HISTORY OF HINDU MATHEMATICS

A SOURCE BOOK

PARTS I AND II

BY

BIBHUTIBHUSAN DATTA

AND

AVADHESH NARAYAN SINGH

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TRANSLITERATION

Vowels

Short :	अ	इ	उ	ऋ	ऌ				
	a	*i*	*u*	*ṛ*	*ḷ*				
Long :	आ	ई	ऊ	ॠ	ॡ	ए	ऐ	ओ	औ
	â	*î*	*û*	*ṝ*	*ḹ*	*e*	*ai*	*o*	*au*

Anusvâra : ं = *ṁ*

Visarga : ः = *ḥ*

Non-aspirant : ऽ = ’

Consonants

Classified :	क्	ख्	ग्	घ्	ङ्
	k	*kh*	*g*	*gh*	*ṅ*
	च्	छ्	ज्	झ्	ञ्
	c	*ch*	*j*	*jh*	*ñ*
	ट्	ठ्	ड्	ढ्	ण्
	ṭ	*ṭh*	*ḍ*	*ḍh*	*ṇ*

	त्	थ्	द्	ध्	न्			
	t	*th*	*d*	*dh*	*n*			
	प्	फ्	ब्	भ्	म्			
	p	*ph*	*b*	*bh*	*m*, *ṃ* = final म्			
Un-classified :	य्	र्	ल्	व्	श्	ष्	स्	ह्
	y	*r*	*l*	*v*	*ś*	*ṣ*	*s*	*h*
Compound :	क्ष्	त्र्	ज्ञ्					
	kṣ	*tr*	*jñ*					
Pâlî :	ळ = *l'*							

LIST OF ABBREVIATIONS

Â	Âryabhaṭîa
ÂJ	Ârṣa-Jyotiṣa
ÂPŚl	Âpastamba Śulba
AV	Atharvaveda
BBi	Bîjagaṇita of Bhâskara II
BCMS	Bulletin of the Calcutta Mathematical Society
BMs	Bakhshâlî Manuscript
BrSpSi	Brâhma-sphuṭa-siddhânta
BŚl	Baudhâyana Śulba
DhGr	Dhyânagrahopadeśa
EI	Epigraphia Indica
GK	Gaṇita-kaumudî
GL	Graha-lâghava
GSS	Gaṇita-sâra-saṁgraha
GT	Gaṇita-tilaka
IA	Indian Antiquary
IHQ	Indian Historical Quarterly
JA	Journal Asiatique
JASB	Journal of the Asiatic Society of Bengal
JIMS	Journal of the Indian Mathematical Society
JRAS	Journal of the Royal Asiatic Society of Great Britain and Ireland
KapS	Kapisthala Saṁhitâ
KK	Khaṇḍa-khâdyaka
KŚl	Kâtyâyana Śulba
KtS	Kâṭhaka Saṁhitâ
L	Lîlâvatî
LBh	Laghu-Bhâskarîya

LIST OF ABBREVIATIONS

LMâ	Laghu-mânasa
MaiS	Maitrâyaṇî Saṁhitâ
MâŚl	Mânava Śulba
MBh	Mahâ-Bhâskarîya
MSi	Mahâ-siddhânta
NBi	Bîjagaṇita of Nârâyaṇa
PâSâ	Pâṭî-sâra of Munîśvara
PLM	Prâcîna-lipi-mâlâ
PSi	Pañca-siddhântikâ
RV	Ṛgveda
ŚBr	Śatapatha Brâhmaṇa
ŚiDVr	Śiṣya-dhî-vṛddhida
SiŚe	Siddhânta-śekhara
SiŚi	Siddhânta-śiromani
SiTVi	Siddhânta-tattva-viveka
SûSi	Sûrya-siddhânta
TBr	Taittirîya Brâhmaṇa
Triś	Triśatikâ
TS	Taittirîya Saṁhitâ
YJ	Yâjuṣa-Jyotiṣa
ZDMG	Zeitschrift der deutschen morgenländischen Gessels-chaft

HISTORY OF HINDU MATHEMATICS

A SOURCE BOOK

PART I

NUMERAL NOTATION AND ARITHMETIC

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इदं नम ऋषिभ्यः पूर्वजेभ्यः पूर्वेभ्यः पथिकृद्भ्यः

(RV, x. 14. 25)

To the Seers, our Ancestors, the first Path-makers

PREFACE

Little is known at present to historians of mathematics regarding the achievements of the early Hindu mathematicians and our indebtedness to them. Though it is now generally admitted that the decimal place-value system of numeral notation was invented and first used by the Hindus, it is not yet fully realized to what extent we are indebted to them for our elementary mathematics. This is due to the lack of a reliable and authentic history of Hindu mathematics. Our object in writing the present book has been to make up for this deficiency by giving a comprehensive account of the growth and development of the science of mathematics in India from the earliest known times down to the seventeenth century of the Christian era.

The subject is treated by topics. Under each topic are collected together and set forth in chronological order translations of relevant Sanskrit texts as found in the Hindu works. The texts have been elucidated, wherever necessary, by adding explanatory notes and comments, and also by illustrative examples culled from original sources. We have tried to avoid repetition as far as has been consistent with our aim. However, on several occasions it has been considered desirable to repeat the same rule in the words of different authors in order to emphasize the continuity or rather the gradual evolution of mathematical thought and terminology in India. Comparative study of this kind has helped us to throw light on certain obscure Sanskrit passages and technical terms whose full significance had not been understood before. In translating the texts we have tried to be as literal and faithful as possible without sacrificing the spirit of the original. Sometimes it has not been possible to find exact parallels to Sanskrit words and technical terms in English. In all such cases we have tried to maintain the spirit of the original in the English version.

The above plan of the book has been adopted in pursuance of our intention to place before those who have no access to the Sanskrit sources all evidence, unfavourable as well as favourable, so that they can judge for themselves the claims of Hindu mathematics, without depending solely on our statements. In order to facilitate comparison with the development of mathematics in other countries the various topics have been arranged generally in accordance with the sequence in Professor D. E. Smith's *History of Mathematics*, Vol. II. This has sometimes necessitated divergence from the arrangement of topics as found in the Hindu works on mathematics.

In search of material for the book we had to examine the literature of the Hindus, non-mathematical as well as mathematical, whether in Sanskrit or in Prâkrit (Pâlî and Ardha Mâgadhî). Very few of the Hindu treatises on mathematics have been printed so far, and even these are not generally known. The manuscript works that exist in the various Sanskrit libraries in India and Europe are still less known. We have not spared labour in collecting as many of these as we could. Sanskrit mathematical works mentioned in the bibliography given at the end of this volume have been specially consulted by us. We are thankful to the authorities of the libraries at Madras, Bangalore, Trivandrum, Trippunithura and Benares, and those of the India Office (London) and the Asiatic Society of

Bengal (Calcutta) for supplying us transcripts of the manuscripts required or sending us manuscripts for consultation. We are indebted also to Dr. R. P. Paranjpye, Vice-Chancellor of the Lucknow University, for help in securing for our use several manuscripts or their transcripts from the state libraries in India and the India Office, London.

It would not have been possible to carry our study as far as has been done without the spade work of previous writers. Foremost among these must be mentioned the late Pandit Sudhakar Dvivedi of Benares, whose editions of the *Lîlâvatî*, *Brâhmasphuṭa-siddhânta*, *Triśatikâ*, *Mahâsiddhânta*, *Siddhânta-tattva-viveka*, etc., have been of immense help. Colebrooke's translations of the arithmetic and algebra of Brahmagupta and Bhâskara II, Kern's edition of the *Âryabhaṭîya* and Rangacarya's edition (with English translation) of the *Gaṇita-sâra-saṁgraha* of Mahâvîra have also been of much use. The recent work of G. R. Kaye, however, has been found to be extremely unreliable. His translation of the *Gaṇitapâda* of the *Âryabhaṭîya* and his edition of the Bakhshâlî Manuscript are full of mistakes and are misleading.

It has been decided to publish the book in three parts. The first part deals with the history of the numeral notation and of arithmetic. The second is devoted to algebra, a science in which the ancient Hindus made remarkable progress. The third part contains the history of geometry, trigonometry, calculus and various other topics such as magic squares, theory of series and permutations and combinations. Each part is complete in itself, so that one interested in any particular branch of mathematics need not consult all of them.

Part I which is now being published contains two chapters. Chapter I gives an account of the various devices employed by the Hindus for denoting numbers. The gradual evolution of the decimal place-value system of notation has been traced and all evidence relating to its use in India collected together for the first time. This evidence shows that the system was in use in India during the earliest centuries of the Christian era, if not earlier. We hope that the facts set forth in this chapter will finally set at rest the controversy about its place of origin. Considerations of space have prevented us from giving details regarding the introduction of the Hindu numerals into Arabia, Northern Africa and Europe. A brief account has, however, been included.

Chapter II deals with arithmetic in general. We have become so familiar with our methods of performing the fundamental arithmetical operations of addition, subtraction, multiplication, division and the extraction of roots that we seldom pause to think how and when these methods were invented. The problem, however, has deep interest for the teacher and historian of mathematics. And an account of the evolution of these methods in the land of their birth should be welcome. We have given details and illustrations of different methods of performing these operations on a *pâṭî*, ("board"), as followed in India from the fifth century onwards. It has been shown that our present methods are simple variations of those of the ancient Hindus. The rule of three, the rules of supposition and false position, and rules relating to calculations involving interest, exchange of commodities, fineness of gold, etc., are all due to the Hindus. In fact, practically the whole of elementary arithmetic can be traced back to them. Thus the importance of chapter II cannot be over-emphasised.

The scheme of transliteration of Sanskrit words and proper names has been indicated in the beginning. Arabic words and proper names that occur in the book have been taken down as found in various secondary sources consulted.

Acknowledgment has been made in footnotes to the texts of various sources from which we have derived assistance. Of the books which have been found specially helpful, we would mention Bühler's Indian Palæography and Ojha's Prâcîna Lipi Mâlâ (in Hindi). We are indebted to Mr. Ojha for permitting us to reproduce from his book the tables of Brâhmî and Kharoṣṭhî numerals. We have pleasure in expressing our obligation to Mr. T. N. Singh for his help in the revision of proof-sheets. He has caused the removal of many obscurities and has made many valuable suggestions of which we have availed ourselves. Our thanks are also due to Mr. R. D. Misra for preparing the index to this volume.

In conclusion we express our thanks to the Law Journal Press for their unfailing courtesy and for expediting the work of printing.

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BIBHUTIBHUSAN DATTA
AVADHESH NARAYAN SINGH

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CHAPTER I

NUMERAL NOTATION

A GLIMPSE OF ANCIENT INDIA

The student of ancient Indian History is struck by the marvellous attainments of the Hindus, both in the Arts and the Sciences, at a very early period. The discoveries at Mohenjo-daro reveal that as early as 3,000 B.C. the inhabitants of the land of the Sindhu—the Hindus—built brick houses, planned cities, used metals such as gold, silver, copper and bronze, and lived a highly organised life. In fact, they were far in advance of any other people of that period. The earliest works available, the *Vedas* (*c*. 3,000 B.C. or probably much earlier), although consisting mainly of hymns of praise and poems of worship, show a high state of civilisation. The *Brâhmaṇa* literature (*c*. 2,000 B.C.) which follows the *Vedas*, is partly ritualistic and partly philosophical. In these works are to be found well-developed systems of metaphysical, social and religious philosophy, as well as the germs of most of the sciences and arts which have helped to make up the modern civilisation. It is here that we find the beginnings of the science of mathematics (arithmetic, geometry, algebra, etc.) and astronomy. This *Brâhmaṇa* period was followed by more than two thousand years of continuous progress and brilliant achievements. Although during this period there were several foreign invasions as well as internal wars and many great kingdoms rose and fell, yet the continuity of intellectual progress was maintained. The constitution of Hindu society was mainly responsible for this. The foreign invaders, instead of being a hindrance, contributed to progress and the strengthening of Hindu society by bringing in new blood. They settled in the land, adopted the religion and customs of the conquered and were completely absorbed into Hindu society. There were a class of people—the Brâhmaṇas—who took the vow of poverty, and devoted themselves, from one generation to another, to the cultivation of the sciences and arts, religion and philosophy. The Brâhmaṇas, thanks to their selflessness and intellectual attainments, were highly respected by the kings and the people alike. They were the law-givers and advisers of the kings. In fact, this body of selfless thinkers and learned men were the real rulers of the land.

The great Epic, the *Râmâyaṇa*, was composed by Vâlmîki, the father of Sanskrit poetry, about 1,000 B.C., Pâṇini, the grammarian, perfected Sanskrit grammar about 700 B.C. and Suśruta wrote on the sciences of medicine and surgery about 600 B.C.[1] A century later, Mahâvîra and Buddha taught their unique systems of religious and moral philosophy, and the doctrine of *Nirvâṇa*. With the spread of these religions evolved the Jaina and Buddhist literatures. Some of the earlier *Purâṇas* and *Dharma-śâstras* were written about this time. The period 400 B.C. to 400 A.D., however, seems to have been a period of great activity and progress. During this period flourished the great Jaina metaphysician Umâsvâti, Patañjali, the grammarian and philosopher, Kauṭilya, the celebrated politician, Nâgârjuna, the chemist, Caraka, the physician, and the immortal poets Aśvaghoṣa, Bhâsa and Kâlidâsa. The great

[1] There is considerable divergence of opinion regarding the dates of the pre-historic works and personalities mentioned in this section. We have given those dates that appear most plausible.

astronomical *Siddhântas*, the *Sûrya*, the *Pitâmaha*, the *Vaśiṣṭha*, the *Parâśara* and others were written during this period and the decimal place-value notation was perfected.

2 HINDUS AND MATHEMATICS

1 **Appreciation of Mathematics.** It is said that in ancient India no science did ever attain an independent existence and was cultivated for its own sake. Whatever of any science is found in Vedic India is supposed to have originated and grown as the handmaid of one or the other of the six "members of the Veda," and consequently with the primary object of helping the Vedic rituals. It is also supposed, sometimes, that any further culture of the science was somewhat discouraged by the Vedic Hindus in suspicion that it might prove a hindrance to their great quest of the knowledge of the Supreme by diverting the mind to other external channels. That is not, indeed, a correct view on the whole. It is perhaps true that in the earlier Vedic Age, sciences grew as help to religion. But it is generally found that the interest of people in a particular branch of knowledge, in all climes and times, has always been aroused and guided by specific reasons. Religion being the prime avocation of the earlier Hindus, it is not unnatural that the culture of other branches of knowledge grew as help to it and was kept subsidiary. But there is enough evidence to show that in course of time all the sciences outgrew their original purposes and were cultivated for their own sake. A new orientation had indeed set in in the latter part of the Vedic Age.

There is a story in the *Chândogya Upaniṣad*[1] whose

[1] *Chândogya Upaniṣad*, vii. 1, 2, 4.

value in support of our view cannot be over-estimated. It is said that once upon a time Nârada approached the sage Sanatkumâra and begged of him the *Brahma-vidyâ* or the supreme knowledge. Sanatkumâra asked Nârada to state what sciences and arts he had already studied so that he (Sanatkumâra) might judge what still remained to be learnt by him. Thereupon Nârada enumerated the various sciences and arts studied by him. This list included astronomy (*nakṣatra-vidyâ*) and arithmetic (*râśi-vidyâ*). Thus the culture of the science of mathematics or of any other branch of secular knowledge, was not considered to be a hindrance to spiritual knowledge. In fact, *Aparâ-vidyâ* ("secular knowledge") was then considered to be a helpful adjunct to *Parâ-vidyâ* ("spiritual knowledge").[1]

Importance to the culture of *Gaṇita* (mathematics) is [redacted] given by the Jainas. Their religious literature is generally classified into four branches, called *anuyoga* ("exposition of principles"). One of them is *gaṇitânu-yoga* ("the exposition of the principles of mathematics"). The knowledge of *Saṁkhyâna* (literally, "the science of numbers," meaning arithmetic and astronomy) is stated to be one of the principal accomplishments of the Jaina priest.[2] In Buddhist literature too, arithmetic (*gaṇanâ*, *saṁkhyâna*) is regarded as the first and the noblest of the arts.[3] All these will give a fair idea of the importance and value set upon the culture of *gaṇita* in ancient India.

The following appreciation of mathematics, although belonging to a much later date, will be found to be interesting, especially, as it comes from the pen

[1] *Muṇḍakopaniṣad*, i. 1, 3-5.

[2] *Bhagavatî-sûtra*, Sûtra 90; *Uttarâdhyayana-sûtra*, xxv. 7, 8, 38.

[3] *Vinaya Piṭaka*, ed. Oldenberg, Vol. IV, p. 7; *Majjhima Nikâya*, Vol. I, p. 85; *Cullaniddesa*, p. 199.

of Mahâvîra (850 A.D.), one of the best mathematicians of his time:

"In all transactions which relate to worldly, Vedic or other similar religious affairs calculation is of use. In the science of love, in the science of wealth, in music and in drama, in the art of cooking, in medicine, in architecture, in prosody, in poetics and poetry, in logic and grammar and such other things, and in relation to all that constitutes the peculiar value of the arts, the science of calculation (*gaṇita*) is held in high esteem. In relation to the movements of the sun and other heavenly bodies, in connection with eclipses and conjunctions of planets, and in connection with the *tripraśna* (direction, position and time) and the course of the moon—indeed in all these it is utilised. The number, the diameter and the perimeter of islands, oceans and mountains; the extensive dimensions of the rows of habitations and halls belonging to the inhabitants of the world, of the interspace between the worlds, of the world of light, of the world of the gods and of the dwellers in hell, and other miscellaneous measurements of all sorts—all these are made out by the help of *gaṇita*. The configuration of living beings therein, the length of their lives, their eight attributes, and other similar things; their progress and other such things, their staying together, etc.—all these are dependent upon *gaṇita* (for their due comprehension). What is the good of saying much? Whatever there is in all the three worlds, which are possessed of moving and non-moving beings, cannot exist as apart from *gaṇita* (measurement and calculation).

" With the help of the accomplished holy sages, who are worthy to be worshipped by the lords of the world, and of their disciples and disciples' disciples, who constitute the well-known series of preceptors, I glean from the great ocean of the knowledge of numbers a little of its essence, in the manner in which gems are picked from the sea, gold is from the stony rock and pearl from the oyster shell; and give out according to the power of my intelligence, the *Sâra-saṁgraha*, a small work on *gaṇita*, which is (however) not small in value."[1]

Mathematics in Hindu Education. The elementary stage in Hindu education lasted from the age of five till the age of twelve. This period slightly differed in the case of sons of kings and noblemen. The main subjects of study were *lipi* or *lekhâ* (alphabets, reading and writing), *rûpa* (drawing and geometry) and *gaṇanâ* (arithmetic). It is said in the *Arthaśâstra* of Kauṭilya (400 B.C.) that having undergone the ceremony of tonsure, the student shall learn the alphabets (*lipi*) and arithmetic (*saṁkhyâna*).[2] We find in the Hâthîgumphâ Inscription[3] that king Khâravela (163 B.C.) of Kaliṅga spent nine years (from the age of sixteen to the age of 25) in learning *lekhâ*, *rûpa* and *gaṇanâ*. Prince Gautama began his education when he was eight years of age "firstly (with) writing and then arithmetic as the most important of the 72 sciences and arts."[4] Mention of *lekhâ*, *rûpa* and *gaṇanâ* is also found in the Jaina canonical works.[5]

[1] *GSS*, i. 9-19.

[2] *Arthaśâstra*, ed. by R. Shamasastri, i. 5, 2; Eng. trans., p. 10.

[3] *Hathigumpha and three other inscriptions*, ed. by Bhagavanlal Indraji, p. 22.

[4] *Antagaḍa-dasâo and Anuttarvavâiya-dasâo*, Eng. trans. by L. D. Barnett, 1907, p. 30; cf. *Kalpasûtra* of Bhadrabâhu, Sûtra 211.

[5] E.g., *Samavâyâṅga-sûtra*, Sûtra 72.

~~SCOPE AND DEVELOPMENT OF HINDU MATHEMATICS~~

Ganita literally means "the science of calculation" and is the Hindu name for mathematics. The term is a very ancient one and occurs copiously in Vedic literature. The *Vedâṅga Jyotiṣa* (*c.* 1,200 B.C.) gives it the highest place of honour among the sciences which form the *Vedâṅga*: "As the crests on the heads of peacocks, as the gems on the hoods of snakes, so is *gaṇita* at the top of the sciences known as the *Vedâṅga*."[1] In ancient Buddhist literature we find mention of three classes of *gaṇita*: (1) *mudrâ* ("finger arithmetic"), (2) *gaṇanâ* ("mental arithmetic") and (3) *saṁkhyâna* ("higher arithmetic in general"). One of the earliest enumerations of these three classes occurs in the *Dîgha Nikâya*,[2] and it is also found in the *Vinaya Piṭaka*,[3] *Divyâvadâna*[4] and *Milindapañho*.[5] The word *saṁkhyâna* has been used for *gaṇita* in several old works.[6] At this remote period *gaṇita* included astronomy, but geometry (*kṣetra-gaṇita*) belonged to a different group of sciences known as *Kalpasûtra*.

It is believed that some time before the beginning of the Christian era, there was a renaissance of Hindu *Gaṇita*.[7] The effect of this revival on the scope of

[1] "Yathâ śikhâ mayurâṇâṁ nâgânâṁ maṇayo yathâ
Tadvadvedâṅgaśâstrâṇâṁ gaṇitaṁ mûrdhani sthitam."
Vedâṅga Jyotiṣa, 4.

[2] I, p. 51.

[3] IV, p. 7.

[4] *Divyâvadâna*, ed. by E. B. Cowell and R. A. Neil, Cambridge, 1886, pp. 3, 26 and 88.

[5] *Milindapañho*, Eng. trans. by Rhys Davids, Oxford, 1890, p. 91.

[6] E.g., *Kalpasûtra* of Bhadrabâhu, ed. by H. Jacobi, Leipzig, 1897; *Bhagavatî-sûtra*, Bombay, 1918, p. 112; *Arthaśâstra*, i. 5. 2.

[7] Bibhutibhusan Datta, "The scope and development of Hindu Gaṇita," *Indian Historical Quarterly*, V, 1929, pp. 479-512.

gaṇita was great. Astronomy (*jyotiṣa*) became a separate subject and geometry (*kṣetra-gaṇita*) came to be included within its scope. The subjects treated in the Hindu *Gaṇita* of the early renaissance period consisted of the following:[1] *Parikarma* ("fundamental operations"), *Vyavahâra* ("determinations"), *Rajju* ("rope," meaning geometry), *Râśi* ("rule of three"), *Kalâsavarṇa* ("operations with fractions"), *Yâvat tâvat* ("as many as," meaning simple equations), *Varga* ("Square," meaning quadratic equations), *Ghana* ("Cube", meaning cubic equations), *Varga-varga* (biquadratic equations) and *Vikalpa* ("permutations and combinations").

Thus *gaṇita* came to mean mathematics in general, while 'finger arithmetic' as well as 'mental arithmetic' were excluded from the scope of its meaning For the calculations involved in *gaṇita*, the use of some writing material was essential. The calculations were performed on a board (*pâṭî*) with a piece of chalk or on sand (*dhûlî*) spread on the ground or on the *pâṭî*. Thus the terms *pâṭî-gaṇita* ("science of calculation on the board") or *dhûlî-karma* ("dust-work"), came to be used for higher mathematics. Later on the section of *gaṇita* dealing with algebra was given the name *Bîja-gaṇita*. The first to effect this separation was Brahmagupta (628), but he did not use the term *Bîja-gaṇita*. The chapter dealing with algebra in his *Brâhma-sphuṭa-siddhânta* is called *Kuṭṭaka*. Śrîdharâcârya (750) regarded *Pâṭî-gaṇita* and *Bîja-gaṇita* as separate and wrote separate treatises on each. This distinction between *Pâṭîgaṇita* and *Bîjagaṇita* has been preserved by later writers.

Having given a brief survey of the position and scope of mathematics in Ancient India, we turn to the

[1] "Parikammaṁ vavahâro rajju râsî kalâsavanne ya ।
Jâvantâvati vaggo ghano tataha vaggavaggo vikappo ta ॥"
Sthânâṅgasûtra, Sûtra 747.

purpose in hand—that of giving a connected account of the development and growth of the different branches of mathematics. The numeration system of the Hindus will engage our attention first.

3 NUMERAL TERMINOLOGY

Scale of Notation. We can definitely say that from the very earliest known times, ten has formed the basis of numeration in India.[1] In fact, there is absolutely no trace of the extensive use of any other base of numeration in the whole of Sanskrit literature. It is also characteristic of India that there should be found at a very early date long series of number names for very high numerals. While the Greeks had no terminology for denominations above the *myriad* (10^4), and the Romans above the *mille* (10^3), the ancient Hindus dealt freely with no less than eighteen denominations. In modern times also, the numeral language of no other nation is as scientific and perfect as that of the Hindus.

In the *Yajurveda Saṁhitâ* (*Vâjasaneyî*)[2] the following list of numeral denominations is given: *Eka* (1), *daśa* (10), *sata* (100), *sahasra* (1000), *ayuta* (10,000), *niyuta* (100,000), *prayuta* (1,000,000), *arbuda* (10,000,000), *nyarbuda* (100,000,000), *samudra* (1,000,000,000), *madhya* (10,000,000,000), *anta* (100,000,000,000), *parârdha* (1,000,000,000,000). The same list occurs at two places in the *Taittirîya Saṁhitâ*.[3] The *Maitrâyaṇî*[4]

[1] Various instances are to be found in the *Ṛgveda;* noted by Macdonell and Keith, *Vedic Index*, Vol. I, p. 343.

[2] *Yajurveda Saṁhitâ*, xvii. 2.

[3] iv. 40. 11. 4; and vii. 2. 20. 1.

[4] ii. 8. 14; the list has *ayuta*, *prayuta*, then again *ayuta*, then *nyarbuda*, *samudra*, *madhya*, *anta*, *parârdha*.

and *Kâṭhaka*[1] *Saṁhitâs* contain the same list with slight alterations. The *Pañcaviṁśa Brâhmaṇa* has the *Yajurveda* list upto *nyarbuda* inclusive, and then follow *nikharva*, *vâdava*, *akṣiti*, etc. The *Śâṅkhyâyana Śrauta Sûtra* continues the series after *nyarbuda* with *nikharva*, *samudra*, *salila*, *antya*, *ananta* (= 10 billions). Each of these denominations is 10 times the preceding, so that they were aptly called *daśaguṇottara saṁjñâ*[2] ("decuple terms").

Coming to later times, *i.e.*, about the 5th century B.C., successful attempts were made to continue the series of number names based on the centesimal scale.[3] ~~We quote below from the *Lalitavistara*~~,[4] a well-known Buddhist work of the first century B.C., ~~the~~ dialogue between Arjuna, the mathematician, and Prince Gautama (Bodhisattva):

"The mathematician Arjuna asked the Bodhisattva, 'O young man, do you know the counting which goes beyond the *koṭi* on the centesimal scale?

Bodhisattva: I know.

Arjuna: How does the counting proceed beyond the *koṭi* on the centesimal scale?

Bodhisattva: Hundred *koṭis* are called *ayuta*, hundred *ayutas* *niyuta*, hundred *niyutas* *kaṅkara*, hundred *kaṅkaras* *vivara*, hundred *vivaras* *kṣobhya*, hundred *kṣobhyas* *vivâha*, hundred *vivâhas* *utsaṅga*, hundred *utsaṅgas* *bahula*, hundred *bahulas* *nâgabala*, hundred *nâgabalas* *tiṭi-*

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[1] xvii. 10; the list is the same with the exception that *niyuta* and *prayuta* change places. In xxxix. 6, after *nyarbuda* a new term *vâdava* intervenes.

[2] Cf. Bhâskara II, *L*, p. 2.

[3] *Śatottara gaṇanâ* or *Śatottara saṁjñâ* (names on the centesimal scale).

[4] *Lalitavistara*, ed. by Rajendra Lal Mitra, Calcutta, 1877,

lambha, hundred *tiṭilambhas vyavasthâna-prajñapti*, hundred *vyavasthâna-prajñaptis hetuhila*, hundred *hetuhilas karahu*, hundred *karahus hetvindriya*, hundred *hetvindriyas samâpta-lambha*, hundred *samâpta-lambhas gaṇanâgati*, hundred *gaṇanâgatis niravadya*, hundred *niravadyas mudrâ-bala*, hundred *mudrâ-balas sarva-bala*, hundred *sarva-balas visaṁjñâ-gati*, hundred *visaṁjñâ-gatis sarvajñâ*, hundred *sarvajñâs vibhutaṅgamâ*, hundred *vibhutaṅgamâs tallakṣaṇa.*[1]"

Another interesting series of number names increasing by multiples of 10 millions is found in Kâccâyana's Pali Grammar.[2] "For example: *dasa* (10) multiplied by *dasa* (10) becomes *sata* (100), *sata* (100) multiplied by ten becomes *sahassa* (1,000), *sahassa* multiplied by ten becomes *dasa sahassa* (10,000), *dasa sahassa* multiplied by ten becomes *sata sahassa*[3] (100,000), *sata sahassa* multiplied by ten becomes *dasa sata sahassa* (1,000,000), *dasa sata sahassa* multiplied by ten becomes *koṭi* (10,000,000). Hundred-hundred-thousand *koṭis* give *pakoṭi.*[4] In this manner the further terms are formed. ~~What are their~~ names? .... hundred hundred-thousands is *koṭi*, hundred-hundred-

[1] Thus *tallakṣaṇa* $=10^{53}$.

This and the following show that the Hindus anticipated Archimedes by several centuries in the matter of evolving a series of number names which "are sufficient to exceed not only the number of a sand-heap as large as the whole earth, but one as large as the universe."

Cf. '*De harenae numero*' in the 1544 edition of the *Opera* of Archimedes; quoted by Smith and Karpinski, *Hindu Arabic Numerals*, Boston, 1911, p. 16.

[2] "Grammaire Pâlie de Kâccâyana," *Journ. Asiatique*, Sixieme Serie, XVII, 1871, p. 411. The explanations to sûtras 51 and 52 are quoted here.

[3] Also called *lakkha* (*lakṣa*).

[4] Also called *koṭi-koṭi*, i.e., $(10{,}000{,}000)^2 = 10^{14}$. The following numbers are in the denomination *koṭi-koṭi*. Compare the *Anuyogadvâra-sûtra*, Sûtra 142.

thousand *koṭis* is *pakoṭi*, hundred-hundred-thousand *pakoṭis* is *koṭippakoṭi*, hundred-hundred-thousand *koṭippakoṭis* is *nahuta*, hundred-hundred-thousand *nahutas* is *ninnahuta*, hundred-hundred-thousand *ninnahutas* is *akkhobhini*; similarly we have *bindu*, *abbuda*, *nirabbuda*, *ahaha*, *ababa*, *aṭaṭa*, *sogandhika*, *uppala*, *kumuda*, *puṇḍarîka paduma*, *kathâna*, *mahâkathâna*, *asaṅkhyeya*."[1]

In the *Anuyogadvâra-sûtra*[2] (*c.* 100 B.C.), a Jaina canonical work written before the commencement of the Christian era, the total number of human beings in the world is given thus: "a number which when expressed in terms of the denominations, *koṭi-koṭi*, etc., occupies twenty-nine places (*sthâna*), or it is beyond the 24th place and within the 32nd place, or it is a number obtained by multiplying sixth square (of two) by (its) fifth square, (*i.e.*, 2^{96}), or it is a number which can be divided (by two) ninety-six times." Another big number that occurs in the Jaina works is the number representing the period of time known as *Śîrṣaprahelikâ*. According to the commentator Hema Candra (b. 1089)[3], this number is so large as to occupy 194 notational places (*aṅka-sthânehi*). It is also stated to be $(8{,}400{,}000)^{28}$.

Notational Places. Later on, when the idea of place-value was developed, the denominations (number names) were used to denote the places which unity would occupy in order to represent them (denominations) in writing a number on the decimal scale. For instance, according to Âryabhaṭa I (499) the denominations are the names of 'places'. He says: "*Eka* (unit) *daśa* (ten), *śata* (hundred), *sahasra* (thousand), *ayuta* (ten thousand), *niyuta* (hundred thousand), *prayuta* (million),

[1] Thus *asaṅkhyeya* is $(10)^{140} = (10{,}000{,}000)^{20}$.

[2] Sûtra 142.

[3] The figures within brackets after the names of authors or works denote dates after Christ.

koṭi (ten million), *arbuda* (hundred million), and *vṛnda* (thousand million) are respectively from *place to place* each ten times the preceding."[1] The first use of the word 'place' for the denomination is met with in the Jaina work quoted above.

In most of the mathematical works, the denominations are called "names of places," and eighteen of these are generally enumerated. Śrîdhara (750) gives the following names:[2] *eka*, *daśa*, *śata*, *sahasra*, *ayuta*, *lakṣa*, *prayuta*, *koṭi*, *arbuda*, *abja*, *kharva*, *nikharva*, *mahâsaroja*, *śaṅkw*, *saritâ-pati*, *antya*, *madhya*, *parârdha*, and adds that the decuple names proceed even beyond this. Mahâvîra (850) gives twenty-four notational places:[3] *eka*, *daśa*, *śata*, *sahasra*, *daśa-sahasra*, *lakṣa*, *daśa-lakṣa*, *koṭi*, *daśa-koṭi*, *śata-koṭi*, *arbuda*, *nyarbuda*, *kharva*, *mahâkharva*, *padma*, *mahâ-padma*, *kṣoṇi*, *mahâ-kṣoṇi*, *śaṅkha*, *mahâ-śaṅkha*, *kṣiti*, *mahâ-kṣiti*, *kṣobha*, *mahâ-kṣobha*.

Bhâskara II's (1150) list agrees with that of Śrîdhara except for *mahâsaroja* and *saritâpati* which are replaced by their synonyms *mahâpadma* and *jaladhi* respectively. He remarks that the names of places have been assigned for practical use by ancient writers.[4]

Nârâyaṇa (1356) gives a similar list in which *abja*, *mahâsaroja* and *saritâpati* are replaced by their synonyms *saroja*, *mahâbja* and *pârâvâra* respectively.

Numerals in Spoken Language. The Sanskrit names for the numbers from one to nine are: *eka*, *dvi*, *tri*, *catur*, *pañca* *ṣaṭ*, *sapta*, *aṣṭa*, *nava*. These with the

[1] *Â*, ii. 2.

[2] *Triś*, R. 2-3; the term used is *daśaguṇâḥ saṁjñâḥ*, i.e., "decuple names."

[3] *GSS*, i. 63-68; "The first place is what is known as *eka*; the second is *daśa*" etc.

[4] *L*, p. 2.

numerical denominations already mentioned suffice to express any required number. In an additive system it is immaterial how the elements of different denominations, of which a number is composed, are spoken. Thus one-ten or ten-one would mean the same. But it has become the usual custom from times immemorial to adhere to a definite mode of arrangement, instead of speaking in a haphazard manner.

In the Sanskrit language the arrangement is that when a number expression is composed of the first two denominations only, the smaller element is spoken first, but when it is composed also of higher denominations, the bigger elements precede the smaller ones, the order of the first two denominations remaining as before. Thus, if a number expression contains the first four denominations, the normal mode of expression would be to say the thousands first, then hundreds, then units and then tens. It will be observed that there is a sudden change of order in the process of formation of the number expression when we go beyond hundred. The change of order, however, is common to most of the important languages of the world.[1] Nothing definite appears to be known as to the cause of this sudden change.

The numbers 19, 29, 39, 49, etc., offer us instances of the use of the *subtractive* principle in the spoken language. In Vedic times we find the use[2] of the terms *ekânna-viṁśati* (one-less-twenty) and *ekânna-catvâriṁśat* (one-less-forty) for nineteen and thirty-nine respectively. In later times (Sûtra period) the *ekânna* was changed to *ekona*, and occasionally even the prefix *eka*

[1] Only in very few languages is the order continuously descending. In English the smaller elements are spoken first in the case of numbers upto twenty only.

[2] *Taittirîya Saṁhitâ*, vii. 2. 11.

was deleted and we have *ûna-viṁśati*, *ûna-triṁśat*, etc.—forms which are used upto the present day. The alternative expressions *nava-daśa* (nine-ten), *nava-viṁśati* (nine-twenty), etc., were also sometimes used.[1]

Practically the whole of Sanskrit literature is in verse, so that for the sake of metrical convenience, various devices were resorted to in the formation of number expressions, the most common being the use of the additive[2] method. The following are a few examples of common occurrence taken from mathematical works:

Subtractive: (1) the number 139 is expressed as $40+100-1$;[3]

(2) 297 is expressed as $300-3$.[4]

Multiplicative: (1) the number eighteen is expressed as 2×9;[5]

(2) twenty-seven is expressed as 3×9 and 12 as 2×6;[6]

(3) 28,483 is expressed as $83+400+(4000\times 7)$.

[1] 19 = *nava-daśa* (*Vâjasaneyî Saṁhitâ*, xiv. 23; *Taittirîya Saṁhitâ*, xiv. 23. 30).
29 = *nava-viṁśati* (*Vâjasaneyî Saṁhitâ*, xiv. 31).
99 = *nava-navati* (*Ṛgveda*, i. 84. 13).

[2] 3339 = *trîṇi śatâni trisahasrâṇi triṁśa ca nava ca*, i.e., "three hundreds and three thousands and thirty and nine." (*Ṛgveda*, iii. 9. 9; also x. 52. 6.)

[3] *GSS*, i. 4: *catvariṁśaścaikona śatâdhika* ("forty increased by one-less-hundred").

[4] *L*, p. 4, *Ex*. 1: *Trihînasya śata-trayasya* ("three less three hundred").

[5] *Â*, ii. 3: *dvi-navaka*.

[6] *Triś*, *Ex*. 43: *tri-navaka* ("three nines"), *dvi-ṣaṭ* ("two sixes").

[7] *GSS*, i. 28: *tryaśîtimiśrâṇi catuśśatâni catussahasraghna nagânvitâni* ("eighty-three combined with four hundred and four thousand multiplied by seven").

The expression of the number 12345654321 in the form "beginning with 1 upto 6 and then diminishing in order" is rather interesting.[1]

What are known as alphabetic and word numerals were generally employed for the expression of large numbers. A detailed account of these numerals will be given later on.

5. THE DEVELOPMENT OF NUMERICAL SYMBOLISM

Writing in Ancient India. It is generally held that numerical symbols were invented after writing had been in use for some time, and that in the early stages the numbers were written out in full in words. This seems to be true for the bigger units, but the signs for the smaller units are as old as writing itself.

Until quite recently historians were divided as to the date when writing was in use in India. There were some who stated that writing was known even in the Vedic age, but the majority following Weber, Taylor, Bühler and others were of opinion that writing was introduced into India from the West about the eighth century B.C. These writers built up theories deriving the ancient Indian script, as found in the inscriptions of Aśoka, from the more ancient writing discovered in Egypt and Mesopotamia. The Semitic origin was first suggested by Sir W. Jones, in the year 1806, and later on supported by Kopp (1821), Lespius (1834), and many others. The supporters of this theory, however, do not completely agree amongst themselves. For, whilst W. Deecke and I. Taylor derive the Indian script from a South-Semitic script, Weber and Bühler derive

[1] *GSS*, i.27: *ekâdiṣaḍantâni krameṇa hînâni.*

it from the Phoenician or a North-Semitic script.[1] Bühler rejects the derivation from a South-Semitic script, stating that the theory requires too many assumptions, and makes too many changes in the letter forms to be quite convincing. He, however, supports Weber's derivation from a North-Semitic script and has given details of the theory.[2] Ojha[3] has examined Bühler's theory in detail and rejects it stating that it is fanciful and that the facts are against it. He states that only one out of the twenty-two letters of the Phoenician (North-Semitic) script resembles a phonetically similar Brâhmî letter. He supports his argument in a most convincing manner by a table of the two alphabets, with phonetically similar letters arranged in a line. He further shows that following Bühler's method of derivation almost any script could be proved to be the parent of another.[4]

Other scholars, who held that writing was known in India as early as the Vedic age, based their conclusion upon literary evidence. The *Vaśiṣṭha Dharmasûtra*, which originally belonged to a school of the *Ṛgveda* offers clear evidence of the use of writing in the Vedic period. Vaśiṣṭha (xvi. 10, 14-15) mentions written documents as legal evidence, and the first of these sûtras

[1] For minor differences in the theories set up by different writers and also for several other theories, see Bühler, *Palaeography*, p. 9; the notes give the references.

[2] Bühler, *l.c.*, pp. 9f.

[3] *PLM*, pp. 18-31.

[4] Recently several other eminent historians have expressed their disagreement with Bühler's derivation. See Bhandarkar, "Origin of the Indian Alphabet," *Sir Asutosh Mukerji Jubilee Volumes*, Vol. III, 1922, p. 493; H. C. Ray, "The Indian Alphabet," *IA*, III, 1924, p. 233; also *Mohenjo-daro and the Indus Valley Civilisation*, 1931, p. 424, where the following remark occurs: "I am convinced that all attempts to derive the Brâhmî alphabet from Semitic alphabets were complete failures."

is a quotation from an older work or from traditional lore. Another quotation from the *Ṛgveda* itself (x. 62. 7), which refers to the writing of the number eight is: *Sahasraṁ me dadato aṣṭakarṇyaḥ*, meaning "gave me a thousand cows on whose ears the number eight was written." The above interpretation, although doubted by some scholars, seems to be correct, as it is supported by Pâṇini.[1] Moreover, the practice of making marks on the ears of cows to denote their relation to their owners, seems to have been prevalent in ancient India.[2]

At another place in the *Ṛgveda* (x. 34), we find mention of a gambler lamenting his lot and saying that "having staked on one,[3] he lost his faithful wife...." Again, in the *Atharvaveda* (vii. 50, (52), 5) we find the mention of the word "written amount".[4] Pâṇini's grammar (*c.* 700 B.C.) contains the terms *yavanâni* ("Semitic writing") and the compounds *lipikâra* and *libikâra* (iii. 2. 21) (writer), which show that writing was known in his time. In addition to these passages, the Vedic works contain some technical terms, such as *akṣara* (a letter of the alphabet), *kâṇḍa* (chapter), *paṭala*, *grantha* (book), etc., which have been quoted as evidence of writing. These specific references to written documents when considered with the advanced state of Vedic civilisation, especially the high development of trade and complicated monetary transactions, the use of prose in the *Brâhmaṇas*, the collection, the methodical arrangement, the numeration, the analysis of the Vedic texts

[1] *Karṇo varṇa lakṣaṇât* (vi. 2. 112) and also (vi. 3. 115) support the interpretation.

[2] *Atharvaveda* (vi. 141) mentions the method of making *mithuna* marks on the ears. In (xii. 4. 6) the practice is denounced. The *Maitrâyaṇî Saṁhitâ* has a chapter dealing with this topic. The method of making such marks is dealt with in iv. 2. 9.

[3] Here 'one' refers to the number stamped on the dice.

[4] *Ajaiṣaṃ tvâ saṁlikhitamajaiṣamuta saṁrudhaṃ.*

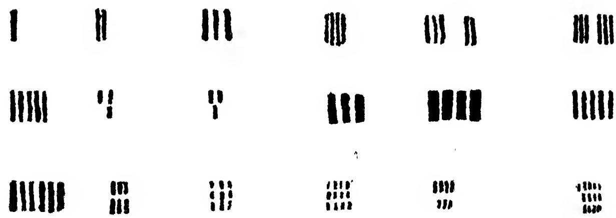
and the phonetic and lexicographic researches found in the *Vedângas*, form sufficient grounds for assigning a very early date to the use of writing in India.[1] Although these arguments possess considerable weight, they were not generally recognised, as will always happen if an *argumentum ex impossibili* is used. R. Shamasastry (1906) has published a derivation based upon ancient Indian hieroglyphic pictures which he believes to be preserved in the *tântric* figures. His learned article has not attracted the attention it deserves.

Recent discoveries have however, sounded the death knell of all theories deriving the Indian script from foreign sources. Pottery belonging to the Megalithic (*c.* 1,500 B.C.) and Neolithic (6,000 B.C.—3,000 B.C.) ages, preserved in the Madras Museum, has been found to be inscribed with writing. And according to Bhandarkar[2] five of these marks are identical with the Brâhmî characters of the time of Aśoka. The excavations at Mohenjo-daro and Harappa have also brought to light written documents, seals and inscriptions, dating from before 3,000 B.C. Thus it would be now absurd to trace the Brâhmî to any Semitic alphabet of the eighth or ninth century B.C.

4 **Earliest Numerals**. The numerical figures contained in the seals and inscriptions of Mohenjo-daro, have not been completely deciphered as yet. The vertical stroke and combinations of vertical strokes arranged side by side, or one group below another, have been found. The numbers one to thirteen seem to have been written by means of vertical strokes, probably, as in the figures given below:[3]

[1] Cf. Bühler, *l.c.*, p. 3. [2] *l.c.*

[3] Marshall, *l.c.*, pp. 450-52. See also "Mohenjo-daro—Indus Epigraphy" by G. R. Hunter (*JRAS*, April, 1932, pp. 470, 478ff.) who is more pronounced about the numerical values of some of the signs.



It is not yet quite certain whether there were special signs for greater numbers such as 20, 30, the hundreds and higher numbers. There are numerous other signs which are believed to represent such numbers, but there seems to be no means of finding out the true values of these signs at present.

Between the finds of Mohenjo-daro and the inscriptions of Aśoka, which contain numerals, there is a gap of 2,700 years or more. ~~No w~~ritten documents containing numerals and belonging to this intervening period have been so far ~~discovered~~. The literary evidence, however, points to the use of numerical symbols at a very early date. ~~The re~~ference to the writing of the number eight in the *Ṛgveda* and the use of numerical denominations as big as 10^{12} in the *Yajurveda Saṁhitâ* and in several other Vedic works, quoted before, offer sufficient grounds for concluding that, even at that remote period, the Hindus must have possessed a well developed system of numerical symbols. The conclusion is supported by the fact that the Greek and the Roman numerical terminologies did not go beyond 10^4, even after writing and a satisfactory numerical symbolism had been in use for several centuries.

The writings on the inscriptions of Aśoka show that in his time the use of numerical symbols in India was quite common.[1] The variations in the forms of the numerical signs suggest that the symbols had been in use for a long time.

Most of the inscriptions of Aśoka and the following period are written in a script which has been called *Brâhmî*, whilst some are in a different script known as *Kharoṣṭhî*. The forms of the numerical symbols in the two scripts are different. We consider them separately.

3·2·1 KHAROṢṬHÎ NUMERALS

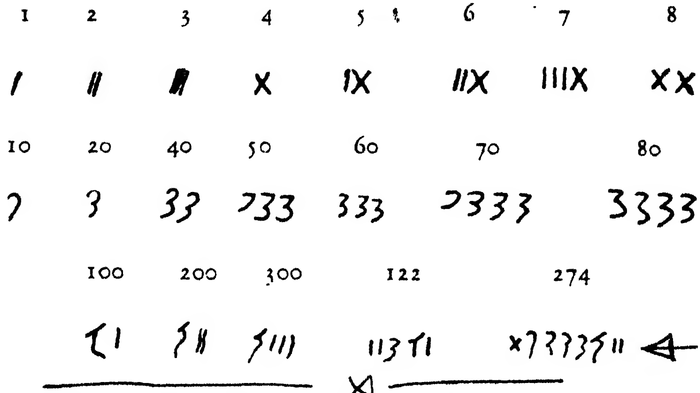
Early Occurrence. The *Kharoṣṭhî lipi* is a script written from right to left. The majority of the Kharoṣṭhî inscriptions have been found in the ancient province of Gândhâra, the modern eastern Afghanistan and the northern Punjab. It was a popular script meant for clerks and men of business. The period during which it seems to have been used in India extends from the fourth century B.C. to the third century A.D. In the Kharoṣṭhî inscriptions of Aśoka only four numerals have been found. These are the primitive vertical marks for one, two, four, and five, thus:

1	2	4	5
/	//	////	/////

More developed forms of these numerals are found in the inscriptions of the Sakas, of the Pârthians and

[1] Megasthenes speaks of mile-stones indicating the distances and the halting places on the roads. The distances must have been written in numerical figures (Bühler, *l.c.*, p. 6; also *Indika of Megasthenes*, pp. 125-26). The complicated system of keeping accounts mentioned in the *Arthaśâstra* of Kauṭilya confirms the conclusion.

of the Kuṣânas, of the 1st century B.C. and the 1st and 2nd centuries A.D., as well as in other probably later documents. The following are some of the numerals of this period:



Forms and their Origin. It cannot be satisfactorily explained why the number four, which was previously represented by four vertical lines came to be represented by a cross later on. The representation of the numbers five to eight follows the additive principle, with four as the base. This method of writing the numbers 4 to 8 is not met with in the early records of the Semites. We do not know how the number nine was written. It is very probable that it was written as IXX, *i.e.*, 4+4+1 (reading from right to left, the order being the same as that of the script). The number 10 has an entirely new sign. The question why it was not written as IIXX, or why the base X (4) was abandoned cannot be satisfactorily answered.

It is accepted by all that the Kharoṣṭhî is a foreign script brought into India from the west. The exact period at which it was imported is unknown. It might have been introduced at the time of the conquest of the Punjab by Darius (*c.* 500 B.C.) or earlier.[1] The numerals given above undoubtedly belong to this script as they proceed from right to left.

The old symbols of the inscriptions of Aśoka, however, seem to have undergone modification in India, especially the numbers from 4 to 19. The symbols for four and ten seem to have been coined in India, in order to introduce simplification and also to bring the Kharoṣṭhî numeral system in line with the Brâhmî notation already in extensive use. The symbol 𐩃 seems to have been derived by turning the Brâhmî symbol 𑁕 which represents 4 in the inscriptions of Aśoka. The inclined cross to represent 4 is found in the Nabatean numerals in use in the earlier centuries of the Christian Era.[2] The Nabatean numerals resemble the Kharoṣṭhî also in the use of the scale of twenty and in the method of formation of the hundreds. It is possible that the Semites might have borrowed the Kharoṣṭhî symbol for 4, although it is not unlikely, as Bühler thinks, that the symbol might have been invented independently by both nations.

[1] The theory of the foreign origin of the script has to be revised in the light of the discoveries at Mohenjo-daro and Harappa, especially in view of the fact that the Mohenjo-daro alphabet ran from right to left.

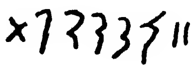
[2] J. Euting, *Nabatäische Inschriften aus Arabien*, Berlin, 1885, pp. 96-97.

The numeral  (10) closely resembles the letter *a* of the Brâhmî alphabet. The symbol for twenty  appears to be a cursive combination of two tens. It resembles one of the early Phoenician forms found in the papyrus Blacas[1] (5th century B.C.). The mode of expressing the numbers 30, 40, etc., by the help of the symbols for 10 and 20, is the same as amongst the early Phoenicians and Aramaeans.

The symbol for 100 resembles the letter *ta* or *tra* of the Brâhmî script, to the right of which stands a vertical stroke.

The symbols for 200, 300, etc., are formed by writing the symbols for 2, 3, etc., respectively to the right of the symbol for 100. This evidently is the use of the multiplicative principle, as is found amongst the early Phoenicians.[2]

The formation of other numbers may be illustrated by the number 274 which is written with the help of the symbols for 2, 100, 20, 10 and 4 arranged as



in the right to left order. The 2 on the right of 100 multiplies 100, whilst the numbers written to the left are added, thus giving 274.

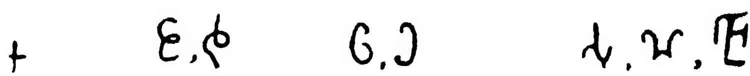
The ancient Kharoṣṭhî numerals are given in Table I.

[1] Bühler, *Palaeography*, p. 77; Ojha, *l.c.*, p. 128; see Table II(*b*).
[2] See Table II(*c*).

BRÂHMÎ NUMERALS

~~**Early Occurrence and Forms.**~~ The Brâhmî inscriptions are found distributed all over India. The Brâhmî script was, thus, the national script of the ancient Hindus. It is undoubtedly an invention of the Brâhmaṇas. The early grammatical and phonetic researches seem to have resulted in the perfection of this script about 1,000 B.C. or earlier. The Brâhmî numerals are likewise a purely Indian invention. Attempts have been made by several writers of note to evolve a theory of a foreign origin of the numerals, but ~~we are convinced that~~ all those attempts were utter failures.[1] These theories will be dealt with at their proper places. Due to the lack of early documents, we are not in a position to say what exactly were the original forms of the Brâhmî symbols. Our knowledge of these symbols goes back to the time of King Aśoka (*c.* 300 B.C.) whose vast dominions included the whole of India and extended in the north upto Central Asia. The forms of these symbols are:

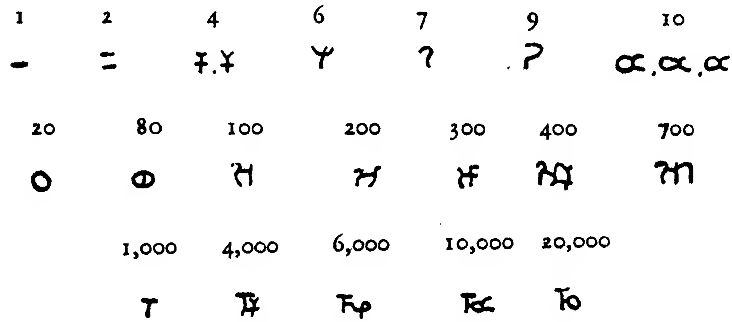
4 6 50 200



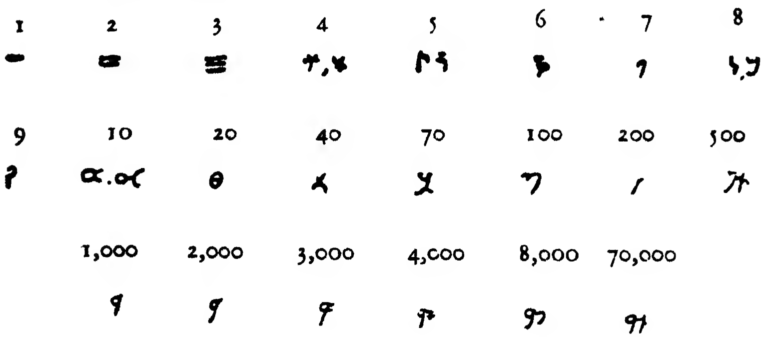
The next important inscription containing numerals is found in a cave on the top of the Nânâghât hill in Central India, about seventy-five miles from Poona. The cave was made as a resting place for travellers by order of a King named Vediśrî, a descendant of King Sâtavâhana. The inscription contains a list of gifts made on the occasion of the performance of several *yajñas* or religious sacrifices. It was first deciphered

[1] *Cf.* Langdon's opinion in *Mohenjo-daro and the Indus Valley Civilisation*, ch. xxiii.

by Pandit Bhagavanlal Indraji who has given the interpretation of the numerical symbols.[1] These occur at about thirty places, and their forms are as below:



A number of inscriptions containing numerals and dating from the first or the second century A.D. are found in a cave in the district of Nasik in the Bombay presidency. These contain a fuller list of numerals. The forms[2] are as follows:



[1] "On Ancient Nâgarî Numeration; from an inscription at Nânâghât," *Journ. of the Bombay Branch of the Royal Asiatic Society*, 1876, Vol. XII, p. 404.

[2] E. Senart, "The inscriptions in the caves at Nasik," *EI*, Vol. VIII, pp. 59-96; "The inscriptions in the cave at Karle," *EI*, Vol. VII, pp. 47-74.

Even after the invention of the zero and the place-value system, the same numerical symbols from 1 to 9, continued to be employed with the zero to denote numbers. Thus the gradual development of these forms can be easily traced. This gradual change from the old system without place-value to the new system with the zero and the place-value is to be met with in India alone. All other nations of the world have given up their indigenous numerical symbols which they had used without place-value and have adopted the zero and a new set of symbols, which were never in use in those countries previously. This fact alone is a strong proof of the Hindu origin of the zero and the place-value system.

The numbers 1, 2 and 3 of the Brâhmî notation were denoted by one, two and three horizontal[1] lines placed one below the other. These forms clearly distinguish the Brâhmî notation from the Kharoṣṭhî and the Semitic systems.

It cannot be said why the strokes were horizontal in Brâhmî and vertical in Kharoṣṭhî and Semitic writings, just as it cannot be said why the writing proceeded from left to right in Brâhmî and from right to left in Kharoṣṭhî and Semitic writings. It appears to us that the Brâhmî and the Kharoṣṭhî (Semitic) numerals have always existed side by side and it cannot be definitely said which of these is the earlier. The difference in writing the symbols 1 to 3, seems to be due to the inherent difference between the two systems of writing. The principles upon which numerical signs are formed in the two systems are quite different.

Difference from other Notations. In the Brâhmî

[1] It has been incorrectly stated by Smith and Karpinski that the Nânâghât forms were vertical. See *Hindu Arabic Numerals*, p. 28.

there are separate signs for each of the numbers 1, 4 to 9 and 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 200, 300,..... 1000, 2000, etc., while in the oldest Kharoṣṭhî and in the earliest Semitic writings, the Hieroglyphic and the Phoenician, the only symbols are those for 1, 10, 20 and 100.

The Hieratic and the Demotic numerals, however, resemble the Brâhmî in having nineteen symbols for the numbers from 1 to 100, but the principle of formation of the numbers 200, 300, 400, 2,000, 3,000 and 4,000 are different, as will appear from Table II(*c*). The method of formation of intermediate and higher numbers is also different in the two systems. While the Brâhmî places the bigger numbers to the left, the arrangement is the reverse of this in the Kharoṣṭhî and Semitic writings. Thus the number 274, is written in Brâhmî with the help of the symbols for 200, 70 and 4 as (200) (70) (4), while in the Kharoṣṭhî and the Semitic numerals it is written as (4) (70) (200).[1]

Theories about their Origin. Quite a large number of theories have been advanced to explain the origin of the Brâhmî numerals. Points of resemblance have been imagined between these numerals and those of other nations. Recourse has been taken by writers to the turning, twisting, adding on or cutting off of parts of the numerals of other nations to fit their pet theories. It is needless to say that each of these theories had its own supporters who were quite convinced of the correctness of their explanations. We give below the outlines of some of these theories:

1. Cunningham[2] believed that writing had been known in India from the earliest known times, and

[1] Compare the same number written in Kharoṣṭhî, p. 24.

[2] Inscriptions of Aśoka, *Corpus Inscriptionum Indicarum*, Vol. I, p. 52.

that the earliest alphabet was pictographic. He suggested that the Brâhmî script was derived from the early pictographic writing. The theory is evidently capable of extension to the numerical signs. Later epigraphists, however, discarded the hypothesis as it appeared too fanciful to them. Cunningham's bold hypothesis regarding the antiquity of writing in India has been more than justified by the recent discovery of the use of a quasi-pictographic script on certain seals and in inscriptions belonging to the fourth millenium B.C. found amongst the excavations at Mohenjo-daro and Harappa. His theory has been revived by Langdon who is of opinion that the Brâhmî alphabet could be derived from the pictographs of Mohenjo-daro.[1] The theory is incomplete as the writings of Mohenjo-daro have not been completely deciphered as yet. It can be called a guess only. As regards the evolution of the Brâhmî numerals, it may be stated that it is at present extremely difficult to differentiate the numerical symbols from the Mohenjo-daro script. If the surmise that the figures, given on p. 19, are numerical symbols be correct, it will not be possible to develop a theory deriving the Brâhmî numerals from them.

2. Bayley[2] asserted that the principles of the Brâhmî system have been derived from the hieroglyphic notation of the Egyptians, and that the majority of the Indian symbols have been borrowed from Phoenician, Bactrian, and Akkadean figures or letters. As has been already remarked[3] the principles of the Brâhmî and the hieroglyphic systems are entirely different and

[1] *Mohenjo-daro etc.*, Chap. xii. This view is strongly supported by Hunter, *l.c.*, p. 490.

[2] *Journal of the Royal Asiatic Soc.*, XV, part I, reprint, London, 1882, pp. 12 and 17. The theory was supported by Taylor, *The Alphabet*, London, 1883, Vol. II, pp. 265-66.

[3] See pages 27-8.

unconnected. The reader will find the hieroglyphic and the Brâhmî systems shown together in Tables II (*a*), (*b*), (*c*), and convince himself of the incorrectness of Bayley's assertion. Moreover, the assumption that the Hindus borrowed from four or five different, partly very ancient and partly more modern, sources, is extremely difficult to believe. Regarding the resemblance between the Bactrian and Akkadean numbers and the Brâhmî forms postulated by Bayley, Bühler[1] remarks that in four cases (four, six, seven and ten) the facts are absolutely against Bayley's hypothesis. Some writers have also criticized Bayley's drawings as being affected by his theory.[2] Under these circumstances his derivation has to be rejected.

3. Burnell[3] pointed out the general agreement of the principles of the Indian system with those of the Demotic notation of the Egyptians. He asserted a resemblance between the Demotic signs for 1 to 9 and the corresponding Indian symbols, and put forward the theory that the Hindus borrowed these signs and later on modified them and converted them into *akṣaras* (letter forms).

4. Bühler[3] has put forward a modification of Burnell's theory. He states, "It seems to me probable that the *Brâhma* numerals are derived from the Egyptian Hieratic figures, and that the Hindus effected their transformation into *Akṣaras*, because they were already accustomed to express numerals by words."

The above theories like the one examined before are not well founded. Tables II (*a*), (*b*), (*c*), show the Hieratic and Demotic symbols together with those of the Brâhmî. An examination of the Tables will reveal

[1] Bühler, *On the Origin of the Indian Brâhma Alphabet*, Strassburg, 1898, pp. 52, 53 foot-note.

[2] *Cf.* Smith and Karpinski, *Hindu Arabic Numerals*, pp. 30-1.

[3] Bühler, *l.c.*, p. 82.

that out of the nineteen symbols to represent the numbers from 1 to 100, only the nine of the Brâhmî resembles the corresponding symbol of the Demotic or the Hieratic. There is absolutely no resemblance between any of the others. To base the derivation on a resemblance between the Hieratic 5 and the Brâhmî 7, as is sought to be done, is absurd. Likewise the changing and twisting of the Demotic and Hieratic forms to suit the theory is unacceptable.

That there is some resemblance between these systems in the fact that each employs the same number of signs, *i.e.*, nineteen, for the representation of numbers upto hundred, cannot be denied. There is, however, a difference in the method of formation of the hundreds and the thousands. In the Brâhmî the numbers 200 and 300 or 2,000 and 3,000, are formed by adding one *mâtṛkâ* and two *mâtṛkâs* to the right of the symbol for hundred or thousand respectively, thus

 = 100,  = 200,  = 300

 = 1,000,  = 2,000,  = 3,000.

The numbers 400 and 4,000 are formed by connecting the symbol for 100 and 1,000 to the number  (4), thus

 = 400 and  = 4,000.*

In the Hieratic the corresponding symbols are:

 = 2
 = 3
 = 4,

 = 100,  = 200,
 = 1,000,  = 2,000,

 = 300,
 = 3,000,  = 400,  = 4,000.

It will be observed that in the Hieratic system the sign for one thousand is not used in the formation of the other thousands. The similarity in principle, even if it were complete, would not force us to conclude that one of these nations copied the other. The use of nineteen signs afforded the easiest and probably the best method of denoting numbers. It is not beyond the limits of probability that what appeared easy to the Egyptians might have also independently occurred to the Hindus.

There are on the other hand some considerations which make us suggest that the Egyptians borrowed the principles of the Hieratic and the Demotic systems from outside, and probably from India—a hypothesis which is not *a priori* impossible as it has been shown that the numeration system of the ancient Hindus based on nineteen signs might have been perfected about 1,000 B.C. It is known that the ancient Egyptian system employed only four signs, those for 1, 10, 20 and 100. Why should there be a sudden change from the old system to one containing nineteen signs cannot be adequately explained except on the hypothesis of foreign influence. Further, the cursive forms for the numbers 2, 3 and 4 are unsuited to the right to left Hieratic or Demotic script. Although these figures are connected with the earlier hieroglyphic and Phoenician figures, yet it is possible that the cursive combinations might have been formed to obtain the nineteen signs necessary for the new system, under the influence of a people with a left to right script. It may be, however, asserted that the hypothesis of an Indian origin of the Hieratic system is a mere suggestion. The two points noted above, by themselves, would not be enough, unless backed by other facts, to put forward a theory. It is expected that further discoveries will throw light on this point.

Relation with Letter Forms. It was suggested by James Princep,[1] as early as 1838, that the numerals were formed after the initial letters of the number names. But knowing the pronunciation of the number names, we find this not to be the case. Other investigators have held that the numeral signs were formed after the letters in the order of the ancient alphabet. Although we find that letters were used to denote numbers as early as the 8th century B.C.,[2] and that many systems of letter-numerals were invented in later times[3] and came into common use, yet we are forced to reject this hypothesis as resemblance between the old numerical forms and the letters in the alphabetic order cannot be shown to exist.

A peculiar numerical notation, using distinct letters or syllables of the alphabet, is found to have been used in the pagination of old manuscripts as well as in some coins and a few inscriptions. The signs are, however, not always the same. Very frequently they are slightly differentiated, probably in order to distinguish the signs with numerical values from those with letter values. The fact that these symbols are letters is also acknowledged by the name *akṣarapallî* which the Jainas occasionally give to this system, in order to distinguish it from the decimal notation, the *aṅkapallî*.[4]

[1] "Examination of inscriptions from Girnar in Gujerat, and Dhauli in Cuttack," *JASB*, 1838.

[2] The method seems to have been used by Pâṇini. See p. 63.

[3] *Vide infra*, pp. 64ff.

[4] Bühler, *l.c.*, p. 78. The details of the *akṣarapallî* are given later on (pp. 72ff).

The use of the letter system, the *akṣarapallî*, suggests that the old Brâhmî numerals also might have been fashioned after the letters or the syllables of the Brâhmî alphabet.

A detailed examination of the numerals and the letter-forms has been made, and the result is tabulated below:[1]

The numerals 1, 2 and 3 which were represented by horizontal strokes do not resemble any letter. They may have been derived from the *mâtṛkâ* signs.

4. The earliest form of this numeral in the inscriptions of Aśoka resembles the letter *ka*; the Nânâghât and Nâsik cave forms resemble the syllable *pka*. While this seems to have been the case with most of the later inscriptions, forms resembling the syllables *pna*, *lka*, *ṭka*, *pkṛ* are also to be met with.

5. In most of the inscriptions it resembles the syllable *tṛ*, whilst forms resembling *ta*, *tâ*, *pu*, *hu*, *ru*, *tṛ*, *tṛâ*, *nâ*, *na*, *hṛ*, *hra*, and *ha* are also found.

6. In most of the inscriptions it resembles *phra*, and in a few *phrâ*, *pha*, *phâ*, *ja* and *hâ*. The resemblance in this case is, however, not close enough.

7. In most of the inscriptions it resembles *gra* or *gu*, whilst n some it is like *ga*.

8. In most cases it resembles the syllable *hra* or *hrâ*. In some inscriptions the form of this numeral cannot be said to resemble any letter or syllable.

9. The oldest forms of this numeral, those in the Nânâghât, the Kuṣâna and the Kṣatrapa inscriptions, cannot be said to resemble any letter or syllable. The later forms, however, resemble the letter *u* or *om*.

[1] For the varying forms of the numerals see Tables III—XIII.

10. The oldest forms of this too do not resemble any letter or syllable, although later forms may be said to resemble *rya*, *ha*, *hṛ*, *kha*, and *ṭhu* or *ṭha*.[1]

20. In all cases it resembles the letter *tha*.
30. Resembles the letter *la* in all cases.
40. Resembles *pta* or *sa* in all cases.
50. It may be said to resemble the *anunâsika*.
60. Resembles *pu*, *pa* and *pra*.
70. Resembles *pu*, *pta*, *pra*, *pna*, *pnâ*, or *hra*.
80. Resembles the *upadhmânîya* sign.
90. Resembles the *upadhmânîya* sign with a central cross.
100. Resembles the syllable *su* in most of the inscriptions. In some it resembles *a*.

Indraji's Theory. The above details show that,

1. No phonetical value can be assigned to the ancient forms of the numerals 6 and 10, and 1, 2 and 3;

2. There is a great deal of variation in the phonetical values of the other units, excepting the cases of 7 and 9; and

3. The phonetical values of the tens are determinate excepting that of 70 which offers great variation.

Thus leaving aside the symbols for 1, 2 and 3, we find that out of the 16 symbols, no resemblance to letter-forms can be satisfactorily shown in the case of two (*i.e.*, 6 and 10),[2] and that in the case of three others (*i.e.*, 7, 9 and 70) there are too many variations in the phonetical values, whilst in the case of the remaining 11 symbols, the phonetical values are more or less determinate. These facts led Pandit Bhagavanlal Indraji

[1] According to Bühler, *l.c.*, p. 80.

[2] The resemblance to *ja*, *sa*, *phra*, etc. and *ṭhû*, *ṭha*, etc., respectively, stated by Bühler (*l. c.*, p. 80), does not appear to us to be close enough.

to put forward the hypothesis that the Brâhmî numerals are derived from the letters or syllables of the Brâhmî script. The Pandit, however, admitted his inability to find the key to the system, nor has it been found by any other scholar upto this time. The problem, in fact, appears to be insoluble, unless further epigraphic material is discovered to show the forms of the numerical symbols anterior to Aśoka. The Aśokan forms as well as those of later inscriptions are in a too well developed state, and are too far away from the time of invention of those symbols, to give us the desired information regarding their origin.

But of all the theories that have been advanced from time to time, that of Pandit Indraji seems to us to be the most plausible. The Hindus knew the art of writing in the fourth millennium B.C. They used numbers as large as 10^9 about 2,000 B.C., and since then their religion and their sciences have necessitated the use of large numbers. Buddha in the sixth century B.C. is stated to have given number names as large as 10^{53} and this number series was continued still further in later times.[1] All these facts reveal a condition that would have been impossible unless arithmetic had attained a considerable degree of progress. It is certain that the Hindus must have felt the necessity of some method of writing these numbers from the earliest known times. It would not be, therefore, against historical testimony to conclude that the Hindus invented the Brâhmî number system. The conclusion is supported by the use, in writing numbers, of the *mâtṛkâ*, the *anunâsika* and the *upadhmânîya* signs which are found only in the Sanskrit script and in no other script, whether ancient or modern. It is further strengthened by Indian tradition, Hindu, Jaina as well as Buddhist, which

[1] Cf. pp. 10-12.

ascribes the invention of the Brâhmî script and the numeral notation to Brahmâ, the Creator, and thereby claims it as a national invention of the remotest antiquity.[1]

Period of Invention. The invention of the system may be assigned to the period 1,000 B.C. to 600 B.C. As the Aśokan numerical figures indicate that the system was common all over India,[2] and that it has had a long history, the lower limit 1,000 B.C. is certainly not placed too early. On the other hand general considerations, such as the high development of the arts and the sciences, the mention of numerical signs and of 64 different scripts in ancient Buddhist literature,[3] and the use of large numbers at a very early period, all point to the date of the invention of the system as being nearer to 1,000 B.C., if not earlier.

Resume. The strength of Pandit Indraji's hypothesis lies in the fact that out of the nineteen signs, eleven definitely resemble the letters or the signs of the Brâhmî alphabet. The resemblance is too striking to be entirely accidental. Moreover, it has been found that the numerical forms closely followed the changing forms of the letters from century to century. This is especially true in the case of the tens and shows that the writers of the ancient inscriptions knew the phonetical values of these symbols. The divergence from letter forms in the case of the signs for the units may be due to the

[1] Bühler (*l.c.*, p. 1, foot-note,) quotes several authorities. Of these the *Nârada Smṛti* and the Jaina canonical work, the *Samavâyânga-sûtra*, belong to the fourth century B.C.

[2] Megasthenes speaks of mile-stones indicating distances and the halting places on the roads. *Indika of Megasthenes*, pp. 125-126; Bühler, *l.c.*

[3] Related in the *Lalitavistara*, both in the Sanskrit text and the Chinese translation of 308 A.D. The Jaina *Samavâyânga-sûtra* (*c.* 300 B.C.) and *Pannavanâ-sûtra* (*c.* 168 B.C.) each gives a list of 18 scripts; see Weber, *Indische Studien*, 16, 280, 399.

fact that they were the first to be invented and were in more common use, so that they acquired special cursive forms and did not follow the changes in the forms of the corresponding letters. We may now summarize the discussion given in this section by saying that (1) the Brâhmî numerical forms were undoubtedly of Indian origin, (2) the form of the tens were derived from certain letters or signs of the alphabet, and (3) the origin of the forms of the units is doubtful. It is probable that they, too, were fashioned after the letters of the alphabet, but there appears to be no means of justifying this assertion unless the forms of these numerals anterior to Aśoka are discovered.

THE DECIMAL PLACE-VALUE SYSTEM

Important Features. The third and most important of the Hindu numeral notations is the decimal place-value notation. In this system there are only ten symbols, those called *aṅka* (literally meaning "mark") for the numbers one to nine, and the zero symbol, ordinarily called *śûnya* (literally, "empty"). With the application of the principle of place-value these are quite sufficient for the writing of all numbers in as simple a way as possible. The scale is, of course, decimal. This system is now commonly used throughout the civilised world. Without the zero and the place-value, the Hindu numerals would have been no better than many others of the same kind, and would not have been adopted by all the civilised peoples of the world. "The importance of the creation of the zero mark," says Professor Halsted, "can never be exaggerated. This giving to airy nothing, not merely a local habitation and a name, a picture, a symbol, but helpful power, is the characteristic of the Hindu race whence it sprang.

It is like coining the *Nirvāṇa* into dynamos. No single mathematical creation has been more potent for the general on-go of intelligence and power."[1]

Forms. A large number of scripts differing from each other are in use in different parts of India today. The forms of the numerical signs in these scripts are also different. Although all the Hindu scripts are derived from a common source—the Brâhmî Script—yet the differences in the forms of the various modern Indian scripts are so great that it would have been difficult to establish any relation between them, if their previous history had not been known. The above remark applies to the numerical signs also, as will appear from a study of the numerical signs in the various vernaculars of India given in Table XV. The great divergence in the forms of the numerical symbols shows that in India, people already knew the use of the zero and the place-value principle before the different scripts came into being, and that the numeral forms were independently modified in various parts of India, just as the letters of the alphabet were modified. And as the changes in the forms in different localities were independent of each other, so there has come about a great divergence in the modern forms. That this divergence already existed in the eleventh century is testified to by Al-Bîrûnî who says, "As in different parts of India, the letters have different shapes the numerical signs, too, which are called *aṅka*, differ."[2]

Nagari Forms. The most important as well as the most widely used of the different symbols are those belonging to the *Nâgarî* script. The present forms of these symbols are:

[1] G. B. Halsted, *On the foundation and technique of Arithmetic*, Chicago, 1912, p. 20.

[2] *Alberuni's India*, I, p. 74.

१, २, ३, ४, ५, ६, ७, ८, ९, ०.

The gradual development of these figures from the Brâhmî numerals is shown in Table XIV.

Epigraphic Instances. The following is a list of inscriptions and grant plates upto the middle of the tenth century, which contain numerals written in the decimal place-value notation. The numerals in the inscriptions and plates after this period, are always given in decimal figures.

1.	595 A.D.	Gurjara grant plate from Sankheda, (*EI*, II, p. 19). The date Saṁvat 346 is given in the decimal place-value notation.
*2.	646 A.D.	Belhari Inscription, (*JA*, 1863).
*3.	674 A.D.	Kanheri Inscription, (*JA*, 1863, p. 392).
4.	8th Century	Ragholi plates of Jaivardhaṇa II, (*EI*, IX, p. 41). The number 30 is written in decimal figures.
5.	725 A.D.	Two Sanskrit Inscriptions in the British Museum, (*IA*, XIII, p. 250). The dates Saṁvat 781 (=723 A.D.) and Saṁvat 783 (=725 A.D.) are given in decimal figures.
*6.	736 A.D.	Dhiniki copper plate grant, (*IA*, XII, p. 155). The date Vikrama Saṁvat 794 is given in decimal figures.
7.	753 A.D.	Ciacole plates of Devendravarmaṇa, (*EI*, III, p. 133). The number 20 is written in decimal figures.
8.	754 A.D.	Râṣṭrakûṭa grant of Dantidurga, (*IA*, XI, p. 108). The date Saṁvat 675 is given in decimal figures.

9.	791 A.D.	Inscription of Sâmanta Devadatta, (*IA*, XIV, p. 351). The date Vikrama Saṁvat 847 is given in decimal figures.
10.	793 A.D.	Daulatabad plates of Śaṅkargaṇa, (*EI*, IX, p. 197). The date Śaka 715 is given in decimal figures.
*11.	813 A.D.	Torkhede plates, (*EI*, III, p. 53; also *IA*, XXV, p. 345). The date Śaka Saṁvat 735 is given in decimal figures.
12.	815 A.D.	Buchkalâ inscription of Nâgbhaṭa, (*EI*, IX, p. 198). The date Saṁvat 872 is given in decimal figures.
13.	837 A.D.	Inscription of Bâuka (Rajputana Museum, *PLM*, p. 127; *EI*, XVIII, p. 87). The date Vikrama Saṁvat 894 is given in decimal figures.
14.	843 A.D.	The inscriptions from Kanheri, No. 43 b., (*IA*, VIII, p. 133). The date Saṁvat 765 is given in decimal figures.
15.	851 A.D.	The inscriptions from Kanheri, No. 15, (*Ibid*). The date[1] Saṁvat 775 is given in decimal figures.
16.	853 A.D.	Pâṇḍukeśvara Plates of Lalitasuradeva, (*IA*, XXV, p. 177). The date Saṁvat 21 of the King's reign is given in decimal figures.
17.	860 A.D.	Ghatiyala Inscription of Kakkuka (*EI*, IX, p. 277). The date Vikrama Saṁvat 918 is given in decimal figures.

[1] For correction of date see *IA*, XX, p. 421.

18.	862 A.D.	Deogarh Jaina Inscription of Bhojadeva, (*EI*, IV, p. 309). The dates Vikrama Saṁvat 919 and the corresponding Śaka Saṁvat 784 are both given in decimal figures.
19.	870 A.D.	Gwalior inscription of the reign of Bhojadeva (*Archaeological Survey of India*, Report, 1903-4, plate 72). Although the date is not given, the ślokas are numbered from 1 to 26 in decimal figures.
20.	876 A.D.	Gwalior inscription of Allah, of the reign of Bhojadeva (*EI*, I, p. 159). The date Vikrama Saṁvat 933, as well as the numbers 270, 187 and 50 are given in decimal figures.
21.	877 A.D.	The inscriptions from Kanheri, No. 43a, (*IA*, XIII, p. 133). The date Saṁvat 799 is given in decimal figures.
22.	882 A.D.	Pehava inscription (*EI*, I, p. 186). The date Saṁvat 276 (Śrî Harṣa Era) is given in decimal figures.
23.	893 A.D.	Grant plate of Balavarmaṇa, (*EI*, IX, p. 1). The date Vallabhî Saṁvat 574 is given in decimal figures.
24.	899 A.D.	Grant plate of Avanîvarmaṇa, (*EI*, IX, p. 1). The date Vikrama Saṁvat 956 is given in decimal figures.
25.	905 A.D.	The Ahar stone inscription (*Journ. United Provinces Hist. Soc.*, 1926, pp. 83 ff) contains several dates written in decimal figures.
26.	910 A.D.	Râṣṭrakûṭa grant of Krishna II (*EI*,

		I, p. 53). The date is given in decimal figures.
27.	917 A.D.	Sanskrit and old Canarese inscriptions, No. 170, (*IA*, XVI, p. 174). The date Saṁvat 974 is given in decimal figures. The number 500 also occurs.
28.	930 A.D.	Cambay plates of Govinda IV, (*EI*, VII, p. 26). The date Śaka Saṁvat 852 is given in decimal figures.
29.	933 A.D.	Sangli plates of Râṣṭrakûṭa Govindarâja IV, (*IA*, XII, p. 249). The date Saṁvat 855 is given in decimal figures.
30.	951 A.D.	Sanskrit and old Canarese inscriptions, No. 135, (*IA*, XII, p. 257). The date Saṁvat 873 is given in decimal figures.
31.	953 A.D.	Inscription of Yaśovarmaṇa, (*EI*, I, p. 122). The date Saṁvat 1011 is given in decimal figures.
32.	968 A.D.	Siyadoni stone inscription (*EI*, I, p. 162). The inscription contains a large number of numerals expressed in decimal figures.
33.	972 A.D.	Râṣṭrakûṭa grant of Amoghavarṣa, (*IA*, XII, p. 263). The date Śaka 894 is given in decimal figures.

Palaeographic evidences of the early use of the decimal place-value system of notation are found in the Hindu colonies of the Far East.[1] The most important ones among these are the three inscriptions of

[1] G. Coedès, "A propos de l'origine des chiffres arabes," *Bull. School of Oriental Studies* (London), VI, 1931, pp. 323-8.

Śrîvijaya, two found at Palembang in Sumatra, and the third in the island of Banka. These contain the dates 605, 606 and 608 of the Śaka Era (corresponding respectively to A.D. 683, 684 and 686) written in numerical figures. Another instance giving the date 605 Śaka is the inscription of Sambor in Cambodia. In an inscription at Po Nagar in Champa, occurs the date 735 Śaka (= 813 A.D.).

Their Supposed Unreliability. The above list contains more than thirty undoubted epigraphic instances of the use of the place-value notation in India. G.R.Kaye,[1] who believes in the theory of the non-Hindu origin of the place-value notation, states that all the early epigraphic evidences of its use in India are unreliable. On the basis of the existence of a few forged grant plates he asserts that in the eleventh century "there occurred a specially great opportunity to regain confiscated endowments and to acquire fresh ones" and thereby concludes that all early epigraphic evidences must be unreliable. Such reasoning is obviously fallacious and needs no refutation.

Most of the copper plates are legal documents recording gifts made by rich persons or kings to Brâhmaṇas on religious occasions. The plates contain details as to the occasion for making the gifts, the names of the donor and the donee, the description of the movable and immovable properties transferred by the gift, and the date of the gift which is always written out in full in words and very often in figures also. The forgeries may be of two kinds: (1) In the original documents, parts relating to either the names of the donor or the donee, or the description of the immovable property may have been obliterated by being beaten out and new

[1] "Notes on Indian Mathematics," *JASB*, (N. S., 1907), III, pp. 482-87.

names or descriptions substituted. All such forgeries are easily detected, because of the uneven surface of the part of the plate that is tampered with and the difference in the writing. (2) An entirely new document may be forged. Such cases, though rare, are also easily detected, because there is obvious divergence as to the date recorded in the document, and that inferred on the basis of the forms of the characters used in the writing. Such forgeries are also marked by an obvious inferiority in execution, and inaccuracies in the statement of genealogies and other historical facts.

Epigraphists have so far found little difficulty in eliminating the spurious grant plates. It might be mentioned that the genuineness of the grant plates included in our list has not been questioned by any epigraphist.[1]

Kaye, in his article quoted above, has given a list of eighteen inscriptions and grant plates and eliminates all but the last two as forgeries. The arguments he has employed and the assertions of facts that he has made are in most cases incorrect and misleading, so that his conclusions cannot be accepted. As an instance of his method, we quote his criticism about the Gurjara grant plate, No. 1 in our list. He writes: "Dr. Bühler quotes this Gurjara inscription of the Chedi year 346 or A.D. 594 as the earliest epigraphic instance of the use of the decimal notation in India. (i) An examination

[1] If any of them is forged, the forgery is so good that it cannot be detected. The writing in such cases, if any, is so well forged as to be indistinguishable from that used in the period to which the plate is said to belong. Therefore, the evidence of these plates as to the method of writing numbers, cannot be rejected, even if they be proved to be spurious at some future date—a contingency which is very unlikely to use. It may also be noted that the list contains several *stone inscriptions* which cannot be spurious.

of the plate (*Ep. Ind.*, II., p. 20) suggests the possibility that the figures were added some time after the plate was engraved. The date is engraved in words as well as in figures. It is 'three hundred years exceeded by forty-six.' The symbols are right at the end of the inscription from which they are marked off by a double bar in a most unusual manner. (ii) The figures are of the type of the period, but they were also in use much later, and in no other example are such symbols used with place-value. (iii) Also there are nine dates written in the old notation (*Ep. Ind.*, V), *e. g.*, there is another grant of the Gurjara of Bharoch in which the date Saṁvat 391 (i.e., A.D. 640) is given in the old notation. Again, there is no other Chedi date, at least before the eleventh century A.D., given in the modern (place-value) notation. (iv) There cannot be the remotest doubt as to the unsoundness of this particular piece of evidence of the early use of the modern system of notation in India."

The following remarks will show to the reader that Kaye's criticism and his conclusion are unfounded and invalid:

(*i*) An examination of the plate, (*EI*, II, p. 19), will convince every one about its genuineness. The writing is bold and clear, the numerical figures occur at the end, as they ought to be, immediately after the words 'three hundred years exceeded by forty-six.' They are separated from the written words by bars, just as they ought to be. There is absolutely nothing suspicious about this method of separation, as it is common custom in India to do so and occurs frequently. That it was the practice to write the date at the end of a document is well known.[1] In fact, the numeral

[1] Many of the plates mentioned in our list contain the date at the end.

figures of the date occasionally mark the end of the document.[1] The double vertical bar, ||, is a sign of inter-punctuation. Although punctuation marks have been in use in India from the earliest known times, yet their use did not become either regular or compulsory till very recent times.[2] Different writers used the various marks differently. In inscriptions, the double vertical bar has been found at the end of sentences, half verses, verses, larger prose sections and documents. In the Junâr inscriptions it occurs after numerals and once after the name of the donor.[3] In manuscripts, the practice of separating numbers by vertical bars is common. It is found in the Bakhshâlî Manuscript[4] and in several others. Thus the occurrence of the numerals at the end and the inter-punctuation mark of the double vertical bar cannot form valid grounds for suspecting the document. The suggestion that the figures were added some time after the plate was engraved is absurd, as there appears to us no reason why one should take the trouble to add the figures when the date was already written in words.

(*ii*) Kaye admits that the figures are of the type of the period. His remark that they were in usé much later is incorrect. The Tables III-V and XII show that the use of three horizontal bars to represent 3 is not

[1] This is so in the Chârgâon plates of Huviṣka (*Arch. Survey Report*, 1908-9, plate 56), in the Inscription of Rudradamana (*IA*, VIII, p. 42) and in others.

[2] There are some copper plate grants which do not contain any punctuation marks; see Bühler, *l.c.*, p. 90.

[3] Bühler, *l.c.*, p. 89.

[4] *E.g.*, | 5 |, 21 r; | 2558 |, 2v; | 330 |, 17v; instances such as these: | 1 | 4 | 9 | 16 |, 16v; and $\underline{|\,2\,|}$, $\underline{|\,4\,|}$, etc., 5v. are very common. Very often, isolated numbers are not separated. The double vertical bar also occurs before and after the words *udâ*, *sûtraṃ*, etc.

found after the eighth century. The figure for 4 used in our grant plate is not found after the sixth century, and the same is true for the figure for 6. The forms of the numerical signs alone fix the date of the writing to the sixth century and not later.

(*iii*) The *Chedi Saṁvat* is one of the thirty-four eras, whose use has been discovered in inscriptions and grant plates. The occurrence of nine dates in the Chedi Saṁvat, written in the old notation after this plate, does not prove the unsoundness of this particular piece of evidence, as Kaye would like us to conclude. It simply shows that in India too, the new system had to fight for supremacy over the older one just as in other countries. In Arabia the new system was introduced in the eighth century, but it did not come into common use until five or six hundred years later. In Europe we find that it was exceptional for common people to use the new system before the sixteenth century—a good witness to this fact being the popular almanacs. Calendars of 1557-96 have generally Roman numerals, while Koebel's Calendar of 1578 gives the Hindu numerals as subordinate to the Roman.[1]

We may, therefore, conclude that the Gurjara grant plate offers us a genuine instance of the use of the new system (with place-value) in India.

Kaye's criticisms regarding the genuineness of some other plates included in our list (marked with asterisks) have been found to be baseless.

Place of Invention of the New System. It has been already stated that the same numeral forms for the numbers 1 to 9, as were in use in India from the earliest known times, have been used in the new system of notation with the place-value. Another noteworthy fact

[1] Smith and Karpiński, *l.c.*, p. 133.

regarding the new system is the arrangement of the *aṅka* (digits). It will be observed that the arrangement in the old system was that the bigger numbers were written to the left of the smaller ones.[1] This same arrangement continues in the new system with place-value, where the digits to the left, due to their place or position, have bigger values. The gradual change from the old system to the new one using the same numerical signs, is to be found in India alone, and this, in our opinion, is one of the strongest arguments in favour of the Hindu origin of the new system. The earliest epigraphic instance of the use of the new system is 594 A.D. No other country in the world offers such an early instance of its use. Epigraphic evidence alone is, therefore, sufficient to assign a Hindu origin to the modern system of notation.

Inventor Unknown. It is not known who the inventor of the new system was, and whether it was invented by some great scholar, or by a conference of sages or by gradual development due to the use of some form of the abacus. Likewise, it is not known to which place, city, district or seat of learning belongs the honour of the invention and its first use. Epigraphic evidence cannot help us in this direction. For the system was used in inscriptions, a very long time after its invention, in fact, when it had become quite popular all over Northern India.

Time of Invention. The grant plates were legal documents. They were written by professional writers. The existence of such writers is mentioned in the southern Buddhist canons and in the Epics.[2] They have

[1] Showing thereby that the place assigned to a numeral depended upon its value. This has been incorrectly thought to be a sort of place-value system by some writers.

[2] Bühler, *l.c.*, p. 5.

been called *lekhaka*, *lipikara* and later on *divira*, *karaṇa*, *kâyastha*, etc. According to Kalhaṇa,[1] the Kings of Kashmir employed a special officer for drafting legal documents. He bore the title of *paṭṭopâdhyâya*, *i.e.*, the teacher (charged with the preparation) of title deeds. The existence of manuals such as the *Lekhapañcâśikâ*, the *Lekhaprakâśa*, which give rules for drafting letters, land grants, treaties, and various kinds of bonds and bills of exchange, show beyond doubt that the writing of grant plates was a specialised art and that the style of writing those documents must always have been centuries behind the times, just as it is even to-day with respect to legal and state documents. The time of invention of the new system must, therefore, be placed several centuries before its first occurrence in a grant plate in the sixth century A.D. The exact period of invention may be roughly deduced from the history of the growth of numerical notations in other countries.

According to Heath,[2] the Greek alphabetic notation was invented in the 7th century B.C., but it came into general use only in the second century A.D. Thus it took about eight hundred years to get popular. In Arabia the new notation was introduced in the 8th century A.D., but it came into common use about five or six hundred years later. The same was the case in Europe. The Arabs got the complete decimal arithmetic, including the method of performing the various operations, at a period when intellectual activity in Arabia was at its greatest height, but they could not make the decimal system common before about five or six hundred years had elapsed.[3] In legal documents

[1] *Râjataraṅgiṇî*, V, pp. 397f.

[2] Heath, *History of Greek Mathematics*, I, Oxford, 1921, p. 34.

[3] The arithmetic written by Al-Kharki in the eleventh century does not use the decimal system, showing that at the time there were two schools amongst the Arab mathematicians, one favouring

and in recording historical dates, the Arabs even now use their old alphabetic notation.

Epigraphic evidences show that the new system was quite common in India in the eighth century and that the old system ceased to exist in Northern India by the middle of the tenth century. This would, therefore, place the invention of our system in the period between the first century B.C. and the third century A.D.

The exact date of the invention, however, would be nearer to the 1st century B.C. or even earlier, because for a long time after its invention, the system must have been looked upon as a mere curiosity and used simply for expressing large numbers. A still longer time must have elapsed before the method of performing the operations of addition, subtraction, multiplication, division and the extraction of roots, could be perfected. It would be only after the perfection of the methods of performing the operations that the system could be used by mathematicians. And then after this it would take about five hundred years, as in Arabia, to become popular. There should, therefore, be a gap of about eight centuries between the time of invention and its coming into popular use, just as was the case with the Greek alphabetic notation. Therefore, on epigraphic evidence alone, the invention of the place-value system must be assigned to the beginning of the Christian era, very probably the 1st century B.C. This conclusion is supported by literary and other evidences which will be given hereafter.

the Hindu numerals, while the other stuck to the old notation. See the article on "Hisâb" by H. Suter in the *Encyclopaedia of Islam*.

9. PERSISTENCE OF THE OLD SYSTEM

The occurrence of the old system of writing numbers, with no place-value, is found generally in inscriptions upto the seventh century A.D., after which it was gradually given up in favour of the new system with place-value. Occasional use of the old system, however, is to be met with in Nepal and in some South Indian inscriptions upto the beginning of the tenth century A.D., but after this period the old system seems to have been forgotten, and completely gone out of use. In the seventh century the new system was in general use, but the old system seems to have been given preference in inscriptions. There are a number of grant plates of the eighth century A.D., in which the dates, although written in the old notation, are incorrectly inscribed, showing thereby that people had already forgotten the old system. In a grant plate of Śîlâditya VI,[1] dated the Gupta year 441 (*c.* 760 A.D.), the sign for 40, instead of the sign for 4, has been subjoined to the sign for 100 to denote 400, *i.e.*, 4,000 has been incorrectly written for 400. There is another grant plate, dated the Gâṅgeya year 183 (*c.* 753 A.D.), in which the figure 183 is wrongly written.[2] This plate is of special interest as it exhibits the use of the old and the new systems in the same document.[3] Another very interesting instance of the use of the old and the new systems in one and the same document is the Ahar stone

[1] *IA*, VI, p. 19, (plate).

[2] *EI*, III, p. 133, (plate). In this the sign of 8 is written for 80 and that of 30 for 3. The number 20 has been written by placing a dot after 2.

[3] For other instances showing admixture of both the old and the new systems, see Fleet Gupta Inscriptions, *Corpus Inscriptionum Indicarum*, III, p. 292; also *IA*, XIV, p. 351, where (800) (4) (9)=849.

inscription.[1] The document records gifts made on several occasions ranging over thirty-seven years, the last entry corresponding to 905 A.D. In this inscription the old notation is used in the first six lines whilst in the following lines it has been discarded and the new place-value notation appears. It is evident from the forms that the writer did not know the old system. For instance, 200 is written by adding the subscript 2 to the letter *su* (100), instead of using a *mâtṛkâ* sign as in the old system. In the same way the sign for 10 is incorrect in so far as a small zero has been affixed to the usual sign for ten. The inscription shows that although the old system had gone out of use completely, yet people tried to use it in inscriptions, probably for the same reason that makes us use the Roman numerals in giving dates, in numbering chapters of books, and in marking the hours on the face of a clock, even upto the present day.

WORD NUMERALS

Explanation of the System. A system of expressing numbers by means of words arranged as in the place-value notation was developed and perfected in India in the early centuries of the Christian era. In this system the numerals are expressed by names of things, beings or concepts, which, naturally or in accordance with the teaching of the *Śâstras*, connote numbers. Thus the number one may be denoted by anything that is markedly unique, *e.g.*, the moon, the earth, etc.; the number two may be denoted by any pair, *e.g.*, the eyes, the hands, the twins, etc.; and similarly others. The zero is denoted by words meaning void, sky, complete, etc.

[1] C. D. Chatterjee, "The Ahar stone inscription," *Journ. United Provinces Hist. Soc.*, 1926, pp. 83-119.

The system is used in works on astronomy, mathematics and metrics, as well as in the dates of inscriptions and in manuscripts. The ancient Hindu mathematicians and astronomers wrote their works in verse. Consequently they strongly felt the need for a convenient method of expressing the large numbers that occur so often in the astronomical works and in the statement of problems in mathematics. The word numerals were invented to fulfil this need and soon became very popular. They are used even upto the present day, whenever big numbers have to be expressed in Sanskrit verse.

The words denoting the numbers from one to nine and zero, with the use of the principle of place-value, give us a very convenient method of expressing numbers by word chronograms. To take a concrete case, the number 1,230 may be expressed in many ways:

1. *kha-guṇa-kara-âdi*,
2. *kha-loka-karṇa-candra*,
3. *âkâśa-kâla-netra-dharâ*, etc.

It will be observed that the same number can be expressed in hundreds of ways by word chronograms. This property makes the word numerals specially suitable for inclusion in metre. To secure still greater variety, the numbers beyond ten are also sometimes denoted by words.

List of Words. The following is a list of words commonly used in this system to denote numbers:

0 is expressed by *śûnya*, *kha*, *gagana*, *ambara*, *âkâśa*, *abhra*, *viyat*, *vyoma*, *antarikṣa*, *nabha*, *jaladharapatha*, *pûrṇa*, *randhra*, *viṣṇupada*, *ananta*, etc.

1 is expressed by *âdi*, *śaśi*, *indu*, *vidhu*, *candra*, *kalâdhara*, *himagu*, *śîtâṁśu*, *kṣapâkara*, *himâṁśu*, *sîtaraśmi*, *prâleyâṁśu*, *soma*, *śaśâṅka*, *mṛgâṅka*, *himakara*, *sudhâṁśu*, *rajanîkara*, *śaśadhara*, *śveta*, *abja*, *bhû*,

bhûmi, *kṣiti*, *dharâ*, *urvarâ*, *go*, *vasundharâ*, *pṛthvî*, *kṣmâ*, *dharaṇî*, *vasudhâ*, *ilâ*, *ku*, *mahî*, *rûpa*, *pitâmaha*, *nâyaka*, *tanu*, etc.

2 is expressed by *yama*, *yamala*, *aśvin*, *nâsatya*, *dasra*, *locana*, *netra*, *akṣi*, *dṛṣṭi*, *cakṣu*, *aṁbaka*, *nayana* *îkṣaṇa*, *pakṣa*, *bâhu*, *kara*, *karṇa*, *kuca*, *oṣṭha*, *gulpha*, *jânu*, *jaṅghâ*, *dvaya*, *dvanda*, *yugala*, *yugma*, *ayana*, *kuṭumba*, *ravicandrau*, *naya*,[1] etc.

3 is expressed by *râma*, *guṇa*, *triguṇa*, *loka*, *trijagat*, *bhuvana*, *kâla*, *trikâla*, *trigata*, *trinetra*, *haranetra*, *sahodarâḥ*, *agni*, *anala*, *vahni*, *pâvaka*, *vaiśvânara*, *dahana*, *tapana*, *hutâśana*, *jvalana*, *śikhin*, *kṛśânu*, *hotṛ*, *pura*, *ratna*[2] (Jaina), etc.

4 is expressed by *veda*, *śruti*, *samudra*, *sâgara*, *abdhi*, *ambhodha*, *ambhodhi*, *jaladhi*, *udadhi*, *jalanidhi*, *salilâkara*, *viṣanidhi*, *vâridhi*, *payodhi*, *payonidhi*, *ambudhi*, *kendra*, *varṇa*, *âśrama*, *yuga*, *turya*, *kṛta*, *aya*, *âya*, *diṣ*, *bandhu*, *koṣṭha*, *gati*, *kaṣâya*, etc.

5 is expressed by *bâṇa*, *śara*, *śastra*, *sâyaka*, *iṣu*, *bhûta*, *parva*, *prâṇa*, *pavana*,[3] *pâṇḍava*, *artha*, *viṣaya*, *mahâbhûta*, *tatva*, *bhâva*, *indriya*, *ratna*, *karaṇîya*,[4] *vrata*, etc.

6 is expressed by *rasa*, *aṅga*, *kâya*, *ṛtu*, *mâsârdha*, *darśana*, *râga*, *ari*, *śâstra*, *tarka*, *kâraka*, *lekhya*, *dravya*,[5] *khara*, *kumâravadana*, *śaṇmukha*, etc.

7 is expressed by *naga*, *aga*, *bhûbhṛt*, *parvata*, *śaila*, *acala*, *adri*, *giri*, *ṛṣi*, *muni*, *yati*, *atri*, *vâra*, *svara*,

[1] Method of comprehending things from particular standpoints—*dravyârthika* and *paryâyârthika*.

[2] Used by Mahâvîra only; others take it for five.

[3] See *SiŚe*, i. 27; *SiŚi*, *gaṇitâdhyâya*, x. 2. Used also for 7 (See the quotations by Bhaṭṭotpala in his commentary on *Bṛhatsaṁhitā*, ch. ii). In Al-Bîrûnî's list it is erroneously put for 9.

[4] That which ought to be done; according to the Jainas—*ahiṁsâ*, *sunṛta*, *asteya*, *brahmacarya*, and *aparigraha*.

[5] Used by Mahâvîra.

dhâtu, *aśva*, *turaga*, *vâji*, *haya*, *chandaḥ*, *dhî*, *kalatra*, *tatva*,[1] *dvípa*, *pannaga*,[2] *bhaya*,[3] *mâtṛkâ*, *vyasana*, etc.

8 is expressed by *vasu*, *ahi*, *nâga*, *gaja*, *danti*, *dvirada*, *diggaja*, *hastin*, *ibha*, *mâtaṅga*, *kuñjara*, *dvîpa*, *puṣkarin*, *sindhura*, *sarpa*, *takṣa*, *siddhi*, *bhûti*, *anuṣṭubha*, *maṅgala*, *anîka*, *karman*,[4] *durita*, *tanu*,[5] *dik*,[6] *mada*,[7] etc.

9 is expressed by *aṅka*, *nanda*, *nidhi*, *graha*, *randhra*, *chidra*, *dvâra*, *go*,[8] *upendra*, *keśava*, *târkṣyadhvaj*, *durgâ*, *padârtha*,[9] *labdha*, *labdhi*, etc.

10 is expressed by *diś*, *dik*, *diśâ*, *âśâ*, *aṅgulî*, *paṅkti*, *kakubh*, *râvaṇaśira*, *avatâra*, *karman*, etc.

11 is expressed by *rudra*, *îśvara*, *mṛḍa*, *hara*, *îśa*, *bhava*, *bharga*, *śulin*, *mahâdeva*, *akṣauhiṇî*, etc.

12 is expressed by *ravi*, *sûrya*, *ina*, *arka*, *mârtaṇḍa*, *dyumaṇi*, *bhânu*, *âditya*, *divâkara*, *mâsa*, *râśi*, *vyaya*, etc.

13 is expressed by *viśvedevâḥ*, *viśva*, *kâma*, *atijagatî*, *aghoṣa*, etc.

14 is expressed by *manu*, *vidyâ*, *indra*, *śakra*, *loka*,[10] etc.

15 is expressed by *tithi*, *ghasra*, *dina*, *ahan*, *pakṣa*,[11] etc.

16 is expressed by *nṛpa*, *bhûpa*, *bhûpati*, *aṣṭi*, *kalâ*, etc.

17 is expressed by *atyaṣṭi*, etc.

[1] Used by Mahâvîra because the Jainas recognise seven *tatvas*; used for five by others.

[2] Used by Mahâvîra.

[3] Used by Mahâvîra.

[4] Used by Mahârvîra for 8 and by others for 10.

[5] Used by Mahâvîra.

[6] This word has been used for 8 as well as for 10. The use of *diś* or *dik* for 4 also occurs.

[7] Used by Mahâvîra only.

[8] This has been used for 1 also.

[9] Used by Mahâvîra only.

[10] Also used for 3.

[11] Also used for 2.

18 is expressed by *dhṛti*, etc.
19 is expressed by *atidhṛti*, etc.
20 is expressed by *nakha*, *kṛti*, etc.
21 is expressed by *utkṛti*, *prakṛti*, *svarga*, etc.
22 is expressed by *kṛti*, *jâti* (?), etc.
23 is expressed by *vikṛti*.
24 is expressed by *gâyatrî*, *jina*, *arhat*, *siddha*, etc.
25 is expressed by *tatva*,[1] etc.
27 is expressed by *nakṣatra*, *uḍu*, *bha*, etc.
32 is expressed by *danta*, *rada*, etc.
33 is expressed by *deva*, *amara*, *tridaśa*, *sura*, etc.
48 is expressed by *jagatî*, etc.
49 is expressed by *tâna*, etc.

Word Numerals without Place-value. In the *Veda* we do not find the use of names of things to denote numbers, but we do find instances of numbers denoting things. For instance, in the *Ṛgveda* the number 'twelve' has been used to denote a year[2] and in the *Atharvaveda* the number 'seven' has been used to denote a group of seven things (the seven seas, etc.).[3] There are instances, however, of fractions having been denoted by word symbols, *e.g.*, *kalâ* $= \frac{1}{16}$, *kuṣṭha* $= \frac{1}{12}$, *śapha* $= \frac{1}{4}$.

The earliest instances of a word being used to denote a whole number are found about 2,000 B.C., in the *Śatapatha Brâhmaṇa*[4] and *Taittirîya Brâhmaṇa*.[5] The

[1] Generally used for 5; also for 7 by Mahâvîra.

[2] "*Deva hitiṃ jugupurdvâdaśasya ṛtuṃ narona praminantyete*......" (vii. 103, 1).

[3] "*Om ye triśapta pariyante*...." (i. 1, 1).

[4] The word *kṛta* has been used for 4.
"*catuṣṭomena kṛtena ayânâṃ*...." (xiii. 3. 2. 1).

[5] "*Ye vai catvâraḥ stomâḥ kṛtaṃ tat*...." (i. 5. 11. 1).

Chândogya Upaniṣad also contains several instances. In the *Vedâṅga Jyotiṣa*[1] (1,200 B.C.) words for numerals have been used at several places. The *Srauta-sûtras* of *Kâtyâyana*[2] and *Lâṭyâyana*[3] have the words *gâyatrî* for 24 and *jagatî* for 48.

At this early stage, however, the word symbols were nothing more than curiosities; their use to denote numbers was rare. Moreover, we find evidences of a certain indefiniteness in the numerical significance attached to certain words. For instance, in the same work, the *Aitareya Brâhmaṇa*, the word *virâṭ* has been used to denote 10 at one place and 30 at another. The principle of place-value being unknown, the word symbols could not be used to denote large numbers, which were usually denoted in terms of the numerical denominations or by breaking the number into parts.[4] The use of the word symbols without place-value is found in the *Piṅgala Chandaḥ-sûtra* composed before 200 B.C. The principle of place-value seems to have been applied to the word numerals between 200 B.C. and 300 A.D.

Word Numerals with Place-value. The earliest instance of the use of the word numerals with place-value in its current form is found in the *Agni-Purâṇa*,[5]

[1] *rûpa* = 1, *aya* = 4, *guṇa* = *yuga* = 12, *bhasamûha* = 27. See (*YJ*. 23, *ÂJ*. 31), (*YJ*. 13, *ÂJ*. 4), (*ÂJ*. 19), (*YJ*. 25) and (*YJ*. 20) respectively.

[2] Weber's edition of *Kâtyâyana Śrauta Sûtra*, p. 1015.

[3] ix. 4. 31.

[4] E.g.,

Daśâyutânâmayutaṁ sahasrâṇi ca viṁśatiḥ
Koṭyaḥ ṣaṣṭiśca ṣaṭ caivà yo'smin rajan-mṛiḍhe hatâḥ

that is, 10 (10000) + 10000+20 (1000) + 60 (10,000,000) + 6 (10,000,000)—*Mahâbhârata*, *Strîparva*, xxvi. 9.

[5] *Agni-Purâṇa*, Baṅgabâsî ed., Calcutta (1314 B.S.), chs. 122-23, 131, 140, 141, 328-335. According to Pargiter, probably the greatest Puranic scholar of modern times, "the *purâṇas* cannot

a work which belongs to the earliest centuries of the Christian era. Bhaṭṭotpala in his commentary on the *Bṛhat-saṁhitâ* has given a quotation from the original *Puliśa-siddhânta*[1] (*c.* 400) in which the word system is used. The number expressed in this quotation is *kha* (0) *kha* (0) *aṣṭa* (8) *muni* (7) *râma* (3) *aśvi* (2) *netra* (2) *aṣṭa* (8) *śara* (5) *râtripâḥ* (1) = 1,582,237,800. There are in this work[2] several other quotations from the *Puliśa-siddhânta*, which contain word numerals. Later astronomical and mathematical manuals such as the *Sûrya-siddhânta* (*c.* 300), the *Pañca-siddhântikâ*[3] (505), the *Mahâ-* and *Laghu-Bhâskarîya*[4] (522), the *Brâhma-sphuṭa-siddhânta*[5] (628), the *Triśatikâ*[6] (*c.* 750), and the *Gaṇita-sâra-saṁgraha*[7] (850), all make use of the word notation.[8]

Word Numerals in Inscriptions. The earliest epigraphic instances of the use of the word numerals are met with in two Sanskrit inscriptions[9] found in Cambodia which was a Hindu colony. They are dated 604

be later than the earliest centuries of the Christian era." (*JRAS*, 1912, pp. 254-55). The *Agni-Purâṇa* is admitted by all scholars to be the earliest of the *Purâṇas*.

[1] *Bṛhat-saṁhitâ*, ed. by S. Dvivedi, Benares, p. 163.

[2] *Ibid*, pages 27, 29, 49, 51, etc. We are, however, not sure whether those quotations are from the original work or from a later redaction of the same.

[3] i. 8; viii. 1, etc.

[4] See *MBh*, ch. 7 and *LBh*, ch. 1.

[5] i. 51-55, etc.

[6] R. 6, Ex. 6, etc.

[7] ii. 7, 9, etc.

[8] In the face of the evidence adduced here, G. R. Kaye's assertion, (*Indian Mathematics*, Calcutta, 1915, p. 31) that the word numerals were introduced into India in the ninth century from the east, shows his ignorance of Indian mathematical works, or is a deliberate misrepresentation.

[9] R. C. Mazumdar, *Ancient Indian colonies in the far east,—Campa*, Vol. I, Lahore, 1927; see inscriptions Nos. 32, 39; also 40, 41, 43 and 44.

A.D. and 625 A.D. Their next occurrence is found in a Sanskrit inscription of Java, belonging to the 8th century.[1]

In India proper, although they were in use amongst the astronomers and mathematicians from the 3rd or 4th century A.D. onwards, it did not become the fashion to use them in inscriptions till a much later date. The earliest Hindu inscriptions using these numerals are dated 813 A.D.[2] and 842 A.D.[3] In the following century they are used in the plates issued by the Eastern Chalukya Amma II, in 943 A.D.[4] In later times the epigraphic instances become more frequent. The notation is also found in several manuscripts in which dates are given in verse.[5]

WORD NUMERALS.

Origin and Early History. It should be noted that the arrangement of words, representing the numbers zero and one to nine, in a word chronogram is contrary to the arrangement that is followed when the same number is written with numerical signs. This fact has misled some scholars to think that the decimal notation and the word numerals were evolved at two different places. G. R. Kaye has gone so far as to suggest that the word numerals were imported into India from the east. This suggestion is incorrect for the simple reason that in no language other than Sanskrit do we find any early use of the word system. Moreover, in no country other than India do we find any trace of the use of a word system of numeration

[1] *IA*, XXI, p. 48.

[2] The Kaḍab plates, *IA*, XII, p. 11; declared by Fleet to be suspicious (Kanarese Dynasties, *Bombay Gazatteer*, I, ii, 399, note 7); cf. Bühler, *l.c.*, p. 86, note 4.

[3] The Dholpur Inscription, *Zeitschrift der Deutschen Morgenländischen Gesellschaft*, XL, p. 42.

[4] *IA*, VII, p. 18.

[5] Bühler, *l.c.*, p. 86, note 7.

as far back as the fourth century A.D., at which period it was in common use amongst the astronomers and mathematicians of India.

During the earlier stages of the development of this system, we find that instead of the word symbols, the number names were used, being arranged from left to right just as the numerical signs. An instance of this is found in the Bakhshâlî Manuscript[1] (*c.* 200), where the number

2 6 5 3 2 9 6 2 2 6 4 4 7 0 6 4 9 9 4....8 3 2 1 8

is expressed as

Ṣaḍviṁśaśca (26) *tripañcâśa* (53) *ekonatriṁśa* (29) *evacha Dvâṣa*[*ṣṭi*](62) *ṣaḍviṁśa* (26) *catuḥcatvâriṁśa* (44) *saptati* (70) *Catuḥṣaṣṭi* (64) *na*[*vanavati*] (99) ... *ṁsanantaram Trirâśiti* (83) *ekaviṁśa* (21) *aṣṭa* (8) ... *pakaṃ*

In the same manuscript, however, the contrary arrangement is used when the number 54 is expressed as *catuḥ* (4) *pañca* (5).[2] Jinabhadra Gaṇi (575) has used word symbols with the left to right arrangement to express numbers.[3] It seems, therefore, that in the beginning opinion was divided as to which method of arrangement should be followed in the word system.

The extensive use of the word numerals by early mathematicians such as Puliśa, Varâhamihira, Lalla and others appears to have set the fashion to write the word numerals with a right to left arrangement, which was generally followed by later writers.

[1] Folio 58, recto. The dots indicate some missing figures. The problem apparently required the expression of a big number in numerical denominations. We do not find a problem of this type in any of the later works. Cf. B. Datta, "The Bakhshālī Mathematics." *BCMS*, XXI, p. 21.

[2] Folio 27, recto.

[3] *Bṛhat-kṣetra-samâsa*, i. 69ff.

No explanation as to why the right to left arrangement was preferred in the word system is to be found in any of the ancient works. The following explanation suggests itself to us, and we believe that it is not far from the truth: The different words forming a number chronogram were to be so selected that the resulting word expression would fit in with the metre used. To facilitate the selection the number was first written down in numerical figures. The selection of the proper words would then, naturally, begin with the figure in the units place, and proceed to the left just as in arithmetical operations. This is in accordance with the rule "*aṅkânâṃ vâmato gatiḥ*," *i.e.*, 'the numerals proceed to the left,' which seems to have been very popular with the Indian mathematicians. The right to left arrangement is thus due to the desire of the mathematicians to look upon the process of formation of the word chronogram as a sort of arithmetical operation.

Date of Invention. The use of the word numerals in the *Agni-Purâṇa* which was composed in the 4th century A.D. or earlier, shows that the word system of numerals must have become quite common in India at that time, the *Purâṇas* being works meant for the common folk. That it was a well developed system in the fourth century is also shown by its extensive use in the *Sûrya-siddhânta* and the *Puliśa-siddhânta*. Its invention consequently must be placed at least two centuries earlier. This would give us the period, 100 A.D. to 200 A.D., as the time of its invention. This conclusion is supported by the epigraphic use of the word notation in 605 A.D., in Cambodia, which shows that by the end of the 6th century A.D., the knowledge of the system had spread over an area roughly of the size of Europe.

It must be pointed out here that the decimal place-value notation and the word numerals were not invented at the same time. The decimal notation must have been in existence and in common use amongst the mathematicians long before the idea of applying the place-value principle to a system of word names could have been conceived. Thus we find that in the beginning (*c.* 200), the place-value principle, as is to be expected, was used with the number names. The word symbols were then substituted for the number names for the sake of metrical convenience. The right to left procedure was finally adopted because of the mathematicians' desire to look upon the formation of the word numeral as a sort of mathematical operation.

The above considerations place the invention of the decimal place-value notation at a period, at least two or three centuries before the invention of the word system. The word notation, therefore, points to the 1st century B.C. as the time of invention of the place-value notation. This conclusion agrees with that arrived on epigraphic evidence alone.

11. ALPHABETIC NOTATIONS

The idea of using the letters of the alphabet to denote numbers can be traced back to Pâṇini (*c.* 700 B.C.) who has used the vowels of the Sanskrit alphabet to denote numbers.[1] No definite evidence of the extensive use of an alphabetic notation is, however, found

[1] In Pâṇini's grammar there are a number of sûtras (rules) which apply to a certain number of sûtras that follow and not to all. Such sûtras are marked by signs according to Pâṇini. Patañjali commenting on sûtra i. 3. 11, says that according to Kâtyâyana (4th century B.C.) a letter (*varṇa*), denoting the number of sûtras upto which a particular rule is to apply, is written over the sûtra. Kaiyyaṭa illustrates this remark by saying that the letter *i* is written above Pâṇini's sûtra, v. 1. 30 to show that it applies only to the next two sûtras. Thus according to Pânini *a* = 1, *i* = 2, *u* = 3,......

upto the 5th century A.D. About this period a number of alphabetic notations were invented by different writers with the sole purpose of being used in verse to denote numbers. The word numerals gave big number chronograms, so that sometimes a whole verse or even more would be devoted to the word chronogram only. This feature of the word system was naturally looked upon with disfavour by some of the Indian astronomers who considered brevity and conciseness to be the main attributes of a scientific composition. Thus the alphabetic notations were invented to replace the word system in astronomical treatises. The various alphabetic systems[1] are simple variations of the decimal place-value notation, using letters of the alphabet in the place of numerical figures. It must be noted here that the Hindu alphabetic systems, unlike those employed by the Greeks or the Arabs, were never used by the common people, or for the purpose of making calculations; their knowledge was strictly confined to the learned and their use to the expression of numbers in verse.

Alphabetic System of Aryabhata I. Âryabhaṭa I (499) invented an alphabetic system of notation, which has been used by him in the *Daśagîtikâ*[2] for enumerating the numerical data of his descriptive astronomy. The

[1] Some alphabetic systems used for the pagination of manuscripts do not use the place-value principle. These systems were the invention of scribes who probably wanted to be pedantic and to show off their learning. Their use was confined to copyists of manuscripts.

[2] The *Daśagîtikâ* as the name implies ought to contain ten stanzas, but actually there are thirteen. Of these the first is an invocation to the Gods, the second is the *paribhâṣâ* ("definition") given above and the thirteenth is of the nature of a colophon. These three stanzas are, therefore, not counted. Cf. W. E. Clark, "Hindu-Arabic Numerals," *Indian Studies in Honour of Charles Rockwell Lanman*, (Harvard Univ. Press), 1929, p. 231.

rule is given in the *Daśagîtikâ* thus:

Vargâkṣarâṇi varge'varge'vargâkṣarâṇi kât ṅmau yaḥ
Khadvinavake svarâ nava varge'varge navântyavarge vâ

The following translation gives the meaning of the rule as intended by the author:

"The *varga*[1] letters beginning with *k* (are used only) in the *varga*[2] places, the *avarga* letters in the *avarga*[3] places, (thus) *ya* equals *ṅmau* (*ṅa* plus *ma*); the nine vowels (are used to denote) the two nines of zeros of *varga* and *avarga* (places). The same (procedure) may be (repeated) after the end of the nine *varga* places."

This rule has been discussed by Whish,[4] Brockhaus,[5] Kern,[6] Barth,[7] Rodet,[8] Kaye,[9] Fleet,[10] Datta,[11] Ganguly,[12] Das,[13] Lahiri[14] and Clark.[15]

The translation of *kha* by "place" (Clark) or by "space" (Fleet) is incorrect. We do not find the word *kha* used in the sense of 'notational place' anywhere in Sanskrit literature. Its meanings are 'void', 'sky', etc., and it has been used for zero, in the mathematical and

[1] *Varga* here means "classed," *i.e.*, the classed letters of the alphabet. The first twenty-five letters of the alphabet are classed in groups of five, the remaining ones are unclassed.

[2] *Varga* here means odd.

[3] *Avarga* here means even.

[4] *Transactions of the Literary Society of Madras*, I, 1827, p. 54.

[5] *Zeitschrifte für die kunde des Morgenländes*, IV, p. 81.

[6] *JRAS*, 1863, p. 380.

[7] *Oeuvres*, III, p. 182.

[8] *JA*, 1880, II, p. 440.

[9] *JASB*, 1907, p. 478; *Indian Mathematics*, Calcutta, 1915, p. 30; *The Bakhshālī Manuscript*, Calcutta, 1927, p. 81.

[10] *JARS*, 1911, p. 109.

[11] *Sâhitya-Pariṣad-Patrikâ*, 1929, p. 22.

[12] *BCMS*, 1926, p. 195.

[13] *IHQ*, III, p. 110.

[14] *History of the World* (in Bengali), Vol. IV, p. 178.

[15] *Âryabhaṭîya of Âryabhaṭa*, Chicago, 1930, p. 2.

astronomical works. We thus replace "the two nines of places" in the translation given by Clark by "the two nines of zeros." Clark has given the following reason for not translating *kha* by zero: "That is equivalent to saying that each vowel adds two zeros to the numerical value of the consonant. This, of course, will work from the vowel *i* on; but the vowel *a* does not add two zeros. It adds no zero or one zero depending on whether it is used with *varga* or *avarga* letters. It seems to me, therefore, more likely that a board divided into columns is implied rather than a symbol for zero, as Rodet thinks." The vowels do not add zeros. The explanation will not work for any of the vowels; for instance, *i*, according to this interpretation, would add two zeros to *g* but three zeros to *y*. What really is implied by *kha* is explained by the commentator Sûryadeva as follows: "*khâni śûnyopalakṣitâni, saṅkhyâvinyâsasthânâni teṣâṃ dvinavakaṃ, khadvinavakaṃ, tasmin khadvinavake śûnyopalakṣitdsthânâṣṭâdaśa* (18) *ityarthaḥ;*" that is, "*kha* denotes zero; the places for putting (writing) the numbers are two nines (*dvinavakaṃ*), therefore, *khadvinavake* means the eighteen places denoted by zeros." It may be mentioned here that the Hindus denote the notational places by zeros. Bhâskara I (522), commenting on *Gaṇitapâda*, 2, which gives the names of ten notational places, says:

"*nyâsaśca sthânânâṃ* oooooooooo."

i.e., "writing down the places we have oooooooooo." Bhâskara I is more explicit in the interpretation of *kha* by zero, for in his comments on the above rule, he states: "*khadvinavake svarâ nava varge*: *kha* means zero (*śûnya*). In two nines of zeros (*kha*), so *khadvinavake*; that is, in the eighteen (places) marked by zeros;......"[1]

[1] Commentary on the *Daśagîtikâ* by Bhâskara I, "*khadvinavake svarâ nava varge khâni śûnyâni, khânâṃ dvinavakaṃ tasmin khadvinavake aṣṭâdaśa śûnyâkṣiteṣu......*"

Thus *kha* must be translated by zero, although the *kha* (zero) here is equivalent to the 'notational places.'[1] What is implied here is certainly the symbol for the zero and not a board divided into columns.

Clark finds great difficulty in translating *navântya-varge vâ*. The reading *hau* instead of *vâ* suggested by Fleet is not acceptable. The translation given by us accords with the several commentaries (by Bhâskara I, Sûryadeva, Parameśvara and Nîlakaṇṭha) consulted by us. They all agree.

Explanation. Ârayabhaṭa's rule gives the method of expressing the alphabetic chronogram in the decimal place-value notation, and *vice versa*. The notational places are indicated as follows:

$$\overbrace{\text{a v}}^{au}\ \overbrace{\text{a v}}^{o}\ \overbrace{\text{a v}}^{ai}\ \overbrace{\text{a v}}^{e}\ \overbrace{\text{a v}}^{\d{l}}\ \overbrace{\text{a v}}^{\d{r}}\ \overbrace{\text{a v}}^{u}\ \overbrace{\text{a v}}^{i}\ \overbrace{\text{a v}}^{a}$$

$$0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0$$

where v stands for *varga* and a for *avarga*.

It will be observed that the eighteen places are denoted by zeros and they are divided into nine pairs, each pair consisting of a *varga* place and an *avarga* place, *i.e.*, odd place and even place.[2] The *varga* letters *k* to *m*[3] are used in *varga* places, *i.e.*, odd places only, and denote the numbers 1, 2,........, 25 in succession. The

[1] Nîlakaṇṭha says: "*khadvinavake*, that is, there are eighteen places, the nine *varga* places and the nine *avarga* places ......" See *Âryabhaṭîya*, ed. by K. Sambasiva Sastri, Trivandrum, 1930, p. 6.

[2] The later Indian treatises use the terms *viṣama* and *sama* for *varga* and *avarga* respectively. *Varga* is also used for a square number or the figure.

[3] These are called *varga* or classified letters, because they are classified into groups of five each.

avarga letters *y* to *h* are used in the *avarga* places, *i.e.*, even places only, and denote the numbers 3, 4, ... , 10 successively. The first *varga* and *avarga* places together constitute the first *varga-avarga* pair, and so on. Nine such *varga-avarga* pairs are denoted by the nine vowels in succession. Thus the first *varga-avarga* pair, *i.e.*, the units and the tens places are denoted by *a*; the second *varga-avarga* pair, *i.e.*, the hundreds and thousands places by *i*; and so on. The vowels thus denote places —zeros according to the Indian usage of denoting the places—and have by themselves no numerical value. When attached to a 'letter-number' a vowel simply denotes the place that the number occupies in the decimal place-value notation. For instance, when the vowel *a* is attached to *y*, it means that the number 3 which *y* denotes is to be put in the first *avarga* place, *i.e.*, the tens place. Thus *ya* is equal to 30. On the other hand when *a* is attached to one of the classed letters, it refers it to the first *varga* place, *i.e.*, the units place. Thus *ṅa* is equal to 5 and *ma* is equal to 25 and *ṅma*[1] is equal to 30. Similarly *yi* denotes that the number 3 is to be put in the thousands place whilst *gi* would mean that the number 3 which *g* represents is to be put in the hundreds place (*g* being a *varga* letter). Thus *yi*=3,000, whilst *gi*=300. It is possible that the zeros already written were rubbed out and the corresponding numerical figures as obtained from a given letter chronogram were substituted in their places. This would automatically give zeros in the vacant places. When this is not done and the numbers are written below the zeros indicating the places, then zeros have

[1] When two consonants are together joined to a vowel, the numbers representing both are referred to the same *varga-avarga* pair. They are added together as in this case, *ṅma* = *ṅa* + *ma* = 5 + 25 = 30.

to be written in the places that remain vacant.[1] The same procedure can be applied to express numbers occupying more than eighteen places, by letting the vowels with *anusvâra* denote the next eighteen places, or by means of any other suitable device.

One advantage of this notation is that it gives very brief chronograms. This advantage is, however, more than counterbalanced by two very serious defects. The first of these is that most of the letter chronograms formed according to this system are very difficult to pronounce. In fact, some of these[2] are so complicated that they cannot be pronounced at all. The second defect is that the system does not allow any great variety in the letter chronograms, as other systems do.

Katapayadi System. In this system the consonants of the Sanskrit alphabet have been used in the place of the numbers 1-9 and zero to express numbers. The conjoint vowels used in the formation of number chronograms, have no numerical significance. It gives brief chronograms, which are generally pleasant sounding words. Skilled writers have been able to coin chronograms which have connected meanings. It is superior to that of Âryabhaṭa I, and also to the word system. Four variants of this system are known to have been used in India. It is probably due to this non-uniformity of notation that the system did not come into general use.

[1] Some examples from the *Âryabhaṭîya* (i. 3):

	ṛ		*u*		*i*		*a*	
	0	0	0	0	0	0	0	0
khyughṛ		*gh*	*y*	*kh*				
		4	3	2	0	0	0	0=4320000
cayagiyiṅuśuchlṛ	*l*	*ch*	*ś*	*ṅ*	*y*	*g*	*y*	*c*
	5	7	7	5	3	3	3	6=5775336

[2] For instance *niśiṣuṇḷkhṣṛ*, *bhadliknukhṛ*, etc.

First Variant: The first variant of the *Kaṭapayâdi* system is described in the following verse taken from the *Sadratnamâlâ*:

Nañâvacaśca śûnyâni saṁkhyâ kaṭapayâdayaḥ
Miśre tûpânta hal saṁkhyâ na ca cintyo halasvaraḥ

"*n*, *ñ* and the vowels denote zeros; (the letters in succession) beginning with *k*, *ṭ*, *p*, and *y*, denote the digits; in a conjoint consonant only the last one denotes a number; and a consonant not joined to a vowel should be disregarded." According to this system, therefore,

1	is denoted	by the	letters	*k*, *ṭ*, *p*, *y*.
2	,,	,,	,,	*kh*, *ṭh*, *ph*, *r*.
3	,,	,,	,,	*g*, *ḍ*, *b*, *l*.
4	,,	,,	,,	*gh*, *ḍh*, *bh*, *v*.
5	,,	,	,,	*ṅ*, *ṇ*, *m*, *ś*.
6	,,	,,	,,	*c*, *t*, *ṣ*.
7	,,	,,	,,	*ch*, *th*, *s*.
8	,,	,,	,,	*j*, *d*, *h*.
9	,,	,,	,,	*jh*, *dh*.
0	,,	,,	,,	*ñ*, *n* and vowels standing by themselves.

The consonants with vowels are used in the places of of the numerical figures just as in the place-value notation. Of conjoint consonants only the last one has numerical significance. A right to left arrangement is employed in the formation of chronograms, just as in the word system, *i.e.*, the letter denoting the units figure is written first, then follows the letter denoting the tens figure and so on. The following examples taken from inscriptions, grant plates and manuscripts will illustrate the system:

$$\overset{2}{\textit{râ}} - \overset{4}{\textit{gha}} - \overset{4}{\textit{vâ}} - \overset{1}{\textit{ya}} \quad = \quad 1442,$$
(1)[1]

[1] *EI*, VI, p. 121.

	4 4 6			
(2)[1]	*bha - va - ti*	=	644,	
	5 1 3 1			
(3)[2]	*śa - ktyâ - lo - ke*	=	1315,	
	6 4 3 1			
(4)[3]	*ta - tvâ - lo - ke*	=	1346,	
	2 3 1 5 6 5 1			
(5)[4]	*kha - go - ntyâ - nme - ṣa - mâ - pe*	=	1565132.	

The origin of this system can be traced back to the fifth century A.D. From a remark[5] made by Sûryadeva in his commentary on the *Âryabhaṭîya*, it appears that the system was known to Âryabhaṭa I (499). Its first occurrence known to us is found in the *Laghu-Bhâskarîya* of Bhâskara I (522).[6]

Second Variant: Âryabhaṭa II (950) has used a modification of the above system. In this variant, the consonants have the same values as above, but the vowels whether standing by themselves or in conjunction with consonants have no numerical significance. Also unlike the first variant, each component of a conjoint consonant has numerical value according to its

[1] *IA*, II, p. 360.
[2] *EI*, III, p. 229.
[3] *EI*, III, p. 38.
[4] The date of the commentary of Ṣaḍguruśiṣya on *Sarvânukramaṇî* is given by this chronogram in the *Kaliyuga* Era: It corresponds to 1184 A.D.
[5] Comments on the *paribhâṣâ-sûtra* of the *Daśagîtikâ*. The author remarks :

"*Vargâkṣarâṇâṁ saṁkhyâ pratipâdane, kaṭapayâditvaṃ nañayośca śûnyâpi siddhaṁ tannirâsârthaṁ kât · grahaṇaṃ.*"

That is, "the letters *kât* have been used to distinguish it (the method of Âryabhaṭa I) from the *Kaṭapayâdi* system of denoting numbers by the help of the *varga* letters, where *n* and *ñ* are equal to zero."
[6] *LBh*, i. 18.

place. The letters are arranged in the left to right order just as in writing numerical figures.[1] The difference between the two variants may be illustrated by the chronogram *ḍha-ja-he-ku-na-he-t-sa-bhâ*.[2] According to Âryabhaṭa II it denotes 488108674, whereas according to the first variant it would denote 47801884.

Third Variant: A third variant of this system is found in some Pâlî manuscripts from Burma.[3] This is in all respects the same as the first variant except that *s*=5, *h*=6 and *l'*=7. The modification in the values of these letters are due to the fact that the Pâlî alphabet does not contain the Sanskrit *ś* and *ṣ*.

Fourth Variant: A fourth variant of the system was in use in South India, and is known as the Kerala System. This is the same as the first variant with the difference that the left-to-right arrangement of letters, just as in writing numerical figures, is employed.

Aksarapalli. Various peculiarities are found in the forms as well as the arrangement of the numerical symbols used in the pagination of old manuscripts. These symbols are known as the *akṣarapallî*, *i.e.*, the letter system.[4] In this system the letters or syllables of the script in which the manuscript is written are used to denote the numbers. The following list gives the phonetic values of the various numerals as found in old manuscripts:[5]

[1] The notation is explained in *MSi*, i. 2:

Rûpât kaṭapayapûrvâ varṇâ varṇakramâdbhavantyaṅkâḥ
Ññau śûnyaṁ prathamârthe â chede e tṛtîyârthe.

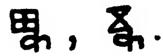
[2] *MSi*, i. 10.

[3] L. D. Barnet, *JRAS*, 1907, pp. 127 ff.

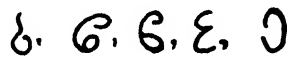
[4] For forms see Tables.

[5] See *PLM*, pp. 107f.

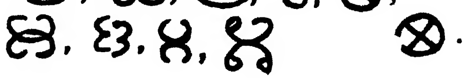
1 = e, sva, rûṁ.
2 = dvi, sti, na.
3 = tri, śrî, maḥ.
4 = ṅka, rṅka, ṅkâ, ṇka, rṇka, ṣka, rṣka,

 (pke),  rphra, pu.

5 = tṛ, rtṛ, rtṛâ, hṛ, nṛ, rnṛ.
6 = phra, rphra, rphru, ghna, bhra, rpu, vyâ, phla.
7 = gra, grâ, rgrâ, rgbhrâ, rggâ, bhra.
8 = hra, rhra, rhrâ, dra.
9 = oṁ, ruṁ, ru, uṁ, ûṁ, a, rnuṁ.
10 = ḷ, ḷa, ṇṭa, ḍa, a, rpta.
20 = tha, thâ, rtha, gha, rgha, pva, va.
30 = la, lâ, rla, rlâ.
40 = pta, rpta, ptâ, rptâ, pna.

50 = s,  e, i, ṇu.

60 = cu, vu, ghu, thu, rthu, rthû, thû, rgha, rghu.
70 = cû, cu, thû, rthü, rghû, rmta.

80 =  pu.
90 = 

100 = su, sû, lu, a.
200 = sû, â, lû, rghû.
300 = stâ, sûâ, ñûâ, sâ, su, suṁ, sû.
400 = sûo, sto, stâ.

It will be observed that to the same numeral there correspond various phonetical values. Very frequently the difference is slight and has been intentionally made, probably to distinguish the signs with numerical values from those with letter values. In some other cases there are very considerable variations, which (according to Bühler) have been caused by misreadings of older signs or dialectic differences in pronunciation. The symbols are written on the margin of each leaf. Due to lack of space they are generally arranged one below the other in the Chinese fashion. This is so in the Bower manuscript which belongs to the sixth century A.D. In later manuscripts the pages are numbered both in the *akṣarapallî* as well as in decimal figures. Sometimes these notations are mixed up as in the following:[1]

		lâ			su			su
33	=	3 ;	100	=	0 ;	102	=	0 ;
					0			2
		su			su			sû
131	=	lâ ;	150	=		209	=	0
		1			0			ruṁ

The *akṣarapallî* has been used in Jaina manuscripts upto the sixteenth century. After this period, the decimal figures are generally used. In Malabar, a system resembling the *akṣarapallî* is in use upto the present day.[2]

[1] Cf. *PLM*, p. 108.

[2] 1 = na, 2 = nna, 3 = nya, 4 = ṣkra,
5 = jhra, 6 = hâ(ha), 7 = gra, 8 = pra,
9 = dre(?), 10 = ma, 20 = tha,
30 = la, 40 = pta, 50 = ba, 60 = tra,
70 = ru (tru), 80 = ca, 90 = ṇa, 100 = ña.
(*Cf. JRAS*, 1896, p. 790)

Other Letter Systems. (*A*) A system of notation in which are employed the sixteen vowels and thirty-four consonants of the Sanskrit alphabet is found in certain manuscripts from Southern India (Malabar and Andhra), Ceylon, Burma and Siam. The thirty-four consonants in order with the vowel *a* denote the numbers from one to thirty-four, then the same consonants with the vowel *â* denote the numbers thirty-five to sixty-eight and so on.[1]

(*B*) Another notation in which the sixteen vowels with the consonant *k* denote the numbers one to sixteen and with *kh* they denote the numbers seventeen to thirty-two, and so on, is found in certain Pâlî manuscripts from Ceylon.[2]

(*C*) In a Pâlî manuscript in the Vienna Imperial Library a similar notation is found with twelve vowels and thirty-four consonants. In this the twelve vowels[3] with *k* denote the numbers from one to twelve, with *kh* they denote the numbers from thirteen to twenty-four, and so on.

These letter systems do not appear to have been in use in Northern India, at least after the third century A.D. They are probably the invention of scribes who copied manuscripts.

THE ZERO SYMBOL

Earliest Use. The zero symbol was used in metrics by Piṅgala (before 200 B.C.) in his *Chandaḥ-sûtra*. He gives the solution of the problem of finding the total number of arrangements of two things in *n* places, repetitions being allowed. The two things considered are

[1] Burnell, *South Indian Palaeography*, London, 1878, p. 79.

[2] *Ibid.*

[3] The vowels *ṛ, ṝ, ḷ, ḹ*, are omitted.

the two kinds of syllables "long" and "short", denoted by *l* and *g* respectively. To find the number of arrangements of long and short syllables in a metre containing *n* syllables, Piṅgala gives the rule in short aphorisms :

"(Place) two when halved;"[1] "when unity is subtracted then (place) zero;"[2] "multiply by two when zero;"[3] "square when halved."[4]

The meaning of the above aphorisms will be clear from the calculations given below for the *Gâyatrî* metre which contains 6 syllables.[5]

	A		B
Place the number	6		
Halve it, result	3	Separately place	2
3 cannot be halved, therefore, subtract 1, result	2	,, ,,	0
Halve it, result	1	,, ,,	2
1 cannot be halved, therefore, subtract 1, result	0	,, ,,	0

The process ends.

The calculation begins from the last number in column B. Taking unity double it at 0, this gives 2; at 2 square this (2), the result is 2^2; then at zero double (2^2), the result is 2^3; ultimately, at 2 square this (2^3), the result is 2^6, which gives the total number of ways

[1] *Piṅgala Chandaḥ-sûtra*, ed. by Sri Sitanath, Calcutta, 1840, viii. 28.

[2] *Ibid*, viii. 29.

[3] *Ibid*, viii. 30.

[4] *Ibid*, viii. 31.

[5] For 7 syllables, the steps are :

Subtract	1	6	place	0	Double $2.2^6 = 2^7$
Halve		3	,,	2	Square 2^6
Subtract	1	2	,,	0	Double $2.2^2 = 2^3$
Halve		1	,,	2	Square 2^2
Subtract	1	0	,,	0	Double $1 = 2$

giving 2^7 as the result.

in which two things can be arranged in 6 places.[1]

It will be observed that two symbols are required in the above calculation to distinguish between two kinds of operations, viz., (1) that of halving and (2) that of the 'absence' of halving and subtraction of unity. These might have been denoted by any two marks arbitrarily chosen.[2] The question arises: why did Piṅgala select the symbols "two" and "zero"? The use of the symbol two can be easily explained as having been suggested by the process of halving—division by the number two. The zero symbol was used probably because of its being associated, at the time, with the notion of 'absence' or 'subtraction.' The use of zero in either sense is found to have been common in Hindu mathematics from early times. The above reference to Piṅgala, however, shows that the Hindus possessed a symbol for zero (*śûnya*), whatever it might have been, before 200 B.C.

The Bakhshâlî Manuscript (*c.* 200) contains the use of zero in calculation. For instance, on folio 56 verso, we have:

$$\text{“}\left|\begin{array}{c} 880 \\ 84 \end{array}\right|\begin{array}{c} 964 \\ 168 \end{array}\left|\ \text{multiplied become}\ \right|\begin{array}{c} 848320 \\ 14112 \end{array}\right|$$

The square of *forty* different places is $\lfloor 1600 \rfloor$. On subtracting this from the number above (numerator), the remainder is $\left|\begin{array}{c} 846720 \\ 14112 \end{array}\right|$. On removal of the common factor, it becomes $\lfloor 60 \rfloor$."

[1] This method of calculation is not peculiar to the *Piṅgala Chandaḥ-sûtra*. It is found in various other works on metrics as well as mathematics. The zero symbol has been similarly employed in this connection in later works also. *Vide infra*.

[2] *E.g.*, Pṛthudakasvâmî uses *va* (from *varga*, "square") and *gu* (from *guṇa*, "multiply"), while Mahâvîra uses the numerals 1 and 0. *Vide infra*.

There are a large number of passages of this kind in the work. It will be noticed that in such passages the sentences would be incomplete without the figures, so the figures must have been put there at the time of the original composition of the text, and cannot be suspected of being later interpolations. For an explicit reference to zero and an operation with it, we take the following instance from the work:

“

0	2	3	4	visible 200
1	1	1	1	1

[1] Adding[2] unity to zero 1 | 2 | 3 | ......”[3]

In the *Pañca-siddhântikâ* (505) zero is mentioned at several places. The following is an instance:

“In Aries the minutes are seven, in the last sign six; in Taurus six (repeated) thrice; five (repeated) twice; four; four; in Gemini they are three, two, one, zero (*śûnya*) (each repeated) twice.”[4]

Zero is here conceived as a number of the same type as three, two or one. It cannot be correctly interpreted otherwise. Addition and subtraction of zero are also used in expressing numbers in this work for the sake of metrical convenience. For instance:

“Thirty-six *increased* by two, three, nine, twelve, nine, three, zero (*śunya*) are the days.”[5]

Instances of the above type all occur in those

[1] The zeros given here are represented in the manuscript by dots. The statement in modern symbols is equivalent to the equation,

$$x + 2x + 3x + 4x = 200.$$

[2] The Sanskrit word is *yutaṁ* meaning literally “adding”, but what is meant is “putting” unity for the unknown (zero).

[3] *BMs*, folio 22, verso.

[4] *PSi*, vi. 12.

[5] *PSi*, xviii. 35; other instances of this nature are in iii. 17; iv. 7; iv. 8; iv. 11; xviii. 44; xviii, 45; xviii. 48; xviii. 51.

sections of the *Pañca-siddhântikâ* which deal with the teachings of Puliśa. It seems, therefore, that such expressions are quotations from the *Puliśa-siddhânta*. As it is known that the word numerals were employed by Puliśa (*c.* 400), it can be safely concluded that he was conversant with the concept of the zero as a numeral.

The writings of Jinabhadra Gaṇi (529-589), a contemporary of Varâhamihira, offer conclusive evidence of the use of zero as a distinct numerical symbol. While mentioning large numbers containing several zeros, he often enumerates, obviously for the sake of abridgement, the number of zeros contained. For instance: 224,400,000,000 is mentioned as "twenty-two forty-four, eight zeros;"[1] 3,200,400,000,000 as "thirty-two two zeros four eight zeros."[2] At another place in his work

$$241960\frac{407150}{483920} = 241960\frac{40715}{48392}$$

is described thus :

"Two hundred thousand forty-one thousand nine hundred and sixty; removing (*apavartana*) the zeros, the numerator is four-zero-seven-one-five, and the denominator four-eight-three-nine-two."[3]

It should be noted that the term *apavartana* means what is known in modern arithmetic as the reduction of a fraction to its lowest terms by removing the common factors from the numerator and the denominator. Hence the zero of Jinabhadra Gaṇi is certainly not a mere concept of nothingness but is a specific numerical symbol used in arithmetical calculation.

[1] *Bṛhat-kṣetra-samâsa*, ed. with the commentary of Malayagiri, Bombay, i. 69.

[2] *Ibid*, i. 71. Other such instances are in i. 90, 97, 102, 108, 113, 119, etc.

[3] *Ibid*, i. 83.

Another contemporary mathematician, Bhâskara I (*c.* 525), refers to the subtraction of zero in his *Mahâbhâskarîya*. In his commentary on the *Âryabhaṭîya* he uses the place-value numerals with zero. As has been pointed out before (p. 66) zero is also used by him to denote the notational places.

Siddhasena Gaṇi who lived in the sixth century, has, in his commentary on the *Tattvârthâdhigama-sûtra* of Umâsvâti, used zero in calculation, as is evidenced by the following two typical instances taken from his work:[1]

"....the remainder is this, 3,534,400,000,000. The square-root of this is extracted; *half of the eight zeros are four zeros*; The root of the ramaining portion is one-eight-eight; hence the resulting root is this, 1,880,000."

"On removing the four zeros, the quotient obtained after that is 100,000."

All known Hindu treatises on arithmetic and algebra contain a section dealing with the fundamental operations with zero, including involution and evolution. Details regarding these operations will be given later on; but it must be pointed out here, that these arithmetical operations with zero, certainly presuppose its existence as a numeral denoted by some specific symbol.[2]

[1] *Tattvârthâdhigama-sûtra* of Umâsvâti, with his own gloss, elucidated by Siddhasena Gaṇi, ed. by H. R. Kapadia, Bombay, 1926, iii. 11 (com.).

[2] Smith and Karpinski (*Hindu-Arabic Numerals*, p. 53) state, "..the Gaṇita-Sâra-Saṅgraha of Mahâvirâchârya (c. 830 A.D.), while it does not use the numerals with place value, has a similar discussion with zero." The first part of the statement is incorrect, because Mahâvîra has always used numerals with place-value. In fact, no trace of numerals without place-value is to be found in the *Gaṇita-sâra-saṁgraha*. J. Tropfke's statement (*Geschichte d. Elementar-Mathematik*, Bd. II, 1926, p. 56) that zero was not regard-

4. THE ZERO SYMBOL

~~**Form of the Symbol.**~~ ~~The above quotations from ancient works prove conclusively that the~~ Zero has been considered as a number in India from the earliest centuries of the Christian era, and that there existed some symbol to denote this number. What exactly was the form of this symbol is doubtful. The Bakhshâlî Manuscript employs a dot for zero, but as the present copy of the work dates from probably the eighth or the ninth century, it cannot be said whether the form of the symbol was the same when the Bakhshâlî work was written, *i.e*, in the third century A.D. or earlier. Evidence as to the use of the dot for zero is also furnished by the writings of Subandhu, a poet litterateur who flourished about the close of the sixth century. In the *Vâsavadattâ* of Subandhu we meet with the following metaphor:

"And at the time of the rising of the moon with its blackness of night, bowing low, as it were, with folded hands under the guise of closing blue lotuses, immediately the stars shone forth,......... like zero dots (*śûnya-bindu*), because of the nullity of metempsychosis, scattered in the sky as if on the ink-blue skin rug of the Creator who reckoneth the sum total with a bit of the moon for chalk."[1]

The term *bindu* ("dot") has been used for zero in word numerals as well as in later literature,[2] when a small circle was in use to denote zero, the dot having

ed as a number before the seventeenth century A.D., is incorrect. *Cf.* B. Datta, "Early literary evidence of the use of the zero in India," *American Math. Monthly*, XXXVIII, 1931, p. 569.

[1] *Vâsavadattâ* of Subandhu, edited by F. Hall (Calcutta, 1859, p. 182) and translated into English by Louis H. Gray (New York, 1913, pp. 99*f*).

[2] *E.g.*, the Hindi poet Bihârî in one of his couplets remarks: "The dot on her forehead is increasing her beauty ten-fold, just as a dot increases a number ten-fold."

been given up long before. The quotation from Subandhu cannot, therefore, be taken as a definite proof of the use of the dot as a symbol for zero in his time. All that we can infer is that at some period before Subandhu, the dot was in use. We may go further and state that very probably, the earliest symbol for zero was a dot and not a small circle.

The earliest epigraphical record of the use of zero is found in the Ragholi plates[1] of Jaivardhana II of the eighth century. The Gwalior inscriptions of the reign of Bhojadeva[2] also contain zero. The form of the symbol in these inscriptions is the small circle. This is the form that has been in common use from quite early times, probably from before the eighth century.

Other Uses of the Symbol. In the present elementary schools in India, the student is taught the names of the several notational places and is made to denote them by zeros arranged in a line. These zeros are written as

.... o o o o o o o o o o

The teacher points out the first zero on the right and says 'units', then he proceeds to the next zero saying 'tens' and so on. The student repeats the names after the teacher. This practice of denoting the notational places by zeros can be traced back to the time of Bhâskara I, who, as already pointed out on page 66, in his commentary on the *Âryabhaṭîya*, *Gaṇita-pâda*,2, says :

'Writing down the places, we have

o o o o o o o o o o."

In all works on arithmetic (*pâṭîgaṇita*) zero has

[1] No. 4 in the list of inscriptions given before.
[2] Nos. 19 and 20 in the list.

been used to denote the unknown. This use of zero can be traced back to the third century A.D. It is used for the unknown in the Bakhshâlî arithmetic. In algebra, however, letters or syllables have been always used for the unknown. It seems that zero for the unknown was employed in arithmetic, really to denote the absence of a quantity, and was not a symbol in the same sense as the algebraic x (*yâ*), for it does not appear in subsequent steps as the algebraic symbols do. This use of zero is mostly found in problems on proportion—the Rule of Five, Rule of Seven, etc. The Arabs also under Hindu influence used zero for the unknown in similar problems. Similar use of zero for the unknown quantity is found in Europe in a Latin manuscript of some lectures by Gottfried Wolack in the University of Erfürt in 1467-68.[1] The dot placed over a number has been used in Hindu *Gaṇita* to denote the negative. In this case it denotes the 'absence' of the positive sign. Similar use of the dot is found in Arabia and Europe obviously under Hindu influence.[2]

13. THE PLACE-VALUE NOTATION IN HINDU LITERATURE

Jaina Canonical Works. The earliest literary evidence of the use of the word "notational place" is furnished by the *Anuyogadvâra-sûtra*,[3] a work written before the Christian era. In this work the total number

[1] Smith and Karpinski, *l.c.*, pp. 53-54.

[2] The occasional use by Al-Battani (929) of the Arabic negative *lâ*, to indicate the absence of minutes (or seconds), noted by Nallino (*Verhandlungen des 5 congresses der Orientalisten*, Berlin, 1882, Vol. II, p. 271), is similar to the use of the zero dot to denote the negative.

[3] The passage has been already quoted in detail (*vide supra* p. 12).

of human beings in the world is given by "a number which when expressed in terms of the denominations, *koṭi-koṭi*, etc., occupies twenty-nine places (*sthâna*)." Reference to the "places of numeration" is found also in a contemporary work, the *Vyavahâra-sûtra*.[1]

Puranas. The Purâṇas which are semi-religious and semi-historical works, also contain references to the notational places. These works were written for the purpose of spreading education on religious and historical matters amongst the common people. Reference to the place-value notation in these works shows the desire of their authors to give prominence to the system. The *Agni-Purâṇa*[2] says:

"In case of multiples from the units place, the value of each place (*sthâna*) is ten times the value of the preceding place."

The *Viṣṇu-Purâṇa*[3] has similarly:

"O *dvija*, from one place to the next in succession, the places are multiples of ten. The eighteenth one of these (places) is called *parârdha*."

The *Vâyu-Purâṇa*[4] observes:

"These are the eighteen places (*sthâna*) of calculation; the sages say that in this way the number of places can be hundreds."

The above three works are the oldest among the Purâṇas and of these the *Agni* and the *Vâyu* Purâṇas in their present form are certainly as old as the fourth century A.D. The *Agni-Purâṇa* is referred by some scholars to the first or second century A.D.

[1] Ch. i.; *cf.* B. Datta, *Scientia*, July, 1931, p. 8.

[2] The *Agni-Purâṇa* contains also the use of the word numerals with place-value (*vide supra* p. 58).

[3] vi. 3.

[4] ci. 102f.

Works on Philosophy. The following simile has been used in *Vyâsa-Bhâṣya*[1] on the *Yoga-sûtra* of Patañjali :

"Thus the same stroke is termed one in the units place, ten in the tens place, and hundred in the hundreds place........"[2]

The same simile occurs in the *Sârîraka-Bhâṣya* of Śaṅkarâcârya :

"Just as, although the stroke is the same, yet by a change of place it acquires the values, one, ten, hundred, thousand, etc...."[3]

The first of the above works cannot be placed later than the sixth century whilst the second one not later than the eighth. The quotations prove conclusively that in the sixth century, the place-value notation was so well known that it could be used as an illustration for a philosophical argument.

Literary Works. A passage from the *Vâsavadattâ* of Subandhu comparing the stars with zero dots has already been mentioned. Several other instances of the use of zero are found in later literature, but they need not be mentioned here.[4]

[1] iii. 13.

[2] The translation is as given by J. H. Woods, *The Yoga System of Patañjali*, p. 216. In a foot-note, it is remarked: "Contrary to Mr. G. R. Kaye's opinion, the following passages show that the place-value system of decimals was known as early as the sixth century A.D." The above passage is also noted by Sir P. C. Ray in his *History of Hindu Chemistry*, Vol. II, p. 117.

[3] III. iii. 17; *cf.* B. Datta, *American Math. Monthly*, XXXIII 1926, pp. 220-1.

[4] *E.g.*, the use of the *śûnya-bindu* in *Naiṣadha-carita* of Śrîharṣa (*c.* 12th century). *Cf.* B. Datta, *Ibid*, pp. 449-454.

14. DATE OF INVENTION OF THE PLACE-VALUE NOTATION

We may now summarise the various evidences regarding the early use of the place-value notation in India :

(1) The earliest palaeographic record of the use of the place-value system belongs to the close of the sixth century A.D.

(2) The earliest use of the place-value principle with the word numerals belongs to the second or the third century A.D. It occurs in the *Agni-Purâṇa*, the Bakhshâlî Manuscript and the *Puliśa-siddhânta*.

(3) The earliest use of the place-value principle with the letter numerals is found in the works of Bhâskara I about the beginning of the sixth century A.D.

(4) The earliest use of the place-value system in a mathematical work occurs in the Bakhshâlî Manuscript about 200 A.D. It occurs in the *Âryabhaṭîya* composed in 499 A.D., and in all later works without exception.

(5) References to the place-value system are found in literature from about 100 B.C. Three references ranging from the second to the fourth century A.D. are found in the Purâṇas.

(6) The use of a symbol for zero is found in Piṅgala's *Chandaḥ-sûtra* as early as 200 B.C.

The reader will observe that the literary and non-mathematical works give much earlier instances of the use of the place-value system than the mathematical works. This is exactly what one should expect. The system when invented must have for some time been used only for writing big numbers. A long time must have elapsed before the methods of performing arithmetical operations with them were invented. The system cannot be expected to occur in a mathematical work before it is in a perfect form. Therefore, the evidences furnished by non-mathematical works should, in fact, be earlier than those of mathematical works.

Mathematical works are not as permanent as religious or literary works. The study of a particular mathematical work is given up as soon as another better work comes into the field. In fact, a new mathematical work is composed with a view to removing the defects of and superseding the older ones. It is quite probable that works employing the place-value notation were written before Âryabhaṭa I, but they were given up and are lost. It will be idle to expect to find copies of such works after a lapse of sixteen hundred years.

In Europe and in Arabia it is still possible to find mss. copies of works using the old numerals or a mixture of the old numerals with the new place-value numerals, but in India absolutely no trace of any such work exists.

In Europe the first definite traces of the place-value numerals are found in the tenth and eleventh centuries, but the numerals came into general use in mathematical text books in the seventeenth century. In India Âryabhaṭa I (499), Bhâskara I (522), Lalla (*c.* 598), and Brahmagupta (628), all use the place-value numerals. There is no trace of any other system of notation in their works. Following the analogy of Europe, we may conclude, on the evidence furnished by Hindu mathematical works alone, that the place-value system might have been known in India about 200 B.C.

As the literary evidence also takes us to that period, we may be certain that the place-value system was known in India about 200 B.C. Therefore we shall not be much in error, if we fix 200 B.C. as the probable date of invention of the place-value system and zero in India. It is possible that further evidence may force us to fix an earlier date.

15. HINDU NUMERALS IN ARABIA[1]

The regular history of the Arabs begins after the flight of Mohammad from Mecca to Medina in A.D. 622. The spread of Islam succeeded in bringing together the scattered tribes of the Arabian Peninsula and creating a powerful nation. The united Arabs, within a short space of time, conquered the whole of Northern Africa and the Spanish Peninsula, and extended their dominions in the east upto the western border of India. They easily put aside their former nomadic life, and adopted a higher civilisation.

The foundations of Arabic literature and science were laid between 750-850 A.D. This was done chiefly with the aid of foreigners and with foreign material. The bulk of their narrative literature came to the Arabs in translation from Persian. Books on the science of war, the knowledge of weapons, the veterinary art, falconry, and the various methods of divination, and some books on medicine were translated from Sanskrit and Persian. They got the exact sciences from Greece and India.

Before the time of Mohammad the Arabs did not possess a satisfactory numeral notation. The numerous computations connected with the financial administration of the conquered lands, however, made the use of a developed numeral notation indispensable. In some localities the numerals of the more civilised conquered nations were used for a time. Thus in Syria, the Greek notation was retained, and in Egypt the

[1] For details consult Cajori's *History of Mathematics*, and Smith and Karpinski's *Hindu Arabic Numerals*.

Coptic. To this early period belongs the Edict of Khalif Walid (699) which forbade the use of the Greek language in public accounts, but made a special reservation in favour of Greek letters as numerical signs, on the ground that the Arabic language possessed no numerals of its own.[1] The Arabic letters gradually replaced the Greek ones in the alphabetic notation and the *abjad* notation came to be used. It is probable that the Arabs had come to know of the Hindu numerals from the writings of scholars like Sebokht, and aslo of their old *ghobâr* forms from other sources. But as their informants could not supply all the necessary information (*e.g.*, the methods of performing the ordinary operations of arithmetic) these numerals had to wait for another century before they were adopted in some of their mathematical works.

During the reign of the Khalif Al-Mansûr (753-774 A.D.) there came embassies from Sindh to Baghdad, and among them were scholars, who brought along with them several works on mathematics including the *Brâhma-sphuṭa-siddhânta* and the *Khaṇḍa-khâdyaka* of Brahmagupta. With the help of those scholars, Al-fazârî, perhaps also Yâkub ibn Târik, translated them into Arabic. Both works were largely used and exercised great influence on Arab mathematics. It was on that occasion that the Arabs first became acquainted with a scientific system of astronomy. It is believed by all writers on the subject that it was at that time that the Hindu numerals were first definitely introduced amongst the Arabs. It also seems that the Arabs at first adopted the *ghobâr* forms of the numerals, which they had already obtained (but without zero) from the

[1] Theophanes (758-818 A.D.), "Chronographia;" *Scriptores Historiae Byzantinae*, Vol. XXXIX, Bonnane, 1839, p. 575; quoted by Smith and Karpinski, *l.c.*, p. 64, note.

Alexandrians, or from the Syrians who were employed as translators by the Khalifs at Baghdad. Al-Khowârîzmî (825), one of the earliest writers on arithmetic among the Arabs, has used the *ghobâr* forms.[1] But not long afterwards, the Arabs realised that the *ghobâr* forms were not suited to their right-to-left script. Then there appears to have been made an attempt to use more convenient forms. But as people had got accustomed to the *ghobâr* forms, they did not like to give them up, and so we find a struggle[2] between the two forms, which continued for about two centuries (10th and 11th) until at last the more convenient ones came into general use. The west Arabs on the other hand did not adopt the modified forms of the east Arabs, but continued to use the *ghobâr* forms, and were thus able to transmit them to awakening Europe. This, perhaps, explains in a better way the divergence in the forms of modern Arabic and modern European numerals, than any theory yet propounded.

In a theory that was advanced by Woepcke, this divergence is explained by assuming that (1) about the second century after Christ, before zero had been invented, the Hindu numerals were brought to Alexandria, whence they spread to Rome and also to west Africa; (2) that in the eighth century, after the notation in India had been already much modified and perfected

[1] Smith and Karpinski, *l.c.*, p. 98.

[2] One document cited by Woepcke is of special interest since it shows the use of the ordinary Arabic forms alongside the *ghobâr* at an early date (970 A.D.). The title of the work is "Interesting and Beautiful Problems on Numbers" copied by Ahmed ibn Mohammed ibn Abdaljalîl Abû Sa'id, al-Sijzî, (951-1024) from a work by a priest and physician, Nazîf ibn Yumn, al-Qass (died 990). Sprenger also calls attention to this fact (in *Zeit. d deutschen-morgenländischen Gesselschaft*, XLV, p. 367). Ali ibn Ahmed Al-Nasâvî (*c.* 1025) tells us that the symbolism of numbers was unsettled in his day (Smith and Karpinski, *l.c.*, p. 98).

by the invention of zero, the Arabs at Baghdad got it from the Hindus; (3) that the Arabs of the west borrowed the Columbus-egg, the zero, from those in the east but retained the old forms of the nine numerals, if for no other reason, simply to be contrary to their political enemies of the east; (4) that the old forms were remembered by the west Arabs to be of Hindu origin, and were hence called *ghobâr* numerals; (5) that, since the eighth century, the numerals in India underwent further changes and assumed the greatly modified forms of modern Devanâgarî numerals.

Now, as to the fact that these figures might have been known in Alexandria in the second century A.D., there is not much doubt. But the question naturally arises: Why should the Alexandrians use and retain a knowledge of these numerals? As far as we know, they did possess numeral notations of their own; why should they give preference to a foreign notation? These questions cannot be satisfactorily answered unless we assume that along with the nine symbols the principle of place-value and probably also the zero was communicated to them. But as they were unprepared for the reception of this abstract conception, they adopted the nine numerals only and used them on the apices. These numerals were then transmitted by them to Rome and to west Africa.

The second assumption that the Hindu numeral figures of the eighth century were adopted by the Arabs is not supported by fact. The figures that are found in the old Arabic manuscripts resemble either the *ghobâr* numerals or the modern Arabic more than the Hindu numerals of the eighth century. In fact, we have every reason to believe that the Arabs knew these *ghobâr* forms, perhaps without the principle of place-value and zero, long before they had direct contact with India, and that they adopted zero only about 750 A.D.

16. HINDU NUMERALS IN EUROPE

Boethius Question. It cannot be definitely said when and how the Hindu numerals reached Europe. Their earliest occurrence is found in a manuscript of the Geometry of Boethius (*c.* 500), said to belong to the tenth century. There are several other manuscripts of this work and they all contain the numerals. Some of these contain the zero whilst the others do not. If these manuscripts (or the portions of them that contain the numerals) be regarded as genuine, it will have to be acknowledged that the Hindu numerals had reached Southern Europe about the close of the fifth century. There are some who consider the passages dealing with the Hindu numerals in the Geometry of Boethius to be spurious. Their arguments can be summarised as below :

(1) The passages in question have no connection with the main theme of the work, which is geometry. The Hindu numerals have not been mentioned in the Arithmetic of Boethius. They have not been used by him anywhere else. Neither Boethius' contemporary Capella (*c.* 475), nor any of the numerous mediaeval writers who knew the works of Boethius makes any reference to the numerals.

(2) The Hindu numeral notation was perfected in India much later than the fifth century, so that the numerals, even if they had been taken to Europe along the trade routes, had no claim to any superiority over the numerals of the west, and so could not have attracted the attention of Boethius.

Of the above arguments, the second is against facts, for it is now established that the Hindu numeral notation with zero was perfected and was in use in India during the earliest centuries of the Christian era. The numerals could have, therefore, easily reached

Europe along the trade routes in the fifth century or even earlier. The first argument is purely speculative and throws doubt on the authenticity of the occurrence of the numerals in Boethius' Geometry. It does not prove anything. It seems to us unfair to question the genuineness of the occurrence of the numerals, when they are found in all manuscripts of the work that are in existence now. Their occurrence in the Geometry can be easily explained on the ground that Boethius' knowledge of those numerals was very meagre. He had obtained the forms from some source—from the Neo-Pythagoreans or direct from some merchant or wandering scholar—but did not know their use. He might have known their use in writing big numbers by the help of the principle of place-value and zero, but he certainly did not know how the elementary operations of arithmetic were to be performed with those numerals. Hence he could make no use of them in his arithmetic or any other work. The writings of Sebokht (*c.* 650) show that the fame of the numerals had reached the west long before they were definitely introduced there. The question of the introduction of the Hindu numerals through the agency of Boethius may, therefore, be regarded as an open one, until further investigations decide it one way or the other.

Definite Evidence. The first writer to describe the *ghobâr* numerals in any scientific way in Christian Europe was Gerbert, a French monk. He was a distinguished scholar, held high ecclesiastical positions in Italy, and was elected to the Papal chair (999). He had also been to Spain for three years. It is not definitely known where he found these numerals. Some say that he obtained them from the Moors in Spain, while others assert that he got them from some other source, probably through the merchants. We find that Gerbert did not appreciate these numerals (and rightly, for there was neither zero nor the place-value), and that in his works, known as the *Regula de abaco computi and the Libellus*, he has used the Roman forms. We thus see that upto the time of Gerbert (died 1003) the principle of place-value was not known in Europe.

As early as 711 A.D., the power of the Goths was shattered at the battle of Jarez de le Frontera, and immediately afterwards the Moors became masters of Spain, and remained so for five hundred years. The knowledge of the modern system of notation which was definitely introduced at Baghdad about the middle of the eighth century must have travelled to Spain and from there made its way into Europe. The schools established by the Moors at Cordova, Granada, and Toledo were famous seats of learning throughout the middle ages, and attracted students from all parts of Europe. Thus although Europe may not be directly indebted to the Moors for its numerical symbols, it certainly is for that important principle which made the ordinary *ghobâr* forms superior to the Roman numerals.

Several instances of the modern system of notation are to be found in Europe in the twelfth century, but no definite attempt seems to have been made for popularizing it before the thirteenth century. Perhaps the most influential in spreading these numerals in Europe was Leonardo Fibonacci of Pisa. Leonardo's father was a commercial agent at Bugia, the modern Bougie, on the coast of Barbary. It had one of the best harbours, and at the close of the twelfth century was the centre of African commerce. Here Leonardo went to school to a Moorish master. On attaining manhood he started on a tour of the Mediterranean and visited Egypt, Syria, Greece, Italy and Provence, meeting with scholars and merchants and imbibing a knowledge of the various systems of numbers in use in the centres of trade. All these systems, he however says, he counted as errors compared with that of the Hindus.[1] Returning to Pisa he wrote his *Liber Abaci* in 1202, rewriting it in 1228.[2] In this the Hindu numerals are explained and used in the usual computations of business. At first Leonardo's book met with a cold reception from the public, because it was too advanced for the merchants and too novel for the universities. However, as time went on people began to realise its importance, and then we find it occupying the highest place among the mathematical classics of the period.

Among other writers whose treatises have helped the spread of the numerals may be mentioned Alexander de Villa Die (*c.* 1240) and John of Halifax (*c.* 1250).

A most determined fight against the spread of these numerals was put up by the abacists who did not use zero but employed an abacus and the apices. But the writings of men like Leonardo succeeded in silencing them, although it took two or three centuries to do so. By the middle of the fifteenth century we find that these numerals were generally adopted by all the nations of western Europe, but they came into common use only in the seventeenth century.

17. MISCELLANEOUS REFERENCES TO THE HINDU NUMERALS

Syrian Reference. The following reference to a passage[3] in a work of Severus Sebokht (662) shows that the fame of the Hindu numerals had reached the banks of

[1] "Sed hoc totum et algorismum atque arcus pictagore quasi errorem computavi respectu modi indorum."

[2] Smith and Karpinski, *l. c.*, p. 131.

[3] Attention was first drawn to this passage by F. Nau. *JA*, II, 1910, pp. 225-227; also see J. Ginsburg, *Bull. American Math. Soc.*, XXIII, 1917, p. 368.

the Euphrates in the beginning of the seventh century. Sebokht was a learned Syrian scholar who lived in the convent of Kenneshre on the Euphrates, in the time of the patriarch Athanasius Gamenale (died 631) and his successor John.[1] Sebokht seems to have been hurt by the arrogance of certain Greek scholars who looked down on the Syrians, and in defending the latter, he claims for them the invention of astronomy. He asserts that the Greeks were merely the pupils of the Chaldeans of Babylon and claims that these same Chaldeans were the very Syrians whom his opponents condemned. It is in this connection that he mentions the Hindus by way of illustration, using the following words:

"I will omit all discussion of the science of the Hindus, a people not the same as the Syrians, their subtle discoveries in the science of astronomy, discoveries that are more ingenious than those of the Greeks and the Babylonians; their computing that surpasses description. I wish only to say that this computation is done by means of nine signs. If those who believe because they speak Greek, that they have reached the limits of science should know these things, they would be convinced that there are also others who know something."

This fragment clearly shows that not only did Sebokht know something of the numerals but that he understood their full significance, and may even have known the zero as Rabbi ben Ezra did (*vide infra* p. 103), in spite of the fact that he too speaks of nine numerals. In fact even today, we speak of nine numerals only, zero not being recognised a numeral.

Arabic References. *Ibn Wahshiya* (855). In his work, *Ancient alphabets and hieroglyphic characters explained*, etc., Ibn Wahshiya gives three forms of

[1] W. Wright, *Short History of Syriac Literature*, pp. 137-139.

Hindu numerals as three species of Hindu alphabets.[1] It seems that the forms were well known in his time in Arabia. In the first set of these forms, dots are used as zeros in the now customary way among the Arabs; in the second set the dots are superposed,[2] while in the third loops are attached to the ordinary symbols for the units.[3] Karpinski thinks that the use of loops was probably the invention of Wahshiya.

Djâhiz (died 869)

The Arab philosopher Djâhiz calls the numerals "figures of Hind" and observes that with these numerals large numbers can be represented with great facility. He asks, "Who is the inventor of the figures of Hind, who are the authors of the *Sindhind* and of the *Arkhand* and the methods of calculation with the ciphers?"[4]

Abul Hasan Al-Masûdî (943)

The Arab historian Al-Masûdî writes, "A congress of sages at the command of the Creator Brahmâ invented the nine figures and also their (the Hindus') astronomy and other sciences."[5] Al-Masûdî's evidence is important as he had first-hand information of India. He was in Multan in 912 A.D. and in Cambay in 916 A.D.

[1] Karpinski, *Bibl. Math.*, XIII, 1912-13, pp. 97-8.

[2] Similar use of dots is found in certain manuscripts in India. Vide R. L. Mitra, *Notices of Sanskrit MSS.*, V, p. 299, MS. No. 1976, Plate No. I; *India Office Catalogue*, MS. Nos. 1946, 1947 and 1871.

[3] This form is similar to the use of ligatures in India when the numerals were written without place-value, (*vide supra*).

[4] Djâhiz's work is available in French translation (Caire, *Du rond et du carré*, 1324). Cf. Carra de Vaux, *Scientia*, XXI, 1921, pp. 273-282.

[5] Reinaud, *Mémoire sur l'Inde*, Paris, 1894, p. 300. This shows that the Hindus in the tenth century had forgotten the inventor or inventors of the numerals, as a very long time had elapsed.

Abû Sahl Ibn Tamim (950)

Ibn Tamim, a native of Kairwân, a village in Tunis in the north of Africa, wrote in his commentary on the *Sefer Yesîrâh*: "The Indians have invented the nine signs for marking the units. I have spoken sufficiently of them in a book that I have composed on the Hindu calculation, known under the name of *Hisâb al-ghobâr*."[1]

Al-Nadîm (987)

In the *Fihrist*, the author Al-Nadîm includes the Hindu numerals in a list of some two hundred alphabets of India (*Hind*.) These numerals are called *hindisah*.[2]

Al-Bîrûnî (1030)

Al-Bîrûnî resided in India for nearly thirteen years (1017-1030) and devoted himself to the study of the arts and sciences of the Hindus. He had also a remarkable knowledge of the Greek sciences and literature, so he was more qualified than any contemporary or even anterior Arab writer to speak with authority about the origin of the numerals. He wrote two books, viz., *Kitâb al-arqam* ("Book of Ciphers") and *Tazkira fi al-hisâb w'al-madd bi al-arqam al-Sind w'al-Hind* ("A treatise on arithmetic and the system of counting with the ciphers of Sindh and India"). In his *Tarikh al-Hind* ("Chronicles of India"), he says: "As in different parts of India the letters have different shapes, the numerical signs too, which are called *aṅka*, differ. The numerical signs which we use are derived from the finest forms of the Hindu signs."[3] At another place he remarks: "The Hindus use the

[1] Reinaud, *l.c.*, p. 399.

[2] *Kitâb al-Fihrist*, ed. G. Flügel, II, pp. 18-19.

[3] *Alberuni's India*, English translation by E. C. Sachau, London 2nd ed., 1910, Vol. I, p. 74.

numeral signs in arithmetic in the same way as we do. I have composed a treatise showing how far possibly, the Hindus are ahead of us in this subject."[1] In his *Athâr-ul-Bâkiya*[2] ("Vestiges of the Past," written in 1000 A.D.) Al-Bîrûnî calls the modern numerals as *al-arqam al-hind*, *i.e.*, "the Indian Ciphers" and he has incidentally referred to their distinction from two other systems of expressing numbers, *viz.*, the sexagesimal system and the alphabetic system (*Harûf al-jumal*).

Abenragel (1048)

It has been stated by Ali bin Abil-Regal Abul-Hasan, called Abenragel, in the preface to his treatise on astronomy that the invention of reckoning with nine ciphers is due to the Hindu philosophers.[3]

Saraf-Eddin (1172)

Mahmûd bin Qajid al-Amûnî Saraf-Eddin of Mecca wrote a treatise, entitled *Fi al-handasa w'al arqam al-hindi* ("On geometry and the Indian ciphers").[4]

Alkalasâdî (died 1486)

In his commentary of the *Talkhis* of Ibn Albanna, Abul Hasan Ali Alkalasâdî states: "These nine signs, called the signs of the *ghobâr* (dust), are those that are employed very frequently in our Spanish provinces and in the countries of Maghrib and of Africa. Their origin is said to have been attributed by tradition to a man of the Indian nation. This man is said to have taken some fine dust, spread it upon a table and taught

[1] *Ibid*, I, p. 177.

[2] *The Chronology of Ancient Nations*, ed. by Sachau, London, 1879, pp. 62 and 132.

[3] J. F. Montucla, *Historie des Mathématiques*, vol. I, p. 376.

[4] H. Suter, *Die Mathematiker und Astronomer der Ararbe und ihre Werke*, Leipzig, 1900, p. 126.

the people multiplication, division and other operations."[1]

Behâ Eddin (*c*. 1600)

Referring to the numerals Behâ Eddin observes: "The Hindu savants have, in fact, invented the nine known characters."[2]

In the quotations from Arab scholars given above, the term *Hind* has been used for India, and *Hindi* for Indian. *Hind* is the term generally used in Arabic and Persian literature for India. In early writings distinction was sometimes made between Sind and Hind. Thus Al-Masûdî and Al-Bîrûnî used Sind to denote the countries to the west of the river Indus. This distinction is clearly in evidence in Ibn Hawkal's map, reproduced in Elliot and Dawson's *History of India*. There were others who did not make this distinction. Thus Istakri (912) uses Hind to denote the whole of India.[3] Again in the *Shâhnâmâ* of Firdausi,[4] Sind has been used for a river as well as for a country, and Hind for the whole of India. In later times this distinction disappeared completely. According to the lexicographers Ibn Seedeh (died 1066) and Firouzâbâdi (1328-1413), Hind is "the name of a well known nation" and according to El-Jowharee (1008) it denotes "the name of a country." Instances of the use of *Hind* to denote India in the literature of the Arabs can be multiplied at pleasure.

Carra de Vaux[5] has suggested that the word *Hind*

[1] *JA*, I, 1863, pp. 59f.

[2] *Kholasât al-hisâb*, translated into French by A. Marre, *Nouv. Ann. Math.*, V, 1864, p. 266.

[3] Elliot and Dawson's *History of India*, II, p. 412.

[4] English translation by A.G. Warner and E. Warner, London, 1906.

[5] Carra de Vaux, *Scientia*, XXI, 1917, p. 273.

does not probably mean India but is really derived from *ènd* (or *hènd*) signifying "measure," "arithmetic" or "geometry." He concludes that the expression "the signs of *hind*" means "the arithmetical signs" and not "the signs of India." As regards the use of the adjective *hindi* by certain scholars in connection with the numerals, he conjectures that it has probably been employed through confusion for *hindasi*.

Carra de Vaux's derivation of the word *hind* from *ènd* or *hènd* cannot be accepted. It has no support from Arabic lexicography. Moreover, the word *hind* is a very ancient one. It occurs in the *Avesta*[1] both in the earlier *Yasna* and in the later (Sassanian) *Vendidad*. The word also occurs in the cuneiform inscriptions of Darius Hystaspes. The Pehlavi writings before the Arab conquest of Iran also show the word *hind*. In all those cases it means India.

The word *hindi* is an adjective formed from *hind* and means "Indian". The fact that in a few isolated cases, it has been confused with the word *hindasi*, cannot make us conclude that this has happened in all cases.

The terms Hindasa, etc. The words *hindasa*, *hindisa*, *handasa*, *hindasi*, *handasi*, etc., have been stated by competent authorities to be adjectives formed from *hind*, meaning "Indian". Kaye[2] and Carra de Vaux[3] oppose this interpretation. Relying on the lexicon of Firouzâbâdi they assert that these terms are derived from the Persian *andâzah*, meaning "measure." There is no doubt that the word *hindasi* denotes "geometrical" in the Arabic language. But

[1] *Yasna*, x. 141; *Yt.*, x. 104 (*Mihir Yast*).

[2] Kaye, *JASB*, III, 1907, p. 489, also *JASB*, VII, 1911, pp. 810f.

[3] Carra de Vaux, *l.c.*

when this term is used in connection with an explanation of the rule of "double false position" or the method of "proof by nines" or in connection with the "numeral notation," we have to admit that it had some other significance also. As the arithmetical rules designated by the term *hindasi* are found in Hindu arithmetic prior to their occurrence in Arabia, it follows that *hindasi* also means Indian. The term *hindasi*, *hindasa* or *handasa* has, therefore, two meanings, one "geometrical" and the other "Indian". The controversy regarding the meaning of this term which was set at rest by Woepcke,[1] has arisen again because Kaye and Carra de Vaux have refused to recognise both meanings of this term.[2] It may be pointed out here that as one of the meanings of *hindasi* is synonymous with *hindi*, there is no wonder that the two words were sometimes confused with each other, especially by scribes who did not understand the text.

European References. *Isidorus of Seville.* The nine characters (of the *ghobâr* type), without zero, are given as an addition to the first chapter of the third book of the *Origines* by Isidorus of Seville in which the Roman numerals are under discussion. Another Spanish copy of the same work (of 992 A.D.) contains the numerals in the corresponding section. The writer ascribes an Indian origin to them in the following words: "Item de figuris arithmetice. Scire debemus in Indos subtilissimum ingenium habere et ceteris gentes eis in arithmetica et geometria et ceteris liberalibus disciplinis concedere. Et hoc manifestum

[1] Woepcke, *JA*, I, 1863, pp. 27f. See also Suter's article on *handasa* in the *Encyclopaedia of Islam* and Rosen's *Algebra of Mohammad Ben Musa*, London, 1831, pp. 196f.

[2] It will not be difficult to point out in any literature words having more than one meaning. Occasionally these meanings have no connection. Whenever such a word is used, the appropriate meaning has to be deduced from the context.

est in nobem figuris, quibus designant unum-quemque gradum cuinslibet gradus. Quarum hec sunt forma."[1]

Rabbi ben Ezra (1092-1167)

Rabbi Abraham ibn Meir ibn Ezra in his work, *Sefer ha-mispar* ("the Book of Number"), gives the Hindu forms of the numerals. He knew of the Hindu origin of the numerals for he states: "that is why the wise people of India have designated all their numbers through nine and have built forms for the nine ciphers."[2]

Leonardo of Pisa

Leonardo of Pisa in his work, *Liber Abaci* (1202), frequently refers to the nine Indian figures. At one place he says: "Ubi ex mirabili magisterio in arte per novem figuras indorum introductus" etc. In another place, as a heading to a separate division, he writes "De cognitione novem figurarum yndorum" etc., "Novem figure indorum he sunt 9 8 7 6 5 4 3 2 1."[3]

Alexander de Villa Dei

Alexander de Villa Dei (*c.* 1240) wrote a commentary on a set of verses called *Carmen de Algorismo*. In this commentary he writes: "This boke is called the boke of algorim or augrym after lewder use. And this boke tretys of the Craft of Nombryng, the quych crafte is called also algorym. Ther was a kyng of Inde the quich heyth Algor & he made this craft...... Algorisms, in the quych we use teen figurys of Inde."[4]

[1] Quoted by Smith and Karpinski, *l.c.*, p. 138.

[2] *Sefer ha-Mispar, Das Buch der zahl, ein hebraisch-arithmetisches Work des R. Abraham ibn Esra*, Moritz Silberberg, Frankfurt a M., 1895, p. 2.

[3] *Liber Abaci*, Rome, 1857; quoted by Smith and Karpinski, *l.c.*, p. 10.

[4] Smith and Karpinski, *l.c.*, p. 11.

Maximus Planudes (*c.* 1330)

Maximus Planudes states that "the nine symbols come from the Indians."[1]

[1] Wäschke's German Translation, Halle, 1878, p. 3.

TABLE I—*Kharoṣṭhî Numerals*

Śaka, Pârthian and Kuṣâna Inscriptions			Aśoka Inscs.	
𐩅𐩅	40	𐩀	𐩀	1
𐩅𐩅𐩄	50	𐩁	𐩁	2
𐩅𐩅𐩅	60	𐩂		3
𐩅𐩅𐩅𐩄	70	𐩃	////	4
𐩅𐩅𐩅𐩅	80	𐩃𐩀	/////	5
𐩀𐩆	100	𐩃𐩁		6
𐩁𐩆	200	𐩃𐩂		7
𐩂𐩆	300	𐩃𐩃		8
𐩀𐩆𐩅𐩁	122	𐩄		10
𐩁𐩆𐩅𐩅𐩅𐩄𐩃	274	𐩅		20

TABLE II(a)—*Semitic and Brâhmî Numerals*

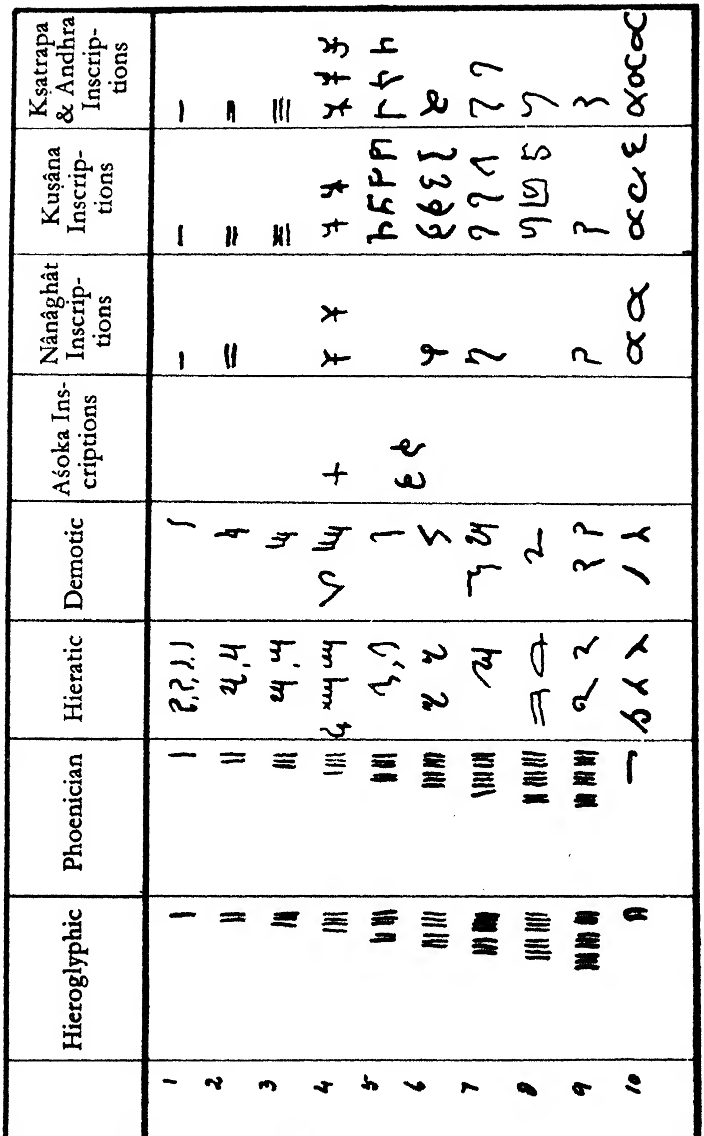


TABLE II(*b*)—*Semitic and Brâhmî Numerals*

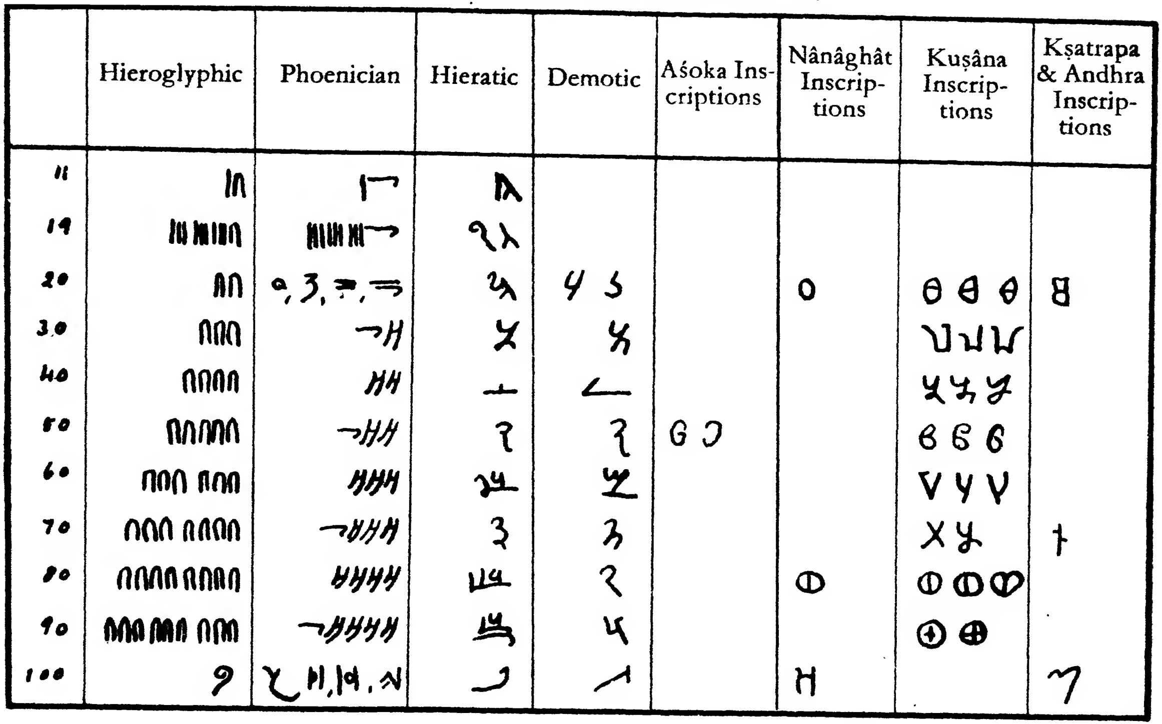


TABLE II(*c*)—*Semitic and Brâhmî Numerals*

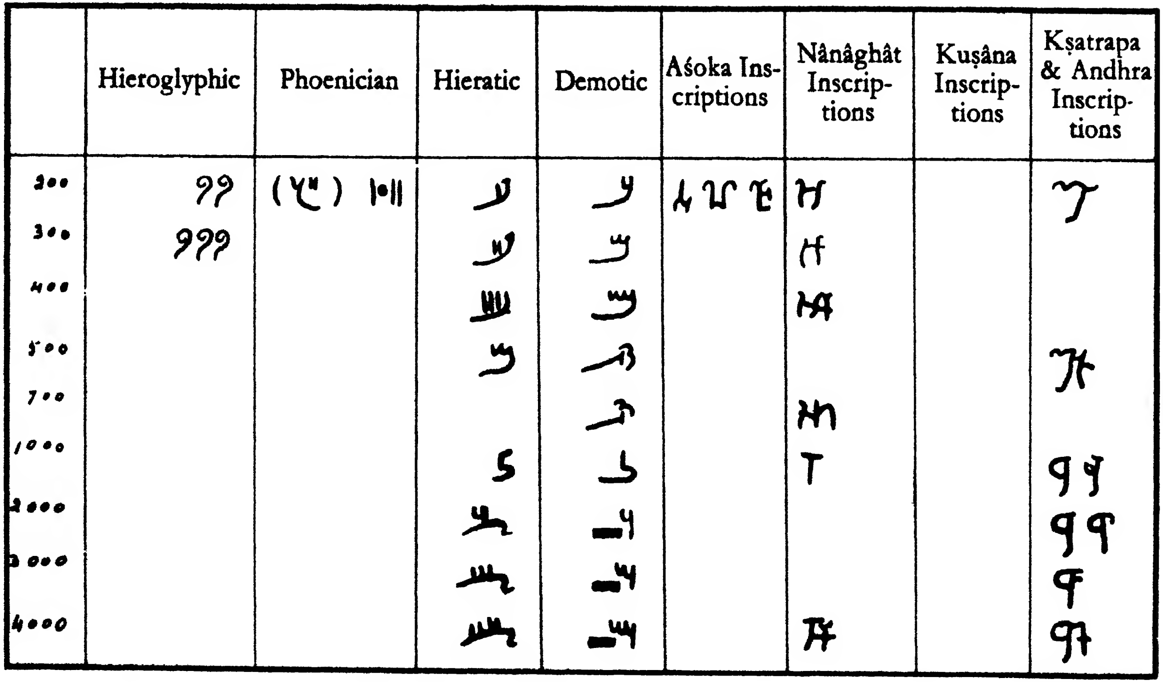


TABLE III—*Brâhmî Numerals*

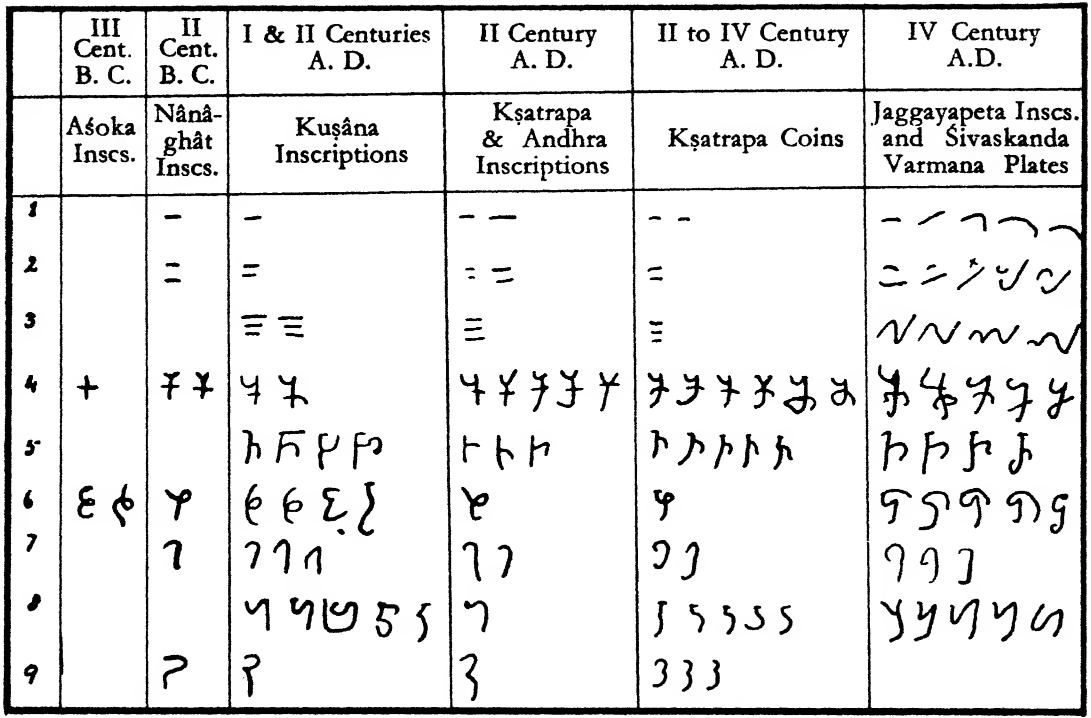


TABLE IV—*Brâhmî Numerals*

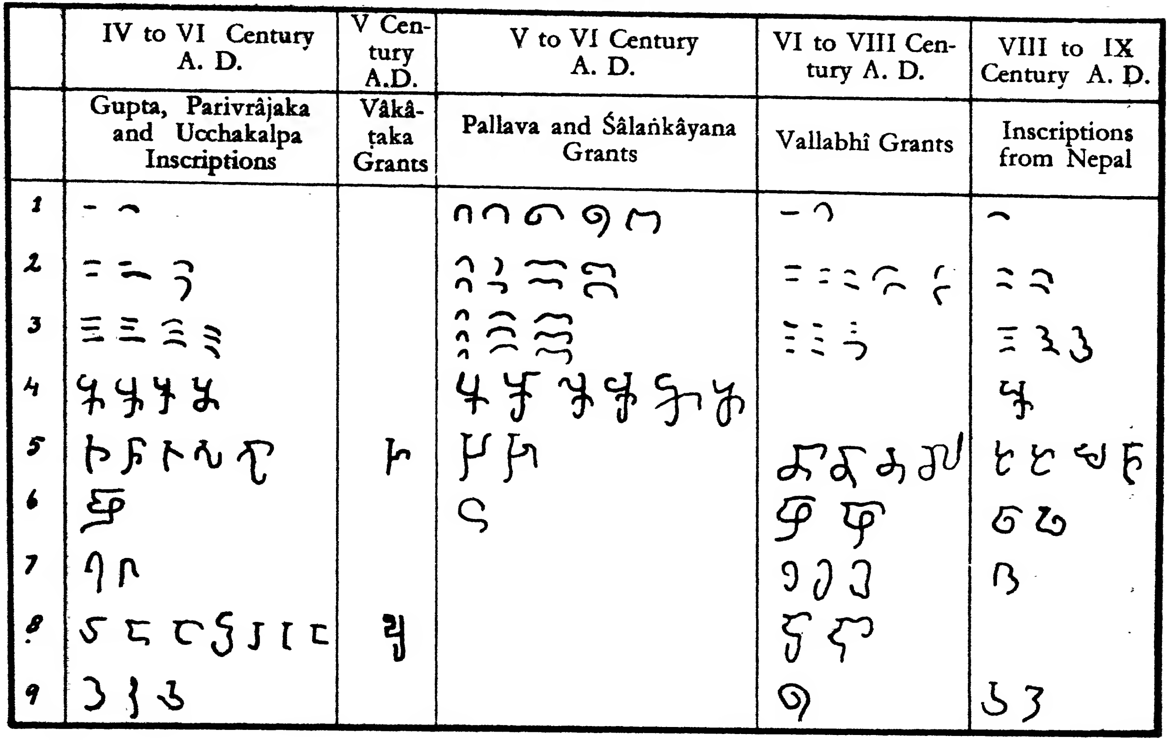


TABLE V—*Brâhmi Numerals*

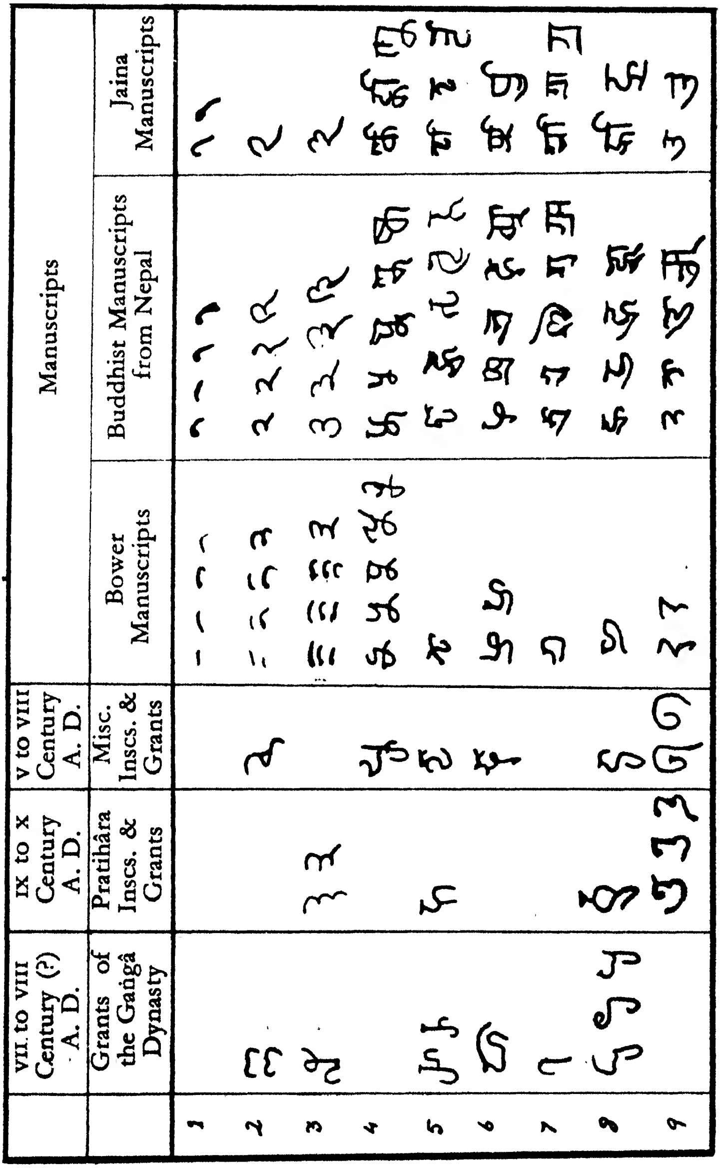


TABLE VI—*Brâhmî Numerals*

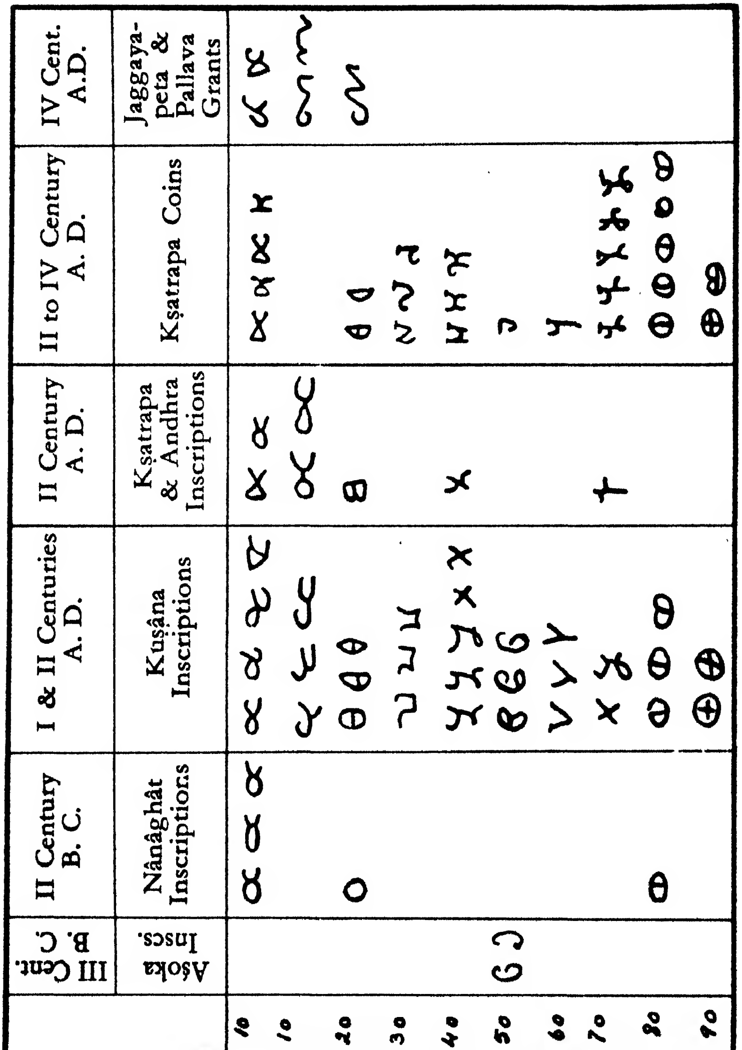


TABLE VII—*Brâhmî Numerals*

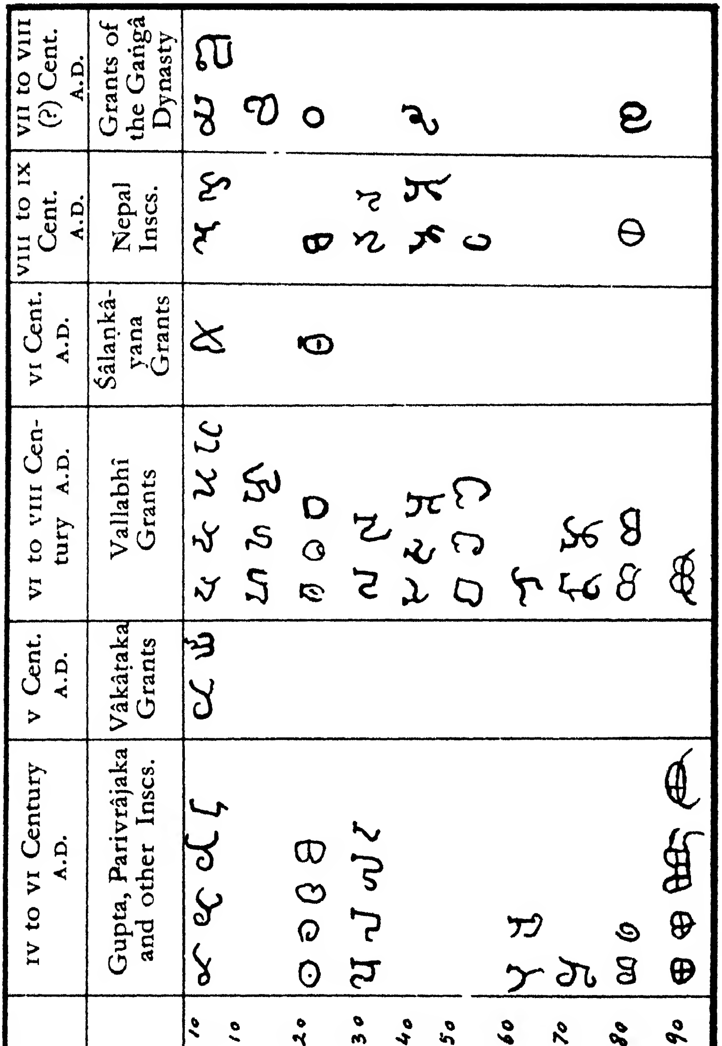


TABLE VIII—*Brâhmî Numerals*

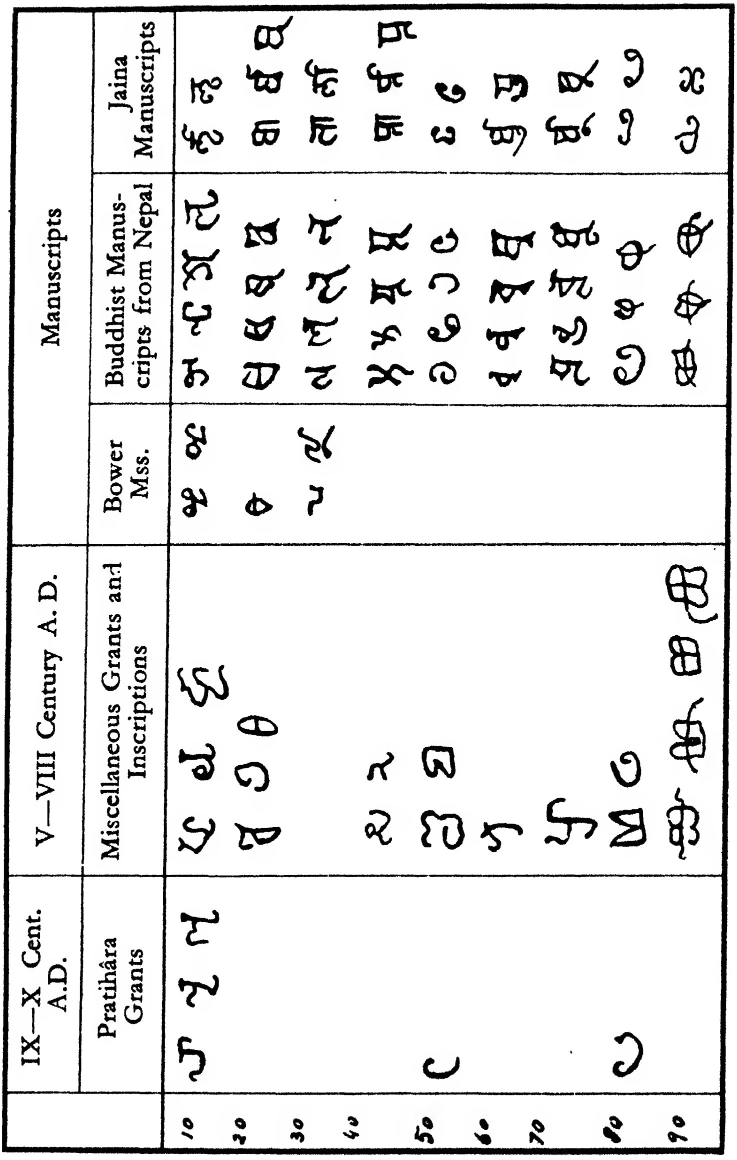


TABLE IX—*Brâhmî Numerals*

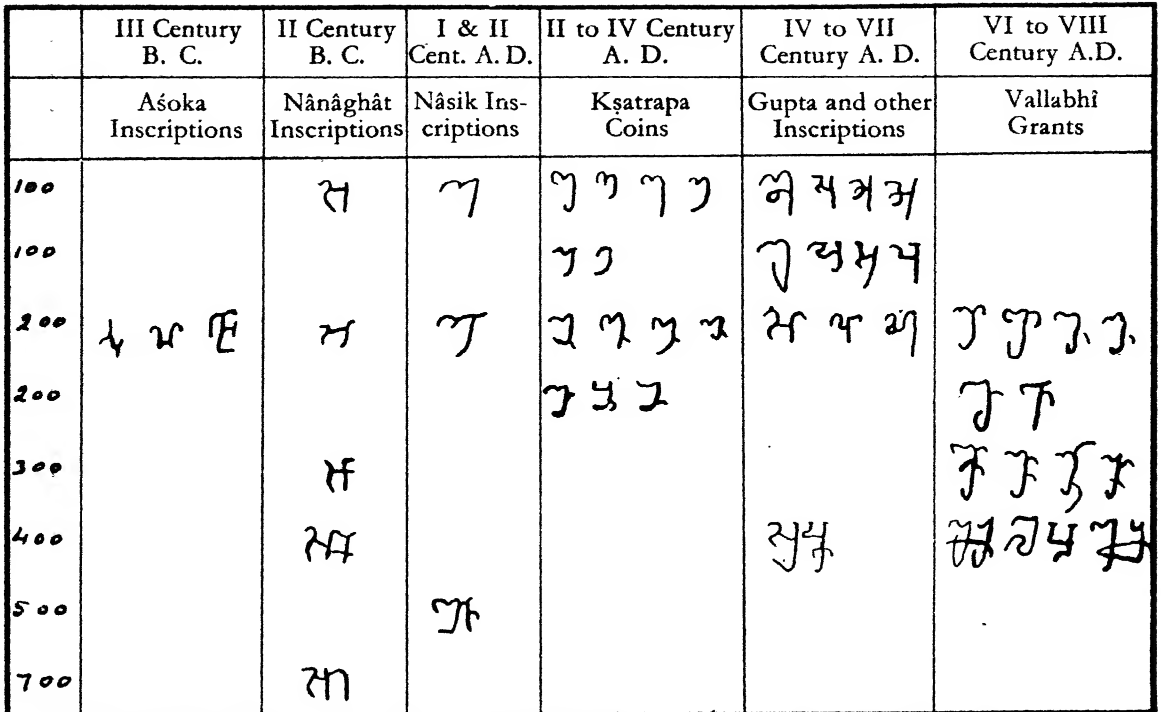


TABLE X—*Brâhmî Numerals*

	VIII to IX Cent. A.D.	VII to VIII Cent. A.D.	IX to X Century A.D.	V—VIII Cent. A.D.	Manuscripts	
	Inscs. from Nepal	Grants of the Gaṅgâ Dynasty	Pratihâra Grants	Misc. Inscriptions	Buddhist Manuscripts	Jaina Manuscripts
100						
200						
300						
400						
500						
600						
700						
800						
900						

TABLE XI—*Brâhmî Numerals*

	II Cent. B.C. Nânâghât Inscs.	I & II Cent. A.D. Nâsik Inscriptions	V Cent. A.D. Vâkâṭaka Grants	Composite numbers in Brâhmî numerals					
1000				15		124		1001	
2000				28		289		1002	
3000				33		301		1101	
4000				45		428		1700	
6000				52		538		6001	
8000				62		679		10001	
10000				79		703		11000	
20000				81		813		24400	
70000				93		955			

TABLE XII—*Early Hindu Numeral Forms*

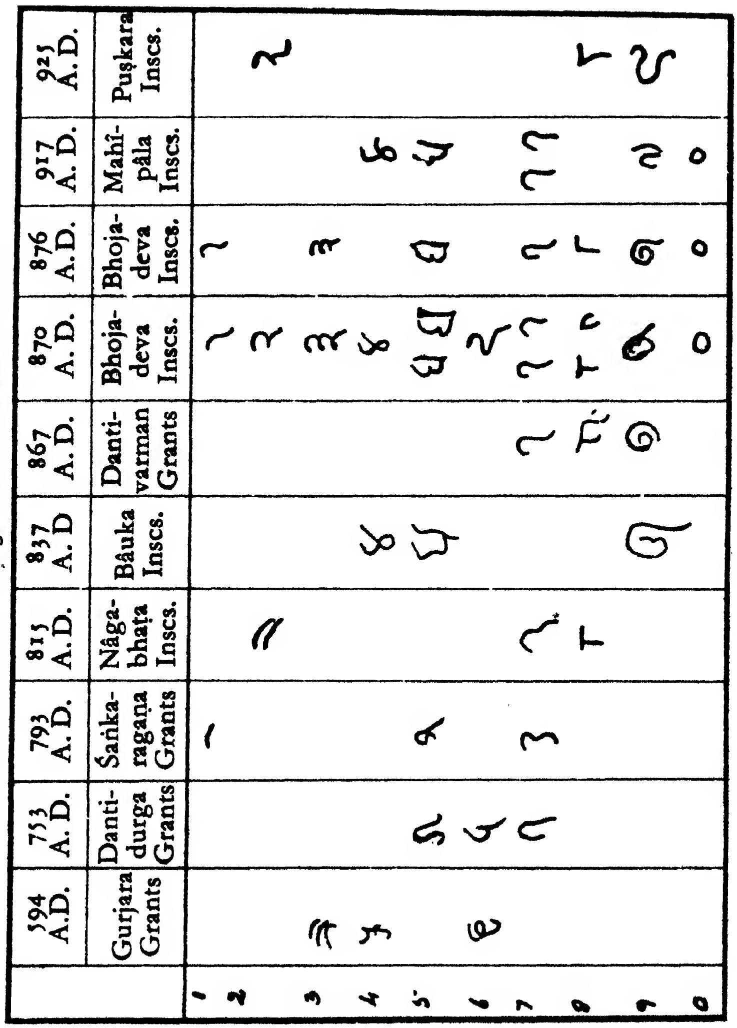


TABLE XIII—*Early Hindu Numeral Forms*

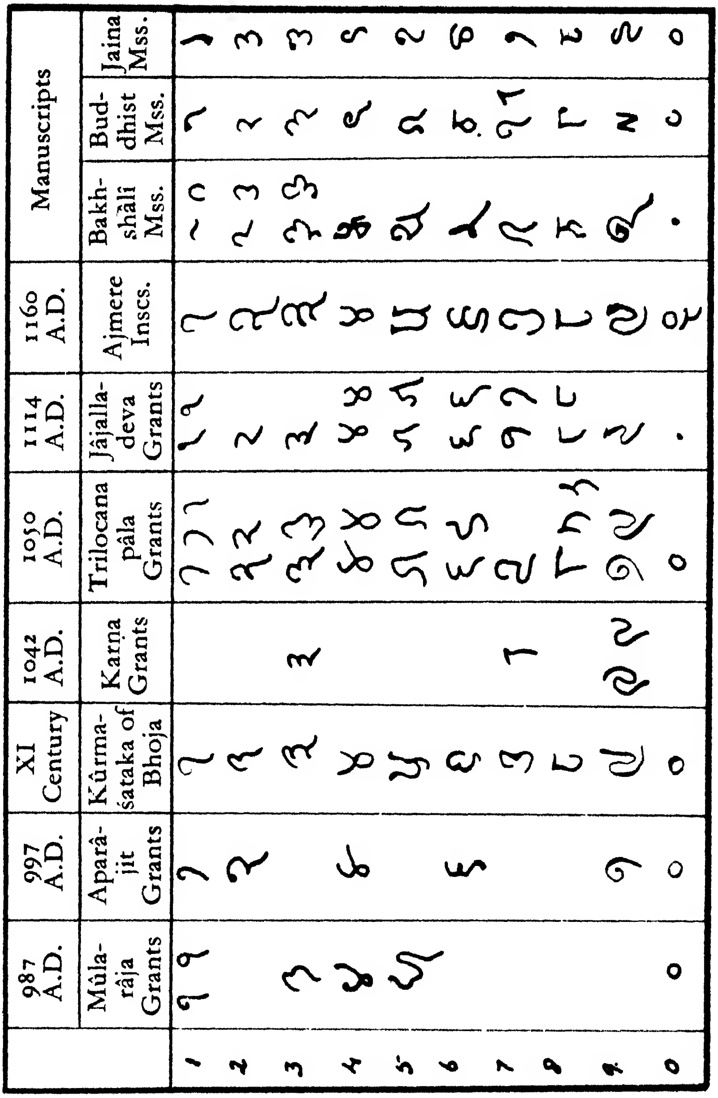


TABLE XIV—*Development of Nâgarî Numerals*

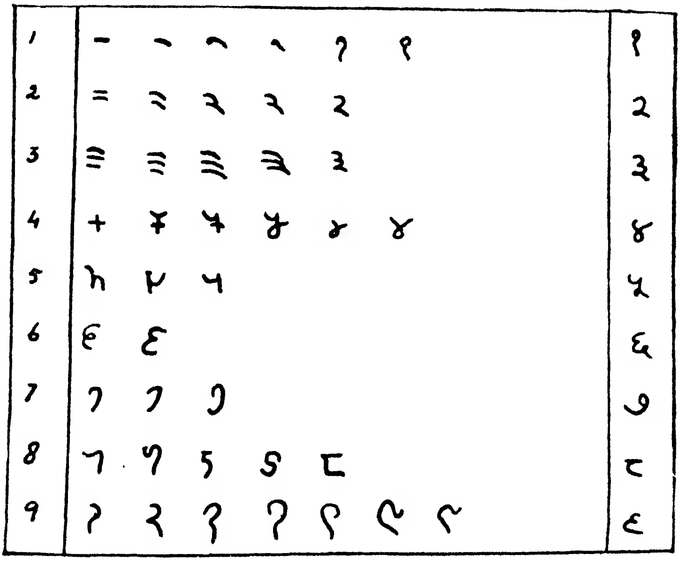
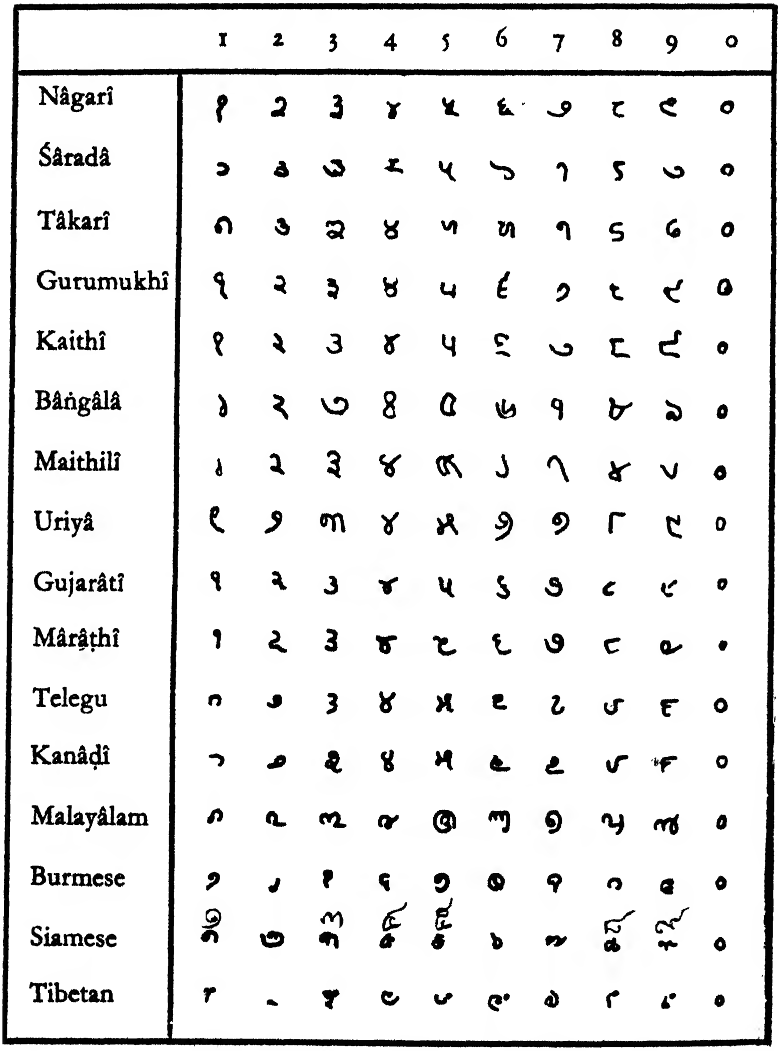


TABLE XV—*Numeral Forms in Modern Hindu Scripts*



CHAPTER II

ARITHMETIC

1. GENERAL SURVEY

Terminology and Scope. Arithmetic forms the major part of the Hindu works on *pâṭîgaṇita*. The word *pâṭîgaṇita* is a compound formed from the words *pâṭî*, meaning "board," and *gaṇita*, meaning "science of calculation;" hence it means the science of calculation which requires the use of writing material (the board).[1] It is believed that this term originated in a non-Sanskrit literature of India, a vernacular of Northern India. The oldest Sanskrit term for the board is *phalaka* or *paṭṭa*, not *pâṭî*. The word *pâṭî* seems to have entered into Sanskrit literature about the beginning of the seventh century A.D.[2] The carrying out of mathematical calculations was sometimes called *dhûlî-karma* ("dust-work"), because the figures were written on dust spread on a board or on the ground. Some later writers have used the term *vyakta-gaṇita* ("the science of calculation by the 'known'") for *pâṭîgaṇita* to distinguish it from algebra which was called *avyakta-gaṇita* ("the science of calculation by the 'unknown'"). The terms *pâṭîgaṇita* and *dhûlî-karma* were translated into Arabic when Sanskrit works were rendered into that language. The Arabic equivalents are *ilm-hisâb-al-takht* ("the science of

[1] Paper being scarce, a wooden board was generally used for making calculations even upto the 19th century.

[2] B. Datta, *American Math. Monthly*, XXXV, p. 526.

calculation on the board") and *hisâb-al-ghobâr* ("calculation on dust") respectively.

Bayley, Fleet and several others suspect that the origin of the term *pâṭî* in Hindu Mathematics lies in the use of the board as an abacus. This conjecture, however, is without foundation, as no trace of the use of any form of the abacus is found in India.

According to Brahmagupta[1] there are twenty operations and eight determinations in *pâṭîgaṇita*. He says:

"He who distinctly and severally knows the twenty logistics, addition, etc., and the eight determinations including (measurement by) shadow is a mathematician."

The twenty logistics, according to Pṛthudakasvâmî, are: (1) *saṁkalita* (addition), (2) *vyavakalita* or *vyutkalita* (subtraction), (3) *guṇana* (multiplication), (4) *bhâgahâra* (division), (5) *varga* (square), (6) *varga-mûla* (square-root), (7) *ghana* (cube), (8) *ghana-mûla* (cube-root), (9-13) *pañca jâti* (the five rules of reduction relating to the five standard forms of fractions), (14) *trairâśika* (the rule of three), (15) *vyasta-trairâśika* (the inverse rule of three), (16) *pañcarâśika* (the rule of five), (17) *saptarâśika* (the rule of seven), (18) *navarâśika* (the rule of nine), (19) *ekâdaśarâśika* (the rule of eleven), and (20) *bhâṇḍa-pratibhâṇḍa* (barter and exchange). The eight determinations are: (1) *miśraka* (mixture), (2) *śreḍhî* (progression or series), (3) *kṣetra* (plane figures), (4) *khâta* (excavation), (5) *citi* (stock), (6) *krâkacika* (saw), (7) *râśi* (mound), and (8) *châyâ* (shadow).

Of the operations named above, the first eight have been considered to be fundamental by Mahâvîra and later writers. The operations of duplation (doubling)

[1] *BrSpSi*, p. 172.

and mediation (halving), which were considered fundamental by the Egyptians, the Greeks and some Arab and western scholars, do not occur in the Hindu mathematical treatises. These operations were essential for those who did not know the place-value system of notation. They are not found in Hindu works, all of which use the place-value notation.

Sources. The only works available which deal exclusively with *pâṭîgaṇita* are: the *Bakhshâlî Manuscript* (*c.* 200), the *Triśatikâ* (*c.* 750), the *Gaṇita-sâra-saṁgraha* (*c.* 850), the *Gaṇita-tilaka* (1039), the *Lîlâvatî* (1150), the *Gaṇita-kaumudî* (1356), and the *Pâṭî-sâra* (1658). These works contain the twenty operations and the eight determinations mentioned above. Examples are also given to illustrate the use of the rules enunciated.

Besides these there are a number of astronomical works, known as *Siddhânta*, each of which contains a section dealing with mathematics. Âryabhaṭa I (499) was the first to include a section on mathematics in his *Siddhânta*, the *Âryabhaṭîya*. Brahmagupta (628) followed Âryabhaṭa in this respect, and after him it became the general fashion to include a section on mathematics in a *Siddhânta* work.[1] The earlier *Siddhânta* works do not possess this feature. The *Sûrya-siddhânta* (*c.* 300) does not contain a section on mathematics. The same is true of the *Vâsiṣṭha*, the *Pitâmaha* and the *Romaka* Siddhântas. Bhâskara I and Lalla,[2] although zealous followers of Âryabhaṭa I, did not emulate him in including a section on mathematics in their astronomical works.

[1] Amongst such works may be mentioned the *Mahâ-siddhânta* (950), the *Siddhânta-śekhara* (1036), the *Siddhânta-tattva-viveka* (1658), etc.

[2] It is stated by Bhâskara II that Lalla wrote a separate treatise on *pâṭîgaṇita*.

Exposition and Teaching. In India conciseness of composition, especially in scientific matters, was highly prized. The more compact and brief the composition, the greater was its value in the eyes of the learned. It is for this reason that the Indian treatises contain only a brief statement of the known formulæ and results, sometimes so concisely expressed as to be hardly understandable. This compactness is more pronounced in the older works; for instance, the exposition in the *Âryabhaṭîya* is more compact than in the later works.

This hankering after brevity, in early times, was due chiefly to the dearth of writing material, the fashion of the time and the method of instruction followed. The young student who wanted to learn *pâṭîgaṇita* was first made to commit to memory all the rules. Then he was made to apply the rules to the solution of problems (also committing the problems to memory). The calculations were made on a *pâṭî* on which dust was spread, the numbers being written on the dust with the tip of the fore-finger or by a wooden style, the figures not required being rubbed out as the calculation proceeded. Sometimes a piece of chalk or soap-stone was used to write on the *pâṭî*. Along with each step in the process of calculation the *sûtra* (rule) was repeated by the student, the teacher supervising and helping the student where he made mistakes. After the student had acquired sufficient proficiency in solving the problems contained in the text he was studying, the teacher set him other problems—a store of graded examples (probably constructed by himself or borrowed from other sources) being the stock-in-trade of every professional teacher. At this stage the student began to understand and appreciate the *rationale* of the easier rules. After this stage was reached the teacher gave proofs of the more difficult formulæ to the pupil.

It will be observed that the method of teaching pursued was extremely defective in so far as it was in the first two stages purely mechanical. A student who did not complete all the three stages knew practically nothing more than the mere mechanical application of a set of formulæ committed to memory; and as he did not know the *rationale* of the formulæ he was using, he was bound to commit mistakes in their application. It may be mentioned that not many teachers themselves could guide a pupil through all the stages of the teaching, and the earnest student, if he had a genuine desire to learn, had to go to some seat of learning or to some celeberated scholar to complete his training.

Mathematics is and has always been the most difficult subject to study, and as a knowledge of higher mathematics could not be turned to material gain there were very few who seriously undertook its study. In India, however, the religious practices of the Hindus required a certain amount of knowledge of astronomy and mathematics. Moreover, there have always been, from very early times, a class of people known as *gaṇaka* whose profession was fortune-telling. These people were astrologers, and in order to impress their clients with their learning, they used to have some knowledge of mathematics and astronomy. Thus it would appear that instruction in mathematics, upto a certain minimum standard, was available almost everywhere in India. As always happens, some of the pupils got interested in mathematics for its own sake, and took pains to make a thorough study of the subject and to add to it by writing commentaries or independent treatises.

Decay of Mathematics. All this was true when the times were normal. In abnormal times when there were foreign invasions, internal warfares or bad government and consequent insecurity, the study of mathematics and, in fact, of all sciences and arts languished. Al-Bîrûnî who visited north-western India after it had been in a very unsettled state due to recurrent Afghan invasions for the sake of plunder and loot complains that he could not find a *paṇḍit* who would explain to him the principles of Indian mathematics. Although Al-Bîrûnî's case was peculiar, for no respectable *paṇḍit* would agree to help a foreigner, especially one belonging to the same class as the invaders and the despoilers of temples, yet we are quite sure that in the Punjab there were very few good scholars at that time. We, however, know of at least one very distinguished mathematician, Śripati, who probably lived in Kashmir at that time.

It is certain, however, that after the 12th century very little original work was done in India. Commentaries on older works were written and some new works brought out, but none of these had sufficient merit as regards exposition or subject matter, so as to displace the works of Bhâskara II, which have held undisputed sway for nine centuries (as standard text books).

The Fundamental Operations. The eight fundamental operations of Hindu *gaṇita* are: (1) addition, (2) subtraction, (3) multiplication, (4) division, (5) square, (6) square-root, (7) cube and (8) cube-root. Most of these elementary processes have not been mentioned in the Siddhânta works. Âryabhaṭa I gives the rules for finding the square- and cube-roots only, whilst Brahmagupta gives the cube-root rule only. In the works on arithmetic (*pâṭīgaṇita*), the methods of addition and subtraction have not been mentioned at all or mentioned very briefly. Names of several methods of multiplication have been mentioned, but the methods themselves have been either very briefly described or not described at all. The modern method of division is briefly described in all the works and so are the methods of squaring, square-root, cubing and cube-root.

Although very brief descriptions of these fundamental operations are available, yet it is not difficult to reconstruct the actual procedure employed in performing these operations in ancient India. These methods have been well-known and taught to children, practically without any change, for the last fifteen hundred years or more. They are still performed in the old fashion on a *pâṭî* ("board") by those who have obtained their primary training in the Sanskrit pâṭhaśâlâ and not in the modern primary school. The details of these methods are also available to us in the various commentaries, viz., the commentary of Pṛthudakasvâmî and the several commentaries on Bhâskara's *Lîlâvatî*.

As already mentioned, the calculations were performed on sand spread on the ground (*dhûlî-karma*[1]) or on a *pâṭî* ("board"). Sometimes a piece of chalk or soap-stone (*pâṇḍu-lekha* or *śvetavarṇî*) was used to write on the *pâṭî*.[2] As the figures written were big, so several lines of figures could not be contained on the board. Consequently, the practice of obliterating figures not required for subsequent work was common. Instances of this would be found in the detailed method of working (the operations) given hereafter.

That all mathematical operations are variations of the two fundamental operations of addition and subtraction was recognised by the Hindu mathematicians

[1] Bhâskara II, *SiŚi*, *candragrahaṇâdhikâra*, 4.

[2] Bhâskara II: *khaṭikâyâ rekhâ ucchâdya*..., *i.e.*, "having drawn lines with a chalk...," quoted by S. Dvivedi in his *History of Mathematics* (in Hindi), Benares, 1910, p. 41.

from early times. Bhâskara I (*c.* 525) states:[1]

"All arithmetical operations resolve into two categories though usually considered to be four.[2] The two main categories are increase and decrease. Addition is increase and subtraction is decrease. These two varieties of operations permeate the whole of mathematics (*gaṇita*). So previous teachers have said: 'Multiplication and evolution are particular kinds of addition; and division and involution of subtraction. Indeed every mathematical operation will be recognised to consist of increase and decrease.' Hence the whole of this science should be known as consisting truly of these two only."

ADDITION

Terminology. Âryabhaṭa II (950) defines addition thus:

"The making into one of several numbers is addition."[3]

The Hindu name for addition is *saṁkalita* (made together). Other equivalent terms commonly used are *saṁkalana* (making together), *miśraṇa* (mixing), *sammelana* (mingling together), *prakṣepaṇa* (throwing together), *saṁyojana* (joining together), *ekîkaraṇa* (making into one), *yukti*, *yoga* (addition) and *abhyâsa*,[4] etc. The word *saṁkalita* has been used by some writers in the general sense of the sum of a series.[5]

The Operation. In all mathematical and astronomical works, a knowledge of the process of addition is

[1] The quotation is from his commentary on the *Âryabhaṭîya*.

[2] *i.e.*, addition, subtraction, multiplication and division.

[3] *MSi*, p. 143.

[4] This word has been used in the sense of addition in the *Śulba* only. It is used for multiplication in later works.

[5] *E.g.*, *Triś*, p.2; *GSS*, p. 17.

taken for granted. Very brief mention of it is made in some later works of elementary character. Thus Bhâskara II says in the *Lîlâvatî*:

"Add the figures in the same places in the direct or the inverse order."[1]

Direct Process. In the direct process of addition referred to above, the numbers to be added are written down, one below the other, just as at present, and a line is drawn at the bottom below which the sum is written. At first the sum of the numbers standing in the units place is written down, thus giving the first figure of the sum. The numbers in the tens place are then added together and their sum is added to the figure in the tens place of the partial sum standing below the line and the result substituted in its place. Thus the figure in the tens place of the sum is obtained; and so on. An alternative method used was to write the biggest addend at the top, and to write the digits of the sum by rubbing out corresponding digits of this addend.[2]

Inverse Process. In the inverse process, the numbers standing in the last place (extreme left) are added together and the result is placed below this last place. The numbers in the next place are then added and so on. The numbers of the partial sum are corrected, if necessary, when the figures in the next vertical line are added. For instance, if 12 be the sum of the numbers in the last place, 12 is put below the bottom line, 2 being directly below the numbers added; then, if the

[1] *L*, p. 2; direct (*krama*), *i.e.*, beginning from the units place; inverse (*utkrama*), *i.e.*, beginning from the last place on the left. The commentator Gaṅgâdhara says: *aṅkânâṃ vâmatogatiriti vitarkeṇa ekasthânâdi yojanaṃ kramaḥ utkramastu antyasthânâdi yojanaṃ*, *i.e.*, "According to the rule 'the numerals increase (in value) towards the left', the addition of units first is the direct method, the addition of figures in the last place first is the inverse method."

[2] Dvivedi, *History of Mathematics*, Benares, 1910, p. 60.

sum of the numbers in the next place is 13 (say), 3 is placed below the figures added and 1 is carried to the left. Thus the figure 2 of the partial sum 12 is rubbed out and substituted by 3.[1]

The Arabs used to separate the places by vertical lines, but this was not done by the Hindus.[2]

6. SUBTRACTION

Terminology. Âryabhaṭa II (950) gives the following definition of subtraction:

"The taking out (of some number) from the *sarvadhana* (total) is subtraction; what remains is called *śeṣa* (remainder)."[3]

The terms *vyutkalita* (made apart), *vyutkalana* (making apart), *śodhana* (clearing), *pâtana* (causing to fall), *viyoga* (separation), etc., have been used for subtraction. The terms *śeṣa* (residue) and *antara* (difference) have been used for the remainder. The minuend has been called *sarvadhana* or *viyojya* and the subtrahend *viyojaka*.

The Operation. Bhâskara II gives the method of subtraction thus:

"Subtract the numbers according to their places in the direct or inverse order."[4]

[1] The *Manorañjana* explains the process of addition thus:
Example. Add 2, 5, 32, 193, 18, 10 and 100.

Sum of units	2,5,2,3,8,0,0	20
Sum of tens	3,9,1,1,0	14
Sum of hundreds	1,0,0,1	2
Sum of sums		360

The horizontal process has been adopted by the commentator so that both the 'direct' and 'inverse' processes may be exhibited by a single illustration. It was never used in practice.

[2] *Cf.* Taylor, *Lîlâwatî*, Bombay, 1816, Introduction, p. 14.

[3] *MSi*, p. 143.

[4] *L*, p. 2.

Direct Process. Sûryadâsa[1] explains the process of subtraction with reference to the example.

$$1000 - 360$$

thus:

"Hence making the subtraction as directed, six cannot be subtracted from the zero standing in the tens place, so taking ten and subtracting six from it, the remainder (four) is placed above (six), and this ten is to be subtracted from the next place. For, as the places of unit, etc., are multiples of ten, so the figure of the subtrahend that cannot be subtracted from the corresponding figure of the minuend is subtracted from ten, the remainder is taken and this ten is deducted from the next place. In this way this ten is taken to the last place until it is exhausted with the last figure. In other words, numbers upto nine occupy one place, the differentiation of places begins from ten, so it is known 'how many tens there are in a given number' and, therefore, the number that cannot be subtracted from its own place is subtracted from the next ten, and the remainder taken."

The above refers to the direct process, in which subtraction begins from the units place.

Inverse Process. The inverse process is similar, the only difference being that it begins from the last place of the minuend, and the previously obtained partial differences are corrected, if required. The process is suitable for working on a *pâṭî* (board) where figures can be easily rubbed out and corrected. This process seems to have been in general use in India, and was considered to be simpler than the direct process.[2]

[1] In his commentary on the *Lîlâvatî*.

[2] According to Gaṅgâdhara, the inverse process of working is easier in the case of subtraction, and the direct in the case of addition.

7. MULTIPLICATION

Terminology. The common Hindu name for multiplication is *guṇana*. This term appears to be the oldest as it occurs in Vedic literature. The terms *hanana*, *vadha*, *kṣaya*, etc. which mean "killing" or "destroying" have been also used for multiplication. These terms came into use after the invention of the new method of multiplication with the decimal place-value numerals; for in the new method the figures of the multiplicand were successively rubbed out (destroyed) and in their places were written the figures of the product.[1] Synonyms of *hanana* (killing) have been used by Ārya-bhaṭa I[2] (499), Brahmagupta (628), Śrîdhara (*c.* 750) and later writers. These terms appear also in the Bakhshâlî Manuscript.[3]

The term *abhyâsa* has been used both for addition and multiplication in the *Śulba* works (800 B.C.). This shows that at that early period, the process of multiplication was made to depend on that of repeated addition. The use of the word *parasparakṛtaṃ* (making together) for multiplication in the Bakhshâlî Manuscript[4] is evidently a relic of olden times. This ancient terminology proves that the definition of multiplication was "a process of addition resting on repetition of the multiplicand as many times as is the number of the multiplicator." This definition occurs in the commentary of the *Āryabhaṭîya* by Bhâskara I. The commentators of the *Lîlâvatî* give the same explanation of the method of multiplication.[5]

[1] See the *kapâṭa-sandhi* method of multiplication, pp. 138ff.

[2] *Ā*, ii. 19, 26, etc.

[3] *BMs*, 65 *verso*.

[4] *BMs*, 3 *verso*.

[5] Colebrooke, *Hindu Algebra*, p. 133.

The multiplicator was termed *guṇya* and the multiplier *guṇaka* or *guṇakâra*. The product was called *guṇana-phala* (result of multiplication) or *pratyutpanna* (lit. "reproduced," hence in arithmetic "reproduced by multiplication"). The above terms occur in all known Hindu works.

Methods of Multiplication. Âryabhaṭa I does not mention the common methods of multiplication, probably because they were too elementary and too well-known to be included in a Siddhânta work. Brahmagupta, however, in a supplement to the section on mathematics in his Siddhânta, gives the names of some methods with very brief descriptions of the processes:

"The multiplicand repeated, as in *gomûtrikâ*, as often as there are digits[1] in the multiplier, is severally multiplied by them and (the results) added (according to places); this gives the product. Or the multiplicand is repeated as many times as there are component parts[2] in the multiplier."[3]

"The multiplicand is multiplied by the sum or the difference of the multiplier and an assumed quantity and, from the result the product of the assumed quantity and the multiplicand is subtracted or added."[4]

Thus Brahmagupta mentions four methods: (1) *gomûtrikâ*, (2) *khaṇḍa*, (3) *bheda* and (4) *iṣṭa*. The common and well-known method of *kapâṭa-sandhi* has been omitted by him.

[1] *khaṇḍa*, translated as "integrant portions" by Colebrooke.

[2] *bheda*, i.e., portions which added together make the whole, or aliquot parts which multiplied together make the entire quantity.

[3] *BrSpSi*, p. 209; Colebrooke, *l.c.*, p. 319.

[4] *BrSpSi*, p. 209. Colebrooke (*l.c.*, p. 320) thinks that this is a method to obtain the true product when the multiplier has been taken to be too great or too small by mistake. This view is incorrect.

Śrîdhara mentions four methods of multiplication: (1) *kapâṭa-sandhi*, (2) *tastha*, (3) *rûpa-vibhâga* and (4) *sthâna-vibhâga*. Mahâvîra mentions the same four. Âryabhaṭa II mentions only the common method of *kapâṭa-sandhi*. Bhâskara II, besides the above four, mentions Brahmagupta's method of *iṣṭa-guṇana*. The five methods given by Bhâskara II were mentioned earlier by Śrîpati in the *Siddhânta-śekhara*. Gaṇeśa[1] (1545) mentions the gelosia method of multiplication under the name of *kapâṭa-sandhi* and adds that the intelligent can devise many more methods of multiplication. The method is also given in the *Gaṇita-mañjarî*. We have designated it as *kapâṭa-sandhi* (*b*).

Seven[2] distinct modes of multiplication employed by the Hindus are given below. Some of these are as old as 200 A.D. These methods were transmitted to Arabia in the eighth century and were thence communicated to Europe, where they occur in the writings of mediæval mathematicians.

Door-junction Method. The Sanskrit term for the method is *kapâṭa-sandhi*. Śrîdhara[3] describes it thus:

"Placing the multiplicand below the multiplier as in *kapâṭa-sandhi*,[4] multiply successively, in the direct or inverse order, moving the multiplier each time. This method is called *kapâṭa-sandhi*."

Âryabhaṭa II[5] (950) gives the following without name:

[1] Commentary on the *Lîlâvatî*, MSS No. I. B. 6. in the Asiatic Soc. of Bengal, Calcutta, pp. 17, 18. In this work only two methods are given, (1) *kapâṭa-sandhi* and (2) *kapâṭa-sandhi* (*b*).

[2] Or ten if we count also the sub-divisions under each head.

[3] *Triś*, pp. 3f.

[4] *kapâṭa* means "door" and *sandhi* means "junction"; hence *kapâṭa-sandhi* means "the junction of doors."

[5] *MSi*, p. 143; the inverse method only has been given.

"Place the first figure of the multiplier over the last figure of the multiplicand, and then multiply successively all the figures of the multiplier by each figure of the multiplicand."

Śrîpati[1] (1039) gives the name *kapâṭa-sandhi* and states:

"Placing the multiplicand below the multiplier as in the junction of two doors multiply successively (the figures of the multiplicand) by moving it (the multiplier) in the direct or inverse order."

Mahâvîra refers to a method known as *kapâṭa-sandhi*, but does not give the details of the process.[2] Bhâskara II gives the method but not the name, while Nârâyaṇa (1356) gives the method in almost the same words as Śrîdhara, and calls it *kapâṭa-sandhi*.

The main features of the method are (*i*) the relative positions of the multiplicand and the multiplier and (*ii*) the rubbing out of figures of the multiplicand and the substitution in their places of the figures of the product. The method owes its name *kapâṭa-sandhi* to the first feature, and the later Hindu terms meaning "killing" or "destroying" for multiplication owe their origin to the second feature. The occurrence of the terms *hanana*, *vadha*, etc., in the works of Âryabhaṭa I and Brahmagupta, and in the Bakhshâlî Manuscript show beyond doubt that this method was known in India about 200 A.D.

The following illustrations[3] explain the two processes of multiplication according to the *kapâṭa-sandhi* plan:

[1] *SiŚe*, xiii. 2; *GT*, 15.

[2] *GSS*, p. 9.

[3] The illustrations are based on the accounts given in the commentaries on the *Lîlâvatî*, especially the *Manorañjana* which gives more details.

Direct Process: This method of working does not appear to have been popular. It has not been mentioned by writers after the 11th century, Śrîpati (1039) being the last writer to mention it.

Example. To multiply 135 by 12.

The numbers are written down on the *pâṭî* thus:

$$\begin{array}{r}12\\135\end{array}$$

The first digit of the multiplicand (5) is taken and multiplied with the digits of the multiplier. Thus $5\times 2=10$; 0 is written below 2, and 1 is to be carried over.[1] Then $5\times 1=5$; adding 1 (carried over), we get 6. 5 which is no longer required is rubbed out and 6 written in its place. Thus we have

$$\begin{array}{r}12\\1360\end{array}$$

The multiplier is then moved one place towards the left, and we have

$$\begin{array}{r}12\\1360\end{array}$$

Now, 12 is multiplied by 3. The details are: $3\times 2=6$; this 6 added to the figure 6 below 2 gives 12. 6 is rubbed out and 2 substituted in its place. 1 is carried over. Then $3\times 1=3$; 3 plus 1 (carried over)$=4$. 3 is rubbed out and 4 substituted. After the multiplier 12 has been moved another place towards the left, the figures on the *pâṭî* stand thus:

$$\begin{array}{r}12\\1420\end{array}$$

Then, $1\times 2=2$; $2+4=6$; 4 is rubbed out and 6 substituted. $1\times 1=1$, which is placed to the left of 6.

[1] For this purpose it was probably noted in a separate portion of the *pâṭî* by the beginner.

As the operation has ended, 12 is rubbed out and the *pâṭî* has

1620

Thus the numbers 12 and 135 have been *killed*[1] and a new number 1620 is born (*pratyutpanna*).[2]

The reader will note that the position of the multiplier and its motion serve two important purposes, *viz*., (*i*) the last figure of the multiplier indicates the digit of the multiplicand by which multiplication is to be performed and, (*ii*) the product is to be added to the number standing underneath the digit of the multiplier multiplied.

Sometimes the product of a digit of the multiplicand and the multiplier extends beyond the last place of the multiplier. In such cases, the last figure of the partial product is noted separately. The reader should note this fact in the case, 135×99, by performing the operation according to the above process.

The beginner was liable to commit mistakes in such cases, (*i*) of not correctly taking into account the separately noted number, or (*ii*) of rubbing out the digit of the multiplicand beyond the last digit of the multiplier. For these reasons, this process was not in general use and the inverse process was preferred.

Inverse Method: There appear to have been two varieties of the inverse method.

(*a*) In the first the numbers are written thus:

12
135

Multiplication begins with the last digit of the multiplicand. Thus $1 \times 2 = 2$; 1 is rubbed out and 2 substi-

[1] This explains the use of the term *hanana* (killing) and its synonyms for multiplication.

[2] Hence the product was termed *pratyutpanna*.

tuted; then $1 \times 1 = 1$, this is written to the left;[1] the multiplier 12 is moved to the next figure. The work on the *pâṭî* stands thus:

$$\begin{array}{r} 12 \\ 1235 \end{array}$$

Then, $3 \times 2 = 6$; 3 is rubbed out and 6 substituted; then $3 \times 1 = 3$ and $3 + 2 = 5$; 2 is rubbed out and 5 substituted in its place. The multiplier having been moved, the work on the *pâṭî* stands thus:

$$\begin{array}{r} 12 \\ 1565 \end{array}$$

Now, $5 \times 2 = 10$; 5 is rubbed out and 0 substituted in its place; then $5 \times 1 = 5$; $5 + 1 = 6$; $6 + 6 = 12$; 6 is rubbed out and 2 substituted, and 1 is carried over; then $1 + 5 = 6$, 5 is rubbed out and 6 substituted in its place. The *pâṭî* has now,

1620

as the product (*pratyutpanna*). The figures to be carried over are noted down on a separate portion of the *pâṭî* and rubbed out after addition.

(*b*) In the second the partial multiplications (*i.e.*, the multiplications by the digits of the multiplicand) are carried out in the direct manner. These partial multiplications, however, seem to have been carried out in the inverse way, this being the general fashion. The following example will illustrate the method of working:

Example. Multiply 324 by 753

The multiplier and the multiplicand are arranged thus:

$$\begin{array}{l} 753 \\ 324 \end{array}$$

[1] Or the alternative plan: $1 \times 1 = 1$ and then $1 \times 2 = 2$, thus giving 12 in the place of 1 in the multiplicand, etc.

Multiplication begins with the last place of the multiplier. 3×7 gives 21; 1 is placed below the 7 of the multiplier and 2 to its left, thus:

753
21 324

Then 3×5 gives 15; 5 is placed below the 5 of the multiplier and 1 carried to the left; the 1 obtained in the previous step is rubbed out and $(1+1)=2$ is substituted, giving

753
225324

Then 3×3 gives 9; the 3 of the multiplicand is rubbed out and 9 substituted. The work on the *pâṭî* now stands thus:

753
225924

The multiplier is now moved one place to the right giving

753
225924

Then multiplying 7 by 2 we get 14. This 14 being set below the 7 gives

753
239924

Multiplying 5 by 2 and setting the result below it, we obtain

753
240924

Finally multiplying 3 by 2 and rubbing out 2, which is required no longer, and substituting 6 in its place, we get

753
240964

The multiplier is then moved one step further giving

753
240964

Multiplying by 4 the digits of the multiplier 753, and setting the results as before we obtain

(*i*)	753		
	243764	multiplying 7×4 and setting the result;
(*ii*)	753	
	243964	multiplying 5×4 and setting the result;
(*iii*)	753	
	243972	multiplying 3×4 and setting the result.

It may be again remarked that the position and motion of the multiplier play a very important part in the above process. The digits of the multiplier are also successively rubbed out in order to avoid confusion, thus 7 is rubbed out at stage (*i*), 5 at (*ii*) and 3 at (*iii*).

The following variation of the above process is also found:[1]

"Multiplicand 135, multiplier 12; the multiplier placed at the last place of the multiplicand gives

12
135

According to the rule 'the numerals progress to the left' the last figure of the multiplicand (the figure 1) is multiplied by 12. Then after moving (12) we get

12
1235

Again, the figure 3 next to the last of the multiplicand being multiplied by the multiplier 12 gives

12
1265
3

[1] *Lîlâvatyudâharaṇa* by Kṛpârâma Daivajña, Asiatic Society of Bengal, Calcutta, Ms. No. III. F. 110. A.

Then after moving (12) we get

$$\begin{array}{r} 12 \\ 1265 \\ 3 \end{array}$$

Again, multiplying the first figure 5 of the multiplicand with the multiplier 12, we get

$$\begin{array}{r} 12 \\ 1260 \\ 36 \end{array}$$

Then rubbing out the multiplier, the numbers

$$\begin{array}{r} 1260 \\ 36 \end{array}$$

being added according to places give 1620."

Transmission to the West. The *kapâṭa-sandhi* method of multiplication was transmitted to the Arabs who learnt the decimal arithmetic from the Hindus. It occurs in the works of Al-Khowârîzmî (825), Al-Nasavî[1] (*c.* 1025) Al-Haṣṣâr[2] (*c.* 1175), Al-Kalasâdî[3] (*c.* 1475) and many others. The following illustration is taken from the work of Al-Nasavî who calls this method *al-amal al-hindî* and *târik al-hindî* ("the method of the Hindus"):

Example. To multiply 324×753

$$\begin{array}{cccccc} & 4 & 3 & & & \\ & \cancel{3} & \cancel{0} & 9 & & \\ & \cancel{2} & \cancel{9} & \cancel{7} & 7 & \\ 2 & \cancel{1} & \cancel{5} & \cancel{9} & \cancel{6} & 2 \\ & & & 3 & 2 & 4 \\ & 7 & 5 & 3 & & \\ & & 7 & 5 & 3 & \\ & & & 7 & 5 & 3 \end{array} \quad \therefore \text{ Product} = 243972.$$

[1] F. Woepeke, I (6), p. 407.

[2] H. Suter, *Bibl. Math.*, II (3), p. 16.

[3] *Ibid*, p. 17.

In the above the arrangement of the multiplicand and multiplier is just the same as in the Hindu method. The multiplier is moved in the same way. As the work is performed on paper, the figures are crossed out instead of being rubbed out.

It may be mentioned that in Europe, the method is found reproduced in the work of Maximus Planudes.

Gelosia Method. The method known as the 'gelosia',[1] has been described in the *Gaṇita-mañjarî* (16th century) as the *kapâṭa-sandhi* method. It appears also in Gaṇeśa's commentary on the *Lîlâvatî*. As the description of the *kapâṭa-sandhi* given by the older mathematicians is incomplete and sketchy, it is difficult to say whether Gaṇeśa is right in identifying the gelosia method with the *kapâṭa-sandhi* of older writers. In our opinion Gaṇeśa's identification is incorrect.[2]

We are at present unable to say definitely whether this method is a Hindu invention or was borrowed from the Arabs who are said to have used it in the 13th century.[3] It occurs in some Arab works of the 14th century, and also in Europe about the same time. Gaṇeśa was undoubtedly one of the best mathematicians of his time and the fact that he identified this method with the *kapâṭa-sandhi* which is the oldest known method shows that the gelosia method must have been in use in India from a long time before him.

The only available description of the method runs as follows:

"(Construct) as many compartments as there are places in the multiplicand and below these as many

[1] We shall designate it as *kapâṭa-sandhi* (*b*) method.

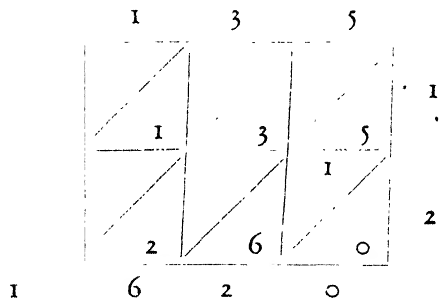
[2] *Cf.* the quotation from Śrîpati given before, p. 137.

[3] Smith, *History*, II, p. 115.

as there are places in the multiplier; the oblique lines in the first, in the one below, and in the other (compartments) are produced. Multiply each place of the multiplicand, by the places of the multiplier (which are) one below the other and set the results in the compartments. The sum taken obliquely on both sides of the oblique lines in the compartments gives the product. This is the *kapâṭa-sandhi*."[1]

The following illustration is taken from Gaṇeśa's commentary on the *Lîlâvatî*:

To Multiply 135 by 12



Cross Multiplication Method. This method has been mentioned by Śrîdhara, Mahâvîra, Śrîpati and some later writers as the *tastha* method. These writers, however, do not explain the method. Śrîdhara simply states: "The next (method) in which (the multiplier) is stationary is the *tastha*."[2] The method is algebraic and has been compared to *tiryak-guṇana* or *vajrâbhyâsa* (cross multiplication) used in algebra.[3] It has been explained by Gaṇeśa (*c.* 1545) thus:

[1] Translated from the *Gaṇita-mañjarî* of Gaṇeśa, son of Ḍhuṇḍhirâja.

[2] *Triś*, p. 3.

[3] Colebrooke, *l.c.*, p. 171, fn. 5.

"That method of multiplication in which the numbers stand in the same place,[1] is called *tastha-guṇana.* It is as follows: after setting the multiplier under the multiplicand multiply unit by unit and note the result underneath. Then as in *vajrâbhyâsa* multiply unit by ten and ten by unit, add together and set down the result in the line. Next multiply unit by hundred, hundred by unit and ten by ten, add together and set down the result as before; and so on with the rest of the digits. This being done, the line of results is the product."[2]

This method was known to the Hindu scholars of the 8th century, or earlier. The method seems to have travelled to Arabia and thence was transmitted to Europe, where it occurs in Paciōli's *Suma*[3] and is stated to be "more fantastic and ingenious than the others." Gaṇeśa has also remarked that "this (method) is very fantastic and cannot be learnt by the dull without the traditional oral instructions."

Multiplication by Separation of Places. This method of multiplication known as *sthâna-khaṇḍa*, is based on the separation of the digits of the multiplicand or of the multiplier. It has been mentioned in all the works from 628 A.D. onwards. Bhâskara II describes the method as follows:

"Multiply separately by the places of figures and add together."[4]

With reference to the example 135×12, Bhâskara II explains the method thus:

[1] In contra-distinction to the method in which the multiplier moves from one place to another.

[2] Gaṇeśa's commentary on the *Līlâvatī*, i, 4-6.

[3] Smith (*l.c.*, II, p. 112) quotes from this work.

[4] *L*, p. 3.

"Taking the digits separately, *viz.*, 1 and 2, the multiplicand being multiplied by them severally, and the products added together according to places, the result is 1620."

Various arrangements appear to have been employed for writing down the working. Some of these are given below:

(*i*)[1]

```
  135
 12
 ----
 12
  36
   60
 ----
 1620
```

(*ii*)[2]

```
12 12 12
 1  3  5
--------
 1260
  36
--------
 1620
```

(*iii*)[3]

```
135  135
  1    2
--------
   270
  135
--------
  1620
```

Zigzag Method. The method is called *gomûtrikâ.*[4] It has been described by Brahmagupta. It is in all

[1] In a manuscript used by Taylor, see his *Lîlâwatî*, pp. 8-9.

[2] This arrangement is found in the commentary of Gaṅgâdhara on the *Lîlâvatî*, in the library of the Asiatic Society of Bengal, Calcutta.

[3] Found in Gaṅgâdhara, *l.c.*

[4] The word *gomûtrikâ*, means "similar to the course of cow's urine," hence "zigzag." Colebrooke's reading *gosûtrikâ* is incorrect. The method of multiplication of astronomical quantities is called *gomûtrikâ* even upto the present day by the paṇḍits.

essentials the same as the *sthâna-khaṇḍa* method. The following illustration is based on the commentary of Pṛthudakasvâmî.

Example. To multiply 1223 by 235

The numbers are written thus:

```
2   1223
3    1223
5     1223
```

The first line of figures is then multiplied by 2, the process beginning at the units place, thus: $2 \times 3 = 6$; 3 is rubbed out and 6 substituted in its place, and so on. After all the horizontal lines have been multiplied by the corresponding numbers on the left in the vertical line, the numbers on the *pâṭî* stand thus:

```
2 4 4 6
  3 6 6 9
    6 1 1 5
2 8 7 4 0 5
```

after being added together as in the present method.

The *sthâna-khaṇḍa* and the *gomûtrikâ* methods resemble the modern plan of multiplication most closely. The *sthâna-khaṇḍa* method was employed when working on paper.

Parts Multiplication Method. This method is mentioned in all the Hindu works from 628 A.D. onwards. Two methods come under this head:

(*i*) The multiplier is broken up into two or more parts whose sum is equal to it. The multiplicand is then multiplied severally by these and the results added.[1]

(*ii*) The multiplier is broken up into two or more aliquot parts. The multiplicand is then multiplied by

[1] Thus $12 \times 135 = (4+8) \times 135 = (4 \times 135) + (8 \times 135)$.

one of these, the resulting product by the second and so on till all the parts are exhausted. The ultimate product is the result.[1]

These methods are found among the Arabs and the Italians, having been obtained from the Hindus. They were known as the "Scapezzo" and "Repiego" methods respectively among the Italians.[2]

Algebraic Method. This method was known as *iṣṭa-guṇana*. Brahmagupta's description of the method has been already quoted. Bhâskara II explains it thus:

"Multiply by the multiplicator diminished or increased by an assumed number, adding or subtracting (respectively) the product of the multiplicand and the assumed number."[3]

This is of two kinds according as we (*i*) add or (*ii*) subtract an assumed number. The assumed number is so chosen as to give two numbers with which multiplication will be easier than with the original multiplier. The two ways are illustrated below:

(*i*) $$135 \times 12 = 135 \times (12+8) - 135 \times 8$$
$$= 2700 - 1080 = 1620$$

(*ii*) $$135 \times 12 = 135 \times (12-2) + 135 \times 2$$
$$= 1350 + 270 = 1620$$

This method was in use among the Arabs[4] and in Europe[5], obviously under Hindu influence.

[1] Thus $12 \times 135 = 3 \times 135 \times 4$.

[2] Smith, *History*, II, p. 117.

[3] *L*, p. 3.

[4] *E.g.*, Behâ Eddin (*c.* 1600). See G. Eneström, *Bibl. Math.*, VII (3), p. 95.

[5] *E.g.*, Widman (1489), Riese (1522), etc. See Smith, *l.c.*, p. 120.

8 DIVISION

Terminology. Division seems to have been regarded as the inverse of multiplication. The common Hindu names for the operation are *bhâgahâra*, *bhâjana*, *haraṇa*, *chedana*, etc. All these terms literally mean "to break into parts," *i.e.*, "to divide," excepting *haraṇa* which denotes "to take away." This term shows the relation of division to subtraction. The dividend is termed *bhâjya*, *hârya*, etc., the divisor *bhâjaka*, *bhâgahara* or simply *hara*, and the quotient *labdhi* "what is obtained" or *labdha*.

The Operation. Division was considered to be a difficult and tedious operation by European scholars even as late as the 15th and 16th centuries;[1] but in India the operation was not considered to be difficult, as the most satisfactory method of performing it had been evolved at a very early period. In fact, no Hindu mathematician seems to have attached any great importance to this operation. Âryabhaṭa I does not mention the method of division in his work. But as he has given the modern methods for extracting square- and cube-roots, which depend on division,[2] we conclude that the method of division was well-known in his time and was not described in the *Âryabhaṭîya* as it was considered to be too elementary. Most Siddhânta writers have followed Âryabhaṭa in excluding the process of division from their works, *e.g.*, Brahmagupta (628), Śrîpati (1039), and some others.

A method of division by removing common factors, seems to have been employed in India before the invention of the modern plan. This removal of common

[1] Smith, *l.c.*, p. 132.
[2] He has used the technical term *labdha* for the quotient.

factors is mentioned in early Jaina works.[1] It has been mentioned by Mahâvîra who knew the modern method, probably because it was considered to be suitable in certain particular cases:

"Putting down the dividend and below it the divisor, and then, having performed division by the method of removing common factors, give out the resulting (quotient)."[2]

The modern method of division is not found in the Bakhshâlî Manuscript, although the name of the operation is found at several places. The absence of the method may be due to the mutilated form of the text, although it is quite possible that the method was not known at that early period (200 A.D.).

The Method of Long Division. The modern method of division is explained in the works on *pâṭîgaṇita*, the earliest of which, Śrîdhara's *Triśatikâ*, gives the method as follows:[3]

"Having removed the common factor, if any, from the divisor and the dividend, divide by the divisor (the digits of the dividend) one after another in the inverse[4] order."

Mahâvîra says:[5]

"The dividend should be divided by the divisor (which is) placed below it, in the inverse order, after having performed on them the operation of removing common factors."

[1] *Tatvârthâdhigama-sûtra*, *Bhâṣya* of Umâsvâti (*c.* 160, ed. by H. R. Kapadia, Bombay, 1926, Part I, ii. 52, p. 225.

[2] *GSS*, p. 11. The method would not give the quotient unless the dividend be completely divisible by the divisor.

[3] *Triś*, p. 4.

[4] *Pratiloma.*

[5] *GSS*, p. 11; *cf.* Rangacarya's translation.

Âryabhaṭa II gives more details of the process:[1]

"Perform division having placed the divisor below the dividend; subtract from (the last digits of the dividend) the proper multiple of the divisior; this (the multiple) is the partial quotient, then moving the divisor divide what remains, and so on."

Bhâskara II,[2] Nârâyaṇa[3] and others give the same method.

The following example will serve to illustrate the Hindu method of performing the operation on a *pâṭî*:

Example. Divide 1620 by 12.

The divisor 12 is placed below the dividend thus :

1620
12

The process begins from the extreme left of the dividend, in this case the figure 16. This 16 is divided by 12. The quotient 1 is placed in a separate line, and 16 is rubbed out and the remainder 4 is substituted in its place. The subtraction is made by rubbing out figures successively as each figure of the product to be subtracted is obtained. Thus, the partial quotient 1, being written, the procedure is

1620	1
12	line of quotients

$1 \times 1 = 1$, so 1 of the dividend is rubbed out (as $1 - 1 = 0$); then $1 \times 2 = 2$, so 4 is substituted in the place of 6 (as $6 - 2 = 4$). The figures on the *pâṭî* are:

420	1
12	line of quotients

[1] *MSi*, p. 144.

[2] Bhâskara gives the process briefly as follows: "That number, by which the divisor being multiplied, balances the last digit of the dividend gives the (partial) quotient, and so on." (*L*, p. 3)

[3] *GK*, i. 16.

The divisor 12 is now moved one place to the right giving

420	1
12	line of quotients

42 is then divided by 12. The resulting quotient 3 is set in the "line of quotients," 42 is rubbed out and the remainder 6 substituted in its place. The figures now stand thus:

60	13
12	line of quotients

Moving the divisor one place to the right, we have

60
12

On division being performed, as before the resulting quotient 5 is set in the "line of quotients" and 60 is rubbed out leaving no remainder. The line of quotients[1] has

135

which is the required result.

The above process, when the figures are not obliterated and the successive steps are written down one below the other, becomes the modern method of long division.

The method seems to have been invented in India about the 4th century A.D., if not earlier. It was transmitted to the Arabs, where it occurs in Arabic works from the 9th century onwards.[2] From Arabia the method travelled to Europe where it came to be known as the galley (*galea*, *batello*) method.[3] In this variation

[1] The "line of quotients" was usually written above the dividend.

[2] Al-Khowârîzmî (*c.* 825), Al-Nasavî (*c.* 1025); cf. Smith, *l.c.*, pp. 138-139.

[3] Also called the 'scratch method'.

of the method, the figures obtained at successive stages are written and crossed out, for the work is carried out on paper (where the figures cannot be rubbed out). The method was very popular in Europe from the 15th to the 18th century.[1] The above example worked on the galley plan would be represented thus:

$$
\begin{array}{lrl}
 & 4 & \\
\mathrm{I} & \cancel{1}\cancel{6}20 & 1 \\
 & \cancel{1}\cancel{2}2 & \\
 & 1 &
\end{array}
$$

$$
\begin{array}{lrl}
\mathrm{II} & \cancel{1} & \\
 & \cancel{4}6 & \\
 & \cancel{1}\cancel{6}\cancel{2}0 & 13 \\
 & \cancel{1}\cancel{2}\cancel{2}2 & \\
 & \cancel{1}1 &
\end{array}
$$

$$
\begin{array}{lrl}
\mathrm{III} & \cancel{1}\cancel{1} & \\
 & \cancel{4}\cancel{6} & \\
 & \cancel{1}\cancel{6}\cancel{2}\cancel{0} & 135 \\
 & \cancel{1}\cancel{2}\cancel{2}\cancel{2} & \\
 & \cancel{1} &
\end{array}
$$

Comparing the successive crossing out of the figures in I, II and III, with the rubbing out of figures in the corresponding steps according to the Hindu plan, it becomes quite clear that the galley method is exactly the same as the Hindu method. The crossing out of figures appears to be more cumbrous than the elegant Hindu plan of rubbing out.

The Hindu plan of moving the divisor as the digits of the quotient were evolved, although not essential, was also copied and occurs in the works of such well-known Arab writers as Al-Khowârîzmî (825), Al-Nasavî (*c.* 1025) and others. The mediæval Latin writers called this feature the *antirioratio*.

[1] For details see Smith, *l.c.*, pp. 136-139.

6. SQUARE

Terminology. The Sanskrit term for square is *varga* or *kṛti*. The word *varga* literally means "rows" or "troops" (of similar things). But in mathematics it ordinarily denotes the square power and also the square figure or its area. Thus Āryabhaṭa I says:[1]

"A square figure of four equal sides[2] and the (number representing its) area are called *varga*. The product of two equal quantities is also *varga*."

How the word *varga* came to be used in that sense has been clearly indicated by Thibaut. He says: "The origin of the term is clearly to be sought for in the graphical representation of a square, which was divided in as many *vargas* or troops of small squares, as the side contained units of some measure. So the square drawn with a side of five *padas* length could be divided into five small *vargas* each containing five small squares, the side of which was one *pada* long."[3] This explanation of the origin of the term *varga* is confirmed by certain passages in the *Śulba* works.[4]

The term *kṛti* literally means "doing," "making" or "action." It carries with it the idea of specific performance, probably the graphical representation.

Both the terms *varga* and *kṛti* have been used in the mathematical treatises, but preference is given to the term *varga*. Later writers, while defining these terms in arithmetic, restrict its meaning. Thus Śrīdhara says:[5]

[1] *A*, ii. 3.

[2] The commentator Parameśvara remarks: "That four sided figure whose sides are equal and both of whose diagonals are also equal is called *samacaturaśra* ("square")."

[3] Thibaut, *Śulba-sūtras*, p. 48.

[4] *ĀpŚl*, iii. 7; *KŚl*, iii. 9; cf. B. Datta, *American Math. Monthly*, XXXIII, 1931, p. 375.

[5] *Triś*, p. 5.

"The product of two equal numbers is *varga.*"

Pṛthudakasvâmî[1], Mahâvîra[2] and others give similar definitions.

The Operation. The occurrence of squaring as an elementary operation is characteristic of Hindu arithmetic. The method, however, is not simpler than direct multiplication. It was given prominence by the Hindu writers probably because the operation of square-root is the exact inverse of that of squaring. Although the method first occurs in the *Brâhma-sphuṭa-siddhânta*, there is no doubt that it was known to Âryabhaṭa I as he has given the square-root method.

Brahmagupta gives the method[3] very concisely thus:

"Combining the product, twice the digit in the less[4] (lowest) place into the several others (digits), with its (*i.e.*, of the digit in the lowest place) square (repeatedly) gives the square."

Śrîdhara (750) is more explicit:[5]

"Having squared the last digit multiply the rest of the digits by twice the last; then move the rest of the digits. Continue the process of moving (the remain-

[1] *Cf.* Colebrooke, *l.c.*, p. 279.

[2] *GSS*, p. 12.

[3] The method is not mentioned in the chapter on Arithmetic, but seems to have been mentioned as an afterthought in the form of an appendix, (*BrSpSi*, p. 212).

[4] *Râśerûnaṃ* has been translated by Colebrooke as "the less portion." This translation is incorrect. He says that "the text is obscure" (p. 322, fn. 9), for according to his translation the rule becomes practically meaningless. The term *râśerûnaṃ* must be translated by "the digit in the lowest place." Dvivedi agrees with the above interpretation (p. 212). The method taught here is "the direct method of squaring."

[5] *Triś*, p. 5. The translation given by Kaye and Ramanujacharia is incorrect. (*Bibl. Math.*, XIII, 1912-13).

ing digits after each operation) to obtain the square."

Mahâvîra[1] (850) gives more details:

"Having squared the last (digit), multiply the rest of the digits by twice the last, (which is) moved forward (by one place). Then moving the remaining digits continue the same operation (process). This gives the square."

Bhâskara II[2] writes:

"Place the square of the last (digit) over itself; and then the products of twice the last (digit) and the others (*i.e.*, the rest) over themselves respectively. Next, moving the number obtained by leaving the last digit (figure), repeat the procedure."

He has remarked that the above process may be begun also with the units place.[3]

The following is the method of working on the *pâṭî*, the process beginning from the last place, according to Śrîdhara, Mahâvîra, Bhâskara II and others:

To square 125.

The number is written down,

125

The last digit is 1. Its square is placed over itself.

1
125

Then twice the last digit $2 \times 1 = 2$; placing it below the rest of the figures (below 2 or below 5 according as the direct or inverse method of multiplication is used)

[1] *GSS*, p. 12.
[2] *L*, p. 4.
[3] *L*, p. 5.

and rubbing out the last digit 1, the work on the *pâṭî* appears as

1
25
2

Performing multiplication by 2 (below) and placing the results over the respective figures, we get

150
25

One round of operation is completed. Next moving the remaining digits, *i.e.*, 25, we have

150
 25

Now, the process is repeated, *i.e.*, the square of the last digit (2) is placed over itself giving

154
 25

Then, placing twice the last digit (*i.e.*, $2\times2=4$) below the rest of the digits and then rubbing out 2, we have

154
 5
 4

Performing multiplication, $4\times5=20$, and placing it over the corresponding figure 5, (*i.e.*, 0 over 5 and 2 carried to the left), the work on the *pâṭî* appears as

1560
 5

Thus a second round of operations is completed.

Then moving 5 we have

1560
 5

Squaring 5 we get 25, and placing it over 5 (*i.e.*, 5 over 5 and 2 carried to the left) we have

15625
5

As there are no 'remaining figures' the work ends. 5 being rubbed out, the *pâṭî* has

15625,

the required square.

According to Brahmagupta and also Bhâskara II, the work may begin from the lowest place (*i.e.*, the units place). The following method is indicated by Brahmagupta:

To square 125.

The number is written down

125

The square of the digit in the least place, *i.e.*, $5^2=25$ is set over it thus:

25
125

Then, $2\times5=10$ is placed below the other digits, and five is rubbed out, thus:

25
12
10

Multiplying by 10 the rest of the digits, *i.e.*, 12, and setting the product over them (the digits), we have

1225
12
10

Then rubbing out 10 which is not required and moving the rest of the digits, *i.e.*, 12, we have

1225
12

Thus one round of operations is completed.

Again, as before, setting the square of 2 above it and $2\times 2=4$ below 1, we have

$$\begin{array}{c}1625\\1\\4\end{array}$$

Multiplying the remaining digit 1 by 4, and setting the product above it, we have

$$\begin{array}{c}5625\\1\end{array}$$

Then, moving the remaining digit 1, we obtain

$$\begin{array}{c}5625\\1\end{array}$$

Thus the second round of operations is completed.

Next setting the square of 1 above it the process is completed, for there are no remaining figures, and the result stands thus:

$$15625$$

Minor Methods of Squaring. The identity

$$(i) \qquad n^2=(n-a)(n+a)+a^2$$

has been mentioned by all Hindu mathematicians as affording a suitable method of squaring in some cases. For instance,

$$15^2=10\times 20+25=225.$$

Brahmagupta says:

"The product of the sum and the difference of the number (to be squared) and an assumed number plus the square of the assumed number give the square."[1]

Śrîdhara (750) gives it thus:

"The square is equal to the product of the sum and the difference of the given number and an assumed

[1] *BrSpSi*, p. 212.

quantity plus the square of the assumed quantity."[1] Mahâvîra, Bhâskara II, Nârâyaṇa and others also give this identity.

The formula

(ii) $(a+b)^2 = a^2+b^2+2ab,$

or its general form

$$(a+b+c+\ldots)^2 = a^2+b^2+c^2+\ldots+2ab+\ldots$$

has been given as a method of squaring. Thus Mahâvîra[2] says:

"The sum of the squares of the two or more portions[3] of the number together with their products each with the others multiplied by two gives the square."

Bhâskara II[4] gives:

"Twice the product of the two parts plus the square of those parts gives the square."

The formula

(iii) $n^2=1+3+5+\ldots$ to n terms

has been mentioned by Srîdhara and Mahâvîra.

Srîdhara[5] says:

"(The square of a number) is the sum of as many terms in the series of which one is the first term and two the common difference."

[1] *Triś*, p. 5.

[2] *GSS*, p. 13.

[3] The word *sthâna* has been used in the original. This word has been generally used in the sense of 'notational place.' Following the commentator, we have rendered it by "portion." As a given number, say, 125, can be broken into parts as 50+40+35 or as 100+20+5, and as the rule applies to both, it is immaterial whether the word '*sthâna*' is translated by 'place' or 'portion.' This rule appears to have been given as an explanation of the Hindu method of squaring used with the place-value numerals.

[4] *L*, p. 4. [5] *Triś*, p. 5.

Mahâvîra uses a similar expression:

"The sum of as many terms of the series of which one is the first term and two the common difference is the square (of the given number)."[1]

Nârâyaṇa[2] has also given the above series. He adds the formula.[3]

$$A^2 = (a+b)^2 = (a-b)^2 + 4ab$$

to find the square of the given number A.

It should be observed that all the above methods of squaring are applied to whole numbers only. The methods of squaring fractions are dealt with in the section on fractions.

7. CUBE

Terminology. The Hindu term for the cube is *ghana*. This term occurs in all the mathematical works. It has been used in the geometrical as well as the arithmetical sense, *i.e.*, to denote the solid cube as well as the continued product of the same number taken three times. Thus Âryabhaṭa I says:

"The continued product of three equals and also the solid having twelve (equal) edges are called *ghana*."[4]

Śrîdhara,[5] Mahâvîra[6] and Bhâskara II[7] each state:

"The continued product of three equal numbers is *ghana*."

[1] *GSS*, p. 12.
[2] *GK*, i. 17-18.
[3] He says: "The square of the difference plus four times the product is the square of the sum."
[4] *Â*, ii. 3.
[5] *Triś*, p. 6.
[6] *GSS*, p. 14.
[7] *L*, p. 5.

Another term for the cube is *bṛnda*, but it is seldom used.

The Operation. A method of cubing applicable to numbers written in the decimal place-value notation, has been in use in India from before the fifth century A.D. Aryabhaṭa I (499) knew the method, although he did not consider it to be as important as the inverse operation of extracting the cube-root which he has explained.

Brahmagupta gives the method in these words:

"Set down the cube of the last;[1] then place at the next place from it, thrice the square of the last multiplied by the succeeding; then place at the next place thrice the square of the succeeding multiplied by the last; and (at the next place) the cube of the succeeding. This gives the cube."

Srîdhara[2] says:

"Set down the cube of the last; then set down at the next place[3] the square of the last multiplied by

[1] The commentator Pṛthudakasvâmî takes 'last' (*antya*) to mean the figure or figures on the extreme left. The number to be cubed is divided into portions of which the cubes are found successively by the application of the process. (See illustration). Thus 'succeeding' would mean here the figure to the right of the 'last.' It does not, however, make any difference if we take 'last' to mean the figure in the units place. The words used by Brahmagupta are '*antya*,' for last and '*uttara*' for succeeding. The rule is:

Sthâpyo'ntya ghano'ntya kṛtistriguṇottarsaṅguṇâ ca tatprathamât ।
Uttarakṛtirantyaguṇâ triguṇâ cottaraghanaśca ghanaḥ ॥

[2] *Triś*, p. 6.

[3] *Sthânâdhikyam*, lit. "increasing one place." This increase is to be made by placing the result one place to the right of the previously noted figures (see illustration). Brahmagupta uses the word *tatprathamât*, i.e., "before that."

thrice the succeeding;[1] then (at the next place) the product of the square of the succeeding and last multiplied by three; and then (at the next place) the cube of the succeeding."

Mahâvîra states:[2]

"The cube of the last, the product of thrice its square and the remaining, the square of the remaining multiplied by thrice the last; placing of these, each one place before the other, constitutes here the process."

Bhâskara II is more explicit:[3]

"Set down the cube of the last; then the square of the last multiplied by three times the succeeding; then the square of the succeeding multiplied by three times the last and then the cube of the succeeding; these placed so that there is difference of a place between one result and the next,[4] and added give the cube. The given number is distributed into portions according to places, one of which is taken for the last and the next as the first and in like manner repeatedly (if there be occasion). Or the same process may be begun from the first place of figures for finding the cube."

The method may be illustrated by the following example:

[1] *Pûrva, âdi,* lit. "preceding". We have rendered them by "succeeding" to be in conformity with the general convention so as to avoid confusion.

[2] *GSS*, p. 15 (47).

It will be observed that the "addition of the cube of the remaining" does not occur in the rule. This has to be understood from the previous stanza which says that the cubes of all the parts are to be added. See the translation of the previous stanza given on pp. 166f.

[3] *L*, p, 5.

[4] *Sthânântaratvena* has been translated by Colebrooke by "according to places." This translation is incorrect and does not give the true significance of the term.

Example. To cube 1234.

The given number has four places, *i.e.*, four portions. First we take the last digit 1 and the succeeding digit 2, *i.e.*, 12, and apply the method of cubing thus :

(i) Cube of the last (1^3)	=	1	
(ii) Thrice the square of the last (3.1^2) multiplied by the succeeding (2) gives ($2.3.1^2$)	=	6	(placing at the next place)
(iii) Thrice the square of the succeeding multiplied by the last gives ($3.2^2.1$)	=	12	(placing at the next place)
(iv) Cube of the succeeding (2^3)	=	8	(placing at the next place)
Thus 12^3 is the sum		1728	

After this we take the next figure 3, *i.e.*, the number 123, and in this consider 12 as the last and 3 as the succeeding. Then the method proceeds thus:

(i) The cube of the last (12^3) as already obtained	=	1728	
(ii) Thrice the square of the last multiplied by the succeeding, *i.e.*, $3.12^2.3$	=	1296	(placing at the next place).
(iii) Thrice the square of the succeeding multiplied by the last, *i.e.*, $3.3^2.12$	=	324	,, ,,
(iv) Cube of the succeeding, *i.e.*, 3^2	=	27	,, ,,
Thus 123^3 is the sum		1860867	

Now the remaining figure 4 is taken, so that the number is 1234, of which 123 is the last and 4 the succeeding. The method proceeds thus:

(i) Cube of the last, *i.e.*, $(123)^3$ as already obtained	=	1860867	
(ii) Thrice the square of the last into the succeeding, *i.e.*, $3.(123)^2.4$	=	181548	(placing at the next place)
(iii) Thrice the square of the succeeding into the last, *i.e.*, $3.4^2.123$	=	5904	" "
(iv) Cube of the last *i.e.*, 4^3	=	64	" "
Thus $(1234)^3$ is the sum		1879080904	

The direct process—that in which the operation begins with the units place—can be similarly performed.

Minor Methods of Cubing. The formula

$$(i) \qquad (a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

and the corresponding result

$$(a+b+c+\dots)^3 = a^3+3a^2(b+c+\dots)+3a(b+c+\dots)^2 + (b+c+\dots)^3$$

are implied in the Hindu method of cubing given above. Mahâvîra[1] gives the following explanation:

"The squares of the last place[2] and the next[3] are taken, and each (square) is multiplied by the other and by three. The sum of these products and the cubes of both (*lit.* all) the places is the cube; the

[1] *GSS*, p. 15.

[2] *Sthâna*, meaning the number represented by the figure standing in that place.

[3] *Anya*, lit. "other," meaning the number represented by the figures standing in the other places.

procedure is repeated (if necessary).[1]

Śrîpati and Bhâskara II[2] state the formula in the form

$$(a+b)^3 = a^3+3ab\,(a+b)+b^3$$

"Thrice the given number multiplied by its two parts, added to the sum of the cubes of those parts, gives the cube."

Nârâyaṇa[3] says

"Thrice the (given) number multiplied by both parts, added to the cubes of the parts, is the cube of the sum."

The formula

$$(ii) \qquad n^3 = n(n+a)(n-a)+a^2(n-a)+a^3$$

has been mentioned by Mâhâvîra[4] in these words:

"The continued product of the given number, the sum and the difference of the given number and an arbitrary quantity, when added to the smaller of these multiplied by the square of the arbitrary number, and the cube of the arbitrary number, give the cube (of the given number)."

Expressions for n^3 involving series have been given by Śrîdhara, Mahâvîra, Śrîpati and Nârâyaṇa. The formula

$$(iii) \qquad n^3 = \sum_{1}^{n} \left\{ 3r\,(r-1)+1 \right\}$$

[1] Thus $(234)^3$ is considered as

$$(200+30+4)^3=(200)^3+3.200^2(30+4) + 3.200(30+4)^2 + (30+4)^3$$

Then the procedure is repeated for obtaining $(30+4)^3$. *Cf.* English translation, p. 17, note.

[2] *GT*, 27; *L*, p. 5.

[3] *GK*, i. 23.

[4] *GSS*, p. 15.

is given by Srîdhara in these words:

"The cube (of a given number) is equal to the series whose terms are formed by applying the rule, 'the last term multiplied by thrice the preceding term plus one,' to the terms of the series whose first term is zero, the common difference is one and the last term is the given number."[1]

Mahâvîra gives the above in the form

$$n^3 = 3\sum_{2}^{n} r(r-1)+n$$

He[2] says:

"In the series, wherein one is the first term as well as the common difference and the number of terms is equal to the given number (n), multiply the preceding term by the immediately following one. The sum of the products so obtained, when multiplied by three and added to the last term (*i.e.*, n) becomes the cube (of n)."

Nârâyaṇa[3] states:

"From the series whose first term and common difference are each one, (the last term being the given number) the sum of the series formed by the last term multiplied by three and the preceding added to one, gives the cube (of the last term)."

Mahâvîra has also mentioned the results,

(*iv*) $x^3 = x+3x+5x+ \ldots.$ to x terms,

(*v*) $x^3 = x^2+(x-1)\{1+3+\ldots.+(2x-1)\}$,

[1] *Triś*, p. 6. The translation given by Kaye and Ramanujacharia (*Bibl.Math.*, III, 1912-13) is incorrect. They admit their inability to follow the meaning (see p. 209, note). S. Dvivedi has misinterpreted the rule, and gives an incorrect explanation in a note on p. 6. The reading *saike* is incorrect.

[2] *GSS*, ii. 45.

[3] *GK*, i. 22.

in these words:[1]

"The cube (of a given number) is equal to the sum of the series whose first term is the given number, the common difference is twice that number, and the number of terms is (equal to) that number.

"Or the square of the given number when added to the product of that number minus one (and) the sum of the series in which the first term is one, the common difference two and the number of terms (is equal to) that number, gives the cube."

8. SQUARE-ROOT

Terminology. The Hindu terms for the "root" are *mûla* and *pada*. The usual meaning of the word *mûla* in Sanskrit literature is "root" of a plant or tree; but figuratively the foot or lowest part or bottom of anything. Its other meanings are "basis," "foundation," "cause," "origin," etc. The word *pada* means "the lower part of the leg" (figuratively the lower part or basis of anything), "foot," "part," "portion," "side," "place," "cause," "a square on a chess-board," etc. The meanings common to both terms are "foot," "the lowest part or basis of anything," "cause" or "origin." It is, therefore, quite clear that the Hindus meant by the term *varga-mûla* ("square-root") "the cause or origin of the square" or "the side of the square (figure)." This is corroborated by the following statement of Brahmagupta:[2]

"The *pada* (root) of a *kṛti* (square) is that of which it is the square."

Of the above terms for the "root," *mûla* is the oldest. It occurs in the *Anuyogadvâra-sûtra* (*c.* 100 B.C.),

[1] *GSS*, ii. 44.
[2] *BrSpSi*, xviii. 35.

and in all the mathematical works. The term *pada* seems to have come into use later on, *i.e.*, from the seventh century A.D. It occurs first in the work of Brahmagupta (628).

The term *mûla* was borrowed by the Arabs who translated it by *jadhr*, meaning "basis of square." The Latin term *radix* also is a translation of the term *mûla*[1].

The word *karanî* is found to have been used in the *Śulba* works and Prâkrta literature as a term for the square-root. In geometry it means a "side." In later times the term is, however, reserved for a surd, *i.e.*, a square-root which cannot be evaluated, but which may be represented by a line.

The Operation. The description of the method of finding the square-root is given in the *Âryabhaṭîya* very concisely thus:

"Always divide[2] the even place by twice the square-root (upto the preceding odd place); after having subtracted from the odd place the square[3] (of the quotient), the quotient put down at the next place (in the line of the root) gives the root."[4]

The method may be illustrated thus:

Example. Find the square-root of 54756.

[1] For further details see Datta, *American Math. Monthly*, XXXIV, pp. 420-423, also XXXVIII, pp. 371-376.

[2] In dividing, the quotient should be taken as great as will allow of the subtraction of its square from the next odd place. This is the force of the Sanskrit text as pointed out by the commentators Bhâskara I, Nîlakantha and others.

[3] The "square" is mentioned and not the "square of the quotient," as in the beginning the greatest possible square is to be subtracted, there being no quotient.

[4] *Â*, ii. 4. Translations of the rule have been given before by Rodet (*JA*, 1880, II), Kaye (*JASB*, 1907 and 1908, III and IV resp.), Singh (*BCMS*, 1927, XVIII), Clark (*Âryabhaṭîya*) and others. Of these Kaye's translation is entirely incorrect.

The odd and even places are marked by vertical and horizontal lines. The different steps are then as indicated below:

	| ‾ | ‾ | 5 4 7 5 6	
Subract square	4	root=2
Divide by twice the root	4) 14 (3 12	Placing quotient at the next place, the root =23
Subtract square of quotient	27 9	
Divide by twice the root	46) 185 (4 184	Placing quotient at the next place, the root =234
Subtract square of quotient	16 16	

The process ends. The root is 234.

It has been stated by G. R. Kaye[1] that Âryabhaṭa's method is algebraic in character, and that it resembles the method given by Theon of Alexandria. Both his statements are incorrect.[2]

The following quotations from Siddhasena Gaṇi (*c.* 550) in his commentary on the *Tatvârthâdhigama-sûtra*[3] will prove conclusively that the Hindu method of extracting the square-root was not algebraic. In connection with the determination of the circumference of a circle of 100,000 *yojanas*, he says:

"The diameter is one hundred thousand *yojanas*; that multiplied by one hundred thousand *yojanas* becomes squared; this is again multiplied by 10 and then

[1] *JASB*, III and IV, in the papers entitled "Notes on Indian Mathematics, I and II."

[2] See Singh, *l.c.*, for details: also Clark, *Âryabhaṭîya*, pp. 23f.

[3] iii. 11.

the square-root (of the product) extracted. The root will be the circumference of the circle. Now to find the number of *yojanas* (by extracting the square-root) we obtain in succession the figures 3,1,6,2,2 and 7 of the root, the number appearing below (that is, as the last divisor) is 632454. This being halved becomes the number three hundred thousand sixteen thousands two hundred and twenty seven. The number in excess as the remainder is this 484471;...."

"Then on multiplication by 4 will be obtained 75600000000000. The square-root of this will be the chord. In finding that (root) will be obtained in succession the figures 2,7,4,9,5 and 4;...."

It is evident that Âryabhaṭa's plan of finding the square-root has been followed in the above cases as the digits of the root are evolved successively one by one.

Later writers give more details of the process. Thus Śrîdhara says:

"Having subtracted the square from the odd place, divide the next (even) place by twice the root which has been separately placed (in a line), and after having subtracted the square of the quotient, write it down in the line; double what has been obtained above (by placing the quotient in the line) and taking this down, divide by it the next even place. Halve the doubled quantity (to get the root)."[1]

Mahâvîra,[2] Âryabhaṭa II[3] and Śrîpati[4] give the rule in the same way as Śrîdhara. Bhâskara II, however, makes a slight variation, for he says:

[1] *Triś*, p. 5. For an illustration of the method of working on a *pâṭî*, see A. N. Singh, *BCMS*, XVIII, p. 129.

[2] *GSS*, p. 13.

[3] *MSi*, p. 145.

[4] *SiŚe*, xiii. 5; *GT*, 23.

"Subtract from the last odd place the greatest square number. Set down double the root in a line, and after dividing by it the next even place subtract the square of the quotient from the next odd place and set down double the quotient in the line. Thus repeat the operation throughout all the figures. Half of the number in the line is the root."[1]

The method of working on the *pâṭî* may be illustrated as below:

Example. Find the square-root of 54756

The given number is written down on the *pâṭî* and the odd and even places are marked by vertical and horizontal lines thus:

$$\begin{matrix} | & - & | & - & | \\ 5 & 4 & 7 & 5 & 6 \end{matrix}$$

Beginning with the last odd place 5, the greatest square number 4 is subtracted. Thus 4 subtracted from 5 gives 1. The number 5 is rubbed out and the remainder 1 substituted in its place. Thus after the first operation is performed, what stands on the *pâṭî* is

$$\begin{matrix} & - & | & - & | \\ 1 & 4 & 7 & 5 & 6 \end{matrix}$$

Double the root 2, *i.e.*, 4, is permanently placed in a separate portion of the *pâṭî* which has been termed *paṅkti* ("line"). Dividing the number upto the next even mark by this number in the line, *i.e.*, dividing 14 by 4 we obtain the quotient 3 and remainder 2. The number 14 is rubbed out and the remainder 2 written

[1] *L*, p. 4. The line in Bhâskara II's method contains the doubled root, whilst in that of Âryabhaṭa I, it contains the root. See Singh, *l.c.*

in its place; thus on the *pâṭî* we have now

4 line of root	$\overset{\mid}{2}7\,\overset{-}{5}\overset{\mid}{6}$	(3 quotient

The square of the quotient $3^2=9$ is subtracted from the figures upto the next odd mark. This gives $(27-9)=18$. 27 is rubbed out and 18 substituted in its place. Double the quotient 3 is now set in the line giving 46. The figures on the *pâṭî* stand thus:

46 line of root	$18\overset{-}{5}\overset{\mid}{6}$	The quotient 3 having been rubbed out.

Dividing the numbers upto the next even mark by the number in the line, *i.e.*, dividing 185 by 46, the quotient is 4 and remainder 1. 185 is rubbed out and the remainder 1 substituted in its place. The figures on the *pâṭî* are now

46 line of root	$1\overset{\mid}{6}$	(4 quotient

Subtracting square of the quotient the remainder is nil, so that 16 is rubbed out. The quotient 4 is doubled and set in the line. The *pâṭî* has now

468 line of root	The quotient 4 having been rubbed out.

Half the number in the line, *i.e.*, $\frac{468}{2}=234$ is the root.

Along with the Hindu numerals, the method of extracting the square-root given above, seems to have been communicated to the Arabs about the middle of the eighth century, for it occurs in precisely the same form in Arabic works on mathematics.[1] In Europe

[1] *E.g.*, Al-Nasavî (1025); see Suter, *Bibl. Math.*, VII, p. 114 and Woepcke, *JA* (6), t. 1, 1863.

it occurs in similar form in the writings of Peurbach (1423-1461), Chuquet (1484), La Roche (1520), Gemma Frisius (1540), Cataneo (1546) and others.[1]

9. CUBE-ROOT

Terminology. The Hindu terms for the cube-root are *ghana-mûla*, *ghana-pada*. These terms have already been discussed before.

The Operation. The first description of the operation of the cube-root is found in the *Aryabhaṭîya*. It is rather too concise:

"Divide the second *aghana* place by thrice the square of the cube-root; subtract from the first *aghana* place the square of the quotient multiplied by thrice the preceding (cube-root); and (subtract) the cube (of the quotient) from the *ghana* place; (the quotient put down at the next place (in the line of the root) gives the root)."[2]

As has been explained by all the commentators, the units place is *ghana*, the tens place is first *aghana*, the hundreds place is second *aghana*, the thousands place is *ghana*, the ten-thousands place is first *aghana* and so on. After marking the places as *ghana*, first *aghana* and second *aghana*, the process begins with the subtraction of the greatest cube number from the figures upto the last *ghana* place. Though this has not been ex-

[1] See Smith, *History*, II, pp. 144-148.

[2] *A*, ii. 5. Translations of this rule have been given by Rodet, Kaye, Singh, Clark, Sengupta and others. Kaye's translation is entirely inaccurate. Other translations, though free, give the correct result. Clark's use of the words "the (preceding) *ghana*" is somewhat misleading. The portion at the end, within brackets, is common to this and the preceding rule for the extraction of the square-root.

plicitly mentioned in the rule, the commentators say that it is implied in the expression "*ghanasya mûla vargeṇa*" etc. ("by the square of the cube-root" etc.) The method may be illustrated as below:

Example. Find the cube-root of 1953125.

The places are divided into groups of three by marking them as below:

	|--|--|	
	1953125	
Subtract cube	1 .... (*c*)	Root = 1
Divide by thrice square of root, *i.e.* 3.1^2	3) 9 (2 .... (*a*)[1]	Placing quotient after the root 1 gives the root 12
Subtract square of quotient multiplied by thrice the previous root, *i. e.*, $2^2.3.1$.	6	
	35	
	12 .... (*b*)	
Subtract cube of quotient, *i.e.*, 2^3	233	
	8 .... (*c*)	
Divide by thrice square of the root, *i.e.*, 3.12^2	432) 2251 (5 . (*a*)	Placing quotient after the root 12 gives the root 125
Subtract square of quotient multiplied by thrice the previous root, *i.e.*, $5^2.3.12$	2160	
	912	
	900 .. (*b*)	
Subtract cube of quotient, *i.e.*, 5^3	125	
	125 .. (*c*)	

Thus the cube-root = 125

It will be evident from the above illustration that the present method of extracting the cube-root is a contraction of Āryabhaṭa's method.

The method given above occurs in all the Hindu mathematical works. For instance, Brahmagupta says:

[1] The quotient in division is to be taken as great as will allow the two subsequent operations (*b*) and (*c*) to be carried out.

"The divisor for the second *aghana* place is thrice the square of the cube-root; the square of the quotient multiplied by three and the preceding (root) must be subtracted from the next (*aghana* place to the right), and the cube (of the quotient) from the *ghana* place; (the procedure repeated gives) the root."[1]

Srîdhara gives more details of the process as actually performed on the *pâṭî*, thus:

"(Divide the digits beginning with the units place into periods of) one *ghana* place and two *aghana* places. From the (last) *ghana* digit subtract the (greatest possible) cube; then taking down the remainder and the third *pada* (*i.e.*, the second *aghana* digit) divide it by thrice the square of the cube-root which has been permanently placed in a separate place; place the quotient in the line; subtract the square of this (quotient) multiplied by thrice the last root from the next (*aghana*) digit. Then as before subtract the cube (of the quotient) from its own place (*i.e.*, the *ghana* digit). Then take down again the third *pada* (*i.e.*, second *aghana* digit), and the rest of the process is as before. (This will give) the root."[2]

Aryabhaṭa II follows Srîdhara and gives details as follows:

"*Ghana* (*i.e.*, the place from which cube is subtracted), *bhâjya* (*i.e.*, the "dividend" place) and *śodhya* (*i.e.*, the "minuend" place) are the denominations (of the places). Subtract the (greatest) cube from its own place (*i.e.*, from the numbers upto the last *ghana* digit); bring down the *bhâjya* digit and divide it[3] by thrice the square of the cube-root which has been permanently

[1] *BrSpSi*, p. 175; *cf.* Colebrooke, *l.c.*, p. 280.
[2] *Triś*, pp. 6f.
[3] Literally, its own place.

12

placed. Place the quotient in the line (of the root). The square of this (quotient) multiplied by thrice the previous root is subtracted from its own place (*i.e.*, the *śodhya* place) and its cube from the *ghana* place. If the above operations be possible then this (*i.e.*, the number in the line) is the root so far. Then bringing down the *bhâjya* digit continue the process as before (till it ends)."[1]

The component digits of the number whose cube-root is to be found are divided into groups of three (one *ghana* and two *aghanas*) each. The digits upto the last *ghana* place (proceeding from left to right) give the first figure of the root (counting from the left). The following period of three digits (to the right) gives the second figure of the root and so on. While working on the *pâṭî*, the digits of the number whose root is to be found are first marked and the method proceeds as follows:

Example. Find the cube-root of 1953125.

The number is written thus:

```
|--|--|
1953125
```

From the last *ghana* digit (marked by a vertical stroke), the greatest cube is subtracted. Here 1^3 being subtracted from 1 gives zero. So 1 is rubbed out. The cube-root of 1^3 is placed in a separate line. The figures on the *pâṭî* stand thus:

```
--|--|            1
953125      ------------
            line of root
```

Then to obtain the second figure of the root, the second *aghana* (*i.e.*, 9) is taken below and divided by

[1] *MSi*, p. 145. The interpretation given by Dvivedi of line 2 of the rule as printed in his edition (p. 145) is incorrect.

thrice the square of the root (*i.e.*, the number in the line). Thus we have

$$\begin{array}{lcll} & & -\,-\,|\,-\,-\,| & \\ & & \underline{953125} & \\ 3\cdot 1^2 & = & 3)\ 9 \quad\ \ (2 & \text{quotient} \\ & & \quad \underline{6} & \\ & & \quad 3 & \end{array}$$

The quotient is taken to be 2, because if it were taken to be 3, the rest of the procedure cannot be carried out. This quotient (2) is set in the line. The first *aghana* is then brought down and we have, on subtracting the square of the quotient multiplied by thrice the previous root, the following:

$$\begin{array}{lcll} & & -\,-\,|\,-\,-\,| & \qquad \frac{12}{\text{line of root}} \\ & & \underline{953125} & \\ 3\cdot 1^2 & = & 3)\ 9 \quad\ \ (2 & \text{quotient} \\ & & \quad \underline{6} & \\ & & \quad 35 & \\ 2^2\cdot 3\cdot 1 & = & \quad \underline{12} & \\ & & \quad 23 & \end{array}$$

On bringing down the *ghana* digit 3, and then subtracting the cube of the quotient we get 225 as below, and the process on the period formed by the digits 953 is completed and the figure 2 of the root is obtained:

$$\begin{array}{lcll} & & -\,-\,|\,-\,-\,| & \\ & & \underline{953125} & \\ & & 3)\ 9 \quad\ \ (2 & \qquad \frac{12}{\text{line of root}} \\ & & \quad \underline{6} & \\ & & \quad 35 & \\ & & \quad \underline{12} & \\ & & \quad 233 & \\ 2^3 & = & \quad \underline{\quad 8} & \\ & & \quad 225 & \end{array}$$

The figures 953 are then rubbed out and the remainder 225 is substituted. After this the process is as before, *i.e.*, thus

	- - |	
	225125	
$12^2 . 3 =$	432) 2251 (5	12
	2160	line of root
	912	
$5^2 . 12 . 3 =$	900	
	125	
$5^3 =$	125	125
	0	line of root

The process ends as all the figures in the number are exhausted. The root is 125, the number in the line of root. As there is no remainder, the root is exact.

The necessity for rubbing out figures arises, as the *pâṭî* is not big enough to contain the whole of the working. The three digits constituting a period are considered together. The figures upto the next second *aghana* have to be brought down and the operation of division performed separately, because the quotient is obtained by trial. As has been already explained, this division is performed by rubbing out the digits of the dividend (and not as in the working given above). If the operations were carried out on the figures of the original number, and if the quotient taken were found to be too big, then it would not be possible to restore the original figures and begin the work again, as will have to be done in case of failure.

10. CHECKS ON OPERATIONS

The earliest available description of a method of checking the results of arithmetical operations, the direct as well as the inverse, is found in the *Mahâsiddhânta*[1] (*c.* 950). It says:

"Add together the own digits of the numbers forming the multiplicand, multiplier, and product upto one place;[2] such should be done with the dividend, divisor, quotient and remainder, etc. Then if the number (of one digit) obtained from the product of those numbers (that have been already obtained) from the multiplicand and the multiplier be equal to that obtained from the product, the multiplication is true. If the number, which results from the product of those obtained from the quotient and the divisor, added to that from the remainder, be equal to that obtained from the dividend, the division is true. Add together the digits of a number, its (nearest) square-root (in integers) and of the remainder. If the number, obtained from the square of that (number) which is obtained from the square-root plus the number obtained from the remainder, be equal to that which results from the given number, the root-extraction is true. If the number, resulting from the cube of the number obtained by adding the digits of the cube-root plus the number obtained from the remainder, be equal to the number resulting from the given number, then the operation is correct. Such are the easy tests for correctness of multiplication etc."

The *rationale* of the above rules will be clear from the following: Let

$$n = d_m d_{m-1} \ldots\ldots d_2 d_1$$

be a number of m digits written in the decimal place-value notation. Let S_1 denote the sum of its digits,

[1] *MSi*, p. 452.

[2] That is, the digits of the number should be added together; the digits of the sum thus obtained should be again added and the process should be continued until there remains a number of one digit only.

S_2 the sum of the digits of S_1, and so on.

Then

$$n = d_1 + 10d_2 + \ldots\ldots + 10^{m-1}d_m,$$
$$S_1 = d_1 + d_2 + d_3 + \ldots\ldots + d_m$$

so that

$$n - S_1 = 9(d_2 + 11d_3 + \ldots\ldots).$$

Therefore,

$$n \equiv S_1 \quad (\text{mod. } 9),$$

Similarly

$$S_1 \equiv S_2 \quad (\text{mod. } 9),$$
$$S_2 \equiv S_3 \quad (\text{mod. } 9),$$
$$\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots$$
$$S_{k-1} \equiv S_k \quad (\text{mod. } 9),$$

where S_k is a number of one digit only, say n', which is certainly less than or equal to 9.

Adding the congruences, we obtain

$$n \equiv n' \quad (\text{mod. } 9).$$

Thus the number obtained by adding the digits of a number repeatedly is equal to the remainder obtained by dividing that number by nine.

Now, if there be a number N which is equal to the continued product of p other numbers $n_1, n_2, n_3, \ldots, n_p$ plus or minus another number R, then we write

$$N = n_1.n_2.n_3 \ldots . n_p \pm R$$

Now, let

$$n_1 \equiv n'_1 \quad (\text{mod. } 9)$$
$$n_2 \equiv n'_2 \quad (\text{mod. } 9)$$
$$\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots$$
$$n_p \equiv n'_p \quad (\text{mod. } 9)$$

Multiplying the congruences, we obtain

$$n.n_2 \ldots . n_p \equiv n'_1.n'_2 \ldots . n'_p \pmod{9}.$$

Further let

$$R \equiv r' \qquad \pmod{9}$$

Therefore

$$n_1.n_2.n_3 \ldots . n_p \pm R \equiv n'_1.n'_2 \ldots . n'_p \pm r' \pmod{9}.$$

Hence

$$N \equiv n'_1.n'_2 \ldots \ldots n'_p \pm r' \pmod{9}.$$

In particular, if

$$n_1 = n_2 = \ldots\ldots\ldots = n_p = n, \text{ say}$$

then will

$$n'_1 = n'_2 = \ldots\ldots\ldots\ldots = n'_p = n'.$$

Therefore,

$$N = n^p \pm R$$

and $$N \equiv n'^p \pm r' \pmod{9}.$$

From the above follow easily the rules of the *Mahâsiddhânta*.

The following rule for testing multiplication is given by Nârâyaṇa[1] (1356):

"The remainders obtained on division of each of the multiplicand and the multiplier by an optional number are multiplied together and then divided by the optional number. If the remainder so obtained be equal to the remainder obtained on dividing the product (of the multiplicand and the multiplier) by the optional number, then, it is correct."

It must be noted here that a complete set of rules for checking by nines is first found in India. Methods for testing multiplication and division were probably

[1] Quoted by S. Dvivedi, *History of Mathematics* (in Hindi), Benares, 1910, p. 79.

known to the Hindus much earlier. But as these tests were not considered to be among the fundamental operations, they were not mentioned in the works on *pâṭî-gaṇita*.[1] Nârâyaṇa seems to be the first Hindu mathematician to give rules for testing operations by the casting out of any desired number.

In the works of early Arab writers the methods of testing multiplication, and division without remainder, by the check of nines are given, while a complete set of rules for testing all operations is found first in the works of Avicenna[2] (*c.* 1020) who calls his method the "Hindu" method. Maximus Planudes[3] also ascribes an Indian origin to the check of nines.

There is thus no doubt as to the Hindu origin of the check of nines. Before Âryabhaṭa II, it was probably used to test multiplication and division only. It was perhaps in this imperfect form when it was communicated to the Arabs. Thereafter, the method was probably perfected independently both in Arabia and India. This would account for the difference in the formulation of the rules by the Arabs and by Âryabhaṭa II, the author of the *Mahâsiddhânta*.[4] It is, however, certain that the Hindus did not borrow the method from the Arabs, because Âryabhaṭa II wrote before Avicenna. Behâ Eddîn[5] (*c.* 1600) gives the check of nines in exactly the same form as Âryabhaṭa II.

[1] Besides the above works, the check of nines is also quoted by Sumatiharṣa (1618) from an anterior writer Bîjadatta, in his commentary on the *Karaṇa-kutûhala* of Bhâskara II, ed. by Mâdhava Śâstrî, Bombay, 1901, i. 7.

[2] F. Woepcke, *JA*(6), I, 1863, pp. 500 *et sq.*

[3] *Vide* Delambre, *Histoire de l'Astronomie Ancienne*, t. I, Paris, 1817, pp. 518 ff.

[4] Noted by B. Datta, *JASB*, XXIII, 1927, p. 265.

[5] *Kholâsat al-hisâb*, French translation by A. Marre, *Nouvelles Annales d. Math.*, t. v, 1846, p. 263.

11. FRACTIONS

Early Use. In India, the knowledge of fractions can be traced back to very early times. In the oldest known work, the *Ṛgveda*, the fractions one-half (*ardha*) and three-fourths (*tri-pâda*[1]) occur. In a passage of the *Maitrâyaṇî Saṁhitâ*[2] are mentioned the fractions one-sixteenth (*kalâ*), one-twelfth (*kuṣṭha*), one-eighth (*śapha*) and one-fourth (*pâda*). In the earliest known mathematical works, the *Sulba-sûtra*, fractions have not only been mentioned, but have been used in the statement and solution of problems.[3]

The ancient Egyptians and Babylonians are known to have used fractions with unit numerators, but there is little evidence of the use by these people of what are called composite fractions. The occurrence of the fraction three-fourths in the *Ṛgveda* is probably the oldest record of a composite fraction known to us. The Sanskrit compound *tri-pâda* literally means "three-feet." Used as a number it denotes that the measure of the part considered bears the same ratio to the whole as three feet of a quadruped bear to the total number of its feet. The term *pâda*, however, is a word numeral for one-fourth, and the compound *tri-pâda* is formed exactly on the same principle as the English term three-fourths.[4] In the *Sulba*, unit fractions are denoted by the use of a cardinal number with the term *bhâga* or *aṁśa*; thus *pañca-daśa-bhâga* ("fifteen-parts") is equivalent to one-fifteenth,[5] *sapta-bhâga* ("seven-parts") is equivalent to one-seventh,[6] and so on. The use of ordinal numbers

[1] *RV*, x. 90. 4.
[2] iii. 7. 7.
[3] B. Datta, *Śulba*, pp. 212 ff.
[4] *tri*=three and *pâda*=fourth.
[5] *ĀpŚl*, x. 3; *KŚl*, v. 8.
[6] *KŚl*, vi. 4.

with the term *bhâga* or *aṁśa* is also quite common, e.g., *pañcama-bhâga* ("fifth part") is equivalent to one-fifth.[1] Sometimes the word *bhâga* is omitted, probably for the sake of metrical convenience.[2] Composite fractions like 3/8 and 2/7 are called *tri-aṣṭama* ("three-eighths") and *dvi-saptama* ("two-sevenths") respectively. In the Bakhshâlî Manuscript the fraction 3/8 is called *tryaṣṭa* ("three-eighths") and $3\frac{3}{8}$ is called *trayastrayaṣṭa* ("three-three-eighths").[3] Instances of the formation of fraction names on the above principle are too numerous in later works to be mentioned here. The present method of expressing fractions is thus derived from Hindu sources and can be traced back to 3,000 B.C.

Weights and Measures. The division of the units of length, weight, money, etc., into smaller units for the sake of avoiding the use of fractional quantities has been common amongst all civilised peoples. It is an index of commercial activity and the development of commercial arithmetic. The Hindus have used systems of weights and measures from the earliest times. The *Śatapatha Brâhmaṇa*[4] (*c.* 2,000 B.C.) gives a very mintute subdivision of time. According to it there are 30 *muhûrta* in a day, 15 *kṣipra* in a *muhûrta*, 15 *itarhi* in a *kṣipra*, 15 *idânî* in an *itarhi* and 15 *prâṇa* in an *idânî*. Thus one *prâṇa* is approximately equivalent to one-seventeenth of a second. It is improbable that the ancient Hindus had any appliance for measuring such small intervals of time. The subdivision is entirely theoretical, and probably made for philosophical reasons. It, nevertheless, shows that the Hindus must

[1] *ĀpŚl*, ix. 7, x. 2; *KŚl*, v. 6.

[2] When the fractions have unit numerators, only the denominators are mentioned. This practice is quite common in later works also, e.g., *ṣaṣṭa* (sixth)$=\frac{1}{6}$ in *L*, p. 7 etc.

[3] *BMs*, 10 *verso*.

[4] xii, 3. 2. 1.

have been in possession of a satisfactory arithmetic of fractions even in those early times. The *Arthaśâstra* of Kauṭilya[1] contains a section dealing with weights and measures which were in use in India in the fourth century B.C. In the *Lalitavistara*[2] Buddha is stated to have given the following system of linear measures:

7	*paramâṇu raja*	=	1	*reṇu*
7	*reṇu*	=	1	*truṭi*
7	*truṭi*	=	1	*vâtâyana raja*
7	*vâtâyana raja*	=	1	*śaśa raja*
7	*śaśa raja*	=	1	*eḍaka raja*
7	*eḍaka raja*	=	1	*go raja*
7	*go raja*	=	1	*likṣâ raja*
7	*likṣâ raja*	=	1	*sarṣapa*
7	*sarṣapa*	=	1	*yava* (breadth of barley)
7	*yava*	=	1	*aṅgulî parva* (breadth of finger)
12	*aṅgulî parva*	=	1	*vitasti*
2	*vitasti*	=	1	*hasta* (cubit)
4	*hasta*	=	1	*dhanu*
1000	*dhanu*	=	1	*krośa*
4	*krośa*	=	1	*yojana*

According to the above table, the smallest Hindu measure of length, a *paramâṇu*[3] $= 1 \cdot 3 \times 7^{-10}$ inches.

All the works on *pâṭîgaṇita* begin with definitions of the weights and measures employed in them. The earlier ones contain a special rule for the reduction of a chain of measures into a proper fraction.[4] It may be mentioned that the systems of weights and measures

[1] The *Arthaśâstra of Kauṭilya*, ed. by R. Shamsastri, Bangalore, 1919.

[2] *Lalitavistara*, ed. R. Mitra, Calcutta, 1877, p. 168.

[3] *Paramâṇu* is the smallest particle of matter. Thus according to the Hindus, the diameter of a molecule is $1 \cdot 3 \times 7^{-10}$.

[4] The process is called *vallî-savarṇana* and occurs in the *Triśatikâ* (p. 12) and the *Gaṇita-tilaka* (p. 39) and not in later works.

given in different works are different from each other. They are the ones current at the time and in the locality in which the book was composed.

Terminology. The Sanskrit term for a fraction is *bhinna*. It means "broken." The European terms *fractio*, *fraction*, *roupt*, *rotto*, or *rocto* etc., are translations of the term *bhinna*, having been derived from the Latin *fractus* (*frangere*) or *ruptus* meaning "broken." The Hindu term *bhinna*, however, had a more general meaning in so far as it included numbers of the form,

$$\left(\frac{a}{b} \pm \frac{c}{d}\right), \quad \left(\frac{a}{b} \text{ of } \frac{c}{d}\right), \quad \left(\frac{a}{b} \pm \frac{c}{d} \text{ of } \frac{a}{b}\right) \quad \text{or} \quad \left(a \pm \frac{b}{c}\right).$$

These forms were termed *jâti*, *i.e.*, "classes," and the Hindu treatises contain special rules for their reduction to proper fractions. Srîdhara and Mahâvîra each enumerate six *jâti*s, while Brahmagupta gives only five and Bhâskara II following Skandasena reduces the number to four. The need for the division of fractions into classes arose out of the lack of proper symbolism to indicate mathematical operations. The only operational symbol used by the Hindus was a dot[1] for the negative sign.

The other terms employed for the fraction are *bhâga* and *aṁśa*, meaning "part" or "portion." The term *kalâ* which originally, in Vedic times, denoted one-sixteenth came to be later on employed for a fraction. Its earliest use as a term for fraction occurs in the *Sulba* works.

Writing of Fractions. From very early times (*c.* 200 A.D.) the Hindus wrote fractions just as we do now, but without the dividing line. When several fractions occurred in the same problem, they were in general separated from each other by vertical and

[1] Generally placed over the number to be subtracted.

horizontal lines. Illustrations of the Hindu method of writing groups of fractions will be found in the examples that will be given hereafter.

Reduction to Lowest Terms. Before performing operations with fractions, it was considered necessary to reduce them to lowest terms. The process of reduction was called *apavartana*, but was not included among the operations. It is not given in the Hindu works, but seems to have been taught by oral instruction. That the method has been in use in India from the earliest centuries of the Christian era, cannot be doubted; for it is mentioned in a non-mathematical work, the *Tattvârthâdhigama-sûtra-bhâṣya*[1] by Umâsvâti (*c.* 150) as a simile to illustrate a philosophical discussion:

"Or, as when the expert mathematician, for the purpose of simplifying operations, removes common factors from the numerator and denominator of a fraction, there is no change in the value of the fraction, so ......"

Reduction to Common Denominator. The operation of reduction to a common denominator[2] is required when fractions are to be added or subtracted. The process is given a prominent place and is generally mentioned along with addition and subtraction. Brhmagupta[3] gives the reduction along with the processes of addition and subtraction thus:

"By the multiplication of the numerator and denominator of each of the (fractional) quantities by the other denominators, the quantities are reduced to a common denominator. In addition, the numerators are united. In subtraction their difference is taken."

[1] ii. 52.
[2] *Kalâ-savarṇana* or *savarṇana*, or *samachbheda-vidhi*.
[3] *BrSpSi*, p. 172.

Srîdhara[1] says:

"To reduce to a common denominator, multiply the numerator and denominator of each (fraction) by the other denominators."

All other works also contain this rule.

Fractions in Combination. It has already been remarked that due to the lack of proper symbolism, the Hindu mathematicians divide combinations of fractions into four classes. They are:

(1)[2] *Bhâga*, *i.e.*, the form $\left(\frac{a}{b} \pm \frac{c}{d} \pm \frac{e}{f} \pm \ldots\right)$.
usually written as

$$\begin{array}{|c|c|c|}\hline a & c & e \\ b & d & f \\ \hline\end{array} \quad \text{or} \quad \begin{array}{|c|c|c|}\hline a & \dot{c} & \dot{e} \\ b & d & f \\ \hline\end{array}$$

where the dots denote subtraction.

(2)[3] *Prabhâga*, *i.e.*, the form $\left(\frac{a}{b} \text{ of } \frac{c}{d} \text{ of } \frac{e}{f} \ \ldots\ldots\right)$, written as

$$\begin{array}{|c|c|c|}\hline a & c & e \\ b & d & f \\ \hline\end{array}$$

(3)[4] *Bhâgânubandha*, *i.e.*, the form

(*i*) $\left(a + \frac{b}{c}\right)$

or (*ii*) $\frac{p}{q} + \frac{r}{s} \text{ of } \frac{p}{q} + \frac{t}{u} \text{ of } \left(\frac{p}{q} + \frac{r}{s} \text{ of } \frac{p}{q}\right) + \ldots$

[1] *Triś*, p. 10. The translation given by Kaye is incorrect.

[2] *BrSpSi*, p. 175; *Triś*, p. 10; *GSS*, p. 33 (55, 56); *MSi*, p. 146; *L*, p. 6.

[3] *Triś*, p. 10; *GSS*, p. 39 (99); *MSi*, p. 146; *L*, p. 6.

[4] *Triś*, p. 10; *GSS*, p. 41 (113); *MSi*, p. 148; *L*, p. 7. These forms are termed *rûpa-bhâgânubandha* ("association of an integer and a fraction") and *bhâga-bhâgânubandha* ("association of fractions of fractions") respectively.

written as

$$(i)\ \begin{array}{|c|}\hline a \\ b \\ c \\ \hline\end{array} \quad \text{or} \quad (ii)\ \begin{array}{|c|}\hline p \\ q \\ \hline r \\ s \\ \hline t \\ u \\ \hline\end{array}$$

(4)[1] *Bhâgâpavâha*, *i.e.*, the form

$$(i)\ \left(a - \frac{b}{c}\right)$$

$$\text{or}\quad (ii)\ \frac{p}{q} - \frac{r}{s} \text{ of } \frac{p}{q} - \frac{t}{u} \text{ of } \left(\frac{p}{q} - \frac{r}{s} \text{ of } \frac{p}{q}\right) - \ldots.$$

written as

$$(i)\ \begin{array}{|c|}\hline a \\ \cdot b \\ c \\ \hline\end{array} \quad \text{or} \quad (ii)\ \begin{array}{|c|}\hline p \\ q \\ \hline \cdot r \\ s \\ \hline \cdot t \\ u \\ \hline\end{array}$$

Besides the above four forms, Srîdhara, Mahâvîra, and some others give two more.

(5)[2] *Bhâga-bhâga*, *i.e.*, the form

$$\left(a \div \frac{b}{c}\right) \text{ or } \left(\frac{p}{q} \div \frac{r}{s}\right)$$

There does not appear to have been any notation for division, such compounds being written as,

[1] *BrSpSi*, p. 176; *GSS*, p. 43 (126); *MSi*, p. 148; *L*, p. 7. These forms are termed *rûpa-bhâgâpavâha* and *bhâga-bhâgâpavâha* respectively.

[2] *Triś*, p. 11; *GSS*, p. 39 (99).

$$\begin{array}{|c|}\hline a\\ b\\ c\\ \hline\end{array} \quad \text{or} \quad \begin{array}{|c|}\hline p\\ q\\ \hline r\\ s\\ \hline\end{array}$$

just as for *bhâgânubandha*. That division is to be performed was known from the problem;[1] *e.g.*, $1 \div \frac{1}{6}$ was written as *ṣaḍ-bhâga-bhâga*,[2] *i.e.*, "one-sixth *bhâga-bhâga*" or "one divided by one-sixth."[3]

(6)[4] *Bhâga-mâtṛ*, *i.e.*, combinations of forms enumerated above. Mahâvîra remarks that there can be twenty-six variations of this type.[5] The following example is given by Srîdhara.[6]

"What is the result when half, one-fourth of one-fourth, one divided by one-third, half plus half of itself, and one-third diminished by half of itself, are added together?"

In modern notation this is

$$\tfrac{1}{2}+(\tfrac{1}{4}\text{ of }\tfrac{1}{4})+(1\div\tfrac{1}{3})+(\tfrac{1}{2}+\tfrac{1}{2}\text{ of }\tfrac{1}{2})+(\tfrac{1}{3}-\tfrac{1}{2}\text{ of }\tfrac{1}{3}).$$

In the old Hindu notation it was written as

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1\\ 2 & 4 & 4 & 1 & 2 & 3\\ \cline{5-6} & & & 3 & 1 & {}^{\cdot}1\\ & & & & 2 & 2\\ \hline\end{array}$$

[1] It is only in the Bakhshâlî Manuscript that the term *bhâ* is sometimes placed before or after the quantity affected.

[2] *Cf.* *Triś*, p. 11.

[3] *GSS*, p. 41 (112) gives $2 \div \frac{3}{4}$ as *tripâda bhaktaṃ dvikaṃ*, i.e., "two divided by three-fourths."

[4] *Triś*, p. 12; *GSS*, p. 45 (138).

[5] As there are five primary classes enumerated by Mahâvîra, so the total number of combinations is

$$^5C_2+{}^5C_3+{}^5C_4+{}^5C_5=26.$$

[6] *Triś*, p. 12.

The defect of the notation is obvious: $\begin{array}{|c|c|}\hline 1 & 1 \\ 4 & 4 \\ \hline\end{array}$ can be read also as $\frac{1}{4}+\frac{1}{4}$, and $\begin{array}{|c|}\hline 1 \\ 1 \\ 3 \\ \hline\end{array}$ can be read also as $1\frac{1}{3}$; so that the exact meaning of the notation can be understood only by a reference to the question.

The rules for the reduction of the first two classes are those of addition or subtraction, and multiplication. The rule for the reduction of the third and fourth classes (from *ii*) are given together by Brahmagupta:

"The (upper) denominator is multiplied by the denominator and the upper numerator by the same (denominator) increased or diminished by its own numerator."[1]

The rule for *bhâgânubandha* is given by Śrîdhara[2] as follows:

(*i*) "In *bhâgânubandha*, add the numerator to the product of the whole number and the denominator."

(*ii*) "Multiply the denominator by the lower denominator and (then) the numerator by the same lower denominator increased by its own numerator."

Other writers give similar rules for reduction in the case of *bhâgânubandha*.

The following example[3] will explain the process of working:

[1] *BrSpSi*, p. 176. The reduction of the form $a \pm \frac{b}{c}$ has been given separately (p. 173).

[2] *Triś*, p. 10. Rule (*i*) is for the reduction of $a+\frac{b}{c}$ and rule (*ii*) is for the reduction of the form

$$\frac{a}{b}+\frac{c}{d} \text{ of } \frac{a}{b}+\frac{e}{f} \text{ of } \left(\frac{a}{b}+\frac{c}{d} \text{ of } \frac{a}{b}\right).$$

[3] *Triś*, p. 11.

Reduce to a proper fraction:

$$3\tfrac{1}{2}+\tfrac{1}{4} \text{ of } 3\tfrac{1}{2}+\tfrac{1}{6} \text{ of } (3\tfrac{1}{2}+\tfrac{1}{4} \text{ of } 3\tfrac{1}{2})+\tfrac{1}{2}+\tfrac{1}{3} \text{ of } \tfrac{1}{2}+\tfrac{1}{4} \text{ of } (\tfrac{1}{2}+\tfrac{1}{3} \text{ of } \tfrac{1}{2}).$$

This was written as

3 1 2	1 2
1 4	1 3
1 6	1 4

Adding denominators to numerators of the lower fractions, and applying rule (*i*) to left-hand top compartment to reduce it to a proper fraction, we get

7 2	1 2
5 4	4 3
7 6	5 4

Now performing multiplication as directed, *i.e.*, multiplying the denominator of the first fraction by all the lower denominators and the numerator by the sum of the numerators and denominators of the lower fractions, we get

$$\tfrac{7}{2}\times\tfrac{5}{4}\times\tfrac{7}{6}=\tfrac{245}{48}, \text{ and } \tfrac{1}{2}\times\tfrac{4}{3}\times\tfrac{5}{4}=\tfrac{20}{24}$$

i.e.,

245 48	20 24

Then making denominators similar (*savarṇana*), we have

245 48	40 48

performing the addition we have $\frac{285}{48}$ or $5\frac{5}{8}$ as the result.

The rule for *bhâgâpavâha* is given in all the works on *pâṭîgaṇita*. It is the same as that for *bhâgânubandha*, except that "addition" or "increase" is replaced by "subtraction" or "decrease" in the enunciation of the rule for *bhâgâpavâha*.

Lowest Common Multiple. Mahâvîra[1] was the first amongst the Indian mathematicians to speak of the lowest common multiple in order to shorten the process. He defines *niruddha* (L. C. M.) as follows:

"The product of the common factors of the denominators and their resulting quotients is called *niruddha*."

The process of reducing fractions to equal denominators is thus described by him:[2]

"The (new) numerators and denominators, obtained as products of multiplication of (each original) numerator and denominator by the (quotient of the) *niruddha* (*i.e.*, L. C. M.) divided by the denominator give fractions with the same denominator."

Bhâskara II[3] does not mention *niruddha* but observes that the process can be shortened. He says:

"The numerator and denominator may be multiplied by the intelligent calculator by the other denominator abridged by the common factor."

The Eight Operations. Operations with fractions were known in India from very early times, the method of performing them being the same as now.

[1] *GSS*, p. 33 (56).
[2] *GSS*, p. 33 (56).
[3] *L*, p. 6.

Although Aryabhaṭa does not mention the elementary operations, there is evidence to show that he knew the method of division by fraction by inverting it. All the operations are found in the Bakhshâlî Manuscript (*c.* 200).

Addition and Subtraction. These operations were performed after the fractions were reduced to a common denominator. Thus Srîdhara says:[1]

"Reduce the fractions to a common denominator and then add the numerators. The denominator of a whole number is unity."

Brahmagupta and Mahâvîra give the method under *Bhâgajâti*. Mahâvîra differs from other writers in giving the methods of the summation of arithmetic and geometric series under the title of addition (*saṁkalita*).[2] Later writers follow Srîdhara.

Multiplication. Brahmagupta says:[3]

"The product of the numerators divided by the product of the denominators is the (result of) multiplication of two or more fractions."

While all other writers give the rule in the same way as Brahmagupta, Mahâvîra refers to cross reduction in order to shorten the work:[4]

"In the multiplication of fractions, the numerators are to be multiplied by the numerators and the denominators by denominators, after carrying out the process of cross reduction,[5] if that be possible."

[1] *Triś*, p. 7.
[2] *Cf. GSS*, pp. 28 (22) ff.
[3] *BrSpSi*, p. 173.
[4] *GSS*, p. 25 (2).
[5] *Vajrâpavartana-vidhi*, i.e., "cancellation crosswise," thus

$$\frac{2}{3} \times \frac{6}{4} = \begin{array}{ccc} 2 & \nwarrow \;\; \nearrow & 6 \\ 3 & \swarrow \;\; \searrow & 4 \end{array} = \frac{2}{2} = 1.$$

Division. Although the elementary operations are not mentioned in the *Aryabhaṭîya*, the method of division by fraction is indicated under the Rule of Three. The Rule of Three states the result as $\frac{f \times i}{p}$.[1] When these quantities are fractional, we get an expression of the form $\frac{\frac{a}{b} \times \frac{c}{d}}{\frac{m}{n}}$, for the evaluation of which Âryabhaṭa I states:

"The multipliers and the divisor are multiplied by the denominators of each other."

As will be explained later on (p. 204) the quantities are written as

a b	m n
c d	

Transferring the denominators we have

a n c	m b d

Performing multiplication, the result is $\frac{anc}{mbd}$.

The above interpretation of a rather obscure line[2] in the *Aryabhaṭîya* is based on the commentaries of Sûryadeva and Bhâskara I. Thus Sûryadeva says:

[1] Where $f =$ *phala*, *i.e.*, "fruit," $i =$ *icchâ*, *i.e.*, "demand or requisition" $p' =$ *pramâṇa*, *i.e.*, "argument."

[2] *A*, p. 43. Previous writers seem to have been misled by the commentary of Parameśvara which is very vague; *cf*; Clark (p. 40) and P. C. Sengupta (p. 25).

"Here by the word *guṇakâra* is meant the multiplier and multiplicand, *i.e.*, the *phala* and *icchâ* quantities that are multiplied together. By *bhâgahâra* is meant the *pramâṇa* quantity. The denominators of the *phala* and *icchâ* are taken to the *pramâṇa*. The denominator of the *pramâṇa* is taken with the *phala* and *icchâ*. Then multiplying these, *i.e.*, (the numerators of) the *phala* and *icchâ* and this denominator, and dividing by (the product of) the numbers standing with the *pramâṇa*, the result is the quotient of the fractions."

Brahmagupta[1] gives the method of division as follows:

"The denominator and numerator of the divisor having been interchanged, the denominator of the dividend is multiplied by the (new) denominator and its numerator by the (new) numerator. Thus division of *proper fractions* is performed."

Śrîdhara[2] adds the following to the method of multiplication:

"Having interchanged the numerator and denominator of the divisor, the operation is the same as before."[3]

Mahâvîra[4] explains the method thus:

"After having made the numerator of the divisor[5] its denominator (and *vice versa*) the operation is the same as in multiplication."

"Or,[6] when (the fractions constituting) the divisor

[1] *BrSpSi*, p. 173.

[2] *Triś*, p. 8.

[3] *i.e.*, the same as that of multiplication.

[4] *GSS*, p. 26 (8).

[5] Mahâvîra uses the term *pramâṇa-râśi* for divisor, showing thereby its connection with the 'rule of three.'

[6] This is similar to the way in which Âryabhaṭa I expresses the method.

and dividend are multiplied by the denominators of each other and these products are without denominators, (the operation) is as in the division of whole[1] numbers."

Square and Square-root. Brahmagupta[2] says:

"The square of the numerator of a proper fraction divided by the square of the denominator gives the square."

"The square-root of the numerator of a proper fraction divided by the square-root of the denominator gives the square-root."

Other works contain the same rules.

Cube and Cube-root. Śrîdhara[3] gives the rule as follows:

"The cube of the numerator divided by the cube of the denominator gives the cube, and the cube-root of the numerator divided by the cube-root of the denominator gives the cube-root."

Other works give the same rules.

Unit Fractions. Mahâvîra has given a number of rules for expressing any fraction as the sum of a number of unit fractions.[4] These rules do not occur in any other work, probably because they were not considered important or useful.

(1) *To express* 1 *as the sum of a number* (n) *of unit fractions.*

The rule for this is:[5]

"When the sum of the different quantities having

[1] The term for whole number is *sakala.*

[2] *BrSpSi*, p. 174.

[3] *Triś*, p. 9.

[4] There is no technical term for unit fraction. The term used is *rûpâṁśaka-râśi*, *i.e.*, "quantity with one as numerator."

[5] *GSS*, p. 36 (75).

one for their numerator is 1, the (required) denominators are such as, beginning with 1, are in order multiplied by 3, the first and the last being multiplied again by 2 and $\frac{2}{3}$."

Algebraically the rule is

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^{n-2}} + \frac{1}{2.3^{n-2}}.$$

(2) *To express* 1 *as the sum of an odd number of unit fractions.*

The rule for this is stated thus:[1]

"When the sum of the quantities (fractions) having one for each of their numerators is one, the denominators are such as, beginning with two, go on rising in value by one, each being further multiplied by that which is (immediately) next to it and then halved."

Algebraically this is

$$1 = \frac{1}{2.3.\frac{1}{2}} + \frac{1}{3.4.\frac{1}{2}} + \dots + \frac{1}{(2n-1).2n.\frac{1}{2}} + \frac{1}{2n.\frac{1}{2}}$$

(3) *To express a unit fraction as the sum of a number of other fractions, the numerators being given.*[2]

The rule for this is:

"The denominator of the first (of the supposed or given numerators) is the denominator of the sum, that of the next is this combined with its numerator and so on; and then multiply (each denominator) by that which is next to it, the last being multiplied by its own numerator. (This gives the required denominators)."

[1] *GSS*, p. 36(77).
[2] Each may be one. *GSS*, p. 36(78).

Algebraically this gives:

$$\frac{1}{n}=\frac{a_1}{n(n+a_1)}+\frac{a_2}{(n+a_1)(n+a_1+a_2)}+\ldots.$$

$$+\frac{a_{r-1}}{(n+a_1+a_2+\ldots.+a_{r-2})(n+a_1+a_2+\ldots.+a_{r-1})}$$

$$+\frac{a_r}{a_r(n+a_1+a_2+\ldots.+a_{r-1})}$$

By taking $a_1=a_2=\ldots=a_r=1$, we get unit fractions. When these are not unity, the fractions may not be in their lowest terms.

(4) *To express any fraction as the sum of unit fractions.*

The rule is:[1]

"The denominator (of the given fraction) when combined with an optionally chosen number and then divided by the numerator so as to leave no remainder, becomes the denominator of the first numerator (which is one); the optionally chosen quantity when divided by this and by the denominator of the sum is the remainder. To this remainder the same process is applied."

Let the number i be so chosen that $\frac{q+i}{p}$ is an integer $=r$; then the rule gives

$$\frac{p}{q}=\frac{1}{r}+\frac{i}{r.q}$$

of which the first is a unit fraction and a similar process can be employed to the remainder to get other unit fractions. In this case the result depends upon the optionally chosen quantities.

[1] *GSS*, p. 37(80).

(5) *To express a unit fraction as the sum of two other unit fractions.*

The following two rules are given:[1]

(*i*) "The denominator of the given sum multiplied by a properly chosen number is the (first) denominator, and this divided by the previously chosen number minus one gives the other; or (*ii*) the two denominators are the factors[2] of the denominator of the sum, each multiplied by their sum."

Expressed algebraically the rules are:

$$(i) \quad \frac{1}{n} = \frac{1}{p.n} + \frac{1}{\dfrac{p.n}{p-1}}\,{}^{3}$$

$$(ii) \quad \frac{1}{a.b} = \frac{1}{a(a+b)} + \frac{1}{b(a+b)}$$

(6) *To express any fraction as the sum of two other fractions whose numerators are given.*

The rule for this is:[4]

"Either numerator multiplied by a chosen number, then combined with the other numerator, then divided by the numerator of the sum so as to leave no remainder, and then divided by the chosen number and multiplied by the denominator of the sum gives rise to one denominator. The denominator corresponding to the other (numerator), however, is this (denominator) multiplied by the chosen quantity."

[1] *GSS*, p. 37(85).

[2] *hâra-hâra-labdha*, lit. "the divisor and quotient by that divisor."

[3] The integer p is so chosen that n is divisible by $(p-1)$.

[4] *GSS*, p. 38(87).

Algebraically the rule is

$$\frac{m}{n}=\frac{a}{\frac{ap+b}{m}\times\frac{n}{p}}+\frac{b}{\frac{ap+b}{m}\times\frac{n}{p}\times p}\ ^{1}$$

A particular[1] case of this would be

$$\frac{m}{n}=\frac{a}{\frac{an+b}{m}}+\frac{b}{\frac{an+b}{m}\times n}$$

provided that $(an+b)$ is divisible by m.

(7) *To express a given fraction as the sum of an even number of fractions whose numerators are previously assigned.*

The rule for this is:[2]

"After splitting up the sum into as many parts, having one for each of their numerators, as there are pairs (among the given numerators), these parts are taken as the sum of the pairs, and (then) the denominators are found according to the rule for finding two fractions equal to a given unit fraction."

12. THE RULE OF THREE

Terminology. The Hindu name for the Rule of Three terms is *trairâśika* ("three terms," hence "the rule of three terms"). The term *trairâśika* can be traced back to the beginning of the Christian era as it occurs in the

[1] Evidently, the chosen number p must be a divisor of n, and such that $\frac{ap+b}{m}$ is an integer.

The solution given does not hold for any values of a and b, but only for such values as allow of an integer p to be so chosen as to satisfy the required conditions.

[2] *GSS*, p. 38(89).

Bakhshâlî Manuscript,[1] in the *Âryabhaṭîya* and in all other works on mathematics. About the origin of the name Bhâskara I (*c.* 525) remarks:[2] "Here three quantities are needed (in the statement and calculation) so the method is called *trairâśika* ("the rule of three terms")."

A problem on the rule of three has the form:

If p yields f, what will i yield?

In the above, the three terms are p, f and i. The Hindus called the term p, *pramâṇa* ("argument"), the term f, *phala* ("fruit") and the term i, *icchâ* ("requisition"). These names are found in all the mathematical treatises. Sometimes they are referred to simply as the first, second and third respectively. Âryabhaṭa II differs from other writers in giving the names *mâna*, *vinimaya* and *icchâ* respectively to the three terms. It has been pointed out by most of the writers that the first and third terms are similar, *i.e.*, of the same denomination.

The Method. Âryabhaṭa I (499) gives the following rule for solving problems on the Rule of Three:

"In the Rule of Three, the *phala* ("fruit"), being multiplied by the *icchâ* ("requisition") is divided by the *pramâṇa* ("argument"). The quotient is the fruit corresponding to the *icchâ*. The denominators of one being multiplied with the other give the multiplier (*i.e.*, numerator) and the divisor (*i.e.*, denominator)."[3]

[1] The term *râśi* is used in the enumeration of topics of mathematics in the *Sthânâṅga-sûtra* (*c.* 300 B.C.) (*Sûtra* 747). There it probably refers to the Rules of Three, Five, Seven, etc.

[2] In his commentary on the *Âryabhaṭîya*.

[3] The above corresponds to *âryâ* 26 and the first half of *âryâ* 27 of the *Gaṇitapâda* of the *Âryabhaṭîya*; compare the working of *Example* I, where the interchange of denominators takes place. See also pp. 195*f*.

Brahmagupta gives the rule thus:

"In the Rule of Three *pramâṇa* ("argument"), *phala* ("fruit") and *icchâ* ("requisition") are the (given) terms; the first and the last terms must be similar. The *icchâ* multiplied by the *phala* and divided by the *pramâṇa* gives the fruit (of the demand)."[1]

Srîdhara states:

"Of the three quantities, the *pramâṇa* ("argument") and *icchâ* ("requisition") which are of the same denomination are the first and the last; the *phala* ("fruit") which is of a different denomination stands in the middle; the product of this and the last is to be divided by the first."[2]

Mahâvîra writes:

"In the Rule of Three, the *icchâ* ("requisition") and the *pramâṇa* ("argument") being similar, the result is the product of the *phala* and *icchâ* divided by the *pramâṇa*."[3]

Âryabhaṭa II introduces a slight variation in the terminology. He says:

"The first term is called *mâna*, the middle term *vinimaya* and the last one *icchâ*. The first and the last are of the same denomination. The last multiplied by the middle and divided by the first gives the result."[4]

Bhâskara II, Nârâyaṇa and others give the rule in the same form as Brahmagupta or Srîdhara.

The Hindu method of working the rule may be illustrated by the following examples taken from the *Triśatikâ*:

[1] *BrSpSi*, p. 178.
[2] *Triś*, p. 15.
[3] *GSS*, p. 58(2).
[4] *MSi*, p. 149.

Example I.[1] "If one *pala* and one *karṣa* of sandal wood are obtained for ten and a half *pana*, for how much will be obtained nine *pala* and one *karṣa*?"

Here 1 *pala* and 1 *karṣa*$=1\frac{1}{4}$ *pala*, and 9 *pala* and 1 *karṣa*$=9\frac{1}{4}$ *pala* are the similar quantities. The "fruit" $10\frac{1}{2}$ *paṇa* corresponding to the first quantity ($1\frac{1}{4}$ *pala*) is given, so that

pramâṇa (argument)	=	$1\frac{1}{4}$
phala (fruit)	=	$10\frac{1}{2}$
icchâ (requisition)	=	$9\frac{1}{4}$

The above quantities are placed in order as

1	10	9
1	1	1
4	2	4

Converting these into proper fractions we have

5	21	37
4	2	4

Multiplying the second and the last and dividing by the first, we have

21 2	5 4
37 4	

$$=\frac{\frac{21}{2}\times\frac{37}{4}}{\frac{5}{4}}$$

Or transferring denominators

21	5
4	2
37	4

$$=\frac{21.4.37}{5.2.4}\ \textit{pala}$$

$=4$ *purâṇa*, 13 *paṇa*, 2 *kâkiṇî* and 16 *varâṭaka*.

In actual working the intermediate step

$$\frac{\frac{21}{2}\times\frac{37}{4}}{\frac{5}{4}}$$

[1] *Triś*, p. 15.

was not written. The denominators of the multipliers were transferred to the side of the divisor and that of the divisor to the multipliers, thus giving at once

$$\frac{21.4.37}{5.2.4}.$$

Example II.[1] "Out of twenty necklaces each of which contains eight pearls, how many necklaces, each containing six pearls, can be made?"

Firstly, we have

1	8	20

The result (performing the operation of the Rule of Three) is 160 pearls.

Secondly, perform the operation of the Rule of Three on the following:

If 6 pearls are contained in one necklace, how many necklaces will contain 160 pearls?

Placing the numbers, we have

6	1	160

Result: necklaces 26, part of necklace $\boxed{\begin{matrix}2\\3\end{matrix}}$.

Inverse Rule of Three. The Hindu name for the Inverse Rule of Three is *vyasta-trairâśika* (lit. "inverse rule of three terms"). After describing the method of the Rule of Three the Hindu writers remark that the operation should be reversed when the proportion is inverse. Thus Śrîdhara observes:

"The method is to multiply the middle term by the first and to divide by the last, in case the proportion is different."[2]

[1] *Triś*, p. 17.
[2] *Triś*, p. 18.

Mahâvîra says:

"In the case of this (proportion) being inverse, the operation is reversed."[1]

Bhâskara II writes:

"In the inverse (proportion), the operation is reversed."[2]

He further observes:

"Where with increase of the *icchâ* (requisition) the *phala* decreases or with its decrease the *phala* increases, there the experts in calculation know the method to be the Inverse Rule of Three."[3]

"Where the value of living beings is regulated by their age; and in the case of gold, where the weight and touch are compared; or when heaps are subdivided,[4] let the Inverse Rule of Three be used."[5]

Example: Example II given under the Rule of Three above has been solved also by the application of the Inverse Rule as follows:

"Statement

8	20	6

Result: necklaces

26
2
3

"

Here if the *icchâ*, i.e., the number of pearls in a necklace, increases, the *phala*, *i.e.*, the number of necklaces, decreases, so that the Inverse Rule of Three is applied.

Appreciation of the Rule of Three. The Rule of Three was highly appreciated by the Hindus because

[1] *GSS*, p. 58(2).

[2] *L*, p. 17.

[3] *L*, p. 17.

[4] "When heaps of grain, which have been meted with a small measure, are again meted with a larger one, the number decreases . . . ." (Com. of Sûryadâsa).

[5] *L*, p. 18.

of its simplicity and its universal application to ordinary problems. The method as evolved by the Hindus gives a ready rule which can be applied even by the "ignorant person" to solve problems involving proportion, without fear of committing errors. Varâhamihira (505) writes:

"If the sun performs one complete revolution in a year, how much does he accomplish in a given number of days? Does not even an ignorant person calculate the sun in such problems by simply scribbling with a piece of chalk?"[1]

Bhâskara II has eulogised the method highly at several places in his work. His remarks are:

"The Rule of Three is indeed, (the essence of) arithmetic."[2]

"As Lord Śri Nârâyaṇa, who relieves the sufferings of birth and death, who is the sole primary cause of the creation of the universe, pervades this universe through His own manifestations as worlds, paradises, mountains, rivers, gods, men, demons, etc., so does the Rule of Three pervade the whole of the science of calculation. ....Whatever is computed whether in algebra or in, this (arithmetic) by means of multiplication and division may be comprehended by the sagacious learned as the Rule of Three. What has been composed by the sages through the multifarious methods and operations such as miscellaneous rules, etc., teaching its easy variations, is simply with the object of increasing the comprehension of the duller intellects like ourselves."[3]

On another occasion Bhâskara II observes:

[1] *PSi*, iv. 37.
[2] *L*, p. 15. The same remark occurs in *SiŚi*, *Golâdhyâya*, *Praśnâdhyâya*, verse 3.
[3] *L*, p. 76.

"Leaving squaring, square-root, cubing and cube-root, whatever is calculated is certainly variation of the Rule of Three, nothing else. For increasing the comprehension of duller intellects like ours, what has been written in various ways by the learned sages having loving hearts like that of the bird *cakora*, has become arithmetic."[1]

Proportion in the West. The history of the Hindu rules of proportion shows how much the West was indebted to India for its mathematics. The Rule of Three occurs in the treatises of the Arabs and mediæval Latin writers, where the Hindu name 'Rule of Three' has been adopted. Although the Hindu names of the terms were discarded, the method of placing the terms in a line, and arranging them so that the first and last were similar, was adopted. Thus Digges (1572) remarked,[2] "Worke by the Rule ensueing ...... Multiplie the last number by the seconde, and diuide the Product by the first number," ... "In the placing of the three numbers this must be observed, that the first and third be of one Denomination." The rule, as has been already stated, was perfected in India in the early centuries of the Christian era. It was transmitted to the Arabs probably in the eighth century and thence travelled to Europe, where it was held in very high esteem[3] and called the "Golden Rule."

Compound Proportion. The Hindu names for compound proportion are the Rule of Five, the Rule of

[1] *SiŚi*, *Golâdhyâya*, *Praśnâdhyâya*, verse 4.

[2] Quoted by Smith, *l.c.* p. 488.

[3] The Arabs, too, held the method in very high esteem as is evidenced by Al-Bîrûnî's writing a separate treatise, *Fi raśikat al-hind* ("On the *raśika* of the Hindus") dealing with the Hindu Rules of Three or more terms. Compare also *India* (I. 313) where an example of *vyasta-trairâśika*, ("the Inverse Rule of Three") is given.

Seven, the Rule of Nine, etc., according to the number of terms involved in the problems. These are sometimes grouped under the general appellation of the "Rule of Odd Terms." The above technical terms as well as the rules were well-known in the time of Âryabhaṭa I (499), although he mentions the Rule of Three only. That the distinction between the Rule of Three and Compound Proportion is more artificial than real was stressed by Bhâskara I (*c.* 525) in his commentary on the *Âryabhaṭîya*. He says:

"Here Âcârya Âryabhaṭa has described the Rule of Three only. How the well-known Rules of Five, etc. are to be obtained? I say thus: The Âcârya has described only the fundamentals of *anupâta* (proportion). All others such as the Rule of Five, etc., follow from that fundamental rule of proportion. How? The Rule of Five, etc., consist of combinations of the Rule of Three .... In the Rule of Five there are two Rules of Three, in the Rule of Seven, three Rules of Three, and so on. This I shall point out in the examples."

Remarks similar to the above concerning the Rules of Five, Seven, etc., have been made by the commentators of the *Lîlâvatî*, especially by Gaṇeśa and Sûryadâsa.[1]

In problems on Compound Proportion, two sets of terms are given. The first set which is complete is called *pramâṇa pakṣa* (argument side) and the second set in which one term is lacking is called the *icchâ pakṣa* (requisition side).

The Method. The rule relating to the solution of problems in compound proportion has been given by Brahmagupta as follows:

"In the case of odd terms beginning with three

[1] Noted by Colebrooke, *l.c.*, p. 35, note.

terms[1] upto eleven, the result is obtained by transposing the fruits of both sides, from one side to the other, and then dividing the product of the larger set of terms by the product of the smaller set. In all the fractions the transposition of denominators, in like manner, takes place on both sides."[2]

Srîdhara says:

"Transpose the two fruits from one side to the other, then having transposed the denominators (also in like manner) and multiplied the numbers (so obtained on each side), divide the side with the larger number of terms by the other (side)."[3]

Mahâvîra[4] and Âryabhaṭa II[5] have given the rule in the same way as Srîdhara. Bhâskara II has given it thus:

"In the rules of five, seven, nine or more terms, after having taken the *phala* (fruit) and *chid*[6] from its

[1] It should be observed that, as stated above, the Rule of Three is a particular case of the above Rule of Odd terms. Brahmagupta is the only Hindu writer to have included the Rule of Three also in the above rule. Some Arab writers have followed him in this respect by not writing the terms of the Rule of Three in a line, but arranging them in compartments, as for the other rules of odd terms.

[2] *BrSpSi*, p. 178.

[3] *Triś*, p. 19.

[4] *GSS*, p. 62 (32).

[5] *MSi*, p. 150, rules 26 and 27 (repeated with a slight variation).

[6] The commentators differ as regards the interpretation of this word. Some take it to mean "divisor," *i.e.*, "denominator," while others say that it means "the fruit of the other side." The rule is, however, correct with either interpretation. The first interpretation, however, brings Bhâskara's version in line with those of his predecessors. It may be mentioned here that Ârya-bhaṭa II repeats the rule twice. At first he does not direct the transposition of denominator, and at the second time he does so.

own side to the other, the product of the larger set of terms divided by the product of the smaller set, gives the result (or produce sought)."[1]

Illustration. We shall illustrate the Hindu method of working by solving the following example taken from the *Lîlâvatî*:

"If the interest of a hundred in one month be five, what will be the interest of 16 in 12 months? Also find the time knowing the interest and principal; and tell the principal knowing the time and interest."

To find interest.

The first set of terms (*pramâṇa pakṣa*) is:

100 *niṣka*, 1 month, 5 *niṣka* (*phala*)

The second set (*icchâ pakṣa*) is:

16 *niṣka*, 12 months, x *niṣka*

The terms are now written in compartments[2] as below:

100	16
1	12
5	0

[3]

[1] *L*, p. 18.

[2] The terms of the same denomination are written in compartments in the same horizontal line.

[3] The figures are written in compartments in order to facilitate the writing of fractions and also to denote the side which contains more terms after transposition of fruits. Sometimes, the compartment corresponding to an absent term is left vacant as we find in a copy of Munîśvara's *Pâṭîsâra* (in the Government Sanskrit Library at Benares). When the terms are written in compartments, the symbol o to denote the unknown or absence of a term is unnecessary. In some commentaries on the *Lîlâvatî* (Asiatic Society of Bengal manuscripts) we find the numbers written without compartments, but in such cases the symbol o is used to denote the absence of a term. After transposition, the side on which o occurs contains a smaller number of terms than the other.

In the above 5 (written lowest) is the "fruit" of the first side, and there is no "fruit" on the second side. Interchanging the fruits we get

100	16
1	12
0	5

The larger set of terms is on the second "side." The product of the numbers is 960. The product of the numbers on the side of the smaller set of terms is 100. Therefore, the required result is $\frac{960}{100}=\frac{48}{5}$, written as $\left|\frac{48}{5}\right|$ or 9 *niṣka*, fraction $\left|\frac{3}{5}\right|$

To find Time:

Here the sides are

100 *niṣka*, 1 month, 5 *niṣka*

and 16 *niṣka*, x months, $\frac{48}{5}$ *niṣka*

The terms are written as

100	16
1	0
5	48 5

Transposing the fruits, *i.e.*, transposing the numbers in the bottom compartment, we get

100	16
1	0
48 5	5

Transposing the denominators we have

100	16
1	0
48	5 5

Here, the larger set of terms is on the first side and their product is 4800. The product of the numbers on the side of the smaller set is 400. Therefore, the result is

$$\begin{array}{|r|}4800\\ 400\end{array} = \frac{4800}{400} = 12 \text{ months.}$$

To know the principal:

The first side is

100 *niṣka*, 1 month, 5 *niṣka*

The second side is

x *niṣka*, 12 months, $\frac{48}{5}$ *niṣka*

This is written as

100	0
1	12
5	48 5

After transposition of fruits (*i.e.*, the terms in the bottom cells) we have

100	0
1	12
48 5	5

Transposing denominators we get

100	0
1	12
48	5 5

The product of the numbers in the larger set divided by the product of the numbers in the smaller set, gives

$$\left|\begin{matrix} 4800 \\ 300 \end{matrix}\right| = 16 \textit{ niṣka}.$$

Rule of Three as a Particular Case. According to Brahmagupta, the above method may be applied to the Rule of Three. Taking the first example solved under the Rule of Three, above, and placing the terms we have

21 2	0
5 4	37 4

[1]

Transposing the fruits, we have

21 2	0
37 4	5 4

Transposing denominators, we get

21	0 2
37 4	5 4

Therefore, the result is $\dfrac{21.37.4}{2.5.4}$ as before.

If we consider the term corresponding to the unknown as the fruit, the terms should be set as below:

[1] Here, we consider $\frac{5}{4}$ *pala* of sandal wood as the "fruit" of $\frac{21}{2}$ *paṇa* (money). The previous method forces us to consider $\frac{21}{2}$ *paṇa* as the "fruit" or the middle term, because the "first" and "third" are directed to be alike. It will be observed that any of the terms may be considered to be the fruit in the alternative method given here.

5	37
4	4
21	
2	0

Hence, as before,[1] the result is $\dfrac{37.4.21}{5.4.2}$

The above method of working the Rule of Three is found among the Arabs,[2] although it does not seem to have been used in India after Brahmagupta. This points to the indebtedness of the Arabs to Brahmagupta especially, for their knowledge of Hindu arithmetic.

Written as above the method of working the Rule of Three appears to be the same as the method of proportion. In the same way the rule of other odd terms, when properly translated into modern symbolism, is nothing but the method of proportion. It has been stated by Smith[3] that the Hindu methods of solution "fail to recognize the relation between the Rule of Three and proportion." This statement appears to have been made without sufficient justification, for the solutions have been evidently obtained by the use of the ideas of proportionality and variation. The aim of the Hindu works is to give a method which can be readily used by common people. For this very reason, the cases in which the variation is inverse have been enumerated. Considered as a method which stimulated the student to think for himself, the method is certainly

[1] The product of the numbers on the side of the larger set is divided by the product of the numbers on the side of the smaller set. o in this case is not a number. It is the symbol for the unknown or absence.

[2] Thus Rabbi ben Ezra wrote $\begin{smallmatrix}4 & 7\\ & 7\end{smallmatrix}$ $\begin{smallmatrix}6 & 3\\ & 0\end{smallmatrix}$ for $47 : 7 = 63 : x$. See Smith, *l.c.*, p. 489f.

[3] *l.c.*, p. 488.

defective, but for practical purposes, it is, in our opinion, the best that could be devised.

13. COMMERCIAL PROBLEMS

Interest in Ancient India. The custom of taking interest is a very old one. In India it can be definitely traced back to the time of Pâṇini (*c.* 700 B.C.) who in his Grammar lays down rules validating the use of the suffix *ka* to number names in case of "an interest, a rent, a profit, a tax or a bribe given."[1] The interest became due every month and the rate of interest was generally given per hundred,[2] although this was not always the case. The rate of interest varied in different localities and amongst different classes of people, but an interest of fifteen per cent per year seems to have been considered just. Thus in Kauṭilya's *Arthaśâstra*, a work of the fourth century B.C., it is laid down: "an interest of a *paṇa* and a quarter per month per cent is just. Five *paṇa* per month per cent is commercial interest. Ten *paṇa* per month per cent prevails in forests. Twenty *paṇa* per month per cent prevails among sea traders."[3] The *Gotama Sûtra* states: "an interest of five *mâṣâ* per twenty (*kârṣâpaṇa*) is just."[4]

Interest in Hindu Ganita. The ordinary problems relating to the finding out of interest, principal or time etc., the other quantities being given, occur in the section dealing with the Rule of Five. The Hindu

[1] Pâṇini's Grammar, v. i. 22, 47, 49.

[2] It has been pointed out by B. Datta that the idea of *per cent* first originated in India. See his article in the *American Mathematical Monthly*, XXXIV, p. 530.

[3] *Arthaśâstra*, edited and translated into English by R. Shamsastry, Mysore, III, ii, p. 214.

[4] *Gotama Sûtra*, xii. 26. Since 20 *mâṣâ* equal a *kârṣâpaṇa*, the rate is 15 per cent annually.

works generally contain a section called *miśraka-vyavahâra* ("calculations relating to mixed quantities") in which occur miscellaneous problems on interest. The contents of this section vary in different works, according to their size and scope. Thus the *Âryabhaṭîya* contains only one rule relating to a problem on interest, whilst the *Gaṇita-sâra-saṁgraha* has a large number of such rules and problems.

Problem involving a Quadratic Equation. Âryabhaṭa I (499) gives a rule for the solution of the following problem:

The principal sum $p(=100)$ is lent for one month (interest unknown$=x$). This unknown interest is then lent out for $t(=\text{six})$ months. After this period the original interest (x) plus the interest on this interest amounts to $A(=\text{sixteen})$. The rate-interest (x) on the principal (p) is required.

The above problem requires the solution of the quadratic equation

$$tx^2+px-Ap=0,$$

which gives $$x=\frac{-p/2\pm\sqrt{(p/2)^2+Apt}}{t}.$$

The negative value of the radical does not give a solution of the problem; so the result is

$$x=\frac{\sqrt{Apt+(p/2)^2}-p/2}{t}.$$

This is stated by Âryabhaṭa I as follows:

"Multiply the sum of the interest on the principal and the interest on this interest (A) by the time (t) and by the principal (p). Add to this result the square of half the principal $\{(p/2)^2\}$. Take the square-root of this. Subtract half the principal ($p/2$) and divide the remainder by the time (t). The result will be the (unknown) interest (x) on the principal."[1]

Brahmagupta (628) gives a more general rule. His problem is:

The principal (p) is lent out for t_1 months and the unknown interest on this ($=x$) is lent out for t_2 months at the same rate and becomes A. To find x.

This gives the quadratic

$$x^2 + \frac{pt_1}{t_2}x - \frac{Apt_1}{t_2} = 0,$$

whose solution is

$$x = \pm\sqrt{\frac{Apt_1}{t_2} + \left(\frac{pt_1}{2t_2}\right)^2} - \frac{pt_1}{2t_2}.$$

The negative value of the radical does not give a solution of the problem, so it is discarded.

Brahmagupta states the formula thus:

"Multiply the principal (p) by its time (t_1) and divide by the other time (t_2) (placing the result) at two places. Multiply the first of these by the mixture (A). Add to this the square of half the other. Take the square-root of this (sum). From the result subtract half the other. This will be the interest (x) on the principal."[2]

Other Problems. Mahâvîra (850) gives two other types of problems on "mixture" requiring the solution of simultaneous equations. As an example of the first type may be mentioned the following:[3]

"It has been ascertained that the interest for $1\frac{1}{2}$ months (t=rate-time) on 60 (c=rate-capital) is $2\frac{1}{2}$ (i=

[1] *A*, p. 41. The Sanskrit terms are: *mûla*=principal, *phala* =interest.

[2] *BrSpSi*, p. 183. This rule is also given by Mahâvîra, *GSS*, p. 71 (44).

[3] *GSS*, p. 69(32).

rate-interest). The interest (on the unknown capital P) for an unknown period (T) is 24 ($=I$), and 60 ($=m=P+T$) is the time combined with the capital lent out. What is the time (T) and what is the capital (P)?"

The problem gives:

$$\frac{iPT}{ct} = I \quad \ldots \quad (1)$$

$$P+T = m \quad \ldots \quad (2)$$

$$\therefore \; P-T = \pm\sqrt{m^2 - \frac{ct}{i} \times 4I}$$

Hence $$P = \tfrac{1}{2}\left(m \pm \sqrt{m^2 - \frac{ct}{i} \times 4I}\right),$$

and $$T = \tfrac{1}{2}\left(m \mp \sqrt{m^2 - \frac{ct}{i} \times 4I}\right)$$

The above result is stated by Mahâvîra thus:

"From the square of the mixture (m) subtract the rate-capital (c) divided by the rate-interest (i) multiplied by the rate-time (t) and four times the given interest (4I). Then the operation of *saṅkramaṇa*[1] is performed in relation to the square-root of this and the mixture (m)."[2]

The second type of problems may be illustrated by the following example:

"The interest on 30 (P) is 5 (I) for an unknown

[1] Given the numbers a and b, the process of *saṅkramaṇa* is the finding out of half their sum and difference *i.e.* $\frac{a+b}{2}$ and $\frac{a-b}{2}$.

[2] *GSS*, p. 68(29). It should be noted that both the signs of the radical are used.

period (T), and at an unknown rate of interest (i) per 100 (c) per $1\frac{1}{2}$ month (t). The mixture ($m=i+T$) is $12\frac{1}{2}$. Find i and T."[1]

The solution is given by

$$T = \tfrac{1}{2}\left(m \pm \sqrt{m^2 - \frac{ctI.4}{P}}\right),$$

and consequently

$$i = \tfrac{1}{2}\left(m \mp \sqrt{m^2 - \frac{ctI.4}{P}}\right).$$

Mahâvîra states the solution thus:

"The rate-capital (c) multiplied by its time (t) and the interest (I) and the square of two ($=4$) is divided by the other capital (P). Then perform the operation of *saṅkramaṇa* in relation to the square-root of the remainder (obtained as the result of subtracting the quotient so obtained) from the square of the mixture (m) and the mixture."[2]

Miscellaneous Problems on Interest. Besides the problems given above various other interesting problems are found in the Hindu works on *pâṭîgaṇita*. Thus Brahmagupta gives the solution of the following problem:

Example. In what time will a given sum s, the interest on which for t months is r, become k times itself?

The rule for the solution of the above is:[3]

"The given sum[4] multiplied by its time and divided

[1] *GSS*, p. 69(34).
[2] *GSS*, p. 69(33).
[3] *BrSpSi*, p. 181.
[4] The Sanskrit term used is *pramâṇa* (argument).

by the interest,[1] being multiplied by the factor[2] less one, is the time (required)."

The *Gaṇita-sâra-saṁgraha* (850) contains a large number of problems relating to interest. Of these may be mentioned the following:

(1) "In this (problem), the (given) capitals are ($c_1=$) 40, ($c_2=$) 30, ($c_3=$) 20 and ($c_4=$) 50; and the months are ($t_1=$) 5, ($t_2=$) 4, ($t_3=$) 3 and ($t_4=$) 6 (respectively). The sum of the interests is ($m=$) 34. (Assuming the rate of interest to be the same in each case, find the amounts of interest in each case)."[3]

Here, if the rate of interest per month for 1 be r, then

$$r = \frac{x_1}{c_1t_1} = \frac{x_2}{c_2t_2} = \frac{x_3}{c_3t_3} = \ldots\ldots.$$

where $x_1, x_2, x_3, \ldots\ldots$ are the interests earned on the capitals $c_1, c_2, c_3, \ldots\ldots$ in $t_1, t_2, t_3 \ldots\ldots$ months respectively.

Therefore,

$$\frac{x_1}{c_1t_1} = \frac{x_2}{c_2t_2} = \frac{x_3}{c_3t_3} = \ldots = \frac{x_1+x_2+x_3+\ldots.}{c_1t_1+c_2t_2+c_3t_3+\ldots}$$

$$= \frac{m}{c_1t_1+c_2t_2+c_3t_3+\ldots.}.$$

or $$x_1 = \frac{m\,c_1t_1}{c_1t_1+c_2t_2+c_3t_3+\ldots}, \quad \text{etc.}$$

This formula is given by Mahâvîra for the solution

[1] The Sanskrit term used is *phala* (fruit).

[2] The Sanskrit term used is *guṇa* (multiple).

[3] *GSS*, p. 70(38).

of the above problem.[1]

(2) "(Sums represented by) 10, 6, 3 and 15 are the (various given) amounts of interest, and 5, 4, 3 and 6 are the (corresponding) months (for which the interests have accrued); the sum of the (corresponding) capital amounts is seen to be 140. (Assuming the rate of interest to be the same in each case, find out these capital amounts)."[2]

(3) "Here (in this problem) the (given) capital amounts are 40, 30, 20 and 50; and 10, 6, 3 and 15 are the (corresponding) amounts of interest; 18 is the quantity representing the mixed sum of the respective periods of time. (Find out these periods separately, assuming the rate of interest to be the same in each case)."[3]

(4) "The interest on 80 for 3 months is unknown; $7\frac{4}{5}$ is the mixed sum of that (unknown quantity taken as the) capital lent out and of the interest thereon for 1 year. What is the capital here and what the interest?"[4]

(5) "The mixed sums (capital+interest) are 50, 58 and 66, and the months (during which interests have accrued) are 5, 7 and 9 (respectively). Find out what

[1] *GSS*, p. 70(37). The formula clearly shows that Mahâvîra knew the algebraic identity

$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots=\frac{a+c+e+\ldots}{b+d+f+\ldots}.$$

[2] *GSS*, p. 70(40). The solution is given by Rule 39 on the same page.

[3] *GSS*, p. 70(43). The solution is given by Rule 42 on the same page.

[4] *GSS*, p. 71(46). This is similar to Âryabhaṭa's problem given before (p. 217).

the interest is (in each case, the capital being the same)?"[1]

(6) "The mixed sums of the capital and periods of interest are 21, 23, and 25; here (in this problem) the amounts of interest are 6, 10 and 14. What is the common capital?"[2]

(7) "Borrowing at the rate of 6 per cent and then lending out at the rate of 9 per cent, one obtains in the way of differential gain 81 at the end of 3 months. What is the capital (utilised here)?"[3]

(8) "The monthly interest on 60 is exactly 5. The capital lent out is 35; the (amount of the) instalment (to be paid) is 15 in (every) 3 months. What is the time of discharge of that debt?"[4]

(9) "The mixed sum (of the capital amounts lent out) at the rates of 2, 6 and 4 per cent per mensem is 4400. Here the capital amounts are such as have equal amounts of interest accruing after 2 months. What (are the capital amounts lent, and what is the equal interest)?"[5]

(10) "A certain person gives once in 12 days an instalment of $2\frac{3}{5}$, the rate of interest being 3 per cent (per mensem). What is the capital amount of the debt discharged in 10 months?"[6]

(11) "The total capital represented by 8520 is invested (in parts) at the (respective) rates of 3, 5 and 8 per cent (per month). Then, in this investment, in 5

[1] *GSS*, p. 71 (48). The solution requires the use of the identity

$$\frac{a}{b} = \frac{c}{d} = \frac{a-c}{b-d}.$$

[2] *GSS*, p. 72(52).
[3] *GSS*, p. 72(55).
[4] *GSS*, p. 73(59).
[5] *GSS*, p. 73(61).
[6] *GSS*, p. 73(65).

months the capital amounts lent out are, on being diminished by the respective amounts of interest, (found to be) equal in value. (What are the respective amounts invested thus?)"[1]

(12) "The total capital represented by 13740 is invested (in parts) at the (respective) rates of 2, 5 and 9 per cent (per month), then, in this investment, in 4 months the capital amounts lent out are, on being combined with the (respective) amounts of interest, (found to be) equal in value. (What are the respective amounts thus invested?)"[2]

(13) "A certain man borrows a certain (unknown) sum of money at an interest of 5 per cent per month. He pays the debt in instalments, due every $\frac{3}{5}$ of a month. The instalments begin with 7 and increase in arithmetical progression, with 7 as the common-difference. 60 is the maximum amount of instalment. He gives in the discharge of his debt the sum of a series in arithmetical progression consisting of $\frac{60}{7}$ terms. After the payment of each instalment, interest is charged only on that part of the principal which remains to be paid. What is the total payment corresponding to the sum of the series, what is the interest (which he paid), what is the time of the discharge of the debt, (and what is the principal sum borrowed)?"[3]

Barter and Exchange. The Hindu name for barter is *bhâṇḍa-prati-bhâṇḍa* ("commodity for commodity"). All the Hindu works on *pâṭîgaṇita* contain problems relating to the exchange of commodities. It is pointed out in these works that problems on barter are cases of compound proportion, and can be solved by the

[1] *GSS*, p. 74(67).
[2] *GSS*, p. 74(67).
[3] *GSS*, pp. 74f, (72-73½). The text of the problem is very obscure. The translation given here is after Rangacarya.

Rule of Five, etc. A typical problem on barter is the following:

"If three hundred mangoes be had in this market for one *dramma*, and thirty ripe pomegranates for a *paṇa*, say quickly, friend, how many (pomegranates) should be had in exchange for ten mangoes?"[1]

Other Types of Commercial Problems. Of various other types of commercial problems found in the Hindu works may be mentioned (1) problems on partnership and proportionate division, and (2) problems relating to the calculation of the fineness of gold.[2] Most of these problems are essentially of an algebraic character, but they are included in *pâṭîgaṇita* (arithmetic). The formulæ giving the solution of each type of examples precede the examples. These formulæ are too numerous to be mentioned. The following examples, however, will illustrate the nature and the scope of such problems:

(1) A horse was purchased by (nine) dealers in partnership, whose contributions were one, etc., upto nine; and was sold by them for five less than five hundred. Tell me what was each man's share of the sale-proceed.

(2) Four colleges, containing an equal number of pupils, were invited to partake of a sacrificial feast. A fifth, a half, a third and a quarter (of the total number of pupils in the college) came from the respective colleges to the feast; and added to one, two, three and four, they were found to amount to eighty-seven; or, with those deducted, they were sixty-seven. Find the actual number of the pupils that came from each college.

[1] *L*, p. 20.

[2] Such problems are found in the *Lîlâvatî*, the *Gaṇita-sâra-saṁgraha*, the *Triśatikâ*, etc.

(3) Three (unequal) jars of liquid butter, of water and of honey, contained thirty-two, sixty and twenty-four *pala* respectively: the whole was mixed together and the jars filled again. Tell me the quantity of butter, of water and of honey in each jar.[1]

(4) According to an agreement three merchants carried out the operations of buying and selling. The capital of the first consisted of six *purâṇa*, that of the second of eight *purâṇa*, but that of the third was unknown. The profit obtained by these men was 96 *purâṇa*. In fact the profit obtained by him (the third person) on his unknown capital happened to be 40 *purâṇa*. What was the amount thrown by him into the transaction and what was the profit of each of the other two merchants?[2]

(5) There were four merchants. Each of them obtained from the others half of what he had with him (at the time of the respective transfers of money). Then they all became possessed of equal amounts of money. What was the measure of money each had to start with?[3]

(6) A great man possessing powers of magical charm and medicine saw a cock fight going on, and spoke separately in confidential language to both the owners of the cocks. He said to one, "If your bird wins, then you give the stake-money to me. If, however, your bird loses then I shall give you two-thirds of that stake-money." He went to the owner of the other cock and promised to give three-fourths (of his stake-money on similar conditions). In each case the gain to him could be only 12 (gold-pieces). Tell me, O ornament

[1] This and the two previous examples are given by Pṛthudakasvâmî to illustrate Rule 16 of the *gaṇitâdhyâya* of the *Brâhma-sphuṭa-siddhânta*.

[2] *GSS*, p. 94(223-5).

[3] *GSS*, p. 99(267½).

on the head of mathematicians, the money each of the cock-owners had staked.[1]

(7) The mixed price of 9 citrons and 7 fragrant wood-apples is 107; again the mixed price of 7 citrons and 9 fragrant wood-apples is 101. O arithmetician, tell me quickly the price of a citron and of a wood-apple, having distinctly separated those prices.[2]

(8) Pigeons are sold at the rate of 5 for 3 (*paṇa*), *sârasa* birds at the rate of 7 for 5 (*paṇa*), swans at the rate of 9 for 7 (*paṇa*) and peacocks at the rate of 3 for 9 (*paṇa*). A certain man was told to bring at these rates 100 birds for 100 (*paṇa*) for the amusement of the king's son, and was sent to do so. What (amount) does he give for each (of the various kinds of birds that he buys)?[3]

(9) There are 1 part (of gold) of 1 *varṇa*, 1 part of 2 *varṇa*, 1 part of 3 *varṇa*, 2 parts of 4 *varṇa*, 4 parts of 5 *varṇa*, 7 parts of 14 *varṇa*, and 8 parts of 15 *varṇa*. Throwing these into the fire, make them all into one (mass), and then (say) what the *varṇa* of the mixed gold is. This mixed gold is distributed among the owners of the foregoing parts. What does each of them get?[4]

(10) Three pieces of gold, of 3 each in weight, and of 2, 3 and 4 *varṇa* (respectively), are added to (an unknown weight of) gold of 13 *varṇa*. The resulting *varṇa* comes to be 10. Tell me, O friend, the measure (of the unknown weight) of gold.[5]

[1] *GSS*, pp. 99-100(270½-2½).
[2] *GSS*, p. 84(140½-2½).
[3] *GSS*, p. 85(152-3).
[4] *GSS*, p. 88 (170-1½).
[5] *GSS*, p. 89(181). Similar examples occur in the *Triśatikâ* (p. 26) and the *Lîlâvatî* (p. 25).

14. MISCELLANEOUS PROBLEMS

Regula Falsi. The rule of false position is found in all the Hindu works.[1] Bhâskara II gives prominence to the method and calls it *iṣṭa-karma* ("rule of supposition"). He describes the method thus:

"Any number, assumed at pleasure, is treated as specified in the particular question, being multiplied and divided, increased or diminished by fractions (of itself); then the given quantity, being multiplied by the assumed number and divided by that (which has been found) yields the number sought. This is called the process of supposition."[2]

Srîdhara takes the assumed number to be one.[3] Mahâvîra gives a large variety of problems to which he applies the rule.[4] Gaṇeśa in his commentary on the *Lîlâvatî* remarks, "In this method, multiplication, division, and fractions only are employed." The following examples will illustrate the nature of the problems solved by the rule of supposition:

(1) Out of a heap of pure lotus flowers, a third, a fifth, a sixth were offered respectively to the gods Siva, Viṣṇu and Sûrya and a quarter was presented to Bhavânî. The remaining six were given to the venerable preceptor. Tell quickly the number of lotuses.[5]

(2) The third part of a necklace of pearls, broken in

[1] The method originated in India and went to Europe through Arabia. There is a mediæval MS., published by Libri in his *Histoire*, I, 304 and possibly due to Rabbi ben Ezra in which the method is attributed to the Hindus. For further details and references, see Smith, *History*, II, p. 437, foot-note 1.

[2] *L*, p. 10.

[3] See the rule on *stambhoddeśa*, *Triś*, p. 13.

[4] These problems occur in chapters iii and iv of the *Gaṇita-sâra-saṁgraha*.

[5] *L*, p. 11. *Cf. GSS*, p. 48 (7).

an amorous struggle, fell to the ground; its fifth part rested on the couch; the sixth part was saved by the wench; and the tenth part was taken by her lover: six pearls remained strung. Say, of how many pearls was the necklace composed?[1]

(3) One-twelfth part of a pillar, as multiplied by $\frac{1}{30}$ part thereof, was to be found under water; $\frac{1}{20}$ of the remainder, as multiplied by $\frac{3}{16}$ thereof, was found buried in the mire below; and 20 *hasta* of the pillar were found in the air (above the water). O friend, give out the length of the pillar.[2]

(4) A number of parrots descended on a paddy field, beautiful with crops bent down through the weight of ripe corn. Being scared away by men, all of them suddenly flew off. One-half of them went to the east, one-sixth went to the south-east; the difference between those that went to the east and those that went to the south-east, diminished by half of itself and again diminished by half of this (resulting difference), went to the south. The difference between those that went to the south and those that went to the south-east diminished by two-fifths of itself went to the south-west; the difference between those that went to the south and those that went to the south-west, went to the west; the difference between those that went to the south-west and those that went to the west, together with three-sevenths of itself went to the north-west; the difference between those that went to the north-west and those that went to the west together with seven-eighths of itself, went to the north; the sum of those that went to the north-west and those that went to the north, diminished by two-thirds of itself went to the north-east; and 280 parrots were found to

[1] *Triś*, p. 14, *cf. GSS*, p. 49 (17-22) for a similar example.
[2] *GSS*, p. 55(60). *Cf. Triś*, p. 13.

remain in the sky above. How many were the parrots in all?[1]

The Method of Inversion. The method of inversion called *vilomagati* ("working backwards") is found to have been commonly used in India from very early times. Thus Âryabhaṭa I says:

"In the method of inversion multipliers become divisors and divisors become multipliers, addition becomes subtraction and subtraction becomes addition."[2]

Brahmagupta's description is more complete. He says:

"Beginning from the end, make the multiplier divisior, the divisor multiplier; (make) addition subtraction and subtraction addition; (make) square square-root, and square-root square; this gives the required quantity."[3]

The following examples will illustrate the nature of problems solved by the above method:

(1) What is that quantity which when divided by 7, (then) multiplied by 3, (then) squared, (then) increased by 5, (then) divided by $\frac{3}{5}$, (then) halved, and then reduced to its square-root happens to be the number 5?[4]

(2) The residue of degrees of the sun less three, being divided by seven, and the square-root of the quotient extracted, and the root less eight multiplied by nine, and to the product one being added, the amount is

[1] *GSS*, pp. 48f (12-16).

[2] *Â*, *Gaṇitapáda*, 28.

[3] *BrSpSi*, p. 301. The method occurs also in *GSS*, p. 102 (286); *MSi*, p. 149; *L*, p. 9; etc.

[4] *GSS*, p. 102 (287). Examples of this type are very common in Hindu arithmetic. They were also very common in Europe. Smith in his *History*, II, quotes two such problems from an American arithmetic of the 16th century.

a hundred. When does this take place on a Wednesday?[1]

Problems on Mixture. The Hindu works on *pâṭîgaṇita* contain a chapter relating to problems on mixture (*miśraka-vyavahâra*). Miscellaneous problems on interest, problems on allegation, and various other types of problems, in which quantities are to be separated from their mixture, form the subject matter of *miśraka-vyavahâra*. A chapter "on mixture" (*De' mescolo*) is found in early Italian works on arithmetic, evidently under Hindu influence.[2]

Some of the problems of this chapter are determinate and some are indeterminate. A few relating to interest and allegation have already been given.[3] The following are some others:

(1) In the interior of a forest, 3 heaps of pomegranates were divided (equally) among 7 travellers, leaving 1 fruit as remainder; 7 (of such heaps) were divided among 9, leaving a remainder of 3 (fruits), again 5 (of such heaps) were divided among 8, leaving 2 fruits as remainder. O mathematician, what is the numerical value of a heap?[4]

(2) On a certain man bringing mango fruits home, his elder son took one fruit first and then half of what remained. The younger son did similarly with what was left. He further took half of what was left thereafter; and the other took the other half. Find the number of fruits brought by the father?[5]

[1] Colebrooke, cha, p. 333 (18).

[2] Smith, *History*, II, p. 588, note 4.

[3] See commercial problems, pp. 216ff; also problems on proportionate division (*prakṣepa-karaṇa*): *Triś*, p. 26; *GSS*, p. 75(79½); *MSi*, pp. 154-155.

[4] *GSS*, p. 82 (128½). Such problems are given under the rule of *vallikâ-kuṭṭîkâra* by Mahâvîra.

[5] *GSS*, p. 82 (131½).

(3) A certain lay follower of Jainism went to a Jina temple with four gate-ways, and having taken (with him) fragrant flowers offered them in worship with devotion (at each gate). The flowers in his hand were doubled, trebled, quadrupled and quintupled (respectively in order) as he arrived at the gates (one after another). The number of flowers offered by him was sixty[1] at each gate. How many flowers were originally taken by him?

(4) The first man has 16 azure-blue gems, the second has 10 emeralds, and the third has 8 diamonds. Each among them gives to each of the others 2 gems of the kind owned by himself; and then all three men come to be possessed of equal wealth. What are the prices of those azure-blue gems, emeralds and diamonds?[2]

(5) In what time will four fountains, being let loose together, fill a cistern, which they would severally fill in a day, in half a day, in a quarter and in a fifth part of a day?[3]

Problems involving Solution of Quadratic Equations. The solution of the quadratic equation has been known in India from the time of Āryabhaṭa I (499). Problems on interest requiring the solution of the quadratic equation have already been mentioned. Mahâvîra and Bhâskara II give many other problems. Mahâvîra divides these problems into two classes: (*i*) those that involve square-roots (*mûla*) and (*ii*) those

[1] *GSS*, p. 79 ($112\frac{1}{2}$-$113\frac{1}{2}$). The printed text has *pañca* ("five"). According to it the answer is 43/12 which appears absurd. There are some other problems in the printed edition which give such absurd results. All those are, we presume, due to the defects of the mss. consulted by the editor. So here we have made the emendation 'sixty.'

[2] *GSS*, p. 87 (165-166).

[3] *BrSpSi*, p. 177 (com.); *L*, p. 23.

that involve the square (*varga*) of the unknown. The first type gives a single positive answer, while the second type has two answers corresponding to the two roots of the quadratic. Bhâskara II deals with the first type of problems only in his *pâṭîgaṇita*, the *Lîlâvatî*. The second type of problems, involving the square of the unknown has been treated by him in his *Bîjagaṇita* (algebra). The following examples will illustrate the nature and scope of such problems:

Problems involving the square-root:

(1) One-fourth of a herd of camels was seen in the forest; twice the square-root of that had gone to mountain slopes; and three times five camels were found to remain on the bank of a river. What was the numerical measure of that herd of camels?[1]

(2) Five and one-fourth times the square-root (of a herd) of elephants are sporting on a mountain slope; five-ninths of the remainder sport on the top of the mountain; five times the square-root of the remainder sport in a forest of lotuses; and there are six elephants then (left) on the bank of a river. How many are the elephants?[2]

(3) In a garden beautified by groves of various kinds of trees, in a place free from all living animals, many

[1] *GSS*, p. 51 (34). The problem belongs to the type of the *mûla-jâti*, and leads to an equation of the form $x-(bx+c\sqrt{x}+a)=0$. The method of solution is given in *GSS*, p. 50 (33).

[2] *GSS*, p. 52 (46). The problem is of the *śeṣa-mûla* variety. It gives the equation

$$x-\tfrac{21}{4}\sqrt{x}-\tfrac{5}{9}\left(x-\tfrac{21}{4}\sqrt{x}\right)-5\sqrt{x-\tfrac{21}{4}\sqrt{x}-\tfrac{5}{9}\left(x-\tfrac{21}{4}\sqrt{x}\right)}=6.$$

Mahâvîra reduces it by putting $z=x-\frac{21}{4}\sqrt{x}-\frac{5}{9}\left(x-\frac{21}{4}\sqrt{x}\right)$. to $z-5\sqrt{z}=6$. In the general case a similar equation is again obtained, which is again reduced, and so on till the equation is reduced to the form, $x-b\sqrt{x}=d$, from which x can be easily obtained.

ascetics were seated. Of them the number equivalent to the square-root of the whole collection were practising *yoga* at the foot of a tree. One-tenth of the remainder, the square-root (of what remained after this), $\frac{1}{9}$ (of what remained after this), then the square-root (of what remained after this), $\frac{1}{8}$ (of what remained after this), the square-root (of what remained after this), $\frac{1}{7}$ (of what remained after this), the square-root (of what remained after this), $\frac{1}{6}$ (of what remained after this), the square-root (of what remained after this), $\frac{1}{5}$ (of what remained after this), the square-root (of what remained after this)—these parts consisted of those who were learned in the teaching of literature, in religious law, in logic, and in politics, as also of those who were versed in controversy, prosody, astronomy, magic, rhetoric and grammar, as well as of those who possessed an intelligent knowledge of the twelve varieties of the *aṅga-śâstra*; and at last 12 ascetics were seen (to remain without being included among those mentioned before). O excellent ascetic, of what numerical value was this collection of ascetics?[1]

(4) A single bee (out of a swarm of bees) was seen in the sky; $\frac{1}{5}$ of the remainder (of the swarm), and $\frac{1}{4}$ of the remainder (left thereafter) and again $\frac{1}{3}$ of the remainder (left thereafter) and a number of bees equal to the square-root of the numerical value of the swarm, were seen in lotuses; and two bees were on a mango tree. How many were there?[2]

(5) Four times the square-root of half the number of a collection of boars went to a forest wherein tigers

[1] *GSS*, p. 52 (42-45). The problem is of the same variety as the above one. The substitution will have to be made 6 times to reduce the resulting equation.

[2] *GSS*, p. 53 (48). This problem is of the *dviragra-śeṣa-mûla* variety.

were at play; 8 times the square-root of $\frac{1}{10}$ of the remainder went to a mountain; and 9 times the square-root of $\frac{1}{2}$ of the (next) remainder went to the bank of a river; and boars equivalent in (numerical) measure to 56 were seen to remain in the forest. Give the numerical measure of all those boars.[1]

(6) The sum of two (quantities, which are respectively equivalent to the) square-root (of the numerical value) of a collection of swans and (the square-root of the same collection) as combined with 68, amounts to 34. How many swans there are in that collection?[2]

(7) Pârtha (Arjuna), irritated in fight, shot a quiver of arrows to slay Karṇa. With half his arrows, he parried those of his antagonist, with four times the square-root of the quiver-full, he killed his horses; with three he demolished the umbrella, standard and bow; and with one he cut off the head of his foe. How many were the arrows, which Arjuna let fly?[3]

(8) The square-root of half the number of a swarm of bees is gone to a shrub of jasmin; and so are eight-ninths of the whole swarm; a female is buzzing to one remaining male that is humming within a lotus, in which he is confined, having been allured to it by its

[1] *GSS*, p. 54 (56). This problem is of the *aṁśa-mûla* variety, wherein fractional parts of square-roots are involved. The problems give equations of the form

$$x-a_1\sqrt{b_1x} - a_2\sqrt{b_2(x-a_1\sqrt{b_1x})}$$
$$-a_3\sqrt{b_3[(x-a_1\sqrt{b_1x}) - a_2\sqrt{b_2(x-a_1\sqrt{b_1x})}} -\ldots=k.$$

By repeated substitutions Mahâvîra reduces the equation to the form $x - A\sqrt{Bx} - c = 0$.

[2] *GSS*, p. 56 (68). This problem is of the *mûla-miśra* variety, wherein the sum of square-roots is involved. It gives an equation of the form $\sqrt{x}+\sqrt{x\pm d} = m$.

[3] *L*, p. 16.

fragrance at night. Say, lovely woman, what is the number of bees.[1]

Problems involving the square of the unknown:

(9) One-twelfth part of a pillar, as multiplied by $\frac{1}{30}$ part thereof, was found under water; $\frac{1}{20}$ of the remainder, as multiplied by $\frac{3}{16}$ thereof, was found buried in the mire, and 20 *hasta* of the pillar were found in the air. O friend, give the measure of the length of the pillar.[2]

(10) A number of elephants (equivalent to) $\frac{1}{10}$ of the herd minus 2, as multiplied by the same ($\frac{1}{10}$ of the herd minus 2), is found playing in a forest of *sallakî* trees. The remaining elephants of the herd equal in number to the square of 6 are moving on a mountain. How many are the elephants?[3]

15. THE MATHEMATICS OF ZERO

It has been shown that the zero was invented in India about the beginning of the Christian era to help the writing of numbers in the decimal scale. The Hindu mind did not rest satisfied till it evolved the complete arithmetic of zero. The Hindus included zero among the numbers (*saṅkhyâ*), and it was used

[1] *L*, p. 16.

[2] *GSS*, p. 55 (60). The problem gives the equation

$$\left(x - \frac{x^2}{12.30}\right) - \frac{1.3}{20.16}\left(x - \frac{x^2}{12.30}\right)^2 = 20.$$

Also solved by *regula falsi*. Mahâvîra puts $\left(x - \frac{1}{12.30}x^2\right) = z$, and then solves the quadratic

$$z - \frac{3}{320}z^2 = 20.$$

The roots of this are then used to get the values of x.

[3] *GSS*, p. 55 (63).

in their arithmetic at the time when the original of the Bakhshâlî Manuscript was written, about the third century A.D. The operation of addition and subtraction of zero are incidentally mentioned in the *Pañca-siddhântikâ* of Varâhamihira (505). The complete decimal arithmetic is found in the commentary of Bhâskara I (*c.* 525) on the *Âryabhaṭîya*. The results of operations by zero are found stated in the work of Brahmagupta (628) and in all later mathematical treatises. The treatment of zero in the arithmetic of the Hindus is different from that found in their algebra. In order, therefore, to bring out this difference clearly, we give separately the results found in *pâṭîgaṇita* (arithmetic) and in *bîjagaṇita* (algebra).

Zero in Arithmetic. The Hindus in their arithmetic define zero as the result of the operation

$$a - a = 0$$

This definition is found in Brahmagupta's work[1] and is repeated in all later works. It is directly used in the operation of subtraction. In carrying out arithmetical operations, the results of the operations of addition, subtraction and multiplication of zero and by zero are required. The Hindus did not recognise the operation of division by zero as valid in arithmetic; but the division of zero by a number was recognised as valid. Nârâyaṇa in his *pâṭîgaṇita* (arithmetic) has clearly stated this distinction:

"Here in *pâṭîgaṇita*, division by zero is not recognised, and therefore, it is not mentioned here. As it is of use in *bîjagaṇita* (algebra), so I have mentioned division by zero in my *Bîjagaṇita*."[2]

[1] *BrSpSi*, p. 309. *Cf.* B. Datta, *BCMS*, XVIII, pp. 165-176 for some other details regarding operations with zero.

[2] *GK*, remark subjoined to i 30.

The following are the results of the operations in relation to zero mentioned in Hindu works on *pâṭî-gaṇita*:

Śrîdhara says:

"In addition cipher makes the sum equal to the additive; when cipher is subtracted (from a number), there is no change in the number. In multiplication and other[1] operations on zero the result is zero. Multiplication of a number by cipher also gives zero."[2]

Âryabhaṭa II, in the chapter on *pâṭîgaṇita* in his *Mahâ-siddhânta*, writes:

"If zero is added to a number, the number is unchanged; the same is true when zero is subtracted. In the operation of multiplication or division of zero (by another number) or in evolution and involution (of zero), the result is zero."[3]

Nârâyaṇa in his *pâṭîgaṇita* states:

"When zero is added to or subtracted from a number, it (the number) remains unchanged; that (number) multiplied by zero becomes zero. In the multiplication and other operations on zero, the result is zero. When a quantity is added to zero the result is that which is added."[4]

Mahâvîra in his *Gaṇita-sâra-saṁgraha* writes:

"A number multiplied by zero is zero, and that number remains unchanged when it is divided by, combined with, or diminished by zero. Multiplication

[1] By other operations are meant: (1) division of zero by a number, (2) squaring and square-root of zero, (3) cubing and cube-root of zero—in the sequence in which these operations are mentioned in Hindu arithmetic.

[2] *Triś*, p. 4. It should be noted that distinction is made between $a \times 0$ and $0 \times a$, although the result in each case is zero.

[3] *MSi*, p. 146.

[4] *GK*, i. 30.

and other operations on zero give rise to zero and in the operation of addition, zero becomes the same as what is added to it."[1]

Mahâvîra's statement that a number remains unchanged when divided by zero is obviously incorrect. The correct result of the operation was known to Brahmagupta who preceded him by more than two centuries. It is, therefore, strange to find Mahâvîra making such a statement. He probably thought that, so far as arithmetic was concerned, division by zero was no division at all.

Zero in Algebra. The earliest treatment of zero in algebra is found in the *Brâhma-sphuṭa-siddhânta*[2] (628):

"Negative less cipher is negative; positive (less cipher is) positive; cipher (less cipher is) nought....

"The product of cipher and negative, or of cipher and affirmative, is cipher; of two ciphers is cipher.

"Cipher, divided by cipher is nought.... Positive or negative divided by cipher is *taccheda* (a fraction with that for denominator), or cipher divided by negative or affirmative (is either zero or is expressed by a fraction with zero as numerator and the finite quantity as denominator)."[3]

Bhâskara II in the *Lîlâvatî* as well as in his *Bîja-gaṇita* gives the result of the operations with zero. In the *Lîlâvatî* he writes:

"In addition cipher makes the sum equal to the additive. In involution and evolution the result is

[1] *GSS*, p. 6. (49).

[2] *BrSpSi*, pp. 309-310, rules 31-35. These rules contain also, the rules of the operations with the positive and the negative in algebra. The results relating to zero only are mentioned here.

[3] The result of the operation given in the bracket is according to the interpretation of S. Dvivedî.

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cipher. A number divided by zero is *kha-hara* (that number with zero as denominator). The product of (a number and) zero is zero, but it must be retained as a multiple of zero (*kha-guṇa*), if any further operations impend. Zero having become a multiplier (of a number), should zero afterwards become a divisor, the number must be understood to be unchanged. So likewise any number, to which zero is added, or from which it is subtracted (is unaltered)."[1]

In the *Bîjagaṇita*, the same results are given with the addition that if a quantity is subtracted from zero, its sign is reversed, while in the case of addition the sign remains the same.

Zero as an Infinitesimal. It will be observed that Brahmagupta directs that the results of the operations $x \div 0$ and $0 \div x$ should be written as $\frac{x}{0}$ and $\frac{0}{x}$ respectively. It is not possible to tell exactly what he actually meant by these forms. It seems that he did not specify the actual value of these forms, because the value of the variable x is not known. Moreover, the zero seems to have been considered by him as an infinitesimal quantity which ultimately reduces to nought. If this surmise be correct, Brahmagupta is quite justified in stating the results as he has done.

The idea of zero as an infinitesimal is more in evidence in the works of Bhâskara II. He says: "The product of (a number and) zero is zero, but the number must be retained as a multiple of zero (*kha-guṇa*), if any further operations impend." He further remarks that this operation is of great use in astronomical calculations. It will be shown in the section on Calculus, that Bhâskara II has actually used quantities which ultimately tend to zero, and has successfully evaluated the differential coefficients of certain functions. He has, moreover,

[1] *L*, p. 8.

used the infinitesimal increment $f'(x)\delta x$ of the function $f(x)$, due to a change δx in x.

The commentator Kṛṣṇa proves the result $0 \times a = 0 = a \times 0$ as follows:

"The more the multiplicand is diminished, the smaller is the product; and, if it be reduced in the utmost degree, the product is so likewise: now the utmost diminution of a quantity is the same with the reduction of it to nothing; therefore, if the multiplicand be nought, the product is cipher. In like manner, as the multiplier decreases, so does the product; and, if the multiplier be nought, the product is so too."

In the above zero is conceived of as the limit of a diminishing quantity.

Infinity. The quotient of division by zero of a finite quantity has been called by Bhâskara II as *kha-hara*, which is synonymous with *kha-cheda* (the quantity with zero as denominator) of Brahmagupta. Regarding the value of the *kha-hara*, Bhâskara II remarks:

"In this quantity consisting of that which has cipher for its divisor, there is no alteration, though many may be inserted or extracted; as no change takes place in the infinite and immutable God, at the period of the destruction or creation of worlds, though numerous orders of beings are absorbed or put forth."[1]

From the above it is evident that Bhâskara II knew that $\frac{a}{0} = \infty$ and $\infty + k = \infty$.

[1] *BBî*, pp. 5-6. G. Thibaut (*Astronomie, Astrology und Mathematik*,, Strasbourg, 1899, p. 72) thought that this passage was an interpolation. There appears no justification for considering this as an interpolation, as the passage occurs in the oldest known commentary and in all copies of the work so far found. *Cf.* Datta, *l.c.*, p. 174.

Gaṇeśa remarks that $\frac{a}{0}$ is "an indefinite and unlimited or infinite quantity: since it cannot be determined how great it is. It is unaltered by the addition or subtraction of finite quantities: since in the preliminary operation of reducing both fractional expressions to a common denominator, preparatory to taking their sum or difference, both numerator and denominator of the finite quantity vanish."

Kṛṣṇa remarks:

"As much as the divisor is diminished, so much is the quotient increased. If the divisor is reduced to the utmost, the quotient is to the utmost increased. But, if it can be specified, that the amount of the quotient is so much, it has not been raised to the utmost: for a quantity greater than that can be assigned. The quotient, therefore, is indefinitely great, and is rightly termed infinite."

Regarding the proof of $\frac{a}{0} \pm k = \frac{a}{0}$ Kṛṣṇa makes the same remarks as Gaṇeśa. He, however, goes a step further when he says that

$$\frac{a}{0} = \frac{b}{0}.$$

This is illustrated by him through the instance of the shadow of a gnomon, which at sun-rise and sun-set is infinite; and is equally so whatever height be given to the gnomon, and whatever number be taken for the radius. "... Thus, if the radius be 120; and the gnomon be 1, 2, 3 or 4; the expression deduced from the proportion, as sine of sun's altitude is to sine of zenith distance, so is gnomon to shadow, becomes $\frac{120}{0}$, $\frac{240}{0}$, $\frac{360}{0}$ or $\frac{480}{0}$. Or, if the gnomon be, as it is usually framed, 12 fingers, and radius be taken as 3438, 120, 100 or 90, the expression will be $\frac{41256}{0}$, $\frac{1440}{0}$, $\frac{1200}{0}$ or $\frac{1080}{0}$, which are all alike infinite."[1]

Indeterminate Forms. Brahmagupta has made the incorrect statement that

$$\frac{0}{0} = 0$$

Bhâskara II has sought to correct this mistake of Brahmagupta. According to him

$$\lim_{\varepsilon \to 0} \frac{a.\varepsilon}{\varepsilon} = a.$$

His language, however, in stating this result is defective, for he calls the infinitesimal ε zero, not being in possession of a suitable technical term. That, in the above case, he actually meant by zero a small quantity tending to the limiting value zero, is abundantly clear from the use he makes of the result in his Astronomy. Taylor[2] and Bapu Deva Śâstrî[3] are also of this opinion.

Bhâskara has given three illustrative examples. They are:

$$(i) \quad \text{Evaluate } \frac{\left(x \times 0 + \frac{x \times 0}{2}\right)}{0} = 63.$$

From this he derives the result $x = 14$, which is correct if we consider $0 = \varepsilon$, a small quantity tending to zero. His other examples are:

$$(ii) \quad \left\{\left(\frac{x}{0} + x - 9\right)^2 + \left(\frac{x}{0} + x - 9\right)\right\} 0 = 90$$

giving $x = 9$; and

[1] All the above passages are taken from the respective commentaries. They have been noted by Colebrooke, *l.c.*

[2] *Lîlâwatî*, Bombay, 1816, p. 29.

[3] His *Bîja-gaṇita* (in Hindi), Pt. I, Benares, 1875, p. 179 *et sq.*

$$(iii)\ \{(x+\tfrac{x}{2})\times 0\}^2+2\{(x+\tfrac{x}{2})\times 0)\}\div 0 = 15,$$

giving $x=2$.[1]

Bhâskara II's result

$$\frac{a}{0}\times 0 = a$$

is, however, not quite correct, as the form is truly indeterminate and may not always have the value a. His attempt, however, at such an early date to assign a meaning to the form $\frac{0}{0}$, and his partial solution of the problem are very creditable, seeing that in Europe mathematicians made similar mistakes upto the middle of the nineteenth century A. D.[2]

[1] The answers of this and the previous example are incorrect because 0^2 has been taken to be equal to 0.

[2] Martin Ohm (1828) says: "If a is not zero, but b is zero, then the quotient a/b has no meaning" for the quotient "multiplied by zero gives only zero and not a, as long as a is not zero." *Lehrbuch der niedern Analysis*, Vol. I, Berlin, 1828, pp. 110, 112.

BIBLIOGRAPHY

OF SANSKRIT MATHEMATICAL WORKS

1. *Âpastamba Śulba* Sûtra *by* Âpastamba (*c.* 400 B.C.). Edited with the commentary of Kapardisvâmî, Karavindasvâmî and of Sundararâja *by* D. Srinivasachar and V.S. Narasimhachar, University of Mysore Sanskrit Series, 1931; *by* A. Bürk, with German translation and note and comments in *Zeitschrift der deutschen morgenländischen Gesselschaft*, LV, 1902, pp. 543-591; LVI, 1903, pp. 327-391.
2. *Ârṣa Jyotiṣa.* Edited *by* Sudhakara Dvivedi with his own commentary, Benares, 1906.
3. *Âryabhaṭîya by* Âryabhaṭa I (499). Edited with the commentary, entitled *Bhatadîpikâ*, of Parameśvara (1430) *by* H. Kern (Leiden, 1875); *by* Udai Narayan Singh, with explanatory notes in Hindi (Muzaffarpur, 1906); with the commentary, entitled *Mahâbhâsya*, of Nîlakaṇṭha (1500) on the last three chapters *by* K. Sambasiva Sastri (Trivandrum, Part I, 1930; Part II, 1931; Part III, *in press*). Translations: *by* P. C. Sengupta ("The Âryabhaṭîyam," in the *Journal of the Department of Letters in the University of Calcutta*, XVI, 1927); *by* W. E. Clark (*The Âryabhaṭîya of Âryabhaṭa*, Chicago, 1930). The second chapter of the book has been translated also *by* L. Rodet ("Leçons de calcul d'Âryabhaṭa," in *Journal Asiatique*, XIII (7), 1878; reprint, Paris, 1879) and *by* G. R. Kaye ("Notes on Indian Mathematics, No. 2—Âryabhaṭa," in *Journal of the Asiatic Society of Bengal*, IV, 1908). Other commentaries in MSS: (i) *by* Bhâskara I (522); (ii) *Bhataprakâśikâ by* Sûryadeva Yajvâ (12th century).
4. *Atharvan Jyotiṣa.* Edited *by* Bhagvad Datta, Lahore, 1924.
5. *Bakhshâlî Manuscript—A Study in Mediæval Mathematics*, Parts I, II and III edited *by* G. R. Kaye, Calcutta, 1927, 1933.
6. *Baudhâyana Śulba* Sûtra *by* Baudhâyana (*c.* 800 B.C.). Edited *by* G. Thibaut, with English translation, critical notes and extracts from the commentary of Dvârakânâtha Yajvâ, in

the *Paṇḍit* (Old Series, IX and X, 1874-5; New Series, I, 1877). The text appears in the edition of the *Baudhâyana Śrauta Sûtra* (as its 30th chapter) *by* W. Caland (in 3 volumes, Calcutta, 1904, 1907, 1913).

7. *Bîjagaṇita of Bhâskara* II (1150). Edited *by* Sudhakara Dvivedi (Benares) and revised by Muralidhara Jha (Benares, 1927); with the commentary, entitled *Navâṅkura*, of Kṛṣṇa (*c.* 1600) *by* D. Apte (Anandasrama Sanskrit Series, Poona, 1930). English translation *by* H. T. Colebrooke (*Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bhascara*, London, 1817). Other commentaries in MSS: (*i*) *Bîjaprabodha by* Râmakṛṣṇa (*c.* 1648); (*ii*) *by* Sûryadâsa (born 1508). References to this treatise in our text are by pages of Muralidhara Jha's revised edition.

8. *Bîjagaṇita* of Jñânarâja (1503). MS.

9. *Bîjagaṇita* of Nârâyaṇa (1350). MS. (incomplete).

10. *Brâhma-sphuṭa-siddhânta by* Brahmagupta (628). Edited with explanatory notes by Sudhakara Dvivedi, Benares, 1902. Chapters xii and xiii of the work dealing respectively with arithmetic and algebra have been translated into English *by* H. T. Colebrooke (*Algebra with Arithmetic and Mensuration* etc.). Commentary *by* Pṛthudakasvâmî (860). MS. (incomplete).

11. *Bṛhajjâtaka* of Varâhamihira (505). Edited with the commentary of Bhaṭṭotpala (966) *by* Rasikmohan Chattopadhyaya (Calcutta, 1300, B.S.); *by* Sitaram Jha (Benares, 1923).

12. *Bṛhat-saṁhitâ by* Varâhamihira (505). Edited *by* H. Kern (Calcutta, 1865); *by* Sudhakara Dvivedi, with the commentary of Bhaṭṭotpala (Vizianagram Sanskrit Series, 2 Vols., Benares, 1895). English translation by H. Kern (See his collected works).

13. *Dhyânagrahopadeśa* of Brahmagupta (628). Edited *by* Sudhakara Dvivedi and published as an appendix to his edition of the *Brâhma-sphuṭa-siddhânta.*

14. *Gaṇita-kaumudî* of Nârâyaṇa (1356). MS.

15. *Gaṇita-mañjarî by* Gaṇeśa, son of Ḍhuṇḍhirâja (1558). MS.

16. *Gaṇita-tilaka by* Śrîpati (1039). Edited with the commentary of Siṁhatilaka Surî (*c.* 1275) *by* H. R. Kapadia (Gaikwad Sanskrit Series, Baroda, 1935).

17. *Gaṇita-sâra-saṁgraha by* Mahâvîra (850). Edited with English translation and notes *by* M. Rangacarya, Madras, 1912.

18. *Graha-lâghava* of Gaṇeśa Daivajña (*c.* 1545). Edited with the commentaries of Mallâri, Viśvanâtha and his own *by* Sudhakara Dvivedi, reprinted, Bombay, 1925.

19. *Karaṇa-kutûhala by* Bhâskara II (1150). Edited with the commentary of Sumatiharṣa *by* Madhava Sastri, Bombay, 1901.

20. *Karaṇa-paddhati by* Pathumana Somayâji (1733). MS.

21. *Kâtyâyana Śulba* Sûtra *by* Kâtyâyana (*c.* 400 B.C.). Edited with explanatory notes *by* Vidyadhar Sharma, Benares, 1928.

22. *Khaṇḍa-khâdyaka by* Brahmagupta (628). Edited with the commentary of Âmarâja (*c.* 1250) *by* Babua Misra (Calcutta, 1925). English translation with notes and comments *by* Prabodh Chandra Sengupta (Calcutta, 1934). Other commentaries in MSS: (*i*) *by* Pṛthudakasvâmî (incomplete); (*ii*) *by* Varuṇa; (*iii*) *by* Bhaṭṭotpala.

23. *Laghu-Bhâskarîya by* Bhâskara I (522). Commentary *by* Śaṅkaranârâyaṇa. MSS.

24. *Laghu-mânasa by* Mañjula (932). Commentaries *by* (*i*) Pṛthudakasvâmî (964); (*ii*) Parameśvara (1430). MSS.

25. *Lîlâvatî by* Bhâskara II (1150). Edited with notes *by* Sudhakara Dvivedi, Benares, 1910. Translations: (*i*) *by* H. T. Colebrooke (*Algebra with Arithemtic and Mensuration* etc.); (*ii*) *by* J. Taylor (*Lilawati*, Bombay, 1816). Colebrooke's translation has been re-edited with critical notes by Haran Chandra Banerji (*Lîlâvatî*, 2nd ed., Calcutta). Commentaries in MSS: (*i*) *Buddhivilâsinî by* Gaṇeśa (1545); (*ii*) *Gaṇitâmṛtasâgarî by* Gaṅgâdhara (1432); (*iii*) *Gaṇitâmṛtalaharî by* Râmakṛṣṇa (1339); (*iv*) *Manorañjana by* Râmakṛṣṇadeva; (*v*) *Gaṇitâmṛtakûpikâ by* Sûryadâsa (1541); (*vi*) *Cintâmaṇi by* Lakṣmîdâsa (1500); (*vii*) *Nisṛṣṭadûtî by* Munîśvara (1608). References to *Lîlâvatî* in our text are by the pages of Dvivedi's edition.

26. *Mânava Śulba* Sûtra *by* Manu, MS. English translation *by* N. K. Mazumdar (in *Journal of the Department of Letters in the University of Calcutta*, VIII, 1922).

27. *Mahâ-Bhâskarîya by* Bhâskara I (522). Commentary *by* (*i*) Sûryadeva (12th century); (*ii*) Parameśvara (1430). MSS.

28. *Mahâ-siddhânta by* Âryabhaṭa II (950). Edited with explanatory notes by Sudhakara Dvivedi, Benares, 1910.

29. *Pañca-siddhântikâ by* Varâhamihira (505). Edited with a commentary in Sanskrit, translation into English and critical notes *by* G. Thibaut and Sudhakara Dvivedi, Benares, 1889.

30. *Pâṭî-sâra by* Munîśvara (born 1603). MS.

31. *Sadratnamâla by* Śaṅkaravarman. MS.

32. *Siddhânta-śekhara by* Śrîpati (1039). Edited *by* Babua Misra, Calcutta, Vol. I (1932), containing chapters i-xii of the text, with the commentary of Makkibhaṭṭa (1377) on chapters i-iv and that of the editor on the rest.

33. *Siddhânta-śiromaṇi by* Bhâskara II (1150). Edited with the author's own gloss (*Vâsanâbhâṣya*) *by* Bapu Deva Sastri, Benares); *by* Muralidhara Jha with the commentaries, *Vâsanâvârtika* of Nṛsiṃha (1621) and *Marîci* of Munîśvara (1635), Vol. I (containing chap. i of the *Gaṇitâdhyâya*) (Benares, 1917); *by* Girija Prasad Dvivedi, with original commentaries in Sanskrit and Hindi, Vols. I and II (Lucknow, 1911, 1926). English translation of the text only by Bapu Deva Sastri and Wilkinson (Calcutta, 1861).

34. *Siddhânta-tattva-viveka by* Kamalâkara (1658). Edited with the *Śeṣa-vâsanâ* of the author, *by* Sudhakara Dvivedi, Benares, 1885.

35. *Śiṣya-dhî-vṛddhida by* Lalla (598). Edited *by* Sudhakara Dvivedi, Benares, 1886. Commentary *by* Mallikârjuna Surî (1179). MS.

36. *Sûrya-siddhânta*. Edited *by* F. E. Hall and Bapu Deva Sastri with the commentary of Raṅganâtha (Calcutta, 1859); *by* Sudhakara Dvivedi with an original commentary in Sanskrit (Calcutta, 1909-11). Translation into English, with critical notes *by* E. Burgess and W. D. Whitney; into Bengali with critical notes *by* Vijñânânanda Svâmî.

37. *Tantra-saṁgraha by* Nîlakaṇṭha (1500). Commentary *by* an unknown writer. MS.

38. *Triśatikâ by* Śrîdhara (750). Edited *by* Sudhakara Dvivedi, Benares, 1899. The principal rules in this text leaving the illustrative examples and their solutions, have been translated into English *by* N. Ramanujachariar and G. R. Kaye and published with notes and comments ("Triśatikâ of Śrîdharâcârya," *Bibl. Math.*, XIII (3), 1912-13, pp. 203ff.). Dvivedi's text is apparently incomplete. The manuscript in our collection, though not perfect, contains a few more rules and examples.

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A SOURCE BOOK

PART II

ALGEBRA

BY

BIBHUTIBHUSAN DATTA
AND
AVADHESH NARAYAN SINGH

इदं नम ऋषिभ्यः पूर्वजेभ्यः पूर्वेभ्यः पथिकृद्भ्यः

(*RV*, x. 14. 25)

To the Seers, our Ancestors, the first Path-makers

PREFACE

The present work forms Part II of our History of Hindu Mathematics and is devoted to the history of Algebra in India. It is intended to be a source book, and the subject is treated topicwise. Under each topic are collected together and set forth in chronological order translations of relevant Sanskrit texts as found in the Hindu mathematical works. This plan necessitates a certain amount of repetition. But it shows to the reader at a glance the improvements made from century to century

To gather materials for the book we have examined all the published mathematical treatises of the Hindus as well as most of the important manuscripts available in Indian libraries, a list of the most important of which has already been included in Part I. We have great pleasure in once more expressing our thanks to the authorities of the libraries at Madras, Bangalore, Trivandrum, Tripunithura, Baroda, Jammu, and Benares, and those of the India Office (London) and the Asiatic Society of Bengal for supplying transcripts of manuscripts or sending them to us for consultation. We are indebted also to Dr. R. P. Paranjpye, Vice-Chancellor of the Lucknow University, for help in securing for our use several manuscripts or their transcripts from the State libraries in India and the India Office.

In translating Sanskrit texts we have tried to be as literal and faithful as possible without sacrificing the spirit of the original, in order to preserve which we have at a few places used literal translations of Sanskrit technical terms instead of modern terminology. For instance, we have used the term 'pulveriser' for the equation $ax+by=1$, and the term 'Square-nature' for the equation $Nx^2+c=y^2$.

The use of symbols—letters of the alphabet to denote unknowns—and equations are the foundations of the science of algebra. The Hindus were the first to make systematic use of the letters of the alphabet to denote unknowns. They were also the first to classify and make a detailed study of equations. Thus they may be said to have given birth to the modern science of algebra.

A portion of the subject matter of this book has been available to scholars through papers by various authors and through Colebrooke's *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bhascara*, but about half of it is being presented here for the first time. For want of space it has not been possible to give a detailed comparison of the Hindu achievements in Algebra with those of other nations. For this the reader is referred to the general works on the history of mathematics by Cantor, Smith, and Tropfke, to Dixon's *History of the Theory of Numbers* and to Neugebauer's *Mathematische Keilschrift-Texte*. A study of this book along with the above standard works will reveal to the reader the remarkable progress in algebra made by the Hindus at an early date. It will also show that we are indebted to the Hindus for the technique and the fundamental results of algebra just as we owe to them the place-value notation and the elements of our arithmetic.

We have pleasure in expressing our thanks to Mr. T. N. Singh and Mr. Ahmad Ali for help in correcting proofs and to Mr. R. D. Misra for preparing the index to this volume.

LUCKNOW — BIBHUTIBHUSAN DATTA
March, 1938 — AVADHESH NARAYAN SINGH

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ALGEBRA

1. GENERAL FEATURES

Name for Algebra. The Hindu name for the science of algebra is *bîjagaṇita*. *Bîja* means "element" or "analysis" and *gaṇita* "the science of calculation." Thus *bîjagaṇita* literally means "the science of calculation with elements" or "the science of analytical calculation." The epithet dates at least as far back as the time of Pṛthûdakasvâmî (860) who used it. Brahmagupta (628) calls algebra *kuṭṭaka-gaṇita*, or simply *kuṭṭaka*.[1] The term *kuṭṭaka*, meaning "pulveriser", refers to a branch of the science of algebra dealing particularly with the subject of indeterminate equations of the first degree. It is interesting to find that this subject was considered so important by the Hindus that the whole science of algebra was named after it in the beginning of the seventh century. Algebra is also called *avyakta-gaṇita* or "the science of calculation with unknowns" (*avyakta*=unknown) in contradistinction to the name *vyakta-gaṇita* or "the science of calculation with knowns" (*vyakta*=known) for arithmetic including geometry and mensuration.

Algebra Defined. Bhâskara II (1150) has defined algebra thus:

"Analysis (*bîja*) is certainly the innate intellect assisted by the various symbols (*varṇa*), which, for the

[1]See Bibhutibhusan Datta, "The scope and development of the Hindu Gaṇita," *IHQ*, V, 1929, pp. 479-512; particularly pp. 489f.

instruction of duller intellects, has been expounded by the ancient sages who enlighten mathematicians as the sun irradiates the lotus; that has now taken the name algebra (*bîjagaṇita*)."[1]

That algebraic analysis requires keen intelligence and sagacity has been observed by him on more than one occasion.

"Neither does analysis consist in symbols, nor are there different kinds of analyses; sagacity alone is analysis, for wide is imagination."[2]

"Analysis is certainly clear intelligence."[3]

"Or intelligence alone is analysis."[4]

In answer to the question, "if (unknown quantities) are to be discovered by intelligence alone what then is the need of analysis ?" he says:

"Because intelligence is certainly the real analysis; symbols are its helps. The innate intelligence which has been expressed for the duller intellects by the ancient sages, who enlighten mathematicians as the sun irradiates the lotus, with the help of various symbols, has now obtained the name of algebra."[5]

Thus, according to Bhâskara II, algebra may be defined as the science which treats of numbers expressed by means of symbols, and in which there is scope and primary need for intelligent artifices and ingenious devices.

Distinction from Arithmetic. What distinguishes algebra from arithmetic, according to the Hindus, will be found to some extent in their special names. Both deal with symbols. But in arithmetic the values of the symbols are *vyakta*, that is, known and definitely determinate,

[1] *BBi*, p. 99.
[2] *BBi*, p. 49; *SiŚi*, *Gola*, xiii. 5.
[3] *L*, p. 15; *SiŚi*, *Gola*, xiii. 3.
[4] *BBi*, p. 49.
[5] *BBi*, p. 100.

while in algebra they are *avyakta*, that is, unknown, indefinite. The relation between these two branches of *gaṇita* is considered by Bhâskara II to be this:

"The science of calculation with unknowns is the source of the science of calculation with knowns."[1]

He has put it more explicitly and clearly thus:

"Algebra is similar to arithmetic in respect of rules (of fundamental operations) but appears as if it were indeterminate. It is not indeterminate to the intelligent; it is certainly not sixfold,[2] but manifold."[3]

The true distinction between arithmetic and algebra, besides that of symbols employed, lies, in the opinion of Bhâskara II, in the demonstration of the rules. He remarks:

"Mathematicians have declared algebra to be computation attended with demonstration: else there would be no distinction between arithmetic and algebra."[4]

The truth of this dictum is evident in the treatment of the *guṇa-karma* in the *Lîlâvatî* and the *madhyamâharaṇa* in the *Bîjagaṇita*. Both are practically treatments of problems involving the quadratic equation. But whereas in the former are found simply the applications of the well-known formulæ for the solution of such equations, in the latter is described also the *rationale* of those formulæ. Similarly we sometimes find included in treatises on arithmetic problems whose solutions require formulæ demonstrated in books on algebra. The method of demonstration has been stated to be "always of two kinds: one geometrical (*kṣetragata*) and

[1] *BBi*, p. 1.

[2] The reference is to the six fundamental operations recognised in algebra as well as to the six subjects of treatment which are essential to analysis.

[3] *L*, p. 15.

[4] *BBi*, p. 127.

the other symbolical (*râśigata*)."[1] We do not know who was the first in India to use geometrical methods for demonstrating algebraical rules. Bhâskara II (1150) ascribes it to "ancient teachers."[2]

Importance of Algebra. The early Hindus regarded algebra as a science of great importance and utility. In the opening verses of his treatise[3] on algebra Brahmagupta (628) observes:

"Since questions can scarcely be known (*i.e.*, solved) without algebra, therefore, I shall speak of algebra with examples.

"By knowing the pulveriser, zero, negative and positive quantities, unknowns, elimination of the middle term, equations with one unknown, factum and the Square-nature, one becomes the learned professor (*âcârya*) amongst the learned."[4]

Similarly Bhâskara II writes:

"What the learned calculators (*sâṁkhyâḥ*) describe as the originator of intelligence, being directed by a wise being (*satpuruṣa*) and which alone is the primal cause (*bîja*) of all knowns (*vyakta*), I venerate that Invisible God as well as that Science of Calculation with Unknowns... Since questions can scarcely be solved without the reasoning of algebra—not at all by those of dull perceptions—I shall speak, therefore, of the operations of analysis."[5]

[1] *BBi*, p. 125. [2] *BBi*, p. 127.

[3] Forming chapter xviii of his *Brâhma-sphuṭa-siddhânta*.

[4] *BrSpSi*, xviii. 1-2.

[5] In the first part of this passage every principal term has been used with a double significance. The term *sâṁkhyâḥ* (literally, "expert calculators") signifies the "Sâṁkhya philosophers" in one sense, "mathematicians" in the other; *satpuruṣa* "the self-existent being of the Sâṁkhya philosophy" or "a wise mathematician"; *vyakta* "manifested universe" or "the science of calculation with knowns."

Nârâyaṇa (1350) remarks :

"I adore that Brahma, also that science of calculation with the unknown, which is the one invisible root-cause of the visible and multiple-qualitied universe, also of multitudes of rules of the science of calculation with the known."[1]

"As out of Him is derived this entire universe, visible and endless, so out of algebra follows the whole of arithmetic with its endless varieties (of rules). Therefore, I always make obeisance to Śiva and also to (*avyakta-*) *gaṇita* (algebra)."[2]

He adds :

"People ask questions whose solutions are not to be found by arithmetic; but their solutions can generally be found by algebra. Since less intelligent men do not succeed in solving questions by the rules of arithmetic, I shall speak of the lucid and easily intelligible rules of algebra."[3]

Scope of Algebra. The science of algebra is broadly divided by the Hindus into two principal parts. Of these the most important one deals with analysis (*bîja*). The other part treats of the subjects which are essential for analysis. They are: the laws of signs, the arithmetic of zero (and infinity), operations with unknowns, surds, the pulveriser (or the indeterminate equation of the first degree), and the Square-nature (or the so-called Pellian equation). To these some writers add concurrence and dissimilar operations, while others include them in arithmetic.[4] At the end of the first section of his treatise on algebra Bhâskara II is found to have

[1] *NBi*, I, R. 1. [2] *NBi*, II, R. 1.

[3] *NBi*, I, R. 5-6.

[4] All writers, except Brahmagupta and Śrîpati, are of the latter opinion.

observed as follows :

"(The section of) this science of calculation which is essential for analysis has been briefly set forth. Next I shall propound analysis, which is the source of pleasure to the mathematician."[1]

Analysis is stated by all to be of four kinds, for equations are classified into four varieties (*vide infra*). Thus each class of equations has its own method of analysis.

Origin of Hindu Algebra. The origin of Hindu algebra can be definitely traced back to the period of the *Śulba* (800-500 B.C.) and the *Brâhmaṇa* (*c.* 2000 B.C.). But it was then mostly geometrical.[2] The geometrical method of the transformation of a square into a rectangle having a given side, which is described in the important *Śulba* is obviously equivalent to the solution of a linear equation in one unknown, *viz.*,

$$ax = c^2.$$

The quadratic equation has its counterpart in the construction of a figure (an altar) similar to a given one but differing in area from it by a specified amount. The usual method of solving that problem was to increase the unit of measure of the linear dimensions of the figure. One of the most important altars of the obligatory Vedic sacrifices was called the *Mahâvedi* (the Great Altar). It has been described to be of the form of an isosceles trapezium whose face is 24 units long, base 30 and altitude 36. If x be the enlarged unit of measure taken in increasing the size of the altar by m units of area, we must have

$$36x \times \frac{(24x + 30x)}{2} = 36 \times \frac{(24 + 30)}{2} + m,$$

[1] *BBi*, p. 43.

[2] Bibhutibhusan Datta, *The Science of the Śulba*, Calcutta, 1932.

or $$972x^2 = 972 + m.$$

Therefore $$x = \sqrt{1 + \frac{m}{972}}.$$

If m be put equal to 972 $(n-1)$, so that the area of the enlarged altar is n times its original area, we get

$$x = \sqrt{n},$$

some particular cases of which are described in the *Śulba*. The particular cases, when $n = 14$ or $14\frac{3}{7}$, are found as early as the *Śatapatha Brâhmaṇa*[1] (*c.* 2000 B.C.).

The most ancient and primitive form of the "Fire-altar for the sacrifices to achieve special objects" was the *Śyenacit* (or "the altar of the form of the falcon").

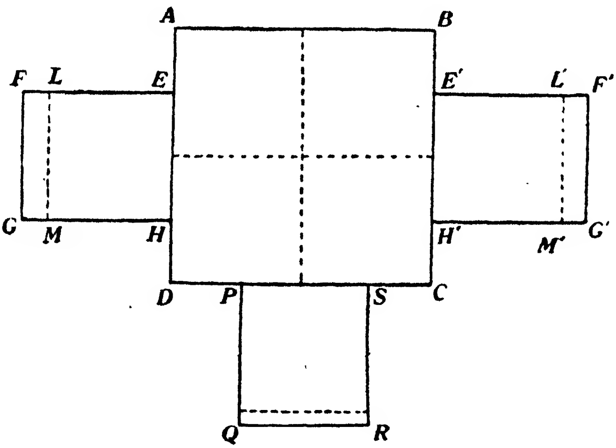

Fig. 1.

Its body (*ABCD*) consists of four squares of one square *puruṣa* each; each of its wings (*EFGH*, *E'F'G'H'*) is a rectangle of one *puruṣa* by one *puruṣa* and a *prâdeśa* (=1/10 of a *puruṣa*). This Fire-altar was enlarged in

[1] *ŚBr*, X. 2. 3. 7ff.

two ways: *first*, in which all the constituent parts were affected in the same proportion; *second*, in which the breadth of the portions $LFGM$ and $L'F'G'M'$ of the wings were left unaffected. If x be the enlarged unit for enlargement in the first case we shall have to solve the quadratic equation

$$2x \times 2x + 2\left\{x \times \left(x + \frac{x}{5}\right)\right\} + x \times \left(x + \frac{x}{10}\right) = 7\tfrac{1}{2} + m,$$

where m denotes the increment of the Fire-altar in size.

Therefore $$x^2 = 1 + \frac{2m}{15}.$$

In particular, when $m = 94$, we shall have

$$x^2 = 13\tfrac{8}{15} = 14 \text{ (approximately)},$$

which occurs in the *Śatapatha Brâhmaṇa*.

In the second case of enlargement the equation for x will be

$$2x \times 2x + 2\{x \times (x + \tfrac{1}{5})\} + x \times (x + \tfrac{1}{10}) = 7\tfrac{1}{2} + m,$$

or $$7x^2 + \tfrac{1}{2}x = 7\tfrac{1}{2} + m,$$

which is a complete quadratic equation.

The problem of altar construction gave rise also to certain indeterminate equations of the second degree such as,

(1) $$x^2 + y^2 = z^2,$$

(2) $$x^2 + a^2 = z^2;$$

and simultaneous indeterminate equations of the type

$$ax + by + cz + dw = p,$$
$$x + y + z + w = q.$$

Further particulars about these equations will be given later on.

2. TECHNICAL TERMS

Coefficient. In Hindu algebra there is no systematic use of any special term for the coefficient. Ordinarily, the power of the unknown is mentioned when the reference is to the coefficient of that power. In explanation of similar use by Brahmagupta his commentator Pṛthûdakasvâmî writes "the number (*aṅka*) which is the coefficient of the square of the unknown is called the 'square' and the number which forms the coefficient of the (simple) unknown is called 'the unknown quantity.' "[1] However, occasional use of a technical term is also met with. Brahmagupta once calls the coefficient *saṁkhyâ*[2] (number) and on several other occasions *guṇaka*, or *guṇakâra* (multiplier).[3] Pṛthûdakasvâmî (860) calls it *aṅka*[4] (number) or *prakṛti* (multiplier). These terms reappear in the works of Śrîpati (1039)[5] and Bhâskara II (1150).[6] The former also used *rûpa* for the same purpose.[7]

Unknown Quantity. The unknown quantity was called in the *Sthânâṅga-sûtra*[8] (before 300 B.C.) *yâvat-tâvat* (as many as or so much as, meaning an arbitrary quantity). In the so-called Bakhshâlî treatise, it was called *yadṛcchâ*, *vâñchâ* or *kâmika* (any desired quantity).[9] This term was originally connected with the Rule of False Position.[10] Âryabhaṭa I (499)

[1] *BrSpSi*, xviii. 44 (*Com*).
[2] *BrSpSi*, xviii. 63.
[3] *BrSpSi*, xviii. 64, 69-71.
[4] *BrSpSi*, xviii. 44 (*Com*).
[5] *SiŚe*, xiv. 33-5.
[6] *BBi*, pp. 33-4.
[7] *SiŚe*, xiv. 19.
[8] *Sûtra* 747; *cf.* Bibhutibhusan Datta, "The Jaina School of Mathematics," *BCMS*, XXI, 1929, pp. 115-145; particularly pp. 122-3.
[9] *BMs*, Folios 22, verso; 23, recto & verso.
[10] Bibhutibhusan Datta, "The Bakshshâlî Mathematics," *BCMS*, XXI, pp. 1-60; particularly pp. 26-8, 66.

calls the unknown quantity *gulikâ* (shot). This term strongly leads one to suspect that the shot was probably then used to represent the unknown. From the beginning of the seventh century the Hindu algebraists are found to have more commonly employed the term *avyakta* (unknown).[1]

Power. The oldest Hindu terms for the power of a quantity, known or unknown, are found in the *Uttarâdhyayana-sûtra* (*c.* 300 B.C. or earlier).[2] In it the second power is called *varga* (square), the third power *ghana* (cube), the fourth power *varga-varga* (square-square), the sixth power *ghana-varga* (cube-square), and the twelfth power *ghana-varga-varga* (cube-square-square), using the multiplicative instead of the additive principle. In this work we do not find any method for indicating odd powers higher than the third. In later times, the fifth power is called *varga-ghana-ghâta* (product of cube and square, *ghâta*=product), the seventh power *varga-varga-ghana-ghâta* (product of square-square and cube) and so on. Brahmagupta's system of expressing powers higher than the fourth is scientifically better. He calls the fifth power *pañca-gata* (literally, raised to the fifth), the sixth power *ṣaḍ-gata* (raised to the sixth); similarly the term for any power is coined by adding the suffix *gata* to the name of the number indicating that power.[3] Bhâskara II has sometimes followed it consistently for the powers one and upwards.[4] In the *Anuyogadvâra-sûtra*[5], a work written before the commencement of the Christian Era, we find certain interesting terms for higher powers, integral as well as fractional, particularly successive squares (*varga*) and square-roots (*varga-mûla*). According to it the term *prathama-varga*

[1] *BrSpSi*, xviii. 2, 41; *SiŚe*, xiv. 1-2; *BBi*, pp. 7ff.
[2] Chapter xxx, 10, 11.
[3] *BrSpSi*, xviii. 41, 42.
[4] *BBi*, p. 56.
[5] *Sûtra* 142.

(first square) of a quantity, say a, means a^2; *dvitîyavarga* (second square) $= (a^2)^2 = a^4$; *tṛtîya-varga* (third square) $= ((a^2)^2)^2 = a^8$; and so on. In general,

$$n\text{th } varga \text{ of } a = a^{2\times 2\times 2\times \dots \text{ to } n \text{ terms}} = a^{2^n}.$$

Similarly, *prathama-varga-mûla* (first square-root) means $\sqrt{a}$; *dvitîya-varga-mûla* (second square-root) $= \sqrt{(\sqrt{a})} = a^{1/4}$; and, in general,

$$n\text{th } varga\text{-}m\hat{u}la \text{ of } a = a^{1/2^n}.$$

Again we find the term *tṛtîya-varga-mûla-ghana* (cube of the third square-root) for $(a^{1/2^3})^3 = a^{3/8}$.

The term *varga* for "square" has an interesting origin in a purely concrete concept. The Sanskrit word *varga* literally means "rows," or "troops" (of similar things). Its application as a mathematical term originated in the graphical representation of a square, which was divided into as many *varga* or troops of small squares, as the side contained units of some measure.[1]

Equation. The equation is called by Brahmagupta (628) *sama-karaṇa*[2] or *samî-karaṇa*[3] (making equal) or more simply *sama*[4] (equation). Pṛthûdakasvâmî (860) employs also the term *sâmya*[5] (equality or equation); and Śrîpati (1039) *sadṛśî-karaṇa*[6] (making similar). Nârâyaṇa (1350) uses the terms *samî-karaṇa*, *sâmya* and *samatva* (equality).[7] An equation has always two *pakṣa* (side). This term occurs in the works of

[1] G. Thibaut, *Sulba-sūtras*, p. 48. Compare also Bibhutibhusan Datta, "On the origin of the Hindu terms for root," *Amer. Math. Mon.*, XXXVIII, 1931, pp. 371-6.

[2] *BrSpSi*, xviii. 63.

[3] *BrSpSi*, xviii, subheading for the section on equations.

[4] *BrSpSi*, xviii. 43.

[5] *BrSpSi*, xii. 66 (*Com*).

[6] *SiŚe*, xiv. 1.

[7] *NBi*, II, R. 2-3.

Srîdhara (*c.* 750), Padmanâbha[1] and others.[2]

Absolute Term. In the Bakhshâlî treatise[3] the absolute term is called *dṛśya* (visible). In later Hindu algebras it has been replaced by a closely allied term *rûpa*[4] (appearance), though it continued to be employed in treatises on arithmetic.[5] Thus the true significance of the Hindu name for the absolute term in an algebraic equation is obvious. It represents the visible or known portion of the equation while its other part is practically invisible or unknown.

3. SYMBOLS

Symbols of Operation. There are no special symbols for the fundamental operations in the Bakhshâlî work. Any particular operation intended is ordinarily indicated by placing the tachygraphic abbreviation, the initial syllable of a Sanskrit word of that import, after, occasionally before, the quantity affected. Thus the operation of addition is indicated by *yu* (an abbreviation from *yuta*, meaning added), subtraction by + which is very probably from *kṣa* (abbreviated from *kṣaya*, diminished), multiplication by *gu* (from *guṇa* or *guṇita*, multiplied) and division by *bhâ* (from *bhâga* or *bhâjita*, divided). Of these again, the most systematically employed abbreviation is that for the operation of subtraction and next comes that of division. In the case of the other two operations the indicatory words

[1] The algebras of Śrîdhara and Padmanâbha are not available now. But the term occurs in quotations from them by Bhâskara II (*BBi*, pp. 61, 67).

[2] *BrSpSi*, xviii. 43 (*Com*); *SiŚe*, xiv. 14, 20; *BBi*, pp. 43-4.

[3] *BMs*, Folio 23, verso; Folio 70, recto and verso (c).

[4] *BrSpSi*, xviii. 43-4; *SiŚe*. xiv, 14, 19; etc.

[5] *Triś*, pp. 11, 12.

are often written in full, or altogether omitted. In the latter case, the particular operations intended to be carried out are understood from the context. We take the following instances from the Bakhshâlî Manuscript:

(i)[1] $\begin{array}{cc}0&5\\1&1\end{array}$ *yu* means $\frac{x}{1}+\frac{5}{1}$, and $\begin{array}{c}11\\1\end{array}$ *yu* $\begin{array}{c}5\\1\end{array}$ means $\frac{11}{1}+\frac{5}{1}$.

(ii)[2] $\left\| \begin{array}{cccccccc}3&3&3&3&3&3&3&10\,gu\\1&1&1&1&1&1&1&1\end{array} \right|$ means $3\times3\times3\times3\times3\times3\times3\times3\times10$.

(iii)[3] $\begin{array}{c}0\\1\end{array} \left\| \begin{array}{cc}1&3\\1&2\end{array} \right\| \begin{array}{cc}2&5\\1&2+\end{array} \left\| \begin{array}{ccc}3&gu&7\\1&&2+\end{array} \right\| \begin{array}{ccc}4&gu&9\\1&&2+\end{array} \right\|$

means

$$x(1+\tfrac{3}{2})+\left\{2x(1+\tfrac{3}{2})-\frac{5x}{2}\right\}+\left\{3x(1+\tfrac{3}{2})-\frac{7x}{2}\right\}$$
$$+\left\{4x(1+\tfrac{3}{2})-\frac{9x}{2}\right\}.$$

(iv)[4] $\left| \begin{array}{cccc}1&1&1&1\,bh\hat{a}\\1+&1&1&1\\2&3&4+&6\end{array} \right\| \begin{array}{c}36\\ \\1\end{array} \right|$ means $\dfrac{36}{(1-\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{4})(1+\frac{1}{6})}$.

(v)[5] $\left| \begin{array}{c}40\;bh\hat{a}\\1\end{array} \right\| \begin{array}{c}160\\1\end{array} \left| \begin{array}{c}13\\1\\2\end{array} \right\|$ means $\dfrac{160}{40}\times13\tfrac{1}{2}$.

In later Hindu mathematics the symbol for subtraction is a dot, occasionally a small circle, which is placed above the quantity, so that $\dot{7}$ or $\mathring{7}$ means -7; other operations are represented by simple juxtaposition.

[1] Folio 59, recto. [2] Folio 47, recto.

[3] Folio 25, verso. The beginning and end of this illustration are mutilated but the restoration is certain.

[4] Folio 13, verso. [5] Folio 42, recto.

Bhâskara II (1150) says, "Those (known and unknown numbers) which are negative should be written with a dot (*bindu*) over them."[1] A similar remark occurs in the algebra of Nârâyaṇa (1350).[2] Their silence about symbols of other fundamental operations proves their non-existence.

Origin of Minus Sign. The origin of · or ° as the minus sign seems to be connected with the Hindu symbol for the zero, o. It is found tc have been used as the sign of emptiness or omission in the early Bakhshâlî treatise as well as in the later treatises on arithmetic (*vide infra*).[3] It is placed over the number affected in order to distinguish it from its use in a purely numerical significance when it is placed beside the number. The origin of the Bakhshâlî minus sign (+) has been the subject of much conjecture. Thibaut suggested its possible connection with the supposed Diophantine negative sign ⋔ (reversed ψ, tachygraphic abbreviation for λεiψis meaning wanting). Kaye believes it. The Greek sign for minus, however, is not ⋔, but ↑. It is even doubtful if Diophantus did actually use it; or whether it is as old as the Bakhshâlî cross.[4] Hoernle[5] presumed the Bakhshâlî minus sign to be the abbreviation *ka* of the Sanskrit word *kanita*, or *nu* (or *nû*) of *nyûna*, both of which mean diminished and both of which abbreviations in the Brâhmî characters would be denoted by a cross. Hoernle was right, thinks Datta,[6] so far as he sought for the origin of + in a tachygraphic abbreviation of some Sanskrit word. But as neither the word *kanita* nor *nyûna* is found to have been used in the Bakhshâlî work in connection with the subtractive

[1] *BBi*, p. 2.
[2] *NBi*, I, R. 7.
[3] p. 16.
[4] *Cf.* Smith, *History*, II, p. 396.
[5] *IA*, XVII, p. 34.
[6] Datta, *Bakh. Math.*, (*BCMS*, XXI), pp. 17-8.

operation, Datta finally rejects the theory of Hoernle and believes it to be the abbreviation *kṣa*, from *kṣaya* (decrease) which occurs several times, indeed, more than any other word indicative of subtraction. The sign for *kṣa*, whether in the Brâhmî characters or in Bakhshâlî characters, differs from the simple cross (+) only in having a little flourish at the lower end of the vertical line. The flourish seems to have been dropped subsequently for convenient simplification.

Symbols for Powers and Roots. The symbols for powers and roots are abbreviations of Sanskrit words of those imports and are placed after the number affected. Thus the square is represented by *va* (from *varga*), cube by *gha* (from *ghana*), the fourth power by *va-va* (from *varga-varga*), the fifth power by *va-gha-ghâ* (from *varga-ghana-ghâta*), the sixth power by *gha-va* (from *ghana-varga*), the seventh power by *va-va-gha-ghâ* (from *varga-varga-ghana-ghâta*) and so on. The product of two or more unknown quantities is indicated by writing *bhâ* (from *bhâvita*, product) after the unknowns with or without interposed dots ; e.g., *yâva-kâgha-bhâ* or *yâvakâghabhâ* means $(yâ)^2\,(kâ)^3$. In the Bakhshâlî treatise the square-root of a quantity is indicated by writing after it *mû*, which is an abbreviation for *mûla* (root). For instance,[1]

11	*yu*	5	*mû*	4
1		1		1

means $\sqrt{11+5}=4$

and

11	7+		*mû*	2
1	1			1

means $\sqrt{11-7}=2.$

In other treatises the symbol of the square-root is *ka* (from *karaṇî*, root or surd) which is usually placed before the quantity affected. For example,[2] *ka* 9 *ka* 450

[1] Folio 59, recto; compare also folio 67, verso.
[2] *BBi*, p. 15.

ka 75 *ka* 54 means $\sqrt{9}+\sqrt{450}+\sqrt{75}+\sqrt{54}$.

Symbols for Unknowns. In the Bakhshâlî treatise there is no specific symbol for the unknown. Consequently its place in an equation is left vacant and to indicate it vividly the sign of emptiness is put there. For instance,[1]

0	2	3	4	*dṛśya* 200
1	1	1	1	1

means $x+2x+3x+4x=200$.

The use of the zero sign to mark a vacant place is found in the arithmetical treatises of later times when the Hindus had a well-developed system of symbols for the unknowns. Thus we find in the *Triśatikâ*[2] of Srîdhara (*c.* 750) the following statement of an arithmetical progression whose first term (*âdiḥ*) is 20, number of terms (*gacchaḥ*) 7, sum (*gaṇitaṁ*) 245 and whose common difference (*uttaraḥ*) is unknown:

| *âdiḥ* 20 | *u* 0 | *gacchaḥ* 7 | *gaṇitaṁ* 245 |

This use of the zero sign in arithmetic was considered necessary as algebraic symbols could not be used there. Lack of an efficient symbolism is bound to give rise to a certain amount of ambiguity in the representation of an algebraic equation especially when it contains more than one known. For instance, in[3]

0	5	*yu*	*mû*	0	*sa*	0	7+	*mû*	0
1	1			1		1	1		1

which means

$$\sqrt{x+5}=s \text{ and } \sqrt{x-7}=t,$$

different unknowns have to be assumed at different vacant places.

[1] *BMs*, Folio 22, verso.
[2] *Triś*, p. 29.
[3] *BMs*, Folio 59, recto.

To avoid such ambiguity, in one instance which contains as many as five unknowns, the abbreviations of ordinal numbers, such as *pra* (from *prathama*, first), *dvi* (from *dvitîya*, second), *tṛ* (from *tṛtîya*, third), *ca* (from *caturtha*, fourth) and *paṁ* (from *pañcama*, fifth), have been used to represent the unknowns; *e.g.*,[1]

9 *pra*	7 *dvi*	10 *tṛ*	8 *ca*	11 *paṁ*	yutam jâtaṁ pratyaika-(kramena)
7 *dvi*	10 *tṛ*	8 *ca*	11 *paṁ*	9 *pra*	16|17|18|19|20

which means

$x_1(=9)+x_2(=7)=16;\ x_2(=7)+x_3(=10)=17;\ x_3(=10)+x_4(=8)=18;\ x_4(=8)+x_5(=11)=19;\ x_5(=11)+x_1(=9)=20.$

Aryabhaṭa I (499) very probably used coloured shots to represent unknowns. Brahmagupta (628) mentions *varṇa* as the symbols of unknowns.[2] As he has not attempted in any way to explain this method of symbolism, it appears that the method was already very familiar. Now, the Sanskrit word *varṇa* means "colour" as well as "letters of the alphabet," so that, in later times, the unknowns are generally represented by letters of the alphabet or by means of various colours such as *kâlaka* (black), *nîlaka* (blue), etc. Again in the latter case, for simplification, only initial letters of the names are generally written. Thus Bhâskara II (1150) observes, "Here (in algebra) the initial letters of (the names of) knowns and unknowns should be written for implying them."[3] It has been stated before that at one time the unknown quantity was called *yâvat-tâvat* (as many

[1] Folio 27, verso. [2] *BrSpSi*, xviii. 2, 42, 51, etc.
[3] *BBi*, p. 2; see also *NBi*, I, R. 7.

as, so much as). In later times this name, or its abbreviation *yâ*, is used for the unknown. According to the celebrated Sanskrit lexicographer Amarasiṁha (*c.* 400 A.D.), *yâvat-tâvat* denotes measure or quantity (*mâna*). He had probably in view the use of that term in Hindu algebra to denote "the measure of an unknown" (*avyakta mâna*). In the case of more unknowns, it is usual to denote the first by *yâvat-tâvat* and the remaining ones by alphabets or colours. Pṛthûdakasvâmî (860) says:

"In an example in which there are two or more unknown quantities, colours such as *yâvat-tâvat*, etc., should be assumed for their values."[1]

He has, indeed, used the colours *kâlaka* (black), *nîlaka* (blue), *pîtaka* (yellow) and *harîtaka* (green).

Śrîpati (1039) writes:

"*Yâvat-tâvat* (so much as) and colours such as *kâlaka* (black), *nîlaka* (blue), etc., should be assumed for the unknowns."[2]

Bhâskara II (1150) says:

"*Yâvat-tâvat* (so much as), *kâlaka* (black), *nîlaka* (blue), *pîta* (yellow), *lohita* (red) and other colours have been taken by the venerable professors as notations for the measures of the unknowns, for the purpose of calculating with them."[3]

"In those examples where occur two, three or more unknown quantities, colours such as *yâvat-tâvat*, etc., should be assumed for them. As assumed by the previous teachers, they are: *yâvat-tâvat* (so much as), *kâlaka* (black), *nîlaka* (blue), *pîtaka* (yellow), *lohitaka* (red), *harîtaka* (green), *śvetaka* (white), *citraka*

[1] *BrSpSi*, xviii. 51 (*Com*). [2] *SiŚe*, xiv. 2.
[3] *BBi*, p. 7.

(variegated), *kapilaka* (tawny), *piṅgalaka* (reddish-brown), *dhûmraka* (smoke-coloured), *pâtalaka* (pink), *śavalaka* (spotted), *śyâmalaka* (blackish), *mecaka* (dark blue), etc. Or the letters of alphabets beginning with *ka*, should be taken as the measures of the unknowns in order to prevent confusion."[1]

The same list with a few additional names of colours appears in the algebra of Nârâyaṇa.[2] This writer has further added,

"Or the letters of alphabets (*varṇa*) such as *ka*, etc., or the series of flavours such as *madhura* (sweet), etc., or the names of dissimilar things with unlike initial letters, are assumed (to represent the unknowns)."

Bhâskara II occasionally employs also the tachygraphic abbreviation of the names of the unknown quantities themselves in order to represent them in an equation. For example,[3] in the following

5 *mâ*	1 *nî*	1 *mu*	1 *va*
1 *mâ*	7 *nî*	1 *mu*	1 *va*
1 *mâ*	1 *nî*	97 *mu*	1 *va*
1 *mâ*	1 *nî*	1 *mu*	2 *va*

mâ stands for *mâṇikya* (ruby), *nî* for (*indra-*)*nîla* (sapphire), *mu* for *muktâphala* (pearl) and *va* for (*sad*)*vajra* (diamond). He has observed in this connection thus:

"(The maxim), 'colours such as *yâvat-tâvat*, etc., should be assumed for the unknowns,' gives (only) one method of implying (them). Here, denoting them

[1] *BBi*, pp. 76f.

[2] *NBi*, I, R. 17-8. These verses have been quoted by Muralidhara Jha in his edition of the *Bîjagaṇita* of Bhâskara II (p. 7, footnote 5).

[3] *BBi*, p. 50; compare also p. 28.

by names, the equations may be formed by the intelligent (calculator)."

It should be noted that *yâvat-tâvat* is not a *varṇa* (colour or letter of alphabet). So in its inclusion in the lists of *varṇa*, as found enumerated in the Hindu algebras—though apparently anomalous—we find the persistence of an ancient symbol which was in vogue long before the introduction of colours to represent unknowns. To avoid the anomaly Muralidhara Jha[1] has suggested the emendation *yâvakastâvat* (*yâvaka* and also; *yâvaka* = red) in the place of *yâvat-tâvat*, as found in the available manuscripts. He thinks that being misled by the old practice, the expression *yâvakastâvat* was confused by copyists with *yâvat-tâvat*. In support of this theory it may be pointed out that *yâvaka* is found to have been sometimes used by Pṛthû lakasvâmî to represent the unknown.[2] Bhâskara II has once used simply *yâvat*.[3] Nârâyana used it on several occasions. The origin of the use of names of colours to represent unknowns in algebra is very probably connected with the ancient use of differently coloured shots for the purpose.

4. LAWS OF SIGNS

Addition. Brahmagupta (628) says:

"The sum of two positive numbers is positive, of two negative numbers is negative; of a positive and a negative number is their difference."[4]

Mahâvîra (850):

"In the addition of a positive and a negative number

[1] See the Preface to his edition of Bhâskara's *Bîjagaṇita*.

[2] *BrSpSi*, xii. 15 (*Com*); xii. 18 (*Com*).

[3] *BBi*, p. 50.

[4] *BrSpSi*, xviii. 30.

(the result) is (their) difference. The addition of two positive or two negative numbers (gives) as much positive or negative numbers respectively."[1]

Srîpati (1039):

"In the addition of two negative or two positive numbers the result is their sum; the addition of a positive and a negative number is their difference."[2]

"The sum of two positive (numbers) is positive; of two negative (numbers) is negative; of a positive and a negative is their difference and the sign of the difference is that of the greater; of two equal positive and negative (numbers) is zero."[3]

Bhâskara II (1150):

"In the addition of two negative or two positive numbers the result is their sum; the sum of a positive and a negative number is their difference."[4]

Nârâyaṇa (1350):

"In the addition of two positive or two negative numbers the result is their sum; but of a positive and a negative number, the result is their difference; subtracting the smaller number from the greater, the remainder becomes of the same kind as the latter."[5]

Subtraction. Brahmagupta writes:

"From the greater should be subtracted the smaller; (the final result is) positive, if positive from positive, and negative, if negative from negative. If, however, the greater is subtracted from the less, that difference is reversed (in sign), negative becomes positive and positive becomes negative. When positive is to be subtracted from negative or negative from positive,

[1] *GSS*, i. 50-1.
[2] *SiSe*, xiv. 3.
[3] *SiŚe*, iii. 28.
[4] *BBi*, p. 2.
[5] *NBi*, I, R. 8.

then they must be added together."[1]

Mahâvîra:

"A positive number to be subtracted from another number becomes negative and a negative number to be subtracted becomes positive."[2]

Śrîpati:

"A positive (number) to be subtracted becomes negative, a negative becomes positive; (the subsequent operation is) addition as explained before."[3]

Bhâskara II:

"A positive (number) while being subtracted becomes negative and a negative becomes positive; then addition as explained before."[4]

Nârâyaṇa:

"Of the subtrahend affirmation becomes negation and negation affirmation; then addition as described before."[5]

Multiplication. Brahmagupta says:

"The product of a positive and a negative (number) is negative; of two negatives is positive; positive multiplied by positive is positive."[6]

Mahâvîra:

"In the multiplication of two negative or two positive numbers the result is positive; but it is negative in the case of (the multiplication of) a positive and a negative number."[7]

Śrîpati:

"On multiplying two negative or two positive

[1] *BrSpSi*, xviii. 31-2.
[2] *GSS*, i. 51.
[3] *SiŚe*, xiv. 3.
[4] *BBi*, p. 3.
[5] *NBi*, I, R. 9.
[6] *BrSpSi*, xviii. 33.
[7] *GSS*, i. 50.

numbers (the product is) positive; in the multiplication of positive and negative (the result is) negative."[1]

Bhâskara II:

"The product of two positive or two negative (numbers) is positive; the product of positive and negative is negative."[2]

The same rule is stated by Nârâyaṇa.[3]

Division. Brahmagupta states:

"Positive divided by positive or negative divided by negative becomes positive. But positive divided by negative is negative and negative divided by positive remains negative."[4]

Mahâvîra:

"In the division of two negative or two positive numbers the quotient is positive, but it is negative in the case of (the division of) positive and negative."[5]

Śrîpati:

"On dividing negative by negative or positive by positive, (the quotient) will be positive, (but it will be) otherwise in the case of positive and negative."[6]

Bhâskara II simply observes: "In the case of division also, such are the rules (*i.e.*, as in the case of multiplication)."[7] Similarly Nârâyaṇa remarks, "What have been implied in the case of multiplication of positive and negative numbers will hold also in the case of division."[8]

Evolution and Involution. Brahmagupta says:

"The square of a positive or a negative number is

[1] *SiŚe*, xiv. 4.
[2] *BBi*, p. 3.
[3] *NBi*, I, R. 9.
[4] *BrSpSi*, xviii. 34.
[5] *GSS*, i. 50.
[6] *SiŚe*, xiv. 4.
[7] *BBi*, p. 3.
[8] *NBi* I, R. 10.

positive. . . . The (sign of the) root is the same as was that from which the square was derived."[1]

As regards the latter portion of this rule the commentator Pṛthûdakasvâmî (860) remarks, "The square-root should be taken either negative or positive, as will be most suitable for subsequent operations to be carried on."

Mahâvîra:

"The square of a positive or of a negative number is positive: their square-roots are positive and negative respectively. Since a negative number by its own nature is not a square, it has no square-root."[2]

Srîpati:

"The square of a positive and a negative number is positive. It will become what it was in the case of the square-root. A negative number by itself is non-square, so its square-root is unreal; so the rule (for the square-root) should be applied in the case of a positive number."[3]

Bhâskara II:

"The square of a positive and a negative number is positive; the square-root of a positive number is positive as well as negative. There is no square-root of a negative number, because it is non-square."[4]

Nârâyaṇa:

"The square of a positive and a negative number is positive. The square-root of a positive number will be positive and also negative. It has been proved that a negative number, being non-square, has no square-root."[5]

[1] *BrSpSi*, xviii. 35. [2] *GSS*, i. 52.
[3] *SiSe*, xiv. 5. [4] *BBi*, p. 4.
[5] *NBi*, I, R. 10.

5. FUNDAMENTAL OPERATIONS

Number of Operations. The number of fundamental operations in algebra is recognised by all Hindu algebraists to be six, *viz.*, addition, subtraction, multiplication, division, squaring and the extraction of the square-root. So the cubing and the extraction of the cube-root which are included amongst the fundamental operations of arithmetic, are excluded from algebra. But the formula

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

or

$$(a + b)^3 = a^3 + 3ab(a + b) + b^3,$$

is found to have been given, as stated before, in almost all the Hindu treatises on arithmetic beginning with that of Brahmagupta (628). By applying it repeatedly, Mahâvîra indicates how to find the cube of an algebraic expression containing more than two terms; thus

$$\begin{aligned}(a + b + c + d + \dots)^3 \\ = a^3 + 3a^2(b + c + d + \dots.) + 3a(b + c + d + \dots)^2 \\ + (b + c + d + \dots.)^3, \\ = a^3 + 3a^2(b + c + d + \dots.) + 3a(b + c + d + \dots)^2 \\ + b^3 + 3b^2(c + d + \dots) + 3b(c + d + \dots\dots)^2 \\ + (c + d + \dots.)^3;\end{aligned}$$

and so on.

Addition and Subtraction. Brahmagupta says:

"Of the unknowns, their squares, cubes, fourth powers, fifth powers, sixth powers, etc., addition and subtraction are (performed) of the like; of the unlike (they mean simply their) statement severally."[1]

Bhâskara II:

"Addition and subtraction are performed of those

[1] *BrSpSi*, xviii. 41.

of the same species (*jâti*) amongst unknowns; of different species they mean their separate statement."[1]

Nârâyaṇa:

"Of the colours or letters of alphabets (representing the unknowns) addition should be made of those which are of the same species; and similarly subtraction. In the addition and subtraction of those of different species the result will be their putting down severally."[2]

Multiplication. Brahmagupta says:

"The product of two like unknowns is a square; the product of three or more like unknowns is a power of that designation. The multiplication of unknowns of unlike species is the same as the mutual product of symbols; it is called *bhâvita* (product or factum)."[3]

Bhâskara II writes:

"A known quantity multiplied by an unknown becomes unknown; the product of two, three or more unknowns of like species is its square, cube, etc.; and the product of those of unlike species is their *bhâvita*. Fractions, etc., are (considered) as in the case of knowns; and the rest (*i.e.*, remaining operations) will be the same as explained in arithmetic. The multiplicand is put down separately in as many places as there are terms in the multiplier and is then severally multiplied by those terms; (the products are then) added together according to the methods indicated before. Here, in the squaring and multiplication of unknowns, should be followed the method of multiplication by component parts, as explained in arithmetic."[4]

The same rules are given by Nârâyaṇa.[5] The fol-

[1] *BBi*, p. 7.
[2] *NBi*, I, R. 19.
[3] *BrSpSi*, xviii. 42.
[4] *BBi*, p. 8.
[5] *NBi*, I, R. 21-2.

lowing illustration amongst others, is given by Bhâskara II:

"Tell at once, O learned, (the result) of multiplying five *yâvat-tâvat* minus one known quantity by three *yâvat-tâvat* plus two knowns.

"Statement: Multiplicand *yâ* 5 *rû* $\dot{1}$; multiplier *yâ* 3 *rû* 2; on multiplication the product becomes *yâ va* 15 *yâ* 7 *rû* $\dot{2}$."[1]

The detailed working of this illustration is shown by the commentator Kṛṣṇa (*c.* 1580) thus:

"*yâ* 3	*yâ* 5	*rû* $\dot{1}$	*yâ va* 15	*yâ* $\dot{3}$		
rû 2	*yâ* 5	*rû* $\dot{1}$		*yâ* 10	*rû* $\dot{2}$"
			yâ va 15	*yâ* 7	*rû* $\dot{2}$

Division. Bhâskara II states:

"By whatever unknowns and knowns, the divisor is multiplied (severally) and subtracted from the dividend successively so that no residue is left, they constitute the quotients at the successive stages."[2]

Nârâyaṇa describes the method of division in nearly the same terms.[3] As an example of division, Bhâskara II proposes to divide $18x^2 + 24xy - 12xz - 12x + 8y^2 - 8yz - 8y + 2z^2 + 4z + 2$ by $-3x - 2y + z + 1$. He simply states the quotient without indicating the different steps in the process. Kṛṣṇa supplies the details of the process which are substantially the same as followed at present.

Squaring. Only one rule for the squaring of an algebraic expression is found in treatises on algebra. It is the same as that stated before in the treatises on arithmetic, *viz.*,

$$(a + b)^2 = a^2 + b^2 + 2ab;$$

[1] *BBi*, p. 8.
[2] *BBi*, p. 9.
[3] *NBi*, I, R. 23.

or, in its general form,

$$(a+b+c+d+\dots)^2=a^2+b^2+c^2+d^2+\dots+2\sum ab.$$

Square-root. For finding the square-root of an algebraic expression Bhâskara II gives the following rule:

"Find the square-root of the unknown quantities which are squares; then deduct from the remaining terms twice the products of those roots two and two; if there be known terms, proceed with the remainder in the same way after taking the square-root of the knowns."[1]

Nârâyaṇa says:

"First find the root of the square terms (of the given expression); then the product of two and two of them (roots) multiplied by two should be subtracted from the remaining terms; (the result thus obtained) is said to be the square-root here (in algebra)."[2]

Jñânarâja writes:

"Take the square-root of those (terms) which are capable of yielding roots; the product of two and two of these (roots) multiplied by two should be deducted from the remaining terms of that square (expression); the result will be the (required) root, so say the experts in this (science)."

6. EQUATIONS

Forming Equations. Before proceeding to the actual solution of an equation of any type, certain preliminary operations have necessarily to be carried out in order to prepare it for solution. Still more preliminary work is that of forming the equation (*samî-karaṇa*, *samî-kâra* or *samî-kriyâ*; from *sama*, equal and

[1] *BBi*, p. 10. [2] *NBi*, I, R. 24.

kṛ, to do; hence literally, making equal) from the conditions of the proposed problem. Such preliminary work may require the application of one or more fundamental operations of algebra or arithmetic. The operations preliminary to the formation of a simple equation have been described by Pṛthûdakasvâmî (860) thus:

"In this case, in the problem proposed by the questioner, *yâvat-tâvat* is put for the value of the unknown quantity; then performing multiplication, division, etc., as required in the problem the two sides shall be carefully made equal. The equation being formed in this way, then the rule (for its solution) follows."[1]

Bhâskara II's descriptions are fuller: He says:

"Let *yâvat-tâvat* be assumed as the value of the unknown quantity. Then doing precisely as has been specifically told—by subtracting, adding, multiplying or dividing[2]—the two equal sides (of an equation) should be very carefully built."[3]

Nârâyaṇa says:

"Of these (four classes of equations), the linear equation in one unknown (will be treated) first. In a problem (proposed), the value of the quantity which is unknown is assumed to be *yâvat*, one, two or any multiple of it, with or without an absolute term, which again may be additive or subtractive. Then on the value thus assumed optionally should be performed, in

[1] *BrSpSi*, xviii. 43 (*Com*).

[2] In his gloss Bhâskara II explains: "Every operation, such as multiplication, division, rule of three, rule of five terms, summation of series, or treatment of plane figures, etc., according to the statement of the problem should be performed......" See *BBi*, p. 44.

[3] *BBi*, p. 43.

accordance with the statement of the problem, the operations such as addition, subtraction, multiplication, division, rule of three, double rule of three, summation, plane figures, excavations, etc. And thus the two sides must be made equal. If the equality of the two sides is not explicitly stated, then one side should be multiplied, divided, increased or decreased by one's own intelligence (according to the problem) and thus the two sides must be made equal."[1]

Plan of Writing Equations. After an equation is formed, writing it down for further operations is technically called *nyâsa* (putting down, statement) of the equation. In the Bakhshâlî treatise the two sides of an equation are put down one after the other in the same line without any sign of equality being interposed.[2] Thus the equations

$$\sqrt{x+5}=s,\quad \sqrt{x-7}=t$$

appear as[3]

0	5	*yu*	*mû*	0	*sa*	0	7+		*mû*	0
1	1			1		1	1			1

The equation

$$x+2x+3\times 3x+12\times 4x=300$$

is stated as[4]

0	2	1	3	3	12	4	*dṛśya*	300.
1	1	1	1	1	1	1		

This plan of writing an equation was subsequently abandoned by the Hindus for a new one in which the two sides are written one below the other without any

[1] *NBi*, II, R. 3 (*Gloss*).
[2] Datta, *Bakh. Math.*, (*BCMS*, XXI), p. 28.
[3] Folio 59, recto. [4] Folio 23, verso.

sign of equality. Further, in this new plan, the terms of similar denominations are usually written one below the other and even the terms of absent denominations on either side are expressly indicated by putting zeros as their coefficients. Reference to the new plan is found as early as the algebra of Brahmagupta (628).[1] Pṛthûdakasvâmî (860) represented the equation[2]

$$10x - 8 = x^2 + 1$$

as follows:

$$\begin{array}{llllll} \textit{yâ va} & 0 & \textit{yâ} & 10 & \textit{rû} & \dot{8} \\ \textit{yâ va} & 1 & \textit{yâ} & 0 & \textit{rû} & 1 \end{array}$$

which means, writing x for *yâ*

$$\begin{array}{l} x^2.0 + x.10 - 8 \\ x^2 \quad + \; x.0 + 1 \end{array}$$

or $$0x^2 + 10x - 8 = x^2 + 0x + 1.$$

If there be several unknowns, those of the same kind are written in the same column with zero coefficients, if necessary. Thus the equation

$$197x - 1644y - z = 6302$$

is represented by Pṛthûdakasvâmî thus:[3]

$$\begin{array}{llllllll} \textit{yâ} & 197 & \textit{kâ} & 164\dot{4} & \textit{nî} & \dot{1} & \textit{rû} & 0 \\ \textit{yâ} & 0 & \textit{kâ} & 0 & \textit{nî} & 0 & \textit{rû} & 6302 \end{array}$$

which means, putting y for *kâ* and z for *nî*,

$$197x - 1644y - z + 0 = 0x + 0y + 0z + 6302.$$

The following two instances are from the *Bîja-gaṇita* of Bhâskara II (1150):[4]

(1) $$\begin{array}{llllllll} \textit{yâ} & 5 & \textit{kâ} & 8 & \textit{nî} & 7 & \textit{rû} & 90 \\ \textit{yâ} & 7 & \textit{kâ} & 9 & \textit{nî} & 6 & \textit{rû} & 62 \end{array}$$

[1] *BrSpSi*, xvii. 43 (*vide infra*, p. 33). Compare also *BBi*, p. 127.
[2] *BrSpSi*, xviii. 49 (*Com*). [3] *BrSpSi*, xviii. 54 (*Com*).
[4] *BBi*, pp. 78, 101.

means (writing x for *yâ*, y for *kâ* and z for *nî*)

$$5x + 8y + 7z + 90 = 7x + 9y + 6z + 62.$$

(2) *yâ gha* 8 *yâ va* 4 *kâ va yâ. bhâ* 10

yâ gha 4 *yâ va* 0 *kâ va yâ. bhâ* 12

means $8x^3 + 4x^2 + 10y^2x = 4x^3 + 0x^2 + 12y^2x,$

or $8x^3 + 4x^2 + 10y^2x = 4x^3 + 12y^2x.$

In the above plan it will be noticed that the terms are ordered according to descending powers of the unknowns. Numerical coefficients are placed after the unknowns; if the coefficient be unity it is noted particularly and if the coefficient be fractional it is kept distinct from the unknown, that is, its denominator is so written as not to come under the unknown;[1] the minus sign is put over the numerical coefficient rather than on the unknown; and the absolute term is invariably put last on either side. As has been observed by Professor Smith,[2] this plan "in one respect was the best that has ever been suggested." For "it shows at a glance the similar terms one above the other, and permits of easy transposition."

The use of the old plan of writing equations is sometimes met with in later works also. For instance, in the MS. of Pṛthûdakasvâmî's commentary on the *Brâhma-sphuṭa-siddhânta*, an incomplete copy of which is preserved in the library of the Asiatic Society of Bengal (No. I B6) we find a statement of equations thus: "first side *yâvargaḥ* 1 *yâvakaḥ* 200 *rû* 0; second side *yâvargaḥ* 0 *yâvakaḥ* 0 *rû* 1500."[3] That is to say,

$$x^2 + 200x + 0 = 0x^2 + 0x + 1500.$$

[1] For instance, see *BBi*, pp. 47 ff.
[2] Smith, *History*, II, pp. 425, 426.
[3] *BrSpSi*, xii. 15 (*Com*).

Preparation of Equations. The operation to be performed on an equation next to its statement (*nyâsa*) is technically known as *samaśodhana* (from *sama*, meaning equal or complete, and *śodhana*, clearance; hence literally meaning, equi-clearance or complete clearance[1]) or simply *śodhana*. The nature of this clearance varies according to the kind of the equation. In the case of an equation in one unknown only, whether linear, quadratic or of higher powers, one side of it is cleared of the unknowns of all denominations and the other side of it of the absolute terms, so that the equation is ultimately reduced to one of the form

$$ax^2 + bx = c,$$

where a, b, c may be positive or negative; some of them may be even zero. Thus Brahmagupta observes:

"From which the square of the unknown and the unknown are cleared, the known quantities are cleared (from the side) *below* that."[2]

Pṛthûdakasvâmî explains:

"This rule[3] has been introduced for that case in which the two sides of the equation having been formed in accordance with the statement of the problem, there are present the square and other powers of the unknown together with the (simple) unknown. The absolute terms should be cleared off from the side opposite to that from which are cleared the square (and other powers) of the unknown and the (simple) unknown. When perfect clearance (*samaśodhana*) has

[1] Colebrooke's rendering of the term as "equal subtraction", though not literally inaccurate, is technically so; at least it is not happy.

[2] *BrSpSi*, xviii. 43.

[3] The reference is to Brahmagupta's rule for the solution of a quadratic equation.

been thus made..."[1]

Śrîpati says:

"From one (side) the square of the unknown and the unknown should be cleared by removing the known quantities; the known quantities (should be cleared) from the side opposite to that."[2]

Similarly Bhâskara II:

"Then the unknown on one side of it (the equation) should be subtracted from the unknown on the other side; so also the square and other powers of the unknown; the known quantities on the other side should be subtracted from the known quantities of another (*i.e.*, the former) side."[3]

Here we give a few illustrations. With reference to the equations from the commentary of Pṛthûdakasvâmî, stated on page 31, the author says:

"Perfect clearance (*samaśodhana*) being made in accordance with the rule, (the equation) will be

yâ va 1 *yâ* $\dot{1}0$

rû $\dot{9}$"

i.e., $x^2 - 10x = -9.$

The following illustration is from the *Bîjagaṇita* of Bhâskara II:[4]

"Thus the two sides are

yâ va 4 *yâ* $3\dot{4}$ *rû* 72

yâ va 0 *yâ* 0 *rû* 90

On complete clearance (*samaśodhana*), the residues of the two sides are

[1] *BrSpSi*, xviii. 44 (*Com*). [2] *SiŚe*, xvi. 17.
[3] *BBi*, p. 44. [4] *BBi*, p. 63.

yâ va 4 *yâ* $\dot{3}\dot{4}$ *rû* 0
yâ va 0 *yâ* 0 *rû* 18"

i.e., $4x^2 - 34x = 18.$

Classification of Equations. The earliest Hindu classification of equations seems to have been according to their degrees, such as simple (technically called *yâvat-tâvat*), quadratic (*varga*), cubic (*ghana*) and biquadratic (*varga-varga*). Reference to it is found in a canonical work of *circa* 300 B. C.[1] But in the absence of further corroborative evidence, we cannot be sure of it. Brahmagupta (628) has classified equations as: (1) equations in one unknown (*eka-varṇa-samîkaraṇa*), (2) equations in several unknowns (*aneka-varṇa-samîkaraṇa*), and (3) equations involving products of unknowns (*bhâvita*). The first class is again divided into two subclasses, *viz.*, (*i*) linear equations, and (*ii*) quadratic equations (*avyakta-varga-samîkaraṇa*). Here then we have the beginning of our present method of classifying equations according to their degrees. The method of classification adopted by Pṛthûdakasvâmî (860) is slightly different. His four classes are: (1) linear equations with one unknown, (2) linear equations with more unknowns, (3) equations with one, two or more unknowns in their second and higher powers, and (4) equations involving products of unknowns. As the method of solution of an equation of the third class is based upon the principle of the elimination of the middle term, that class is called by the name *madhyamâharaṇa* (from *madhyama*, "middle", *âharaṇa* "elimination", hence meaning "elimination of the middle term"). For the other classes, the old names given by Brahmagupta have been retained. This method of classification has been followed by subsequent writers.

[1] *Sthânâṅga-sûtra*, Sûtra 747. For further particulars see Datta, *Jaina Math.*, (*BCMS*, XXI), pp. 119ff.

Bhâskara II distinguishes two types in the third class, *viz.*, (*i*) equations in one unknown in its second and higher powers and (*ii*) equations having two or more unknowns in their second and higher powers. According to Kṛṣṇa (1580) equations are primarily of two classes: (1) equations in one unknown and (2) equations in two or more unknowns. The class (1), again, comprises two subclasses: (*i*) simple equations and (*ii*) quadratic and higher equations. The class (2) has three subclasses: (*i*) simultaneous linear equations, (*ii*) equations involving the second and higher powers of unknowns, and (*iii*) equations involving products of unknowns. He then observes that these five classes can be reduced to four by including the second subclasses of classes (1) and (2) into one class as *madhyamâharaṇa*.

7. LINEAR EQUATIONS IN ONE UNKNOWN

Early Solutions. As already stated, the geometrical solution of a linear equation in one unknown is found in the *Śulba*, the earliest of which is not later than 800 B.C. There is a reference in the *Sthânâṅga-sûtra* (*c.* 300 B.C.) to a linear equation by its name (*yâvat-tâvat*) which is suggestive of the method of solution[1] followed at that time. We have, however, no further evidence about it. The earliest Hindu record of doubtless value of problems involving simple algebraic equations and of a method for their solution occurs in the Bakhshâlî treatise, which was written very probably about the beginning of the Christian Era. One problem is:[2]

"The amount given to the first is not known. The second is given twice as much as the first; the third

[1] Datta, *Jaina Math.*, (*BCMS*, XXI), p. 122.
[2] *BMs*, Folio 23, recto.

thrice as much as the second; and the fourth four times as much as the third. The total amount distributed is 132. What is the amount of the first?"

If x be the amount given to the first, then according to the probelm,

$$x + 2x + 6x + 24x = 132.$$

Rule of False Position. The solution of this equation is given as follows:

" 'Putting any desired quantity in the vacant place'; any desired quantity is || 1 ||; 'then construct the series'

$$\left|\begin{array}{c|cc|cc|cc|}1 & 2 & & 2 & 3 & 6 & 4\\ 1 & 1 & & 1 & 1 & 1 & 1\end{array}\right|$$

'multiplied' || 1 | 2 | 6 | 24 |; 'added' 33. 'Divide the visible quantity' $\left|\begin{array}{c}132\\33\end{array}\right|$; (which) on reduction becomes $\left|\begin{array}{c}4\\1\end{array}\right|$. (This is) the amount given (to the first)."[1]

The solution of another set of problems in the Bakhshâlî treatise, leads ultimately to an equation of the type[2]

$$ax + b = p.$$

The method given for its solution is to put any arbitrary value g for x, so that

$$ag + b = p', \text{ say}.$$

Then the correct value will be

$$x = \frac{p - p'}{a} + g.$$

[1] *Ibid.* [2] *Vide infra*, pp. 48f.

The above method of solution of a linear equation was known in the middle ages, amongst Arab and European algebraists, by the name of the Rule of False Position. It is noteworthy that the terms *yaddṛcchâ*, *vâñcchâ*, and *kâmika* of the Bakhshâlî treatise are equivalent to the term *yâvat-tâvat*. So the origin of this latter term seems to be connected with the Rule of False Position. It is interesting to find that the rule was once given so much importance in Hindu algebra that the section dealing with it was named after it.

Disappearance from Later Algebra. The Rule of False Position bespeaks of an early stage of development of the science of algebra when there was no symbol for the unknown. It naturally disappears with the introduction of a system of notations.[1] This will account for the complete disappearance of the Rule of False Position from the later Hindu treatises on algebra beginning with that of Âryabhaṭa I (499). There are found, however, very limited applications of it in the arithmetical treatises of Śrîdhara (*c.* 750), Mahâvîra (850) and Bhâskara II (1150). This can be accounted for easily. The problems which have been solved by those writers with the help of the Rule of False Position are really of algebraic nature, though incorporated into arithmetical treatises. But as the use of algebraic symbols and notations is not permissible in arithmetic, recourse had to be taken to that Rule. For instance, we take the following problem from the *Gaṇita-sâra-saṁgraha* of Mahâvîra:

"The sum of $\frac{1}{8}$, $\frac{1}{4}$ of $\frac{1}{3}$, $\frac{1}{5}$ of $\frac{1}{2}$, $\frac{1}{5}$ of $\frac{3}{4}$ of $\frac{1}{6}$, of a certain number is equal to $\frac{1}{2}$. What is that unknown (number)?"[2]

Mahâvîra gives the following rule for finding out

[1] Smith, *History*, II, p. 437. [2] *GSS*, iii. 108.

the unknown in a problem of this kind:

"Put down one for the value of the unknown; then in accordance with the previous rule (find) the sum (of its parts); divide the known (number) by that (sum); the quotient will be (the value of) the unknown in compound fractions."[1]

Operation with an Optional Number. Bhâskara II describes a method called *Iṣṭa-karma* or "operation with an optional number." This may be illustrated by the following example:

"What is that number which multiplied by five, diminished by its third part and (then) divided by ten, will become, together with its one-third, half and one-fourth parts, equal to seventy minus two?"[2]

i.e., $$\frac{5x - 5x/3}{10} + \frac{x}{3} + \frac{x}{2} + \frac{x}{4} = 70 - 2.$$

Bhâskara assumes $x = 3$ and then calculates

$$\frac{5 \times 3 - 5 \times 3/3}{10} + \tfrac{3}{3} + \tfrac{3}{2} + \tfrac{3}{4} = \tfrac{17}{4}.$$

He then states

$$x = 68 \times 3 \div \tfrac{17}{4} = 48.$$

He observes: "Similarly, in every example, by whatever the (required) number is multiplied or divided, by whatever fraction of the number it is found to have been increased or diminished, assuming an optional number, on it perform the same operations in accordance with the statement of the problem; by that, which results, divide the known number multiplied by the assumed number; the quotient will be the (required) number."[3]

[1] *GSS*, iii. 107. [2] *L*, p. 10.
[3] *L*, p. 11.

Solution of Linear Equations. Âryabhaṭa I (499) says:

"The difference of the known "amounts" relating to the two persons should be divided by the difference of the coefficients of the unknown.[1] The quotient will be the value of the unknown, if their possessions be equal."[2]

This rule contemplates a problem of this kind: Two persons, who are equally rich, possess respectively a, b times a certain unknown amount together with c, d units of money in cash. What is that amount?

If x be the unknown amount, then by the problem

$$ax + c = bx + d.$$

Therefore $$x = \frac{d - c}{a - b}.$$

Hence the rule.

Brahmagupta says:

"In a (linear) equation in one unknown, the difference of the known terms taken in the reverse order, divided by the difference of the coefficients of the unknown (is the value of the unknown)."[3]

Śrîpati writes:

"First remove the unknown from any one of the sides (of the equation) leaving the known term; the reverse (should be done) on the other side. The difference of the absolute terms taken in the reverse order

[1] The original is *gulikântara* which literally means "the difference of the unknowns." But what is implied is "the difference of the coefficients of the unknown." As has been observed by Pṛthûdakasvâmî, according to the usual practice of Hindu algebra, "the coefficient of the square of the unknown is called the square (of the unknown) and the coefficient of the (simple) unknown is called the unknown." *BrSpSi*, xviii. 44 (*Com*).

[2] *Â*, ii. 30.

[3] *BrSpSi*, xviii. 43.

divided by the difference of the coefficients of the unknown will be the value of the unknown."[1]

Bhâskara II states:

"Subtract the unknown on one side from that on the other and the absolute term on the second from that on the first side. The residual absolute number should be divided by the residual coefficient of the unknown; thus the value of the unknown becomes known."[2]

Nârâyaṇa writes:

"From one side clear off the unknown and from the other the known quantities; then divide the residual known by the residual coefficient of the unknown. Thus will certainly become known the value of the unknown."[3]

For illustration we take a problem proposed by Brahmagupta:

"Tell the number of elapsed days for the time when four times the twelfth part of the residual degrees increased by one, plus eight will be equal to the residual degrees plus one."[4]

It has been solved by Pṛthûdakasvâmî as follows:

"Here the residual degrees are (put as) *yâvat-tâvat*, *yâ*; increased by one, *yâ* 1 *rû* 1; twelfth part of it, $\dfrac{\textit{yâ}\ 1\quad \textit{rû}\ 1}{12}$; four times this, $\dfrac{\textit{yâ}\ 1\quad \textit{rû}\ 1}{3}$; plus the absolute quantity eight, $\dfrac{\textit{yâ}\ 1\quad \textit{rû}\ 25}{3}$. This is equal to the residual degrees plus unity. The statement of both sides tripled is

yâ 1 *rû* 25
yâ 3 *rû* 3

[1] *SiŚe*, xiv. 15.
[2] *BBi*, p. 44.
[3] *NBi*, II, R. 5.
[4] *BrSpSi*, xviii. 46.

The difference between the coefficients of the unknown is 2. By this the difference of the absolute terms, namely 22, being divided, is produced the residual of the degrees of the sun, 11. These residual degrees should be known to be irreducible. The elapsed days can be deduced then, (proceeding) as before."

In other words, we have to solve the equation

$$\tfrac{4}{12}(x + 1) + 8 = x + 1,$$

which gives $\qquad x + 25 = 3x + 3,$

or $\qquad 2x = 22.$

Therefore $\qquad x = 11.$

The following problem and its solution are from the *Bījagaṇita* of Bhâskara II :

"One person has three hundred coins and six horses. Another has ten horses (each) of similar value and he has further a debt of hundred coins. But they are of equal worth. What is the price of a horse ?

"Here the statement for equi-clearance is :

$$6x + 300 = 10x - 100.$$

Now, by the rule, 'Subtract the unknown on one side from that on the other etc.,' unknown on the first side being subtracted from the unknown on the other side, the remainder is $4x$. The absolute term on the second side being subtracted from the absolute term on the first side, the remainder is 400. The residual known number 400 being divided by the coefficient of the residual unknown $4x$, the quotient is recognised to be the value of x, (namely) 100."[1]

There are a few instances in the Bakhshâlî work where a method similar to that of later writers appears

[1] *BBi*, pp. 44f.

to have been followed for the solution of a linear equation. One such problem is: Two persons start with different initial velocities (a_1, a_2); travel on successive days distances increasing at different rates (b_1, b_2). But they cover the same distance after the same period of time. What is the period?

Denoting the period by x, we get

$$a_1 + (a_1 + b_1) + (a_1 + 2b_1) + \ldots \text{ to } x \text{ terms}$$
$$= a_2 + (a_2 + b_2) + (a_2 + 2b_2) + \ldots \text{ to } x \text{ terms},$$

or $$\left\{ a_1 + \left(\frac{x-1}{2}\right) b_1 \right\} x = \left\{ a_2 + \left(\frac{x-1}{2}\right) b_2 \right\} x;$$

whence $$x = \frac{2(a_2 - a_1)}{b_1 - b_2} + 1,$$

which is the solution given in the Bakhshâlî work:

"Twice the difference of the initial terms divided by the difference of the common differences, is increased by unity. The result will be the number of days in which the distance moved will be the same."[1]

8. LINEAR EQUATIONS WITH TWO UNKNOWNS

Rule of Concurrence. One topic commonly discussed by almost all Hindu writers goes by the special name of *saṅkramaṇa* (concurrence). According to Nârâyaṇa (1350), it is also called *saṅkrama* and *saṅkrâma*.[2] Brahmagupta (628) includes it in algebra while others consider it as falling within the scope of arithmetic. As explained by the commentator Gaṅgâdhara (1420), the subject of discussion here is "the investigation of two quantities concurrent or grown together in the form of their sum and difference."

[1] *BMs*, Folio 4, verso. [2] *GK*, i. 31.

Or, in other words, *saṅkramaṇa* is the solution of the simultaneous equations

$$x + y = a,$$
$$x - y = b.$$

So Brahmagupta and Śrîpati are perfectly right in thinking that concurrence is truly a topic for algebra.

Brahmagupta's rule for solution is :

"The sum is increased and diminished by the difference and divided by two ; (the result will be the two unknown quantities) : (this is) concurrence."[1]

The same rule is restated by him on a different occasion in the form of a problem and its solution.

"The sum and difference of the residues of two (heavenly bodies) are known in degrees and minutes. What are the residues ? The difference is both added to and subtracted from the sum, and halved ; (the results are) the residues."[2]

Similar rules are given also by other writers.[3]

Linear Equations. Mahâvîra gives the following examples leading to simultaneous linear equations together with rules for the solution of each.

Example. "The price of 9 citrons and 7 fragrant wood-apples taken together is 107 ; again the price of 7 citrons and 9 fragrant wood-apples taken together is 101. O mathematician, tell me quickly the price of a citron and of a fragrant wood-apple quite separately."[4]

If x, y be the prices of a citron and of a fragrant

[1] *BrSpSi*, xviii. 36. [2] *BrSpSi*, xviii. 96.
[3] *GSS*, vi. 2 ; *MSi*, xv. 21 ; *SiŚe*, xiv. 13 ; *L*, p. 12 ; *GK*, i. 31.
[4] *GSS*, vi. 140½–142½.

wood-apple respectively, then

$$9x + 7y = 107,$$
$$7x + 9y = 101.$$

Or, in general,
$$ax + by = m,$$
$$bx + ay = n.$$

Solution. "From the larger amount of price multiplied by the (corresponding) bigger number of things subtract the smaller amount of price multiplied by the (corresponding) smaller number of things. (The remainder) divided by the difference of the squares of the numbers of things will be the price of each of the bigger number of things. The price of the other will be obtained by reversing the multipliers."[1]

Thus $$x = \frac{am - bn}{a^2 - b^2}, \quad y = \frac{an - bm}{a^2 - b^2}.$$

Example. "A wizard having powers of mystic incantations and magical medicines seeing a cock-fight going on, spoke privately to both the owners of the cocks. To one he said, 'If your bird wins, then you give me your stake-money, but if you do not win, I shall give you two-thirds of that.' Going to the other, he promised in the same way to give three-fourths. From both of them his gain would be only 12 gold pieces. Tell me, O ornament of the first-rate mathematicians, the stake-money of each of the cock-owners."[2]

i.e., $$x - \tfrac{3}{4}y = 12, \quad y - \tfrac{2}{3}x = 12.$$

Or, in general,

$$x - \frac{c}{d}y = p, \quad y - \frac{a}{b}x = p.$$

[1] *GSS*, vi. 139½.

[2] *GSS*, vi. 270-2½.

Solution:[1]

$$x = \frac{b(c+d)}{(c+d)b - (a+b)c}\,p,$$

$$y = \frac{d(a+b)}{(a+b)d - (c+d)a}\,p.$$

The following example with its solution is taken from the *Bîjagaṇita* of Bhâskara II :

Example. "One says, 'Give me a hundred, friend, I shall then become twice as rich as you.' The other replies, 'If you give me ten, I shall be six times as rich as you.' Tell me what is the amount of their (respective) capitals ?"[2]

The equations are

$$x + 100 = 2(y - 100), \qquad (1)$$

$$y + 10 = 6(x - 10). \qquad (2)$$

Bhâskara II indicates two methods of solving these equations. They are substantially as follows :

First Method.[3] Assume

$$x = 2z - 100, \quad y = z + 100,$$

so that equation (1) is identically satisfied. Substituting these values in the other equation, we get

$$z + 110 = 12z - 660;$$

whence $z = 70$. Therefore, $x = 40$, $y = 170$.

Second Method.[4] From equation (1), we get

$$x = 2y - 300,$$

and from equation (2)

$$x = \tfrac{1}{6}(y + 70).$$

[1] *GSS*, vi. 268½-9½.
[2] *BBi*, p. 41.
[3] *BBi*, p. 46.
[4] *BBi*, pp. 78f.

Equating these two values of x, we have

$$2y - 300 = \tfrac{1}{6}(y + 70),$$

or

$$12y - 1800 = y + 70;$$

whence $y=170$. Substituting this value of y in any of the two expressions for x, we get $x = 40$.

It is noteworthy that the second method of solution of the problem under consideration is described by Bhâskara II in the section of his algebra dealing with "linear equations with several unknowns," while the first method in that dealing with "linear equations in one unknown." In this latter connection he has observed that the solution of a problem containing two unknowns can sometimes be made by ingenious artifices to depend upon the solution of a simple linear equation.

9. LINEAR EQUATIONS WITH SEVERAL UNKNOWNS

A Type of Linear Equations. The earliest Hindu treatment of systems of linear equations involving several unknowns is found in the Bakhshâlî treatise. One problem in it runs as follows :

"[Three persons possess a certain amount of riches each.] The riches of the first and the second taken together amount to 13 ; the riches of the second and the third taken together are 14 ; and the riches of the first and the third mixed are known to be 15. Tell me the riches of each."[1]

If x_1, x_2, x_3 be the wealths of the three merchants respectively, then

$$x_1 + x_2 = 13, \; x_2 + x_3 = 14, \; x_3 + x_1 = 15. \qquad (1)$$

Another problem is

[1] *BMs*, Folio 29, recto. The portions within [ ] in this and the following illustration have been restored.

"[Five persons possess a certain amount of riches each. The riches of the first] and the second mixed together amount to 16; the riches of the second and the third taken together are known to be 17; the riches of the third and the fourth taken together are known to be 18; the riches of the fourth and the fifth mixed together are 19; and the riches of the first and the fifth together amount to 20. Tell me what is the amount of each."[1]

i.e., $x_1 + x_2 = 16,\ x_2 + x_3 = 17,\ x_3 + x_4 = 18,$

$$x_4 + x_5 = 19,\ x_5 + x_1 = 20. \qquad (2)$$

There are in the work a few other similar problems.[2] Every one of them belongs to a system of linear equations of the type

$$x_1 + x_2 = a_1,\ x_2 + x_3 = a_2,\ \ldots,\ x_n + x_1 = a_n, \qquad (I)$$

n being *odd*.

Solution by False Position. A system of linear equations of this type is solved in the Bakhshâlî treatise substantially as follows:

Assume an arbitrary value p for x_1 and then calculate the values of x_2, x_3, ... corresponding to it. Finally let the calculated value of $x_n + x_1$ be equal to b (say). Then the true value of x_1 is obtained by the formula

$$x_1 = p + \tfrac{1}{2}(a_n - b).$$

In the particular case (1) the author[3] assumes the arbitrary value 5 for x_1; then are successively calculated the values $x'_2 = 8$, $x'_3 = 6$ and $x'_3 + x'_1 = 11$. The correct values are, therefore,

$$x_1 = 5 + (15 - 11)/2 = 7,\ x_2 = 6,\ x_3 = 8.$$

[1] *BMs*, Folios 27 and 29, verso.
[2] *BMs*, Folio 30, recto; also see Kaye's Introduction, p. 40.
[3] *BMs*, Folio, 29, recto.

Rationale. By the process of elimination we get from equations (I)

$$(a_2-a_1)+(a_4-a_3)+\ldots.+(a_{n-1}-a_{n-2})+2x_1=a_n.$$

Assume $x_1=p$; so that

$$(a_2-a_1)+(a_4-a_3)+\ldots..+(a_{n-1}-a_{n-2})+2p=b, \text{ say.}$$

Subtracting $\quad 2(x_1-p)=a_n-b.$

Therefore $\quad x_1=p+\frac{1}{2}(a_n-b).$

Second Type. A particular case of the type of equations (I) for which $n=3$, may also be looked upon as belonging to a different type of systems of linear equations, *viz.*,

$$\Sigma x-x_1=a_1, \Sigma x-x_2=a_2, \ldots, \Sigma x-x_n=a_n, \quad \text{(II)}$$

where Σx stands for $x_1+x_2+\ldots+x_n$. But it will not be proper to say that equations of this type have been treated in the Bakhshâlî treatise.[1] They have, however, been solved by Âryabhaṭa (499) and Mahâvîra (850). The former says:

"The (given) sums of certain (unknown) numbers, leaving out one number in succession, are added together separately and divided by the number of terms less one; that (quotient) will be the value of the whole."[2]

i.e., $$\Sigma x=\sum_{r=1}^{n} a_r/(n-1).$$

Mahâvîra states the solution thus:

"The stated amounts of the commodities added together should be divided by the number of men less

[1] The example cited by Kaye (*BMs*, Introd., p. 40, Ex. vi) which conforms to this type of equations is based upon a misapprehension of the text.

[2] *Â*, ii. 29.

one. The quotient will be the total value (of all the commodities). Each of the stated amounts being subtracted from that, (the value) in the hands (of each will be found)."[1]

In formulating his rule Mahâvîra had in view the following example:

"Four merchants were each asked separately by the customs officer about the total value of their commodities. The first merchant, leaving out his own investment, stated it to be 22; the second stated it to be 23, the third 24 and the fourth 27; each of them deducted his own amount in the investment. O friend, tell me separately the value of (the share of) the commodity owned by each."[2]

Here $x_1 + x_2 + x_3 + x_4 = \dfrac{22+23+24+27}{4-1} = 32.$

Therefore $x_1 = 10, \quad x_2 = 9, \quad x_3 = 8, \quad x_4 = 5.$

Nârâyaṇa says:

"The sum of the depleted amounts divided by the number of persons less one, is the total amount. On subtracting from it the stated amounts severally will be found the different amounts."[3]

The above type of equations is supposed by some modern historians of mathematics[4] to be a modification of the type considered by the Greek Thymaridas and solved by his well known rule *Epanthema*, namely,[5]

$$x + x_1 + x_2 + \dots + x_{n-1} = s,$$
$$x + x_1 = a_1,\; x + x_2 = a_2, \dots,\; x + x_{n-1} = a_{n-1}.$$

[1] *GSS*, vi. 159. [2] *GSS*, vi. 160-2.

[3] *GK*, ii. 28.

[4] Cantor, *Vorlesungen über Geschichte der Mathematik* (referred to hereafter as Cantor, *Geschichte*), I, p. 624; Kaye, *Ind. Math.*, p. 13; *JASB*, 1908, p. 135.

[5] Heath, *Greek Math.*, I, p. 94.

The solution given is

$$x = \frac{(a_1 + a_2 + \ldots + a_{n-1}) - s}{n - 2}.$$

But that supposition has been disputed by others.[1] Sarada Kanta Ganguly has shown that it is based upon a misapprehension. It will be noticed that in the Thymaridas type of linear equations, the value of the sum of the unknowns is given whereas in the Âryabhaṭa type it is not known. In fact, Âryabhaṭa determines only that value.

Third Type. A more generalised system of linear equations will be

$$b_1 \Sigma x - c_1 x_1 = a_1, \quad b_2 \Sigma x - c_2 x_2 = a_2, \ldots, \\ b_n \Sigma x - c_n x_n = a_n. \qquad \text{(III)}$$

Therefore $$\Sigma x = \frac{\Sigma (a/c)}{\Sigma (b/c) - 1}.$$

Hence $$x_r = \frac{b_r}{c_r} \cdot \frac{\Sigma (a/c)}{\Sigma (b/c) - 1} - \frac{a_r}{c_r}, \qquad (1)$$

$$r = 1, 2, 3, \ldots, n.$$

A particular case of this type is furnished by the following example of Mahâvîra:

"Three merchants begged money mutually from one another. The first on begging 4 from the second and 5 from the third became twice as rich as the others. The second on having 4 from the first and 6 from the third became thrice as rich. The third man on begging 5 from the first and 6 from the second became five times as rich as the others. O mathematician, if you know

[1] Rodet, *Leçons de Calcul d'Âryabhaṭa*, *JA*, XIII (7), 1878; Sarada Kanta Ganguly, "Notes on Aryabhaṭa," *Jour. Bihar and Orissa Research Soc.*, XII, 1926, pp. 88ff.

the *citra-kuṭṭaka-miśra*,[1] tell me quickly what was the amount in the hand of each."[2]

That is, we get the equations

$$x + 4 + 5 = 2(y + z - 4 - 5),$$
$$y + 4 + 6 = 3(z + x - 4 - 6),$$
$$z + 5 + 6 = 5(x + y - 5 - 6);$$

or

$$2(x + y + z) - 3x = 27,$$
$$3(x + y + z) - 4y = 40,$$
$$5(x + y + z) - 6z = 66;$$

a particular case of the system (III). Substituting in (1), we get

$$x = 7, \quad y = 8, \quad z = 9.$$

In general, suppose $a_{r,1}, a_{r,2}, \ldots a_{r,r-1}, a_{r,r+1} \ldots a_{r,n}$ to be the amounts begged by the rth merchant from the others; and x_r the amount that he had initially. Then

$$x_1 + \Sigma' a_{1,m} = b_1(\Sigma x - x_1 - \Sigma' a_{1,m}),$$
$$x_2 + \Sigma' a_{2,m} = b_2(\Sigma x - x_2 - \Sigma' a_{2,m}),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_n + \Sigma' a_{n,m} = b_n(\Sigma x - x_n - \Sigma' a_{n,m}),$$

where $\Sigma' a_{r,m}$ denotes summation from $m = 1$ to $m = n$ excluding $m = r$. Therefore

$$\Sigma x + (b_1 + 1)\Sigma' a_{1,m} = (b_1 + 1)(\Sigma x - x_1),$$
$$\Sigma x + (b_2 + 1)\Sigma' a_{2,m} = (b_2 + 1)(\Sigma x - x_2),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\Sigma x + (b_n + 1)\Sigma' a_{n,m} = (b_n + 1)(\Sigma x - x_n).$$

Let $k_r = (b_r + 1)\Sigma' a_{r,m}, \quad r = 1, 2, 3, \ldots, n.$

[1] This is the name given by Mahâvîra to problems involving equations of type (III).

[2] *GSS*, vi. 253½-5½.

Then dividing the foregoing equations by b_1+1, $b_2+1, \ldots$, respectively, and adding together, we get

$$\Sigma x\left(\frac{1}{b_1+1}+\frac{1}{b_2+1}+\ldots+\frac{1}{b_n+1}\right)$$
$$+\left(\frac{k_1}{b_1+1}+\frac{k_2}{b_2+1}+\ldots+\frac{k_n}{b_n+1}\right)=(n-1)\Sigma x.$$

$$\therefore \quad \Sigma x=\left(\frac{k_1}{b_1+1}+\frac{k_2}{b2+1}+\ldots+\frac{k_n}{b_n+1}\right)$$
$$\div\left(\frac{b_1}{b_1+1}+\frac{b_2}{b_2+1}+\ldots+\frac{b_n}{b_n+1}-1\right).$$

Whence

$$x_r=\left\{\frac{k_r+b_rk_1}{b_1+1}+\frac{k_r+b_rk_2}{b_2+1}+\ldots+\frac{k_r+b_rk_{r-1}}{b_{r-1}+1}\right.$$
$$\left.+\frac{k_r+b_rk_{r+1}}{b_{r+1}+1}+\ldots+\frac{k_r+b_rk_n}{b_n+1}-(n-2)k_r\right\}$$
$$\div(b_r+1)\left(\frac{b_1}{b_1+1}+\frac{b_2}{b_2+1}+\ldots+\frac{b_n}{b_n+1}-1\right).$$

Mahâvîra describes the solution thus:

"The sum of the amounts begged by each person is multiplied by the multiple number relating to him as increased by unity. With these (products), the amounts at hand are determined according to the rule *Iṣṭaguṇa-ghna* etc.[1] They are reduced to a common denominator, and then divided by the sum diminished by unity of the multiple numbers divided by themselves as increased by unity. (The quotients) should be known to be the amounts in the hands of the persons."[2]

Problems of the above kind have been treated also by Nârâyaṇa (1357). He says:

[1] The reference is to rule vi. 241.
[2] *GSS*, vi. 251½-252½.

"Multiply the sum of the monies received by each person by his multiple number plus unity. Then proceed as in the method for "the purse of discord." Divide the multiple number related to each by the same as increased by unity. By the sum diminished by unity of these quotients, divide the results just obtained. The quotients will be the several amounts in their possession."[1]

One particular case, where $b_1 = b_2 = \ldots . b_n = 1$ and $c_1 = c_2 = \ldots = c_n = c$, was treated by Brahmagupta (628). He gave the rule:

"The total value (of the unknown quantities) plus or minus the individual values (of the unknowns) multiplied by an optional number being severally (given), the sum (of the given quantities) divided by the number of unknowns increased or decreased by the multiplier will be the total value; thence the rest (can be determined)."[2]

$$\Sigma x \pm cx_1 = a_1,\ \Sigma x \pm cx_2 = a_2, \ldots, \Sigma x \pm cx_n = a_n.$$

Therefore $$\Sigma x = \frac{a_1 + a_2 + \ldots + a_n}{n \pm c}.$$

Hence $$x_1 = \frac{1}{c}\left(\pm a_1 \mp \frac{a_1 + a_2 + \ldots + a_n}{n \pm c} \right);$$

and so on.

Brahmagupta's Rule. Brahmagupta (628) states the following rule for solving linear equations involving several unknowns:

"Removing the other unknowns from (the side of) the first unknown and dividing by the coefficient of the first unknown, the value of the first unknown (is obtained). In the case of more (values of the first unknown),

[1] *GK*, ii. 33f. [2] *BrSpSi*, xiii. 47.

two and two (of them) should be considered after reducing them to common denominators. And (so on) repeatedly. If more unknowns remain (in the final equation), the method of the pulveriser (should be employed). (Then proceeding) reversely (the values of other unknowns can be found)."[1]

Pṛthûdakasvâmî (860) has explained it thus:

"In an example in which there are two or more unknown quantities, colours such as *yâvat-tâvat*, etc., should be assumed for their values. Upon them should be performed all operations conformably to the statement of the example and thus should be carefully framed two or more sides and also equations. Equi-clearance should be made first between two and two of them and so on to the last: from one side one unknown should be cleared, other unknowns reduced to a common denominator and also the absolute numbers should be cleared from the side opposite. The residue of other unknowns being divided by the residual coefficient of the first unknown will give the value of the first unknown. If there be obtained several such values, then with two and two of them, equations should be formed after reduction to common denominators. Proceeding in this way to the end find out the value of one unknown. If that value be (in terms of) another unknown then the coefficients of those two will be reciprocally the values of the two unknowns. If, however, there be present more unknowns in that value, the method of the pulveriser should be employed. Arbitrary values may then be assumed for some of the unknowns."

It will be noted that the above rule embraces the indeterminate as well as the determinate equations. In fact, all the examples given by Brahmagupta in illustra-

[1] *BrSpSi*, xviii. 51.

tion of the rule are of indeterminate character. We shall mention some of them subsequently at their proper places. So far as the determinate simultaneous equations are concerned, Brahmagupta's method for solving them will be easily recognised to be the same as our present one.

Mahâvîra's Rules. Certain interest problems treated by Mahâvîra lead to simple simultaneous equations involving several unknowns. In these problems certain capital amounts (c_1, c_2, c_3, . . .) are stated to have been lent out at the same rate of interest (r) for different periods of time (t_1, t_2, t_3, . . .). If (i_1, i_2, i_3, . . .) be the interests accrued on the several capitals,

$$i_1 = \frac{rt_1c_1}{100}, \quad i_2 = \frac{rt_2c_2}{100}, \quad i_3 = \frac{rt_3c_3}{100}, \ldots .$$

(i) If $i_1 + i_2 + i_3 + \ldots . = I$, c_r and t_r be known (for $r = 1, 2, \ldots$), we have

$$i_1 = \frac{Ic_1t_1}{c_1t_1 + c_2t_2 + c_3t_3 + \ldots .},$$

with similar values for i_2, i_3, ...

(ii) Or, if $c_1 + c_2 + c_3 + \ldots = C$, i_r and t_r be known (for $r = 1, 2, \ldots$), we have

$$c_1 = \frac{Ci_1/t_1}{i_1/t_1 + i_2/t_2 + \ldots . .},$$

and so on.

(iii) Or, if we are given the sum of the periods $t_1 + t_2 + \ldots = T$, c_r and i_r, then

$$t_1 = \frac{Ti_1/c_1}{i_1/c_1 + i_2/c_2 + \ldots . .},$$

with similar values for t_2, t_3, ...

Mahâvîra has given separate rules for the solution of problems of each of the above three kinds.[1]

Bhâskara's Rule. Bhâskara II has given practically the same rule as that of Brahmagupta for the solution of simultaneous linear equations involving several unknowns. We take the following illustrations from his works.

Example 1. "Eight rubies, ten emeralds and a hundred pearls which are in thy ear-ring were purchased by me for thee at an equal amount; the sum of the price rates of the three sorts of gems is three less than the half of a hundred. Tell me, O dear auspicious lady, if thou be skilled in mathematics, the price of each."[2]

If x, y, z be the prices of a ruby, emerald and pearl respectively, then

$$8x = 10y = 100z,$$
$$x + y + z = 47.$$

Assuming the equal amount to be w, says Bhâskara II, we shall get

$$x = w/8,\ y = w/10,\ z = w/100.$$

Substituting in the remaining equation, we easily get $w = 200$. Therefore

$$x = 25,\ y = 20,\ z = 2.$$

Example 2. "Tell the three numbers which become equal when added with their half, one-fifth and one-ninth parts, and each of which, when diminished by those parts of the other two, leaves sixty as remainder."[3]

Here we have the equations

$$x + x/2 = y + y/5 = z + z/9, \qquad (1)$$

$$x - \frac{y}{5} - \frac{z}{9} = y - \frac{z}{9} - \frac{x}{2} = z - \frac{x}{2} - \frac{y}{5} = 60. \qquad (2)$$

[1] *GSS*, vi. 37, 39, 42. [2] *BBi*, p. 47.
[3] *BBi*, p. 52.

Bhâskara puts w for each of the equal quantities in (1), so that

$$x = \tfrac{2}{3}w,\ y = \tfrac{5}{6}w,\ z = \tfrac{9}{10}w.$$

Substituting these values in (2), any one of them will give

$$\tfrac{2}{5}w = 60;$$

whence $w = 150$. Therefore

$$x = 100,\ y = 125,\ z = 135.$$

It should be noted that the equations (2) are sufficient for the determination of the unknowns.

Example 3. Another type of problem is: Three portions (x, y, z) of a sum of money (c) were lent out at three different rates of interest (r_1, r_2, r_3 per cent per month) for three different periods (t_1, t_2, t_3 months). The interests accrued on them severally were the same. What were those portions?[1]

i.e.,
$$x + y + z = c, \tag{1}$$

$$\frac{xr_1t_1}{100} = \frac{yr_2t_2}{100} = \frac{zr_3t_3}{100} = I. \tag{2}$$

From (2)

$$x = \frac{100\,I}{r_1t_1},\quad y = \frac{100\,I}{r_2t_2},\quad z = \frac{100\,I}{r_3t_3}.$$

Substituting in (1), we get

$$100\,I\left(\frac{1}{r_1t_1} + \frac{1}{r_2t_2} + \frac{1}{r_3t_3}\right) = c.$$

Therefore
$$I = \frac{c}{100\left(\frac{1}{r_1t_1} + \frac{1}{r_2t_2} + \frac{1}{r_3t_3}\right)}.$$

[1] *L*, p. 22.

Hence $$x = \frac{\dfrac{100 \times 1}{r_1 t_1} \times c}{\dfrac{100 \times 1}{r_1 t_1} + \dfrac{100 \times 1}{r_2 t_2} + \dfrac{100 \times 1}{r_3 t_3}},$$

with similar values for y and z. Bhâskara II says:

"The arguments multiplied by their respective times are divided by the fruit taken into elapsed times. They being divided by their sum and multiplied by the total amount give the portions severally lent out."[1]

10. QUADRATIC EQUATIONS

Early Treatment. It has been shown before that the altar-construction of the Vedic Hindus involved the solution of the complete quadratic equation

$$ax^2 + bx = c, \qquad \text{(I)}$$

as well as of the pure quadratic $ax^2 = c$. The equation that had to be solved was

$$7x^2 + \tfrac{1}{2}x = 7\tfrac{1}{2} + m,$$

which gives $$x = \tfrac{1}{28}(\sqrt{841 + 112m} - 1),$$

or $$x^2 = \tfrac{1}{784}\{842 + 112m - 2\sqrt{841 + 112m}\}.$$

Simplifying and neglecting higher powers of m, we get

$$x^2 = 1 + \frac{4m}{29}, \text{ approximately.}$$

Kâtyâyana gives the value[2]

$$x^2 = 1 + \frac{m}{7}.$$

[1] *L*, pp. 21f. See also *GT*, 123. If the rate of interest be given as r per p per t months, then p is the argument, t the time and r the fruit. *Cf.* Part I, p. 204; also pp. 225-226.

[2] Datta, *Śulba*, p. 167.

The geometrical solution of the simple quadratic equation

$$4h^2 - 4dh = -c^2$$

is found in the early canonical works of the Jainas (500-300 B. C.) and also in the *Tattvârthâdhigama-sûtra* of Umâsvâti (*c.* 150 B. C.),[1] as

$$h = \tfrac{1}{2}(d - \sqrt{d^2 - c^2}).$$

Bakhshâlî Treatise. The solution of the quadratic equation was certainly known to the author of the Bakhshâlî treatise (*c.* 200). In this work there are some problems of the following type: A certain person travels s *yojana* on the first day and b *yojana* more on each successive day. Another who travels at the uniform rate of S *yojana* per day, has a start of t days. When will the first man overtake the second?

If x be the number of days after which the first overtakes the second, then we shall have

$$S(t + x) = x\left\{s + \left(\frac{x-1}{2}\right)b\right\},$$

or
$$bx^2 - \{2(S - s) + b\}x = 2tS.$$

Therefore

$$x = \frac{\sqrt{\{2(S-s)+b\}^2 + 8btS} + \{2(S-s)+b\}}{2b},$$

which agrees exactly with the solution as stated in the Bakhshâlî treatise.

"The daily travel [S] diminished by the march of the first day [s] is doubled; this is increased by the common increment [b]. That (sum) multiplied by itself is designated {as the *kṣepa* quantity}. The product of the daily

[1] Datta, "Geometry in the Jaina Cosmography," *Quellen und Studien Zur Ges. d. Math.*, Ab. B, Bd. 1 (1931), pp. 245-254.

travel and the start [t] being multiplied by eight times the common increment, the *kṣepa* quantity is added. The square-root of this {is increased by the *kṣepa* quantity; the sum divided by twice the common increment will give the required number of days}."[1]

Nearly the whole of the detailed working of the particular example in which $S = 5$, $t = 6$, $s = 3$ and $b = 4$, is preserved.[2] It is substantially as follows:

$St = 30$; $S - s = 5 - 3 = 2$; $2(S - s) + b = 8$; $\{2(S - s) + b\}^2 = 64$; $8St = 240$; $8Stb = 960$; $\{2(S - s) + b\}^2 + 8Stb = 1024$; $\sqrt{1024} = 32$; $32 + 8 = 40$; $40 \div 8 = 5 = x$.

For another problem[3] $S = 7$, $t = 5$, $s = 5$, $b = 3$;

then $$x = \frac{7 + \sqrt{889}}{6}.$$

The formula for determining the number of terms (n) of an A.P. whose first term (a), common difference (b) and sum (s) are known, is stated in the form

$$n = \frac{\sqrt{8bs + (2a - b)^2} - (2a - b)}{2b}.$$

The working of the particular example in which $s = 60$, $a = 1$, $b = 1$ is preserved substantially as follows:[4]

$8bs = 480$; $2a = 2$; $2a - b = 1$; $(2a - b)^2 = 1$;

$8bs + (2a - b)^2 = 481$; $n = \frac{1}{2}(-1 + \sqrt{481})$, etc.

Âryabhaṭa I. To find the number of terms of an A.P., Âryabhaṭa I (499) gives the following rule:

[1] *BMs*, Folio 5, recto.

[2] *BMs*, Folio 5, verso; Compare also Kaye's Introduction, pp. 37, 45.

[3] *BMs*, Folio 6, recto and verso.

[4] *BMs*, Folio 65 verso. Working of this example has been continued on folios 56, verso and recto, and 64, recto.

"The sum of the series multiplied by eight times the common difference is added by the square of the difference between twice the first term and the common difference; the square-root (of the result) is diminished by twice the first term and (then) divided by the common difference: half of this quotient plus unity is the number of terms."[1]

That is to say,

$$n = \frac{1}{2}\left\{\frac{\sqrt{8bs + (2a - b)^2} - 2a}{b} + 1\right\}.$$

The solution of a certain interest problem involves the solution of the quadratic

$$tx^2 + px - Ap = 0.$$

Āryabhaṭa gives the value of x in the form[2]

$$x = \frac{\sqrt{Apt + (p/2)^2} - p/2}{t}.$$

Though Āryabhaṭa I has nowhere indicated any method of solving the quadratic, it appears from the above two forms that he followed two different methods in order to make the unknown side of the equation $ax^2 + bx = c$, a perfect square. In one case he multiplied both the sides of the equation by $4a$ and in the other simply by a.

Brahmagupta's Rules. Brahmagupta (628) has given two specific rules for the solution of the quadratic. His first rule is as follows:

"The quadratic: the absolute quantities multiplied by four times the coefficient of the square of the unknown are increased by the square of the coefficient of the middle (*i.e.*, unknown); the square-root of the result being diminished by the coefficient of the middle

[1] *Ā*, ii. 20.

[2] *Ā*, ii. 25; vide Part I, pp. 219f.

and divided by twice the coefficient of the square of the unknown, is (the value of) the middle."[1]

$$i.e., \qquad x = \frac{\sqrt{4ac + b^2} - b}{2a}.$$

The second rule runs as:

"The absolute term multiplied by the coefficient of the square of the unknown is increased by the square of half the coefficient of the unknown; the square-root of the result diminished by half the coefficient of the unknown and divided by the coefficient of the square of the unknown is the unknown."[2]

$$i.e., \qquad x = \underline{\sqrt{ac + (b/2)^2} - (b/2)}.$$

The above two methods of Brahmagupta are identical with those employed before him by Ārya-bhaṭa I (499). The root of the quadratic equation for the number of terms of an A.P. is given by Brahma-gupta in the first form:[3]

$$n = \frac{\sqrt{8bs + (2a - b)^2} - (2a - b)}{2b}.$$

For the solution of the quadratic Brahmagupta uses also a third formula which is similar to the one now commonly used. Though it has not been expressly described in any rule, we find its application in a few

[1] *BrSpSi*, xviii. 44. It will be noted that in this rule Brahma-gupta has employed the term *madhya* (middle) to imply the simple unknown as well as its coefficient. The original of the term is doubtless connected with the mode of writing the quadratic equation in the form

$$ax^2 + bx + 0 = 0x^2 + 0x + c,$$

so that there are three terms on each side of the equation.

[2] *BrSpSi*, xviii. 45. [3] *BrSpSi*, xii. 18.

instances. One of them is an interest problem: A certain sum (p) is lent out for a period (t_1); the interest accrued (x) is lent out again at this rate of interest for another period (t_2) and the total amount is A. Find x.

The equation for determining x is

$$\frac{t_2}{pt_1}x^2 + x = A.$$

Hence, we have

$$x = \sqrt{\left(\frac{pt_1}{2t_2}\right)^2 + \frac{Apt_1}{t_2}} - \frac{pt_1}{2t_2};$$

which is exactly the form in which Brahmagupta states the result.[1]

There is a certain astronomical problem which involves the quadratic equation[2]

$$(72 + a^2)x^2 \mp 24apx = 144\left(\frac{R^2}{2} - p^2\right),$$

where a = *agrâ* (the sine of the amplitude of the sun), b = *palabhâ* (the equinoctial shadow of a gnomon 12 *aṅguli* long), R = radius, and x = *koṇaśaṅku* (the sine of the altitude of the sun when his altitude is 45°). Dividing out by $(72 + a^2)$, we have

$$x^2 \mp 2mx = n,$$

where

$$m = \frac{12ap}{72 + a^2}, \quad n = \frac{144(R^2/2 - p^2)}{72 + a^2}.$$

Therefore, we have

$$x = \sqrt{m^2 + n} \pm m,$$

as stated by Brahmagupta. This result is given also in the *Sûryasiddhânta*[3] (*c*. 300) and by Śrîpati (1039).[4]

[1] *BrSpSi*, xii. 15. Vide Part I, p. 220.
[2] *BrSpSi*, iii. 54-55.
[3] *SûSi*, iii. 30-1.
[4] *SiŚe*, iv. 74.

Śrîdhara's Rule. Śrîdhara (*c.* 750) expressly indicates his method of solving the quadratic equation. His treatise on algebra is now lost. But the relevant portion of it is preserved in quotations by Bhâskara II[1] and others.[2] Śrîdhara's method is :

"Multiply both the sides (of an equation) by a known quantity equal to four times the coefficient of the square of the unknown; add to both sides a known quantity equal to the square of the (original) coefficient of the unknown: then (extract) the root."[3]

That is, to solve $ax^2 + bx = c$,

we have $4a^2x^2 + 4abx = 4ac$,

or $(2ax + b)^2 = 4ac + b^2$.

Therefore $2ax + b = \sqrt{4ac + b^2}$.

Hence $$x = \frac{\sqrt{4ac + b^2} - b}{2a}.$$

An application of this rule is found in Śrîdhara's *Triśatikâ*, in connection with finding the number of terms of an A.P.[4]

i.e., $$n = \frac{\sqrt{8bs + (2a - b)^2} - 2a + b}{2b},$$

[1] *BBi*, p. 61.

[2] Jñânarâja (1503) in his *Bîjagaṇita* and Sûryadâsa (1541) in his commentary on Bhâskara's *Bîjagaṇita*.

[3] "Caturâhatavargasamai rûpaiḥ pakṣadvayaṁ guṇayet,
Avyaktavargarûpairyuktau pakṣau tato mûlam."

This is the reading of Śrîdhara's rule as stated by Jñânarâja and Sûryadâsa and accepted also by Sudhakara Dvivedi. But according to the reading of Kṛṣṇa (*c.* 1580) and Râmakṛṣṇa (*c.* 1648), which has been accepted by Colebrooke, the second line of the verse should be

"Pûrvâvyaktasya kṛteḥ samarûpâṇi kṣipet tayoreva"

or "add to them known quantities equal to the square of the original coefficient of the unknown."

[4] *Triś*, R. 41.

where a is the first term, b the common difference and s the sum of n terms.

Mahâvîra. The only work of Mahâvîra (850) which is available now is the *Gaṇita-sâra-saṁgraha*. As it is admittedly devoted to arithmetic we cannot expect to find in it a rule for solving the quadratic. But there are in it several problems whose solutions presuppose a knowledge of the roots of the quadratic. One problem is as follows:

"One-fourth of a herd of camels was seen in the forest; twice the square-root of the herd had gone to the mountain-slopes; three times five camels were on the bank of the river. What was the number of those camels?"[1]

If x be the number of camels in the herd, then

$$\tfrac{1}{4}x + 2\sqrt{x} + 15 = x.$$

Or, in general, the equation to be solved is

$$\frac{a}{b}x + c\sqrt{x} + d = x,$$

or
$$\left(1 - \frac{a}{b}\right)x - c\sqrt{x} = d.$$

Mahâvîra gives the following rule for the solution of this equation:

"Half the coefficient of the root (of the unknown) and the absolute term should be divided by unity minus the fraction (*i.e.*, the coefficient of the unknown). The square-root of the sum of the square of the coefficient of the root (of the unknown) and the absolute term (treated as before) is added to the coefficient of the root (of the unknown treated as before). The sum squared is the (unknown) quantity in this *mûla* type of problems."[2]

[1] *GSS*, iv. 34.

[2] *GSS*, iv. 33.

i.e., $$x = \left\{\frac{c/2}{1 - a/b} + \sqrt{\left(\frac{c/2}{1 - a/b}\right)^2 + \frac{d}{1 - a/b}}\right\}^2,$$

which shows that Mahâvîra employed the modern rule for finding the root of a quadratic. His solution for the interest problem treated by Brahmagupta is exactly the same as that of the latter.[1] We shall presently show that he knew that the quadratic has two roots.

Âryabhaṭa II. The formula for the number of terms (n) of an A.P. whose first term (a), common difference (b) and sum (s) are known is given by Âryabhaṭa II (*c.* 950) as follows :[2]

$$n = \frac{\sqrt{2bs + (a - b/2)^2} - a + b/2}{b},$$

which shows that for solving the quadratic he followed the second method of Âryabhaṭa I and Brahmagupta.

Śrîpati's Rules. Śrîpati (1039) indicates two methods of solving the quadratic. There is a lacuna in our manuscript in the rule describing the first method, but it can be easily recognised to be the same as that of Śrîdhara.

"Multiply by four times the coefficient of the square of the unknown and add the square of the coefficient of the unknown; (then extract) the square-root......... divided by twice the coefficient of the square of the unknown, is said to be (the value of) the unknown."

"Or multiplying by the coefficient of the square of the unknown and adding the square of half the coefficient of the unknown, (extract) the square-root. Then (proceeding) as before, it is diminished by half the coefficient of the unknown and divided by the coefficient

[1] *GSS*, vi. 44. [2] *MSi*, xv. 50.

of the square of the unknown. This (quotient) is said to be (the value of) the unknown."[1]

i.e., $$ax^2 + bx = c,$$

or $$a^2x^2 + abx + (b/2)^2 = ac + (b/2)^2.$$

Therefore $$ax + b/2 = \sqrt{ac + (b/2)^2}.$$

Hence $$x = \frac{\sqrt{ac + (b/2)^2} - b/2}{a}.$$

Bhâskara II's Rules. Bhâskara II (1150) says:

"When the square of the unknown, etc., remain, then, multiplying the two sides (of the equation) by some suitable quantities, other suitable quantities should be added to them so that the side containing the unknown becomes capable of yielding a root (*pada-prada*). The equation should then be formed again with the root of this side and the root of the known side. Thus the value of the unknown is obtained from that equation."[2]

This rule has been further elucidated by the author in his gloss as follows:

"When after perfect clearance of the two sides, there remain on one side the square, etc., of the unknown and on the other side the absolute term only, then, both the sides should be multiplied or divided by some suitable optional quantity; some equal quantities should further be added to or subtracted from both the sides so that the unknown side will become capable of yielding a root. The root of that side must be equal to the root of the absolute terms on the other side. For, by simultaneous equal additions, etc., to the two equal sides the equality remains. So forming an equation again with these roots the value of the unknown is found."[3]

[1] *SiŚe*, xiv. 17-8, 19.
[2] *BBi*, p. 59.
[3] *BBi*, p. 61.

It may be noted that in his treatise on arithmetic Bhâskara II has always followed the modern method of dividing by the coefficient of the square of the unknown.[1]

Jñânarâja (1503) and Gaṇeśa (1545) describe the same general methods for solving the quadratic as Bhâskara II.

Elimination of the Middle Term. The method of solving the quadratic was known amongst the Hindu algebraists by the technical designation *madhyamâharaṇa* or "The Elimination of the Middle" (from *madhyama* = middle and *âharaṇa* = removal, or destroying, that is, elimination). The origin of the name will be easily found in the principle underlying the method. By it a quadratic equation which, in its general form, contains three terms and so has a middle term, is reduced to a pure quadratic equation or a simple equation involving only two terms and so having no middle term. Thus the middle term of the original quadratic is eliminated by the method generally adopted for its solution. And hence the name. Bhâskara II has observed, "It is also specially designated by the learned teachers as the *madhyamâharaṇa*. For by it, the removal of one of the two[2] terms of the quadratic, the middle one, (takes place)."[3] The name is, however, employed also in an extended sense so as to embrace the methods for solving the cubic and the biquadratic, where also

[1] *L*, pp. 15ff.

[2] Referring to the two terms on the unknown side of the complete quadratic. Or the text *varga-râśâvekasya* may be rendered as "of one out of the unknown quantity and its square." According to the commentators Sûryadâsa (1541) and Kṛṣṇa (1580), it implies "of one between the two square terms," *viz.*, the square of the unknown and the square of the absolute number.

[3] *BBi*, p. 59.

certain terms are eliminated. It occurs as early as the works of Brahmagupta (628).[1]

Two Roots of the Quadratic. The Hindus recognised early that the quadratic has generally two roots. In this connection Bhâskara II has quoted the following rule from an ancient writer of the name of Padmanâbha whose treatise on algebra is not available now:

"If (after extracting roots) the square-root of the absolute side (of the quadratic) be less than the negative absolute term on the other side, then taking it negative as well as positive, two values (of the unknown) are found."[2]

Bhâskara points out with the help of a few specific illustrations that though these double roots of the quadratic are theoretically correct, they sometimes lead to incongruity and hence should not always be accepted. So he modifies the rule as follows:

"If the square-root of the known side (of the quadratic) be less than the negative absolute term occurring in the square-root of the unknown side, then making it negative as well as positive, two values of the unknown should be determined. This is (to be done) occasionally."[3]

Example 1. "The eighth part of a troop of monkeys, squared, was skipping inside the forest, being delightfully attached to it. Twelve were seen on the hill delighting in screaming and rescreaming. How many were they?"[4]

[1] *BrSpSi*, xviii. 2.

[2] "Vyaktapakṣasya cenmûlamanyapakṣarṇarûpataḥ
Alpaṁ dhanarṇagaṁ kṛtvâ dvividhotpadyate mitiḥ"—*BBi*, p. 67.

[3] *BBi*, p. 59; also compare the author's gloss on the same (p. 61).

[4] *BBi*, p. 65.

Solution. "Here the troop of monkeys is x. The square of the eighth part of this together with 12, is equal to the troop. So the two sides are[1]

$$\tfrac{1}{64}x^2 + 0x + 12 = 0x^2 + x + 0.$$

Reducing these to a common denominator and then deleting the denominator, and also making clearance, the two sides become

$$x^2 - 64x + 0 = 0x^2 + 0x - 768.$$

Adding the square of 32 to both sides and (extracting) square-roots, we get

$$x - 32 = \pm (0x + 16).$$

In this instance the absolute term on the known side is smaller than the negative absolute term on the side of the unknown; hence it is taken positive as well as negative; the two values of x are found to be 48, 16."

Example 2. "The fifth part of a troop of monkeys, leaving out three, squared, has entered a cave; one is seen to have climbed on the branch of a tree. Tell how many are they?"[2]

Solution. "In this the value of the troop is x; its fifth part less three is $\frac{1}{5}x - 3$; squared, $\frac{1}{25}x^2 - \frac{6}{5}x + 9$; this added with the visible (number of monkeys), $\frac{1}{25}x^2 - \frac{6}{5}x + 10$, is equal to the troop. Reducing to a common denominator, then deleting the denominator and making clearance, the two sides become

$$x^2 - 55x + 0 = 0x^2 + 0x - 250.$$

Multiplying these by 4, adding the square of 55, and

[1] We have here followed the modern practice of writing the two sides of an equation in a line with the sign of equality interposed, at the same time, preserving the other salient feature of the Hindu method of indicating the absent terms, if any, by putting zeros as their coefficients.

[2] *BBi*, pp. 65ff.

extracting roots, we get

$$2x - 55 = \pm (0x + 45).$$

In this case also, as in the previous, two values are obtained: 50, 5. But, in this case, the second (value) should not be accepted as it is inapplicable. *People have no faith in the known becoming negative.*"

The implied significance of this last observation is this: If the troop consists of only 5 monkeys, its fifth part will be 1 monkey. How can we then leave out 3 monkeys? Again, how can the remainder be said to have entered the cave? It seems to have also a wider significance.

Example 3. "The shadow of a gnomon of twelve fingers being diminished by a third part of the hypotenuse, becomes equal to fourteen fingers. O mathematician, tell it quickly."[1]

Solution. "Here the shadow is (taken to be) x. This being diminished by a third part of the hypotenuse becomes equal to fourteen fingers. Hence conversely, fourteen being subtracted from it, the remainder, a third of the hypotenuse, is $x - 14$. Thrice this, which is the hypotenuse, is $3x - 42$. The square of it, $9x^2 - 252x + 1764$, is equal to the square of the hypotenuse, $x^2 + 144$. On making equi-clearance, the two sides become

$$8x^2 - 252x + 0 = 0x^2 + 0x - 1620.$$

Multiplying both these sides by 2 and adding the square of 63, the roots are

$$4x - 63 = \pm (0x + 27).$$

On forming an equation with these sides again, and (proceeding) as before, the values of x are $45/2$, 9.

[1] *BBi*, pp. 66f.

(Thus) the value of the shadow is 45/2 or 9. The second value of the shadow is less than 14, so, on account of impracticability, it should not be accepted. Hence it has been said 'twofold values occasionally.' This will be an exception to what has been stated in the algebra of Padmanâbha, *viz*..."

Known to Mahâvîra. It has been stated before that Mahâvîra (850) knew that the quadratic has two roots. We shall now substantiate it by the following rules and illustrations from his work.

"One-sixteenth of a collection of peacocks multiplied by itself, was on the mango tree; $\frac{1}{9}$ of the remainder multiplied by itself together with 14 were on the *tamâla* tree. How many were they?"[1]

If x be the number of peacocks in the collection, the problem leads to the quadratic equation

$$\frac{x}{16} \times \frac{x}{16} + \frac{15x}{16 \times 9} \times \frac{15x}{16 \times 9} + 14 = x.$$

This is a particular case of the type of equations contemplated by the author

$$\frac{a}{b} x^2 - x + c = 0.$$

The following rule has been given for its solution.

"The quotient of its denominator divided by its numerator, less four times the remainder, is multiplied by that denominator (as divided by the numerator). The square-root of this should be *added to* and *subtracted from* that denominator (as divided by the numerator); half that is the total quantity."[2]

Thus $$x = \frac{b/a \pm \sqrt{(b/a - 4c)b/a}}{2}.$$

[1] *GSS*, iv. 59. [2] *GSS*, iv. 57.

Certain other problems[1] lead to equations of the type

$$\left(\frac{a}{b}x \mp d\right)^2 + c = x.$$

The solution is given as

"Half the denominator divided by its numerator is increased or diminished by the quantity to be subtracted or added. The square of this is diminished by the square of the quantity to be subtracted or added and by the remainder. The square-root of the result *added to* or *subtracted from* the square-root (of the square obtained before) and divided by the fractional part, will be the value (of the unknown)."[2]

$$i.e.,\ x = \left\{\left(\frac{b}{2a} \pm d\right) \pm \sqrt{\left(\frac{b}{2a} \pm d\right)^2 - d^2 - c}\right\} \div \frac{a}{b}.$$

We need not add further instances to prove that Mahâvîra recognised both the roots of a quadratic equation.[3] There are, however, a few problems in which he has taken into consideration only one of the roots.[4] For instance, take the equation on page 66,

$$\tfrac{1}{4}x + 2\sqrt{x} + 15 = x,$$

or $$\tfrac{3}{4}x - 2\sqrt{x} = 15.$$

Therefore $$\sqrt{x} = \left\{\tfrac{4}{3} \pm \sqrt{(\tfrac{4}{3})^2 + \frac{4.15}{3}}\right\},$$

or $$\sqrt{x} = (\tfrac{4}{3} \pm \tfrac{14}{3}) = 6 \text{ or } -\tfrac{10}{3}.$$

$$\therefore \quad x = 36.$$

The root can not be negative, hence the negative value of the radical is neglected in the rule.

Brahmagupta. The existence of two roots of a

[1] *GSS*, iv. 62-4. [2] *GSS*, iv. 61.

[3] More instances will be found in *GSS*, vi. 29ff.

[4] See *GSS*, iv. 33-52.

quadratic equation appears to have been known also to Brahmagupta (628). In illustration of his rules for the solution of the quadratic, he has stated two problems involving practically the same equation.

(1) "The square-root of the residue of the revolution of the sun less 2 is diminished by 1, multiplied by 10 and added by 2: when will this be equal to the residue of the revolution of the sun less 1, on Wednesday?"[1]

(2) "When will the square of one-fourth the residue of the exceeding months less three, be equal to the residue of the exceeding months?"[2]

Following Pṛthûdakasvâmî let us take in *Example* 1 the residue of the revolution of the sun to be $x^2 + 2$; then by the question

$$10(x - 1) + 2 = x^2 + 1,$$

or $$x^2 - 10x = - 9.$$

In *Example* 2, put $4x$ for the residue of the exceeding months; then

$$(x - 3)^2 = 4x,$$

or $$x^2 - 10x = - 9.$$

Now, by the second rule of Brahmagupta, retaining both the signs of the radical, we get

$$x = 5 \pm \sqrt{25 - 9} = 9 \text{ or } 1.$$

As shown by Pṛthûdakasvâmî, the first value is taken for *Example* 1 and second value for *Example* 2. Thus it appears that Brahmagupta uses sometimes the positive and at other times the negative sign with the radical. Hence it seems that Brahmagupta knew that a quadratic equation has two roots, though from considerations of utility in his problems, he retains only one of them.

[1] *BrSpSi*, xviii. 49. [2] *BrSpSi*, xviii. 50.

11. EQUATIONS OF HIGHER DEGREES

Cubic and Biquadratic. The Hindus did not achieve much in the solution of the cubic and biquadratic equations. Bhâskara II (1150) attempted the application of the method of the *madhyamâharaṇa* (elimination of the middle) to those equations also so as to reduce them by means of advantageous transformations and introduction of auxiliary quantities to simple and quadratic equations respectively. He thus anticipated one of the modern methods of solving the biquadratic. "If, however," observes Bhâskara II, "due to the presence of the cube, biquadrate, etc., the work (of reduction) cannot proceed any further, after the performance of such operations, for want of a root of the unknown side (of an equation), then the value of the unknown must be obtained by the ingenuity (of the mathematician)."[1] He has given two examples, one of the cubic and the other of the biquadratic, in which such reduction is possible.

Example 1. "What is that number, O learned man, which being multiplied by twelve and increased by the cube of the number, is equal to six times the square of the number added with thirty-five.

Solution. "Here the number is x. This multiplied by twelve and increased by the cube of the number becomes $x^3 + 12x$. It is equal to $6x^2 + 35$. On making clearance, there appears on the first side $x^3 - 6x^2 + 12x$; on the other side 35. Adding negative eight to both the sides and extracting cube-roots, we get

$$x - 2 = 0x + 3.$$

And from this equation the number is found to be 5."[2]

Example 2. "What is that number which being

[1] *BBi*, p. 61. [2] *BBi*, p. 64.

multiplied by 200 and added to the square of the number, and then multiplied by 2 and subtracted from the fourth power of the number will become one myriad less unity? Tell that number if thou be conversant with the operations of analysis.

Solution. "Here the number is x; multiplied by 200 it becomes $200x$; added to the square of the number, becomes $x^2 + 200x$; this being multiplied by two, $2x^2 + 400x$; by this being diminished the fourth power of the number, namely, this x^4, becomes $x^4 - 2x^2 - 400x$. This is equal to a myriad less unity. Equi-clearance having been made, the two sides will be

$$x^4 - 2x^2 - 400x = 0x^4 + 0x^2 + 0x + 9999.$$

Here on adding four hundred x plus unity to the first side, the root can be extracted, but on adding the same to the other side, there will be no root of it. Thus the work (of reduction) does not proceed. Hence here ingenuity (is called for). Here adding to both the sides four times the square of x, four hundred x and unity and then extracting roots, we get

$$x^2 + 0x + 1 = 0x^2 + \dot{2}x + 100.$$

Again, forming equation with these and proceeding as before, the value of x is obtained as 11. In similar instances the value of the unknown must be determined by the ingenuity of the mathematician."[1]

Higher Equations. Mahâvîra considered certain simple equations of higher degrees in connection with the treatment of the geometric series. They are of the type

$$(i) \quad ax^n = q,$$

$$(ii) \quad a\frac{x^n - 1}{x - 1} = p;$$

[1] *BBi*, pp. 64f.

where a is the first term of a G. P., q its *guṇadhana*, i.e., $(n+1)$th term, p its sum and x the unknown common ratio.

To solve equation (*i*) Mahâvîra says, "That which on multiplication by itself as many times as the number of terms becomes equal to the *guṇadhana* divided by the first term, is the common ratio."[1]

i.e., $$x = \sqrt[n]{q/p}.$$

In other words x is the nth root of q/p. But how to find such a root he does not attempt to indicate. His rule for solving an equation of the type (*ii*) is as follows :

"That by which the sum divided by the first term is divisible again and again, subtracting unity every time, is the common ratio."[2]

The method will be better understood from the solution of the following example :

"(Of a certain series in G. P.) the first term is 3, number of terms 6 and sum 4095. What is the common ratio ?"[3]

Thus $$3\frac{x^6-1}{x-1} = 4095,$$

or $$3(x^5+x^4+x^3+x^2+x+1) = 4095,$$

a quintic equation. Here dividing 4095 by 3 we get 1365. Now let us try with the divisor 4; we have $(1365-1)/4 = 341$; $(341-1)/4 = 85$; $(85-1)/4 = 21$; $(21-1)/4 = 5$; $(5-1)/4 = 1$; $(1-1)/4 = 0$. So the number 1365 is exhausted on 6 successive divisions by 4, in the way indicated in the rule. Hence $x = 4$. What suggested the method is clearly this :

$$a\frac{x^n-1}{x-1} \div a = \frac{x^n-1}{x-1}; \quad \frac{x^n-1}{x-1} - 1 = x\left(\frac{x^{n-1}-1}{x-1}\right);$$

[1] *GSS*, ii. 97. [2] *GSS*, ii. 101.
[3] *GSS*, ii. 102; compare also Rangacarya's note thereto.

which is divisible by x. However, the solution is obtained in every case by trial only.

Mahâvîra has treated some equations of the following general type:

$$a_1\sqrt{b_1x} + a_2\sqrt{b_2(x - a_1\sqrt{b_1x})}$$
$$+ a_3\sqrt{b_3\{(x - a_1\sqrt{b_1x}) - a_2\sqrt{b_2(x - a_1\sqrt{b_1x})}\}}$$
$$+ \dots + R = x;$$

$$\text{or } (x - a_1\sqrt{b_1x}) - a_2\sqrt{b_2(x - a_1\sqrt{b_1x})}$$
$$- a_3\sqrt{b_3\{(x - a_1\sqrt{b_1x}) - a_2\sqrt{b_2(x - a_1\sqrt{b_1x})}\}}$$
$$- \dots = R.$$

If there be r terms on the left hand side, then on rationalisation, we shall have an equation of 2^rth degree in x. By proper substitutions, the equation will be ultimately reduced to a quadratic equation of the form

$$X - A\sqrt{BX} = R,$$

whose solution is given by Mahâvîra as

$$X = \left\{\frac{A + \sqrt{A^2 + 4R/B}}{2}\right\}^2 \times B.$$

This result has been termed by him, the "essence" (*sâra*) of the general equation.[1] Mahâvîra gives two problems involving equations of the above type.

(1) "(Of a herd of elephants) nine times the square-root of the two-thirds plus six times the square-root of the three-fifths of the remainder (entered the deep forest); (the remaining) 24 elephants with their round temples wet with the stream of exuding ichor, were seen by me in a forest. How many were the elephants (in

[1] *GSS*, iv. 51, 52.

the herd)?"[1]

If x be the number of elephants in the herd, then by the statement of the problem

$$9\sqrt{\frac{2x}{3}} + 6\sqrt{\tfrac{3}{5}\left(x - 9\sqrt{\frac{2x}{3}}\right)} + 24 = x.$$

Put $y = x - 9\sqrt{\frac{2x}{3}}$; then the equation becomes

$$y - 6\sqrt{3y/5} = 24.$$

Therefore $\quad y = 60 \text{ or } \frac{48}{5}.$

Hence $\quad x - 9\sqrt{\frac{2x}{3}} = 60,$

whence $\quad x = 150, 24.$

Again $\quad x - 9\sqrt{\frac{2x}{3}} = \frac{48}{5},$

whence $\quad x = \frac{3}{5}(61 \pm 3\sqrt{385}).$

Of the four values of x obtained above, only the value $x = 150$ can satisfy all the conditions of the problem; others are inapplicable. That will explain why Mahâvîra has retained in his solution only the positive sign of the radical.

(2) "Four times the square-root of the half of a collection of boars went into a forest where tigers were at play; twice the square-root of the tenth part of the remainder multiplied by 4 went to a mountain; 9 times the square-root of half the remainder went to the bank of a river; boars numbering seven times eight were seen in the forest. Tell their number."[2]

[1] *GSS*, iv. 54-5. [2] *GSS*, iv. 56.

If x be the total number of boars in the collection,

$$4\sqrt{x/2} + 8\sqrt{\tfrac{1}{10}(x - 4\sqrt{x/2})}$$
$$+ 9\sqrt{\tfrac{1}{2}\{(x - 4\sqrt{x/2}) - 8\sqrt{\tfrac{1}{10}(x - 4\sqrt{x/2})}\}}$$
$$+ 56 = x.$$

Put $y = x - 4\sqrt{x/2}$; then

$$y - 8\sqrt{y/10} - 9\sqrt{(y - 8\sqrt{y/10})/2} = 56.$$

Again put $z = y - 8\sqrt{y/10}$; then

$$z - 9\sqrt{z/2} = 56.$$

Therefore $z = \left(\dfrac{9 + \sqrt{81 + 4.2.56}}{2}\right)^2 \times \tfrac{1}{2} = 128.$

Then $y - 8\sqrt{y/10} = 128;$

whence $y = \left(\dfrac{8 + \sqrt{64 + 10.4.128}}{2}\right)^2 \times \tfrac{1}{10} = 160.$

Again $x - 4\sqrt{x/2} = 160;$

hence $x = \left(\dfrac{4 + \sqrt{16 + 4.2.160}}{2}\right)^2 \times \tfrac{1}{2} = 200.$

Note that according to the problem the positive value of the radical has always to be taken.

12. SIMULTANEOUS QUADRATIC EQUATIONS

Common Forms. Various problems involving simultaneous quadratic equations of the following forms have been treated by Hindu writers :

$$\left.\begin{aligned} x - y &= d \\ xy &= b \end{aligned}\right\}\ldots(i) \qquad \left.\begin{aligned} x + y &= a \\ xy &= b \end{aligned}\right\}\ldots(ii)$$

$$\left.\begin{aligned} x^2 + y^2 &= c \\ xy &= b \end{aligned}\right\}\ldots(iii) \qquad \left.\begin{aligned} x^2 + y^2 &= c \\ x + y &= a \end{aligned}\right\}\ldots(iv)$$

For the solution of (*i*) Âryabhaṭa I (499) states the following rule:

"The square-root of four times the product (of two quantities) added with the square of their difference, being added and diminished by their difference and halved gives the two multiplicands."[1]

i.e., $x = \frac{1}{2}(\sqrt{d^2 + 4b} + d), \ y = \frac{1}{2}(\sqrt{d^2 + 4b} - d)$.

Brahmagupta (628) says:

"The square-root of the sum of the square of the difference of the residues and two squared times the product of the residues, being added and subtracted by the difference of the residues, and halved (gives) the desired residues severally."[2]

Nârâyaṇa (1357) writes:

"The square-root of the square of the difference of two quantities plus four times their product is their sum."[3]

"The square of the difference of the quantities together with twice their product is equal to the sum of their squares. The square-root of this result plus twice the product is the sum."[4]

For the solution of (*ii*) the following rule is given by Mahâvîra (850):

"Subtract four times the area (of a rectangle) from the square of the semi-perimeter; then by *saṅkramaṇa*[5] between the square-root of that (remainder) and the semi-perimeter, the base and the upright are obtained."[6]

[1] *Â*, ii. 24. [2] *BrSpSi*, xviii. 99.
[3] *GK*, i. 35. [4] *GK*, i. 36.
[5] Given *a* and *b*, the process of *saṅkramaṇa* is the finding of half their sum and difference, *i.e.*, $\frac{a+b}{2}$ and $\frac{a-b}{2}$ (see pp. 43f).
[6] *GSS*, vii. $129\frac{1}{2}$.

i.e., $x = \frac{1}{2}(a + \sqrt{a^2 - 4b}), y = \frac{1}{2}(a - \sqrt{a^2 - 4b})$.

Nârâyaṇa says :

"The square-root of the square of the sum minus four times the product is the difference."[1]

For (*iii*) Mahâvîra gives the rule :

"Add to and subtract twice the area (of a rectangle) from the square of the diagonal and extract the square-roots. By *saṅkramaṇa* between the greater and lesser of these (roots), the side and upright (are found)."[2]

$$i.e., \quad x = \tfrac{1}{2}(\sqrt{c + 2b} + \sqrt{c - 2b}),$$
$$y = \tfrac{1}{2}(\sqrt{c + 2b} - \sqrt{c - 2b}).$$

For equations (*iv*) Âryabhaṭa I writes :

"From the square of the sum (of two quantities) subtract the sum of their squares. Half of the remainder is their product."[3]

The remaining operations will be similar to those for the equations (*ii*); so that

$$x = \tfrac{1}{2}(a + \sqrt{2c - a^2}), \ y = \tfrac{1}{2}(a - \sqrt{2c - a^2}).$$

Brahmagupta says :

"Subtract the square of the sum from twice the sum of the squares ; the square-root of the remainder being added to and subtracted from the sum and halved, (gives) the desired residues."[4]

Mahâvîra,[5] Bhâskara II[6] and Nârâyaṇa[7] have also treated these equations.

Nârâyaṇa has given two other forms of simul-

[1] *GK*, i. 35.
[2] *GSS*, vii. 127½.
[3] *A*, ii. 23.
[4] *BrSpSi*, xviii. 98.
[5] *GSS*, vii. 125½.
[6] *L*, p. 39.
[7] *GK*, i. 37.

taneous quadratic equations, namely,

$$\left.\begin{aligned} x^2 + y^2 &= c \\ x - y &= d \end{aligned}\right\} \dots (v) \qquad \left.\begin{aligned} x^2 - y^2 &= m \\ xy &= b \end{aligned}\right\} \dots (vi)$$

For the solution of (v) he gives the rule :

"The square-root of twice the sum of the squares decreased by the square of the difference is equal to the sum."[1]

i.e., $$x + y = \sqrt{2c - d^2}.$$

Therefore

$$x = \tfrac{1}{2}(\sqrt{2c - d^2} + d), \quad y = \tfrac{1}{2}(\sqrt{2c - d^2} - d).$$

For (vi) Nârâyaṇa writes :

"Suppose the square of the product as the product (of two quantities) and the difference of the squares as their difference. From them by *saṅkrama* will be obtained the (square) quantities. Their square-roots severally will give the quantities (required)."[2]

We have

$$\left.\begin{aligned} x^2 - y^2 &= m \\ x^2y^2 &= b^2 \end{aligned}\right\}$$

These are of the form (i). Therefore

$$x^2 = \tfrac{1}{2}(\sqrt{m^2 + 4b^2} + m), \quad y^2 = \tfrac{1}{2}(\sqrt{m^2 + 4b^2} - m).$$

Whence we get the values of x and y.

Rule of Dissimilar Operations. The process of solving the following two particular cases of simultaneous quadratic equations was distinguished by most Hindu mathematicians by the special designation *viṣama-karma*[3] (dissimilar operation) :

[1] *GK*, i. 33. [2] *GK*, i. 34.

[3] The name *viṣama-karma* originated obviously in contradistinction to the name *saṅkramaṇa*. This is evident from the term *viṣama-saṅkramaṇa* used by Mahâvîra for *viṣama-karma*.

$$\left.\begin{aligned} x^2 - y^2 &= m \\ x - y &= n \end{aligned}\right\} \dots (i) \qquad \left.\begin{aligned} x^2 - y^2 &= m \\ x + y &= p \end{aligned}\right\} \dots (ii)$$

These equations are found to have been regarded by them as of fundamental importance. The solutions given are:

$$\text{for } (i) \qquad x = \tfrac{1}{2}\left(\frac{m}{n} + n\right), \quad y = \tfrac{1}{2}\left(\frac{m}{n} - n\right);$$

$$\text{for } (ii) \qquad x = \tfrac{1}{2}\left(p + \frac{m}{p}\right), \quad y = \tfrac{1}{2}\left(p - \frac{m}{p}\right).$$

Thus Brahmagupta says:

"The difference of the squares (of the unknowns) is divided by the difference (of the unknowns) and the quotient is increased and diminished by the difference and divided by two; (the results will be the two unknown quantities); (this is) dissimilar operation."[1]

The same rule is restated by him on a different occasion in the course of solving a problem.

"If then the difference of their squares, also the difference of them (are given): the difference of the squares is divided by the difference of them, and this (latter) is added to and subtracted from the quotient and then divided by two; (the results are) the residues; whence the number of elapsed days (can be found)."[2]

Mahâvîra states:

"The *saṅkramaṇa* of the divisor and the quotient of the two quantities is dissimilar (operation); so it is called by those who have reached the end of the ocean of mathematics."[3]

Similar rules are given also by other writers.[4]

[1] *BrSpSi*, xviii. 36. [2] *BrSpSi*, xviii. 97.
[3] *GSS*, vi. 2.
[4] *MSi*, xv. 22; *SiŚe*, xiv. 13; *L*, pp. 13, 37; *GK*, i. 32.

Mahâvîra's Rules. Mahâvîra (850) has treated certain problems involving the simultaneous quadratic equations:

$$u + x = a, \qquad urw = \alpha x,$$
$$u + y = b. \qquad usw = \alpha y.$$

Here $$\frac{r}{s} = \frac{x}{y} = \frac{a - u}{b - u}.$$

Therefore $$u = \frac{rb - sa}{r - s}.$$

Hence $$x = \left(\frac{a - b}{r - s}\right)r, \quad y = \left(\frac{a - b}{r - s}\right)s, \quad w = \left(\frac{a - b}{rb - sa}\right)\alpha.$$

In the above equations x, y are the interests accrued on the principal u in the periods r, s respectively and w is the rate of interest per α.

Mahâvîra states the result thus:

"The difference of the mixed sums [a, b] multiplied by each other's periods [r, s], being divided by the difference of the periods, the quotient is known as the principal [u]."[1]

Again, there are problems involving the equations:

$$u + x = p, \qquad uxw = \alpha m,$$
$$u + y = q. \qquad uyw = \alpha n.$$

Where x, y are the periods for which the principal u is lent out at the rate of interest w per α and m, n are the respective interests.

Here $$\frac{m}{n} = \frac{x}{y} = \frac{p - u}{q - u}.$$

Therefore $$u = \frac{mq - np}{m - n}.$$

[1] *GSS*, vi. 47.

Hence $x = \left(\frac{p-q}{m-n}\right) m, \quad y = \left(\frac{p-q}{m-n}\right) n,$

$$w = \frac{\alpha(m-n)^2}{(p-q)(mq-np)}.$$

Mahâvîra gives the rule :

"On the difference of the mixed sums multiplied by each other's interests, being divided by the difference of the interests, the quotient, the wise men say, is the principal."[1]

13. INDETERMINATE EQUATIONS OF THE FIRST DEGREE

General Survey. The earliest Hindu algebraist to give a treatment of the indeterminate equation of the first degree is Āryabhaṭa I (born 476). He gave a method for finding the general solution in positive integers of the simple indeterminate equation

$$by - ax = c$$

for integral values of a, b, c and further indicated how to extend it to get positive integral solutions of simultaneous indeterminate equations of the first degree. His disciple, Bhâskara I (522), showed that the same method might be applied to solve $by - ax = -c$ and further that the solution of this equation would follow from that of $by - ax = -1$. Brahmagupta and others simply adopted the methods of Āryabhaṭa I and Bhâskara I. About the middle of the tenth century of the Christian Era, Āryabhaṭa II improved them by pointing out how the operations can in certain cases be abridged considerably. He also noticed the cases of failure of the methods for an equation of the form

[1] *GSS*, vi. 51.

$by - ax = \pm c$. These results reappear in the works of later writers.[1]

Its Importance. It has been observed before that the subject of indeterminate analysis of the first degree was considered so important by the ancient Hindu algebraists that the whole science of algebra was once named after it. That high estimation of the subject continued undiminished amongst the later Hindu mathematicians. Āryabhaṭa II enumerates it distinctively along with the sciences of arithmetic, algebra, and astronomy.[2] So did Bhâskara II and others. As has been remarked by Gaṇeśa,[3] the separate mention of the subject of indeterminate analysis of the first degree is designed to emphasize its difficulty and importance. On account of its special importance, the treatment of this subject has been included by Bhâskara II in his treatise of arithmetic also, though it belongs particularly to algebra.[4] It is also noteworthy that there is a work exclusively devoted to the treatment of this subject. Such a special treatise is a very rare thing in the mathematical literature of the ancient Hindus. This work, entitled *Kuṭṭâkâra-śiromaṇi*,[5] is by one Devarâja, a commentator of Āryabhaṭa I.

[1] For "India's Contribution to the Theory of Indeterminate Equations of the First Degree," see the comprehensive article of Professor Sarada Kanta Ganguly in *Journ. Ind. Math. Soc.*, XIX, 1931, *Notes and Questions*, pp. 110-120, 129-142; see also XX, 1932, *Notes and Questions*. Compare also the Dissertation of D. M. Mehta on "Theory of simple continued fractions (with special reference to the history of Indian Mathematics)."

[2] *MSi*, i. 1.

[3] Vide his commentary on the *Līlâvatî* of Bhâskara II

[4] Bhâskara's treatment of the pulveriser in his *Bîjagaṇita* is repeated nearly word for word in his *Lîlâvatî*.

[5] There are four manuscript copies of this work in the Oriental Library, Mysore.

Three Varieties of Problems. Problems whose solutions led the ancient Hindus to the investigation of the simple indeterminate equation of the first degree were distinguished broadly into three varieties. The problem of one variety is to find a number (N) which being divided by two given numbers (a, b) will leave two given remainders (R_1, R_2). Thus we have

$$N = ax + R_1 = by + R_2.$$

Hence $$by - ax = R_1 - R_2.$$

Putting $$c = R_1 \sim R_2,$$

we get $$by - ax = \pm c$$

the upper or lower sign being taken according as $R_1 >$ or $< R_2$. In a problem of the second kind we are required to find a number (x) such that its product with a given number (α) being increased or decreased by another given number (γ) and then divided by a third given number (β) will leave no remainder. In other words we shall have to solve

$$\frac{\alpha x \pm \gamma}{\beta} = y$$

in positive integers. The third variety of problems similarly leads to equations of the form

$$by + ax = \pm c.$$

Terminology. The subject of indeterminate analysis of the first degree is generally called by the Hindus *kuṭṭaka*, *kuṭṭâkâra*, *kuṭṭîkâra* or simply *kuṭṭa*. The names *kuṭṭâkâra* and *kuṭṭa* occur as early as the *Mahâ-Bhâskarîya* of Bhâskara I (522).[1] In the commentary of the *Âryabhaṭîya* by this writer we find the terms *kuṭṭaka* and *kuṭṭâkâra*. Brahmagupta has used *kuṭṭaka*,[2] *kuṭṭâkâra*,[3] and *kuṭṭa*.[4] Mahâvîra, it appears, had a

[1] *MBh*, i. 41, 49.

[2] *BrSpSi*, xviii. 2, 11, etc.

[3] *BrSpSi*, xviii. 6, 15, etc.

[4] *BrSpSi*, xviii. 20, 25, etc.

preferential liking for the name *kuṭṭîkâra*.[1]

In a problem of the first variety the quantities (a, b) are called "divisors" (*bhâgahâra*, *bhâjak*, *cheda*, etc.) and (R_1, R_2) "remainders" (*agra*, *śeṣa*, etc.), while in a problem of the second variety, β is ordinarily called the "divisor" and γ the "interpolator" (*kṣepa*, *kṣepaka*, etc.); here α is called the "dividend" (*bhâjya*), the unknown quantity to be found (x) the "multiplier" (*guṇaka*, *guṇakâra*, etc.) and y the quotient (*phala*). The unknown (x) has been sometimes called by Mahâvîra as *râśi* (number) implying "an unknown number."[2]

Origin of the name. The Sanskrit words *kuṭṭa*, *kuṭṭaka*, *kuṭṭâkâra* and *kuṭṭîkâra* are all derived from the root *kuṭṭ* "to crush", "to grind," "to pulverise" and hence etymologically they mean the act or process of "breaking", "grinding", "pulverising" as well as an instrument for that, that is, "grinder", "pulveriser". Why the subject of the indeterminate analysis of the first degree came to be designated by the term *kuṭṭaka* is a question which will be naturally asked. Gaṇeṣa (1545) says: "*Kuṭṭaka* is a term for the multiplier, for multiplication is admittedly called by words importing 'injuring,' 'killing.' A certain given number being multiplied by another (unknown quantity), added or subtracted by a given interpolator and then divided by a given divisor leaves no remainder; that multiplier is the *kuṭṭaka*: so it has been said by the ancients. This is a special technical term."[3] The same explanation as to the origin of the name *kuṭṭaka* has been offered by Sûryadâsa (1538), Kṛṣṇa (*c.* 1580) and Raṅganâtha (1602).[4]

[1] *GSS*, vi. 79½, etc. [2] *GSS*, vi. 115½ff.

[3] Vide his commentary on the *Lîlâvatî* of Bhâskara II.

[4] Vide the commentaries of Sûryadâsa on *Lîlâvatî* and *Bîja-gaṇita*, of Kṛṣṇa on *Bîjagaṇita*, and of Raṅganâtha on *Siddhânta-śiromaṇi*.

But it is one-sided inasmuch as it has admittedly in view a problem of the second variety where we have indeed to find an unknown multiplier. But the rules of the earlier algebraists such as Âryabhaṭa I and Brahmagupta were formulated with a view to the solution of a problem of the first variety. So the considerations which led those early writers to adopt the name *kuṭṭaka* must have been different. Mahâvîra has once stated that, according to the learned, *kuṭṭîkâra* is another name for "the operation of *prakṣepaka*" (lit., throwing, scattering, implying division into parts).[1] In fact, his writing led his translator to interpret *kuṭṭîkâra* as "proportionate division", "a special kind of division or distribution."[2] Bhâskara I, who had in view a problem of the second variety, once remarked, "the number is obtained by the operation of pulverising (*kuṭṭana*), when it is desired to get the multiplier (*guṇakâra*). . . ."[3] It will be presently shown that the Hindu method of solving the equation $by - ax = \pm c$ is essentially based on a process of deriving from it successively other similar equations in which the values of the coefficients (a, b) become smaller and smaller.[4] Thus the process is indeed the same as that of breaking a whole thing into smaller pieces. In our opinion, it is this that led the ancient mathematicians to adopt the name *kuṭṭaka* for the operation.

Preliminary Operations. It has been remarked by most of the writers that in order that an equation

[1] "Prakṣepaka-karaṇamidaṁ. . . . .kuttîkâro budhaissamuddiṣṭaṃ"—*GSS*, vi. 79½.

[2] Vide *GSS* (English translation), pp. 117, 300.

[3] "Kṛta-kuṭṭana-labdha-râśimeṣâṃ
Guṇakâraṃ samuśanti. . . . . ."—*MBh*, i. 48.

[4] It has been expressly stated by Sûryadeva Yajvâ that the process must be continued "yâvaddharabhâjyayoralpatâ."

of the form

$$by - ax = \pm c \quad \text{or} \quad by + ax = \pm c$$

may be solvable, the two numbers a and b must not have a common divisor; for, otherwise, the equation would be absurd, unless the number c had the same common divisor. So before the rules adumbrated hereafter can be applied, the numbers a, b, c must be made prime (*dṛḍha* = firm, *niccheda* = having no divisor, *nirapavarta* = irreducible) to each other.

Thus Bhâskara I observes:

"The dividend and divisor will become prime to each other on being divided by the residue of their mutual division. The operation of the pulveriser should be considered in relation to them."[1]

Brahmagupta says:

"Divide the multiplier and the divisor mutually and find the last residue; those quantities being divided by the residue will be prime to each other."[2]

Āryabhaṭa II has made the preliminary operations in successive stages. These will be described later on.[3]

Śrîpati states:

"The dividend, divisor and interpolator should be divided by their common divisor, if any, so that it may be possible to apply the method to be described."[4]

"If the dividend and divisor have a common divisor, which is not a divisor of the interpolator then the problem would be absurd."[5]

Bhâskara II writes:

"As preparatory to the method of the pulveriser,

[1] *MBh*, i. 41.
[2] *BrSpSi*, xviii. 9.
[3] *Vide infra*, p. 104.
[4] *SiŚe*, xiv. 22.
[5] *SiŚe*, xiv. 26.

the dividend, divisor and interpolator must be divided by a common divisor, if possible. If the number by which the dividend and divisor are divisible, does not divide the interpolator then the problem is absurd. The last residue of the mutual division of two numbers is their common divisor. The dividend and divisor, being divided by their common divisor, become prime to each other."[1]

Rules similar to these have been given also by Nârâyaṇa,[2] Jñânarâja and Kamalâkara.[3] So in our subsequent treatment of the Hindu methods for the solution in positive integers of the equation $by \pm ax = \pm c$, we shall always take, unless otherwise stated, a, b prime to each other.

Solution of $by - ax = \pm c$

Âryabhaṭa I's Rule. The rule of Âryabhaṭa I (499)[4] is rather obscure inasmuch as all the operations intended to be carried out have not been described fully and clearly. So it has been misunderstood by many writers.[5] Following the interpretation of the rule by Bhâskara I (525), a direct disciple of Âryabhaṭa I, Bibhutibhusan Datta has recently given the following translation:[6]

[1] *L*, p. 76; *BBi*, pp. 24f.

[2] *NBi*, I, R. 53-4.

[3] *SiTVi*, xiii. 179ff.

[4] *Â*, ii. 32-3.

[5] L. Rodet, "Leçons de calcul d'Âryabhatta," *JA*, XIII, 1878, pp. 303ff; G. R. Kaye, "Notes on Indian Mathematics. No. 2—Âryabhaṭa," *JASB*, IV, 1908, pp. 111ff; *BCMS*, IV, p. 55; N. K. Mazumdar, "Aryyabhatta's rule in relation to Indeterminate Equations of the First Degree," *BCMS*, III, pp 11-9; P. C. Sen Gupta, "Âryabhaṭiyam," *Jour. Dept. Let. Cal. Univ.*, XVI, 1927; reprint, p. 27.; S. K. Ganguly, *BCMS*, XIX, 1928, pp. 170ff; W. E. Clark, *Âryabhaṭîya of Âryabhaṭa*, Chicago, 1930, pp. 42ff.

[6] Bibhutibhusan Datta, "Elder Âryabhaṭa's rule for the solution of indeterminate equations of the first degree," *BCMS*, XXIV, 1932, pp. 35-53.

"Divide the divisor corresponding to the greater remainder by the divisor corresponding to the smaller remainder. The residue (and the divisor corresponding to the smaller remainder) being mutually divided, the last residue should be multiplied by such an optional integer that the product being added (in case the number of quotients of the mutual division is even) or subtracted (in case the number of quotients is odd) by the difference of the remainders (will be exactly divisible by the last but one remainder. Place the quotients of the mutual division successively one below the other in a column; below them the optional multiplier and underneath it the quotient just obtained). Any number below (*i.e.*, the penultimate) is multiplied by the one just above it and then added by that just below it. Divide the last number (obtained by doing so repeatedly[1]) by the divisor corresponding to the smaller remainder; then multiply the residue by the divisor corresponding to the greater remainder and add the greater remainder. (The result will be) the number corresponding to the two divisors."

He has further shown that it can be rendered also as follows:

"Divide the divisor corresponding to the greater remainder by the divisor corresponding to the smaller remainder. The residue (and the divisor corresponding to the smaller remainder) being mutually divided (until the remainder becomes zero), the last quotient should be multiplied by an optional integer and then added (in case the number of quotients of the mutual division is even) or subtracted (in case the number of quotients is odd) by the difference of the remainders. (Place the other quotients of the mutual division succes-

[1] The process implied here is shown in detail in the working of the example on pages 113f.

sively one below the other in a column; below them the result just obtained and underneath it the optional integer). Any number below (*i.e.*, the penultimate) is multiplied by the one just above it and then added by that just below it. Divide the last number (obtained by doing so repeatedly) by the divisor corresponding to the smaller remainder; then multiply the residue by the divisor corresponding to the greater remainder and add the greater remainder. (The result will be) the number corresponding to the two divisors."

Āryabhaṭa's problem is : To find a number (N) which being divided by two given numbers (a, b) will leave two given remainders (R_1, R_2).[1] This gives:

$$N = ax + R_1 = by + R_2.$$

Denoting as before by c the difference between R_1 and R_2, we get

$$(i) \quad by = ax + c, \text{ if } R_1 > R_2,$$

or

$$(ii) \quad ax = by + c, \text{ if } R_2 > R_1$$

the equation being so written as to keep c always positive. Hence the problem now reduces to making either

$$\frac{ax + c}{b} \text{ or } \frac{by + c}{a}$$

according as $R_1 > R_2$ or $R_2 > R_1$, a positive integer. So Āryabhaṭa says: "Divide the divisor corresponding to the greater remainder etc."

[1] It has already been stated (p. 90) that in a problem of the first variety which gives an equation of the above form (and in which $R_1 > R_2$).

a = divisor corresponding to greater remainder,
b = divisor corresponding to lesser remainder,
R_1 = greater remainder,
R_2 = lesser remainder.

Suppose $R_1 > R_2$; then the equation to be solved will be

$$ax + c = by \qquad \text{(I)}$$

a, b being prime to each other.

Let

$$
\begin{array}{l}
b)\; a\; (q \\
\quad \underline{bq} \\
\quad r_1)\; b\; (q_1 \\
\qquad \underline{r_1 q_1} \\
\qquad r_2)\; r_1\; (q_2 \\
\qquad\quad \underline{r_2 q_2} \\
\qquad\quad r_3 \\
\qquad \underline{\cdots\cdots} \\
\qquad\quad r_{m-1})\; r_{m-2}\; (q_{m-1} \\
\qquad\qquad \underline{r_{m-1} q_{m-1}} \\
\qquad\qquad r_m)\; r_{m-1}\; (q_m \\
\qquad\qquad\quad \underline{r_m q_m} \\
\qquad\qquad\quad r_{m+1}
\end{array}
$$

Then, we get[1]

$$
\begin{aligned}
a &= bq + r_1, \\
b &= r_1 q_1 + r_2, \\
r_1 &= r_2 q_2 + r_3, \\
r_2 &= r_3 q_3 + r_4, \\
\cdots & \quad \cdots \\
r_{m-2} &= r_{m-1} q_{m-1} + r_m, \\
r_{m-1} &= r_m q_m + r_{m+1}.
\end{aligned}
$$

Now, substituting the value of a in the given equation (I), we get

$$by = (bq + r_1)x + c.$$

Therefore

$$y = qx + y_1,$$

[1] When $a < b$, we shall have $q = 0$, $r_1 = a$.

where $$by_1 = r_1x + c.$$

In other words, since $a = bq + r_1$, on putting

$$y = qx + y_1 \tag{1}$$

the given equation (I) reduces to

$$by_1 = r_1x + c. \tag{I. 1}$$

Again, since $\quad b = r_1q_1 + r_2$,

putting similarly $\quad x = q_1y_1 + x_1$

the equation (I. 1) can be further reduced to

$$r_1x_1 = r_2y_1 - c \tag{I. 2}$$

and so on.

Writing down the successive values and reduced equations in columns, we have

(1)	$y = qx + y_1$,	$by_1 = r_1x + c$,	(I. 1)
(2)	$x = q_1y + x_1$,	$r_1x_1 = r_2y_1 - c$,	(I. 2)
(3)	$y_1 = q_2x_1 + y_2$,	$r_2y_2 = r_3x_1 + c$,	(I. 3)
(4)	$x_1 = q_3y_2 + x_2$,	$r_3x_2 = r_4y_2 - c$,	(I. 4)
(5)	$y_2 = q_4x_2 + y_3$,	$r_4y_3 = r_5x_2 + c$,	(I. 5)
(6)	$x_2 = q_5y_3 + x_3$,	$r_5x_3 = r_6y_3 - c$,	(I. 6)
	....	....	
$(2n-1)$	$y_{n-1} = q_{2n-2}x_{n-1} + y_n$,	$r_{2n-2}y_n = r_{2n-1}x_{n-1} + c$,	(I. $2n-1$)
$(2n)$	$x_{n-1} = q_{2n-1}y_n + x_n$,	$r_{2n-1}x_n = r_{2n}y_n - c$,	(I. $2n$)
$(2n+1)$	$y_n = q_{2n}x_n + y_{n+1}$,	$r_{2n}y_{n+1} = r_{2n+1}x_n + c$,	(I. $2n+1$)

Now the mutual division can be continued either (*i*) to the finish or (*ii*) so as to get a certain number of quotients and then stopped. In either case the number of quotients found, neglecting the first one (q), as is usual with Āryabhaṭa, may be even or odd.

Case i. First suppose that the mutual division is continued until the zero remainder is obtained. Since a, b are prime to each other, the last but one remainder is unity.

Subcase (*i.* 1). Let the number of quotients be *even*. We then have

$$r_{2n} = 1, \; r_{2n+1} = 0, \; q_{2n} = r_{2n-1}.$$

The equations (I. $2n$) and (I. $\overline{2n+1}$), therefore, become

$$y_n = q_{2n}x_n + c$$

and

$$y_{n+1} = c$$

respectively. Giving an arbitrary integral value (t) to x_n, we get an integral value of y_n. From that we can find the value of x_{n-1} by ($2n$). Proceeding backwards step by step we ultimately find the values of x and y in positive integers. So that the equation (I) is solved.

Subcase (*i*. 2). If the number of quotients be *odd*, we shall have

$$r_{2n-1} = 1,\ r_{2n} = 0,\ q_{2n-1} = r_{2n-2}.$$

The equations ($2n+1$) and (I. $\overline{2n+1}$) will then be absent and the equations (I. $\overline{2n-1}$) and (I. $2n$) will be reduced respectively to

$$x_{n-1} = q_{2n-1}y_n - c$$

and

$$x_n = -c.$$

Giving an arbitrary integral value (t') to y_n we get an integral value of x_{n-1}. Then proceeding backwards as before we can calculate the values of x and y.

Case ii. Next suppose that the mutual division is stopped after having obtained an even or odd number of quotients.

Subcase (*ii*. 1). If the number of quotients obtained be *even*, the reduced form of the original equation is

$$r_{2n}y_{n+1} = r_{2n+1}x_n + c,$$

or

$$y_{n+1} = \frac{r_{2n+1}x_n + c}{r_{2n}}.$$

Giving a suitable integral value (t) to x_n as will make

$$y_{n+1} = \frac{r_{2n+1}t + c}{r_{2n}} = \text{an integral number},$$

we get an integral value for y_n by ($2n+1$). The values of x and y can then be calculated by proceeding as before.

Subcase (*ii*. 2). If the number of quotients be *odd*, the reduced form of the quotient is

$$r_{2n-1}x_n = r_{2n}y_n - c,$$

or
$$x_n = \frac{r_{2n}y_n - c}{r_{2n-1}}.$$

Putting $y_n = t'$, where t' is an integer, such that

$$x_n = \frac{r_{2n}t' - c}{r_{2n-1}} = \text{a whole number},$$

we get an integral value of x_{n-1} by ($2n$). Whence can be calculated the values of x and y in integers.

If $x = \alpha$, $y = \beta$ be the least integral solution of $ax + c = by$, we shall have

$$a\alpha + c = b\beta.$$

Therefore $$a(bm + \alpha) + c = b(am + \beta),$$

m being any integer. Therefore, in general,

$$x = bm + \alpha.$$

But we have calculated before that

$$x = q_1 y_1 + x_1;$$

$$\therefore \quad q_1 y_1 + x_1 = bm + \alpha.$$

Thus it is found that the minimum value α of x is equal to the remainder left on dividing its calculated value by b. Whence we can calculate the minimum value of $N (= a\alpha + R_1)$. This will explain the *rationale* of the operations described in the latter portion of the rule of Âryabhaṭa I.

Bhâskara I's Rules. Bhâskara I (522) writes:

"Set down the dividend above and the divisor below. Write down successively the quotients of their mutual division, one below the other, in the form of a chain. Now find by what number the last remainder should be multiplied, such that the product being subtracted by the (given) residue (of the revolution) will be exactly divisible (by the divisor corresponding to that remainder). Put down that optional number below the chain and then the (new) quotient underneath. Then multiply the optional number by that quantity which stands just above it and add to the product the (new) quotient (below). Proceed afterwards also in the same way. Divide the upper number (*i.e.*, the multiplier) obtained by this process by the divisor and the lower one by the dividend; the remainders will respectively be the desired *ahargaṇa* and the revolutions."[1]

The equation contemplated in this rule is[2]

$$\frac{ax - c}{b} = \text{a positive integer.}$$

This form of the equation seems to have been chosen by Bhâskara I deliberately so as to supplement the form of Âryabhaṭa I in which the interpolator is always made positive by necessary transposition. Further b is taken to be greater than a, as is evident from the following rule. So the first quotient of mutual division of a by b is always zero. This has not been taken into consideration. Also the number of quotients in the chain is taken to be even.

[1] *MBh*, i. 42-4.

The above rule has been formulated with a view to its application in astronomy.

[2] As already stated on p. 90, when the equation is stated in this *second* form

a = dividend,
b = divisor,
c = interpolator,
x = multiplier,
y = quotient.

He further observes:

"When the dividend is greater than the divisor, the operations should be made in the same way (*i.e.*, according to the method of the pulveriser) after deleting the greatest multiple of the divisor (from the dividend). Multiply the (new) multiplier thus obtained by that multiple and add the (new) quotient; the result will be the quotient here (required)."[1]

That is to say, if in the equation

$$ax \pm c = by,$$

$a = mb + a'$, we may neglect the portion mb of the dividend and proceed at once with the solution of

$$a'x \pm c = by.$$

Let $x = \alpha$, $y = \beta$ be a solution of this equation. Then

$$a'\alpha \pm c = b\beta;$$

$$\therefore \qquad (mb + a')\alpha \pm c = b(m\alpha + \beta),$$

or

$$a\alpha \pm c = b(m\alpha + \beta).$$

Hence $x = \alpha$, $y = m\alpha + \beta$ is a solution of the given equation.

Brahmagupta's Rules. For the solution of Āryabhaṭa's problem Brahmagupta (628) gives the following rule:

"What remains when the divisor corresponding to the greater remainder is divided by the divisor corresponding to the smaller remainder—that (and the latter divisor) are mutually divided and the quotients are severally set down one below the other. The last residue (of the reciprocal division after an even number of quotients has been obtained[2]) is multiplied by

[1] *MBh*, i. 47.

[2] Compare the next rule: "Such is the process when the quotients (of mutual division) are even etc."

such an optional integer that the product being added with the difference of the (given) remainders will be exactly divisible (by the divisor corresponding to that residue). That optional multiplier and then the (new) quotient just obtained should be set down (underneath the listed quotients). Now, proceeding from the lowermost number (in the column), the penultimate is multiplied by the number just above it and then added by the number just below it. The final value thus obtained (by repeating the above process) is divided by the divisor corresponding to the smaller remainder. The residue being multiplied by the divisor corresponding to the greater remainder and added to the greater remainder will be the number in view."[1]

He further observes:

"Such is the process when the quotients (of mutual division) are even in number. But if they be odd, what has been stated before as negative should be made positive or as positive should be made negative."[2]

Regarding the direction for dividing the divisor corresponding to the greater remainder by the divisor corresponding to the smaller remainder, Pṛthûdakasvâmî (860) observes that it is not absolute, rather optional; so that the process may be conducted in the same way by starting with the division of the divisor corresponding to the smaller remainder by the divisor corresponding to the greater remainder. But in this case of inversion of the process, he continues, the difference of the remainders must be made negative.

That is to say, the equation

$$by = ax + c$$

can be solved by transforming it first to the form

$$ax = by - c,$$

[1] *BrSpSi*, xviii. 3-5. [2] *BrSpSi*, xviii. 13.

so that we shall have to start with the division of b by a.

Mahâvîra's Rules. Mahâvîra (850) formulates his rules with a view to the solution of

$$\frac{ax \pm c}{b} = y,$$

in positive integers. He says:

"Divide the coefficient of the unknown by the given divisor (mutually); reject the first quotient and then set down the other quotients of mutual division one below the other. When the residue has become sufficiently small, multiply it by an optional number such that the product, being combined with the interpolator, which if positive must be made negative (and *vice versa*) in case (the number of quotients retained is) odd, will be exactly divisible (by the divisor corresponding to that residue). Place that optional number and the resulting quotient in order under the chain of quotients. Now add the lowermost number to the product of the next two upper numbers. The number (finally obtained by this process) being divided by the given divisor, (the remainder will be the least value of the unknown)."[1]

This method has been redescribed by Mahâvîra in a slightly modified form. Here he continues the mutual division until the remainder zero is obtained and further takes the optional multiplier to be zero.

"With the dividend, divisor and remainder reduced (by their greatest common factor the operations should be performed). Reject the first quotient and set down the other quotients of mutual division (one below the other) and underneath them the zero[2] and the given remainder

[1] *GSS*, vi. 115½ (first portion).

[2] We have emended *sâgra* of the printed text to *khâgra*.

(as reduced) in succession. The remainder, being multiplied by positive or negative as the number of quotients is even or odd, should be added to the product of the next two upper numbers. The number (finally obtained by the repeated application of this process) whether positive or negative, being divided by the divisor, the remainder will be (the least value of) the multiplier."[1]

Âryabhaṭa II. The details of the process adopted by Âryabhaṭa II (950) in finding the general solution of $(ax \pm c)/b = y$ in positive integers have been described by him thus:

"Set down the dividend, interpolator and divisor as stated (in a problem): this is the *first operation.*

"Divide them by their greatest common divisor so as to make them without a common factor: this is the *second operation.*

"Divide the dividend and interpolator by their greatest common divisor: the *third operation.*

"Divide the interpolator and divisor by their greatest common divisor: the *fourth operation.*

"Divide the dividend and interpolator, then the interpolator (thus reduced) and divisor by their respective different greatest common divisors: the *fifth operation.*

"On forming the chain from these (reduced numbers), if the remainder becomes unity, then the object (of solving the problem) will be realised; but if the remainder in it be zero, the questioner does not know the method of the pulveriser.

"Divide the (reduced) dividend and divisor reciprocally until the remainder becomes unity. (The quotients placed one below the other successively will form)

[1] *GSS*, vi. 136½ (first portion). Our interpretation differs from those of Rangacharya and Ganguly.

the (auxiliary) chain. Note down whether the number of quotients is even or odd. Multiply by the ultimate the number just above it and then add unity. The chain formed on replacing the penultimate by this result is the corrected one. Multiply by the un-destroyed (*i.e.*, corrected) penultimate the number just above it, then add the ultimate number; (now) destroy the ultimate. On proceeding thus (repeatedly) we shall finally obtain two numbers which are (technically) called *kuṭṭa*. I shall speak (later on) of those two quantities as obtained in the case of an odd number of quotients. If on dividing the dividend by the divisor once only the residue becomes unity, then the quotient is known to be the upper *kuṭṭa* and the remainder (*i.e.*, unity) the lower *kuṭṭa*.

"The upper and lower *kuṭṭa* thus obtained, being both multiplied by the interpolator of the given equation and then divided respectively by its dividend and divisor, the residues will be the quotient and multiplier respectively.

"In the case of the third operation (having been performed before) multiply the upper *kuṭṭa* by the interpolator of the question and the lower *kuṭṭa* by the interpolator as reduced by the greatest common divisor. The same should be done reversely in the case of the fourth operation. In the case of these two operations, the *kuṭṭa* after being multiplied as indicated should be divided respectively by the dividend and divisor stated by the questioner, the residues will be the quotient and multiplier respectively.

"In the fifth operation, multiply the upper *kuṭṭa* by the greatest common divisor of the dividend and the interpolator, and the lower one by the other (*i.e.*, the greatest common divisor of the given divisor and the reduced interpolator). The products are the intermediate quotient and multiplier. Multiply the divisor of the question by the intermediate quotient and also its dividend by the intermediate multiplier. Difference of these products is the required intermediate divider. The intermediate quotient and multiplier are multiplied by the interpolator of the question and then divided by the intermediate divider. The quotients thus obtained being divided respectively by the dividend and divisor of the question, the residues will be the quotient and multiplier (required).

"The quotient and multiplier are obtained correctly by the process just described in the case of a positive interpolator when the chain is even and in the case of a negative interpolator if the chain is odd. In the case of an even chain and negative interpolator, also of an odd chain and positive interpolator, the quotient and multiplier thus obtained are subtracted respectively from the dividend and divisor made prime to each other and the residues give them correctly."[1]

The *rationale* of these rules will be easily found to be as follows:

(*i*) It will be noticed that to solve

$$by = ax \pm c, \tag{1}$$

in positive integers, Āryabhaṭa II first finds the solution of

$$by = ax \pm 1.$$

If $x = \alpha$, $y = \beta$ be a solution of this equation, we get

$$b\beta = a\alpha \pm 1,$$

or

$$b(c\beta) = a(c\alpha) \pm c.$$

Therefore $x = c\alpha$, $y = c\beta$ is a solution of (1).

(*ii*) Let $a = a'g$, $c = c'g$; then (1) reduces to

$$by' = a'x \pm c',$$

[1] *MSi*, xviii. 1-14.

where $y' = y/g$.

Let $x = \alpha$, $y' = \beta$ be a solution of

$$by' = a'x \pm 1,$$

so that we have

$$b\beta = a'\alpha \pm 1.$$

Hence
$$bgc'\beta = a'gc'\alpha \pm c'g;$$

or
$$b(c\beta) = a(c'\alpha) \pm c.$$

Therefore $x = c'\alpha$, $y = c\beta$ is a solution of (1).

(*iii*) Let $b = g'b'$, $c = g'c''$; then (1) reduces to

$$b'y = ax' \pm c'',$$

where $x' = x/g'$. If $x' = \alpha$, $y = \beta$ be a solution of

$$b'y = ax' \pm 1,$$

we have

$$b'\beta = a\alpha \pm 1.$$

Therefore
$$b'g'c''\beta = ag'c''\alpha \pm g'c'',$$

or
$$b(c''\beta) = a(c\alpha) \pm c.$$

Hence $x = c\alpha$, $y = c''\beta$ is a solution of (1).

(*iv*) Let $a = a'g$, $c = c'g$, $b = b''g''$ and $c' = c''g''$.
Then the given equation $by = ax \pm c$ reduces to

$$b''y' = a'x' \pm c'',$$

where $x' = x/g''$, $y' = y/g$. Now, if $x' = \alpha$, $y' = \beta$ be a solution of

$$b''y' = a'x' \pm 1,$$

we shall have, multiplying both sides by gg'',

$$b''gg''\beta = a'gg''\alpha \pm gg'',$$

or
$$b(g\beta) = a(g''\alpha) \pm gg'',$$

or
$$b\left\{\frac{c(g\beta)}{gg''}\right\} = a\left\{\frac{c(g''\alpha)}{gg''}\right\} \pm c.$$

Since $gg'' = a(g''\alpha) \sim b(g\beta)$, we get

$$b\left\{\frac{c(g\beta)}{a(g''\alpha) \sim b(g\beta)}\right\} = a\left\{\frac{c(g''\alpha)}{a(g''\alpha) \sim b(g\beta)}\right\} \pm c.$$

Therefore

$$x = \frac{c(g''\alpha)}{a(g''\alpha) \sim b(g\beta)}, \quad y = \frac{c(g\beta)}{a(g''\alpha) \sim b(g\beta)},$$

is a solution of the given equation $by = ax \pm c$. Since $c = c''gg'' = c''\{a(g''\alpha) \sim b(g\beta)\}$, both these values are integral.

In each of the above cases the minimum values of x, y satisfying the equation $by = ax \pm c$ are given by the residues left on dividing the values of x, y as calculated above by b and a respectively, provided the two quotients are equal.

Let $x = P$, $y = Q$ be the solution as calculated above; further suppose that

$$P = mb + p, \; Q = na + q;$$

where m, n are integers such that $p < b$, $q < a$.

If $m \neq n$, the minimum solution is either

$$\left.\begin{aligned} x &= p, \\ y &= (n - m)\,a + q \end{aligned}\right\}(1) \quad \text{or} \quad \left.\begin{aligned} x &= (m - n)b + p \\ y &= q \end{aligned}\right\}(2)$$

according as $m <$ or $> n$. Now, if the interpolator c is positive, it can be shown that (2) is not a solution. For, if it were,

$$\begin{aligned} \frac{bq - c}{a} &= x, \quad \text{an integer}, \\ &= (m - n)b + p > b. \end{aligned}$$

But $q < a$, therefore,

$$\frac{bq - c}{a} < b,$$

which is absurd. Therefore, (1) must be the minimum solution in this case, not (2).

Similarly, if the interpolator c is negative, it can be shown that (2) is the minimum solution, not (1).

Hence the following rule of Āryabhaṭa II:

"If the quotients (m, n) obtained in the case of any proposed question be not equal, then the (derived) value for the multiplier should be accepted and that of the quotient rejected, if the interpolator is positive. On the other hand when the interpolator is negative, then the (derived) value for the quotient should be accepted and that for the multiplier rejected. How to obtain the quotient from the multiplier and the multiplier from the quotient correctly in all cases, I shall explain now. Multiply the (accepted) value of the multiplier by the dividend of the proposed question, add its interpolator and then divide by the divisor of the proposed question; the quotient is the corrected one. The product of the proposed divisor and the (accepted) quotient being added by the reverse of the interpolator and then divided by the dividend of the proposed question, the quotient is the (correct) multiplier."[1]

He has further indicated how to get all positive integral solutions of the equation $by = ax \pm c$ after having obtained the minimum solution.

"The (minimum) quotient and multiplier being added respectively with the dividend and divisor as stated in the question or as reduced, after multiplying both by an optional number, give various other values."[2]

That is to say, if $x = \alpha$, $y = \beta$ be the minimum solution, the general solution will be

$$x = bm + \alpha, \ y = am + \beta.$$

[1] *MSi*, xviii. 15-8. [2] *MSi*, xviii. 20.

Śrîpati's Rule. Śrîpati (1039) writes:

"Divide the dividend and divisor reciprocally until the residue is small. Set down the quotients one below the other in succession; then underneath them an optional number and below it the corresponding quotient, the optional number being determined thus: (the number) by which the last residue must be multiplied such that the product being subtracted by the interpolator and then divided by the divisor (corresponding to that residue), leaves no remainder. It is to be so when the number of quotients is even; in the case of an odd number of quotients the interpolator, if negative, must be first made positive and conversely, if positive, must be made negative; so it has been taught by the learned in this (branch of analysis). Now multiply the term above the optional number by it (the optional number) and then add the quotient below. Proceeding upwards such operation should be performed again and again until two numbers are obtained. The first one being divided by the divisor, (the residue) will give (the least value of) the multiplier; similarly the second being divided by the dividend, will give (the least value) of the quotient."[1]

Bhâskara II's Rules. Bhâskara II (1150) describes the method of the pulveriser thus:

"Divide mutually the dividend and divisor made prime to each other until unity becomes the remainder in the dividend. Set down the quotients one under the other successively; beneath them the interpolator and then cipher at the bottom. Multiply by the penultimate the number just above it and add the

[1] *SiŚe*, xiv. 22-25.
This rule is the same as that of Bhâskara I and holds under the same conditions. (See pp. 99f).

ultimate; then reject that ultimate. Do so repeatedly until only a pair of numbers is left. The upper one of these being divided by the reduced dividend, the remainder is the quotient; and the lower one being divided by the reduced divisor, the remainder is the multiplier. Such is precisely the process when the quotients (of mutual division) are even in number. But when they are odd, the quotient and multiplier so obtained must be subtracted from their respective abraders and the residues will be the true quotient and multiplier."[1]

Bhâskara II then shows how the process of solving a problem by the method of the pulveriser can sometimes be abbreviated to a great extent. He says:

"The multiplier is found by the method of the pulveriser after reducing the additive and dividend by their common divisor. Or, if the additive (previously reduced or not) and the divisor be so reduced, the multiplier found (by the method) being multiplied by their common measure will be the true one.

"Such is the process of finding the multiplier and quotient, when the interpolator is positive. On subtracting them from their respective abraders will be obtained the result for the subtractive interpolator."[2]

Kṛṣṇa (*c.* 1580) gives the following *rationale* of these rules:

We shall have to solve in positive integers

$$by = ax \pm c. \tag{1}$$

(*i*) Suppose g is the greatest common measure of a and c, so that $a = a'g$, $c = c'g$. Then

$$by = a'gx \pm c'g,$$

or

$$by' = a'x \pm c', \tag{1.1}$$

where $y' = y/g$. If $x = \alpha$, $y' = \beta$ be a solution of (1.1),

[1] *BBi*, pp. 25f; *L*, p. 77. [2] *BBi*, p. 26; *L*, pp. 78, 79.

then clearly $x = \alpha$, $y = g\beta$ is a solution of (1).

(*ii*) Let $b = g'b'$, $c = g'c''$; then equation (1) reduces to

$$b'y = ax' \pm c'', \tag{1.2}$$

where $x' = x/g'$. If $x' = \alpha'$, $y = \beta'$ be a solution of (1.2), then clearly $x = g'\alpha$, $y = \beta'$ is a solution of (1).

(*iii*) Let $a = a'g$, $c = c'g$; also $b = b''g''$, $c' = c''g''$; then equation (1) reduces to

$$b''y' = a'x' \pm c'', \tag{1.3}$$

where $x' = x/g''$, $y' = y/g$. Then if $x' = \alpha$, $y' = \beta$ be a solution of (1.3), we shall have $x = g''\alpha$, $y = g\beta$ as a solution of (1).

Now, let the minimum solution of $by = ax + c$ be $x = \alpha$, $y = \beta$. Then

$$b\beta = a\alpha + c.$$

Hence $$b(a - \beta) = a(b - \alpha) - c.$$

Therefore, $x = b - \alpha$, $y = a - \beta$ is a solution of $by = ax - c$. Since $\alpha < b$, $\beta < a$, provided $c < a, b$, this solution is positive. Thus we find that the minimum solution of the equation $by = ax - c$ can be derived from that of the equation $by = ax + c$, as has been stated by Bhâskara II.

Bhâskara II further observes :[1]

"In abrading the (calculated values of) the multiplier and the quotient (by the divisor and the dividend respectively) the intelligent should take out the same multiple (of them).

"The multiplier and quotient may be found as before after abrading the interpolator by the divisor; the quotient (obtained), however, must be increased by the abrading quotient in case the interpolator is positive, but, if it is negative, the abrading quotient

[1] *BBi*, p. 26; *L*, pp. 79, 81.

must be subtracted.

"Or the multiplier may be found as before after abrading both the dividend and the interpolator by the divisor; from (this multiplier) the quotient may be found by multiplying (it) by the dividend, adding (the interpolator) and then dividing (the sum by the divisor).[1]

"Those (minimum values of) the multiplier and the quotient being added by any (optionally chosen) multiple of their respective abraders become manifold."

We take the following illustrative example with the different methods of its solution from Bhâskara II:

To solve, in positive integers,

$$\frac{100x + 90}{63} = y.$$

First Method. Statement :

$$\begin{array}{rl} \text{Dividend} = 100 & \\ & \text{Additive} = 90 \\ \text{Divisor} = 63 & \end{array}$$

Dividing mutually 100 by 63, we have

$$\begin{array}{l}
63)\ 100\ (1 \\
\quad\ \underline{63} \\
\quad\ 37)\ 63\ (1 \\
\qquad\quad \underline{37} \\
\qquad\quad 26)\ 37\ (1 \\
\qquad\qquad\ \ \underline{26} \\
\qquad\qquad\ \ 11)\ 26\ (2 \\
\qquad\qquad\qquad\ \ \underline{22} \\
\qquad\qquad\qquad\ \ 4)\ 11\ (2 \\
\qquad\qquad\qquad\qquad\ \ \underline{8} \\
\qquad\qquad\qquad\qquad\ \ 3)\ 4\ (1 \\
\qquad\qquad\qquad\qquad\qquad \underline{3} \\
\qquad\qquad\qquad\qquad\qquad 1
\end{array}$$

[1] *i.e.*, by substituting the value of the multiplier in the original equation.

Then, forming the chain as directed in the rule, we get

$$\begin{array}{c} 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 1 \\ 90 \\ 0 \end{array}$$

By the rule, "Multiply by the penultimate the number just above it etc.," the two numbers obtained finally are 2430 and 1530.[1] Dividing these by 100 and 63 respectively, the remainders are 30 and 18. Hence $x = 18$, $y = 30$.

Second Method. Reducing the dividend and the additive by their greatest common divisor (10), we have the statement:

$$\begin{array}{r} \text{Dividend} = 10 \\ \text{Divisor} = 63 \end{array} \quad \text{Additive} = 9$$

Since

$$\begin{array}{l} 63)\ 10\ (0 \\ \quad\ \ \underline{0} \\ \quad 10)\ 63\ (6 \\ \qquad\quad \underline{60} \\ \qquad\quad 3)\ 10\ (3 \\ \qquad\qquad\ \ \underline{9} \\ \qquad\qquad\ \ 1 \end{array}$$

[1] Successive operations in the application of the rule are :

1		1		1		1		1		$\cancel{1}$	2430
1		1		1		1		$\cancel{1}$	1530	$\cancel{1}$	1530
1		1		1		$\cancel{1}$	900	$\cancel{1}$	900	$\cancel{1}$	$9\cancel{0}\cancel{0}$
2		2		$\cancel{2}$	630	$\cancel{2}$	630	$\cancel{2}$	$\cancel{6}\cancel{3}\cancel{0}$	$\cancel{2}$	$\cancel{6}\cancel{3}\cancel{0}$
2		$\cancel{2}$	270	$\cancel{2}$	270	$\cancel{2}$	$\cancel{2}\cancel{7}\cancel{0}$	$\cancel{2}$	$\cancel{2}\cancel{7}\cancel{0}$	$\cancel{2}$	$\cancel{2}\cancel{7}\cancel{0}$
$\cancel{1}$	90	$\cancel{1}$	90	$\cancel{1}$	$9\cancel{0}$	$\cancel{1}$	$9\cancel{0}$	$\cancel{1}$	$9\cancel{0}$	$\cancel{1}$	$9\cancel{0}$
90		$9\cancel{0}$		$9\cancel{0}$		$9\cancel{0}$		$9\cancel{0}$		$9\cancel{0}$	
$\cancel{0}$		$\cancel{0}$		$\cancel{0}$		$\cancel{0}$		$\cancel{0}$		$\cancel{0}$	

we get the chain

$$\begin{matrix} 0 \\ 6 \\ 3 \\ 9 \\ 0 \end{matrix}$$

By the rule, "Multiply by the penultimate etc.," we obtain finally the numbers 27 and 171. Dividing them respectively by 10 and 63, we get the residues 7 and 45. Since the number of quotients of the mutual division is odd, subtracting 7 and 45 from the corresponding abraders 10 and 63, we get 3 and 18. In this case we neglect 3. So $x = 18$; whence by the given equation $y = 30$. Or, multiplying the quotient 3 as obtained above by the greatest common divisor 10, we get the same result $y = 30$.

Third Method. Reducing the divisor and the additive by their greatest common divisor (9), the statement is :

$$\begin{matrix} \text{Dividend} = 100 \\ \text{Divisor} = 7 \end{matrix} \qquad \text{Additive} = 10$$

Since

$$\begin{array}{l} 7)\ 100\ (14 \\ \quad\ \ \underline{98} \\ \qquad 2)\ 7\ (3 \\ \qquad\quad\ \ \underline{6} \\ \qquad\quad\ \ 1 \end{array}$$

we get the chain

$$\begin{matrix} 14 \\ 3 \\ 10 \\ 0 \end{matrix}$$

By the rule, "Multiply by the penultimate etc.," we obtain the two numbers 430 and 30. Dividing them by 100 and 7 respectively, the residues are 30 and 2.

Multiplying the latter by the greatest common divisor 9, we get $x = 18$ and $y = 30$.

Fourth Method. Dividing the divisor and the additive by their common measure (9) and again the dividend and the reduced additive by their common measure (10), we have

$$\begin{array}{rl} \text{Dividend} = 10 & \\ & \text{Additive} = 1 \\ \text{Divisor} = 7 & \end{array}$$

Since

$$\begin{array}{l} 7)\ 10\ (1 \\ \quad\ \ \underline{7} \\ \quad\ \ 3)\ 7\ (2 \\ \qquad\ \ \underline{6} \\ \qquad\ \ 1 \end{array}$$

we get the chain

$$\begin{array}{c} 1 \\ 2 \\ 1 \\ 0 \end{array}$$

By the rule, "Multiply by the penultimate etc.," we have finally the numbers 3 and 2. Dividing them by 10 and 7 respectively, the residues are the same. Multiplying them respectively by the common measure 10 of the dividend and reduced additive, and 9 of the divisor and additive, we get as before $x = 18$ and $y = 30$.

Adding to these minimum values (18, 30) of (x, y) optional multiples of the corresponding abraders (63, 100), we get the general solution of $100x + 90 = 63y$ in positive integers as $x = 63m + 18$, $y = 100m + 30$, where m is any integer.

Rules similar to those of Bhâskara II have been given by Nârâyaṇa,[1] Jñânarâja and Kamalâkara.[2]

[1] *NBi*, I, R. 55-60. [2] *SiTVi*, xiii. 183-190.

Solution of $by = ax \pm 1$

Constant Pulveriser. Though the simple indeterminate equation $by = ax \pm 1$ is solved exactly in the same way as the equation $by = ax \pm c$ and is indeed a particular case of the latter, yet on account of its special use in astronomical calculations[1] it has received separate consideration at the hands of most of the Hindu algebraists. It may, however, be noted that the separate treatment was somewhat necessitated by the physical conditions of the problems involving the two types. In the case of $by = ax \pm c$ the conditions are such that the value of either y or x, more particularly of the latter, has to be found and the rules for solution are formulated with that object. But in the case of the other ($by = ax \pm 1$) the physical conditions require the values of both y and x.

The equation $by = ax \pm 1$ is generally called by the name of *sthira-kuṭṭaka* or the "constant pulveriser" (from *sthira*, meaning constant, steady). Pṛthûdakasvâmî (860) sometimes designates it also as *dṛḍha-kuṭṭaka* (from *dṛḍha* = firm). But that name disappeared from later Hindu algebras because the word *dṛḍha* was employed by later writers[2] as equivalent to *niccheda* (having no divisor) or *nirapavarta* (irreducible). The origin of the name "constant pulveriser" has been explained by Pṛthûdakasvâmî as being due to the fact that the interpolator (± 1) is here invariable. Gaṇeśa[3] (1545) explains it in detail thus: In astronomical problems involving

[1] Thus Bhâskara II observes, "This method of calculation is of great use in mathematical astronomy." (*BBi*, p. 31). He then points out how the solutions of various astronomical problems can be derived from the solution of the same indeterminate equation. (*BBi*, p. 32; *L*, p. 81).

[2] This special technical use of the word *dṛḍha* occurs before Brahmagupta (628) in the works of Bhâskara I (522).

[3] *Vide* his commentary on the *Lîlâvatî* of Bhâskara II.

equations of the type $by - ax = \pm c$, the physical conditions are such that the dividend (a) and the divisor (b) are constant but the interpolator (c) always varies; so for their solution different sets of operations will have to be performed if we start directly to solve them all. But starting with the equation $by - ax = \pm 1$, we can derive the necessary solutions of all our equations from a constant set of operations. Hence the name is very significant. A similar explanation has been given by Kṛṣṇa (*c.* 1580).

Bhâskara I's Rule. Bhâskara I (522) writes:

"The method of the pulveriser is applied also after subtracting unity. The multiplier and quotient are respectively the numbers above and underneath. Multiplying those quantities by the desired number, divide by the reduced divisor and dividend; the residues are in this case known to be the (elapsed) days and (residues of) revolutions respectively."[1]

In other words, it has been stated that the solution of the equation

$$\frac{ax - c}{b} = y,$$

can be obtained by multiplying the solution of

$$\frac{ax - 1}{b} = y,$$

by c and then abrading as before. In general, the solution of the equation $by = ax \pm c$ in positive integers can be easily derived from that of $by = ax \pm 1$. If $x = \alpha$, $y = \beta$ be a solution of the latter equation, we shall have

$$b\beta = a\alpha \pm 1.$$

Then $$b(c\beta) = a(c\alpha) \pm c.$$

[1] *MBh*, i. 45.

Hence $x = c\alpha$, $y = c\beta$ is a solution of the former. The minimum solution will be obtained by abrading the values of x and y thus computed by b and a respectively, as indicated before.

Brahmagupta's Rule. To solve the equation $by = ax - 1$, Brahmagupta gives the following rule :

"Divide them (*i.e.*, the abraded coefficient of the multiplier and the divisor) mutually and set down the quotients one below the other. The last residue (of the reciprocal division after an even[1] number of quotients has been obtained) is multiplied by an optional integer such that the product being diminished by unity will be exactly divisible (by the divisor corresponding to that residue). The (optional) multiplier and then this quotient should be set down (underneath the listed quotients). Now proceeding from the lowermost term to the uppermost, by the penultimate multiply the term just above it and then add the lowermost number. (The uppermost number thus calculated) being divided by the reduced divisor, the residue (is the quantity required). This is the method of the constant pulveriser."[2]

Bhâskara II's Rule. Bhâskara II (1150) writes :

"The multiplier and quotient determined by supposing the additive or subtractive to be unity, multiplied severally by the desired additive or subtractive and then divided by their respective abraders, (the residues) will be those quantities corresponding to them (*i.e.*, desired interpolators)."[3]

This rule has been reproduced by Nârâyaṇa.[4] We take the following illustrative example with its solution

[1] In view of the rule in *BrSpSi*, xviii. 13.

[2] *BrSpSi*, xviii. 9-11.

[3] *BBi*, p. 31; *L*, p. 81.

[4] *NBi*, I, R. 65.

from Bhâskara II :[1]

$$\frac{221x + 65}{195} = y.$$

On dividing by the greatest common divisor 13, we get

$$\frac{17x + 5}{15} = y.$$

Now, by the method of the pulveriser the solution of the equation

$$\frac{17x + 1}{15} = y$$

is found to be $x=7, y=8$. Multiplying these values by 5 and then abrading by 15 and 17 respectively, we get the required minimum solution $x=5, y=6$.

Again a solution of

$$\frac{17x - 1}{15} = y$$

will be found to be $x=8$, $y=9$. Multiplying these quantities by 5 and abrading by 15 and 17, we get the solution of

$$\frac{17x - 5}{15} = y$$

to be $x=10$, $y=11$.

Solution of $by + ax = \pm c$

An equation of the form $by + ax = \pm c$ was generally transformed by Hindu algebraists into the form $by = - ax \pm c$ so that it appeared as a particular case of $by = ax \pm c$ in which a was negative.

Brahmagupta's Rule. Such an equation seems to

[1] *BBi*, pp. 28, 31; *L*, pp. 77, 81.

have been solved first by Brahmagupta (628). But his rule is rather obscure: "The reversal of the negative and positive should be made of the multiplier and interpolator."[1] Pṛthûdakasvâmî's explanation does not throw much light on it. He says, "If the multiplier be negative, it must be made positive; and the additive must be made negative: and then the method of the pulveriser should be employed." But he does not indicate how to derive the solution of the equation

$$by = -ax + c \qquad (1)$$

from that of the equation

$$by = ax - c \qquad (2)$$

The method, however, seems to have been this:

Let $x = \alpha$, $y = \beta$ be the minimum solution of (2). Then we get

$$b\beta = a\alpha - c$$

or

$$b(a - \beta) = -a(\alpha - b) + c.$$

Hence $x = \alpha - b$, $y = a - \beta$ is the minimum solution of (1). This has been expressly stated by Bhâskara II and others.

Bhâskara II's Rule. Bhâskara II says:

"Those (the multiplier and quotient) obtained for a positive dividend being treated in the same manner give the results corresponding to a negative dividend."[2]

The treatment alluded to in this rule is that of subtraction from the respective abraders. He has further elaborated it thus:

"The multiplier and quotient should be determined by taking the dividend, divisor and interpolator as positive. They will be the quantities for the additive interpolator. Subtracting them from their

[1] *BrSpSi*, xviii. 13. [2] *BBi*, p. 26.

respective abraders, the quantities for a negative interpolator are found. If the dividend or divisor be negative, the quotient should be stated as negative."[1]

Nârâyaṇa. Nârâyaṇa (1350) says:

"In the case of a negative dividend find the multiplier and quotient as in the case of its being positive and then subtract them from their respective abraders. One of these results, either the smaller one or the greater one, should be made negative and the other positive."[2]

Illustrative Examples. Examples with solutions from Bhâskara II:[3]

Example 1. $13y = -60x \pm 3$.

By the method described before we find that the minimum solution of

$$13y = 60x + 3$$

is $x = 11$, $y = 51$. Subtracting these values from their respective abraders, namely 13 and 60, we get 2 and 9. Then by the maxim. "In the case of the dividend and divisor being of different signs, the results from the operation of division should be known to be so," making the quotient negative we get the solution of

$$13y = -60x + 3$$

as $x = 2$, $y = -9$. Subtracting these values again from their respective abraders (13, 60), we get the solution of

$$13y = -60x - 3$$

as $x = 11$, $y = -51$.

Example 2. $-11y = 18x \pm 10$.

Proceeding as before we find the minimum solution of

$$11y = 18x + 10$$

[1] *BBi*, p. 29. [2] *NBi*, I, R. 63.

[3] *BBi*, pp. 29, 30.

to be $x = 8$, $y = 14$. These will also be the values of x and y in the case of the negative divisor but the quotient for the reasons stated before should be made negative. So the solution of

$$-11y = 18x + 10$$

is $x = 8$, $y = -14$. Subtracting these (*i.e.*, their numerical values) from their respective abraders, we get the solution of

$$-11y = 18x - 10$$

as $x = 3$, $y = -4$.

"Whether the divisor is positive or negative, the numercial values of the quotient and multiplier remain the same: when either the divisor or the dividend is negative, the quotient must always be known to be negative."

The following example with its solution is from the algebra of Nârâyaṇa :[1]

$$7y = -30x \pm 3.$$

The solution of

$$7y = 30x + 3$$

is $x = 2$, $y = 9$. Subtracting these values from the respective abraders, namely 7 and 30, and making one of the remainders negative, we get $x = 5$, $y = -21$ and $x = -5$, $y = 21$ respectively as solutions of

$$7y = -30x \pm 3.$$

Particular Cases. The Hindus also tound special types of general solutions of certain particular cases of the equation $by + ax = c$. For instance, we find in the *Gaṇita-sâra-saṅgraha* of Mahâvîra (850) problems of the following type :

"The *varṇa* (or colours) of two pieces of gold weighing 16 and 10 are unknown, but the mixture of

[1] *NBi*, I, Ex. 29.

them has the *varṇa* 4; what is the *varṇa* of each piece of gold?"[1]

If x, y denote the required *varṇa*, then we shall have

$$16x + 10y = 4 \times 26;$$

or in general

$$ax + by = c(a + b).$$

Therefore $$a(x - c) = b(c - y);$$

whence $$x = c \pm m/a, \quad y = c \mp m/b,$$

where m is an arbitrary integer.

Hence the following rule of Mahâvîra:

"Divide unity (severally) by the weights of the two ingots of gold. The resulting *varṇa* being set down at two places, increase or decrease it at one place and do reversely at the other place, by the unity divided by its own quantity of gold (the results will be the corresponding *varṇa*)."[2]

He has also remarked that "assuming an arbitrary value for one of the *varṇa*, the other can be found as before."[3]

A variation of the above problem is found in the *Lîlâvatî* of Bhâskara II:

"On mixing up two ingots of gold of *varṇa* 16 and 10 is produced gold of *varṇa* 12; tell me, O friend, the weights of the original ingots."[4]

That is to say, we shall have to solve the equation

$$16x + 10y = 12(x + y);$$

or in general

$$ax + by = c(x + y).$$

Hence $$x = m(c - b), \quad y = m(a - c),$$

where m is an arbitrary integer.

[1] *GSS*, vi. 188. [2] *GSS*, vi. 187.
[3] *GSS*, vi. 189. [4] *L.*, p. 26.

Hence the rule of Bhâskara II:

"Subtract the resulting *varṇa* from the higher *varṇa* and diminish it by the lower *varṇa*; the remainders multiplied by an optional number will be the weights of gold of the lower and higher *varṇa* respectively."[1]

In the above example $c - b = 2$, $a - c = 4$. So that, taking $m = 1$, 2, or $1/2$, Bhâskara II obtains the values of (x, y) as $(2, 4)$, $(4, 8)$ or $(1, 2)$. He then observes that in the same way numerous other sets of values can be obtained.

14. ONE LINEAR EQUATION IN MORE THAN TWO UNKNOWNS

To solve a linear equation involving more than two unknowns the usual Hindu method is to assume arbitrary values for all the unknowns except two and then to apply the method of the pulveriser. Thus Brahmagupta remarks, "The method of the pulveriser (should be employed), if there be present many unknowns (in an equation)."[2] Similar directions have been given by Bhâskara II and others.[3]

One of the astronomical problems proposed by Brahmagupta[4] leads to the equation:

$$197x - 1644y - z = 6302.$$

Hence
$$x = \frac{1644y + z + 6302}{197}.$$

The commentator assumes $z = 131$. Then

$$x = \frac{1644y + 6433}{197};$$

[1] *L*, p. 25.
[2] *BrSpSi*, xviii. 51.
[3] *BBi*, p. 76.
[4] *BrSpSi*, xviii. 55.

hence by the method of the pulveriser

$$x = 41, \quad y = 1.$$

The following example with its solution is from the algebra of Bhâskara II:

"The numbers of flawless rubies, sapphires, and pearls with one person are respectively 5, 8 and 7; and O friend, another has 7, 9 and 6 respectively of the same gems. In addition they have coins to the extent of 90 and 62. They are thus equally rich. Tell quickly, O intelligent algebraist, the price of each gem."[1]

If x, y, z represent the prices of a ruby, sapphire and pearl respectively, then by the question

$$5x + 8y + 7z + 90 = 7x + 9y + 6z + 62.$$

Therefore $$x = \frac{-y + z + 28}{2}.$$

Assume $z = 1$; then

$$x = \frac{-y + 29}{2};$$

whence by the method of the pulveriser, we get

$$x = 14 - m, \quad y = 2m + 1,$$

where m is an arbitrary integer. Putting $m = 0, 1, 2, 3, \ldots$ we get the values of (x, y, z) as (14, 1, 1), (13, 3, 1), (12, 5, 1), (11, 7, 1), etc. Bhâskara II then observes, "By virtue of a variety of assumptions multiplicity of values may thus be obtained."

Sometimes the values of most of the unknowns present in an equation are assumed arbitrarily or in terms of any one of them, so as to reduce the equation to a simple determinate one. Thus Bhâskara II says:

"In case of two or more unknowns, x multiplied by 2 etc. (*i.e.*, by arbitrary known numbers), or divided,

[1] *BBi*, p. 77.

increased or decreased by them, or in some cases (simply) any known values may be assumed for the other unknowns according to one's own sagacity. Knowing these (the rest is an equation in one unknown)."[1]

The above example has been solved again by Bhâskara II in accordance with this rule thus :[2]

(1) Assume $x = 3z$, $y = 2z$. Then the equation reduces to

$$38z + 90 = 45z + 62.$$

Therefore $z = 4$. Hence $x = 12$, $y = 8$.

(2) Or assume $y = 5$, $z = 3$. Then the equation becomes

$$5x + 151 = 7x + 125.$$

Whence $x = 13$.

15. SIMULTANEOUS INDETERMINATE EQUATIONS OF THE FIRST DEGREE

Śrîpati's Rule. We have described before the rule of Brahmagupta for the solution of simultaneous equations of the first degree.[3] In the latter portion of that rule there are hints for the solution of simultaneous indeterminate equations by the application of the method of the pulveriser. Similar rules have been given by later Hindu algebraists. Thus Śrîpati (1039) says :

"Remove the first unknown from any one side of an equation leaving the rest, and remove the rest from the other side. Then find the value of the first by dividing the other side by its coefficient. If there be found thus several values (of the first unknown), the same (opera-

[1] *BBi*, p. 44.
[2] *BBi*, p. 46.
[3] See pp. 54f.

tions) should be made again (by equating two and two of those values) after reducing them to a common denominator. (Proceed thus repeatedly) until there results a single value for an unknown. Now apply the method of the pulveriser; and from the values (determined in this way) the other unknowns will be found by proceeding backwards. In the pulveriser the multiplier will be the value of the unknown associated with the dividend and the quotient, of that with the divisor."[1]

Bhâskara II's Rule. Bhâskara II (1150) writes:

" Remove the first unknown from the second side of an equation and the others as well as the absolute number from the first side. Then on dividing the second side by the coefficient of the first unknown, its value will be obtained. If there be found in this way several values of the same unknown, from them, after reduction to a common denominator and then dropping it, values of another unknown should be determined. In the final stage of this process, the multiplier and quotient obtained by the method of the pulveriser will be the values of the unknowns associated with the dividend and the divisor (respectively). If there be several unknowns in the dividend, their values should be determined after assuming values of all but one arbitrarily. Substituting these values and proceeding reversely, the values of the other unknowns can be obtained. If on so doing there results a fractional value (at any stage), the method of the pulveriser should be employed again. Then determining the (integral) values of the latter unknowns accordingly and substituting them, the values of the former unknowns should be found proceeding reversely again."[2]

A similar rule has been given by Jñânarâja.

[1] *SiŚe*, xiv. 15-6. [2] *BBi*, p. 76.

Example from Bhâskara II:

"(Four merchants), who have horses 5, 3, 6 and 8 respectively; camels 2, 7, 4 and 1; whose mules are 8, 2, 1 and 3; and oxen 7, 1, 2 and 1 in number; are all owners of equal wealth. Tell me instantly the price of a horse, etc."[1]

If x, y, z, w denote respectively the prices of a horse, a camel, a mule and an ox, and W be the total wealth of each merchant, we have

$$5x + 2y + 8z + 7w = W \qquad (1)$$
$$3x + 7y + 2z + w = W \qquad (2)$$
$$6x + 4y + z + 2w = W \qquad (3)$$
$$8x + y + 3z + w = W \qquad (4)$$

Then $x = \frac{1}{2}(5y - 6z - 6w)$, from (1) and (2)

$= \frac{1}{3}(3y + z - w)$, from (2) and (3)

$= \frac{1}{2}(3y - 2z + w)$, from (3) and (4)

From the first and second values of x, we get

$$y = \tfrac{1}{9}(20z + 16w);$$

and from the second and third values, we have

$$y = \tfrac{1}{3}(8z - 5w).$$

Equating these two values of y and simplifying,

$$20z + 16w = 24z - 15w.$$

Therefore $z = \dfrac{31w}{4}.$

Take $w = 4t$; then

$$z = 31t, \quad y = 76t, \quad x = 85t.$$

Special Rules. Bhâskara II observes that the physical conditions of problems may sometimes be such that the ordinary method of solving simultaneous in-

[1] *BBi*, p. 79.

determinate equations of the first degree, which has been just explained, will fail to give the desired result. One such problem has been described by him as follows:

"Tell quickly, O algebraist, what number is that which multiplied by 23 and severally divided by 60 and 80 leaves remainders whose sum is 100."[1]

Let the number be denoted by x; the quotients by u, v; and the remainders by s, t. Then we have

$$\frac{23x - s}{60} = u, \quad \frac{23x - t}{80} = v;$$

also $$s + t = 100.$$

Therefore $$x = \frac{60u + s}{23} = \frac{80v + t}{23}.$$

Hence $$x = \frac{60u + 80v + s + t}{46},$$

or $$x = \frac{30u + 40v + 50}{23}.$$

For the solution of the above he observes:

"Here, (although) there is more than one quotient (u, v) in the dividend, the value of any should not be arbitrarily assumed; for on so doing the process will fail."[2] "In a case like this," continues he, "the (given) sum of the remainders should be so broken up that each remainder will be less than the divisor corresponding to it and further that impossibility will not arise; then must be applied the usual method."

In the present example we thus suppose $s = 40$, $t = 60$. Hence we have

$$60u + 40 = 80v + 60$$

[1] *BBi*, p. 91. [2] *BBi*, p. 91f.

or $$u = \frac{80v + 20}{60} = \frac{4v + 1}{3};$$

whence by the method of the pulveriser, we get

$$v = 3w + 2, \ u = 4w + 3.$$

Therefore $$x = \frac{240w + 220}{23}.$$

Again, applying the method of the pulveriser in order to obtain an integral value of x, we have

$$w = 23m + 1, \quad x = 240m + 20.$$

If we take $s = 30$, $t = 70$, we shall find, by proceeding in the same way, another value of x as $240m + 90$.

General Problem of Remainders. One type of simultaneous indeterminate equations of the first degree is furnished by the general problem of remainders, *viz.*, to find a number N which being severally divided by $a_1, a_2, a_3, \ldots, a_n$ leaves as remainders $r_1, r_2, r_3, \ldots, r_n$ respectively.

In this case, we have the equations

$$N = a_1x_1 + r_1 = a_2x_2 + r_2 = a_3x_3 + r_3 = \ldots$$
$$= a_nx_n + r_n.$$

The method of solution of these equations was known to Āryabhaṭa I (499). For this purpose the term *dvicchedâgraṃ* occurring in his rule for the pulveriser must be explained in a different way so that the last line of the translations given before (pp. 94-5) will have to be replaced by the following: "(The result will be) the remainder corresponding to the product of the two divisors."[1] This explanation is, in fact, given by Bhâskara I, the direct disciple and earliest commentator of Āryabhaṭa I. Such a rule is expressly stated by

[1] See Bibhutibhusan Datta, *BCMS*, XXIV, 1932.

Brahmagupta.[1]

The *rationale* of this method is simple: Starting with the consideration of the first two divisors, we have

$$N = a_1x_1 + r_1 = a_2x_2 + r_2.$$

By the method described before we can find the minimum value α of x_1 satisfying this equation. Then the minimum value of N will be $a_1\alpha + r_1$. Hence the general value of N will be given by

$$\begin{aligned} N &= a_1(a_2t + \alpha) + r_1, \\ &= a_1a_2t + a_1\alpha + r_1, \end{aligned}$$

where t is an integer. Thus $a_1\alpha + r_1$ is the remainder left on dividing N by a_1a_2, as stated by Āryabhaṭa I and Brahmagupta. Now, taking into consideration the third condition, we have

$$N = a_1a_2t + a_1\alpha + r_1 = a_3x_3 + r_3,$$

which can be solved in the same way as before. Proceeding in this way successively we shall ultimately arrive at a value of N satisfying all the conditions.

Pṛthûdakasvâmî remarks:

"Wherever the reduction of two divisors by a common measure is possible, there 'the product of the divisors' should be understood as equivalent to the product of the divisor corresponding to the greater remainder and quotient of the divisor corresponding to the smaller remainder as reduced (*i.e.*, divided) by the common measure.[2] When one divisor is exactly divisible by the other then the greater remainder is the (required) remainder and the divisor corresponding to

[1] *BrSpSi*, xviii. 5.

[2] *i.e.*, if p be the L.C.M. of a_1 and a_2, the general value of N satisfying the above two conditions will be

$$N = pt + a_1\alpha + r_1$$

instead of

$$N = a_1a_2t + a_1\alpha + r_1.$$

the greater remainder is taken as 'the product of the divisors.' (The truth of) this may be investigated by an intelligent mathematician by taking several symbols."

Examples from Bhâskara I:

(1) "Find that number which divided by 8 leaves 5 as remainder, divided by 9 leaves 4 as remainder and divided by 7 leaves 1 as remainder."[1]

That is to say, we have to solve

$$N = 8x + 5 = 9y + 4 = 7z + 1.$$

The solution is given substantially thus: The minimum value of N satisfying the first two conditions

$$N = 8x + 5 = 9y + 4$$

is found by the method of the pulveriser to be 13. This is the remainder left on dividing the number by the product 8.9. Hence

$$N = 72t + 13 = 7z + 1.$$

Again, applying the same method we find the minimum number satisfying all the conditions to be 85.

(2) "Tell me at once, O mathematician, that number which leaves unity as remainder when divided by any of the numbers from 2 to 6 but is exactly divisible by 7."

By the same method, says Bhâskara I (522), the number is found to be 721. By a different method Sûryadeva Yajvâ obtains the number 301. It is interesting to find that this very problem was afterwards treated by Ibn-al-Haitam (*c.* 1000) and Leonardo Fibonacci of Pisa (*c.* 1202).[2]

To solve a problem of this kind Bhâskara II adopts

[1] See his commentary on *Â*, ii. 32-3.

[2] L.E. Dickson, *History of the theory of Numbers*, Vol. II, referred to hereafter as Dickson, *Numbers* II, pp. 59, 60.

two methods. One is identical with the method of Âryabhaṭa I and the other follows from his general rule for the solution of simultaneous indeterminate equations of the first degree. They will be better understood from his applications to the solution[1] of the following problem which, as Pṛthûdakasvâmî (860) observes,[2] was popular amongst the Hindus:

To find a number N which leaves remainders 5, 4, 3, 2 when divided by 6, 5, 4, 3 respectively.

i.e., $N = 6x + 5 = 5y + 4 = 4z + 3 = 3w + 2.$

(1) We have

$$x = \frac{5y - 1}{6},\quad y = \frac{4z - 1}{5},\quad z = \frac{3w - 1}{4}.$$

Now by the method of the pulveriser, we get from the last equation

$$w = 4t + 3,\quad z = 3t + 2,$$

where t is an arbitrary integer. Substituting in the second equation, we get

$$y = \frac{12t + 7}{5}.$$

To make this integral, we again apply the method of the pulveriser, so that

$$t = 5s + 4,\quad y = 12s + 11.$$

This value of y makes x a whole number. Hence we have finally

$$w = 20s + 19,\ z = 15s + 14,\ y = 12s + 11,\ x = 10s + 9.$$

$$\therefore\ N = 60s + 59.$$

(2) Or we may proceed thus:

Since $N = 6x + 5 = 5y + 4,$

[1] *BBi*, pp. 85f.

[2] *Vide* his commentary on *BrSpSi*, xviii. 3-6.

we have $$x = \frac{5y - 1}{6}.$$

But x must be integral, so $y = 6t + 5$, $x = 5t + 4$.

Hence $$N = 30t + 29.$$

Again $$N = 30t + 29 = 4z + 3.$$

$$\therefore \qquad t = \frac{2z - 13}{15}.$$

Since t must be integral, we must have $z = 15s + 14$; hence $t = 2s + 1$. Therefore

$$N = 60s + 59.$$

The last condition is identically satisfied. Pṛthûdakasvâmî followed this second method to solve the above problem.

Conjunct Pulveriser. The foregoing system of indeterminate equations of the first degree can be put into the form[1]

$$\left.\begin{aligned} by_1 &= a_1x \pm c_1 \\ by_2 &= a_2x \pm c_2 \\ by_3 &= a_3x \pm c_3 \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \qquad (1)$$

On account of its important applications in mathematical astronomy this modified system has received special treatment at the hands of Hindu algebraists from Âryabhaṭa II (950) onwards. It is technically called

[1] For, we have

$$a_1x_1 + r_1 = a_2x_2 + r_2 = a_3x_3 + r_3 = \ldots = a_nx_n + r_n.$$

Then
$$a_2x_2 = a_1x_1 + (r_1 - r_2),$$
$$a_2x_3 = \frac{a_1a_2}{a_3}x_1 + \frac{a_2}{a_3}(r_1 - r_3),$$
$$a_2x_4 = \frac{a_1a_2}{a_4}x_1 + \frac{a_2}{a_4}(r_1 - r_4),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

saṁśliṣṭakuṭṭaka or the "conjunct pulveriser" (from, *kuṭṭaka* = pulveriser and *saṁśliṣṭa* = joined together, related).

For the solution of the above system of equations Âryabhaṭa II lays down the following rule :

"In the solution of simultaneous indeterminate equations of the first degree with a common divisor, the dividend will be the sum of the multipliers[1] and the interpolator the sum of the given interpolators."[2]

A similar rule is given by Bhâskara II. He says :

"If the divisor be the same but the multipliers different then making the sum of the multipliers the dividend and the sum of residues the residue (of a pulveriser), the investigation is carried on according to the foregoing method. This true method of the pulveriser is called the conjunct pulveriser."[3]

Rationale. If the equations (1) are satisfied by some value a of x, then the same value will satisfy the equation

$$b(y_1 + y_2 + \ldots) = (a_1 + a_2 + \ldots)x + (c_1 + c_2 + \ldots) \quad (2).$$

Thus, if we can find the general value of x satisfying equation (2), one of these values, at least, will satisfy all the equations (1).

To illustrate the application of the above Bhâskara II gives the following example :[4]

$$\left.\begin{aligned} 63y_1 &= 5x - 7 \\ 63y_2 &= 10x - 14 \end{aligned}\right\} \qquad (A)$$

Adding up the equations and dividing by the common factor 3, we get

$$21Y = 5x - 7,$$

[1] In the equations (1), a_1, a_2, ... are called multipliers.

[2] *MSi*, xviii. 48.

[3] *BBi*, p. 33 ; *L*, p. 82.

[4] *BBi*, p. 33 ; *L*, p. 82.

where $Y = y_1 + y_2$. By the method of the pulveriser the least positive value of x satisfying this equation is $x = 14$. This value of x is found to satisfy both the equations (A).

Generalised Conjunct Pulveriser. A generalised case of the conjunct pulveriser is that in which the divisors as well as the multipliers vary. Thus we have

$$b_1y_1 = a_1x \pm c_1,$$
$$b_2y_2 = a_2x \pm c_2,$$
$$b_3y_3 = a_3x \pm c_3,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Simultaneous indeterminate equations of this type have been treated by Mahâvîra (850) and Śrîpati (1039). Mahâvîra says :

"Find the least solutions of the first two equations. Divide the divisor corresponding to the greater solution by the other divisor (and as in the method of the pulveriser find the least value of) the multiplier with the difference of the solutions as the additive. That multiplied by the divisor (corresponding to the greater solution) and then added by the greater solution (will be the value of the unknown satisfying the two equations)."[1]

A similar rule is given by Śrîpati :

"Find the least solutions of the first two equations. Dividing the divisor corresponding to the greater solution by the divisor corresponding to the smaller solution, the residue (and its divisor) should be mutually divided. Then taking the difference of the numbers as the additive, determine (the least value of) the multiplier of the divisor corresponding to the greater solution in the manner explained before. Multiply that value by the

[1] *GSS*, vi. 115½, 136½ (last lines).

latter divisor and then add the solution (corresponding to it). The resulting number (severally) multiplied by the two multipliers and divided by the corresponding divisors will leave remainders as stated."[1]

The *rationale* of these rules will be clear from the following:

Taking the first two equations, we have

$$b_1 y_1 = a_1 x \pm c_1,$$
$$b_2 y_2 = a_2 x \pm c_2.$$

Suppose α_1 to be the least value of x satisfying the first equation as found by the method of the pulveriser. Then $b_1 m + \alpha_1$, where m is an arbitrary integer, will be the general value of x satisfying that equation. Similarly, we shall find from the second equation the general value of x as $b_2 n + \alpha_2$. If the same value of x satisfies both the equations we must have

$$b_2 n + \alpha_2 = b_1 m + \alpha_1,$$

or $$b_2 n = b_1 m + (\alpha_1 - \alpha_2);$$

supposing $\alpha_1 > \alpha_2$. Solving this equation, we can find the value of m and hence of $b_1 m + \alpha_1$ of x satisfying both the equations. The general value of x derived from this may be equated to the value of x from the third equation and the resulting equation solved again, and so on.

In illustration of his rule Mahâvîra proposed several problems. One of these has already been given (Part I, p. 233). Here are two others:

(1) "Five (heaps of fruits) added with two (fruits) were divided (equally) between nine travellers; six (heaps) added with four (fruits) were divided amongst eight; four (heaps) increased by one (fruit) were divided

[1] *SiŚe*, xiv. 28.

amongst seven. Tell the number (of fruits in each heap)."[1]

This gives the equations:

$$9y_1 = 5x + 2, \quad 8y_2 = 6x + 4, \quad 7y_3 = 4x + 1.$$

(2) "The (dividends) are the sixteen numbers beginning with 35 and increasing successively by three; divisors are 32 and others successively increasing by 2; and 1 increasing by 3 gives the remainders positive and negative. What is the unknown multiplier?"[2]

This gives the equations:

$$32y_1 = 35x \pm 1, \quad 34y_2 = 38x \pm 4, \quad 36y_3 = 41x \pm 7, \ldots$$

Alternative Method. In four palm-leaf manuscript copies of the *Lîlâvatî* of Bhâskara II Sarada Kanta Ganguly discovered a rule describing an alternative method for the solution of the generalised conjunct pulveriser.[3] There is also an illustrative example. The genuineness of this rule and example is accepted by him; but it has been questioned by A. A. Krishnaswami Ayyangar[4] who attributes them to some commentator of the work. His arguments are not convincing.[5] The chief points against the presumption, which have been noted also by Ganguly, are: (1) the rule and example in question have not been mentioned by the earlier commentators of the *Lîlâvatî* and (2) they have not been so far traced in any manuscript of the *Bîjagaṇita*, though the treatment of the pulveriser occurs nearly word for

[1] *GSS*, vi. 129½. [2] *GSS*, vi. 138½.

[3] S. K. Ganguly, "Bhâskarâcârya and simultaneous indeterminate equations of the first degree," *BCMS*, XVII, 1926, pp. 89-98.

[4] A. A. Krishnaswami Ayyangar, "Bhâskara and samslishta Kuttaka," *JIMS*, XVIII, 1929.

[5] For Ganguly's reply to Ayyangar's criticism see *JIMS*, XIX, 1931.

word in the two works. Still we are in favour of accepting Ganguly's conclusion.[1] The rule in question is this:

"If the divisors as well as the multipliers be different, find the value of the unknown answering to the first set of them. That value being multiplied by the second dividend and then added by the second interpolator will be the interpolator (of a new *kuṭṭaka*); the product of the second dividend and first divisor will be the dividend there and the divisor will be the second divisor. The value of the unknown multiplier determined from the *kuṭṭaka* thus formed being multiplied by the first divisor and added by the previous value of the unknown multiplier will be the value (answering to the two divisors). The dividend (for the next step) has been stated to be equal to the product of the two divisors. So proceed in the same way with the third divisor. And so on with the others, if there be many."

The *rationale* of this rule is as follows: Let α_1 be the least value of x satisfying the first equation of the system, *viz.*,

$$b_1y_1 = a_1x \pm c_1.$$

Hence the general value is $x = b_1t + \alpha_1$, where t is any integer. Substituting this value in the second equation, we get

$$b_2y_2 = a_2b_1t + (a_2\alpha_1 \pm c_2).$$

If $t = \tau$ be a solution of this equation, a value of x

[1] Of the four manuscripts containing the rule and example in question two are from Puri, in Oriya characters, with the commentary of Śrîdhara Mahâpâtra (1717); the other two, in Andhra characters and without any commentary, are preserved in the Oriental Libraries of Madras and Mysore. So these four manuscript copies do not appear to have been drawn from the same source. This is a strong point in favour of the genuineness of the rule and example.

satisfying both the equations will be $\alpha_2 = b_1\tau + \alpha_1$ as stated in the rule. Now the general value of t will be $t = b_2m + \tau$, where m is an integer. Hence $x = b_1t + \alpha_1 = b_1b_2m + b_1\tau + \alpha_1 = b_1b_2m + \alpha_2$. Substituting this value in the third equation we can find the least value of m and hence a value of x answering to the three equations. And so on for the other equations.

The example runs thus :

"'Tell me that number which multiplied by 7 and then divided by 62, leaves the remainder 3. That number again when multiplied by 6 and divided by 101 leaves the remainder 5 ; and when multiplied by 8 and divided by 17 leaves the remainder 9. Also (give) at once the process of the pulveriser for (finding) the number with the remainders all positive."

Symbolically, we have

(1) $62y_1 = 7x - 3,\ 101y_2 = 6x - 5,\ 17y_3 = 8x - 9;$

(2) $62y_1 = 7x + 3,\ 101y_2 = 6x + 5,\ 17y_3 = 8x + 9.$

16. SOLUTION OF $Nx^2 + 1 = y^2$

Square-nature. The indeterminate quadratic equation

$$Nx^2 \pm c = y^2,$$

is called by the Hindus *Varga-prakṛti* or *Kṛti-prakṛti*, meaning the "Square-nature."[1] Bhâskara II (1150) states that the absolute number should be *rûpa*,[2] which means "unity" as well as "absolute number" in general. Kamalâkara (1658) says :

[1] *Varga* = *kṛti* = "square" and *prakṛti* = "nature," "principle," "origin," etc. Colebrooke has rendered the term *varga-prakṛti* as "Affected Square."

[2] "Tatra rûpakṣepapadârthaṁ tâvat"—*BBi*, p. 33.

"Hear first the nature of the *varga-prakṛti* : in it the square (of a certain number) multiplied by a multiplier and then increased or diminished by an interpolator becomes capable of yielding a square-root."[1]

It was recognised that the most fundamental equation of this class is

$$Nx^2 + 1 = y^2,$$

where N is a non-square integer.

Origin of the Name. As regards the origin of the name *varga-prakṛti*, Kṛṣṇa (1580) says : "That in which the *varga* (square) is the *prakṛti* (nature) is called the *varga-prakṛti* ; for the square of *yâvat*, etc., is the *prakṛti* (origin) of this (branch of) mathematics. Or, because this (branch of) mathematics has originated from the number which is the *prakṛti* of the square of *yâvat*, etc., so it is called the *varga-prakṛti*. In this case the number which is the multiplier of the square of *yâvat*, etc., is denoted by the term *prakṛti*. (In other words) it is the coefficient of the square of the unknown."[2] This double interpretation has been evidently suggested by the use of the term *prakṛti* by Bhâskara II in two contexts. He has denoted by it sometimes the quantity N of the above equation as in "There the number which is (associated) with the square of the unknown is the *prakṛti*;"[3] and at other times x^2, as in "Supposing the square of one of the two unknowns to be the *prakṛti*."[4] Other Hindu algebraists have, however, consistently

[1] *SiTVi*, xiii. 208.

[2] See his commentary on the *Bîjagaṇita* of Bhâskara II.

[3] "Tatra varṇavarge yo'ṅkaḥ sâ prakṛtiḥ" (*BBi*, p. 100). Compare also "Tatra yâvattâvadvarge yo'ṅkaḥ sâ prakṛtiḥ" (p. 107) ; "Iṣṭaṁ hrasvaṁ tasya vargah prakṛtyâ kṣuṇṇo..." (p. 33).

[4] "Tatraikâṁ varṇakṛtiṁ prakṛtiṁ prakalpya..." (*BBi*, p. 106). Compare also "Sarûpake varṇakṛti tu yatra tatrecchaikâṁ prakṛtiṁ prakalpya..." (p. 105).

employed the term *prakṛti* to denote N only.[1] Brahmagupta (628) uses the term *guṇaka* (multiplier) for the same purpose.[2] This latter term, together with its variation *guṇa*, appears occasionally also in later works.[3]

We presume that the name *varga-prakṛti* originated from the following consideration: The principle (*prakṛti*) underlying the calculations in this branch of mathematics is to determine a number (or numbers) whose nature (*prakṛti*) is such that its (or their) square (or squares, *varga*) or the simple number (or numbers) after certain specified operations will yield another number (or numbers) of the nature of a square. So the name is, indeed, very significant. This interpretation seems to have been intended, at any rate, by the earlier writers who used the term in a wider sense.[4] It is perhaps noteworthy that we do not find in the works of Brahmagupta the use of the word *prakṛti* either in the sense of N or of x^2.

Technical Terms. Of the various technical terms which are ordinarily used by the Hindu algebraists in connection with the Square-nature we have already dealt with the most notable one, *prakṛti*, together with its synonyms. Others have been explained by Pṛthûdakasvâmî (860) thus:

"Here are stated for ordinary use the terms which

[1] For instance, Pṛthûdakasvâmî (860) writes: "The multiplier (of the square of the unknown) is known as the *prakṛti*;" Śrîpati (1039): "Kṛter-guṇako prakṛtirbhṛśoktaḥ" (*SiŚe*, xiv. 32); Kamalâkara: "Guṇo yo râśi-vargasya saiva prakṛtirucyate."

[2] *BrSpSi*, xviii. 64.

[3] For instance, Śrîpati employs the term *guṇaka* (*SiŚe*, xiv. 32); Bhâskara II and Nârâyaṇa use *guṇa* (*BBi*, p. 42; *NBi*, I, R. 84).

[4] For instance, Brahmagupta seems to have considered the scope of the subject wide enough to include such equations as

$$x + y = u^2, \ x - y = v^2, \ xy + 1 = w^2,$$

amongst others (cf. *BrSpSi*, xviii. 72).

are well known to people. The number whose square, multiplied by an optional multiplier and then increased or decreased by another optional number, becomes capable of yielding a square-root, is designated by the term the lesser root (*kaniṣṭha-pada*) or the first root (*âdya-mûla*). The root which results, after those operations have been performed, is called by the name the greater root (*jyeṣṭha-pada*) or the second root (*anya-mûla*). If there be a number multiplying both these roots, it is called the augmenter (*udvartaka*); and, on the contrary, if there be a number dividing the roots, it is called the abridger (*apavartaka*)."[1]

Bhâskara II (1150) writes:

"An optionally chosen number is taken as the lesser root (*hrasva-mûla*). That number, positive or negative, which being added to or subtracted from its square multiplied by the *prakṛti* (multiplier) gives a result yielding a square-root, is called the interpolator (*kṣepaka*). And this (resulting) root is called the greater root (*jyeṣṭha-mûla*)."[2]

Similar passages occur in the works of Nârâyaṇa,[3] Jñânarâja and Kamalâkara.[4]

The terms 'lesser root' and 'greater root' do not appear to be accurate and happy. For if $x = m$, $y = n$ be a solution of the equation $Nx^2 + c = y^2$, m will be less than n, if N and c are both positive. But if they are of opposite signs, the reverse will sometimes happen.[5]

[1] See Pṛthûdakasvâmî's commentary on *BrSpSi*, xviii. 64. In the equation $Nx^2 \pm c = y^2$, $x =$ lesser root, $y =$ greater root, $N =$ multiplier, and $c =$ interpolator.

[2] *BBi*, p. 33.

[3] *NBi*, I, R. 72.

[4] *SiTVi*, xiii. 209.

[5] For instance, take the following example from Bhâskara II (*BBi*, p. 43):

Therefore, in the latter case, where $m > n$, it will be obviously ambiguous to call m the lesser root and n the greater root, as was the practice in later Hindu algebra. This defect in the prevalent terminology was noticed by Kṛṣṇa (1580). He explains it thus: "These terms are significant. Where the greater root is sometimes smaller than the lesser root owing to the interpolator being negative, there also it becomes greater than the lesser root after the application of the Principle of Composition."[1] The earlier terms, 'the first root' (*ādya-mūla*) for the value of x and 'the second root' or 'the last root' (*antya-mūla*) for the value of y, are quite free from ambiguity. Their use is found in the algebra of Brahmagupta (628).[2] The later terms appear in the works of his commentator Pṛthūdakasvāmī (860).

The interpolator is called by Brahmagupta *kṣepa*, *prakṣepa* or *prakṣepaka*.[3] Śrīpati occasionally employs the synonym *kṣipti*.[4] When negative, the interpolator is sometimes distinguished as 'the subtractive' (*śodhaka*).

$$13x^2 - 13 = y^2.$$

One solution of it is given by the author as $x = 1, y = 0$; so that here the lesser root is greater than the greater root. The same is the case in the solution $x = 2, y = 1$ of his example (*BBī*, p. 43)

$$-5x^2 + 21 = y^2.$$

Brahmagupta gives the example (*BrSpSi*, xviii. 77)

$$3x^2 - 800 = y^2,$$

which has a solution ($x = 20, y = 20$) where the two roots are equal.

[1] For example, by composition of the solution (1, 0) of the equation $13x^2 - 13 = y^2$ with the solution $(\frac{3}{2}, \frac{11}{2})$ of the equation $13x^2 + 1 = y^2$, we obtain, after Bhâskara II, a new solution $(\frac{11}{2}, \frac{39}{2})$ of the former, in which the greater root is greater than the lesser root. Similarly, by composition of the solution (2, 1) of the equation $-5x^2 + 21 = y^2$ with the solution $(\frac{1}{3}, \frac{2}{3})$ of the equation $-5x^2 + 1 = y^2$, we get a new solution (1, 4) of the former satisfying the same condition.

[2] *BrSpSi*, xviii. 64, 66f.

[3] *BrSpSi*, xviii. 65.

[4] *SiŚe*, xiv. 32.

The positive interpolator is then called 'the additive.'[1]

Brahmagupta's Lemmas. Before proceeding to the general solution of the Square-nature Brahmagupta has established two important lemmas. He says:

"Of the square of an optional number multiplied by the *guṇaka* and increased or decreased by another optional number, (extract) the square-root. (Proceed) twice. The product of the first roots multiplied by the *guṇaka* together with the product of the second roots will give a (fresh) second root; the sum of their cross-products will be a (fresh) first root. The (corresponding) interpolator will be equal to the product of the (previous) interpolators."[2]

The rule is somewhat cryptic because the word *dvidhâ* (twice) has been employed with double implication. According to one, the earlier operations of finding roots are made on two optional numbers with two optional interpolators, and with the results thus obtained the subsequent operations of their composition are performed. According to the other implication of the word, the earlier operations are made with one optionally chosen number and one interpolator, and the subsequent ones are carried out after the repeated statement of those roots for the second time. It is also implied that in the composition of the quadratic roots their products may be added together or subtracted from each other.

That is to say, if $x = \alpha$, $y = \beta$ be a solution of the equation

$$Nx^2 + k = y^2,$$

and $x = \alpha'$, $y = \beta'$ be a solution of

$$Nx^2 + k' = y^2,$$

then, according to the above,

[1] *BrSpSi*, xviii. 64-5.

[2] *BrSpSi*, xviii. 64-5.

$$x = \alpha\beta' \pm \alpha'\beta,\ y = \beta\beta' \pm N\alpha\alpha'$$

is a solution of the equation

$$Nx^2 + kk' = y^2.$$

In other words, if

$$N\alpha^2 + k = \beta^2,$$
$$N\alpha'^2 + k' = \beta'^2,$$

then

$$N(\alpha\beta' \pm \alpha'\beta)^2 + kk' = (\beta\beta' \pm N\alpha\alpha')^2. \tag{I}$$

In particular, taking $\alpha = \alpha'$, $\beta = \beta'$, and $k = k'$, Brahmagupta finds from a solution $x = \alpha$, $y = \beta$ of the equation

$$Nx^2 + k = y^2,$$

a solution $x = 2\alpha\beta$, $y = \beta^2 + N\alpha^2$ of the equation

$$Nx^2 + k^2 = y^2.$$

That is, if

$$N\alpha^2 + k = \beta^2,$$

then

$$N(2\alpha\beta)^2 + k^2 = (\beta^2 + N\alpha^2)^2. \tag{II}$$

This result will be hereafter called *Brahmagupta's Corollary*.

Description by Later Writers. Brahmagupta's Lemmas have been described by Bhâskara II (1150) thus :

"Set down successively the lesser root, greater root and interpolator ; and below them should be set down in order the same or another (set of similar quantities). From them by the Principle of Composition can be obtained numerous roots. Therefore, the Principle of Composition will be explained here. (Find) the two cross-products of the two lesser and the two greater roots ; their sum is a lesser root. Add the product of the two lesser roots multiplied by the *prakṛti* to the product of the two greater roots ; the sum will be a greater root. In that (equation) the interpolator will be the product of the two previous interpolators. Again the difference of the two cross-products is a lesser root. Subtract the product of the two lesser roots multiplied by the *prakṛti* from the product of the two greater roots; (the difference) will be a greater root. Here also, the interpolator is the product of the two (previous) interpolators."[1]

Statements similar to the above are found in the works of Nârâyaṇa[2] (1350), Jñanarâja (1503) and Kamalâkara[3] (1658).

Principle of Composition. The above results are called by the technical name, *Bhâvanâ* (demonstration or proof, meaning anything demonstrated or proved, hence theorem, lemma; the word also means composition or combination). They are further distinguished as *Samâsa Bhâvanâ* (Addition Lemma or Additive Composition) and *Antara Bhâvanâ* (Subtraction Lemma or Subtractive Composition). Again, when the *Bhâvanâ* is made with two equal sets of roots and interpolators, it is called *Tulya Bhâvanâ* (Composition of Equals) and when with two unequal sets of values, *Atulya Bhâvanâ* (Composition of Unequals). Kṛṣṇa has observed that when it is desired to derive roots of a Square-nature, larger in value, one should have recourse to the Addition Lemma and for smaller roots one should use the Subtraction Lemma.

Brahmagupta's Lemmas were rediscovered and recognised as important by Euler in 1764 and by Lagrange in 1768.

Proof. The proof of Brahmagupta's Lemmas has been given by Kṛṣṇa substantially as follows :

[1] *BBi*, p. 34. [2] *NBi*, I, R. 72-75½.
[3] *SiTVi*, xiii. 210-214.

We have

$$N\alpha^2 + k = \beta^2,$$
$$N\alpha'^2 + k' = \beta'^2.$$

Multiplying the first equation by β'^2, we get

$$N\alpha^2\beta'^2 + k\beta'^2 = \beta^2\beta'^2.$$

Now, substituting the value of the factor β'^2 of the interpolator from the second equation, we get

$$N\alpha^2\beta'^2 + k(N\alpha'^2 + k') = \beta^2\beta'^2,$$

or

$$N\alpha^2\beta'^2 + Nk\alpha'^2 + kk' = \beta^2\beta'^2.$$

Again, substituting the value of k from the first equation in the second term of the left-hand side expression, we have

$$N\alpha^2\beta'^2 + N\alpha'^2(\beta^2 - N\alpha^2) + kk' = \beta^2\beta'^2,$$

or

$$N(\alpha^2\beta'^2 + \alpha'^2\beta^2) + kk' = \beta^2\beta'^2 + N^2\alpha^2\alpha'^2.$$

Adding $\pm 2N\alpha\beta\alpha'\beta'$ to both sides, we get

$$N(\alpha\beta' \pm \alpha'\beta)^2 + kk' = (\beta\beta' \pm N\alpha\alpha')^2.$$

Brahmagupta's Corollary follows at once from the above by putting $\alpha' = \alpha$, $\beta' = \beta$ and $k' = k$.

General Solution of the Square-Nature. It is clear from Brahmagupta's Lemma (I) that when two solutions of the Square-nature,

$$Nx^2 + 1 = y^2,$$

are known, any number of other solutions can be found. For, if the two solutions be (a, b) and (a', b'), then two other solutions will be

$$x = ab' \pm a'b, \quad y = bb' \pm Naa'.$$

Again, composing this solution with the previous ones, we shall get other solutions. Further, it follows from Brahmagupta's Corollary that if (a, b) be a solution of the equation, another solution of it is $(2ab,\ b^2 + Na^2)$. Hence, in order to obtain a set of solutions of the

Square-nature it is necessary to obtain only one solution of it. For, after having obtained that, an infinite number of other solutions can be found by the repeated application of the Principle of Composition. Thus Śrîpati (1039) observes: "There will be an infinite (set of two roots)."[1] Bhâskara II (1150) remarks: "Here (*i.e.*, in the solution of the Square-nature) the roots are infinite by virtue of (the infinitely repeated application of) the Principle of Composition as well as of (the infinite variety of) the optional values (of the first roots)."[2] Nârâyaṇa (1350) writes, "By the Principle of Composition of equal as well as unequal sets of roots, (will be obtained) an infinite number of roots."[3]

Modern historians of mathematics are incorrect in stating that Fermat (1657) was the first to assert that the equation $Nx^2 + 1 = y^2$, where N is a non-square integer, has an unlimited number of solutions in integers.[4] The existence of an infinite number of integral solutions was clearly mentioned by Hindu algebraists long before Fermat.

Another Lemma. Brahmagupta says:

"On dividing the two roots (of a Square-nature) by the square-root of its additive or subtractive, the roots for the interpolator unity (will be found)."[5]

That is to say, if $x = \alpha$, $y = \beta$ be a solution of the equation

$$Nx^2 + k^2 = y^2,$$

then $x = \alpha/k$, $y = \beta/k$ is a solution of the equation

$$Nx^2 + 1 = y^2.$$

This rule has been restated in a different way thus:

[1] *SiŚe*, xiv. 33.
[2] "Ihânantyaṁ bhâvanâbhistatheṣṭataḥ"—*BB*, p. 34.
[3] *NBi*, I, R. 78. Compare also *SiTVi*, xiii. 217.
[4] Smith, *History*, II, p. 453.
[5] *BrSpSi*, xviii. 65.

"If the interpolator is that divided by a square then the roots will be those multiplied by its square-root."[1]

That is, suppose the Square-nature to be

$$Nx^2 \pm p^2d = y^2,$$

so that its interpolator p^2d is exactly divisible by the square p^2. Then, putting therein $u = x/p$, $v = y/p$, we derive the equation

$$Nu^2 \pm d = v^2,$$

whose interpolator is equal to that of the original Square-nature divided by p^2. It is clear that the roots of the original equation are p times those of the derived equation.

Bhâskara II writes:

"If the interpolator (of a Square-nature) divided by the square of an optional number be the interpolator (of another Square-nature), then the two roots (of the former) divided by that optional number will be the roots (of the other). Or, if the interpolator be multiplied, the roots should be multiplied."[2]

The same rule has been stated in slightly different words by Nârâyaṇa[3] and Kamalâkara.[4] Jñânarâja simply observes:

"If the interpolator (of a Square-nature) be divided by the square of an optional number then its roots will be divided by that optional number."

Thus we have, in general, if $x = \alpha$, $y = \beta$ be a solution of the equation

$$Nx^2 \pm k = y^2,$$

[1] *BrSpSi*, xviii. 70.
[2] *BBi*, p. 34.
[3] *NBi*, I, R. 76-76½.
[4] *SiTVi*, xiii. 215.

$x = \alpha/m$, $y = \beta/m$ is a solution of the equation

$$Nx^2 \pm k/m^2 = y^2;$$

and $x = n\alpha$, $y = n\beta$ is a solution of the equation

$$Nx^2 \pm n^2k = y^2,$$

where m, n are arbitrary rational numbers.

By this Lemma, the solutions of the Square-natures

$$(i) \quad 6x^2 + 12 = y^2,$$

$$(ii) \quad 6x^2 + 75 = y^2,$$

$$\text{and} \quad (iii) \quad 6x^2 + 300 = y^2,$$

can be derived, as shown by Bhâskara II,[1] from those of

$$6x^2 + 3 = y^2,$$

since $12 = 2^2.3$, $75 = 5^2.3$, and $300 = 10^2.3$. How to solve this latter equation will be indicated later on.

Rational Solution. In order to obtain a first solution of $Nx^2 + 1 = y^2$ the Hindus generally suggest the following tentative method: Take an arbitrary small rational number α, such that its square multiplied by the *guṇaka* N and increased or diminished by a suitably chosen rational number k will be an exact square. In other words, we shall have to obtain empirically a relation of the form

$$N\alpha^2 \pm k = \beta^2,$$

where α, k, β are rational numbers. This relation will be hereafter referred to as the *Auxiliary Equation*. Then by Brahmagupta's Corollary, we get from it the relation

$$N(2\alpha\beta)^2 + k^2 = (\beta^2 + N\alpha^2)^2,$$

or

$$N\left(\frac{2\alpha\beta}{k}\right)^2 + 1 = \left(\frac{\beta^2 + N\alpha^2}{k}\right)^2.$$

[1] *BBi*, p. 41.

Hence, one rational solution of the equation $Nx^2 + 1 = y^2$ is given by

$$x = \frac{2\alpha\beta}{k}, \quad y = \frac{\beta^2 + N\alpha^2}{k}. \qquad (A)$$

Srîpati's Rational Solution. Srîpati (1039) has shown how a rational solution of the Square-nature can be obtained more easily and directly without the intervention of an auxiliary equation. He says:

"Unity is the lesser root. Its square multiplied by the *prakṛti* is increased or decreased by the *prakṛti* combined with an (optional) number whose square-root will be the greater root. From them will be obtained two roots by the Principle of Composition."[1]

If m^2 be a rational number optionally chosen, we have the identity

$$N.1^2 + (m^2 - N) = m^2,$$

or

$$N.1^2 - (N - m^2) = m^2.$$

Then, applying Brahmagupta's Corollary to either, we get

$$N(2m)^2 + (m^2 \sim N)^2 = (m^2 + N)^2;$$

$$\therefore \quad N\left(\frac{2m}{m^2 \sim N}\right)^2 + 1 = \left(\frac{m^2 + N}{m^2 \sim N}\right)^2.$$

Hence $$x = \frac{2m}{m^2 \sim N}, \quad y = \frac{m^2 + N}{m^2 \sim N}, \qquad (B)$$

where m is any rational number, is a solution of the equation

$$Nx^2 + 1 = y^2.$$

The above solution reappears in the works of later Hindu algebraists. Bhâskara II says:

[1] *SiŚe*, xiv. 33.

"Or divide twice an optional number by the difference between the square of that optional number and the *prakṛti*. This (quotient) will be the lesser root (of a Square-nature) when unity is the additive. From that (follows) the greater root."[1]

Nârâyaṇa states:

"Twice an optional number divided by the difference between the square of that optional number and the *guṇaka* will be the lesser root. From that with the additive unity determine the greater root."[2]

Similar statements are found also in the works of Jñânarâja and Kamalâkara.[3]

If m be an optional number, it is stated that $\dfrac{2m}{m^2 \sim N}$ is a lesser root of $Nx^2 + 1 = y^2$. Then, substituting that value of x in the equation, we get

$$y^2 = N\left(\frac{2m}{m^2 \sim N}\right)^2 + 1,$$

$$= \left(\frac{m^2 + N}{m^2 \sim N}\right)^2.$$

Hence the greater root is

$$y = \frac{m^2 + N}{m^2 \sim N}.$$

The same solution will be obtained by assuming

$$y = mx - 1.$$

Kṛṣṇa points out that it can also be found thus:

$$4Nm^2 = (m^2 + N)^2 - (m^2 \sim N)^2, \text{ identically.}$$

$$\therefore\quad 4Nm^2 + (m^2 \sim N)^2 = (m^2 + N)^2,$$

[1] *BBi*, p. 34.
[2] *NBi*, I, R. 77f.
[3] *SiTVi*, xiii. 216.

or $$N\left(\frac{2m}{m^2 \sim N}\right)^2 + 1 = \left(\frac{m^2 + N}{m^2 \sim N}\right)^2.$$

His remark that this method does not require the help of the Principle of Composition shows that Bhâskara II and others obtained the solution in the way indicated by Srîpati.

The above rational solution of the Square-nature has been hitherto attributed by modern historians of mathematics to Bhâskara II. But it is now found to be due to an anterior writer, Srîpati (1039). It was rediscovered in Europe by Brouncker (1657).

Illustrative Examples. In illustration of the foregoing rules we give the following examples with their solutions from Bhâskara II.

Examples. "Tell me, O mathematician, what is that square which multiplied by 8 becomes, together with unity, a square; and what square multiplied by 11 and increased by unity, becomes a square."[1]

That is to say, we have to solve

$$(1) \quad 8x^2 + 1 = y^2,$$
$$(2) \quad 11x^2 + 1 = y^2.$$

Solutions. "In the second example assume 1 as the lesser root. Multiplying its square by the *prakṛti*, namely 11, subtracting 2 and then extracting the square-root, we get the greater root as 3. Hence the statement for composition is[2]

$m = 11$	$l = 1$	$g = 3$	$i = -2$
	$l = 1$	$g = 3$	$i = -2$

[1] *BBi*, p. 35.

[2] The abbreviations are: m = multiplier, l = lesser root, g = greater root and i = interpolator. In the original they are respectively *pra*, *ka*, *jye*, and *kṣe*, the initial syllables of the corresponding Sanskrit terms.

Proceeding as before we obtain the roots for the additive 4: $l = 6$, $g = 20$, (for) $i = 4$. Then by the rule, 'If the interpolator (of a Square-nature) divided by the square of an optional number etc.,'[1] are found the roots for the additive unity: $l = 3$, $g = 10$ (for) $i = 1$. Whence by the Principle of Composition of Equals, we get the lesser and greater roots: $l = 60$, $g = 199$ (for) $i = 1$. In this way an infinite number of roots can be deduced.

"Or, assuming 1 for the lesser root, we get for the additive 5: $l = 1$, $g = 4$, (for) $i = 5$. Whence by the Principle of Composition of Equals, the roots are $l = 8$, $g = 27$, (for) $i = 25$. Then by the rule, 'If the interpolator (of a Square-nature) divided by the square of an optional number etc.,' taking 5 as the optional number, we get the roots for the additive unity: $l = 8/5$, $g = 27/5$, (for) $i = 1$. The statement of these for composition with the previous roots is

$$\begin{array}{llll} m = 11 & l = 8/5 & g = 27/5 & i = 1 \\ & l = 3 & g = 10 & i = 1 \end{array}$$

By the Principle of Composition the roots are obtained as: $l = 161/5$, $g = 534/5$ (for) $i = 1$.

"Or composing according to the rule, 'The difference of the two cross-products is a lesser root etc.,' we get the roots: $l = 1/5$, $g = 6/5$ (for) $i = 1$. And so on in many ways.

"The two roots for the additive unity will now be found in a different way by the rule, 'Or divide twice an optional number by the difference between the square of that optional number and the *prakṛti* etc.' Here, in the first example, assume the optional number to be 3. Its square is 9; multiplier is 8; their difference is 1;

[1] *Vide supra*, p. 151.

dividing by this twice the optional number, namely 6, we get the lesser root for the additive unity as 6. Whence, proceeding as before, the greater root comes out as 17.

"In the same way, in the second example also, assuming the optional number to be 3, the lesser and greater roots are found to be (3, 10).

"Thus, by virtue of (the infinite variety of) the optional values as well as of (the infinitely repeated application of) the Principle of Additive and Subtractive Compositions, an infinite number of roots (may be found)."[1]

Solution in Positive Integers. As has been stated before, the aim of the Hindus was to obtain solutions of the Square-nature in positive integers; so its first solution must be integral. But neither the tentative method of Brahmagupta nor that of Srîpati is of much help in this direction, for they do not *always* yield the desired result. These authors, however, discovered that if the interpolator of the auxiliary equation in the tentative method be ± 1, ± 2 or ± 4, an integral solution of the equation $Nx^2 + 1 = y^2$ can always be found. Thus Srîpati (1039) expressly observes, "If 1, 2 or 4 be the additive or subtractive (of the auxiliary equation) the lesser and greater roots will be integral (*abhinna*).[2]

(*i*) If $k = \pm 1$; then the auxiliary equation will be[3]

$$N\alpha^2 \pm 1 = \beta^2,$$

[1] The original is, "Evamiṣṭavaśât samâsântarabhâvanâbhyâṁ ca padânâmânantyam." (*BBi*, p. 36).

[2] "Dvyekâmbudhikṣepaviśodhanâbhyâṃ
Syâtâmabhinne laghuvṛddhamûle."—*SiSe*, xiv. 32.
The Sanskrit word *abhinna* literally means "non-fractional."

[3] The special treatment of the equation $Nx^2 - 1 = y^2$ is given later on.

where α, β are integers. Then by Brahmagupta's Corollary, we get

$$x = 2\alpha\beta,\ \ y = \beta^2 + N\alpha^2$$

as the required first solution in positive integers of the equation $Nx^2 + 1 = y^2$.

(*ii*) Let $k = \pm 2$; then the auxiliary equation is

$$N\alpha^2 \pm 2 = \beta^2.$$

By Brahmagupta's Corollary, we have

$$N(2\alpha\beta)^2 + 4 = (\beta^2 + N\alpha^2)^2,$$

or $$N(\alpha\beta)^2 + 1 = \left(\frac{\beta^2 + N\alpha^2}{2}\right)^2.$$

Hence the required first solution is

$$x = \alpha\beta,\ \ y = \tfrac{1}{2}(\beta^2 + N\alpha^2).$$

Since $$N\alpha^2 = \beta^2 \mp 2,$$

we have $$\tfrac{1}{2}(\beta^2 + N\alpha^2) = \beta^2 \mp 1 = \text{a whole number}.$$

(*iii*) Now suppose $k = +4$; so that

$$N\alpha^2 + 4 = \beta^2.$$

With an auxiliary equation like this the first integral solution of the equation $Nx^2 + 1 = y^2$ is

$$x = \tfrac{1}{2}\alpha\beta,$$
$$y = \tfrac{1}{2}(\beta^2 - 2);$$

if α is even; or

$$x = \tfrac{1}{2}\alpha(\beta^2 - 1),$$
$$y = \tfrac{1}{2}\beta(\beta^2 - 3);$$

if β is odd.

Thus Brahmagupta says:

"In the case of 4 as additive the square of the second root diminished by 3, then halved and multiplied by the second root will be the (required) second root; The square of the second root diminished by unity and then divided by 2 and multiplied by the first root will be the (required) first root (for the additive unity)."[1]

The *rationale* of this solution is as follows:

Since $$N\alpha^2 + 4 = \beta^2, \tag{1}$$

we have $$N\left(\frac{\alpha}{2}\right)^2 + 1 = \left(\frac{\beta}{2}\right)^2. \tag{2}$$

Then, by Brahmagupta's Corollary, we get

$$N\left(\frac{\alpha\beta}{2}\right)^2 + 1 = \left(\frac{\beta^2}{4} + N\frac{\alpha^2}{4}\right)^2.$$

Substituting the value of N in the right-hand side expression from (1), we have

$$N\left(\frac{\alpha\beta}{2}\right)^2 + 1 = \left(\frac{\beta^2 - 2}{2}\right)^2. \tag{3}$$

Composing (2) and (3),

$$N\left\{\frac{\alpha}{2}(\beta^2 - 1)\right\}^2 + 1 = \left\{\frac{\beta}{2}(\beta^2 - 3)\right\}^2.$$

Hence $$x = \tfrac{1}{2}\alpha\beta, \quad y = \tfrac{1}{2}(\beta^2 - 2);$$

and $$x = \tfrac{1}{2}\alpha(\beta^2 - 1), \quad y = \tfrac{1}{2}\beta(\beta^2 - 3);$$

are solutions of

$$Nx^2 + 1 = y^2.$$

If β be even, the first values of (x, y) are integral. If β be odd, the second values are integral.

(*iv*) Finally, suppose $k = -4$; the auxiliary equation is

$$N\alpha^2 - 4 = \beta^2.$$

Then the required first solution in positive integers of $Nx^2 + 1 = y^2$ is

$$x = \tfrac{1}{2}\alpha\beta(\beta^2 + 3)(\beta^2 + 1),$$
$$y = (\beta^2 + 2)\{\tfrac{1}{2}(\beta^2 + 3)(\beta^2 + 1) - 1\}.$$

[1] *BrSpSi*, xviii. 67.

Brahmagupta says:

"In the case of 4 as subtractive, the square of the second is increased by three and by unity; half the product of these sums and that as diminished by unity (are obtained). The latter multiplied by the first sum less unity is the (required) second root; the former multiplied by the product of the (old) roots will be the first root corresponding to the (new) second root."[1]

The *rationale* of this solution is as follows:

$$N\alpha^2 - 4 = \beta^2. \qquad (1)$$

$$\therefore \qquad N\left(\frac{\alpha}{2}\right)^2 - 1 = \left(\frac{\beta}{2}\right)^2.$$

Hence by Brahmagupta's Corollary, we get

$$N\left(\frac{\alpha\beta}{2}\right)^2 + 1 = \left(\frac{\beta^2}{4} + N\frac{\alpha^2}{4}\right)^2$$
$$= \{\tfrac{1}{2}(\beta^2 + 2)\}^2. \qquad (2)$$

Again, applying the Corollary, we have

$$N\{\tfrac{1}{2}\alpha\beta(\beta^2 + 2)\}^2 + 1 = \{\tfrac{1}{2}(\beta^4 + 4\beta^2 + 2)\}^2. \qquad (3)$$

Now, by the Lemma, we obtain from (2) and (3)

$$N\{\tfrac{1}{2}\alpha\beta(\beta^2 + 3)(\beta^2 + 1)\}^2 + 1$$
$$= [(\beta^2 + 2)\{\tfrac{1}{2}(\beta^2 + 3)(\beta^2 + 1) - 1\}]^2.$$

Hence $\qquad x = \tfrac{1}{2}\alpha\beta(\beta^2 + 3)(\beta^2 + 1),$

$$y = (\beta^2 + 2)\{\tfrac{1}{2}(\beta^2 + 3)(\beta^2 + 1) - 1\},$$

is a solution of $Nx^2 + 1 = y^2$.

It can be proved easily that these values of x, y are integral. For, if β is even, $\beta^2 + 2$ is also even. Therefore, the above values of x, y are integral. If on the contrary β is odd, β^2 is also odd; then $\beta^2 + 1$ and $\beta^2 + 3$ are even. Hence in this case also the above values are integral.

[1] *BrSpSi*, xviii. 68.

Putting $p = \alpha\beta$, $q = \beta^2 + 2$ we can write the above solution in the form

$$x = \tfrac{1}{2}p(q^2 - 1),$$
$$y = \tfrac{1}{2}q(q^2 - 3),$$

in which it was found by Euler.

17. CYCLIC METHOD

Cyclic Method. It has been just shown that the most fundamental step in Brahmagupta's method for the general solution in positive integers of the equation

$$Nx^2 + 1 = y^2,$$

where N is a non-square integer, is to form an auxiliary equation of the kind

$$Na^2 + k = b^2,$$

where a, b are positive integers and $k = \pm 1$, ± 2 or ± 4. For, from that auxiliary equation, by the Principle of Composition, applied repeatedly whenever necessary, one can derive, as shown above, one positive integral solution of the original Square-nature. And thence, again by means of the same principle, an infinite number of other solutions in integers can be obtained. How to form an auxiliary equation of this type was a problem which could not be solved completely and satisfactorily by Brahmagupta. In fact, he could not do it otherwise than by trial. But Bhâskara II succeeded in evolving a very simple and elegant method by means of which one can derive an auxiliary equation having the required interpolator ± 1, ± 2 or ± 4, simultaneously with its two integral roots, from another auxiliary equation empirically formed with any simple integral value of the interpolator, positive or negative. This method is called by the technical name *Cakravâla* or the "Cyclic Method."[1]

The purpose of the Cyclic Method has been defined by Bhâskara II thus: "By this method, there will appear two integral roots corresponding to an equation with ± 1, ± 2 or ± 4 as interpolator."[2]

Bhâskara's Lemma. The Cyclic Method of Bhâskara II is based upon the following Lemma:

If $$Na^2 + k = b^2,$$

where a, b, k are integers, k being positive or negative, then

$$N\left(\frac{am + b}{k}\right)^2 + \frac{m^2 - N}{k} = \left(\frac{bm + Na}{k}\right)^2,$$

where m is an arbitrary whole number.

The *rationale* of this Lemma is simple: We have

$$Na^2 + k = b^2,$$

and $$N.1^2 + (m^2 - N) = m^2, \text{ identically.}$$

Then by Brahmagupta's Lemma, we get

$$N(am + b)^2 + k(m^2 - N) = (bm + Na)^2.$$

$$\therefore \quad N\left(\frac{am + b}{k}\right)^2 + \frac{m^2 - N}{k} = \left(\frac{bm + Na}{k}\right)^2.$$

Bhâskara's Rule. Bhâskara II (1150) says:

"Considering the lesser root, greater root and interpolator (of a Square-nature) as the dividend, addend and divisor (respectively of a pulveriser), the (indeterminate) multiplier of it should be so taken as will make the residue of the *prakṛti* diminished by the square of that multiplier or the latter minus the *prakṛti* (as the case

[1] The Sanskrit word *Cakravâla* means "circle," especially "horizon." The method is so called, observes Sûryadâsa, because it proceeds as in a circle, the same set of operations being applied again and again in a continuous round.

[2] *BBi*, p. 38.

may be) the least. That residue divided by the (original) interpolator is the interpolator (of a new Square-nature); it should be reversed in sign in case of subtraction from the *prakṛti*. The quotient corresponding to that value of the multiplier is the (new) lesser root; thence the greater root. The same process should be followed repeatedly putting aside (each time) the previous roots and the interpolator. This process is called *Cakravâla* (or the 'Cyclic Method').[1] By this method, there will appear two integral roots corresponding to an equation with ± 1, ± 2 or ± 4 as interpolator. In order to derive integral roots corresponding to an equation with the additive unity from those of the equation with the interpolator ± 2 or ± 4 the Principle of Composition (should be applied)."[2]

Suppose we have an equation of the form

$$Na^2 + k = b^2, \tag{1}$$

where a, b, k are simple integers, relatively prime, k being positive or negative. Then by Bhâskara's Lemma

$$N\left(\frac{am+b}{k}\right)^2 + \frac{m^2-N}{k} = \left(\frac{bm+Na}{k}\right)^2, \tag{2}$$

where m is an arbitrary integral number. In the above rule, m has been styled the indeterminate multiplier. Now, by means of the pulveriser, its value is determined so that

$$\frac{am+b}{k} \text{ is a whole number.}$$

[1] The original text is *cakravâlamidaṁ jaguḥ*. The commentator Kṛṣṇa explains, "âcâryâ etadgaṇitaṁ cakravâlamiti jaguḥ" or "The learned professors call this method of calculation the *Cakravâla*." So Bhâskara II appears to have taken the Cyclic Method from earlier writers. But it is not found in any work anterior to him so far known.

[2] *BBi*, pp. 36ff.

Again, of the various such values, Bhâskara II chooses that one which will make $|m^2 - N|$ as small as possible. Let that value of m be n. Now let

$$a_1 = \frac{an + b}{k},$$

$$b_1 = \frac{bn + Na}{k},$$

$$k_1 = \frac{n^2 - N}{k}.$$

The numbers a_1, b_1, k_1 are all integral. The equation (2) then becomes

$$Na_1^2 + k_1 = b_1^2. \qquad (3)$$

Proceeding exactly in the same way, we can obtain from (3) a new equation of the same kind,

$$Na_2^2 + k_2 = b_2^2,$$

where again a_2, b_2, k_2 are whole numbers. By repeating the process, we shall ultimately arrive at an equation, states Bhâskara II, in which the interpolator k will reach the value ± 1, ± 2 or ± 4, and in which (a, b) will be integers.

Nârâyaṇa's Rule. The above rule of Bhâskara II has been reproduced by Nârâyaṇa (1350). He writes :

"Making the lesser root, greater root and interpolator (of a Square-nature) the dividend, addend and divisor (respectively of a pulveriser), the (indeterminate) multiplier of it should be determined in the way described before. The *prakṛti* being subtracted from the square of that or the square of the multiplier being subtracted from the *prakṛti*, the remainder divided by the (original) interpolator is the interpolator (of a new Square-nature); and it will be reversed in sign in case of subtraction of the square of the multiplier. The quotient (corresponding to that value of the multiplier) is the lesser root (of the new Square-nature); and that multiplied by the multiplier and diminished by the product of the previous lesser root and (new) interpolator will be its greater root. By doing so repeatedly will be obtained two integral roots corresponding to the interpolator ± 1, ± 2 or ± 4. In order to derive integral roots for the additive unity from those answering to the interpolator ± 2 or ± 4, the Principle of Composition (should be adopted)."[1]

It will be noticed that Nârâyaṇa does not expressly state that the value of the indeterminate multiplier m should be so chosen as will make $|m^2 - N|$ least. It is perhaps particularly noteworthy that he recognised the relation

$$b_1 = a_1 n - k_1 a.$$

For
$$b_1 = \frac{bn + Na}{k},$$

$$= \frac{n(a_1 k - an) + Na}{k}, \quad [\because \quad a_1 k = an + b]$$

$$= a_1 n - \left(\frac{n^2 - N}{k}\right) a,$$

$$= a_1 n - k_1 a,$$

$$\therefore \quad a = \frac{a_1 n - b_1}{k_1}.$$

Similarly, it will be found that

$$b = \frac{b_1 n - N a_1}{k_1}.$$

For
$$b_1 n = a_1 n^2 - k_1 a n,$$

[1] *NBi*, I, R. 79-82.

$$= a_1 (N + kk_1) - k_1 an,$$
$$[\because \quad kk_1 = n^2 - N]$$
$$= a_1 N + k_1 b, \qquad [\because \quad a_1 k = an + b]$$
$$\therefore \qquad b = \frac{b_1 n - N a_1}{k_1}.$$

Illustrative Examples. In illustration of the Cyclic Method, Bhâskara II works out in detail the following examples :

"What is that number whose square multiplied by 67 or 61 and then added by unity becomes capable of yielding a square-root ? Tell me, O friend, if you have a thorough knowledge of the method of the Square-nature."[1]

That is to say, we are to solve

$$(i) \quad 67x^2 + 1 = y^2,$$
$$(ii) \quad 61x^2 + 1 = y^2.$$

Leaving out the details of the operations in connection with the process of the pulveriser, Bhâskara's solutions are substantially as follows :

$$(i) \qquad 67x^2 + 1 = y^2.$$

We take the auxiliary equation

$$67 . 1^2 - 3 = 8^2.$$

Then, by the Lemma,

$$67\left(\frac{1 . m + 8}{-3}\right) + \frac{m^2 - 67}{-3} = \left(\frac{8m + 67 . 1}{-3}\right)^2. \quad (1)$$

By the method of the *Kuṭṭaka* the solution of

$$\frac{m + 8}{-3} = \text{an integer},$$

[1] *BBi*, p. 38.

It is remarkable that the equation $61x^2 + 1 = y^2$ was proposed by Fermat to Frénicle in a letter of February, 1657. Euler solved it in 1732.

is $m = -3t + 1$. Putting $t = -2$, we get $m = 7$ which makes $|m^2 - 67|$ least. On substituting this value, the equation (1) reduces to

$$67.5^2 + 6 = 41^2.$$

Again, by the Lemma, we have

$$67\left(\frac{5n + 41}{6}\right)^2 + \frac{n^2 - 67}{6} = \left(\frac{41n + 67.5}{6}\right)^2. \quad (2)$$

The solution of

$$\frac{5n + 41}{6} = \text{a whole number},$$

is $n = 6t + 5$. $|n^2 - 67|$ will be least for the value $t = 0$, that is, when $n = 5$. The equation (2) then becomes

$$67.11^2 - 7 = 90^2.$$

Now, we form

$$67\left(\frac{11p + 90}{-7}\right)^2 + \frac{p^2 - 67}{-7} = \left(\frac{90p + 67.11}{-7}\right)^2. \quad (3)$$

The solution of

$$\frac{11p + 90}{-7} = \text{an integral number},$$

is $p = -7t + 2$. Taking $t = -1$, we have $p = 9$; and this value makes $|p^2 - 67|$ least. Substituting that in (3) we get

$$67.27^2 - 2 = 221^2.$$

By the Principle of Composition of Equals, we get from this equation

$$67\,(2.27.221)^2 + 4 = (221^2 + 67.27^2)^2,$$

or $$67(11934)^2 + 4 = (97684)^2.$$

Dividing out by 4, we have

$$67\,(5967)^2 + 1 = (48842)^2.$$

Hence $x = 5967$, $y = 48842$ is a solution of (*i*).

$$(ii) \qquad 61x^2 + 1 = y^2.$$

Here we start with the auxiliary equation

$$61 . 1^2 + 3 = 8^2.$$

By the Lemma, we have

$$61\left(\frac{m+8}{3}\right)^2 + \frac{m^2-61}{3} = \left(\frac{8m+61}{3}\right)^2. \qquad (1)$$

Now the solution of

$$\frac{m+8}{3} = \text{an integer},$$

is $m = 3t + 1$. Putting $t = 2$, we get the value $m = 7$ which makes $| m^2 - 61 |$ least. On substituting this value in (1), it becomes

$$61 . 5^2 - 4 = 39^2.$$

Dividing out by 4, we get

$$61(\tfrac{5}{2})^2 - 1 = (\tfrac{39}{2})^2. \qquad (2)$$

By the Principle of Composition of Equals, we have

$$61(2 . \tfrac{5}{2} . \tfrac{39}{2})^2 + 1 = \{(\tfrac{39}{2})^2 + 61(\tfrac{5}{2})^2\}^2,$$

or

$$61(\tfrac{195}{2})^2 + 1 = (\tfrac{1523}{2})^2. \qquad (3)$$

Combining (2) and (3),

$$61(3805)^2 - 1 = (29718)^2.$$

Composing this with itself, we get

$$61(226153980)^2 + 1 = (1766319049)^2.$$

Hence $x = 226153980$, $y = 1766319049$ is a solution of (*ii*).

The following two examples have been cited by Nârâyaṇa :

$$(iii) \quad 103x^2 + 1 = y^2,$$

$$(iv) \quad 97x^2 + 1 = y^2.$$

Their solutions are given substantially as follows:
For (*iii*) we have the auxiliary equation

$$103 \cdot 1^2 - 3 = 10^2.$$

By the Lemma, we get

$$103\left(\frac{m+10}{-3}\right)^2 + \frac{m^2 - 103}{-3} = \left(\frac{10m + 103}{-3}\right)^2.$$

The general solution of

$$\frac{m+10}{-3} = \text{an integer},$$

is $m = -3t + 2$. Putting $t = -3$, we get $m = 11$. Then

$$103 \cdot 7^2 - 6 = 71^2.$$

Again, by the Lemma,

$$103\left(\frac{7n+71}{-6}\right)^2 + \frac{n^2 - 103}{-6} = \left(\frac{71n + 103 \cdot 7}{-6}\right)^2.$$

The solution of

$$\frac{7n+71}{-6} = \text{a whole number},$$

is $n = -6t + 1$. Taking $t = -1$, we get

$$103 \cdot 20^2 + 9 = 203^2.$$

Next, we have

$$103\left(\frac{20p + 203}{9}\right)^2 + \frac{p^2 - 103}{9} = \left(\frac{203p + 103 \cdot 20}{9}\right)^2.$$

Now, $$\frac{20p + 203}{9} = \text{an integral number}$$

for $p = 9t + 2$. When $t = 1$, $p = 11$. On taking this value we find

$$103 \cdot 47^2 + 2 = 477^2.$$

Applying the Principle of Composition of Equals, we get

$$103(2.47.477)^2 + 4 = (477^2 + 103.47^2)^2,$$

or
$$103(44838)^2 + 4 = (455056)^2.$$

Hence
$$103(22419)^2 + 1 = (227528)^2,$$

which gives $x = 22419$, $y = 227528$ as a solution of (*iii*).

For the solution of (*iv*) the auxiliary equation is

$$97.1^2 + 3 = 10^2.$$

Therefore

$$97\left(\frac{m + 10}{3}\right)^2 + \frac{m^2 - 97}{3} = \left(\frac{10m + 97}{3}\right)^2.$$

The solution of

$$\frac{m + 10}{3} = \text{an integer},$$

is $m = 3t + 2$. Taking $t = 3$, we have $m = 11$. Then

$$97.7^2 + 8 = 69.^2$$

Next, we have

$$97\left(\frac{7n + 69}{8}\right)^2 + \frac{n^2 - 97}{8} = \left(\frac{69n + 97.7}{8}\right)^2.$$

The solution of

$$\frac{7n + 69}{8} = \text{an integer},$$

is $n = 8t + 5$. Taking $t = 1$, that is, $n = 13$, we get

$$97.20^2 + 9 = 197^2.$$

Whence

$$97\left(\frac{20p + 197}{9}\right)^2 + \frac{p^2 - 97}{9} = \left(\frac{197p + 97.20}{9}\right)^2.$$

The solution of

$$\frac{20p + 197}{9} = \text{a whole number},$$

is $p = 9t + 5$. Putting $t = 1$, we get $p = 14$. With this value of p we have

$$97 \cdot 53^2 + 11 = 522^2.$$

Whence

$$97\left(\frac{53q + 522}{11}\right)^2 + \frac{q^2 - 97}{11} = \left(\frac{522q + 97 \cdot 53}{11}\right)^2.$$

The solution of

$$\frac{53q + 522}{11} = \text{an integer},$$

is $q = 11t + 8$. The appropriate value of q is given by $t = 0$. So, taking $q = 8$, we have

$$97 \cdot 86^2 - 3 = 847^2.$$

Next, we find

$$97\left(\frac{86r + 847}{-3}\right) + \frac{r^2 - 97}{-3} = \left(\frac{847r + 97 \cdot 86}{-3}\right)^2.$$

The solution of

$$\frac{86r + 847}{-3} = \text{a whole number},$$

is $r = 3t + 1$. Putting $t = -3$, we get $r = 10$. Taking this value, we have

$$97 \cdot 569^2 - 1 = 5604^2.$$

By the Principle of Composition of Equals, we find

$$97(6377352)^2 + 1 = (62809633)^2.$$

Hence $x = 6377352$, $y = 62809633$ is a solution of (*iv*).

Proofs. It has been stated by Bhâskara II that:

(1) when a_1 is an integer, k_1 and b_1 are each a whole number;

(2) his Cyclic Method will in every case lead to the desired result.

He has not adduced proofs. We presume that he knew a proof at least of the first proposition. For he must have recognised the simple relation

$$b_1 = a_1 n - k_1 a,$$

which has been expressly stated by Nârâyaṇa (1350). This shows at once that b_1 will be a whole number, if k_1 is so. This is also evident from the equation, $Na_1^2 + k_1 = b_1^2$, itself. Hence, it now remains to prove that k_1 is an integral number.

Eliminating b between

$$a_1 = \frac{an + b}{k},$$

and

$$b_1 = \frac{bn + Na}{k},$$

we have

$$k(a_1 n - b_1) = a(n^2 - N),$$

or

$$\frac{k}{a}(a_1 n - b_1) = n^2 - N.$$

Therefore

$$\frac{k}{a}(a_1 n - b_1) \text{ is an integer.}$$

Since k and a have no common factor, a must divide $a_1 n - b_1$; that is

$$\frac{a_1 n - b_1}{a} = \frac{n^2 - N}{k} = k_1 = \text{an integer.}$$

Hence b_1 also is a whole number.[1]

[1] *Hankel's Proof*: Hankel proves these two results thus:

Since $a_1 k = an + b$ and $k = b^2 - Na^2$,

we get

$$a_1(b^2 - Na^2) = an + b,$$

or

$$\frac{b}{a}(a_1 b - 1) = (n + Naa_1).$$

Since a, b have no common factor, a must divide $a_1 b - 1$; that is,

$$\frac{a_1 b - 1}{a} = \text{an integer.}$$

18. SOLUTION OF $Nx^2 \pm c = y^2$

The general solution of the indeterminate quadratic equation

$$Nx^2 \pm c = y^2$$

in positive integers was first given by Brahmagupta (628). He says:

"From two roots (of a Square-nature) with any given additive or subtractive, by making (combination) with the roots for the additive unity, other first and second roots (of the equation having) the given additive or subtractive (can be found)."[1]

Eliminating n between

$$a_1 k = an + b, \quad b_1 k = bn + Na,$$

we get $\quad a_1 b - ab_1 = 1.$

Hence $\quad b_1 = \dfrac{a_1 b - 1}{a} = \text{a whole number.}$

Now
$$n^2 - N = \frac{(a_1 k - b)^2 - Na^2}{a^2}$$
$$= \frac{a_1^2 k^2 - 2bka_1 + k}{a^2}$$
$$= \frac{k(a_1^2 k - 2ba_1 + 1)}{a^2}.$$

Therefore $\quad \dfrac{k}{a^2}(a_1^2 k - 2ba_1 + 1)$ is a whole number.

Since a, k have no common factor, it follows that

$$\frac{a_1^2 k - 2ba_1 + 1}{a^2} = \frac{n^2 - N}{k} = k_1 = \text{an integer.}$$

Also
$$k_1 = \frac{n^2 - N}{k} = \frac{a_1^2 k - 2ba_1 + 1}{a^2}$$
$$= \frac{a_1^2(b^2 - Na^2) - 2ba_1 + 1}{a^2}$$
$$= \left(\frac{a_1 b - 1}{a}\right)^2 - Na_1^2.$$

[1] *BrSpSi*, xviii, 66.

Thus having known a single solution in positive integers of the equation $Nx^2 \pm c = y^2$, says Brahmagupta, an infinite number of other integral solutions can be obtained by making use of the integral solutions of $Nx^2 + 1 = y^2$. If (p, q) be a solution of the former equation found empirically and if (α, β) be an integral solution of the latter then, by the Principle of Composition,

$$x = p\beta \pm q\alpha, \quad y = q\beta \pm Np\alpha$$

will be a solution of the former. Repeating the operations we can easily deduce as many solutions as we like.

This method reappears in later Hindu algebras. Bhâskara II says :

"In (a Square-nature) with the additive or subtractive greater (than unity), one should find two roots by his own intelligence only ; then by their composition with the roots obtained for the additive unity an infinite number of roots (will be found)."[1]

Nârâyaṇa writes similarly :

"When the additive or subtractive is greater than unity, two roots should be determined by one's own intelligence. Then, by combining them with the roots for the additive unity, an infinite number of roots can be obtained."[2]

We take the following illustrative examples with solutions from Nârâyaṇa :

Example. "Tell me that square which being multiplied by 13 and then increased or diminished by 17 or 8 becomes capable of yielding a square root."[3]

[1] *BBi*, p. 42. [2] *NBi*, I, R. 86.
[3] *NBi*, I, *Ex.* 44.

That is, solve

$$(1) \quad 13x^2 \pm 17 = y^2,$$
$$(2) \quad 13x^2 \pm 8 = y^2.$$

Solution. "In the first example it is stated that the multiplier = 13 and interpolator = 17.

"Now the roots for the interpolator 3 are (1, 4). And for the interpolator 51, the roots are (1, 8). For the composition of these with the previous roots (1, 4) the statement will be

$$\begin{array}{llll} m = 13 & l = 1 & g = 8 & i = 51 \\ & l = 1 & g = 4 & i = 3 \end{array}$$

So, by the Addition Lemma, we get the roots corresponding to the interpolator 153 as (12, 45). The rule says, 'If the interpolator (of a Square-nature) be divided by the square of an optional number etc.' Now take the optional number to be 3, so that the interpolator may be reduced to 17. For $3^2 = 9$ and $153/9 = 17$. Therefore, dividing the roots just obtained by the optional number 3, we get the required roots (4, 15).

"Applying the Subtraction Lemma and proceeding similarly we get the roots for the interpolator 17 as (4/3, 19/3).

"In the second example the statement is: multiplier = 13, interpolator = − 17. Proceeding as before we get (by the Addition Lemma) the roots (147, 530); and (by the Subtraction Lemma), the roots (3, 10)."[1]

Form $Mn^2x^2 \pm c = y^2$. Brahmagupta says:

"If the multiplier is that divided by a square, the first root is that divided by its root."[2]

[1] Our MS. does not contain the solution of the equations $13x^2 \pm 8 = y^2$.

[2] *BrSpSi*, xviii. 70.

That is to say, suppose the equation to be

$$Mn^2x^2 \pm c = y^2, \tag{1}$$

so that the multiplier (*i.e.*, coefficient of x^2) is divisible by n^2. Putting $nx = u$, we get

$$Mu^2 \pm c = y^2. \tag{2}$$

Then clearly the first root of (1) is equal to the first root of (2) divided by n. The corresponding second root will be the same for both the equations.

The same rule is taught by Bhâskara II[1] and Nârâyaṇa. The latter says:

"Divide the multiplier (of a Square-nature) by an arbitrary square number so that there is left no remainder. Take the quotient as the multiplier (of another Square-nature). The lesser root (of the reduced equation) divided by the square-root of the divisor will be the lesser root (of the original equation)."[2]

Form $a^2x^2 \pm c = y^2$. For the solution of a Square-nature of this particular form, Brahmagupta gives the following rule:

"If the multiplier be a square, the interpolator divided by an optional number and then increased and decreased by it, is halved. The former (of these results) is the second root; and the other divided by the square-root of the multiplier is the first root."[3]

Thus, it is stated that

$$x = \frac{1}{2a}\left(\frac{\pm c}{m} - m\right),$$

$$y = \frac{1}{2}\left(\frac{\pm c}{m} + m\right),$$

[1] *BBi*, p. 42. [2] *NBi*, I, R. 84.
[3] *BrSpSi*, xviii. 69.

where m is an arbitrary number, is a solution of the equation

$$a^2x^2 \pm c = y^2.$$

The same solution has been given by Bhâskara II and Nârâyaṇa.[1] Bhâskara's rule runs as follows:

"The interpolator divided by an optional number is set down at two places; the quotient is diminished (at one place) and increased (at the other) by that optional number and then halved. The former is again divided by the square-root of the multiplier. (The quotients) are respectively the lesser and greater roots."[2]

The *rationale* of the above solution has been given by the commentators Sûryadâsa and Kṛṣṇa substantially as follows:

$$\pm c = y^2 - a^2x^2$$
$$= (y - ax)(y + ax).$$

Assume $y - ax = m$, m being an arbitrary rational number. Then

$$y + ax = \frac{\pm c}{m}.$$

Whence by the rule of concurrence, we get

$$x = \frac{1}{2a}\left(\frac{\pm c}{m} - m\right),$$

$$y = \tfrac{1}{2}\left(\frac{\pm c}{m} + m\right).$$

Form $\mathbf{c - Nx^2 = y^2}$. Though the equation of the form $c - Nx^2 = y^2$ has not been considered by any Hindu algebraist as deserving of special treatment, it occurs incidentally in examples. For instance, Bhâskara II has proposed the following problem:

[1] *NBi*, I, R. 85. [2] *BBi*, p. 42.

"What is that square which being multiplied by -5 becomes, together with 21, a square? Tell me, if you know, the method (of solving the Square-nature) when the multiplier is negative."[1]

Thus it is required to solve

$$-5x^2 + 21 = y^2. \tag{1}$$

Nârâyaṇa has a similar example, *viz.*,[2]

$$-11x^2 + 60 = y^2. \tag{2}$$

Two obvious solutions of (1) are (1, 4) and (2, 1). Composing them with the roots of

$$-5x^2 + 1 = y^2,$$

says Bhâskara II, an infinite number of roots of (1) can be derived.

Form $Nx^2 - k^2 = y^2$. Bhâskara II observes:

"When unity is the subtractive the solution of the problem is impossible unless the multiplier is the sum of two squares."[3]

Nârâyaṇa writes:

"In the case of unity as the subtractive, the multiplier must be the sum of two squares. Otherwise, the solution is impossible."[4]

Thus it has been said that a rational solution of

$$Nx^2 - 1 = y^2,$$

and consequently of

$$Nx^2 - k^2 = y^2$$

is not possible unless N is the sum of two squares.

[1] *BBi*, p. 43. [2] *NBi*, I, *Ex*. 43.

[3] "Rûpaśuddhau khiloddiṣṭaṁ vargayogo guṇo na cet"—*BBi*, p. 40.

[4] *NBi*, I, R. 83.

For, if $x = p/q$, $y = r/s$ be a possible solution of the equation, we have

$$N(p/q)^2 - k^2 = (r/s)^2,$$

or
$$N = (qr/ps)^2 + (qk/p)^2.$$

Bhâskara II then goes on :

"In case (the solution is) not impossible when unity is the subtractive, divide unity by the roots of the two squares and set down (the quotients) at two places. They are two lesser roots. Then find the corresponding greater roots at the two places. Or, when unity is the subtractive, the roots should be found as before."

Thus, according to Bhâskara II, two rational solutions of

$$Nx^2 - 1 = y^2,$$

where $N = m^2 + n^2$, will be

$$\left.\begin{array}{l} x = \dfrac{1}{m} \\ y = \dfrac{n}{m} \end{array}\right\}, \qquad \left.\begin{array}{l} x = \dfrac{1}{n} \\ y = \dfrac{m}{n} \end{array}\right\}.$$

So two rational solutions of

$$(m^2 + n^2)\, x^2 - k^2 = y^2,$$

will be

$$\left.\begin{array}{l} x = \dfrac{k}{m} \\ y = \dfrac{kn}{m} \end{array}\right\}, \qquad \left.\begin{array}{l} x = \dfrac{k}{n} \\ y = \dfrac{km}{n} \end{array}\right\}.$$

The following illustrative example of Bhâskara II[1] is also reproduced by Nârâyaṇa :[2]

$$13x^2 - 1 = y^2.$$

[1] *BBi*, p. 41. [2] *NBi*, I, *Ex.* 58.

The former solves it substantially in the following ways :

(1) Since $13 = 2^2 + 3^2$ two rational solutions are $(1/2, 3/2)$ and $(1/3, 2/3)$.

(2) An obvious solution of

$$13x^2 - 4 = y^2$$

is $x = 1$, $y = 3$. Then dividing out by 4, as shown before, we get a solution of the equation $13x^2 - 1 = y^2$ as $(1/2, 3/2)$.

(3) Again, since an obvious solution of

$$13x^2 - 9 = y^2$$

is $x = 1, y = 2$, we get, on dividing out by 9, a solution of our equation as $(1/3, 2/3)$.

(4) From these fractional roots, we may derive *integral roots* by the Cyclic Method. Since

$$13(\tfrac{1}{2})^2 - 1 = (\tfrac{3}{2})^2,$$

we have, by Bhâskara's Lemma, m being an indeterminate multiplier,

$$13\left(\frac{m/2 + 3/2}{-1}\right)^2 + \frac{m^2 - 13}{-1} = \left(\frac{3m/2 + 13/2}{-1}\right)^2,$$

or $$13\left(\frac{m + 3}{-2}\right)^2 + \frac{m^2 - 13}{-1} = \left(\frac{3m + 13}{-2}\right)^2.$$

The suitable value of m which will make $(m+3)/2$ an integer and $|m^2 - 13|$ minimum is 3. So that we have

$$13 . 3^2 + 4 = 11^2.$$

From this again we get the relation

$$13\left(\frac{3n + 11}{4}\right)^2 + \frac{n^2 - 13}{4} = \left(\frac{11n + 13.3}{4}\right)^2.$$

The appropriate value of the indeterminate multiplier in this case is $n = 3$. Substituting this value, we have

$$13 . 5^2 - 1 = 18^2.$$

Hence an integral solution of our equation $13x^2 - 1 = y^2$ is $(5, 18)$.

"In all cases like this an infinite number of roots can be derived by composition with the roots for the additive unity."[1]

Nârâyaṇa states the methods (2) and (3) only.

19. GENERAL INDETERMINATE EQUATIONS OF THE SECOND DEGREE: SINGLE EQUATIONS

The earliest mention of the solution of the general indeterminate equation of the second degree is found in the *Bîjagaṇita* of Bhâskara II (1150). But there are good grounds to believe that he was not its first discoverer, for he is found to have taken from certain ancient authors a few illustrative examples the solutions of which presuppose a knowledge of the solution of such equations.[2] Neither those illustrations nor a treatment of equations of those types occurs in the algebra of Brahmagupta or in any other extant work anterior to Bhâskara II.

Bhâskara II distinguishes two kinds of indeterminate equations: *Sakṛt samîkaraṇa* (Single Equations) and *Asakṛt samîkaraṇa* (Multiple Equations).[3]

Solution. For the solution of the general indeterminate equation of the second degree, Bhâskara II (1150) lays down the following rule:

[1] "Iha sarvatra padânâṁ rûpakṣepapadâbhyâṁ bhâvanayâ'-nantyam"—*BBi*, p. 41.

[2] *Vide infra*, pp. 267f.

[3] *BBi*, pp. 106, 110.

"When the square, etc., of the unknown are present (in an equation), after the equi-clearance has been made, (find) the square-root of one side by the method described before for it, and the root of the other side by the method of the Square-nature. Then (apply) the method of (simple) equations to these roots. If (the other side) does not become a case for the Square-nature, then, putting it equal to the square of another unknown, the other side and so the value of the other (*i.e.*, the new) unknown should be obtained in the same way as in the Square-nature; and similarly the value of the first unknown. The intelligent should devise various artifices so that it may become a matter for (the application of) the Square-nature."[1]

He has further elucidated the rule thus:

"When, after the clearance of the two sides has been made, there remain the square, etc., of the unknown, then, by multiplying the two sides with a suitable number and by the help of other necessary operations as described before, the square-root of one side should be extracted. If there be present on the other side the square of the unknown with an absolute term, then the two roots of that side should be found by the method of the Square-nature. There the number associated with the square of the unknown is the *prakṛti* ('multiplier'), and the absolute number is to be considered as the interpolator. What is obtained as the lesser root in this way will be the value of the unknown associated with the multiplier (*prakṛti*); the greater root is (again) the root of that square (formed on the first side). Hence making an equation of this with the square-root of the first side, the value of the unknown on the first side should be determined.

[1] *BBi*, p. 99.

"But if there be present on the second side the square of the unknown together with (the first power of) the unknown, or only the (simple) unknown with or without an absolute number, then it is not a case for the Square-nature. How then is the root to be found in that case? So it has been said: 'If (the other side) does not become a case for the Square-nature etc.' Then, putting it equal to the square of another unknown, the square-root of one side should be found in the way indicated before, and the two roots of the other side should then be determined by the method of the Square-nature. There again the lesser root is the value of the unknown associated with the *prakṛti* and the greater root is equal to the square-root of that side of the equation. Forming proper equations with the roots, the values of the unknowns should be determined.

"If, however, even after the second side has been so treated, it does not turn out to be a case for the Square-nature, then the intelligent (mathematicians) should devise by their own sagacity all such artifices as will make it a case for the method of the Square-nature and then determine the values of the unknowns."[1]

Having thus indicated in a general way the broad outlines of his method for the solution of the general indeterminate equation of the second degree, Bhâskara II discusses the different types of equations severally, explaining the rules in every case in greater detail with the help of illustrative examples.

(*i*) *Solution of* $ax^2 + bx + c = y^2$

For the general solution of the quadratic indeterminate equation

$$ax^2 + bx + c = y^2, \qquad (1)$$

[1] *BBi*, p. 100.

Bhâskara II gives the following particular rule:

"On taking the square-root of one side, if there be on the second side only the square of the unknown together with an absolute number, in such cases, the greater and lesser roots should be determined by the method of the Square-nature. Of these two, the greater root is to be put equal to the square-root of the first side mentioned before, and thence the value of the first unknown should be determined. The lesser will be the value of the unknown associated with the *prakṛti*. In this way, the method of the Square-nature should be applied to this case by the intelligent."[1]

As an illustration of this rule Bhâskara II works out in detail the following example:

"What number being doubled and added to six times its square, becomes capable of yielding a square-root? O ye algebraist, tell it quickly."[2]

Solution. "Here let the number be x. Doubled and together with six times its square, it becomes $6x^2 + 2x$. This is a square. On forming an equation with the square of y, the statement is

$$6x^2 + 2x + 0y^2 = 0x^2 + 0x + y^2.$$

On making equi-clearance in this the two sides are $6x^2 + 2x$ and y^2.

"Then multiplying these two sides by 6 and superadding 1, the root of the first side, as described before, is $6x + 1$.

"Now on the second side of the equation remains $6y^2 + 1$. By the method of the Square-nature, its roots are: the lesser 2 and the greater 5, or the lesser 20 and the greater 49. Equating the greater root with the square-root of the first side, *viz.*, $6x + 1$, the value of

[1] *BBi*, pp. 100-1.

[2] *BBi*, p. 101.

x is found to be $2/3$ or 8. The lesser root, 2 or 20, is the value of y, the unknown associated with the *prakṛti*. In this way, by virtue of (the multiplicity of) the lesser and greater roots, many solutions can be obtained."[1]

In other words the method described above is this :

Completing the square on the left-hand side of the equation $ax^2 + bx + c = y^2$, we have

$$(ax + \tfrac{1}{2}b)^2 = ay^2 + \tfrac{1}{4}(b^2 - 4ac).$$

Putting $z = ax + \frac{1}{2}b$, $k = \frac{1}{4}(b^2 - 4ac)$, we get

$$ay^2 + k = z^2. \tag{1.1}$$

If $y = l$, $z = m$ be found empirically to be a solution of this equation, another solution of it will be

$$y = lq \pm mp,$$
$$z = mq \pm alp,$$

where $ap^2 + 1 = q^2$. Hence a solution of (1) is

$$x = -\frac{b}{2a} + \frac{1}{a}(mq \pm alp),$$
$$y = lq \pm mp.$$

Now suppose $x = r$, when $z = m$; that is, let $m = ar + b/2$. Substituting in the above expressions, we get the required solution of (1) as

$$\left.\begin{aligned} x &= \frac{1}{2a}(bq - b) + qr \pm lp, \\ y &= lq \pm (apr + \tfrac{1}{2}bp); \end{aligned}\right\} \tag{1.2}$$

where $ap^2 + 1 = q^2$ and $ar^2 + br + c = l^2$.

Thus having known *one* solution of $ax^2 + bx + c = y^2$, an infinite number of other solutions can be

[1] *BBi*, p. 101.

easily obtained by the method of Bhâskara II. The method is, indeed, a very simple and elegant one. It has been adopted by later Hindu algebraists. As the relevant portion of the algebra of Nârâyaṇa (1350) is now lost, we cannot reproduce his description of the method. Jñânarâja (1503) says:

"(Find) the square-root of the first side according to the method described before and, by the method of the Square-nature, the roots of the other side, where the coefficient of the square of the unknown is considered to be the *prakṛti* and the interpolator is an absolute term. Then the greater root will be equal to the previous square-root and the other (*i.e.*, the lesser root) to the unknown associated with the *prakṛti*."

The above solution (1.2), but with the upper sign only, was rediscovered in 1733 by Eluer.[1] His method is indirect and cumbrous. Lagrange's (1767) method begins in the same way as that of Bhâskara II. by completing the square on the left-hand side of the equation.[2]

(*ii*) *Solution of* $ax^2 + bx + c = a'y^2 + b'y + c'$

Bhâskara II has treated the more general type of quadratic indeterminate equations:

$$ax^2 + bx + c = a'y^2 + b'y + c'. \tag{2}$$

His rule in this connection runs as follows:

"If there be the square of the unknown together with the (simple) unknown and an absolute number, putting it equal to the square of another unknown its root (should be investigated). Then on the other side (find)

[1] Leonard Euler, *Opera Mathematica*, vol. II, 1915, pp. 6-17; Compare also pp. 576-611.

[2] Additions to *Elements of Algebra by Leonard Euler*, translated into English by John Hewlett, 5th edition, London, 1840, pp. 537ff.

the roots by the method of the Square-nature, as has been stated before. Put the lesser root[1] equal to the root of the first side and the greater root equal to that of the second."[2]

He further elucidates the rule thus:

"In this case, on taking the square-root of the first side, there remain on the other side the square of the unknown and the (simple) unknown with or without an absolute number. In that case forming an equation of the second side with the square of another unknown, the roots (should be found). Of these (roots just determined), making the lesser equal to the root of the first side (of the given equation) and the greater to the root of the second side, the values of the unknowns should be determined."

Example. "Say what is the number of terms of a series (in A. P.) whose first term is 3, the common difference is 2; but whose sum multiplied by 3 is equal to the sum of a different number of terms."[3]

Solution. "Here the statements of the series are: first term $= 3$, common difference $= 2$, number of terms $= x$; first term $= 3$, common difference $= 2$, number of terms $= y$. The two sums are (respectively) $x^2 + 2x$, $y^2 + 2y$. Making three times the first equal to the second, the statement for clearance is

$$3x^2 + 6x = y^2 + 2y.$$

After the clearance, multiplying the two sides (of the equation) by 3 and superadding 9, the square-root of the first side is $3x + 3$. On the second side of the

[1] The meaning of the terms 'lesser root', 'greater root', etc., as used here, will be clear from the illustration and the general solution given below.

[2] *BBi*, p. 104. [3] *BBi*, p. 104.

equation stands $3y^2 + 6y + 9$. Forming an equation of this with z^2, and similarly multiplying the sides by 3 and superadding -18, the root of it is $3y + 3$. Then the roots of the other side, $3z^2 - 18$, by the method of the Square-nature are the lesser $= 9$ and greater $= 15$, or the lesser $= 13$ and greater $= 57$. Equating the lesser root with the square-root of the first side, namely, $3x + 3$, and the greater root with the square-root of the second side, namely, $3y + 3$, the values of x, y are found to be $(2, 4)$ or $(10, 18)$. So in every case."

In general, on completing the square on the left-hand side, equation (2) becomes

$$(ax + \tfrac{1}{2}b)^2 = aa'y^2 + ab'y + ac' + (\tfrac{1}{4}b^2 - ac).$$

Put
$$ax + \tfrac{1}{2}b = z, \tag{2.1}$$

and then complete the square on the right-hand side. Thus the given equation is finally reduced to

$$aa'z^2 - \beta = w^2, \tag{2.2}$$

where
$$w = aa'y + \tfrac{1}{2}ab', \tag{2.3}$$

and
$$\beta = a^2a'c' + (\tfrac{1}{4}b^2 - ac)\,aa' - (\tfrac{1}{2}ab')^2.$$

Now, if $z = l$, $w = m$ be a solution of the equation (2.2), another solution will be

$$z = lq \pm mp,$$
$$w = mq \pm aa'lp\,;$$

where $a'ap^2 + 1 = q^2$. Substituting in (2.1) and (2.3), we get

$$\left.\begin{aligned} x &= -\frac{b}{2a} + \frac{1}{a}(lq \pm mp), \\ y &= -\frac{b'}{2a'} + \frac{1}{aa'}(mq \pm aa'lp). \end{aligned}\right\} \tag{2.4}$$

Now, let $l = ar + \frac{1}{2}b$ and $m = aa's + \frac{1}{2}ab'$. Substituting in the above expressions, we get the required solution of (2) in the form:

$$\left.\begin{aligned} x &= \frac{1}{2a}(qb \pm pab' - b) + qr \pm pa's, \\ y &= \frac{1}{2a'}(qb' \pm pa'b - b') + qs \pm par; \end{aligned}\right\} \qquad (2.5)$$

where $$aa'p^2 + 1 = q^2,$$

and $$ar^2 + br + c = a's^2 + b's + c'.$$

The form (2.5) shows that having found empirically *one* solution of $ax^2 + bx + c = a'y^2 + b'y + c'$ Bhâskara could find an infinite number of other solutions of it.

Jñânarâja (1503) says:

"If on the other side be present the square as well as the linear power of the unknown together with an absolute term, put it equal to the square of another unknown and then determine the lesser and greater roots. The lesser root will be equal to the first square-root and the greater to the second square-root."

He gives with solution the following illustrative example:

$$3(x^2 + 4x) = y^2 + 4y,$$

or $$(3x + 6)^2 = 3y^2 + 12y + 36.$$

Putting $3x + 6 = z$, where z is the "first square-root" of Jñânarâja, we get

$$z^2 = 3y^2 + 12y^2 + 36,$$

or $$3z^2 = (3y + 6)^2 + 72.$$

Now put $3y + 6 = w$, where w is the "second square-root." Then

$$3z^2 - 72 = w^2.$$

Therefore, by the method of the Square-nature, $z = 18$, $w = 30$. Whence $x = 4$, $y = 8$, is a solution

(iii) *Solution of* $ax^2 + by^2 + c = z^2$

Bhâskara II followed several devices for the solution of the equation

$$ax^2 + by^2 + c = z^2. \tag{3}$$

In every case his object was to transform the equation into the form of the Square-nature. He says :

"In such cases, where squares of two unknowns with (or without) an absolute number are present, supposing either of them optionally as the *prakṛti*, the rest (of the terms) should be considered as the interpolator. Then the roots should be investigated in the way described before. If there be more equations than one (the process will be especially helpful)."[1]

He then explains further :

"Where on finding the square root of the first side, there remain on the other side squares of two unknowns with or without an absolute number, there consider the square of one of the unknowns as the *prakṛti* ; the remainder will then be the interpolator. Then by the rule: 'An optionally chosen number is taken as the lesser root, etc.,'[2] the unknown in the interpolator multiplied by one, etc., and added with one, etc., or not, according to one's own sagacity, should be assumed for the lesser root ; then determine the greater root."[3]

There are thus indicated two artifices for solving the equation (3). They are :

(*i*) Set $x = my$; so that equation (3) transforms into

[1] *BBi*, pp. 105f.

[2] The reference is to the rule for solving the Square-nature (*vide supra* p. 144) (*BBi*, p. 33).

[3] *BBi*, p. 106.

$$z^2 = (am^2 + b)y^2 + c$$
$$= \alpha y^2 + c,$$

where $\alpha = am^2 + b$. Hence the required solution of $ax^2 + by^2 + c = z^2$ is

$$x = my = m(rq \pm ps),$$
$$y = rq \pm ps,$$
$$z = sq \pm \alpha pr;$$

where $s^2 = \alpha r^2 + c$ and $q^2 = \alpha p^2 + 1$.

(*ii*) Set $x = my \pm n$; then the equation reduces to

$$z^2 = \alpha y^2 \pm 2amny + \gamma$$

where $\alpha = am^2 + b$ and $\gamma = an^2 + c$.

Completing the square on the right-hand side of this, we get

$$\alpha z^2 - \beta = w^2,$$

where $w = \alpha y \pm amn$ and $\beta = \gamma\alpha - a^2m^2n^2 = a(bn^2 + cm^2) + bc$.

If $z = s$, $w = r$ be a solution of this equation, another solution will be

$$z = sq \pm rp,$$
$$w = rq \pm \alpha sp;$$

where $q^2 = \alpha p^2 + 1$. Hence the solution of $ax^2 + by^2 + c = z^2$ is

$$x = \frac{m}{\alpha}(rq \pm \alpha sp \mp amn) \pm n,$$

$$y = \frac{1}{\alpha}(rq \pm \alpha sp \mp amn),$$

$$z = sq \pm rp;$$

where $q^2 = \alpha p^2 + 1$, $r^2 = \alpha s^2 - \beta$, $\alpha = am^2 + b$ and $\beta = a(bn^2 + cm^2) + bc$.

In working certain problems, Bhâskara II is found to have occasionally followed other artifices also for the solution of the equation (3). For instance:

(*iii*)[1] Set $w^2 = by^2 + c$. Then equation (3) becomes

$$z^2 - w^2 = ax^2.$$

Whence $$z = \tfrac{1}{2}\left(\frac{a}{m} + m\right)x,$$

and $$w = \tfrac{1}{2}\left(\frac{a}{m} - m\right)x;$$

where m is an arbitrary number. Therefore

$$x = \frac{2mw}{a - m^2},$$

$$z = \left(\frac{a + m^2}{a - m^2}\right)w.$$

Now, if $y = l$, $w = r$ be a solution of

$$w^2 = by^2 + c,$$

another solution of it will be

$$y = lq \pm pr,$$
$$w = rq \pm blp;$$

where $ap^2 + 1 = q^2$. Therefore, the solution of (3) will be

$$x = \frac{2m}{a - m^2}(rq \pm blp),$$

$$y = lq \pm pr,$$

$$z = \frac{a + m^2}{a - m^2}(rq \pm blp);$$

where $ap^2 + 1 = q^2$ and $bl^2 + c = r^2$.

(*iv*)[2] Suppose $c = 0$; then the equation to be solved will be

[1] See *BBi*, p. 108. [2] *BBi*, p. 106.

$$ax^2 + by^2 = z^2.$$

In this case set $x = uy$, $z = vy$; so that u, v will be given by

$$au^2 + b = v^2,$$

which can be solved by the method of the Square-nature. Some of these devices were followed also by later Hindu algebraists, for instance, Jñânarâja (1503) and Kamalâkara (1658).[1]

Example from Kamalâkara:[2]

$$7x^2 + 8y^2 = z^2.$$

This is one of a double equation by Bhâskara II.[3]

To solve $ax^2 + by^2 + c = z^2$, Kamalâkara observes:

"In this case, suppose the coefficient of the square of the first unknown as the *prakṛti* and the coefficient of the square of the other unknown together with the absolute number as the interpolator to that. The two roots can thus be determined in several ways."[4]

And again:

"(Suppose) the coefficient of the square of one of the unknowns as the *prakṛti* and the rest comprising two terms, the square of an unknown and an absolute number, as the interpolator. Then assume the value of the lesser root to be equal to the other unknown together with an absolute term."[5]

He seems to have indicated also a slightly different method:

"Or assume the value of the lesser root to be equal to another unknown plus or minus an absolute number

[1] *SiTVi*, xiii. 260-1.
[2] *SiTVi*, xiii. 258.
[3] *BBi*, p. 106.
[4] *SiTVi*, xiii. 264.
[5] *SiTVi*, xiii. 267 f.

and similarly also the value of the greater root. The remaining operations should be performed by the intelligent in the way described by Bhâskara in his algebra."[1]

That is to say, assume

$$x = mw \pm \alpha, \quad z = nw \pm \beta.$$

Substituting in the equation $ax^2 + by^2 + c = z^2$, we get

$$(am^2 - n^2)w^2 \pm 2w\,(am\alpha \mp n\beta) + by^2 + (c + a\alpha^2 - \beta^2) = 0.$$

Putting $\lambda = am^2 - n^2$, $\mu = am\alpha \mp n\beta$, $\nu = c + a\alpha^2 - \beta^2$, this equation can be reduced to

$$-\lambda by^2 + (\mu^2 - \nu\lambda) = u^2,$$

where $u = \lambda w \pm \mu$.

Kamalâkara gives also some other methods which are applicable only in particular cases.

Case i. Suppose that b and c are of different signs.[2] Two sub-cases arise:

(1) *Form* $ax^2 + by^2 - c = z^2$.

First find u, v, says Kamalâkara, such that

$$au^2 - c = v^2.$$

Assuming
$$x = \sqrt{\frac{b}{ac}}\,vy + u,$$
we have

$$ax^2 + by^2 - c = \frac{b}{c}v^2y^2 + 2\sqrt{\frac{ab}{c}}\,uvy + by^2 + (au^2 - c)$$

$$= \frac{b}{c}(au^2 - c)y^2 + 2\sqrt{\frac{ab}{c}}\,uvy + by^2 + v^2$$

[1] *SiTVi*, xiii. 265. [2] *SiTVi*, xiii. 285-7.

$$= \frac{ab}{c} u^2y^2 + 2\sqrt{\frac{ab}{c}}\, uvy + v^2$$

$$= \left(\sqrt{\frac{ab}{c}}\, uy + v\right)^2.$$

Hence $$z = \sqrt{\frac{ab}{c}}\, uy + v.$$

The following illustrative example and its solution are given :[1]

$$5x^2 + 16y^2 - 20 = z^2.$$

Its solution is

$$x = \tfrac{2}{5}vy + u,$$
$$z = 2uy + v;$$

where $5u^2 - 20 = v^2$. An obvious solution of this equation is given by $u = 3$, $v = 5$. Hence, we get a solution of the given equation as

$$x = 2y + 3,$$
$$z = 6y + 5.$$

Therefore $(x, y) = (5, 1), (7, 2), (9, 3), \ldots\ldots$

(2) *Form* $ax^2 - by^2 + c = z^2$.

In this case first solve

$$au'^2 + c = v'^2.$$

Then the required solution is

$$x = \sqrt{\frac{b}{ac}}\, v'y + u',$$

$$z = \sqrt{\frac{ab}{c}}\, u'y + v'.$$

Example from Kamalâkara :[2]

$$5x^2 - 20y^2 + 16 = z^2.$$

[1] *SiTVi*, xiii. 279.   [2] *SiTVi*, xiii. 279.

Then

$$x = \tfrac{1}{2}v'y + u',$$
$$z = \tfrac{5}{2}u'y + v';$$

where $5u'^2 + 16 = v'^2$. One solution of this equation is $u' = 2$, $v' = 6$. The corresponding solution of the given equation is

$$x = 3y + 2,$$
$$z = 5y + 6.$$

Therefore $(x, y) = (5, 1), (8, 2), (11, 3)$, etc.

Case ii. Let the two terms of the interpolator be of the same sign and positive.

Example from Kamalâkara:[1]

$$5x^2 + 8y^2 + 23 = z^2.$$

Assume arbitrarily a value of x or y and then find the other by the method of the Square-nature.[2]

(*iv*) *Solution of* $a^2x^2 + by^2 + c = z^2$

Let the coefficient of x^2 (or y^2) be a square number. The equation is of the form

$$a^2x^2 + by^2 + c = z^2.$$

For this case Bhâskara II observes :

"If the *prakṛti* is a square, then obtain the roots by the rule: 'The interpolator divided by an optional number is set down at two places, etc.' "[3]

Thus, according to Bhâskara II, the solution of the above equation is

$$x = \frac{1}{2a}\left(\frac{by^2 + c}{m}\right) - m,$$

[1] *SiTVi*, xiii. 296.
[2] *SiTVi*, xiii. 298.
[3] *BBi*, p. 106.

$$z = \tfrac{1}{2}\left(\frac{by^2 + c}{m} + m\right);$$

where m is an arbitrary number.

Kamalâkara divides equations of this form into two classes according as c is or is not a square.[1]

(1) Let c be a square ($= d^2$, say). That is to say, we have to solve

$$a^2x^2 + by^2 + d^2 = z^2.$$

The solution of this particular case, says Kamalâkara, is given by

$$x = \frac{b}{2ad}y^2.$$

For, with this value, we have

$$z^2 = \frac{b^2y^4}{4d^2} + by^2 + d^2$$

$$= \frac{1}{4d^2}(by^2 + 2d^2)^2.$$

Hence $$z = \frac{by^2}{2d} + d.$$

(2) When c is not a square, Kamalâkara first finds α, β such that

$$\alpha^2 + c = \beta^2.$$

He next obtains n such that the value of

$$\frac{a\beta n - b/2}{a^2 . \alpha/a}$$

is also n; and then says that

$$x = ny^2 + \frac{\alpha}{a};$$

[1] *Vide* his gloss on *SiṪVi*, xiii. 275.

whence will follow the value of z.

Since $$\frac{a\beta n - b/2}{a\alpha} = n,$$

we get $$n = \frac{b}{2a(\beta - \alpha)};$$

so that $$x = \frac{by^2}{2a(\beta - \alpha)} + \frac{\alpha}{a}.$$

Therefore $$z^2 = a^2x^2 + by^2 + c$$

$$= \frac{b^2y^4}{4(\beta - \alpha)^2} + \alpha^2 + \frac{\alpha by^2}{(\beta - \alpha)} + by^2 + c$$

$$= \frac{b^2y^4}{4(\beta - \alpha)^2} + \frac{\beta by^2}{(\beta - \alpha)} + \beta^2$$

$$= \left\{\frac{by^2}{2(\beta - \alpha)} + \beta\right\}^2.$$

Hence $$z = \frac{by^2}{2(\beta - \alpha)} + \beta.$$

Example from Kamalâkara :

$$4x^2 + 48y^2 + 20 = z^2.$$

Since $4^2 + 20 = 6^2$, and the solution of

$$\frac{12n - 24}{8} = n$$

is $n = 6$, we get the required solution of the given equation as

$$x = 6y^2 + 2,$$
$$z = 12y^2 + 6.$$

It may be noted that the solution stated by Kamalâkara follows easily from that of Bhâskara II, on putting therein $m = \beta - \alpha$, where $\alpha^2 + c = \beta^2$.

In particular, if we put $c = 0$ and $a = 1$, Bhâskara's solution reduces to

$$x = \tfrac{1}{2}\left(\frac{b}{m}y^2 - m\right),$$

$$z = \tfrac{1}{2}\left(\frac{b}{m}y^2 + m\right);$$

where m is arbitrary.

If $b = \alpha\beta$, taking $m = 2\beta p^2$, we easily arrive at

$$x = \alpha q^2 - \beta p^2,$$
$$y = 2pq,$$
$$z = \alpha q^2 + \beta p^2,$$

where p, q are arbitrary integers, as the solution in positive integers of $z^2 = x^2 + by^2$. This solution was given by A. Desboves (1879).[1]

Taking $m = 2\nu^2$, we can derive Matsunago's (*c.* 1735) solution of $z^2 = x^2 + by^2$, *viz.*,

$$x = b\mu^2 - \nu^2,$$
$$y = 2\mu\nu,$$
$$z = b\mu^2 + \nu^2;$$

where μ, ν are arbitrary integers.

(*v*) *Solution of* $ax^2 + bxy + cy^2 = z^2$

For the solution of the equation

$$ax^2 + bxy + cy^2 = z^2 \qquad (5)$$

Bhâskara II lays down the following rule:

"When there are squares of two unknowns together with their product, having extracted the square-root of one part, it should be put equal to half the difference of the remaining part divided by an optional number

[1] Dickson, *Numbers*, II, p. 405.

and the optional number."[1]

It has been again elucidated thus:

"Where there exists also the product of the unknowns (in addition to their squares), by the rule, 'when there are squares etc.,' the square-root of as much portion of it as affords a root, should be extracted. The remaining portion, divided by an optional number and then diminished by that optional number and halved, should be put equal to that square-root."[2]

The above rule, in fact, contemplates a particular case of equation (5) in which a or c is a square number.

(*i*) Suppose $a = p^2$. The equation to be solved is then

$$p^2x^2 + bxy + cy^2 = z^2. \tag{5.1}$$

Therefore $$\left(px + \frac{by}{2p}\right)^2 + y^2\left(c - \frac{b^2}{4p^2}\right) = z^2.$$

Putting $px + \dfrac{by}{2p} = w$, we get

$$z^2 - w^2 = y^2\left(c - \frac{b^2}{4p^2}\right).$$

Whence $$z - w = \lambda,$$

$$z + w = \frac{y^2}{\lambda}\left(c - \frac{b^2}{4p^2}\right),$$

where λ is an arbitrary rational number. So

$$w = \tfrac{1}{2}\left\{\frac{y^2}{\lambda}\left(c - \frac{b^2}{4p^2}\right) - \lambda\right\},$$

as stated in the rule. Therefore,

$$x = \frac{1}{2p}\left\{\frac{y^2}{\lambda}\left(c - \frac{b^2}{4p^2}\right) - \lambda\right\} - \frac{by}{2p^2},$$

[1] Y. Mikami, *The Development of Mathematics in China and Japan*, Leipzig, 1913, p. 231.

[2] *BBi*, p. 106.

and $$z = \tfrac{1}{2}\left\{\frac{y^2}{\lambda}\left(c - \frac{b^2}{4p^2}\right) + \lambda\right\}.$$

Now, if we suppose $y = m/n$, where m, n are arbitrary integers, we get the solution of (5.1) as

$$x = \frac{1}{8\lambda p^2 n^2}\{m^2(4cp^2 - b^2) - 4\lambda^2 p^2 n^2 - 4\lambda bmn\},$$

$$y = \frac{m}{n},$$

$$z = \frac{1}{8\lambda p^2 n^2}\{m^2(4cp^2 - b^2) + 4\lambda^2 p^2 n^2\}.$$

Since the given equation is homogeneous, any multiple of these values of x, y, z will also be its solution. Therefore, multiplying by $8\lambda p^2 n^2$, we get the following solution of the equation $p^2x^2 + bxy + cy^2 = z^2$ in integers :

$$\left.\begin{aligned} x &= m^2(4cp^2 - b^2) - 4\lambda^2 p^2 n^2 - 4\lambda bmn, \\ y &= 8\lambda mnp^2, \\ z &= m^2(4cp^2 - b^2) + 4\lambda^2 p^2 n^2, \end{aligned}\right\} \quad (5.2)$$

where m, n, are arbitrary integers.

In particular, putting $a = b = c = 1$, and $\lambda = p = 1$ in (5.2), we get

$$x = 3m^2 - 4n(n + m),$$
$$y = 8mn,$$
$$z = 3m^2 + 4n^2,$$

as the solution of the equation

$$x^2 + xy + y^2 = z^2.$$

Dividing out by $8n$, the above solution can be put into the form

$$x = \tfrac{1}{2}\left(\frac{3m^2}{4n} - n - m\right),$$

$$y = m,$$

$$z = \frac{1}{8n}(3m^2 + 4n^2);$$

as has been stated by Nârâyaṇa:

"An arbitrary number is the first. Its square less by its (square's) one-fourth, is divided by an optional number and then diminished by the latter and also by the first. Half the remainder is the second number. The sum of their squares together with their product is a square."[1]

It is noteworthy that in practice Nârâyaṇa approves of only integral solutions of his equation. For instance, he says:

"'Any arbitrary number is the first.' Suppose it to be 12. Then with the optional number unity, are obtained the numbers (12, 95/2). For *integral values*, they are doubled (24, 95). With the optional number 2, are obtained (12, 20). It being possible, these are reduced by the common factor 4 to (3, 5). In this way, owing to the varieties of the optional number, an infinite number of solutions can be obtained."[2]

(*ii*) If neither a nor c be a square, the solution can be obtained thus:

Multiplying both sides of the equation (5) by a and then completing a square on the left-hand side, the equation transforms into

$$(ax + \tfrac{1}{2}by)^2 + (ac - \tfrac{1}{4}b^2)y^2 = az^2.$$

Putting $ax + \frac{1}{2}by = w$ and $\beta = \frac{1}{4}(b^2 - 4ac)$,

we get $$w^2 = az^2 + \beta y^2. \qquad (5\cdot3)$$

[1] *GK*, i. 55.

[2] See the example in illustration of the same.

The method of the solution of an equation of this form, according to Bhâskara II, has been described before.

Assume $w = vy$, $z = uy$; so that the values of u, v will be given by

$$v^2 = au^2 + \beta. \tag{5.4}$$

If $u = m$, $v = n$ be a solution of (5.4), another solution will be

$$u = mq \pm pn,$$
$$v = nq \pm amp;$$

where $ap^2 + 1 = q^2$. Therefore, a solution of (5) is

$$x = \frac{y}{2a}\{2(nq \pm amp) - b\},$$
$$z = y(mq \pm pn);$$

where $ap^2 + 1 = q^2$ and $am^2 + \beta = n^2$.

Put $n = ar + \frac{1}{2}b$ and $y = \frac{s}{t}$; then we have

$$x = \frac{s}{2at}\{q(2ar + b) \pm 2amp - b\},$$
$$y = \frac{s}{t},$$
$$z = \frac{s}{2t}\{2mq \pm p(2ar + b)\}.$$

Multiplying by $2at$, we get the following solution of $ax^2 + bxy + cy^2 = z^2$ in integers:

$$x = s\{q(2ar + b) \pm 2amp - b\},$$
$$y = 2as,$$
$$z = as\{2mq \pm p(2ar + b)\};$$

where $ap^2 + 1 = q^2$ and $m^2 = ar^2 + br + c$.

20. RATIONAL TRIANGLES

Rational Right Triangles: Early Solutions. The earliest Hindu solutions of the equation

$$x^2 + y^2 = z^2 \qquad (1)$$

are found in the *Śulba*. Baudhâyana (*c.* 800 B.C.), Âpastamba and Kâtyâyana (*c.* 500 B.C.)[1] give a method for the transformation of a rectangle into a square, which is the equivalent of the algebraical identity

$$mn = \left(m - \frac{m-n}{2}\right)^2 - \left(\frac{m-n}{2}\right)^2,$$

where m, n are any two arbitrary numbers. Thus we get

$$(\sqrt{mn})^2 + \left(\frac{m-n}{2}\right)^2 = \left(\frac{m+n}{2}\right)^2.$$

Substituting p^2, q^2 for m, n respectively, in order to eliminate the irrational quantities, we get

$$p^2q^2 + \left(\frac{p^2 - q^2}{2}\right)^2 = \left(\frac{p^2 + q^2}{2}\right)^2,$$

which gives a rational solution of (1).

For finding a square equal to the sum of a number of other squares of the same size, Kâtyâyana gives a very elegant and simple method which furnishes us with another solution of the rational right triangle. Kâtyâyana says:

"As many squares (of equal size) as you wish to combine into one, the transverse line will be (equal to) one less than that; twice a side will be (equal to) one more than that; (thus) form (an isosceles) triangle. Its arrow (*i.e.*, altitude) will do that."[2]

[1] *BŚl*, i. 58; *ĀpŚl*, ii. 7; *KŚl*, iii. 2. For details of the construction see Datta, *Śulba*, pp. 83f, 178f.

[2] *KŚl*, vi. 5; Compare also its *Pariśiṣṭa*, verses 40-1.

Thus for combining n squares of sides a each, we form the isosceles triangle ABC, such that $AB = AC = (n+1)a/2$,

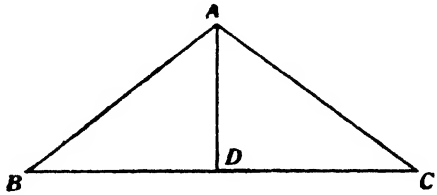

Fig. 2

and $BC = (n-1)a$. Then $AD^2 = na^2$. This gives the formula

$$a^2(\sqrt{n})^2 + a^2\left(\frac{n-1}{2}\right)^2 = a^2\left(\frac{n+1}{2}\right)^2.$$

Putting m^2 for n in order to make the sides of the right-angled triangle free from the radical, we have

$$m^2a^2 + \left(\frac{m^2-1}{2}\right)^2 a^2 = \left(\frac{m^2+1}{2}\right)a^2,$$

which gives a rational solution of (1).

Tacit assumption of the following further generalisation is met with in certain constructions described by Āpastamba :[1]

If the sides of a rational right triangle be increased by any rational multiple of them, the resulting figure will be a right triangle.

In particular, he notes

$$3^2 + 4^2 = 5^2,$$
$$(3 + 3\cdot3)^2 + (4 + 4\cdot3)^2 = (5 + 5\cdot3)^2,$$
$$(3 + 3\cdot4)^2 + (4 + 4\cdot4)^2 = (5 + 5\cdot4)^2;$$

[1] *ĀpŚl*, v. 3, 4. Also compare Datta, *Śulba*, pp. 65f

$$5^2 + 12^2 = 13^2,$$
$$(5 + 5.2)^2 + (12 + 12.2)^2 = (13 + 13.2)^2.$$

Âpastamba also derives from a known right-angled triangle several others by changing the unit of measure of its sides and vice versa.[1] In other words, he recognised the principle that if (α, β, γ) be a rational solution of $x^2 + y^2 = z^2$, then other rational solutions of it will be given by $(l\alpha, l\beta, l\gamma)$, where l is any rational number. This is clearly in evidence in the formula of Kâtyâyana in which a is any quantity. It is now known that all rational solutions of $x^2 + y^2 = z^2$ can be obtained without duplication in this way.

Later Rational Solutions. Brahmagupta (628) says:

"The square of the optional (*iṣṭa*) side is divided and then diminished by an optional number; half the result is the upright, and that increased by the optional number gives the hypotenuse of a rectangle."[2]

In other words, if m, n be any two rational numbers, then the sides of a right triangle will be

$$m, \ \tfrac{1}{2}\left(\frac{m^2}{n} - n\right), \ \tfrac{1}{2}\left(\frac{m^2}{n} + n\right).$$

The Sanskrit word *iṣṭa* can be interpreted as implying "given" as well as "optional". With the former meaning the rule will state how to find rational right triangles having a given leg. Such is, in fact, the interpretation which has been given to a similar rule of Bhâskara II.[3]

[1] Datta, *Sulba*, p. 179. [2] *BrSpSi*, xii. 35.

[3] *Vide infra* p. 211; H. T. Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bhascara*, London, 1817, (referred to hereafter as, Colebrooke, *Hindu Algebra*), p. 61 footnote.

A similar rule is given by Śrîpati (1039):

"Any optional number is the side; the square of that divided and then diminished by an optional number and halved is the upright; that added with the previous divisor is the hypotenuse of a right-angled triangle. For, so it has been explained by the learned in the matter of the rules of geometry."[1]

Karavindasvamî a commentator of the *Âpastamba Sulba*, finds the solution

$$m, \left(\frac{n^2+2n}{2n+2}\right)m, \left(\frac{n^2+2n+2}{2n+2}\right)m,$$

by generalising a rule of the *Sulba*.[2]

Integral Solutions. Brahmagupta was the first to give a solution of the equation $x^2+y^2=z^2$ in integers. It is

$$m^2-n^2,\ 2mn,\ m^2+n^2,$$

m, n being any two unequal integers.[3]

Mahâvîra (850) says :

"The difference of the squares (of two elements) is the upright, twice their product is the base and the sum of their squares is the diagonal of a *generated* rectangle."[4]

He has re-stated it thus :

"The product of the sum and difference of the elements is the upright. The *saṅkramaṇa*[5] of their squares gives the base and the diagonal. In the operation of generating (geometrical figures), this is the process."[6]

[1] *SiŚe*, xiii. 41.
[2] *ÂpŚl*, 1.2 (*Com.*); also see Datta, *Śulba*, pp. 14-16.
[3] *BrSpSi*, xii. 33 ; *vide infra*, p. 222. [4] *GSS*, vii. 90½.
[5] For the definition of this term see pp. 43f.
[6] *GSS*, vii. 93½.

Bhâskara II (1150) writes:

"Twice the product of two optional numbers is the upright; the difference of their squares is the side; and the sum of their squares is the hypotenuse. (Each of these quantities is) rational (and integral)."[1]

It has been stated before that the early Hindus recognised that fresh rational right triangles can be derived from a known one by multiplying or dividing its sides by any rational number. The same principle has been used by Mahâvîra and Bhâskara II in their treatment of the solution of rational triangles and quadrilaterals. Gaṇeśa (1545) expressly states:

"If the upright, base and hypotenuse of a rational right-angled triangle be multiplied by any arbitrary rational number, there will be produced another right-angled triangle with rational sides."

Hence the most general solution of $x^2 + y^2 = z^2$ in integers is

$$(m^2 - n^2)l,\ 2mnl,\ (m^2 + n^2)l$$

where m, n, l are integral numbers.

Mahâvîra's Definitions. A triangle or a quadrilateral whose sides, altitudes and other dimensions can be expressed in terms of *rational* numbers is called *janya* (meaning generated, formed or that which is generated or formed) by Mahâvîra.[2] Numbers which

[1] *L*, p. 36.

[2] *GSS*, introductory line to vii. 90½. The section of Mahâvîra's work devoted to the treatment of rational triangles and quadrilaterals bears the sub-title *janya-vyavahâra* (*janya* operation) and it begins as "Hereafter we shall give out the *janya* operations in calculations relating to measurement of areas." Mahâvîra's treatment of the subject has been explained fully by Bibhutibhusan Datta in a paper entitled: "On Mahâvîra's solution of rational triangles and quadrilaterals," *BCMS*, XX, 1928-9, pp. 267-294.

are employed in forming a particular figure are called its *bîja-saṁkhyâ* (element-numbers) or simply *bîja* (element or seed). For instance, Mahâvîra has said: "Forming O friend! the generated figure from the *bîja* 2, 3,"[1] "forming another from half the base of the figure (rectangle) from the *bîja* 2, 3,"[2] etc. Thus, according to Mahâvîra, "forming a rectangle from the *bîja* m, n" means taking a rectangle with the upright, base and diagonal as $m^2 - n^2$, $2mn$, $m^2 + n^2$ respectively. It is noteworthy that Mahâvîra's mode of expression in this respect very closely resembles that of Diophantus who also says, "Forming now a right-angled triangle from 7, 4," meaning "taking a right-angled triangle with sides $7^2 - 4^2$, $2.7.4$, $7^2 + 4^2$ or 33, 56, 65."[3] It should also be noted that Mahâvîra never speaks of "right-angled triangle." What Diophantus called "forming a right-angled triangle from m, n," Mahâvîra calls "forming a longish quadrilateral or rectangle from m, n."

Right Triangles having a Given Side. In the *Śulba* we find an attempt to find rational right triangles having a given side, that is, rational solutions of

$$x^2 + a^2 = z^2.$$

In particular, we find mention of two such right triangles having a common side a, *viz.*, (a, $3a/4$, $5a/4$) and (a, $5a/12$, $13a/12$).[4] The principle underlying these solutions will be easily detected to be that of the reduction of the sides of any rational right triangle in the ratio of the given side to its corresponding

[1] "Bîje dve trîṇi sakhe kṣetre janye tu saṁsthâpya"—*GSS*, vii. 92½.

[2] "He dvitribîjakasya kṣetrabhujârdhena cânyamutthâpya"—*GSS*, vii. 111½.

[3] *Arithmetica*, Book III, 19; T. L. Heath, *Diophantus of Alexandria*, p. 167.

[4] Datta, *Śulba*, p. 180.

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side. This principle of finding rational right triangles having a given side has been followed explicitly by Mahâvîra (850).[1]

It has been stated before that one rule of Brahmagupta[2] can be interpreted as giving rational solutions of $x^2 + a^2 = z^2$ as

$$a, \ \tfrac{1}{2}\left(\frac{a^2}{n} - n\right), \ \tfrac{1}{2}\left(\frac{a^2}{n} + n\right),$$

where n is any rational number. In fact, he has used this solution in finding rational isosceles triangles having a given altitude.[3] This solution has been expressly stated by Mahâvîra (850). He says:

"The *saṅkramaṇa* between any optional divisor of the square of the given upright or the base and the (respective) quotient gives the diagonal and the base (or upright)."[4]

He has restated the solution thus:

"The *saṅkramaṇa* between any (rational) divisor of the upright and the quotient gives the elements; or any (rational) divisor of half the side and the quotient are the elements."[5]

The right triangles formed according to the first half of this rule are:[6]

$$a, \ \tfrac{1}{2}\left(\frac{a^2}{p^2} - p^2\right), \ \tfrac{1}{2}\left(\frac{a^2}{p^2} + p^2\right),$$

[1] *Vide infra*, p. 213
[2] *Vide supra*, p. 206.
[3] *Vide infra*, p. 223
[4] *GSS*, vii. 97½.
[5] *GSS*, vii. 95½.
[6] The "elements" here are $\frac{1}{2}(a/p + p)$, $\frac{1}{2}(a/p - p)$, where p is an optional number.

and those according to the second half are :[1]

$$\frac{a^2}{4q^2} - q^2, \ a, \ \frac{a^2}{4q^2} + q^2.$$

Bhâskara II gives two solutions one of which is the same as that of Brahmagupta. He says :

"The side is given : from that multiplied by twice an optional number and divided by the square of that optional number minus unity, is obtained the upright; this again multiplied by the optional number and diminished by the given side becomes the hypotenuse. This triangle is a right-angled triangle.

"Or the side is given : its square divided by an optional number is put down at two places ; the optional number is subtracted (at one place) and added (at another) and then halved ; these results are the upright and the hypotenuse. Similarly from the given upright can be obtained the side and the hypotenuse."[2]

That is to say, the two solutions are

$$a, \ \frac{2na}{n^2 - 1}, \ n\left(\frac{2na}{n^2 - 1}\right) - a,$$

and $$a, \ \tfrac{1}{2}\left(\frac{a^2}{n} - n\right), \ \tfrac{1}{2}\left(\frac{a^2}{n} + n\right).$$

Bhâskara II illustrates this by finding four right triangles having a side equal to 12, *viz.*, (12, 35, 37), (12, 16, 20), (12, 9, 15) and (12, 5, 13).[3]

The *rationale* of the first solution has been given by Śûryadâsa (1538) thus : Starting with the rational right triangle $n^2 - 1$, $2n$, $n^2 + 1$, he observes that if x, y, z

[1] The "elements" here are q, $a/2q$, where q is an optional number.

[2] *L*, p. 34.

[3] *L*, pp. 34f.

be the corresponding sides of another right triangle, then

$$\frac{x}{n^2-1} = \frac{y}{2n} = \frac{z}{n^2+1} = k \text{ (say)}.$$

Hence $\quad x = k(n^2-1),\ y = 2nk,\ z = k(n^2+1).$

Therefore $\quad x + z = 2kn^2 = ny.$

If now we have $x = a$, then

$$k = \frac{a}{n^2-1}.$$

Hence $$y = \frac{2na}{n^2-1},$$

and $$z = ny - a = n\left(\frac{2na}{n^2-1}\right) - a.$$

The second rule has been demonstrated by Sûryadâsa, Gaṇeśa and Raṅganâtha thus:

Since $\quad x^2 + a^2 = z^2,$

we have $\quad a^2 = z^2 - x^2 = (z-x)(z+x).$

Assume $z - x = n$, where n is any rational number; then

$$z + x = \frac{a^2}{n}.$$

$$\therefore \quad z = \tfrac{1}{2}\left(\frac{a^2}{n} + n\right),\ x = \tfrac{1}{2}\left(\frac{a^2}{n} - n\right).$$

Generalising the method of the *Âpastamba Sulba* the commentators obtained the solution[1]

$$a,\ \left(\frac{m^2+2m}{2m+2}\right)a,\ \left(\frac{m^2+2m+2}{2m+2}\right)a.$$

[1] Datta, *Śulba*, p. 16.

Right Triangles having a Given Hypotenuse. For finding all rational right triangles having a given hypotenuse (c), that is, for rational solutions of

$$x^2 + y^2 = c^2,$$

Mahâvîra gives three rules. The first rule is :

"The square-root of half the sum and difference of the diagonal and the square of an optional number are they (the elements)."[1]

In other words, the required solution will be obtained from the "elements" $\sqrt{(c + p^2)/2}$ and $\sqrt{(c - p^2)/2}$, where p is any rational number. Hence the solution is

$$p^2, \sqrt{c^2 - p^4}, \ c.$$

The second rule is :

"Or the square-root of the difference of the squares of the diagonal and of an optional number, and that optional number are the upright and the base."[2]

That is, the solution is

$$p, \sqrt{c^2 - p^2}, \ c.$$

These solutions are defective in the sense that $\sqrt{c^2 - p^4}$ or $\sqrt{c^2 - p^2}$ might not be rational unless p is suitably chosen. Mahâvîra's third rule is of greater importance. He says :

"Each of the various figures (rectangles) that can be formed from the elements are put down ; by its diagonal is divided the given diagonal. The perpendicular, base and the diagonal (of this figure) multiplied by this quotient (give rise to the corresponding sides of the figure having the given hypotenuse)."[3]

[1] *GSS*, vii. 95½.
[2] *GSS*, vii. 97½.
[3] *GSS*, vii. 122½.

Thus having obtained the general solution of the rational right triangle, *viz.*, $m^2 - n^2$, $2mn$, $m^2 + n^2$, Mahâvîra reduces it in the ratio $c/(m^2 + n^2)$, so that all rational right triangles having a given hypotenuse c will be given by

$$\left(\frac{m^2 - n^2}{m^2 + n^2}\right)c,\ \left(\frac{2mn}{m^2 + n^2}\right)c,\ c.$$

By way of illustration Mahâvîra finds four rectangles (39, 52), (25, 60), (33, 56) and (16, 63) having the same diagonal 65.[1]

This method was later on rediscovered in Europe by Leonardo Fibonacci of Pisa (1202) and Vieta. It has been pointed out before that the origin of the method can be traced to the *Sulba*.

Bhâskara II (1150) says:

"From the given hypotenuse multiplied by an optional number and doubled and then divided by the square of the optional number added to unity, is obtained the upright; this is again multiplied by the optional number; the difference between that (product) and the given hypotenuse is the side.

"Or divide twice the hypotenuse by the square of an optional number added to unity. The hypotenuse minus the quotient is the upright and the quotient multiplied by the optional number is the side."[2]

Thus, according to the above, the sides of a right-angled triangle whose hypotenuse is c are:

$$\frac{2mc}{m^2 + 1},\ m\left(\frac{2mc}{m^2 + 1}\right) - c,\ c;$$

or

$$\frac{2mc}{m^2 + 1},\ c - \frac{2c}{m^2 + 1},\ c.$$

[1] *GSS*, vii. 123-124½. [2] *L*, pp. 35, 36.

By way of illustration Bhâskara II finds two right triangles (51, 68) and (40, 75) having the same hypotenuse 85.[1]

Sûryadâsa demonstrates the above substantially thus :

If (x, y, z) be the sides of the right triangle, we have

$$\frac{x}{m^2 - 1} = \frac{y}{2m} = \frac{z}{m^2 + 1} = k \text{ (say)},$$

where m is any rational integer. Then

$$x = k(m^2 - 1), \; y = 2mk, \; z = k(m^2 + 1).$$

Therefore $$x + z = 2km^2 = my.$$

Since z is given to be equal to c, we have

$$k = \frac{c}{m^2 + 1}.$$

Hence $$y = \frac{2mc}{m^2 + 1},$$

and $$x = my - z = m\left(\frac{2mc}{m^2 + 1}\right) - c.$$

Problems Involving Areas and Sides. Mahâvîra proposes to find rational rectangles (or squares) in which the area will be *numerically* (*saṁkhyayâ*) equal to any multiple or submultiple of a side, diagonal or perimeter, or of any linear combination of two or more of them. Expressed symbolically, the problem is to solve

$$\left.\begin{aligned} x^2 + y^2 &= z^2, \\ mx + ny + pz &= rxy; \end{aligned}\right\} \qquad (1)$$

m, n, p, r being any rational numbers $(r \neq 0)$. For the solution of this problem he gives the following rule :

[1] *L*, pp. 35f.

"Divide the sides (or their sum) of any generated square or other figure as multiplied by their respective given multiples by the area of that figure taken into its given multiple. The sides of that figure multiplied by this quotient will be the sides of the (required) square or other figure."[1]

That is to say, starting with any rational solution of

$$x'^2 + y'^2 = z'^2, \tag{2}$$

we shall have to calculate the value of

$$mx' + ny' + pz' = Q, \text{ say.} \tag{3}$$

Then the required solution of (1) will be obtained by reducing the values of x', y', z' in the ratio of $Q/rx'y'$. Thus

$$\left.\begin{aligned} x &= x'Q/rx'y' = Q/ry', \\ y &= y'Q/rx'y' = Q/rx', \\ z &= z'Q/rx'y'. \end{aligned}\right\} \tag{4}$$

Mahâvîra gives several illustrative examples some of which are very interesting:

"In a rectangle the area is (numerically) equal to the perimeter; in another rectangle the area is (numerically) equal to the diagonal. What are the sides (in each of these cases)?"[2]

Algebraically, we shall have to solve

$$\left.\begin{aligned} x^2 + y^2 &= z^2, \\ 2(x + y) &= xy, \end{aligned}\right\} \tag{1.1}$$

and

$$\left.\begin{aligned} x^2 + y^2 &= z^2, \\ xy &= z. \end{aligned}\right\} \tag{1.2}$$

Starting with the solution $s^2 - t^2$, $2st$, $s^2 + t^2$ of (2) and putting $m = n = 2$, $p = 0$, $r = 1$ in (4), we get

[1] *GSS*, vii. 112½.   [2] *GSS*, vii. 115½.

the solution of (1.1) as

$$\frac{2(s^2-t^2)+4st}{2st},\quad \frac{2(s^2-t^2)+4st}{s^2-t^2},$$
$$\left\{\frac{2(s^2-t^2)+4st}{2st\,(s^2-t^2)}\right\}(s^2+t^2).$$

And putting $m = n = 0$, $p = r = 1$, in (4), we have the solution of (1.2) as

$$\frac{s^2+t^2}{2st},\quad \frac{s^2+t^2}{s^2-t^2},\quad \frac{(s^2+t^2)^2}{2st(s^2-t^2)}.$$

Bhâskara II solves a problem similar to the second one above:

Find a right triangle whose area equals the hypotenuse.[1]

He starts with the rational right triangle ($3x$, $4x$, $5x$); then by the condition, area = hypotenuse, finds the value $x = 5/6$. So that a right triangle of the required type is ($5/2$, $10/3$, $25/6$). He then observes: "In like manner, by virtue of various assumptions, other right triangles can also be found."[2] The general solution in this case is

$$\frac{s^2+t^2}{st},\quad \frac{2(s^2+t^2)}{s^2-t^2},\quad \frac{(s^2+t^2)^2}{st(s^2-t^2)}.$$

Another example of Mahâvîra runs as follows:

"(Find) a rectangle of which twice the diagonal, thrice the base, four times the upright and twice the perimeter are together equal to the area (numerically)."[3]

Problems Involving Sides but not Areas. Mahâvîra also obtained right triangles whose sides multiplied

[1] *BBi*, p. 56.
[2] "Evamiṣṭavaśâdanye'pi"—*BBi*, p. 56.
[3] *GSS*, vii. 117½.

by arbitrary rational numbers have a given sum. Algebraically, the problems require the solution of

$$\left.\begin{aligned} x^2 + y^2 &= z^2, \\ rx + sy + tz &= A; \end{aligned}\right\}$$

where r, s, t, A are known rational numbers. His method of solution is the same as that described above. Starting with the general solution of

$$x'^2 + y'^2 = z'^2$$

we are asked to calculate the value

$$rx' + sy' + tz' = P, \text{ say.}$$

Then, says Mahâvîra, the required solution is

$$x = x'A/P, \quad y = y'A/P, \quad z = z'A/P.$$

One illustrative problem given by Mahâvîra is:

"The perimeter of a rectangle is unity. Tell me quickly, after calculating, what are its base and upright."[1]

Starting with the rectangle $m^2 - n^2$, $2mn$, $m^2 + n^2$, we have in this case $P = 2(m^2 - n^2 + 2mn)$. Hence all rectangles having the same perimeter unity will be given by

$$\frac{m^2 - n^2}{2(m^2 - n^2 + 2mn)}, \quad \frac{mn}{m^2 - n^2 + 2mn};$$

m, n being any rational numbers.

The isoperimetric right triangles will be given by

$$\left(\frac{m-n}{2m}\right)p, \quad \frac{np}{m+n}, \quad \left\{\frac{m^2+n^2}{2m(m+n)}\right\}p;$$

where p is the given perimeter.

Another example is:

"(Find) a rectangle in which twice the diagonal, thrice the base, four times the upright and the perimeter together equal unity."[2]

[1] *GSS*, vii. 118½. [2] *GSS*, vii. 119½.

Pairs of Rectangles. Mahâvîra found "pairs of rectangles such that

(*i*) their perimeters are equal but the area of one is double that of the other, or

(*ii*) their areas are equal but the perimeter of one is double that of the other, or

(*iii*) the perimeter of one is double that of the other and the area of the latter is double that of the former."

These are particular cases of the following general problem contemplated in his rule:

To find (x, y) and (u, v) representing the base and upright respectively of two rectangles which are related, such that

$$\left.\begin{aligned} 2m(x+y) &= 2n(u+v), \\ pxy &= quv; \end{aligned}\right\} \qquad (A)$$

where m, n, p, q are known integers.

His rule for the solution of this general problem is:

"Divide the greater multiples of the area and the perimeter by the (respective) smaller ones. The square of the product of these ratios multiplied by an optional number is the upright of one rectangle. That diminished by unity will be its base, when the areas are equal. Otherwise, multiply the bigger ratio of the areas by that optional number and subtract unity; three times the upright diminished by this (difference) will be the base. The upright and base of the other rectangle should be obtained from its area and perimeter (thus determined) with the help of the rule, 'From the square of half the perimeter, etc.,' described before."[1]

[1] *GSS*, vii. 131½-133. The reference in the concluding line is to rule vii. 129½.

In other words, to solve (A), assume

$$y = s\{(\text{ratio of perimeters})(\text{ratio of areas})\}^2, \tag{1}$$

and $x = y - 1$, if $p = q$, (2)

or $x = 3[y - \{s\,(\text{ratio of areas}) - 1\}]$, if $p \neq q$, (2′)

where s is an arbitrary number, and the ratios are to be so presented as always to remain greater than or equal to unity.

Let $m \geqslant n$, $q \geqslant p$. Then we shall have to assume

$$\left.\begin{aligned} y &= s\frac{m^2q^2}{n^2p^2}, \\ x &= 3\left(s\frac{m^2q^2}{n^2p^2} - s\frac{q}{p} + 1\right). \end{aligned}\right\} \tag{3}$$

Substituting these values in (A), we get

$$u + v = \frac{m}{n}\left(4s\frac{m^2q^2}{n^2p^2} - 3s\frac{q}{p} + 3\right), \tag{4}$$

$$uv = 3s\frac{m^2q}{n^2p}\left(s\frac{m^2q^2}{n^2p^2} - s\frac{q}{p} + 1\right).$$

Then

$$(u - v)^2 = \frac{m^2}{n^2}\left\{\left(4s\frac{m^2q^2}{n^2p^2} - \frac{9sq}{2p} + 3\right)^2 + \frac{3sq}{4p}\left(\frac{sq}{p} - 4\right)\right\}.$$

Now, if the arbitrary multiplier s be chosen such that

$$\frac{sq}{p} = 4, \tag{5}$$

we have

$$u - v = \frac{m}{n}\left(4s\frac{m^2q^2}{n^2p^2} - \frac{9sq}{2p} + 3\right). \tag{6}$$

From (4) and (6) we get

$$\left.\begin{aligned} u &= \frac{m}{n}\left(4s\frac{m^2q^2}{n^2p^2} - \frac{15sq}{4p} + 3\right), \\ v &= \frac{3smq}{4np}. \end{aligned}\right\} \tag{7}$$

Substituting the value of s from (5) in (3) and (7) we have finally the solution of (A), when $m \geqslant n$, $q \geqslant p$, as

$$\left.\begin{aligned} y &= 4\frac{m^2q}{n^2p}, & v &= 3\frac{m}{n}, \\ x &= 3\left(4\frac{m^2q}{n^2p} - 3\right), & u &= 4\frac{m}{n}\left(4\frac{m^2q}{n^2p} - 3\right). \end{aligned}\right\} \quad \text{(I)}$$

Mahâvîra has observed that "when the areas are equal" we are to assume[1]

$$y = s\frac{m^2}{n^2},$$

$$x = s\frac{m^2}{n^2} - 1.$$

[1] Bibhutibhusan Datta has shown that this restriction is *not necessary*. In fact, starting with the assumption

$$\left.\begin{aligned} y &= s\,\frac{m^2q^2}{n^2p^2}, \\ x &= s\,\frac{m^2q^2}{n^2p^2} - 1\,; \end{aligned}\right\} m \geqslant n,\ q \geqslant p,$$

and proceeding in the same way as above, he has obtained another solution of (A) in the form

$$\left.\begin{aligned} y &= 2\,\frac{m^2q}{n^2p}, & v &= \frac{m}{n}, \\ x &= 2\,\frac{m^2q}{n^2p} - 1, & u &= \frac{m}{q}\left(4\,\frac{m^2q}{n^2p} - 2\right). \end{aligned}\right\} \quad \text{(II)}$$

Datta finds two general solutions of (A), *viz.*

$$\left.\begin{aligned} y &= \frac{rm^2q^2}{n^2p^2} + t, \\ x &= \frac{rm^2q^2}{tn^2p^2}\left(\frac{rm^2q^2}{n^2p^2} - \frac{rq}{p} + t\right), \\ v &= \frac{rmq}{np}, \\ u &= \frac{m}{nt}\left(\frac{rm^2q^2}{n^2p^2} + t\right)\left(\frac{rm^2q^2}{n^2p^2} - \frac{rq}{p} + t\right); \end{aligned}\right\} \quad \text{(III)}$$

Isosceles Triangles with Integral sides. Brahmagupta says :

"The sum of the squares of two unequal numbers is the side ; their product multiplied by two is the altitude, and twice the difference of the squares of those two unequal numbers is the base of an isosceles triangle."[1]

Mahâvîra gives the following rule for obtaining an isosceles triangle from a single generated rectangle:

"In the isosceles triangle (required), the two diagonals (of a generated rectangle[2]) are the two sides, twice its side is the base, the upright is the altitude, and the area (of the generated rectangle) is the area."[3]

Thus if m, n be two integers such that $m \neq n$, the sides of all rational isosceles triangles with integral sides are :

$$(i) \quad m^2 + n^2, \; m^2 + n^2, \; 2(m^2 - n^2);$$

or

$$(ii) \quad m^2 + n^2, \; m^2 + n^2, \; 4mn.$$

and

$$\left.\begin{aligned} y &= \frac{rm^2q^2}{n^2p^2} - t, & v &= \frac{m}{n}\left(\frac{rq}{p} - t\right), \\ x &= \frac{rm^2q^2}{n^2p^2}\left(\frac{rq}{tp} - 1\right), & u &= \frac{rmq}{np}\left(\frac{rm^2q^2}{tn^2p^2} - 1\right) \end{aligned}\right\} \quad \text{(IV)}$$

where $m \geqslant n$, $q \geqslant p$ and r, t are any two integers.

See Datta, "On Mahâvîra's solution of rational triangles and quadrilaterals," *BCMS*, XX, 1928-9, pp. 267-294; particularly p. 285.

[1] *BrSpSi*, xii. 33.

[2] A rectangle generated from the numbers m and n has its sides equal to $m^2 - n^2$ and $2mn$ and its diagonal equal to $m^2 + n^2$. *Cf.* pp. 208-9.

[3] *GSS*, vii. $108\frac{1}{2}$.

The altitude of the former is $2mn$ and of the latter $m^2 - n^2$ and the area in either case is the same, that is, $2mn(m^2 - n^2)$.

Juxtaposition of Right Triangles. It will be noticed that the device employed by Brahmagupta and Mahâvîra to find the above solutions is to juxtapose two rational right triangles—equal in this case—so as to have a common leg. It is indeed a very powerful device. For, every rational triangle or quadrilateral may be formed by the juxtaposition of two or four rational right triangles. So, in order to construct such rational figures, it suffices to know only the complete solution of $x^2 + y^2 = z^2$ in integers. The beginning of this principle is found as early as the *Baudhâyana Śulba*[1] (800 B.C.) wherein is described the formation of a kind of brick, called *ubhayî* (born of two), by the juxtaposition of the eighths of two suitable rectangular bricks of the same breadth (and thickness) but of different lengths.

Isosceles Triangles with a Given Altitude. Brahmagupta gives a rule to find all rational isosceles triangles having the same altitude. He says :

"The (given) altitude is the producer (*karaṇî*). Its square divided by an optional number is increased and diminished by that optional number. The smaller is the base and half the greater is the side."[2]

That is to say, the sides and bases of rational isosceles triangles having the same altitude a are respectively,

$$\tfrac{1}{2}\left(\frac{a^2}{m} + m\right),\ \tfrac{1}{2}\left(\frac{a^2}{m} + m\right) \text{ and } \left(\frac{a^2 - m}{m}\right),$$

where m is any rational number.

[1] *BŚl*, iii. 122 ; Compare Datta, *Śulba*, p. 45, where necessary figures are given.

[2] *BrSpSi*, xviii. 37.

In particular, let the given altitude be 8. Then taking $m = 4$ Pṛthûdakasvâmî (860) obtains the rational isosceles triangle (10, 10, 12).

Pairs of Rational Isosceles Triangles. Mahâvîra gives the following rule for finding two isosceles triangles whose perimeters, as also their areas, are related in given proportions :

"Multiply the square of the ratio-numbers of the perimeters by the ratio-numbers of the areas mutually and then divide the larger product by the smaller. Multiply the quotient by 6 and 2 (severally) and then diminish the smaller by unity : again (find severally) the difference between the results, and twice the smaller one : these are the two sets of elements for the figures to be generated. From them the sides, etc., can be obtained in the way described before."[1]

If (s_1, s_2) and $(\triangle_1, \triangle_2)$ denote the perimeters and areas of two rational isosceles triangles, such that

$$s_1 : s_2 = m : n, \qquad \triangle_1 : \triangle_2 = p : q, \qquad (1)$$

where the ratio-numbers m, n, p, q are known integers, then the triangles will be obtained, says Mahâvîra, from the rectangles *generated* from

$$\left(6\frac{n^2p}{m^2q}, 2\frac{n^2p}{m^2q} - 1\right) \text{ and } \left(4\frac{n^2p}{m^2q} + 1, 4\frac{n^2p}{m^2q} - 2\right),$$

where $n^2p > m^2q$, when the dimensions of the first are multiplied by m and those of the second by n.

The dimensions of the isosceles triangle formed from the first set of *bîja* are :

$$\text{side} = m\left\{\left(6\frac{n^2p}{m^2q}\right)^2 + \left(2\frac{n^2p}{m^2q} - 1\right)^2\right\},$$

[1] *GSS*, vii. 137.

$$\text{base} = 24m\,\frac{n^2p}{m^2q}\left(2\,\frac{n^2p}{m^2q} - 1\right),$$

$$\text{altitude} = m\left\{\left(6\,\frac{n^2p}{m^2q}\right)^2 - \left(2\,\frac{n^2p}{m^2q} - 1\right)^2\right\};$$

and from the second set

$$\text{side} = n\left\{\left(4\,\frac{n^2p}{m^2q} + 1\right)^2 + \left(4\,\frac{n^2p}{m^2q} - 2\right)^2\right\},$$

$$\text{base} = 4n\left(4\,\frac{n^2p}{m^2q} + 1\right)\left(4\,\frac{n^2p}{m^2q} - 2\right),$$

$$\text{altitude} = n\left\{\left(4\,\frac{n^2p}{m^2q} + 1\right)^2 - \left(4\,\frac{n^2p}{m^2q} - 2\right)^2\right\}.$$

It can be easily verified that the perimeters and areas of the isosceles triangles thus obtained satisfy the conditions (1).

In particular, putting $m = n = p = q = 1$, we have two isosceles triangles of sides, bases and altitudes (29, 40, 21) and (37, 24, 35) which have equal perimeters (98) and equal areas (420). This particular case was treated by Frans van Schooten the Younger (1657), J. H. Rahn (1697) and others.[1]

It is evident that multiplying the above values by m^4q^2 we get pairs of isosceles triangles whose dimensions are integral.

Rational Scalene Triangles. Brahmagupta says:

"The square of an optional number is divided twice by two arbitrary numbers; the moieties of the sums of the quotients and (respective) optional numbers are the sides of a scalene triangle; the sum of the moieties of the differences is the base."[2]

[1] Dickson, *Numbers*, II, p. 201. [2] *BrSpSi*, xii. 34.

That is to say, the sides of a rational scalene triangle are

$$\tfrac{1}{2}\left(\frac{m^2}{p}+p\right),\ \tfrac{1}{2}\left(\frac{m^2}{q}+q\right),\ \tfrac{1}{2}\left(\frac{m^2}{p}-p\right)+\tfrac{1}{2}\left(\frac{m^2}{q}-q\right)$$

where m, p, q are any rational numbers. The altitude (m), area and segments of the base of this triangle are all rational.

Mahâvîra gives the rule:

"Half the base of a derived rectangle is divided by any optional number. With this divisor and the quotient is obtained another rectangle. The sum of the uprights (of these two rectangles) will be the base of the scalene triangle, the two diagonals, its sides and the base (of either rectangle) its altitude."[1]

If m, n be any two rational numbers, the rational rectangle ($AB'BH$)

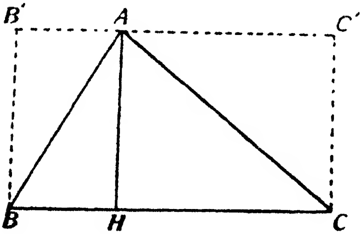

Fig. 3

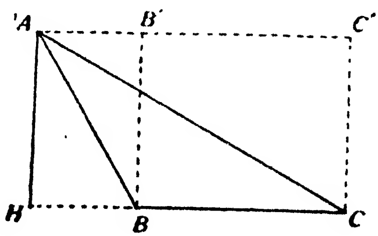

Fig. 4

formed from them is

$$m^2-n^2,\ 2mn,\ m^2+n^2.$$

If p, q be any two rational factors of mn, that is, if $mn=pq$, the second rectangle ($AC'CH$) is

$$p^2-q^2,\ 2pq,\ p^2+q^2.$$

[1] *GSS*, vii. 110½.

Now, juxtaposing these two rectangles so that they do not overlap (Fig. 3), the sides of the rational scalene triangle are obtained as

$$p^2 + q^2,\ m^2 + n^2,\ \{(p^2 - q^2) + (m^2 - n^2)\},$$

where $mn = pq$. Evidently the two rectangles can be juxtaposed so as to overlap (Fig. 4). So the general solution will be

$$p^2 + q^2,\ m^2 + n^2,\ \{(p^2 - q^2) \pm (m^2 - n^2)\}.$$

The altitude of the rational scalene triangle thus obtained is $2mn$ or $2pq$, its area $pq(p^2 - q^2) \pm mn(m^2 - n^2)$ and the segments of the base are $p^2 - q^2$ and $m^2 - n^2$.

In particular, putting $m = 12$, $p = 6$, $q = 8$ in Brahmagupta's general solution, Pṛthûdakasvâmî derives a scalene triangle of which the sides (13, 15), base (14), altitude (12), area (84) and the segments of the base (5, 9) are all integral numbers.

In order to get the above solutions of the rational scalene triangle the method employed was, it will be noticed, the juxtaposition of two rational right triangles so as to have a common leg. In Europe, it is found to have been employed first by Bachet (1621). The credit for the discovery of this method of finding rational scalene triangles should rightly go to Brahmagupta (628), but not to Bachet as is supposed by Dickson.[1]

Triangles having a Given Area. Mahâvîra proposes to find all triangles having the same given area A. His rules are:

"Divide the square of four times the given area by three; The quotient is the square of the square of a side of the equilateral triangle."[2]

[1] Dickson, *Numbers*, II, p. 192.
[2] *GSS*, vii. 154½.

"Divide the given area by an optional number; the square-root of the sum of the squares of the quotient and the optional number is a side of the isosceles triangle formed. Twice the optional number is its base and the area divided by the optional number is the altitude."[1]

"The cube of the square-root of the sum of eight times the given area and the square of an optional number is divided by the product of the optional number and that square-root; the quotient is diminished by half the optional number which is the base (of the required triangle). The *saṅkramaṇa* between this remainder and the quotient of the square of the optional number divided by twice that square-root will give the two sides."[2]

The last rule has been re-stated differently.[3]

21. RATIONAL QUADRILATERALS

Rational Isosceles Trapeziums. Brahmagupta has shown how to obtain an isosceles trapezium whose sides, diagonals, altitude, segments and area are all rational numbers. He says:

"The diagonals of the rectangle (generated) are the flank sides of an isosceles trapezium; the square of its side is divided by an optional number and then lessened by that optional number and divided by two; (the result) increased by the upright is the base and lessened by it is the face."[4]

That is to say, we shall have (Fig. 5)

$$CD = \tfrac{1}{2}\left(\frac{4m^2n^2}{p} - p\right) + (m^2 - n^2),$$

[1] *GSS*, vii. 156½.
[2] *GSS*, vii. 158½.
[3] *GSS*, vii. 160½-161½.
[4] *BrSpSi*, xii. 36.

$$AB = \tfrac{1}{2}\left(\frac{4m^2n^2}{p} - p\right) - (m^2 - n^2),$$
$$AD = BC = m^2 + n^2\,;$$

also

$$DH = m^2 - n^2,$$
$$HC = \tfrac{1}{2}\left(\frac{4m^2n^2}{p} - p\right),$$
$$AC = BD = \tfrac{1}{2}\left(\frac{4m^2n^2}{p} + p\right),$$
$$AH = 2mn,$$

area $$ABCD = mn\left(\frac{4m^2n^2}{p} - p\right).$$

By choosing the values of m, n and p suitably, the values of all the dimensions of the isosceles trapezium can be made integral. Thus, starting with the rectangle (5, 12, 13) and taking $p = 6$, Pṛthûdakasvâmî finds, by way of illustration, the isosceles trapezium whose flank sides = 13, base = 14, and face = 4. Its altitude (12), segments of base (5, 9), diagonals (15) and area (108) are also integers.

Mahâvîra writes:

"For an isosceles trapezium the sum of the perpendicular of the first generated rectangle and the perpendicular of the second rectangle which is generated from any (rational) divisor of half the base of the first and the quotient, will be the base; their difference will be the face; the smaller of the diagonals (of the generated rectangles) will be the flank side; the smaller perpendicular will be the segment; the greater diagonal will be the diagonal (of the isosceles trapezium); the greater area will be the area and the base (of either rectangle) will be the altitude."[1]

[1] *GSS*, vii. 99½.

The first rectangle ($AA'DH$) generated from m, n is

$$m^2 - n^2,\ 2mn,\ m^2 + n^2.$$

If p, q be any two rational factors of half the base of this rectangle, that is, if $pq = mn$, the second rectangle ($AB'CH$) from these factors will be

$$p^2 - q^2,\ 2pq,\ p^2 + q^2.$$

By judiciously juxtaposing these two rectangles, we shall obtain an isosceles trapezium of the type required ($ABCD$):

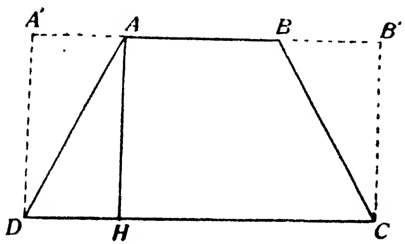

Fig. 5

$$\begin{aligned}
CD &= (p^2 - q^2) + (m^2 - n^2),\\
AB &= (p^2 - q^2) - (m^2 - n^2),\\
AD &= BC = m^2 + n^2, \text{ if } m^2 + n^2 < p^2 + q^2,\\
DH &= m^2 - n^2, \qquad\qquad \text{if } m^2 - n^2 < p^2 - q^2,\\
AC &= BD = p^2 + q^2, \text{ if } p^2 + q^2 > m^2 + n^2,\\
AH &= 2mn = 2pq,
\end{aligned}$$

area $ABCD = 2pq\,(p^2 - q^2)$,

$$\text{if } 2pq\,(p^2 - q^2) > 2mn(m^2 - n^2).$$

The necessity of the conditions $m^2 + n^2 < p^2 + q^2$, $m^2 - n^2 < p^2 - q^2$, etc., will be at once realised from a glance at Figs. 5 and 6. The above specifications of the dimensions of a rational isosceles trapezium will give Fig. 5. But when the conditions are reversed so that

$m^2+n^2>p^2+q^2$, $m^2-n^2>p^2-q^2$, $2pq\,(p^2-q^2) < 2mn(m^2-n^2)$, the dimensions of the isosceles trapezium (Fig. 6) are:

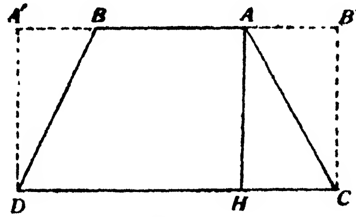

Fig. 6

$$\begin{aligned} CD &= (m^2-n^2)+(p^2-q^2),\\ AB &= (m^2-n^2)-(p^2-q^2),\\ AC &= BD = p^2+q^2,\\ DH &= m^2-n^2,\\ AD &= BC = m^2+n^2,\\ AH &= 2mn = 2pq,\\ \text{area } ABCD &= 2mn(m^2-n^2). \end{aligned}$$

Pairs of Isosceles Trapeziums. Mahâvîra gives the following rule for finding the face, base and equal sides of an isosceles trapezium having an area and altitude exactly equal to those of another isosceles trapezium whose dimensions are known:

"On performing the *viṣama-saṅkramaṇa* between the square of the perpendicular (of the known isosceles trapezium) and an optional number, the greater result will be the equal sides of the (required) isosceles trapezium; half the sum and difference of the smaller result and the moieties of the face and base (of the known figure) will be the base and face (respectively of the required figure)."[1]

[1] *GSS*, vii. 173½.

Let a, b, c, h, denote respectively the face, base, equal sides and altitude of the known isosceles trapezium and let a', b', c', h', denote the corresponding quantities of the required isosceles trapezium. Then, since the two trapeziums are equal in area and altitude, we must have

$$h' = h,$$

$$b' + a' = b + a, \tag{1}$$

and
$$c'^2 - \left(\frac{b' - a'}{2}\right)^2 = h^2,$$

or
$$\{c' + (b' - a')/2\}\{c' - (b' - a')/2\} = h^2,$$

whence
$$c' - (b' - a')/2 = r,$$

and
$$c' + (b' - a')/2 = h^2/r,$$

r being any rational number. Then

$$c' = \tfrac{1}{2}(h^2/r + r), \tag{2}$$

$$b' - a' = (h^2/r - r). \tag{3}$$

From (1) and (2), we get

$$b' = (b + a)/2 + (h^2/r - r)/2, \tag{4}$$

$$a' = (b + a)/2 - (h^2/r - r)/2. \tag{5}$$

If $a = 4$, $b = 14$, $c = 13$, $h = 12$, taking $r = 10$, we shall have[1] $a' = 34/5$, $b' = 56/5$, $c' = 61/5$.

It has been stated above that, if m, n, p, q are rational numbers such that $m^2 \pm n^2 < p^2 \pm q^2$, we must have

$$a = (p^2 - q^2) - (m^2 - n^2),$$
$$b = (p^2 - q^2) + (m^2 - n^2),$$
$$c = m^2 + n^2,$$
$$h = 2mn = 2pq.$$

[1] *GSS*, vii. 174½.

Substituting these values in (2), (4), (5) we get the dimensions of the equivalent isosceles trapezium as

$$a' = (p^2 - q^2) - (4p^2q^2/r - r)/2,$$
$$b' = (p^2 - q^2) + (4p^2q^2/r - r)/2,$$
$$c' = (4p^2q^2/r + r)/2.$$

If $m^2 \pm n^2 > p^2 \pm q^2$, the sides of the pair of isosceles trapeziums equal in area and altitude will be

$$a = (m^2 - n^2) - (p^2 - q^2),$$
$$b = (m^2 - n^2) + (p^2 - q^2),$$
$$c = p^2 + q^2;$$
$$a' = (m^2 - n^2) - (4m^2n^2/r - r)/2,$$
$$b' = (m^2 - n^2) + (4m^2n^2/r - r)/2,$$
$$c' = (4m^2n^2/r + r)/2.$$

These two isosceles trapeziums will also have equal diagonals.

Rational Trapeziums with Three Equal Sides. This problem is nearly the same as that of the rational isosceles trapezium with this difference that in this case one of the parallel sides also is equal to the slant sides. Brahmagupta states the following solution of the problem :

"The square of the diagonal (of a *generated* rectangle) gives three equal sides; the fourth (is obtained) by subtracting the square of the upright from thrice the square of the side (of that rectangle). If greater, it is the base; if less, it is the face."[1]

The rectangle generated from m, n is given by

$$m^2 - n^2,\ 2mn,\ m^2 + n^2.$$

[1] *BrSpSi*, xii. 37.

If $ABCD$ be a rational trapezium whose sides AB, BC, AD are equal, then

$$AB = BC = AD = (m^2 + n^2)^2,$$

$$CD = 3(2mn)^2 - (m^2 - n^2)^2 = 14m^2n^2 - m^4 - n^4,$$

or $$CD = 3(m^2 - n^2)^2 - (2mn)^2 = 3m^4 + 3n^4 - 10m^2n^2.$$

In particular, putting $m = 2$, $n = 1$, Pṛthûdakasvâmî deduces two rational trapeziums with three equal sides, *viz.*, (25, 25, 25, 39) and (25, 25, 25, 11).

The first solution is also given by Mahâvîra who indicates the method for obtaining it. He says :

"For a trapezium with three equal sides (proceed) as in the case of the isosceles trapezium with (the rectangle formed from) the quotient of the area of a generated rectangle divided by the square-root of its side multiplied by the difference of its elements and divisor; and that (formed) from the side and upright."[1]

That is to say, from any rectangle $(m^2 - n^2, 2mn, m^2 + n^2)$, calculate

$$\frac{2mn(m^2 - n^2)}{\sqrt{2mn(m - n)}} = \sqrt{2mn}\,(m + n).$$

Then from $\sqrt{2mn(m - n)}$, $\sqrt{2mn(m + n)}$ form the rectangle

$$8m^2n^2,\ 4mn(m^2 - n^2),\ 4mn(m^2 + n^2). \qquad (1)$$

Again from $2mn$, $m^2 - n^2$ form another rectangle

$$6m^2n^2 - m^4 - n^4,\ 4mn\,(m^2 - n^2),\ (m^2 + n^2)^2 \qquad (2)$$

By the juxtaposition of the rectangles (1) and (2) we get Brahmagupta's rational trapezium with three

[1] *GSS*, vii. 101½.

equal sides:

$$CD = 8m^2n^2 + (6m^2n^2 - m^4 - n^4) = 14m^2n^2 - m^4 - n^4,$$
$$AB = 8m^2n^2 - (6m^2n^2 - m^4 - n^4) = (m^2 + n^2)^2 = AD$$
$$= BC, \quad \text{if} \quad m^2 + n^2 < 4mn.$$

The segment (CH), altitude (AH), diagonals (AC, BD) and area of this trapezium are also rational, for

$$CH = 6m^2n^2 - m^4 - n^4,$$
$$AH = 4mn(m^2 - n^2),$$
$$AC = BD = 4mn(m^2 + n^2),$$
$$\text{area} \quad ABCD = 32m^3n^3(m^2 - n^2).$$

Rational Inscribed Quadrilaterals. Brahmagupta formulated a remarkable proposition: To find all quadrilaterals which will be inscribable within circles and whose sides, diagonals, perpendiculars, segments (of sides and diagonals by perpendiculars from vertices as also of diagonals by their intersection), areas, and also the diameters of the circumscribed circles will be expressible in integers. Such quadrilaterals are called Brahmagupta quadrilaterals.

The solution given by Brahmagupta is as follows:

"The uprights and bases of two right-angled triangles being reciprocally multiplied by the diagonals of the other will give the sides of a quadrilateral of unequal sides: (of these) the greatest is the base, the least is the face, and the other two sides are the two flanks."[1]

Taking Brahmagupta's integral solution, the sides of the two right triangles of reference are given by

$$m^2 - n^2, \; 2mn, \; m^2 + n^2;$$
$$p^2 - q^2, \; 2pq, \; p^2 + q^2;$$

[1] *Br.Sp.Si.*, xii. 38.

where m, n, p, q are integers. Then the sides of a Brahmagupta quadrilateral are

$$\left.\begin{array}{c}(m^2 - n^2)(p^2 + q^2), \quad (p^2 - q^2)(m^2 + n^2), \\ 2mn(p^2 + q^2), \quad 2pq(m^2 + n^2).\end{array}\right\} \quad (A)$$

Mahâvîra says:

"The base and the perpendicular (of the smaller and the larger derived rectangles of reference) multiplied reciprocally by the longer and the shorter diagonals and (each again) by the shorter diagonal will be the sides, the face and the base (of the required quadrilateral). The uprights and bases are reciprocally multiplied and then added together; again the product of the uprights is added to the product of the bases; these two sums multiplied by the shorter diagonal will be the diagonals. (These sums) when multiplied respectively by the base and perpendicular of the smaller figure of reference will be the altitudes; and they when multiplied respectively by the perpendicular and the base will be the segments of the base. Other segments will be the difference of these and the base. Half the product of the diagonals (of the required figure) will be the area."[1]

If the rectangle generated from m, n be smaller than that from p, q, then, according to Mahâvîra, we obtain the rational inscribed quadrilateral of which the sides are

$$(m^2 - n^2)(p^2 + q^2)(m^2 + n^2), \quad (p^2 - q^2)(m^2 + n^2)^2,$$
$$2mn(p^2 + q^2)(m^2 + n^2), \quad 2pq(m^2 + n^2)^2;$$

whose diagonals are

$$\{2pq(m^2 - n^2) + 2mn(p^2 - q^2)\}(m^2 + n^2),$$
$$\{(m^2 - n^2)(p^2 - q^2) + 4mnpq\}(m^2 + n^2);$$

[1] *GSS*, vii. 103½.

whose altitudes are

$$\{2pq(m^2 - n^2) + 2mn(p^2 - q^2)\}2mn,$$
$$\{(m^2 - n^2)(p^2 - q^2) + 4mnpq\}(m^2 - n^2);$$

whose segments are

$$\{2pq(m^2 - n^2) + 2mn(p^2 - q^2)\}(m^2 - n^2),$$
$$\{(m^2 - n^2)(p^2 - q^2) + 4mnpq\}2mn;$$

and whose area is

$$\tfrac{1}{2}\{2pq(m^2 - n^2) + 2mn(p^2 - q^2)\}\{(m^2 - n^2)(p^2 - q^2) + 4mnpq\}(m^2 + n^2)^2.$$

Śrîpati writes:

"Of the two right triangles the sides and uprights are reciprocally multiplied by the hypotenuses; of the products the greatest is the base, the smallest is the face and the rest are the two flank sides of a quadrilateral with unequal sides."[1]

Bhâskara II gives the rule:

"The sides and uprights of two optional right triangles being multiplied by their reciprocal hypotenuses become the sides: in this way has been derived a quadrilateral of unequal sides. There the two diagonals can be obtained from those two triangles. The product of the uprights, added with the product of the sides, gives one diagonal; the sum of the reciprocal products of the uprights and sides is the other."[2]

Bhâskara II[3] illustrates by taking the right triangles (3, 4, 5) and (5, 12, 13) so that the resulting cyclic quadrilateral is (25, 39, 60, 52). The same example was

[1] *SiŚe*, xiii. 42. [2] *L*, p. 51.
[3] *L*, p. 52.

given before by Mahâvîra[1] and Pṛthûdakasvâmî.[2] This cyclic quadrilateral also appears in the *Triśatikâ* of Śrîdhara[3] and in the commentary of the *Âryabhaṭîya* by Bhâskara I (522). The diagonals of this quadrilateral are, states Bhâskara II, 56 (= 3.12 + 4.5) and 63 (= 4.12 + 3.5) (Fig. 7). He then observes:

"If the figure be formed by changing the arrangement of the face and flank then the second diagonal will be equal to the product of the hypotenuses of the two right triangles (of reference), *i.e.*, 65." (Fig. 8).

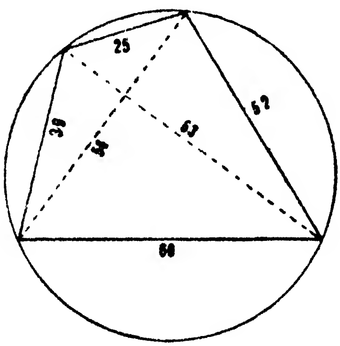

Fig. 7

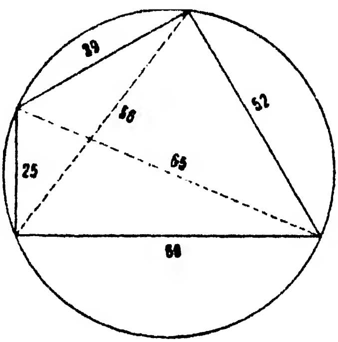

Fig. 8

By taking the right triangles (3, 4, 5) and (15, 8, 17) Bhâskara II gets another cyclic quadrilateral (68, 51, 40, 75), whose diagonals are (77, 85), altitude is 308/5, segments are 144/5 and 231/5, and area is 3234.[4] (Fig. 9). With the sequence of the sides (68, 40, 51, 75) the

[1] *GSS*, vii. 104½.
[2] *BrSpSi*, xii. 38 (*Com.*).
[3] *Triś*, *Ex*. 80.
[4] *L*, pp. 46ff.

diagonals are (77, 84) (Fig. 10), and with (68, 40, 75, 51) they are (84, 85). (Fig. 11).

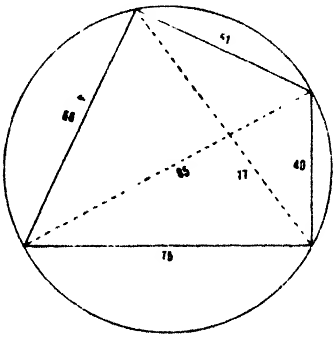

Fig. 9

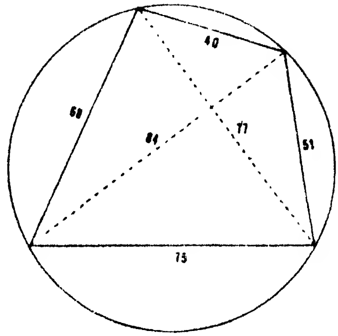

Fig. 10

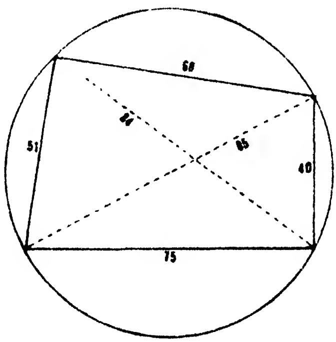

Fig. 11

The deep significance of Brahmagupta's results has been demonstrated by Chasles[1] and Kummer.[2]

[1] M. Chasles, *Aperçu historique sur l'origine et development des méthodes en géométrie*, Paris, 1875, pp. 436ff.

[2] E. E. Kummer, "Über die Vierecke, deren Seiten und Diogonalen rational sind," *Journ. für Math.*, XXXVII, 1848, pp. 1-20.

In fact, according to the sequence in which the quantities (A) are taken, there will be two varieties of Brahmagupta quadrilaterals having them as their sides: (1) one in which the two diagonals intersect at right angles and (2) the other in which the diagonals intersect obliquely. The arrangement (A) gives a quadrilateral of the first variety. For the oblique variety, the sides are in the following order:

$$\left.\begin{array}{ll}(p^2-q^2)(m^2+n^2), & (m^2-n^2)(p^2+q^2),\\ 2mn(p^2+q^2), & 2pq(m^2+n^2);\end{array}\right\} \quad (B)$$

or

$$\left.\begin{array}{ll}(p^2-q^2)(m^2+n^2), & 2mn(p^2+q^2),\\ (m^2-n^2)(p^2+q^2), & 2pq(m^2+n^2).\end{array}\right\} \quad (C)$$

Bhâskara II points out that the diagonals of the Brahmagupta quadrilateral are in the (A) variety,

$$2pq(m^2-n^2)+2mn(p^2-q^2), \quad 4mnpq+(p^2-q^2)(m^2-n^2);$$

in (B),

$$2pq(m^2-n^2)+2mn(p^2-q^2), \quad (p^2+q^2)(m^2+n^2);$$

and in (C),

$$4mnpq+(p^2-q^2)(m^2-n^2), \quad (p^2+q^2)(m^2+n^2).$$

The diameter of the circumscribed circle in every case is $(p^2+q^2)(m^2+n^2)$.

Gaṇeśa (1545) shows that the quadrilateral is formed by the juxtaposition of four right triangles obtained by multiplying the sides of each of two rational right triangles by the upright and base of the other. He writes:

"A quadrilateral is divided into four triangles by its intersecting diagonals. So by the juxtaposition of four triangles a quadrilateral will be formed. For that purpose the four triangles are assumed in this manner: Take two right triangles formed in the way indicated

Compare also L. E. Dickson, "Rational Triangles and Quadrilaterals," *Amer. Math. Mon.*, XXVIII, 1921, pp. 244-250.

before. If the upright, base and hypotenuse of a rational right triangle be multiplied by any arbitrary rational number, there will be produced another right triangle with rational sides. Hence on multiplying the sides of each of the two right triangles by an optional number equal to the base of the other and again by an optional nnmber equal to the upright of the other, four right triangles will be obtained by the judicious juxtaposition of which the quadrilateral will be formed."

He then shows with the help of specific examples (see Figs. 12, 13 & 14) that we can obtain in this way

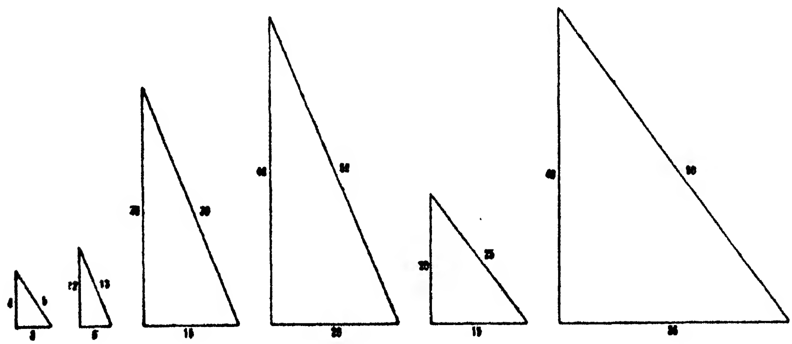

Fig. 12

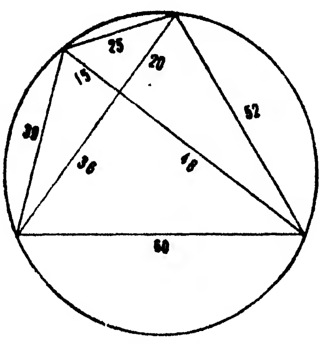

Fig. 13

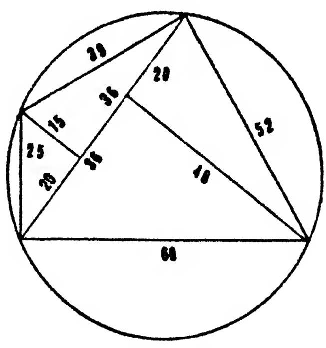

Fig. 14

from the same set of two rational right triangles two varieties of rational convex quadrilaterals: One in which the diagonals intersect each other perpendicularly; and the other in which they do so obliquely.

Inscribed Quadrilaterals having a Given Area. Mahâvîra proposes to find all rational inscribed rectangles having the same given area (A, say). He says:

"The square-root of the exact area is a side of the square. The quotient of the area by an optional number and that optional number will be the base and upright of the rectangle."[1]

For finding all inscribed rational isosceles trapeziums having the same area A his rule is:

"The given area multiplied by the square of an optional number is diminished by the area of a generated rectangle and then divided by the base of that rectangle; the quotient divided by the optional number is the face; the quotient added with twice the upright and divided by the optional number gives the base; the base (of the generated rectangle) divided by the optional number is the altitude; and the diagonal divided by the optional number gives the two flank sides."[2]

That is to say, if $m^2 - n^2$, $2mn$, $m^2 + n^2$ be the upright, base and diagonal of a rectangle formed from m, n, the dimensions of the isosceles trapezium will be

$$\text{face} = \frac{s^2A - 2mn(m^2 - n^2)}{2mns},$$

$$\text{base} = \frac{1}{s}\left\{\frac{s^2A - 2mn(m^2 - n^2)}{2mn} + 2(m^2 - n^2)\right\}$$

$$= \frac{s^2A + 2mn(m^2 - n^2)}{2mns},$$

[1] *GSS*, vii. 146. [2] *GSS*, vii. 148.

$$\text{altitude} = \frac{2mn}{s},$$
$$\text{side} \quad = \frac{m^2 + n^2}{s};$$

where s is an arbitrary rational number chosen such that $s^2 A > 2mn(m^2 - n^2)$.

For the construction of an inscribed trapezium of three equal sides Mahâvîra gives the following rule :

"The square of the given area is divided by the cube of an optional number and then increased by that optional number ; half the result gives the (equal) sides of a generated trapezium of three equal sides (having the given area) ; twice the optional number diminished by the side is the base ; and the given area divided by the optional number is the altitude."[1]

In other words, the dimensions of an inscribed trapezium of three equal sides having a given area A will be

$$\text{side} \quad = \tfrac{1}{2}\Big(\frac{A^2}{s^3} + s\Big),$$
$$\text{base} \quad = 2s - \tfrac{1}{2}\Big(\frac{A^2}{s^3} + s\Big),$$
$$\text{altitude} = \frac{A}{s}.$$

To find inscribed quadrilaterals having a given area Mahâvîra gives the following rule :

"Break up the square of the given area into any four arbitrary factors. Half the sum of these factors is diminished by them (severally). The remainders are the sides of an (inscribed) quadrilateral with unequal sides."[2]

[1] *GSS*, vii. 150.

[2] *GSS*, vii. 152. This result follows from the fact that the area of a cyclic quadrilateral is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$.

Triangles and Quadrilaterals having a Given Circum-Diameter. Mahâvîra proposes to find all rational triangles and quadrilaterals inscribable in a circle of given diameter. His solution is:

"Divide the given diameter of the circle by the calculated diameter (of the circle circumscribing any generated figure of the required kind). The sides of that generated figure multiplied by the quotient will be the sides of the required figure."[1]

In other words, we shall have to find first a rational triangle or cyclic quadrilateral; then calculate the diameter of its circum-circle and divide the given diameter by it. Dimensions of the optional figure multiplied by this quotient will give the dimensions of the required figure of the type.

It has been found before (p. 227) that the sides of a rational triangle are proportional to

$$m^2 + n^2, \ p^2 + q^2, \ (p^2 - q^2) \pm (m^2 - n^2)$$

and its altitude is proportional to $2mn$ (or $2pq$), m, n, p, q being any rational numbers such that $mn = pq$. The diameter of the circle circumscribed about this triangle is proportional to

$$\frac{(m^2 + n^2)(p^2 + q^2)}{2mn}.$$

Then the sides of a rational triangle inscribed in a circle of diameter D will be

$$\frac{2mnD}{p^2 + q^2}, \ \frac{2mnD}{m^2 + n^2}, \ 2mnD\frac{(p^2 - q^2) \pm (m^2 - n^2)}{(m^2 + n^2)(p^2 + q^2)};$$

and its altitude will be

$$\frac{D(2mn)^2}{(m^2 + n^2)(p^2 + q^2)}.$$

[1] *GSS*, vii. 221½.

The dimensions of a rational inscribed quadrilateral, as stated by Mahâvîra, have been noted before (p. 236). The diameter of its circum-circle is

$$(p^2 + q^2)(m^2 + n^2)^2.$$

Then, according to Mahâvîra, the sides of a rational quadrilateral inscribed in a circle of diameter D, are

$$D\left(\frac{2mn}{m^2+n^2}\right),\ D\left(\frac{m^2-n^2}{m^2+n^2}\right),\ D\left(\frac{2pq}{p^2+q^2}\right),\ D\left(\frac{p^2-q^2}{p^2+q^2}\right);$$

its diagonals are

$$\{2pq\,(m^2-n^2) + 2mn\,(p^2-q^2)\}\frac{D}{(p^2+q^2)(m^2+n^2)},$$

$$\{(m^2-n^2)(p^2-q^2) + 4mnpq\}\frac{D}{(p^2+q^2)(m^2+n^2)};$$

and its area is

$$\frac{D^2}{2(p^2+q^2)(m^2+n^2)}\{2pq(m^2-n^2) + 2mn(p^2-q^2)\}$$
$$\times\{(m^2-n^2)(p^2-q^2) + 4mnpq\};$$

so that the sides, diagonals and area are all rational.

22. SINGLE INDETERMINATE EQUATIONS OF HIGHER DEGREES

The Hindus do not seem to have paid much attention to the treatment of indeterminate equations of degrees higher than the second. Some interesting examples involving such equations are, however, found in the works of Mahâvîra (850), Bhâskara II (1150) and Nârâyaṇa (1350).

Mahâvîra's Rule. One problem of Mahâvîra is as follows:

Given the sum (s) of a series in A.P., to find its first term (a), common difference (b) and the number of terms (n).

In other words, it is required to solve in rational numbers the equation

$$\{a + (\frac{n-1}{2})b\}n = s,$$

containing three unknowns a, b and n, and of the third degree. The following rule is given for its solution :

"Here divide the sum by an optional factor of it ; that divisor is the number of terms. Subtract from the quotient another optional number ; the subtrahend is the first term. The remainder divided by the half of the number of terms as diminished by unity is the common difference."[1]

Bhâskara's Method. Bhâskara II proposes the problems :

"Tell those four numbers which are unequal but have a common denominator, whose sum or the sum of whose cubes is equal to the sum of their squares."[2]

If x, y, z, w be the numbers, then

(1) $x + y + z + w = x^2 + y^2 + z^2 + w^2$,

(2) $x^3 + y^3 + z^3 + w^3 = x^2 + y^2 + z^2 + w^2$.

Let the numbers be x, $2x$, $3x$, $4x$, says Bhâskara II. That is, suppose $y = 2x$, $z = 3x$, $w = 4x$ in the above

[1] *GSS*, vii. 78.

There are also other problems where instead of s, the given quantity is $s + a$, $s + b$, $s + n$ or $s + a + b + n$. (*GSS*, ii. 83 ; *cf.* also vi. 80). For such problems also the method of solution is the same as before, *i.e.*, to assume suitable arbitrary values for two of the unknowns so that the indeterminate cubic equation is thereby reduced to a determinate linear equation in one unknown. (*GSS*, ii. 82 ; vi. 317).

[2] *BBi*, p. 55.

equations. Then by (1) we get

$$10x = 30x^2.$$

$$\therefore \quad x = \tfrac{1}{3}.$$

Hence $\qquad x, y, z, w = \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{3}{3}, \tfrac{4}{3},$

is a solution of (1).

Again, with the same assumption, the equation (2) reduces to

$$100x^3 = 30x^2.$$

$$\therefore \quad x = \tfrac{3}{10}.$$

Hence $\qquad x, y, z, w = \tfrac{3}{10}, \tfrac{6}{10}, \tfrac{9}{10}, \tfrac{12}{10},$

is a solution of (2).

The following problem has been quoted by Bhâskara II from an ancient author :

"The square of the sum of two numbers added with the cube of their sum is equal to twice the sum of their cubes. Tell, O mathematician, (what are those two numbers)."[1]

If x, y be the numbers, then by the statement of the question

$$(x + y)^2 + (x + y)^3 = 2(x^3 + y^3).$$

"Here, so that the operations may not become lengthy," says Bhâskara II, "assume the two numbers to be $u + v$ and $u - v$." So on putting

$$x = u + v, \; y = u - v,$$

the equation reduces to

$$4u^3 + 4u^2 = 12uv^2,$$

or $\qquad 4u^2 + 4u = 12v^2,$

or $\qquad (2u + 1)^2 = 12v^2 + 1.$

[1] *BBi*, p. 101.

Solving this equation by the method of the Square-nature we get values of u, v. Whence the values of (x, y) are found to be (5, 1), (76, 20), etc.

Nârâyaṇa's Rule. Nârâyaṇa gives the rule :

"Divide the sum of the squares, the square of the sum and the product of any two optional numbers by the sum of their cubes and the cube of their sum, and then multiply by the two numbers (severally). (The results) will be the two numbers, the sum of whose cubes and the cube of whose sum will be equal to the sum of their squares, the square of the sum and the product of them."[1]

That is to say, the solution of the equations

(1) $x^3 + y^3 = x^2 + y^2$, (4) $(x + y)^3 = x^2 + y^2$,

(2) $x^3 + y^3 = (x + y)^2$, (5) $(x + y)^3 = (x + y)^2$,

(3) $x^3 + y^3 = xy$, (6) $(x + y)^3 = xy$,

are respectively

$$(1.1)\quad \begin{cases} x = \dfrac{(m^2 + n^2)m}{m^3 + n^3}, \\[2ex] y = \dfrac{(m^2 + n^2)n}{m^3 + n^3}; \end{cases} \qquad (4.1)\quad \begin{cases} x = \dfrac{(m^2 + n^2)m}{(m + n)^3}, \\[2ex] y = \dfrac{(m^2 + n^2)n}{(m + n)^3}; \end{cases}$$

$$(2.1)\quad \begin{cases} x = \dfrac{(m + n)^2 m}{m^3 + n^3}, \\[2ex] y = \dfrac{(m + n)^2 n}{m^3 + n^3}; \end{cases} \qquad (5.1)\quad \begin{cases} x = \dfrac{(m + n)^2 m}{(m + n)^3}, \\[2ex] y = \dfrac{(m + n)^2 n}{(m + n)^3}; \end{cases}$$

$$(3.1)\quad \begin{cases} x = \dfrac{m^2 n}{m^3 + n^3}, \\[2ex] y = \dfrac{mn^2}{m^3 + n^3}; \end{cases} \qquad (6.1)\quad \begin{cases} x = \dfrac{m^2 n}{(m + n)^3}, \\[2ex] y = \dfrac{mn^2}{(m + n)^3}; \end{cases}$$

[1] *GK*, i. 58.

where m, n are rational numbers.

It will be noticed that the equation (2) can be reduced, by dividing out by $x+y$, to

$$x^2 - xy + y^2 = x + y\,;$$

and similarly (5) can be reduced to

$$x + y = 1.$$

With $m = 1$, $n = 2$ Nârâyaṇa gives the following sets of particular values:

$$\begin{array}{ll} (1.2)\ x, y = \frac{5}{9}, \frac{10}{9} & (4.2)\ x, y = \frac{5}{27}, \frac{10}{27} \\ (2.2)\ x, y = 1, 2 & (5.2)\ x, y = \frac{1}{3}, \frac{2}{3}, \\ (3.2)\ x, y = \frac{2}{9}, \frac{4}{9} & (6.2)\ x, y = \frac{2}{27}, \frac{4}{27}. \end{array}$$

He then observes: "In this way one can find by his own intelligence two numbers for the case of *difference*, etc."

Form $\mathbf{ax^{2n+2} + bx^{2n} = y^2}$. For the solution of an equation of the form

$$ax^{2n+2} + bx^{2n} = y^2,$$

where n is an integer, Bhâskara II gives the following rule:

"Removing a square factor from the second side, if possible, the two roots should be investigated in this case. Then multiply the greater root by the lesser. Or, if a biquadratic factor has been removed, the greater root should be multiplied by the square of the lesser root. The rest of the operations will then be as before."[1]

Suppose $ax^2 + b = z^2$; then the equation becomes

$$y^2 = x^{2n}z^2.$$

$$\therefore \quad y = x^n z.$$

The method of solving $ax^2 + b = z^2$ in positive integers has been described before.

[1] *BBi*, p. 102.

Two examples of equations of this form occur in the *Bîjagaṇita* of Bhâskara II :[1]

(1) $5x^4 - 100x^2 = y^2$,

(2) $8x^6 + 49x^4 = y^2$.

It may be noted that the second equation appears in the course of solving another problem.

Equation $\mathbf{ax^4 + bx^2 + c = u^3}$. One very special case of this form arises in the course of solving another problem. It is[2]

$$(a + x^2)^2 + a^2 = u^3,$$

or $$x^4 + 2ax^2 + 2a^2 = u^3.$$

Let $u = x^2$, supposes Bhâskara II, so that we get

$$x^6 - x^4 = 2a^2 + 2ax^2,$$

or $$x^4(2x^2 - 1) = (2a + x^2)^2,$$

which can be solved by the method indicated before.

Another equation is[3]

$$5x^4 = u^3.$$

In cases like this "the assumption should be always such," remarks Bhâskara II, "as will make it possible to remove (the cube of) the unknown." So assume $u = mx$; then

$$x = \tfrac{1}{5}m^3.$$

23. LINEAR FUNCTIONS MADE SQUARES OR CUBES

Square-pulveriser. The indeterminate equation of the type

$$bx + c = y^2$$

[1] *BBi*, pp. 103, 107. [2] *BBi*, p. 103 ; also *vide infra*, p. 280.
[3] *BBi*, p. 50 ; also *vide infra*, p. 278.

is called *varga-kuṭṭaka* or the "Square-pulveriser,"[1] inasmuch as, when expressed in the form

$$\frac{y^2 - c}{b} = x,$$

the problem reduces to finding a square (*varga*) which being diminished by c will be exactly divisible by b, which closely resembles the problem solved by the method of the pulveriser (*kuṭṭaka*).

For the solution in *integers* of an equation of this type, the method of the earlier writers appears to have been to assume suitable arbitrary values for y and then to solve the equation for x. Brahmagupta gives the following problems:

"The residue of the sun on Thursday is lessened and then multiplied by 5, or by 10 Making this (result) an exact square, within a year, a person becomes a mathematician."[2]

"The residue of any optional revolution lessened by 92 and then multiplied by 83 becomes together with unity a square. A person solving this within a year is a mathematician."[3]

That is to say, we are to solve the equations:

(1) $5x - 25 = y^2$,
(2) $10x - 100 = y^2$,
(3) $83x - 7635 = y^2$.

Pṛthûdakasvâmî solves them thus:

(1.1) Suppose $y = 10$; then $x = 125$. Or, put $y = 5$; then $x = 10$.

(2.1) Suppose $y = 10$; then $x = 20$.

(3.1) Assume $y = 1$; then $x = 92$.

[1] *BBi*, p. 122.
[2] *BrSpSi*, xviii 76.
[3] *BrSpSi*, xviii. 79.

He then remarks that by virtue of the multiplicity of suppositions there will be an infinitude of solutions in every case. But no method has been given either by Brahmagupta or by his commentator Pṛthûdakasvâmî to obtain the general solution.

The above method is reproduced by Bhāskara II.[1] He has also given the following rule:

"If a simple unknown be multiplied by the number which is the divisor of a square, etc., (on the other side) then, in order that its value may in such cases be integral, the square, etc., of another unknown should be put equal to (the other side). The rest (of the operations) will be as described before."[2]

His gloss on this rule runs as follows:

"In those cases, such as the Square-pulveriser, etc., where on taking the root of one side of the equation there remains on the other side a simple unknown multiplied by the number which was the divisor of the square, etc., the square, etc., of another unknown plus or minus an absolute term should be assumed for (the value of this other side) in order that its value may be integral. The rest (of the operations) will be as taught before."

Bhâskara has also quoted from an ancient author the following rule:

"(Find) a number whose square is exactly divisible by the divisor, as also its product by twice the square-root of the absolute term. An unknown multiplied by that number and superadded by the square-root of the absolute term should be assumed (for the unknown on the other side). If the absolute term does not yield a square-root, then after dividing it by the divisor, the

[1] *Vide infra*, p. 255f.

[2] *BBi*, p. 120.

remainder should be increased by so many times the divisor as will make a square. If this is not possible, then the problem is not soluble."[1]

Case i Let c be a square, equal to β^2, say. Then we have to solve

$$bx + \beta^2 = y^2.$$

The rule says, find p such that

$$p^2 = bq, \quad 2p\beta = br.$$

Then assume $\quad y = pu + \beta\,;$

whence we get $\quad x = qu^2 + ru.$

Bhâskara II prefers the assumption

$$y = bv + \beta,$$

so that we have $\quad x = bv^2 + 2\beta v.$

Case ii. If c is not a square, suppose $c = bm + n$. Then find s such that

$$n + sb = r^2.$$

Now assume $\quad y = bu \pm r.$

Substituting this value in the equation $bx + c = y^2$, we get

$$\begin{aligned} bx + c &= (bu \pm r)^2 \\ &= b^2u^2 \pm 2bru + r^2, \end{aligned}$$

or $\quad bx + c - r^2 = b^2u^2 \pm 2bru,$

or $\quad bx + b(m - s) = b^2u^2 \pm 2bru.$

$\therefore \quad x = bu^2 \pm 2ru - (m - s).$

Example from Bhâskara II :[2]

$$7x + 30 = y^2.$$

On dividing 30 by 7 the remainder is found to be 2; we also know that $2 + 7 . 2 = 4^2$. Therefore, we

[1] *BBi*, p. 121. [2] *BBi*, pp. 120, 121.

assume in accordance with the above rule

$$y = 7u \pm 4;$$

whence we get $$x = 7u^2 \pm 8u - 2,$$

which is the general solution.

Cube-pulveriser. The indeterminate equation of the type

$$bx + c = y^3$$

is called the *ghana-kuṭṭaka* or the "Cube-pulveriser."[1] For its solution in integers Bhâskara II says:

"A method similar to the above may be applied also in the case of a cube thus: (find) a number whose cube is exactly divisible by the divisor, as also its product by thrice the cube-root of the absolute term. An unknown multiplied by that number and superadded by the cube-root of the absolute term should be assumed. If there be no cube-root of the absolute term, then after dividing it by the divisor, so many times the divisor should be added to the remainder as will make a cube. The cube-root of that will be the root of the absolute number. If there cannot be found a cube, even by so doing, that problem will be insoluble."[2]

Case i. Let $c = \beta^3$. Then we shall have to find p such that

$$p^3 = bq, \quad 3p\beta = br.$$

Now assume $$y = pv + \beta.$$

Substituting in the equation $bx + \beta^3 = y^3$ we get

$$bx + \beta^3 = (pv + \beta)^3$$

$$= p^3v^3 + 3pv\beta(pv + \beta) + \beta^3,$$

or $$bx = bqv^3 + brv(pv + \beta).$$

$\therefore$ $$x = qv^3 + rv(pv + \beta).$$

[1] *BBi*, p. 122. [2] *BBi*, p. 121.

Or, if we assume $y = bv + \beta$, we shall have

$$x = b^2v^3 + 3\beta v(bv + \beta).$$

Case ii. $c \neq$ a cube. Suppose $c = bm + n$; then find s such that

$$n + sb = r^3.$$

Now assume $y = bv + r$, whence we get

$$x = b^2v^3 + 3rv(bv + r) - (m - s),$$

as the general solution.

Example from Bhâskara II :[1]

$$5x + 6 = y^3.$$

Since $6 = 5 . 1 + 1$ and $1 + 43 . 5 = 6^3$,

we assume $y = 5v + 6.$

Therefore $x = 25v^3 + 18v(5v + 6) + 42$,

is the general solution.

Equation $\mathbf{bx \pm c = ay^2}$. To solve an equation of the type

$$ay^2 = bx \pm c,$$

Bhâskara II says :

"Where the first side of the equation yields a root on being multiplied or divided[2] (by a number), there also the divisor will be as stated in the problem but the absolute term will be as modified by the operations."[3]

[1] *BBi*, p. 122.

[2] The printed text has *bitvâ kṣiptâ* (subtracting or adding). After collating with several copies Colebrooke accepted the reading *batvâ kṣiptâ* (multiplying or adding). But we think that the correct reading should be *batvâ hṛtvâ* (multiplying or dividing) For in his gloss Bhâskara II has employed the terms *guṇito vibhakto va* (multiplied or divided). Our emendation is further supported by the *rationale* of the rule.

[3] *BBi*, p. 121.

What is implied is this: Multiplying both sided of the given equation by a, we get

$$a^2y^2 = abx \pm ac,$$

Put $u = ay$, $v = ax$. Then the equation reduces to

$$u^2 = bv \pm ac,$$

which can be solved in the way described before.

We take the following illustrative example with its solution from Bhâskara II :[1]

$$5y^2 + 3 = 16x.$$

Multiplying by 5, and putting $u = 5y$, $v = 5x$, we get

$$u^2 = 16v - 15.$$

The solution of this is

$$u = 8w \pm 1,$$
$$v = 4w^2 \pm w + 1.$$

Therefore, we have

$$(1) \quad 5y = 8w + 1,$$

or $$(2) \quad 5y = 8w - 1.$$

Now, solving by the method of the pulveriser, we get the solution of (1) as

$$y = 8t + 5,$$
$$w = 5t + 3;$$

and that of (2) as

$$y = 8t + 3,$$
$$w = 5t + 2;$$

where t is any rational number.

Equation $bx \pm c = ay^n$ After describing the above methods Bhâskara II observes, *ityagre'pi yojyamiti śeṣaḥ* or "the same method can be applied further on

[1] *BBi*, p. 121.

(*i.e.*, to the cases of higher powers)"[1] Again at the end of the section he has added *evaṁ buddhimadbhiranyadapi yathâsambhavaṁ yojyaṃ*, *i e*, "similar devices should be applied by the intelligent to further cases as far as practicable."[2] What is implied is as follows:

(1) To solve $\dfrac{x^n \pm c}{b} = y.$

Put $x = mz \pm k$. Then

$$\frac{x^n \pm c}{b} = \frac{1}{b}\left\{m^n z^n \pm n m^{n-1} z^{n-1} k + \frac{n(n-1)}{2} m^{n-2} z^{n-2} k^2 \pm \right.$$

$$\left. \ldots + nmz\,(\pm k)^{n-1} + (\pm k)^n \pm c\right\}$$

$$= \frac{1}{b}\left\{m^n z^n \pm n m^{n-1} z^{n-1} k + \ldots + nmz\,(\pm k)^{n-1}\right\}$$

$$+ (\pm)^n \left(\frac{k^n \pm c}{b}\right)$$

Now, if

$$\frac{k^n \pm c}{b} = \text{a whole number},$$

$\dfrac{x^n \pm c}{b}$ will be an integral number when (1) $m = b$ or (2) b is a factor of m^n, $nm^{n-1}k$, etc. Or, in other words, knowing one integral solution of (1) an infinite number of others can be derived.

(2) To solve $\dfrac{ax^n \pm c}{b} = y.$

Multiplying by a^{n-1}, we get

$$\frac{a^n x^n \pm ca^{n-1}}{b} = ya^{n-1}.$$

[1] *BBi*, p. 121. [2] *BBi*, p. 122.

Putting $u = ax$, $v = ya^{n-1}$, we have

$$\frac{u^n \pm ca^{n-1}}{b} = v,$$

which is similar to case (1).

24. DOUBLE EQUATIONS OF THE FIRST DEGREE

The earliest instance of the solution of the simultaneous indeterminate quadratic equation of the type

$$\left.\begin{aligned} x \pm a = u^2, \\ x \pm b = v^2, \end{aligned}\right\}$$

is found in the Bakhshâlî treatise. The portion of the manuscript containing the rule is mutilated. The example given in illustration can, however, be restored as follows:

"A certain number being added by five {becomes capable of yielding a square-root}; the same number {being diminished by} seven becomes capable of yielding a square-root. What is that number is the question."[1]

That is to say, we have to solve

$$\sqrt{x+5} = u, \qquad \sqrt{x-7} = v.$$

The solution given is as follows:

"The sum of the additive and subtractive is $\boxed{12}$; its half $\boxed{6}$; minus two $\boxed{4}$; its half is $\boxed{2}$; squared $\boxed{4}$. 'Should be increased by the subtractive'; {the subtractive is} $\boxed{7}$; adding this we get $\boxed{11}$. This is the number (required)."

From this it is clear that the author gives the

[1] *BMs*, Folio 59, recto.

solution of the equations

$$x + a = u^2, \quad x - b = v^2;$$

as
$$x = \left\{\tfrac{1}{2}\left(\frac{a+b}{m} - m\right)\right\}^2 + b,$$

where m is any integer.[1]

Brahmagupta (628) gives the solution of the general case. He says:

"The difference of the two numbers by the addition or subtraction of which another number becomes a square, is divided by an optional number and then increased or decreased by it. The square of half the result diminished or increased by the greater or smaller (of the given numbers) is the number (required)."[2]

i.e.,
$$x = \left\{\tfrac{1}{2}\left(\frac{a-b}{m} \pm m\right)\right\}^2 \mp a,$$

or
$$x = \left\{\tfrac{1}{2}\left(\frac{a-b}{m} \mp m\right)\right\}^2 \mp b,$$

where m is an arbitrary integer.

The *rationale* is very simple. Since

$$u^2 = x \pm a, \quad v^2 = x \pm b,$$

we have
$$u^2 - v^2 = \pm a \mp b.$$

Therefore
$$u - v = m,$$

and
$$u + v = \frac{\pm a \mp b}{m},$$

where m is arbitrary. Hence

$$u = \tfrac{1}{2}\left(\frac{\pm a \mp b}{m} + m\right) = \pm \tfrac{1}{2}\left(\frac{a-b}{m} \pm m\right).$$

[1] In the above solution m is taken to be 2.
[2] *BrSpSi*, xviii. 74.

Since it is obviously immaterial whether u is taken as positive or negative, we have

$$u = \tfrac{1}{2}\left(\frac{a-b}{m} \pm m\right).$$

Similarly $$v = \tfrac{1}{2}\left(\frac{a-b}{m} \mp m\right).$$

Therefore $$x = \left\{\tfrac{1}{2}\left(\frac{a-b}{m} \pm m\right)\right\}^2 \mp a,$$

or $$x = \left\{\tfrac{1}{2}\left(\frac{a-b}{m} \mp m\right)\right\}^2 \mp b,$$

where m is an arbitrary number.

Brahmagupta gives another rule for the particular case :

$$x + a = u^2,$$
$$x - b = v^2.$$

"The sum of the two numbers the addition and subtraction of which make another number (severally) a square, is divided by an optional number and then diminished by that optional number. The square of half the remainder increased by the subtractive number is the number (required)."[1]

i.e., $$x = \left\{\tfrac{1}{2}\left(\frac{a+b}{m} - m\right)\right\}^2 + b.$$

Nârâyaṇa (1357) says :

"The sum of the two numbers by which another number is (severally) increased and decreased so as to make it a square is divided by an optional number and then diminished by it and halved ; the square of the result added with the subtrahend is the other number."[2]

He further states :

[1] *BrSpSi*, xviii. 73.

[2] *GK*, i. 52.

"The difference of the two numbers by which another number is increased twice so as to make it a square (every time), is increased by unity and then halved. The square of the result diminished by the greater number is the other number."[1]

i.e., $$x = \left(\frac{a - b + 1}{2}\right)^2 - a$$

is a solution of

$$x + a = u^2,\ x + b = v^2,\ a > b.$$

"The difference of the two numbers by which another number is diminished twice so as to make it a square (every time), is decreased by unity and then halved. The result multiplied by itself and added with the greater number gives the other."[2]

i.e., $$x = \left(\frac{a - b - 1}{2}\right)^2 + a$$

is a solution of

$$x - a = u^2,\ x - b = v^2,\ a > b.$$

The general case

$$\left.\begin{aligned} ax + c &= u^2, \\ bx + d &= v^2, \end{aligned}\right\} \qquad (1)$$

has been treated by Bhâskara II. He first lays down the rule:

"In those cases where remains the (simple) unknown with an absolute number, there find its value by forming an equation with the square, etc., of another unknown plus an absolute number. Then proceed to the solution of the next equation comprising the simple unknown with an absolute number by substituting in it the root obtained before."[3]

[1] *GK*, i. 53. [2] *GK*, i. 54.
[3] *BBi*, pp. 117-8.

He then proceeds to explain it further :

"In those cases where on taking the square-root of the first side, there remains on the other side the (simple) unknown with or without an absolute number, find there the value of that unknown by forming an equation with the square of another unknown plus an absolute number. Having obtained the value of the unknown in this way and substituting that value (in the next equation) further operations should be proceeded with. If, however, on equating the root of the first with another unknown plus an absolute number, no further operations remain to be done, then the equation has to be made with the square, etc., of a known number."

(*i*) Set $u = mw + \alpha$; then substituting in the first equation, we get

$$x = \frac{1}{a}(m^2w^2 + 2mw\alpha + \alpha^2 - c).$$

Substituting this value of x in the next equation, we have

$$\frac{b}{a}(m^2w^2 + 2mw\alpha + \alpha^2 - c) + d = v^2, \qquad (1.1)$$

which can be solved by the method of the Square-nature.

(*ii*) In the course of working out an example[1] Bhâskara II is found to have followed also a different procedure, which was subsequently adopted by Lagrange.[2]

Eliminate x between the two equations. Then

$$bu^2 + (ad - bc) = av^2,$$

or

$$abu^2 + k = w^2, \qquad (1.2)$$

where $w = av$, $k = a^2d - abc$.

[1] *Vide infra*, p. 265.
[2] *Addition to Euler's Algebra*, p. 547.

If $u = r$, $w = s$ be a solution of this transformed equation, another solution of it will be

$$u = rq \pm ps,$$
$$w = qs \pm abrp\,;$$

where $abp^2 + 1 = q^2$. Therefore, the general solution of (1) is

$$x = \frac{1}{a}(rq \pm ps)^2 - \frac{c}{a},$$
$$u = rq \pm ps,$$
$$v = \frac{1}{a}(qs \pm abrp)\,;$$

where $abp^2 + 1 = q^2$ and $abr^2 + a^2d - abc = s^2$.

Now, a rational solution of the equation $abp^2 + 1 = q^2$ is

$$p = \frac{2t}{t^2 - ab}, \quad q = \frac{t^2 + ab}{t^2 - ab},$$

where t is any rational number. Therefore, the above general solution becomes

$$\left.\begin{aligned} x &= \frac{1}{a(t^2 - ab)^2}\left\{ r(t^2 + ab) \pm 2st \right\}^2 - \frac{c}{a}, \\ u &= \frac{1}{(t^2 - ab)}\left\{ r(t^2 + ab) \pm 2st \right\}, \\ v &= \frac{1}{a(t^2 - ab)}\left\{ s(t^2 + ab) \pm 2abrt \right\}, \end{aligned}\right\} \quad (1.3)$$

where $abr^2 + a^2d - abc = s^2$.

(*iii*) Suppose c and d to be squares, so that $c = \alpha^2$, $d = \beta^2$. Then we shall have to solve

$$ax + \alpha^2 = u^2,$$
$$bx + \beta^2 = v^2.$$

The auxiliary equation $abr^2 + a^2d - abc = s^2$ in this case becomes

$$abr^2 + (a^2\beta^2 - ab\alpha^2) = s^2.$$

The same equation is obtained by proceeding as in case (*i*) with the assumption $v = by + \beta$.

An obvious solution of it is $r = \alpha$, $s = a\beta$. Hence in this case the general solution (1.3) becomes

$$x = \frac{1}{a(t^2 - ab)^2}\{\alpha(t^2 + ab) \pm 2a\beta t\}^2 - \frac{\alpha^2}{c},$$

$$u = \frac{1}{(t^2 - ab)}\{\alpha(t^2 + ab) \pm 2a\beta t\},$$

$$v = \frac{1}{(t^2 - ab)}\{\beta(t^2 + ab) \pm 2b\alpha t\},$$

where t is any rational number.

Putting $\alpha = \beta = 1$, $t = a$, and taking the positive sign only, we get a particular solution of the equations

$$\left.\begin{aligned} ax + 1 &= u^2 \\ bx + 1 &= v^2 \end{aligned}\right\}$$

as

$$x = \frac{8(a + b)}{(a - b)^2}, \quad u = \frac{3a + b}{a - b}, \quad v = \frac{a + 3b}{a - b}.$$

This solution has been stated by Brahmagupta (628):

"The sum of the multipliers multiplied by 8 and divided by the square of the difference of the multipliers is the (unknown) number. Thrice the two multipliers increased by the alternate multiplier and divided by their difference will be the two roots."[1]

It has been described partly by Nârâyaṇa (1357) thus :

[1] *BrSpSi*, xviii. 71.

"The two numbers by which another number is multiplied at two places so as to make it (at every place), together with unity, a square, their sum multiplied by 8 and divided by the square of their difference is the other number."[1]

We take an illustrative example with its solution from Bhâskara II :

"If thou be expert in the method of the elimination of the middle term, tell the number which being severally multiplied by 3 and 5, and then added with unity, becomes a square."[2]

That is to say, we have to solve

$$\left.\begin{aligned} 3x + 1 &= u^2, \\ 5x + 1 &= v^2. \end{aligned}\right\}$$

Bhâskara II solves these equations substantially as follows :

(1) Set $u = 3y + 1$; then from the first equation,

$$x = 3y^2 + 2y.$$

Substituting this value in the other equation, we get

$$15y^2 + 10y + 1 = v^2,$$

or $$(15y + 5)^2 = 15v^2 + 10.$$

By the method of the Square-nature we have the solutions of this equation as

$$\left.\begin{aligned} v &= 9 \\ 15y + 5 &= 35 \end{aligned}\right\}, \quad \left.\begin{aligned} v &= 71 \\ 15y + 5 &= 275 \end{aligned}\right\}\ldots,$$

Therefore $\quad y = 2, 18, \ldots$

Hence $\quad x = 16, 1008, \ldots$

(2) Or assume the unknown number to be

$$x = \tfrac{1}{3}(u^2 - 1),$$

[1] *GK*, i. 51. [2] *BBi*, p. 118.

so that the first condition of the problem (*i.e.*, the first equation) is identically statisfied. Then by the second condition

$$\tfrac{5}{3}(u^2 - 1) + 1 = v^2,$$

$$\text{or} \qquad (5u)^2 = 15v^2 + 10.$$

Now, by the method of the Square-nature, we get the values of (u, v) as $(7, 9)$, $(55, 71)$, etc. Therefore, the values of x are, as before, 16, 1008, etc.

The following example is by Nârâyaṇa :

"O expert in the art of the Square-nature, tell me the number which being severally multiplied by 4 and 7 and decreased by 3, becomes capable of yielding a square-root."[1]

That is, solve :

$$\left.\begin{aligned} 4x - 3 &= u^2, \\ 7x - 3 &= v^2. \end{aligned}\right\}$$

Nârâyaṇa says : "By the method indicated before the number is 1, 21, or 1057."

25. DOUBLE EQUATIONS OF THE SECOND DEGREE

First Type. The double equations of the second degree considered by the Hindus are of two general types. The first of them is

$$\left.\begin{aligned} ax^2 + by^2 + c &= u^2, \\ a'x^2 + b'y^2 + c' &= v^2. \end{aligned}\right\}$$

Of these the more thoroughly treated particular cases are as follows :

$$\textit{Case i.} \quad \left\{\begin{aligned} x^2 + y^2 + 1 &= u^2, \\ x^2 - y^2 + 1 &= v^2 ; \end{aligned}\right.$$

[1] *GK*, p. 40.

Case ii. $\begin{cases} x^2 + y^2 - 1 = u^2, \\ x^2 - y^2 - 1 = v^2. \end{cases}$

It should be noted that though the earliest treatment of these equations is now found in the algebra of Bhâskara II (1150), they have been admitted by him as being due to previous authors (*âdyodâharaṇaṃ*).

For the solution of (*i*) Bhâskara II assumes[1]

$$x^2 = 5z^2 - 1, \quad y^2 = 4z^2,$$

so that both the equations are satisfied. Now, by the method of the Square-nature, the solutions of the equation $5z^2 - 1 = x^2$ are (1, 2), (17, 38),...Therefore, the solutions of (*i*) are

$$\left.\begin{matrix} x = 2 \\ y = 2 \end{matrix}\right\}, \quad \left.\begin{matrix} x = 38 \\ y = 34 \end{matrix}\right\}, \quad \ldots$$

Similarly, for the solution of (*ii*), he assumes

$$x^2 = 5z^2 + 1, \quad y^2 = 4z^2,$$

which satisfy the equations. By the method of the Square-nature the values of (z, x) in the equation $5z^2 + 1 = x^2$ are (4, 9), (72, 161), etc. Hence the solutions of (*ii*) are

$$\left.\begin{matrix} x = 9 \\ y = 8 \end{matrix}\right\}, \quad \left.\begin{matrix} x = 161 \\ y = 144 \end{matrix}\right\}, \quad \ldots$$

Bhâskara II further says that for the solution of equations of the forms (*i*) and (*ii*) a more general assumption will be

$$x^2 = pz^2 \mp 1, \quad y^2 = m^2 z^2;$$

where p, m are such that

$$p \pm m^2 = \text{a square},$$

[1] *BBi*, p. 108.

the upper sign being taken for Case *i* and the lower sign for Case *ii*. Both the equations are then identically satisfied. Suppose

$$p + m^2 = s^2,$$
$$p - m^2 = t^2.$$

Whence
$$s = \tfrac{1}{2}\left(\frac{2m^2}{n} + n\right),$$
$$t = \tfrac{1}{2}\left(\frac{2m^2}{n} - n\right),$$

where n is any rational number. Therefore

$$p = \tfrac{1}{4}\left(\frac{4m^4}{n^2} + n^2\right).$$

Here he observes that m^2 should be so chosen that p will be an integer.

Hence
$$\left.\begin{aligned} x^2 &= \tfrac{1}{4}\left(\frac{4m^4}{n^2} + n^2\right)z^2 \mp 1, \\ y^2 &= m^2z^2\,; \end{aligned}\right\} \qquad (1)$$

the upper sign being taken for Case *i* and the lower sign for Case *ii*.

Whence
$$u = \tfrac{1}{2}\left(\frac{2m^2}{n} + n\right)z,$$
$$v = \tfrac{1}{2}\left(\frac{2m^2}{n} - n\right)z.$$

Or, we may proceed in a different way, says Bhâskara II :

Since

$$(p^2 + q^2) \pm 2pq$$

is always a square, we may assume

$$x^2 = (p^2 + q^2)w^2 \mp 1,$$
$$y^2 = 2pqw^2.$$

For a rational value of y, $2pq$ must be a square. So we take

$$p = 2m^2, \ q = n^2.$$

Hence we have the assumption

$$\left.\begin{aligned} x^2 &= (4m^4 + n^4)w^2 \mp 1, \\ y^2 &= 4m^2n^2w^2; \end{aligned}\right\} \qquad (2)$$

the upper sign being taken for Case *i* and the lower sign for Case *ii*.

Whence

$$u = (2m^2 + n^2)w,$$
$$v = (2m^2 - n^2)w.$$

It will be noticed that the equations (1) follow from (2) on putting $w = z/2n$. So we shall take the latter as our fundamental assumption for the solution of the equations (*i*) and (*ii*). Then, from the solutions of the subsidiary equations

$$(4m^4 + n^4)w^2 \mp 1 = x^2,$$

by the method of the Square-nature, observes Bhâskara II, an infinite number of integral solutions of the original equations can be derived.[1]

Now, one rational solution of

$$(4m^4 + n^4)w^2 + 1 = x^2$$

is

$$w = \frac{2r}{(4m^4 + n^4) - r^2},$$

$$x = \frac{(4m^4 + n^4) + r^2}{(4m^4 + n^4) - r^2}.$$

Therefore, we have the general solution of

$$\left.\begin{aligned} x^2 + y^2 - 1 &= v^2 \\ x^2 - y^2 - 1 &= u^2 \end{aligned}\right\}$$

[1] Cf. *BBi*, p. 110.

as

$$\left.\begin{aligned} x &= \frac{(4m^4+n^4)+r^2}{(4m^4+n^4)-r^2}, & u &= \frac{2r(2m^2+n^2)}{(4m^4+n^4)-r^2}, \\ y &= \frac{4mnr}{(4m^4+n^4)-r^2}, & v &= \frac{2r(2m^2-n^2)}{(4m^4+n^4)-r^2}; \end{aligned}\right\} \quad (A)$$

where m, n, r are rational numbers.

For $r = s/t$, we get Genocchi's solution.[1]

In particular, put $m = 2t$, $n = 1$, $r = 8t^2 - 1$ in (A). Then, we get the solution

$$\left.\begin{aligned} x &= \tfrac{1}{2}\left(\frac{8t^2-1}{2t}\right)^2 + 1, & u &= \frac{64t^4-1}{8t^2}, \\ y &= \frac{8t^2-1}{2t}, & v &= \tfrac{1}{2}\left(\frac{8t^2-1}{2t}\right)^2 \end{aligned}\right\} \quad (a)$$

Putting $m = t$, $n = 1$, $r = 2t^2 + 2t + 1$ in (A), we have[2]

$$\left.\begin{aligned} x &= t + \frac{1}{2t}, & u &= t + \frac{1}{2t}, \\ y &= 1, & v &= t - \frac{1}{2t}. \end{aligned}\right\} \quad (b)$$

Again, if we put $m = t$, $n = 1$, $r = 2t^2$ in (A), we get

$$\left.\begin{aligned} x &= 8t^4 + 1, & u &= 4t^2(2t^2+1), \\ y &= 8t^3, & v &= 4t^2(2t^2-1). \end{aligned}\right\} \quad (c)$$

These three solutions have been stated by Bhâskara II in his treatise on arithmetic. He says,

[1] *Nouv. Ann. Math.*, X, 1851, pp. 80-85; also Dickson, *Numbers*, II, pp. 479. For a summary of important Hindu results in algebra see the article of A. N. Singh in the *Archeon*, 1936.

[2] Here, and also in (c), we have overlooked the negative sign of x, y, u and v.

"The square of an optional number is multiplied by 8, decreased by unity, halved and then divided by that optional number. The quotient is one number. Half its square plus unity is the other number. Again, unity divided by twice an optional number added with that optional number is the first number and unity is the second number. The sum and difference of the squares of these two numbers minus unity will be (severally) squares."[1]

"The biquadrate and the cube of an optional number is multiplied by 8, and the former product is again increased by unity. The results will be the two numbers (required)."[2]

Nârâyaṇa writes:

"The cube of any optional number is the first number; half the square of its square plus unity is the second. The sum and difference of the squares of these two numbers minus unity become squares."[3]

That is, if m be an optional number, one solution of (*ii*), according to Nârâyaṇa, is

$$x = \frac{m^4}{2} + 1, \qquad u = (m^2 + 2)\frac{m^2}{2},$$

$$y = m^3, \qquad v = (m^2 - 2)\frac{m^2}{2}.$$

It will be noticed that this solution follows easily from the solution (*c*) of Bhâskara II, on putting $t = m/2$. This special solution was found later on by E. Clere (1850).[4]

[1] *L*, p. 13. [2] *L*, p. 14.
[3] *GK*, i. 46.
[4] *Nouv. Ann. Math.*, IX, 1850, pp. 116-8; also Dickson, *Numbers*, II, p. 479; Singh, l. c.

Now, let us take into consideration the equation

$$(4m^4 + n^4)w^2 - 1 = x^2.$$

Its solutions are known to be

$$\left.\begin{aligned} w &= \frac{1}{n^2} \\ x &= \frac{2m^2}{n^2} \end{aligned}\right\}, \qquad \left.\begin{aligned} w &= \frac{1}{2m^2} \\ x &= \frac{n^2}{2m^2} \end{aligned}\right\}$$

From these, by the Principle of Composition, we get respectively two other solutions

$$\left.\begin{aligned} w &= \frac{16m^4 + n^4}{n^6} \\ x &= \frac{32m^6 + 6m^2n^4}{n^6} \end{aligned}\right\}, \qquad \left.\begin{aligned} w &= \frac{m^4 + n^4}{2m^6} \\ x &= \frac{n^6 + 3n^2m^4}{2m^6} \end{aligned}\right\}$$

Therefore, the general solutions of

$$\left.\begin{aligned} x^2 + y^2 + 1 &= u^2, \\ x^2 - y^2 + 1 &= v^2; \end{aligned}\right\}$$

are

$$\left.\begin{aligned} x &= \frac{2m^2}{n^2}, \qquad & u &= \frac{2m^2 + n^2}{n^2}, \\ y &= \frac{2m}{n}, \qquad & v &= \frac{2m^2 - n^2}{n^2}; \end{aligned}\right\} \qquad (a')$$

$$\left.\begin{aligned} x &= \frac{1}{n^6}(32m^6 + 6m^2n^4), \\ y &= \frac{2m}{n^5}(16m^4 + n^4), \\ u &= \frac{1}{n^6}(16m^4 + n^4)(2m^2 + n^2), \\ v &= \frac{1}{n^6}(16m^4 + n^4)(2m^2 - n^2); \end{aligned}\right\} \qquad (a'')$$

$$\left.\begin{aligned} x &= \frac{n^2}{2m^2}, & u &= \frac{2m^2+n^2}{2m^2}, \\ y &= \frac{n}{m}, & v &= \frac{2m^2-n^2}{2m^2}; \end{aligned}\right\} \qquad (b')$$

and

$$\left.\begin{aligned} x &= \frac{1}{2m^6}(n^6+3n^2m^4), \\ y &= \frac{n}{m^5}(m^4+n^4), \\ u &= \frac{1}{2m^6}(m^4+n^4)(2m^2+n^2), \\ v &= \frac{1}{2m^6}(m^4+n^4)(2m^2-n^2). \end{aligned}\right\} \qquad (b'')$$

Putting $n = 1$ in (a') and (a''), we have the integral solutions

$$\left.\begin{aligned} x &= 2m^2, & u &= 2m^2+1, \\ y &= 2m, & v &= 2m^2-1; \end{aligned}\right\} \qquad (a'.1)^1$$

and

$$\left.\begin{aligned} x &= 2m^4(16m^2+3), \\ y &= 2m(16m^4+1), \\ u &= (16m^4+1)(2m^2+1), \\ v &= (16m^4+1)(2m^2-1). \end{aligned}\right\} \qquad (a''.1)$$

Similarly, if we put $m = 1$ in (b') and (b''), we get

$$\left.\begin{aligned} x &= \tfrac{1}{2}n^2, & u &= \tfrac{1}{2}(n^2+2), \\ y &= n, & v &= \tfrac{1}{2}(n^2-2); \end{aligned}\right\} \qquad (b'.1)$$

and

$$\left.\begin{aligned} x &= \tfrac{1}{2}n^2(n^4+3), & u &= \tfrac{1}{2}(n^4+1)(n^2+2), \\ y &= n(n^4+1), & v &= \tfrac{1}{2}(n^4+1)(n^2-2). \end{aligned}\right\} \qquad (b''.1)$$

[1] This solution was given by Drummond (*Amer. Math. Mon.*, IX, 1902, p. 232).

The solution $(b'.1)$ is stated by Nârâyaṇa thus :

"Any optional number is the first and half its square is the second. The sum and difference of the squares of these two numbers with unity become capable of yielding a square-root."[1]

Case iii. Form

$$\left.\begin{aligned} ax^2 + by^2 &= u^2, \\ a'x^2 + b'y^2 + c' &= v^2. \end{aligned}\right\}$$

For the solution of double equations of this form Bhâskara II adopts the following method :

The solution of the first equation is $x = my$, $u = ny$; where

$$am^2 + b = n^2.$$

Substituting in the second equation, we get

$$(a'm^2 + b')y^2 + c' = v^2,$$

which can be solved by the method of the Square-nature.

Example from Bhâskara II :[2]

$$\left.\begin{aligned} 7x^2 + 8y^2 &= u^2 \\ 7x^2 - 8y^2 + 1 &= v^2 \end{aligned}\right\}.$$

He solves it substantially as follows :

In the first equation suppose $x = 2y$; then $u = 6y$. Putting $x = 2y$, the second equation becomes

$$20y^2 + 1 = v^2.$$

By the method of the Square-nature the values of y satisfying this equation are 2, 36, etc. Hence the solutions of the given double equation are

$$x = 4, 72, \ldots$$
$$y = 2, 36, \ldots$$

[1] *GK*, i. 45. [2] *BBi*, p. 119.

Case iv. Form

$$\left.\begin{aligned} a(x^2 \pm y^2) + c &= u^2, \\ a'(x^2 \pm y^2) + c' &= v^2. \end{aligned}\right\}$$

Putting $x^2 \pm y^2 = z$ Bhâskara II reduces the above equations to

$$\left.\begin{aligned} az + c &= u^2, \\ a'z + c' &= v^2; \end{aligned}\right\}$$

the method for the solution of which has been given before.

Example with solution from Bhâskara II :[1]

$$\left.\begin{aligned} 2(x^2 - y^2) + 3 &= u^2 \\ 3(x^2 - y^2) + 3 &= v^2 \end{aligned}\right\}.$$

Set $\quad x^2 - y^2 = z; \quad$ then

$$2z + 3 = u^2,$$

$$3z + 3 = v^2.$$

Eliminating z we get

$$3u^2 = 2v^2 + 3,$$

or $$(3u)^2 = 6v^2 + 9.$$

Whence $$v = 6, 60, \ldots$$

$$3u = 15, 147, \ldots$$

Therefore $$u = 5, 49, \ldots$$

Hence $$x^2 - y^2 = z = 11, 1199, \ldots$$

Therefore, the required solutions are

$$\left.\begin{aligned} x &= \tfrac{1}{2}\left(\frac{11}{m} + m\right) \\ y &= \tfrac{1}{2}\left(\frac{11}{m} - m\right) \end{aligned}\right\}, \quad \left.\begin{aligned} x &= \tfrac{1}{2}\left(\frac{1199}{m} + m\right) \\ y &= \tfrac{1}{2}\left(\frac{1199}{m} - m\right) \end{aligned}\right\}, \quad \ldots$$

where m is an arbitrary rational number.

[1] *BBi*, p. 119.

For $m = 1$, the values of (x, y) will be $(6, 5)$, $(600, 599), \ldots$

For $m = 11$, we get the solution $(60, 49), \cdots$

Case v. For the solution of the double equation of the general form

$$\left.\begin{aligned} ax^2 + by^2 + c &= u^2 \\ a'x^2 + b'y^2 + c' &= v^2 \end{aligned}\right\}$$

Bhâskara II's hint[1] is: Find the values of x, u in the first equation in terms of y, and then substitute that value of x in the second equation so that it will be reduced to a Square-nature. He has, however, not given any illustrative example of this kind.

Second Type. Another type of double equation of the second degree which has been treated is

$$\left.\begin{aligned} a^2x^2 + bxy + cy^2 &= u^2, \\ a'x^2 + b'xy + c'y^2 + d' &= v^2. \end{aligned}\right\}$$

The solution of the first equation has been given before to be

$$x = \frac{1}{2a}\left\{\frac{y^2}{\lambda}\left(c - \frac{b^2}{4a^2}\right) - \lambda\right\} - \frac{by}{2a^2},$$

$$u = \tfrac{1}{2}\left\{\frac{y^2}{\lambda}\left(c - \frac{b^2}{4a^2}\right) + \lambda\right\},$$

where λ is an arbitrary rational number. Putting $\lambda = y$, we have

$$x = \frac{y}{2a}\left(c - \frac{b^2}{4a^2} - 1\right) - \frac{by}{2a^2} = \alpha y,$$

$$u = \frac{y}{2}\left(c - \frac{b^2}{4a^2} + 1\right);$$

where $$\alpha = \frac{1}{2a}\left(c - \frac{b^2}{4a^2} - 1\right) - \frac{b}{2a^2}.$$

[1] *Vide supra*, pp. 190f.

Substituting in the second equation, we get

$$(a'\alpha^2 + b'\alpha + c')y^2 + d' = v^2,$$

which can be solved by the method of the Square-nature. This method is equally applicable if the unknown part in the second equation is of a different kind but still of the second degree.

Bhâskara II gives the following illustrative example together with its solution :[1]

$$\left.\begin{aligned} x^2 + xy + y^2 &= u^2 \\ (x+y)u + 1 &= v^2 \end{aligned}\right\}.$$

Multiplying the first equation by 36, we get

$$(6x + 3y)^2 + 27y^2 = 36u^2.$$

Whence
$$6x + 3y = \tfrac{1}{2}\left(\frac{27y^2}{\lambda} - \lambda\right),$$

$$6u = \tfrac{1}{2}\left(\frac{27y^2}{\lambda} + \lambda\right),$$

where λ is an arbitrary rational number. Taking $\lambda = y$, we have

$$6x + 3y = 13y,$$

or
$$x = \tfrac{5}{3}y,$$

and
$$u = \tfrac{7}{3}y.$$

Substituting in the second equation, we get

$$56y^2 + 9 = 9v^2.$$

By the method of the Square-nature the values of y are 6, 180, ...

Hence the required values of (x, y) are (10, 6), (300, 180), ...

[1] *BBi*, pp. 107f.

26. DOUBLE EQUATIONS OF HIGHER DEGREES

There are a few problems which involve double equations of degrees higher than the second. The following examples are taken from Bhâskara II :

Example 1. "The sum of the cubes (of two numbers) is a square and the sum of their squares is a cube. If you know them, then I shall admit that you are a great algebraist."[1]

We have to solve the equations

$$\left.\begin{aligned} x^2 + y^2 &= u^3, \\ x^3 + y^3 &= v^2. \end{aligned}\right\}$$

The solution of this problem by Bhâskara II is as follows :

"Here suppose the two numbers to be z^2, $2z^2$. The sum of their cubes is $9z^6$. This is by itself a square and its square-root is $3z^3$.

"Now the sum of the squares of those two numbers is $5z^4$. This must be a cube. Assuming it to be equal to the cube of an optional multiple of $5z$ and removing the factor z^3 from both sides (we get $z = 25p^3$, where p is an optional number); so, as stated before, the numbers are (putting $p = 1$) 625, 1250. The assumption should be always such as will make it possible to remove (the cube of) the unknown."[2]

In general, assume $x = mz^2$, $y = nz^2$; substituting in the second equation, we have

$$x^3 + y^3 = (m^3 + n^3)z^6 = v^2.$$

If $\quad m^3 + n^3 = \text{a square} = p^2$, say,

then $\quad v = pz^3$.

[1] *BBi*, p. 56. [2] *BBi*, pp. 56f.

Now, from the first equation, we have

$$(m^2 + n^2)z^4 = u^3.$$

Assume $u = rz$; then

$$z = \frac{r^3}{m^2 + n^2}.$$

Hence we get

$$x = \frac{mr^6}{(m^2 + n^2)^2}, \quad y = \frac{nr^6}{(m^2 + n^2)^2};$$

where r is any integer and m, n are such that

$$m^3 + n^3 = \text{a square}.$$

One obvious solution[1] of this equation is $m = 1$, $n = 2$. Hence we get the solution

$$x = \frac{r^6}{25}, \quad y = \frac{2r^6}{25}.$$

This particular solution has been given by Nârâyaṇa, who says:

"The square of the cube of an optional number is the one and twice it is the other. These divided by 25 will be the two numbers, the sum of whose squares will

[1] Now $m^3 + n^3$ can be made a square by putting

$$m = (p^3 + q^3)p, \quad n = (p^3 + q^3)q,$$

so that $$m^3 + n^3 = (p^3 + q^3)^4.$$

Hence the solution of our equation will be

$$x = \frac{pr^6}{(p^2 + q^2)^2 (p^3 + q^3)^3},$$

$$y = \frac{qr^6}{(p^2 + q^2)^2 (p^3 + q^3)^3}.$$

Putting $r = (p^2 + q^2)(p^3 + q^3)s$, we have the solution in positive integers as

$$x = p(p^2 + q^2)^4 (p^3 + q^3)^3 s^6,$$
$$y = q(p^2 + q^2)^4 (p^3 + q^3)^3 s^6;$$

where p, q, s are any integral numbers.

be a cube and the sum of whose cubes will be a square."[1] He then adds by way of illustration:

"With the optional number 1, we get the two numbers (1/25, 2/25); with 2, (64/25, 128/25); with 5, (625, 1250); with 1/2, (1/1600, 1/800); with 1/3, (1/18225, 2/18225). Thus by virtue of (the multiplicity of) the optional number many solutions can be found."

Example 2. "O most learned algebraist, tell me those various pairs of whole numbers whose difference is a square and the sum of whose squares is a cube."[2]

That is to say, solve in positive integers

$$\left.\begin{aligned} y - x &= u^2, \\ y^2 + x^2 &= v^3. \end{aligned}\right\}$$

Bhâskara II's process of solving this problem is as follows:

"Let the two numbers be x, y. Putting their difference, $y - x$, equal to u^2, we get the value of x as $y - u^2$. Having thus found the value of x, the two numbers become $y - u^2$, y.

"The sum of their squares $= 2y^2 - 2yu^2 + u^4$. This is a cube. Making it equal to u^6 and transposing we get

$$u^6 - u^4 = 2y^2 - 2yu^2.$$

Multiplying both sides by 2 and superadding u^4, we get the square-root of the second side $= 2y - u^2$, and the first side $= 2u^6 - u^4$. Dividing out by u^4 (and putting w for $2y/u^2 - 1$), we get

$$2u^2 - 1 = w^2.$$

By the method of the Square-nature the roots of this equation are

$$u = 5,\ 29,\ \ldots$$
$$w = 7,\ 41,\ \ldots$$

[1] *GK*, i. 50.

[2] *BBi*, p. 103.

"Then by the rule, 'Or, if a biquadratic factor has been removed, the greater root should be multiplied by the square of the lesser root,'[1] we get

$$2y - 25 = 175,$$

or $$2y - 841 = 34481.$$

Therefore $$y = 100, 17661, \ldots$$

"Finding the respective values of the numbers, they are (75, 100), (16820, 17661), etc."

Example 3. "Bring out quickly those two numbers of which the sum of the cube (of one) and the square (of the other) becomes a square and whose sum also is a square."[2]

That is to say, solve

$$\begin{cases} x^3 + y^2 = u^2, & (1) \\ x + y \ \ = v^2. & (2) \end{cases}$$

This problem has been solved by Bhâskara II in two ways, which are substantially as follows:

First method. From (1) we get

$$u = \tfrac{1}{2}\left(\frac{x^3}{\lambda} + \lambda\right), \quad y = \tfrac{1}{2}\left(\frac{x^3}{\lambda} - \lambda\right),$$

where λ is an arbitrary number. Putting $\lambda = x$, we get

$$u = \tfrac{1}{2}(x^2 + x), \ y = \tfrac{1}{2}(x^2 - x).$$

Substituting this value of y in (2), we get

$$x^2 + x = 2v^2,$$

or $$(2x + 1)^2 = 8v^2 + 1.$$

[1] The reference is to the rule on p. 249.
[2] *BBi*, p. 107.

By the method of the Square-nature we have

$$\left.\begin{array}{r} v = 6 \\ 2x + 1 = 17 \end{array}\right\}, \qquad \left.\begin{array}{r} v = 35 \\ 2x + 1 = 99 \end{array}\right\}, \ldots$$

Whence the values of (x, y) are $(8, 28)$, $(49, 1176)$, ...

Second Method. Assume $x = 2w^2$, $y = 7w^2$. Then

$$x + y = 9w^2 = (3w)^2.$$

So the equation (2) is satisfied. Now, substituting those values in (1) we get

$$8w^6 + 49w^4 = u^2,$$

or $$w^4(8w^2 + 49) = u^2.$$

If $$8w^2 + 49 = z^2,$$

then $$u = zw^2.$$

Now the values of w making $8w^2 + 49$ a square are 2, 3, 7... Hence the required numbers (x, y) are $(8, 28)$, $(18, 63)$, $(98, 343)$, ...

Example 4. "What is that number which multiplied by three and added with unity becomes a cube; the cube-root squared and multiplied by three becomes, together with unity, a square."[1]

That is to say, solve

$$\begin{cases} 3x + 1 = u^3, & (1) \\ 3u^2 + 1 = v^2. & (2) \end{cases}$$

It has been solved by Bhâskara II thus:

From (2), by the method of the Square-nature, we get the values of (u, v) as $(1, 2)$, $(4, 7)$, $(15, 26)$, ... Whence the values of x are 21, $3374/3$, ...

[1] *BBi*, p. 119. This problem is admittedly taken by Bhâskara II from an earlier writer.

Alternatively[1] we assume $u = 3y + 1$; then from the equation (1) we get

$$x = 3y(3y^2 + 3y + 1).$$

Also from (2) we have

$$27y^2 + 18y + 4 = v^2$$
$$= (my - 2)^2, \text{ say.}$$

Hence
$$y = \frac{18 + 4m}{m^2 - 27}.$$

Therefore, the required value of x is

$$x = 9\left(\frac{18 + 4m}{m^2 - 27}\right)^3 + 9\left(\frac{18 + 4m}{m^2 - 27}\right)^2 + 3\left(\frac{18 + 4m}{m^2 - 27}\right),$$

where m is a rational number greater than 5.

The first of the previous solutions is given by $m = 9$.

Double Equations in Several Unknowns. To solve a double equation involving several unknowns, Bhâskara II gives the following hints:

"When there are square and other powers of three or more unknowns, leaving out any two unknowns at pleasure, the values of others should be arbitrarily assumed and the roots investigated."[2]

For the case of a single equation, he says:

"But when there is only one equation, the roots should be determined as before after assuming optional values for all the unknowns except one."

27. MULTIPLE EQUATIONS

There are some very elegant problems in which three or more functions, linear or quadratic, of the unknowns have to be made squares or cubes. The

[1] See *BBi*, p. 121. [2] *BBi*, p. 106.

following example occurs in the *Laghu-Bhâskarîya* of Bhâskara I[1] (522):

Example 1. To find two numbers x and y such that the expressions $x+y$, $x-y$, $xy+1$ are each a perfect square.

Brahmagupta gives the following solution:

"A square is increased and diminished by another. The sum of the results is divided by the square of half their difference. Those results multiplied (severally) by this quotient give the numbers whose sum and difference are squares as also their product together with unity."[2]

Thus the solution is:

$$x = P(m^2 + n^2),$$
$$y = P(m^2 - n^2),$$

where $P = \dfrac{(m^2+n^2)+(m^2-n^2)}{[\frac{1}{2}\{(m^2+n^2)-(m^2-n^2)\}]^2}$, m, n being any rational numbers.

Nârâyaṇa (1357) says:

"The square of the square of an optional number is set down at two places. It is decreased by the square (at one place) and increased (at another), and then doubled. The sum and difference of the results are squares and so also their product together with unity."[3]

That is,
$$x = 2(p^4 + p^2),$$
$$y = 2(p^4 - p^2),$$

where p is any rational number.

[1] *LBh*, viii. 17. [2] *BrSpSi*, xviii. 72.
[3] *GK*, i. 47.

The *rationale* of this solution is as follows:

Suppose

$$x = 2z^2(m^2 + n^2), \quad y = 2z^2(m^2 - n^2),$$

so that $x \pm y$ are already squares. We have, therefore, only to make

$$xy + 1 = 4z^4(m^2 + n^2)(m^2 - n^2) + 1 = \text{a square.}$$

Now

$$4z^4(m^4 - n^4) + 1 = (2z^2m^2 - 1)^2 + 4z^2(m^2 - z^2n^4).$$

Hence, in order that $xy + 1$ may be a square, one sufficient condition is

$$m^2 = z^2n^4.$$

Therefore $$2z^2 = \frac{2m^2}{n^4} = \frac{(m^2+n^2)+(m^2-n^2)}{[\frac{1}{2}\{(m^2+n^2)-(m^2-n^2)\}]^2}.$$

Again $$x = 2z^2(m^2 + n^2) = 2(z^4n^4 + z^2n^2),$$

or $$x = 2(p^4 + p^2), \text{ if } p = zn.$$

Therefore $$y = 2(p^4 - p^2).$$

Example 2. "If thou be expert in mathematics, tell me quickly those two numbers whose sum and difference are squares and whose product is a cube."[1]

That is, solve

$$\left.\begin{aligned} x \pm y &= \text{squares,} \\ xy &= \text{a cube.} \end{aligned}\right\}$$

Bhâskara II says:

"Here let the two numbers be $5z^2$, $4z^2$. *They are assumed such as will make their sum and difference both squares.* Their product is $20z^4$. This must be a cube. Putting it equal to the cube of an optional multiple[2] of $10z$ and removing the common factor z^3 from the sides as before, (we shall ultimately find) the numbers to be 10000, 12500."

[1] *BBi*, p. 56. [2] *G.K*, i. 49.

In general, let us assume, as directed by Bhâskara II,

$$x = (m^2 + n^2)z^2, \quad y = 2mnz^2,$$

which will make $x \pm y$ squares. We have, therefore, only to make

$$2mn(m^2 + n^2)z^4 = \text{a cube.}$$

Let $$2mn(m^2 + n^2)z^4 = p^3z^3.$$

Then $$z = \frac{p^3}{2mn(m^2 + n^2)}.$$

Therefore $$x = \frac{(m^2 + n^2)p^6}{\{2mn(m^2 + n^2)\}^2},$$

$$y = \frac{2mnp^6}{\{2mn(m^2 + n^2)\}^2},$$

where m, n, p are arbitrary.

This general solution has been explicitly stated by Nârâyaṇa thus :

"The square of the cube of an optional number is divided by the square of the product of the two numbers stated above and then severally multiplied by those numbers. (Thus will be obtained) two numbers whose sum and difference are squares and whose product is a cube."[1]

The two numbers stated above[2] are $m^2 + n^2$ and $2mn$ whose sum and difference are squares.

In particular, putting $m = 1$, $n = 2$, $p = 10$, Nârâyaṇa finds $x = 12500$, $y = 10000$. With other values of m, n, p he obtains the values (3165/16, 625/4), (62500/117, 250000/507), (15625/1872, 15625/2028); and observes: "thus by virtue of (the multiplicity of) the optional numbers many values can be found."

[1] *GK*, i. 49. [2] The reference is to rule i. 48.

Example 3. To find numbers such that each of them severally added to a given number becomes a square; and so also the product of every contiguous pair increased by another given number.

For instance, let it be required to find *four* numbers such that

$$x + \alpha = p^2, \qquad xy + \beta = \xi^2,$$
$$y + \alpha = q^2, \qquad yz + \beta = \eta^2,$$
$$z + \alpha = r^2, \qquad zw + \beta = \zeta^2.$$
$$w + \alpha = s^2,$$

The method for the solution of a problem of this kind is indicated in the following rule quoted by Bhâskara II (1150) from an earlier writer, whose name is not known:

"As many multiple (*guṇa*) as the product-interpolator (*vadha-kṣepa*) is of the number-interpolator (*râśi-kṣepa*), with the square-root of that as the common difference are assumed certain numbers; these squared and diminished by the number-interpolator (severally) will be the unknowns."[1]

In applying this method to solve a particular problem, to be stated presently, Bhâskara II observes by way of explanation:

"In these cases, that which being added to an (unknown) number makes it a square is designated as the number-interpolator. The number-interpolator multiplied by the square of the difference of the square-roots pertaining to the numbers, is equal to the product-interpolator. For the product of those two numbers added with the latter certainly becomes a square. The products of two and two contiguous of the square-roots pertaining to the numbers diminished by the

[1] *BBi*, p. 68.

number-interpolator are the square-roots corresponding to the products of the numbers."[1]

Since $x = p^2 - \alpha$, $y = q^2 - \alpha$, we get

$$xy + \beta = (p^2 - \alpha)(q^2 - \alpha) + \beta$$
$$= (pq - \alpha)^2 + \{\beta - \alpha(q - p)^2\}.$$

In order that $xy + \beta$ may be a square, a sufficient condition is

$$\alpha(q - p)^2 = \beta,$$

or $\quad q = p \pm \sqrt{\beta/\alpha} = p \pm \gamma$, where $\gamma = \sqrt{\beta/\alpha}$.

Then $\quad xy + \beta = (pq - \alpha)^2$.

Hence $\quad \xi = pq - \alpha$.

Similarly $\quad r = q \pm \gamma, \quad s = r \pm \gamma$.

Thus, it is found that the square-roots p, q, r, s form an A.P. whose common difference is γ $(= \sqrt{\beta/\alpha})$.

Further, we have

$$x = p^2 - \alpha,$$
$$y = (p \pm \gamma)^2 - \alpha,$$
$$z = (p \pm 2\gamma)^2 - \alpha,$$
$$w = (p \pm 3\gamma)^2 - \alpha,$$

as stated in the rule.

These values of the unknowns, it will be easily found, satisfy all the conditions about their products. For

$$xy + \beta = \{p(p \pm \gamma) - \alpha\}^2,$$
$$yz + \beta = \{(p \pm \gamma)(p \pm 2\gamma) - \alpha\}^2,$$
$$zw + \beta = \{(p \pm 2\gamma)(p \pm 3\gamma) - \alpha\}^2.$$

[1] *BBi*, p. 67.

Thus we have

$$\xi = p(p \pm \gamma) - \alpha,$$
$$\eta = (p \pm \gamma)(p \pm 2\gamma) - \alpha,$$
$$\zeta = (p \pm 2\gamma)(p \pm 3\gamma) - \alpha;$$

as stated by Bhâskara II.

It has been observed by him that the above principle is well known in mathematics. But we do not find it in the works anterior to him, which are available to us.

It is noteworthy that the above principle will hold even when all the β's are not equal. For, suppose that in the above instance the second set of conditions is replaced by the following:

$$xy + \beta_1 = \xi^2,$$
$$yz + \beta_2 = \eta^2,$$
$$zw + \beta_3 = \zeta^2.$$

Then, proceeding in the same way, we find that

$$q = p \pm \sqrt{\beta_1/\alpha}, \quad r = q \pm \sqrt{\beta_2/\alpha}, \quad s = r \pm \sqrt{\beta_3/\alpha},$$

and $\xi = pq - \alpha, \quad \eta = qr - \alpha, \quad \zeta = rs - \alpha.$

It should also be noted that in order that $xy + \beta$ or $p^2q^2 - \alpha(p^2 + q^2) + \alpha^2 + \beta$ may be a square, there may be other values of q besides the one specified above, namely $q = p \pm \sqrt{\beta/\alpha}$. We may, indeed, regard

$$p^2q^2 - \alpha(p^2 + q^2) + \alpha^2 + \beta = \xi^2$$

as an indeterminate equation in q. Since we know one solution of it, namely $q = p \pm \gamma$, $\xi = p(p \pm \gamma) - \alpha$, we can find an infinite number of other solutions by the method of the Square-nature.

Now, suppose that another condition is imposed on the numbers, *viz.*,

$$wx + \beta' = \mu^2.$$

On substituting the values of x and w this condition transforms into

$$p^4 \pm 6\gamma p^3 + (9\gamma^2 + 2\alpha)p^2 \pm 6\alpha\gamma p + \alpha^2 - 9\beta + \beta' = \mu^2,$$

an indeterminate equation of the fourth degree in p.

In the following example and its solution from Bhâskara II we find the application of the above principle :

Example. "What are those four numbers which together with 2 become capable of yielding square-roots ; also the products of two and two contiguous of which added by 18 yield square-roots ; and which are such that the square-root of the sum of all the roots added by 11 becomes 13. Tell them to me, O algebraist friend."[1]

Solution. "In this example, the product-interpolator is 9 times the number-interpolator. The square-root of 9 is 3. Hence the square-roots corresponding to the numbers will have the common difference 3. Let them be

$$x, x + 3, x + 6, x + 9.$$

"Now the products of two and two contiguous of these minus the number-interpolator are the square-roots pertaining to the products of the numbers as increased by 18. So these square-roots are

$$x^2 + 3x - 2,$$
$$x^2 + 9x + 16,$$
$$x^2 + 15x + 52.$$

"The sum of these and the previous square-roots all together is $3x^2 + 31x + 84$. This added with 11

[1] *BBi*, p. 67.
It will be noticed that by virtue of the last condition the problem becomes, in a way, determinate.

becomes equal to 169. Hence

$$3x^2 + 31x + 95 = 0x^2 + 0x + 169.$$

"Multiplying both sides by 12, superadding 961, and then extracting square-roots, we get

$$6x + 31 = 0x + 43.$$
$$\therefore \quad x = 2.$$

"With the value thus obtained, we get the values of the square roots pertaining to the numbers to be 2, 5, 8, 11. Subtracting the number-interpolator from the squares of these, we have the (required) numbers as 2, 23, 62, 119."

Example 4. To find two numbers such that

$$\begin{aligned} x - y + k &= u^2, \\ x + y + k &= v^2, \\ x^2 - y^2 + k' &= s^2, \\ x^2 + y^2 + k'' &= t^2. \end{aligned}$$

Bhâskara II says:

"Assume first the value of the square-root pertaining to the difference (of the numbers wanted) to be any unknown with or without an absolute number. The root corresponding to the sum will be equal to the root pertaining to the difference together with the square-root of the quotient of the interpolator of the difference of the squares divided by the interpolator for the sum or difference of the numbers. The squares of these two less their interpolator are the sum and difference of the numbers. From them the two numbers can be found by the rule of concurrence."[1]

[1] *BBi*, pp. 111ff.

That is to say, if w is any rational number, we assume

$$u = w \pm \alpha,$$

where α is an absolute number which may be o. Then

$$v = (w \pm \alpha) + \sqrt{k'/k}.$$

Now
$$\begin{aligned} x^2 - y^2 + k' &= (x-y)(x+y) + k' \\ &= (u^2 - k)(v^2 - k) + k' \\ &= u^2v^2 - k(u^2 + v^2) + k^2 + k'. \end{aligned}$$

One sufficient condition that the right-hand side may be a square is

$$k(v - u)^2 = k',$$

or
$$v = u + \sqrt{k'/k},$$

which is stated in the rule. Therefore,

$$\begin{aligned} x - y &= (w \pm \alpha)^2 - k, \\ x + y &= (w \pm \alpha + \sqrt{k'/k})^2 - k. \end{aligned}$$

Hence
$$\begin{aligned} x &= \tfrac{1}{2}\{(w \pm \alpha)^2 + (w \pm \alpha + \sqrt{k'/k})^2 - 2k\}, \\ y &= \tfrac{1}{2}\{(w \pm \alpha + \sqrt{k'/k})^2 - (w + \alpha)^2\}. \end{aligned}$$

Now, if γ denotes $\sqrt{k'/k}$, we get

$$\begin{aligned} x^2 + y^2 = u^4 + 2\gamma u^3 &+ (3\gamma^2 - 2k)u^2 \\ &+ 2\gamma(\gamma^2 - k)u + \tfrac{1}{2}k^2 + \tfrac{1}{2}(\gamma^2 - k)^2. \end{aligned}$$

So it now remains to solve

$$\begin{aligned} u^4 + 2\gamma u^3 + (3\gamma^2 - 2k)u^2 &+ 2\gamma(\gamma^2 - k)u \\ &+ \tfrac{1}{2}k^2 + \tfrac{1}{2}(\gamma^2 - k)^2 + k'' = t^2, \end{aligned}$$

which is an indeterminate equation in u.

Applications. We take an illustrative example with its solution from Bhâskara II.

"O thou of fine intelligence, state a pair of numbers, other than 7 and 6, whose sum and difference (severally) added with 3 are squares; the sum of their squares decreased by 4 and the difference of the squares increased by 12 are also squares; half their product together with the smaller one is a cube; again the sum of all the roots plus 2 is a square."[1]

That is to say, if $x > y$, we have to solve

$$x - y + 3 = u^2,$$
$$x + y + 3 = v^2,$$
$$x^2 - y^2 + 12 = s^2,$$
$$x^2 + y^2 - 4 = t^2,$$
$$\tfrac{1}{2}xy + y = p^3,$$
$$u + v + s + t + p + 2 = q^2.$$

This problem has been solved in two ways:

First Method. As directed in the above rule, assume

$$u = w - 1.$$

Then $\qquad x - y = (w - 1)^2 - 3 = w^2 - 2w - 2,$

$$x + y = (\overline{w - 1} + 2)^2 - 3 = w^2 + 2w - 2.$$

Therefore $\qquad x = w^2 - 2, \quad y = 2w.$

Now, we find that

$$x^2 - y^2 + 12 = (w^2 - 4)^2,$$
$$x^2 + y^2 - 4 = w^4,$$
$$\tfrac{1}{2}xy + y = w^3.$$

So all the equations except the last one are already satisfied. This remaining equation now reduces to

$$2w^2 + 3w - 2 = q^2.$$

Completing the square on the left-hand side of this equation, we get

$$(4w + 3)^2 = 8q^2 + 25.$$

[1] *BBi*, p. 115.

By the method of the Square-nature its solutions are

$$\left.\begin{array}{r}q=5\\ 4w+3=15\end{array}\right\},\qquad \left.\begin{array}{r}q=175\\ 4w+3=495\end{array}\right\},\quad\ldots$$

Therefore $\qquad w=3,\ 123,\ldots$

Hence the values of (x, y) are $(7, 6)$, $(15127, 246)$, ...

Second Method. Or assume[1]

$$x-y+3=w^2,$$

then $\qquad x+y+3=w^2+4w+4=(w+2)^2.$

Whence $\qquad x=w^2+2w-1,\quad y=2w+2.$

Now, we find that

$$x^2-y^2+12=(w^2+2w-3)^2,$$
$$x^2+y^2-4=(w^2+2w+1)^2,$$
$$\tfrac{1}{2}xy+y=(w+1)^3.$$

Then the remaining condition reduces to

$$2w^2+7w+3=q^2.$$

Completing the square on the left-hand side, we get

$$(4w+7)^2=8q^2+25.$$

Whence by the method of the Square-nature, we get

$$\left.\begin{array}{r}q=5\\ 4w+7=15\end{array}\right\},\qquad \left.\begin{array}{r}q=175\\ 4w+7=495\end{array}\right\},\quad\ldots$$

Therefore $\qquad w=2,\ 122,\ldots$

Hence $\qquad (x, y)=(7, 6),\ (15127, 246),\ldots$

Another very interesting example which has been borrowed by Bhâskara II from an earlier writer is the following:[2]

[1] This is clearly equivalent to the supposition, $u=w$, $v=w+2$.

[2] The text is *kasyâpyudâharaṇam* ("the example of some one"). This observation appears to indicate that this particular example was borrowed by Bhâskara II from a secondary source; its primary source was not known to him.

"Tell me quickly, O sound algebraist, two numbers, excepting 6 and 8, which are such that the cube-root of half the sum of their product and the smaller one, the square-root of the sum of their squares, the square-roots of the sum and difference of them (each) increased by 2, and of the difference of their squares plus 8, all being added together, will be capable of yielding a square-root."[1]

That is to say, if $x > y$, we have to solve

$$\sqrt[3]{\tfrac{1}{2}(xy + y)} + \sqrt{x^2 + y^2} + \sqrt{x^2 - y^2 + 8}$$
$$+ \sqrt{x + y + 2} + \sqrt{x - y + 2} = q^2$$

In every instance of this kind, remarks Bhâskara II, "the values of the two unknown numbers should be so assumed in terms of another unknown that all the stipulated conditions will be satisfied." In other words, the equation will have to be resolved into a number of other equations all of which have to be satisfied simultaneously. Thus we shall have to solve

$$x - y + 2 = u^2,$$
$$x + y + 2 = v^2,$$
$$x^2 - y^2 + 8 = s^2,$$
$$x^2 + y^2 = t^2,$$
$$\tfrac{1}{2}(xy + y) = p^3,$$
$$u + v + s + t + p = q^2.$$

The last equation represents the original one.

There have been indicated several methods of solving these equations.

(*i*) Set $x = w^2 - 1$, $y = 2w$; then we find that

$$x - y + 2 = (w - 1)^2,$$
$$x + y + 2 = (w + 1)^2,$$

[1] *BBi*, p. 110.

$$x^2 - y^2 + 8 = (w^2 - 3)^2,$$
$$x^2 + y^2 = (w^2 + 1)^2,$$
$$\tfrac{1}{2}(xy + y) = w^3.$$

So all the equations except the last one are identically satisfied. This last equation now becomes

$$2w^2 + 3w - 2 = q^2.$$

Completing the square on the left-hand side, we get

$$(4w + 3)^2 = 8q^2 + 25.$$

Solutions of this are

$$\left.\begin{array}{r} q = 5 \\ 4w + 3 = 15 \end{array}\right\}, \quad \left.\begin{array}{r} q = 30 \\ 4w + 3 = 85 \end{array}\right\}, \quad \left.\begin{array}{r} q = 175 \\ 4w + 3 = 495 \end{array}\right\}, \ldots$$

Therefore, we have the solutions of our problem as

$$(x, y) = (8, 6), (1677/4, 41), (15128, 246), \ldots$$

Or set

$$(ii) \quad \begin{cases} x = w^2 + 2w, \\ y = 2w + 2; \end{cases}$$

$$(iii) \quad \begin{cases} x = w^2 - 2w, \\ y = 2w - 2; \end{cases}$$

$$\text{or} \quad (iv) \quad \begin{cases} x = w^2 + 4w + 3, \\ y = 2w + 4. \end{cases}$$

In conclusion Bhâskara II remarks: "Thus there may be a thousandfold artifices; since they are hidden to the dull, a few of them have been indicated here out of compassion for them."[1]

It will be noticed that in devising the various artifices noted above for the solution of the problem, Bhâskara II has been in each case guided by the result that if $u = w \pm \alpha$, then, $v = w \pm \alpha + \sqrt{k'/k}$. He has simply taken different values of α in the different cases.

[1] *BBi*, p. 110.

28. SOLUTION OF $axy = bx + cy + d$

Bakhshâlî Treatise. The earliest instance of a quadratic indeterminate equation of the type $axy = bx + cy + d$, in Hindu mathematics occurs in the Bakhshâlî Treatise (*c.* 200).[1] The text is very mutilated. But the example that is preserved is

$$xy = 3x + 4y \mp 1,$$

of which the solutions preserved are

$$x = \frac{3 \cdot 4 - 1}{1} + 4 = 15,$$

$$y = 1 + 3 = 4;$$

and

$$x = 1 + 4 = 5,$$

$$y = \frac{3 \cdot 4 + 1}{1} + 3 = 16.$$

Hence, in general, the solutions of the equation

$$xy = bx + cy + d,$$

which appear to have been given are:

$$\left.\begin{array}{l} x = \dfrac{bc + d}{m} + c, \\ y = m + b; \end{array}\right\} \quad \text{or} \quad \left.\begin{array}{l} x = m + c, \\ y = \dfrac{bc + d}{m} + b; \end{array}\right\}$$

where m is an arbitrary number.

An Unknown Author's Rule. Brahmagupta (628) has described the following method taken from an author who is not known now.[2]

[1] *BMs*, Folio 27, recto; compare also Kaye's Introduction §82.

[2] Pṛthûdakasvâmî (860) says that the method is due to a writer other than Brahmagupta. This is further corroborated by Brahmagupta's strictures on it (*vide infra*, p. 299).

"The product of the coefficient of the factum and the absolute number together with the product of the coefficients of the unknowns is divided by an optional number. Of the optional number and the quotient obtained, the greater is added to the lesser (of the coefficients of the unknowns) and the lesser to the greater (of the coefficients), and (the sums) are divided by the coefficient of the factum. (The results will be values of the unknowns) in the reverse order."[1]

As has been observed by Pṛthûdakasvâmî, this rule is to be applied to an equation containing the factum after it has been prepared by transposing the factum term to one side and the absolute term together with the simple unknown terms to the other. Then the solutions will be, m being an arbitrary rational number,

$$x = \frac{1}{a}(m + c),$$

$$y = \frac{1}{a}\left(\frac{ad + bc}{m} + b\right),$$

if $b > c$ and $m > \frac{ad + bc}{m}$. If these conditions be reversed then x and y will have their values interchanged.

The *rationale* of the above solutions can be easily shown to be as follows:

$$axy = bx + cy + d,$$

or $$a^2xy - abx - acy = ad,$$

or $$(ax - c)(ay - b) = ad + bc.$$

Suppose $$ax - c = m, \text{ a rational number};$$

then $$ay - b = \frac{ad + bc}{m}.$$

[1] *BrSpSi*, xviii, 60.

Therefore $$x = \frac{1}{a}(m + c),$$

$$y = \frac{1}{a}\left(\frac{ad + bc}{m} + b\right).$$

Or, we may put $$ay - b = m;$$

then we shall have $$ax - c = \frac{ad + bc}{m};$$

whence $$x = \frac{1}{a}\left(\frac{ad + bc}{m} + c\right),$$

$$y = \frac{1}{a}(m + b).$$

It will thus be found that the restrictive condition of adding the greater and lesser of the numbers m and $(ad + bc)/m$ to the lesser and greater of the numbers b and c respectively as adumbrated in the above rule is quite unnecessary.

Brahmagupta's Rule. Brahmagupta gives the following rule for the solution of a quadratic indeterminate equation involving a factum :

"With the exception of an optional unknown, assume arbitrary values for the rest of the unknowns, the product of which forms the factum. The sum of the products of these (assumed values) and the (respective) coefficients of the unknowns will be absolute quantities. The continued products of the assumed values and of the coefficient of the factum will be the coefficient of the optionally (left out) unknown. Thus the solution is effected without forming an equation of the factum. Why then was it done so?"[1]

The reference in the latter portion of this rule is to the method of the unknown writer. The principle

[1] *BrSpSi*, xviii. 62-3, *vide supra*, p. 297.

underlying Brahmagupta's method is to reduce, like the Greek Diophantus (*c.* 275),[1] the given indeterminate equation to a simple determinate one by assuming arbitrary values for all the unknowns except one. So it is undoubtedly inferior to the earlier method. Brahmagupta gives the following illustrative example :

"On subtracting from the product of signs and degrees of the sun, three and four times (respectively) those quantities, ninety is obtained. Determining the sun within a year (one can pass as a proficient) mathematician."[2]

If x denotes the signs and y the degrees of the sun, then the equation is

$$xy - 3x - 4y = 90.$$

Thus this problem, as that of Bhâskara II (*infra*), appears to have some relation with that of the Bakhshâlî work. Pṛthûdakasvâmî solves it in two ways. *Firstly*, he assumes the arbitrary number to be 17, then

$$x = \frac{1}{1}\left(\frac{90.1 + 3.4}{17} + 4\right) = 10,$$

$$y = \frac{1}{1}(17 + 3) = 20.$$

Secondly, he assumes arbitrarily $y = 20$. On substituting this value in the above equation, it reduces to

$$20x - 3x = 170\,;$$

whence $x = 10$.

Mahâvîra's Rule. Mahâvîra (850) has not treated equations of this type. There are, however, two problems in his *Gaṇita-sâra-saṁgraha* which involve similar equations. One of them is to find the increase or

[1] Heath, *Diophantus*, pp. 192-4, 262.
[2] *BrSpSi*, xviii. 61.

decrease of two numbers (a, b) so that the product of the resulting numbers will be equal to another optionally given number (d). Thus we are to solve

$$(a \pm x)(b \pm y) = d,$$

or

$$xy \pm (bx + ay) = d - ab.$$

The rule given for solving this is :

"The difference between the product of the given numbers and the optional number is put down at two places. It is divided (at one place) by one of the given numbers increased by unity and (at the other) by the optional number increased by the other given number. These will give in the reverse order the values of the quantities to be added or subtracted."[1]

That is to say,

$$\left.\begin{aligned} x &= \frac{d \sim ab}{d + b} \\ y &= \frac{d \sim ab}{a + 1} \end{aligned}\right\}, \quad \text{or} \quad \left.\begin{aligned} x &= \frac{d \sim ab}{b + 1} \\ y &= \frac{d \sim ab}{d + a} \end{aligned}\right\}$$

Thus the solutions given by Mahâvîra are much cramped. The other problem considered by him is to separate the capital, interest and time when their sum is given : If x be the capital invested and y the period of time in months, then the interest will be mxy, where m is the rate of interest per month. Then the problem is to solve

$$mxy + x + y = p.$$

Mahâvîra solves this equation by assuming arbitrary values for y.[2]

[1] *GSS*, vi. 284.

[2] *GSS*, vi. 35.

Śrîpati's Rule. Śrîpati (1039) gives the following rule :

"Remove the factums from one side, the (simple) unknowns and the absolute numbers from the other. The product of the coefficients of the unknowns being added to the product of the absolute quantity and the coefficient of the factum, (the sum) is divided by an optional number. The quotient and the divisor should be added arbitrarily to the greater or smaller of the coefficients of the unknowns. These divided by the coefficient of the factum will be the values of the unknowns in the reverse order."[1]

$$i.e.,\ \left.\begin{aligned} x &= \frac{1}{a}(m+c) \\ y &= \frac{1}{a}\left(\frac{ad+bc}{m}+b\right) \end{aligned}\right\},\quad \left.\begin{aligned} x &= \frac{1}{a}\left(\frac{ad+bc}{m}+c\right) \\ y &= \frac{1}{a}(m+b) \end{aligned}\right\},$$

where m is arbitrary.

Bhâskara II's Rule. Bhâskara II (1150) has given two rules for the solution of a quadratic indeterminate equation containing the product of the unknowns. His first method is the same as that of Brahmagupta:

"Leaving one unknown quantity optionally chosen, the values of the other should be assumed arbitrarily according to convenience. The factum will thus be reduced and the required solution can then be obtained by the first method of analysis."[2]

Bhâskara's aim was to obtain *integral* solutions. The above method is, however, not convenient for the purpose. He observes :

"On assuming in this way an arbitrary known value for one of the unknowns, the *integral* values of the

[1] *SiŚe*, xiv. 20-1. [2] *BBi*, p. 123.

two unknowns can be obtained with much difficulty."[1] So he describes a second method "by which they can be obtained with little difficulty."

"Transposing the factum from one side chosen at pleasure, and the (simple) unknowns and the absolute number from the other side (of the equation), and then dividing both the sides by the coefficient of the factum, the product of the coefficients of the unknowns together with the absolute number is divided by an optional number. The optional number and that quotient should be increased or diminished by the coefficients of the unknowns at pleasure. They (results thus obtained) should be known as the values of the two unknowns reciprocally."[2]

This rule has been elucidated by the author thus:

"From one of the two equal sides the factum being removed, and from the other the unknowns and the absolute number; then dividing the two sides by the coefficients of the factum, the product of the coefficients of the unknowns on the other side added to the absolute number, is divided by an optional number. The optional number and the quotient being arbitrarily added to the coefficients of the unknowns, should be known as the values of the unknowns in the reciprocal order. That is, the one to which the coefficient of the *kâlaka* (the second unknown) is added, will be the value of the *yâvat-tâvat* (the first unknown) and the one to which the coefficient of the *yâvat-tâvat* is added, will be the value of the *kâlaka*. But if, after that has been done, owing to the magnitude, the statements (of the problem) are not fulfilled, then

[1] "Evamekasmin vyakte râśau kalpite sati bahûnâyâsenabhinnau râśî jñâyete"—*BBi*, p. 124.
[2] *BBi*, pp. 124f.

from the optional number and the quotient, the coefficients of the unknowns should be subtracted, and (the remainders) will be the values of the unknowns in the reciprocal order."

Thus Bhâskara's solutions are

$$\left.\begin{aligned} x &= \frac{c}{a} \pm m' \\ y &= \frac{b}{a} \pm n' \end{aligned}\right\} \qquad \left.\begin{aligned} x &= \frac{c}{a} \pm n' \\ y &= \frac{b}{a} \pm m' \end{aligned}\right\}$$

where m' is any arbitrary number and $n' = \frac{1}{m'}\left(\frac{bc}{a^2} + \frac{d}{a}\right)$.

The *rationale* of these solutions is as follows:

$$axy = bx + cy + d,$$

or $$xy - \frac{b}{a}x - \frac{c}{a}y = \frac{d}{a},$$

or $$\left(x - \frac{c}{a}\right)\left(y - \frac{b}{a}\right) = \frac{d}{a} + \frac{bc}{a^2} = m'n', \text{ say.}$$

Then, either

$$\left.\begin{aligned} x - \frac{c}{a} &= \pm m' \\ y - \frac{b}{a} &= \pm n' \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} x - \frac{c}{a} &= \pm n' \\ y - \frac{b}{a} &= \pm m' \end{aligned}\right\};$$

whence the solutions.

Bhâskara's Proofs. The same *rationale* of the above solutions has been given also by Bhâskara II with the help of the following illustrative example. He observes that the proof "is twofold in every case: one geometrical (*kṣetragata*), the other algebraic (*râśigata*)."[1]

Example. "The sum of two numbers multiplied by four and three, added by two is equal to the product

[1] *BBi*, p. 125.

of those numbers. Tell me, if thou knowest, those two numbers."[1]

Solution. "Having performed the operations as stated, the sides are

$$xy = 4x + 3y + 2.$$

The product of the coefficients of the unknowns plus the absolute term is 14. Dividing this by an optional number (say) unity, the optional number and the quotient are 1, 14. To these being arbitrarily added 4, 3, the coefficients of the unknowns, the values of (x, y) are (4, 18) or (17, 5). (Dividing) by (the optional number) 2, (other values will be) (5, 11) or (10, 6.)"[2]

Geometrical Proof. "The second side of the equation is equal to the factum. But the factum is the area of an oblong quadrilateral of which the base and upright are the unknown quantities. Within this figure (Fig. 15) exist four x's, three y's and the absolute number 2. From this figure on taking off four x's and y minus four multiplied by its own coefficient, (*i.e.*, 3), it becomes this (Fig. 16).

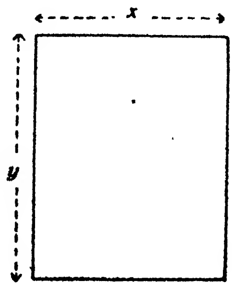

Fig. 15

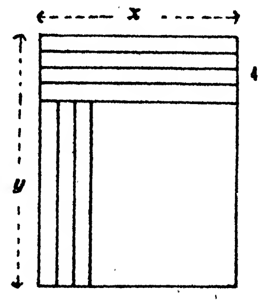

Fig. 16

The other side of the equation being so treated there-

[1] *BBi*, pp. 123, 125. [2] *BBi*, p. 125.

results 14. This must be the area of the figure remaining at the corner (see Fig. 16) within the rectangle representing the factum, and is the product of its base and upright. But these are (still) to be known here. Therefore, assuming an optional number for the base, the upright will be obtained on dividing the area 14 by it. One of these, base and upright, being increased by 4, the coefficient of x will be the upright of the figure representing the factum, because when four x's were separated from the factum-figure, its upright was lessened by 4. Similarly the other being increased by 3, the coefficient of y, will be the base They are precisely the values of x and y."[1]

Algebraic Proof "This is also geometrical in origin. In this the values of the base and upright of the smaller rectangle within the rectangle whose base and upright are x and y respectively, are assumed to be two other unknowns u and v.[2] One of them being increased by the coefficient of x will be the value of the upright of the outer figure and the other being increased by the coefficient of y will be taken to be the value of the base of the outer figure. Thus $y = u + 4$, $x = v + 3$. Substituting these values of the unknowns x, y, on both sides of the equation, the upper side will be $3u + 4v + 26$ and the factum side will be $uv + 3u + 4v + 12$. On making perfect clearance between these sides, the lower side becomes uv and the upper side 14. This is the area of that inner rectangle and it is equal to the product of the coefficients of the unknowns plus the absolute number. How the values of the unknowns are to be thence deduced, has been already explained."[3]

[1] *BBi*, p. 126.

[2] In the original text they are respectively *nī* (for *nīlaka*) and *pī* (for *pītaka*).

[3] *BBi*, p. 127.

Bhâskara II further observes :

"Thus the proof of the solution of the factum has been shown to be of two kinds. What has been said before—the product of the coefficients of the unknowns together with the absolute number is equal to the area of another rectangle inside the rectangle representing the factum and lying at a corner—is sometimes otherwise. For, when the coefficients of the unknowns are negative, the factum-rectangle will be inside the other rectangle at one corner ; and when the coefficients of the unknowns are greater than the base and upright of the factum-rectangle, and are positive, the other will be outside the factum rectangle and at a corner, as (Figs. 17, 18).

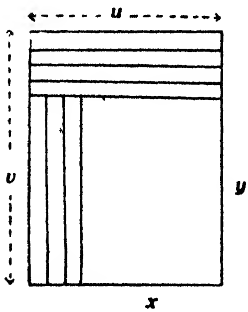

Fig. 17

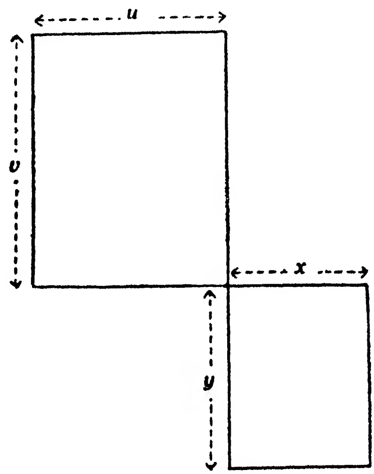

Fig. 18

When it is so, the coefficients of the unknowns lessened by the optional number and the quotient, will be the values of x and y."[1]

[1] *BBi*, p. 127.