In case the circuit degenerates into a simple series impedance, that is, $Z = Z \underline{/\beta}$, then equations 146 and 148 become:

$$\left.\begin{array}{l} P_r = \dfrac{E_s E_r}{Z} \cos(\phi - \beta) - \dfrac{E_r^2}{Z} \cos \beta \\[2mm] Q_r = \dfrac{E_s E_r}{Z} \sin(\phi - \beta) + \dfrac{E_r^2}{Z} \sin \beta \\[2mm] P_s = -\dfrac{E_s E_r}{Z} \cos(\phi + \beta) + \dfrac{E_s^2}{Z} \cos \beta \\[2mm] Q_s = \dfrac{E_s E_r}{Z} \sin(\phi + \beta) + \dfrac{E_s^2}{Z} \sin \beta \end{array}\right\} \quad (149)$$

Finally, for the rather special case in which the series impedance is taken as inductive only, that is, $Z = Z \underline{/90} = jX$, then equations 149 become:

$$\left.\begin{array}{l} P_r = \dfrac{E_s E_r}{X} \cos(\phi - 90) = \dfrac{E_s E_r}{X} \sin \phi \\[2mm] Q_r = -\dfrac{E_s E_r}{X} \cos \phi + \dfrac{E_r^2}{X} \\[2mm] P_s = \dfrac{E_s E_r}{X} \sin \phi \\[2mm] Q_s = \dfrac{E_s E_r}{X} \cos \phi - \dfrac{E_s^2}{X} \end{array}\right\} \quad (150)$$

Coming back to the original equations 146 and 148, it is of interest to notice what conditions correspond to maximum received and maximum sending power. In equation 146, P_r will be a maximum when

$$\phi = \beta \qquad (151)$$

Also, from equation 148, P_s will be a maximum when

$$\phi = 180 - \beta \qquad (152)$$

Substituting these requirements for maximum power into the proper equations leads to the following results:

$$\left.\begin{array}{l} P_{r\,max} = \dfrac{E_s E_r}{B} - \dfrac{A}{B} E_r^2 \cos(\beta - \alpha) \\[2mm] P_{s\,max} = \dfrac{E_s E_r}{B} + \dfrac{D}{B} E_s^2 \cos(\beta - \Delta) \end{array}\right\} \quad (153)$$

The implications involved in equations 153 are of tremendous importance, and cannot be emphasized too strongly. Notice that, in arriving at the expressions for maximum power, there was no indication of the available load or generator capacity. From the consideration of the four-terminal network it is found that a certain maximum power can be delivered to the load, and a certain maximum power is accepted by the input of the network, regardless of the amount of generating capacity available at the input terminals. In reality there might be tremendous amounts of power at the input end, and yet the network has a maximum limit. More will be said about this maximum power limit in Chapter 9.

53. Equivalent Simple Series Impedance Line.—For some problems of analysis, as well as for computational ease, it is often desirable to replace the actual circuit of generalized constants A, B, C, and D by a simple series impedance. Notice, particularly, that the original circuit can be quite complex, including long lines with distributed parameters and terminal transformers or other equipment. This complex circuit may be replaced by a π equivalent by putting the expressions of equations 130 equal to the values of A, B, C, and D for the actual circuit. Thus:

$$1 + ZY_r = A$$
$$Z = B$$
$$1 + ZY_s = D$$

from which:

$$\left. \begin{array}{l} Z = B \\ Y_r = \dfrac{A-1}{B} \\ Y_s = \dfrac{D-1}{B} \end{array} \right\} \qquad (154)$$
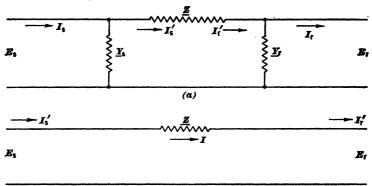
By this expedient the actual complex circuit is changed to that of Fig. 45 (a). If Y_r is considered as part of the received load, and Y_s as part of the generator equipment, a new simple series impedance is obtained, as in Fig. 45 (b). Here:

$$\left.\begin{array}{l} I'_r = I_r + E_r Y_r \\ I'_s = I_s - E_s Y_s \\ I'_s = I'_r = I \end{array}\right\} \qquad (155)$$

The circuit may now be solved as a simple series circuit, making suitable correction on the final results to account for Y_r and Y_s. The vector diagram of Fig. 45 (b) is quite simple, being represented by a voltage triangle of sides E_r, E_s, and IZ, as shown in Fig. 46.

(a)

(b)

Fig. 45. Simple Series Impedance Line.
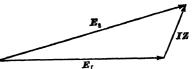
Let it be required to obtain the conditions for maximum power delivered when the ratio $\dfrac{E_s}{E_r}$ is known but the exact values of E_s and E_r are still to be chosen. This solution can be carried out graphically as follows: Having the value of impedance $Z = Z/\underline{\beta}$, construct the impedance triangle $R + jX = Z$, as shown in Fig. 47. Erect a perpendicular bisector of Z at S, and then lay off the line aO making an angle of $(90 - \beta)$ with the impedance line Z. This line aO will intersect the perpendicular bisector of Z at O. With O as a center, draw a circle with radius $Oa = Ob$.

Assume a series of values of E_s and E_r, all satisfying the particular ratio $\dfrac{E_s}{E_r}$ which is desired. From a as a center, strike arcs equal to E_s; and, from b as a center, strike arcs equal to E_r.

These arcs will intersect at points on a curve of E_s and E_r, for constant ratio $\dfrac{E_s}{E_r}$.   The desired operating point is obtained at the intersection of this curve with the circle at c.   The proper values of E_s and E_r are given by the lengths ca and cb, respectively.

Up to the present, Fig. 47 is to the scale of ohms impedance, since it was started by laying off the line $ab = Z$,

Fig. 46.   Vector Diagram for Fig. 45.

the impedance of the line.   Now it is possible to choose any particular values of E_s and E_r, provided these values satisfy the ratio $\dfrac{E_s}{E_r}$ which was initially assumed.   As soon as the choice of E_s is made, a new scale of voltage is automatically available for Fig. 47.   Thus, the distance ab is now the impedance drop along the line.   Dividing this voltage drop by the impedance Z will
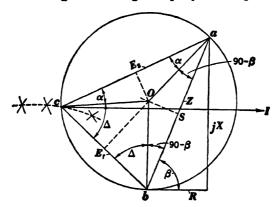
Fig. 47.   Conditions for Maximum Power.

give the current.   This current is obviously located parallel to the resistance line and perpendicular to the reactance line, as shown.

The receiver power factor, as well as the sending power factor, can readily be found from Fig. 47, so that the power

and reactive volt-amperes for both ends of the line are available. As previously indicated, corrections should be applied to these results to account for the receiver and sending shunt admittances shown in Fig. 45 (a).

In order to completely justify the construction of Fig. 47, it is still necessary to prove that the angle acb is equal to β; that is, the torque angle of the system should be equal to the line impedance angle for maximum power delivered. The necessary proof is based on the geometry of Fig. 47. In the triangle abc, the center O of the circle passing through a, b, and c is located at the intersection of the perpendicular bisectors of the sides of the triangle. The lengths of the lines connecting the point O to the corners of the triangle are obviously all equal. Thus, we have three isosceles triangles, aOb, bOc, and cOa; and therefore the corresponding angles at each corner are equal, as shown. The internal angles of the triangle abc must add up to 180 degrees; thus:

$$2\alpha + 2\Delta + 2(90 - \beta) = 180$$
$$(\alpha + \Delta) + 90 - \beta = 90$$
$$\alpha + \Delta = \phi = \beta$$

Another equivalent simple series circuit may be obtained by replacing the entire network by an equivalent generator and suitable series impedance. Such a change can be readily made by an application of Thevenin's Theorem. Looking back from the receiver terminals toward the generator, we must first determine the equivalent open-circuit voltage. This can readily be obtained by putting $I_r = 0$ in the following general equation:

$$E_s = A E_r + B I_r$$

from which:

$$E_{ro} = \frac{E_s}{A}$$

Next replace the actual network by a single impedance, which is equal to the looking-back impedance of the network. To obtain the looking-back impedance of the network of Fig. 45, we must apply a source of voltage to the receiver terminals

and connect an impedance across the sending terminals; and this impedance must be equal to the internal generator impedance. For power flow from the receiver to the sending end,

$$E_r = DE_s + BI_s$$
$$I_r = CE_s + AI_s$$

Therefore, the looking-back impedance is $Z_{LB} = \dfrac{DE_s + BI_s}{CE_s + AI_s}$.

Since the generator impedance is $Z_g = \dfrac{E_s}{I_s}$,

$$Z_{LB} = \frac{DZ_g + B}{CZ_g + A}$$

The equivalent circuit is shown in Fig. 48, for which

$$\frac{E_s}{A} = E_r + I_r Z_{LB}$$

54. Method of Obtaining Generalized Constants from Test.* —The problem of obtaining the generalized constants A, B, C, and D by computational methods is of course the only procedure in the case of an engineering design project in which the circuit is not as yet in existence. However, when the problem is to obtain these constants for circuits or networks which are already in operation, there is a possibility that either computational or test methods may be chosen.
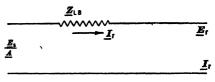
Fig. 48.   Equivalent Impedance Line.

Computational methods can be no more reliable than the accuracy of the circuit data which are available. Furthermore, if the circuit includes a very long line, the variations in circuit layout, the changes in topography, and other items of construction may greatly complicate the possibility of obtaining a high degree of accuracy in the data used for computational

*See also "Measured Electrical Constants of 270-Mile, 154 KV Transmission Line," by Streifus, Roadhaus, and Gow, *AIEE Trans.*, Vol. 63, July, 1944, p. 538.

work. Particularly in the case of zero sequence networks, the actual computational methods become quite complex and so much of the work depends on the proper choice of initial data. Thus, whenever possible, the test method should at least be considered and carried out so as to reinforce the results obtained by calculation.

To obtain the generalized constants, four measurements of impedance are desirable. In communication work these tests are quite common and are generally made by means of an impedance bridge; however, in the power field the ammeter, voltmeter, and wattmeter method must generally be used. In case the circuits contain transformers or other iron-cored devices, the test should be made as nearly at normal operating voltage as possible If this is not possible, then a sufficient amount of data should be taken so that the observer can extend his calculations to include the conditions which would exist at rated voltage. These tests include the measurement of impedances as follows:

(a) Z_{so} = sending-end impedance, with receiver open-circuited;

(b) Z_{sc} = sending-end impedance, with receiver short-circuited;

(c) Z_{ro} = receiver-end impedance, with sending end open-circuited;

(d) Z_{rc} = receiver-end impedance, with sending end short-circuited.

These measured conditions are now imposed on the two fundamental circuit equations, which are:

$$E_s = AE_r + BI_r$$
$$I_s = CE_r + DI_r$$

(a) *Receiver end open-circuited:* Under this condition, $I_r = 0$ and the general equations become:

$$E_s = AE_r \text{ and } I_s = CE_r \qquad (156)$$

Dividing E_s by I_s, we have:

$$\frac{A}{C} = Z_{so} \tag{157}$$

Furthermore, the magnitudes of A and C can readily be obtained as follows:

$$A = \frac{E_s}{E_r} \text{ and } C = \frac{I_s}{E_r} \tag{158}$$

Ordinarily, in a field test, it would be impossible to obtain the relative phase relations between E_s and E_r or between I_s and E_r; and the results of equation 158 could only be applied as to magnitude. However, even this limited application will be of extreme value as a check on the constants as obtained later. .

(b) *Receiver end short-circuited:* Obviously E_r must be equal to zero for this case, and the general equations become:

$$E_s = BI_r \text{ and } I_s = DI_r \tag{159}$$

Dividing E_s by I_s gives:

$$\frac{B}{D} = Z_{sc} \tag{160}$$

Also, in magnitude only:

$$B = \frac{E_s}{I_r} \text{ and } D = \frac{I_s}{I_r} \tag{161}$$

(c) *Sending end open-circuited:* The general equations for a four-terminal network, with power supplied to the receiver end instead of at the sending end, are similar to the equations already given; but there is an interchange of sending and receiver quantities, and also an interchange in the roles played by the constants A and D. Thus:

$$E_r = DE_s + BI_s$$
$$I_r = CE_s + AI_s$$

For the particular conditions of this test, $I_s = 0$. Therefore:

$$E_r = DE_s \text{ and } I_r = CE_s \tag{162}$$

and

$$\frac{D}{C} = Z_{ro} \tag{163}$$

As before, in magnitude only:

$$D = \frac{E_r}{E_s} \text{ and } C = \frac{I_r}{E_s} \tag{164}$$

(d) *Sending end short-circuited:* In this case, $E_s = 0$ and the general equations become:

$$E_r = BI_s \text{ and } I_r = AI_s \tag{165}$$

and
$$\frac{B}{A} = Z_{rc} \tag{166}$$

Moreover, in magnitude only:

$$B = \frac{E_r}{I_s} \text{ and } A = \frac{I_r}{I_s} \tag{167}$$

The four tests performed lead to the following relations:

$$\left.\begin{array}{ll} \dfrac{A}{C} = Z_{so} & , \quad \dfrac{B}{D} = Z_{sc} \\[2ex] \dfrac{D}{C} = Z_{ro} & , \quad \dfrac{B}{A} = Z_{rc} \end{array}\right\} \tag{168}$$

From these four results, we notice that

$$C = \pm \frac{1}{\sqrt{Z_{ro}(Z_{so} - Z_{sc})}} \tag{169}$$

Having obtained equation 169, it is then quite simple to obtain the values of A, B, and D by substitution into the proper forms of equation 168.

In checking equation 169, it will be found that the quantity $AD - BC$ was taken equal to unity, this being true for a linear and bilateral four-terminal network.

In the next chapter, use will be made of certain ratios, which can be expressed in terms of the test constants of equations 168. These ratios are:

$$\frac{A}{B} = \frac{1}{Z_{rc}} \text{ and } \frac{D}{B} = \frac{1}{Z_{sc}} \tag{170}$$

CHAPTER 9

CIRCLE DIAGRAMS

55. Introduction.—Consider the two generalized circuit equations, namely,

$$E_s = AE_r + BI_r$$
$$I_s = CE_r + DI_r \qquad \bullet$$

As already indicated, the constants A, B, C, and D are known quantities for any particular four-terminal network.

An elementary problem may involve the determination of E_s and I_s for given values of E_r and I_r. Such a solution is perfectly sensible and illustrates the general mechanism of the four-terminal network; but, from a practical point of view, it has exceedingly small importance, since the particular load conditions E_r and I_r which were used may not again repeat themselves for a considerable length of time. In other words, an engineering solution of a problem must include the entire behavior of the system. Such a solution must give all possible trends and answers. Obviously, the best method of showing these complete results is to develop suitable graphic plots or a family of curves between major variables.

One of the most interesting results concerning the general behavior of single frequency circuits is found in the fact that, invariably, circuits will be found to behave according to a circular function. To illustrate this principle, consider the simple series circuit of a constant inductive reactance and variable resistance connected to a source of constant voltage E of constant frequency. The important problem involved in such a circuit is to determine the envelope curve of the current vector as the resistance is varied from zero to infinity. Such an envelope curve will indicate at a glance all the possibilities of the circuit and also the impossibilities.

To prove this, we will start with ohm's law for the circuit. Thus:

$$I = \frac{E}{Z}$$
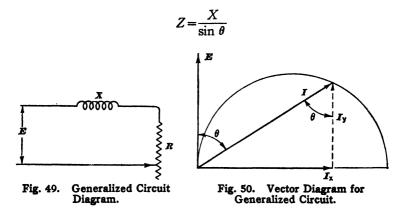
where $Z = \sqrt{R^2+X^2}$;

    E = constant applied voltage;

    R = variable resistance;

    X = constant inductive reactance.

The circuit diagram and the vector diagram are shown in Figs. 49 and 50. Notice from Fig. 50 that the circuit power-factor angle θ has such a value that

$$Z = \frac{X}{\sin \theta}$$

Fig. 49. Generalized Circuit Diagram.

Fig. 50. Vector Diagram for Generalized Circuit.

Thus ohm's law may now be written as follows:

$$I = \frac{E \sin \theta}{X}$$

This is the equation of a circle in polar coordinates, and the result may be expressed in rectangular coordinates in the following manner.

Multiplying each member of the last equation by I gives:

$$I^2 = \frac{EI \sin \theta}{X}$$

But from Fig. 50 the current I can be expressed in terms of its two components I_x and I_y. Thus:

$$I_x^2 + I_y^2 = \frac{E}{X} I_x$$

Rearranging this expression, we have:

$$I_x^2 - \frac{E}{X}I_x + \left(\frac{E}{2X}\right)^2 + I_y^2 = \left(\frac{E}{2X}\right)^2$$

$$\left(I_x - \frac{E}{2X}\right)^2 + I_y^2 = \left(\frac{E}{2X}\right)^2$$

This last expression is recognized as the equation of a circle, the center of which has the coordinates $I_x = \frac{E}{2X}$ and $I_y = 0$ and the radius of which is $\frac{E}{2X}$.

The envelope circle is shown in Fig. 50, illustrating that the current vector must always terminate on the circle. If conditions exist in which equivalent negative resistances are to be included, then we need only complete the circle of Fig. 50, thus having at a glance the complete operation of such a circuit for $-\infty < R < +\infty$. It is not the purpose of this volume to justify the meaning of the negative half-circle. Let it be noticed, however, that the complete circle in Fig. 50 forms the basis of the induction-motor and induction-generator circle diagrams, which are of great assistance in the analysis of such machines.

The simple case just used can be generalized to any four-terminal network which is linear and bilateral and is loaded with variable resistance, variable inductance, or variable capacity. Notice particularly that, as long as only one parameter is variable, the others being constant, a circle function is possible. Other circle-diagram plots also can be made, such as those for looking-in impedance and looking-in admittance. In communication networks the looking-in impedance type of circle diagram is very valuable, because in that field the engineer must always have a clear picture of all circuit impedances under all operating conditions.

In general terms, therefore, we must obtain a method of analysis which will indicate a complete picture of the entire range of the circuit operation, and this is not possible by the original generalized equations. It is indeed a very fortunate

circumstance that circle diagrams are a possibility. The ease with which circles can be drawn makes the final problem of steady-state analysis of a system quite simple and straight-forward, even though a complete picture of conditions may require many circles.

Four-terminal power networks are operated at sensibly constant input and output voltages. As will be shown in the next article, one may define the operation of such a network—when operating at constant voltages—with the aid of two circles, one circle for the receiver end and one circle for the sending end. The particular scales used may depend on particular requirements.

56. Receiver and Sending Circles.—*Receiver Circles:* Consider the fundamental equation of sending voltage, or

$$E_s = A E_r + B I_r$$

in which $A = A\underline{/\alpha}$ and $B = B\underline{/\beta}$;

$$E_r = E_r\underline{/0} \text{ and } I_r = I_r\underline{/\overline{\theta_r}};$$

$$E_s = E_s\underline{/\phi}.$$

The relations of the foregoing equation are distinctly shown in Fig. 51 (a). The vector relations chosen refer to E_r as the reference vector. For special reasons, which will be apparent later, the vector E_r is not drawn along the X-axis.

In drawing the vector diagram of Fig. 51 (a), the vector BI_r is located lagging the X-axis by the receiver power-factor angle θ_r, and the other components are located accordingly. By simple geometry it can be shown that E_r makes an angle β with the X-axis and AE_r makes an angle $(\beta - \alpha)$ with the X-axis. Thus, point M can be located by merely drawing the vector AE_r from the origin O, at an angle $(\beta - \alpha)$ with the negative X-axis. It is therefore seen that the point M is independent of the receiver power factor.

The important part of the vector diagram of Fig. 51 (a) is the triangle MON. For constant E_s, constant E_r, and variable receiver current I_r, it will be seen that the only degree of
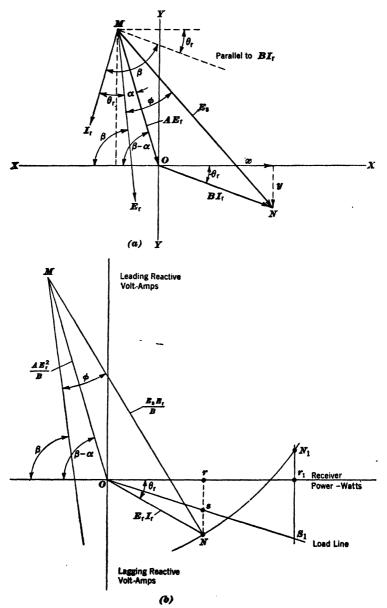
Fig. 51.   Vector Relations for Receiver Circles.

freedom in the triangle MON is for the point N to move in a circle whose radius is equal to E_s. This can only be accomplished if the receiver power factor is free to vary, which means that the receiving end of the line must be equipped with synchronous condensers operating in parallel with the load.

A more rigorous mathematical proof of the circle diagram may be given. From Fig. 51 (a) it follows that:

$$E_s = -AE_r[-\cos{(\beta-a)} + j\sin{(\beta-a)}] + BI_r[\cos{\theta_r} - j\sin{\theta_r}]$$

Now let $AE_r \cos{(\beta-a)} = h$;
$\qquad AE_r \sin{(\beta-a)} = k$;
$\qquad BI_r \cos{\theta_r} = x$;
$\qquad BI_r \sin{\theta_r} = y$.

Then, by substitution:

$$E_s = (x+h) + j(y-k)$$

The conjugate of this expression is:

$$\bar{E}_s = (x+h) - j(y-k)$$

and thus:

$$E_s^2 = (x+h)^2 + (y-k)^2 \qquad (171)$$

Equation 171 is the equation of a circle with its center M having the coordinates $x = -h$ and $y = k$ and with its radius equal to E_s.

Fig. 51 (a) is drawn to the scale of voltage. However, it is possible to change this scale to that of volt-amperes by multiplying the different vectors by the term $\dfrac{E_r}{B}$. Thus, Fig. 51 (a) is changed to Fig. 51 (b), in which:

$$MO = \frac{AE_r^2}{B}, \; ON = E_rI_r, \text{ and } MN = \frac{E_sE_r}{B}$$

The advantage of Fig. 51 (b) is rather obvious, since the projection of the point N on the X-axis indicates phase power received, while the projection of point N on the Y-axis indicates the reactive phase volt-amperes at the receiver.

The synchronous condenser capacity may be easily determined if a load line is introduced. In Fig. 51 (*b*) this line is drawn at an angle with the *X*-axis which corresponds to the load power factor. Thus for a receiver load indicated by the point *r*, there is required a total lagging reactive volt-amperes of *rN*. Of this amount the load takes care of an amount equal to *sN*. In this case the condenser is operating under lagging power factor.

Consider, however, a load indicated by r_1 watts and r_1S_1 lagging reactive volt-amperes. For this particular load, the synchronous condenser must operate at a leading power factor with a total volt-amperes capacity indicated by the distance N_1S_1, and must therefore counterbalance the load lagging reactive volt-amperes. Thus it is possible to determine the synchronous condenser requirements for any particular power load. Furthermore, the net power factor of the receiver end of the line is always given by the angle between the *X*-axis and line *ON* as *N* moves around the circle.

The effect of different voltages may be conveniently determined by including in Fig. 51 (*b*) several circles at certain selected voltages E_s and E_r which are thought worth while. This feature will be discussed in greater detail later.
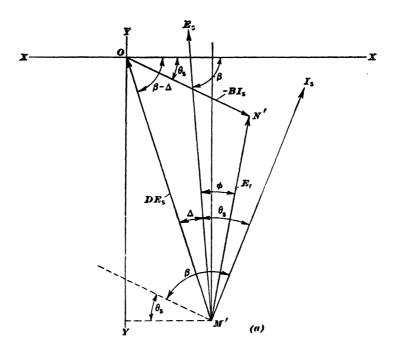
Sending Circles: The data obtained from Fig. 51 (*a*) refers entirely to the conditions at the receiver end of the line. To obtain the operating characteristics at the sending end, it is necessary to carry out a similar development, starting with equation 127 or

$$E_r = DE_s - BI_s$$

where the coefficients D and B are defined in Art. 44. In this equation, let

$$E_s = E_s \underline{/0} \text{ and } I_s = I_s \underline{/\theta_s};$$

$$D = D \underline{/\Delta} \text{ and } B = B \underline{/\beta};$$

$$E_r = E_r \underline{/\phi}$$

The angle ϕ used here is the same angle as that used in the discussion on receiver circles in connection with the expression $E_s = E_s \underline{/\phi}$. We may now plot the vector diagram in Fig. 52 (*a*) as follows:
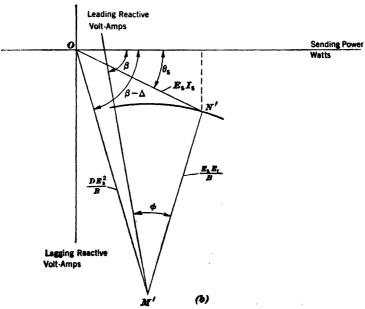
Fig. 52. Vector Relations for Sending Circles.

The vector $-BI_s$ is located lagging the X-axis by the sending power factor angle θ_s, and the other components are located accordingly.   In a manner similar to that used in Fig. 51 (a) it can be easily shown that the vector E_s makes an angle β with the X-axis, while the vector DE_s makes an angle $(\beta - \Delta)$ with the X-axis.   In other words, the point M' can be located without reference to the sending power-factor angle θ_s.

Considering that E_s and E_r are held constant, there is left only one degree of freedom in the triangle $M'ON'$, namely, in the length BI_s..   The point N' can travel in a circle having the radius E_r and the center at point M'.

It will be observed that the projection of the vector BI_s onto the X-axis is proportional to power input.   To convert the triangle $M'ON'$ to a volt-ampere scale, a multiplier $\dfrac{E_s}{B}$ must be applied; thus, in Fig. 52 (b) we have the sending diagram plotted to a volt-ampere base, in which the scale of the abscissas is power input and the scale of the ordinates is reactive volt-amperes at the sending end.

Figs. 51 (b) and 52 (b) may now be combined as indicated in Fig. 53.   In this figure, points M and M' are located in the manner indicated in Fig. 51 (b) and Fig. 52 (b).   The radii of both the receiver circle and the sending circle are equal to $\dfrac{E_s E_r}{B}$.

Consider a certain load power point, as P, on the load line. Since the receiver must operate on the circle, the length of line PN represents the leading reactive volt-amperes which must be supplied by a synchronous condenser operating in parallel with the load.   Also, the receiver power is given by the abscissa of the point P_r on the X-axis.

To obtain the operating point on the sending circle, lay off the angle ϕ from the sending reference line as it also appears between the receiver reference line and line MN.   Drawing line $M'N'$, the sending operating point is determined.   The sending power is obtained by the abscissa of the point P_s which is the projection of point N' on the X-axis.   The efficiency of
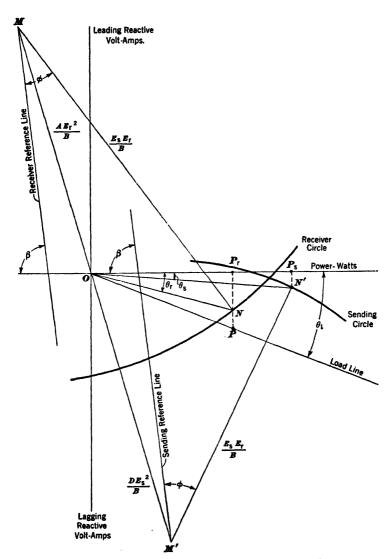
Fig. 53.   Receiver and Sending Circles Combined.

the line is obviously equal to $\dfrac{OP_r}{OP_s} \times 100$. The sending power factor is also found graphically as the angle θ_s.

57. Example.—To illustrate the application of the circle diagram, the following example is chosen.

A three-phase 110-Kv line is built with five types of construction, as follows:

Section	Length Miles	Conductor	Equivalent Spacing Feet	Z per Mile per Circuit	Y per Mile per Circuit
1	26.92		21.4	$0.235 + j0.796$	$j5.18 \times 10^{-6}$
2	40.4	397,500	17.6	$0.235 + j0.773$	$j5.35 \times 10^{-6}$
3	9.35	ACSR	17.6	$0.235 + j0.773$	$j5.35 \times 10^{-6}$
4	20.75		16.3	$0.235 + j0.763$	$j5.41 \times 10^{-6}$
5	27.99	2/0 H.D.S. 7 strands	11.34	$0.4403 + j0.826$	$j5.194 \times 10^{-6}$

The first step in the solution of this problem is to calculate the generalized circuit constants A, B, C, and D for each section, as indicated in equations 108. The next step is to combine the five sections, thus obtaining the four generalized constants of the entire line, as indicated in equation 132. The results obtained for this example are:

$$A = 0.95572 + j0.009147$$
$$B = 28.4 + j85.95$$
$$C = 0 + j0.000802$$
$$D = 0.973 + j0.00977$$

The following data are assumed:

> Line voltage at sending end = 113 Kv;
> Line voltage at receiver end = 105 Kv;
> Power factor of load = 95% lagging.

Using these data, the circle diagrams of Fig. 54 were obtained. From the results of Fig. 54, it is now possible to determine the detailed characteristics of the line, as follows:

Receiver		Angle Between Sending and Receiver Voltages Degrees	Sending	
Load Kw	Synch. Condenser Kva		Power Kw	Reactive Power Kva
0	− 15,100	− 1.5	300	− 6,700
10,000	− 8,300	2.9	10,500	− 3,500
20,000	− 400	7.8	21,500	− 1,000
30,000	8,500	13.0	33,500	200
40,000	18,000	18.5	45,400	1,000
50,000	29,700	24.3	58,000	800
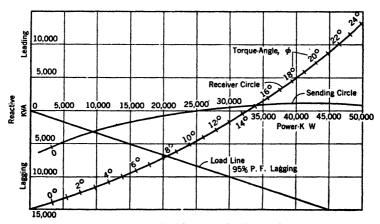
Fig. 54.   Circle Diagrams for Example.

In the tabulation, lagging Kva is indicated as a negative quantity.   Also, a positive angle indicates the angle by which the receiver voltage lags the sending voltage.

This problem can be solved for other values of sending and receiver voltages.   As a suggestion, 5 per cent above and below the values used might be tried.

58.   Special Form of Circle Diagram.—When several send-ing and receiver voltages are to be investigated, there is a special modification of the scales of Fig. 53 that is quite useful. The several distances of Fig. 53 are as follows:

$$OM = \frac{AE_r^2}{B}, \quad MN = \frac{E_s E_r}{B}, \quad ON = E_r I_r$$

$$OM' = \frac{DE_s^2}{B}, \quad M'N' = \frac{E_s E_r}{B}, \quad ON' = E_s I_s$$

If the first set is divided by E_r^2 and the second set is divided by E_s^2, then:

$$OM = \frac{A}{B}, \quad MN = \frac{1}{B} \times \frac{E_s}{E_r}, \quad ON = \frac{E_r I_r}{E_r^2}$$

$$OM' = \frac{D}{B}, \quad M'N' = \frac{1}{B} \times \frac{E_r}{E_s}, \quad ON' = \frac{E_s I_s}{E_s^2}$$

By referring to Fig. 53 it will be realized that, if a family of circles are drawn to a volt-ampere scale, they will overlap each other, thus giving a resultant diagram which is hard to use. By using the modified scale, the distances from the origin O to the two circle centers are constant, and circles for different voltages will be concentric, as shown in Fig. 55. As just shown,

$$ON = \frac{E_r I_r}{E_r^2} = \frac{\text{Receiver Volt-Amperes}}{E_r^2}$$

and

$$ON' = \frac{E_s I_s}{E_s^2} = \frac{\text{Sending Volt-Amperes}}{E_s^2}$$

Then the projections of ON and ON' on the X-axis and Y-axis will be as follows:

$$ON \cos \theta_r = \frac{P_r}{E_r^2} = \frac{\text{Receiver Power}}{E_r^2}$$

$$ON' \cos \theta_s = \frac{P_s}{E_s^2} = \frac{\text{Sending Power}}{E_s^2}$$
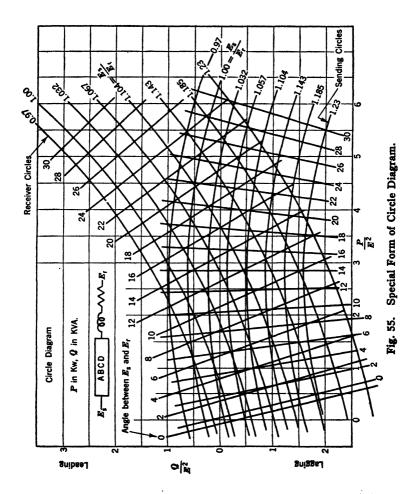
$$ON \sin \theta_r = \frac{Q_r}{E_r^2} = \frac{\text{Receiver Reactive V.I.}}{E_r^2}$$

$$ON' \sin \theta_s = \frac{Q_s}{E_s^2} = \frac{\text{Sending Reactive V.I.}}{E_s^2}$$

Circles may be drawn for different voltage ratios. Thus:

$$K_{sr} = \frac{E_s}{E_r} \quad \text{and} \quad K_{rs} = \frac{E_r}{E_s}, \quad \text{where} \quad K_{sr} = \frac{1}{K_{rs}}$$

Inspection of these results will indicate that this change reduces the circles to the scale of admittances.

Fig. 55. Special Form of Circle Diagram.

The method of using a diagram like that in Fig. 55 is illustrated by the following example.

59. Example.—In this example a transmission line with step down transformers at the receiver end is considered.

The data for the transmission line are as follows:

Normal voltage = 154 Kv;
Length = 84.74 miles;
Conductor = 397,500 ACSR;
Equivalent spacing = 21.4 ft;
Impedance = $0.235 + j0.796$ per mile;
Admittance = $0 + j5.19 \times 10^{-6}$ per mile;
$A = 0.985 + j0.00435$;
$B = 19.65 + j67.1$;
$C = 0 + j0.000438$;
$D = A$

The data for the transformers are:

Single phase, auto-transformer with tertiary windings;
Voltages = 89,000/63,600 volts;
Reactance = 5 per cent;
Resistance = 0.35 per cent;
Capacity = 75,000 Kva for three phases;
Exciting current negligible.

The first step in the solution is to combine the transmission-line constants with those of the transformer, and to draw the circle diagrams of Fig. 55 from these results. Such a method of drawing the circle diagrams is quite advantageous, due to obvious symmetry, which lends itself well to interpretation. It will be noted that the specific voltages are not fixed, and that any curve is applicable to any sending and receiver voltages as long as the ratio of voltages corresponds to the particular circle.

To illustrate the use of Fig. 55, assume that $\dfrac{P_r}{E_r^2} = 4$ and the ratio $\dfrac{E_s}{E_r} = 1.104$.

Then:

$$\frac{Q_r}{E_r^2} = +0.2$$

The angle between E_s and $E_r = 17.6°$.

$$\frac{P_s}{E_s^2} = 3.55$$

$$\frac{Q_s}{E_s^2} = -0.5$$

For $E_r = 105$ Kv on the low side of the transformer or 147 Kv on the high side of the transformer,

$$E_s = 147 \times 1.104 = 162 \text{ Kv}$$

The corresponding phase voltages are:

$$E_r = 85 \text{ Kv and } E_s = 93.6 \text{ Kv}$$

Then:

$$P_r = 4 \times \overline{85}^2 = 28,900 \text{ Kw}$$

$$P_s = 3.55 \times \overline{93.6}^2 = 31,200 \text{ Kw}$$

$$Q_r = +0.2 \times \overline{85}^2 = 1,445 \text{ leading Kva}$$

$$Q_s = -0.5 \times \overline{93.6}^2 = 4,380 \text{ lagging Kva}$$

From these figures, the following pertinent results may be obtained:

$$\text{Efficiency of line} = \frac{28,900}{31,200} \times 100 = 92.6 \text{ per cent}$$

$$\text{Receiver power factor} = \cos\left(\tan^{-1}\frac{1,445}{28,900}\right) = 0.999$$

$$\text{Sending power factor} = \cos\left(\tan^{-1}\frac{4,380}{31,200}\right) = 0.99$$

Synchronous condenser capacity for a 95% power factor load is

$$1,445 + \frac{28,900}{0.95} \sin \theta_l = 1,445 + 30,400 \times 0.312$$

$$= 10,845 \text{ Kva per phase}$$

60. Another Form of Circle Diagram.—Another modification of the circle diagram of Fig. 53 may be developed. This is of particular value for problems in which either the sending voltage or the receiver voltage is to be held absolutely constant.

Thus, suppose a new line is designed to connect a new hydro-electric development to an existing high-tension system. The receiver voltage will, therefore, be sensibly constant; but, with tap changing transformers at the sending end, the sending voltage E_s will be adjustable to any of several values.

In this case the distances of Fig. 53 may be divided by the square of the constant voltage or E_r^2. The results are:

$$OM = \frac{A}{B}, \; MN = \frac{1}{B} \times \frac{E_s}{E_r}, \; ON = \frac{E_r I_r}{E_r^2}$$

$$OM' = \frac{D}{B} \times \left(\frac{E_s}{E_r}\right)^2, \; M'N' = \frac{1}{B} \times \frac{E_s}{E_r}, \; ON' = \frac{E_s I_s}{E_r^2}$$

This method has the advantage that the x- and y-scales of the circle diagrams may still be retained as power and reactive volt-amperes, since E_r^2 is a constant. The receiver circles will be concentric, while the sending circles will overlap.

Still another modification can be obtained for the condition of E_s being constant while E_r is considered as a fundamental variable. The proper scales and the appearance of the diagram are rather obvious from the preceding discussion.

61. Maximum Power Limit.—Insofar as the electrical characteristics are concerned, a transmission line does not differ a great deal from a synchronous motor. Just as a synchronous motor has a maximum power output limit, at which it pulls out of step, so does the transmission line have a power limit. Such a power limit is entirely independent of the power supply; for constant sending and receiver voltages, it is merely a function of the constants of the line.

The value of the maximum power limit can be readily obtained from the circle diagram of Fig. 51 (b), which is modified as shown in Fig. 56. As seen from Fig. 56, the maximum power delivered for a particular sending and receiver voltage is obviously indicated by the value P_{max}, which is:

$$P_{max} = \frac{E_s E_r}{B} - \frac{A E_r^2}{B} \cos (\beta - \alpha) \qquad (172)$$

The torque angle between the sending and receiver voltages is given by the angle ϕ of Fig. 51 (b). For the condition of maximum receiver power, this angle becomes ϕ_{max}, Fig. 56, which is seen to be equal to the angle β. It will be recalled that the angle β is defined by the generalized circuit coefficient B. Thus:

$$\underline{B} = B\underline{/\beta} = b_1 + jb_2$$
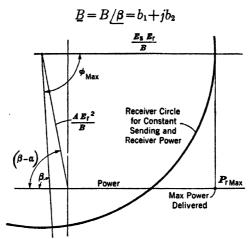
Fig. 56. Conditions for Maximum Power.

In commercial lines the value of b_2 is generally quite large compared to b_1; hence, the angle β is less than 90°, approaching 90° as a theoretical limit.

It will be noticed that the result of equation 172 is exactly the same as was previously derived in equation 153, Art. 52.

Similar graphical interpretation for the maximum input power can be obtained by analysis of the sending circles of Fig. 52 (b).

62. Loss Circle Diagrams.—Circle diagrams for receiver as well as sending power have been given in the last few articles. It is now proposed to derive circle equations for constant power loss in four-terminal networks. This proof starts with the two fundamental circuit equations, or:

$$E_s = AE_r + BI_r$$
$$I_s = CE_r + DI_r$$

As a preliminary problem, it is required to prove the following relation:

$$P_s = \frac{1}{2}(\bar{E}_s I_s + E_s \bar{I}_s)$$

where P_s = sending power.

Let:

$$E_s = E_s \underline{/\delta}$$
$$\bar{E}_s = E_s \underline{/\bar{\delta}}$$
$$I_s = I_s \underline{/\phi}$$
$$\bar{I}_s = I_s \underline{/\bar{\phi}}$$

Therefore:

$$P_s = \frac{1}{2}\Big(E_s \underline{/\bar{\delta}} I_s \underline{/\phi} + E_s \underline{/\delta} I_s \underline{/\bar{\phi}} \Big)$$
$$= \frac{1}{2} E_s I_s \Big(\underline{/\bar{\delta-\phi}} + \underline{/\delta-\phi} \Big)$$
$$= \frac{1}{2}(2 E_s I_s \cos \theta_s) = E_s I_s \cos \theta_s$$

where $\theta_s = \delta - \phi$ = sending power-factor angle.

Thus applying the foregoing proposition:

$$P_s = \frac{1}{2}[\bar{E}_s I_s + E_s \bar{I}_s]$$

$$= \frac{1}{2}[(\bar{A}\bar{E}_r + \bar{B}\bar{I}_r)(CE_r + DI_r) + (AE_r + BI_r)(\bar{C}\bar{E}_r + \bar{D}\bar{I}_r)]$$

$$= \frac{1}{2}[\bar{A}CE_r^2 + \bar{B}DI_r^2 + \bar{A}D\bar{E}_r I_r + \bar{B}CE_r \bar{I}_r$$
$$+ A\bar{C}E_r^2 + B\bar{D}I_r^2 + A\bar{D}E_r \bar{I}_r + B\bar{C}\bar{E}_r I_r]$$

$$= \frac{1}{2}[2AC \cos (\tau - \alpha)E_r^2 + 2BD \cos (\beta - \Delta)I_r^2$$
$$+ (P_r + jQ_r)(\bar{A}D + B\bar{C}) + (P_r - jQ_r)(A\bar{D} + \bar{B}C)]$$

$$= \frac{1}{2}[2AC \cos (\tau - \alpha)E_r^2 + 2BD \cos (\beta - \Delta)I_r^2$$
$$+ P_r(\bar{A}D + A\bar{D} + B\bar{C} + \bar{B}C) - jQ_r(\bar{A}D - A\bar{D} + B\bar{C} - \bar{B}C)]$$

$$= \frac{1}{2}\{2AC \cos (\tau - \alpha)E_r^2 + 2BD \cos (\beta - \Delta)I_r^2$$
$$+ P_r[2AD \cos (\alpha - \Delta) + 2BC \cos (\beta - \tau)]$$
$$- jQ_r[2AD \, j\sin(\Delta - \alpha) + 2BC \, j\sin(\beta - \tau)]\}$$

$$P_s = AC \cos (\tau - a)E_r^2 + BD \cos (\beta - \Delta)I_r^2$$
$$+ [AD \cos (\Delta - a) + BC \cos (\beta - \tau)]P_r$$
$$- [AD \sin (\Delta - a) + BC \sin (\beta - \tau)]Q_r$$

or
$$P_s = hE_r^2 + kI_r^2 + pP_r + qQ_r \tag{173}$$

where $C = C/\tau$;

$h = AC \cos (\tau - a)$;

$k = BD \cos (\beta - \Delta)$;

$p = AD \cos (\Delta - a) + BC \cos (\beta - \tau)$;

$q = -AD \sin (\Delta - a) - BC \sin (\beta - \tau)$.

The power loss P of a line is equal to the input power minus the received power. Thus, $P = P_s - P_r$, or

$$P = hE_r^2 + kI_r^2 + (p-1)P_r + qQ_r \tag{174}$$

Since $I_r^2 = \dfrac{P_r^2 + Q_r^2}{E_r^2}$,

$$PE_r^2 = hE_r^4 + k(P_r^2 + Q_r^2) + (p-1)E_r^2 P_r + qE_r^2 Q_r$$

or
$$P_r^2 + (p-1)\frac{E_r^2}{k}P_r + \left(Q_r^2 + q\frac{E_r^2}{k}Q_r\right) = -\frac{h}{k}E_r^4 + P\frac{E_r^2}{k}$$

Completing squares in the above equation, we have:

$$\left[P_r + \frac{p-1}{2k}E_r^2\right]^2 + \left[Q_r + \frac{q}{2k}E_r^2\right]^2 = E_r^4\left[-\frac{h}{k} + \frac{(p-1)^2}{4k^2} + \frac{q^2}{4k^2}\right] + P\frac{E_r^2}{k}$$

from which

$$\left[P_r + \frac{p-1}{2k}E_r^2\right]^2 + \left[Q_r + \frac{q}{2k}E_r^2\right]^2 =$$
$$\frac{E_r^2}{k^2}\left[\left\{-hk + \frac{(p-1)^2 + q^2}{4}\right\}E_r^2 + kP\right]$$

Finally:
$$[P_r + m]^2 + [Q_r + n]^2 = R^2 \tag{175}$$

where $m = \dfrac{p-1}{2k}E_r^2$;

$n = \dfrac{q}{2k}E_r^2$;

$$R = \frac{E_r}{k}\left[kP + \left\{ \frac{(p-1)^2+q^2}{4} - hk \right\} E_r^2 \right]^{\frac{1}{2}} = E_r \sqrt{\frac{P-m}{k}}$$

Equation 175 represents a circle plotted to the scale of P_r and Q_r with the coordinates of its center $-m$ and $-n$ and its radius equal to R. In equation 175 all quantities are assumed to be on a per-phase basis, as seen from the initial equations used in setting up the proof. P_r represents the receiver power in watts per phase; Q_r, the receiver reactive volt-amperes per phase; and P, the loss in watts per phase.

Since the π circuit is quite often used in transmission-line and network problems, it is well to apply equation 175 to this special case. Thus, for the equivalent π circuit, we have:

$$A = 1 + ZY_R$$
$$B = Z$$
$$C = (Y_R + Y_s + ZY_R Y_s)$$
$$D = (1 + ZY_s)$$

Also,
$$Z = R + jX$$
$$Y_r = G_R + jB_R$$
$$Y_s = G_s + jB_s$$

By using these constants we may now obtain the necessary data for equation 175, as follows:

$$h = G_R + G_s + 2RG_R G_s - 2XB_R G_s + Y_r^2 k$$

$$k = BD \cos{(\beta - \Delta)} = R + Z^2 G_s$$

$$p = AD \cos{(\Delta - a)} + BC \cos{(\beta - \tau)}$$

$$= 1 + 2RG_R + 2RG_s + 2Z^2 G_s G_R$$

$$= 1 + 2RG_s + 2G_R k$$

$$q = -2RB_R + 2XG_s - 2Z^2 G_s B_R = 2XG_s - 2B_R k$$

$$hk = G_R k + G_s k + 2RG_R G_s k - 2XB_R G_s k + Y_r^2 k^2$$

$$m = \left(\frac{RG_s}{R + Z^2 G_s} + G_R \right)$$

$$n = \left(\frac{XG_s}{R + Z^2 G_s} - B_R \right)$$

$$\frac{(p-1)^2 + q^2}{4} - hk = -RG_s - G_R(R + Z^2 G_s)$$

Therefore, substituting in equation 175, we obtain:

$$\left[P_r + \left(\frac{RG_s}{R + Z^2 G_s} + G_R \right) E_r^2 \right]^2 + \left[Q_r + \left(\frac{XG_s}{R + Z^2 G_s} - B_R \right) E_r^2 \right]^2 =$$

$$\frac{E_r^2 [(R + Z^2 G_s) P - \{ RG_s + G_R (R + Z^2 G_s) \} E_r^2]}{(R + Z^2 G_s)^2} \qquad (176)$$

63. Efficiency Circle Diagram.—Circle diagrams for constant efficiency may be obtained in somewhat the same general manner as were the constant loss circles. Thus, efficiency is:

$$\eta = \frac{P_r}{P_s} = \frac{P_r}{h E_r^2 + k I_r^2 + p P_r + q Q_r}$$

Rearranging:

$$P_r^2 + \left(p - \frac{1}{\eta} \right) \frac{E_r^2}{k} P_r + Q_r^2 + \frac{q}{k} E_r^2 Q_r = -\frac{h}{k} E_r^4$$

Completing squares:

$$\left[P_r + \left(p - \frac{1}{\eta} \right) \frac{E_r^2}{2k} \right]^2 + \left[Q_r + q \frac{E_r^2}{2k} \right]^2 =$$

$$\left[\left(p - \frac{1}{\eta} \right)^2 + q^2 \right] \frac{E_r^4}{4k^2} - \frac{h}{k} E_r^4 \qquad (177)$$

Equation 177 represents a circle for constant efficiency η. The coordinates of the center of this circle are:

$$-\left(p - \frac{1}{\eta} \right) \frac{E_r^2}{2k} \text{ and } -q \frac{E_r^2}{2k}$$

and its radius is equal to:

$$\left[\frac{\left(p - \frac{1}{\eta} \right)^2 + q^2}{4} - hk \right]^{\frac{1}{2}} \frac{E_r^2}{k}$$

Applying equation 177 to the special case of a π network, we have:

$$\left[P_r+\left(1-\frac{1}{\eta}+2RG_s+2G_Rk\right)\frac{E_r^2}{2k}\right]^2+\left[Q_r+\left(\frac{XG_s}{k}-B_r\right)E_r^2\right]^2=$$

$$\left[\frac{\left(1-\frac{1}{\eta}\right)^2}{4}-\frac{1}{\eta}(RG_s+G_Rk)\right]\frac{E_r^4}{k^2}$$

Since $k=R+Z^2G_s$,

$$\left[P_r+\left(\frac{\frac{\eta-1}{2\eta}+RG_s}{R+Z^2G_s}+G_R\right)E_r^2\right]^2+\left[Q_r+\left(\frac{XG_s}{R+Z^2G_s}-B_r\right)E_r^2\right]^2=$$

$$\left[\frac{(\eta-1)^2}{4\eta}-\frac{RG_s+RG_r+ZG_sG_R}{\eta}\right]\frac{E_r^4}{(R+Z^2G_s)^2} \tag{178}$$

For constant efficiency η, equation 178 will give a circle. The values of P_r, Q_r, and E_r are, of course, for a single phase.

The particular value of equations 176 and 177 is that when these constant-loss and constant-efficiency circles are superimposed on the sending and receiver power circles of Fig. 53, there is available at a glance the general trend of loss and efficiency as the operating point N of the receiver circle shifts.
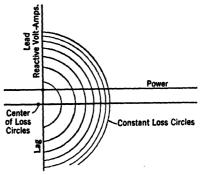
Fig. 57.   Loss Circles.

The general appearances of the loss and efficiency circles are indicated in Figs. 57 and 58. It is evident from equation 176 that the loss circles are all concentric, while the efficiency circles have different centers, as indicated.

In case the scale of the receiver power circle is changed to

that of $\dfrac{P_r}{E_r^2}$ and $\dfrac{Q_r}{E_r^2}$ as used in Fig. 55, it is then advisable to make the same general transformation in the loss and efficiency circles.
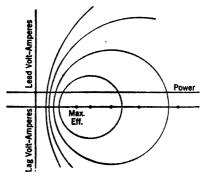
Fig. 58. Efficiency Circles.

Modifying equations 175 and 177, we have:

$$\left[\frac{P_r}{E_r^2}+\frac{p-1}{2k}\right]^2+\left[\frac{Q_r}{E_r^2}+\frac{q}{2k}\right]^2=\frac{1}{k^2}\left[\frac{(p-1)^2+q^2}{4}-hk+k\frac{P}{E_r^2}\right] \quad (179)$$

and

$$\left[\frac{P_r}{E_r^2}+\frac{p-\dfrac{1}{\eta}}{2k}\right]^2+\left[\frac{Q_r}{E_r^2}+\frac{q}{2k}\right]^2=\frac{1}{k^2}\left[\frac{\left(p-\dfrac{1}{\eta}\right)^2+q^2}{4}-hk\right] \quad (180)$$

CHAPTER 10

SYMMETRICAL COMPONENTS

64. Introduction.—The method of Symmetrical Components is in a general sense a special application of the general proposition of "Superposition," which is applied in many different forms in the computations of electrical circuits and in the analysis of electrical machinery. As first proposed by Dr. C. L. Fortescue in 1918, the method was applicable to an n-phase system, but the discussion in this volume will be limited to three-phase systems.* For most practical circuits, linearity of the electrical parameters can be assumed. In electrical machinery, linearity is not quite true because of effects of magnetic saturation and other factors; but average constants are often used, approximating actual linearity quite closely.

Fig. 59. **Highly Unbalanced Voltages or Currents of 3-Phase System.**

Consider the vectors A, B, and C in Fig. 59 as representing highly unbalanced voltages or currents of a three-phase system. For convenience, the vector A is drawn along the reference axis, the other vectors leading A by angles β and ϕ, respectively. Thus,

$$\left.\begin{array}{l} A = A\underline{/0} \\ B = B\underline{/\beta} \\ C = C\underline{/\phi} \end{array}\right\} \tag{181}$$

The solutions of some problems involving unbalanced systems of vectors may become quite cumbersome or nearly impossible with fundamental circuit methods. However, by the method of Symmetrical Components it is proposed first to break up the unbalanced system of vectors into groups of

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*"Symmetrical-Component Analysis of a Four-Phase System," by A. Boyajian, *AIEE Trans.*, Vol. 63, p. 48.

balanced units, then to solve the circuit for each of these balanced groups, and finally to superimpose the individual answers, thus obtaining a solution of the unbalanced problem.

The general circuit considered in the following introductory discussion is, with the exception of the actual faulted condition, assumed to be composed of balanced impedances. Modifications to take into account circuits of different impedances are discussed later. In other words, the system of vectors of Fig. 59 can be broken up into three individual systems, which are known as the Positive-Sequence System, the Negative-Sequence System, and the Zero-Sequence System.
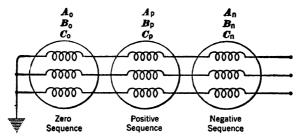
Fig. 60.   Individual Systems of General Three-Phase Circuit.

Before developing any mathematical relation, let us assume a three-phase circuit, as shown in Fig. 60, which is composed of three generators operating at absolutely the same frequency, the first generator being of the single-phase type but having three parallel circuits while the other generators are three-phase but of opposite sequences. The vector diagrams of the phase voltages of these machines might be given by Fig. 61. Adding the voltages of the three windings gives the resultant diagram in Fig. 62.

From Fig. 62 it is observed that the zero-sequence component indicates the presence of uni-phase voltages or currents; and the positive and negative sequences are individually balanced, as shown in Fig. 61. Mathematically the resultant vectors are:

$$\left. \begin{array}{l} A = A_0 + A_p + A_n \\ B = B_0 + B_p + B_n \\ C = C_0 + C_p + C_n \end{array} \right\} \tag{182}$$

65. Zero Sequence.—To determine the value of the zero-sequence component, it is only necessary to take one-third of the sum of A, B, and C.   Thus:

$$\frac{1}{3}(A+B+C)=\frac{1}{3}[(A_0+B_0+C_0)+(A_p+B_p+C_p)+(A_n+B_n+C_n)]$$

Since $A_p+B_p+C_p=0$ and $A_n+B_n+C_n=0$,

$$A_0=B_0=C_0=\frac{1}{3}(A+B+C) \tag{183}$$
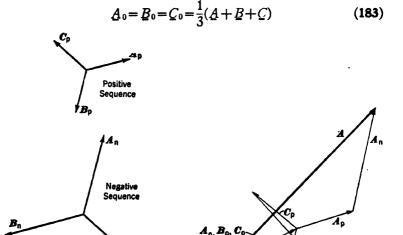
Fig. 61.   Vector Diagrams of Phase Voltages.

Fig. 62.   Resultant Vectors.

Equation 183 indicates that each zero-sequence vector is equal to one-third of the sum of the three original unbalanced vectors.

66. Positive Sequence.—We may define the positive-sequence vectors as follows:

$$\left.\begin{array}{l} A_p=A_p \\ B_p=A_p\underline{/120} \\ C_p=A_p\underline{/120} \end{array}\right\} \tag{184}$$

The negative-sequence vectors can be defined similarly, as follows:

$$\left.\begin{array}{l} A_n = A_n \\ B_n = A_n \underline{/120} \\ C_n = A_n \overline{/120} \end{array}\right\} \tag{185}$$

Also, remembering that $A_0 = B_0 = C_0$ and substituting in equation 182, we have the following:

$$\left.\begin{array}{l} A = A_0 + A_p + A_n \\ B = A_0 + A_p \overline{/120} + A_n \underline{/120} \\ C = A_0 + A_p \underline{/120} + A_n \overline{/120} \end{array}\right\} \tag{186}$$

Equation 186 may be modified as follows:

$$\left.\begin{array}{l} A \quad\quad = A_0 \quad\quad + A_p + A_n \\ B\underline{/120} = A_0 \underline{/120} + A_p + A_n \underline{/240} \\ C\overline{/120} = A_0 \overline{/120} + A_p + A_n \overline{/240} \end{array}\right\} \tag{187}$$

Adding these three equations, we have:

$$A + B\underline{/120} + C\overline{/120} = 3A_p$$

from which:

$$A_p = \frac{1}{3}\left(A + B\underline{/120} + C\overline{/120} \right) \tag{188}$$

Equation 188 may be solved analytically if the three unbalanced vectors are expressed in complex form; otherwise, a graphical solution may be obtained. In either case it will be noticed that the vector B is first rotated 120° counter-clockwise, and the vector C is rotated 120° clockwise. To illustrate this procedure graphically, the vectors of Fig. 62 are handled according to equation 188 in Fig. 63.

67. Negative Sequence.—Equation 186 is now modified in such a manner that the zero-sequence and positive-sequence terms disappear. Thus:

$$\left.\begin{array}{l} \underline{A} = \underline{A}_0 + \underline{A}_p + \underline{A}_n \\ \underline{B}/\underline{120} = \underline{A}_0/\underline{120} + \underline{A}_p/\underline{240} + \underline{A}_n \\ \underline{C}/\underline{120} = \underline{A}_0\underline{/120} + \underline{A}_p\underline{/240} + \underline{A}_n \end{array}\right\} \qquad (189)$$

from which:

$$\underline{A}_n = \frac{1}{3}\Big(\underline{A} + \underline{B}/\underline{120}$$

$$+ \underline{C}\underline{/120}\Big) \qquad (190)$$

The solution of equation 190 is shown graphically in Fig. 64.

68. Sequence Networks.—In the preceding articles the general basic theory of Symmetrical Components has been de-
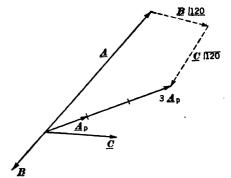
Fig. 63.   Positive-Sequence Component.

veloped.   In analyzing unbalanced systems for a particular type of fault, it is also necessary to determine in what manner the sequence components are to be superimposed.   In other words, it is now necessary to determine in what parts of the system each of the three components is effective and how the respective networks are to be treated.   To establish the proper network for each sequence, it is first necessary to determine what impedances are effective to the flow of sequence currents by our standard devices in power systems.   Thus, Chapters 11 and 12 deal with sequence impedances of alternators and transformers; Chapters 13 and 14, with transmission-line impedances; and Chapters 15 and 16, with sequence networks.   Later chapters cover additional details.

69. Nature of Impedance.—The theory of Symmetrical Components is used fundamentally for the computation of sustained short-circuit currents, the study of the operation of rotating machines, and relay operation.   Since most equipment of power systems has considerably more inductive reactance than resistance, it is sometimes satisfactory to neglect the resistance

in problems of system faults. Also, in order to simplify the illustrative problems, resistance has generally been neglected. The sustained current is then obtained on the basis that the system impedance is purely inductive. The next problem is to determine the reactances of different types of equipment
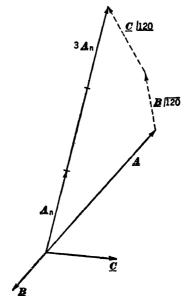
Fig. 64. Negative-Sequence Component.

which make up a system. The different parts of a system will be taken up in the natural order from the generating source toward the load; namely, alternators, transformers, and transmission lines.

Since this is an introductory text, the detailed inner workings of electrical equipment are given in very brief form, and often in a general qualitative manner. As indicated in the preface of this volume, an attempt is made to acquaint the reader with an important general field of system operation, without investigating any one branch in detail.

CHAPTER 11

ALTERNATOR IMPEDANCES

70. Alternator as a Coupled Circuit.—For the present, we shall consider a three-phase alternator delivering balanced currents. This condition corresponds to positive-sequence operation of the machine, and such a machine corresponds to a peculiar type of coupled circuit. On the stator or armature we find three windings, while on the revolving element there is
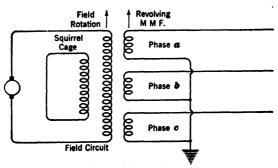
Fig. 65. Coupled Circuit for Three-Phase Alternator.

located the field winding. These four windings constitute a four-unit coupled circuit, as shown in Fig. 65.

With balanced armature currents in the three phases, the joint magnetic effect of the armature windings is to produce a constant amplitude sinusoidal mmf which revolves at synchronous speed in the same direction as the field pole rotates. Thus the relative mmf reaction between the armature and field circuit is fixed and constant for a given balanced current. Different internal power-factor currents will merely change the orientation of the armature mmf with respect to the field mmf.

From the field circuit as the observation point, one sees only direct-current reactions, namely, constant current and a constant mutual flux. However, observing the reactions from any one-phase winding, one sees a typical alternating-current

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coupled circuit, with an alternating current flowing in the winding and alternating flux present in the magnetic circuit.

To make the equivalent coupled circuit complete, another winding should be added to Fig. 65 to represent the action of amortisseur or damper windings often found on the rotor of an alternator.

71. Physical Nature of Reactance.—The inductive reactance of a circuit is given by the following basic equation:

$$X = 2\pi f L \tag{191}$$

where X = reactance, in ohms;

f = frequency, in cycles per second;

L = inductance, in henries.

It has already been brought out in the first few chapters of this volume that inductance is a measure of the flux linkages per ampere. Thus:

$$L = \frac{N\phi}{I} 10^{-8} \text{ henries} \tag{192}$$

where $N\phi$ = flux linkages;

I = current, in amperes.

This equation is not quite correct because it implies that the magnetic circuit is non-saturable. In reality,

$$L = \frac{d(N\phi)}{di} 10^{-8}$$

The immediate problem is one of a qualitative discussion of the different types of machine reactances. Therefore equation 192 will be the one considered. For actual quantitative measurements it is recognized that due precautions must be taken to reach proper saturation effects in the iron circuits.

The many different types of alternator reactances which have been coined to aid in the proper statement of machine operation depend entirely on the particular number of flux linkages which are involved in the particular case. In all cases

the reactances are defined from the basis of a single-phase winding of the armature. Because of the peculiar type of coupled circuit, one element of which is revolving, it is found convenient to consider flux linkages of several different types, primarily due to particular machine conditions and also to particular magnetic reluctances.

We will consider the extreme case of a salient pole alternator, rather than the non-salient pole machine. The magnetic circuit of such a machine is quite complex. The radial reluc-
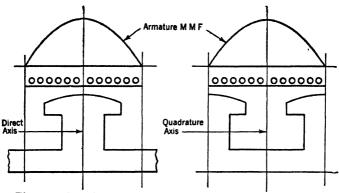
Fig. 66.   Zero Per Cent
Lagging Power Factor.

Fig. 67.   Unity Power Factor.

tance across the air gap varies from a minimum at the center lines of poles to a maximum half way between poles. This variation in air-gap reluctance introduces the first basic problem in defining machine inductances. As the power factor of the machine is changed, there will be introduced a shift in the position of the armature mmf with respect to the poles of the machine, with a corresponding change in flux linkages per ampere. From the fundamental machine theory we know that at zero per cent lagging power the armature mmf is demagnetizing, and at unity power factor the armature mmf is crossmagnetizing. This difference is shown in Figs. 66 and 67. In other words, as the power factor varies from zero per cent to unity, the degree of coupling of the circuit of Fig. 65 changes considerably. Thus we are led to expect that the reactances are functions of the power factor. To define the reactances in

this way would be a complicated process, the advantage of which would be questionable. However, for intermediate power factors it is quite common practice to resolve the armature current or mmf into two components, one acting along the center line of poles or the direct axis (see Fig. 66) and the other along the quadrature axis half way between poles. Following this basic plan of mmf components, we proceed to define two extreme values of reactances, namely, direct and quadrature reactances. Direct reactance is defined on the basis of Fig. 66, in which the armature mmf is in direct line with the center line of poles; and quadrature reactance is defined on the basis of Fig. 67, in which the armature mmf is in line with the quadrature axis.

72. Direct and Quadrature Synchronous Reactances.* The term synchronous reactance might be called the looking-back reactance, observed from a particular phase winding. It is therefore the total reactance of the coupled circuit. In any standard text on machinery there will be found more or less elaborate discussions on the peculiarities and methods of measurement of this reactance. In this analysis we will assume that the armature is being supplied with balanced three-phase currents from some external source. The rotor will be turned at synchronous speed in the same direction as the revolving armature mmf. By adjusting the armature currents, it is possible to attain any desired degree of saturation in the magnetic circuit. This can be done without the presence of direct current in the field, and therefore it is convenient to leave the field circuit open.

The two extreme conditions may now be obtained by producing an alignment of the armature mmf first with the direct axis and then with the quadrature axis. The complete flux paths are illustrated in Figs. 68 and 69. The corresponding reactances are equal to $2\pi f$ multiplied by the flux linkages per ampere for each extreme case. The test procedure for obtain-

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*The method of analysis offered in Arts. 72 to 75 follows the general plan presented by Mr. Loyd P. Shildneck in "Synchronous Machine Reactances," *General Electric Review*, Nov., 1932, p. 560.

ing these reactances would require that the rotor be operated by an auxiliary synchronous motor with the same number of poles as the machine under test. If the machine under test is equipped with a movable stator operated by a suitable crank mechanism, it would be a simple matter to obtain the two extreme conditions of Figs. 68 and 69. The total impedance of the machine could then be obtained by means of phase meters
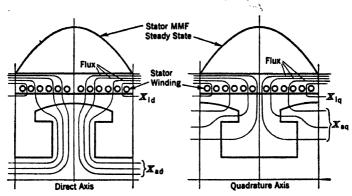
Fig. 68.   Direct Synchronous   Fig. 69. Quadrature Synchronous
        Reactance Flux Paths.   Reactance Flux Paths.

reading power, current, and voltage. The impedance would obviously be:

$$Z = \frac{E}{I}(\cos\ \theta + j\ \sin\ \theta) \tag{193}$$

where E = phase voltmeter reading;
    I = phase ammeter reading;
    θ = power factor angle.

Finally the reactances would be equal to $\frac{E}{I}\ \sin\ \theta$. The standard notation used for these two constants is:

    X_d = direct synchronous reactance;
    X_q = quadrature synchronous reactance.

73.   Direct and Quadrature Leakage Reactances.—In basic machine theory it is generally brought out that leakage in-

ductance involves only that portion of the armature flux linkages which is localized around the armature conductors and does not extend into the body of the poles, so as to link also with the field winding. The same general point of view is taken in defining primary and secondary leakage inductances of transformers. Thus in the present case we define direct leakage reactance (X_{ld}) and quadrature leakage reactance (X_{lq}) as those portions of the synchronous reactances X_d and X_q which are due to the leakage armature fluxes only. These leakage reactances are illustrated in Figs. 68 and 69.
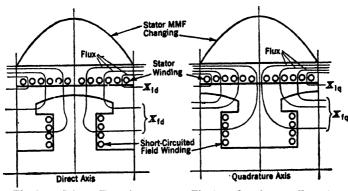
Fig. 70. Direct Transient Reactance Flux Paths.

Fig. 71. Quadrature Transient Reactance Flux Paths.

The remainders of the synchronous reactances are due to the mutual fluxes. These remainders are indicated as X_{ad} and X_{aq}. The final relationship between these several components is as follows:

$$\left.\begin{array}{l} X_d = X_{ld} + X_{ad} \\ X_q = X_{lq} + X_{aq} \end{array}\right\} \qquad (194)$$

The two components X_{ad} and X_{aq} account for the effect of armature reaction in the direct axis and quadrature axis, respectively.

74. Direct and Quadrature Transient Reactances.—Consider the same detailed set-up as suggested in Art. 72, with the only exception that the field circuit is now short-circuited. It will be assumed that the machine under consideration has no

amortisseur windings or any other short-circuited circuit on the field structure other than the field winding. When the armature circuits are energized, there will be set up in the magnetic circuit a transient component of armature mmf. The armature mmf must build up from zero to the final value indicated in Figs. 68 and 69. During this transient period there will be induced opposing mmf's in the field winding. The effect of this opposing field mmf is to force the mutual flux into a somewhat different path from that in Figs. 68 and 69, as shown in Figs. 70 and 71.

With the two extreme orientations of the armature mmf, we obtain in this case two additional reactances, known as transient reactances. These are X'_d in the direct axis and X'_q in the quadrature axis. Comparison of Figs. 68 and 70 will indicate that the direct transient reactance must be considerably less than the direct synchronous reactance. On the other hand, the flux paths of Figs. 69 and 71 are somewhat similar, so that it would be expected that the quadrature transient reactance and the quadrature synchronous reactance are of about equal magnitude.

As before, the leakage reactances may be thought of as component parts of the transient reactances. Thus:

$$\left. \begin{array}{l} X'_d = X_{ld} + X_{fd} \\ X'_q = X_{lq} + X_{fq} \end{array} \right\} \tag{195}$$

In these relations, X_{fd} and X_{fq} indicate the contributions of the mutual flux which threads the field winding.

75. Direct and Quadrature Subtransient Reactances.—In the case of alternators equipped with amortisseur or damper windings, all results outlined in Art. 74 are somewhat modified because of the presence of a second short-circuited winding on the field structure. Qualitatively, the results are of the same general nature as outlined in Art. 74, but the flux patterns and the final reactance values are somewhat smaller in magnitude than those obtained for the conditions of Art. 74. The flux patterns of this case are indicated in Figs. 72 and 73.

These reactances are defined as subtransient reactances X_d'' and X_q'', respectively, and they may be broken up into parts, as follows:

$$\left.\begin{array}{l} X_d'' = X_{ld} + X_{Dd} \\ X_q'' = X_{ld} + X_{Dq} \end{array}\right\} \tag{196}$$

The terms X_{Dd} and X_{Dq} take into account the effects of mutual flux within the short-circuited field and damper windings.

76. Determination of Direct and Quadrature Synchronous Reactances.—One method of determining the synchronous re-
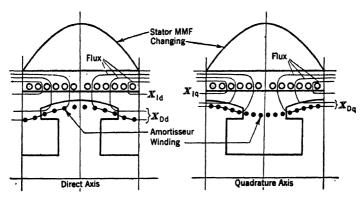
Fig. 72. Direct Sub-
transient Reactance Flux
Paths.

Fig. 73. Quadrature Sub-
transient Reactance Flux
Paths.

actances has already been suggested in Art. 72.   This method requires special equipment and therefore may not be possible with most alternators.

Another method quite commonly used, which gives reasonable results, is to connect the three-phase armature to a reduced-voltage source of standard frequency.   The rotor of the machine is then driven at slightly below synchronous speed, in the same direction as the revolving armature mmf, so that the unit will have a small speed slip.   The field circuit should be left open during the test.   Suitable meters should be inserted in the armature circuit for the measurement of armature impedance,

namely, a voltmeter, an ammeter, and a wattmeter. As the rotor slips with respect to the synchronously revolving armature mmf, the armature mmf axis will line up alternately with the direct axis and the quadrature axis. The magnetic reluctance will be a minimum when the mmf axis coincides with the axis of poles, and will be a maximum when the two axes are at quadrature. Oscillograms of the voltage, the current, and the voltage across the field-circuit terminals are shown in Fig. 74.
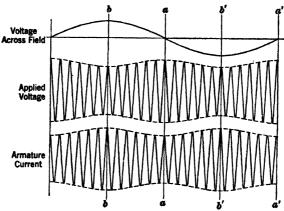
Fig. 74. Oscillograms for Synchronous Reactances.

When the magnetic reluctance is least, the required flux is set up with minimum current; while, when the magnetic reluctance is greatest, maximum current will be observed. With a certain amount of care, it is possible to detect the modulated swing of Fig. 74 on the ammeter, voltmeter, and wattmeter. The unsaturated impedances can be computed from either the observed meter readings or the oscillograms.

Since the test is made at reduced voltage, the values obtained are unsaturated reactances. Saturated direct synchronous reactances may be approximated reasonably well by the method of "adjusted synchronous reactance" or still better by the method of "saturated synchronous reactance." Both of these methods involve operating the machine as a generator, instead of in the manner previously described. These two tests give a measure of the direct synchronous reactance because they are

applied to the alternator operating at zero per cent lagging power factor, in which case the armature mmf is lined up with the direct axis of the machine, as illustrated in Figs. 66 and 68.

77. Adjusted Synchronous Reactance Method of Determining Direct Synchronous Reactance.—The first step involves the determination of the no-load saturation curve, like that shown in Fig. 75. Next the alternator is loaded with as near perfect inductances as possible. The load should, of course, be perfectly balanced between the three phases. As the field current is changed, the load inductance is also changed so as to maintain rated armature current. The resultant curve plotted between terminal phase voltage and field current is known as the zero per cent lagging power factor, full-load characteristic. For any field current I_f the vertical difference between the two curves of Fig. 75 can be taken equal to the IX_d drop or the difference between the excitation voltage E_0 and the terminal voltage E. It will be left for the student to show that the resistance will not ma-

terially affect the result. Finally, any vertical difference in Fig. 75 can be divided by the rated current of the machine to obtain the corresponding direct synchronous reactance, X_d.

Prior to 1940 this method was very commonly accepted, but more recent develop-ments have shown the method to be some-what incorrect. It
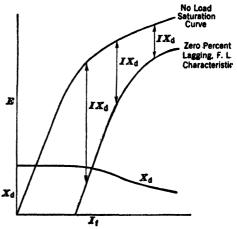
Fig. 75. Adjusted Synchronous Reactance Curves.

places the emphasis on the "adjusted" synchronous reactance as a function of field excitation and therefore does not correctly picture the proper degree of saturation.

78. Saturated Synchronous Reactance Method for Determining Direct Synchronous Reactance.—The adjusted synchronous reactance method was discussed in Art 77, and its great simplicity of application was outlined. It was, however, pointed out that the method was not correct because it did not indicate the true magnetic saturation. One method of obtaining a corrected saturated direct synchronous reactance is as follows: In Fig. 76 is shown a no-load saturation curve, or induced emf characteristic curve. The point P represents any particular induced voltage E_M. If the machine operated at constant magnetic flux saturation, as indicated by this point P, then the machine characteristic would be given by a straight line passing through O and P. To obtain the saturated direct synchronous reactance corresponding to operation at point P and a particular load current, locate the Potier* triangle to the right of point P, as shown in Fig. 76. The distance ab is the saturated direct synchronous reactance drop for the particular load current of the Potier triangle. Thus:

$$X_d = \frac{ab}{I} \quad (197)$$

where I = armature current corresponding to the Potier triangle.

The same procedure can be repeated for several points along the induced emf curve, and complete data of saturated direct synchronous reactance can be obtained for all values of induced emf, as indicated in Fig. 77.
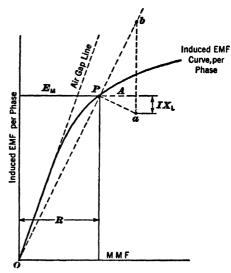
Fig. 76.   Determination of Saturated Direct Synchronous Reactance.

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*For discussions of the Potier method of analysis, consult any standard textbook on a-c machinery.

79. Saturated Quadrature Synchronous Reactances.

Saturated quadrature synchronous reactances are somewhat difficult to obtain. It might be suggested that, having determined the two synchronous reactances by the method of Art. 72 or Art. 76 and then having obtained the saturated direct synchronous reactance by the method of Art. 77 or Art. 78, it would be satisfactory to multiply the value of unsaturated quadrature synchronous reactance by the ratio of the saturated and unsaturated direct synchronous reactances.

For salient pole machines the magnetic reluctance of the quadrature axis is quite high because of the large air gap (see Fig. 69). Also, because of this large air gap, there is less tendency for saturation effects, and the unsaturated quadrature synchronous reactance probably is reasonably accurate.

For non-salient pole machines the difference between the direct and quadrature synchronous reactances should be small, since there is little difference between the magnetic reluctances of these two axes. Typical ratios of quadrature to direct synchronous reactances are 0.6 to 0.7 for salient pole machines and 0.9 to 0.95 for non-salient pole machines.

80. Method of Determining Transient and Subtransient Reactances.

These reactances define transient operation in which fundamental frequency cur-
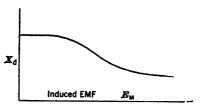
Fig. 77. Saturated Direct Synchronous Reactance.

rents are induced in the field winding. When the three armature currents are initially set up, there will be created a transient component of armature mmf (see Chapter 20) which is stationary in space. This transient mmf will gradually decrease to zero in a few cycles. The transient stationary mmf will produce a stationary transient flux which will induce fundamental frequency voltages in the field windings. Thus, to determine the values of these four reactances, it is first necessary to arrange a test in which the coupled circuit of Fig. 65 will operate with fundamental frequency currents in all windings.

The method of test is the same for the transient and sub-transient reactances.   Transient reactances apply if the rotor has no amortisseur windings, while subtransient reactances apply in case amortisseur windings or any forms of damper windings are present on the rotor.

The test corresponding to the foregoing requirements involves connecting only one phase of the armature to a source of reduced voltage and blocking the rotor, with the field winding short-circuited.   Meter readings of current, voltage, and power will give the transient or subtransient impedance.   Thus:

$$Z = \frac{E}{I}(\cos\,\theta + j\,\sin\,\theta) \qquad (198)$$

where $\cos\,\theta = \dfrac{W}{EI}$;

> W = power, in watts;
> E = applied phase voltage;
> I = phase current, in amperes.

The reactance is obviously given by the imaginary term of equation 198.

If a series of readings are taken for different rotor positions, the complete variation in reactance will be obtained.   The reactance is highest along the quadrature axis and lowest along the direct axis.   For a generator equipped with amortisseur windings, the values obtained will generally be much lower and also there will be very little difference between the direct and quadrature reactances.   These results are illustrated by Fig. 78.

81.   Positive-Sequence Impedance.*—In a general sense, all the reactances discussed are of the positive-sequence variety. However, we will define the positive-sequence reactance as a synchronous reactance which is involved in obtaining sustained

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*The nomenclature used in Arts. 81, 82, and 83 is not correct in the strict sense.   These three impedances should be referred to as impedance to the flow of positive-sequence current, impedance to the flow of negative-sequence current, and impedance to the flow of zero-sequence current. The interpretation given in these articles is acceptable for physically balanced three-phase systems.   For the general case, see Chapter 19.

short-circuit currents. The question involved is merely which of the two synchronous reactances (direct or quadrature) will come nearest to representing the conditions of an alternator under system faults. If the short circuit happens at the generator terminals, the direct reactance would be the choice, since the machine power factor is practically zero per cent lagging. However, the operating power factor of an alternator might be considerably better than zero per cent lagging for the
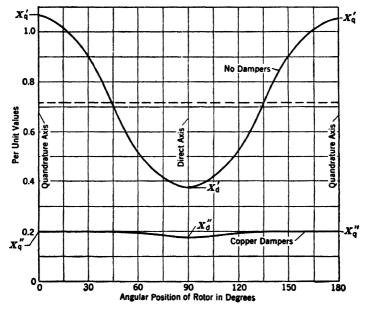
Fig. 78. Transient and Subtransient Reactances.

condition of a remote fault. The quadrature reactance will never be chosen because the operating power factor will hardly be equal to unity under any system fault. The direct synchronous reactance seems to be the best choice for the positive-sequence reactance.

82. Negative-Sequence Reactance.—Negative-sequence currents will produce a synchronously revolving mmf in an alternator, just like the effect produced by positive-sequence

currents.   In this case the rotor will be turning at synchronous speed in the opposite direction to the revolving armature mmf.

Considering the machine as a coupled circuit (see Fig. 65), negative-sequence operation produces a changing magnetic-circuit reluctance, with alternating currents of double frequency being induced in the field winding and in other damper or amortisseur windings if they are present.   Thus, the negative-sequence reactance must be of a transient type and must vary through the cycle of values shown in Fig. 78.

For purposes of short-circuit calculations, about the only sensible method is to take some average value of reactance. Neglecting wave-shape distortion, it is a general policy to take the value of negative-sequence reactance as follows:

$$X_n = \tfrac{1}{2}(X_d'' + X_q'') \tag{199}$$

That is, X_n = average of direct and quadrature subtransient reactances.

In case there are no dampers or amortisseur windings, then transient reactances are used instead of subtransient reactances.

83. Zero-Sequence Reactance.—Zero-sequence operation assumes that in-phase currents exist in each of the three phases of an alternator.   Such currents would produce a stationary mmf, reacting with a revolving rotor to induce double-frequency voltages in the field winding.

A convenient method of test is to connect the three phase windings in series.   This combination is then supplied from a single-phase source, with wattmeter, ammeter, and voltmeter connections.   The field winding is short-circuited.   This reactance is quite small and may be considered as the average of the measured values for the rotor blocked in the direct axis and in the quadrature axis, as suggested in previous tests.   Since the three windings are in series, the zero-sequence impedance per phase is taken as one-third of the measured value.

84. Average Values of Sequence Reactances.—From the foregoing discussions, it is seen that a number of factors enter into the evaluation of sequence reactances and, therefore,

average values should be used carefully. The values given in Table 5 are obtained from page 99 of the book "Symmetrical Components" by Wagner and Evans. All values are percentages, and in each case the two values indicate the range of the constant.

TABLE 5
AVERAGE VALUES OF SEQUENCE REACTANCES

Machine	Positive Sequence	Negative Sequence	Zero Sequence	Direct Transient	Direct Sub-transient
Turbine generator.....	95–145	7–17	1–14	12–26	7–17
Salient pole motors and generators (with damper winding).............	60–145	13–35	2–20	20–51	13–35
Waterwheel generators (no damper winding)...	60–145	30–70	4–22	20–45	17–38
Condensers...........	150–220	17–37	2–15	27–55	18–38

85. Short-Circuit Method for Determining Negative-Sequence Impedance.—There are several methods of determining the negative-sequence impedance, the one given here being about the most straightforward. In this method the alternator is subjected to a line-to-line fault with ammeter, volt-
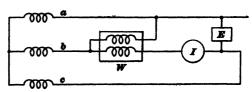
Fig. 79.   Circuit for Negative-Sequence Impedance.

meter, and wattmeter connected as shown in Fig. 79. The field current should be adjusted to a value sufficient to cause at least rated current to circulate through the armature.

From Fig. 79:

$$I_a = 0, \ I_b = I, \ I_c = -I$$

$$E_{ab} = -E, \ E_{bc} = 0, \ E_{ca} = E$$

From fundamental negative-sequence laws:

$$I_{an} = \frac{1}{3}(I_a + I_b\underline{/240} + I_c\underline{/120})$$

Thus:

$$I_{an} = \frac{1}{3}(0 + I\underline{/240} - I\underline{/120}) = \frac{I}{\sqrt{3}}\underline{/90} \qquad \textbf{(200)}$$

Similarly:

$$E_{bcn} = \frac{1}{3}(E_{bc} + E_{ca}\underline{/240} + E_{ab}\underline{/120})$$

from which:

$$E_{bcn} = \frac{1}{3}(0 + E\underline{/240} - E\underline{/120}) = \frac{E}{\sqrt{3}}\underline{/90} \qquad \textbf{(201)}$$
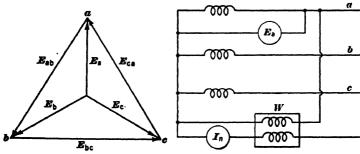
Fig. 80.   Vector Diagram of
Negative-Sequence Voltages.

Fig. 81.   Circuit for Zero-Sequence
Impedance.

The vector diagram of negative-sequence voltages is shown
in Fig. 80.   From this diagram:

$$E_{an} = \frac{E_{bcn}}{\sqrt{3}}\underline{/90}$$

From equation 201 it follows that:

$$E_{an} = \frac{E}{3}$$

Then:

$$Z_n = \frac{E_{an}}{I_{an}} = \frac{\sqrt{3}E}{3I}\underline{/90} = \frac{E}{\sqrt{3}I}\underline{/90} \qquad \textbf{(202)}$$

If $\theta = \cos^{-1}\left(\dfrac{W}{EI}\right)$,

$$I = I\underline{/\theta}$$

and
$$Z_n = \frac{E\underline{/90}}{\sqrt{3}I\underline{/\theta}} = \frac{E}{\sqrt{3}I}(\sin\theta + j\cos\theta) \tag{203}$$

86. Short-Circuit Method for Determining Zero-Sequence Impedance.—In Fig. 81 is shown a generator with a line to line to neutral short circuit. A voltmeter E_a indicates the voltage of the open phase, an ammeter gives the neutral current, and a wattmeter is connected with the current element in the neutral and with the voltage element across the open phase.

From fundamental sequence relations, the zero-sequence voltage is:

$$E_0 = \frac{1}{3}E_a$$

Furthermore:

$$I_0 = \frac{I_n}{3}$$

Since
$$Z_0 = \frac{E_0}{I_0},$$

$$Z_0 = \frac{E_a}{I_n} = \frac{E_a}{I_n}(\cos\theta + j\sin\theta) \tag{204}$$

where $\cos\theta = \dfrac{W}{E_a I}$.

87. Summary of Alternator Impedances.—In this chapter, methods are given for the calculation of the several types of alternator impedance. A complete discussion of other test methods, with example data for a large range of machine types and sizes, is given in "Determination of Synchronous Machine Constants from Test," by S. H. Wright, *A I E E Transactions*, Vol. 50, page 1331.

CHAPTER 12

TWO-WINDING TRANSFORMER IMPEDANCES

88. Series Impedances.—As an introduction to the transformer, simple series lumped impedances will be considered. In Fig. 82 is shown a three-phase line with impedances in each line. As long as these impedances are magnetically independent, it is evident that the positive-, negative-, and zero-sequence impedances are the same. Therefore:

$$Z_p = Z_n = Z_0 = Z \tag{205}$$

where Z_p = positive-sequence impedance;

Z_n = negative-sequence impedance;

Z_0 = zero-sequence impedance;

Z = circuit impedance.

89. Shunt Impedances.—For three-phase circuits, loads may be of either Y or delta type. Delta impedances should be converted to equivalent Y loads, in order that they may be handled on a per-phase basis. As long as these impedances are magnetically independent, all sequence impedances are equal, as indicated by
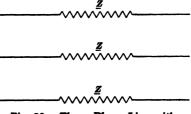
Fig. 82.   Three-Phase Line with Series Impedances.

equation 205; but the zero-sequence impedance can exist only if the neutral is grounded.

90. Neutral Impedance.—A neutral impedance implies that the neutrals of Y-connected circuits are grounded through a single impedance, as illustrated in Fig. 83. It should be kept in mind that positive- and negative-sequence currents will not flow through the neutral to ground. Hence, neutral impedances are not involved in positive- and negative-sequence net-

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works. The three zero-sequence currents, however, will flow to ground, as shown in Fig. 83. It will be noticed that the current through the neutral impedance is equal to $3I_0$, and the voltage drop in the neutral is equal to $3I_0Z$. The phase zero-sequence neutral impedance is therefore equal to $3Z$.

91. Single-Phase, Two-Winding Transformers.—All transformers in power networks are reducible to equivalent circuits, either of the T type, as illustrated in Fig. 84, or with a series impedance, as in Fig. 85, if the exciting current can be neglected. This last assumption is generally satisfactory for most steady-state fault current calculations.

The values of impedances indicated in Figs. 84 and 85 are the impedance Z_A of the primary, the
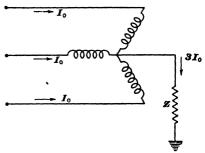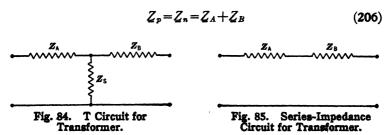
Fig. 83.   Three-Phase Line with Neutral Impedance.

impedance Z_B of the secondary, and the exciting impedance Z_s, all being based on one winding. For most power transformers it is permissible to neglect the exciting current when making short-circuit calculations. Hence, the positive- and negative-sequence impedances are equal. Thus:

$$Z_p = Z_n = Z_A + Z_B \qquad (206)$$

Fig. 84.   T Circuit for Transformer.

Fig. 85.   Series-Impedance Circuit for Transformer.

Typical values of transformer impedances, as given by Wagner and Evans on page 113 of the book "Symmetrical Components," are shown in Table 6.

The zero-sequence impedance of single-phase, two-winding transformers is the same as the positive- and negative-sequence impedances; but in three-phase circuits the zero-sequence network depends on the specific type of transformer connection. Several possible connections are taken up here.

TABLE 6

TYPICAL TRANSFORMER IMPEDANCES

Distribution.............................	3	per cent
Network................................	5	per cent
Power:		
up to 66 Kv......................	5–7	per cent
88 and 110 Kv...................	6–9	per cent
132 and 154 Kv....	8–10	per cent
187 and 220 Kv..................	10–14	per cent
Direct-current resistance..............	0.35–0.5	per cent
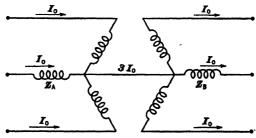
Fig. 86.   Three-Phase Circuit with Y-Y Connection and Interconnected Neutrals.

Case 1:   Y-Y connection with primary and secondary neutrals connected together but not grounded.—This type of connection is shown in Fig. 86.   Such a connection might not be encountered in exactly the form shown; but, assuming such a possibility, the connection serves as an interesting introduction to the field of transformers.   This circuit assumes that the primary and secondary neutrals are solidly connected together but are not grounded.   Remembering that the primary and secondary windings are magnetically coupled through a common magnetic circuit, the magnitude of the resultant impedance to zero-

sequence currents will depend on whether the transformers are of additive or subtractive polarity. Thus, the primary and secondary will act as a series choke coil insofar as the zero-sequence currents are affected, and the resultant inductance will be of the following magnitude:*

$$L_p + L_s + 2M \text{ or } L_p + L_s - 2M$$

where L_p = primary self-inductance;
L_s = secondary self-inductance;
M = mutual inductance.

In case the primary and secondary coils are connected in an additive polarity relation, the inductance is given by the first relation. Such a value is quite high and is equivalent to that for an open circuit, the impedance being greater than the exciting impedance. However, if the subtractive-polarity relation is used, the value of the resultant inductance is much lower, as given by the second equation. It will be helpful to recall that

$$M = k \sqrt{L_p L_s}$$
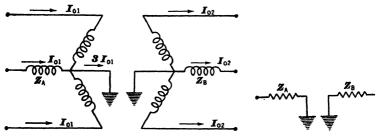
where k is slightly less than unity.

Fig. 87. Single-Phase Circuit Equivalent to Fig. 86.

Thus, the zero-sequence impedance equivalent of Fig. 86 is represented by the series circuit of Fig. 87, the impedance Z being very high for additive polarity and quite low for subtractive polarity.

Case 2: Y-Y connection with neutrals grounded.—In this case, which is illustrated in Fig. 88, the zero-sequence current may flow between the fault and the secondary neutral, and a primary equivalent current may flow between grounded neutrals in the primary circuit. Thus the transformer does not act as a simple series choke coil, as in the preceding case, but maintains the transformer ratio features. Since currents may flow through both primary and secondary, the zero-sequence impedance is the normal series transformer impedance, the equivalent single-phase circuit being that of Fig. 89.

*For details of this statement, see "Principles of Electrical Engineering" by Timbie and Bush, p. 216.

Case 3: Y-Y connection with only one neutral grounded.
Insofar as the zero-sequence current is concerned, this connection, which is shown in Fig. 90, corresponds to three transformers operating with open-circuited secondaries. Hence, the only current flowing in the primary must be the exciting current,
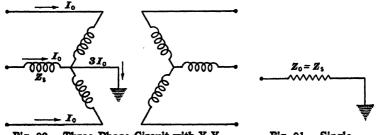
Fig. 88.   Three-Phase Circuit with Y-Y Connection and Both Neutrals Grounded.

Fig. 89.   Single-Phase Circuit Equivalent to Fig. 88.

and the zero-sequence impedance is equal to the open-circuit, or shunt, impedance Z_s. The equivalent single-phase circuit is shown in Fig. 91.

As has been stated, the open-circuit impedance Z_s is so large that the network in Fig. 91 may be assumed to be an open circuit when the flow of zero-sequence current is considered.

Fig. 90.   Three-Phase Circuit with Y-Y Connection and One Neutral Grounded.

Fig. 91.   Single-Phase Circuit Equivalent to Fig. 90.

Case 4: Y-Y connection with both neutrals ungrounded.
Such a circuit is obviously an open circuit in regard to the flow of zero-sequence current.

Case 5: Grounded Y to delta.—See Fig. 92 for the circuit representing this type of connection.   Single-phase voltages
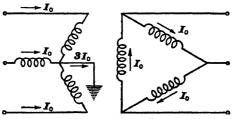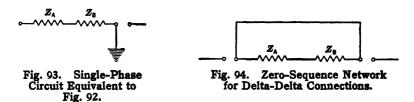
Fig. 92.   Three-Phase Circuit with
Grounded Y-Delta Connection.

acting in the closed delta will cause zero-sequence currents to flow in the closed circuit; but, obviously, no current can flow in the line connections to the delta winding.   The equivalent circuit is given in Fig. 93.

Case 6: Ungrounded Y-delta connections.—Since zero-sequence currents cannot flow in the Y side of such a trans-

Fig. 93.   Single-Phase
Circuit Equivalent to
Fig. 92.

Fig. 94.   Zero-Sequence Network
for Delta-Delta Connections.

former, the impedance to zero-sequence current in the delta winding must be equal to the exciting impedance; or, in case such impedance is to be omitted, then an ungrounded Y-delta transformer corresponds to an open circuit for zero-sequence currents.

Case 7: Delta-delta connections.—Insofar as the external connecting lines are concerned, the delta-delta windings correspond to an open circuit for the flow of zero-sequence currents, since zero-sequence line currents must have a path to ground.   However, zero-sequence currents may, under certain conditions, circulate within the deltas, the zero-sequence network being shown in Fig. 94.

92. Three-Phase, Two-Winding Transformers.—When considering three-phase transformers, it is first necessary to analyze the magnetic circuits as found in practice. In Fig. 95 are shown three single-phase transformers, and in Fig. 96 is shown a shell type three-phase transformer. Assuming that only the primaries are excited, it will be noticed that the
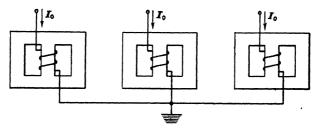
Fig. 95.   Three Single-Phase Transformers.

magnetic fluxes set up will be independent of each other and will follow the paths indicated.
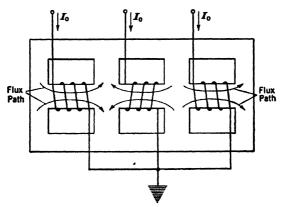
Fig. 96.   Three-Phase Shell Transformer.

Both the single-phase units and the three-phase shell transformer have complete magnetic circuits for each phase, whether the currents flowing in the primary are of positive, negative, or zero sequence. Therefore, the exciting impedance to the flow of positive-, negative-, and zero-sequence currents is the same. The series impedance of these two types of trans-

formers is also the same for all sequences.   A three-phase core-type transformer, however, offers somewhat different conditions.

In Fig. 97 are shown the general arrangement of windings and the magnetic circuit of such a transformer.   For the case of positive- and negative-sequence currents, the magnetic flux of the third phase is equal in magnitude to the vector sum of the other two phase fluxes.   Thus, the positive- and negative-
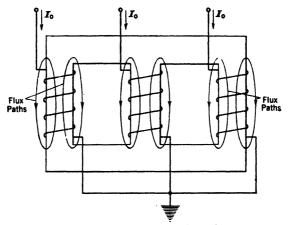
Fig. 97.   Three-Phase Core Transformer.

sequence exciting impedances are quite high and are similar to the value found in single-phase units.   Also, the positive-and negative-sequence series impedances are similar to those found in single-phase units.   However, when zero-sequence currents flow in the windings of a core transformer, the three iron cores will carry in-phase fluxes, which will be seen to oppose each other.   Thus, the exciting fluxes must complete their circuits through the leakage paths between iron cores, and the zero-sequence exciting impedance is therefore considerably less than that for single-phase units.   The zero-sequence series impedance is also less than that for positive- and negative-sequence currents.

CHAPTER 13

SHORT TRANSMISSION LINE IMPEDANCES

93. Positive- and Negative-Sequence Impedances.—Short transmission lines include all lines in which the charging capacity current is small enough to be neglected. Such lines can be treated as simple series impedances, while in long lines the capacity current becomes important. Long lines are treated in Chapters 7 and 14.

In Chapter 19 some discussion is given on the general phase of the work dealing with three-phase circuits with unbalanced impedances. However, up to the present point in this discussion of symmetrical components, it has been assumed that we were dealing with three-phase circuits composed of balanced impedances (with the exception of the faulted point impedances), and that the unbalance referred to voltages or currents. We will assume that transmission circuits are perfectly balanced or, if they are unbalanced, that the conductors have been suitably transposed, as is quite customary in major transmission-line practice. In case the lines are not transposed or are not balanced, it is still possible to obtain an approximate result by assuming an average value for the impedance on the basis of transposition. Thus, from equation 41, Art. 20, the positive- and negative-sequence impedances can be expressed as follows:

$$Z_1 = Z_2 = R + j4.657 \times 10^{-3} f \log \frac{D}{r} \text{ ohms/mile/phase} \quad (207)$$

in which R and $j\,4.657 \times 10^{-3} f \log \frac{D}{r}$ are, respectively, the resistance and the inductive reactance per mile per phase.

There are a considerable number of variables in an actual installation which influence the accuracy of the solution, and for exact results all these variables become quite involved and difficult to apply. To any solution of this type there should be coupled a large amount of engineering experience and judgment before final results are accepted.

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94. Zero-Sequence Impedances.—The three power conductors of a three-phase system act as parallel wires insofar as the zero-sequence currents are concerned, as shown in Fig. 98.

Equations 60 and 61 give the impedances of a single-phase circuit with earth return. We may adapt those equations to our case by converting the three parallel conductors to an equivalent single conductor. The resistance of such a single conductor would be *one-third* the resistance of one single wire. The zero-sequence voltage drop may be obtained as the product
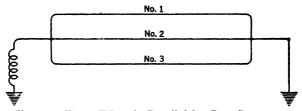
Fig. 98.   Three Wires in Parallel for Zero Sequence.

of the ground current and the impedance of the *single-conductor equivalent*, or as the product of the *conductor current* and an impedance which is three times as great as the equivalent single-circuit impedance. Since all three-phase circuits are solved on a per-phase basis, this last method is the common solution. Hence:

$$Z_0 = R + 4.764 \times 10^{-3} f + j13.97 \times 10^{-3} f \log \frac{D_e^2}{r} \text{ ohms/phase} \quad (208)$$

where R is the resistance of one conductor.

Also:

$$Z_M = 4.764 \times 10^{-3} f + j13.97 \times 10^{-3} f \log \frac{D_e^2}{d} \text{ ohms/phase} \quad (209)$$

In comparing equations 208 and 209 with equations 60 and 61, a multiplier of 3 will be noticed. This takes into account the fact that in the present case three conductors are in parallel, and the impedance of one of these parallel circuits is three times the impedance of the single-conductor equivalent.

95. Short Transmission Line (No Capacity) Zero-Sequence Networks.—Before considering the zero-sequence network, it

is well to investigate the circuit of a single conductor and ground wire, as shown in Fig. 99.   Such a circuit is fundamentally that of a single-turn transformer, illustrated in Fig. 100.   The circuit equations are as follows:

$$\left.\begin{aligned} E_a &= I_a Z_a + I_b Z_M \\ E_b &= I_b Z_b + I_a Z_M \end{aligned}\right\} \tag{210}$$

where Z_a = self-impedance of conductor a, which is the equivalent power conductor;

$\quad Z_b$ = self-impedance of conductor b, which represents the ground wire;

$\quad Z_M$ = mutual impedance between conductors a and b.
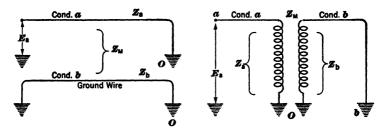
Fig. 99.   Single Conductor and
Ground Wire.

Fig. 100.   Single-Turn
Transformer.

In this case, $E_b = 0$.   Therefore:

$$0 = I_b Z_b + I_a Z_M$$

and

$$I_b = -\frac{I_a Z_M}{Z_b}$$

Also:

$$E_a = I_a Z_a - I_a \frac{Z_M^2}{Z_b} = I_a \left(Z_a - \frac{Z_M^2}{Z_b} \right)$$

and

$$\frac{E_a}{I_a} = Z_0 = Z_a - \frac{Z_M^2}{Z_b} \tag{211}$$

The transformer equivalent circuit involving leakage reactances is shown in Fig. 101.   For a single conductor and ground wire:

$$Z'_a = Z_a - Z_M = R_a + 1.588 \times 10^{-3} f + j4.657 \times 10^{-3} f \log \frac{D_e^2}{r_a}$$

$$- 1.588 \times 10^{-3} f - j4.657 \times 10^{-3} f \log \frac{D_e^2}{d_{ab}}$$

$$Z'_a = R_a + j4.657 \times 10^{-3} f \log \frac{d_{ab}}{r_a} \tag{212}$$

Similarly:

$$Z'_b = R_b + j4.657 \times 10^{-3} f \log \frac{d_{ab}}{r_b} \tag{213}$$
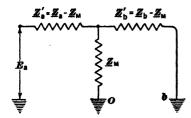
Fig. 101.   Transformer Equivalent
Circuit Involving Leakage
Reactances.
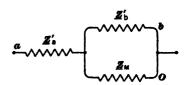
Fig. 102.   Resultant-
Impedance Circuit.

The resultant circuit as given by equation 211 is shown in Fig. 102.   This circuit has a resultant impedance of:

$$Z_0 = Z'_a + \frac{Z'_b Z_M}{Z'_b + Z_M} = \frac{Z'_a Z'_b + Z'_a Z_M + Z'_b Z_M}{Z'_b + Z_M}$$

$$= \frac{(Z_a - Z_M)(Z_b - Z_M) + (Z_a - Z_M)Z_M + (Z_b - Z_M)Z_M}{Z_b - Z_M + Z_M}$$

$$= \frac{Z_a(Z_b - Z_M) + (Z_a - Z_M)Z_M}{Z_b}$$

$$Z_0 = \frac{Z_a Z_b - Z_M^2}{Z_b} = Z_a - \frac{Z_M^2}{Z_b} \tag{214}$$

For the case of polyphase circuits in which zero-sequence impedance networks are involved, the same performances will apply per phase if the values of Z_a, Z_b, and Z_M are those given in equations 208 and 209.

For three-phase circuits:

$$Z'_a = R_a + j0.01397 f \log \frac{d_{ab}}{r_a} \tag{215}$$

$$Z'_b = R_b + j0.01397\, f \log \frac{d_{ab}}{r_b}$$

$$Z_M = 0.004764\, f + j0.01397\, f \log \frac{D_e^2}{d_{ab}}$$

(216)

Several examples will be given to clarify the process. These examples are obtained from the book "Symmetrical Components" by Wagner and Evans.

96. Example 1. *Single circuit, three-phase line with one ground wire.*

DATA: Conductors and ground wire = 397,500 c.m. ACSR;
GMR of conductors = 0.0277 ft;
Resistance per mile of conductor = 0.235 ohm;
Frequency = 60;
Damp earth, $\rho = 100$.
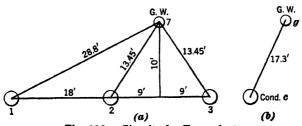
Fig. 103. Circuits for Example 1.

SOLUTION: (a) Convert the actual conductor configuration of Fig. 103 (a) to an equivalent single conductor c and a ground wire, as in Fig. 103 (b).

$$r_c = \sqrt[9]{r_1 r_2 r_3 d_{12}^2 d_{23}^2 d_{31}^2}$$

$$= \sqrt[9]{(0.0277)^3 (18)^2 (18)^2 (36)^2} = 2.43 \text{ ft}$$

and $\qquad d_{cg} = \sqrt[3]{28.8 \times 13.45 \times 13.45} = 17.3 \text{ ft}$

(b) Self and mutual impedances for the circuit in Fig. 104 (a), as obtained from equations 215 and 216, are:

$$Z'_c = R_c + j0.01397\, f \log \frac{d_{cg}}{r_c}$$

$$Z'_c = 0.235 + j0.8382 \log \frac{17.3}{2.43} = 0.235 + j0.714$$

$$Z'_g = 3R_g + j0.01397 f \log \frac{d_{cg}}{r_g}$$

$$= 3 \times 0.235 + j0.8382 \log \frac{17.3}{0.0277} = 0.705 + j2.35$$

$$Z_M = 0.004764 f + j0.01397 f \log \frac{D_e^2}{d_{cg}}$$

$$= 0.28584 + j0.8382 \log \frac{2,800}{17.3} = 0.286 + j1.85$$

(c) For the equivalent circuit in Fig. 104 (b):

$$Z_0 = Z'_c + \frac{Z'_g Z_M}{Z'_g + Z_M} = 0.46 + j1.75$$

(a) Equivalent Circuit   (b) Resultant Circuit
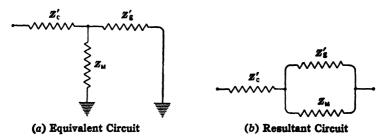
Fig. 104.   Equivalent and Resultant Circuits for Example 1.

Example 2.   *Single circuit, three-phase line with two ground wires.*

DATA: Conductors and ground wires = 397,500 c.m. ACSR;
GMR = 0.0277 ft;
Neglect resistance;
Frequency = 60;
$\rho = 100$.

SOLUTION: The actual conductor configuration is shown in Fig. 105 (a), and the equivalent circuit in Fig. 105 (b).

$r_c = 2.43$ ft (see Example 1)

$$r_g = \sqrt[4]{r_a r_b d_{ab} d_{ab}} = \sqrt[4]{0.0277 \times 18} = 0.706 \text{ ft}$$

$$d = \sqrt[6]{13.45 \times 13.45 \times 28.8 \times 13.45 \times 13.45 \times 28.8} = 17.3 \text{ ft}$$

Therefore:

$$Z'_c = j0.8382 \log \frac{17.3}{2.43} = j0.714$$

$$Z'_g = j0.8382 \log \frac{17.3}{0.706} = j1.16$$

$$Z_M = j0.8382 \log \frac{2,800}{17.3} = j1.85$$

Fig. 105. Circuits for Example 2.

According to the equivalent circuit of Fig. 105 (b):

$$Z_0 = j0.714 + \frac{(j1.16)(j1.85)}{j1.16 + j1.85} = j1.43$$

Fig. 106. Circuits for Example 3.

Example 3. *Double circuit, three-phase line with two ground wires.*

DATA: Conductors and ground wire = 397,500 c.m. ACSR; GMR = 0.0277 ft; $\rho = 100$; $f = 60$; Neglect resistance.

SOLUTION: Fig. 106 (a) and (b) show the actual conductor configuration and the equivalent circuit.

$$r_g = \sqrt[3]{0.077 \times 24} = 0.816 \text{ ft}$$

$$r_c = \sqrt[36]{(0.0277)^6 (18)^4 (22.47)^8 (24.08)^4 (16)^4 (8.544)^8 (24)^2} = 5.64 \text{ ft}$$

$$d_{cg} = \sqrt[12]{(5.83 \times 13 \times 21.21 \times 21.59 \times 27.30 \times 29.70)^2}$$

By proceeding as in Examples 2 and 1, the impedances are found to be:

$$Z'_c = j0.41, \ Z'_g = j1.11, \text{ and } Z_M = j1.10$$

Also:

$$Z_0 = j0.41 + \frac{(j1.11)(j1.85)}{j1.11 + j1.85} = j1.10$$

97. Zero-Sequence Impedance by Self and Mutual Voltages, for Three Conductors.—An alternate method which may be used for the deter- mination of the impedance to zero-sequence current involves the application of fundamental self-impedance and mutual- impedance voltage equations to the original circuits without resorting to the initial use of geometric mean equivalents. To illustrate this method, con- sider the case of a three-phase circuit of conductors 1, 2, and 3, the GMR of each of which is r, as illustrated in Fig. 107. A common earth equivalent

Fig. 107.   Three Conductors with Earth Return.

return conductor, the GMR of which is $r_b = 1$ foot, is assumed. The equivalent distance D_e is assumed to be the same for each of the conductors 1, 2, and 3.

Fig. 107 may be interpreted as three single-turn windings of a coupled circuit, with a voltage V_0 applied to each winding and a current I_0 in each winding. Therefore, from funda- mental self- and mutual-impedance laws,

$$V_0 = I_0 Z_{11} + I_0 Z_{12} + I_0 Z_{31} \quad \text{(for winding 1)}$$
$$V_0 = I_0 Z_{22} + I_0 Z_{12} + I_0 Z_{23} \quad \text{(for winding 2)}$$
$$V_0 = I_0 Z_{33} + I_0 Z_{31} + I_0 Z_{23} \quad \text{(for winding 3)}$$

in which Z_{11}, Z_{22}, and Z_{33} are self impedances which can be evaluated by a direct application of equation 60; and Z_{12}, Z_{23} and Z_{31} are mutual impedances which can be evaluated by the method of equation 61.

Solving the foregoing equations for the ratio of V_0 to I_0, we have:

$$Z_0 = \frac{V_0}{I_0} = \frac{Z_{11} + Z_{22} + Z_{33}}{3} + 2\frac{Z_{12} + Z_{23} + Z_{31}}{3}$$

This relation indicates that the zero-sequence impedance is equal to the average of the three self impedances plus twice the average mutual impedance. One may interpret the result just obtained to represent the zero-sequence impedance of a three-phase line with transposed conductors.

For transposed three-phase lines or for average values of self and mutual impedances, equations 60 and 61 may be used, provided that r_a of equation 60 is interpreted as the geometric mean of the conductor GMR's and provided that d_{ab} of equation 61 is interpreted as the geometric mean of the spacings between conductors, or $\sqrt[3]{d_{12}d_{23}d_{31}}$.

Considering average impedances or a transposed line, we may, therefore, write that

$$Z_0 = Z_{11} + 2Z_{12} \qquad \cdot \qquad \text{(217 a)}$$

The student should determine that the foregoing result corresponds to equation 208 in which the term r is equal to the geometric mean equivalent of the three conductors.

98. Self- and Mutual-Impedance Solution with Ground Wires.

Example 1. *Single circuit, three-phase line with one ground wire.*

SOLUTION: Referring to Fig. 103 in which the ground wire is labeled 7, we may write the following voltage equations:

$$V_0 = I_0 Z_{11} + I_0 Z_{12} + I_0 Z_{31} + I_7 Z_{17}$$
$$V_0 = I_0 Z_{22} + I_0 Z_{12} + I_0 Z_{23} + I_7 Z_{27}$$

$$V_0 = I_0 Z_{33} + I_0 Z_{31} + I_0 Z_{23} + I_7 Z_{37}$$
$$0 = I_7 Z_{77} + I_0 Z_{17} + I_0 Z_{27} + I_0 Z_{37}$$

As an average, or if conductors 1, 2, and 3 are transposed,

$$Z_{11} = Z_{22} = Z_{33}$$
$$Z_{12} = Z_{23} = Z_{31}$$
$$Z_{17} = Z_{27} = Z_{37}$$

Then:

$$V_0 = I_0 Z_{11} + 2 I_0 Z_{12} + I_7 Z_{17}$$
$$0 = I_7 Z_{77} + 3 I_0 Z_{17}$$

from which:

$$Z_0 = Z_{11} + 2 Z_{12} - 3 \frac{Z_{17}^2}{Z_{77}} \tag{217 b}$$

Example 2. *Single circuit, three-phase line with two ground wires.*

SOLUTION: The conditions are illustrated in Fig. 105. This example will involve five equations, as follows:

$$V_0 = I_0 Z_{11} + I_0 Z_{12} + I_0 Z_{31} + I_7 Z_{17} + I_8 Z_{18}$$
$$V_0 = I_0 Z_{22} + I_0 Z_{12} + I_0 Z_{23} + I_7 Z_{27} + I_8 Z_{28}$$
$$V_0 = I_0 Z_{33} + I_0 Z_{31} + I_0 Z_{23} + I_7 Z_{37} + I_8 Z_{38}$$
$$0 = I_7 Z_{77} + I_8 Z_{78} + I_0 (Z_{17} + Z_{27} + Z_{37})$$
$$0 = I_8 Z_{88} + I_7 Z_{78} + I_0 (Z_{18} + Z_{28} + Z_{38})$$

In addition to the same previous assumption of average impedances for transposed conductors, the following additional simplifications are made:

$$Z_{17} = Z_{27} = Z_{37} = Z_{18} = Z_{28} = Z_{38}$$

and

$$Z_{77} = Z_{88}$$

Then:

$$V_0 = I_0 Z_{11} + 2 I_0 Z_{12} + (I_7 + I_8) Z_{17}$$
$$0 = (I_7 + I_8)(Z_{77} + Z_{78}) + 6 I_0 Z_{17}$$

from which:

$$Z_0 = Z_{11} + 2 Z_{12} - 6 \frac{Z_{17}^2}{Z_{77} + Z_{78}} \tag{217 c}$$

Example 3. *Double circuit, three-phase line with six conductors, 1, 2, 3, 4, 5, and 6, and two ground wires, 7 and 8; see Fig. 106.*

SOLUTION: This example will involve eight voltage equations, each equation having eight voltage-drop terms.   Transposition of conductors is assumed within each circuit, but not between circuits.   Assuming that the zero-sequence current is divided equally between the two circuits, we may write sample voltage equations as follows:

For conductor 1,

$$V_0 = \frac{I_0}{2}(Z_{11}+Z_{12}+Z_{13}+Z_{14}+Z_{15}+Z_{16})+I_7Z_{17}+I_8Z_{18}$$

For conductor 4,

$$V_0 = \frac{I_0}{2}(Z_{44}+Z_{41}+Z_{42}+Z_{43}+Z_{45}+Z_{46})+I_7Z_{47}+I_8Z_{48}$$

For ground wire 7,

$$0 = I_7Z_{77}+I_8Z_{78}+\frac{I_0}{2}(Z_{71}+Z_{72}+Z_{73}+Z_{74}+Z_{75}+Z_{76})$$

For ground wire 8,

$$0 = I_8Z_{88}+I_7Z_{87}+\frac{I_0}{2}(Z_{81}+Z_{82}+Z_{83}+Z_{84}+Z_{85}+Z_{86})$$

Equations for conductors 2, 3, 5, and 6 follow patterns similar to those shown for conductors 1 and 4.

Assuming transposition or average values of impedances, we may write the following relations:

$$Z_{11}=Z_{22}=Z_{33}=Z_{44}=Z_{55}=Z_{66}$$
$$Z_{12}=Z_{23}=Z_{31}=Z_{45}=Z_{56}=Z_{64}$$
$$Z_{14}=Z_{15}=Z_{16}=Z_{24}=Z_{25}=Z_{26}=Z_{34}=Z_{35}=Z_{36}$$
$$Z_{17}=Z_{27}=Z_{37}=Z_{48}=Z_{58}=Z_{68}$$
$$Z_{18}=Z_{28}=Z_{38}=Z_{47}=Z_{57}=Z_{67}$$
$$Z_{77}=Z_{88}$$

Therefore:

$$V_0 = \frac{I_0}{2}(Z_{11}+2Z_{12}+3Z_{14})+I_7Z_{17}+I_8Z_{18}$$

$$V_0 = \frac{I_0}{2}(Z_{11}+2Z_{12}+3Z_{14})+I_7Z_{18}+I_8Z_{17}$$

$$0 = I_7Z_{77}+I_8Z_{78}+3\frac{I_0}{2}(Z_{17}+Z_{18})$$

$$0 = I_8 Z_{77} + I_7 Z_{78} + 3\frac{I_0}{2}(Z_{17} + Z_{18})$$

From the last four equations, it follows that:

$$2V_0 = I_0(Z_{11} + 2Z_{12} + 3Z_{14}) + (I_7 + I_8)(Z_{17} + Z_{18})$$
$$(I_7 + I_8)(Z_{77} + Z_{78}) = -3I_0(Z_{17} + Z_{18})$$

Hence,

$$2V_0 = I_0(Z_{11} + 2Z_{12} + 3Z_{14}) - 3I_0\frac{(Z_{17} + Z_{18})^2}{Z_{77} + Z_{78}}$$

Also, the zero-sequence impedance of the parallel combination of six power conductors is:

$$Z_0 = \frac{V_0}{I_0} = \frac{1}{2}\left[Z_{11} + 2Z_{12} + 3Z_{14} - 3\frac{(Z_{17} + Z_{18})^2}{Z_{77} + Z_{78}}\right]$$

If Z_{17} is taken as an average value of the impedances Z_{17} and Z_{18}, then

$$Z_0 = \frac{1}{2}\left(Z_{11} + 2Z_{12} + 3Z_{14} - 12\frac{Z_{17}^2}{Z_{77} + Z_{78}}\right) \qquad \textbf{(217 d)}$$

Correct values of geometric mean distances must be used in applying the results of equations 60 and 61 in each of the examples outlined in this article. These values are indicated in the following table.

Impedance	GMD	
Z_{12}	$\sqrt[3]{d_{12}d_{23}d_{31}}$	
Z_{17}	$\sqrt[3]{d_{17}d_{27}d_{37}}$	(one ground wire)
	$\sqrt[6]{d_{17}d_{27}d_{37}d_{47}d_{57}d_{67}}$	(two ground wires)
Z_{14}	$\sqrt[9]{d_{14}d_{25}d_{36}(d_{15}d_{26}d_{34})^2}$	

CHAPTER 14

LONG LINE IMPEDANCES

99. Positive- and Negative-Sequence Constants.—The first step in the solution of any power network is the development of the three sequence circuits (see examples in Chapter 16). A long line must, therefore, be reduced to a form suitable for representation in such a circuit. Equations 107, derived in Chapter 7, are not in a convenient form, since they involve distributed admittances to neutral (or ground), which cannot be handled along with other circuit elements involving lumped constants. To overcome this difficulty, long lines are treated in terms of equivalent π or T circuits, as derived in Arts. 41 and 42.

Positive- and negative-sequence constants of balanced long lines are the same, since these components represent balanced systems of voltages and currents. Either π or T equivalents may be used, the choice depending on particular features of the other component parts of the sequence networks.

Since long lines are quite often transposed, the solution of equations 107 and corresponding equivalent π or T circuits are carried out on a per-phase basis. The presence of ground wires does not affect the distribution of the dielectric field sufficiently to warrant any correction of the fundamental equations, insofar as the positive and negative sequences are concerned. Thus, the procedure is as follows:

(a) Calculate the reactance per mile per phase according to equation 41, which is:

$$X_3 = 4.657 \times 10^{-3} f \log \frac{D_e}{r} \text{ ohms/mile} \tag{218}$$

(b) Calculate the capacity and susceptance per phase according to equation 84. Thus:

$$C_{ao} = \frac{0.03882}{\log \dfrac{D_e}{r}} \text{ mfds/mile} \tag{219}$$

from which $B = 2\pi f C_{ao}$

(*c*) After determining the conductor resistance and assuming the shunt conductance equal to zero (perfect insulation assumed), obtain the constants A, B, C, and D of equations 108.

(*d*) From these values, the final step is to compute the equivalent constants of the π or T circuit (see equations 114 and 115, and 118 and 119), as the problem may demand. With these last values, it is then possible to sketch the positive- and negative-sequence networks, from which the network impedances can be determined by the step-by-step simplification of the circuits.
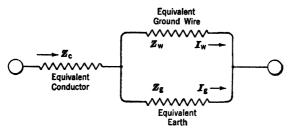
Fig. 108. Zero-Sequence Series Impedance for Long Line.

100. Zero-Sequence Constants.—The zero-sequence constants are handled somewhat in the same manner as the other sequence constants; namely, by determining the four generalized constants A, B, C, and D (see equation 108) and then converting these constants into an equivalent π or T circuit as in Arts. 41 and 42. However, in the positive- and negative-sequence networks, each phase may be treated separately; while, for the zero-sequence case, all conductors are in parallel as one side of the circuit and the ground wires and the earth are in parallel as the other side of the circuit. It is therefore necessary to use special means in determining the series impedance and shunt admittance which go into the calculation of the four generalized constants.

101. Zero-Sequence Series Impedance.—The method outlined in Chapter 13 is used in the determination of the zero-sequence impedance for a long line, but the impedance must in this case be calculated on a per-mile basis. As outlined in

Chapter 13, the several power conductors are replaced by an equivalent single conductor; similarly, the ground wires, if there are more than one, are also replaced by a single ground wire. Finally an equivalent circuit is developed in which the impedances of the equivalent ground wire and earth return are connected in parallel, the combination being in series with the impedance of the equivalent conductors. This is shown in Fig. 108. The three component impedances of Fig. 108 are as given by equations 215, 216, and 217. Thus:

$$Z_c = R_a + j0.01397\, f \log \frac{d_{ab}}{r_a} \tag{220}$$

$$Z_w = R_b + j0.01397\, f \log \frac{d_{ab}}{r_b} \tag{221}$$

$$Z_g = 0.004764\, f + j0.01397\, f \log \frac{D^2_e}{d_{ab}} \tag{222}$$

From these results it is obvious that:

$$Z_0 = Z_c + \frac{Z_w Z_g}{Z_w + Z_g} \tag{223}$$

From Fig. 108 it is possible to determine how the zero-sequence current will divide between the ground wire and the earth return.

Let K_g = ratio of earth current to total current;
and K_w = ratio of ground-wire current to total current.

Then:

$$K_g = \frac{Z_w}{Z_w + Z_g} \tag{224}$$

$$K_w = \frac{Z_g}{Z_w + Z_g} \tag{225}$$

It should be noticed that the ratios given in equations 224 and 225 are in complex form, since the impedances are complex quantities.

102. Zero-Sequence Capacity of Three-Phase Lines Without Ground Wires.*—As suggested in Art. 100, the conductors

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*The solutions for zero-sequence capacity given in this volume were suggested by Edith Clarke of the General Electric Company.

of a power line must be considered as connected in parallel insofar as the zero-sequence action is concerned, the current dividing equally among these conductors and then returning through the earth. A standard method of calculating capacity of earth return circuits is to consider the conductor images located at a distance below the surface of zero potential which is exactly equal to the height of the conductor above this zero-potential surface. For general purposes, this distance may be considered as the height of the conductors above the surface of the earth. The ideal case would involve a horizontal arrangement of conductors, so that the capacity offered to the three conductors would therefore be symmetrical. However, since three-phase lines are generally transposed, it is sufficiently accurate to consider an average condition of all three conductors, even for the conditions of unsymmetrical arrangements of conductors.

Fig. 109. Conditions for Earth Return Circuit.

Since the same current flows in the three conductors, we may define the charge per conductor as Q_0, namely, the zero-sequence charge per phase. Obviously the charge on the image conductors will be $-Q_0$. The general notation is shown in Fig. 109, distances between conductors being denoted by the letter D with suitable subscripts, and distances between conductors and images being denoted by the letter H with suitable subscripts. Radii of conductors are denoted by r.

Using equation 73 as a guide, we may write the results for the voltages between conductors and their respective images as follows:

$$E_{aa} = \frac{1}{2\pi K}\left[2Q_0 \operatorname{Ln} \frac{H_a}{r} + 2Q_0 \operatorname{Ln} \frac{H_{ab}}{D_{ab}} + 2Q_0 \operatorname{Ln} \frac{H_{ca}}{D_{ca}} \right]$$

$$E_{bb} = \frac{1}{2\pi K}\left[2Q_0 \operatorname{Ln} \frac{H_b}{r} + 2Q_0 \operatorname{Ln} \frac{H_{bc}}{D_{bc}} + 2Q_0 \operatorname{Ln} \frac{H_{ab}}{D_{ab}} \right] \Bigg\} \quad (226)$$

$$E_{cc} = \frac{1}{2\pi K}\left[2Q_0 \operatorname{Ln} \frac{H_c}{r} + 2Q_0 \operatorname{Ln} \frac{H_{ca}}{D_{ca}} + 2Q_0 \operatorname{Ln} \frac{H_{bc}}{D_{bc}} \right]$$

The average voltage between conductors and images is:

$$\frac{1}{3}(E_{aa} + E_{bb} + E_{cc}) = \frac{1}{3\pi K}\left[Q_0 \operatorname{Ln} \frac{H_a H_b H_c (H_{ab}H_{bc}H_{ca})^2}{r^3(D_{ab}D_{bc}D_{ca})^2} \right]$$

But the zero-sequence voltage is equal to one-half of the average voltage between conductors and images. Hence:

$$E_0 = \frac{Q_0}{2\pi K} \operatorname{Ln} \sqrt[3]{\frac{H_a H_b H_c (H_{ab}H_{bc}H_{ca})^2}{r^3(D_{ab}D_{bc}D_{ca})^2}}$$

from which the zero-sequence capacity per phase is:

$$C_0 = \frac{Q_0}{E_0} = \frac{2\pi K}{\operatorname{Ln} \sqrt[3]{\dfrac{H_a H_b H_c (H_{ab}H_{bc}H_{ca})^2}{r^3(D_{ab}D_{bc}D_{ca})^2}}} \text{ mfds/cm} \quad (227)$$

where $K = 0.08842 \times 10^{-6}$ mfds/cm³. Substituting the value of K and reducing equation 227 to the basis of one mile, we have:

$$*C_0 = \frac{0.03882 \times 10^{-6}}{\log \sqrt[3]{\dfrac{H_a H_b H_c (H_{ab}H_{bc}H_{ca})^2}{r^3(D_{ab}D_{bc}D_{ca})^2}}} \text{ fds/mile/phase} \quad (228)$$

103.   Zero-Sequence Capacity of Three-Phase Lines with Ground Wires.—The procedure outlined in Art. 102 can be followed also in case ground wires are present. These wires introduce additional charges, but the voltage between ground wires and their respective images is zero, since the wires are solidly connected to ground throughout the length of the line.

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*The results of equation 228 and of the other cases which are considered in this chapter can be reduced to the form given on page 1189 of the "Standard Handbook for Electrical Engineers," eighth edition.

Case A: One ground wire W.—In this case, equations 226 will be modified as follows:

$$E_{aa}=\frac{1}{2\pi K}\left(2Q_0\ Ln\ \frac{H_a}{r}+2Q_0\ Ln\ \frac{H_{ab}}{D_{ab}}+2Q_0\ Ln\ \frac{H_{ca}}{D_{ca}}+2Q_w\ Ln\ \frac{H_{aw}}{D_{aw}}\right)$$

$$E_{bb}=\frac{1}{2\pi K}\left(2Q_0\ Ln\ \frac{H_b}{r}+2Q_0\ Ln\ \frac{H_{bc}}{D_{bc}}+2Q_0\ Ln\ \frac{H_{ab}}{D_{ab}}+2Q_w\ Ln\ \frac{H_{bw}}{D_{bw}}\right)$$

$$E_{cc}=\frac{1}{2\pi K}\left(2Q_0\ Ln\ \frac{H_c}{r}+2Q_0\ Ln\ \frac{H_{ca}}{D_{ca}}+2Q_0\ Ln\ \frac{H_{bc}}{D_{bc}}+2Q_w\ Ln\ \frac{H_{cw}}{D_{cw}}\right)$$

$$E_{ww}=0=\frac{1}{2\pi K}\left(2Q_w Ln\frac{H_w}{r_w}+2Q_0 Ln\frac{H_{aw}}{D_{aw}}+2Q_0 Ln\frac{H_{bw}}{D_{bw}}+2Q_0 Ln\frac{H_{cw}}{D_{cw}}\right)$$

But the zero-sequence voltage above ground is equal to one-half of the average of the first three of these equations. Thus:

$$E_0=\tfrac{1}{2}\left(\frac{E_{aa}+E_{bb}+E_{cc}}{3}\right)$$

$$=\frac{1}{2\pi K}\left(Q_0\ Ln\sqrt[3]{\frac{H_aH_bH_c(H_{ab}H_{bc}H_{ca})^2}{r^3(D_{ab}D_{bc}D_{ca})^2}}+Q_w\ Ln\sqrt[3]{\frac{H_{aw}H_{bw}H_{cw}}{D_{aw}D_{bw}D_{cw}}}\right)$$

From equation 227, the following relation is obtained for a three-phase line without ground wires:

$$Ln\sqrt[3]{\frac{H_aH_bH_c(H_{ab}H_{bc}H_{ca})^2}{r^3(D_{ab}D_{bc}D_{ca})^2}}=\frac{2\pi K}{C_0}$$

where C_0 is the zero-sequence capacity of a line without ground wires.

Substituting this value in the last equation for E_0 gives:

$$E_0=\frac{Q_0}{C_0}+\frac{1}{2\pi K}Q_w\ Ln\ \frac{H_{ew}}{D_{ew}} \tag{229}$$

where $H_{ew}=\sqrt[3]{H_{aw}H_{bw}H_{cw}}$;

and $D_{ew}=\sqrt[3]{D_{aw}D_{bw}D_{cw}}.$

But, from the expression for E_{ww}, we have:

$$Q_w \, \mathrm{Ln} \, \frac{H_w}{r_w} = -Q_0 \, \mathrm{Ln} \, \frac{H_{aw}H_{bw}H_{cw}}{D_{aw}D_{bw}D_{cw}}$$

$$= -3Q_0 \, \mathrm{Ln} \, \sqrt[3]{\frac{H_{aw}H_{bw}H_{cw}}{D_{aw}D_{bw}D_{cw}}} = -3Q_0 \, \mathrm{Ln} \, \frac{H_{ew}}{D_{ew}}$$

Substituting this result in equation 229 gives:

$$E_0 = Q_0 \left[\frac{1}{C_0} - \frac{3}{2\pi K} \frac{\left(\mathrm{Ln}\dfrac{H_{ew}}{D_{ew}}\right)^2}{\mathrm{Ln}\dfrac{H_w}{r_w}} \right]$$

Let C'_0 = zero-sequence capacity with one ground wire. Then:

$$C'_0 = \frac{Q_0}{E_0} = \cfrac{1}{\dfrac{1}{C_0} - \dfrac{3}{2\pi K} \cfrac{\left(\mathrm{Ln}\dfrac{H_{ew}}{D_{ew}}\right)^2}{\mathrm{Ln}\dfrac{H_w}{r_w}}}$$

Rearranging:

$$C'_0 = \frac{C_0}{1 - \dfrac{C_0}{A'}} \; \text{mfds/cm} \qquad (230)$$

where:

$$A' = \frac{2\pi K}{3} \frac{\mathrm{Ln}\dfrac{H_w}{r_w}}{\left(\mathrm{Ln}\dfrac{H_{ew}}{D_{ew}}\right)^2} \; \text{mfds/cm} \qquad (231)$$

To convert equation 230 to read farads per mile, it is only necessary that C_0 be given in farads per mile (equation 228) and that the value of A' be modified as follows:

$$A' = 0.01294 \times 10^{-6} \frac{\log\dfrac{H_w}{r_w}}{\left(\log\dfrac{H_{ew}}{D_{ew}}\right)^2} \; \text{fds/mile} \qquad (232)$$

Case B: Two ground wires W and V.—Examination of the equations for E_{aa}, E_{bb}, and E_{cc} for the case of one ground wire will indicate that the effect of the ground wire is introduced by the last terms in those three equations. If an additional

ground wire V is introduced, another term must be added to each equation. Thus equation 229 for the zero-sequence voltage will become:

$$E_0 = \frac{Q_0}{C_0} + \frac{Q_w}{2\pi K} \text{ Ln } \frac{H_{ew}}{D_{ew}} + \frac{Q_v}{2\pi K} \text{ Ln } \frac{H_{ev}}{D_{ev}} \qquad (233)$$

where $H_{ev} = \sqrt[3]{H_{av}H_{bv}H_{cv}}$;

$D_{ev} = \sqrt[3]{D_{av}D_{bv}D_{cv}}$.

Furthermore:

$$E_{ww} = 0 = \frac{1}{2\pi K}\left(2Q_w \text{ Ln } \frac{H_w}{r_w} + 2Q_v \text{ Ln } \frac{H_{wv}}{D_{wv}} + 6Q_0 \text{ Ln } \frac{H_{ew}}{D_{ew}} \right)$$

$$E_{vv} = 0 = \frac{1}{2\pi K}\left(2Q_v \text{ Ln } \frac{H_v}{r_v} + 2Q_w \text{ Ln } \frac{H_{wv}}{D_{wv}} + 6Q_0 \text{ Ln } \frac{H_{ev}}{D_{ev}} \right)$$

From these relations:

$$\left.\begin{aligned}
Q_w \text{ Ln } \frac{H_w}{r_w} + Q_v \text{ Ln } \frac{H_{wv}}{D_{wv}} &= -3Q_0 \text{ Ln } \frac{H_{ew}}{D_{ew}} \\
Q_v \text{ Ln } \frac{H_v}{r_v} + Q_w \text{ Ln } \frac{H_{wv}}{D_{wv}} &= -3Q_0 \text{ Ln } \frac{H_{ev}}{D_{ev}}
\end{aligned}\right\} \qquad (234)$$

For simplicity, introduce the following notation:

$$L_w = \text{Ln } \frac{H_w}{r_w}, \ L_v = \text{Ln } \frac{H_v}{r_v};$$

$$L_{wv} = \text{Ln } \frac{H_{wv}}{D_{wv}};$$

$$L_{ew} = \text{Ln } \frac{H_{ew}}{D_{ew}}, \ L_{ev} = \text{Ln } \frac{H_{ev}}{D_{ev}}.$$

Then equations 234 become:

$$\left.\begin{aligned}
Q_w L_w + Q_v L_{wv} &= -3Q_0 L_{ew} \\
Q_v L_v + Q_w L_{wv} &= -3Q_0 L_{ev}
\end{aligned}\right\} \qquad (235)$$

Multiplying the first equation by L_v and the second equation by L_{wv}, and then subtracting the second from the first, gives:

$$Q_w = 3Q_0\frac{L_{ew}L_v - L_{ev}L_{wv}}{L_wL_v - L_{wv}^2} \tag{236}$$

Again, multiplying the first equation by L_{wv} and the second equation by L_w, and subtracting the second from the first, gives:

$$Q_v = -3Q_0\frac{L_{ew}L_{wv} - L_{ev}L_w}{L_{wv}^2 - L_vL_w} \tag{237}$$

Substituting the results from equations 236 and 237 in equation 233 gives:

$$E_0 = Q_0\left[\frac{1}{C_0} - \frac{3}{2\pi K}\left\{\frac{L_{ew}(L_{ew}L_v - L_{ev}L_{wv})}{L_wL_v - L_{wv}^2} + \frac{L_{ev}(L_{ew}L_{wv} - L_{ev}L_w)}{L_{wv}^2 - L_vL_w}\right\}\right]$$

$$= Q_0\left[\frac{1}{C_0} - \frac{3}{2\pi K}\left\{\frac{L_{ew}^2L_v + L_{ev}^2L_w - 2L_{ew}L_{ev}L_{wv}}{L_wL_v - L_{wv}^2}\right\}\right]$$

Finally, let C_0'' = zero-sequence capacity with two ground wires. Then:

$$C_0'' = \frac{Q_0}{E_0} = \frac{C_0}{1 - \dfrac{C_0}{A''}} \text{ fds/mile} \tag{238}$$

where:

$$A'' = \frac{0.01294\times10^{-6}\left[\log\dfrac{H_w}{r_w}\log\dfrac{H_v}{r_v} - \left(\log\dfrac{H_{wv}}{D_{wv}}\right)^2\right]}{\left(\log\dfrac{H_{ew}}{D_{ew}}\right)^2\log\dfrac{H_v}{r_v} + \left(\log\dfrac{H_{ev}}{D_{ev}}\right)^2\log\dfrac{H_w}{r_w} - 2\log\dfrac{H_{ew}}{D_{ew}}\log\dfrac{H_{ev}}{D_{ev}}\log\dfrac{H_{wv}}{D_{wv}}}$$

fds/mile \hfill (239)

Case C: Special case of two identical ground wires.—It will be assumed that the ground wires are located above horizontally spaced power conductors; and that the ground wires are equidistant from the center power conductor and are at the same height above the ground. For these conditions:

$$r_w = r_v,\ H_w = H_v,\ D_{bw} = D_{bv},\ H_{bw} = H_{bv};$$
$$D_{aw} = D_{cv},\ H_{aw} = H_{cv},\ D_{ab} = D_{bc},\ H_{ab} = H_{bc};$$
$$H_a = H_b = H_c,\ H_{av} = H_{cw},\ D_{av} = D_{aw};$$
$$H_{cv} = H_{aw},\ D_{cv} = D_{cw},\ H_{ev} = H_{ew};\ D_{ev} = D_{ew}.$$

Substituting these values in equation 239 will give for A'' the following:

$$A'' = 0.01294 \times 10^{-6} \left[\frac{\left(\log\dfrac{H_w}{r_w}\right)^2 - \left(\log\dfrac{H_{wv}}{D_{wv}}\right)^2}{2\left(\log\dfrac{H_{ew}}{D_{ew}}\right)^2 \log\dfrac{H_w}{r_w} - 2\left(\log\dfrac{H_{ew}}{D_{ew}}\right)^2 \log\dfrac{H_{wv}}{D_{wv}}} \right]$$

$$= \frac{0.01294 \times 10^{-6}}{2} \left[\frac{\log\dfrac{H_w}{r_w} + \log\dfrac{H_{wv}}{D_{wv}}}{\left(\log\dfrac{H_{ew}}{D_{ew}}\right)^2} \right] \text{fds/mile} \qquad (240)$$

The value of A'' from equation 240 can now be used in equation 238 in the same manner as the more general form of equation 239.

CHAPTER 15

SEQUENCE NETWORKS

104. Introduction.—In Chapter 10 it was indicated that the method of Symmetrical Components involved the application of a form of "superposition" in which unbalanced three-phase voltages and currents are considered in terms of the three components; namely, positive, negative, and zero sequences. In the present chapter it is proposed to develop specific laws of superposition for the most common types of commercial applications. The method of Symmetrical Components is adaptable to a good many applications of unbalance, but its use in the present case will be limited to the computation of fault currents.

The actual three-phase network, with the exception of the faulted point, will be assumed to be composed of balanced impedances. A later chapter will give a brief discussion of impedance unbalance.
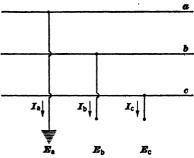
Fig. 110.   Three-Phase Line with One Conductor Grounded.

The common types of faults on a three-phase system are:

1. Single line to ground fault.
2. Double line to ground fault.
3. Line to line fault.
4. Three-phase fault.

105. Single Line to Ground Fault.—In Fig. 110 is shown a three-phase line with conductor a grounded. Let I_a, I_b, and I_c be the ground currents of the lines, and E_a, E_b, and E_c be the voltages of the conductors above ground. Since conductor a is the only one that is grounded, it is obvious that:

$$\left.\begin{array}{l} E_a = 0 \\ I_b = 0 \\ I_c = 0 \end{array}\right\} \qquad (241)$$

From equation 183 we may obtain the zero-sequence current, as follows:

$$I_{a0} = \frac{1}{3}[I_a + I_b + I_c] = \frac{1}{3}I_a \qquad (242)$$

From equation 188 the positive-sequence current is:

$$I_{ap} = \frac{1}{3}[I_a + I_b\underline{/120} + I_c\underline{/120}] = \frac{1}{3}I_a \qquad (243)$$

Similarly, according to equation 190, the negative-sequence current is:

$$I_{an} = \frac{1}{3}[I_a + I_b\overline{/120} + I_c\underline{/120}] = \frac{1}{3}I_a \qquad (244)$$

From equations 242, 243, and 244, it is evident that:

$$I_{a0} = I_{ap} = I_{an} = \frac{1}{3}I_a \qquad (245)$$

Also, according to equation 182, the total voltage of conductor a above ground is:

$$E_a = 0 = E_{ap} + E_{an} + E_{a0} \qquad (246)$$

The only network which can possibly satisfy equations 245 and 246 is a series circuit like that shown in Fig. 111. As will be noticed in Fig. 111, three-phase balanced generators contain only positive-sequence induced voltage, there being no generator source in the negative- and zero-sequence networks. From Fig. 111:
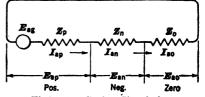
Fig. 111.   Series Circuit for Sequence Components.

$$I_{ap} = I_{an} = I_{a0} = \frac{E_{ag}}{Z_p + Z_n + Z_0} \qquad (247)$$

Also, from fundamental laws, the terminal voltage is equal to the induced voltage minus the voltage drops.   Thus:

$$\left.\begin{array}{l}E_{ap}=E_{ag}-I_{ap}Z_p\\E_{an}=0-I_{an}Z_n\\E_{a0}=0-I_{a0}Z_0\end{array}\right\}\qquad(248)$$

Having determined the sequence components, the actual line values are as follows:

$$\left.\begin{array}{l}I_a=I_{ap}+I_{an}+I_{a0}\\I_b=I_{bp}+I_{bn}+I_{b0}=I_{ap}\underline{/120}+I_{an}\underline{/120}+I_{a0}=0\\I_c=I_{cp}+I_{cn}+I_{c0}=I_{ap}\underline{/120}+I_{an}\underline{/120}+I_{a0}=0\end{array}\right\}\qquad(249)$$

Similarly:

$$\left.\begin{array}{l}E_a=E_{ap}+E_{an}+E_{a0}\\E_b=E_{ap}\underline{/120}+E_{an}\underline{/120}+E_{a0}\\E_c=E_{ap}\underline{/120}+E_{an}\underline{/120}+E_{a0}\end{array}\right\}\qquad(250)$$

But, with conductor a grounded, it is obvious that $E_a=0$. This can be shown to be true by proper substitution (see equations 248 and 250). Thus:

$$E_a=E_{ag}-I_{ap}Z_p-I_{an}Z_n-I_{a0}Z_0$$

Also, from equation 247:

$$E_a=E_{ag}-\left(\frac{E_{ag}}{Z_p+Z_n+Z_0}\right)(Z_p+Z_n+Z_0)=E_{ag}-E_{ag}=0$$

106. Example of Line to Ground Fault.—To illustrate the foregoing theory, consider a generator with the following data:

Induced voltage, $E_{ag}=100\underline{/0}$ volts;
Positive-sequence impedance, $Z_p=1.0\underline{/90}$ ohms;
Negative-sequence impedance, $Z_n=0.5\underline{/90}$ ohms;
Zero-sequence impedance, $Z_0=0.1\underline{/90}$ ohms;
Generator neutral solidly grounded.

For a line to ground fault, we must solve the network of Fig. 112, which is seen to be the same as Fig. 111. The calculations are as follows:

$$I_{ap} = I_{an} = I_{a0} = \frac{100\,\underline{/0}}{1.0\,\underline{/90} + 0.5\,\underline{/90} + 0.1\,\underline{/90}}$$

$$= \frac{100\,\underline{/0}}{1.6\,\underline{/90}} = 62.5\,\underline{/90}$$

$$E_{ap} = 100\,\underline{/0} - 62.5\,\underline{/90} \times 1.0\,\underline{/90}$$

$$= 100\,\underline{/0} - 62.5\,\underline{/0} = 37.5\,\underline{/0}$$

$$E_{an} = -62.5\,\underline{/90} \times 0.5\,\underline{/90} = -31.25\,\underline{/0}$$

$$E_{a0} = -62.5\,\underline{/90} \times 0.1\,\underline{/90} = -6.25\,\underline{/0}$$
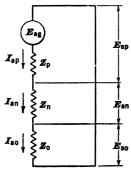
Fig. 112. Conditions for Example in Art. 106.
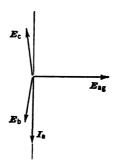
Fig. 113. Results of Example in Art. 106.

From these results, the following values are obtained:

$$I_a = 62.5\,\underline{/90} + 62.5\,\underline{/90} + 62.5\,\underline{/90} = 187.5\,\underline{/90}$$

$$I_b = 0 \text{ and } I_c = 0$$

$$E_a = 0$$

$$E_b = 37.5\,\underline{/120} - 31.25\,\underline{/120} - 6.25\,\underline{/0}$$

$$= -18.75 - j32.5 + 15.625 - j27.1 - 6.25 = -9.375 - j59.6$$

$$E_c = 37.5\,\underline{/120} + 31.25\,\underline{/120} - 6.25\,\underline{/0}$$

$$= -18.75 + j32.5 + 15.625 + j27.1 - 6.25 = -9.375 + j59.6$$

The results are plotted in Fig. 113.

107. Line to Ground Fault, Including Ground Impedance.
In this case, which is represented in Fig. 114, the relations that apply are:

$$E_a = I_a Z = (I_{ap} + I_{an} + I_{a0})Z \qquad (251)$$

and

$$I_b = I_c = 0 \qquad (252)$$
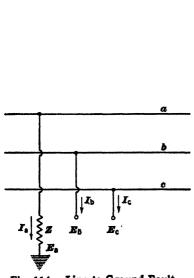
Fig. 114.   Line to Ground Fault,
with Ground Impedance.
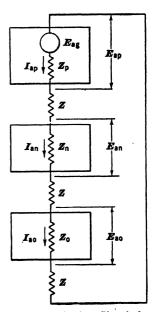
Fig. 115.   Series Circuit for
Sequence Components.

The current relations will be the same as given by equation 245. Thus:

$$I_{ap} = I_{an} = I_{a0} = \frac{1}{3} I_a \qquad (253)$$

This means that the three networks are in series, as shown in Fig. 115, with the impedance Z introduced in series with each sequence network (see equation 251). In this case:

$$I_{ap} = I_{an} = I_{a0} = \frac{E_{ag}}{Z_p + Z_n + Z_0 + 3Z} \qquad (254)$$

Also:

$$\left.\begin{array}{l}E_{ap}=E_{ag}-I_{ap}Z_p\\ E_{an}=\quad\;\;-I_{an}Z_n\\ E_{a0}=\quad\;\;-I_{a0}Z_0\end{array}\right\} \tag{255}$$

Furthermore:

$$E_a=E_{ap}+E_{an}+E_{a0}=E_{ag}-I_{ap}(Z_p+Z_n+Z_0)$$

$$=E_{ag}-\frac{E_{ag}(Z_p+Z_n+Z_0)}{Z_p+Z_n+Z_0+3Z}=\frac{E_{ag}(3Z)}{Z_p+Z_n+Z_0+3Z}=3I_{ap}Z$$

This result is seen to check with equation 251, in the light of the results of equation 254.

Having determined the sequence values of current and voltage in phase a, other values can be obtained by direct application of equations 249 and 250.
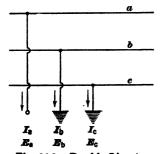
Fig. 116.   Double Line to
          Ground Fault.
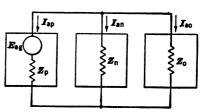
Fig. 117.   Network for Sequence
          Components.

108. Double Line to Ground Fault.—It will be assumed that phases b and c are both grounded and conductor a is free, as in Fig. 116.   From the conditions of Fig. 116:

$$\left.\begin{array}{l}E_b=E_c=0\\ I_a=0\end{array}\right\} \tag{256}$$

From the fundamental sequence relations:

$$\left.\begin{array}{l}E_{ap}=\dfrac{1}{3}(E_a+E_b\underline{/120}+E_c\underline{/120})=\dfrac{1}{3}E_a\\[2mm] E_{an}=\dfrac{1}{3}(E_a+E_b\underline{/120}+E_c\underline{/120})=\dfrac{1}{3}E_a\\[2mm] E_{a0}=\dfrac{1}{3}(E_a+E_b+E_c)=\dfrac{1}{3}E_a\end{array}\right\} \tag{257}$$

From equation 257 it follows tnat:

$$E_{ap} = E_{an} = E_{a0} \qquad (258)$$

Therefore, the sequence networks must be considered as connected in parallel, as in Fig. 117. From this figure, we have the following:

$$I_{ap} = \frac{E_{ag}}{Z_p + \dfrac{Z_n Z_0}{Z_n + Z_0}} \qquad (259)$$

$$I_{ap} = -I_{an} - I_{a0} \qquad (260)$$

$$E_{ap} = E_{ag} - I_{ap} Z_p \qquad (261)$$

From equation 258 we may write:

$$E_{ap} = E_{an} = E_{a0} = -I_{an} Z_n = -I_{a0} Z_0$$

from which:

$$\left. \begin{array}{l} I_{an} = -\dfrac{E_{ap}}{Z_n} \\[2mm] I_{a0} = -\dfrac{E_{ap}}{Z_0} \end{array} \right\} \qquad (262)$$

The final phase values are obtained according to equations 249 and 250.

109. Example of Double Line to Ground Fault.—The *b* and *c* phases of the generator of Art. 106 will be considered grounded. The network circuit is given in Fig. 117. The calculations follow:

$$I_{ap} = \frac{100\underline{/0}}{1.0\underline{/90} + \dfrac{(0.5\underline{/90})(0.1\underline{/90})}{0.5\underline{/90} + 0.1\underline{/90}}} = \frac{100\underline{/0}}{1.0833\underline{/90}} = 92.3\overline{\underline{/90}}$$

$$E_{ap} = 100\underline{/0} - 92.3\overline{\underline{/90}} \times 1.0\underline{/90} = 100\underline{/0} - 92.3\underline{/0} = 7.7\underline{/0}$$

$$E_{an} = E_{a0} = 7.7\underline{/0}$$

$$I_{an} = -\frac{7.7\underline{/0}}{0.5\underline{/90}} = -15.4\overline{\underline{/90}}$$

$$I_{a0} = -\frac{7.7\underline{/0}}{0.1\underline{/90}} = -77.0\underline{/90}$$

$$I_{ap} = -I_{an} - I_{a0} = 92.4\underline{/90}$$

From these values, the following results are obtained:

$$E_a = E_{ap} + E_{an} + E_{a0} = 3E_{ap} = 23.1\underline{/0}$$

$$E_b = E_c = 0$$

$$I_a = I_{ap} + I_{an} + I_{a0} = 0$$

$$I_b = I_{ap}\underline{/120} + I_{an}\underline{/120} + I_{a0}$$

$$= 92.4\underline{/210} - 15.4\underline{/30} - 77\underline{/90}$$

$$= -80 + j46.2 - 13.35 - j7.7 + j77 = -93.35 + j115.5$$

$$I_c = I_{ap}\underline{/120} + I_{an}\underline{/120} + I_{a0}$$

$$= 92.4\underline{/30} - 15.4\underline{/210} - 77\underline{/90}$$

$$= 80 + j46.2 + 13.35 - j7.7 + j77 = 93.35 + j115.5$$
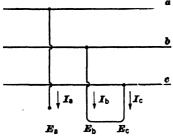
Fig. 118. Line to Line Fault.
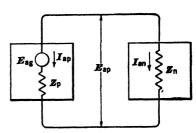
Fig. 119. Network for Sequence Components.

The ground current is:

$$I_g = I_b + I_c = j231$$

It will be noticed that the ground current is equal to three times the zero-sequence current.

This solution assumes that the neutral is solidly grounded. If this is not the case, then the double line to ground fault becomes a line to line fault.

110. Line to Line Fault.—Such a type of fault is illustrated in Fig. 118. The fundamental equations are seen to be as follows:

$$I_a = 0 \tag{263}$$

$$E_b = E_c \tag{264}$$

$$I_b = -I_c \tag{265}$$

Also,

$$I_{a0} = \frac{1}{3}(I_a + I_b + I_c) = 0 \tag{266}$$

and

$$E_{a0} = 0 \tag{267}$$

$$I_{ap} = \frac{1}{3}(I_a + I_b\underline{/120} + I_c\underline{/\overline{120}}) = \frac{I_b}{3}(\underline{/120} - \underline{/\overline{120}})$$

$$I_{ap} = \frac{I_b}{3}(\underline{/120} + \underline{/60}) = \frac{I_b}{3}(\sqrt{3}\underline{/90}) = \frac{I_b}{\sqrt{3}}\underline{/90} \tag{268}$$

$$I_{an} = \frac{1}{3}(I_a + I_b\underline{/\overline{120}} + I_c\underline{/120}) = \frac{I_b}{3}(\underline{/\overline{120}} - \underline{/120})$$

$$I_{an} = \frac{I_b}{\sqrt{3}}\underline{/\overline{90}} = -I_{ap} \tag{269}$$

$$E_{ap} = \frac{1}{3}(E_a + E_b\underline{/120} + E_c\underline{/\overline{120}}) = \frac{1}{3}[E_a + E_b(\underline{/120} + \underline{/\overline{120}})]$$

$$E_{ap} = \frac{1}{3}(E_a - E_b) \tag{270}$$

$$E_{an} = \frac{1}{3}(E_a + E_b\underline{/\overline{120}} + E_c\underline{/120}) = \frac{1}{3}(E_a - E_b) \tag{271}$$

Inspecting equations 268, 269, 270, and 271, we find that Fig. 119 will satisfy the requirements of line to line fault. From this figure we can now complete the necessary solution. Thus:

$$I_{ap} = \frac{E_{ag}}{Z_p + Z_n} \tag{272}$$

$$I_{an} = -I_{ap} \tag{273}$$

$$\left.\begin{array}{l} E_{ap} = E_{ag} - I_{ap}Z_p \\ E_{an} = E_{ap} \end{array}\right\} \tag{274}$$

The actual line currents and voltages can now be obtained as in previous articles.

111. Example of Line to Line Fault.—This type of fault is illustrated with the generator data of Art. 106. The calculations follow:

$$I_{ap} = \frac{100\underline{/0}}{1.0\underline{/90}+0.5\underline{/90}} = \frac{100\underline{/0}}{1.5\underline{/90}} = 66.6\underline{/90}$$

$$I_{an} = -66.6\underline{/90}$$

$$I_{a0} = 0$$

$$E_{a0} = 0$$

$$E_{ap} = 100\underline{/0} - 66.6\underline{/90} \times 1.0\underline{/90}$$

$$= 100\underline{/0} - 66.6\underline{/0} = 33.4\underline{/0}$$

$$E_{an} = E_{ap} = 33.4\underline{/0}$$

Therefore:

$$E_a = E_{a0} + E_{ap} + E_{an} = 33.4\underline{/0} + 33.4\underline{/0} = 66.8\underline{/0}$$

$$E_b = E_{a0} + E_{ap}\underline{/120} + E_{an}\underline{/120}$$

$$= 33.4\underline{/120} + 33.4\underline{/120} = -33.4\underline{/0}$$

$$E_c = E_{a0} + E_{ap}\underline{/120} + E_{an}\underline{/120}$$

$$= 33.4\underline{/120} + 33.4\underline{/120} = -33.4\underline{/0}$$

$$I_a = I_{a0} + I_{ap} + I_{an} = 0$$

$$I_b = I_{a0} + I_{ap}\underline{/120} + I_{an}\underline{/120}$$

$$= 66.6\underline{/210} - 66.6\underline{/30} = 115.5\underline{/180}$$

$$I_c = I_{a0} + I_{ap}\underline{/120} + I_{an}\underline{/120}$$

$$= 66.6\underline{/30} - 66.6\underline{/210} = 115.5\underline{/0}$$

112. Three-Phase Faults.—Fault calculations, as considered in this chapter, have involved circuits of balanced impedances in the three phases. A three-phase fault is there-

fore a case of absolutely balanced conditions, and only positive-sequence quantities will be involved. This is illustrated in Figs. 120 and 121.

The solution of Fig. 121 is quite obvious, the final currents and voltages being equal to the positive-sequence quantities. The fault of Fig. 120 could have been grounded without changing the interpretation of the problem.
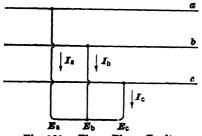
Fig. 120.   Three-Phase Fault.   Fig. 121.   Network for Sequence Components.
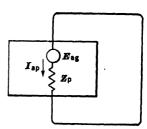
113. Double Line to Ground Through Impedance.—From Fig. 122 we have the following:

$$\left.\begin{array}{c} I_a = 0 \\ E_b = E_c = (I_b + I_c)\, Z \end{array}\right\} \tag{275}$$

$$E_{a0} = \frac{1}{3}(E_a + E_b + E_c) = \frac{1}{3}(E_a + 2E_b) \tag{276}$$

$$E_{ap} = \frac{1}{3}(E_a + E_b \underline{/120} + E_c \overline{/120})$$

$$E_{ap} = \frac{1}{3}[E_a + E_b(\underline{/120} + \overline{/120})] = \frac{1}{3}(E_a - E_b) \tag{277}$$

$$E_{an} = \frac{1}{3}[E_a + E_b \overline{/120} + E_c \underline{/120}]$$

$$E_{an} = \frac{1}{3}[E_a + E_b(\overline{/120} + \underline{/120})] = \frac{1}{3}(E_a - E_b) \tag{278}$$

From equations 277 and 278:

$$E_{ap} = E_{an} \tag{279}$$

Equation 279 indicates that the positive- and negative-sequence networks should be connected in parallel.

Equation 276 may be changed to:

$$E_{a0} = \frac{1}{3}(E_a - E_b + 3E_b) = \frac{1}{3}(E_a - E_b) + E_b$$

From equations 275 and 277:

$$E_{a0} = E_{ap} + (I_b + I_c)Z \qquad (280)$$

Also,

$$I_{a0} = \frac{1}{3}(I_a + I_b + I_c) = \frac{1}{3}(I_b + I_c) \qquad (281)$$

Equation 280 becomes:

$$E_{a0} = E_{ap} + 3I_{a0}Z \qquad (282)$$
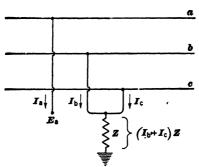
Fig. 122.   Double Line to Ground
Through Impedance.

Analyzing equations 279 and 282 will show that the sequence networks must be in parallel with an impedance equal to $3Z$ in the zero-sequence leg, as shown in Fig. 123. From Fig. 123, it is evident that:

$$I_{ap} = \frac{E_{ag}}{Z_p + \dfrac{Z_n(Z_0 + 3Z)}{Z_n + Z_0 + 3Z}} \qquad (283)$$

$$E_{ap} = E_{ag} - I_{ap}Z_p = E_{an} = -I_{an}Z_n \qquad (284)$$

$$I_{an} = -\frac{E_{ap}}{Z_n} \qquad (285)$$

$$I_{a0} = -\frac{E_{ap}}{Z_0 + 3Z}$$

$$I_{a0} = -I_{ap} - I_{an}$$

$$(286)$$

$$E_{a0} = -I_{a0}Z_0$$

$$E_{a0} = E_{ap} + 3Z I_{a0}$$

$$(287)$$
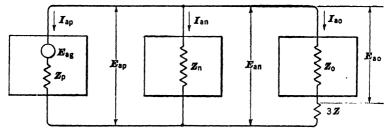
Fig. 123.   Network for Sequence Components
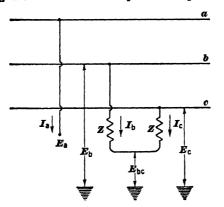
Fig. 124.   Line to Line Fault
Through Impedance.

Having determined the sequence components of current and voltage, the actual circuit currents and voltages can be obtained according to equations 249 and 250.

114. Line to Line Fault Through Impedance.—This type of fault is shown in Fig. 124, from which the following equations are obtained:

$$\left.\begin{array}{l} I_a = 0 \\ E_{bc} = E_b - I_b Z = E_c - I_c Z \\ I_b = -I_c \end{array}\right\} \tag{288}$$

Also:

$$I_{a0} = \frac{1}{3}(I_a + I_b + I_c) = 0 \tag{289}$$

$$I_{ap} = \frac{1}{3}(I_a + I_b \underline{/120} + I_c \underline{/\overline{120}})$$

$$I_{ap} = \frac{1}{3}[I_b(\underline{/120} - \underline{/\overline{120}})] = \frac{I_b}{\sqrt{3}} \underline{/90} \tag{290}$$

$$I_{an} = \frac{1}{3}(I_a + I_b \underline{/\overline{120}} + I_c \underline{/120})$$

$$I_{an} = \frac{I_b}{\sqrt{3}} \underline{/90} = -I_{ap} \tag{291}$$
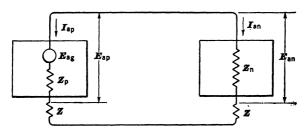
Fig. 125.   Network for Sequence Components.

Since $I_{a0} = 0$, it is obvious that $E_{a0} = 0$.   But,

$$E_{ap} = \frac{1}{3}(E_a + E_b \underline{/120} + E_c \underline{/\overline{120}})$$

in which:

$$E_b = E_{bc} + I_b Z$$

and

$$E_c = E_{bc} + I_c Z = E_{bc} - I_b Z$$

Therefore:

$$E_{ap} = \frac{1}{3}[E_a + (E_{bc} + I_b Z)\underline{/120} + (E_{bc} - I_b Z)\underline{/\overline{120}}]$$

$$= \frac{1}{3}[E_a - E_{bc} + \sqrt{3} I_b Z \underline{/90}]$$

$$E_{ap} = \frac{1}{3}(E_a - E_{bc}) + I_{ap}Z \tag{292}$$

$$E_{an} = \frac{1}{3}[E_a - E_{bc} + \sqrt{3}I_b Z \underline{/90}]$$

$$E_{an} = \frac{1}{3}(E_a - E_{bc}) + I_{an}Z \tag{293}$$

The circuit network which satisfies equations 289 to 293 is given in Fig. 125. The solution of Fig. 125 is carried out in the same manner as shown in preceding articles.

CHAPTER 16

SHORT-CIRCUIT CALCULATIONS OF POWER NETWORKS

115. Introduction.—In the last few chapters the general theory of symmetrical components and sequence impedances has been introduced. It is now proposed to apply this material to the solution of a few rather simple network problems. As has been previously indicated, the present interest in this method of circuit calculation is for the solution of steady-state currents and voltages in faulted networks. The general procedure may be outlined as follows:

(a) Inspect the original network, and prepare detailed individual sequence networks.

(b) Reduce each sequence circuit to the simplest form, according to the method of equivalent parallel and series circuits and equivalent Y-delta transformations.
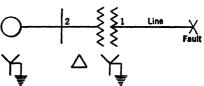
Fig. 126. Simple Network of Generator, Transformer, and Transmission Line.

(c) Connect the three sequence networks according to the requirements of the particular fault (see Chapter 15).

(d) Obtain the total sequence currents and voltages.

(e) Obtain the distribution of the sequence current and voltage in all parts of the original circuit.

(f) Combine the sequence currents and voltages, obtaining currents and voltages in each phase at all points of the original network.

116. Simple Network of Generator, Transformer, and Transmission Line.—In Fig. 126 it will be assumed that the generator is Y-connected and the neutral is grounded. Also, the transformer bank is composed of single-phase units, delta-Y connected with solidly grounded neutral. The first step in

analyzing the network is to determine how the three sequence circuits are set up. These circuits are shown in Fig. 127.

The positive- and negative-sequence networks will naturally include the impedances for all the component units of Fig. 126. The zero-sequence circuit, however, will not include the gen-

(a) Positive-Sequence Network

(b) Negative-Sequence Network

(c) Zero-Sequence Network
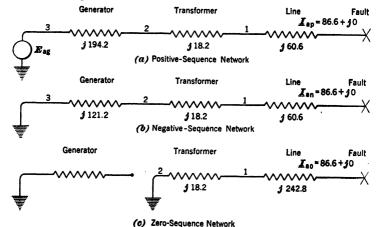
Fig. 127. Sequence Circuits for Fig. 126.

erator. Even though the generator neutral is grounded, it is impossible for zero-sequence currents to flow between the generator and the delta winding of the transformer. This is obviously true because zero-sequence currents must find a return circuit through the ground. However, since the transformer neutral is grounded, zero-sequence currents may flow to the transmission line (see Art. 91, case 5).

117. Example for Fig. 126.—The following data may be assumed for the network in Fig. 126:

Device	Kva Capacity	Line Voltage	Per Cent Reactances		
			Positive	Negative	Zero
Generator.....	50,000	11,000	80*	50	20
Transformer....	40,000	11,000/110,000	6	6	6
Line..........	20,000	110,000	10	10	40

* Synchronous reactances.

The given values of per cent reactances are based on the capacity and voltage of each particular device. It will be noticed that the positive- and negative-sequence transmission-line reactances have the same value, whereas the zero-sequence reactance is somewhat larger. The first step is to convert all per cent reactances to a common Kva base, preferably that of the generator. These values are tabulated as follows:

Device	Per Cent Reactances Based on 50,000 Kva		
	Positive	Negative	Zero
Generator................	80	50	20
Transformer..............	7.5	7.5	7.5
Line....................	25	25	100

As will be noticed in Fig. 127, the positive-, negative-, and zero-sequence networks are made up of components in series. Therefore, the total reactances are:

Positive sequence ......% $X_p = 80 + 7.5 + 25 = 112.5\%$
Negative sequence......% $X_n = 50 + 7.5 + 25 = 82.5\%$
Zero sequence .... % $X_0 = 7.5 + 100 = 107.5\%$

Commercial problems of this type are quite often carried out in the percentage or per-unit notation. However, as the first introduction to network calculations, it is probably worth while to adopt the basic impedance method. The novice will find this method somewhat easier to follow; and, after he has mastered the basic features, he will find very little trouble in adjusting himself to other methods of notation.
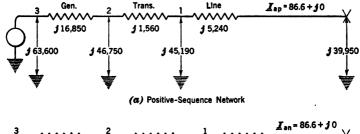
The values of per cent reactance can now be converted into ohmic reactances by applying the relation:
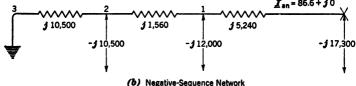
$$X = \frac{(\%X)E}{100 I_n}$$

where $I_n = \dfrac{50,000}{\sqrt{3} \times 110} = 262$ amps.

Therefore, the ohmic reactances are:

$$X_p = j273 \text{ ohms}$$
$$X_n = j200 \text{ ohms}$$
$$X_0 = j261 \text{ ohms}$$

(a) Positive-Sequence Network

(b) Negative-Sequence Network

(c) Zero-Sequence Network
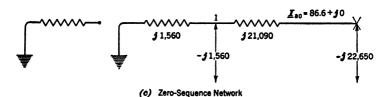
Fig. 128.   Sequence Voltage Drops and Phase Voltages.

118.   Single Line to Ground Fault.—The conditions are similar to those considered in Arts. 105 and 106 and in Fig. 111. For the network in Figs. 126 and 127:

$$I_{ap} = I_{an} = I_{a0} = \frac{j63,600}{i(273+200+261)} = 86.6 \text{ amps}$$

Also:

$$I_a = 259.8 \text{ amps}$$
$$I_b = 0$$
$$I_c = 0$$

According to equations 248, the sequence voltages are:

$$E_{ap} = j63{,}600 - 86.6 \times j273 = j39{,}950$$
$$E_{an} = -86.6 \times j200 = -j17{,}300$$
$$E_{a0} = -86.6 \times j261 = -j22{,}650$$

The resultant voltages at the fault point $\times$ are:

$$E_{ax} = j39{,}950 - j17{,}300 - j22{,}650 = 0$$

$$E_{bx} = j39{,}950\overline{/120} - j17{,}300\underline{/120} - j22{,}650$$

$$= 34{,}600 - j19{,}975 + 15{,}000 + j8{,}650 - j22{,}650$$

$$= 49{,}600 - j33{,}975$$

$$E_{cx} = j39{,}950\underline{/120} - j17{,}300\overline{/120} - j22{,}650$$

$$= -34{,}600 - j19{,}975 - 15{,}000 + j8{,}650 - i22{,}650$$

$$= -49{,}600 - j33{,}975$$

These three voltages represent the potentials above ground of the lines at the point of fault. The three line voltages at the point of fault are:

$$E_{abx} = E_b - E_a = 49{,}600 - j33{,}975$$
$$E_{bcx} = E_c - E_b = -99{,}200 + j0$$
$$E_{cax} = E_a - E_c = 49{,}600 + j33{,}975$$

To determine the potential of each line at points 1, 2, and 3 in Figs. 126 and 127, the impedance drop for each sequence must be computed. The actual ohmic impedance of each part of Fig. 127 is given in the following table:

	Ohmic Impedance Based on 50,000 Kva		
	Positive	Negative	Zero
Generator................	$j194.2$	$j121.2$	not in circuit
Transformer..............	$j18.2$	$j18.2$	$j18.2$
Line.....................	$j60.6$	$j60.6$	$j242.8$
Total..................	$j273.0$	$j200.0$	$j261.0$

The sequence component voltage drops and the conductor voltages above ground are shown in Fig. 128.

The actual conductor voltages above ground at points 1, 2, and 3, Fig. 128, may now be calculated. Thus:

$E_{a1} = j45,190 - j12,060 - j1,560 = j31,570$

$E_{a2} = j46,750 - j10,500 = j36,250$

$E_{a3} = j63,600$

$E_{b1} = j45,190\underline{/120} - j12,060\underline{/120} - j1,560 = 49,600 - j18,125$

$E_{b2} = j46,750\underline{/120} - j10,500\underline{/120} = 49,600 - j18,125$

$E_{b3} = j63,600\underline{/120} = 55,100 - j31,800$

$E_{c1} = j45,190\underline{/120} - j12,060\underline{/120} - j1,560 = -49,600 - j18,125$

$E_{c2} = j46,750\underline{/120} - j10,500\underline{/120} = -49,600 - j18,125$

$E_{c3} = j63,600\underline{/120} = -55,100 - j31,800$
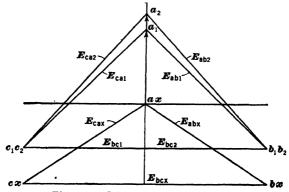
Fig. 129.   Phase and Line Voltages.

The line voltages at these several points are:

$E_{ab1} = 49,600 - j49,695$
$E_{ab2} = 49,600 - j54,375$
$E_{bc1} = -99,200 + j0$
$E_{bc2} = -99,200 + j0$
$E_{ca1} = 49,600 + j49,695$
$E_{ca2} = 49,600 + j54,375$

These several phase and line voltages are shown in Fig. 129. It is particularly interesting to examine this figure and detect how the line-voltage triangle shifts from an isosceles triangle at the point of fault to a balanced equilateral triangle within the generator.

In computing the line currents, all three sequence impedances were included for the transmission line and the transformer, but only positive- and negative-sequence impedances in the generator circuits, since the generator feeds into a delta-connected transformer bank. Thus, the equivalent Y transformer primary currents are:

$$I_a = I_{ap} + I_{un} = 173.2 + j0$$
$$I_b = 86.6\underline{/120} + 86.6\underline{/120} = -86.6 + j0$$
$$I_c = 86.6\underline{/120} + 86.6\underline{/120} = -86.6 + j0$$

Fig. 130.   Transformer Connections.

As would be expected, the sum of these three currents must be equal to zero.

In the solution thus far, it has been assumed that the entire circuit was at the same voltage as the transmission line. Also, no account has been taken of the phase angles involved in a delta-Y transformation. In other words the entire solution has been based on equivalent Y connections of all circuits. It should be remembered that the transformers of Fig. 126 were connected in delta-Y, as indicated in Fig. 130, corresponding windings being indicated by the letters x, y, and z.

The primary delta currents in the x, y, and z primary windings can be obtained by multiplying the secondary Y currents by the inverse ratio of turns, namely, 5.8. Thus:

$$I_x = 5.8(173.2 + j0) = 1,004 + j0$$
$$I_y = 5.8(-86.6 + j0) = -502 + j0$$
$$I_z = 5.8(-86.6 + j0) = -502 + j0$$

The corresponding line currents are:

$$I_1 = I_x - I_z = 1,004 + j0 + 502 - j0 = 1,506 + j0$$
$$I_2 = I_y - I_x = -502 + j0 - 1,004 - j0 = -1,506 - j0$$
$$I_3 = I_z - I_y = -502 + j0 + 502 - j0 = 0$$

The same values can be obtained from the resultant current, $I_a = 259.8$, flowing in the transmission line. Thus:

$$I_{12} = 259.8 \times 5.8 = 1,506 + j0$$
$$I_1 = 1,506 + j0$$
$$I_2 = -1,506 - j0$$

Again, applying the turns ratio of the transformer, the primary voltages at the generator may be found as follows:

$$E_{12} = \frac{E_{a3}}{5.8} = \frac{j63,600}{5.8} = 11,000\underline{/90}$$

$$E_{23} = \frac{E_{b3}}{5.8} = \frac{j63,600\underline{/120}}{5.8} = 11,000\underline{/30}$$

$$E_{31} = \frac{E_{c3}}{5.8} = \frac{j63,600\underline{/120}}{5.8} = 11,000\underline{/210}$$

119. Double Line to Ground Fault.—See Arts. 108 and 109 and Fig. 117 for typical conditions. As indicated in Fig. 116, conductors a and b are assumed to be grounded. Thus, considering the same circuit as in Arts. 117 and 118, we have, according to equation 259:

$$I_{ap} = \frac{j63,600}{j273 + \frac{j200 \times j261}{j200 + j261}} = \frac{j63,600}{j386.2} = 164.5$$

From equation 261:

$$E_{ap} = j63,600 - 164.5 \times j273 = j63,600 - j45,000 = j18,600$$

and, from equation 258:

$$E_{an} = E_{a0} = j18,600$$

Also, by equations 262:

$$I_{an} = -\frac{j18,600}{j200} = -93.0$$

$$I_{a0} = -\frac{j18,600}{j261} = -71.5$$

The sequence current and voltage distributions are shown in Fig. 131, the detail computations and notation being similar to those in the example of Art. 118.

(a) Positive-Sequence Network

(b) Negative-Sequence Network

(c) Zero-Sequence Network
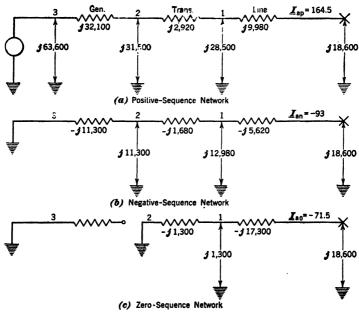
Fig. 131. Sequence Currents and Voltages.

The actual currents in the three transmission lines are:

$$I_a = 164.5 - 93 - 71.5 = 0$$

$$I_b = 164.5\underline{/120} - 93\underline{/120} - 71.5 = -82.25 - j142.5 + 46.5$$
$$-j80.5 - 71.5 = -107.25 - j223$$

$$I_c = 164.5\underline{/120} - 93\underline{/120} - 71.5 = -82.25 + j142.5 + 46.5$$
$$+j80.5 - 71.5 = -107.25 + j223$$

The equivalent Y transformer primary currents are:

$$I_a = 164.5 - 93 = 71.5 + j0$$

$$I_b = 164.5\underline{/120} - 93\underline{/120} = -35.75 - j223$$

$$I_c = 164.5\underline{/120} - 93\overline{\underline{/120}} = -35.75 + j223$$

The actual primary delta currents are found as follows:

$$I_{12} = 5.8(71.5 + j0) = 415 + j0$$
$$I_{23} = 5.8(-35.75 - j223) = -207.5 - j1,294$$
$$I_{31} = 5.8(-35.75 + j223) = -207.5 + j1,294$$

From these currents:

$$I_1 = 415 + j0 + 207.5 - j1,294 = 622.5 - j1,294$$
$$I_2 = -207.5 - j1,294 - 415 + j0 = -622.5 - j1,294$$
$$I_3 = -207.5 + j1,294 + 207.5 + j1,294 = 0 + j2,588$$

These values can also be obtained directly from the actual secondary transformer currents, as follows:

$$I_{12} = 5.8(0 + j0) = 0$$
$$I_{23} = 5.8(-107.25 - j223) = -622.5 - j1,294$$
$$I_{31} = 5.8(-107.25 + j223) = -622.5 + j1,294$$

From these results:

$$I_1 = 0 + 622.5 - j1,294 = 622.5 - j1,294$$
$$I_2 = -622.5 - j1,294 - 0 = -622.5 - j1,294$$
$$I_3 = -622.5 + j1,294 + 622.5 + j1,294 = 0 + j2,588$$

The corresponding phase voltages are:

$$E_{ax} = 3 \times j18,600 = j55,800$$

$$E_{bx} = j18,600\underline{/120} + j18,600\overline{\underline{/120}} + j18,600 = 0$$

$$E_{cx} = j18,600\underline{/120} + j18,600\overline{\underline{/120}} + j18,600 = 0$$

Other voltages on points 1 and 2 of Fig. 131 are:

$$E_{a1} = j28,580 + j12,980 + j1,300 = j42,860$$

$$E_{b1} = j28,580\overline{\underline{/120}} + j12,980\underline{/120} + j1,300 = 13,500 - j19,480$$

$$E_{c1} = j28,580\underline{/120} + j12,980\overline{\underline{/120}} + j1,300 = -13,500 - j19,480$$

$E_{a2} = j31,500 + j11,300 = j42,800$

$E_{b2} = j31,500 \underline{/120} + j11,300 \underline{/120} = 17,500 - j21,400$

$E_{c2} = j31,500 \underline{/120} + j11,300 \underline{/120} = -17,500 - j21,400$

$E_{12} = j11,000 = 11,000 \underline{/90}$

$E_{23} = 11,000 \underline{/30}$

$E_{31} = 11,000 \underline{/210}$

120. Solution of Network.—To further illustrate the application of symmetrical components, the network illustrated in Fig. 132 will be analyzed. Generator A, transformer a, and transmission line x will be considered the same as in the example relating to Fig. 126. The connections of other component circuits are indicated in Fig. 132, and the corresponding data are tabulated as follows:

Device	Kva Capacity	Line Voltage	Per Cent Reactance		
			Positive	Negative	Zero
Generator A.....	50,000	11,000	80*	50	20
Transformer a....	40,000	11,000/110,000	6	6	6
Line x..........	20,000	110,000	10	10	40
Generator B.....	50,000	11,000	90*	40	30
Transformer b....	50,000	11,000/110,000	10	10	10
Line y..........	25,000	110,000	15	15	40
Line z..........	10,000	110,000	5	5	10
Transformer c....	40,000	110,000/44,000	12	12	12

* Synchronous reactances.

In practice, two-winding Y-Y connected transformers are not generally used, but they are chosen here to illustrate the procedure, since the treatment of three-winding transformers will be given in a later chapter.

The first step is to convert all the given per cent reactances to a common basis of Kva. For convenience the basis is taken equal to 100,000 Kva, which is the total generator capacity.

The converted per cent reactances are as follows:

Device	Per Cent Reactances Based on 100,000 Kva		
	Positive	Negative	Zero
Generator A.............	160	100	40
Transformer a............	15	15	15
Line x..................	50	50	200
Generator B.............	180	80	60
Transformer b............	20	20	20
Line y..................	60	60	160
Line z..................	50	50	100
Transformer c............	30	30	30
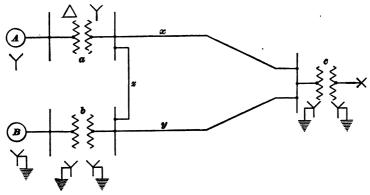
Fig. 132.   Typical Power Network.

The corresponding ohmic reactances are shown in the following table.

Device	Ohmic Reactances Based on 44,000 Volts		
	Positive	Negative	Zero
Generator A.............	31.1	19.4	7.76
Transformer a............	2.81	2.81	2.81
Line x..................	9.8	9.8	38.8
Generator B.............	34.9	15.5	11.65
Transformer b............	3.88	3.88	3.88
Line y..................	11.65	11.65	31.1
Line z..................	9.8	9.8	19.4
Transformer c............	5.82	5.82	5.82

The three sequence networks are represented in Figs. 133, 134, and 135, in which the corresponding reactances are shown with each device of the circuit.
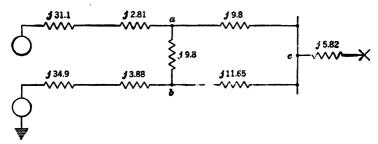
Fig. 133.   Positive-Sequence Network.
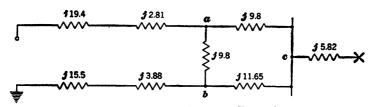
Fig. 134.   Negative-Sequence Network.
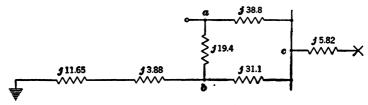
Fig. 135.   Zero-Sequence Network.

As done in the preceding example, it is now necessary to simplify the three networks into an equivalent simple series reactance.   Examination of Figs. 133 and 134 reveals that these sequence networks involve circuits composed of the three transmission lines x, y, and z.   This combination must be converted to an equivalent star or T circuit by the use of the following equations (which can be obtained from any textbook on three-phase-circuit theory):

$$Z_{a0} = \frac{Z_{ab}Z_{ca}}{Z_{ab}+Z_{bc}+Z_{ca}}$$

$$Z_{b0} = \frac{Z_{bc}Z_{ab}}{Z_{ab}+Z_{bc}+Z_{ca}}$$

$$Z_{c0} = \frac{Z_{ca}Z_{bc}}{Z_{ab}+Z_{bc}+Z_{ca}}$$

Fig. 136.   Positive-Sequence Network.
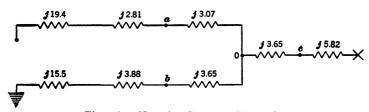
Fig. 137.   Negative-Sequence Network.

By applying these equations to the three networks, it is possible to convert the networks to the forms given in Figs. 136 and 137.   The zero-sequence network does not require transformation since the zero-sequence circuit in Fig. 135 is open between point a and generator A.

By series and parallel conversions, sequence networks will resolve themselves into the forms shown in Figs. 138, 139, and 140.

Finally Figs. 138, 139, and 140 can be reduced to the following single impedances:

$$Z_p = 0 + j29.22$$
$$Z_n = 0 + j21.55$$
$$Z_0 = 0 + j41.55$$

To continue the problem, a single line to ground fault will be assumed on the load side of transformer c, Fig. 132. The solution follows the outline given in Arts. 105 and 106. The network circuit is as given in Fig. 141.
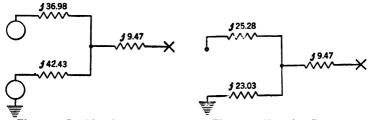
Fig. 138. Positive-Sequence Network.

Fig. 139. Negative-Sequence Network.
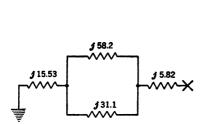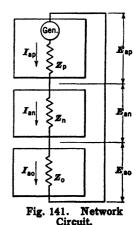
Fig. 140. Zero-Sequence Network.

Fig. 141. Network Circuit.

From equation 247:

$$I_{ap} = I_{an} = I_{a0} = \frac{\frac{44{,}000}{\sqrt{3}}}{(j29.22 + j21.55 + j41.55)} = \frac{25{,}400}{j92.32} = 276\underline{/90}$$

From equation 248:

$$E_{ap} = 25{,}400 - 276\underline{/90} \times 29.22\underline{/90} = 17{,}350\underline{/0}$$

$$E_{an} = -276\underline{/90} \times 21.55\underline{/90} = -5{,}950\underline{/0}$$

$$E_{a0} = -276\underline{/90} \times 41.55\underline{/90} = -11{,}400\underline{/0}$$

From equation 249:

$$I_a = 3 \times 276 \underline{/90} = 828 \underline{/90}$$

$$I_b = 0 \text{ and } I_c = 0$$

From equation 250, the corresponding conductor voltages at the faulted point are:

$$E_a = 17,350 \underline{/0} - 5,920 \underline{/0} - 11,430 \underline{/0} = 0$$

$$E_b = 17,350 \underline{/120} + 5,920 \underline{/300} - 11,430 \underline{/0}$$

$$= -8,675 - j15,200 + 2,960 - j5,130 - 11,430 = -17,145 - j20,330$$

$$E_c = 17,350 \underline{/120} + 5,920 \underline{/60} - 11,430 \underline{/0}$$

$$= -8,675 + j15,200 + 2,960 + j5,130 - 11,430 = -17,145 + j20,330$$
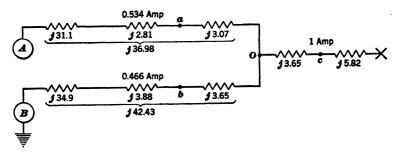
Fig. 142.   Basic Current Distribution for Positive-Sequence Network.

The results obtained for I_a, I_b, and I_c and also for E_a, E_b, and E_c are, for the conditions in Fig. 132, the sustained fault currents and line to ground voltages of the conductors at the point of fault.

The next logical step in the solution of this type of problem is to determine the current distribution throughout the entire network.   It is more convenient to obtain the relative current distribution first, as follows.

Consider the positive-sequence network of Fig. 136, which is reproduced in Fig. 142.   For 1 ampere delivered to the fault, it is obvious that 1 ampere must flow from point O, through. point c to the fault $\times$.   Assuming that the voltages of gen-

erators A and B are equal and in time phase, it then follows that the currents flowing from points a and b must be in the inverse ratio of the total impedances in the two branches.   Thus:

$$I_a = 1 \times \frac{42.43}{42.43 + 36.98} = 0.534$$

$$I_b = 1 \times \frac{36.98}{42.43 + 36.98} = 0.466$$

This assumption is not absolutely true; as a matter of fact, the generating stations of a power network seldom have equal and in time phase excitation voltages.   However, the assumption is justified in a problem of this type, since general current trends are more important than exact values.   The results obtained form the basis upon which the engineer must then use judgment in the selection of equipment.   In case the actual absolute values and phase displacements of the generator voltages are known, exact values of I_a and I_b can be obtained.

The section of the circuit included between the limits a, b, and c of Fig. 142 is the Y equivalent of a delta as seen in Fig. 133.   The currents in the delta of Fig. 133 must have values such that the voltage drops between the three points a, b, and c of Fig. 133 are equal to the corresponding voltage drops in Fig. 142.   Thus:

$$V_{ac} = 0.534 \times 3.07 + 3.65 \times 1 = 5.29 \text{ volts}$$

and the current between a and c, Fig. 133, is:

$$I_{ac} = \frac{5.29}{9.8} = 0.54 \text{ amp}$$

Similarly, from Fig. 142:

$$V_{bc} = 0.466 \times 3.65 + 3.65 \times 1 = 5.35 \text{ volts}$$

and the current between b and c, Fig. 133, is:

$$I_{bc} = \frac{5.35}{11.65} = 0.46 \text{ amp}$$

These values are indicated on Fig. 143.

Examination of Fig. 143 will indicate that the current in the line ab must flow from point b to point a and is equal to 0.006 ampere.

Following the same procedure, the current distribution for the negative-sequence network is shown in Figs. 144 and 145.

The current distribution in the zero-sequence network is shown in Fig. 146.
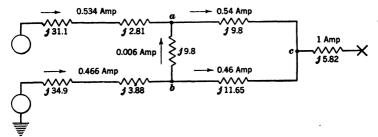
Fig. 143.   Current Distribution for Positive-Sequence Network.
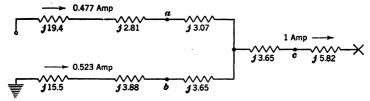
Fig. 144.   Basic Current Distribution for Negative-Sequence Network.
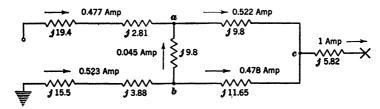
Fig. 145.   Current Distribution for Negative-Sequence Network.

The actual sequence currents can now be obtained from the relative values shown in Figs. 143, 145, and 146 and the values of positive-, negative-, and zero-sequence currents at the fault. These results are shown in Figs. 147, 148, and 149.

The currents in the three lines may now be obtained by the use of equations 249.   These results are shown in Fig. 150.

Since the actual circuit of Fig. 132 includes three voltages and transformers connected delta-Y and Y-Y, it is now

necessary to bring these particulars into the final values of currents. The required modifications are shown in Fig. 151.

The line currents between generator A and the delta transformer primary can be determined by the aid of Fig. 152. Since
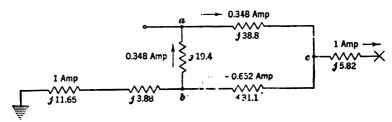
Fig. 146.   Current Distribution for Zero-Sequence Network.
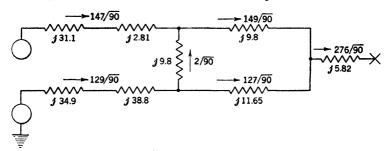
Fig. 147.   Positive-Sequence Currents.
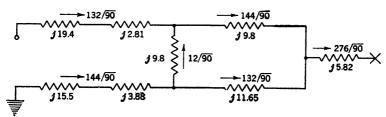
Fig. 148.   Negative-Sequence Currents.

the turns ratio of the transformer is $\dfrac{\frac{110}{\sqrt{3}}}{11} = 5.77$, it follows that the transformer primary currents are:

$$I_{at} = 646\underline{/90}$$
$$I_{bt} = 30 - j323$$
$$I_{ct} = -30 - j323$$

It will be noticed that the primary currents have been reversed in phase. This feature is not absolutely important and may not be needed. It is introduced in the problem to
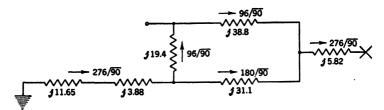
Fig. 149.   Zero-Sequence Currents.
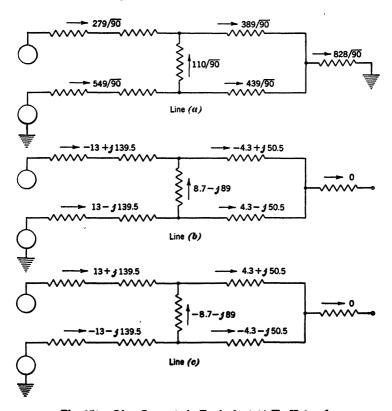
Fig. 150.   Line Currents in Equivalent 44-Kv Network.

bring out the fundamental fact that the primary and secondary mmf's of a transformer are in opposition.
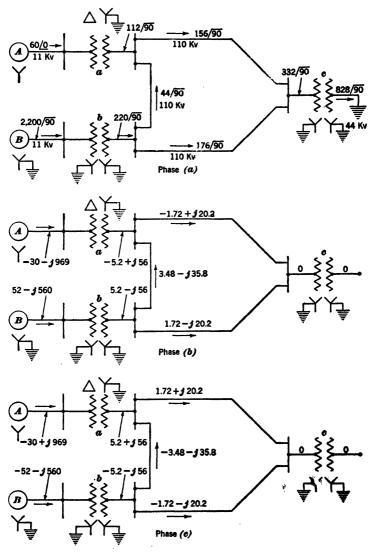
Fig. 151. Actual Line Currents.

Using the notation of Fig. 152, the three generator line currents are:

$$I_{aL} = 30 - j323 + 30 + j323 = 60\underline{/0}$$
$$I_{bL} = -30 - j323 - j646 = -30 - j969$$
$$I_{cL} = j646 - 30 + j323 = -30 + j969$$

These results are introduced in Fig. 151.

The ground or neutral currents are tabulated as follows:

Generator A . . . . $I_{0A} = 60 - 30 - j969 - 30 + j969 = 0$
Generator B . . . . $I_{0B} = -j2,200 + 52 - j560 - 52 - j560 = -j3,320$
Transformer a, secondary . $I_{0a_2} = -j112 - 5.2 + j56 + 5.2 + j56 = 0$
Transformer b, primary . . . . $I_{0b_1} = I_{0B} = -j3,320$
Transformer b, secondary

$$I_{0b_2} = -j220 + 5.2 - j56 - 5.2 - j56 = -j332$$

Transformer c, primary . . $I_{0c_1} = -j332 + 0 + 0 = -j332$
Transformer c, secondary . . $I_{0c_2} = -j828 + 0 + 0 = -j828$
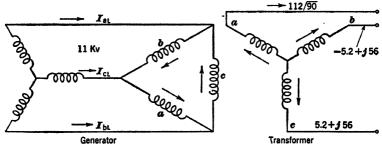
Fig. 152. Line Currents Between Generator and Transformer Primary.

The sequence-component voltage drops on the basis of 44 Kv may be obtained by determining the products of the currents and reactances, as indicated in Figs. 147, 148, and 149. These voltage drops are shown in Fig. 153.

The voltages above ground for each sequence at all points of the three networks of Fig. 153, all voltages being based on a 44-Kv system, may now be obtained by adding successive voltage drops to the sequence potentials at the point of fault. These results are shown in Fig. 154. The sequence voltages at the fault were found to be as follows:

$$E_{ap} = 17{,}350\underline{/0}, \; E_{an} = -5{,}950\underline{/0}, \text{ and } E_{a0} = -11{,}400\underline{/0}$$

The corresponding voltages above ground of each phase can now be obtained by the application of equations 250 which are:

$$E_a = E_{ap} + E_{an} + E_{a0}$$

$$E_b = E_{ap}\underline{/120} + E_{an}\underline{/120} + E_{a0}$$

$$E_c = E_{ap}\underline{/120} + E_{an}\underline{/120} + E_{a0}$$

(a) Positive Sequence

(b) Negative Sequence

(c) Zero Sequence
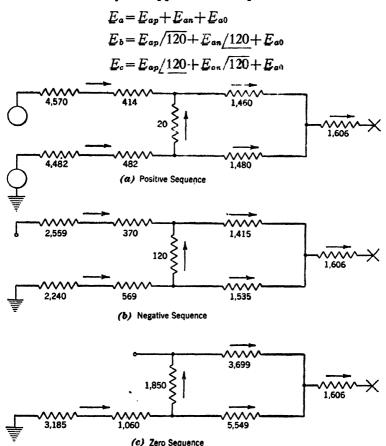
Fig. 153. Sequence-Component Voltage Drops on Basis of 44 Kv.

Applying this procedure will give the voltages indicated in Fig. 155.

The values of Fig. 155 can now be corrected to take into account the several voltage transformations and the particular connections. These results are shown in the three circuits of Fig. 156.

One additional modification of Fig. 156 is still necessary before the final results are correct. It will be noticed that generator A is connected to the 110-Kv system through a delta-Y bank of transformers. The values of voltages given for generator A and point d in Fig. 156 are on the basis of a

(a) Positive Sequence

(b) Negative Sequence

(c) Zero Sequence
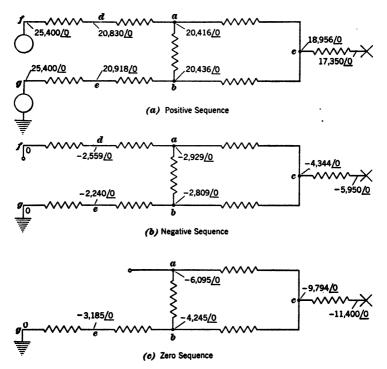
Fig. 154. Sequence Potentials Above Ground on Basis of 44 Kv.

Y-Y transformer. A ratio of $\frac{11}{44} = 0.25$ was used in obtaining the voltages at point d. Since the primary is connected in delta, the turns ratio is $\dfrac{11}{\frac{44}{\sqrt{3}}} = 0.433$ and the voltages obtained from this ratio are line voltages. A correction factor of $\dfrac{0.433}{0.25} = 1.73$ is

therefore necessary.   This modification will give the following
final values of line voltages:

$$E_{ba} = E_{c\Delta} = -(-2,284+j5,055)1.73 = 3,950 - j8,760$$

$$E_{ac} = E_{b\Delta} = -(-2,284-j5,055)1.73 = 3,950 + j8,760$$

$$E_{cb} = E_{a\Delta} = -(4,568\underline{/0})1.73 = -7,900\underline{/0}$$
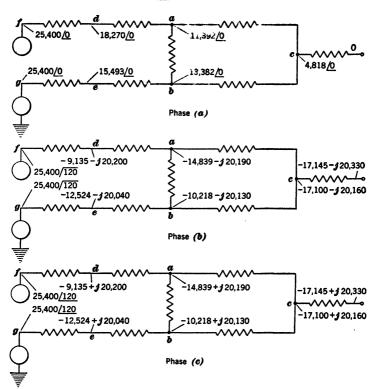
Fig. 155.   Phase Voltages above Ground.

The notation used in Fig. 157 corresponds to that used in
Fig. 152.   The generator phase voltages have magnitudes of
6,350 volts, as shown in Fig. 156, but these phase voltages must
be shifted from the phase relations indicated because of the
shifting action of the delta-Y transformation.
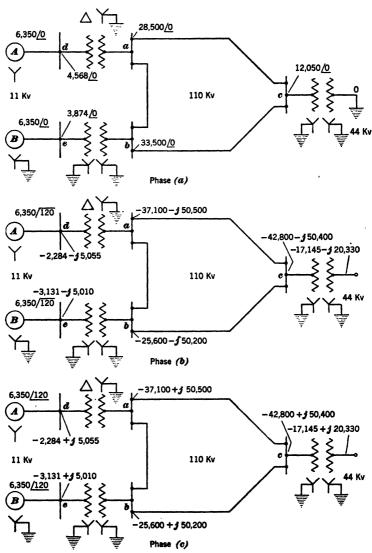
Fig. 156.   Phase Voltages under Single Line to Ground Fault.

This problem seems quite long, but it represents a rather simple example of the problem of fault calculation in power
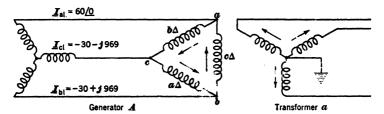
Fig. 157.   Voltages Between Lines for Generator and Transformer.

networks.   Other types of faults might be imposed on the same circuit, the results being determined according to the methods outlined in Arts. 106 to 114.

CHAPTER 17

THREE-WINDING TRANSFORMERS

121. Types of Three-Winding Transformers.—There are two common types of three-winding transformers; namely, the transformer of three entirely separate windings and the auto-

(a) Three Separate Windings   *(b)* Auto-Transformer

Fig. 158.   Three-Winding Transformers.

(a) Delta—Delta-Y

(b) Y-Delta-Y
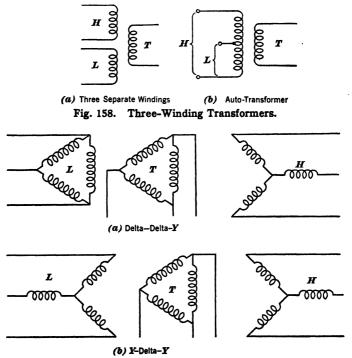
Fig. 159.   Connections Used for Three-Winding Transformers.

transformer with tertiary winding.   These two types are illustrated in Fig. 158.   The notation used in both types is the same; namely, H represents the high-tension circuit, T the tertiary winding, and L the low-tension circuit.   The common com-

mercial applications of these two types of transformers when applied to three-phase circuits are shown in Figs. 159 and 160.

In no case are the three windings connected in Y, since such a connection would not provide a suitable path for the third harmonic exciting current. The auto-transformer is connected into an equivalent Y-Y-delta, the tertiary winding being in delta. In both types, as shown in Figs. 159 and 160, the tertiary winding is quite often used for no other purpose than
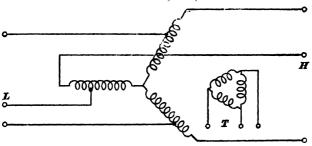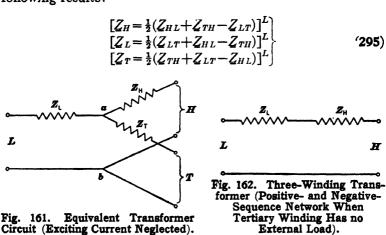
Fig. 160. Auto-Transformer Connections (Tertiary Winding in Delta).

to provide the necessary third harmonic exciting current, although it may be designed to carry an external load. In the problems worked out in this text, the tertiary will be assumed to be without external load.

122. Winding Impedances and Equivalent Circuit of Each Phase.—The series characteristics of either type of transformer in Fig. 158 are determined in the same manner. Three tests are made, in each of which the impedance is measured across one set of terminals, with a second winding short-circuited and the third winding open-circuited. These three measured values are then referred to one of the three windings, the L winding being chosen as the reference circuit in this chapter. Details covering the methods of these tests are found in any standard text on machinery. The results of these three tests will give the following data:

$$\left.\begin{array}{l} [Z_{HL} = Z_H + Z_L]^L \\ [Z_{LT} = Z_L + Z_T]^L \\ [Z_{TH} = Z_T + Z_H]^L \end{array}\right\} \qquad (294)$$

In equation 294 the subscripts refer to particular windings designated by those letters. The exponent L used outside the bracket is used to indicate that all impedances are based on the L winding. Solving the three equations 294 will lead to the following results:

$$\left.\begin{array}{l}[Z_H = \tfrac{1}{2}(Z_{HL} + Z_{TH} - Z_{LT})]^L \\ [Z_L = \tfrac{1}{2}(Z_{LT} + Z_{HL} - Z_{TH})]^L \\ [Z_T = \tfrac{1}{2}(Z_{TH} + Z_{LT} - Z_{HL})]^L\end{array}\right\} \qquad (295)$$

Fig. 161. Equivalent Transformer Circuit (Exciting Current Neglected).

Fig. 162. Three-Winding Transformer (Positive- and Negative-Sequence Network When Tertiary Winding Has no External Load).

From the results of equations 295, we have the approximate equivalent transformer circuit shown in Fig. 161. This circuit does not include the shunted impedance, which accounts for the exciting current. In most cases the exciting current is small enough to be negligible, but for special cases it can be introduced between the points a and b.

123. Positive- and Negative-Sequence Impedance.—The values of impedance as given in equations 295 apply to all balanced three-phase operation regardless of sequence; hence, they apply for positive- and negative-sequence components. In case a load is connected to the tertiary winding, the three-winding transformer resolves itself to a simple junction in the positive- and negative-sequence networks. However, in case there is no external load connected to the tertiary, then Fig. 161 resolves itself into a simple series circuit insofar as the positive- and negative-sequence networks are concerned, as indicated in Fig. 162.

124. Zero-Sequence Network.—In order to permit the evaluation of the zero-sequence impedance, it is necessary to specify the connections involved and to show the presence of

(a) Three Windings (Cana Delta–Y)

(b) Three Windings (Y–Delta–Y, L and H Grounded)

(c) Three Windings (Y–Delta–Y, H Grounded)

(d) Auto Transformer (Y–Delta–Y)
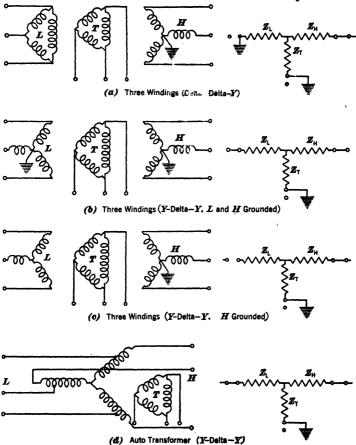
Fig. 163. Equivalent Zero-Sequence Networks.

other equipment such as generators or other transformers. Also line impedances or neutral impedances will modify the zero-sequence network. The simple equivalent zero-sequence network for three-winding transformers or auto-transformers, without taking into effect other equipment, is shown in Fig. 163.

The circuits of Fig. 163 (*a*), (*b*), and (*c*) can be incorporated into the general zero-sequence network, from which a simplified value of impedance can be obtained without any special trouble, the methods outlined in Chapters 15 and 16 being applicable. The solution of auto-transformers will, however, lead to somewhat more involved results because of the effect of the common
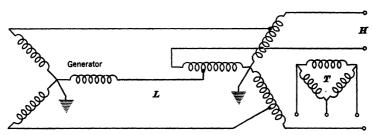
Fig. 164.   Y-Connected Generator and Auto-Transformer.

magnetic circuit.   Particularly, the auto-transformer without neutral ground is very special.   This case is considered later in this Chapter.   Several common arrangements of auto-transformers are taken up in the following articles.

125.   Zero-Sequence Equivalent Impedance of Auto-Transformers.—Consider a Y-connected generator with grounded

Fig. 165.   Equivalent Zero-Sequence Circuit for Fig. 164.

Fig. 166.   Equivalent Network of Fig. 164, with Impedance Z_{ng} in the Generator Neutral.

neutral, connected through zero impedance lines to the L winding, the auto-transformer being grounded, as in Fig. 164.   A fault is assumed on the high-tension line H.

As will be noticed from Fig. 163, the tertiary delta corresponds to a grounded impedance.   The term Z_g represents the zero-sequence generator impedance.   It will be recalled that

the generator does not act as the source of zero-sequence voltage. Zero-sequence current flows in networks due to zero-sequence voltage components which are present at the faulted point. Hence, in Fig. 164 the tertiary and low-tension circuits must be considered as being in parallel. Thus:

$$\left[Z_0 = Z_H + \frac{Z_T(Z_L + Z_0)}{Z_T + Z_L + Z_0}\right]^L \tag{296}$$

The equivalent circuit of equation 206 is represented in Fig. 165.

If an impedance Z_{ng} is introduced in the generator neutral,
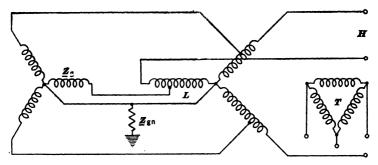
Fig. 167. Generator and Transformer Neutrals Solidly Connected and Jointly Grounded Through Impedance Z_{gn}.

then the equivalent circuit is changed to that of Fig. 166. For this circuit:

$$\left[Z_0 = Z_H + \frac{Z_T(Z_L + Z_0 + 3Z_{ng})}{Z_T + Z_L + Z_0 + 3Z_{ng}}\right]^L \tag{297}$$

The neutral impedance Z_{ng} is multiplied by three as explained in Art. 90.

Another modification of Fig. 164 is shown in Fig. 167, in which the generator neutral is connected solidly to the transformer neutral and is grounded through an impedance Z_{gn}. The equivalent network is shown in Fig. 168, and the resultant zero-sequence impedance is as follows:

$$\left[Z_0 = 3Z_{gn} + \frac{Z_T(Z_0 + Z_L)}{Z_T + Z_0 + Z_L} + Z_H\right]^L \tag{298}$$

In case impedances are introduced in lines H and L, then equation 298 will become:

$$\left[Z_0 = 3Z_{gn} + \frac{Z_T(Z_g + Z_l + Z_L)}{Z_T + Z_g + Z_l + Z_L} + Z_H + Z_h \right]^L \tag{299}$$

where Z_l = impedance in line L;
$\qquad Z_h$ = impedance in line H.
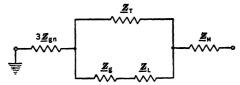
Fig. 168.   Equivalent Zero-Sequence Circuit
for Fig. 167.

126.   Impedance in Transformer Neutral.—In Fig. 169 is shown another possible condition which may at first sight not appear to offer any special treatment.   In case the neutral impedance is incorporated in the transformer circuit when the tests for Z_H, Z_T, and Z_L are made, then no additional problem is introduced, since in such a method the effect of the neutral
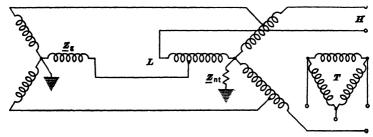
Fig. 169.   Impedance in Transformer Neutral.

impedance is absorbed as part of the transformer.   However, if such an impedance is considered independent of the originally measured transformer impedances, then the solution is somewhat more complicated.

In Fig. 170 are shown the circuits of an auto-transformer and a three-winding transformer with neutral impedance Z_{nt}.

In these diagrams the values of Z_H, Z_L, and Z_T are seen to apply to the transformer circuit alone, Z_{nt} being added.   To determine the proper relations, three impedance tests must be performed with the circuit of Fig. 170, as follows:

Test No. 1: Low-tension winding short-circuited and tertiary winding open.

Test No. 2: Tertiary winding short-circuited and low-tension winding open.
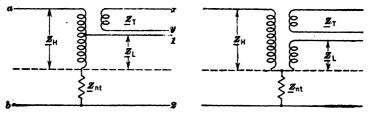
Fig. 170.   Circuits of Transformers with Neutral Impedance.

Test No. 3:   High-tension winding short-circuited and low-tension winding open.

The detail calculations are conducted in the foregoing order.   The neutral impedance is considered on the basis of the low-tension winding.   The letters H, L, and T, when used as exponents of Z, indicate the particular voltage reference of the impedance.

(a) *Test No. 1.* The conditions are shown in Fig. 171.   Let $n =$ ratio of turns of high-tension and low-tension windings. Then $I_L = nI_H$.   The current through the neutral impedance is:

Fig. 171.   Conditions for Test No. 1.

$$I_c = I_L - I_H = (n-1)I_H$$

The terminal voltage of the low-tension winding is:

$$E_{db} = I_c Z_{nt} = (n-1)I_H Z_{nt}$$

where Z_{nt} is to the scale of the low-tension winding.

The induced voltage in the low-tension winding is equal to:

$$(n-1)I_H Z_{nt}+nI_H Z_L$$

Also the induced voltage in the high-tension winding is:

$$E_{ab}=[(n-1)I_H Z_{nt}+nI_H Z_L]n$$

and the applied voltage of the high-tension winding is:

$$[(n-1)I_H Z_{nt}+nI_H Z_L]n+Z_H^H I_H$$
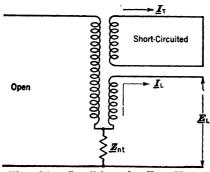
Fig. 172. Conditions for Test No. 2.

As indicated in Fig. 171,

$$E_H = E_{ab} - E_{bc}$$

$$= [(n-1)I_H Z_{nt}+nI_H Z_L]n+Z_H^H I_H - (n-1)I_H Z_{nt}$$

$$= [n(n-1)Z_{nt}+n^2 Z_L+Z_H^H - (n-1)Z_{nt}]I_H$$

Since $Z_H^H = Z_H^L n^2$,

$$Z_{HL}^H = n(n-1)Z_{nt}+n^2 Z_L+n^2 Z_H^L - (n-1)Z_{nt}$$

$$= (n-1)^2 Z_{nt}+n^2 Z_L+n^2 Z_H^L$$

Also:

$$\left[Z_{HL}=\frac{(n-1)^2}{n^2}Z_{nt}+Z_H+Z_L \right]^L \qquad (300)$$

(b) *Test No. 2.* Fig. 172 represents the conditions for this test. The tertiary winding is assumed to have the same number

of turns as the low-tension winding. Thus, $I_T = I_L$. Then:

$$E_L = I_T Z_T^L + I_L Z_L + I_L Z_{nt}$$

and
$$[Z_{LT} = Z_T^L + Z_L + Z_{nt}]^L \qquad (301)$$

(c) *Test No. 3.* For the conditions in Fig. 173:

$$E_T = \frac{I_H Z_H^H + I_H Z_{nt}}{n} + I_T Z''$$
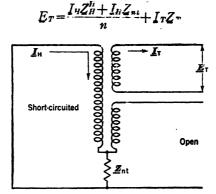
Fig. 173.   Conditions for Test No. 3.

Since $I_H = \dfrac{I_T}{n}$,

$$E_T = I_T\left(\frac{Z_H^H}{n^2} + \frac{Z_{nt}}{n^2} + Z_T\right)$$

$$= I_T\left(Z_H^L + Z_T + \frac{Z_{nt}}{n^2}\right)$$

Then:
$$\left[Z_{TH} = Z_H + Z_T + \frac{Z_{nt}}{n^2}\right]^L \qquad (302)$$

Applying equations 295:

$$[Z_{He} = \tfrac{1}{2}(Z_{HL} + Z_{TH} - Z_{LT})]^L$$

Substituting equations 300, 301, and 302, we have:

$$\left[Z_{He} = Z_H - \frac{n-1}{n^2} Z_{nt}\right]^L \qquad (303)$$

Also:
$$[Z_{Le}=\tfrac{1}{2}(Z_{HL}+Z_{LT}-Z_{TH})]^L$$

from which:
$$\left[Z_{Le}=Z_L+\frac{n-1}{n}Z_{nt}\right]^L \qquad (304)$$

Likewise:
$$[Z_{Te}=\tfrac{1}{2}(Z_{TH}+Z_{LT}-Z_{HL})]^L$$

or
$$\left[Z_{Te}=Z_T+\frac{Z_{nt}}{n}\right]^L \qquad (305)$$
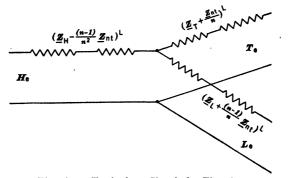
Fig. 174.   Equivalent Circuit for Fig. 170.

Therefore, the equivalent circuit of Fig. 170 is given by Fig. 174, the impedances having values given by equations 303, 304, and 305.

The equivalent circuit of Fig. 169 can now be obtained by using Fig. 165, in which Z_{He}, Z_{Te}, and Z_{Le} are substituted for the impedances Z_H, Z_T, and Z_L, as shown in Fig. 175. It will be noticed that the impedance Z_{nt} is multiplied by three. This is due to the fact that three times zero-sequence current flows through this impedance, causing a voltage drop of $3I_0Z_{nt}$. Thus, the zero-sequence impedance of Fig. 169 is:

$$\left[Z_0=Z_H-3\frac{n-1}{n^3}Z_{nt}+\frac{\left(Z_T+3\frac{Z_{nt}}{n}\right)\left(Z_L+Z_g+3\frac{n-1}{n}Z_{nt}\right)}{Z_T+3\frac{Z_{nt}}{n}+Z_L+Z_g+3\frac{n-1}{n}Z_{nt}}\right]^L \qquad (306)$$

127. Zero-Sequence Impedance of Auto-Transformer Grounded Through Impedance Z_{nt} and Generator Grounded Through Impedance Z_{ng}.—This case is the same as given in Fig. 175 with the addition of the term $3Z_{ng}$ in series with Z_g. The zero-sequence impedance is as follows:

$$\left[Z_0 = Z_H - 3\frac{n-1}{n^2}Z_{nt} + \frac{\left(Z_T + 3\frac{Z_{nt}}{n}\right)\left(Z_L + Z_g + 3Z_{ng} + 3\frac{n-1}{n}Z_{nt}\right)}{Z_T + 3\frac{Z_{nt}}{n} + Z_L + Z_g + 3Z_{ng} + 3\frac{n-1}{n}Z_{nt}} \right]^L \tag{307}$$
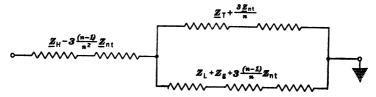
Fig. 175. Equivalent Circuit for Fig. 169.

For $Z_{nt} = 0$, equation 307 reduces to:

$$\left[Z_0 = Z_H + \frac{Z_T(Z_L + Z_g + 3Z_{ng})}{Z_T + Z_L + Z_g + 3Z_{ng}} \right]^L$$

This is the same as equation 297.

128. Zero-Sequence Impedance of Isolated Auto-Transformer Neutral with Grounded Generator Neutral.—In this case $Z_{nt} = \infty$. If Z_{nt} is put equal to infinity in equation 307, the result will be indeterminate. Equation 307 may, however, be slightly modified as follows: Let $Z_0 = Z_H + Z_A$ and $Z = Z_L + Z_g + 3Z_{ng}$. Then:

$$Z_A = -\frac{3Z_{nt}}{n} + \frac{3Z_{nt}}{n^2} + \frac{\left(Z_T + \frac{3Z_{nt}}{n}\right)\left(Z + 3Z_{nt} - \frac{3Z_{nt}}{n}\right)}{Z_T + \frac{3Z_{nt}}{n} + (Z + 3Z_{nt}) - \frac{3Z_{nt}}{n}}$$

This reduces to:

$$Z_A = \frac{3Z_T Z_{nt}(n^2 - 2n + 1) + Z_T Z n^2 + 3ZZ_{nt}}{n^2(Z_T + Z + 3Z_{nt})}$$

$$= \frac{\dfrac{(n-1)^2}{n}Z_T + \dfrac{Z_T Z}{3Z_{nt}} + \dfrac{Z}{n^2}}{\dfrac{Z_T + Z}{3Z_{nt}} + 1}$$

Putting $Z_{nt} = \infty$,

$$Z_A = \left(\frac{n-1}{n}\right)^2 Z_T + \frac{Z}{n^2}$$

and

$$\left[Z_0 = Z_H + \left(\frac{n-1}{n}\right)^2 Z_T + \frac{Z_L + Z_g + 3Z_{ng}}{n^2}\right]^L \tag{308}$$

CHAPTER 18

CIRCUITS FOR MEASUREMENT OF COMPONENT CURRENTS, VOLTAGES, AND POWER

129. Measurement of Zero-Sequence Voltages.—It has been shown that:

$$E_0 = \frac{1}{3}(E_a + E_b + E_c) \qquad (309)$$

where the voltages E_a, E_b. and E_c are the respective line to neutral voltages of conductors a, b, and c. If three potential transformers are connected as indicated in Fig. 176, with the
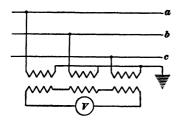
Fig. 176. Measurement of Zero-Sequence Voltages.
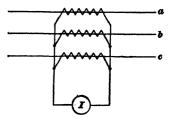
Fig. 177. Measurement of Zero-Sequence Current.

secondaries connected in series through a voltmeter, a potential reading is obtained which is proportional to the results of equation 309.

130. Measurement of Zero-Sequence Current.—The zero-sequence current of a three-phase system is:

$$I_0 = \frac{1}{3}(I_a + I_b + I_c) \qquad (310)$$

A circuit which will comply with equation 310 is shown in Fig. 177. Since most systems require potential and current transformers in order that standard 110-volt and 5-ampere meters can be used, it is obvious that the turns ratio chosen for Figs.

176 and 177 should be three times the value used for standard installations.

131. Measurement of Positive- and Negative-Sequence Voltages.—The fundamental equation defining the relationship of positive-sequence phase voltage in terms of the three phase voltages to ground is as follows:

$$E_p = \frac{1}{3}(E_a + E_b \underline{/120} + E_c \underline{/240}) \tag{311}$$
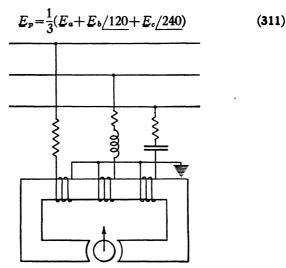
Fig. 178.   Measurement of Positive- or Negative-Sequence Voltages.

This equation may be changed to:

$$E_p = \frac{1}{3}(E_a - E_b \underline{/60} - E_c \underline{/60})$$

or

$$E_p = \frac{1}{3}\left(E_a - \frac{E_b}{\underline{/60}} - \frac{E_c}{\overline{/60}}\right)$$

Dividing this equation by an impedance Z, we have:

$$I = \frac{E_p}{Z} = \frac{1}{3}\left(\frac{E_a}{Z} - \frac{E_b}{Z\underline{/60}} - \frac{E_c}{Z\overline{/60}}\right)$$

If $Z = Z_a$, $Z\underline{/60} = Z_b$, and $Z\overline{/60} = Z_c$, then:

$$I = \frac{1}{3}(I_a - I_b - I_c) \tag{312}$$

Equation 312 is the basis of the circuit of Fig. 178, in which a common meter core has three windings, passing currents I_a, I_b, and I_c, respectively. To take care of the negative signs in equation 312, the connections to the b and c windings are reversed. It should be noticed that the impedances of the three meter circuits must be of the same modulus but of different angles. Thus if Z_a is made up of pure resistance, then:

$$Z_a = R$$

$$Z_b = R\underline{/60} = R(0.5 + j0.866)$$

$$Z_c = R\overline{/60} = R(0.5 - j0.866)$$

Similarly, the negative-sequence voltage is defined as follows:

$$E_n = \frac{1}{3}(E_a + E_b\underline{/240} + E_c\underline{/120}) \tag{313}$$

This can be changed to:

$$E_n = \frac{1}{3}(E_a - E_b\underline{/60} - E_c\overline{/60})$$

or

$$E_n = \frac{1}{3}\left(E_a - \frac{E_b}{\overline{/60}} - \frac{E_c}{\underline{/60}}\right)$$

It is therefore evident that the same meter of Fig. 178 can be changed to read the negative-sequence voltage by merely interchanging the impedances Z_b and Z_c.

In the connections just outlined, line to neutral voltages are used. It is, however, possible to read positive- and negative-sequence components of voltage with a network utilizing line voltages. Obviously, the three line voltages of a three-phase system add up to zero, even if they are unbalanced, and therefore zero-sequence voltages are not present in line voltages.

Consider the circuit of Fig. 179 (a), in which two potential transformers are used, the secondaries being connected to a voltmeter of internal impedance Z_m. Z_a and Z_b indicate impedances used in the secondary circuits. The secondary circuit is drawn in somewhat simpler form in Fig. 179 (b), from which we have the following relations:

$$I_m = I_{ab} + I_{bc}$$
$$E_{ab} = I_m Z_m + I_{ab} Z_a$$
$$E_{bc} = I_m Z_m + I_{bc} Z_b$$

Using the current equation, we may eliminate I_{bc} in the second voltage equation. Thus:

$$E_{bc} = I_m Z_m + I_m Z_b - I_{ab} Z_b$$
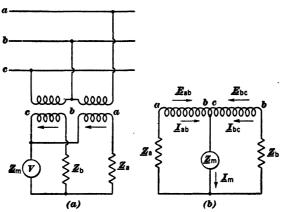
Fig. 179. Circuit with Two Potential Transformers.

The current I_{ab} may now be eliminated by considering this last voltage equation and the equation for E_{ab}, the result being:

$$I_m(Z_a Z_b + Z_b Z_m + Z_m Z_a) = E_{ab} Z_b + E_{bc} Z_a$$

Finally:

$$I_m = \frac{E_{ab} Z_b + E_{bc} Z_a}{Z_a Z_b + Z_b Z_m + Z_m Z_a} \qquad (314)$$

From fundamental relations, the positive-sequence component of voltage between lines a and b is:

$$E_{abp} = \frac{1}{3}(E_{ab} + E_{bc}\underline{/120} + E_{ca}\underline{/240})$$

Since $E_{ca} = -E_{ab} - E_{bc}$,

$$E_{abp} = \frac{1}{3}(\sqrt{3}E_{ab}\underline{/30} + \sqrt{3}E_{bc}\underline{/90}) \qquad (315)$$

But, from a simple positive-sequence vector diagram, it follows that:

$$E_{abp} = \sqrt{3} E_{ap} \underline{/30}$$

where E_{ap} = positive-sequence phase voltage of conductor a above neutral. Thus:

$$E_{ap} = \frac{1}{3}(E_{ab} + E_{bc}\underline{/60}) \tag{316}$$

If the meter impedance Z_m is small compared to impedances Z_a and Z_c, it can be neglected, and equation 314 becomes:

$$I_m = \frac{E_{ab}Z_b + E_{bc}Z_a}{Z_a Z_b}$$

from which:

$$I_m = \frac{E_{ab}}{Z_a} + \frac{E_{bc}}{Z_b} \tag{317}$$

Suppose that $Z_a = Z$ and $Z_b = Z\underline{/60}$. Then:

$$I_m = \frac{1}{Z}(E_{ab} + E_{bc}\underline{/60}) = \frac{3E_{ap}}{Z} \tag{318}$$

Comparing equations 316 and 318, it is obvious that the meter current I_m will be proportional to the positive-sequence voltage E_{ap}, provided that $Z_a = Z$ and $Z_b = Z\underline{/60}$. It is quite interesting to notice that the magnitudes of Z_a and Z_b must be the same, but there is an unlimited choice in power factors which can be employed.

Consider now the fundamental negative-sequence component of line voltage. Thus:

$$E_{abn} = \frac{1}{3}(E_{ab} + E_{bc}\underline{/120} + E_{ca}\underline{/240})$$

This equation reduces to:

$$E_{abn} = \frac{1}{3}(\sqrt{3}E_{ab}\underline{/30} + \sqrt{3}E_{bc}\underline{/90}) \tag{319}$$

But, since $E_{abn} = \sqrt{3}E_{an}\underline{/30}$,

$$E_{an} = \frac{1}{3}(E_{ab} + E_{bc}\underline{/60}) \tag{320}$$

In equation 317, suppose that $Z_a = Z$ and $Z_b = Z\underline{/60}$.   Then:

$$I_m = \frac{1}{Z}(E_{ab} + E_{bc}\underline{/60}) \tag{321}$$

Comparison of equations 320 and 321 will indicate that I_m is proportional to E_{an}, provided that $Z_a = Z$ and $Z_b = Z\underline{/60}$.   From these results it will be noticed that only one set of transformers is necessary to obtain readings of positive- and negative-sequence voltages.   The connections are shown in Fig. 180.
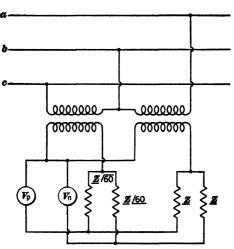
Fig. 180.   Connections for Positive- and Negative-Sequence Voltages.

132.   Measurement of Positive- and Negative-Sequence Currents.—The positive-sequence current is:

$$I_p = \frac{1}{3}(I_a + I_b\underline{/120} + I_c\underline{/240}) \tag{322}$$

where I_a, I_b, and I_c are the three line currents.   By proceeding as in Art. 131, it can be shown that the meter current must be:

$$I_p = \frac{1}{3}(I_a - I_b\underline{/60} - I_c\underline{/60}) \tag{323}$$

A circuit based on this equation is shown in Fig. 181.

Referring equation 323 to Fig. 181, the equivalent parallel impedances of the meter windings and the corresponding shunts must have values which introduce the phase rotation required. Thus the meter will read the following:

$$I_M = I_{aM} - I_{bM} - I_{cM}$$
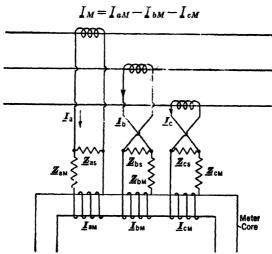
Fig. 181.   Circuit for Positive- and Negative-Sequence Currents.

If $I_{aM} = KI_a$, $I_{bM} = KI_b\underline{/60}$, and $I_{cM} = KI_c\ \underline{/60}$, K being a convenient numerical coefficient less than 1, then the meter will read:

$$I_M = KI_a - KI_b\underline{/60} - KI_c\underline{/60} \tag{324}$$

From Fig. 181:

$$I_a = I_{as} + I_{aM}$$
$$I_b = I_{bs} + I_{bM}$$
$$I_c = I_{cs} + I_{cM}$$

where the subscripts a, b, and c refer to the total current transformer currents, the subscript s indicates the shunts, and the subscript M stands for the meter windings.   Therefore:

$$I_{as} = (1-K)I_a$$
$$I_{bs} = (1-K\underline{/60})I_b$$
$$I_{cs} = (1-K\underline{/60})I_c$$

Also:

$$\frac{I_{aM}}{I_{as}} = \frac{K}{1-K} = \frac{Z_{as}}{Z_{aM}}$$

$$\frac{I_{bM}}{I_{bs}} = \frac{K\underline{/60}}{1-K\underline{/60}} = \frac{Z_{bs}}{Z_{bM}}$$

$$\frac{I_{cM}}{I_{cs}} = \frac{K\underline{\backslash 60}}{1-K\underline{\backslash 60}} = \frac{Z_{cs}}{Z_{cM}}$$
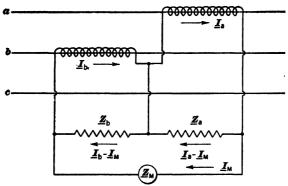
Fig. 182. Ungrounded System.

From the last three equations:

$$Z_{aM}=1-K \qquad\qquad Z_{as}=K$$
$$Z_{bM}=1-K\underline{/60} \qquad Z_{bs}=K\underline{/60}$$
$$Z_{cM}=1-K\underline{\backslash 60} \qquad Z_{cs}=K\underline{\backslash 60}$$

If $K=0.5$, then:

$$Z_{aM}=0.5+j0 \qquad\qquad Z_{as}=0.5+j0$$
$$Z_{bM}=0.75+j4.33 \qquad Z_{bs}=0.25-j4.33$$
$$Z_{cM}=0.75-j4.33 \qquad Z_{cs}=0.25+j4.33$$

For the negative-sequence ammeter connection, the impedances Z_{bM} and Z_{bs} are interchanged with the impedances Z_{cM} and Z_{cs}.

The current circuit of Fig. 181 has no limitation as to its applicability. It is therefore satisfactory, whether the three-phase system is grounded or not. For ungrounded systems (zero-sequence component current not present), simpler methods may be used. An example is shown in Fig. 182.

As indicated in Fig. 182, the current through impedance Z_a is $I_a - I_m$, and the current through impedance Z_b is $I_b - I_m$. Therefore, the voltage across the meter terminals is:

$$(I_a - I_m)Z_a + (I_b + I_m)Z_b = I_m Z_m$$

Solving for I_m, we have:

$$I_m = \frac{I_a Z_a + I_b Z_b}{Z_a + Z_b + Z_m} \tag{325}$$

But, from the fundamental relationship of sequence components, the positive-sequence component of current in line a is:

$$I_{ap} = \frac{1}{3}(I_a + I_b\underline{/120} + I_c\underline{/240})$$

Neglecting the zero-sequence component of current, $I_c = -I_a - I_b$. Therefore:

$$I_{ap} = \frac{1}{3}(\sqrt{3}I_a\underline{/30} + \sqrt{3}I_b\underline{/90})$$

or

$$I_{ap} = \frac{\underline{/30}}{\sqrt{3}}(I_a + I_b\underline{/60}) \tag{326}$$

In equation 325, let $Z_a = Z$ and $Z_b = Z\underline{/60}$. Then:

$$I_m = \frac{I_a + I_b\underline{/60}}{\sqrt{3}\underline{/30} + \dfrac{Z_m}{Z}} \tag{327}$$

If Z_m is made quite small compared to Z, so that the ratio $\dfrac{Z_m}{Z}$ is negligible, then:

$$I_m = \frac{I_a + I_b\underline{/60}}{\sqrt{3}\underline{/30}} = I_{ap}\underline{/60} \tag{328}$$

Thus it is seen that the meter reading will indicate a current equal to the positive-sequence component. There is, however, a negative angular shift of 60 degrees. This angular shift is of no particular importance unless the circuit is used in connection with some positive-sequence voltage network for the purpose of reading positive-sequence power.

Considering the negative-sequence component of current, we have:

$$I_{an} = \frac{1}{3}(I_a + I_b\underline{/120} + I_c\underline{/240})$$

Again neglecting the zero-sequence component of current, we have:

$$I_{an} = \frac{1}{3}(\sqrt{3}\,I_a\underline{/30} + \sqrt{3}\,I_b\underline{/90}) = \frac{\underline{/30}}{\sqrt{3}}(I_a + I_b\underline{/60}) \quad (329)$$
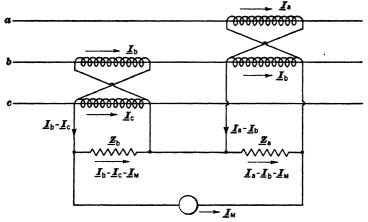
Fig. 183.   Circuit with Four Current Transformers.

In equation 325, let $Z_a = Z$ and $Z_b = Z\underline{/60}$.   Then:

$$I_m = \frac{I_a + I_b\underline{/60}}{\sqrt{3}\underline{/30} + \dfrac{Z_m}{Z}} \quad (330)$$

Again letting $\dfrac{Z_m}{Z}$ be negligible, then:

$$I_m = \frac{I_a + I_b\underline{/60}}{\sqrt{3}\underline{/30}} = I_{an}\underline{/60} \quad (331)$$

Thus, it is seen that the meter of Fig. 182 can be made to read negative-sequence currents, provided that $Z_a = Z$ and $Z_b = Z\underline{/60}$; and to read positive-sequence currents, provided that $Z_a = Z$ and $Z_b = Z\underline{/60}$.

For systems with grounded neutrals, it is necessary to use either the method of Fig. 181 or some other method such as that illustrated in Fig. 183, in which four current transformers are used.

From Fig. 183, the potential drop across the meter is:

$$(I_a - I_b - I_m)Z_a + (I_b - I_c - I_m)Z_b = I_m Z_m$$

Solving this equation for the meter current, we have:

$$I_m = \frac{(I_a - I_b)Z_a + (I_b - I_c)Z_b}{Z_a + Z_b + Z_m} = \frac{I_a Z_a - I_b(Z_a - Z_b) - I_c Z_b}{Z_a + Z_b + Z_m} \quad (332)$$

The positive-sequence component of current in line a is:

$$I_{ap} = \frac{1}{3}(I_a + I_b \underline{/120} + I_c \underline{/240})$$

or

$$I_{ap} = \frac{1}{3}(I_a - I_b \underline{/\overline{60}} - I_c \underline{/60}) \quad (333)$$

Let $Z_a = Z$ and $Z_b = Z\underline{/60}$, from which $Z_a - Z_b = Z\underline{/\overline{60}}$. Then, from equation 332:

$$I_m = \frac{I_a - I_b \underline{/\overline{60}} - I_c \underline{/60}}{\sqrt{3}\underline{/30} + \frac{Z_m}{Z}} \quad (334)$$

If, as before, the ratio $\frac{Z_m}{Z}$ is made small enough to be negligible, then the meter will read:

$$I_m = \sqrt{3}I_{ap}\underline{/30} \quad (335)$$

Also, the negative-sequence component of current in line a is:

$$I_{an} = \frac{1}{3}(I_a + I_b \underline{/\overline{120}} + I_c \underline{/\overline{240}})$$

or

$$I_{an} = \frac{1}{3}(I_a - I_b \underline{/60} - I_c \underline{/\overline{60}}) \quad (336)$$

Comparing this equation with equation 332, and putting $Z_a = Z$ and $Z_b = Z\underline{/\overline{60}}$, from which $Z_a - Z_b = Z\underline{/60}$, we find that:

$$I_m = \frac{I_a - I_b \underline{/60} - I_c \underline{/60}}{\sqrt{3}\underline{/30} + \frac{Z_m}{Z}}$$

If we neglect the ratio $\dfrac{Z_m}{Z}$, we have:

$$I_m = \sqrt{3} I_{an} \underline{/30} \qquad\qquad (337)$$

The proper impedances to be used in Fig. 183 are seen to be the same as those used in Fig. 182, but the meter calibrations and the angular relationships are different.
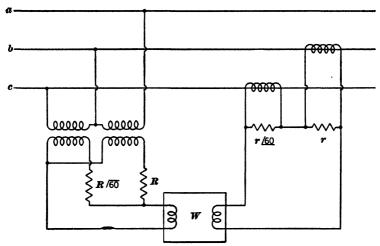
Fig. 184.   Positive-Sequence Power.

133. Measurement of Positive- and Negative-Sequence Power.—If the voltage and current of a circuit are given by the vectors E and I, then the power is given by the real part of the vector product $\bar{E}I$, and the reactive volt-amperes are given by the imaginary part of the product $\bar{E}I$, where $\bar{E}$ is the conjugate of E.   In general:

$$P - jQ = EI \cos \beta - jEI \sin \beta$$

where P = real power;
$\qquad Q$ = reactive volt-amperes;
$\qquad \beta$ = power factor angle (lagging).

Thus, to measure the positive-sequence power, we must devise a circuit which will read the real part of the product $\bar{E}_{ap}I_{ap}$.

Suppose that a wattmeter is used, the potential element being excited from the circuit of Fig. 179 while the current element is supplied from the circuit of Fig. 182. Such a wattmeter will read the real part of the product $\bar{E}_m I_m$; or, as seen from equations 318 and 328,

$$P - jQ = \frac{3\bar{E}_{ap}}{Z} \times I_{ar}/\overline{60} = 3\bar{E}_{ar} I_{ap} \frac{/\overline{60}}{Z}$$
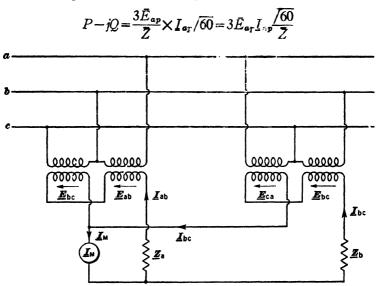
Fig. 185.   Potential Network for Positive-Sequence Power.

In case Z is made entirely resistive, that is, $Z = R + j0$, there will be an extra phase shift of 60 degrees which must be introduced in either the potential circuit or the current circuit. Since such a meter circuit reads phase power of a balanced system, it is possible to introduce any one of the three phase currents by proper placement of the sequence networks in the actual circuit. Thus suppose that the current network of Fig. 182 is connected so as to read the current I_{bp} instead of I_{ap}. Then the meter would read:

$$P - jQ = \frac{3}{R} \bar{E}_{ap} I_{bp} /\overline{60}$$

Since $I_{bn} = I_{ap}/\overline{120}$, then:

$$P - jQ = \frac{3}{R} \bar{E}_{ap} I_{ap} /\overline{180} \tag{338}$$

Finally the current element connections can be reversed so that the wattmeter will read positive. The proper connections for this purpose are shown in Fig. 184. It should be remembered, however, that this circuit cannot be used unless zero-sequence current is absent.

The scheme of Fig. 184 is by no means the only arrangement possible. For example, the order of the potential connections of Fig. 184 may be interchanged, a corresponding change being made in the type of impedance in the metering circuit.

For grounded systems, the current network of Fig. 183 must be used, since it eliminates the zero-sequence current. As before, from equations 318 and 335:

$$P-jQ = \frac{3\bar{E}_{ap}}{\bar{Z}} \times \sqrt{3}I_{ap}\underline{/30}$$

In this case, if Z is made entirely resistive, that is, $Z = R + j0$, there will be some added complications in eliminating the additional 30-degree angle. However, there is no reason why Z cannot be modified so as to cause a cancellation of the 30 degrees. The only limitation to the value of Z is that Z_a and Z_b of the circuit of Fig. 179 shall not have negative real values. Thus, if $\bar{Z} = Z\underline{/30}$, then:

$$P-jQ = \frac{3\sqrt{3}}{Z}\bar{E}_{ap}I_{ap} \tag{339}$$

and the meter will read an amount proportional to positive-sequence power. In this case the impedances in the voltage network of Fig. 179 will be $Z_a = Z\underline{/30}$ and $Z_b = Z\underline{/30}$.

In case a resistance is used for the value of $\bar{Z}$, then a circuit must be used in the potential network to counterbalance the 30-degree shift in the current network. Such a scheme is shown in Fig. 185. From this figure, it follows that:

$$I_m = I_{ab} + I_{bc}$$
$$E_{ab} - E_{bc} = I_m Z_m + I_{ab} Z_a$$
$$E_{bc} - E_{ca} = I_m Z_m + I_{bc} Z_b$$

Eliminating the current I_{bc} from the second voltage equation, we have:

$$E_{bc} - E_{ca} = I_m(Z_m + Z_b) - I_{ab}Z_b$$

The current I_{ab} can now be eliminated by simultaneous solution of this last expression for $E_{bc} - E_{ca}$ and the equation for $E_{ab} - E_{bc}$. Thus:

$$I_m(Z_a Z_b + Z_b Z_m + Z_m Z_a) = (E_{ab} - E_{bc})Z_b + (E_{bc} - E_{ca})Z_a$$

and

$$I_m = \frac{(E_{ab} - E_{bc})Z_b + (E_{bc} - E_{ca})Z_a}{Z_a Z_b + Z_b Z_m + Z_m Z_a} \tag{340}$$

Equation 340 is very much like equation 314. As a matter of fact it might have been obtained by direct analyses with equation 314 applied to Fig. 185 instead of Fig. 179.

If the meter impedance Z_m is quite small compared to the impedances Z_a and Z_b, then equation 340 will reduce to:

$$I_m = \frac{(E_{ab} - E_{bc})Z_b + (E_{bc} - E_{ca})Z_a}{Z_a Z_b} = \frac{(E_{ab} - E_{bc})}{Z_a} + \frac{(E_{bc} - E_{ca})}{Z_b}$$

Now, let $Z_a = Z$ and $Z_b = Z\underline{/60}$. Then:

$$I_m = \frac{1}{Z}(E_{ab} - E_{bc} + E_{bc}\underline{/60} - E_{ca}\underline{/60})$$

or

$$I_m = \frac{1}{Z}(E_{ab} + E_{bc}\underline{/120} + E_{ca}\underline{/240}) \tag{341}$$

The expression in the parenthesis in equation 341 is recognizable as $3E_{abp}$. Thus:

$$I_m = \frac{3}{Z}E_{abp}$$

But, since $E_{abp} = \sqrt{3}E_{ap}\underline{/30}$,

$$I_m = \frac{3\sqrt{3}}{Z}E_{ap}\underline{/30} \tag{342}$$

Using this voltage circuit and the current circuit of Fig. 183 and equation 335, we would obtain:

$$P - jQ = \left(\frac{3\sqrt{3}}{Z}E_{ap}\underline{/30}\right)\sqrt{3}I_{ap}\underline{/30}$$

Such a combination of potential and current networks would obviously introduce a 60-degree phase shift. To overcome this trouble, the current network may be shifted to indicate I_{bp} instead of I_{ap}. The power reading would then be:

$$P - jQ = \left(\frac{3}{Z} \frac{\sqrt{3}}{2} \bar{E}_{ap} \underline{/30} \right) \sqrt{3} I_{bp} \underline{/30}$$

Remembering that $I_{bp} = I_{ap} \underline{/120}$, we have:

$$P - jQ = \frac{9}{Z} \bar{E}_{ap} I_{ap} \underline{/180} \qquad (343)$$
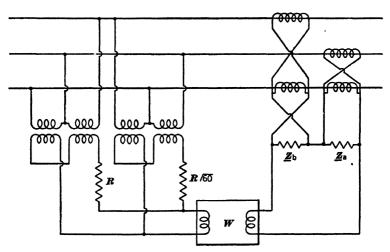
Fig. 186.   Positive-Sequence Power for Grounded Systems.

Finally, the current terminals can be reversed so that the indication of equation 343 is made positive. This connection is indicated in Fig. 186.

Corresponding negative-sequence wattmeter connections can readily be obtained by using suitable negative-sequence potential and current networks in a manner similar to that used for the measurement of positive-sequence power.

CHAPTER 19

CIRCUITS WITH UNBALANCED IMPEDANCES

134. Introduction.—The entire discussion so far given under the method of symmetrical components has involved only balanced three-phase circuits; that is, the problems have been set up on a per-phase basis, each phase having the sequence components of voltage or current but having equal impedances and admittances throughout the entire network from generators to the faulted point. True, the fault introduces special types of unbalanced conditions, but all lines and circuits from the fault back to the power sources have been taken as balanced. This limitation may at first appear unfortunate, and thus render a fine method of approach somewhat narrow in its field of application. As a matter of fact, however, the method of symmetrical components is of most importance in the solution of heavy power generating and transmitting networks, in which, because of the wide diversity of loads in distribution systems, the phases can be considered as carrying balanced resultant loads and furthermore all transmission-line conductors can be taken as transposed to eliminate unbalanced impedances and admittances. For the sake of completeness, it is wise to look into the problem of an unbalanced set of impedances and to see how the method of symmetrical components might be introduced.

135. Unbalanced Y Set of Impedances.—Let three unbalanced impedances Z_a, Z_b, and Z_c be connected in Y, the neutral being grounded. The corresponding voltages to ground and the phase currents will be denoted by E_a, E_b, E_c, I_a, I_b and I_c. Then:

$$E_a = I_a Z_a, \; E_b = I_b Z_b, \text{ and } E_c = I_c Z_c$$

Also, according to the theory of symmetrical components:

$$E_a = E_0 + E_p + E_n$$
$$E_b = E_0 + E_p\underline{/120} + E_n\underline{/120}$$
$$E_c = E_0 + E_p\underline{/120} + E_n\overline{/120}$$
$$I_a = I_0 + I_p + I_n$$
$$I_b = I_0 + I_p\overline{/120} + I_n\underline{/120}$$
$$I_c = I_0 + I_p\underline{/120} + I_n\overline{/120}$$

Therefore:

$$E_0 + E_p + E_n = Z_a(I_0 + I_p + I_n) \qquad (a)$$
$$E_0 + E_p\overline{/120} + E_n\underline{/120} = Z_b(I_0 + I_p\overline{/120} + I_n\underline{/120}) \qquad (b) \qquad \textbf{(344)}$$
$$E_0 + E_p\underline{/120} + E_n\overline{/120} = Z_c(I_0 + I_p\underline{/120} + I_n\overline{/120}) \qquad (c)$$

Adding equations 344 (a), (b), and (c) gives:

$$3E_0 = I_0(Z_a + Z_b + Z_c) + I_p(Z_a + Z_b\overline{/120} + Z_c\underline{/120}) +$$
$$I_n(Z_a + Z_b\underline{/120} + Z_c\overline{/120}) \qquad \textbf{(345)}$$

Multiplying equation 344 (b) by $\underline{/120}$ and equation 344 (c) by $\overline{/120}$, and then adding the results gives:

$$3E_p = I_0(Z_a + Z_b\underline{/120} + Z_c\overline{/120}) + I_p(Z_a + Z_b + Z_c) +$$
$$I_n(Z_a + Z_b\overline{/120} + Z_c\underline{/120}) \qquad \textbf{(346)}$$

Multiplying equation 344 (b) by $\overline{/120}$ and equation 344 (c) by $\underline{/120}$, and adding the results gives:

$$3E_n = I_0(Z_a + Z_b\overline{/120} + Z_c\underline{/120}) + I_p(Z_a + Z_b\underline{/120} + Z_c\overline{/120})$$
$$+ I_n(Z_a + Z_b + Z_c) \qquad \textbf{(347)}$$

We now define the following as the sequence-component impedances:

$$\left. \begin{aligned} Z_0 &= \frac{1}{3}(Z_a + Z_b + Z_c) \\ Z_p &= \frac{1}{3}(Z_a + Z_b\underline{/120} + Z_c\overline{/120}) \\ Z_n &= \frac{1}{3}(Z_a + Z_b\overline{/120} + Z_c\underline{/120}) \end{aligned} \right\} \qquad \textbf{(348)}$$

Substituting these results in equations 345, 346, and 347 gives:

$$E_0 = I_0 Z_0 + I_p Z_n + I_n Z_p$$
$$E_p = I_0 Z_p + I_p Z_0 + I_n Z_n \tag{349}$$
$$E_n = I_0 Z_n + I_p Z_p + I_n Z_0$$

It is apparent from equation 349 that all sequence-component currents are involved in setting up each of the three sequence-component voltages.

From the equations defining the three sequence-component impedances, we may also determine the following:

$$Z_a = Z_0 + Z_p + Z_n$$
$$Z_b = Z_0 + Z_p \underline{/120} + Z_n \underline{/120} \tag{350}$$
$$Z_c = Z_0 + Z_p \underline{/120} + Z_n \underline{/120}$$

Fig. 187.   System for Replacing Three Unbalanced Impedances.

Furthermore, in case the three Y impedances are all balanced:

$$Z_0 = Z, \; Z_p = 0, \text{ and } Z_n = 0 \tag{351}$$

where $Z = Z_a = Z_b = Z_c$.

In other words a balanced system can be considered as having only zero-sequence impedances, positive and negative sequences showing up only when the system impedances are unbalanced.   Thus a set of three unbalanced impedances can be replaced by the system of Fig. 187.

In case no zero-sequence current is flowing, that is, $I_0 = 0$, then:

$$E_0 = I_p Z_n + I_n Z_p$$
$$E_p = I_p Z_0 + I_n Z_n \tag{352}$$
$$E_n = I_p Z_p + I_n Z_0$$

The results of equations 349 for E_0, E_p, and E_n may be remembered by the aid of the simple cyclic chart shown in Fig. 188. Notice that the order of currents in the three voltage equations is I_0, I_p, and I_n. This is illustrated by the clockwise order of the three points 0, p, and n in Fig. 188. It will then be noticed that the cyclic order of the impedances is given by counter-clockwise order of the same symbols in Fig. 188, starting in each case with the proper voltage subscript. Thus, for E_0 the impedances are in the order Z_0, Z_n, and Z_p; for E_p the impedances are in the order Z_p, Z_0, and Z_n; and for E_n, in the order Z_n, Z_p, and Z_0.

136. Power Relations.—From the preceding article it is apparent that in unbalanced circuits all sequence currents produce their corresponding effects on each of the three sequence-component voltages. It is now proposed to develop
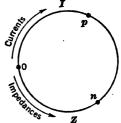
Fig. 188. Cyclic Chart for Equations 349.

the general power relationships in unbalanced circuits. From fundamental circuit laws it has been shown that:

$$P - jQ = \bar{E}I$$

where P = real power;

   Q = reactive (lagging) volt-amperes.

Therefore, for phase a:

$$P_a - jQ_a = \bar{E}_a I_a = (\bar{E}_0 + \bar{E}_p + \bar{E}_n)(I_0 + I_p + I_n)$$

or $\quad P_a - jQ_a = (\bar{E}_0 I_0 + \bar{E}_p I_p + \bar{E}_n I_n) + (\bar{E}_0 I_p + \bar{E}_p I_n + \bar{E}_n I_0)$
$$+ (\bar{E}_0 I_n + \bar{E}_p I_0 + \bar{E}_n I_p) \qquad (353)$$

Similarly for phase b:

$$P_b - jQ_b = \bar{E}_b I_b = (\bar{E}_0 + \bar{E}_p \underline{/120} + \bar{E}_n \underline{/\overline{120}})(I_0 + I_p \underline{/\overline{120}} + I_n \underline{/120})$$

or $P_b - jQ_b = (\bar{E}_0 I_0 + \bar{E}_p I_p + \bar{E}_n I_n) + (\bar{E}_0 I_p + \bar{E}_p I_n + \bar{E}_n I_0) \underline{/\overline{120}} +$
$$(\bar{E}_0 I_n + \bar{E}_p I_0 + \bar{E}_n I_p) \underline{/120} \qquad (354)$$

Also for phase c:

$$P_c - jQ_c = \bar{E}_c \bar{I}_c = (\bar{E}_0 + \bar{E}_p \underline{/120} + \bar{E}_n \underline{/120})(\bar{I}_0 + \bar{I}_p \underline{/120} + \bar{I}_n \underline{/120})$$

or $\quad P_c - jQ_c = (\bar{E}_0 \bar{I}_0 + \bar{E}_p \bar{I}_p + \bar{E}_n \bar{I}_n) + (\bar{E}_0 \bar{I}_p + \bar{E}_p \bar{I}_n + \bar{E}_n \bar{I}_0) \underline{/120} +$

$$(\bar{E}_0 \bar{I}_n + \bar{E}_p \bar{I}_0 + \bar{E}_n \bar{I}_p) \underline{/120} \tag{355}$$

Inspection of equations 353, 354, and 355 shows that all sequence components are effective in setting up both real and reactive power in each of the three phases. However, if equations 353, 354, and 355 are added, the total three-phase power is found to be:

$$P_t - jQ_t = 3(\bar{E}_0 \bar{I}_0 + \bar{E}_p \bar{I}_p + \bar{E}_n \bar{I}_n) \tag{356}$$

Equation 356 indicates that the total real and reactive powers are due to the interaction of corresponding sequence voltages and currents, voltages and currents of different sequences not entering into the total results. The second and third parenthesis terms of equations 353, 354, and 355 indicate, therefore, the amount of power transfer from phase to phase.

CHAPTER 20

ALTERNATOR TRANSIENTS

137. Simple A-C Transient.—The actual alternator transient may become quite complicated under certain conditions of operation. The particular problem of present interest concerns the transient behavior of the full three-phase short-circuit of an alternator, the initial state being that of no load. The analysis which is given in this chapter is not rigorous and exact. Several simplifying assumptions are made with the view of presenting the qualitative picture, and finally certain equations as produced by Doherty and Nickle are given and discussed.

Fig. 189. Simple Series Circuit.

In Fig. 189 is shown a simple series circuit of resistance and inductance, which is supplied from a source of constant sinusoidal voltage. Obviously there can be no current flow through
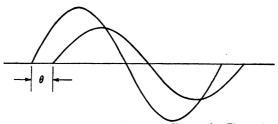
Fig. 190. Voltage and Current Waves for Fig. 189.

the circuit when the switch is open. This condition of the circuit may be defined as *steady state number one*. In this steady state there is, obviously, no energy stored in the inductance of the circuit.

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Now, let the switch be closed. After the transient has vanished, the current will take up a definite position with respect to the voltage, as indicated in Fig. 190, the angle between the voltage and current being determined by the circuit power factor. This condition may be defined as *steady state number two*. If the switch were closed at an instant of the cycle corresponding to normal steady state zero current, then there would be no transient, since at these instants there is no energy stored in the magnetic circuit. However, suppose that the switch is closed at an instant corresponding to normal maximum current. With maximum current, the circuit would have a maximum of stored energy. Obviously the stored energy cannot be suddenly
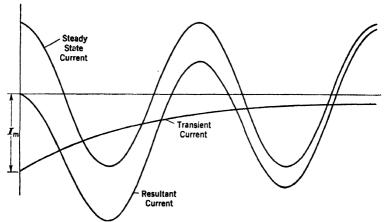
Fig. 191. Transient in R and L Circuit.

changed from zero to maximum value, without an adjustment taking place. This is shown in Fig. 191.

The initial value of the transient current is equal to the current of steady state number one (in this case equal to zero) minus the instantaneous current of steady state number two. In the case of Fig. 191 the current before switch closure was equal to zero; and, after switch closure, it should be equal to the maximum value. Therefore the initial value of the transient current must be equal to the negative maximum value. The transient current cannot sustain itself since there does not

exist any corresponding voltage.   Hence, its value will gradually decrease to zero, in accordance with the following equation:

$$i_t = I_m \epsilon^{-\frac{R}{L}t} \tag{357}$$

where R = resistance of the circuit, L = inductance, and t = time in seconds measured from the instant of switch closure.   The resultant current of the circuit is the sum of the transient current and the current of steady state number two, as shown in Fig. 191.
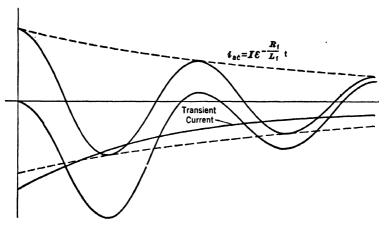
Fig. 192.   Envelope of Current.

In this discussion, the applied voltage was assumed to remain constant; that is, successive amplitudes of the wave have the same magnitude.   Thus the current wave of Fig. 191 is found to be symmetrical with respect to the decremental transient current axis.   If it is assumed, however, that the envelope of the voltage decreases in magnitude according to the exponential law, then the current envelope will also decrease, as shown in Fig. 192.

138. Derivation of A-C Transient.—Consider the circuit of Fig. 189. The instantaneous voltage drop across the circuit is:

$$e = Ri + L\frac{di}{dt} \tag{358}$$

Rearranging this expression gives:

$$\frac{di}{dt}+ai=\frac{e}{L}$$

where $a=\frac{R}{L}$.

Multiplying this expression by ϵ^{at} gives:

$$\frac{di}{dt}\epsilon^{at}+ai\epsilon^{at}=\frac{e}{L}\epsilon^{at}$$

Now let $u=i\epsilon^{at}$. Then:

$$\frac{du}{dt}=\frac{di}{dt}\epsilon^{at}+ai\epsilon^{at}$$

and

$$\frac{du}{dt}=\frac{e}{L}\epsilon^{at} \text{ or } du=\frac{e}{L}\epsilon^{at}dt$$

Integrating this expression leads to the following:

$$u=\int\frac{e}{L}\epsilon^{at}dt+K$$

where K is a constant of integration, the value of which must be determined from the terminal conditions of the problem.

Since $u=i\epsilon^{at}$,

$$i=\epsilon^{-at}\int\frac{e}{L}\epsilon^{at}dt+K\epsilon^{-at} \tag{359}$$

Equation 359 is perfectly general in that the form of the voltage e is not as yet specified. For the present we will assume that this voltage is a simple harmonic with respect to time. Thus:

$$e=E_m \sin (\omega t+a) \tag{360}$$

It is convenient, from a mathematical point of view, to express this voltage in exponential form, as follows:

$$e=E_m\frac{\epsilon^{j(\omega t+a)}-\epsilon^{-j(\omega t+a)}}{2j}$$

Substituting this expression for e in equation 359 gives:

$$i=\frac{E_m}{2jL}\epsilon^{-at}\int(\epsilon^{ja}\epsilon^{(a+j\omega)t}-\epsilon^{-ja}\epsilon^{(a-j\omega)t})dt+K\epsilon^{-at}$$

Integrating this expression gives:

$$i = \frac{E_m}{2jL}\epsilon^{-at}\left(\frac{\epsilon^{ja}\epsilon^{(a+j\omega)t}}{a+j\omega} - \frac{\epsilon^{-ja}\epsilon^{(a-j\omega)t}}{a-j\omega}\right) + K\epsilon^{-at}$$

This may be rearranged as follows:

$$i = \frac{E_m}{2j}\epsilon^{-at}\left(\frac{\epsilon^{at}\epsilon^{j(\omega t+a)}}{aL+j\omega L} - \frac{\epsilon^{at}\epsilon^{-j(\omega t+a)}}{aL-j\omega L}\right) + K\epsilon^{-at}$$

Since $a = \frac{R}{L}$, it is obvious that $aL = R$. Then:

$$aL+j\omega L = Z\epsilon^{j\phi} \text{ and } aL-j\omega L = Z\epsilon^{-j\phi}$$

where $Z = \sqrt{R^2+\omega^2 L^2} =$ impedance in ohms; and $\phi =$ power factor angle.

Therefore:

$$i = \frac{E_m}{2j}\left(\frac{\epsilon^{j(\omega t+a)}}{Z\epsilon^{j\phi}} - \frac{\epsilon^{-j(\omega t+a)}}{Z\epsilon^{-j\phi}}\right) + K\epsilon^{-at}$$

This expression may be changed to:

$$i = \frac{E_m}{Z}\left(\frac{\epsilon^{j(\omega t+a-\phi)} - \epsilon^{-j(\omega t+a-\phi)}}{2j}\right) + K\epsilon^{-at}$$

Finally:

$$i = \frac{E_m}{Z}\sin\,(\omega t+a-\phi) + K\epsilon^{-at} \tag{361}$$

The first term of equation 361 is recognizable as the current for steady state number two, while the second term is obviously the transient term.

To determine the value of K, it is now necessary to apply the proper terminal conditions to equation 361. The current will be assumed to be zero at $t=0$. Obviously at $t=0$, it follows from equation 360 that $e = E_m \sin\,a$. Substituting these conditions in equation 361 gives:

$$0 = \frac{E_m}{Z}\sin\,(a-\phi) + K$$

from which:

$$K = -\frac{E_m}{Z}\sin\,(a-\phi)$$

Therefore, for the particular conditions assumed, the general equation 361 becomes:

$$i = \frac{E_m}{Z} \sin (\omega t + a - \phi) - \frac{E_m}{Z} \epsilon^{-at} \sin (a - \phi) \qquad (362)$$

The first term of equation 362 is seen to be the steady-state term, while the second term is the transient term. This equation will conform to the qualitative discussion associated with Fig. 191.

139. A-C Transient with Decremental Voltage.—The actual mechanism of operation going on inside of an alternator during transients is quite complicated, as will be shown by the brief discussions in succeeding articles. The air-gap flux does not
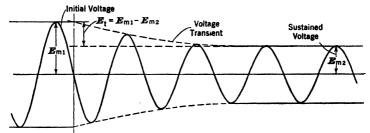
Fig. 193. Generator Induced Voltage.

remain constant as a balanced three-phase fault progresses, but gradually decreases from an initial value to some lower value, according to some exponential law. Thus the induced voltage per phase must drop off according to the same law. To illustrate the general nature of the problem, a hypothetical generator of decremental voltage will be assumed connected to the circuit of Fig. 189. This is illustrated in Fig. 193. From this figure it is obvious that the equation of the alternator induced voltage can be written as follows:

$$e = (E_t \epsilon^{-\beta t} + E_{m2}) \sin (\omega t + a) \qquad (363)$$

where $E_t = E_{m1} - E_{m2}$;

E_{m1} = maximum value of initial voltage;

E_{m2} = maximum value of final voltage;

β = armature voltage decrement.

By the theorem of superposition, we can consider the voltage e as due to the effect of the following two voltages:

$$E_t\epsilon^{-\beta t} \sin (\omega t+a) \text{ and } E_{m2} \sin (\omega t+a)$$

The current due to the second voltage can be written directly from equation 362 and is:

$$i_2 = \frac{E_{m2}}{Z} \sin (\omega t+a-\phi) - \frac{E_{m2}}{Z}\epsilon^{-at} \sin (a-\phi) \qquad (364)$$

To determine the current due to the first component of voltage, it is necessary to go back to equation 359. As before, it is convenient to express this voltage in exponential form. Thus:

$$e_1 = \frac{E_t}{2j}\epsilon^{-\beta t}(\epsilon^{j(\omega t+a)} - \epsilon^{-j(\omega t+a)})$$

or

$$e_1 = \frac{E_t}{2j}[\epsilon^{ja}\epsilon^{(-\beta+j\omega)t} - \epsilon^{-ja}\epsilon^{(-\beta-j\omega)t}] \qquad (365)$$

Substituting this in equation 359 will give:

$$i_1 = \frac{E_t}{2jL}\epsilon^{-at} \int (\epsilon^{ja}\epsilon^{(a-\beta+j\omega)t} - \epsilon^{-ja}\epsilon^{(a-\beta-j\omega)t})dt + K\epsilon^{-at}$$

Integrating this expression gives:

$$i_1 = \frac{E_t}{2jL}\epsilon^{-at}\left(\frac{\epsilon^{ja}\epsilon^{(a-\beta+j\omega)t}}{a-\beta+j\omega} - \frac{\epsilon^{-ja}\epsilon^{(a-\beta-j\omega)t}}{a-\beta-j\omega}\right) + K\epsilon^{-at}$$

Let $(a-\beta+j\omega)L = S\epsilon^{jr}$;

and $(a-\beta-j\omega)L = S\epsilon^{-jr}$.

Then:

$$i_1 = \frac{E_t}{2j}\epsilon^{-\beta t}\left(\frac{\epsilon^{j(\omega t+a)}}{S\epsilon^{jr}} - \frac{\epsilon^{-j(\omega t+a)}}{S\epsilon^{-jr}}\right) + K\epsilon^{-at}$$

Rearranging:

$$i_1 = \frac{E_t}{S}\epsilon^{-\beta t}\left(\frac{\epsilon^{j(\omega t+a-r)} - \epsilon^{-j(\omega t+a-r)}}{2j}\right) + K\epsilon^{-at}$$

Finally:

$$i_1 = \frac{E_t}{S}\epsilon^{-\beta t} \sin (\omega t+a-r) + K\epsilon^{-at} \qquad (366)$$

Again, K may be determined by applying the proper terminal conditions to equation 366. Thus, for $t=0$, i_1 will be assumed equal to zero; and, from equation 365, $e=E_t \sin a$. Substituting these conditions in equation 366 gives:

$$0 = \frac{E_t}{S} \sin (a-\tau) + K$$

from which:

$$K = -\frac{E_t}{S} \sin (a-\tau)$$

Substituting this result in equation 366 gives, for the particular condition of the problem:

$$i_1 = \frac{E_t}{S} \epsilon^{-\beta t} \sin (\omega t + a - \tau) - \frac{E_t}{S} \epsilon^{-at} \sin (a-\tau) \qquad (367)$$

Finally, the actual current transient, which is due to the sum of equations 367 and 364, is:

$$i = \frac{E_{m2}}{Z} \sin (\omega t + a - \phi) + \frac{E_{m1} - E_{m2}}{S} \epsilon^{-\beta t} \sin (\omega t + a - \tau)$$

$$- \left(\frac{E_{m2}}{Z} \sin (a - \phi) + \frac{E_{m1} - E_{m2}}{S} \sin (a - \tau) \right) \epsilon^{-at}$$

Assuming that the inductive reactance is several times as great as the resistance, it is then sufficiently accurate to consider that the magnitude of the current is determined by the reactances, and that the power factor angles ϕ and τ are equal to 90 degrees. Thus:

$$i = \frac{E_{m2}}{X} \cos(\omega t + a) + \frac{E_{m1} - E_{m2}}{X} \epsilon^{-\beta t} \cos(\omega t + a) - \frac{E_{m1}}{X} \epsilon^{-at} \cos a \qquad (368)$$

The first term of equation 368 is the steady-state current, the second term is due to the generator voltage decrement term, while the last term is the transient or direct current decrement term.

140. Three-Phase Alternator Short-Circuit Transients.

In obtaining equation 368, the circuit of Fig. 189 was used. It was assumed to possess constant reactance as well as constant

resistance. An alternator is not a simple static circuit of constant parameters, but is a rather complex coupled circuit; in some cases, the degree of coupling changes continuously as in a salient pole machine. Thus we encounter a difference in reactances between the direct axis and the quadrature axis. Furthermore, the reactances at the instant of short-circuit are quite different from those under steady-state conditions. Thus it is to be expected that the equations for a three-phase short-circuit on an alternator should be more complicated. The equations that follow contain, however, the same types of terms as are given in equation 368 and also an additional term of double frequency. These equations are found in a paper by R. E. Doherty and C. A. Nickle*, and the following discussion is taken from that paper.

"These equations cover the case of dead three-phase short-circuit at the terminals of the machine. Only the internal resistance of the machine windings is involved and this is entirely negligible, in most cases, in determining the magnitude of the short-circuit currents.

"The general method used is to set up the expressions for the magnetomotive forces, A_d and A_q, in the direct and quadrature axes of the machine for any time t after the short-circuit occurs. This is done by summing up the mmf's in each axis due to the phase currents, i_a, i_b, and i_c, each total phase mmf being multiplied by a function of time to obtain the component acting at any particular instant. The flux linkages in each axis are then determined from the expressions for A_d and A_q. In the next step the total flux linkages in each phase are found. These expressions for flux linkages at any instant after short-circuit are then equated to the known linkages which existed at the instant of short-circuit. This step is made according to the principle of constant linkages since the resistance is assumed to be negligible. The resulting three equations are then solved for the phase currents i_a, i_b, and i_c. This results in expressions for the currents, each made up of a *d-c component, a positive-phase sequence fundamental, and a positive-phase sequence second*

*R. E. Doherty and C. A. Nickle, *Three-Phase Short-Circuit Synchronous Machines—V*, 1930.

harmonic. Each of these components actually exists only at the initial instant of short-circuit and the next step is, therefore, to determine the decrement factors which govern their decay. *The positive-phase sequence fundamental does not decay to zero but to the sustained value of current.* This sustained current is found and subtracted from the expression for the fundamental component of the initial current so that the decrement factor may be applied to the remainder which decays to zero. To make the expressions for total currents hold true, the sustained current is then added so that the final expressions have four component parts, each in terms of e_0, the terminal voltage existing before short-circuit, X_d and X_q, the synchronous reactances in the direct and quadrature axes, and X'_d, the transient reactance of the direct axis. These expressions are:

$$i_a = \frac{e_0}{X_d} \cos(\omega t - a) + e_0 \frac{X_d - X'_d}{X_d X'_d} \epsilon^{-\beta t} \cos(\omega t - a)$$

$$- e_0 \frac{X'_d + X_q}{2X'_d X_q} \epsilon^{-at} \cos a + e_0 \frac{X'_d - X_q}{2X'_d X_q} \epsilon^{-at} \cos(2\omega t - a) \quad \textbf{(369)}$$

$$i_b = \frac{e_0}{X_d} \cos\left(\omega t - a - \frac{2\pi}{3}\right) + e_0 \frac{X_d - X'_d}{X_d X'_d} \epsilon^{-\beta t} \cos\left(\omega t - a - \frac{2\pi}{3}\right)$$

$$- e_0 \frac{X'_d + X_q}{2X'_d X_q} \epsilon^{-at} \cos\left(a + \frac{2\pi}{3}\right)$$

$$+ e_0 \frac{X'_d - X_q}{2X'_d X_q} \epsilon^{-at} \cos\left(2\omega t - a - \frac{2\pi}{3}\right) \quad \textbf{(370)}$$

$$i_c = \frac{e_0}{X_d} \cos\left(\omega t - a - \frac{4\pi}{3}\right) + e_0 \frac{X_d - X'_d}{X_d X'_d} \epsilon^{-\beta t} \cos\left(\omega t - a - \frac{4\pi}{3}\right)$$

$$- e_0 \frac{X'_d + X_q}{2X'_d X_q} \epsilon^{-at} \cos\left(a + \frac{4\pi}{3}\right)$$

$$+ e_0 \frac{X'_d - X_q}{2X'_d X_q} \epsilon^{-at} \cos\left(2\omega t - a - \frac{4\pi}{3}\right) \quad \textbf{(371)}$$

"An expression for the field current is obtained by finding the currents induced in the field winding by the mmf in the

direct axis.   This is found in terms of the currents i_a, i_b, and i_c, whence the expressions given above are substituted to obtain the current in terms of e_0, X_d, and X'_d.

"The final expression is:

$$I = e_0 \frac{X_d}{X'_d} - e_0 \frac{X_d - X'_d}{X'_d} \cos \omega t"$$

Fig. 194.   Air-Gap Flux.

In the preceding expressions, the letters have the following meanings:

$$\beta = \frac{X_d}{X'_d} \times \frac{R}{L_{of}};$$

$$a = \frac{r_a(X'_d + X_q)}{2X'_d X_q};$$

R = resistance of the field circuit;
L_{of} = inductance of the field circuit with the armature open-circuited;
r_a = resistance of the armature;
t = time;
a = displacement between the armature winding axis and the axis of the field pole at the instant of short-circuit.

141.   General Nature of Alternator Transient.—The equations given in the preceding article have been checked with actual oscillographic records and found to give very accurate results.   However, it will be worth-while to examine qualitatively the internal workings of a simple alternator during a *balanced three-phase short-circuit*.

If a three-phase generator is subjected to a three-phase short-circuit, the initial currents will be quite large, and will decrease gradually to the sustained value. Assuming that the setting of the field rheostat is not changed, this decrease in current is due primarily to the demagnetizing action of the armature. Thus if Fig. 192 represents the current of one phase of a three-
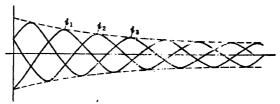
Fig. 195. Alternating Armature Currents (Transient Currents Neglected).

phase machine, then the decrease in the current envelop i_{ac} is due to the decrease in generated voltage which in turn is due to the decrease in air-gap flux. The decrease in air-gap flux will follow some curve similar to Fig. 194.

The change in flux can be represented by the following equation:

$$\phi = \phi_s + (\phi_0 - \phi_s)\epsilon^{-\frac{R_f}{L_f}t} \tag{372}$$

where R_f and L_f are the resistance and the inductance of the field circuit.

Fig. 196. A-C Armature MMF.

Fig. 194 neglects the effect of the transient currents entirely, and it is assumed that the three alternating armature currents of Fig. 195 were instantaneously established, thus producing the armature mmf curve of Fig. 196.

With a sudden rise of armature demagnetizing mmf, as indicated by the ordinate F_0 of Fig. 196, there would be induced in the field circuit a voltage which would cause a corresponding

sudden rise in field current to counterbalance the demagnetizing action; the resultant field current would then gradually decrease to its original value in the same manner as the flux curve of Fig. 194. The change in field current is illustrated in Fig. 197.

142. Armature MMF Transient.—The mmf produced by steady state balanced three-phase armature currents is known to be of constant magnitude and to revolve at constant synchronous speed. This statement can be verified by consulting any standard text on machinery.

$$i = I_1 + I_0 \epsilon^{-\frac{R_f}{L_f}t}$$
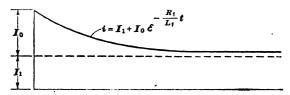
Fig. 197. Field Current Transient (Armature Transient Currents Neglected).

Consider now the three current waves of Fig. 195. These current waves differ only from steady state waves in that all three decrease in magnitude according to the following law:

$$i_{ac} = I\epsilon^{-\frac{R_a}{L_a}t} \tag{373}$$

Thus these three current waves, flowing through three windings which are 120 degrees apart in space location, produce a resultant mmf which will rotate at synchronous speed and will decrease in magnitude as indicated by Fig. 196. Both of these actions may be illustrated by the polar diagram of Fig. 198.

The change of revolving mmf may last for several cycles, the number depending on the constants of the machine; in Fig. 198, two and one-quarter cycles are indicated. However, the three currents of Fig. 195 cannot be developed in the armature without the corresponding transient currents similar to the action indicated in Figs. 191 and 192. The three alternating currents and the three transient currents are shown in Fig. 199 (a) and (b).

The three transient currents will also flow in the armature. These three currents, not being alternating, cannot produce a revolving mmf, but they will produce a transient mmf that is
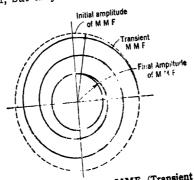
Fig. 198. Resultant MMF (Transient Currents Neglected).

(a) Three Alternating Currents

(b) Three Transient Currents

Fig. 199. Component Armature Currents.
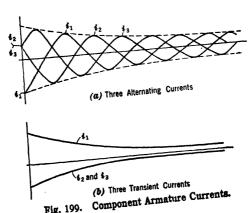
stationary in space and of decreasing amplitude. Its amplitude varies according to the following law:

$$F_t = F_0 \epsilon^{-\frac{R_a}{L_a}t} \tag{374}$$

The initial value of this transient mmf will be equal and opposite to the initial revolving mmf. This is true because the initial transient currents are equal and opposite to the corresponding

initial values of the alternating currents, as shown in Fig. 199.

The resultant armature mmf may be developed by the aid of the polar diagram of Fig. 198, to which is added the correspond-
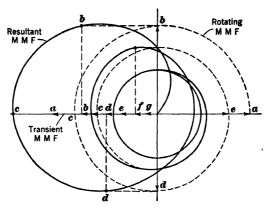
Fig. 200.   Resultant Armature MMF.

ing values of transient mmf.   The transient mmf is assumed to vanish in two cycles.   This construction is shown in Fig. 200.

The resultant values of armature mmf are obtained by adding vectorially the corresponding rotating and transient mmf's.   The resultant envelope of Fig. 200 may now be trans-
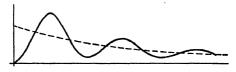
Fig. 201.   Resultant Armature MMF.

ferred to rectangular coordinates, as shown in Fig. 201.   It is seen that the resultant armature mmf has a fundamental frequency component.   This fundamental frequency component of armature mmf will introduce a corresponding voltage in the field circuit, and will produce an alternating component in the field current.   Thus, the field current of Fig. 197 is modified to the form shown in Fig. 202.

143.   Single-Phase Armature MMF.—The armature current of a single-phase generator or of a three-phase generator

under single-phase operation may be represented by the wave shown in Fig. 203, provided that all harmonics are omitted. Since only one phase is carrying current, the mmf produced in this case can be considered as made up of a unidirectional component as produced by the transient current, with a superimposed pulsating component due to the sine-wave component of current.
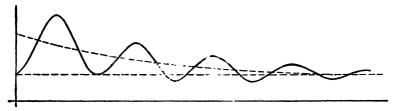
Fig. 202. Resultant Field Current for Three-Phase Short-Circuit.

The transient component of current will induce a current of fundamental frequency in the rotor circuit, while the sine-wave component of armature current will induce a double frequency component in the field circuit. These two com-

Fig. 203. Single-Phase Current.

ponents will result in a field-current wave like that indicated in Fig. 204. The air-gap flux will obviously pulsate at the composite frequencies shown in Fig. 204, and will induce third harmonic voltages in the armature, thus causing a distortion in the current wave of Fig. 203.

144. Leakage, Transient, and Subtransient Reactance. Consider the case of a three-phase short-circuit as discussed in Arts. 141 and 142. Neglecting the presence of the field circuit and the effect of the damper winding, the initial effective alternating current will be determined by the leakage reactance of the armature. Thus:

$$I_0 = \frac{E}{X_{ld}} \qquad (375)$$

where $E =$ phase induced voltage; and $X_{ld} =$ leakage reactance per phase.

Equation 375 would hold, under the particular assumptions made, provided the current transient were perfectly symmetrical, that is, if it did not contain a d-c decremental component. For a completely offset transient the effective initial value will depend on the magnitude of this d-c decrement.

Since the unidirectional current has an initial value equal to the maximum a-c amplitude (maximum fault current), then the resultant effective initial current is:

$$I_i = \sqrt{(I_0)^2 + (\sqrt{2}I_0)^2} = \sqrt{3}I_0 \qquad (376)$$
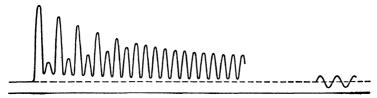
Fig. 204.   Field Current under Single-Phase Fault.

However, as seen in Chapter 11, the reactance of an alternator at the initial instant of short-circuit is not the leakage reactance, but is the transient or subtransient reactance, the last one being used in case the machine is equipped with damper windings. Furthermore, the armature mmf is practically all demagnetizing under short-sircuit conditions of modern generators. Hence, direct components of transient and subtransient reactances should be used. Thus, for a machine without damper winding, equation 375 should give way to the following more accurate expression:

$$I_0 = \frac{E}{X'_d} \qquad (377)$$

Finally for a machine having damper windings:

$$I_0 = \frac{E}{X''_d} \qquad (378)$$

The value of the voltage E used in equations 375, 377, and 378 corresponds to open-circuit conditions.

145. Sustained Short-Circuit Current.—After the transient effect of a three-phase fault disappears, there remains simply the sustained fault current, which is limited by the synchronous reactance X_d. This reactance is made up of the leakage reactance and the effect of the demagnetizing action of armature reaction (see Chapter 11). The sustained current for a balanced three-phase fault is determined as follows:

$$I_s = \frac{E}{X_d} \tag{379}$$
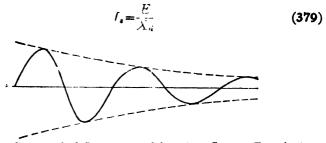
Fig. 205. Symmetrical Component of Armature Current Transient.

Unbalanced sustained currents are defined in terms of sequence impedances, as discussed in Chapter 11.

146. Determination of Transient and Subtransient Reactance.—Transient and subtransient reactances may be determined by design constants or from test. The test procedure involves taking oscillograms of short-circuit currents. For conditions of single-phase faults or other unsymmetrical conditions, transient and subtransient reactances must be further subdivided into direct and quadrature components. The same subdivision is, of course, also applicable to synchronous reactance. The terms direct and quadrature refer to Blondel's theory of mmf action along the axis of field poles (direct), and mmf action along the axis midway between poles (quadrature).*

* For a complete analysis of these components, see "The Reactances of Synchronous Machines," by Park and Robertson, *A I E E Trans.*, Vol. 47, p. 514; "The Calculation of the Armature Reactance of Synchronous Machines," by Alger, *A I E E Trans.*, Vol. 47, p. 493; "Determination of Synchronous Machine Constants by Test," by Wright, *A I E E Trans.*, Vol. 50, p. 1331.

The discussion of this article will be confined to the initial currents of balanced three-phase faults, which involve only the direct transient and subtransient reactances. Furthermore it will be assumed that the short-circuit takes place from the initially open-circuited condition. The solution for transient and subtransient reactances from the oscillograms of a short-circuit from loaded conditions is slightly more involved.

In Fig. 205 is shown a typical symmetrical transient of armature current. As has been indicated previously, the decay of current in a circuit of resistance and inductance follows an exponential law. If only one resistance and one inductance are involved in the decay of current in Fig. 205, the envelope of this wave, when transferred to semi-logarithmic paper, will plot as a straight line. However, during the first few cycles, the current changes much faster than indicated by a single decrement. This is illustrated by the semi-logarithmic plot in Fig. 206. If the straight line is extended back to the left, it will give the maximum value of the initial alternating current which would flow, neglecting the effect of damper windings. Let this maximum current be designated as I_{tm}. Then, if E is the effective open-circuited phase voltage, the transient reactance becomes:

$$X'_d = \frac{\sqrt{2}E}{I_{tm}} \tag{380}$$

Also, if I_{stm} equals the maximum initial value of current, then the subtransient reactance per phase is:

$$X''_d = \frac{\sqrt{2}E}{I_{stm}} \tag{381}$$

The equation of the envelope of Fig. 205 is:

$$i = \sqrt{2}I_s + (I_{tm} - I_{sm})\epsilon^{-\frac{t}{T_t}} + (I_{stm} - I_{tm})\epsilon^{-\frac{t}{T_{st}}} \tag{382}$$

where I_{sm} = maximum value of sustained current;
$\quad I_s$ = effective value of sustained current;
$\quad T_t$ = transient time constant;
$\quad T_{st}$ = subtransient time constant

Equation 382 can be changed to:

$$i = \sqrt{2}I_s + \Delta I_t \epsilon^{-\frac{t}{T_t}} + \Delta I_{st} \epsilon^{-\frac{t}{T_{st}}} \tag{383}$$

where $\Delta I_t = (I_{tm} - I_{sm})$;
$\Delta I_{st} = (I_{stm} - I_{tm})$.
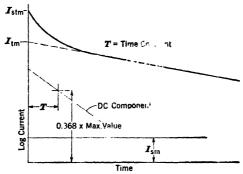
Fig. 206.   Semi-Logarithmic Plot of Envelope of Fig. 205.

At $t=0$, the transient and subtransient increments of currents are ΔI_t and ΔI_{st}, respectively.   At $t=T_t$ and $t=T_{st}$, the relations are:

$$\Delta I_t' = \Delta I_t \epsilon^{-1} \text{ or } \frac{\Delta I_t'}{\Delta I_t} = \frac{1}{\epsilon} = 0.368$$

$$\Delta I_{st}' = \Delta I_{st} \epsilon^{-1} \text{ or } \frac{\Delta I_{st}'}{\Delta I_{st}} = \frac{1}{\epsilon} = 0.368$$

Thus the time constants may be obtained by observing the time t corresponding to 36.8 per cent of the corresponding increment currents ΔI_t and ΔI_{st}.

The decrement of the direct-current component may also be plotted as a straight line on semi-logarithmic paper, and from the plot the direct-current time constant may be obtained in the same manner as just outlined.

CHAPTER 21

SYSTEM STABILITY

147. Transient Stability.—The steady state behavior of transmission networks is very conveniently analyzed by means of circle diagrams, as outlined in Chapter 9. Certain basic facts concerning maximum power limits were outlined in Art. 61. Furthermore, in Art. 52, certain power relations were derived, the receiver and sending powers being given by equations 146 and 148. Thus:

$$P_r = \frac{E_s E_r}{B} \cos(\phi - \beta) - \frac{A}{B} E_r^2 \cos(\beta - \alpha)$$

$$P_s = -\frac{E_s E_r}{B} \cos(\phi + \beta) + \frac{D}{B} E_s^2 \cos(\beta - \Delta)$$

The notation used in these equations is the same as that used throughout the text and is not again reviewed. However, under normal operation the receiver and sending voltages may be said to be constant, the only degree of freedom in the foregoing equations being in the angle ϕ between E_s and E_r. This angle has already been referred to as the system torque angle.

Maximum power received has already been given in Art. 52 and again in Art. 61, and will occur when the torque angle ϕ becomes equal to the generalized constant angle β (see equation 151). The power limit of equation 172 can only be obtained provided the receiver load is increased very slowly so as to avoid all possible effects of speed transients. When a load is suddenly applied to a line, the angle ϕ between the sending and receiver voltages tends to increase, and possibly to travel beyond the final value. This is illustrated in Fig. 207, the angle ϕ being plotted against time t.

The maximum steady state torque angle cannot exceed the angle β (see Art. 61). It might, therefore, be concluded that the maximum transient torque angle (ϕ_{tm}, Fig. 207) cannot be

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greater than β. This conclusion is, however, not true; in fact, maximum torque angles considerably in excess of the angle β are possible.

Transient stability might be defined as that value of maximum power which can be transmitted over a particular system without the system falling out of step because of the effect of line faults or sudden changes of load. Transient stability is rather an indefinite quantity, and depends on a large number of conditions of the local circuits and also of the terminal machines. The manner in which the system is operated will
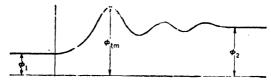
Fig. 207. Relation Between Torque Angle and Time.

have very marked effects on the results obtained. Automatic speed control and automatic regulation materially affect the results. When machines are involved, there are three important elements which influence the system behavior. These elements are:

(a) the inertia of rotating parts;
(b) the induction motor damping effect;
(c) the static maximum tangential torque which the system will transmit.

148. Theoretical Considerations.—Consider a power system composed of several generator stations and also several load centers. Under normal operation there is a definite angle displacement between the excitation voltage vectors of all generators or synchronous motors, as well as a definite angular displacement between all voltage vectors at all different points in the network. The maximum angular displacement between any two revolving units (holding constant voltage) under steady state conditions has been shown to be limited to the value of β (see Art. 61).

Transient instability may be due to several causes, such as the sudden application of additional load or the sudden loss of load, or it may be due to any one of the many types of faults which may take place in practice. When a fault takes place in such a network, the angles between all excitation voltages are set into oscillation, coming to rest at some new value. This oscillation is due to the effect of the stored energy in all revolving masses and also the retarding torques developed between stators and rotors. It is also due to the sudden attempt to change the stored electro-potential and kinetic energy of the electrical system. All governors are somewhat sluggish and take an appreciable time to actuate. Therefore, at the first instant after the disturbance, constant input power may be assumed.

Consider, for example, the sudden application of a load to a system. The increase in load must be supplied from the stored energy in all revolving masses. This will cause a deceleration of rotors and a resultant drop in voltage due to the sudden heavy flow of current. During the deceleration of the rotors, a reaction is set up in the damper windings or even in the field windings, and this reaction tends to oppose the deceleration. In the meanwhile the governors may cause an increase in input power, or the voltage regulators may cause an increase in exciting flux, thus increasing the generator driving torques which may actually cause an acceleration of the rotors; or the governors in some cases may merely overcome the decelerating effects of the load. As has already been pointed out, there is a critical limit to which the angular displacement between the voltages at the extreme ends of a particular line can swing; if this critical value is exceeded, the system may fall out of step.

One method which has been used for the study of instability is that of torque-angle swing curves. This method is confined entirely to the first alternation of the torque-angle transient of the system. It involves the plotting of torque-angle curves (from equations 146 and 148) for each different condition of operation under investigation; and, by further means, noting how a sudden transfer is made from one curve to the next. A simple set of torque-angle swing curves consists of those

obtained between a generator and a synchronous motor, these two machines being connected by a short line of zero resistance.

Let E_g and E_m be the excitation voltages of the generator and the motor, respectively. If the resistances of the generator and motor are neglected, the conditions involved will be given by the vector diagram of Fig. 208.

If x = total reactance of the circuit, then:

$$I = \frac{e}{x}$$

Also, the power delivered is:

$$P = E_g I \cos \theta_1 = E_g \frac{e \cos \theta_1}{x} = E_g \frac{e'}{x}$$
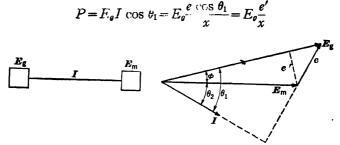
Fig. 208. Generator and Motor Connected by Simple Line.

But, from Fig. 208,

$$e' = E_m \sin \phi$$

where ϕ = torque angle between generator and motor. Therefore:

$$P = \frac{E_g E_m}{x} \sin \phi \qquad (384)$$

The results of equation 384 are illustrated in Fig. 209 for three conditions of excitation and reactance. The scale of ordinates is given as power, while the scale of abscissas is in degrees. Since an ideal system of zero resistance has been chosen, it is obvious that maximum power corresponds to an angle of 90°.

As a simple example, consider such a system undergoing a change due to the sudden application of additional power, ΔP, the initial load being indicated by P_0, as in Fig. 210. It

should be kept in mind that infinite prime-mover capacity is assumed. This means that generated capacity is immediately available, the instant an additional load is thrown onto the system. For convenience the motor may be considered as an infinite bus. Thus, assume that E_m is fixed and that E_s must forge ahead, thus increasing the angle ϕ. If this interpretation
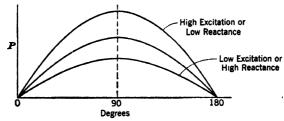
Fig. 209.   Power Delivered to Motor.
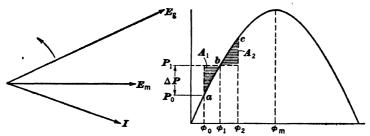
Fig. 210.   System Subjected to Sudden Application of Additional Load.

seems awkward, the generator may be assumed as an infinite bus, holding its speed absolutely constant, while the synchronous motor drops back, thus increasing the angle ϕ. Thus the net power increase during the transient period is:

$$P_n = P_1 - \frac{E_g E_m}{x} \sin \phi$$

The torque angle must accelerate along the curve in Fig. 210. During this process there is a change in the stored energy of the revolving masses of the rotors, according to the fundamental relation $\frac{1}{2}Mv^2$, and the work done on the rotors is represented by the shaded area A_1. Because of this acceleration, the rotor will travel beyond the angle ϕ_1, decelerating in speed until it

reaches some angle ϕ_2. The area A_2 indicates the amount of energy previously stored in the revolving mass of the rotor and now being returned to the circuit. Obviously the areas A_1 and A_2 must be equal, and the machine will oscillate along the curve *abc*, finally settling at point *b*. At that point:

$$P_1 = \frac{E_g E_m}{x} \sin \phi_1$$

It should be noticed that ϕ_1 should be less than ϕ_m, the critical angle for steady stability.

In contrast with the preceding case, consider the system operating at a load

$$P_2 = \frac{E_g E_m}{x} \sin \phi_2$$

as illustrated in Fig. 211.

Now let the load be increased to P_3 and let the net power increase be:

$$P_n = P_3 - \frac{E_g E_m}{x} \sin \phi$$

As previously mentioned the area A_1 indicates the amount of energy stored during acceleration, while the area A_2 indicates the energy returned to the circuit during deceleration. If $A_2 < A_1$, the rotor will fall out of step and the system reaches a condition of instability. It will be noticed that the torque angle may momentarily exceed 90 degrees without causing permanent instability.
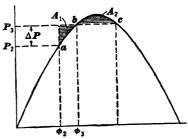
Fig. 211.   Modification of System of Fig. 210.

Consider now the generator supplying an infinite bus through two parallel lines, as in Fig. 212. First consider switch *s* closed. The equivalent reactance is:

$$x = \frac{x_1 x_2}{x_1 + x_2}$$

and the circuit is changed to that of Fig. 213.   Again:

$$P = \frac{E_g E_m}{x} \sin \phi$$

The swing curve of the system of Fig. 213 is similar to that of a simple single line.   Curve A in Fig. 214 is a typical curve for a parallel system, and curve B is for a single line.   The
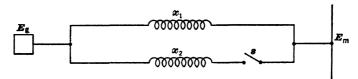
Fig. 212.   Generator Supplying Infinite Bus.
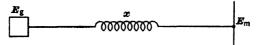
Fig. 213.   Simple Circuit for Fig. 212.
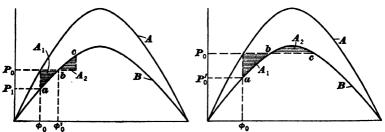
Fig. 214.   Swing Curves for Parallel   Fig. 215.   Modification of System
      System and Single Line.   of Fig. 214.

reactance of a single line is obviously greater than the reactance of the parallel system, and curve B is therefore lower.

Suppose the parallel system is operating at ϕ_0 and P_0.   With constant load, the switch is suddenly opened.   The original power which can be transmitted at an angle displacement of ϕ_0 and on curve B is shown as P_1.   If the load power remains at P_0, the angular displacement must increase along the line abc and will oscillate between points a and c and will come to rest at b with the areas A_1 and A_2 being equal.   In case these two

areas are not equal, a condition of instability will be obtained. The resultant displacement angle will be ϕ_0'.

Suppose, however, that the conditions accompanying the opening of the switch are as indicated in Fig. 215. In this case, $A_1 > A_2$ and the system becomes unstable.

It should be noticed that the transient angle may exceed 90 degrees. The problem is merely that of transfer of energy through acceleration and deceleration of moving parts.
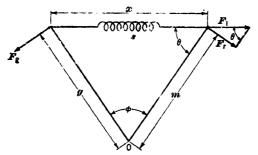
Fig. 216. Diagram for Mechanical Equivalent.

149. Mechanical Equivalent.—It has already been shown that the power transfer for a simple case is:

$$P = \frac{E_m E_g}{x} \sin \phi \qquad (385)$$

where E_m = motor excitation voltage;

   E_g = generator excitation voltage;

   x = total reactance between machines;

   ϕ = angle between E_m and E_g.

Now consider two torque arms pivoted at a common point O, and connected together by means of a spring s, as shown in Fig. 216. The elongation of the spring will be assumed directly proportional to the spring tension F_l. Thus:

$$F_l = Kx \qquad (386)$$

The left arm represents the generator with force F_g applied, while the right arm represents the receiver of the system with a force F_r applied. From Fig. 216 it follows that:

$$F_r = F_l \sin \theta \qquad (387)$$

Substituting from equation 386,

$$F_r = Kx \sin \theta$$

The receiver applied torque is:

$$T = F_r m = Kmx \sin \theta = Kmgx \frac{\sin \theta}{g}$$

By fundamental trigonometry,

$$\frac{\sin \theta}{g} = \frac{\sin \phi}{x}$$

Therefore:

$$T = Kgm \sin \phi = T_m \sin \phi \qquad (388)$$

where T_m = maximum torque which can be delivered by such a system.

It will be noticed that this maximum torque will occur when $\phi = 90°$. Beyond this angle, the torque actually decreases. The mechanical device here considered corresponds to the electric circuit without resistance or damping. To complete the analogy, the torque arms g and m should be considered as radii of flywheels representing the masses of the rotating equipment.

Assume now that the system of Fig. 216 is suddenly subjected to an increase in torque, the magnitude of which is:

$$\Delta T = T - T_m \sin \phi \qquad (389)$$

The inertia reaction set up in the mechanical system must be equal to the change in torque. Thus,

$$I \frac{d^2\phi}{dt^2} = T - T_m \sin \phi \qquad (390)$$

where I = moment of inertia of the system.

In case resilience (capacity) and damping or friction (resistance), as well as inertia (inductance), are present in such a system, then the fundamental differential equation becomes:

$$I \frac{d^2\phi}{dt^2} + K_d \frac{d\phi}{dt} + K\phi = T - T_m \sin \phi \qquad (391)$$

where K_d and K are constants. The particular peculiarity of this last equation is the right hand member, $T - T_m \sin \phi$. Except for this member, equation 391 is similar to the differential equation of an oscillatory electric circuit, which is:

$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{c} = e \tag{392}$$

In terms of power instead of torques, equations 390 and 391 become:

$$P_a \frac{d^2\phi}{dt^2} = P - P_m \sin \phi = \Delta P \tag{393}$$

$$P_a \frac{d^2\phi}{dt^2} + P_d \frac{d\phi}{dt} + P_c\phi = P - P_m \sin \phi = \Delta P \tag{394}$$

where P_a, P_d, and P_c are constants.

The resilience term is of no practical importance. Therefore, equation 394 becomes:

$$P_a \frac{d^2\phi}{dt^2} + P_d \frac{d\phi}{dt} = P - P_m \sin \phi \tag{395}$$

If radians and seconds of time are used in equation 395, then P_a = watts per radian per second per second, or power per degree of angular acceleration; and P_d = watts per radian per second, or power per degree of angular velocity.

The term $P_a \dfrac{d^2\phi}{dt^2}$ represents the power component due to the inertia of the revolving masses; $P_d \dfrac{d\phi}{dt}$ represents the power component due to damping action by induction motor operation; and $P_m \sin \phi$ represents the synchronous motor power component. Unfortunately, equation 395 does not lend itself to mathematical analysis; the standard method employed in its evaluation is by the use of the "Integraph."

150. Transient Torque-Angle Equation in Terms of Per-Unit Torques.—Equation 390, which considers only the inertia reaction set up in the mechanical system, may be written in the following form:

$$I \frac{d^2\phi}{dt^2} + T_m \sin \phi = T$$

In the mechanical analogue of Art. 149, for which this equation applies, the angle ϕ refers to *mechanical* radians and I = the moment of inertia of the mechanical system. In an electrical system it is necessary to consider electrical radians. Hence, if ϕ is now taken in electrical radians, the fundamental equation becomes:

$$\frac{2}{p} I \frac{d^2\phi}{dt^2} + T_m \sin \phi = T$$

It should be noticed that the term $\sin \phi$ remains unchanged when converting ϕ to electrical radians, since this term is a function of the vector relations of the machine and does not depend on the mechanical reactions.

Dividing the preceding equation throughout by T_m gives the general form of the transient torque-angle equation in per-unit torque notation. Thus:

$$\frac{2}{p} \frac{I}{T_m} \frac{d^2\phi}{dt^2} + \sin \phi = \frac{T}{T_m} = K \qquad (396)$$

Now, let the power associated with the maximum torque T_m be denoted by P_m. Then:

$$P_m = 0.746 \left(\frac{2\pi N T_m}{33,000} \right) = \frac{\left(\frac{2\pi N}{60} \right) T_m}{737.6} \qquad (397)$$

where N = revolutions per minute.

From equation 397,

$$T_m = \frac{737.6 P_m}{\left(\frac{2\pi N}{60} \right)}$$

Therefore, the coefficient of $\frac{d^2\phi}{dt^2}$ is:

$$\frac{2I}{p T_m} = \frac{2I}{p} \left(\frac{\frac{2\pi N}{60}}{737.6 P_m} \right) = \frac{I \left(\frac{2\pi N}{60} \right)^2}{737.6 P_m \left(\frac{\pi p N}{60} \right)}$$

Let $M = \dfrac{I\left(\dfrac{2\pi N}{60}\right)^2}{737.6 P_m} = $ inertia constant in seconds. Since $f = \dfrac{p}{2}\dfrac{N}{60} = $ normal frequency in cycles per second,

$$\frac{2I}{pT_m} = \frac{M}{2\pi f}$$

Therefore, equation 396 may be modified as follows:

$$\frac{M}{2\pi f}\frac{d^2\phi}{dt^2} + \sin\,\phi = K \qquad (398)$$

Theoretical discussions of transient stability are generally carried on from equation 398, since it is in a per-unit torque form. This equation has its limitations. The most important is that it considers the moment of inertia effect as concentrated in a single rotating machine; whereas, in an actual system, rotating masses are located at all generating stations and at those load points which have rotating equipment.

In the simple case of a generator connected to a synchronous motor through a transmission line, it has been found that equation 398 can be used, provided that

$$M = \frac{M_1 M_2}{M_1 + M_2}$$

where $M_1 = $ inertia constant of generator;

$M_2 = $ inertia constant of motor.

151. Energy Stored in Accelerating Machine.—When a sudden change in load takes place, the readjustment of the rotor position of all generators is mainly due to the amount of stored energy in the revolving masses, since the governor mechanisms cannot change the input power instantaneously. From mechanics, the energy in foot-pounds stored in a revolving mass is:

$$E = \tfrac{1}{2}I\omega^2$$

where $I = $ moment of inertia

Since $\omega = 2\pi\dfrac{N}{60}$,

$$E = \tfrac{1}{2}I\left(\frac{2\pi N}{60}\right)^2$$

Expressing this stored energy in Kw-sec, we have:

$$E = \tfrac{1}{2}I\left(\frac{2\pi N}{60}\right)^2 \times 3.766 \times 10^{-7} \times 3{,}600 = \frac{\tfrac{1}{2}I\left(\frac{2\pi N}{60}\right)^2}{737.6}\text{Kw-sec} \quad \textbf{(399)}$$

By introducing the inertia constant M, we may express the kinetic energy as:

$$E = \tfrac{1}{2}M P_m \text{ Kw-sec} \qquad\qquad \textbf{(400)}$$

where M is the time in seconds required to accelerate the rotor from standstill to normal speed N when maximum input power P_m is suddenly applied.

Since $\omega = 2\pi f =$ rotor angular velocity in electrical radians per second, then the rotor angular velocity in electrical degrees per second is:

$$\omega_0 = \frac{2\pi f}{2\pi} \times 360 = 360 f$$

Hence, the angular acceleration of the rotor is obtained as:

$$a = \frac{360 f}{M} \text{ electrical degrees per second per second}$$

Solving for M from equation 400 and substituting its value in the expression for acceleration gives $a = \dfrac{360 f}{2E} \times P_m$ or

$$a = \frac{180 f}{E} \times P_m \text{ electrical degrees per second per second} \qquad \textbf{(401)}$$

Equation 401 can be applied to any transient condition following a sudden change of power, ΔP, as follows:

$$a = \frac{180 f}{E} \times \Delta P \qquad\qquad \textbf{(402)}$$

The term E refers, of course, to the rotor energy stored at the time of the disturbance, and can be computed from equation 399. For some calculations it may be more convenient to change equation 399 to the following form:

$$E = 2.3WR^2N^210^{-7} \text{ Kw-sec} \qquad \textbf{(403)}$$

where $WR^2 =$ moment of inertia in pounds $\times$ feet squared.

In the studies of stability problems, it is natural to assume that the machines are operating at their rated speeds prior to the disturbance. Hence, the stored energy obtained from equation 403 can be considered as a machine constant, so that equation 402 may be rewritten as follows:

$$\alpha = C \, \Delta P \text{ electrical degrees per second per second} \qquad \textbf{(404)}$$

where $C = \dfrac{180f}{E} =$ acceleration constant.

In terms of the inertia constant M, we find that the acceleration constant is:

$$C = \frac{360f}{MP_m} = \text{electrical degrees per second per second per Kw} \qquad \textbf{(405)}$$

In other words, C is the acceleration per unit of power disturbance.

152. Special Form of Transient Equation Suitable to Step-by-Step Solution.—Again referring to Equation 390, we have:

$$I \frac{d^2\phi}{dt^2} = T - T_m \sin \phi = \Delta T \qquad \textbf{(406)}$$

in which ϕ is measured in mechanical radians. In equation 406, ΔT is the change in torque produced by the transient imposed on the system. Assuming that ΔT is held constant over a small angular swing, then equation 406 becomes:

$$I \frac{d^2\phi}{dt^2} = \Delta T$$

Converting ϕ to electrical radians gives:

$$\frac{2}{p} I \frac{d^2\phi}{dt^2} = \Delta T \qquad \textbf{(407)}$$

But, from the results associated with the inertia constant M, it was found that

$$\frac{2}{p} I = \frac{M}{2\pi f} T_m$$

Therefore, equation 407 becomes:

$$\frac{M}{2\pi f} T_m \frac{d^2\phi}{dt^2} = \Delta T$$

For small angular swings the speed may be assumed constant. Therefore, the torques T_m and ΔT can be taken as directly proportional to power, and we have:

$$\frac{M}{2\pi f} P_m \frac{d^2\phi}{dt^2} = \Delta P$$

If now the angle ϕ is converted to electrical degrees instead of electrical radians, we have:

$$\frac{M P_m}{360 f} \frac{d^2\phi}{dt^2} = \Delta P$$

From equation 405, the preceding expression becomes:

$$\frac{1}{C} \frac{d^2\phi}{dt^2} = \Delta P$$

Finally:

$$\frac{d^2\phi}{dt^2} = a = C \Delta P \text{ electrical degrees per second per second} \qquad \textbf{(408)}$$

This equation is, of course, identical with equation 404, which was developed in the preceding article.

153. General Swing Curves.—The power delivered by a transmission line of simple series reactance was given in equation 384, which is:

$$P = \frac{E_g E_m}{x} \sin \phi$$

in which E_g = generator voltage and E_m = receiving motor voltage. This result could have been obtained directly from the first expression of equations 149 by putting $Z = x$, $E_s = E_g$, $E_r = E_m$, and $\beta = 90°$.

For general four-terminal networks defined by the constants A, B, C, and D, the swing curves of power, as derived in Art. 52, are:

$$P_r = \frac{E_s E_r}{B} \cos(\phi - \beta) - \frac{A}{B} E_r^2 \cos(\beta - \alpha)$$

$$P_s = -\frac{E_s E_r}{B} \cos(\phi + \beta) + \frac{D}{B} E_s^2 \cos(\beta - \Delta)$$

$$\tag{409}$$

in which the notation is the same as that used in Art. 52.

In case the four-terminal network can be represented by a simple series impedance, then equations 409 become:

$$P_r = \frac{E_s E_r}{Z} \cos(\phi - \beta) - \frac{E_r^2}{Z} \cos \beta$$

$$P_s = -\frac{E_s E_r}{Z} \cos(\phi + \beta) + \frac{E_s^2}{Z} \cos \beta$$

$$\tag{410}$$

By application of equations 409 and 410, curves of the general shape given in Fig. 209 can be drawn for any particular four-terminal network.

The general problem of transient torque-angle fluctuation under suddenly applied loads or other disturbances is quite involved, and at present a rigorous mathematical solution is not available. The solutions of important problems of this type are, therefore, obtained by actual test on artificial models. These models or equivalent circuits are generally known as A-C Analyzers or Calculating Boards.

154. The Alternating-Current Calculating Board.*—This article describes the mechanical and electrical features of the various component parts of the Westinghouse Alternating-Current Calculating Board. Other types of boards will be similar in general operation and arrangement of parts.

The board consists of an assembly of resistors, reactors, condensers, power units, transformers, and metering equipment. It provides means for reproducing actual power systems on a miniature scale and for obtaining voltage, current, and phase-angle readings produced by assumed normal, abnormal, and transient conditions on a phase to neutral basis. Unbalanced conditions are taken care of through the method of Symmetrical

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* This material has been furnished by the Westinghouse Electric and Manufacturing Company. .

Fig. 217.   A-C Calculating Board.

Components. Power for the board is supplied from a 440-cycle, 220-volt, three-phase, sine-wave alternator.

The A-C Calculating Board, shown in Fig. 217, consists of a meter desk and an assembly of several vertical steel cabinets, in which are located the various units of resistance, reactance, and capacitance; auto-transformers; "power·units" for simulating synchronous apparatus; cord-and-jack assemblies to permit reproducing the system network to be studied; and suitable current and voltage receptacles whereby the metering equipment may be inserted into any circuit.

The metering equipment may be connected to any circuit on the board through a three-wire metering bus connecting the meter desk with the three-point polarized receptacles located on the posts of the cord-and-jack cabinets. Connection between these receptacles and the circuits is made by means of a three-wire flexible cord which is fitted on either end with suitable plugs, one for inserting into the polarized receptacles and the other for plugging into the meter switches.

The load impedance units contain a resistance adjustable in 2-ohm steps from 2 to 3,998 ohms and a reactance consisting of a tapped reactor with steps of 20 ohms up to 280 ohms. These units may be put in series with a variable air gap reactor adjustable from 1 to 20 ohms and provided with a scale calibrated in fractional steps. Small toggle switches are used to obtain the desired setting of the resistors and the 20-ohm steps on the fixed air gap reactor.

Banks of line and load impedances are similar in general appearance to the load impedances, but the magnitudes are quite smaller.

Power units to represent generators or other sources of emf consist of a three-phase, 220-volt, 100% induction regulator and a three-phase shifter with single-phase output. These units are shown at the left in Fig. 217.

The base voltage to use for any given problem is determinde from the conversion factors which establish the constants of proportionality between the actual system quantities and the calculating board quantities. The adjustment of the voltage regulators and phase shifters to give the desired bus voltages

and distribution of current is quite simple and is very similar to that procedure which goes on in the actual power system.

High-impedance reactor circuits are used to simulate the magnetizing current of a large bank of transformers or the reactive component of loads. These circuits are made up of a tapped reactor which can be adjusted from 0 to 2,100 ohms in 300-ohm steps. Condenser circuits have a range from 0 to 4.1 microfarads in 0.01-microfarad steps.

Auto-transformers are used to represent tap-changing equipment or the conditions where a double-loop system is closed through two sets of transformers with different turn ratios. They are especially designed, being made with very thin Hypernik punchings so that the exciting current can be neglected for all normal problems. The range of settings available is from 85 to $119\frac{1}{2}\%$ in steps of $\frac{1}{2}\%$.

Instead of using the ordinary type of ammeter and voltmeter, two coil dynamometer-wattmeters are used to measure current and voltage, the greater per cent of the energy necessary to operate them being supplied in their "field" coils from a phase shifter. The arrangements for making the measurements are shown in Fig. 218. The use of the phase shifter to supply the "field" energy for the meters also makes possible its use to measure the phase angle. Real and reactive components of current and voltage with respect to any reference direction can be obtained, since the phase position of the "field" or standardizing current is controlled by the position of the phase-shifter rotor. These two meters, the ammeter and the voltmeter, together with the phase shifter, suffice to take care of all measurements which it is necessary to make for a system study. Both real and reactive power are easily obtained from readings of in-phase and out-of-phase current components.

In addition to the voltmeter and the ammeter, a stardardizing current milliammeter is also supplied. This measures the current which supplies the energy to the field coils of the ammeter and voltmeter, and should always be kept at the prescribed mark. Adjustment of this current is obtained by means of the standardizing current rheostat, which is in series with the phase-shifter output.
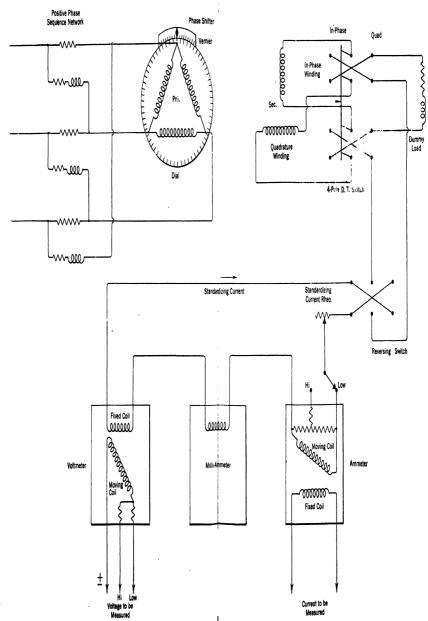
Fig. 218. Schematic Diagram of Measuring Circuit.

The phase shifter has a three-phase primary and a two-phase, four-wire secondary. The secondary is connected to the phase-shifter secondary switch, by which the phase position of the standardizing current can be shifted 90° by throwing this switch from the "in-phase" position to the "quadrature" position.

There are two methods which have been in general use for transferring actual system quantities to calculating board quantities. These are the *Kva base*, in which the impedances are expressed in per cent; and the *Kv base*, in which the impedances are expressed in ohms.

A single-line diagram of the system should be prepared, with the impedances expressed as ohms to neutral on a common voltage base. Actual impedances as obtained from the lengths, sizes, spacings, and kinds of conductors are transferred from their own voltage bases to another base by multiplying by the *square of the turn ratio* of the transformers connecting them. Thus:

$$Z_1 = Z_2 \times \frac{N_1^2}{N_2^2} \tag{411}$$

where N_1 = turns of primary of transformer;

N_2 = turns of secondary of transformer;

Z_1 = actual ohms at primary base voltage;

Z_2 = actual ohms at secondary base voltage.

It will be noted that the term "turn ratio" is specifically used, and *not nominal voltage ratio*. This latter expression is often used for short-circuit studies, for which it is sufficiently accurate, since very large voltage drops are involved and the difference between the nominal voltage ratio and the turn ratio would introduce only a very small error. However, when the voltage drops considered are comparatively small, as for voltage regulation studies, this is not the case; and, in order to avoid large errors, turn ratios must be used.

In order to set up the system on the calculating board, it is necessary to establish definite ratios between the actual system quantities and the similitude quantities of the calculating board. In general, this is done by increasing the impedances and

decreasing the voltages so that the current and power handled on the board will be very small proportions of the actual values and will be within the ratings of the various units. Each of the four multipliers that are used to establish the ratios between the actual and the calculating board quantities will be designated by the capital letter K with a subscript to indicate that it refers to current, voltage, power, or impedance. Thus, K_A is the current multiplier; K_E is the voltage multiplier; K_p is the power multiplier; and K_z is the impedance multiplier. These multipliers are defined as follows:

$$\left.\begin{array}{l} K_A i = I = \text{line current} \\ K_E e = E_L = \text{line voltage} \\ K_p e i = \text{Kva} \\ K_z Z \text{ (actual ohms to neutral)} = z \end{array}\right\} \quad (412)$$

where small letters refer to calculating board quantities and capital letters refer to actual system quantities. Note that the first three of these multipliers are factors by which calculating board quantities are multiplied to get actual quantities; and the impedance conversion factor, K_z, is so defined that, when it is multiplied by the actual ohms to neutral, the product will be the calculating board ohms.

The relations among these multipliers are contained in two independent equations. For three-phase systems, these are:

$$K_p = \frac{\sqrt{3} K_E K_A}{1,000} \quad (413)$$

$$K_z = \frac{K_A \sqrt{3}}{K_E} \quad (414)$$

On account of the interdependence of the quantities, *it is not always possible to have simple multipliers for all four quantities*; therefore, it is desirable to pick out those quantities which will be most useful in the solution of the problem and to assign to them convenient multipliers. The other quantities will then be determined from the equations.
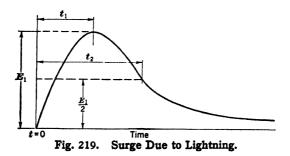
In general, the A-C Calculating Board is used for solving all types of problems where phase-angle considerations have to

be taken into account.   The four main classes into which such types of problems can be divided are:

(*a*) voltage regulation and load control studies;
(*b*) steady state stability problems;
(*c*) transient stability problems;
(*d*) short-circuit studies (especially for ground relay application).

CHAPTER 22

TRANSMISSION-LINE SURGES

155. Types of Transients to Be Studied.—In this chapter we are particularly interested in a study of those transients or surges generally caused by lightning. Switching transients due to the application of sinusoidal voltages are not included in the scope of this text. Surges due to lightning are generally of a non-oscillatory nature with a very steep wave front, and are of gradually decreasing tail, as shown in Fig. 219. The maximum voltage and its corresponding time t_1, together with

Fig. 219. Surge Due to Lightning.

half of the maximum voltage on the decreasing tail with its associated time t_2, supply enough data to designate the general shape of such a wave. Waves are therefore labeled by a notation such as 1,000 Kv/3/20, which means that the maximum voltage is 1,000 Kv at 3 micro-seconds from the forward toe of the wave and that at 20 micro-seconds the voltage has dropped to 500 Kv.

For any mathematical study it is first necessary that such a wave be represented by a suitable mathematical equation. An absolutely correct equation would involve an infinite number of terms, but such a degree of accuracy is not necessary, since a much simpler approximation is possible which is well within the accuracy required in practical problems. As a matter

of fact, the exact solution of the transient differential equations is not as yet known except in special cases. Hence, exact notation for the impressed potential would be of no avail.

The curve of Fig. 219 can be approximated by a double exponential function, as follows:

$$e = E(\epsilon^{-at} - \epsilon^{-bt}) \qquad (415)$$

A plot of equation 415, indicating how such an equation will give a curve similar to Fig. 219, is shown in Fig. 220.

Equation 415 applies only for values of time $t \geqq 0$, as the
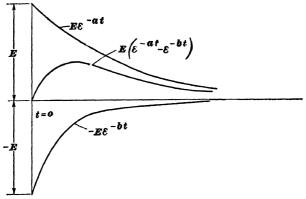
Fig. 220.   Equation of Curve of Fig. 219.

voltage $e = 0$ for $t < 0$. To completely define the curve, taking into account the fact that $e = 0$ for $t < 0$, it is convenient to introduce Heaviside's unit function. Thus:

$$e = E(\epsilon^{-at} - \epsilon^{-bt})1 \qquad (416)$$

156. Determination of Constants a, b, and E from an Observed Wave.—It has already been indicated that the curve of Fig. 219 has been obtained in actual field tests on transmission lines. For purposes of mathematical analysis it is then necessary to obtain the constants a, b, and E of equation 416 which will satisfy the oscillographic test results. The procedure is as follows:

From Fig. 219 it is obvious that the slope of the curve is zero at the maximum value of voltage.   Applying this fact to equation 415, we have:

$$\frac{de}{dt} = 0 = -a\epsilon^{-at} + b\epsilon^{-bt}$$

from which:

$$a\epsilon^{-at_1} = b\epsilon^{bt_1}$$

$$\frac{\epsilon^{-at_1}}{\epsilon^{-bt_1}} = \frac{b}{a}$$

$$\epsilon^{(b-a)t_1} = \frac{b}{a}$$

$$(b-a)t_1 = \mathrm{Ln}\frac{b}{a}$$

$$t_1 = \frac{\mathrm{Ln}\dfrac{b}{a}}{b-a} = \frac{1}{a}\left[\frac{\mathrm{Ln}\dfrac{b}{a}}{\dfrac{b}{a}-1}\right] = \frac{B}{a} \qquad (417)$$

where $B = \dfrac{\mathrm{Ln}\dfrac{b}{a}}{\dfrac{b}{a}-1}$.

Substituting equation 417 into equation 415, we have:

$$E_1 = E(\epsilon^{-B} - \epsilon^{-\frac{b}{a}B})$$

where E_1 is the peak voltage at instant t_1.   Thus:

$$y_1 = \frac{E_1}{E} = \epsilon^{-B} - \epsilon^{-\frac{b}{a}B} \qquad (418)$$

Furthermore, at the point at which the voltage is half of the maximum, the relation is:

$$\frac{E_1}{2} = E(\epsilon^{-at_2} - \epsilon^{-bt_2})$$

from which:

$$\tfrac{1}{2}\left(\frac{E_1}{E}\right) = \epsilon^{-at_2} - \epsilon^{-bt_2} = \epsilon^{-at_1\frac{t_2}{t_1}} - \epsilon^{-bt_1\frac{t_2}{t_1}}$$

Since $at_1 = B$ and $bt_1 = \dfrac{b}{a}B$,

$$y_2 = \frac{E_1}{E} = 2\left(\epsilon^{-B\frac{t_2}{t_1}} - \epsilon^{-\frac{b}{a}B\frac{t_2}{t_1}} \right) \tag{419}$$

To apply this material to a particular oscillographic record, proceed as follows:

(a) Prepare a curve for the relation $B = \dfrac{\ln\dfrac{b}{a}}{\dfrac{b}{a} - 1}$ for assumed values of $\dfrac{b}{a}$.

(b) From the actual oscillogram, record the values of t_2 and t_1, which are then substituted in equation 419.

(c) Prepare curves for equations 418 and 419 for assumed values of $\dfrac{b}{a}$. The intersection point of these two curves will indicate the correct values of $\dfrac{b}{a}$, B, and E which apply to the particular oscillogram studied.

(d) Since $at_1 = B$ and $bt_1 = \dfrac{b}{a}B$, it is now possible to obtain the proper values of a and b, so that the complete equation may then be written as follows:

$$e = E(\epsilon^{-at} - \epsilon^{-bt})1$$

157. Fundamental Differential Equations.—The steady state differential equations of a transmission line with sinusoidal emf's and currents were given as equations 93 and 94 in Art. 38. Thus:

$$\left. \begin{array}{l} \dfrac{d^2E}{ds^2} = m^2E \\[2mm] \dfrac{d^2I}{ds^2} = m^2I \end{array} \right\} \tag{420}$$

Since sinusoidal quantities were considered in these relations, there was involved only one fundamental variable, namely, the

distance s, measured from the receiver end.   The time t is also a variable; but, by the choice of sinusoidal emf's and currents, the time variation enters only in the vectorial expression of the voltages and currents.

In the case of transients, equations 420 take on a somewhat more complicated form, since the voltages and currents must be expressed as instantaneous quantities and there is a definite dependance between the distance s and the time t.   Another difference is introduced in the present problem; namely, the distance s will be measured from the sending end.   This choice
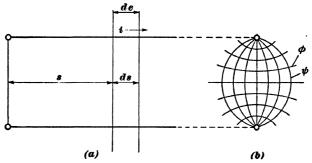
Fig. 221.   Conditions for Transmission Line.

is more logical in this case, since the point at which the transient is initiated is more conveniently considered as the sending end of the line.   Consider, therefore, the segment of line indicated in Fig. 221 (a).

The actual phenomenon of a line transient is more involved than merely the establishment of current along the conductors and potential between conductors.   The current within the conductors and the voltage between the conductors are, in a sense, merely the means whereby more important phenomena take place.   Associated with the current in a conductor is a magnetic field surrounding the conductor, and associated with the potential between conductors is a dielectric field terminating at the two conductors.   These fields are shown in Fig. 221 (b).   If a potential is suddenly applied at the sending end, there will be set up a magnetic field and a dielectric field which will be propagated over the extent of the line.   This latter phenomenon will

take place in the surrounding space, and the conductors form merely a "track" which guides the direction of this wave travel. On reaching the receiver end of the line, there is a reflection of this electromagnetic wave. Successive reflections between the receiver and sending ends may take place before the transient state finally converges into the steady state of the circuit.

Before starting any derivations, we must define the following terms:

R = resistance per mile,
L = inductance per mile;
C = capacity per mile;
G = conductance per mile.

The resistance and inductance are series constants, while G and C are shunt constants, in the same sense that $Z = R + jX$ was defined in Art. 38 as the series impedance and $Y = G + jB$ as the shunt admittance. The parameters for an element of line of length ds are $R\,ds$, $L\,ds$, $G\,ds$, and $C\,ds$.

Also, we will use the following notation:

ϕ = electromagnetic flux;
ψ = electrostatic or dielectric flux;
$d\phi$ = electromagnetic flux linking current i within the element of length ds;
$d\psi$ = electrostatic flux within the element of length ds.

In Fig. 221 there will be a voltage drop de across the distance ds which will be made up of the resistance drop and also the drop as set up by the rate of change of the magnetic flux $d\phi$. Thus:

$$-de = iR\,ds + \frac{\partial}{\partial t}(d\phi) \tag{421}$$

Similarly, there will be a current flowing between conductors within the elementary length ds, which is due to the conductance and capacity susceptance paths taken in parallel. Thus:

$$-di = eG\,ds + \frac{\partial}{\partial t}(d\psi) \tag{422}$$

From the fundamental concepts of inductance and capacity, these two constants, for the element ds, are:

$$L \, ds = \frac{d\phi}{i} \text{ and } C \, ds = \frac{d\psi}{e}$$

or
$$d\phi = iL \, ds \text{ and } d\psi = eC \, ds \tag{423}$$

Substituting these values in equations 421 and 422 gives:

$$\left. \begin{array}{l} -de = iR \, ds + \dfrac{\partial}{\partial t}(iL \, ds) \\[2em] -di = eG \, ds + \dfrac{\partial}{\partial t}(eC \, ds) \end{array} \right\} \tag{424}$$

Equations 424 may now be written in somewhat simplified operational form as follows:

$$\left. \begin{array}{l} -de = \left(R + L\dfrac{\partial}{\partial t} \right) i \, ds = Z(p)i \, ds \\[2em] -di = \left(G + C\dfrac{\partial}{\partial t} \right) e \, ds = Y(p)e \, ds \end{array} \right\} \tag{425}$$

where
$$\left. \begin{array}{l} Z(p) = R + L\dfrac{\partial}{\partial t} \\[2em] Y(p) = G + C\dfrac{\partial}{\partial t} \end{array} \right\} \tag{426}$$
and

In equations 425 the partial derivatives apply to the current i and the voltage e, respectively.

But the voltage de is equal to the rate of change of voltage with respect to s multiplied by the length ds, or:

$$de = \frac{\partial e}{\partial s} \times ds$$

Similarly
$$di = \frac{\partial i}{\partial s} \times ds$$

Therefore, from equations 425, we have:

$$\left. \begin{array}{l} -\dfrac{\partial e}{\partial s} = Z(p)i \\[2em] -\dfrac{\partial i}{\partial s} = Y(p)e \end{array} \right\} \tag{427}$$

It is of interest to compare these two expressions with those shown in equations 89 and 90. A definite similarity will be noticed, the difference being merely that now we must consider instantaneous values of voltage and current while in the previous solution vector quantities were used. Solving the two expressions in equations 427, we obtain:

$$\left.\begin{array}{l} \dfrac{\partial^2 e}{\partial s^2} = m^2 e \\[2mm] \dfrac{\partial^2 i}{\partial s^2} = m^2 i \end{array}\right\} \qquad (428)$$

where $m^2 = Z(p)Y(p) = \left[R + L\dfrac{\partial}{\partial t}\right]\left[G + C\dfrac{\partial}{\partial t}\right]$.

Again, equations 428 are similar to equations 93 and 94. In the operational notation of equations 426, p indicates the partial derivative with respect to time, or:

$$p = \frac{\partial}{\partial t}$$

Hence:

$$m^2 = (R + Lp)(G + Cp) = Z(p)Y(p)$$
or $$m^2 = RG + (GL + CR)p + LCp^2 \qquad (429)$$

The general solution of equations 428 is somewhat similar to that carried out for the steady state in Art. 38 and given in equation 95. Thus:

$$E = K_1\left(1 + \frac{m^2 s^2}{\underline{/2}} + \frac{m^4 s^4}{\underline{/4}} + \ldots\right) + K_2\left(ms + \frac{m^3 s^3}{\underline{/3}} + \ldots\right)$$

which can be modified to the following:

$$E = K_1\left(\frac{\epsilon^{ms} + \epsilon^{-ms}}{2}\right) + K_2\left(\frac{\epsilon^{ms} - \epsilon^{-ms}}{2}\right)$$

Rearranging this expression, we have:

$$E = \left(\frac{K_1 + K_2}{2}\right)\epsilon^{ms} + \left(\frac{K_1 - K_2}{2}\right)\epsilon^{-ms}$$

Finally:

$$E = \epsilon^{-ms}f_1(t) + \epsilon^{ms}f_2(t) \tag{430}$$

where $f_1(t) = \dfrac{K_1 - K_2}{2}$

$f_2(t) = \dfrac{K_1 + K_2}{2}.$

As was seen in Art. 38, K_1 and K_2 were found to be equal to the sinusoidal or vector expressions of receiver voltage and impedance voltage drop, respectively; or these terms were found to be functions of time. In the same manner we conclude that the voltage obtained from equation 428 must be:

$$e = \epsilon^{-ms}f_1(t) + \epsilon^{ms}f_2(t) \tag{431}$$

This result differs from equation 430 in two respects. First, the value of m is now defined by equation 429, and is not a complex number; second, the functions of time, $f_1(t)$ and $f_2(t)$, cannot be definitely assigned, since these functions may take on several forms.

That equation 431 is the solution of equation 428 may be checked by actually carrying out the second derivative of equation 431 and determining whether this derivative satisfies equation 428. Thus, the first derivative of equation 431 is:

$$\frac{\partial e}{\partial s} = -m\epsilon^{-ms}f_1(t) + m\epsilon^{ms}f_2(t)$$

and the second derivative is:

$$\frac{\partial^2 e}{\partial s^2} = m^2[\epsilon^{-ms}f_1(t) + \epsilon^{ms}f_2(t)] = m^2 e$$

This result is seen to fulfill the requirements of equation 428.

To determine the solution for the current, we need merely apply equation 427 to equation 431. Thus:

$$i = -\frac{1}{Z(p)}\frac{\partial e}{\partial s} = -\frac{1}{Z(p)}\frac{\partial}{\partial s}[\epsilon^{-ms}f_1(t) + \epsilon^{ms}f_2(t)]$$

or $\qquad\qquad i = \dfrac{m}{Z(p)}[\epsilon^{-ms}f_1(t) - \epsilon^{ms}f_2(t)] \tag{432}$

In equations 431 and 432, $f_1(t)$ and $f_2(t)$ are functions of time only, their general form depending on the nature of the terminal conditions, or the applied potential. Space or distance variation is indicated by the exponential terms. From equation 429, the expression for the exponent m is:

$$m = \sqrt{LC} \sqrt{p^2 + \left(\frac{R}{L} + \frac{G}{C}\right)p + \frac{RG}{LC}} \tag{433}$$

It will be noticed that the term $= \frac{\partial}{\partial t}$ appears twice in this expression for m. Since m is an exponent, the problem is so complicated that there is no general mathematical solution for equations 431 and 432. However, by making a slight assumption, which will not greatly affect the accuracy of the problems involved in commercial lines, it is possible to obtain a more definite answer to the problem; at least a simpler form is obtained which can then be interpreted, and the general trend of the transient behavior can be followed to an understandable conclusion.

Notice that the quantity which appears under the second radical in equation 433 can be converted into a complete quadratic if the following assumption is made:

$$\frac{R}{L} = \frac{G}{C}$$

Making this assumption corresponds to assuming that commercial power lines are distortionless lines. Equation 433 then becomes:

$$m = \sqrt{LC}\left(p + \frac{R}{L}\right) \tag{434}$$

ore or less associated with the assumption just made is
ict that, in the case of standard lines, the effects of R and
quite small as compared to those of L and C. Thus, we
n the following relation:

$$\frac{m}{Z(p)} = \frac{\sqrt{Z(p)Y(p)}}{Z(p)} = \sqrt{\frac{Y(p)}{Z(p)}} = \sqrt{\frac{G + Cp}{R + Lp}} = \sqrt{\frac{C}{L}} \tag{435}$$

Substituting equations 434 and 435 into equations 431 and 432, we have:

$$\left.\begin{array}{l} e = \epsilon^{-\sqrt{LC}\left(p+\frac{R}{L}\right)s} f_1(t) + \epsilon^{\sqrt{LC}\left(p+\frac{R}{L}\right)s} f_2(t) \\[2ex] i = \sqrt{\frac{C}{L}}\left[\epsilon^{-\sqrt{LC}\left(p+\frac{R}{L}\right)s} f_1(t) - \epsilon^{\sqrt{LC}\left(p+\frac{R}{L}\right)s} f_2(t)\right] \end{array}\right\} \quad (436)$$

The general trouble with equations 436 is that most non-mathematically minded individuals do not have much of a conception as to the meaning of the exponential, $p = \dfrac{\partial}{\partial t}$. Thus, our problem is now to convert the expressions to a more understandable form. The term which causes the trouble is:

$$\epsilon^{\pm\sqrt{LC}\left(p+\frac{R}{L}\right)s}$$

As already indicated, the difficulty is due to the presence of the operator $p = \dfrac{\partial}{\partial t}$ as an exponent. This difficulty may be overcome by the following transformation:

$$\epsilon^{ap} f(t) = f(t+a)$$

The correctness of the transformation may be proved by applying Taylor's expansion theorem. Thus:

$$f(t+a) = f(t) + a\frac{\partial}{\partial t}f(t) + \frac{a^2}{2!}\frac{\partial^2}{\partial t^2}f(t) + \cdots\cdots$$

$$= \left(1 + ap + \frac{a^2}{2!}p^2 + \cdots\cdots\right)f(t)$$

$$= \epsilon^{ap} f(t)$$

If $a = \sqrt{LC}\,s$,

$$\epsilon^{-\sqrt{LC}\,ps} f_1(t) = f_1(t - \sqrt{LC}\,s)$$

$$\epsilon^{\sqrt{LC}\,ps} f_2(t) = f_2(t + \sqrt{LC}\,s)$$

Using these relations, equations 436 become:

$$\left.\begin{array}{l} e = \epsilon^{-\sqrt{LC}\frac{R}{L}s} f_1(t - \sqrt{LC}\,s) + \epsilon^{\sqrt{LC}\frac{R}{L}s} f_2(t + \sqrt{LC}\,s) \\[2ex] i = \sqrt{\frac{C}{L}}\left[\epsilon^{-\sqrt{LC}\frac{R}{L}s} f_1(t - \sqrt{LC}\,s) - \epsilon^{\sqrt{LC}\frac{R}{L}s} f_2(t + \sqrt{LC}\,s)\right] \end{array}\right\} \quad (437)$$

By this mathematical device, we have eliminated the operator $p = \frac{\partial}{\partial t}$, which looked quite troublesome.

The next question involved is the significance of the two terms of equation 437.

Returning to the initial basic expressions in equations 427, we have:

$$-\frac{\partial e}{\partial s} = \left(R + L\frac{\partial}{\partial t} \right) i$$

$$-\frac{\partial i}{\partial s} = \left(G + C\frac{\partial}{\partial t} \right) e$$

In accordance with the previous assumption, R and G are quite small and may be neglected. Then:

$$\frac{\partial e}{\partial s} = -L\frac{\partial i}{\partial t} \text{ or } \frac{\partial e}{\partial i} = -L\frac{\partial s}{\partial t} = -Lv$$

where $v = \frac{\partial s}{\partial t} =$ velocity of forward wave.

Also, $\qquad \frac{\partial i}{\partial s} = -C\frac{\partial e}{\partial t}$ or $\frac{\partial e}{\partial i} = -\frac{1}{C}\frac{\partial t}{\partial s} = -\frac{1}{Cv}$

Therefore:

$$Lv = \frac{1}{Cv} \text{ and } v = \pm\frac{1}{\sqrt{LC}} \qquad \textbf{(438)}$$

Since the velocity is constant, $v = \frac{\partial s}{\partial t} = \frac{s}{t}$ or $\pm\frac{1}{\sqrt{LC}} = \frac{s}{t}$. Therefore:

$$\sqrt{LC}\, s = \pm t \qquad \textbf{(439)}$$

It should be remembered that s is measured from the sending end. Therefore, a positive value of v refers to a positive wave or a wave traveling from the sending end to the receiver end of the line. Conversely, a negative velocity must refer to a wave which has been reflected and is traveling from the receiver end to the sending end of the line. In equation 439, the plus sign is used for the forward wave, and the minus sign for the reflected wave. Thus, the voltage and current equations become:

$$e = \epsilon^{-\frac{R}{L}t}\left[f_1\left(t - \frac{s}{v} \right) + f_2\left(t + \frac{s}{v} \right) \right]$$
$$i = \sqrt{\frac{C}{L}}\,\epsilon^{-\frac{R}{L}t}\left[f_1\left(t - \frac{s}{v} \right) - f_2\left(t + \frac{s}{v} \right) \right]$$

$$\left.\right\} \quad (440)$$

These equations may be written as follows:

$$e = e_1 + e_r$$
$$i = i_1 + i_r$$

$$\left.\right\} \quad (441)$$

where $e_1 = \epsilon^{-\frac{R}{L}t} f_1\left(t - \frac{s}{v} \right)$ = forward or incident wave:

$$e_r = \epsilon^{-\frac{R}{L}t} f_2\left(t + \frac{s}{v} \right) = \text{reflected wave;}$$

and similarly for currents, except that i_r is negative.

Furthermore, the term $\sqrt{\dfrac{L}{C}}$ is the surge impedance of the line and is generally denoted by Z_0.

In equations 440, the attenuation is obtained by the exponential function $\epsilon^{-\frac{R}{L}t}$. In deriving the foregoing expressions, an ideal line was assumed; that is:

$$\frac{R}{L} = \frac{G}{C}$$

The attenuation factor could have been written as $\epsilon^{-\frac{G}{C}t}$; or one should actually consider the energy in the traveling wave as being dissipated through both the resistance R and the conductance G. Actually the electromagnetic (kinetic) energy will dissipate itself in the resistance R while the electro-potential energy will dissipate itself in the conductance. Hence, the two damping factors should be taken as follows:

$$\epsilon^{-\frac{R}{L}t} \text{ for the electromagnetic energy}$$

and $\qquad \epsilon^{-\frac{G}{C}t}$ for the electro-potential energy

For the ideal line, for which $\frac{R}{L}=\frac{G}{C}$, we may state that the damping factor for the entire energy content is:

$$\epsilon^{-\frac{1}{2}\left(\frac{R}{L}+\frac{G}{C}\right)t}$$

This assumption considers that the energy of a traveling wave is dissipated only in the resistance and conductance of the line. However, in power lines under the action of surges, there is general indication that the damping takes place through other mediums, mainly in the corona which becomes very important for voltage surges exceeding the magnitude of the critical voltage breakdown.

There is not enough test evidence or theoretical information available at present which indicates definitely the law of energy dissipation in the corona. Consequently one must be satisfied in writing the basic differential equations as follows:

$$\left.\begin{array}{l} e=f(t)\left[f_1\left(t-\frac{s}{v}\right)+f_2\left(t+\frac{s}{v}\right)\right] \\ i=\frac{f(t)}{Z_0}\left[f_1\left(t-\frac{s}{v}\right)-f_2\left(t+\frac{s}{v}\right)\right] \end{array}\right\} \quad (442)$$

in which $f(t)$ is the damping term. For the special case in which $\frac{R}{L}=\frac{G}{C}$ and the corona is neglected, the damping term is:

$$f(t)=\epsilon^{-\frac{1}{2}\left(\frac{R}{L}+\frac{G}{C}\right)t}=\epsilon^{-\frac{R}{L}t}=\epsilon^{-\frac{G}{C}t}$$

For lines in which $\frac{R}{L}\neq\frac{G}{C}$, one may use the foregoing damping factor as representing an average value.

158. Velocity of Wave Propagation.—In the preceding article it was shown that the velocity of wave propagation in a distortionless line has a value equal to $v=\frac{1}{\sqrt{LC}}$. It is of interest to investigate this relation further. The inductance per centimeter of a single-phase line is given by equation 35. If the internal conductor flux linkages are neglected,

$$L = \frac{\mu}{\pi}\left(\text{Ln}\,\frac{D}{a}\right)10^{-8} \text{ henry}$$

where D = conductor spacing, in centimeters.

The capacity per centimeter of a single-phase line is given by equation 68. Since the conductor radius is quite small compared to the conductor spacing, this equation becomes:

$$C = \frac{\pi K}{\text{Ln}\,\dfrac{D}{a}} \text{ farads}$$

Therefore:
$$v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\left(\dfrac{\mu}{\pi} \times \pi K\right)10^{-8}}}$$

Since $\mu = \dfrac{4\pi}{10}$ and $K = \dfrac{10 \times 10^{-12}}{36\pi}$,

$$v = \frac{1}{\sqrt{\left(\dfrac{10^{-12}}{9}\right)10^{-8}}} = 3 \times 10^{10} \text{ cm/sec}$$

$$= 985 \text{ ft/micro-sec}$$

For single-phase cables the approximate velocity is:

$$v = \frac{3 \times 10^{10}}{\sqrt{K_0}} \text{ cm/sec}$$

where K_0 = relative dielectric constant.

159. Forward (or Incident) and Reflected Waves.—In equations 441 it was stated that:

$$e = e_1 + e_r$$

$$i = i_1 + i_r$$

where $e_1 = \epsilon^{-\frac{R}{L}t} f_1\left(t - \frac{s}{v}\right)$ = forward wave of voltage;

$e_r = \epsilon^{-\frac{R}{L}t} f_2\left(t + \frac{s}{v}\right)$ = reflected wave of voltage;

$$i_1 = \sqrt{\frac{C}{L}} \, \epsilon^{-\frac{R}{L}t} f_1\left(t - \frac{s}{v}\right) = \text{forward wave of current;}$$

$$i_r = -\sqrt{\frac{C}{L}} \, \epsilon^{-\frac{R}{L}t} f_2\left(t + \frac{s}{v}\right) = \text{reflected wave of current.}$$

Notice that the surge impeaance was taken equal to $\sqrt{\frac{L}{C}}$. Thus:

$$i_1 = \frac{e_1}{Z_0} \text{ and } i_r = -\frac{e_r}{Z_0} \tag{443}$$
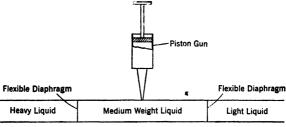
Fig. 222. Analogue of Electric Surge.

Considering the lightning surge as discussed in Art. 155, then the first forward wave of voltage is:

$$e_1 = \epsilon^{-\frac{R}{L}t}[E(\epsilon^{-at} - \epsilon^{-bt})1]$$

Bewley, in "Traveling Waves on Transmission Systems," has worked out a number of typical examples based on the foregoing equation of incident voltage. In this text it is proposed to cover only the more practical aspects of traveling waves.

160. Nature of Surge on Conductors.—A lightning surge is analogous to the surge produced in the device of Fig. 222. This diagram represents an elastic pipe completely filled with three kinds of liquids of different weights; one heavy, one of medium weight, and one light. These three liquids are separated by means of flexible diaphragms. A large piston gun suddenly injects a surge of liquid into the pipe at some point,

thus causing waves of pressure and velocity to travel toward the ends of the pipe.   Reflection waves will be set up at all points of discontinuity, such as at the flexible diaphragms.

Consider two circuits of surge impedances Z_1 and Z_2 connected in series as indicated in Fig. 223.   If voltage and current surges are introduced into circuit 1, waves of voltage and cur-

Fig. 223.   Circuits of Surge Impedances in Series.

rent will travel down conductor 1 toward the point of discontinuity.   Let these waves be denoted by e_1 and i_1, respectively. Waves e_2 and i_2 will be transmitted through the point of discontinuity into circuit number 2, while at the point of discontinuity reflections will take place in both voltage and current.   Let these reflected waves be denoted by e_r and i_r.
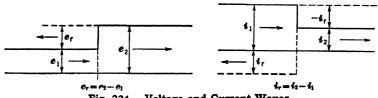
Fig. 224.   Voltage and Current Waves.

The foregoing results are illustrated in Fig. 224.   The fundamental relations existing between the various quantities are:

$$e_1 = i_1 Z_1 \qquad\qquad e_2 = i_2 Z_2 \qquad\qquad e_r = -i_r Z_1$$
$$e_r = e_2 - e_1 \qquad\qquad i_r = i_2 - i_1$$

From these relations it follows that:

$$e_2 = e_r + e_1 = -i_r Z_1 + e_1 = (i_1 - i_2)Z_1 + e_1$$

$$= \left(\frac{e_1}{Z_1} - \frac{e_2}{Z_2}\right)Z_1 + e_1 = 2e_1 - e_2\frac{Z_1}{Z_2}$$

Finally:

$$e_2 = e_1\frac{2Z_2}{Z_1 + Z_2} \qquad\qquad\qquad \textbf{(444)}$$

It therefore follows that:

$$e_r = e_1 \frac{Z_2 - Z_1}{Z_1 + Z_2} \tag{445}$$

From equation 444, we obtain:

$$i_2 Z_2 = i_1 Z_1 \frac{2Z_2}{Z_1 + Z_2}$$

or

$$i_2 = i_1 \frac{2Z_1}{Z_1 + Z_2} \tag{446}$$

Hence:

$$i_r = i_1 \frac{Z_1 - Z_2}{Z_1 + Z_2} \tag{447}$$
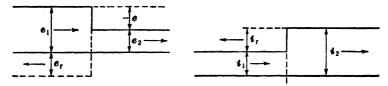
Fig. 225. Voltage and Current Waves.

Several cases of equations 444 to 447 are of particular interest. These follow:

Case (a): $Z_1 < Z_2$. The conditions of this case are illustrated in Fig. 224, in which e_2 and e_r are both positive, e_2 being greater than e_1. The current wave i_2 is less than i_1, and i_r is negative.

Case (b): $Z_2 = Z_1$. In this case:

$$e_2 = e_1 \text{ and } e_r = 0$$
$$i_2 = i_1 \text{ and } i_r = 0$$

Case (c): $Z_2 < Z_1$. In this case, $e_2 < e_1$ and e_r will be negative, while $i_2 > i_1$ and i_r will be positive. These conditions are illustrated in Fig. 225.

Case (d): $Z_2 = 0$. In this case:

$$e_2 = 0 \qquad\qquad e_r = -e_1$$
$$i_2 = 2i_1 \qquad\qquad i_r = i_1$$

Case (e): $Z_2 = $ infinity. From equation 446 it follows that $i_2 = 0$, from which $i_r = -i_1$. Also, $e_2 = 2e_1$ and $e_r = e_1$.

Equations 444 to 447 may be abbreviated as follows:

$$e_2 = e_1 M \text{ and } i_2 = i_1 N$$
$$e_r = e_1 M' \text{ and } i_r = i_1 N'$$

where M and N are the transmission coefficients, while M' and N' are the reflection coefficients.
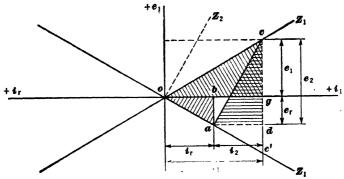
Fig. 226.   Star Diagram for $Z_2 > Z_1$.

161. Energy of Traveling Waves.—The surge energy may be stored as electrostatic or electromagnetic energy. These two forms of stored energy are:

Electrostatic energy $= \frac{1}{2} Ce^2$ watt-sec
Electromagnetic energy $= \frac{1}{2} Li^2$ watt-sec

But, since $e = iZ = i\sqrt{\dfrac{L}{C}}$, these energies are equal.   Thus:

$$\tfrac{1}{2} Ce^2 = \tfrac{1}{2} Li^2$$

Professor Karapetoff suggests the use of so-called star diagrams* as a means of illustrating the actions of traveling waves, as shown in Fig. 226.   Currents and voltages are measured along the x and y axes, as indicated.   The diagonals are drawn with slopes equal to the surge impedance Z_1. Another line of slope Z_2 is drawn through the origin.

Let $e_1 =$ forward voltage and $i_1 =$ forward current.   For these conditions, c shows the operating point on the Z_1 line.   Through

c, draw a line parallel to Z_2, and complete triangle acd. Then:

$$cd = e_1 + e_r = e_2 = \text{transmitted voltage}$$
and
$$ad = i_1 + i_r = i_2 = \text{transmitted current wave}$$

Also the triangle acd is proportional to the power transmitted; triangle ocg is proportional to initial forward wave power; while triangle oab is proportional to power reflected

Fig. 226 corresponds to the case when $Z_2 > Z_1$, and therefore $e_2 > e_1$ while $i_2 < i_1$. Consider now the case of $Z_2 < Z_1$, as in Fig. 227, all construction details being the same as in Fig. 226.
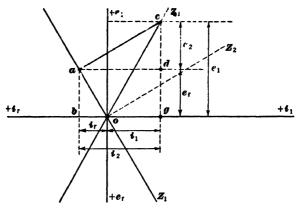
Fig. 227. Star Diagram for $Z_2 < Z_1$.

This case and three other special cases, shown in Figs. 228, 229, and 230, are left to the reader to analyze.

162. Traveling Waves at a Junction.—For the conditions shown in Fig. 231, a voltage surge e_1 is initiated on line No. 1 and travels toward a junction O. With respect to the incoming wave, all outgoing feeders can be considered in parallel. Therefore, the looking-in impedance at the junction is:

$$Z_0(p) = \cfrac{1}{\sum_2^n \cfrac{1}{Z_k(p)}} \qquad (448)$$

In case the feeders are transmission lines of distributed constants, then the line surge impedance is:

$$Z_k(p) = \sqrt{\frac{L}{C}} \text{ approximately}$$
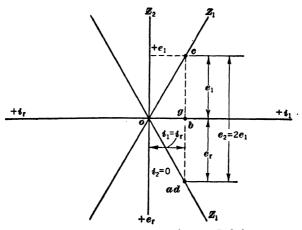
Fig. 228. Star Diagram for $Z_2 = $ Infinity.
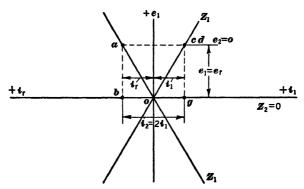
Fig. 229. Star Diagram for $Z_2 = $ Zero.

In applying equation 448, there is one marked difference from the standard application of the law of parallel impedances. Normally, the looking-in impedance of a network should include all impedance elements which are connected in series or series multiple to a single set of terminals. Thus, in Fig. 231 any one

of the n circuits might be made up of several series elements. In normal sinusoidal steady state problems, the looking-in impedance of any one of these lines would have to consist of all series impedances. However, in the present theory we are considering *lumped surges*, which for a particular instant of
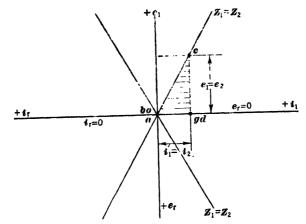
Fig. 230. Star Diagram for $Z_2 = Z_1$.
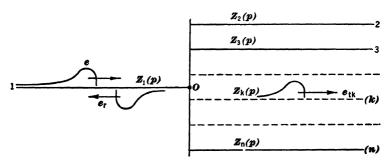
Fig. 231. Traveling Waves at Junction.

time are localized, so to speak, over a comparatively short length of line. As a surge strikes a particular piece of equipment, the only reaction offered to the progress of the surge is the reaction due to the surge impedance of that particular piece of equipment. Thus, in Fig. 231 and equation 448, the n impedances connected to the junction may include the surge

impedances of transformers, or the surge impedance of a lightning arrester, or even the surge impedance of a tower.

Incidentally, it should be emphasized that the surge impedance of equipment such as a transformer or some other iron-cored device is not to be confused with the commonly specified power frequency impedance. Much of the confusion and misunderstanding of the problem of surges is due to a poor understanding of what is involved in the surge impedance. For example, at normal frequencies, a transformer is thought to possess only resistance and inductance. For a very steep wave front such as we are considering, the inter-turns capacity effect of the transformer coils becomes important.

For Fig. 231 the total potential at the transition point is:

$$e = iZ_0(p) = (i_1 + i_r)Z_0(p)$$

or

$$e = e_1 + e_r = (i_1 + i_r)Z_0(p)$$

Since $i_1 = \dfrac{e_1}{Z_1(p)}$ and $i_r = -\dfrac{e_r}{Z_1(p)}$,

$$e_1 + e_r = \left[\frac{e_1}{Z_1(p)} - \frac{e_r}{Z_1(p)} \right] Z_0(p)$$

$$(e_1 + e_r)Z_1(p) = (e_1 - e_r)Z_0(p)$$

$$e_r = \frac{Z_0(p) - Z_1(p)}{Z_0(p) + Z_1(p)} e_1 \qquad (449)$$

The total potential at the junction is:

$$e = e_1 + e_r = e_1 + \frac{Z_0(p) - Z_1(p)}{Z_0(p) + Z_1(p)} e_1$$

or

$$e = \frac{2Z_0(p)}{Z_0(p) + Z_1(p)} e_1 \qquad (450)$$

The total current wave transmitted to the feeders is:

$$i_t = \frac{e}{Z_0(p)} = \frac{2}{Z_0(p) + Z_1(p)} e_1 \qquad (451)$$

Since $e_1 = i_1 Z_1(p)$,

$$i_t = \frac{2Z_1(p)}{Z_0(p) + Z_1(p)} i_1 \qquad (452)$$

The transmitted voltage wave is:

$$e_t = i_t Z_0(p) = \frac{2Z_1(p)Z_0(p)}{Z_0(p)+Z_1(p)}i_1$$

Since $i_1 Z_1(p) = e_1$,

$$e_t = \frac{2Z_0(p)}{Z_0(p)+Z_1(p)}e_1 = e \quad \text{(see equation 450)}$$

The incident or forward wave of current is:

$$i_1 = \frac{e_1}{Z_1(p)} \tag{453}$$

The reflected current is:

$$i_r = -\frac{e_r}{Z_1(p)} = -\frac{Z_0(p)-Z_1(p)}{Z_0(p)+Z_1(p)}\times\frac{e_1}{Z_1(p)}$$

or

$$i_r = -\frac{Z_0(p)-Z_1(p)}{Z_0(p)+Z_1(p)}i_1 \tag{454}$$

The transmitted current wave to one of the n lines leaving the junction is:

$$i_{tk} = \frac{e_t}{Z_k(p)} = \frac{2Z_0(p)}{Z_0(p)+Z_1(p)}\times\frac{e_1}{Z_k(p)} \tag{455}$$

The total transmitted current is:

$$i_t = \sum_2^n i_{tk} = \frac{2Z_0(p)}{Z_0(p)+Z_1(p)}\times e_1\left[\sum_2^n\frac{1}{Z_k(p)}\right]$$

Since $\sum_2^n\dfrac{1}{Z_k(p)} = \dfrac{1}{Z_0(p)}$,

$$i_t = \frac{2}{Z_0(p)+Z_1(p)}\times e_1 \quad \text{(see equation 451)}$$

It is seen that all currents and voltages can be expressed in terms of the *incident* or *forward wave*.

163. Example of Lightning Surge.—To illustrate the application of the foregoing theory of incident and reflected waves, consider the case of a direct lightning stroke to a *ground wire* at exactly mid-span, as indicated in Fig. 232, the ground wire

construction being symmetrical in both directions. The lightning surge introduces potential energy into the ground wire. It will travel both ways from mid-span, dividing in inverse proportion to the surge looking-in impedances in the two directions. At a point of discontinuity at a tower, part of the surge will be transmitted to the second span and part to the tower footing; and the remainder reflected back to mid-span. A reflection will occur at each point of discontinuity, until all the energy is dissipated in circuit resistance elements. The following notation is used.

Z_s = surge impedance of lightning stroke;

Z = surge impedance of line;

Z_t = surge impedance of tower;

Z_f = surge impedance of tower footing.
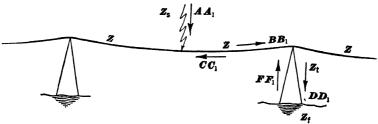
Fig. 232.   Lightning Surge.

From equations 444 and 445, the transmission and reflection coefficients are defined as follows:

$$\text{Transmission coefficient} = \frac{2Z_2}{Z_1+Z_2}$$

$$\text{Reflection coefficient} = \frac{Z_2-Z_1}{Z_1+Z_2} = \text{Trans. coeff.} - 1$$

where Z_1 = surge impedance of circuit number 1;

Z_2 = surge impedance of second circuit, or group of circuits, to which wave is transmitted.

For this problem, transmission coefficients are indicated A, B, C, etc.; while reflection coefficients are indicated A_1, B_1, C_1, etc. The following coefficients must be defined:

A and A_1 at mid-span for lightning striking ground wire;

B and B_1 at tower top for waves traveling from line to tower;

C and C_1 at mid-span for waves returning to mid-span from the towers;

D and D_1 at tower footing for waves traveling from tower top to tower footing;

F and F_1 at tower top for waves traveling up the tower.

The transmission coefficients are

$$A = \frac{2Z}{2Z_s + Z}$$

$$B = \frac{2Z_t}{2Z_t + Z}$$

$$C = \frac{4Z_s}{2Z_s + Z} \tag{456}$$

$$D = \frac{2Z_f}{Z_t + Z_f}$$

$$F = \frac{2Z}{Z + 2Z_t}$$

In arriving at the transmission coefficients, certain particulars are involved, as follows:

At mid-span the looking-in impedance from the lightning stroke to the line involves the two half-spans in parallel, or

$$Z_2 = \frac{Z}{2}$$

At the top of a tower, in determining the coefficient B,

$$Z_2 = \frac{ZZ_t}{Z + Z_t}$$

At the top of a tower, in determining the coefficient F,
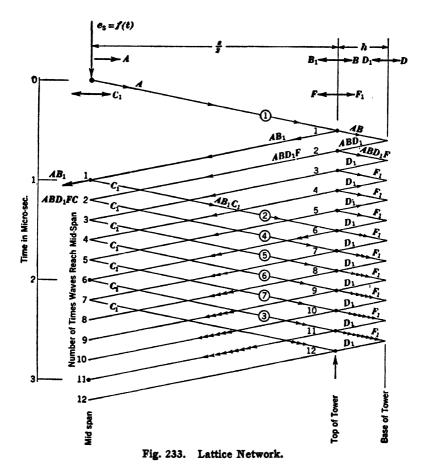
$$Z_2 = \frac{Z}{2}$$

In determining the coefficient C,

$$Z_2 = \frac{Z_s Z}{Z_s + Z}$$

However, reflected waves will return to mid-span from both

directions, and the potential at mid-span will therefore be doubled. To account for waves returning to mid-span from both directions, the coefficient C has included an extra 2.

Fig. 233. Lattice Network.

Each reflection coefficient can be obtained by subtracting unity from the corresponding transmission coefficient.

The initial potential of the stroke will be transmitted along the conductor in both directions toward the towers. At the towers, this wave strikes a point of discontinuity, and a reflection wave will start back to mid-span, while two components

will be transmitted beyond the top of the tower; one of these waves is transmitted to the base of the tower, while the other component is transmitted to the second span. The wave flowing down the tower will in turn be reflected at the base. Thus, there will be a series of reflections up and down the tower, and each time a wave reaches the top a certain component will be transmitted back to mid-span. Similarly, there will be a series of waves traveling back and forth between mid-span and the top of the tower.
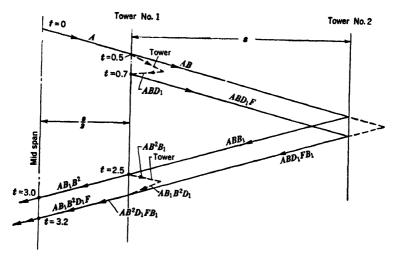
Fig. 234. Lattice Network.

In order to determine the resultant potential at any one point in the system, it is necessary that some suitable book-keeping method be adopted for taking care of all the waves and their many reflections. Mr. Bewley, in "Traveling Waves on Transmission Systems," has suggested the use of the so-called "lattice network." This chart, as illustrated for the particular problem chosen, is shown in Fig. 233. Along the horizontal are laid off the length of one half-span and also the height of the tower, as indicated by $\frac{s}{2}$ and h. The vertical scale of the chart indicates time in micro-seconds.

The stroke potential $e_s = f(t)$ is applied at mid-span, and is indicated at the origin of the chart. The potential wave transmitted to the conductor and traveling toward the tower is Ae_s. This wave is illustrated by the sloping line from the origin to the vertical line marked "top of tower." The slope of this line is obviously the velocity of wave propagation. At the top of the tower, this wave strikes a point of discontinuity and a certain component is reflected, while transmission takes place to the next span and also to the base of the tower. The components transmitted to the next span are left out of Fig. 233 and are shown in detail in Fig. 234. The reflected wave is e_sAB', while the wave transmitted to the base of the tower is $.e_sAB$.
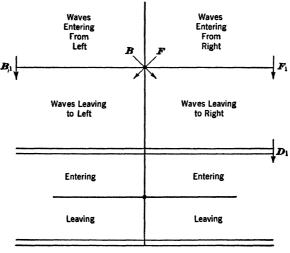
Fig. 235.   Quadrant Chart.

For all practical purposes it is sufficient to include in such a chart only those waves which return to the point at which the resultant potential is required within a time period of about 3 micro-seconds. Furthermore, to simplify the work, attenuation coefficients may be neglected within such a short period of time.

Another form of chart which can be successfully used to keep track of the many waves is illustrated by the form shown in Fig. 235. This chart is based on the fact that any point of

discontinuity between two circuits may have waves approaching the point from both directions, and waves will leave the point of discontinuity in either of two directions. To keep track of all these waves, a series of X and Y axes is drawn for each time a wave reaches the particular point of discontinuity. Waves entering the particular point are recorded in the first and second quadrants, while waves leaving are recorded in the third and fourth quadrants. Furthermore, the second and third quadrants pertain to waves in one circuit, while the first and fourth quadrants pertain to waves in the second circuit.

Diagonal vectors through the origin of these X and Y coordinates indicate transmitted waves; and, for convenience, the respective transmission coefficients might well be indicated along with these diagonal vectors. Reflection of waves takes place along vertical lines in this chart, as illustrated by vectors marked B_1, D_1, and F_1.

Such a chart carried out in detail for the problem in question is shown in Fig. 236.

From Figs. 233 and 236, it is apparent that the distance $\frac{s}{2}$ has been taken equal to five times the height of the tower and that the velocities of propagation along the conductor and along the tower are assumed to be equal.

From Fig. 233 it is seen that a second wave will reach the top of the tower at 1.5 micro-seconds and that additional waves reach this point at successive time intervals of 0.1 micro-second. Thus, in the "quadrant chart," the first wave reaching the top of the tower is shown as A and the next reaching the top of the tower from the left is AB_1C in the sixth line. This wave is related to the first wave leaving the tower to the left by the reflection coefficient C_1.

To avoid lengthy combinations of terms, certain combinations of coefficients have been abbreviated with the letters L and T with proper subscripts. The waves leaving the top of the tower will move toward mid-span, and the voltages at that point are obtained by multiplying the values of such waves by C, as indicated in the following table with the corresponding times in micro-seconds.

No. of
Times Waves
Reach Tower

Waves

Top of Tower

A B

Entering

1 B_1

AB_1

AB Leaving

D_1

ABD_1

F

2 F_1

ABD_1F

ABD_1F_1

D_1

$ABD_1^2 F_1$

3 F_1

$ABD_1^2 F_1 F$

C_1

$ABD_1^2 F_1^2$

D_1

$ABD_1^3 F_1^2$

4 F_1

$ABD_1^3 F_1^2 F$

$ABD_1^3 F_1^3$

D_1

$ABD_1^4 F_1^3$

5 F_1

$ABD_1^4 F_1^3 F$

C_1

$ABD_1^4 F_1^4$

D_1

AB_1C_1 (B)

$ABD_1^5 F_1^4$

6 B_1

F_1

$L_6 = (AB_1^2 C_1 + ABD_1^5 F_1^4 F)$

$AB_1C_1B + ABD_1^5 F_1^5 = T_6$

D_1

ABD_1FC_1

$T_6 D_1$

7 F_1

$L_7 = ABB_1C_1D_1F + T_6 D_1 F$

$AB^2 D_1 FC_1 + T_6 D_1 F_1 = T_7$

D_1

$ABD_1^2 F_1 FC_1$

$T_7 D_1$

8 F_1

$L_8 = ABD_1^2 F_1 FC_1B_1 + T_7 D_1 F$

$ABD_1^2 F_1 FC_1B + T_7 D_1 F_1 = T_8$

D_1

$ABD_1^3 F_1^2 FC_1$

$T_8 D_1$

9 F_1

$L_9 = ABD_1^3 F_1^2 FC_1B_1 + T_8 D_1 F$

$T_8 D_1 F_1 + ABD_1^3 F_1^2 FC_1B = T_9$

D_1

$ABD_1^4 F_1^3 FC_1$

$T_9 D_1$

10 F_1

$L_{10} = ABD_1^4 F_1^3 FC_1B_1 + T_9 D_1 F$

$T_9 D_1 F_1 + ABD_1^4 F_1^3 FC_1B = T_{10}$

D_1

$L_6 C_1$

$T_{10} D_1$

11 F_1

$L_{11} = L_6 B_1 C_1 + T_{10} D_1 F$

$T_{10} D_1 F_1 + L_6 C_1 B = T_{11}$

D_1

$L_7 C_1$

$T_{11} D_1$

12 F_1

$L_7 C_1 B_1 + T_{11} D_1 F$

Fig. 236.   Application of Quadrant Chart.

TIME	WAVE
0	A
1.0	$A B_1 C$
1.2	$A B C D_1 F$
1.4	$A B C D_1 F (D_1 F_1)$
1.6	$A B C D_1 F (D_1 F_1)^2$
1.8	$A B C D_1 F (D_1 F_1)^3$
2.0	$L_6 C$
2.2	$L_7 C$
2.4	$L_8 C$
2.6	$L_9 C$
2.8	$L_{10} C$
3.0	$L_{11} C$
3.2	$L_{12} C$

For purposes of calculation, the abbreviated values for L and T are as follows:

$$T_6 = A B B_1 C_1 + A B (D_1 F_1)^5$$

$$T_7 = A B^2 C_1 D_1 F + A B B_1 C_1 (D_1 F_1) + A B (D_1 F_1)^6$$

$$T_8 = A B^2 C_1 D_1^2 F F_1 + A B^2 C_1 D_1 F (D_1 F_1) + A B B_1 C_1 (D_1 F_1)^2 \\ + A B (D_1 F_1)^7$$

$$T_9 = 2 A B^2 C_1 D_1^2 F F_1 (D_1 F_1) + A B^2 C_1 D_1 F (D_1 F_1)^2 \\ + A B B_1 C_1 (D_1 F_1)^3 + A B (D_1 F_1)^8$$

$$T_{10} = 3 A B^2 C_1 D_1^2 F F_1 (D_1 F_1)^2 + A B^2 C_1 D_1 F (D_1 F_1)^3 \\ + A B B_1 C_1 (D_1 F_1)^4 + A B (D_1 F_1)^9$$

$$T_{11} = 4 A B^2 C_1 D_1^2 F F_1 (D_1 F_1)^3 + A B^2 C_1 D_1 F (D_1 F_1)^4 \\ + A B B_1 C_1 (D_1 F_1)^5 + A B (D_1 F_1)^{10} + A B B_1^2 C_1^2$$

$$L_6 = A B_1^2 C_1 + A B D_1^5 F_1^4 F$$

$$L_7 = A B B_1 C_1 D_1 F + A B B_1 C_1 D_1 F + A B D_1 F (D_1 F_1)^5$$

$$L_8 = A B B_1 C_1 D_1^3 F_1 F + A B^2 C_1 D_1^2 F^2 + A B B_1 C_1 D_1 F (D_1 F_1) \\ + A B D_1 F (D_1 F_1)^6$$

$$L_9 = A B B_1 C_1 D_1^3 F_1^2 F + A B^2 C_1 D_1^3 F^2 F_1 \\ + A B^2 C_1 D_1^2 F^2 (D_1 F_1) + A B B_1 C_1 D_1 F (D_1 F_1)^2 \\ + A B D_1 F (D_1 F_1)^7$$

$$L_{10} = ABB_1C_1D_1^4F_1^3F + 2AB^2C_1D_1^3F^2F_1(D_1F_1)$$
$$+ AB^2C_1D_1^2F^2(D_1F_1)^2 + ABB_1C_1D_1F(D_1F_1)^3$$
$$+ ABD_1F(D_1F_1)^8$$

$$L_{11} = AB_1^3C_1^2 + ABB_1C_1D_1^5F_1^4F + 3AB^2C_1D_1^3F^2F_1(D_1F_1)^2$$
$$+ AB^2C_1D_1^2F^2(D_1F_1)^3 + ABB_1CD_1F(D_1F_1)^4$$
$$+ ABD_1F(D_1F_1)$$

$$L_{12} = ABB_1^2C_1^2D_1F + ABB_1^2C_1^2D_1F + ABB_1C_1D_1F(D_1F_1)^5$$
$$+ 4AB^2C_1D_1^3F^2F_1(D_1F_1)^3 + AB^2C_1D_1^2F^2(D_1F_1)^4$$
$$+ ABB_1C_1D_1F(D_1F_1)^5 + ABD_1F(D_1F_1)^{10}$$
$$+ ABB_1^2C_1^2D_1F$$

Therefore the waves reaching mid-span from $t = 2.0$ to 3.2 are as follows:

TIME	WAVE
2.0	$ABCD_1F(D_1F_1)^4 + AB_1^2CC_1$
2.2	$ABCD_1F[(D_1F_1)^5 + 2B_1C_1]$
2.4	$ABCD_1F[(D_1F_1)^6 + 2B_1C_1(D_1F_1) + BC_1D_1F]$
2.6	$ABCD_1F[(D_1F_1)^7 + 2B_1C_1(D_1F_1)^2 + 2BC_1(D_1F_1)(D_1F)]$
2.8	$ABCD_1F[(D_1F_1)^8 + 2B_1C_1(D_1F_1)^3 + 3BC_1(D_1F_1)^2(D_1F)]$
3.0	$ABCD_1F[(D_1F_1)^9 + 2B_1C_1(D_1F_1)^4$ $+ 4BC_1(D_1F_1)^3(D_1F)] + AB_1C(B_1C_1)^2$
3.2	$ABCD_1F[(D_1F_1)^{10} + 2B_1C_1(D_1F_1)^5$ $+ 5BC_1(D_1F_1)^4(D_1F) + 3(B_1C_1)^2]$

To the preceding list must be added the waves which are transmitted to the second tower and reflected back toward the first tower and thence to mid-span. For the wave transmitted to the second tower and reflected back toward tower No. 1 and thence to mid-span, the values are:

$$t = 3.0 \ldots \ldots \ldots AB_1CB^2$$
$$t = 3.2 \ldots \ldots \ldots AB_1CB^2D_1F$$
$$t = 3.2 \ldots \ldots \ldots ABCD_1FB_1B$$

For the wave transmitted from mid-span to the top of the second tower, thence to the footing of the second tower and reflected back to the first tower and then to mid-span, the value is:

$$t = 3.2 \ldots \ldots \ldots \ldots ABCD_1F(B^2)$$

Thus, to the previous values must be added the following waves:

$$t = 3.0 \ldots \ldots \ldots AB_1C(B^2)$$
$$t = 3.2 \ldots \ldots \ldots ABCD_1F(2BB_1 + B^2)$$
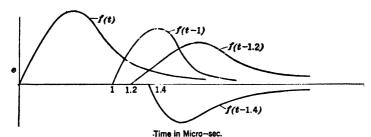
Fig. 237. Total Voltage at Mid-Span.

If the initial stroke is denoted as $e = f(t)$, then the respective traveling waves reaching mid-span would contribute the following to the voltage at that point:

TIME	COMPONENT VOLTAGE
0	$f(t)$
1.0	$AB_1C\, f(t-1.0)$
1.2	$ABCD_1F\, f(t-1.2)$
1.4	$ABCD_1F(D_1F_1)\, f(t-1.4)$
1.6	$ABCD_1F(D_1F_1)^2\, f(t-1.6)$
1.8	$ABCD_1F(D_1F_1)^3\, f(t-1.8)$
2.0	$L_6C\, f(t-2.0)$
2.2	$L_7C\, f(t-2.2)$
2.4	$L_8C\, f(t-2.4)$
2.6	$L_9C\, f(t-2.6)$
2.8	$L_{10}C\, f(t-2.8)$
3.0	$(L_{11}C + AB_1CB^2)\, f(t-3.0)$
3.2	$[L_{12}C + ABCD_1F(2BB_1 + B^2)]\, f(t-3.2)$

In the preceding table it is understood that any one term such as $K f(t-x)$ holds only for values of $t > x$; and that, for $t \leqq x$,

$$K f(t-x) = 0$$

A graphical representation of the total voltage at mid-span is therefore obtained by the summation of the several functions previously tabulated or as illustrated in Fig. 237.

As a result of a negative reflection or transmission coefficient, it is possible for some of the waves to be negative. This in turn may cause the potential at mid-span to take on positive and negative values, and the potential may actually oscillate from positive to negative values.

As already indicated, attenuation was completely neglected in the present problem. In case attenuation were taken into account, it would have been necessary to apply the damping coefficient for the conductors and tower each time a wave traveled over these particular parts of the circuit. However, the result obtained with attenuation would be somewhat lower than the value shown, so that the solution given is at least on the side of safety.

164. Comments on Surge Calculations.—The work outlined in Chapter 22 serves merely to introduce the student to the problem of surge calculations. The reader is no doubt impressed with the complexity of the problem and should have noticed the lack of engineering data and of tried mathematical methods.

Probably the most troublesome part of the solution is found in the fact that no definite and accurate method is available for calculating the surge impedances of towers, grounds, and even lumped circuits. In dealing with lumped circuits, such as transformers or choke coils, there is a general temptation to use normal power frequency impedances for the surge impedances. Such a practice is very dangerous, as the impedance under high frequencies is quite different from that at power frequencies. Under normal power frequency operation, the effect of the circuit inductance is very often the only important parameter. Under high-frequency operation, the distributed

winding capacity cannot be neglected. Furthermore the circuit resistance is also quite different under the action of very high frequencies.

PROBLEMS

CHAPTER 2

1. In Fig. 6 the power wave consumed in a resistance is shown to be sinusoidal and of double frequency. Thus

$$p = E_{rm} I_m \sin^2 \omega t$$

where E_{rm} = maximum voltage across resistance; I_m = maximum current flowing; $\omega = 2\pi f$; f = frequency; and t = time in seconds.

Derive equations for the instantaneous power waves of each phase of a three-phase balanced load of power factor angle θ, and then show that the total three-phase power is independent of time. What important deduction do you make from this result?

2. The Boulder Dam-Los Angeles transmission line has a resistance per mile per phase of 0.113 ohm and an inductance per mile per phase of 0.00212 henry. The frequency of the line is 60 cycles per second, and the length of the complete line is 266 miles. Neglecting the line capacity, calculate the sending voltage, sending power, sending power factor, and efficiency required to deliver 100,000 Kw at 275,000 line volts for power factors of 80% lagging, unity, and 80% leading. Plot these results against receiver power factor.

3. The capacity per mile per phase of the line of Problem 2 is 0.0136×10^{-6} farads. Consider the total capacity of the 266 miles concentrated at the mid-point of the line. Calculate the sending voltage, sending power, sending power factor, and efficiency required to deliver 100,000 Kw at 275,000 line volts, for receiver power factors of 80% lagging, unity, and 80% leading. Plot these results against receiver power factor. Determine also the receiver voltage and the charging Kva for each of the particular sending voltages just obtained.

4. Repeat Problem 3 with the total capacity divided equally between the sending and receiver ends.

NOTE: It will be of interest to compare the results of Problems 2, 3, and 4 of this chapter with the exact line solution of Problem 13, Chapter 9.

5. Determine the maximum energy storage in the line inductance and capacity for the particular conditions of Problems 2, 3, and 4.

6. Table 1 represents the general impedance laws for the electric, magnetic, and dielectric phases of the electrical circuit. Remembering the analogies indicated in Chapter 2, see how far you can build up a similar table for the field of dynamics.

7. Calculate the capacitance of a condenser made up of alternate layers of tin-foil and paper, alternate sheets of tin-foil being connected

together as the two line terminals. There are 126 sheets of tin-foil and 125 sheets of paper having a thickness of 3 mils and an area of 6 square inches. The dielectric constant of paper is 3.

8. A storage battery is short-circuited by a conductor of uniform size that has a length of x miles. Define current density and voltage gradient along such a wire. Outline a series magnetic circuit and explain clearly the similar relations of flux density and mmf gradient. Consider next the magnetic field produced by a long straight conductor. Refer to Art. 105 of "Principles of Electrical Engineering" by Timbie and Bush, and contrast the meaning of the "Circuital Law" with the example of voltage gradient along the battery circuit, and indicate under what conditions

$$\oint H\, dl = NI \text{ and } \oint H\, dl = 0$$

Is the circuital law a form of Kirchhoff's law? Explain.

9. In Table 1 the following relations are given for the electric and dielectric phenomena:

$$D = \frac{I}{A}, \; D = \frac{Q}{A}, \text{ and } G = \frac{E}{L}$$

Contrast these relations and explain their meaning on the basis of electron motion. Also contrast the general phenomena of electric and dielectric behavior with the mechanical behavior of an elastic body.

10. A conductor is shaped like the frustum of a cone, the two extreme cross-sectional areas being A_1 and A_2. The total length of the conductor is X. Uniform current density is assumed throughout the conductor cross-section. Derive an equation for the resistance voltage drop from end to end of the conductor.

11. A toroid of rectangular cross-section is wound with N turns of wire, carrying a current of I amperes. The outer radius r_2 is much larger than the inner radius r_1. The width of the toroid may be denoted by w. Assuming uniform permeability of the magnetic circuit, derive the equation for the total flux within the toroid. Compare this result with the value of flux obtained by using the mean circumference. Repeat the solution, assuming that the permeability is directly proportional to the radius.

CHAPTER 3

1. A cylindrical tube has an internal radius r_1 and an outer radius r_2. Derive equations for the internal and external flux linkages, assuming a current I flowing in the tube and uniform current density.

2. A solid round return conductor of radius r_0 is placed inside of the tube of Problem 1. Assuming uniform current density, derive an equation for the inductance of this circuit per centimeter length of conductor.

3. A toroidal ring of permeability μ is uniformly wound with N turns of wire in which I amperes flow. The inner and outer radii of the ring are r_1 and r_2, respectively, and the radius of the circular cross-section of the toroid is $\rho = \dfrac{r_2 - r_1}{2}$. Assuming uniform permeability throughout the cross-section of the ring, determine its inductance in henries.

4. The leakage flux about a conductor inside of a slot, as indicated in Fig. 238, may be considered as being divided into three parts: (1) that part which crosses the slot above the conductor, (2) that part which crosses the slot below the upper edge of the conductor, and (3) that part which passes between tooth tips. The mmf set up by the current in the conductor will be assumed as concentrated in the air or the non-magnetic parts of the circuit, since the reluctance of the flux paths is almost entirely concentrated in the air (or non-magnetic material) of the slot and across the tooth tips. Uniform current density in the conductor is assumed.
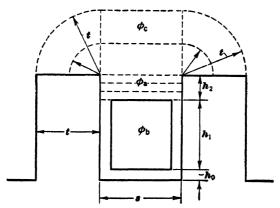
Fig. 238. Conditions in Problem 4.

(a) Derive an equation for the inductance L_a above the conductor, per centimeter of conductor. (b) Derive an equation for the inductance L_b for the partial linkage section, per centimeter of conductor. (c) Derive an equation for the inductance L_c across the tooth tips, per centimeter of conductor.

NOTE: The flux path is assumed to be circular as the flux leaves the tooth tips, and to be straight over the slot width s.

5. A rectangular copper conductor of thickness t and width w, Fig. 239, carries a current I, which is assumed to be uniformly distributed over the cross-sectional area. Assuming that the internal flux follows a rectangular path similar to the conductor rectangle, derive an equation for the internal inductance per centimeter length of conductor.

6. A three-phase line has its conductors a, b, and c arranged so that $D_{ab} = D_{bc} = D$ and $D_{ca} = KD$, where K is a constant not greater than 2.

The GMR of the conductors is r. Determine equations for the inductance in henries per mile for each conductor.

7. A single-phase circuit is built with three conductors a, b, and c. Conductors b and c are connected in parallel to form the return circuit. Let $r_a = r_b = r = $ GMR of conductors a and b, while $r_c = $ GMR of conductor c. The separations of the conductors are D_{ab}, D_{bc} and D_{ca}. Determine the ratio of current division between conductors b and c, and then determine the inductance of the circuit in henries per mile.

8. Two parallel three-phase lines have the six conductors located at the corners of a hexagon, as indicated in Fig. 240. Determine the equation for the inductance of each conductor.
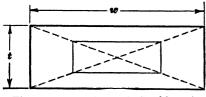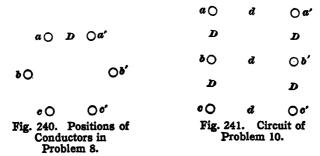
Fig. 239.   Conductor in Problem 5.

9. Solve Problem 8 with the conductors a' and c' interchanged, thus putting corresponding phase conductors at diametrically opposite corners of the hexagon.

10. Determine the inductance per conductor in the parallel three-phase circuit shown in Fig. 241.

Fig. 240.   Positions of Conductors in Problem 8.

Fig. 241.   Circuit of Problem 10.

11. Solve Problem 10 with the conductors a and c interchanged.

12. Consider four equally spaced conductors a, b, c, and d, Fig. 242. Conductors a and b are in parallel, and conductors c and d are in parallel. All conductors are solid wires for which GMR $= \rho$. Determine the ratio of current division between parallel conductors and also the impedance of the circuit. (See "Electric Power Transmission and Distribution" by Woodruff.)

Fig. 242.   Conductors of Problem 12.

13. Check several of the values of inductive reactance in the table of Appendix VI.

14. Check values of the inductive reactance given in Chapter 1 for the Boulder Dam-Los Angeles line.

15. A three-phase, 60-cycle, 220,000-volt line with horizontally spaced conductors 32 feet apart is carrying a balanced load of 100,000 Kw at unity power factor. Neglect the capacity current. As indicated in Fig. 243, two conductors, 6 inches apart, are located in either position (a) or position (b). Determine the resultant flux produced by the power line and linking with one mile of the parallel circuit in each of the positions shown. What voltage is set up per mile in this circuit?
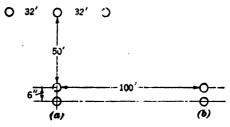
Fig. 243. Conductors of Problem 15.

16. A coil is made up of six turns, arranged as in Fig. 8. If 60% of the total flux links with all the conductors, 80% links with four turns, while 100% links with only the two center turns, compute the inductance ratio, that is, the ratio of the actual coil inductance to the maximum inductance for a concentrated winding.

17. In the case of a three-phase line with horizontally spaced conductors a, b, and c, the conductors are arranged so that distance D_{ab} = distance $D_{bc} = D$ and distance $D_{ac} = 2D$. Currents are as follows: $I_a = I\underline{/120}$, $I_b = I\underline{/0}$, $I_c = I\underline{/120}$. Determine: (a) an equation for the total flux linkages about each conductor; (b) an expression for the total inductance of each conductor; (c) the voltage drop for each conductor, the resistance being neglected.

18. Sketch a vector diagram showing the three voltage drops and the three currents in Problem 17. (a) Determine an expression for each power, in watts, obtained from the product of the corresponding voltage drop, current, and cosine of the angle between the voltage and the current. (b) Explain these results.

19. Repeat Problem 17 for the reverse sequence of currents.

20. Repeat Problem 18 for the reverse sequence of currents.

21. The Boulder Dam-Los Angeles line operates with balanced, positive-sequence line voltages of 278,000 volts at the generator end of the line. The conductors have a diameter of 1.4 inches. For the purpose of this problem assume that these conductors are solid round wires, although they actually are of the segmental hollow type (see Art. 2). Conductors are horizontally spaced 32½ feet apart, the phase-a conductor being at the left and the phase-c conductor at the right. Resistance of conductors is

0.0214 ohm per 1,000 feet at 25° C and frequency of 60 cycles per second. Neglect the capacitance of the line.

Assume a balanced load of 150,000 Kva at unity power factor at the sending end of this line.   Use the following current notation: $I_a = I\underline{/120}$, $I_b = I\underline{/0}$, $I_c = I\underline{/240}$.

Calculate the resistance and inductance voltage drops and receiver line voltages for the following two cases:

(a) Conductors fully transposed.

(b) Conductors not transposed.

What is the nature of the load at the receiver end for these two cases?

22.   Repeat Problem 21 for the case of balanced, positive-sequence line voltages of 275,000 volts at the receiver end and a balanced load at the receiver end of 150,000 Kva at unity power factor, solving for the same quantities applied to the sending end.

CHAPTER 4

1.   A single-phase line is built with conductor a as one side of the circuit and with conductors b and c operating in parallel as the other or return side of circuit.   Considering the geometric mean radii as r_a, r_b, and r_c and the distances between conductors as D_{ab}, D_{bc}, and D_{ca}, derive an equation for the total single-phase inductance per centimeter of line, by the use of equivalent GMR and GMD, according to the theory of Chapter 4.   What limitations are involved in such a solution?

NOTE: Uniform current density throughout the conductor was assumed in Art. 21.

2.   The terminal connections of a three-phase generator are made with six conductors located as in Fig. 244.   Applying the theory of GMR and GMD equivalents, derive an equation for the inductance of each phase. All conductors have the same GMR.   Compare the result with the answer for Problem 10, Chapter 3.

a ◯ d ◯ a'

D D

b ◯ d ◯ b'

D D

c ◯ d ◯ c'
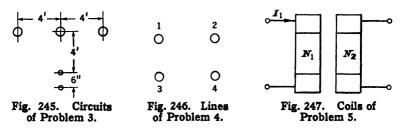
Fig. 244.   Conductors
of Problem 2.

3.   The Boulder Dam-Los Angeles line uses a hollow copper conductor with an outside diameter of 1.4 inches and an approximate wall thickness of 100 mils.   Assuming perfectly smooth outside and inside surfaces, apply the method of Appendix II and determine the conductor GMR. Check this result with the value in Table 3.

4. Assuming that the center strand of a standard seven-strand cable is left out, derive the equation for the resultant geometric mean radius as done in Art. 24.

5. Check the value in Table 3 for a 19-strand cable.

6. A single-phase line is composed of four conductors located at the corners of a perfect square. Each side of the single-phase circuit is composed of two conductors on one side of the square. Because of the symmetry of the arrangement, the current in each conductor may be assumed to be equal to one-half of the total current. Using the basic law of flux linkages of Chapter 3, derive equations for the inductance per conductor, the inductance per pair of parallel conductors, and the circuit inductance. Also derive an equation for the circuit inductance by the law of GMR's.

CHAPTER 5

1. A current of 50 amperes, at 60 cycles, flows through conductor a, the return conductor being so far away that its effect can be neglected.

Fig. 245. Circuits of Problem 3.

Fig. 246. Lines of Problem 4.

Fig. 247. Coils of Problem 5.

A second circuit composed of conductors y and y' is installed so that these conductors are 3 feet and 3.5 feet from conductor a. Determine the voltage induced in the second circuit per mile of circuit.

2. Suppose the positions of y and y' in Problem 1 are interchanged at every mile. What will be the resulting effect of conductor a?

3. A three-phase circuit with horizontal spacing, as in Fig. 245, carries currents as follows:

$$I_a = 100\underline{/0}, \ I_b = 100\underline{/120}, \text{ and } I_c = 100\underline{/240}$$

Located directly below the center conductor is a telephone circuit of two wires. Determine the resultant induced voltage in the telephone circuit per mile of circuit for a frequency of 60 cycles per second.

4. Two single-phase power lines are installed on the same poles, as indicated in Fig. 246. Conductors 1 and 2 belong to one line and conductors 3 and 4 belong to the second line. Derive equations for the mutual inductance of each circuit upon the second circuit. To be perfectly general, assume that all conductor spacings are different.

5. Two coils of turns N_1 and N_2, Fig. 247, are arranged so that their axes coincide. Coil 1 carries a current of I_1 amperes. (a) Write equa-

tions for the induced emf in both coils in terms of the time rate of change of flux and also in terms of the self and mutual inductances. (*b*) Repeat (*a*) for the case of coil 2 carrying current I_2. In both (*a*) and (*b*) neglect the effect of partial linkages and assume that the mutual flux is equal to K times the total flux of each coil, where $K =$ coefficient of coupling. (*c*) Assume now that both coils are connected in series and carry the same current I. Write equations for the total induced voltages for the conditions of additive and subtractive coupling and derive equations for total inductances (see Case 1, Art 91).

CHAPTER 6

1. Derive an equation for the capacity of a single conductor concentric lead-covered cable having a single homogeneous dielectric of constant k.

2. Repeat Problem 1 for the case in which three types of dielectrics with constants k_1, k_2, and k_3 are used. The dielectrics are assumed to be applied in concentric layers.

3. Check several values of capacitive susceptance given in the table of Appendix VI.

4. Check the capacitive susceptance given in Chapter 1 for the Boulder Dam-Los Angeles line.

5. Three conductors 1.4 inches in diameter are located 32 feet apart horizontally and 70 feet above the surface of the ground. Assuming transposition of conductors and balanced three-phase voltages, determine general equations for the capacity per phase; first neglect the effect of the earth, and then take the earth into account. What is the earth correction factor? From these equations determine the capacities in farads per mile for the line with and without the effect of the earth.

6. Calculate the charging current per conductor for the Boulder Dam-Los Angeles line, using data in Chapter 1. At a line voltage of 287,000 volts, how many Kva of charging capacity are required for one three-phase line?

7. Equations 66, 67, and 68 are all applicable for the computation of capacity between two wires. Compare the results of these three equations for two wires that are 5 feet apart and 0.5 inch in diameter.

8. A double-circuit tower has its six conductors disposed at the corners of a hexagon, each side of which has a length of D centimeters. The two lines operate in parallel. The two top wires form one phase, the two middle wires form the second phase, and the two bottom wires form the third phase. The conductors of each circuit are transposed, but there is no transposition between opposite sides of the tower. Derive the equation for the capacitance per conductor to neutral. The current per conductor is equal to one-half the total phase current.

9.   Repeat Problem 8 for the case in which corresponding phase wires are diametrically opposite.

10.   Plot a diagram similar to Fig. 22 for the case of two conductors having a radius equal to 1 centimeter and spacing equal to 10 centimeters. The voltage between conductors is 1,000 volts. Equipotential line at 100 volts difference should be drawn.

CHAPTER 7

1.   A three-phase, 60-cycle, 275,000-volt transmission line has the following data:

$$\text{Resistance per mile} \ldots \ldots \ldots \ldots = 0.1125 \text{ ohm};$$
$$\text{Capacity per mile} \ldots \ldots \ldots \ldots \ldots = 0.0136 \text{ microfarad};$$
$$\text{Inductance per mile} \ldots \ldots \ldots \ldots = 0.00212 \text{ henry}.$$

Determine: (a) The constant m (see equation 94); and (b) the characteristic or surge impedance Z_0.

2.   By substituting the identities of equation 110 into equation 106, derive equations for E and I in terms of the exponentials $\epsilon^u \epsilon^{jv}$ and $\epsilon^{-u} \epsilon^{-jv}$ and explain the significance of these results.

3.   Modify the equations obtained in Problem 2, for the special case when $E_r = Z_0 I_r$.   Explain the significance of these results.

4.   Tabulate the values of $\epsilon^u \epsilon^{jv}$ and $\epsilon^{-u} \epsilon^{-jv}$ for the line of Problem 1 and the lengths of Problems 2 and 3, and plot a family of vectors showing the initial, reflected, and resultant components of voltage and current as the length of line is increased.

5.   Determine the generalized constants A, B, C, and D for lengths of 100, 200, 300, 400, and 500 miles for the data of Problem 1.

6.   Determine the equivalent π and T circuits for the line of Problem 1 for lengths of 100, 200, 300, 400, and 500 miles.

7.   For the data of Problem 1 and a receiver voltage of 275,000 volts across lines and a receiver power of 100,000 Kw at 90% lagging power factor, determine the sending voltages for lengths of 100, 200, 300, 400, and 500 miles.

8.   From equation 106 derive formulas for the receiver voltage and current in terms of the sending voltage and current.

9.   For a receiver current equal to zero, determine equations for the receiver voltage and sending current in terms of the sending voltage. From these equations determine, for the line of Problem 1, the receiver voltage, sending current, and sending Kva required to maintain a sending voltage of 275,000 volts across lines for lengths of 100, 200, 300, 400, and 500 miles.

10.   With a line voltage of 110,000 volts applied to line number 8 of Appendix VII, determine the receiver voltage for the open-circuited condition.

11.   What would be the sending voltage for line number 8 of Appendix VII, in order to maintain 110,000 volts at the receiver at no load?

12.   Repeat Problem 11 for 10,000 lagging reactive Kva at the receiver end operating at 110,000 volts.

13.   A three-phase line has the following data:

> Resistance per phase per mile..... = 0.113 ohm;
> Inductance per phase per mile.... = 0.00212 henry;
> Capacity per phase per mile...... = 0.0136 × 10⁻⁶ farad;
> Frequency..................... = 60 cycles per second.

Calculate the sending voltage, sending power, sending power factor, and efficiency for a load of 100,000 Kw at 275,000 line volts at the receiver, for power factors of 80% lagging, unity, and 80% leading.   Plot these results against the receiver power factor.   Determine also the receiver voltage and charging Kva for each of the particular sending voltages just determined.   (Compare these results with those of Problems 2, 3, and 4 of Chapter 2.)

14.   In Art. 40 the following equations are given for a transmission line?

$$E_s = \frac{E_r}{2}(\epsilon^u \epsilon^{jv} + \epsilon^{-u}\epsilon^{-jv}) + \frac{I_r Z_0}{2}(\epsilon^u \epsilon^{jv} - \epsilon^{-u}\epsilon^{-jv})$$

$$I_s = \frac{I_r}{2}(\epsilon^u \epsilon^{jv} + \epsilon^{-u}\epsilon^{-jv}) + \frac{E_r Y_0}{2}(\epsilon^u \epsilon^{jv} - \epsilon^{-u}\epsilon^{-jv})$$

Assuming a line of zero attenutation (that is, $u = 0$) and a line of quarter wave length (that is, $v = \frac{\pi}{2}$), derive equations for the receiver current and voltage in terms of sending quantities, and explain the results.   Notice that zero attenuation is obtained if the series resistance and shunt conductance are equal to zero, so that $ms = j(\sqrt{xb})s = jv$ and $Z_0 = \sqrt{\frac{x}{b}}$. Determine also the equation for s in terms of the line inductance and capacity.   What length in miles will this correspond to for a 60-cycle line?

15.   Write equations for the voltage and current of a line for the special case of line loaded with an impedance which is equal to the surge impedance of the line.   Derive equations for the power and reactive volt-ampere efficiency of such a line.

CHAPTER 8

1.   A transmission line with general circuit constants A, B, C, and D is connected to a receiver transformer of series impedance Z_{TR} (shunt admittance omitted).   Determine the generalized constants for the combination.

2. Solve Problem 1 for the case of a line in series with a sending-end transformer of impedance Z_{TS} (neglect shunt admittance).

3. Solve Problem 1 for the case of a line with sending and receiver transformer series impedance.

4. Two transmission lines with constants A_1, B_1, C_1, D_1 and A_2, B_2, C_2, D_2 are connected in series with a balanced load of impedance Z_M tapped off at the junction point of the two lines. Determine the general constants for the combination.

5. Three transmission lines are connected in series, line 1 being at the sending end and line 3 at the receiver end. Determine general constants for the combination.

6. Two lines are connected in parallel. Determine general constants for the combination.

7. Determine enough data to plot curves for P_r, P_s, Q_r, and Q_s (see equations 146 and 148) against torque angle for line number 7 in Appendix VII when $E_s = E_r = 115,000$ line volts.

8. Repeat Problem 7 for line number 1 for the following line voltages: $E_s = 278,000$ and $E_r = 275,000$ volts.

9. Determine the constants A, B, C, and D for one of the lines of Appendix VII.

10. Determine the Thevenin equivalent simple series line for one of the lines of Appendix VII.

11. The following test data were obtained on a laboratory model transmission line at the University of Tennessee:

Sending end impedance, receiver open .... $= 18.48 \underline{/84.8}$
Sending end impedance, receiver short-circuited.... $= 13.67 \underline{/77.1}$
Receiver end impedance, sending end open .... $= 18.65 \underline{/85.2}$

Determine the generalized constants A, B, C, and D.

12. From the results of Problem 11, obtain data and plot curves of P_r, Q_r, P_s, and Q_s for $E_s = E_r = 127$ volts (see equations 146 and 148).

13. A four-terminal network is set up in the laboratory. In such a case both the sending and receiver terminals are accessible. Devise a method for obtaining the constants A, B, C, and D from the two sending-end impedances, namely, Z_{so} and Z_{sc} (see Art. 54).

14. For the circuit in Fig. 39 of Chapter 8, assume the following data:

$$Z_1 = 1.25 + j5.5, \quad Z_2 = 1 + j5.0 \quad \text{and} \quad Y_0 = \frac{1}{Z_0} = 0.05 - j0.08$$

Determine the constants A, B, C, and D for this circuit. If the receiver voltage and current are given as $E_r = 2,300 + j0$ and $I_r = 80 - j60$, determine the sending voltage and current: (a) by using generalized circuit equations; and (b) by direct solution of the T circuit.

15. The impedance and admittances of Fig. 40, Chapter 8, are as follows:

$$Z = 1.25 + j2.5 \text{ and } Y_s = Y_r = 0.02 - j0.08$$

Determine the generalized constants A, B, C, and D. If the receiver voltage and current are $E_r = 2,300 + j0$ and $I_r = 100 - j50$, determine the sending voltage and current: (a) by using the generalized circuit equations; and (b) by direct solution of the π circuit.

16. Derive equations for the generalized constants A, B, C, and D for the simple series circuit in Fig. 38. See equations 128, Art. 45.

17. Derive equations for the generalized constants A, B, C, and D for the T circuit in Fig. 39. See equations 129, Art. 46.

18. Derive equations for the generalized constants A, B, C, and D for the π circuit in Fig. 40. See equations 130, Art. 47.

(a) Synchronous Motor Equivalent   (b) Alternator Equivalent

(c)
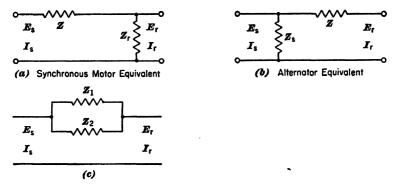
Fig. 248.   Circuits for Problem 19.

19. Derive equations for the generalized constants A, B, C, and D for the circuits shown in Fig. 248. For each of the three circuits, verify the statement that $N = AD - BC = 1$ (see equation 126).

20. Verify that $N = AD - BC = 1$ for each circuit of Arts. 45 to 48, inclusive.

21. A transmission line is defined by the following constants (see equations 131):

$$A = \cosh ms, \quad B = Z_0 \sinh ms,$$
$$C = Y_0 \sinh ms, \text{ and } D = A$$

Determine the equivalent T circuit for this line.

22. Solve problem 21 for an equivalent π circuit.

23. Verify that $N = AD - BC = 1$ for the case of tap-changing transformers and phase-shifting units (see Art. 51).

CHAPTER 9

1. Draw circle diagrams for the line of the example of Art. 57, but assume a sending line voltage of 125 Kv and a receiver line voltage of 115 Kv. From these circles determine the synchronous condenser capacity, torque angle, sending power, sending power factor, and efficiency as functions of the receiver power. What is the static power limit of this line?

2. Using the line data of the example of Art. 57, set up circle diagrams, according to the method outlined in Arts. 58 and 59, for voltage ratios of 1.1, 1.0, and 0.9. Obtain all pertinent data for a receiver line voltage of 115,000 volts.

3. Solve Problem 2 by the method outlined in Art. 60 for a receiver line voltage of 115,000 volts.

4. Start with the following generalized equations:

$$E_s = A E_r + B I_r$$
and
$$I_s = C E_r + D I_r$$

Also, utilize the following relation for the receiver power and reactive volt-amperes:

$$P_r - jQ_r = \bar{E}_r I_r$$

where $\bar{E}_r$ = conjugate receiver voltage.

Solve for $P_r - jQ_r$. Then, taking the term for receiver power, determine the maximum power limit and the corresponding torque angle by obtaining a solution of the derivative of P_r with respect to the torque angle ϕ. Check this result with that of Art. 61.

5. A circuit with constants A, B, C, and D is loaded on an impedance Z so that $E_r = I_r Z$. Obtain an equation for I_s in terms of E_s, the constants A, B, C, and D, and the impedance Z; then show that the resultant equation represents a circle diagram for constant voltage E_s.

6. A simple line of impedance $Z = R + jX$ is loaded with a variable resistance r. With constant voltage applied at the sending end, determine conditions for maximum power delivered to the receiver end.

NOTE: Set up the expression for the receiver power in terms of the variable load resistance, and then set the derivative of the receiver power with respect to the load resistance equal to zero.

7. Repeat Problem 6 for the case of a load of variable impedance Z_L of variable power factor.

8. Solve Problem 6 when the line is defined by the constants A, B, C, and D.

9. Solve Problem 7 when the line is defined by the constants A, B, C, and D.

10. In power transmission, the torque angle must not exceed the angle β (see Art. 61); whereas, in the field of electrical communication,

there seems to be no such limitation of the electrical length of a line. Why is this true?

11. Determine the Thevenin equivalent simple series circuit for the line of Art. 59.

12. Determine the necessary data and plot loss and efficiency circles for the line of Art. 57.

13. Plot circle diagrams for line number 1 of Appendix VII for $E_s = 287,000$ and $E_r = 275,000$ line volts. From this chart, obtain data and draw curves of sending power, sending power factor, receiver power factor, efficiency, reactive Kva at sending end, reactive Kva at receiver end, and torque angle, all these quantities being plotted against receiver power.

14. The line of Problem 13 is supplied at the generator end with a bank of single-phase transformers with 120,000 Kva total capacity and 10% impedance and 1% resistance (based on 120,000 Kva). The generator voltage is 18,000 volts across lines. The normal high-tension voltage of the transformer is 275,000 volts. The transformers are equipped with taps in steps of 5 and 10 per cent, plus and minus. Construct a family of circle diagrams for the composite system for each of the foregoing transformer ratios, the generator voltage being held constant at 18,000 and the receiver voltage being held constant at 275,000 volts. From this chart indicate changes in operation obtained by tap changing.

15. The system of Problem 14 is also supplied with a phase shifter between the generator and the sending-end transformer. Assume no additional impedance. The phase shifter gives plus and minus 10-degree shifts in voltage. For normal voltage taps, develop circle diagrams for ± 10 and ± 5 degree phase shifts for constant generator and receiver voltages. Contrast this chart with the results of Problem 14 and discuss the feasibility of phase shifters.

16. A four-terminal network is loaded with a variable resistance R. Using the two generalized equations and noting that $E_r = I_r R$ and $AD - BC = 1$, determine the equation for the looking-in impedance and show that such an equation represents a circle envelope, in which resistance appears along the x-axis and reactances appear along the y-axis.

17. Solve Problem 16 for the case of a simple series circuit of constant inductive reactance and variable resistance (see Fig. 49).

18. Develop a circle diagram of current similar to Fig. 50 for a series circuit of constant capacity reactance and variable resistance. Include a circle showing the individual voltage drops.

19. Solve Problem 18 for a series circuit of constant resistance and variable reactance, both inductive and capacitative.

20. In Art. 56 the following equation based on the receiver circle of Fig. 51 (a) is given:

$$E_s = A E_r[\cos (\beta - \alpha) - j \sin (\beta - \alpha)] + B I_r[\cos \theta_r - j \sin \theta_r]$$

Using this equation, prove the following (which is another form for the receiver circle equation):

$$(P_r + lE_r^2)^2 + (Q_r - mE_r^2)^2 = \frac{E_s^2 E_r^2}{B^2}$$

where P_r = receiver power;

Q_r = receiver reactive volt-amperes;

$$l = \frac{A}{B} \cos (\beta - \alpha) \text{ and } m = \frac{A}{B} \sin (\beta - \alpha).$$

21. Using the result of Problem 20, determine the equation of the envelope of all possible receiver circles for variable receiver voltage and constant sending voltage.

$$Answer. \quad (mP_r + lQ_r)^2 + \frac{E_s^2}{B^2}(lP_r - mQ_r) = \frac{E_s^4}{4B^4}$$

Note: This can be accomplished by differentiating the result of Problem 20 with respect to E_r. From the derivative, an expression for E_r may be obtained, which should then be substituted in the result of Problem 20.

22. A transmission system has the following constants:

$$A = 0.917 + j0.017$$
$$B = 30.6 + j151.05$$

For a value of $E_s = 89,000$ volts determine the necessary data and plot the receiver circle envelope to the scale of P_r and Q_r.

23. If the receiver power factor angle is θ_r, the value of which is such that $Q_r = -P_r \tan \theta_r$, obtain the equation of the receiver envelope in terms of the receiver power factor.

$$Answer: \quad P_r = \frac{E_s^2 \left[-(l + m \tan \theta_r) \pm \dfrac{A}{B \cos \theta_r} \right]}{2B^2(m - l \tan \theta_r)^2}$$

24. Using the data of Problem 22 and taking E_s as 89,000 volts, determine the data and plot receiver power against power factor angle from the result of Problem 23.

25. Using the result of Problem 20, determine an equation of the envelope of all possible receiver circles for variable receiver voltage and constant sending voltage, expressing the resultant receiver power in terms of E_s and E_r and eliminating Q_r.

$$Answer: \quad P_r = -lE_r^2 + \frac{lE_s^2 \pm m\sqrt{4A^2E_s^2E_r^2 - E_s^4}}{2A^2}$$

Note: This can be accomplished by differentiating the result of Problem 20 with respect to E_r. From the derivative an expression of Q_r may be obtained, which should then be substituted in the result of Problem 20.

26. Using the data of Problem 22 and taking E_s as 89,000 volts, determine data for receiver power-voltage curves, based on the result of

Problem 25. Repeat the solution for $E_s = 80$ per cent, 90 percent, 110 per cent, and 120 per cent, assuming $E_s = 89,000$ volts to be 100 per cent. Plot these data as curves, using voltages as the ordinates and powers as the abscissas.

27. Obtain an equation similar to the result of Problem 20, but for sending power P_s and sending reactive volt-amperes Q_s. See Fig. 52 (*b*).

28. Repeat Problem 21, but for the sending circles.

29. Repeat Problem 23, but for the sending circles.

30. Repeat Problem 25, but for the sending circles.

CHAPTER 10

1. The three line voltages of a three-phase system have been obtained by voltmeter readings as 2,100 volts between conductors *a* and *b*, 2,300 volts between conductors *b* and *c*, and 2,500 volts between conductors *c* and *a*. Positive sequence is assumed. Determine the sequence components, and check the results vectorially.

2. A three-phase system is delivering power to a single-phase load. In this case, $I_a = I$ and $I_b = -I$. Determine the sequence components of current.

3. A three-phase balanced set of positive sequence voltages is applied to a circuit, the line impedances being $Z = 0.1 + j0.5$. The supply voltages, neutral to lines, are: $E_{0a} = 6,350/0$, $E_{0b} = 6,350/120$, and $E_{0c} = 6,350/120$. The currents are: $I_a = 1,000/0$, $I_b = 1,200/120$, and $I_c = 1,500/270$. Determine: (*a*) the receiver voltages, using the given currents in computing voltage drops; (*b*) the sequence component currents; and (*c*) the sequence component voltage drops and the corresponding receiver voltages.

4. A three-wire single-phase system has the following voltages: $E_{ab} = 110/0$, $E_{bc} = 110/0$, and $E_{ca} = 220/180$. Determine sequence component voltages.

5. A three-wire single-phase system may be considered as a special type of three-phase system with phase voltages as follows: $E_{0a} = 110/180$, $E_{0b} = 0$, and $E_{0c} = 110/0$. Determine the sequence component phase voltages.

6. The phase *a* of a three-phase circuit is open, so that $I_b = -I_c$. Determine the sequence component current.

7. A balanced delta-connected distribution system is grounded at one corner of the delta. Determine the equivalent sequence phase voltages.

8. The currents in a three-wire system are: $I_a = 100/0$ and $I_b = I_c = 50/180$. Determine the sequence currents.

9. In a laboratory test, three voltage sources are connected in series as indicated in Fig. 60 of Chapter 10. The sequence voltages of phase *a* are: $E_{a0} = 100/0$, $E_{ap} = 100/30$, and $E_{an} = 100/0$. Determine the resultant phase

voltages and line voltages.   From these line voltages, calculate the sequence phase and line voltages.

CHAPTER 11

1.   The saturation curve of an iron-cored reactor is given by the following data:

I in amperes....	5	10	20	34	55	85
ϕ in lines........	8,000	12,000	14,000	15,000	16,000	19,000

Neglecting hysteresis, determine and plot values of inductance against current.   The reactor may be assumed to have 500 turns.   What are the maximum, minimum, and average values of the inductance?

2.   (a) Plot the no-load characteristics and the zero per cent lagging power factor full-load characteristics for the three-phase alternators listed in the accompanying table.   Then determine values of adjusted direct synchronous reactance.   Plot the results against field current and also against induced emf (induced emf = terminal voltage plus leakage reactance drop).   (b) Determine values of saturated direct synchronous reactance for each machine listed in the table.   Plot the results against induced emf.   Compare values of saturated and adjusted synchronous reactances.   (c) Specify the positive-sequence reactance for each machine.

NOTE: The Potier triangle must be used in plotting zero per cent lagging power factor characteristics.

THREE-PHASE ALTERNATOR DATA

Machine	A		B		C		D	
Type	Oil Engine		Steam Engine		Vert. Water Wheel		Vert. Water Wheel	
Capacity, Kva	500		938		750		16,000	
Terminal line volts	2,300		2,300		600		11,000	
Speed, rpm	225		180		200		133.3	
Frequency, cycles/sec	60		60		60		60	
Pole arc/pole pitch	0.62		0.614		0.648		0.71	
	I_f	E_p	I_f	E_p	I_f	E_p	I_f	E_p
	23	867	33	810	34	251	100	3,700
	29	1,040	60	1,270	51	318	150	5,200
No-load Sat.	38	1,270	80	1,500	64	376	250	7,050
I_f = field current	50	1,500	125	1,730	87	433	356	8,060
E_p = phase voltage	71.5	1,730	163	1,850	132	490	500	8,950
	90.0	1,850	220	1,960	164	520	600	9,400
	118	1,960						
Zero % F. L.								
Lagging	35	0	51	0	57	0	190	0
P. F. Sat.	103	1,500	164	1,500	147	376	550	6,350

3. The test data in the accompanying tabulation were obtained on a 11-Kva, 220-volt, 1,200-rpm, 60-cycle, salient pole alternator. Determine values of adjusted synchronous reactance (X_d), and plot the results against field current. What is the unsaturated synchronous reactance? Plot the same results against induced voltage.

SALIENT POLE ALTERNATOR DATA

NO-LOAD SATURATION CHARACTERISTIC							
Field current	1.2	2.0	3.0	4.0	5.2	7.2	9.0	10.75
Phase Voltage	39.0	65.5	95.0	120	140	163	177	186

ZERO PER CENT LAGGING POWER FACTOR FULL-LOAD (30 AMPERES) SATURATION CHARACTERISTIC							
Field Current	3.3	4.0	5.0	6.0	7.0	8.0	9.0	10.0
Phase Voltage	0	21.0	52.0	80	100	114	124	132

4. Using the data of Problem 3, determine the Potier triangle, and obtain the leakage reactance and armature reaction. Determine values of saturated synchronous reactance (X_d), and plot the results against induced voltage.

5. The generator of Problem 3 was operated at synchronous speed, while the stator was connected to a three-phase source of power. The field current was left open-circuited. In this manner a synchronously revolving mmf was provided, which rotated in the same direction as the field structure. The alternator was equipped with suitable means of varying the relative angular position between the revolving armature mmf and the field poles. Input power to the armature was measured by means of a voltmeter, ammeter, and two wattmeters. The following data were obtained for different relative angles between the revolving mmf and the field poles:

Angle, elec. deg.	0	10	20	30	40	50	60	70	80	90
Phase voltage	68	68	68	68	68	68	68	68	68	68
Current, amperes	19.0	20.0	21.1	22.9	24.8	26.7	28.3	29.6	30.3	30.4
Watts	514	820	1,110	1,300	1,400	1,360	1,180	960	634	220

Determine values of synchronous impedance, and plot values of synchronous reactance against relative angular position. Specify the values of direct and quadrature synchronous reactances. How do these values

of direct synchronous reactance compare with the results obtained in Problems 1 and 2?

6. The alternator of Problem 3 was tested for subtransient reactance according to the method outlined in Art. 80. The test data are as follows:

Angle, elec. deg.	0	10	20	30	40	50	60	70	80	90
Current, amperes	30	30	30	30	30	30	30	30	30	30
Phase voltage	14.1	14.2	14.3	14.45	15.4	16.6	17.45	18.25	19.0	19.0
Watts	204	208	212	228	258	300	336	364	390	395

Determine the corresponding values of subtransient reactances and plot the results against angular displacement. What are the direct and quadrature subtransient reactances? From these values determine the negative-sequence reactance

7. The alternator of Problem 3 was tested according to the method of Art. 85 for negative-sequence impedance. The following data were obtained:

Field Current	1.6	1.9	2.1	2.35	2.4	2.7	2.9
Voltage	23.7	28.0	31.1	33.8	36.3	40.8	44.1
Current	22.0	25.7	28.6	31.0	33.0	37.2	40.0
Watts	440	610	754	890	1020	1295	1500

Determine values of negative-sequence impedance.

8. The alternator of Problem 3 was tested according to the method of Art. 86 for zero-sequence impedance. The following data were obtained:

Field Current	1.6	2.0	2.2	2.4	2.7	3.2	3.6	4.0
Voltage	9.3	12.0	12.7	14.2	15.8	19.0	21.6	23.8
Current	29.7	37.9	40.4	45.0	50.0	61.0	68.6	75.6
Watts	180	290	330	418	520	745	962	1,180

Determine values of zero-sequence reactance.

CHAPTER 12

1. Determine the transformer T circuit for t ansformer .... from the data in the following table: (a) based on the high-voltage side; (b) based on the low-voltage side.
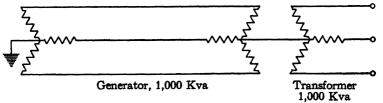
2. Determine the positive- and negative-sequence impedance for transformer .... from the table. Specify the impedance to the scale of the high-tension and low-tension windings.

3. Determine the zero-sequence impedance based on the high-tension and low-tension windings for transformer .... for each of the following three-phase connections:

(a) Y–Y, both neutrals grounded;
(b) Y–Y, primary neutral grounded, secondary neutral open;
(c) Y–Y, both neutrals ungrounded;
(d) Y–delta, neutral grounded;
(e) Y–delta, neutral ungrounded;
(f) Delta–delta.

For each of these cases indicate how the transformer impedance is introduced into the system network.

TRANSFORMER DATA

Trans-former	Kva	Voltage		No-Load Losses Watts	Per Cent Exc. Current	Load Copper Losses Watts	Per Cent Series Imped-ance
		H. V.	L. V.				
A	667	2,300	240	2,500	7.0	7,500	5
B	1,000	13,200	600	3,500	6.0	10,500	6
C	2,000	22,000	2,300	6,300	5.5	18,900	6
D	5,000	38,200	12,000	14,300	5.5	42,900	8
E	10,000	76,400	11,500	27,000	5.0	81,000	9
F	25,000	127,000	13,800	70,000	5.0	182,000	15

4. A generator is connected to a load through a Y–Y bank of transformers with interconnected neutrals, as indicated in Fig. 249. The

Generator, 1,000 Kva   Transformer
                              1,000 Kva

Fig. 249. Connections for Problem 4.

primary line voltage is 13,200 volts, while the secondary line voltage is 2,300 volts. The generator zero-sequence reactance is 10 per cent. The transformer exciting current may be taken as 7 per cent on either the high-or low-voltage windings. The transformer coefficient of coupling may be

taken as 90 per cent. Neglecting all transformer resistances, determine the zero-sequence impedance of the circuit for both additive and subtractive polarity transformers.

5. Solve Problem 4 with a resistance of 10 ohms inserted in the generator neutral.

6. A three-phase transformer of the shell type has the following rating: 10,000 Kva, 11,500 to 132,000 line voltage, no-load loss per phase = 27,000 watts, per cent exciting current = 5, load copper losses per phase = 81,000 watts, and per cent series impedance per phase = 9. The low-voltage windings are delta-connected while the high-voltage windings are Y-connected with neutral grounded. Determine the positive-, negative-, and zero-sequence impedances and indicate the sequence circuits.

7. Repeat Problem 6 for a core-type three-phase transformer having the same detail rating.

CHAPTER 13

1. The zero-sequence self reactance of a three-phase transmission line is:

$$X_0 = 0.01397f \log \frac{D_e^2}{r} \text{ (see equation 208)}$$

The mutual reactance to the ground wire is:

$$X_M = 0.01397f \log \frac{D_e^2}{d} \text{ (see equation 209)}$$

Determine the equation for the coefficient of coupling between the power conductors and the ground wire.

2. Two coils have self inductances of 0.4 henry and 0.2 henry and mutual inductance of 0.15 henry. Determine their coefficient of coupling.

3. Two coils have the following self impedances:

$$Z_a = 2 + j15 \text{ and } Z_b = 2 + j20$$

The mutual impedance is:

$$Z_m = 1.5 + j10.0$$

Determine the equivalent T circuit and the resultant looking-in impedance with the winding b short-circuited.

4. The coupled circuit of Problem 3 has a load applied across the terminals of the coil b. The voltage and current delivered by the coil b are:

$$E_b = 50 + j0 \text{ and } I_b = 10 - j10$$

Determine the applied voltage E_a and the current I_a.

5. Three conductors are located in a horizontal arrangement with a spacing of 20 feet. A single ground wire is located 15 feet directly above

the middle power conductor.   All power and ground wires are of the same type with GMR =0.03 foot.   The conductor resistance =0.235 ohm per mile.   The frequency is 25 cycles per second.   The line crosses a desert with $\rho = 500$.   Determine the zero-sequence impedance per mile.

6.   Repeat Problem 5 for $\rho = 10$.

7.   Repeat Problem 5 for $\rho = 100$.

8.   Repeat Problem 5 for 60 cycles per second.

9.   Repeat Problem 6 for 60 cycles per second.

10.   Repeat Problem 7 for 60 cycles per second.

11.   Three horizontally spaced power conductors of GMR =0.03 foot have a spacing of 20 feet.   Two ground wires are located 15 feet above and 15 feet to each side of the two outside conductors.   The ground wires are of the same material and size as the power conductors.   Resistance per mile =0.235 ohm.   If $\rho = 100$ and $f = 60$ cycles per second, determine the zero-sequence impedance.

12.   Derive an equation for the zero-sequence impedance of a parallel two-circuit three-phase line without ground wires by the method outlined in Art. 98.

13.   Repeat Problem 12 for the case of one ground wire.

14.   By applying the basic laws given in Art. 98, develop expressions for the ratio of the ground-wire current to the total fault current, for single-circuit and double-circuit three-phase lines with one ground wire and also with two ground wires.   (See *Standard Handbook for Electrical Engineers*, eighth edition, page 1189.)

15.   Prove that the result given in Art. 97 for the zero-sequence impedance of a three-phase line without ground wires is identical to equation 208.

16.   Prove that the results obtained in Art. 98 for each example are identical to the solutions of these same examples in Art. 97.

17.   One section of the Boulder Dam-Los Angeles double-circuit line has its conductors located on two towers with horizontal disposition of conductors.   Each tower is equipped with two ground wires.   Assuming that the two circuits are close enough to each other to involve mutual coupling effects between circuits, write the necessary voltage equations, as outlined in Art. 98, for the determination of the zero-sequence imped-ance of the parallel combination.

CHAPTER 14

1. A three-phase line has its conductors located in a horizontal plane 60 feet above the ground, the two outer conductors being 20 feet from the center conductor. Conductors are 397,500 cir. mil ACSR, with a GMR of 0.0277 foot and an outside diameter of 0.806 inch. The resistance per mile of conductor is 0.235 ohm. For a frequency ot 60 cycles, calculate the positive- and negative-sequence constants A, B, C, and D for a line 200 miles long. Transposition of conductors is assumed.

2. Determine the positive- and negative-sequence π-circuit equivalent for the line of Problem 1

3. Determine the positive- and negative-sequence T-circuit equivalent for the line of Problem 1.

4. Assuming an earth resistivity ρ of 100 meter-ohms, determine the zero-sequence impedance per mile for the line of Problem 1.

5. Determine the zero-sequence capacity per mile for the line of Problem 1.

6. Determine the zero-sequence constants A, B, C, and D for the line of Problem 1.

7. Determine the equivalent zero-sequence π and T circuits from the data of Problem 6.

8. Solve Problem 4 for the case of one ground wire of the same material as the power conductor, but located 12 feet above the horizontal plane of the power conductors, and half-way between the center and right-hand conductors.

9. Determine the zero-sequence capacity per mile for the data of Problem 8.

10. Solve Problem 6 for the data of Problems 8 and 9.

11. Solve Problem 7 for the data of Problems 8 and 9.

12. Repeat Problem 8 for two ground wires, symmetrically placed 12 feet above the horizontal power conductors and 20 feet apart.

13. Repeat Problem 9 for the data of Problem 12.

14. Repeat Problem 10 with the data of Problems 12 and 13.

15. Repeat Problem 11 with the data of Problems 12 and 13.

16. Show that the results of equations 228 and 230 are in agreement with those given on page 1189 of the "Standard Handbook for Electrical Engineers," eighth edition, for corresponding circuits.

CHAPTER 15

1. The balanced induced emf of a generator is 6,350 volts from line to neutral (Y connected). Under a three-phase short-circuit, it delivers a current of 1,000 amperes; with a line-to-line fault, it delivers 1,500 amperes; and, with a line to neutral short-circuit, a current of 2,100 amperes is obtained. Determine the positive-, negative-, and zero-sequence impedances. Neglect resistance.

2. A generator is operated, as shown in Fig. 250, with phases b and c connected together and then connected to terminal a through an impedance Z. Determine the equivalent circuit and derive equations for the positive-sequence, negative-sequence, and zero-sequence currents. (See Wagner and Evans, "Symmetrical Components," page 51, example 4.)
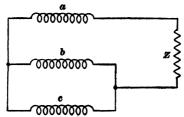
Fig. 250.   Connections for Problem 2.

3. Using the data of the example of Art. 106, obtain a complete solution for a line-to-ground fault with a ground or neutral resistance $Z = 1.0\underline{/0}$ ohm.

4. Using the data of the example of Art. 106, obtain a complete solution for a double line-to-ground fault with a ground impedance $Z = 1.0\underline{/0}$.

5. Using the data of the example of Art. 106, obtain a complete solution for a line-to-line fault through an impedance $Z = 1.0\underline{/90}$.

6. Solve Problem 5 for a line-to-line impedance $Z = 1.0\underline{/0}$.

CHAPTER 16

1. The circuit in Fig. 251 is subjected to a line-to-ground fault at ✕. The data follow.

Generator: 50,000 Kva; 11,000 volt;

$$\% X_p = 80\%; \% X_n = 50\%; \% X_0 = 20\%.$$

Sending Transformer: 40,000 Kva; 11,000/154,000 volts;

$$\% X = 10\%.$$

Receiver Transformer: 40,000 Kva; 154,000/44,000 volts;

$$\% X = 12\%.$$

Load: 25,000 Kva; 44,000 volt; 80% lagging power factor.

Neglect the resistances of the generator and the transformers. Using the π circuit of Problem 7, Chapter 14, for the transmission line, determine all currents and voltages for the line-to-ground fault at $\times$.

2. Repeat Problem 1, using the transmission line T circuit of Problem 7, Chapter 14.

3. Repeat Problem 1, using the line data of Problem 11, Chapter 14.

4. Repeat Problem 2, using the line data of Problem 11, Chapter 14.
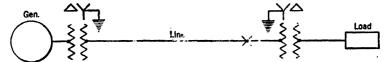
Fig. 251. Circuit for Problem 1.

5. Solve Problem 1, using the line data of Problem 15, Chapter 14.

6. Solve Problem 2, using the line data of Problem 15, Chapter 14.

7. Solve Problem 1 for a line-to-line fault at $\times$.

8. Solve Problem 1 for a line-to-line ground fault at $\times$.

9. Solve the network of the example of Arts. 117 to 119 for a line-to-ground fault with a ground impedance $Z = 100 + j0$.

10. Solve the network of the example of Arts. 117 to 119 for a line-to-line fault.
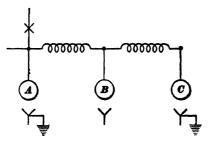
Fig. 252. Conditions for Problem 13.

11. Solve Problem 10 for the case of a double line-to-ground fault through an impedance $Z = 100 + j0$ (see Art. 113).

12. Solve Problem 10 for the case of a line-to-line fault through an impedance $Z = 100 + j0$ (see Art. 114).

13. Given a generating station with the circuit as shown in Fig. 252. Generators have the following ratings: $A = 15,000$ Kva, $B = 35,000$ Kva, and $C = 50,000$ Kva. The positive-sequence reactance of each machine is

equal to 80%, based on the rating of that machine.   Two reactors of 15% reactance, based on 100,000 Kva, are introduced in the bus.   The rated no-load bus voltage is 11,000 volts across lines.   Determine the current flowing into a three-phase fault at ✕.   Neglect the resistances of the machines and reactors.

14.   The negative- and zero-sequence reactances, in per cent, for the circuit of Problem 13 are as follows:

GENERATOR	NEGATIVE	ZERO	BASE KVₐ
A	60	50	15,000
B	60	50	35,000
C	60	50	50,000
Reactors	15	15	100,000

Determine the line current and the voltages above ground throughout the circuit for a line-to-ground fault at ✕.

(a)

(b)
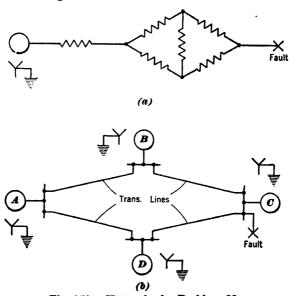
Fig. 253.   Networks for Problem 23.

15.   Repeat Problem 14 for a line-to-line fault at ✕.

16.   Repeat Problem 14 for a line-to-line ground fault at ✕.

17.   Solve the network of the example of Art. 120 for a line-to-ground fault through an impedance $Z = 20 + j0$ (see Art. 107).

18.   Solve Problem 17 for the case of a double line-to-ground fault (see Art. 108).

19. Solve Problem 17 for the case of a line-to-line fault (see Art. 110).

20. Solve Problem 17 for a three-phase fault (see Art. 112).

21. Solve Problem 17 for a double line-to-ground fault through an impedance $Z = 20 + j0$ (see Art. 113).

22. Solve problem 17 for a line-to-line fault through an impedance $Z = 20 + j0$ (see Art. 114).

23. Outline in detail the various steps in the solution for the network of each of the sketches in Fig. 253 for the different types of faults.

24. Solve Problem 23 for the circuit in Fig. 253 (b), for faults half-way between stations B and C.

25. Solve Problem 23 for the circuit in Fig. 253 (b), with faults half-way between stations B and C, but with a tie line between stations B and D.

CHAPTER 17

1. A three-winding single-phase transformer has the following data:

$$\text{High voltage} = 38,200 \text{ volts;}$$
$$\text{Low voltage} = 6,360 \text{ volts;}$$
$$\text{Tertiary} = 2,300 \text{ volts.}$$

Short-circuit reactances are as follows: H to L, 12 per cent on 20,000 Kva; H to T, 5 per cent on 10,000 Kva; L to T, 15 per cent on 1,000 Kva. Determine equivalent star diagrams based on each of the three voltages. Magnetizing current is neglected.

2. Auto-transformers with tertiaries are used to connect a 275,000-volt system to a 220,000-volt system. The tertiary winding is designed for delta connection, while the other common winding is designed for Y conductors. The following data apply:

$$H \text{ winding} \dots\dots\dots\dots 20,000 \text{ Kva, } 159 \text{ Kv;}$$
$$L \text{ winding} \dots\dots\dots\dots 15,000 \text{ Kva, } 127 \text{ Kv;}$$
$$T \text{ winding} \dots\dots\dots\dots 10,000 \text{ Kva, } 11 \text{ Kv.}$$

The short-circuit reactances are as follows: H to L, 10%; H to T, 15%; L to T, 15%, based on each particular Kva capacity. Determine equivalent star diagrams based on each of the three voltages.

3. A 50,000-Kva, 11,000-volt generator is connected to a 22,000-volt system by means of a bank of auto-tertiary transformers of 60,000 Kva total capacity, as shown in Fig. 164. The impedances are as follows: Generator, zero-sequence reactance = 7%, based on 50,000 Kva; L to H short-circuit reactance = 15% based on 20,000 Kva; L to T short-circuit reactance = 10%, H to T short-circuit reactance = 12% Determine the zero-sequence reactance for the combination.

4. A resistance of 0.25 ohm is now introduced in the generator neutral of Problem 3. Determine the zero-sequence impedance of the combination.

5. If the resistance of 0.25 ohm is connected as shown in Fig. 167, compile the resultant circuit zero-sequence impedance. All data are the same as given in Problem 3.

6. Determine the zero-sequence impedance of the circuit of Fig. 169 with a resistance of 0.25 ohm in the transformer neutral. All other data are to be the same as given in Problem 3.

7. Solve Problem 6 with an additional resistance of 0.25 ohm in the generator neutral.

8. Determine the zero-sequence impedance of the circuit of Fig. 169 with the transformer neutral isolated and the generator neutral solidly grounded. Use data of Problem 3.

CHAPTER 20

1. A d-c source of voltage E is suddenly impressed upon a circuit of R and L in series. Determine the equation for the current of the circuit, the voltage drop across R, the voltage drop across L, the power input to resistance, the power input to inductance, and the total energy stored in inductance.

2. Repeat Problem 1 with a capacity in place of the inductance.

3. Solve Problem 1 for a circuit of resistance, inductance, and capacity, obtaining corresponding data for all parts of the circuit. Note that there are three special cases for this problem; namely, non-oscillatory, critical, and oscillatory.

4. A sudden change in the exciter field causes the exciter voltage to increase according to the following law: $e = E - E\epsilon^{-qt}$. Determine the equation for the increase in the alternator field current. The field circuit is assumed to have constant R and L.

5. The voltage of an exciter is decreased according to the law $e = E\epsilon^{-bt}$. Determine the equation for the decrease in the alternator field current.

6. An alternator short-circuit was observed to have the following properties: The sustained short-circuit current was 100 amperes, maximum value; the initial projected value of the oscillographic envelope indicated a current of 2,000 amperes; the transient was completely offset so that the d-c decrement appeared to be 1,000 amperes. Plotting the d-c component and the difference between the a-c envelope and the maximum sustained current, lead to the information in the following table. Determine X'_d, X''_d, X_d, T_t, T_{st}, and the d-c decrement time constant. Write a complete equation for the entire current envelope, including the d-c decrement term.

RECORD OF TRANSIENT CURRENT

T	0	0.2	0.4	0.6	0.8	2.8
$\log i_{dc}$	3			1.08		
$\log \Delta i_{ac}$	2.9542	2.5	2.3	2.1	2.0	1.4

Logarithms are to 10 as base.

Δi_{ac} = maximum alternating current − maximum sustained alternating current.

7.   The contacts of a vibrating type alternator voltage regulator are closing and opening at a certain definite tempo, resulting in the exciter voltage wave shown in Fig. 254. During the closing period, the voltage
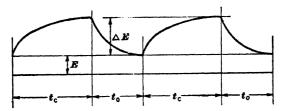
Fig. 254.   Exciter Voltage Wave in Problem 7.

is given by the equation $e_c = E + \Delta E(1 - \epsilon^{-\alpha t})$; and, during the opening period, by the equation $e_o = E + \Delta E \epsilon^{-\beta t}$. Determine the current wave in the field circuit of the alternator. By the process of wave analysis, find the equivalent direct current and the first three frequency alternating component currents.

CHAPTER 21

1.   An artificial three-phase line at the University of Tennessee is available for torque angular demonstrations. On this line, 5 Kva at 230 volts across lines is approximately equivalent to 300,000 Kva at 275,000 line volts. Determine the approximate ratio of actual to artificial line current, and also the ratio of artificial to actual impedance. On the assumption that this line was built to represent the Boulder Dam-Los Angeles line, the data for which are given in Chapter 1, determine the artificial line series impedance and the shunt susceptance per mile. Also determine the artificial line generalized constants A, B, C, and D.

2.   A 60-cycle, 266-mile transmission line of negligible series resistance and negligible shunt susceptance has an inductance per mile per phase of 0.002 henry. Sketch the swing curves for $E_s = 166,000$ and $E_r = 159,000$

volts. With the line initially unloaded, determine the transient power limit by sudden application of loads. Solve this problem graphically as outlined in Figs. 210 and 211. Plot the curve between the load applied and the peak torque-angle swing. What are the limitations of this solution?

3. Two identical lines having data as of Problem 2 are operating in parallel. By the equal-area method of Figs. 214 and 215, determine the transient power limit if one line is suddenly opened. Use the graphical construction. Plot the line power against the peak angular swing.

4. Sketch receiver swing curves for actual line number 1 of Appendix VII (see Equations 409), and carry out a solution similar to Problem 2.

5. Repeat Problem 3 with the data of Problem 4. State and explain the limitations to the method used in the solutions of Problems 4 and 5.

6. Replace the actual circuit of line 1, Appendix VII, with its Thevenin equivalent (see Art. 53 and Fig. 48). Carry out the same solution for this Thevenin equivalent circuit as called for in Problem 2.

7. In Art. 150 it is suggested that an equivalent inertia constant $M = \dfrac{M_1 M_2}{M_1 + M_2}$ can be used in the application of equation 398 to the case of finite machines at each end of the transmission network. Prove this statement.

HINT. Rewrite equation 398 as the following two expressions:

$$\frac{M_1}{2\pi f}\frac{d^2\phi_1}{dt^2} = K_1 - T_1 \text{ and } \frac{M_2}{2\pi f}\frac{d^2\phi_2}{dt^2} = K_2 - T_2$$

in which ϕ_1 = torque angle of machine 1, and ϕ_2 = torque angle of machine 2, both being measured from a synchronously revolving reference; and T_1 and T_2 indicate unit magnetic torques, $K_1 - T_1$ and $K_2 - T_2$ representing the unit accelerating torques for the machines.

8. Determine a similar equation in which the torque angle ϕ is now equal to $\phi_1 - \phi_2$. (See "Power System Stability," Vol. I, by Kimbark, page 132.)

9. Consider the circuit of Fig. 212, with the following operating schedule:

    Initial condition, both lines operating in parallel.
    Second condition, one line with a three-phase fault at its center.
    Final condition, faulty line cleared by the opening of circuit breakers
        at both ends of the line.

Sketch typical swing curves similar to Figs. 214 and 215 for these three cases, and indicate conditions for stability by the equal-area criterion. Derive an equation for the critical clearing torque angle for a given fixed system load. (The critical clearing torque angle locates the dividing line between the two areas involved in the equal-area criterion.)

CHAPTER 22

1.   A voltage surge of 1,000 Kv/3/20 is to be represented by the following equation:

$$e = E(\epsilon^{-at} - \epsilon^{-bt})$$

Determine the values of E, a, and b.

2.   The voltage equation ot a transmission line under sinusoidal wave operation is:

$$E = \left(\frac{E_r + I_r Z_0}{2}\right)\epsilon^{u}\epsilon^{jv} + \left(\frac{E_r - I_r Z_0}{2}\right)\epsilon^{-u}\epsilon^{-jv}$$

where $u + jv = ms$, $m = \sqrt{ZY}$, $Z_0 = \sqrt{\frac{Z}{Y}}$, $Z = R + jx$, and $Y = G + jB$.

Modify this equation for a distortionless line, for which $\frac{R}{L} = \frac{G}{C}$, and explain what important results are involved.

3.   Give a clear statement comparing the physical properties of simple harmonic impedance, or $Z = R + jX$, and operational impedance, or $Z(p) = R + pL$.

4.   In surge problems the term surge impedance is used.   Compare this term with the surge impedance Z_0 of sinusoidally operating transmission lines.

5.   Contrast the meaning of the following exponentials: $\epsilon^{\pm x}$, $\epsilon^{\pm j\theta}$, and $\epsilon^{\pm ap}$.

6.   Prove that $\epsilon^{ap} f(t) = f(t + a)$.

7.   Plot curves representing the function

$$f = A \cos (\theta \pm \omega t)$$

where θ = distance or space measured in radians;
    t = time in seconds;
    ω = angular velocity.

Explain what such a function represents.

8.   Plot curves representing the function

$$f = A \cosh (\theta \pm \omega t)$$

where θ = hyperbolic space radians;
    t = seconds;
    ω = hyperbolic radians per second.

Indicate what such a function represents.

9.   Repeat Problem 8 for the very general form

$$f = (s \pm vt)$$

where s = space;
    t = time;
    v = velocity.

10.   Prove that the velocity of wave propagation, for the case of negligible resistance and conductance, is $v = \pm\dfrac{1}{\sqrt{LC}}$.

11.   Plot curves for the function $f = \epsilon^{-at}\cos(\theta \pm vt)$, and explain their meaning.

12.   If $f(t) = E(\epsilon^{-at} - \epsilon^{-bt})1$, what is the form of $f\left(t - \dfrac{s}{v}\right)$?   Sketch this function for different values of s.   What is the form of the function $\epsilon^{-\frac{R}{L}t}f\left(t - \dfrac{s}{v}\right)$?   Explain.

13.   A surge passes from a circuit with surge impedance of unity to a circuit with impedance twice as great.   Sketch the star diagram and explain all features of the diagram.

14.   In Fig. 226 the energy of each wave is represented by the following areas:

> Initial forward wave............... = area *ogc*;
> Reflected wave.................... = area *oba*;
> Transmitted wave................. = area *acd*.

Prove that area *ogc* = area *oba* + area *acd*, thus demonstrating that the energy in the initial forward wave is equal to the reflected energy plus the transmitted energy.

APPENDIX I

GEOMETRIC INTERPRETATION OF HYPERBOLIC FUNCTIONS

In the unit circle in Fig. 255, with its center at the origin O and its radius OA equal to unity, consider the central angle AOP, whose value in radians we denote by θ. Obviously,

$$\tan \theta = \frac{AR}{OA} = AR$$

and

$$\tan \theta = \frac{PT}{OP} = PT$$

Therefore:

$$AR = PT$$
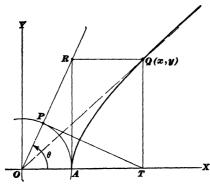
Fig. 255. Hyperbolic Functions.

Construct TQ so that $TQ = AR$ and let the coordinates of Q be (x,y). Then,

$$\frac{OT}{OP} = OT = \sec \theta \text{ or } x = \sec \theta$$

and

$$y = TQ = AR = \frac{AR}{OA} = \tan \theta \text{ or } y = \tan \theta$$

But, $\sec^2 \theta = 1 + \tan^2 \theta$, and hence $x^2 = 1 + y^2$ or

$$x^2 - y^2 = 1$$

The point Q therefore moves on the rectangular hyperbola $x^2 - y^2 = 1$ as the point P moves on the unit circle.

By definition:

$$\sinh u = \frac{e^u - e^{-u}}{2}$$

$$\cosh u = \frac{e^u + e^{-u}}{2}$$

Hence, $\cosh^2 u - \sinh^2 u = \frac{1}{4}(e^{2u} + 2 + e^{-2u}) - (e^{2u} - 2 + e^{-2u})$ or

$$\cosh^2 u - \sinh^2 u = 1$$

We have then: $x^2 - y^2 = 1$ and $\cosh^2 u - \sinh^2 u = 1$. If we let $x = \cosh u$ and $y = \sinh u$, then x and y satisfy the *necessary* condition that $x^2 - y^2 = 1$. Hence, a geometric interpretation of $\sinh u$ and $\cosh u$ is as follows:

$$\sinh u = TQ = y$$
$$\cosh u = OT = x$$

This, however, leaves unanswered the question as to what u itself is, that is, what the geometric interpretation of u is. We have:

$$\sinh u = \frac{e^u - e^{-u}}{2} = y$$

$$\cosh u = \frac{e^u + e^{-u}}{2} = x$$

Hence, $$x = \frac{e^u + e^{-u}}{2}$$

and $$y = \frac{e^u - e^{-u}}{2}$$

From these relations:
$$x + y = e^u$$
and $$x - y = e^{-u}$$

Now, consider the area of sector OAQ, which is the figure bounded by the segments OA and OQ and the arc AQ. Denote this area by S. Then:

$$S = \tfrac{1}{2}xy - \int_1^x y \, dx$$

Since $y = \sqrt{x^2 - 1}$,

$$2S = xy - 2\int_1^x \sqrt{x^2 - 1} \, dx = xy - 2\int_1^x \sqrt{t^2 - 1} \, dt$$

$$2S = xy - \left[t\sqrt{t^2 - 1} - \text{Ln}\,(t + \sqrt{t^2 - 1}) \right]_{t=1}^{t=x}$$

$$2S = x\sqrt{x^2 - 1} - x\sqrt{x^2 - 1} + \text{Ln}\,(x + \sqrt{x^2 - 1})$$

$$2S = \text{Ln}\,(x + \sqrt{x^2 - 1})$$

But, $x + \sqrt{x^2 - 1} = x + y = e^u$; and $\text{Ln}\,(x + \sqrt{x^2 - 1}) = \text{Ln}\,e^u = u$. Hence, $2S = u$, or $u = 2$ area OAQ. We see, then, that u measures twice the area of the hyperbolic sector OAQ.

On the other hand, if we let K = area of the circular sector OAP (with radius 1), we have:

$$K = \tfrac{1}{2}\theta$$

Hence, θ measures twice the area of the circular sector OAP. The analogy between θ and u is then expressed through the areas of the circular sector OAP and the hyperbolic sector OAQ. The angle θ is called the "Gudermannian" of u, denoted by gd u. It is also called the "hyperbolic amplitude" of u, denoted by amh u.

APPENDIX II

GEOMETRIC PROOF OF THE GMR FOR A CIRCULAR AREA

In Fig. 256, consider two small areas ΔA_1 and ΔA_2, such that $\Delta A_1 = \Delta A_2$. Then:

$$n = \frac{\pi a^2}{\Delta A_1} = \frac{\pi a^2}{\Delta A_2} = \text{number of small areas within area } \pi a^2$$

and

$$n^2 = \frac{\pi^2 a^4}{\Delta A_1 \Delta A_2}$$

From Fig. 256:

$$d = \sqrt{u^2 + v^2 - 2uv \cos \theta}$$
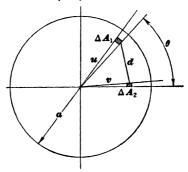
Fig. 256.   GMR of Circular Area.

By definition, the geometric mean radius R is:

$$R = \sqrt[n^2]{\prod^{n^2} d}$$

Taking logarithms of both sides:

$$\operatorname{Ln} R = \operatorname{Ln} \sqrt[n^2]{\prod^{n^2} d} = \frac{1}{n^2} \operatorname{Ln} \prod^{n^2} d = \frac{1}{n^2} \sum^{n^2} \operatorname{Ln} d$$

Substituting the value of n^2, we have:

$$\text{Ln } R = \frac{\Delta A_1 \Delta A_2}{\pi^2 a^4} \sum^{n^2} \text{Ln } d$$

or

$$\pi^2 a^4 \text{ Ln } R = \sum^{n^2} \text{Ln } d(\Delta A_1 \Delta A_2)$$

Now, as n^2 approaches infinity, the areas ΔA_1 and ΔA_2 will approach the infinitesimals $u\ du\ d\theta$ and $v\ dv\ d\phi$, and the summation expression will converge into an integral. Thus:

$$\pi^2 a^4 \text{ Ln } R = 2 \int \int \int \int (\text{Ln } d) u\ du\ d\phi\ v\ dv\ d\theta$$

In the transfer from a finite summation to an integral form, it is necessary to consider the length from ΔA_1 to ΔA_2 as well as the length from ΔA_2 to ΔA_1. Since the integral form includes this distance only once, the total expression must be multiplied by 2.

Substituting the value of d and introducing the proper limits, we have:

$$\pi^2 a^4 \text{ Ln } R = \int_0^a u\ du \int_0^u v\ dv \int_0^{2\pi} d\phi \int_0^{2\pi} \text{Ln}(u^2+v^2-2uv \cos \theta)\ d\theta$$

Since $\int_0^{2\pi} \text{Ln } (a \pm b \cos \theta)d\theta = 2\pi \text{ Ln } \dfrac{a+\sqrt{a^2-b^2}}{2}$, when $a \geqq b$, then:

$$\pi^2 a^4 \text{ Ln } R = 2\pi \int_0^a u\ du \int_0^u v\ dv \int_0^{2\pi} d\phi \left[\text{Ln } \frac{u^2+v^2+\sqrt{(u^2+v^2)^2-4u^2v^2}}{2} \right]$$

or

$$a^4 \text{ Ln } R = 4 \int_0^a u\ du \int_0^u v\ dv \text{ Ln } u^2 \qquad (1)$$

It should be noticed that there are two possible solutions of the first integral, obtainable as the positive and negative values of the radical. These results are, respectively, $\text{Ln } u^2$ and $\text{Ln } v^2$. Continuing with the first of these possibilities, we have, as the next step:

$$a^4 \text{ Ln } R = 4 \int_0^a u \text{ Ln } u^2\ du \left(\frac{v^2}{2}\right)_0^u = 4 \int_0^a u^3 \text{ Ln } u\ du$$

Finally:

$$a^4 \text{ Ln } R = 4 \left[u^4\left(\frac{\text{Ln } u}{4} - \frac{1}{16}\right) \right]_0^a = a^4\left(\text{Ln } a - \frac{1}{4}\right)$$

Returning to the second possibility previously indicated, we have:

$$a^4 \text{ Ln } R = 4 \int_0^a v\ dv \int_0^v u\ du \text{ Ln } v^2 \qquad (2)$$

The solution of this expression will give the same result as before.

It should be noticed that the expressions (1) and (2) evaluate the integral over a circular area; in the first case the radius is u, while in the second case the radius is v. Thus, we finally have:

$$\text{Ln } R = \text{Ln } a - \frac{1}{4} = \text{Ln } a - \text{Ln } \epsilon^{0.25} = \text{Ln } \frac{a}{\epsilon^{0.25}}$$

and

$$R = \frac{a}{\epsilon^{0.25}} = a\epsilon^{-0.25}$$

which corresponds to equation 23 in Chapter 3.

APPENDIX III

SIGN OF REACTIVE POWER

In Art. 52 the expression for "vector power" was computed in the following manner:

$$P + jQ = \bar{E}I = EI \cos \theta + jEI \sin \theta$$

in which $E = E\underline{/\theta_e}$, $\bar{E} = E\underline{/\theta_e}$, $I = I\underline{/\theta_a}$, and $\theta = \theta_a - \theta_e$. In this derivation a positive value of jQ signifies capacitive reactive vars, while a negative value of jQ signifies inductive vars.

A similar solution can be obtained by taking the product of the vector voltage and the conjugate vector current, the result being:

$$P - jQ = E\bar{I} = EI \cos \theta - jEI \sin \theta$$

In this case a positive value of jQ signifies inductive reactive vars, while a negative value of jQ signifies capacitive reactive vars.

The first convention, namely, positive reactive power for capacitive reactive power (leading current), is the one used throughout this text. This convention has been in accordance with the ASA standards; however, recent studies presented in the November, 1946, issue of *Electrical Engineering* indicate a tendency to reverse previously accepted practice, inductive reactive vars being considered as the positive quantity. This convention is the normal result obtained from the product of vector voltage and conjugate vector current.

The foregoing two forms for "vector power" are defined without any reference to a particular convention of the direction of power flow. Either form can be applied with equal ease to the input or output of a network. In the study of a single transmission line it is often convenient to define the input to the line as $P_s + jQ_s$ and to define the output of the line as $P_r + jQ_r$ (see equations 145 and 147). In this case the transmission line as a network is considered as having input at one terminal point and output at the other terminal point. However, in problems dealing with large networks it is more convenient to consider all loads as component parts of the network and to treat all synchronous machines as sources.

In these problems the "vector power" expression in either one of the foregoing conventions is considered as an input quantity. Negatives of either of those conventions would obviously signify "vector power" output.

The reader should clearly realize that real power and reactive vars are quite different intrinsically. Real or true power having a definite average value per cycle can very properly be considered as having direction of flow. On the other hand, the average value of the reactive vars per cycle is zero, and the reactive vars represent a process of cyclic oscillation of energy interchange into and out of a given network in the period of one-half cycle. Thus, any convention of sign or any convention of "in" and "out" reactive power depends entirely on the particular definition chosen.
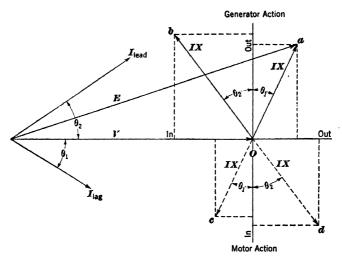
Fig. 257.   Vector Diagram for Synchronous Machine.

The question of relative sign of reactive power is therefore of minor importance from the point of view of the power-system operating engineer who is concerned with metering these quantities. A standard notation of "in" and "out" power seems to be desirable when applied to real power, but such a notation is not so definite when applied to reactive power. In fact, the convention as applied to reactive power is just as arbitrary as the convention chosen for the sign of reactive power. It is quite in order to assign a direction to real power flow, since average real power as indicated by a wattmeter is a definite quantity which is delivered by the generating sources to the loads of the system. In contrast, the average value of reactive power is always zero, irrespective of whether it is capacitive or inductive in nature.

To illustrate how the convention of "in" and "out" reactive power may be established, consider a synchronous machine operating with terminal

voltage V and current I. For simplicity the armature resistance is neglected. For the generator action, the voltage behind synchronous reactance is obtained by adding the synchronous reactance drop to the terminal voltage. For motor action the reactance drop must be subtracted from the terminal voltage. The operation of a synchronous machine as a generator or as a motor may be illustrated in one common vector diagram, as indicated in Fig. 257, in which the terminal voltage V is used as the reference vector. A set of coordinate reference axes is chosen with the origin at O. The IX voltage drops for lagging and leading currents are shown, full lines being used for generator action and dashed lines for motor action. From the geometry of the diagram it will be found that the angles between the IX vectors and the vertical axis or y axis are equal, respectively, to the power-factor angles θ, and θ_2.
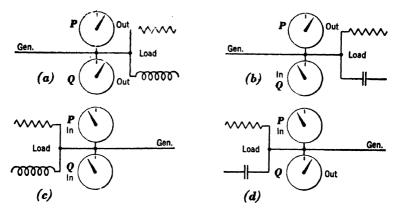
Fig. 258.   Metering at Transfer Point in Transmission System.

The scale of the lines Oa, Ob, Oc, and Od may be changed from that of voltage to that of volt-amperes by the application of a constant multiplier equal to $\dfrac{V}{X}$. It will then be noticed that the projections of lines Oa, Ob, Oc, and Od on the y axis indicate true power ($VI \cos \theta$), while the corresponding projections of these lines on the x axis will indicate reactive power ($VI \sin \theta$).

In line with standard convention, projections on the positive x axis and positive y axis are considered positive, while projections on the negative x axis and negative y axis are considered negative quantities.

Furthermore, generator real power is very logically power output or "out" power, while motor real power is power input or "in" power. Thus, the projections of the lines Oa and Ob on the positive y axis indicate "out" power, while projections of the lines Oc and Od on the negative y axis indicate "in" power.

If the same geometric pattern is followed, it is logical to say that projections of lines Oa and Od on the positive x axis indicate "out" reactive power, while projections of lines Ob and Oc on the negative x axis indicate "in" reactive power.

Inspection of the vector diagram will indicate that "out" reactive power is obtained when the synchronous machine is over-excited, while "in" reactive power is obtained when the synchronous machine is under-excited.

Consider now the specific problem of metering at a transfer point in a transmission system, as indicated in Fig. 258. The use of zero-center wattmeters and reactive meters is implied. It is assumed that the connections of the meters to the circuit are made in such a manner that both read to the right or in the "out" direction in diagram (a), which indicates a generating source supplying power to a composite load of resistance and inductance. Meter deflections are shown for three other circuit conditions in (b), (c), and (d), respectively.

It has been suggested that inductances may be considered as "sinks" of reactive power, while condensers may be considered as "sources" of reactive vars. As previously pointed out, this concept is erroneous; but it does supply a rule of thumb by which one may interpret the meanings of the meter deflections. Thus, the wattmeter and the reactive meter will deflect in the same direction when both components of a load are considered as "sinks," such as resistance and inductance or their equivalents. These two meters will deflect in opposite directions when the load applied to the system is composed of resistance and capacity or their equivalents.

HYPERBOLIC, EXPONENTIAL, AND TRIGONOMETRIC FUNCTIONS*

x (Radians)	Cosh x	Sinh x	Tanh x	e^x	e^-x	Cos x	Sin x	Tan x	Degrees	x
0.00	1.0000	0.0000	0.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.00°	0.00
0.01	1.0001	0.0100	0.0100	1.0101	0.9900	1.0000	0.0100	0.0100	0.57°	0.01
0.02	1.0002	0.0200	0.0200	1.0202	0.9802	0.9998	0.0200	0.0200	1.15°	0.02
0.03	1.0005	0.0300	0.0300	1.0305	0.9704	0.9996	0.0300	0.0300	1.72°	0.03
0.04	1.0008	0.0400	0.0400	1.0408	0.9608	0.9992	0.0400	0.0400	2.29°	0.04
0.05	1.0013	0.0500	0.0500	1.0513	0.9512	0.9988	0.0500	0.0500	2.86°	0.05
0.06	1.0018	0.0600	0.0599	1.0618	0.9418	0.9982	0.0600	0.0601	3.44°	0.06
0.07	1.0025	0.0701	0.0699	1.0725	0.9324	0.9976	0.0699	0.0701	4.01°	0.07
0.08	1.0032	0.0801	0.0798	1.0833	0.9231	0.9968	0.0799	0.0802	4.58°	0.08
0.09	1.0041	0.0901	0.0898	1.0942	0.9139	0.9960	0.0899	0.0902	5.16°	0.09
0.10	1.0050	0.1002	0.0997	1.1052	0.9048	0.9950	0.0998	0.1003	5.73°	0.10
0.11	1.0061	0.1102	0.1096	1.1163	0.8958	0.9940	0.1098	0.1104	6.30°	0.11
0.12	1.0072	0.1203	0.1194	1.1275	0.8869	0.9928	0.1197	0.1206	6.88°	0.12
0.13	1.0085	0.1304	0.1293	1.1388	0.8781	0.9916	0.1296	0.1307	7.45°	0.13
0.14	1.0098	0.1405	0.1391	1.1503	0.8694	0.9902	0.1395	0.1409	8.02°	0.14
0.15	1.0113	0.1506	0.1489	1.1618	0.8607	0.9888	0.1494	0.1511	8.59°	0.15
0.16	1.0128	0.1607	0.1587	1.1735	0.8521	0.9872	0.1593	0.1614	9.17°	0.16
0.17	1.0145	0.1708	0.1684	1.1853	0.8437	0.9856	0.1692	0.1717	9.74°	0.17
0.18	1.0162	0.1810	0.1781	1.1972	0.8353	0.9838	0.1790	0.1820	10.31°	0.18
0.19	1.0181	0.1912	0.1878	1.2093	0.8270	0.9820	0.1889	0.1923	10.89°	0.19

HYPERBOLIC, EXPONENTIAL, AND TRIGONOMETRIC FUNCTIONS—(Continued)

x (Radians)	Cosh x	Sinh x	Tanh x	e^x	e^{-x}	Cos x	Sin x	Tan x	Degrees	x
0.20	1.0201	0.2013	0.1974	1.2214	0.8187	0.9801	0.1987	0.2027	11.46°	0.20
0.21	1.0221	0.2116	0.2070	1.2337	0.8106	0.9780	0.2085	0.2131	12.03°	0.21
0.22	1.0243	0.2218	0.2165	1.2461	0.8025	0.9759	0.2182	0.2236	12.61°	0.22
0.23	1.0266	0.2320	0.2260	1.2586	0.7945	0.9737	0.2280	0.2341	13.18°	0.23
0.24	1.0289	0.2423	0.2355	1.2712	0.7866	0.9713	0.2377	0.2447	13.75°	0.24
0.25	1.0314	0.2526	0.2449	1.2840	0.7788	0.9689	0.2474	0.2553	14.32°	0.25
0.26	1.0340	0.2629	0.2543	1.2969	0.7711	0.9664	0.2571	0.2660	14.90°	0.26
0.27	1.0367	0.2733	0.2636	1.3100	0.7634	0.9638	0.2667	0.2768	15.47°	0.27
0.28	1.0395	0.2837	0.2729	1.3231	0.7558	0.9611	0.2764	0.2876	16.04°	0.28
0.29	1.0424	0.2941	0.2821	1.3364	0.7483	0.9582	0.2860	0.2984	16.62°	0.29
0.30	1.0453	0.3045	0.2913	1.3499	0.7408	0.9553	0.2955	0.3093	17.19°	0.30
0.31	1.0484	0.3150	0.3004	1.3634	0.7334	0.9523	0.3051	0.3203	17.76°	0.31
0.32	1.0516	0.3255	0.3095	1.3771	0.7261	0.9492	0.3146	0.3314	18.33°	0.32
0.33	1.0550	0.3360	0.3185	1.3910	0.7189	0.9460	0.3240	0.3425	18.91°	0.33
0.34	1.0584	0.3466	0.3275	1.4049	0.7118	0.9428	0.3335	0.3537	19.48°	0.34
0.35	1.0619	0.3572	0.3364	1.4191	0.7047	0.9394	0.3429	0.3650	20.05°	0.35
0.36	1.0655	0.3678	0.3452	1.4333	0.6977	0.9359	0.3523	0.3764	20.63°	0.36
0.37	1.0692	0.3785	0.3540	1.4477	0.6907	0.9323	0.3616	0.3879	21.20°	0.37
0.38	1.0731	0.3892	0.3627	1.4623	0.6839	0.9287	0.3709	0.3994	21.77°	0.38
0.39	1.0770	0.4000	0.3714	1.4770	0.6771	0.9249	0.3802	0.4111	22.35°	0.39
0.40	1.0811	0.4108	0.3800	1.4918	0.6703	0.9211	0.3894	0.4228	22.92°	0.40
0.41	1.0852	0.4216	0.3885	1.5068	0.6637	0.9171	0.3986	0.4346	23.49°	0.41
0.42	1.0895	0.4325	0.3969	1.5220	0.6570	0.9131	0.4078	0.4466	24.06°	0.42
0.43	1.0939	0.4434	0.4053	1.5373	0.6505	0.9090	0.4169	0.4586	24.64°	0.43
0.44	1.0984	0.4543	0.4136	1.5527	0.6440	0.9048	0.4259	0.4708	25.21°	0.44

0.45	1.1030	0.4653	0.4219	1.5683	0.6376	0.9005	0.4350	0.4831	25.78°	0.45
0.46	1.1077	0.4764	0.4301	1.5841	0.6313	0.8961	0.4440	0.4954	26.36°	0.46
0.47	1.1125	0.4875	0.4382	1.6000	0.6250	0.8916	0.4529	0.5080	26.93°	0.47
0.48	1.1174	0.4987	0.4462	1.6161	0.6188	0.8870	0.4618	0.5206	27.50°	0.48
0.49	1.1225	0.5098	0.4542	1.6323	0.6126	0.8823	0.4706	0.5334	28.07°	0.49
0.50	1.1276	0.5211	0.4621	1.6487	0.6065	0.8776	0.4794	0.5463	28.65°	0.50
0.51	1.1329	0.5324	0.4700	1.6653	0.6005	0.8727	0.4882	0.5594	29.22°	0.51
0.52	1.1383	0.5438	0.4777	1.6820	0.5945	0.8678	0.4969	0.5726	29.79°	0.52
0.53	1.1438	0.5552	0.4854	1.6989	0.5886	0.8628	0.5055	0.5859	30.37°	0.53
0.54	1.1494	0.5666	0.4930	1.7160	0.5827	0.8577	0.5141	0.5994	30.94°	0.54
0.55	1.1551	0.5782	0.5005	1.7333	0.5769	0.8525	0.5227	0.6131	31.51°	0.55
0.56	1.1609	0.5897	0.5080	1.7507	0.5712	0.8473	0.5312	0.6269	32.09°	0.56
0.57	1.1669	0.6014	0.5154	1.7683	0.5655	0.8419	0.5396	0.6410	32.66°	0.57
0.58	1.1730	0.6131	0.5227	1.7860	0.5599	0.8365	0.5480	0.6552	33.23°	0.58
0.59	1.1792	0.6248	0.5299	1.8040	0.5543	0.8309	0.5564	0.6692	33.80°	0.59
0.60	1.1855	0.6367	0.5371	1.8221	0.5488	0.8253	0.5646	0.6841	34.38°	0.60
0.61	1.1919	0.6485	0.5441	1.8404	0.5434	0.8197	0.5729	0.6939	34.95°	0.61
0.62	1.1984	0.6605	0.5511	1.8589	0.5379	0.8139	0.5810	0.7139	35.52°	0.62
0.63	1.2051	0.6725	0.5581	1.8776	0.5326	0.8080	0.5891	0.7291	36.10°	0.63
0.64	1.2119	0.6846	0.5649	1.8965	0.5273	0.8021	0.5972	0.7445	36.67°	0.64
0.65	1.2188	0.6968	0.5717	1.9155	0.5220	0.7961	0.6052	0.7602	37.24°	0.65
0.66	1.2258	0.7090	0.5784	1.9348	0.5169	0.7900	0.6131	0.7761	37.82°	0.66
0.67	1.2330	0.7213	0.5850	1.9542	0.5117	0.7838	0.6210	0.7923	38.39°	0.67
0.68	1.2403	0.7336	0.5915	1.9739	0.5066	0.7776	0.6288	0.8087	38.96°	0.68
0.69	1.2477	0.7461	0.5980	1.9937	0.5016	0.7713	0.6365	0.8253	39.53°	0.69
0.70	1.2552	0.7586	0.6044	2.0138	0.4966	0.7648	0.6442	0.8423	40.11°	0.70
0.71	1.2628	0.7712	0.6107	2.0340	0.4916	0.7584	0.6518	0.8595	40.68°	0.71
0.72	1.2706	0.7838	0.6169	2.0544	0.4868	0.7518	0.6594	0.8771	41.25°	0.72
0.73	1.2785	0.7966	0.6231	2.0751	0.4819	0.7452	0.6669	0.8949	41.83°	0.73
0.74	1.2865	0.8094	0.6292	2.0959	0.4771	0.7385	0.6743	0.9131	42.40°	0.74

HYPERBOLIC, EXPONENTIAL, AND TRIGONOMETRIC FUNCTIONS—(Continued)

x (Radians)	Cosh x	Sinh x	Tanh x	e^x	e^-x	Cos x	Sin x	Tan x	Degrees	x
0.75	1.2947	0.8223	0.6352	2.1170	0.4724	0.7317	0.6816	0.9316	42.97°	0.75
0.76	1.3030	0.8353	0.6411	2.1383	0.4677	0.7248	0.6889	0.9505	43.54°	0.76
0.77	1.3114	0.8484	0.6469	2.1598	0.4630	0.7179	0.6961	0.9697	44.12°	0.77
0.78	1.3199	0.8615	0.6527	2.1815	0.4584	0.7109	0.7033	0.9893	44.69°	0.78
0.79	1.3286	0.8748	0.6584	2.2034	0.4538	0.7039	0.7104	1.0092	45.26°	0.79
0.80	1.3374	0.8881	0.6640	2.2255	0.4493	0.6967	0.7174	1.0296	45.84°	0.80
0.81	1.3464	0.9015	0.6696	2.2479	0.4449	0.6895	0.7243	1.0505	46.41°	0.81
0.82	1.3555	0.9150	0.6751	2.2705	0.4404	0.6822	0.7312	1.0717	46.98°	0.82
0.83	1.3647	0.9286	0.6805	2.2933	0.4360	0.6749	0.7379	1.0934	47.56°	0.83
0.84	1.3740	0.9423	0.6858	2.3164	0.4317	0.6675	0.7446	1.1156	48.13°	0.84
0.85	1.3835	0.9561	0.6911	2.3396	0.4274	0.6600	0.7513	1.1383	48.70°	0.85
0.86	1.3932	0.9700	0.6963	2.3632	0.4232	0.6524	0.7578	1.1616	49.27°	0.86
0.87	1.4029	0.9840	0.7014	2.3869	0.4190	0.6448	0.7643	1.1853	49.85°	0.87
0.88	1.4128	0.9981	0.7064	2.4109	0.4148	0.6372	0.7707	1.2097	50.42°	0.88
0.89	1.4229	1.0122	0.7114	2.4351	0.4107	0.6294	0.7771	1.2346	50.99°	0.89
0.90	1.4331	1.0265	0.7163	2.4596	0.4066	0.6216	0.7833	1.2602	51.57°	0.90
0.91	1.4434	1.0409	0.7211	2.4843	0.4025	0.6138	0.7895	1.2864	52.14°	0.91
0.92	1.4539	1.0554	0.7259	2.5093	0.3985	0.6058	0.7956	1.3133	52.71°	0.92
0.93	1.4645	1.0700	0.7306	2.5345	0.3946	0.5978	0.8016	1.3409	53.29°	0.93
0.94	1.4753	1.0847	0.7352	2.5600	0.3906	0.5898	0.8076	1.3692	53.86°	0.94
0.95	1.4862	1.0995	0.7398	2.5857	0.3867	0.5817	0.8134	1.3984	54.43°	0.95
0.96	1.4973	1.1144	0.7443	2.6117	0.3829	0.5735	0.8192	1.4284	55.00°	0.96
0.97	1.5085	1.1294	0.7487	2.6379	0.3791	0.5653	0.8249	1.4592	55.58°	0.97
0.98	1.5199	1.1446	0.7531	2.6645	0.3753	0.5570	0.8305	1.4910	56.15°	0.98
0.99	1.5314	1.1598	0.7574	2.6912	0.3716	0.5487	0.8360	1.5237	56.72°	0.99

1.00	57.30°	1.5574	0.8415	0.5403	0.3679	2.7183	0.7616	1.1752	1.5431	1.00
1.01	57.87°	1.5922	0.8468	0.5319	0.3642	2.7456	0.7658	1.1907	1.5549	1.01
1.02	58.44°	1.6281	0.8521	0.5234	0.3606	2.7732	0.7699	1.2063	1.5669	1.02
1.03	59.01°	1.6652	0.8573	0.5148	0.3570	2.8011	0.7739	1.2220	1.5790	1.03
1.04	59.59°	1.7036	0.8624	0.5062	0.3535	2.8292	0.7779	1.2379	1.5913	1.04
1.05	60.16°	1.7433	0.8674	0.4976	0.3499	2.8577	0.7818	1.2539	1.6038	1.05
1.06	60.73°	1.7844	0.8724	0.4889	0.3465	2.8864	0.7857	1.2700	1.6164	1.06
1.07	61.31°	1.8270	0.8772	0.4801	0.3430	2.9154	0.7895	1.2862	1.6292	1.07
1.08	61.88°	1.8712	0.8820	0.4713	0.3396	2.9447	0.7932	1.3025	1.6421	1.08
1.09	62.45°	1.9171	0.8866	0.4625	0.3362	2.9743	0.7969	1.3190	1.6553	1.09
1.10	63.03°	1.9648	0.8912	0.4536	0.3329	3.0042	0.8005	1.3357	1.6685	1.10
1.11	63.60°	2.0143	0.8957	0.4447	0.3296	3.0344	0.8041	1.3524	1.6820	1.11
1.12	64.17°	2.0660	0.9001	0.4357	0.3263	3.0649	0.8076	1.3693	1.6956	1.12
1.13	64.74°	2.1198	0.9044	0.4257	0.3230	3.0957	0.8110	1.3863	1.7093	1.13
1.14	65.32°	2.1759	0.9086	0.4176	0.3198	3.1268	0.8144	1.4035	1.7233	1.14
1.15	65.89°	2.2345	0.9128	0.4085	0.3166	3.1582	0.8178	1.4208	1.7374	1.15
1.16	66.46°	2.2958	0.9168	0.3993	0.3135	3.1899	0.8210	1.4382	1.7517	1.16
1.17	67.04°	2.3600	0.9203	0.3902	0.3104	3.2220	0.8243	1.4558	1.7662	1.17
1.18	67.61°	2.4273	0.9246	0.3809	0.3073	3.2544	0.8275	1.4736	1.7808	1.18
1.19	68.18°	2.4979	0.9284	0.3717	0.3042	3.2871	0.8306	1.4914	1.7957	1.19
1.20	68.75°	2.5722	0.9320	0.3624	0.3012	3.3201	0.8337	1.5095	1.8107	1.20
1.21	69.33°	2.6503	0.9356	0.3530	0.2982	3.3535	0.8367	1.5276	1.8258	1.21
1.22	69.90°	2.7328	0.9391	0.3437	0.2952	3.3872	0.8397	1.5460	1.8412	1.22
1.23	70.47°	2.8198	0.9425	0.3342	0.2923	3.4212	0.8426	1.5645	1.8568	1.23
1.24	71.05°	2.9119	0.9458	0.3248	0.2894	3.4556	0.8455	1.5831	1.8725	1.24
1.25	71.62°	3.0096	0.9490	0.3153	0.2865	3.4903	0.8483	1.6019	1.8884	1.25
1.26	72.19°	3.1133	0.9521	0.3058	0.2837	3.5254	0.8511	1.6209	1.9045	1.26
1.27	72.77°	3.2236	0.9551	0.2963	0.2808	3.5609	0.8538	1.6400	1.9208	1.27
1.28	73.34°	3.3413	0.9580	0.2867	0.2780	3.5966	0.8565	1.6593	1.9373	1.28
1.29	73.91°	3.4672	0.9608	0.2771	0.2753	3.6328	0.8591	1.6788	1.9540	1.29

HYPERBOLIC, EXPONENTIAL, AND TRIGONOMETRIC FUNCTIONS—(Continued)

x (Radians)	Cosh x	Sinh x	Tanh x	ϵ^x	ϵ^{-x}	Cos x	Sin x	Tan x	Degrees	x
1.30	1.9709	1.6984	0.8617	3.6693	0.2725	0.2675	0.9636	3.6021	74.48°	1.30
1.31	1.9880	1.7182	0.8643	3.7062	0.2698	0.2579	0.9662	3.7471	75.06°	1.31
1.32	2.0053	1.7381	0.8668	3.7434	0.2671	0.2482	0.9687	3.9033	75.63°	1.32
1.33	2.0228	1.7583	0.8693	3.7810	0.2645	0.2385	0.9712	4.0723	76.20°	1.33
1.34	2.0404	1.7786	0.8717	3.8190	0.2618	0.2288	0.9735	4.2556	76.78°	1.34
1.35	2.0583	1.7991	0.8741	3.8574	0.2592	0.2190	0.9757	4.4552	77.35°	1.35
1.36	2.0764	1.8198	0.8764	3.8962	0.2567	0.2092	0.9779	4.6734	77.92°	1.36
1.37	2.0947	1.8406	0.8787	3.9354	0.2541	0.1995	0.9799	4.9131	78.50°	1.37
1.38	2.1132	1.8617	0.8810	3.9749	0.2516	0.1896	0.9819	5.1774	79.07°	1.38
1.39	2.1320	1.8829	0.8832	4.0149	0.2491	0.1798	0.9837	5.4707	79.64°	1.39
1.40	2.1509	1.9043	0.8854	4.0552	0.2466	0.1700	0.9855	5.7979	80.21°	1.40
1.41	2.1701	1.9259	0.8875	4.0960	0.2441	0.1601	0.9871	6.1654	80.79°	1.41
1.42	2.1894	1.9477	0.8896	4.1371	0.2417	0.1502	0.9887	6.5811	81.36°	1.42
1.43	2.2090	1.9697	0.8917	4.1787	0.2393	0.1403	0.9901	7.0555	81.93°	1.43
1.44	2.2288	1.9919	0.8937	4.2207	0.2369	0.1304	0.9915	7.6018	82.51°	1.44
1.45	2.2488	2.0143	0.8957	4.2631	0.2346	0.1205	0.9927	8.2381	83.08°	1.45
1.46	2.2691	2.0369	0.8977	4.3060	0.2322	0.1106	0.9939	8.9886	83.65°	1.46
1.47	2.2896	2.0597	0.8996	4.3492	0.2299	0.1006	0.9949	9.8874	84.22°	1.47
1.48	2.3103	2.0827	0.9015	4.3929	0.2276	0.0907	0.9959	10.983	84.80°	1.48
1.49	2.3312	2.1059	0.9033	4.4371	0.2254	0.0807	0.9967	12.350	85.37°	1.49
1.50	2.3524	2.1293	0.9052	4.4817	0.2231	0.0707	0.9975	14.101	85.94°	1.50
1.51	2.3738	2.1529	0.9069	4.5267	0.2209	0.0608	0.9982	16.428	86.52°	1.51
1.52	2.3955	2.1768	0.9087	4.5722	0.2187	0.0508	0.9987	19.670	87.09°	1.52
1.53	2.4174	2.2008	0.9104	4.6182	0.2165	0.0408	0.9992	24.498	87.66°	1.53
1.54	2.4395	2.2251	0.9121	4.6646	0.2144	0.0308	0.9995	32.461	88.24°	1.54

1.55	2.4619	2.2496	0.9138	4.7115	0.2122	0.0208	0.9998	48.078	88.81°	1.55
1.56	2.4845	2.2743	0.9154	4.7588	0.2101	0.0108	0.9999	92.621	89.38°	1.56
1.57	2.5074	2.2993	0.9170	4.8066	0.2080	0.0008	1.0000	1255.8	89.95°	1.57
1.58	2.5305	2.3245	0.9186	4.8550	0.2060	−0.0092	1.0000	−108.65	90.53°	1.58
1.59	2.5538	2.3499	0.9202	4.9037	0.2039	−0.0192	0.9998	−52.067	91.10°	1.59
1.60	2.5775	2.3756	0.9217	4.9530	0.2019	−0.0292	0.9996	−34.233	91.67°	1.60
1.61	2.6014	2.4015	0.9232	5.0028	0.1999	−0.0392	0.9992	−25.495	92.25°	1.61
1.62	2.6255	2.4276	0.9246	5.0531	0.1979	−0.0492	0.9988	−20.307	92.82°	1.62
1.63	2.6499	2.4540	0.9261	5.1039	0.1959	−0.0592	0.9982	−16.871	93.39°	1.63
1.64	2.6746	2.4806	0.9275	5.1552	0.1940	−0.0691	0.9976	−14.427	93.97°	1.64
1.65	2.6995	2.5075	0.9289	5.2070	0.1920	−0.0791	0.9969	−12.599	94.54°	1.65
1.66	2.7247	2.5346	0.9302	5.2593	0.1901	−0.0891	0.9960	−11.181	95.11°	1.66
1.67	2.7502	2.5620	0.9316	5.3122	0.1882	−0.0990	0.9951	−10.047	95.68°	1.67
1.68	2.7760	2.5896	0.9329	5.3656	0.1864	−0.1090	0.9940	−9.1208	96.26°	1.68
1.69	2.8020	2.6175	0.9342	5.4195	0.1845	−0.1189	0.9929	−8.3492	96.83°	1.69
1.70	2.8283	2.6456	0.9354	5.4739	0.1827	−0.1288	0.9917	−7.6966	97.40°	1.70
1.71	2.8549	2.6741	0.9367	5.5290	0.1809	−0.1388	0.9903	−7.1373	97.98°	1.71
1.72	2.8818	2.7027	0.9379	5.5845	0.1791	−0.1487	0.9889	−6.6524	98.55°	1.72
1.73	2.9090	2.7317	0.9391	5.6407	0.1773	−0.1585	0.9874	−6.2281	99.12°	1.73
1.74	2.9364	2.7609	0.9402	5.6973	0.1755	−0.1684	0.9857	−5.8534	99.69°	1.74
1.75	2.9642	2.7904	0.9414	5.7546	0.1738	−0.1782	0.9840	−5.5204	100.27°	1.75
1.76	2.9922	2.8202	0.9425	5.8124	0.1720	−0.1881	0.9822	−5.2221	100.84°	1.76
1.77	3.0206	2.8503	0.9436	5.8709	0.1703	−0.1979	0.9802	−4.9534	101.41°	1.77
1.78	3.0493	2.8806	0.9447	5.9299	0.1686	−0.2077	0.9782	−4.7101	101.99°	1.78
1.79	3.0782	2.9113	0.9458	5.9895	0.1670	−0.2175	0.9761	−4.4887	102.56°	1.79
1.80	3.1075	2.9422	0.9468	6.0496	0.1653	−0.2272	0.9738	−4.2863	103.13°	1.80
1.81	3.1371	2.9734	0.9478	6.1104	0.1637	−0.2369	0.9715	−4.1005	103.71°	1.81
1.82	3.1669	3.0049	0.9488	6.1719	0.1620	−0.2466	0.9691	−3.9294	104.28°	1.82
1.83	3.1972	3.0367	0.9498	6.2339	0.1604	−0.2563	0.9666	−3.7712	104.85°	1.83
1.84	3.2277	3.0689	0.9508	6.2965	0.1588	−0.2660	0.9640	−3.6245	105.42°	1.84

HYPERBOLIC, EXPONENTIAL, AND TRIGONOMETRIC FUNCTIONS—(Continued)

x (Radians)	Cosh x	Sinh x	Tanh x	e^x	e^{-x}	Cos x	Sin x	Tan x	Degrees	x
1.85	3.2585	3.1013	0.9518	6.3598	0.1572	−0.2756	0.9613	−3.4881	106.00°	1.85
1.86	3.2897	3.1340	0.9527	6.4237	0.1557	−0.2852	0.9585	−3.3609	106.57°	1.86
1.87	3.3212	3.1671	0.9536	6.4883	0.1541	−0.2948	0.9556	−3.3219	107.14°	1.87
1.88	3.3531	3.2005	0.9545	6.5535	0.1526	−0.3043	0.9526	−3.1304	107.72°	1.88
1.89	3.3852	3.2342	0.9554	6.6194	0.1511	−0.3138	0.9495	−3.0257	108.29°	1.89
1.90	3.4177	3.2682	0.9562	6.6859	0.1496	−0.3233	0.9463	−2.9271	108.86°	1.90
1.91	3.4506	3.3025	0.9571	6.7531	0.1481	−0.3327	0.9430	−2.8341	109.43°	1.91
1.92	3.4838	3.3372	0.9579	6.8210	0.1466	−0.3421	0.9396	−2.7463	110.01°	1.92
1.93	3.5173	3.3722	0.9587	6.8895	0.1451	−0.3515	0.9362	−2.6632	110.58°	1.93
1.94	3.5512	3.4075	0.9595	6.9588	0.1437	−0.3609	0.9326	−2.5843	111.15°	1.94
1.95	3.5855	3.4432	0.9603	7.0287	0.1423	−0.3702	0.9290	−2.5095	111.73°	1.95
1.96	3.6201	3.4792	0.9611	7.0993	0.1409	−0.3795	0.9252	−2.4383	112.30°	1.96
1.97	3.6551	3.5156	0.9619	7.1707	0.1395	−0.3887	0.9214	−2.3705	112.87°	1.97
1.98	3.6904	3.5523	0.9626	7.2427	0.1381	−0.3979	0.9174	−2.3058	113.45°	1.98
1.99	3.7261	3.5894	0.9633	7.3155	0.1367	−0.4070	0.9134	−2.2441	114.02°	1.99
2.00	3.7622	3.6269	0.9640	7.3891	0.1353	−0.4161	0.9093	−2.1850	114.59°	2.00
2.01	3.7987	3.6647	0.9647	7.4633	0.1340	−0.4252	0.9051	−2.1285	115.16°	2.01
2.02	3.8355	3.7028	0.9654	7.5383	0.1327	−0.4343	0.9008	−2.0744	115.74°	2.02
2.03	3.8727	3.7414	0.9661	7.6141	0.1313	−0.4432	0.8964	−2.0224	116.31°	2.03
2.04	3.9103	3.7803	0.9668	7.6906	0.1300	−0.4522	0.8919	−1.9725	116.88°	2.04
2.05	3.9483	3.8196	0.9674	7.7679	0.1287	−0.4611	0.8874	−1.9246	117.46°	2.05
2.06	3.9867	3.8593	0.9680	7.8460	0.1275	−0.4699	0.8827	−1.8784	118.03°	2.06
2.07	4.0255	3.8993	0.9687	7.9248	0.1262	−0.4787	0.8780	−1.8340	118.60°	2.07
2.08	4.0647	3.9398	0.9693	8.0045	0.1249	−0.4875	0.8731	−1.7911	119.18°	2.08
2.09	4.1043	3.9806	0.9699	8.0849	0.1237	−0.4962	0.8682	−1.7498	119.75°	2.09

2.10	120.32°	−1.7099	0.8632	−0.5048	0.1225	8.1662	0.9705	4.0219	4.1443	2.10
2.11	120.89°	−1.6713	0.8581	−0.5135	0.1212	8.2482	0.9710	4.0635	4.1847	2.11
2.12	121.47°	−1.6340	0.8529	−0.5220	0.1200	8.3311	0.9716	4.1056	4.2256	2.12
2.13	122.04°	−1.5979	0.8477	−0.5305	0.1188	8.4149	0.9722	4.1480	4.2669	2.13
2.14	122.61°	−1.5629	0.8423	−0.5390	0.1177	8.4994	0.9727	4.1909	4.3086	2.14
2.15	123.19°	−1.5290	0.8369	−0.5474	0.1165	8.5849	0.9732	4.2342	4.3507	2.15
2.16	123.76°	−1.4961	0.8314	−0.5557	0.1153	8.6711	0.9738	4.2779	4.3932	2.16
2.17	124.33°	−1.4642	0.8258	−0.5640	0.1142	8.7583	0.9743	4.3221	4.4362	2.17
2.18	124.90°	−1.4332	0.8201	−0.5722	0.1130	8.8463	0.9748	4.3666	4.4797	2.18
2.19	125.48°	−1.4031	0.8143	−0.5804	0.1119	8.9352	0.9753	4.4117	4.5236	2.19
2.20	126.05°	−1.3738	0.8085	−0.5885	0.1108	9.0250	0.9757	4.4571	4.5679	2.20
2.21	126.62°	−1.3453	0.8026	−0.5966	0.1097	9.1157	0.9762	4.5030	4.6127	2.21
2.22	127.20°	−1.3176	0.7966	−0.6046	0.1086	9.2073	0.9767	4.5494	4.6580	2.22
2.23	127.77°	−1.2906	0.7905	−0.6125	0.1075	9.2999	0.9771	4.5962	4.7037	2.23
2.24	128.34°	−1.2643	0.7843	−0.6204	0.1065	9.3933	0.9776	4.6434	4.7499	2.24
2.25	128.92°	−1.2366	0.7781	−0.6282	0.1054	9.4877	0.9780	4.6912	4.7966	2.25
2.26	129.49°	−1.2136	0.7718	−0.6359	0.1044	9.5831	0.9785	4.7394	4.8437	2.26
2.27	130.06°	−1.1892	0.7654	−0.6436	0.1033	9.6764	0.9789	4.7880	4.8914	2.27
2.28	130.63°	−1.1653	0.7589	−0.6512	0.1023	9.7767	0.9793	4.8372	4.9395	2.28
2.29	131.21°	−1.1420	0.7523	−0.6588	0.1013	9.8749	0.9797	4.8868	4.9881	2.29
2.30	131.78°	−1.1192	0.7457	−0.6663	0.1003	9.9742	0.9801	4.9370	5.0372	2.30
2.31	132.35°	−1.0969	0.7390	−0.6737	0.0993	10.074	0.9805	4.9876	5.0868	2.31
2.32	132.93°	−1.0751	0.7322	−0.6811	0.0983	10.176	0.9809	5.0387	5.1370	2.32
2.33	133.50°	−1.0538	0.7254	−0.6883	0.0973	10.278	0.9812	5.0903	5.1876	2.33
2.34	134.07°	−1.0329	0.7185	−0.6956	0.0963	10.381	0.9816	5.1425	5.2388	2.34
2.35	134.65°	−1.0125	0.7115	−0.7027	0.0954	10.486	0.9820	5.1951	5.2905	2.35
2.36	135.22°	−0.9924	0.7044	−0.7098	0.0944	10.591	0.9823	5.2483	5.3427	2.36
2.37	135.79°	−0.9728	0.6973	−0.7168	0.0935	10.697	0.9827	5.3020	5.3954	2.37
2.38	136.36°	−0.9535	0.6901	−0.7237	0.0926	10.805	0.9830	5.3562	5.4487	2.38
2.39	136.94°	−0.9346	0.6828	−0.7306	0.0916	10.913	0.9834	5.4109	5.5026	2.39

HYPERBOLIC, EXPONENTIAL, AND TRIGONOMETRIC FUNCTIONS—(Continued)

x (Radians)	Cosh x	Sinh x	Tanh x	e^x	e^-x	Cos x	Sin x	Tan x	Degrees	x
2.40	5.5570	5.4662	0.9837	11.023	0.0907	−0.7374	0.6755	−0.9160	137.51°	2.40
2.41	5.6119	5.5221	0.9840	11.134	0.0898	−0.7441	0.6681	−0.8978	138.08°	2.41
2.42	5.6674	5.5785	0.9843	11.246	0.0889	−0.7508	0.6606	−0.8799	138.66°	2.42
2.43	5.7235	5.6354	0.9846	11.359	0.0880	−0.7573	0.6530	−0.8623	139.23°	2.43
2.44	5.7801	5.6929	0.9849	11.473	0.0872	−0.7638	0.6454	−0.8450	139.80°	2.44
2.45	5.8373	5.7510	0.9852	11.588	0.0863	−0.7702	0.6378	−0.8280	140.37°	2.45
2.46	5.8951	5.8097	0.9855	11.705	0.0854	−0.7766	0.6300	−0.8113	140.95°	2.46
2.47	5.9535	5.8689	0.9858	11.822	0.0846	−0.7828	0.6222	−0.7949	141.52°	2.47
2.48	6.0125	5.9288	0.9861	11.941	0.0837	−0.7890	0.6144	−0.7787	142.09°	2.48
2.49	6.0721	5.9892	0.9864	12.061	0.0829	−0.7951	0.6065	−0.7627	142.67°	2.49
2.50	6.1323	6.0502	0.9866	12.182	0.0821	−0.8011	0.5985	−0.7470	143.24°	2.50
2.51	6.1931	6.1118	0.9869	12.305	0.0813	−0.8071	0.5904	−0.7316	143.81°	2.51
2.52	6.2545	6.1741	0.9871	12.429	0.0805	−0.8130	0.5823	−0.7163	144.39°	2.52
2.53	6.3166	6.2369	0.9874	12.554	0.0797	−0.8187	0.5742	−0.7013	144.96°	2.53
2.54	6.3793	6.3004	0.9876	12.680	0.0789	−0.8244	0.5660	−0.6865	145.53°	2.54
2.55	6.4426	6.3645	0.9879	12.807	0.0781	−0.8301	0.5577	−0.6719	146.10°	2.55
2.56	6.5066	6.4293	0.9881	12.936	0.0773	−0.8356	0.5494	−0.6575	146.68°	2.56
2.57	6.5712	6.4946	0.9884	13.066	0.0765	−0.8410	0.5410	−0.6432	147.25°	2.57
2.58	6.6365	6.5607	0.9886	13.197	0.0758	−0.8464	0.5325	−0.6292	147.82°	2.58
2.59	6.7024	6.6274	0.9888	13.330	0.0750	−0.8517	0.5240	−0.6153	148.40°	2.59
2.60	6.7690	6.6947	0.9890	13.464	0.0743	−0.8569	0.5155	−0.6016	148.97°	2.60
2.61	6.8363	6.7628	0.9892	13.599	0.0735	−0.8620	0.5069	−0.5881	149.54°	2.61
2.62	6.9043	6.8315	0.9895	13.736	0.0728	−0.8670	0.4983	−0.5747	150.11°	2.62
2.63	6.9729	6.9009	0.9897	13.874	0.0721	−0.8720	0.4896	−0.5615	150.69°	2.63
2.64	7.0423	6.9709	0.9899	14.013	0.0714	−0.8768	0.4808	−0.5484	151.26°	2.64

2.65	7.1123	7.0417	0.9901	14.154	0.0707	−0.8816	0.4720	−0.5354	151.83°	2.65
2.66	7.1831	7.1132	0.9903	14.296	0.0699	−0.8863	0.4632	−0.5226	152.41°	2.66
2.67	7.2546	7.1854	0.9905	14.440	0.0693	−0.8909	0.4543	−0.5100	152.98°	2.67
2.68	7.3268	7.2583	0.9906	14.585	0.0686	−0.8953	0.4454	−0.4974	153.55°	2.68
2.69	7.3998	7.3319	0.9908	14.732	0.0679	−0.8998	0.4364	−0.4850	154.13°	2.69
2.70	7.4735	7.4063	0.9910	14.880	0.0672	−0.9041	0.4274	−0.4727	154.70°	2.70
2.71	7.5479	7.4814	0.9912	15.029	0.0665	−0.9083	0.4183	−0.4606	155.27°	2.71
2.72	7.6231	7.5572	0.9914	15.180	0.0659	−0.9124	0.4092	−0.4485	155.84°	2.72
2.73	7.6991	7.6338	0.9915	15.333	0.0652	−0.9165	0.4001	−0.4365	156.42°	2.73
2.74	7.7758	7.7112	0.9917	15.487	0.0646	−0.9204	0.3909	−0.4247	156.99°	2.74
2.75	7.8533	7.7894	0.9919	15.643	0.0639	−0.9243	0.3817	−0.4129	157.56°	2.75
2.76	7.9316	7.8683	0.9920	15.800	0.0633	−0.9281	0.3724	−0.4013	158.14°	2.76
2.77	8.0107	7.9480	0.9922	15.959	0.0627	−0.9318	0.3631	−0.3897	158.71°	2.77
2.78	8.0905	8.0285	0.9923	16.119	0.0620	−0.9353	0.3538	−0.3782	159.28°	2.78
2.79	8.1712	8.1098	0.9925	16.281	0.0614	−0.9398	0.3414	−0.3668	159.86°	2.79
2.80	8.2527	8.1919	0.9926	16.445	0.0608	−0.9422	0.3350	−0.3555	160.43°	2.80
2.81	8.3351	8.2749	0.9928	16.610	0.0602	−0.9455	0.3256	−0.3443	161.00°	2.81
2.82	8.4182	8.3586	0.9929	16.777	0.0596	−0.9487	0.3161	−0.3332	161.57°	2.82
2.83	8.5022	8.4432	0.9931	16.945	0.0590	−0.9519	0.3066	−0.3221	162.15°	2.83
2.84	8.5871	8.5287	0.9932	17.116	0.0584	−0.9549	0.2970	−0.3111	162.72°	2.84
2.85	8.6728	8.6150	0.9933	17.288	0.0578	−0.9578	0.2875	−0.3002	163.29°	2.85
2.86	8.7594	8.7021	0.9935	17.462	0.0573	−0.9606	0.2779	−0.2893	163.87°	2.86
2.87	8.8469	8.7902	0.9936	17.637	0.0567	−0.9633	0.2683	−0.2785	164.44°	2.87
2.88	8.9352	8.8791	0.9937	17.814	0.0561	−0.9660	0.2586	−0.2677	165.01°	2.88
2.89	9.0244	8.9689	0.9938	17.993	0.0556	−0.9685	0.2490	−0.2570	165.58°	2.89
2.90	9.1146	9.0596	0.9940	18.174	0.0550	−0.9710	0.2392	−0.2464	166.16°	2.90
2.91	9.2056	9.1512	0.9941	18.357	0.0545	−0.9733	0.2295	−0.2358	166.73°	2.91
2.92	9.2976	9.2437	0.9942	18.541	0.0539	−0.9756	0.2198	−0.2253	167.30°	2.92
2.93	9.3905	9.3371	0.9943	18.728	0.0534	−0.9777	0.2100	−0.2148	167.88°	2.93
2.94	9.4844	9.4315	0.9944	18.916	0.0529	−0.9798	0.2002	−0.2044	168.45°	2.94

HYPERBOLIC, EXPONENTIAL, AND TRIGONOMETRIC FUNCTIONS—(Continued)

x (Radians)	Cosh x	Sinh x	Tanh x	ϵ^x	ϵ^{-x}	Cos x	Sin x	Tan x	Degrees	x
2.95	9.5792	9.5268	0.9945	19.106	0.0523	−0.9817	0.1904	−0.1940	169.02°	2.95
2.96	9.6749	9.6231	0.9946	19.298	0.0518	−0.9836	0.1806	−0.1836	169.60°	2.96
2.97	9.7716	9.7203	0.9948	19.492	0.0513	−0.9853	0.1708	−0.1733	170.17°	2.97
2.98	9.8693	9.8185	0.9949	19.688	0.0508	−0.9870	0.1609	−0.1630	170.74°	2.98
2.99	9.9680	9.9177	0.9950	19.886	0.0503	−0.9885	0.1510	−0.1528	171.31°	2.99
3.0	10.068	10.018	0.9951	20.086	0.0498	−0.9900	0.1411	−0.1426	171.89°	3.0
3.1	11.122	11.077	0.9960	22.198	0.0450	−0.9991	0.0416	−0.0416	177.62°	3.1
3.2	12.287	12.246	0.9967	24.533	0.0408	−0.9983	−0.0584	0.0585	183.35°	3.2
3.3	13.575	13.538	0.9973	27.113	0.0369	−0.9875	−0.1577	0.1598	189.08°	3.3
3.4	14.999	14.965	0.9978	29.964	0.0334	−0.9668	−0.2555	0.2643	194.81°	3.4
3.5	16.573	16.543	0.9982	33.115	0.0302	−0.9365	−0.3508	0.3746	200.54°	3.5
3.6	18.313	18.286	0.9985	36.598	0.0273	−0.8968	−0.4425	0.4935	206.26°	3.6
3.7	20.236	20.211	0.9988	40.447	0.0247	−0.8481	−0.5298	0.6247	211.99°	3.7
3.8	22.362	22.339	0.9990	44.701	0.0224	−0.7910	−0.6119	0.7736	217.72°	3.8
3.9	24.711	24.691	0.9992	49.402	0.0202	−0.7259	−0.6878	0.9474	223.45°	3.9
4.0	27.308	27.290	0.9993	54.598	0.0183	−0.6536	−0.7568	1.1578	229.18°	4.0
4.1	30.178	30.162	0.9995	60.340	0.0166	−0.5748	−0.8183	1.4235	234.91°	4.1
4.2	33.351	33.336	0.9996	66.686	0.0150	−0.4903	−0.8716	1.7778	240.64°	4.2
4.3	36.857	36.843	0.9996	73.700	0.0136	−0.4008	−0.9162	2.2859	246.37°	4.3
4.4	40.732	40.719	0.9997	81.451	0.0123	−0.3073	−0.9516	3.0963	252.10°	4.4
4.5	45.014	45.003	0.9998	90.017	0.0111	−0.2108	−0.9775	4.6373	257.83°	4.5
4.6	49.747	49.737	0.9998	99.484	0.0101	−0.1122	−0.9937	8.8602	263.56°	4.6
4.7	54.978	54.969	0.9998	109.95	0.0091	−0.0124	−0.9999	80.713	269.29°	4.7
4.8	60.759	60.751	0.9999	121.51	0.0082	0.0875	−0.9962	−11.385	275.02°	4.8
4.9	67.149	67.141	0.9999	134.29	0.0074	0.1865	−0.9825	−5.2675	280.75°	4.9

x										x
5.0	74.210	74.203	0.9999	148.41	0.0067	0.2837	−0.9589	−3.3805	286.48°	5.0
5.1	82.014	82.008	0.9999	164.02	0.0061	0.3780	−0.9258	−2.4494	292.21°	5.1
5.2	90.639	90.633	0.9999	181.27	0.0055	0.4685	−0.8835	−1.8856	297.94°	5.2
5.3	100.17	100.17	1.0000	200.34	0.0050	0.5544	−0.8323	−1.5013	303.67°	5.3
5.4	110.71	110.70	1.0000	221.41	0.0045	0.6347	−0.7728	−1.2175	309.40°	5.4
5.5	122.35	122.34	1.0000	244.69	0.0041	0.7087	−0.7055	−0.9956	315.13°	5.5
5.6	135.22	135.21	1.0000	270.43	0.0037	0.7756	−0.6313	−0.8139	320.86°	5.6
5.7	149.44	149.43	1.0000	298.87	0.0033	0.8347	−0.5507	−0.6597	326.59°	5.7
5.8	165.15	165.15	1.0000	330.30	0.0030	0.8855	−0.4646	−0.5247	332.32°	5.8
5.9	182.52	182.52	1.0000	365.04	0.0027	0.9275	−0.3739	−0.4031	338.05°	5.9
6.0	201.72	201.71	1.0000	403.43	0.0025	0.9602	−0.2754	−0.2910	343.77°	6.0

APPENDIX V

COMMON LOGARITHMS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	.0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

COMMON LOGARITHMS OF NUMBERS—(*Continued*)

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

APPENDIX VI

WIRE TABLES*

BARE CONCENTRIC-LAY CABLES OF STANDARD ANNEALED COPPER

Size of Cable		Ohms per 1000 Feet		Pounds per 1000 Feet	Standard Concentric Stranding			Flexible Concentric Stranding		
Circular Mils	A.W.G. No.	25° C. (=77° F.)	65° C. (=149° F.)		Number of Wires	Diameter of Wires, in Mils	Outside Diameter, in Mils	Number of Wires	Diameter of Wires, in Mils	Outside Diameter, in Mils
2,000,000		0.00539	0.00622	6180.0	127	125.5	1631.0	169	108.8	1632.0
1,900,000		0.00568	0.00655	5870.0	127	122.3	1590.0	169	106.0	1590.0
1,800,000		0.00599	0.00692	5560.0	127	119.1	1548.0	169	103.2	1548.0
1,700,000		0.00634	0.00732	5250.0	127	115.7	1504.0	169	100.3	1504.0
1,600,000		0.00674	0.00778	4940.0	127	112.2	1459.0	169	97.3	1460.0
1,500,000		0.00719	0.00830	4630.0	91	128.4	1412.0	127	108.7	1413.0
1,400,000		0.00770	0.00889	4320.0	91	124.0	1364.0	127	105.0	1365.0
1,300,000		0.00830	0.00958	4010.0	91	119.5	1315.0	127	101.2	1315.0
1,200,000		0.00899	0.0104	3710.0	91	114.8	1263.0	127	97.2	1264.0
1,100,000		0.00981	0.0114	3400.0	91	109.9	1209.0	127	93.1	1210.0
1,000,000		0.0108	0.0124	3090.0	61	128.0	1152.0	91	104.8	1153.0
950,000		0.0114	0.0131	2930.0	61	124.8	1123.0	91	102.2	1124.0
900,000		0.0120	0.0138	2780.0	61	121.5	1093.0	91	99.4	1094.0
850,000		0.0127	0.0146	2620.0	61	118.0	1062.0	91	96.6	1063.0
800,000		0.0135	0.0156	2470.0	61	114.5	1031.0	91	93.8	1031.0
750,000		0.0144	0.0166	2320.0	61	110.9	998.0	91	90.8	999.0
700,000		0.0154	0.0178	2160.0	61	107.1	964.0	91	87.7	965.0
650,000		0.0166	0.0192	2010.0	61	103.2	929.0	91	84.5	930.0
600,000		0.0180	0.0207	1850.0	61	99.2	893.0	91	81.2	893.0
550,000		0.0196	0.0226	1700.0	61	95.0	855.0	91	77.7	855.0
500,000		0.0216	0.0249	1540.0	37	116.2	814.0	61	90.5	815.0
450,000		0.0240	0.0277	1390.0	37	110.3	772.0	61	85.9	773.0
400,000		0.0270	0.0311	1240.0	37	104.0	728.0	61	81.0	729.0

Size of Cable		Ohms per 1000 Feet		Pounds per 1000 Feet	Standard Concentric Stranding			Flexible Concentric Stranding		
Circular Mils	A.W.G. No.	25° C. (=77° F.)	65° C. (=149° F.)		Number of Wires	Diameter of Wires, in Mils	Outside Diameter, in Mils	Number or Wires	Diameter of Wires, in Mils	Outside Diameter, in Mils
350,000		0.0308	0.0356	1080.0	37	97.3	681.0	61	75.7	682.0
300,000		0.0360	0.0415	926.0	37	90.0	630.0	61	70.1	631.0
250,000		0.0431	0.0498	772.0	37	82.2	575.0	61	64.0	576.0
212,000	0000	0.0509	0.0587	653.0	19	105.5	528.0	37	75.6	533.0
168,000	000	0.0642	0.0741	518.0	19	94.0	470.0	37	67.3	471.0
133,000	00	0.0811	0.0936	411.0	19	83.7	418.0	37	60.0	420.0
106,000	0	0.102	0.117	326.0	19	74.5	373.0	37	53.4	374.0
83,700	1	0.129	0.149	258.0	19	66.4	332.9	37	47.6	333.0
66,400	2	0.162	0.187	205.0	7	97.4	292.0	19	59.1	296.0
52,600	3	0.205	0.237	163.0	7	86.7	260.0	19	52.6	263.0
41,700	4	0.259	0.299	129.0	7	77.2	232.0	19	46.9	234.0
33,100	5	0.326	0.376	102.0	7	68.8	206.0	19	41.7	209.0
26,300	6	0.410	0.473	81.0	7	61.2	184.0	19	37.2	186.0
20,800	7	0.519	0.599	64.3	7	54.5	164.0	19	33.1	166.0
16,500	8	0.654	0.755	51.0	7	48.6	146.0	19	29.5	147.0

NOTE 1. The fundamental resistivity used in calculating the table is the International Annealed Copper Standard, viz. 0.15328 ohm (meter, gram) at 20° C. (increased by 2 per cent to allow for increased length due to stranding). The density is 8.89 grams per cubic centimeter.

NOTE 2.— This table is in accord with standards adopted by the Standards Committee of the American Institute of Electrical Engineers, both in respect to the "Number of wires" and in respect to the correction for increase of resistance and mass due to the twist of the wires. The values given for "Ohms per 1000 feet" and "Pounds per 1000 feet" are 2 per cent greater than for a solid rod of cross section equal to the total cross section of the wires of the cable. This increment of 2 per cent means that the values are correct for cables having a lay of 1 in 15.7. For any other lay, equal to 1 in n, resistance or mass may be calculated by increasing the above tabulated values by

$$\left(\frac{493.}{n^2} - 2\right)\%$$

STANDARD ANNEALED COPPER WIRE
American Wire Gage (B. & S.)

| Gage No. | Diameter in Mils | Cross Section | | Ohms per 1000 Feet | | Pounds per 1000 Feet |
		Circular Mils	Square Inches	25° C. (= 77° F.)	65° C. (= 149° F.)	
0000	460.0	212,000.0	0.166	0.0500	0.0577	641.0
000	410.0	168,000.0	0.132	0.0630	0.0727	508.0
00	365.0	133,000.0	0.105	0.0795	0.0917	403.0
0	325.0	106,000.0	0.0829	0.100	0.116	319.0
1	289.0	83,700.0	0.0657	0.126	0.146	253.0
2	258.0	66,400.0	0.0521	0.159	0.184	201.0
3	229.0	52,600.0	0.0413	0.201	0.232	159.0
4	204.0	41,700.0	0.0328	0.253	0.292	126.0
5	182.0	33,100.0	0.0260	0.319	0.369	100.0
6	162.0	26,300.0	0.0206	0.403	0.465	79.5
7	144.0	20,800.0	0.0164	0.508	0.586	63.0
8	128.0	16,500.0	0.0130	0.641	0.739	50.0
9	114.0	13,100.0	0.0103	0.808	0.932	39.6
10	102.0	10,400.0	0.00815	1.02	1.18	31.4
11	91.0	8,230.0	0.00647	1.28	1.48	24.9
12	81.0	6,530.0	0.00513	1.62	1.87 ·	19.8
13	72.0	5,180.0	0.00407	2.04	2.36	15.7
14	64.0	4,110.0	0.00323	2.58	2.97	12.4
15	57.0	3,260.0	0.00256	3.25	3.75	9.86
16	51.0	2,580.0	0.00203	4.09	4.73	7.82
17	45.0	2,050.0	0.00161	5.16	5.96	6.20
18	40.0	1,620.0	0.00128	6.51	7.51	4.92
19	36.0	1,290.0	0.00101	8.21	9.48	3.90
20	32.0	1,020.0	0.000802	10.4	11.9	3.09
21	28.5	810.0	0.000636	13.1	15.1	2.45
22	25.3	642.0	0.000505	16.5	19.0	1.94
23	22.6	509.0	0.000400	20.8	24.0	1.54
24	20.1	404.0	0.000317	26.2	30.2	1.22
25	17.9	320.0	0.000252	33.0	38.1	0.970
26	15.9	254.0	0.000200	41.6	48.0	0.769
27	14.2	202.0	0.000158	52.5	60.6	0.610
28	12.6	160.0	0.000126	66.2	76.4	0.484
29	11.3	127.0	0.0000995	83.4	96.3	0.384
30	10.0	101.0	0.0000789	105.0	121.0	0.304
31	8.9	79.7	0.0000626	133.0	153.0	0.241
32	8.0	63.2	0.0000496	167.0	193.0	0.191
33	7.1	50.1	0.0000394	211.0	243.0	0.152
34	6.3	39.8	0.0000312	266.0	307.0	0.120
35	5.6	31.5	0.0000248	335.0	387.0	0.0954
36	5.0	25.0	0.0000196	423.0	488.0	0.0757
37	4.5	19.8	0.0000156	533.0	616.0	0.0600
38	4.0	15.7	0.0000123	673.0	776.0	0.0476
39	3.5	12.5	0.0000098	848.0	979.0	0.0377
40	3.1	9.9	0.0000078	1070.0	1230.0	0.0299

NOTE 1.—The *fundamental resistivity* used in calculating the tables is the International Annealed Copper Standard, viz., 0.15328 ohm (meter, gram) at 20° C. The *temperature coefficient* for this particular resistivity is $a_{20} = 0.00393$, or $a_0 = 0.00427$. However, the temperature coefficient is proportional to the conductivity, and hence the change of *resistivity* per degree C. is a constant, 0.000597 ohm (meter, gram). The "constant mass" temperature coefficient of any sample is

$$a_t = \frac{0.000597 + 0.000005}{\text{resistivity in ohms (meter, gram) at } t° C.}.$$

The *density* is 8.89 grams per cubic centimeter.

NOTE 2.—The values given in the table are only for annealed copper of the standard resistivity. The user of the table must apply the proper correction for copper of any other resistivity. Hard-drawn copper may be taken as about 2.7 per cent higher resistivity than annealed copper.

NOTE 3.—Ohms per mile, or pounds per mile, may be obtained by multiplying the respective values above by 5.28.

ALUMINUM CABLE STEEL REINFORCED*
(A.C.S.R.)

B. & S. Gage No. (A.W.G.)	A.C.S.R. Aluminum Area Circular Mils	Aluminum Area Square Inches	Copper Equivalent C.M. or No.	Copper Equivalent Square Inches	Usual Stranding (Inches) Aluminum	Usual Stranding (Inches) Steel	Elastic Limit, Lb.	Ultimate Str'th, Lb.	Ohms per 1000 Feet (61%)	Diam. In.	Weight Per 1000 Feet Total	Per 1000 Feet Al.	Per 1000 Feet Steel	Per Mile Total	Per Mile Al.	Per Mile Steel
.....	900000	.7060	566000	.4442	54X.1291	7X.1291	21800	31600	.0193	1.162	1158.0	844.0	314.0	6120	4462	1658
.....	795000	.6244	500000	.3927	54X.1214	7X.1214	19250	27950	.0217	1.093	1024.0	747.0	277.0	5407	3944	1463
.....	715500	.5620	450000	.3532	54X.1151	7X.1151	17360	25200	.0241	1.036	920.0	671.0	249.0	4857	3542	1315
.....	605000	.4750	380500	.2987	54X.1059	7X.1059	14675	21270	.0286	.953	779.0	568.0	211.0	4113	2999	1114
.....	500000	.3927	314500	.2468	30X.1291	7X.1291	17400	24080	.0347	.904	783.0	469.0	314.0	4135	2477	1658
.....	397500	.3122	250000	.1962	30X.1151	7X.1151	13250	19170	.0435	.806	622.0	373.0	249.0	3284	1969	1315
.....	336400	.2642	No. 4/0	.1662	30X.1059	7X.1059	11715	16200	.0515	.741	527.6	316.0	211.0	2783	1669	1114
.....	266800	.2094	No. 3/0	.1318	6X.2108	7X.0705	6470	9385	.0648	.633	343.0	250.0	93.0	1811	1319	492
No. 4/0	211600	.1662	No. 2/0	.1045	6X.1880	1X.1880	5940	8435	.0816	.564	295.0	199.0	96.0	1556	1052	504
No. 3/0	167805	.1318	No. 1/0	.0829	6X.1670	1X.1670	4690	6660	.1026	.501	232.5	157.0	75.5	1227	830	397
No. 2/0	133079	.1045	No. 1	.0657	6X.1490	1X.1490	3730	5300	.1294	.447	185.0	125.0	60.0	977	660	317
No. 1/0	105534	.0829	No. 2	.0521	6X.1327	1X.1327	2960	4200	.1639	.398	147.0	99.5	47.5	776	525	251
No. 1	83694	.0657	No. 3	.0413	6X.1182	1X.1182	2355	3340	.2070	.355	117.0	79.0	38.0	617	417	200
No. 2	66373	.0521	No. 4	.0328	6X.1052	1X.1052	1860	2660	.2610	.316	92.4	62.5	29.9	488	330	158
No. 3	52634	.0413	No. 5	.0260	6X.0938	1X.0938	1480	2100	.3291	.281	73.4	49.7	23.7	387	262	125
No. 4	41742	.0328	No. 6	.0206	6X.0834	1X.0834	1170	1665	.4655	.250	58.0	39.3	18.7	306	207	99
No. 5	33102	.0260	No. 7	.0163	6X.0743	1X.0743	930	1315	.5217	.223	46.0	31.0	15.0	243	164	79
No. 6	26250	.0206	No. 8	.0130	6X.0661	1X.0661	735	1045	.6577	.198	36.4	24.6	11.8	192	130	62
No. 7	20816	.0163	No. 9	.0103	6X.0586	1X.0586	575	820	.8293	.176	28.5	19.3	9.2	151	102	49
No. 8	16509	.0130	No. 10	.0082	6X.0525	1X.0525	465	660	1.0450	.158	23.0	15.6	7.4	121	82	39

Courtesy of Aluminum Company of America.

ALL ALUMINUM CABLE

B. & S. Gage No. (A.W.G.)	Aluminum Area Cir. Mils	Aluminum Area Sq. Inches	Copper Equivalent C. M. or No.	Copper Equivalent Square Inches	Usual Stranding (Inches)	Elastic Limit, Lb.	Ultimate Strength, Lb.	Ohms per 1000 Feet (61%)	Diam., Inches	Weight—Lb. Per 1000 Feet	Weight—Lb. Per Mile
....	1590000	1.249	1000000	.7854	61×.1615	17500	30000	.0109	1.454	1493	7880
....	1515000	1.190	950000	.7460	61×.1577	16650	28600	.0114	1.419	1423	7510
....	1431000	1.124	900000	.7065	61×.1533	15750	27000	.0121	1.380	1345	7100
....	1351500	1.061	850000	.6672	61×.1490	14850	25500	.0127	1.341	1270	7105
....	1272000	.9990	800000	.6280	61×.1445	14000	24000	.0135	1.301	1195	6310
....	1192500	.9366	750000	.5890	61×.1398	13100	22500	.0145	1.257	1120	5915
....	1113000	.8742	700000	.5494	61×.1351	12250	21000	.0155	1.215	1046	5520
....	1033500	.8117	650000	.5102	37×.1672	11350	19500	.0167	1.170	971	5120
....	954000	.7493	600000	.4710	37×.1606	10500	18000	.0181	1.124	896	4730
....	874500	.6868	550000	.4318	37×.1538	9600	16500	.0197	1.077	822	4350
....	795000	.6244	500000	.3927	37×.1465	8750	15000	.0217	1.026	747	3940
....	750000	.5890	472000	.3707	37×.1425	8250	14140	.0230	.994	705	3720
....	715500	.5620	450000	.3533	37×.1391	7870	13500	.0241	.974	672	3550
....	636000	.4995	400000	.3140	37×.1312	7000	12000	.0271	.918	598	3155
....	556500	.4371	350000	.2748	19×.1711	6120	10500	.0311	.856	523	2760
....	500000	.3927	314500	.2469	19×.1623	5500	9420	.0347	.810	469	2480
....	477000	.3746	300000	.2356	19×.1585	5240	9000	.0363	.793	448	2365
....	397500	.3122	250000	.1962	19×.1447	4370	7490	.0435	.724	373	1970
....	336400	.2642	No. 4/0	.1662	19×.1330	3700	6350	.0515	.657	316	1668
....	300000	.2356	188800	.1482	19×.1256	3300	5650	.0578	.621	282	1489
....	266800	.2094	No. 3/0	.1318	7×.1953	2940	5040	.0648	.586	251	1325
No. 4/0	211600	.1662	No. 2/0	.1045	7×.1740	2330	3970	.0816	.522	199	1050
No. 3/0	167805	.1318	No. 1/0	.0829	7×.1548	1845	3180	.1026	.464	158	834
No. 2/0	133079	.1045		.0657	7×.1380	1465	2520	.1294	.414	125	655
No. 1/0	105534	.0829	No. 2	.0521	7×.1228	1160	1990	.1639	.368	99.2	524
No. 1	83694	.0657	No. 3	.0413	7×.1093	920	1580	.2070	.328	78.6	415
No. 2	66373	.0521	No. 4	.0328	7×.0975	730	1250	.2610	.293	62.4	329
No. 3	52634	.0413	No. 5	.0260	7×.0868	580	992	.3291	.258	49.5	261
No. 4	41742	.0328	No. 6	.0206	7×.0772	460	786	.4150	.232	39.2	207

INDUCTIVE REACTANCE IN OHMS PER MILE AT 60~ OF ONE CONDUCTOR OF A SINGLE-PHASE OR EQUILATERAL THREE-PHASE LINE

Cir. Mils or A.W.G.	Interaxial Spacing in Feet												
	1	1.5	2	3	4	5	6	8	10	12	15	20	25
1,200,000	0.397	0.444	0.483	0.530	0.566	0.592	0.617	0.650	0.675	0.698	0.725	0.762	0.788
1,100,000	0.403	0.451	0.487	0.536	0.571	0.599	0.621	0.656	0.684	0.703	0.732	0.766	0.796
1,000,000	0.409	0.458	0.493	0.542	0.577	0.605	0.627	0.662	0.690	0.709	0.738	0.771	0.800
900,000	0.415	0.465	0.500	0.548	0.583	0.612	0.632	0.668	0.695	0.715	0.743	0.778	0.806
800,000	0.421	0.473	0.507	0.555	0.590	0.618	0.638	0.675	0.703	0.723	0.751	0.784	0.813
700,000	0.428	0.479	0.515	0.563	0.598	0.625	0.647	0.683	0.710	0.729	0.758	0.791	0.819
600,000	0.437	0.488	0.523	0.573	0.608	0.633	0.658	0.693	0.718	0.738	0.770	0.800	0.828
500,000	0.451	0.500	0.535	0.584	0.619	0.647	0.669	0.703	0.730	0.752	0.780	0.812	0.842
400,000	0.464	0.514	0.548	0.598	0.632	0.660	0.682	0.716	0.743	0.755	0.793	0.827	0.853
300,000	0.482	0.532	0.566	0.615	0.650	0.677	0.699	0.734	0.760	0.723	0.810	0.845	0.873
250,000	0.493	0.542	0.577	0.626	0.661	0.688	0.711	0.745	0.772	0.794	0.821	0.863	0.884
0000	0.510	0.558	0.594	0.642	0.677	0.704	0.726	0.762	0.788	0.810	0.838	0.872	0.901
000	0.524	0.572	0.608	0.656	0.692	0.718	0.740	0.776	0.802	0.824	0.852	0.885	0.915
00	0.538	0.586	0.622	0.670	0.706	0.732	0.754	0.790	0.816	0.838	0.866	0.900	0.929
0	0.552	0.600	0.636	0.684	0.720	0.746	0.768	0.804	0.830	0.852	0.880	0.914	0.943
1	0.566	0.614	0.649	0.698	0.734	0.760	0.782	0.818	0.844	0.866	0.894	0.929	0.957
2	0.580	0.628	0.664	0.712	0.748	0.774	0.796	0.832	0.858	0.880	0.908	0.944	0.971
3	0.594	0.642	0.678	0.726	0.762	0.788	0.810	0.846	0.872	0.894	0.922	0.958	0.985
4	0.608	0.657	0.692	0.740	0.776	0.803	0.824	0.860	0.886	0.908	0.936	0.972	0.999
5	0.622	0.671	0.706	0.754	0.790	0.817	0.838	0.874	0.900	0.922	0.953	0.987	1.013
6	0.636	0.684	0.720	0.768	0.804	0.831	0.853	0.888	0.915	0.936	0.964	1.001	1.027
7	0.650	0.698	0.734	0.782	0.818	0.845	0.867	0.902	0.929	0.950	0.978	1.015	1.041
8	0.663	0.712	0.748	0.796	0.832	0.859	0.880	0.916	0.943	0.964	0.992	1.029	1.055
9	0.677	0.726	0.762	0.810	0.846	0.873	0.894	0.930	0.957	0.978	1.006	1.043	1.069
10	0.691	0.740	0.776	0.823	0.860	0.886	0.908	0.944	0.971	0.992	1.020	1.057	1.083

NOTE.—Tabular values for No. 0000 and smaller are for solid wires; for larger sizes, the wires are stranded.

60~ CAPACITIVE SUSCEPTANCE TO NEUTRAL IN MICROMHOS PER MILE

Circular Mils or A.W.G.	Interaxial Spacing in Feet, Single Phase or Equilateral Three Phase												
	1	1.5	2	3	4	5	6	8	10	12	15	20	25
1,200,000	11.5	10.1	9.27	8.33	7.84	7.45	7.10	6.73	6.40	6.21	5.97	5.69	5.46
1,100,000	11.4	9.95	9.15	8.26	7.76	7.33	7.05	6.70	6.37	6.17	5.93	5.63	5.44
1,000,000	11.1	9.80	9.05	8.14	7.62	7.28	6.97	6.60	6.32	6.12	5.88	5.59	5.39
900,000	10.9	9.62	8.92	8.05	7.57	7.17	6.89	6.53	6.26	6.05	5.81	5.54	5.33
800,000	10.7	9.50	8.75	7.94	7.45	7.05	6.80	6.45	6.19	5.99	5.77	5.49	5.29
700,000	10.5	9.33	8.65	7.83	7.36	7.00	6.73	6.38	6.11	5.91	5.68	5.42	5.25
600,000	10.3	9.12	8.45	7.68	7.20	6.88	6.65	6.29	6.03	5.84	5.61	5.37	5.17
500,000	10.1	8.96	8.30	7.55	7.09	6.76	6.53	6.15	5.93	5.74	5.53	5.30	5.09
400,000	9.70	8.66	8.08	7.34	6.91	6.61	6.36	6.02	5.82	5.62	5.42	5.21	5.02
300,000	9.32	8.36	7.80	7.11	6.71	6.41	6.19	5.88	5.68	5.49	5.31	5.08	4.90
250,000	9.04	8.18	7.65	6.98	6.59	6.31	6.10	5.81	5.59	5.42	5.24	5.01	4.86
0000	8.52	7.73	7.25	6.67	6.31	6.06	5.87	5.59	5.39	5.23	5.06	4.86	4.70
000	8.28	7.53	7.08	6.52	6.18	5.93	5.75	5.48	5.29	5.14	4.97	4.77	4.63
00	8.05	7.34	6.91	6.38	6.05	5.82	5.64	5.38	5.19	5.05	4.89	4.68	4.55
0	7.83	7.16	6.75	6.24	5.93	5.70	5.53	5.28	5.10	4.97	4.81	4.61	4.48]
1	7.63	6.98	6.60	6.11	5.81	5.59	5.43	5.19	5.02	4.88	4.73	4.53	4.42
2	7.44	6.82	6.45	5.98	5.69	5.49	5.33	5.10	4.93	4.80	4.65	4.46	4.35
3	7.25	6.67	6.31	5.86	5.58	5.39	5.23	5.01	4.85	4.72	4.58	4.40	4.28
4	7.07	6.52	6.18	5.75	5.48	5.29	5.14	4.92	4.77	4.65	4.51	4.34	4.22
5	6.90	6.38	6.05	5.64	5.38	5.19	5.05	4.84	4.69	4.58	4.44	4.26	4.15
6	6.74	6.24	5.92	5.53	5.28	5.10	4.97	4.76	4.62	4.50	4.37	4.19	4.09
7	6.59	6.11	5.80	5.44	5.19	5.01	4.88	4.69	4.55	4.44	4.31	4.13	4.03
8	6.45	5.98	5.69	5.33	5.10	4.93	4.80	4.61	4.48	4.37	4.25	4.07	3.97
9	6.32	5.86	5.59	5.23	5.02	4.85	4.72	4.53	4.41	4.31	4.19	4.01	3.91
10	6.20	5.75	5.50	5.13	4.94	4.77	4.65	4.45	4.34	4.25	4.14	3.95	3.85

NOTE.—Tabular values are for solid wires for No. 0000 and smaller; for stranded wires for larger sizes.

REPRESENTATIVE POWER TRANSMISSION LINES IN U. S. A.*

Transmission Line No.	Company	Location of Line	Date of Construction
1	City of Los Angeles	Boulder Dam, Nevada, to Los Angeles, Calif.	1936
2	Southern California Edison Co. Ltd.	Vincent Line (California)	1926
3	Southern California Edison Co. Ltd.	Big Creek, Calif., to Eagle Rock, Calif	1923
4	Pacific Gas and Electric Co.	Pit River, Calif., to Vaca-Dixon, Calif.	1922
5	City of Seattle	Gorge Plant to Seattle, Wash.	1924
6	Utah Power and Light Co.	Grace, Idaho, to Salt Lake City, Utah	1914
7	Mississippi River Power Co.	Keokuk, Iowa, to St. Louis, Mo.	1912
8	Alabama Power Co. and Mississippi Power Co.	Jordan Dam, Ala., to Meridan, Miss.	1926

REPRESENTATIVE POWER TRANSMISSION LINES* IN U. S. A.

Line No.	Line-to-Line Potential, Kilovolts	Length of Line, Miles	Frequency, Cycles per Second	Number of Three-Phase Circuits per Tower
1	Sending 278 Receiving 275	266 { (a) 225, (b) 41 }	60	1
2	220	237	50	2
3	220	240	50	1
4	220	202 { (a) 27.5, (b) 32.5, (c) 142 }	60	1
5	165	100	60	1
6	132	135	60	2
7	115	144	25	2
8	110	151	60	1

* All are three-phase lines.

Line No.	Conductors		Normal Arrangement	Spacing Inches
	Material and Stranding	Total Cross Section, Circular Mils		
1 (a)	Copper	512,000	`˙D ˙D˙  ˙D ˙D˙`	D=390
(b)	Segmental* hollow tube Spiral pitch 28 inches	Outside diameter 1.40 inches	`· D ·` `·d· ·d·`	D=486 d=294
2	A.C.S.R. { 54 aluminum / 7 steel	1,033,500	`· D · D ·`	D=267
3	A.C.S.R. { 54 aluminum / 7 steel	605,000	`· D · D ·`	D=207
4 (a)	A.C.S.R. { 42 aluminum / 19 steel	518,000	(a) and (b)	D=228
(b)	Copper 7×7 Rope-stranded	500,000	`˙D ˙D˙   ˙D ˙D˙`	D=288
(c)	Copper 7×7 Rope-stranded	500,000	(c) `·d· ·d·`	d=180
5	A.C.S.R. { 26 aluminum / 7 steel	477,000	`· D ·`	D=180
6	Copper 12 wires	250,000	`· D ·  ·d· ·d·`	D=252 d=156
7	Copper 19 strand concentric	300,000	`· D ·  ·d· ·d·`	D=222 d=120
8	A.C.S.R. { 30 aluminum / 7 steel	397,500	`· D · D ·`	D=168

* See *Electrical Engineering*, May 1935, Fig. 5.

Line No.	Constants per Mile per Phase			Propagation Constant m				Characteristic Impedance Z_0			
				Polar Form		Rectangular Form		Polar Form		Rectangular Form	
	R Ohms	L Henries	C Farads	m Magnitude	θ_p Degrees	u Hyperbolic Radian per mile	v Circular Radian per mile	Magnitude Ohms	θ_{Z_0} Degrees	R Component, Ohms	X Component, Ohms
1 (a)	0.113*	0.00212	0.0136×10^{-6}	0.00203	86.0°	0.000143	0.00203	397	−4.03°	396	−27.9
1 (b)	0.0566**	0.000093	0.0295×10^{-6}	0.00205	85.7°	0.000154	0.00205	185	−4.30°	184	−13.8
2	0.0913	0.00209	0.0142×10^{-6}	0.00172	86.0°	0.000119	0.00172	385	−3.96°	385	−26.6
3	0.157	0.00211	0.0142×10^{-6}	0.00174	83.3°	0.000202	0.00173	391	−6.66°	388	−45.3
4 (a)	0.180*	0.00218	0.0141×10^{-6}	0.00208	83.6°	0.000231	0.00207	392	−6.38°	389	−43.5
4 (b)	0.122*	0.00102	0.0139×10^{-6}	0.00209	85.8°	0.000154	0.00208	398	−4.22°	397	−29.3
4 (c)	0.0610**	0.00208	0.0294×10^{-6}	0.00208	85.5°	0.000163	0.00207	187	−4.51°	187	−14.7
5	0.196	0.00208	0.0143×10^{-6}	0.00209	83.0°	0.000255	0.00207	387	−7.02°	384	−47.3
6	0.118	0.00106	0.0284×10^{-6}	0.00211	81.8°	0.000302	0.00209	197	−8.23°	195	−28.3
7	0.0976	0.00101	0.0298×10^{-6}	0.000934	74.2°	0.000254	0.000899	200	−15.80°	192	−54.3
8	0.235	0.00208	0.0143×10^{-6}	0.00210	81.7°	0.000305	0.00208	390	−8.34°	386	−56.6

Values of R, L, and C obtained as discussed in Chapters III and VI.
Assumptions:
Temperature, 25° C.
Conductivity; for hard-drawn copper 97.3%, for hard-drawn aluminum 61.0%.
For aluminum cables,

R and L values those presented in Electrical Characteristics of A.C.S.R., by Aluminum Company of America.
G taken as zero for all lines.

* Constants for each of the separate 3-phase lines.
** Constants for the double-circuit part.

Line No.	Wave Length, Miles	Velocity of Propagation, Miles per second	Maximum kv-a., delivered under steady-state operating conditions. (Prior to January 1, 1936)	Power factor at receiver end for maximum kv-a. delivered
1 { (a)	3100	186,000	*	
{ (b)	3070	184,000		
2	3660	183,000	**	
3	3630	181,000	163,000	0.94 leading
4 { (a)	3040	182,000	140,000	0.99 leading
{ (b)	3020	181,000	140,000	0.99 leading
{ (c)	3030	182,000	140,000	0.99 leading
5	3030	182,000	58,000	0.93 lagging
6	3010	180,000	68,000	0.90 lagging
7	6990	175,000	110,000	0.90 lagging
8	3020	181,000	12,000	1.0

* The calculated value of the reliable operating capacity of the two thr.e phase lines in parallel is 235,000-240,000 kw. The calculated value of the emergency operating capacity is 300,000 kw. (See paper by E. F. Scattergood in *Electrical Engineering*, May, 1935, page 510.)

** This line, prior to January 1, 1936, was not used separately, and therefore the maximum kv-a. delivered under steady-state operating conditions and the corresponding power factor were not obtainable.

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