Differential Equations

By Harry W. Reddick

DIFFERENTIAL EQUATIONS
Second Edition

By Harry W. Reddick and Frederic H. Miller

ADVANCED MATHEMATICS FOR ENGINEERS
Second Edition

Differential Equations

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SECOND EDITION

NEW YORK · JOHN WILEY & SONS, INC.
LONDON · CHAPMAN & HALL, LIMITED

SECOND EDITION

Second Printing, July, 1950

PRINTED IN THE UNITED STATES OF AMERICA

Preface

This book deals with methods of solving ordinary differential equations and with problems in applied mathematics involving ordinary differential equations; it includes no treatment of partial differential equations.

It has been my aim, in both the theory and the numerous illustrative examples, to achieve a clarity of explanation which will enable any student who desires to understand actually to do so. In the physical applications the importance of setting forth clearly the physical units involved is stressed, so that numerical results with proper units attached may be obtained.

Changes in the second edition include a new chapter on the linear equation of second order and additional material on hyperbolic functions, systems of curves, and vibratory motion. The text has been rewritten and amplified in several places, and illustrative exercises together with nine new figures have been added in an attempt to improve the exposition.

There has been a considerable increase in the number of problems. It should be a source of satisfaction to the student that answers are given to all of them.

I wish to express my appreciation to Mr. William C. Shachmut, of the National Lead Company Research Laboratories, Brooklyn, N. Y., for working over with me the material of Chapter 7. He has also proposed problems, which have been included in the second edition, particularly some of those involving electric circuits and vibratory motion.

The book is suitable for courses of various lengths in both engineering and liberal arts schools. For a short course of two hours per week for one semester in an engineering school, the following selection has been found appropriate: Chapter 1; Arts. 11–17 and 21–24 of Chapter 3; Chapter 4 (omitting Art. 37); Chapter 5; and Arts. 45–47 of Chapter 6.

HARRY W. REDDICK

New York, N. Y.
June, 1949

Contents

Chapter 1

PRELIMINARY IDEAS

Chapter 2

THE FORMATION OF DIFFERENTIAL EQUATIONS

Chapter 3

DIFFERENTIAL EQUATIONS OF FIRST ORDER

Chapter 4

LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Chapter 5

SOME SPECIAL HIGHER ORDER EQUATIONS

Chapter 6

SIMULTANEOUS EQUATIONS

Chapter 7

THE LINEAR EQUATION OF SECOND ORDER

Chapter 8

SERIES SOLUTIONS

Chapter 1

PRELIMINARY IDEAS

1. Ordinary and partial equations; order and degree. The subject, differential equations, is a natural outgrowth of differential and integral calculus. Differential calculus is concerned primarily with finding the derivative of a known function of a single variable. In integral calculus the inverse problem is paramount—that of finding the function of a single variable when its derivative is known. If y is an unknown function of x, but its derivative dy/dx is a known function $f(x)$, the problem is: Given $dy/dx = f(x)$; find y, that is, $\int f(x)\,dx$.

Now suppose that dy/dx is a known function of both x and y, or that a relation connecting x with y and its derivatives of any order is known; the problem of finding y is now a much broader one. We are thus led to the subject, differential equations, whose basic problem is: Given an equation connecting x, y, dy/dx, d^2y/dx^2, $\cdots$, d^ny/dx^n; find a relation between x and y, free of the derivatives.

We shall call an equation containing one or more ordinary derivatives or differentials an *ordinary differential equation;* if the derivatives involved are partial derivatives, the equation is called a *partial differential equation.* Such equations are of great importance in applied mathematics, inasmuch as many problems arising in mathematical physics and in all fields of engineering are conveniently expressed in the language of differential relations.

Examples of ordinary differential equations are

$$\frac{dy}{dx} = x, \tag{1}$$

$$\frac{d^2y}{dx^2} = x, \tag{2}$$

$$\frac{d^2y}{dx^2} + y = 0, \tag{3}$$

$$x\,dy + y\,dx + y^2\,dy = 0, \tag{4}$$

$$x\,dx + y\,dy = \tfrac{1}{2}\,dt, \tag{5}$$

$$\left(\frac{dy}{dx}\right)^2 + 3xy = 7x, \tag{6}$$

$$2x^3\,\frac{d^4y}{dx^4} + y\left(\frac{dy}{dx}\right)^2 + 6x = \sin y. \tag{7}$$

Equation (4) appears as an equation involving differentials, but by dividing through by dx it may be written in the equivalent form,

$$(x + y^2)\,\frac{dy}{dx} + y = 0. \tag{4'}$$

Similarly, equation (5) may be written in the equivalent form involving derivatives,

$$2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} = 1. \tag{5'}$$

Examples of partial differential equations are

$$\frac{\partial^2 y}{\partial x^2} = K\,\frac{\partial^2 y}{\partial t^2}, \tag{8}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{9}$$

In the first of these, y is a function of the two independent variables x and t; in the second, u is a function of three independent variables x, y, z.

Many problems of mathematical physics require partial differential equations for their formulation. For example, suppose that we wish to study the motion of a stretched string vibrating in a plane. We may take the x-axis along the string when at rest, with the origin at one end of the string, then denote by x, y, the coordinates of any point P on the string. The displacement y of the point P depends not only on x but also on the time t, so that y is a function of the two independent variables x and t. The law which expresses this functional relation is given by the partial differential equation (8), whose solution exhibits y explicitly in terms of x and t.

Equation (9) is *Laplace's equation;* it arises in various problems of mathematical physics. Partial differential equations are of great importance but will not be considered further in this book; * the term *differential equation* will henceforth mean ordinary differential equation.

The *order* of a differential equation is the order of the derivative of highest order involved in the equation. For example, equations (1), (4) or (4′), (5) or (5′), and (6) are of first order; equations (2) and (3) are of second order; equation (7) is of fourth order.

Sometimes the term *degree* is used in connection with a differential equation. In equation (6) the derivative dy/dx is squared. The equation is of second degree in dy/dx, but it is equivalent to the two first degree equations

$$\frac{dy}{dx} = \sqrt{7x - 3xy}, \qquad \frac{dy}{dx} = -\sqrt{7x - 3xy}. \tag{6'}$$

Equation (7), although containing $(dy/dx)^2$, is an equation of first degree, since it is of the first degree in its highest derivative. In dealing with methods of solving differential equations, we shall confine ourselves to ordinary differential equations of first degree in the highest derivative involved, except for equations like (6) where the equivalent first degree equations can easily be obtained.

2. Solutions. A *solution* of a differential equation is a functional relation among the variables, free of derivatives,

* For a derivation of equations (8) and (9) together with a discussion of their applications, the reader is referred to Reddick and Miller's *Advanced Mathematics for Engineers*, 2nd Ed., John Wiley & Sons, Chapters VII, VIII, and X.

which satisfies the differential equation. The most general such relation is called the *general solution.* By solving a differential equation we shall mean, unless other conditions are specified, finding the general solution. A differential equation is solved by the process of integration.

EXAMPLE 1. Solve

$$\frac{dy}{dx} = x. \tag{1}$$

We first multiply by dx and obtain

$$dy = x\,dx.$$

Now, integrating both sides, we have

$$y = \tfrac{1}{2}x^2 + C, \tag{2}$$

where C is an arbitrary constant. In integrating we do not add an arbitrary constant to each side, for that would give

$$y + C_1 = \tfrac{1}{2}x^2 + C_2,$$

or

$$y = \tfrac{1}{2}x^2 + C_2 - C_1 = \tfrac{1}{2}x^2 + C,$$

which is the same result as before, since the difference of two arbitrary constants is another arbitrary constant.

A solution of (1) is $y = \frac{1}{2}x^2$; another solution is $y = \frac{1}{2}x^2 + 3$, etc.; but the general solution is $y = \frac{1}{2}x^2 + C$, where C is an arbitrary constant. To verify that the relation (2) is a solution, we substitute it in (1), obtaining

$$\frac{d}{dx}\left(\tfrac{1}{2}x^2 + C\right) = x,$$

$$x \equiv x,$$

an identity which verifies the solution.

EXAMPLE 2. Solve

$$\frac{d^2y}{dx^2} = x. \tag{3}$$

First multiplying by dx, we have

$$\frac{d^2y}{dx^2}\,dx = x\,dx.$$

Integration of both sides now gives

$$\frac{dy}{dx} = \tfrac{1}{2}x^2 + C_1.$$

Multiplying by dx and integrating again, we find

$$y = \tfrac{1}{6}x^3 + C_1x + C_2, \tag{4}$$

which is the general solution, C_1 and C_2 being arbitrary constants. Other solutions which are not general are, for example,

$$y = \tfrac{1}{6}x^3 + Cx, \qquad y = \tfrac{1}{6}x^3 + x + 6.$$

An important inference may be drawn from the two preceding examples. We note that in Example 1 the differential equation is of first order, one integration is performed, and the general solution contains one arbitrary constant. In Example 2 the differential equation is of second order, two integrations are performed, and the general solution contains two arbitrary constants. We might infer that, for a differential equation of nth order, since n integrations are involved in obtaining its general solution, the general solution must contain n arbitrary constants. The inference is correct and we state it as follows: *The general solution of a differential equation of* n*th order contains* n *essential arbitrary constants.*

This principle should be kept in mind when solving differential equations, and used as a check on the number of arbitrary constants in the solution. Care should be exercised in order that the arbitrary constants appearing in a solution shall all be essential, that is, shall not be reducible in number by a mere change of notation. For instance, $y = (C_1 + C_2)x$ cannot be the general solution of a differential equation of second order, for, although there are apparently two arbitrary constants, C_1 and C_2, the sum $C_1 + C_2$ is also an arbitrary constant and can be denoted by K. By a change of notation the solution reduces to $y = Kx$, which contains only one arbitrary constant and is the general solution of a differential equation of first order. Likewise $ax + by + c = 0$ is not the general solution of a differential equation of third order, for the arbitrary constants a, b, c are not all essential. Dividing through by one of them, say c, and writing $a/c = A$, $b/c = B$, reduces the equation to $Ax + By + 1 = 0$, which contains two essential arbitrary constants and is the general solution of a differential equation of second order. We now see that a solution of a differential equation of order n may contain any number of arbitrary constants from 0 to n inclusive; but only if it contains n essential arbitrary constants can it be the general solution.

A *particular solution* or *particular integral* of a differential equation is obtained by assigning particular values to the arbitrary constants in the general solution. Thus

$$y = \tfrac{1}{6}x^3 + 1, \qquad y = \tfrac{1}{6}x^3 + 2x + 5,$$

are particular solutions of differential equation (3). It will appear in Chapter 4 that particular solutions may be obtained also in other ways.

After reading the solutions of Examples 1 and 2 of this article, the thought may have occurred to the student: Why do we need a book dealing with methods of solving differential equations, since all that is necessary is to integrate? What is this more than integral calculus? These would be fair questions if all differential equations were as easy to solve as the two just solved. It usually happens, however, that the differential equation cannot be integrated as it stands. The chief difficulty in solving it is in preparing it for integration. As we proceed we shall see that, for differential equations of various types, methods can be developed for putting the equation into such a form that integration may be applied and the general solution obtained.

EXAMPLE 3. Solve

$$\frac{d^2y}{dx^2} + y = 0. \tag{5}$$

This is an equation which a student familiar only with integral calculus might have difficulty in integrating. He might integrate the first term after multiplying through by dx, and obtain dy/dx, but how would he integrate the second term, $y\,dx$? The equation can be prepared for integration as follows.

We first multiply the equation through by $2\,dy$ (inserting in the first term a factor dx/dx, which equals unity) and obtain

$$2\frac{dy}{dx}\frac{d^2y}{dx^2}dx + 2y\,dy = 0.$$

Now the first term is the differential of $(dy/dx)^2$ and the second term is the differential of y^2, so that integration yields

$$\left(\frac{dy}{dx}\right)^2 + y^2 = C_1^2,$$

where C_1^2 is an arbitrary constant. We choose C_1^2 instead of C_1 as the arbitrary constant in order to make the next integration come out in neater form.

We now transpose the term y^2 and take the square root of both sides, obtaining

$$\frac{dy}{dx} = \pm\sqrt{C_1^2 - y^2},$$

or

$$\pm\frac{dy}{\sqrt{C_1^2 - y^2}} = dx. \tag{6}$$

Integration gives

$$\pm \text{arc sin}\,\frac{y}{C_1} = x + C_2.$$

If we multiply by ± 1 and take the sine of both sides, we have

$$\frac{y}{C_1} = \sin\,[\pm(x + C_2)],$$

or

$$y = \pm C_1 \sin\,(x + C_2). \tag{7}$$

Changing the arbitrary constants, $\pm C_1$ to a, C_2 to b, puts the solution in the form

$$y = a \sin (x + b). \tag{8}$$

This equation (8) contains two arbitrary constants, a and b, and is the general solution of the differential equation (5) of second order.

When a differential equation is solved by different methods, different-looking solutions, which at first glance do not seem to be equivalent, may result; they may result also from allowing arbitrary constants to enter in different ways or from using different formulas of integration. If these different-looking solutions contain the proper number of arbitrary constants, however, and are all correct, they are equivalent. Any one of them can be reduced (at least theoretically) to any other one by a mere change of notation with respect to the arbitrary constants, leaving their number unchanged. For instance, if in the preceding example the first arbitrary constant had been chosen as C_1 instead of C_1^2 we would have had instead of (7),

$$y = \pm\sqrt{C_1} \sin (x + C_2).$$

Then, by changing $\pm\sqrt{C_1}$ to a, C_2 to b, we obtain (8).

Again, suppose that, on arriving at equation (6), the left member had been integrated into an arc cosine. Then we would have had

$$\mp \text{arc} \cos \frac{y}{C_1} = x + C_2,$$

$$\frac{y}{C} = \cos [\mp(x + C_2)],$$

$$y = C_1 \cos (x + C_2). \tag{9}$$

Now at first glance (9) may not seem equivalent to (8). But if in (9) we change the arbitrary constants, C_1 to $-a$, C_2 to $b + \pi/2$, we obtain (8).

We shall see in Chapter 4 that this same differential equation (5) may be solved by an entirely different method and the following solution obtained,

$$y = A \sin x + B \cos x, \tag{10}$$

where A and B are arbitrary constants. Let us show that (10) is equivalent to (8). We multiply the right member of (10) by $\sqrt{A^2 + B^2}/\sqrt{A^2 + B^2}$ and obtain

$$y = \sqrt{A^2 + B^2}\left[\sin x \cdot \frac{A}{\sqrt{A^2 + B^2}} + \cos x \cdot \frac{B}{\sqrt{A^2 + B^2}}\right]. \tag{11}$$

We now make a change of notation on the arbitrary constants, changing from the old pair A, B to a new pair a, b, related as follows:

$$\sqrt{A^2 + B^2} = a, \qquad \frac{A}{\sqrt{A^2 + B^2}} = \cos b, \qquad \frac{B}{\sqrt{A^2 + B^2}} = \sin b.$$

It is permissible to call one of these fractions a cosine and the other a sine of the same angle since the sum of their squares is 1. Equation (11) now takes the form

$$y = a(\sin x \cos b + \cos x \sin b),$$

or

$$y = a \sin (x + b),$$

which is equation (8).

Hence (8), (9), and (10) are equivalent forms; any one of them may be taken as the general solution of differential equation (5).

3. Hyperbolic functions. Frequently we shall have occasion to use hyperbolic functions, particularly the hyperbolic sine, cosine, and tangent. The hyperbolic functions, hyperbolic sine, hyperbolic cosine, etc., are connected with a hyperbola in a manner analogous to that in which the circular functions, sine, cosine, etc., are connected with a circle. Corresponding to the formulas of ordinary trigonometry, an analogous set of formulas can be developed in hyperbolic trigonometry. In order to exhibit this analogy, the idea of sector area, rather than of angle, is fundamental.

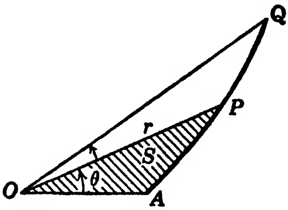

Fig. 1

Given a sector (Fig. 1) formed by two straight lines OA and OP, drawn from the origin O, and the arc AP of a curve. Let r = length of OP, θ = angle AOP, and S = area of the sector. If the sector takes an increment POQ due to an infinitesimal increase in the angle θ, the differential sector area is given by a formula from calculus:

$$dS = \tfrac{1}{2}r^2\,d\theta.$$

If OA is taken as x-axis and x, y are the rectangular coordinates of the point P, we have

$$r^2 = x^2 + y^2, \qquad d\theta = d\tan^{-1}\frac{y}{x} = \frac{x\,dy - y\,dx}{x^2 + y^2},$$

and hence

$$dS = \tfrac{1}{2}(x\,dy - y\,dx).$$

Now consider a unit circle, $x^2 + y^2 = 1$, and a unit equilateral hyperbola, $x^2 - y^2 = 1$ (Fig. 2). Representing by u

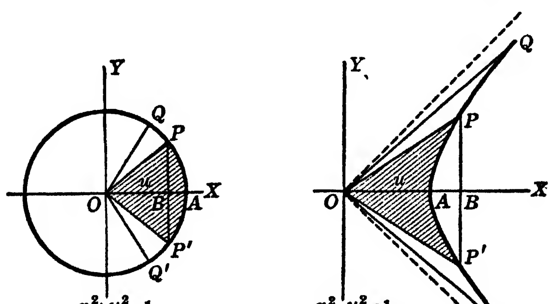

Fig. 2

the sector area $OPAP'$, with $OA = 1$, we shall express the rectangular coordinates x, y of P in terms of u. As the sector opens out, P moves to Q and P' to Q'. Thus the increment in the sector area u is twice the area of POQ, and the differential sector area is

$$du = x\,dy - y\,dx.$$

Substituting first the value of y from the equation of the circle, then the value of y from the equation of the hyperbola, we have

For the Circle

$$du = x\,d\sqrt{1-x^2} - \sqrt{1-x^2}\,dx$$

$$= \left(\frac{-x^2}{\sqrt{1-x^2}} - \sqrt{1-x^2}\right)dx$$

$$= \frac{-dx}{\sqrt{1-x^2}},$$

$$u = \int_1^x \frac{-dx}{\sqrt{1-x^2}} = \cos^{-1} x,$$

$$x = \cos u.$$

For the Hyperbola

$$du = x\,d\sqrt{x^2-1} - \sqrt{x^2-1}\,dx$$

$$= \left(\frac{x^2}{\sqrt{x^2-1}} - \sqrt{x^2-1}\right)dx$$

$$= \frac{dx}{\sqrt{x^2-1}},$$

$$u = \int_1^x \frac{dx}{\sqrt{x^2-1}} = \ln\left(x + \sqrt{x^2-1}\right),$$

$$e^u = x + \sqrt{x^2-1},$$

$$e^{2u} - 2xe^u + x^2 = x^2 - 1,$$

$$x = \frac{e^u + e^{-u}}{2} = \cosh u.$$

Thus, in order to express x in terms of u, we are led in the case of the circle to the familiar *circular* function, cosine. Usually we think of a cosine as being the cosine of an angle, but we could just as well think of it as being the cosine of the number representing an area. For instance, if $OA = 1$ in., the number of square inches in the circular sector u equals the number of radians in the angle POA, so that the length OB in the circle can be regarded as the cosine of either number. In the case of the hyperbola we arrive at the fact that x is the particular function of u: $(e^u + e^{-u})/2$. It is natural to call this, by analogy, the *hyperbolic* function, hyperbolic cosine, of u. Thus we have the first definition in hyperbolic trigonometry:

$$\cosh u = \frac{e^u + e^{-u}}{2}. \qquad (1)$$

Consider again the meaning of the equations

$$x = \cos u, \qquad x = \cosh u.$$

How do these equations check dimensionally? One does not take the cos or the cosh of square inches and obtain inches. Suppose that instead of the unit circle and unit hyperbola we had used the equations

$$x^2 + y^2 = a^2, \qquad x^2 - y^2 = a^2;$$

then we would have obtained

$$x = a \cos \frac{u}{a^2}, \qquad x = a \cosh \frac{u}{a^2}.$$

Now, for a (in.) and u (in.^{2}), the ratio u/a^2 is a pure number whose cos or cosh is also dimensionless, so that x has the dimension of a. When $a = 1$ (in.) we could write

$$x = 1 \cos \frac{u}{1^2}, \qquad x = 1 \cosh \frac{u}{1^2},$$

which check dimensionally, but it is customary to write merely

$$x = \cos u, \qquad x = \cosh u,$$

with the understanding that x represents the number of inches in OB, and u the number of square inches in the sector area.

In order to express y in terms of u, we have

For the Circle	For the Hyperbola
$y = \sqrt{1 - x^2}$	$y = \sqrt{x^2 - 1} = \sqrt{\cosh^2 u - 1}$
$= \sqrt{1 - \cos^2 u}$	$= \sqrt{\dfrac{e^{2u} + 2 + e^{-2u}}{4} - 1}$
$= \sin u.$	$= \sqrt{\dfrac{e^{2u} - 2 + e^{-2u}}{4}}$
	$= \dfrac{e^u - e^{-u}}{2} = \sinh u.$

The second definition in hyperbolic trigonometry is

$$\sinh u = \frac{e^u - e^{-u}}{2}. \tag{2}$$

The other four hyperbolic functions, hyperbolic tangent, hyperbolic cotangent, hyperbolic secant, hyperbolic cosecant, we define as follows, by analogy to the circular functions:

$$\tanh u = \frac{\sinh u}{\cosh u}, \tag{3}$$

$$\operatorname{sech} u = \frac{1}{\cosh u}, \tag{5}$$

$$\coth u = \frac{1}{\tanh u}, \tag{4}$$

$$\operatorname{csch} u = \frac{1}{\sinh u}. \tag{6}$$

We regard these six definitions as defining the six hyperbolic functions also when u is negative. From these definitions the formulas of hyperbolic trigonometry may be derived. Following are some of the formulas frequently used which are derivable from definitions (1), (2), (3), and (5):

$$\cosh^2 u - \sinh^2 u = 1, \tag{7}$$

$$\sinh (u + v) = \sinh u \cosh v + \cosh u \sinh v, \tag{8}$$

$$\cosh (u + v) = \cosh u \cosh v + \sinh u \sinh v, \tag{9}$$

$$\frac{d}{du} \sinh u = \cosh u, \tag{10}$$

$$\frac{d}{du} \cosh u = \sinh u, \tag{11}$$

$$\frac{d}{du} \tanh u = \operatorname{sech}^2 u, \tag{12}$$

$$\int \sinh u \, du = \cosh u, \tag{13}$$

$$\int \cosh u \, du = \sinh u, \tag{14}$$

$$\int \tanh u \, du = \ln \cosh u. \tag{15}$$

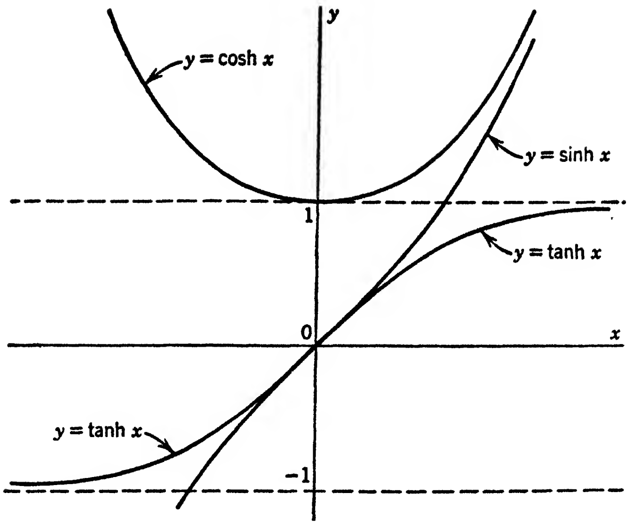
y
y = cosh x
y = sinh x
1
y = tanh x
0
x
y = tanh x
−1

FIG. 3

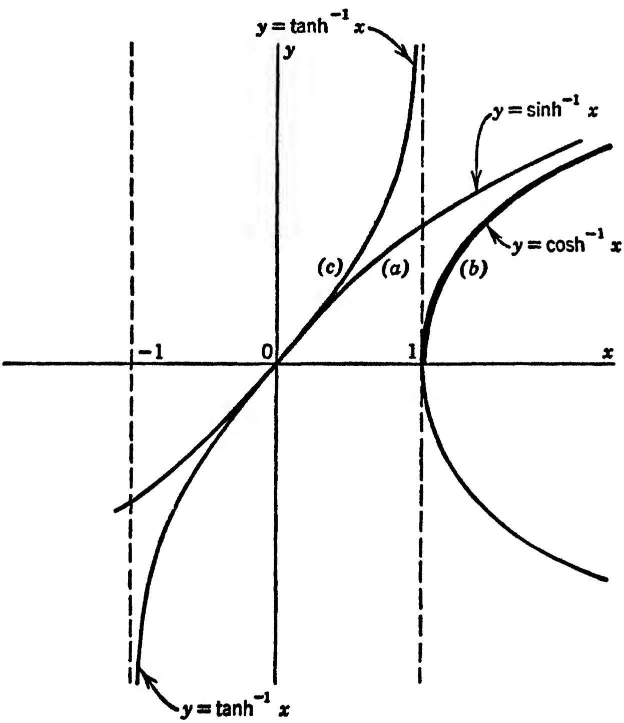
y = tanh⁻¹ x
y
y = sinh⁻¹ x
y = cosh⁻¹ x
(c)
(a)
(b)
−1
0
1
x
y = tanh⁻¹ x

FIG. 4

4. Graphs of the hyperbolic and inverse hyperbolic functions. Tables giving the numerical values of the hyperbolic functions may be employed to construct the graphs of the functions. Thus with x and y as coordinates the curves $y = \sinh x$, $y = \cosh x$, $y = \tanh x$ are obtained as represented in Fig. 3.

Reflection of these curves in the line $y = x$ gives the inverse functions $y = \sinh^{-1} x$, $y = \cosh^{-1} x$, $y = \tanh^{-1} x$, as in Fig. 4.

All the functions in Figs. 3 and 4 are single-valued with the exception of $\cosh^{-1} x$, Fig. 4(*b*). In order to make this function single-valued, which is desirable in problems where differentiation or integration is involved, we choose the *principal value*, i.e., the value of the function on the upper half of the curve in Fig. 4(*b*), as the value of the function. The upper half of the curve [heavy line in Fig. 4(*b*)] is called the *principal branch* of the function.

5. Some formulas involving inverse hyperbolic functions. Just as the hyperbolic functions can be represented in exponential form, so the inverse functions have equivalent logarithmic forms:

$$\sinh^{-1} x = \ln (x + \sqrt{x^2 + 1}), \tag{1}$$

$$\cosh^{-1} x = \ln (x + \sqrt{x^2 - 1}), \quad \text{principal value,} \tag{2}$$

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{1 + x}{1 - x}. \tag{3}$$

We now derive formula (2), leaving the derivation of (1) and (3) as Prob. 11.

If

$$y = \cosh^{-1} x,$$

then

$$x = \cosh y = \frac{e^y + e^{-y}}{2}.$$

Hence

$$e^y - 2x + e^{-y} = 0$$

or, multiplying by e^y,

$$e^{2y} - 2xe^y + 1 = 0.$$

Solving for e^y by the quadratic formula, we obtain

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}.$$

Choosing the larger of these values of e^y to obtain the principal value of y, we have

$$y = \ln (x + \sqrt{x^2 - 1}),$$

that is,

$$\cosh^{-1} x = \ln (x + \sqrt{x^2 - 1}).$$

The derivatives of these inverse functions and the corresponding integral formulas are

$$\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1 + x^2}}, \qquad \int \frac{dx}{\sqrt{1 + x^2}} = \sinh^{-1} x, \qquad (4)$$

$$\frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}}, \qquad \int \frac{dx}{\sqrt{x^2 - 1}} = \cosh^{-1} x, \qquad (5)$$

$$\frac{d}{dx} \tanh^{-1} x = \frac{1}{1 - x^2}, \qquad \int \frac{dx}{1 - x^2} = \tanh^{-1} x. \qquad (6)$$

We derive formula (5), leaving the derivation of (4) and (6) as Prob. 12.

If

$$y = \cosh^{-1} x,$$

then

$$x = \cosh y$$

and

$$\frac{dx}{dy} = \sinh y = \pm\sqrt{\cosh^2 y - 1} = \pm\sqrt{x^2 - 1}.$$

Choosing the upper sign, since the slope of the upper half of the curve, Fig. 4(*b*), is positive, we have

$$\frac{dy}{dx} = \frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}},$$

and

$$\int \frac{dx}{\sqrt{x^2 - 1}} = \cosh^{-1} x.$$

The same result could be obtained, of course, by differentiating the equivalent logarithmic form:

$$\frac{d}{dx}\cosh^{-1}x = \frac{d}{dx}\ln\left(x + \sqrt{x^2-1}\right)$$
$$= \frac{1 + \dfrac{x}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}}.$$

PROBLEMS

1. Solve

$$(a)\ \frac{dy}{dx} = \cos x; \qquad (b)\ \frac{d^3y}{dx^3} = 24x.$$

2. Solve

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2},$$

obtaining the general solution in algebraic form.

3. Determine the order of the differential equation of which each of the following equations is the general solution:

(*a*) $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ (the general equation of a conic);
(*b*) $x\cos^2\alpha + y = cx^2 - x\sin^2\alpha$, where α and c are arbitrary constants;
(*c*) the general equation of a circle.

4. Solve

$$\frac{dy}{ay+b} + \frac{dx}{ax+b} = 0.$$

Here a and b are given constants and will appear, together with the arbitrary constant, in the solution.

5. Solve

$$\frac{d^2y}{dx^2} = \frac{1}{y^3}.$$

6. Derive formulas (7) to (15) of Art. 3.

7. Solve

$$\frac{d^2y}{dx^2} - y = 0.$$

Show that

$$(a)\ y = Ae^x + Be^{-x}, \qquad (b)\ y = M\sinh x + N\cosh x,$$
$$(c)\ y = a\sinh(x+b), \qquad (d)\ y = c\cosh(x+d),$$

are all equivalent forms of the general solution.

8. Find the area in the first quadrant between the curves

$$y = \sinh x, \qquad y = \cosh x.$$

9. Find the distance from the origin to the point on the y-axis from which tangents drawn to the curve, $y = \cosh x$, are perpendicular to each other.

10. Find the area in the first quadrant between the curve $y = \tanh x$ and its asymptote.

11. Derive formulas (1) and (3) of Art. 5.

12. Derive formulas (4) and (6) of Art. 5.

13. Find the area bounded by $y = \sinh x$, $y = \tanh x$, $x = \sinh^{-1} 1$.

14. Find the logarithmic equivalent of $\coth^{-1} x$, draw the curve $y = \coth^{-1} x$, and find the values of y and dy/dx when $x = -2$.

15. Find the coordinates of the point of inflection on the principal branch of the curve $y = a \cosh^{-1} (a/x)$.

Chapter 2

THE FORMATION OF DIFFERENTIAL EQUATIONS

6. An inverse problem. Before proceeding with the development of methods for solving differential equations, we shall consider in this chapter the inverse process, that is, the formation of the differential equation by eliminating the arbitrary constants from its general solution. For the case of two variables the problem may be stated thus: Given an equation involving two variables and n arbitrary constants; find the differential equation (of nth order) of which the given equation is the general solution. This is usually an easier problem than the direct process of solving a given differential equation.

There are, of course, other ways in which a differential equation may be obtained. For example, the expression of a geometric property of a family of curves may lead directly to a differential equation. Furthermore, the exact formulation of a physical problem from mechanical principles often produces a differential equation. Both these processes will be encountered in our later work.

We now take some of the examples of Chapter 1, and proceed from the general solution to the formation of the differential equation; the process may be regarded as a method of proving the correctness of the general solution. We shall also form some differential equations for which the methods of solution are given in succeeding chapters.

7. Elimination by differentiation and combination. Some examples will illustrate the method.

EXAMPLE 1. Eliminate the arbitrary constant from the equation

$$y = \tfrac{1}{2}x^2 + C.$$

Differentiating once, we obtain

$$\frac{dy}{dx} = x.$$

In Example 1, Art. 2, we solved the second equation, obtaining the first as solution. Here we perform the inverse process and form the differential equation from its solution.

EXAMPLE 2. Eliminate the arbitrary constants from the following equation, or, in other words, obtain the differential equation of which the following equation is the general solution:

$$y = \tfrac{1}{6}x^3 + C_1x + C_2.$$

Differentiating twice, since there are two arbitrary constants to eliminate, we have

$$\frac{dy}{dx} = \tfrac{1}{2}x^2 + C_1,$$

$$\frac{d^2y}{dx^2} = x,$$

thus forming by differentiation the differential equation whose solution was found in Example 2, Art. 2, by the process of integration.

If a differential equation could always be solved like $d^2y/dx^2 = x$, by mere integration and addition of an arbitrary constant at each integration, without reduction or change of form, then the reverse process could be carried out as in the previous example; differentiation of the solution as many times as there are arbitrary constants to be eliminated would produce at once the differential equation free of arbitrary constants. However, in solving a differential equation, we usually apply processes of simplification at various stages of the solution, which mix up the arbitrary constants and variables in such a way that mere differentiation of the solution will not automatically eliminate the arbitrary constants and produce the differential equation.

EXAMPLE 3. Form a differential equation by eliminating the arbitrary constants from the equation

$$y = a \sin (x + b). \tag{1}$$

Two differentiations give

$$\frac{dy}{dx} = a \cos (x + b), \tag{2}$$

$$\frac{d^2y}{dx^2} = -a \sin (x + b). \tag{3}$$

We notice that the right member of (3) is the negative of the right member of (1), so that by adding these two equations we eliminate a and b at once, obtaining

$$\frac{d^2y}{dx^2} + y = 0, \tag{4}$$

which is the differential equation whose solution (1) was found in Example 3, Art. 2.

In the above example the arbitrary constants did not disappear automatically after two differentiations of the original equation. We had three equations, the original (1) and the two obtained from it by differentiation, (2) and (3), from which it was necessary to eliminate the constants a and b. In this case the elimination was effected by combining (1) and (3) without using (2); usually all the equations must be used, that is, the original equation and those obtained from it by differentiation.

It may be noticed that in this example the inverse process of finding the differential equation when the solution is given is much shorter than the direct process shown in Art. 2, of finding the solution when the differential equation is given. Given the solution (1), we find the differential equation (4) in three steps. One might say, "Why not reverse the steps, starting with the differential equation (4)?" But the first step backward cannot be taken; the differential equation was found by elimination, and elimination is a process which covers up its tracks—it is irreversible.

EXAMPLE 4. Eliminate the arbitrary constants from

$$y = Ae^{2x} + Be^{-x}. \tag{5}$$

Differentiating twice and using primes to denote differentiation with respect to x, we have

$$y' = 2Ae^{2x} - Be^{-x}, \tag{6}$$

$$y'' = 4Ae^{2x} + Be^{-x}. \tag{7}$$

Adding (5) and (6), then (6) and (7), we thus eliminate B:

$$y + y' = 3Ae^{2x}, \tag{8}$$

$$y'' + y' = 6Ae^{2x}. \tag{9}$$

Now multiplying equation (8) by 2 and subtracting from equation (9) eliminates A and gives

$$y'' - y' - 2y = 0, \tag{10}$$

which is the required differential equation whose solution is (5).

This process may be regarded as a method of proving that (5) is the general solution of (10). However, if (5) and (10) were both given, we could verify that (5) satisfies (10) by merely differentiating (5) and substituting in (10), obtaining an identity. Thus, substituting (5), (6), and (7) in (10), we have

$$4Ae^{2x} + Be^{-x} - 2Ae^{2x} + Be^{-x} - 2Ae^{2x} - 2Be^{-x} \equiv 0.$$

The foregoing examples illustrate the method called *elimination by differentiation and combination,* which consists in combining the original equation and those obtained from it by differentiation, in such a way as to eliminate the arbitrary constants. It is understood that differentiation of the original equation must be performed exactly the number of times equal to the number of arbitrary constants to be eliminated. Of course, only as many of the equations so formed need to be combined as are necessary to effect the elimination. The first two examples illustrate the special case where only one of these equations itself expresses the result of the elimination.

8. Isolation of constants. We now give a variation in the solution of Example 4, Art. 7, in which at each stage before differentiation one of the arbitrary constants is isolated, i.e., stands in a term unaccompanied by a variable, so that it will disappear on differentiation.

EXAMPLE 1. (Second solution of Example 4, Art. 7.) Eliminate the arbitrary constants from

$$y = Ae^{2x} + Be^{-x}.$$

Multiplying by e^x we isolate B:

$$e^x y = Ae^{3x} + B.$$

Differentiation gives

$$e^x(y' + y) = 3Ae^{3x}.$$

Multiplying by e^{-3x}, we have

$$e^{-2x}(y' + y) = 3A.$$

Differentiating again,

$$e^{-2x}(y'' + y' - 2y' - 2y) = 0,$$

or

$$y'' - y' - 2y = 0.$$

This last step illustrates the fact that, when we are forming a differential equation, we may cancel out a factor not containing a derivative.

In some cases it may be desirable not to isolate one of the constants, yet by multiplying the equation through by a factor we may obtain another equation from which it is simpler to eliminate the constants.

EXAMPLE 2. Find the differential equation whose general solution is

$$y = e^{-2x}(A \cos x + B \sin x). \tag{1}$$

Multiplying by e^{2x}, we have

$$e^{2x}y = A \cos x + B \sin x. \tag{2}$$

Two differentiations give

$$e^{2x}(y' + 2y) = -A \sin x + B \cos x, \tag{3}$$

$$e^{2x}(y'' + 4y' + 4y) = -A \cos x - B \sin x. \tag{4}$$

Thus, the effect of multiplying the original equation by e^{2x} was an equation (2) whose right member reproduced itself, apart from sign, after two differentiations. Adding equations (2) and (4), we obtain the required differential equation

$$y'' + 4y' + 5y = 0. \tag{5}$$

In the previous examples y is expressed explicitly in terms of x and arbitrary constants. It may happen, however, as in the next example, that the given equation expresses an implicit relation between the variables, which it may not be practicable to solve for one of the variables.

Example 3. Eliminate the arbitrary constant C from

$$xy = C(x^5 + y^5).$$

We may isolate C by dividing by $x^5 + y^5$ or by Cxy; the latter gives

$$\frac{x^4}{y} + \frac{y^4}{x} = \frac{1}{C}.$$

Differentiating, we have

$$\frac{4x^3y - x^4y'}{y^2} + \frac{4xy^3y' - y^4}{x^2} = 0,$$

or

$$x(4y^5 - x^5)y' + y(4x^5 - y^5) = 0,$$

which is the required differential equation.

9. Elimination by determinants. This method depends on a proposition in algebra: In order for a system of $n + 1$ linear equations in n unknowns to be consistent, the determinant formed from the coefficients of the unknowns and the terms free of the unknowns must vanish.

In the case, say, of four linear equations in three unknowns, the proposition can be stated thus: In order that the system of equations

$$\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0, \\ a_2x + b_2y + c_2z + d_2 &= 0, \\ a_3x + b_3y + c_3z + d_3 &= 0, \\ a_4x + b_4y + c_4z + d_4 &= 0, \end{aligned} \tag{1}$$

be consistent, we must have

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0. \tag{2}$$

In other words, (2) may be regarded as the result of eliminating the unknowns x, y, z from (1).

Suppose now that we have an equation involving x, y, and, say, three arbitrary constants, A, B, C, from which the three constants are to be eliminated. In order to apply the above proposition to our problem we differentiate this equation three times, having then, in all, four equations from which the three quantities, A, B, C, are to be eliminated. The quantities A, B, C, to be eliminated, take the place of x, y, z in equations (1); the coefficients of A, B, C, and the terms free of A, B, C are the elements of the determinant (2).

Example 1. Eliminate the arbitrary constants from

$$y = Ax^2 + Be^x + Ce^{-2x}. \tag{3}$$

Differentiating three times, we have

$$\begin{aligned} y' &= 2Ax + Be^x - 2Ce^{-2x}, \\ y'' &= 2A + Be^x + 4Ce^{-2x}, \\ y''' &= \qquad\quad Be^x - 8Ce^{-2x}. \end{aligned}$$

The result of eliminating A, B, C from these four equations is

$$\begin{vmatrix} y & x^2 & e^x & e^{-2x} \\ y' & 2x & e^x & -2e^{-2x} \\ y'' & 2 & e^x & 4e^{-2x} \\ y''' & 0 & e^x & -8e^{-2x} \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} y & x^2 & 1 & 1 \\ y' & 2x & 1 & -2 \\ y'' & 2 & 1 & 4 \\ y''' & 0 & 1 & -8 \end{vmatrix} = 0. \tag{4}$$

In forming this determinant we notice that all the coefficients of any one of the letters A, B, C must appear in the same column, but the order in which the columns are set down is immaterial, since any change in order of columns could at most change only the sign of the determinant; also any factor not containing a derivative may be cancelled from a column or row, since this factor could be taken outside as a factor of the determinant.

Equation (4) is really the differential equation required, but in determinant form. If we wish the result in the usual form of a differential equation we must develop the determinant. We may obtain two more 0's in the fourth row by the following manipulation: Replace the first column by itself minus y''' times the third column; replace the fourth column by itself plus 8 times the third column; now all the elements of the fourth row are 0 except the third one which is 1, and (4) reduces to

$$\begin{vmatrix} y - y''' & x^2 & 9 \\ y' - y''' & 2x & 6 \\ y'' - y''' & 2 & 12 \end{vmatrix} = 0.$$

Dividing 3 out of the third column, and developing with respect to the first column, we have

$$(y - y''')(8x - 4) - (y' - y''')(4x^2 - 6) + (y'' - y''')(2x^2 - 6x) = 0,$$

or

$$(x^2 - x - 1)y''' + (x^2 - 3x)y'' - (2x^2 - 3)y' + (4x - 2)y = 0,$$

which is the differential equation whose general solution is (3).

PROBLEMS

Eliminate the arbitrary constants from the following equations, encountered in Chapter 1, and thus obtain the corresponding differential equations.

1. $y = C_1 \cos(x + C_2)$.	Eq. (9), Art. 2.
2. $y = A \sin x + B \cos x$.	Eq. (10), Art. 2.
3. $y = x^4 + C_1x^2 + C_2x + C_3$.	Prob. 1(*b*), Art. 5.
4. $y = x + C(1 + xy)$.	Prob. 2, Art. 5.
5. $x \cos^2 \alpha + y = cx^2 - x \sin^2 \alpha$.	Prob. 3(*b*), Art. 5.
6. $(x - a)^2 + (y - b)^2 = r^2$.	Prob. 3(*c*), Art. 5.
7. $(ax + b)(ay + b) = c$ (c arbitrary).	Prob. 4, Art. 5.
8. $(ax + b)(ay + b) = c$ (a, b, c arbitrary).	
9. $y = Ae^x + Be^{-x}$.	Prob. 7(*a*), Art. 5.
10. $y = a \sinh(x + b)$.	Prob. 7(*c*), Art. 5.

Find the differential equations from the following general solutions.

11. $(x^2 + a^2)y^2 = a^2x^2$.

12. $y = Ae^{x+y} + Be^{-x+y}$.

13. $y = Ae^x + Be^{-2x} + Ce^{3x}$.

14. $(x + 5y + 9)^4 = C(x + 2y + 3)$.

15. $\tanh\left(\frac{x}{4} + \frac{y}{2}\right) = \sqrt{3} \tan\left(\frac{\sqrt{3}}{4}x + C\right)$.

16. $y = Ae^{1/\sqrt{x}} + Be^{-1/\sqrt{x}}$.

17. $y = A\sqrt{1 + x^2} + Bx$.

18. $y = A\left(1 + x + \frac{x^2}{2}\right) + Be^x$.

19. $y = A(\cos x + x \sin x) + B(\sin x - x \cos x)$.

20. $y = A\frac{\sinh x}{x} + B\frac{\cosh x}{x}$.

21. $y = A \cos 2x \cosh(2x + \alpha) + B \sin 2x \sinh(2x + \beta)$, where A, B, α, and β are arbitrary constants.

22. $y = e^{x^2}\left(C_1 + C_2 \int e^{-x^2}\,dx\right)$.

23. $y = C_1x \int \frac{e^{x^3/8}\,dx}{x^2} + C_2x$.

24. Find the differential equation whose general solution is

$$y = Ax^n + Bx^{1-n},$$

(*a*) if A, B are arbitrary; (*b*) if A, B, n are arbitrary.

10. The differential equation of a family of curves. The equation

$$x^2 + y^2 = C \tag{1}$$

represents a family of circles. For instance, if $C = 1$ it is a circle of radius 1; if $C = 2$ it is a circle of radius $\sqrt{2}$, etc. If C is an arbitrary constant, capable of taking on an infinity of different values, the equation represents an infinite family of concentric circles centered at the origin. If all the circles are to be real, C is arbitrary in a certain range, namely, $C \geqq 0$. A symbol, such as C in equation (1), which is constant for any one curve, but which can take on an arbitrary number of different values, is called a parameter. A family of curves, such as (1), which contains one parameter, is called a one-parameter family, or a singly infinite family, or a family of ∞^1 curves, the exponent on the ∞ indicating the number of parameters in the equation of the family. Thus we may say that in a plane there are ∞^1 concentric circles with fixed center.

How many circles are there in the xy-plane with centers on the x-axis? The answer is ∞^2, since the equation of such a family is $(x - a)^2 + y^2 = r^2$, where a and r are arbitrary constants or parameters. The differential equation of the family is of second order, since two arbitrary constants will have to be eliminated to produce it.

On the other hand there are only ∞^1 circles of radius 1 with centers on the x-axis; the equation of this family is $(x - a)^2 + y^2 = 1$, containing only one parameter, and the differential equation will be of first order. The phrase, "in the xy-plane," is always understood in such an example, since we are dealing with only two variables.

In general, an n-parameter family of curves will be represented by a differential equation of nth order. To obtain the differential equation of the family, we write, by analytic geometry, the equation of the family containing the proper number of essential parameters, then eliminate the parameters by one of the methods explained in the preceding articles.

Example 1. Find the differential equation of the family of circles concentric about the origin.

The equation of the family of circles is

$$x^2 + y^2 = C.$$

Differentiating, we have

$$x\,dx + y\,dy = 0,$$

or

$$\frac{dy}{dx} + \frac{x}{y} = 0,$$

which is the differential equation required.

Example 2. Find the differential equation of all circles with centers on the x-axis.

The equation of the family is

$$(x - a)^2 + y^2 = r^2.$$

Differentiating twice, we have

$$x - a + yy' = 0,$$

then

$$1 + yy'' + y'^2 = 0,$$

which is the required differential equation.

Example 3. Find the differential equation of (*a*) all vertical parabolas, i.e., parabolas with vertical axes; (*b*) all vertical parabolas with vertices on the line $y = x$.

There are ∞^3 vertical parabolas. The equation of the family may be written in either of two ways:

$$y - k = a(x - h)^2, \qquad y = Ax^2 + Bx + C.$$

Using the second form, and differentiating three times, we have

$$y' = 2Ax + B,$$

$$y'' = 2A,$$

$$y''' = 0.$$

The last equation is the answer to part (*a*).

For part (*b*) we use the first form, (h, k) being the vertex of the parabola. The condition requires that $k = h$; hence the equation is

$$y - h = a(x - h)^2,$$

which represents a two-parameter family. Differentiating twice, we have

$$y' = 2a(x - h),$$
$$y'' = 2a,$$

from which

$$a = \frac{y''}{2}, \qquad x - h = \frac{y'}{y''}, \qquad y - h = y - x + \frac{y'}{y''}.$$

Substituting these values in the original equation, we have

$$y - x + \frac{y'}{y''} = \frac{y'^2}{2y''},$$

or

$$2(y - x)y'' = y'^2 - 2y',$$

which is the answer to part (*b*).

PROBLEMS

Find the differential equations of the following families of curves.

1. Circles with centers on the y-axis.
2. Circles of fixed radius r, with centers on the x-axis.
3. Circles with centers on the line $y = x$.
4. Circles passing through the origin, with centers on the line $y = x$.
5. Circles tangent to the y-axis at the origin.
6. Circles tangent to both coordinate axes, with centers on the line $y = x$.
7. Circles with centers on the x-axis, and tangent to the lines $y = \pm x$.
8. Circles with centers on the y-axis, and tangent to the lines $y = \pm x$.
9. Parabolas with axis on the x-axis.
10. Parabolas with axis on the x-axis and focus at the origin.
11. Horizontal parabolas tangent to the y-axis.
12. Vertical parabolas passing through the point (a, b).
13. The probability curves, $y = (h/\sqrt{\pi})e^{-h^2x^2}$.
14. The family of tractrices

$$\pm(x + C) = \sqrt{k^2 - y^2} - k \cosh^{-1}\frac{k}{y},$$

where k is a fixed constant and C is arbitrary.

15. Find the differential equation of (*a*) the doubly infinite family of catenaries, $y = a \cosh [(x - b)/a]$; (*b*) the triply infinite family of catenaries, $y = a \cosh [(x - b)/a] + c$.

16. Find the differential equation of all parabolas tangent to the y-axis at the origin.

17. Write (*a*) in rectangular coordinates, (*b*) in polar coordinates, the equation of the family of all circles passing through the origin. Find the differential equation in each case, and transform one differential equation into the other.

18. Find the differential equation of the family of hyperbolas having (*a*) asymptotes parallel to the coordinate axes; (*b*) asymptotes parallel to the coordinate axes and centers on the line $y = x$. Show, from the two differential equations, without using their general solutions, that every solution of the second differential equation is a solution of the first.

19. In Prob. 3(*a*), Art. 5, we found from the general equation of a conic that the differential equation of all conics is of the fifth order. This differential equation may be found by dividing out one of the constants in the general equation (or, what amounts to the same thing, putting one constant equal to unity), then differentiating five times, and eliminating the five essential constants from the six equations by means of a determinant of the sixth order. The result comes out surprisingly simple. There is a shorter method, however, due to Halphen. (See Goursat-Hedrick-Dunkel, *Mathematical Analysis*, Vol. II, Part II, p. 5.) Find the differential equation of the conics by one of these two methods.

20. Find the differential equation of all parabolas. Show, from this differential equation and that of the conics, Prob. 19, that every solution of the former is a solution of the latter.

Chapter 3

DIFFERENTIAL EQUATIONS OF FIRST ORDER

11. Introduction. In this chapter we consider various types of differential equations of first order and the methods of solving them, together with their physical and geometric applications. A first order differential equation involving only two variables x and y, and of first degree in the derivative dy/dx, can be written, in differential form,

$$M\,dx + N\,dy = 0, \tag{1}$$

where M and N are, in general, functions of x and y.

12. Separable equations. If, in equation (1), Art. 11, M is a function of x only (or a constant) and N is a function of y only (or a constant), the equation has its variables separated and can be integrated at once. A differential equation which can be written in the form

$$M(x)\,dx + N(y)\,dy = 0 \tag{1}$$

is called *separable.*

EXAMPLE 1. Solve

$$\frac{dy}{dx} = \frac{x}{y}.$$

Separating the variables, we have

$$x\,dx - y\,dy = 0.$$

Integration gives

$$\frac{x^2}{2} - \frac{y^2}{2} = \frac{C}{2},$$

or

$$x^2 - y^2 = C,$$

which is the general solution. The constant of integration can be written as any function of C; here we chose $C/2$.

EXAMPLE 2. Solve the following differential equation, obtaining the general solution in algebraic form, and prove that the solution is correct:

$$(1 + y^2)\,dx + (1 + x^2)\,dy = 0. \tag{2}$$

We first write the equation in the form

$$\frac{dx}{1 + x^2} + \frac{dy}{1 + y^2} = 0.$$

Integration then gives

$$\text{arc tan } x + \text{arc tan } y = \text{arc tan } C.$$

Here we choose the constant of integration in the form arc tan C, since in the next step we are going to take the tangent of both sides in order to express the result in algebraic form. In doing this we apply the formula for the tangent of the sum of two angles, namely,

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta},$$

with $\alpha = \text{arc tan } x$, $\beta = \text{arc tan } y$, and obtain

$$\frac{x + y}{1 - xy} = C, \tag{3}$$

or

$$x + y = C(1 - xy), \tag{4}$$

which is the required solution.

The solution may be proved correct either by eliminating C from the solution (4) and producing the differential equation (2), or by substituting in (2) the values of y and dy/dx obtained from (4), thus producing an identity in x.

By the first method, writing (4) in the form (3) and differentiating, we have

$$(1 - xy)(dx + dy) + (x + y)(x\,dy + y\,dx) = 0,$$

$$(1 + y^2)\,dx + (1 + x^2)\,dy = 0,$$

which is equation (2).

By the second method we have, from equation (4),

$$y = \frac{C - x}{1 + Cx}, \qquad \frac{dy}{dx} = -\frac{1 + C^2}{(1 + Cx)^2},$$

$$1 + y^2 = \frac{(1 + Cx)^2 + (C - x)^2}{(1 + Cx)^2} = \frac{(1 + C^2)(1 + x^2)}{(1 + Cx)^2}.$$

Substitution of these values for $1 + y^2$ and dy/dx in equation (2) produces an identity.

In the next example and frequently thereafter we shall have occasion to take antilogarithms of both sides of an equation in order to reduce it to an equation free of logarithms. Suppose, for example, that we wish to remove the logarithms from the equation

$$2 \ln x - 3 \ln y = \ln C + \sin x \tag{5}$$

by taking antilogarithms of both sides. This means that we must write an equation such that, if we should take logarithms of both sides of it, we would have equation (5).

Let us see whence each term of equation (5) would come, by the process of taking logarithms: $2 \ln x$ from x^2, $3 \ln y$ from y^3, the *difference* $2 \ln x - 3 \ln y$ from the *fraction* x^2/y^3; $\ln C$ from C, $\sin x$ from $e^{\sin x}$, the *sum* $\ln C + \sin x$ from the *product* $Ce^{\sin x}$. We can therefore remove the logarithms from an equation by properly changing coefficients to exponents, differences to fractions, sums to products, and terms free of logarithms to exponentials. Taking antilogarithms, equation (5) becomes

$$\frac{x^2}{y^3} = Ce^{\sin x}. \tag{6}$$

In applying this process to an equation having one member zero, it must be remembered that antilog $0 = 1$. Thus, if equation (5) is written in the form

$$2 \ln x - 3 \ln y - \sin x - \ln C = 0,$$

we have, on taking antilogarithms,

$$\frac{x^2 e^{-\sin x}}{Cy^3} = 1,$$

which is the same as equation (6).

EXAMPLE 3. Solve

$$2y \cos y \, dy = y \sin y \, dx + \sin y \, dy.$$

Transposing the last term to the left side and dividing by $y \sin y$, we have

$$\left(2 \cot y - \frac{1}{y}\right) dy = dx.$$

Integration yields

$$2 \ln \sin y - \ln y = x + \ln C.$$

Then, taking antilogarithms, we obtain the general solution in the form

$$\sin^2 y = Cye^x.$$

13. Particular solutions and particular values. It happens frequently that we are more interested in a particular solution of a differential equation, which satisfies a prescribed condition, than in the general solution; or our chief interest may be in finding a particular value of one of the variables corresponding to an assigned value of the other.

EXAMPLE 1. Suppose that we wish to find the solution of the differential equation

$$\frac{dx}{dt} = 1 + 2x, \tag{1}$$

which satisfies the condition $x = \frac{1}{2}$ when $t = 0$, and that we also wish to find the value of x when $t = 1$.

We may proceed as follows: Separating the variables, we have

$$\frac{dx}{1 + 2x} = dt. \tag{2}$$

Integration then gives

$$\tfrac{1}{2} \ln (1 + 2x) = t + C. \tag{3}$$

The condition, $x = \frac{1}{2}$, $t = 0$, determines C:

$$\tfrac{1}{2}\ln 2 = C.$$

Equation (3) then becomes

$$\frac{1}{2}\ln\frac{1+2x}{2} = t,$$

$$1 + 2x = 2e^{2t},$$

$$x = e^{2t} - \tfrac{1}{2}, \tag{4}$$

the required particular solution. We then find the value of x corresponding to $t = 1$:

$$x]_{t=1} = e^2 - \tfrac{1}{2} = 6.89.$$

Instead of finding the general solution (3) and determining the value of the arbitrary constant, we could have integrated equation (2) between limits. Integrating between the known pair of values $(\frac{1}{2}, 0)$ and the general pair (x, t), we have

$$\int_{1/2}^{x}\frac{dx}{1+2x} = \int_0^t dt,$$

$$\tfrac{1}{2}\ln(1+2x)\Big]_{1/2}^{x} = t\Big]_0^t,$$

$$\frac{1}{2}\ln\frac{1+2x}{2} = t,$$

$$x = e^{2t} - \tfrac{1}{2},$$

$$x]_{t=1} = 6.89.$$

If, however, we are interested only in the value of x when $t = 1$, it is not necessary to find the particular solution (4). We can integrate equation (2) between the limits $(\frac{1}{2}, 0)$ and $(x, 1)$, thus:

$$\int_{1/2}^{x}\frac{dx}{1+2x} = \int_0^1 dt,$$

$$\frac{1}{2}\ln\frac{1+2x}{2} = 1,$$

$$x = e^2 - \tfrac{1}{2} = 6.89.$$

Example 2. Suppose that the derivative dx/dt is proportional to x, that $x = 5$ when $t = 0$, and that $x = 10$ when $t = 5$. What is the value of x when $t = 12$?

The differential equation is

$$\frac{dx}{dt} = kx,$$

where k is an unknown constant of proportionality which, together with the constant of integration, makes two constants to be determined from the two given conditions. We shall solve the problem by two methods. In the first method, we determine both k and the constant of integration; in the second method we neither determine the value of k nor use a constant of integration.

First method. Separating the variables, we have

$$\frac{dx}{x} = k\,dt. \tag{5}$$

Integrating,

$$\ln x = kt + C. \tag{6}$$

The condition, $x = 5$, $t = 0$, gives $\ln 5 = C$, so that (6) becomes

$$\ln\frac{x}{5} = kt. \tag{7}$$

The condition, $x = 10$, $t = 5$, now determines $k = \frac{1}{5}\ln 2$, and (7) becomes

$$\ln\frac{x}{5} = \frac{t}{5}\ln 2,$$

or

$$x = 5\cdot 2^{t/5}.$$

Then

$$x\Big]_{t=12} = 5\cdot 2^{2.4} = 26.4.$$

Second method. Integrating equation (5) between (5, 0) and (10, 5), then between (5, 0) and (x, 12), we have

$$\int_5^{10}\frac{dx}{x} = k\int_0^5 dt,$$

$$\int_5^{x}\frac{dx}{x} = k\int_0^{12} dt.$$

Dividing the second result by the first, we find

$$\frac{\ln \frac{x}{5}}{\ln 2} = \frac{12}{5} = 2.4,$$

$$x = 5 \cdot 2^{2.4} = 26.4.$$

EXAMPLE 3. Given the differential equation

$$\left(\frac{dy}{dx}\right)^2 = y,$$

find (*a*) the general solution; (*b*) two curves through the point (1, 4) satisfying the differential equation.

(*a*) Taking the square root of both sides and separating the variables, we have

$$\frac{dy}{\pm\sqrt{y}} = dx. \tag{8}$$

Integration gives the general solution

$$\pm 2\sqrt{y} = x + C, \tag{9}$$

or

$$y = \left(\frac{x + C}{2}\right)^2. \tag{10}$$

We could solve equations (8) separately and obtain

$$2\sqrt{y} = x + C_1, \qquad -\sqrt{2}\, y = x + C_2. \tag{11}$$

It might seem then that the general solution of the original differential equation contains two arbitrary constants, C_1 and C_2. But giving C_1 and C_2 arbitrary values in (11) yields the same system of curves that is obtained by giving C arbitrary values in (9) or (10). Hence C_1 and C_2 are not both essential and can be replaced by C. To each value of C correspond two curves of the general solution.

(*b*) Letting $x = 1$, $y = 4$ in (10) we obtain the values of C for which the curves of the general solution pass through the point (1, 4):

$$4 = \left(\frac{1 + C}{2}\right)^2, \qquad 1 + C = \pm 4, \qquad C = 3 \text{ or } -5.$$

Hence the required curves are

$$y = \left(\frac{x+3}{2}\right)^2, \qquad y = \left(\frac{x-5}{2}\right)^2.$$

PROBLEMS

Solve the following differential equations, and prove that the solutions are correct.

1. $\dfrac{dy}{dx} = \dfrac{3y}{x}$.
2. $2 \sin x \cos y \, dy = \cos x \sin y \, dx$.
3. $\sqrt{1 - y^2}\, dx + \sqrt{1 - x^2}\, dy = 0$ (algebraic solution).
4. $\dfrac{dy}{dx} = e^{x+y}$.
5. $y(x^2 \, dy - y^2 \, dx) = x^2 \, dy$.

Find the particular solutions of the following differential equations, satisfying the prescribed conditions.

6. $\dfrac{dy}{dx} + x = \dfrac{x}{y}$ $(x = 2, y = 0)$.
7. $x \dfrac{dy}{dx} = y - xy$ $(x = 1, y = 2)$.
8. $a \dfrac{dx}{dt} = b + cx$ $\left(\dfrac{dx}{dt} = 1, t = 0\right)$. Express x in terms of t.
9. $x \cos y \, dx + x^2 \sin y \, dy = a^2 \sin y \, dy$ $\left(x = \dfrac{a}{2}, y = \dfrac{\pi}{6}\right)$.
10. $\dfrac{dx}{dt} = k(18 - 9x + x^2)$ $(x = 0, t = 0)$ $(x = 2, t = 10)$.

11. If $\dfrac{dx}{dt} = 6 - 3x$, and $x = 0$ when $t = 0$, find the value of x when $t = 1$.

12. If $\dfrac{x}{y}\dfrac{dy}{dt} + 1 = 3y^2$, and $y = 1$ when $x = 2$, find the positive value of y corresponding to $x = \frac{2}{3}$.

13. If $\dfrac{dx}{dt} = (1 - x)(3 - x)$, and $x = 0$ when $t = 0$, (*a*) find the value of t when $x = \frac{1}{2}$; (*b*) find the value of x when $t = \frac{1}{2}$.

14. Given $\dfrac{dx}{dt} = k(6 - x)^2$, with $x = 0$ when $t = 0$ and $x = 2$ when $t = 10$. Find the value of x when $t = 15$.

15. If $\dfrac{du}{dt} + ku = 0$, $u = u_0$ when $t = 0$, $u = u_0/2$ when $t = 16$, $u = nu_0$ when $t = 2$, find the value of n.

16. Find a curve through the origin whose slope is given by

(*a*) $\dfrac{dy}{dx} = -\dfrac{y^2 + 2y + 4}{x^2 + 2x + 4}$;

(*b*) $\dfrac{dy}{dx} = -\dfrac{y + 2}{x + 2}$.

Why are the answers to (*a*) and (*b*) the same?

17. Find a curve satisfying the following differential equation in polar coordinates and passing through the point $(3, \pi/3)$:

$$\frac{d\rho}{\rho\, d\theta} = \frac{\rho^2 - 1}{\rho^2 + 1} \tan \theta.$$

18. Solve

$$\frac{dy}{dx} = \frac{x - xy^2}{8y + 2x^2y}.$$

19. Solve

$$e^x \frac{dy}{dx} = 1 - \tan x + \tan^2 x.$$

20. Solve

$$(1 - xy)(dx + dy) + (x - y)(dx - dy) = 0.$$

21. Solve

$$\frac{y\, dx}{1 + y + y^2} = x\, dy + y\, dx.$$

22. Solve

$$2 \cosh x \frac{dy}{dx} = 1 + y^2,$$

and show that the particular solution which satisfies the condition ($x = 0$, $y = 0$) is

$$y = \tanh \frac{x}{2}.$$

23. Find the abscissa of the point where the curve $y = e^x$ cuts the curve represented by the particular solution of

$$\frac{dy}{dx} = \frac{y}{x \ln x} + y \qquad (x = e,\ y = 1).$$

Show that the ordinate of the point is a number of more than 1,650,000 digits.

24. Given the differential equation

$$\left(\frac{dy}{dx}\right)^2 = 9xy,$$

find (*a*) the general solution; (*b*) two curves through the point $(4^{-1/3}, \frac{1}{4})$ satisfying the equation.

25. Find two curves through the point (1, 2) satisfying the differential equation

$$\left(\frac{dy}{dx}\right)^2 = 7x - 3xy.$$

14. Dynamics. We consider now some physical problems which require the solution of separable differential equations.

Newton's second law of motion states that the rate of change of momentum of a particle is proportional to the force acting on it and is in the same direction as the force. Thus, if a particle of constant mass m moves with varying velocity v under the action of a force F, we have

$$F = k\frac{d}{dt}(mv) = km\frac{dv}{dt}, \tag{1}$$

where dv/dt is the acceleration of the particle and k is a constant of proportionality depending on the system of units employed. When cgs (centimeter-gram-second) units are used, so that mass is measured in grams and acceleration in centimeters per second per second, we may take F in dynes, so that $k = 1$, or we may take F in grams, so that $k = 1/g$, where g is the gravitational constant, approximately equal to 980.5 cm/sec^2. When fps (foot-pound-second) units are used, so that mass is measured in pounds and acceleration in feet per second per second, F may be measured in poundals, making $k = 1$, or in pounds, making $k = 1/g$, where now $g = 32.17$ ft/sec^2, approximately.

If we consider a moving mass as concentrated at its center of gravity, we may set up the differential equation of its motion by equating the expression for force, given in (1), to the resultant force in the line of motion.

Example 1. A sled weighing 100 lb is being pushed in a straight line against the wind by a force of 10 lb. Suppose that friction is negligible but that there is an air resistance to motion whose magnitude in pounds is equal to twice the velocity of the sled in feet per second. If the sled starts from rest, find the velocity and the distance traveled at the end of 1 sec.

Let x (ft), positive in the direction of motion, be the displacement from the initial position ($x = 0$), and let v (ft/sec) be the velocity, at time t (sec). We equate the expression for force, given by (1), to the resultant force, which is positive and equal in magnitude to $10 - 2v$, thus forming the differential equation of the motion,

$$\frac{100}{g}\frac{dv}{dt} = 10 - 2v, \tag{2}$$

where $g = 32.17$ ft/sec^2. Separating the variables, we get

$$\frac{dv}{5 - v} = \frac{g}{50}\,dt. \tag{3}$$

In order to obtain v in terms of t, we may integrate (3), adding a constant of integration, and then determine this constant from the fact that when $t = 0$ we must have $v = 0$. Or we may use definite integrals, thus avoiding a constant of integration, and integrate from the known pair of values, $v = 0$, $t = 0$, to the general pair of values v, t. Using the latter method, we have

$$\int_0^v \frac{dv}{5 - v} = \frac{g}{50}\int_0^t dt,$$

$$\ln(5 - v)\Big]_0^v = -\frac{g}{50}\,t\Big]_0^t,$$

$$\ln\frac{5 - v}{5} = -\frac{gt}{50},$$

$$\frac{5 - v}{5} = e^{-gt/50},$$

$$v = 5(1 - e^{-gt/50}) \text{ ft/sec.} \tag{4}$$

The relation (4) gives the velocity at any time t; for $t = 1$ sec this becomes

$$v\Big]_{t=1} = 5(1 - e^{-32.17/50}) = 5(1 - e^{-0.6434})$$

$$= 5(1 - 0.5255) = 2.37 \text{ ft/sec.}$$

We may notice from (4) that v does not increase indefinitely with t but approaches a *limiting velocity* of 5 ft/sec. The limiting velocity can be found, however, simply by setting dv/dt equal to 0 in equation (2);

$$0 = 10 - 2v, \qquad v = 5.$$

In order to obtain x in terms of t, we replace v by dx/dt in equation (4) and integrate between (0, 0) and (x, t), since $x = 0$ when $t = 0$. We thus obtain

$$\int_0^x dx = 5\int_0^t (1 - e^{-gt/50})\, dt,$$

$$x = 5\left[t + \frac{50}{g}(e^{-gt/50} - 1)\right] \text{ ft.} \tag{5}$$

Formula (5) is the general x, t relation; for $t = 1$ sec we have

$$x\Big]_{t=1} = 5\left[1 + \frac{50(0.5255 - 1)}{32.17}\right] = 1.31 \text{ ft.}$$

15. Trial and error. In applied mathematics it is often necessary to solve equations such as $\ln x = x - 2$, $\sin\theta = 1 - 2\theta$, $\varphi + 3e^{-\varphi} = 5$, etc. In such equations we do not find the exact value of the unknown quantity but obtain an approximate value to the required degree of accuracy. The process that is usually the most convenient is known as the method of *trial and error.*

As an illustration suppose that in Example 1, Art. 14, we wish to find the time required for the sled to move 5 ft from its starting position. From equation (5), Art. 14, we have

$$1 = t + \frac{50}{g}(e^{-gt/50} - 1), \tag{1}$$

an equation to be solved for t. We start by guessing a value. It may be rather difficult to make a good guess for an equation in its original form; for this reason, and for ease in carrying out the process, it is expedient to reduce the equation to as simple a form as possible. If in equation (1) we let

$$\frac{gt}{50} = \varphi, \qquad \text{or} \qquad t = \frac{50\varphi}{g}, \tag{2}$$

it reduces to

$$1 = \frac{50}{g}(\varphi + e^{-\varphi} - 1),$$

$$\frac{g}{50} + 1 = \varphi + e^{-\varphi}.$$

Putting $g = 32.17$, there results the equation

$$\varphi + e^{-\varphi} = 1.643 \tag{3}$$

to be solved for φ; then t can be found from equation (2). This is a much easier process than finding t directly from equation (1).

From Peirce's "Tables" we find

φ	$e^{-\varphi}$	$\varphi + e^{-\varphi}$
1.4	0.247	1.647
1.3	0.273	1.573
1.395	0.248	1.643

The value 1.4 for φ gives a value for $\varphi + e^{-\varphi}$ which is too large by 0.004, and the value $\varphi = 1.3$ makes $\varphi + e^{-\varphi}$ too small by 0.070, so we decrease 1.4 by $(4/74) \times 0.1 = 0.005$ and try $\varphi = 1.395$, which checks.

Equation (2) now gives

$$t = \frac{50(1.395)}{32.17} = 2.17 \text{ sec},$$

the required time.

We could have let

$$e^{-gt/50} = \theta, \qquad \text{or} \qquad t = -\frac{50}{g}\ln\theta, \tag{4}$$

thus reducing equation (1) to

$$1 = \frac{50}{g}(-\ln\theta + \theta - 1),$$

$$\theta - \ln\theta = 1.643, \tag{5}$$

in which event we would solve equation (5) for θ, then find t from equation (4).

Care must be exercised, however, in solving equation (5), in order to choose the right solution. One might find from Peirce's "Tables" that $\theta = 2.598$ is a solution of equation (5), but this value of θ, when substituted in (4), would give a negative value for t. Before solving equation (5) it should be noticed that we must have $0 < \theta < 1$, from equation (4), since t must be positive. This vigilance was not demanded in solving equation (3) since there it was only necessary to take φ positive. From Peirce's "Tables" we find

θ	$\ln\theta$	$\theta - \ln\theta$
0.25	−1.386	1.636
0.24	−1.427	1.667
0.248	−1.394	1.642

Then equation (4) gives, as before,

$$t = \frac{-50(-1.394)}{32.17} = 2.17 \text{ sec.}$$

16. Chemical reactions; first and second order processes. In a chemical reaction the rate of change of a substance may be proportional to the amount of that substance present at a given time. Such a reaction is known as a *first order process.*

If x is the amount of the substance present at time t, the differential equation representing the process is

$$\frac{dx}{dt} = kx, \tag{1}$$

where k is a constant of proportionality.

Suppose, on the other hand, that a molecule of one substance A combines with one molecule of a second substance B so as to form one molecule of C. If a and b are the amounts of A and B respectively at time $t = 0$, and x is the amount of C at time t, then $a - x$ and $b - x$ are the amounts of A and B present respectively at time t, and $x = 0$ when $t = 0$. When the rate of change of x is proportional to the product of the amounts of A and B remaining at any given time, the reaction is known as a *second order process* and is represented by the differential equation

$$\frac{dx}{dt} = k(a - x)(b - x). \tag{2}$$

It is the process which is known as second order; the differential equation is of the first order.

Example 1. Suppose that, in the process represented by equation (2), $a = 5$ and $b = 4$, and that $x = 1$ when $t = 5$ min; find the value of x when $t = 10$ min.

Separating the variables, we have

$$\frac{dx}{(5 - x)(4 - x)} = \left(\frac{1}{4 - x} - \frac{1}{5 - x}\right) dx = k\,dt.$$

Integration between limits, using the pairs of values (0, 0), (1, 5), (x, 10), gives

$$\ln \frac{5 - x}{4 - x}\Bigg]_0^1 = kt\Big]_0^5, \tag{3}$$

$$\ln \frac{5 - x}{4 - x}\Bigg]_0^x = kt\Big]_0^{10}. \tag{4}$$

Dividing (4) by (3), we obtain

$$\frac{\ln\left(\dfrac{5-x}{4-x}\cdot\dfrac{4}{5}\right)}{\ln\left(\dfrac{4}{3}\cdot\dfrac{4}{5}\right)} = \frac{10}{5} = 2,$$

or

$$\frac{5-x}{4-x} = \frac{5}{4}\left(\frac{16}{15}\right)^2 = \frac{64}{45},$$

$$225 - 45x = 256 - 64x,$$

$$19x = 31,$$

$$x = 1.63.$$

17. Steady-state heat flow. Experimental study of the flow of heat has shown that the rate of flow across an area is proportional to the product of the area and the temperature gradient. The temperature gradient is the rate of change of the temperature with respect to distance normal to the area. We now state this law in the form of a differential equation, confining ourselves to the case in which the flow has settled down into a steady state and the rate of flow is therefore independent of time. Furthermore, we consider here only the case in which the temperature depends on only one space coordinate, x, measured in the direction of flow.* If the flow is radially outward from the axis of a cylindrical surface of radius x, all points of the surface will have the same temperature; i.e., the cylinder will be an *isothermal surface.* If the flow is radially outward from a point, x will be the radius of an isothermal sphere, whereas, if the flow takes place in parallel straight lines, the isothermal surfaces will be parallel planes perpendicular to the lines of flow, x measuring the distance of one of these planes from a fixed plane of reference.

* The more general problems of heat flow at a rate varying with the time, and of steady-state flow in which the temperature depends on more than one coordinate, lead to partial differential equations. See Reddick and Miller, *Advanced Mathematics for Engineers*, 2nd Ed., Chapter VII, Arts. 71, 72.

Let q (cal/sec) be the constant rate at which heat is flowing in a body perpendicularly through an isothermal surface of area A (cm^2), all points of which have the coordinate x (cm) and the temperature u(°C). Then q will be proportional to $A\,du/dx$, and we have the differential equation

$$q = -kA\frac{du}{dx}. \tag{1}$$

The constant of proportionality, k (cal/cm deg sec), is a property of the material of the body and is called the *thermal conductivity*. Since the temperature decreases as x increases, the temperature gradient, du/dx, is negative and the minus sign must be used in equation (1) in order to make q positive.

EXAMPLE 1. A pipe 10 cm in diameter contains steam at 100°C. It is insulated with a coating of magnesia 3 cm thick, for which $k = 175 \times 10^{-6}$ cal/cm deg sec. If the outer surface of the insulation is kept at 40°C, find the rate of heat loss from a meter length of pipe, and the temperature halfway through the insulation. The thickness of the pipe is not taken into account; it is assumed that the inner surface of the insulation is at 100°C.

By letting u represent the temperature on an isothermal cylindrical surface of radius x, within the insulation, then $A = 2\pi xL$ will be the area of a portion of the surface of length L. For the rate of heat loss, which is equal to the rate of flow across A, equation (1) gives

$$q = -k\cdot 2\pi xL \cdot \frac{du}{dx}. \tag{2}$$

Separating the variables and integrating from the inner surface where $x = 5$, $u = 100$, to the outer surface where $x = 8$, $u = 40$, we have

$$q\int_5^8 \frac{dx}{x} = -2\pi kL\int_{100}^{40} du,$$

$$q \ln x\Big]_5^8 = -2\pi kLu\Big]_{100}^{40},$$

$$q \ln 1.6 = 120\pi kL, \tag{3}$$

or, when $k = 175 \times 10^{-6}$ and $L = 100$, in cgs units,

$$q = \frac{2.1\pi}{\ln 1.6} = \frac{6.597}{0.4700} = 14.0 \text{ cal/sec.}$$

To find the temperature halfway through the insulation, substitute in (2) the value of q given by (3), then integrate from $x = 5$, $u = 100$, to $x = 6.5$, $u = u$:

$$\frac{120\pi kL}{\ln 1.6} = -2\pi kLx\frac{du}{dx},$$

$$\int_{100}^{u} du = -\frac{60}{\ln 1.6}\int_{5}^{6.5}\frac{dx}{x},$$

$$u - 100 = -\frac{60 \ln 1.3}{\ln 1.6} = -\frac{15.74}{0.4700} = -33.5; \qquad u = 66.5°\text{C}.$$

PROBLEMS

1. A 20-lb weight moves in a horizontal straight line under the joint action of a constant force of 12 lb, in the direction of motion, and a resisting force whose magnitude in pounds is equal to four times the instantaneous velocity in feet per second. If the weight starts from rest, find its velocity and the distance traveled after $\frac{1}{2}$ sec.

2. If $x\,dx + 2\,dy = y(x\,dx + dy)$, and $y = 0$ when $x = 2$, find by the method of trial and error the value of y when $x = \sqrt{6}$.

3. If, from 0 to x, the area between the curve $y = \tanh x$ and the x-axis equals the area between the curve and its asymptote, find the value of x.

4. A weight of w lb falls from rest. If the resistance of the air is proportional to the velocity, and if the limiting velocity is 173 ft/sec, find (*a*) the velocity at the end of 10 sec; (*b*) the distance fallen at the end of 10 sec; (*c*) the time when the velocity is half the limiting velocity; (*d*) the time when the distance fallen is 173 ft.

5. A ship weighing 8×10^7 lb, and starting from rest, reaches a limiting velocity V ft/sec under a force of 2×10^5 lb exerted by the propellers. Assuming a resistance to the motion proportional to the square of the velocity, find (*a*) the time required to attain half the limiting velocity; (*b*) the time required for the velocity to increase from one-third to two-thirds of the limiting velocity.

6. A body falls from rest in a medium offering resistance proportional to the square of the velocity. The limiting velocity is 80 ft/sec. When the velocity has reached 56 ft/sec, find (*a*) the time elapsed; (*b*) the distance traversed. $\left(\text{Note that, since } v = \frac{dx}{dt}, \frac{dv}{dt} = v\frac{dv}{dx}.\right)$

7. A charged sphere, of weight w lb, falls from rest under the combined forces of gravity and an electric field, the latter producing an upward force ot E lb. If the air resistance is proportional to the speed and the limiting speed is V ft/sec, find the time required to reach p per cent of the limiting speed. Use g (ft/sec^2) for acceleration due to gravity.

8. A body falling from rest in a liquid acquires a velocity which approaches 10 ft/sec as a limit. Assuming the resistance of the medium to be proportional to the velocity, and the specific gravity of the body to be 3 times that of the liquid, find (*a*) the velocity at the end of 1 sec; (*b*) the time required to traverse the first 6 ft; (*c*) the distance fallen at the end of 1 sec.

9. A body falls from rest in a liquid whose density is $1/n$ that of the body. The liquid offers resistance proportional to the velocity. If the velocity at the end of 1 sec is 20 ft/sec and at the end of 4 sec is one-half the limiting velocity, find the limiting velocity and the value of n.

10. Work Prob. 9 if the resistance is proportional to the square of the velocity.

11. In a certain chemical reaction, the rate of conversion of a substance at time t is proportional to the quantity of the substance still untransformed at that instant. If one-quarter the original amount of the substance has been converted when $t = 3$ min and if an amount A has been converted when $t = 6$ min, find the original amount of the substance.

12. If radium decomposes at a rate proportional to the amount present, and half the original quantity disappears in n years, (*a*) what percentage will disappear in $n/2$ years? (*b*) in how many years will 25 per cent disappear?

13. According to Newton's law of cooling, the rate at which heat is lost by a heated body is proportional to the difference in temperature between the body and the surrounding medium. If a thermometer is removed from a room in which the temperature is 70°F into the open where the temperature is 30°F, and if its reading is 60°F at the end of $\frac{1}{2}$ min, (*a*) how long after the removal will the reading be 40°F; (*b*) how long will it take for the reading to drop from 50°F to 35°F; (*c*) what is the temperature drop during the first 2 min after the removal?

14. In the chemical reaction represented by equation (2), Art. 16, suppose that $x = 2$ when $t = 10$ min. Find x at the end of 15 min (*a*) when $a = 6$, $b = 3$; (*b*) when $a = b = 6$.

15. In the chemical reaction represented by equation (2), Art. 16, suppose that $a = 4$, $b = 3$, and that $x = 1$ when $t = 3$. Find the value of t when $x = 2$.

16. In equation (2), Art. 16, given $a = b$ and $x = a/n$ when $t = t_1$; find the value of x when $t = 2t_1$.

17. Assume the differential equation

$$\frac{dp}{dh} = -\rho$$

connecting the atmospheric pressure p (lb/ft²), the density of the atmosphere ρ (lb/ft³), and the height h (ft) above the surface of the earth. Then, if a relation involving two or three of the variables p, ρ, h is known, one of the variables may be eliminated to obtain a relation between the other two.

If

$$p = 192\rho(144 - 0.001h),$$

and the pressure at sea level is 2120 lb/ft², find the pressure at a height of 10,000 ft.

18. In Prob. 17, at what height will the pressure be half the pressure at sea level?

19. If $dp/dh = -\rho$ and p is equal to 14.7 and 12.0 lb/in.² at the surface of the earth and at a height of 1 mile respectively, find p at a height of 2.5 miles, assuming that $p = k\rho$.

20. In Prob. 19, find ρ at a height of 2 miles, assuming that $p = k\rho^{3/2}$.

21. If $dp/dh = -\rho$ and p is equal to 14.7 and 10.1 lb/in.² at heights 0 and 10,000 ft respectively, at what height is p equal to 8 lb/in.²? Assume $\rho = p^{4/5}$.

22. If $dp/dh = -\rho$, how much will the barometer reading decrease in ascending 1000 ft above sea level under the following assumptions?

The pressure of the atmosphere is proportional to the density.
The barometer reading at sea level is 29.9 in. of mercury.
The density of air at sea level is 0.0807 lb/ft³.
The specific gravity of mercury is 13.6.
The density of water is 62.4 lb/ft³.

23. If the density of sea water under a pressure of p lb/ft² is

$$\rho = 64(1 + 2 \times 10^{-8}p) \text{ lb/ft}^3,$$

find the density and pressure at a depth of 2 miles below sea level. Use $dp/dh = \rho$.

24. Find the heat loss in calories per day from 20 meters of pipe 30 cm in diameter, containing steam at 100°C, if the pipe is covered with a layer of concrete 10 cm thick and the outer surface of the concrete is kept at 35°C. Find also the temperature halfway through the concrete. (Assume $k = 225 \times 10^{-5}$ cal/cm deg sec.)

25. Work Example 1, Art. 17, replacing the meter length of pipe by a hollow sphere 1 meter in diameter.

26. A steam pipe of radius 10 in. is covered with a 2-in. layer of insulation. Find how much the thickness of the insulation must be increased to cut down the heat loss by 40 per cent if the temperature at the outer surface of the insulation is kept constant.

27. A steam pipe has inner and outer radii, r_1 and r_2 respectively, and the temperatures at its inner and outer surfaces are respectively u_1 and u_2.

(*a*) Derive a formula for the steady-state temperature u at radial distance $r(r_1 < r < r_2)$. (*b*) Compute the value of u if $r_1 = 5$ cm, $r_2 = 10$ cm, $u_1 = 100°C$, $u_2 = 80°C$, $r = 8$ cm.

28. A hollow spherical shell has inner and outer radii, r_1 and r_2 respectively, and the temperatures of its inner and outer surfaces are respectively u_1 and u_2. (*a*) Derive a formula for the steady-state temperature u at radial distance $r(r_1 < r < r_2)$. (*b*) Compute the value of u if $r_1 = 10$ cm, $r_2 = 20$ cm, $u_1 = 100°C$, $u_2 = 64°C$, $r = 15$ cm.

29. A cylindrical pipe and a hollow spherical shell both have inside diameters of 3 cm and outside diameters of 5 cm. If the thermal conductivity of the material of the pipe is $\frac{7}{10}$ that of the shell, find how long the pipe must be in order that the quantities of heat conducted through pipe and shell walls shall be the same under the same inner and same outer temperatures in the respective cases.

30. If a flat furnace wall of thickness t_1 cm, conductivity k_1 cal/cm deg sec, and area A cm^2 is covered by an insulating material of thickness t_2 cm and conductivity k_2 cal/cm deg sec, find the heat loss in calories per second through the wall if its inner surface is at u_1°C and the outer surface of the insulation is at u_2°C.

31. A steam pipe of length L cm and radius x_0 cm is covered by n layers of insulation of conductivities $k_1, k_2, \cdots, k_n$ (cal/cm deg sec) and outer radii $x_1, x_2, \cdots, x_n$ (cm) respectively. If the steam is at u_0°C and the outer surface of the outside layer of insulation is at u_n°C, find the heat loss in calories per second through the insulation.

32. A steam pipe of radius x_0 is to have two layers of different insulating materials of outer radii x_1, x_2 $(x_0 < x_1 < x_2)$. Show that, for maximum efficiency of insulation, the better insulator should form the inside or outside layer according as $x_1 \gtrless \sqrt{x_0 x_2}$, but that if $x_1 = \sqrt{x_0 x_2}$ the two insulating materials are interchangeable.

33. Assuming that the velocity of efflux of water (volume per unit time) through an orifice in the bottom of a tank is proportional to the product of the area of the orifice and the square root of the depth of the water, the differential equation is

$$\frac{A\,dh}{dt} = -kB\sqrt{h},$$

where h (ft) is the depth of the water and A (ft^2) is the area of the water surface at any time t (sec), and B (ft^2) is the area of the orifice. The constant of proportionality, k (ft$^{1/2}$/sec), may be determined empirically.

Find the time required to empty a cubical tank whose edge is 4 ft. The tank has a hole 2 in. in diameter in the bottom and is originally full of water. (Take $k = 4.8$.)

34. Suppose that the tank of Prob. 33 is originally empty and that water runs into it at the rate of V ft^3/sec. Show that, if the tank is to fill, it is necessary to have $V > \pi/15$. Find the time required for the tank to fill if $V = 2\pi/15$ ft^3/sec.

35. A funnel has the shape of a right circular cone with vertex down and is full of water. If half the volume of water runs out in time t, find the time required to empty.

36. A tank in the form of a prism with equilateral triangular ends rests on one of its rectangular faces and is full of water. It has an outlet in the bottom and one of equal size in the top. Show that the tank will empty twice as fast if it is inverted.

37. If the tank in Prob. 36 is filled only halfway up and then inverted, find the ratio of the time required to empty it in the inverted position to the time which would be required in the original position.

38. A cylindrical tank 8 ft long and 4 ft in diameter is full of water. Find the time required to empty through a hole 1 in. in diameter in the bottom (*a*) if the cylinder is standing on end, (*b*) if the cylinder lies horizontal. (Take $k = 4.8$.)

39. For the tank of Prob. 38 in horizontal position, find (*a*) the time required to half empty, (*b*) the depth of water after 30 min.

40. Show that a cylindrical tank full of water will empty faster through a hole of given diameter in the bottom if it stands on end or if it lies horizontal, according as $(D/L) \lessgtr (64/9\pi^2)$, where D and L are, respectively, the diameter and the length of the tank.

41. A parabolic bowl 18 ft across the top and 9 ft deep is full of water. Find the time required to empty it through a hole of 2-in. diameter in the bottom. (Take $k = 4.8$.)

42. A parabolic bowl of equal depth and radius, and a hemispherical bowl of the same radius, are full of water. Find the ratio of the times required to empty through a hole of the same size in the bottom.

43. Ether flows at the rate of 4π ft^3/hr into a bowl which is a paraboloid of revolut· 10 ft across the top and 10 ft deep. If the rate of evaporation is proportional to the area A (ft^2) of the liquid surface, the constant of proportionality being $k = \frac{1}{10}$ ft/hr, find the time required to fill the bowl.

44. A body with initial velocity u ft/sec downward falls under gravity in a medium offering resistance proportional to the square of the velocity. Show that the distance fallen in t sec is

$$y = \frac{V^2}{g} \ln \left[\cosh \frac{gt}{V} + \frac{u}{V} \sinh \frac{gt}{V} \right] \text{ ft},$$

where V ft/sec is the limiting velocity and g ft/sec^2 is the acceleration due to gravity.

18. Integrable combinations. The differential equations encountered so far in this chapter have been of such a nature that the variables could be separated. We shall see now that a differential equation in which the variables are not separable can be solved if it can be arranged in integrable combinations. The process will be illustrated by some examples.

EXAMPLE 1. Solve

$$x\,dy = (3x^2 - y)\,dx. \tag{1}$$

This equation cannot be separated into terms containing x only and terms containing y only, but, by transposing the term $y\,dx$, we have

$$x\,dy + y\,dx = 3x^2\,dx.$$

The right member is now free of y and integrates at once into x^3. The left member contains both x and y, but it is an integrable combination of x and y; we recognize it as the exact differential of xy, so that it integrates, not term by term, but as a whole, into xy. The general solution of (1) is therefore

$$xy = x^3 + C.$$

If we make the substitution $xy = u$ in (1), a substitution suggested by the combination $x\,dy + y\,dx$, the equation takes the form

$$du = 3x^2\,dx,$$

in which the variables u and x are separated, and integration gives

$$u = x^3 + C,$$

or

$$xy = x^3 + C.$$

Differential equations integrable by combinations can always be reduced to separable equations in new variables by means of substitutions, but we usually integrate the combinations which are recognizable as exact differentials without substituting new variables.

Some of the frequently occurring integrable combinations and the functions of which they are exact differentials are:

$$x\,dy + y\,dx = d(xy),$$

$$\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right),$$

$$\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right),$$

$$\frac{2xy\,dy - y^2\,dx}{x^2} = d\left(\frac{y^2}{x}\right),$$

$$\frac{2xy\,dx - x^2\,dy}{y^2} = d\left(\frac{x^2}{y}\right),$$

$$\frac{x\,dy - y\,dx}{x^2 + y^2} = \frac{\dfrac{x\,dy - y\,dx}{x^2}}{1 + \left(\dfrac{y}{x}\right)^2} = d\left(\tan^{-1}\frac{y}{x}\right),$$

$$\frac{x\,dy - y\,dx}{x^2 - y^2} = \frac{\dfrac{x\,dy - y\,dx}{x^2}}{1 - \left(\dfrac{y}{x}\right)^2} = d\left(\frac{1}{2}\ln\frac{x+y}{x-y}\right) = d\tanh^{-1}\frac{y}{x}.$$

EXAMPLE 2. Solve

$$x\,dy - y\,dx = y\,dy. \tag{2}$$

If we multiply both sides by $1/x^2$, the left side becomes an integrable combination but the right side is not integrable. However, multiplying by $1/y^2$ and changing signs, we have

$$\frac{y\,dx - x\,dy}{y^2} = -\frac{dy}{y}. \tag{3}$$

Integration now gives

$$\frac{x}{y} = C - \ln y.$$

A differential equation such as (3), whose terms are exact differentials or form combinations which are exact differentials, is called an *exact* differential equation. The function $1/y^2$ is called an *integrating factor* of equation (2), since multiplication by this factor produces the exact equation (3). Rules can be devised for finding integrating factors of certain types of differential equations, but the method of finding these factors by inspection will suffice for our needs.

EXAMPLE 3. Solve

$$x\,dx + y\,dy = \sqrt{x^2 + y^2}\,dx. \tag{4}$$

Division by the radical on the right gives

$$\frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}} = dx, \tag{5}$$

an equation whose left member is an integrable combination of the form $du/2\sqrt{u}$, where $u = x^2 + y^2$, and therefore integrates into $\sqrt{u}$; thus $1/\sqrt{x^2 + y^2}$ is an integrating factor of (4). Integration of equation (5) yields

$$\sqrt{x^2 + y^2} = x + C,$$

the general solution of equation (4).

EXAMPLE 4. Solve

$$\frac{dy}{dx} = \frac{y}{x} - \frac{1}{2y}.$$

Clearing of fractions, we have

$$2xy\,dy = (2y^2 - x)\,dx,$$

or

$$2y(x\,dy - y\,dx) = -x\,dx.$$

If, now, we divide by x^2y, i.e., try $1/x^2y$ as an integrating factor, we have an integrable combination on the left but the right side is not integrable; also, a similar situation occurs if we divide by y^3. However, division by x^3 gives

$$2\left(\frac{y}{x}\right)\frac{x\,dy - y\,dx}{x^2} = -\frac{dx}{x^2}.$$

The left side is now an integrable combination of the form $2u\,du$, where $u = y/x$, and the right side is also integrable. Integrating, we have

$$\left(\frac{y}{x}\right)^2 = \frac{1}{x} + C,$$

or

$$y^2 = x + Cx^2.$$

19. Homogeneous equations. A function $f(x,y)$ is homogeneous of degree n if the replacement of x by kx and y by ky, where k is any quantity, constant or variable, multiplies the function by k^n, that is, if

$$f(kx, ky) = k^n f(x, y).$$

In particular, if $n = 0$ the function is homogeneous of degree zero and is unchanged by replacing x by kx and y by ky; such a function, if we take $k = 1/x$, can be written as a function of the combination y/x, as follows:

$$f(x, y) = f(kx, ky) = f\left(1, \frac{y}{x}\right) = F\left(\frac{y}{x}\right).$$

As illustrations, observe the functions

$$(a)\ f(x, y) = x^2 - 2xy + y^2,$$

$$(b)\ f(x, y) = x - 7y + 6,$$

$$(c)\ f(x, y) = y \ln x,$$

$$(d)\ f(x, y) = y(\ln x - \ln y),$$

$$(e)\ f(x, y) = \frac{x^2y + xy^2}{3x^3 - 4y^3}.$$

(*a*) $f(kx, ky) = k^2x^2 - 2kx \cdot ky + k^2y^2 = k^2(x^2 - 2xy + y^2) = k^2 f(x, y)$; hence the function $x^2 - 2xy + y^2$ is homogeneous of second degree.

(*b*) $f(kx, ky) = kx - 7ky + 6$. The k cannot be factored out; the function $x - 7y + 6$ is not homogeneous. If $f(x, y)$ is a polynomial, it is homogeneous only if each of its terms is of the same degree in x and y.

(*c*) $f(kx, ky) = ky \ln kx$; the function $y \ln x$ is not homogeneous.

(*d*) $f(kx, ky) = ky(\ln kx - \ln ky) = ky \ln x/y = kf(x, y)$; hence $f(x, y)$ is homogeneous of first degree.

(*e*) $f(kx, ky) = \dfrac{k^3(x^2y + xy^2)}{k^3(3x^3 - 4y^3)} = f(x, y)$. The function $f(x, y)$

is homogeneous of degree zero and therefore can be written as a function of y/x; division of both numerator and denominator by x^3 gives

$$\frac{x^2y + xy^2}{3x^3 - 4y^3} = \frac{\dfrac{y}{x} + \left(\dfrac{y}{x}\right)^2}{3 - 4\left(\dfrac{y}{x}\right)^3}.$$

The quotient of two functions, both homogeneous of the same degree, is a homogeneous function of degree zero, since, when x is replaced by kx and y by ky, the power of k occurring as factor of the numerator is the same as in the denominator and cancels out.

A differential equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

is called homogeneous if $f(x, y)$ is a homogeneous function of degree zero. If the differential equation is written in the form

$$M(x, y)\, dx + N(x, y)\, dy = 0, \tag{2}$$

it will then be homogeneous if M and N are homogeneous functions of the same degree. If a differential equation is homogeneous it can be solved by either of the following methods:

(*a*) *Substitute* y = vx. *The resulting equation is separable in* v *and* x; *solve it, and replace* v *by* y/x.

(*b*) *Substitute* x = vy. *The resulting equation is separable in* v *and* y; *solve it, and replace* v *by* x/y.

We have to prove that the substitution $y = vx$ reduces a homogeneous differential equation to one that is separable in v and x. The analogous proof for method (*b*) is left as an exercise for the student in the first problem of the next group.

Since the differential equation is homogeneous, it may be written in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right). \tag{3}$$

Letting $y = vx$, hence $dy = v\,dx + x\,dv$, equation (3) takes the form

$$v\,dx + x\,dv = F(v)\,dx,$$

or

$$\frac{dv}{F(v) - v} = \frac{dx}{x},$$

an equation in which v and x are separated. In the exceptional case where $F(v) = v$, equation (3) is itself separable: $dy/dx = y/x$.

EXAMPLE 1. Solve

$$2xy\,dx + (x^2 + y^2)\,dy = 0.$$

Method (*a*). Let $y = vx$, $dy = v\,dx + x\,dv$. It will be noticed that, since the differential equation is homogeneous of second degree, x^2 will cancel out of each term after the substitution $y = vx$. In general x^n will cancel out if n is the degree of homogeneity. Similarly y^n will cancel out after the substitution $x = vy$. The substitution and cancellation can be made at the same time. We have

$$2v\,dx + (1 + v^2)(v\,dx + x\,dv) = 0,$$

or

$$(v^3 + 3v)\,dx + x(1 + v^2)\,dv = 0.$$

Separating the variables and multiplying by 3 in order to make the second numerator the exact differential of its denominator, we get

$$\frac{3\,dx}{x} + \frac{(3v^2 + 3)\,dv}{v^3 + 3v} = 0.$$

Integrating,

$$3 \ln x + \ln (v^3 + 3v) = \ln C.$$

Taking antilogarithms,

$$x^3(v^3 + 3v) = C.$$

Replacing v by y/x,

$$y^3 + 3x^2y = C.$$

Method (*b*). Letting $x = vy$, $dx = v\,dy + y\,dv$, then cancelling y^2, we have

$$2v(v\,dy + y\,dv) + (v^2 + 1)\,dy = 0,$$

or

$$(3v^2 + 1)\,dy + 2vy\,dv = 0.$$

Separating the variables and multiplying by 3, we get

$$\frac{3\,dy}{y} + \frac{6v\,dv}{3v^2 + 1} = 0.$$

Integrating,

$$3 \ln y + \ln (3v^2 + 1) = \ln C.$$

Taking antilogarithms,

$$y^3(3v^2 + 1) = C.$$

Replacing v by x/y,

$$3x^2y + y^3 = C.$$

Shorter method. We have used this problem as an illustration of the standard methods for solving a homogeneous equation, but there is a shorter way to solve Example 1. Writing the equation in the form

$$x^2\,dy + 2xy\,dx + y^2\,dy = 0,$$

we see that the first two terms form an integrable combination, the differential of x^2y, so that integration yields at once

$$x^2y + \frac{y^3}{3} = \frac{C}{3},$$

or

$$3x^2y + y^3 = C.$$

It is desirable to use the shorter method of integrable combinations whenever possible, provided that too much time need not be spent in recognizing the integrable combination.

EXAMPLE 2. Solve

$$(x + \sqrt{y^2 - xy})\, dy - y\, dx = 0.$$

In Example 1 there was little to choose between the methods (*a*) and (*b*), although (*b*) is perhaps a trifle simpler. In this example, however, method (*b*) is definitely easier, as the student may verify by solving the problem using method (*a*). There is no general rule for determining beforehand which method is easier, but usually it is better to substitute for the variable whose differential has the simpler coefficient. Here dx has the simpler coefficient and we substitute $x = vy$. Then, after division by y, the equation becomes

$$(v + \sqrt{1 - v})\, dy - (v\, dy + y\, dv) = 0.$$

Separating the variables, we get

$$\frac{dy}{y} - \frac{dv}{\sqrt{1 - v}} = 0.$$

Integrating,

$$\ln y + 2\sqrt{1 - v} = \ln C.$$

Taking antilogarithms,

$$ye^{2\sqrt{1-v}} = C.$$

Replacing v by x/y,

$$ye^{2\sqrt{1-x/y}} = C.$$

20. The equivalence of solutions. Suppose that we have obtained by different methods two general solutions of the same differential equation and we wish to show that they are equivalent. One way is to show that, by changing the C of one solution into the proper function of the C of the other, the solutions become identical. For instance, if the two solutions are

$$x + Cxy + y = 0, \qquad C'(x + y) - xy = 0,$$

we show that they are equivalent by letting $C = -1/C'$, whereupon the first reduces to the second.

On the other hand, if the functional relation between the C's is not obvious, we may show that the solutions are equivalent without finding this relation. Let

$$u(x, y) = C, \qquad v(x, y) = C', \tag{1}$$

be two general solutions of the same differential equation; then they must give the same value for dy/dx. The values of dy/dx obtained by differentiating these implicit functions (1) are respectively

$$\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}, \qquad \frac{dy}{dx} = -\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}},$$

and the condition that these two values of dy/dx are identical is

$$\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \cdot \frac{\partial u}{\partial y} \equiv 0,$$

or

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \equiv 0. \qquad (2)$$

The determinant J is called the Jacobian of the functions u, v with respect to x, y. The two solutions (1) may be shown to be equivalent by showing that their Jacobian vanishes identically. This means that there is a functional relation * between C and C', but it does not tell what the relation is; the search for the relation is sometimes an interesting exercise.

Example 1. Show that the two following solutions of differential equation (2), Art. 12, are equivalent:

$$\text{arc tan}\, x + \text{arc tan}\, y = C, \qquad \frac{x + y}{1 - xy} = C'. \qquad (3)$$

Here we can see the functional relation between C and C', namely, $C = \text{arc tan}\, C'$; for, if we substitute $C = \text{arc tan}\, C'$ in the first equation and take the tangent of both sides, we get the second equation.

* For a proof that the vanishing of the Jacobian is a necessary and sufficient condition for functional dependence, see Goursat-Hedrick, *Mathematical Analysis*, Vol. I, p. 52.

But, if this relation were not apparent, we could, without finding it, show the solutions equivalent as follows: Let

$$u = \arc\tan x + \arc\tan y, \qquad v = \frac{x+y}{1-xy};$$

then, forming the Jacobian (2), we have

$$J = \begin{vmatrix} \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \\ \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \end{vmatrix} = \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} \equiv 0.$$

Hence the solutions (3) are equivalent; they are general solutions of the same differential equation. The vanishing of the Jacobian of u, v does not prove that the solutions $u = C$, $v = C'$ are correct, but it shows that if one is correct the other is also; if both solutions were incorrect and the Jacobian vanished, they would still be equivalent solutions of *some* differential equation.

We now consider an example in which the functional relation between the C's is not so obvious.

EXAMPLE 2. Solve the differential equation

$$(x^2 + y^2)\,dx + x^2\,dy = 0 \tag{4}$$

by substituting $y = vx$, then by substituting $x = vy$, and prove the solutions equivalent. Also find the functional relation between the C's.

First solution. Letting $y = vx$ in (4), we have

$$(1 + v^2)\,dx + v\,dx + x\,dv = 0,$$

$$\frac{dx}{x} + \frac{dv}{1+v+v^2} = 0,$$

$$\ln x + \frac{2}{\sqrt{3}}\arc\tan\frac{2v+1}{\sqrt{3}} = C,$$

$$\ln x + \frac{2}{\sqrt{3}}\arc\tan\frac{2y+x}{\sqrt{3}\,x} = C. \tag{5}$$

Second solution. Letting $x = vy$ in (4), we have

$$(v^2 + 1)(v\,dy + y\,dv) + v^2\,dy = 0,$$

$$\frac{dy}{y} + \frac{(v^2 + 1)\,dv}{v(v^2 + v + 1)} = 0,$$

$$\frac{dy}{y} + \left(\frac{1}{v} - \frac{1}{v^2 + v + 1}\right) dv = 0,$$

$$\ln y + \ln v - \frac{2}{\sqrt{3}} \arctan \frac{2v + 1}{\sqrt{3}} = C',$$

$$\ln x - \frac{2}{\sqrt{3}} \arctan \frac{2x + y}{\sqrt{3}\,y} = C'. \tag{6}$$

We have to prove that the two solutions (5) and (6) are equivalent. Letting f and g stand for the left members of (5) and (6) respectively, we shall prove that these two solutions are equivalent by showing that the Jacobian of f, g with respect to x, y is identically zero. We find

$$\frac{\partial f}{\partial x} = \frac{x^2 + y^2}{x(x^2 + xy + y^2)} = \frac{\partial g}{\partial x},$$

$$\frac{\partial f}{\partial y} = \frac{x}{x^2 + xy + y^2} = \frac{\partial g}{\partial y};$$

hence

$$J = \begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\[2ex] \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{vmatrix} \equiv 0,$$

since the two rows of the determinant are identical.

To find the functional relation between C and C', subtract equation (6) from equation (5):

$$\arctan \frac{2y + x}{\sqrt{3}x} + \arctan \frac{2x + y}{\sqrt{3}y} = \frac{\sqrt{3}}{2}(C - C'). \tag{7}$$

Taking the tangent of both sides of (7), we find

$$\frac{\sqrt{3}(2y^2 + xy + 2x^2 + xy)}{3xy - (2y + x)(2x + y)} = \tan\frac{\sqrt{3}}{2}(C - C'),$$

or

$$\frac{2\sqrt{3}(x^2 + xy + y^2)}{-2(x^2 + xy + y^2)} = -\sqrt{3} = \tan\frac{\sqrt{3}}{2}(C - C');$$

hence

$$\frac{\sqrt{3}}{2}(C - C') = \text{arc tan}\,(-\sqrt{3}) = -\frac{\pi}{3},$$

and the required functional relation is

$$C = C' - \frac{2\pi}{3\sqrt{3}}.$$

PROBLEMS

1. Prove that the homogeneous differential equation (1) or (2) of Art. 19 becomes separable upon substituting $x = vy$, thus verifying method (*b*).

Solve each of the following differential equations.

2. $(x - 2y^3)\,dy = y\,dx.$

3. $\dfrac{dy}{dx} = 1 + \dfrac{y}{x}.$

4. $x\,dy = (x^2 + y^2 + y)\,dx.$

5. $\dfrac{dy}{dx} = \dfrac{e^y}{2y - xe^y}.$

6. $(x^2 + y^2)\,dx = 2xy\,dy.$

7. $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{y^2}{x^2}.$

8. $\dfrac{dy}{dx} = \dfrac{2y}{x} + \dfrac{3x}{2y}.$

9. $\dfrac{dy}{dx} = \dfrac{xy}{x^2 - xy + y^2}.$

10. $\dfrac{dy}{dx} + \dfrac{y}{x} = e^{xy}.$

11. $x\dfrac{dy}{dx} = y(\ln y - \ln x).$

12. $\dfrac{dy}{dx} = \dfrac{x^2 - y^2 - 2y}{y^2 - x^2 - 2x}.$

13. $\dfrac{dy}{dx} = \dfrac{x - y}{x + y}.$

14. $\dfrac{dy}{dx} + \dfrac{3xy}{3x^2 + y^2} = 0.$

15. $\left(x + \dfrac{y}{x}\right)\dfrac{dy}{dx} = y.$

16. $3xy^2\,dy = (x^3 + 2y^3)\,dx.$

17. $3x\,dy = 2y\,dx - xy\cos x\,dx.$

18. $y\,dx + x\left(\ln\dfrac{y}{x} - 2\right)dy = 0.$

19. $\left(2x\tan\dfrac{y}{x} + y\right)dx = x\,dy.$

20. $\dfrac{x\,dy}{y\,dx} + \dfrac{3x^2 - y^2}{3y^2 - x^2} = 0.$

21. $\dfrac{dy}{dx} + \dfrac{7x + 3y + 5}{3x + 11y + 17} = 0.$

22. $\dfrac{x\,dy}{y\,dx} + \dfrac{2x^3 - x^2y - y^3}{2y^3 - xy^2 - x^3} = 0.$

23. $\dfrac{y\,dx - x\,dy}{y^2} = \left(\dfrac{1}{x} + \dfrac{1}{y}\right) dy.$

24. $x(y + \sqrt{x^2 + y^2})\,dy = y^2\,dx.$

25. $\sqrt{y^2 - 1}(1 - y\sqrt{x^2 - 1})\,dx + \sqrt{x^2 - 1}(1 - x\sqrt{y^2 - 1})\,dy = 0.$

26. $\sqrt{xy}\,dx = (x - y + \sqrt{xy})\,dy.$

27. $x(x^2 + y^2)\,dy = y(x^2 + y\sqrt{x^2 + y^2} + y^2)\,dx.$

28. Find a solution of

$$\frac{dy}{dx} = \frac{e^x y}{e^x + 2y}$$

such that $y = 1$ when $x = 0$.

29. Find a curve through the point $(1, -2)$ satisfying the differential equation

$$\frac{dy}{dx} = \frac{y(xy + 1)}{y(1 - x^2) - x}$$

30. Find a curve through the point $(\frac{1}{2}, 1)$ whose slope at each point is $xy/(x + y)^2$. For what value of x will the slope of the curve equal $\frac{1}{4}$?

31. Show that the general solution of the differential equation

$$\frac{dy}{dx} - 2 = \frac{x + 2}{y - 2(x + 1)}$$

represents a family of hyperbolas whose asymptotes are the lines

$$y = x, \qquad y = 3x + 4.$$

32. Find two curves satisfying the differential equation

$$\left(\frac{dy}{dx} + \frac{y}{x}\right)^2 + x^2 = 2,$$

each curve having slope $\frac{1}{2}$ at a point where it crosses the parabola $y = x^2/2$.

33. Show that for the differential equation

$$\frac{dy}{dx} + \frac{x}{y} = 1$$

(*a*) the substitution $y = vx$ leads to the general solution

$$\frac{\sqrt{3}}{2} \ln (x^2 - xy + y^2) + \arctan \frac{2y - x}{\sqrt{3}\,x} = C_1;$$

(*b*) the substitution $x = vy$ leads to the general solution

$$\frac{\sqrt{3}}{2} \ln (x^2 - xy + y^2) - \arctan \frac{2x - y}{\sqrt{3}\,y} = C_2.$$

Show that the two above solutions are equivalent, and find the relation connecting C_1 and C_2.

34. Solve the differential equation

$$\frac{dy}{dx} = \cosh\left(\tfrac{1}{2}x + y\right)$$

(*a*) by means of the substitution $\frac{1}{2}x + y = v$, obtaining the solution

$$\tanh\left(\frac{x}{4} + \frac{y}{2}\right) = \sqrt{3}\tan\left(\frac{\sqrt{3}}{4}x + C_1\right);$$

(*b*) by means of the substitution $\frac{1}{2}x + y = \ln v$, obtaining the solution

$$2e^{(x/2)+y} + 1 = \sqrt{3}\tan\left(\frac{\sqrt{3}}{4}x + C_2\right).$$

Show that the two above solutions are equivalent, and find the relation connecting C_1 and C_2.

35. Show that

$$e^{x+y} = C_1(\sqrt{9 + e^{2x}} - 3), \qquad e^{x-y} = C_2(\sqrt{9 + e^{2x}} + 3)$$

are equivalent solutions of the same differential equation. Find the differential equation and the relation connecting C_1 and C_2.

21. Linear equations. The standard form of the linear differential equation of first order is

$$\frac{dy}{dx} + Py = Q, \tag{1}$$

where P and Q are functions of x. It is called linear because it is of the first degree in the dependent variable and its derivative, that is, of the first degree in y and dy/dx. Either P or Q may be a constant, which can be regarded as a special case of a function of x, but if their ratio is constant the equation is separable.

Writing equation (1) in the form

$$dy + Py\,dx = Q\,dx, \tag{2}$$

let us try to integrate it by changing the left side into an integrable combination. We multiply equation (2) by an unknown function of x, say R, obtaining

$$R\,dy + yRP\,dx = RQ\,dx. \tag{3}$$

The right side of (3) is in integrable form, since Q is a known function of x, and R when it is found will be a function of x.

Can we determine R so that the left side of (3) will be an integrable combination? The combination

$$R\,dy + y\,dR = d(Ry) \tag{4}$$

is at once suggested. Comparing the left member of (3) with this exact differential (4), we see that they will be identical provided that

$$dR = RP\,dx, \tag{5}$$

in which case the left member of (3) will integrate into Ry. Now (5) is a separable equation determining the integrating factor R:

$$\frac{dR}{R} = P\,dx,$$

$$\ln R = \int P\,dx, \tag{6}$$

$$R = e^{\int P\,dx}. \tag{7}$$

The constant of integration in (6) is omitted, or taken equal to zero, since the simplest value of R is desired.

With the value of R given by (7), the left member of (3) integrates into Ry and the integral of the right member is expressed by $\int RQ\,dx + C$, so that the general solution of the linear equation (1) is

$$Ry = \int RQ\,dx + C, \tag{8}$$

where

$$R = e^{\int P\,dx}.$$

Therefore, to solve the linear equation (1), first compute the integrating factor R, then insert it in (8), perform the indicated integration, and simplify the result if possible.

Example 1. Solve

$$x\,dy = 2(x^4 + y)\,dx.$$

The equation is linear, but it must be put into standard form (1) before using formula (8), thus:

$$\frac{dy}{dx} - \frac{2}{x}y = 2x^3.$$

Here $P = -2/x$, $Q = 2x^3$. Computing R, we have

$$R = e^{\int -(2/x)\,dx} = e^{-2 \ln x} = e^{\ln x^{-2}} = x^{-2}.$$

Substitution in (8) gives

$$x^{-2}y = \int x^{-2} \cdot 2x^3\,dx + C = x^2 + C,$$

or

$$y = x^4 + Cx^2.$$

EXAMPLE 2. Solve

$$(x - \sin y)\,dy + \tan y\,dx = 0,$$

and find the particular solution satisfying the condition $y = \pi/6$ when $x = 1$.

At first glance this equation does not seem to be linear, for, if we write dy/dx as the first term, the equation takes the form

$$\frac{dy}{dx} + \frac{\tan y}{x - \sin y} = 0,$$

which is not in standard form (1). The equation is not linear with y as dependent variable, that is, it is not linear in y and dy/dx; but, if x is taken as the dependent variable and the equation is written in the form

$$\frac{dx}{dy} + \cot y \cdot x = \cos y,$$

we see that it is linear in x and dx/dy. Hence it can be solved by using formula (8) with x and y interchanged, P and Q now being functions of y, namely, $P = \cot y$, $Q = \cos y$.

We have

$$R = e^{\int \cot y\,dy} = e^{\ln \sin y} = \sin y.$$

Then formula (8) gives

$$\sin y \cdot x = \int \sin y \cos y \, dy + C = \tfrac{1}{2} \sin^2 y + C,$$

or, changing C to $C/2$,

$$2x \sin y = \sin^2 y + C,$$

which is the general solution of the given equation.

To find the required particular solution of the given equation, substitute $x = 1$, $\sin y = \sin (\pi/6) = \frac{1}{2}$; then

$$2 \cdot (\tfrac{1}{2}) = \tfrac{1}{4} + C, \qquad C = \tfrac{3}{4},$$

and

$$8x \sin y = 4 \sin^2 y + 3.$$

Sometimes a differential equation may be solved in more than one way. For instance, the above example may be solved also by use of an integrable combination.

Shorter solution. The differential equation may be written

$$x\,dy + \tan y\,dx = \sin y\,dy.$$

Multiplication by the integrating factor $\cos y$ produces on the left the integrable combination $x \cos y\,dy + \sin y\,dx$ or $d(x \sin y)$; hence

$$x \cos y\,dy + \sin y\,dx = \sin y \cos y\,dy,$$

$$x \sin y = \tfrac{1}{2} \sin^2 y + C.$$

When $x = 1$, $y = \pi/6$, $\frac{1}{2} = \frac{1}{8} + C$, $C = \frac{3}{8}$, and the required particular solution is

$$8x \sin y = 4 \sin^2 y + 3.$$

PROBLEMS

Solve the following differential equations.

1. $\dfrac{dy}{dx} = e^x + y.$

2. $x\dfrac{dy}{dx} + 3y = x^2 + 1.$

3. $\dfrac{dy}{dx} = \dfrac{xy - 6}{2x^2}.$

4. $2x^3\,dy + 3(x^2y + 1)\,dx = 0.$
5. $(1 + x^2)\,dy = (x^3 - xy + x)\,dx.$
6. $\dfrac{dy}{dx} = \dfrac{xy + 1}{1 - x^2}.$
7. $x\,dy - 2y\,dx = 2y^4\,dy.$
8. $x\left(\dfrac{dy}{dx} + y\right) = x - y.$
9. $2x\,dy = (y + 2x^2 \ln x)\,dx.$
10. $\dfrac{dy}{dx} + x = \dfrac{y}{x}.$
11. $(x - 1)\,dy = [y + (x - 2)e^x]\,dx.$
12. $\dfrac{dy}{dx} - (x + y) = 1.$
13. $x^2\dfrac{dy}{dx} + y = x^2 + 2xy.$
14. $(2x + y^2)y\,dy = dx.$
15. $3\dfrac{dy}{dx} = \dfrac{100 + 2x - y}{50 + x}.$
16. $\dfrac{dy}{dx} - \dfrac{y}{x} = \dfrac{x + y}{x - 1}.$
17. $\dfrac{dy}{dx} - (x + 1)y = x^2 + 4x + 2.$

18. Solve the following differential equation by three different methods—by regarding it as homogeneous, by regarding it as linear, and by using an integrable combination:

$$\frac{dy}{dx} = \frac{y}{x + 2y}.$$

19. Find the curve, passing through the point $(\pi/6, -\frac{1}{2})$, whose slope at any point is given by the function $(y + \sin^3 x)/(\sin x \cos x)$.

20. Obtain a solution of the differential equation

$$\frac{dy}{dx} - y \cot x = \sin 2x$$

such that y will vanish when $x = \pi/2$. Find the maximum and minimum values of y, and sketch the curve representing the solution from $x = 0$ to $x = 2\pi$.

21. If

$$\frac{dy}{dx} + 2x^2 + 3 = 2y,$$

and $y = 2$ when $x = 0$, find the value of y when $x = 1$.

22. If

$$\frac{dy}{dx} - \frac{y}{x} = y - x,$$

and $y = 2$ when $x = 1$, find the value of y when $x = 2$.

23. If

$$\frac{dy}{dx} = \cos 3x + 2y \tan 2x,$$

and $y = 1$ when $x = 0$, find the value of y when $x = \pi/2$.

24. If

$$\frac{dy}{dx} + 2(x - y) + 1 = 0,$$

and $y = 2$ when $x = 0$, find the value of y when $x = 1$.

25. If

$$\frac{dy}{dx} = \frac{y \ln y}{x + \ln y},$$

and $y = e$ when $x = 0$, find the value of x when $y = 3$.

26. If

$$x\,dy - 2y\,dx = 2y^3\,dy,$$

and $y = 1$ when $x = 0$, find the value of x when $y = \frac{1}{4}$.

27. A curve satisfies the differential equation

$$\frac{dy}{dx} + 2xy = 2x^5$$

and passes through the point (0, 2). Show that this point is a maximum point on the curve and that the minimum value of y is 1.

22. Chemical solutions. We shall consider now some physical problems leading to differential equations of the first order which are not separable.

Example 1. A tank contains 100 gal of brine in which 50 lb of salt are dissolved. Suppose that brine containing 2 lb/gal of salt runs into the tank at the rate of 3 gal/min and that the mixture, kept uniform by stirring, runs out of the tank at the rate of 2 gal/min. Find the amount of salt in the tank at the end of 30 min.

We set up the differential equation according to the following scheme:

$$\begin{Bmatrix}\text{Rate of increase}\\ \text{of salt (lb/min)}\end{Bmatrix} = \begin{Bmatrix}\text{Rate of inflow}\\ \text{of salt (lb/min)}\end{Bmatrix} - \begin{Bmatrix}\text{Rate of outflow}\\ \text{of salt (lb/min)}\end{Bmatrix}.$$

Let x be the number of pounds of salt in the tank at the end of t min; then

$$\text{Rate of increase of salt} = \frac{dx}{dt}\left(\frac{\text{lb}}{\text{min}}\right).$$

The brine flows in at the rate of 3 gal/min and each gallon contains 2 lb of salt; hence

$$\text{Rate of inflow of salt} = 3\left(\frac{\text{gal}}{\text{min}}\right)\cdot 2\left(\frac{\text{lb}}{\text{gal}}\right) = 6\left(\frac{\text{lb}}{\text{min}}\right).$$

Since the brine runs into the tank 1 gal/min faster than it runs out, the amount of brine in the tank at the end of t min is $100 + t$ gal, and the concentration of salt present at that time is $x/(100 + t)$ lb/gal; hence

$$\text{Rate of outflow of salt} = 2\left(\frac{\text{gal}}{\text{min}}\right)\cdot\frac{x}{100+t}\left(\frac{\text{lb}}{\text{gal}}\right) = \frac{2x}{100+t}\left(\frac{\text{lb}}{\text{min}}\right).$$

Therefore the differential equation is

$$\frac{dx}{dt} = 6 - \frac{2x}{100+t}.$$

When the last term is transposed to the left side, the differential equation takes the standard linear form

$$\frac{dx}{dt} + \frac{2}{100+t}\cdot x = 6, \qquad (1)^*$$

which we solve by the method of Art. 21:

$$R = e^{\int 2\,dt/(100+t)} = e^{2\ln(100+t)} = (100+t)^2,$$

$$(100+t)^2 x = \int 6(100+t)^2\,dt + C,$$

$$(100+t)^2 x = 2(100+t)^3 + C. \qquad (2)$$

The constant of integration is determined from the condition that $x = 50$ when $t = 0$, so that

$$C = 100^2\cdot 50 - 2\cdot 100^3 = -100^2\cdot 150.$$

* The substitution $x = c(100 + t)$, where c is the concentration of salt present at time t, would reduce equation (1) to an equation separable in the variables c and t. (See Prob. 1, Art. 24.)

Hence

$$(100 + t)^2 x = 2(100 + t)^3 - 100^2 \cdot 150,$$

or

$$x = 2(100 + t) - \frac{150}{\left(1 + \frac{t}{100}\right)^2}, \tag{3}$$

which gives the amount of salt in the tank at any time t; for $t = 30$ min we have

$$x\Big]_{t=30} = 260 - \frac{150}{1.69} = 171 \text{ lb.} \tag{4}$$

Instead of adding and evaluating a constant of integration, we could have integrated equation (1) between limits, writing, instead of equation (2),

$$(100 + t)^2 x\Big]_{\substack{t=0\\x=50}}^{\substack{t=30\\x=x}} = 2(100 + t)^3\Big]_0^{30}.$$

Then

$$130^2 x - 100^2 \cdot 50 = 2(130^3 - 100^3),$$

$$x = \frac{2 \cdot 130^3 - 150 \cdot 100^2}{130^2} = 260 - \frac{150}{1.69} = 171 \text{ lb.}$$

Variations of Example 1. In certain cases, problems of this type lead to separable differential equations. For instance, suppose that in the example just solved the brine is running out at the same rate that it is running in, namely, 3 gal/min. The amount of brine in the tank remains constant, 100 gal, and the differential equation is

$$\frac{dx}{dt} = 6 - \frac{3x}{100}.$$

Separation of variables gives

$$\frac{dx}{200 - x} = \frac{3\,dt}{100},$$

and the solution proceeds as follows:

$$\ln(200 - x)\Big]_{50}^{x} = \frac{-3t}{100}\Big]_{0}^{30},$$

$$\ln\frac{200 - x}{150} = -0.9,$$

$$200 - x = 150e^{-0.9} = 150 \times 0.4066,$$

$$x = 200 - 61 = 139 \text{ lb},$$

the amount of salt in the tank at the end of 30 min.

Again, suppose that the conditions of the original problem are the same except that pure water is running into the tank. There is now no inflow of salt, and the differential equation is

$$\frac{dx}{dt} = -\frac{2x}{100 + t}.$$

Separating the variables, and solving, we have

$$\frac{dx}{x} = -\frac{2\,dt}{100 + t},$$

$$\ln x\Big]_{50}^{x} = -2\ln(100 + t)\Big]_{0}^{30},$$

$$\ln\frac{x}{50} = -2\ln\frac{130}{100} = -2\ln 1.3 = \ln\frac{1}{1.69}.$$

$$x = \frac{50}{1.69} = 29.6 \text{ lb},$$

which in this case is the amount of salt in the tank at the end of 30 min.

23. Electric circuits. In the subsequent differential equations for electric circuits the following symbols will be used: R (ohms) = resistance; L (henries) = inductance; C (farads) = capacitance; e (volts) = electromotive force (emf), q (coulombs) = quantity of electricity, or charge on a condenser, and i (amperes) = current, at time t (sec). The capital letters are constants; the small letters are, in general, variables.

Under certain conditions the circuit equation is a linear differential equation of first order, a special case of which reduces to a separable equation.

For a simple circuit containing a resistance and an inductance in series with a source of emf *

$$L\frac{di}{dt} + Ri = e. \tag{1}$$

In general, when e is a function of t, the equation is linear, but when e is a constant the equation is separable.

For a simple circuit containing a resistance and a capacitance in series with an emf

$$R\frac{dq}{dt} + \frac{1}{C}q = e, \tag{2}$$

or upon differentiating, since $i = dq/dt$,

$$R\frac{di}{dt} + \frac{1}{C}i = \frac{de}{dt}. \tag{3}$$

In general, when the right member of (2) or (3) is a function of t, the equation is linear, but it becomes separable if the right member is constant. If R, C, and e are given, q may be determined from (2), and then i can be obtained from the relation $i = dq/dt$; or i may be determined directly from equation (3).

The general solutions of equations (1), (2), and (3) can be written down in a form involving an integral, but, unless e is constant, the integrations cannot be carried out until the form of e as a function of t is given.

For example, to obtain the general solution of (1), where $e = f(t)$, we have

$$\frac{di}{dt} + \frac{R}{L}i = \frac{1}{L}f(t). \tag{4}$$

* The differential equation for a circuit containing resistance, inductance, and capacitance will be found in Art. 39. For a derivation of these differential equations, see Bedell and Crehore, *Alternating Currents*, or Reddick and Miller, *Advanced Mathematics for Engineers*, 2nd Ed., Arts. 7 and 14.

The integrating factor * is $\epsilon^{\int (R/L)\,dt} = \epsilon^{(R/L)t}$; hence

$$\epsilon^{(R/L)t} \cdot i = \frac{1}{L} \int \epsilon^{(R/L)t} f(t)\, dt + K,$$

where K is a constant of integration, and the general solution is

$$i = \frac{1}{L}\, \epsilon^{-(R/L)t} \int \epsilon^{(R/L)t} f(t)\, dt + K\epsilon^{-(R/L)t}. \tag{5}$$

The integration cannot be carried out until the form of $f(t)$ is specified. Suppose that the emf is a simple harmonic function of the time, $e = f(t) = E \sin \omega t$. Here E is the amplitude or maximum value of the emf, and ω (rad/sec) is the angular velocity, or 2π times the frequency in cycles per second, so that, when t is given in seconds, the angle ωt is in radians. Equation (5) now becomes (Peirce, 414)

$$i = \frac{1}{L}\, \epsilon^{-(R/L)t} \left[\frac{E}{\left(\frac{R}{L}\right)^2 + \omega^2}\, \epsilon^{(R/L)t} \left(\frac{R}{L} \sin \omega t - \omega \cos \omega t \right) \right] + K\epsilon^{-(R/L)t},$$

or

$$i = \frac{E}{R^2 + L^2\omega^2} (R \sin \omega t - L\omega \cos \omega t) + K\epsilon^{-(R/L)t}. \tag{6}$$

The value of K is determined when a corresponding pair of values (i, t) is known; thus if $i = 0$ when $t = 0$, we have $K = EL\omega/(R^2 + L^2\omega^2)$, and equation (6) reduces to

$$i = \frac{E}{R^2 + L^2\omega^2} (R \sin \omega t - L\omega \cos \omega t) + \frac{EL\omega}{R^2 + L^2\omega^2}\, \epsilon^{-(R/L)t}. \tag{7}$$

Hence, for a simple circuit containing a resistance R and an inductance L, in series with an impressed emf of the form $e = E \sin \omega t$, equation (7) gives the current i at any time t after the introduction of the emf under the condition that $i = 0$ when $t = 0$. This condition means that the circuit was

* We now write ϵ instead of e for the constant $2.718\cdots$ to distinguish it from the electromotive force e.

idle, i.e., no current was flowing, just before the emf was introduced, so that at the time when the emf is applied the current starts at zero. There can be no sudden jump in the current when the emf is introduced, for that would make di/dt infinite and equation (1) would no longer be valid. The last term o (7) is called the *transient* term, since it dies out with increasing t; theoretically it becomes zero only when t is infinite, but practically it is negligible for t fairly small. The remaining term on the right is called the *steady-state* term, since it gives the value of the current when a steady state has been reached, i.e., after the transient has died out.

We now apply the method used in Art. 2 to reduce the equation of the steady-state current to another form:

$$i = \frac{E}{R^2 + L^2\omega^2}(R \sin \omega t - L\omega \cos \omega t)$$

$$= \frac{E}{\sqrt{R^2 + L^2\omega^2}}\left[\frac{R}{\sqrt{R^2 + L^2\omega^2}} \sin \omega t - \frac{L\omega}{\sqrt{R^2 + L^2\omega^2}} \cos \omega t\right],$$

or

$$i = \frac{E}{\sqrt{R^2 + L^2\omega^2}} \sin\left(\omega t - \tan^{-1}\frac{L\omega}{R}\right). \tag{8}$$

Equation (8) shows that the maximum value, or amplitude, of the steady-state current is $E/\sqrt{R^2 + L^2\omega^2}$, which is less than E/R, the value it would have if there were no inductance. The expression $\sqrt{R^2 + L^2\omega^2}$ is called the *impedance*; it represents apparent resistance. The steady-state current is alternately positive and negative and oscillates between $+E/\sqrt{R^2 + L^2\omega^2}$ and $-E/\sqrt{R^2 + L^2\omega^2}$; it is a simple harmonic function of the time, having the same frequency as the impressed emf.

Now consider the case where $e = E$, a constant. Equation (1) reduces to

$$\frac{di}{dt} = \frac{E - Ri}{L},$$

a separable equation whose solution proceeds as follows:

$$\frac{di}{E - Ri} = \frac{1}{L}\,dt,$$

$$\ln(E - Ri) = -\frac{R}{L}t + \ln K,$$

$$E - Ri = K\epsilon^{-(R/L)t},$$

$$i = \frac{1}{R}(E - K\epsilon^{-(R/L)t}), \tag{9}$$

which is the general solution of equation (1) for a constant emf. The condition $i = 0$ when $t = 0$ would give $K = E$ and reduce the equation to the form

$$i = \frac{E}{R}(1 - \epsilon^{-(R/L)t}). \tag{10}$$

As t increases, the exponential term becomes negligible and the current takes on its Ohm's law value, E/R.

EXAMPLE 1. A simple electric circuit contains a resistance of 10 ohms and an inductance of 4 henries in series with an impressed emf of 100 sin 200t volts. If the current $i = 0$ when $t = 0$, find (a) the current when $t = 0.01$ sec; (b) the ratio of the steady-state current when $t = 4$ sec, to its maximum value.

(a) We could use equation (7) as a formula and substitute the given values of the letters, but instead let us start with the differential equation and integrate between limits:

$$4\frac{di}{dt} + 10i = 100 \sin 200t,$$

$$\frac{di}{dt} + 2.5i = 25 \sin 200t,$$

$$\epsilon^{2.5t}i\Big]_{\substack{t=0\\i=0}}^{\substack{t=0.01\\i=i}} = 25\int_0^{0.01} \epsilon^{2.5t} \sin 200t\,dt,$$

$$\epsilon^{0.025}i = \frac{25}{2.5^2 + 200^2}\,\epsilon^{2.5t}(2.5 \sin 200t - 200 \cos 200t)\Big]_0^{0.01},$$

$$\epsilon^{0.025}i = \frac{1}{8}\,\epsilon^{2.5t}\left(\frac{1}{80} \sin 200t - \cos 200t\right)\Big]_0^{0.01}$$

(where 1/8 is written instead of 1/8.00125),

$$i = \tfrac{1}{8}\left(\tfrac{1}{80}\sin 2 - \cos 2 + \epsilon^{-0.025}\right),$$

$$i = \frac{1}{8}\left(\frac{0.9093}{80} + 0.4161 + 0.9753\right),$$

$$i = \tfrac{1}{8}(1.403) = 0.175 \text{ amp.}$$

(*b*) From equation (8) the ratio of the steady-state current, when $t = 4$ sec, to its maximum value is $\sin(800 - \tan^{-1} 80)$. Expressing the angle in degrees, we have

$$\tan^{-1} 80 = 89.28°,$$

$$800 \text{ rad} = 800 \times 57.29578° = 45{,}836.62°,$$

$$800 - \tan^{-1} 80 \text{ rad} = 45{,}747.34°.$$

Reducing this angle by $127 \times 360° = 45{,}720°$, we obtain for the required ratio

$$\sin 27.34° = 0.459.$$

Without using degrees, the computation could have been made as follows:

$$\tan^{-1} 80 = 1.5583,$$

$$800 - \tan^{-1} 80 = 798.4417.$$

Reducing this angle by $254\pi = 797.9645$, we obtain for the required ratio

$$\sin(0.4772) = 0.459.$$

24. Rope wound on a cylinder. A rope is wound on a rough circular cylinder whose axis is horizontal. Let μ represent the coefficient of friction, a (ft) the radius of the cylinder, T (lb) the tension in the rope at any point P, ρ (lb/ft) the linear density of the rope, and θ the angle POA. (See Fig. 5.) Let Q be a point on the rope somewhat to the left of P, $\Delta\theta$ the angle POQ, Δs the length of arc PQ, and M the midpoint of PQ. Suppose that the rope is on the point of slipping from right to left; then the tension at Q will be $T + \Delta T$, a quantity larger by ΔT than the tension at P.

Let us resolve along the tangent line at M the forces acting on the small portion of the rope, PQ. The tension $T + \Delta T$, acting tangentially to the cylinder at Q, yields toward the left along the tangent line a force $(T + \Delta T) \cos (\Delta\theta/2)$. We equate this to the sum of all forces toward the right along the tangent line. Notice that the frictional force along the tangent line is toward the right, since it opposes the motion, and that it is equal to μ times the resultant along the normal MO of the tension T, the tension $T + \Delta T$, and the weight $\rho\Delta s$ of PQ. The resulting equation is

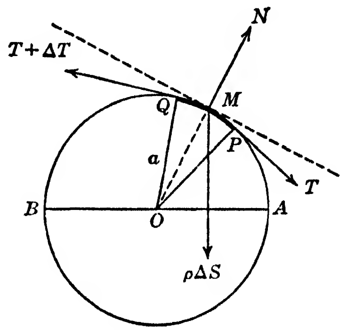

FIG. 5

$$(T + \Delta T) \cos \frac{\Delta\theta}{2} = T \cos \frac{\Delta\theta}{2} + \rho\Delta s \cos\left(\theta + \frac{\Delta\theta}{2}\right) +$$
$$\mu\left[T \sin \frac{\Delta\theta}{2} + (T + \Delta T) \sin \frac{\Delta\theta}{2} + \rho\Delta s \sin\left(\theta + \frac{\Delta\theta}{2}\right)\right]. \quad (1)$$

If we let $\Delta\theta \to 0$, and hence $\Delta s \to 0$ and $\Delta T \to 0$, in order to obtain the tension T at P, we obtain $T = T$; but if, after cancelling the equal terms, $T \cos (\Delta\theta/2)$, we divide equation (1) through by $\Delta\theta$ and then take limits as $\Delta\theta \to 0$, we obtain the differential equation which determines T as a function of θ:

$$\frac{\Delta T}{\Delta\theta} \cos \frac{\Delta\theta}{2} = \rho \frac{\Delta s}{\Delta\theta} \cos\left(\theta + \frac{\Delta\theta}{2}\right) +$$
$$\mu\left[\frac{2T + \Delta T}{2} \frac{\sin \frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}} + \rho \frac{\Delta s}{\Delta\theta} \sin\left(\theta + \frac{\Delta\theta}{2}\right)\right],$$

or, since

$$\frac{\Delta s}{\Delta \theta} = a \qquad \text{and} \qquad \lim_{\Delta\theta \to 0} \frac{\sin \dfrac{\Delta \theta}{2}}{\dfrac{\Delta \theta}{2}} = 1,$$

$$\frac{dT}{d\theta} = \rho a \cos \theta + \mu[T + \rho a \sin \theta].$$

The linear differential equation

$$\frac{dT}{d\theta} - \mu T = \rho a(\cos \theta + \mu \sin \theta) \tag{2}$$

therefore states the law connecting the tension T at any point P with the angle corresponding to this point.

The general solution of equation (2) may be found by the method of Art. 21; it is

$$T = \frac{\rho a}{1 + \mu^2} [(1 - \mu^2) \sin \theta - 2\mu \cos \theta] + Ce^{\mu\theta}, \tag{3}$$

where C is an arbitrary constant. (See Problem 25 at the end of this article.)

Two special cases occur: when T is so large compared to ρ that ρ can be taken equal to zero, and when friction is negligible so that μ can be taken equal to zero. These cases are represented respectively by the following differential equations and general solutions:

$$\frac{dT}{d\theta} = \mu T, \qquad T = Ce^{\mu\theta}; \tag{4}$$

$$\frac{dT}{d\theta} = \rho a \cos \theta, \qquad T = \rho a \sin \theta + C. \tag{5}$$

Example 1. On a rough circular cylinder with horizontal axis and a diameter of 8 ft and in a plane perpendicular to the axis lies a chain with one end at the level of the axis of the cylinder. If the coefficient of friction between the chain and cylinder is $\frac{3}{5}$, how far below the axis must the other end of the chain hang down so that the chain is on the point of slipping?

Referring to Fig. 5, suppose that the right end of the chain is at A and that the left end hangs down a distance L below B. Equation (2) is the differential equation for this problem. Assuming that the general solution (3) has been obtained, the constant C is determined from the condition $T = 0$ when $\theta = 0$ as follows:

$$0 = \frac{4\rho}{1 + \frac{9}{25}}\left[0 - \frac{6}{5}\right] + C,$$

$$C = \frac{100\rho}{34} \cdot \frac{6}{5} = \frac{60\rho}{17}.$$

Also $T = \rho L$ when $\theta = \pi$, so that we have from (3):

$$\rho L = \frac{100\rho}{34}\left[\frac{16}{25}\sin\pi - \frac{6}{5}\cos\pi\right] + \frac{60\rho}{17}e^{3\pi/5},$$

$$L = \tfrac{60}{17}\,(1 + e^{3\pi/5}) = \tfrac{60}{17}(1 + 6.586) = 26.8 \text{ ft.}$$

The same result is obtained by substituting $\mu = \frac{3}{5}$, $a = 4$, in the formula found for L in Problem 26 of the following group.

Example 2. A rope weighing $\frac{1}{2}$ lb/ft is wrapped once around a smooth horizontal cylinder 6 ft in diameter. If the rope is cut at its lowest point so that the ends hang down, find the maximum tension in the rope.

Assuming that the cylinder is so smooth that friction is negligible, equation (5) is applicable:

$$T = \tfrac{1}{2}\cdot 3\cdot \sin\theta + C.$$

The length of rope hanging down on each side is equal to a quarter of a circumference or $\frac{1}{4}\cdot 6\pi$ ft and its weight is $(6\pi/4)\frac{1}{2} = \frac{3}{4}\pi$ lb, so that C is determined by the condition $T = \frac{3}{4}\pi$ when $\theta = 0$ or π; hence $C = \frac{3}{4}\pi$, and

$$T = \tfrac{3}{2}\sin\theta + \tfrac{3}{4}\pi.$$

The maximum tension occurs at the top where $\theta = \pi/2$; hence

$$T_{\max} = \tfrac{3}{2} + \tfrac{3}{4}\pi = 3.86 \text{ lb.}$$

PROBLEMS

1. In Ex. 1, Art. 22, what is the concentration of salt in the tank at the end of 30 min [obtained from equation (4)]? Obtain the same result by solving the separable equation mentioned in the footnote to equation (1).

2. A tank contains 100 gal of fresh water. Brine containing 1 lb/gal of salt runs into the tank at the rate of 2 gal/min, and the mixture, kept uniform by stirring, runs out at the rate of 1 gal/min. Find (*a*) the amount of salt present when the tank contains 125 gal of brine; (*b*) the concentration of salt in the tank at the end of 1 hr.

3. A tank contains 100 gal of brine holding 25 lb of salt in solution. Three gallons of brine, each containing 1 lb of dissolved salt, run into the tank per minute, and the mixture, kept uniform by stirring, runs out of the tank at the rate of 1 gal/min. Find (*a*) the amount of salt in the tank at the end of 40 min; (*b*) the time when the concentration of salt in the tank is $\frac{3}{4}$ lb/gal.

4. (*a*) Brine, containing 1 lb of salt per gallon, runs into a 200-gal tank initially full of brine containing 3 lb of salt per gallon, at the rate of 4 gal/min. If the mixture runs out at the same rate, when will the concentration of salt in the tank reach 1.01 lb/gal? (*b*) Work Prob. 4(*a*) if the mixture runs out at the rate of 5 gal/min.

5. A tank contains 100 gal of saturated brine (3 lb/gal of salt). Brine containing 2 lb/gal of salt runs into the tank at the rate of 5 gal/min, and the mixture runs out at the rate of 4 gal/min. Find the minimum content of salt in the tank and the time required to reach the minimum.

6. Show that in Prob. 5, if the mixture runs out at the rate of 3 gal/min, the content of salt in the tank continually increases.

7. A tank contains 100 gal of fresh water. Brine containing 1 lb/gal of salt runs into the tank at the rate of 1 gal/min, and the mixture runs out at the rate of 3 gal/min. Find the maximum amount of salt in the tank and the time required to reach the maximum.

8. A tank contains V gal of fresh water. Brine containing c lb/gal of salt runs into the tank, and the mixture runs out, the ratio of the rate of inflow to the rate of outflow being $r(<1)$. Find the maximum amount of salt in the tank and the corresponding volume of brine in the tank.

9. A resistance of 3 ohms and an inductance of 1 henry are connected in series with an emf of 10 sin 10t volts. If the current is zero when $t = 0$, find (*a*) the current when $t = 0.1$ sec; (*b*) the current when $t = 3$ sec; (*c*) the ratio of the numerical value of the steady-state current when $t = 3$ sec to its maximum value.

10. A resistance of 20 ohms and an inductance of 2 henries are connected in series with an emf of e volts. If the current is zero when $t = 0$, find the current at the end of 0.01 sec if (*a*) $e = 100$; (*b*) $e = 100 \sin 150t$.

11. Find the general solution of equation (2), Art. 23, if $e = f(t)$. From this result obtain the general solution when (*a*) $e = E$, a constant; (*b*) $e = E \sin \omega t$.

12. Obtain the general solution of equation (3), Art. 23, for an emf of $E \sin \omega t$ volts, (*a*) by substituting $e = E \sin \omega t$ in equation (3) and integrating; (*b*) by differentiating the result of Prob. 11(*b*).

13. An emf is introduced into a circuit containing in series a resistance of 10 ohms and an uncharged condenser of capacitance 5×10^{-4} farad. Find the current and the charge on the condenser when $t = 0.01$ sec, (*a*) if $e = 100$ volts; (*b*) if $e = 100 \sin 120\pi t$ volts.

14. An emf of 100 volts is introduced into a circuit containing in series a resistance of 10 ohms and an uncharged condenser of capacitance 5×10^{-4} farad. When a steady state has been reached, the emf is removed from the circuit. Find the current and the charge on the condenser 0.01 sec after the removal of the emf.

15. An emf of $100 \sin 120\pi t$ volts is introduced into a circuit containing in series a resistance of 100 ohms and a condenser of capacitance 5×10^{-4} farad. There is already a charge on the condenser at the time $t = 0$, when the emf is introduced, such that the current at that time is 1 amp (positive). Find the current 0.1 sec later.

16. An inductance of 1 henry and a resistance of 100 ohms are connected in series with a constant source E (volts) through a switch. The switch is closed, and 0.01 sec later the current is $\frac{1}{2}$ amp. Find E.

17. A resistance of 10 ohms is put in series with an inductance of L henries. The circuit is connected through a switch to a constant source of E volts. If the current reaches $\frac{3}{4}$ of its steady-state value in 0.1 sec, find L.

18. (*a*) A condenser of capacitance C is in series with a resistance R. The circuit is connected to a constant source E through a switch. If the condenser is uncharged at the instant the switch is thrown, and after t sec has a charge Q, what source was applied?

(*b*) Find E if $t = 0.01$ sec, $R = 100$ ohms, $C = 5 \times 10^{-5}$ farad, $Q = 5.2 \times 10^{-3}$ coulomb.

19. A certain relay is designed so as to close a circuit when 60 volts are impressed across its terminals (i.e., when $Ri = 60$ volts). The relay coil has an inductance of $\frac{1}{2}$ henry and operates from a 120-volt d-c source. If the circuit closes 0.05 sec after being connected to the source, find (*a*) the resistance of the relay; (*b*) the current when the circuit closes. Neglect the resistance of the leads.

20. If in Prob. 19 the relay is designed to operate when the current is 6 amp, in the same time, 0.05 sec, find the resistance of the relay coil.

21. An uncharged condenser of capacitance C (farad) is charged from a source of constant voltage through a resistance R (ohms). When will the current (amp) be equal in magnitude to the charge (coulomb) on the condenser?

22. Show that in Prob. 21 the charge will reach 63.2 per cent of its maximum value in RC sec and that it would reach its maximum value in this same time if it increased continually at its original rate.

23. A condenser of capacitance 4×10^{-4} farad is discharged through a resistance of 10 ohms. If the current is 1 amp at the end of 0.01 sec, what was the initial charge on the condenser? How much resistance should be taken out of the circuit in order to obtain half the current in the same time?

24. A series circuit consists of a resistance of 120 ohms and an inductance of $6/\pi$ henries. A 220-volt d-c generator is put in series with a 220-volt a-c generator (60 cycles frequency), and the combination is connected to the circuit through a switch. Find (*a*) the current at time t after the switch is closed; (*b*) the current after $1/20\pi$ sec; (*c*) the steady-state current; (*d*) the voltage across the inductance and the voltage across the resistance when $t = 1/20\pi$ sec.

25. Obtain the general solution (3) of the differential equation (2), Art. 24.

26. In the problem leading to differential equation (2), Art. 24, assume that the right end of the rope is at the level of the axis of the cylinder and that the other end hangs down a distance L below the axis; find L, in terms of μ and a, when the rope is on the point of slipping.

27. In the problem leading to differential equation (2), Art. 24, assume that the right end of the rope is at the top of the cylinder and that the other end is at the level of the axis. (*a*) Find the equation which determines the coefficient of friction if the rope is on the point of slipping. (*b*) Solve the equation of part (*a*) by the method of trial and error, obtaining the value of μ to three figures.

28. A rough circular cylinder with horizontal axis has a piece of flexible cable lying across the uppermost quarter of its circular cross section. Find the force which, applied tangentially at one end of the cable, would just cause it to slip. Radius of cylinder = 2 ft. Weight of cable = $\frac{1}{2}$ lb/ft. Coefficient of friction = $\frac{2}{3}$.

29. A flexible cable weighing 1 lb/ft hangs a distance of L ft below the horizontal axis of a drum of radius 6 in. and reaches three-quarters of the distance around the upper half of a circular cross section. Assuming that the cable is on the point of slipping and that the coefficient of friction is $\frac{1}{2}$, find L.

30. A piece of cable weighing 1.5 lb/ft rests along the circumference of a smooth cylinder of radius 4 ft. If the maximum tension in the cable is 3 lb, what is the length of the cable?

31. A ship, pulling with a force of 10,500 lb, is held by 2.5 turns of rope wound around a post. If the coefficient of friction is 0.4, with what force must a man pull at the other end of the rope in order to hold the ship? How many turns of rope would be necessary if the man pulls with a force of 40 lb?

32. (*a*) A rope 35 ft long weighing $\frac{1}{4}$ lb/ft is thrown over a rough ($\mu = 0.5$) cylinder 3 ft in diameter whose axis is horizontal and 20 ft above ground.

On one end of the rope a piece of iron is suspended. The other end of the rope is held by a man standing on the ground who uses 2 ft of the rope as a grip by winding it around his hands. If the man pulls straight downward with a force of 50 lb at a distance of 4 ft from the ground, how much does the piece of iron weigh if it is on the point of dropping?

(*b*) Work part (*a*) neglecting the weight of the rope.

(*c*) Work part (*a*) assuming that the cylinder is smooth.

33. On a smooth cylinder rests a 40-lb weight attached to a chain which passes over the cylinder and hangs down 2 ft below the horizontal diameter of the cylinder. A 20-lb weight is attached to the lower end of the chain. The diameter of the cylinder is 12 ft, and the weight of the chain is 5 lb/ft. Find the position in which the 40-lb weight rests in equilibrium.

34. A flexible cable weighing ρ lb/ft hangs a distance of 6 in. below the horizontal axis of a drum 5 ft in diameter and reaches around the upper half of a circular cross section for a distance equal to one-third the circumference of the drum. Assuming that the cable is on the point of slipping, obtain a formula for finding the coefficient of friction μ, and show that $\mu = 0.44$ satisfies it closely.

35. Assuming $\mu = 0.7324$ in Prob. 27, find the angle BOP, where B is the lower end of the rope, O is the center of the cylinder, and P is the point on the rope where the tension is a maximum.

36. A trough is 2 ft long and of width $1/(10 - x)$ ft at a distance of x ft from its flat base. Ether flows into the trough, originally empty, at the rate of $10/(10 + t)$ ft^3/hr, where t (hr) is the time after the flow starts. If the rate of evaporation is proportional to the area A (ft^2) of the liquid surface, the constant of proportionality being $k = \frac{1}{10}$ ft/hr, find the depth of the liquid in the trough after 2 hr. How much greater would the depth have been if none of the liquid had evaporated?

37. Tests, taken after an overhaul of a plastic molding machine producing 100 articles per second, showed approximately the following rate of production of defective articles: (1) 20 per min due to uncontrollable causes, (2) 1 per hr increase in defectives per 10^4 additional articles, due to steady wear of machine parts, (3) 2 per hr increase in defectives per 10^3 additional defective articles. Find the total number of defectives produced during 10 hr after an overhaul.

25. Substitutions. A differential equation that is not in one of the forms previously discussed may be capable of transformation into one of these forms by substituting one or two new variables. The new equation may then be solved and the result transformed back in terms of the original variables. We shall now consider a few equations of this kind.

(*a*) *Bernoulli's equation.* A differential equation of the form

$$\frac{dy}{dx} + Py = Qy^n, \tag{1}$$

where P and Q are functions of x, and n is any constant except 0 or 1, is called a Bernoulli equation. It is assumed that n is neither 0 nor 1, for then the equation would be linear or separable, respectively.

We shall show that equation (1) can be transformed into a linear equation by substituting a new dependent variable in place of y; but, in order to see the appropriate substitution, we first multiply equation (1) by y^{-n} so that a function of x alone appears on the right, thus:

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q.$$

We notice now that, if the equation is multiplied by $1 - n$, the first term becomes the derivative of y^{1-n}.

$$(1 - n)y^{-n}\frac{dy}{dx} + (1 - n)Py^{1-n} = (1 - n)Q.$$

Hence the substitution $y^{1-n} = z$ produces the equation

$$\frac{dz}{dx} + (1 - n)Pz = (1 - n)Q,$$

which is linear in z and dz/dx and can be solved in terms of z and x by formula (8) of Art. 21, after which the replacement of z by y^{1-n} will give the general solution of equation (1)

However, in practice we do not need to use the letter z, but proceed as follows to solve equation (1). First multiply through by y^{-n} and by the new exponent of y in the second term. Then apply formula (8), Art. 21, replacing P by the new coefficient in the second term, Q by the new right member, and y by the new power of y in the second term.

If, instead of equation (1), we had

$$\frac{dx}{dy} + Px = Qx^n, \tag{2}$$

where P and Q are functions of y, we would follow the same procedure with x and y interchanged.

EXAMPLE 1. Solve

$$\frac{dy}{dx} = x^3y^2 + xy.$$

Written in the standard form (1), the equation is

$$\frac{dy}{dx} - xy = x^3y^2.$$

Multiplying by y^{-2} and by the new exponent of y in the second term, which is -1, we have

$$-y^{-2}\frac{dy}{dx} + xy^{-1} = -x^3.$$

Since the first term is the derivative of y^{-1}, the equation is linear with y^{-1} as dependent variable. Using formula (8) of Art. 21, we have

$$R = e^{\int x\,dx} = e^{x^2/2},$$

$$e^{x^2/2}y^{-1} = \int e^{x^2/2}(-x^3\,dx) + C = -2\int \frac{x^2}{2}\,e^{x^2/2}(x\,dx) + C.$$

Integration (Peirce, 402), with x replaced by $x^2/2$, gives

$$e^{x^2/2}y^{-1} = -2e^{x^2/2}\left(\frac{x^2}{2} - 1\right) + C.$$

Multiplying by $e^{-x^2/2}y$ and rearranging, we have

$$(2 - x^2 + Ce^{-x^2/2})y = 1.$$

(*b*) *Equations reducible to homogeneous.* A differential equation of the form

$$(a_1x + a_2y + a_3)\,dx + (b_1x + b_2y + b_3)\,dy = 0, \tag{3}$$

where the a's and b's are constants, can be reduced to a homogeneous equation by making a substitution for both variables, x and y. The method would not be used in the special cases where the constants have values that render the equation solvable by one of the previous methods; for instance,

When $a_2 = b_1 = 0$, or $a_1 = b_2 = 0$, or $a_1/b_1 = a_2/b_2 = a_3/b_3$, the equation is separable.

When $a_2 = b_1 \neq 0$, the equation is solvable by means of the integrable combination $x\,dy + y\,dx = d(xy)$.

When $a_3 = b_3 = 0$, the equation is homogeneous.

When $a_1 = 0$, or $b_2 = 0$, the equation is linear.

We shall employ a transformation which moves the origin to the point (h, k), then choose the point (h, k) so that the resulting equation is homogeneous. Let $x = X + h$, $y = Y + k$ (making $dx = dX$, $dy = dY$), and substitute in (3):

$$(a_1X + a_2Y + a_1h + a_2k + a_3)\,dX + (b_1X + b_2Y + b_1h + b_2k + b_3)\,dY = 0.$$

Now choose h and k so that

$$\begin{aligned} a_1h + a_2k + a_3 &= 0, \\ b_1h + b_2k + b_3 &= 0. \end{aligned} \tag{4}$$

The result is

$$(a_1X + a_2Y)\,dX + (b_1X + b_2Y)\,dY = 0, \tag{5}$$

a homogeneous equation which we can solve in terms of X and Y, and then go back to the original variables by means of the substitution

$$X = x - h, \qquad Y = y - k.$$

In practice, however, it is not necessary actually to change to the new variables X, Y and afterwards change X to $x - h$ and Y to $y - k$. The same effect is produced if, after dropping a_3 and b_3 from equation (3), we solve the resulting equation and then change x to $x - h$ and y to $y - k$, h and k having been determined previously by solving equations (4). The method can be stated as follows:

1. *Solve equations (4) for* h *and* k.
2. *Drop* a_3, b_3 *from (3) and solve the resulting equation.*
3. *In the result of step 2 change* x *to* x − h, y *to* y − k.

EXAMPLE 2. Solve

$$(x + y + 1)\,dx + (6x + 10y + 14)\,dy = 0.$$

Solving

$$h + k + 1 = 0,$$
$$3h + 5k + 7 = 0,$$

we obtain $h = 1$, $k = -2$.

We now solve

$$(x + y)\,dx + (6x + 10y)\,dy = 0.$$

The substitution $x = vy$ gives

$$(v + 1)(v\,dy + y\,dv) + (6v + 10)\,dy = 0.$$

Separating the variables,

$$\frac{dy}{y} + \frac{(v + 1)\,dv}{(v + 5)(v + 2)} = 0.$$

Resolving the second fraction into partial fractions,

$$\frac{dy}{y} + \left[\frac{4}{3(v + 5)} - \frac{1}{3(v + 2)}\right] dv = 0.$$

Multiplying by 3 and integrating,

$$3 \ln y + 4 \ln (v + 5) - \ln (v + 2) = \ln C.$$

Taking antilogarithms,

$$y^3(v + 5)^4 = C(v + 2).$$

Replacing v by x/y,

$$(x + 5y)^4 = C(x + 2y).$$

Finally we change x to $x - 1$, y to $y + 2$, and obtain

$$(x + 5y + 9)^4 = C(x + 2y + 3),$$

which is the required solution.

There is one exceptional case in which equations (4) cannot be solved for h and k. This occurs when in equation (3)

$$b_1x + b_2y = m(a_1x + a_2y),$$

where m is a constant. It happens, however, that this exceptional case is more easily treated than the regular one, for, if in equation (3) we substitute $a_1x + a_2y = v$, $b_1x + b_2y = mv$, and the value of either dx or dy from the differential relation $a_1\,dx + a_2\,dy = dv$, we obtain a separable equation.

EXAMPLE 3. Solve

$$\frac{dy}{dx} = \frac{x + y + 4}{2x + 2y - 1}.$$

Substituting $x + y = v$, $dx = dv - dy$, we have

$$(2v - 1)\,dy = (v + 4)(dv - dy),$$

or

$$(3v + 3)\,dy = (v + 4)\,dv,$$

which is separable:

$$3\,dy = \frac{v + 4}{v + 1}\,dv = \left(1 + \frac{3}{v + 1}\right)dv.$$

Integrating,

$$3y + C = v + 3\ln(v + 1).$$

Replacing v by $x + y$,

$$x - 2y + 3\ln(x + y + 1) = C.$$

(c) *Substitution suggested by the form of equation.* Sometimes a substitution of variables, suggested by the form of a differential equation, will reduce the equation to a solvable form in the new variables.

EXAMPLE 4. Solve

$$\frac{dy}{dx} = \sin(x + y).$$

A little reflection discloses that this equation does not belong to one of the types previously considered, nor does it help to expand the right member and write

$$\frac{dy}{dx} = \sin x \cos y + \cos x \sin y;$$

but the form of the equation suggests letting

$$x + y = v, \qquad dy = dv - dx.$$

The result of this substitution is

$$\frac{dv}{dx} - 1 = \sin v,$$

an equation separable in the variables v, x. We have

$$\frac{dv}{1 + \sin v} = dx.$$

Integration (Peirce, 294) gives

$$C - \tan\left(\frac{\pi}{4} - \frac{v}{2}\right) = x,$$

or, replacing v by $x + y$.

$$x = C - \tan\left(\frac{\pi}{4} - \frac{x + y}{2}\right).$$

The above example suggests the more general case: A differential equation of the form

$$\frac{dy}{dx} = f(ax + by) \tag{6}$$

is reducible to a separable equation by the substitution $ax + by = v$.

The proof is immediate, for, using this substitution together with the value of dy obtained therefrom, equation (6) becomes

$$\frac{1}{b}\left(\frac{dv}{dx} - a\right) = f(v),$$

or

$$\frac{dv}{bf(v) + a} = dx,$$

in which the variables are separated.

PROBLEMS

Solve each of the following differential equations as a Bernoulli equation and also by another method.

1. $x\,dy = y(xy - 1)\,dx.$
2. $\dfrac{dy}{dx} = \dfrac{4x^3y}{x^4 + y^2}.$
3. $\dfrac{x\,dy}{y\,dx} + 1 = 3y^2.$
4. $\dfrac{dx}{dy} + \dfrac{y}{x} = \dfrac{x}{y}.$
5. $\dfrac{y\,dx}{x\,dy} = xy^3 + y^2 - 1.$

Solve the following differential equations.

6. $\dfrac{dy}{dx} + y = \dfrac{x}{y}.$
7. $3x\dfrac{dy}{dx} + y + x^2y^4 = 0.$
8. $x^3\,dy + 3x^2\,dx = 2\cos y\,dy.$
9. $\dfrac{y}{x} = (2 - x^2y^2)\dfrac{dy}{dx}.$
10. $\cos x\dfrac{dy}{dx} - y\sin x + y^2 = 0.$
11. $2\dfrac{dy}{dx} = \dfrac{y(2x + 3y^2)}{x^2}.$
12. $\cos x\dfrac{dy}{dx} = y\sin x + y^2\tan x.$
13. $\dfrac{dy}{dx} = \dfrac{4\sin^2 y}{x^5 + x\tan y}.$

14. $2x^2 \frac{dy}{dx} + y = 4y^3.$

15. $\frac{dy}{dx} = \frac{6x + y - 12}{6x - y - 12}.$

16. $\frac{dy}{dx} - \frac{1}{x + y} = 1.$

17. $\frac{dy}{dx} = \frac{x + y - 1}{x - y - 1}.$

18. $2\frac{dy}{dx} + \frac{x + y + 1}{2x + 3y - 1} = 0.$

19. $(x + 2y)\, dx + (y - 1)\, dy = 0.$

20. $\frac{dy}{dx} + \frac{3(y - 3)}{x + y} = 2.$

21. $3\frac{dy}{dx} = \frac{12y - 6x + 26}{y + 2x - 2}.$

22. $\frac{dy}{dx} + \frac{1 + xy^3}{1 + x^3y} = 0.$ (Hint: Let $x + y = u$, $xy = v$.)

23. $\frac{dy}{dx} = \cos(5x - 3y).$

24. $\frac{dy}{dx} = \frac{y^3}{e^{2x} + y^2}.$

25. $\frac{dy}{dx} = (x + 1)^2 + (4y + 1)^2 + 8xy + 1.$

26. $x\frac{dy}{dx} - y = 2x^2y(y^2 - x^2).$

27. Find the solution of

$$(4 - x^2 + y^2)\, dx + 4y\, dy = 0$$

for which $y = 1$ when $x = 2$.

28. Solve

$$x^2 \frac{dy}{dx} - 2xy = 3y^n,$$

where n is a constant. For what values of n is the solution valid? Find the solution for any exceptional values of n.

29. Solve

$$\frac{dy}{dx} = \frac{1 - x - 2y}{y + 3},$$

if $x = 4$ when $y = 1$.

30. If

$$\frac{dy}{dx} = \frac{e^y + 3x}{x^2}$$

and $y = 0$ when $x = 1$, find the value of y when $x = 1.5$.

31. Solve

$$x\frac{dy}{dx} + y \ln \frac{y}{x} = y(1 + x \sin x)$$

by using each of the following substitutions:

(*a*) Let $y/x = v$, obtaining an integrable combination in v and x.
(*b*) Let $\ln (y/x) = v$, obtaining an equation linear in v and dv/dx.

32. Show that the differential equation

$$\frac{x\,dy}{y\,dx} = f(xy)$$

is reducible to an equation separable in v and x by the substitution $xy = v$, hence solve

(*a*) $y(1 + x^2y^2)\,dx = x\,dy$; (*b*) $\dfrac{x\,dy}{y\,dx} = \dfrac{2 + x^2y^2}{2 - x^2y^2}$.

33. Show that the differential equation

$$\frac{dy}{dx} = f\left(\frac{a_1x + a_2y + a_3}{b_1x + b_2y + b_3}\right)$$

is reducible to a homogeneous equation by the substitution $x = X + h$, $y = Y + k$, if h and k are chosen so that

$$a_1h + a_2k + a_3 = 0,$$

$$b_1h + b_2k + b_3 = 0.$$

Solve

(*a*) $\dfrac{dy}{dx} = \left(\dfrac{x + 4y}{4x - 4}\right)^2$; (*b*) $\dfrac{dy}{dx} = \left(\dfrac{x + y + 1}{x + y}\right)^3$.

34. The general form of *Riccati's equation* is

$$\frac{dy}{dx} = P + Qy + Ry^2,$$

where P, Q, and R are functions of x. Solve the following special cases:

(*a*) $P = x^3$, $Q = 2/x$, $R = 1/x$. Let $y = x^2v$.
(*b*) $P = \sec^2 x$, $Q = \tan x$, $R = -1$. Let $y = \tan x + (1/v)$.

35. Solve Prob. 34(*b*) using each of the following substitutions by which the equation becomes separable:

$$y = v \sec x; \qquad y = v \cos x + \tan x.$$

26. Curves determined from geometric properties. It often happens that a family of curves may be characterized by a geometric property stated in terms of the coordinates of a point on one of the curves and the first derivative of one of the coordinates with respect to the other. Such a statement will be a differential equation of first order whose general solution represents the family of curves each of which possesses the given property. Then, if an additional condition is given, such as the requirement that the curve shall pass through a given

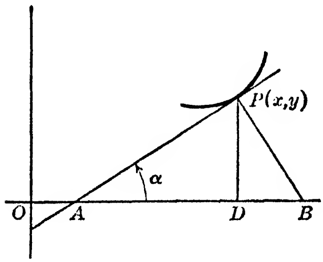

Fig. 6

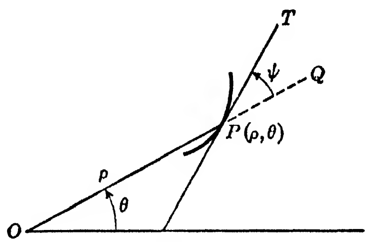

Fig. 7

point, this condition may be used to determine the arbitrary constant in the general solution. Using this value for the arbitrary constant, there is obtained the particular solution which represents a curve possessing the given geometric property and satisfying the required condition. The geometric property may be stated either in rectangular coordinates or in polar coordinates.

If x, y are the rectangular coordinates of a point P on a curve (x = abscissa, y = ordinate), dy/dx represents the slope of the curve at that point, that is, the tangent of the angle measured counterclockwise from the positive direction of the x-axis around to the tangent line at P. Denoting this angle by α (Fig. 6),

$$\frac{dy}{dx} = \tan \alpha. \tag{1}$$

Thus dy/dx will be positive when α is an acute angle, and negative when α is obtuse.

If ρ, θ are the polar coordinates of a point P on a curve

(ρ = radius vector, θ = vectorial angle), $d\rho/d\theta$ does not represent an important geometric concept such as that expressed by equation (1) in rectangular coordinates. The combination $\rho\, d\theta/d\rho$, however, is shown in calculus to represent the tangent of the angle QPT (Fig. 7), measured counterclockwise from the radius vector OP (extended) around to the tangent line at P. Denoting this angle by ψ,

$$\rho \frac{d\theta}{d\rho} = \tan \psi. \tag{2}$$

By means of relations (1) and (2) we can set up differential equations in rectangular and polar coordinates which define systems of curves having certain geometric properties; sometimes we shall bring in other formulas from calculus, such as those for area under a curve, distance along a curve, and area of a surface of revolution. We shall make a distinction between the terms "length" and "distance." The length of a line segment or arc will mean its actual length, a positive number of linear units, irrespective of the direction in which it is measured, whereas distance, if positive in one direction, is negative in the opposite direction. When dealing with lengths, the terms "tangent" and "normal" will mean the length of the segment of tangent line and normal line, respectively, between a point P on a curve and the x-axis.

EXAMPLE 1. At every point of a curve the projection of the normal on the x-axis has the length k. Find the family of curves possessing this property.

The length of DB in Fig. 6 is $y \tan \alpha = y(dy/dx)$. For P in any of the four quadrants and α either acute or obtuse, the length of DB is always either $y(dy/dx)$ or $-y(dy/dx)$. The differential equation for the required family of curves is therefore

$$\pm y \frac{dy}{dx} = k.$$

Integration gives

$$y^2 = \pm 2kx + C,$$

a family of parabolas each of which possesses the required property. The axis of x is the axis of all the parabolas; they open toward the right or left according as the $+$ or $-$ sign is used, and shift to right or left with varying C.

EXAMPLE 2. Find the curves for which the length of any arc equals the area under the arc and above the x-axis.

The statement that length equals area means that the number of inches of arc equals the number of square inches of area. If the integral representing the arc length from any fixed abscissa a to a variable abscissa x is equated to the integral representing the corresponding area under this arc and above the x-axis, we have

$$\int_a^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^x y\, dx. \qquad (y > 0.) \tag{3}$$

Differentiating with respect to x,

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = y, \tag{4}$$

or

$$\left(\frac{dy}{dx}\right)^2 = y^2 - 1,$$

whence

$$\pm \frac{dy}{\sqrt{y^2 - 1}} = dx. \tag{5}$$

Integrating,

$$\pm \cosh^{-1} y = x + C \qquad (y \neq 1), \tag{6}$$

or

$$y = \cosh (x + C). \tag{7}$$

Equation (7) represents a family of catenaries obtained by shifting the catenary, $y = \cosh x$, to the right and left (Fig. 8). Each catenary has the property that the length of any arc equals the area between the arc and the x-axis. Direct substitution of (7) into (3) verifies that the curves (7) satisfy the required condition. It should be noticed that equation (6) does not follow from (5) if $y = 1$, so that this value of y should be tested to see whether it satisfies equation (3). It is seen that $y = 1$ does satisfy (3) and hence also pos-

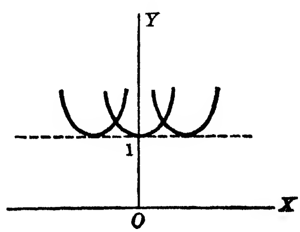

FIG. 8

sesses the required property; this line is the envelope of the curves (7). The complete solution of the problem is therefore the family of curves (7) together with their envelope, $y = 1$.

EXAMPLE 3. At every point P of a curve the radius vector and the tangent at P intersect at the same angle, $\cot^{-1} k$. Find the curves having this property.

Referring to Fig. 7 we see that $\psi = \cot^{-1} k$ or $180° - \cot^{-1} k$, so that $\tan \psi = \pm 1/k$; then from equation (2) the differential equation of the required family of curves is

$$\frac{\rho\, d\theta}{d\rho} = \pm \frac{1}{k},$$

or

$$\frac{d\rho}{\rho} = \pm k\, d\theta.$$

Integration gives

$$\ln \rho = \pm k\theta + \ln C,$$

or

$$\rho = Ce^{\pm k\theta},$$

two singly infinite families of equiangular spirals, so named because of the equal angles between radius vector and tangent at all points of the curves.

27. The range of the parameter. The general solution of a differential equation of first order in two variables, x, y, is an equation which involves x, y, and an arbitrary constant or parameter, say C, and is represented geometrically as a singly infinite or one-parameter family of curves. This solution satisfies the differential equation for any value of C, real or imaginary. If, for a given value of C, we can assign real values to one variable, within a certain interval, and obtain corresponding real values for the other variable, we have a real solution or real curve. In most applications, whether physical or geometrical, it is demanded that the solution be real. Such is the case when we are seeking the family of curves characterized by given geometrical properties.

In the three examples of Art. 26 the real curves satisfying the conditions are obtained merely by letting C take on all real values. But in some cases C may enter the general solution in such a way that a more restricted range of values for C will yield all the real solutions, or a more extensive range, including imaginary values of C, may be necessary in order to obtain all the real solutions.

EXAMPLE 1. Find the family of curves characterized by the property that at each point the tangent line is perpendicular to the line joining the point to the origin.

The differential equation is

$$\frac{dy}{dx} = -\frac{x}{y},$$

or

$$x\,dx + y\,dy = 0,$$

whose general solution is

$$x^2 + y^2 = C. \tag{1}$$

In order to satisfy the geometric condition and obtain real curves, the range of C must be restricted to positive real values. The solution of the problem is represented by the family of circles concentric about the origin:

$$x^2 + y^2 = C \qquad (C > 0). \tag{2}$$

If C^2 were written instead of C in the general solution (1), the solution of the problem would be

$$x^2 + y^2 = C^2 \qquad (C > 0),$$

with the same range for C. For negative real values of C the curves would be duplicated.

EXAMPLE 2. At any point of a curve the slope is equal to the reciprocal of the abscissa. Find the equation of the family of curves having this property.

The differential equation is

$$\frac{dy}{dx} = \frac{1}{x},$$

or

$$dy = \frac{dx}{x}.$$

Integrating, we have

$$y = \ln x + C. \tag{3}$$

For $x > 0$ and C taking on all real values, equation (3) represents the curves to the right of the y-axis in Fig. 9. If $x < 0$, $\ln x$ is imagi-

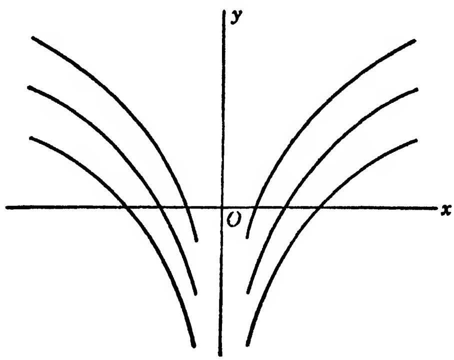

FIG. 9

nary, and with C real we get no curves to the left of the y-axis. But the curves on the left of the y-axis in Fig. 9, symmetrical to those on the right, possess the required property. At any point on one of these curves the slope equals the reciprocal of the abscissa. Equation (3) with C real does not represent the complete solution of the problem. We may change the form of equation (3), or we may extend the range of C in equation (3), in order to obtain the family of curves shown in Fig. 9.

The integral of dx/x is sometimes written $\int dx/x = \ln|x| + C$ in order to take care of the case $x < 0$. With this notation we could write, instead of (3),

$$y = \ln|x| + C, \tag{4}$$

and this equation, with C real, would represent the curves both to the right and left of the y-axis in Fig. 9.

Another form, instead of (3), could be obtained by changing C to $\ln C$, namely,

$$y = \ln Cx,$$

or

$$e^y = Cx. \tag{5}$$

Equation (5), with C ranging over all real values except 0, also represents the curves both to the right and left of the y-axis in Fig. 9.

However, the original equation (3) also can represent the complete family of curves in Fig. 9 if C is not restricted to being real. For $x > 0$ and C real in equation (3), we have the curves to the right of the y-axis in Fig. 9. Now reflect these curves in the y-axis by changing x to $-x$, and we get

$$y = \ln(-x) + C. \tag{6}$$

But, for $x > 0$,

$$\ln(-x) = \ln x + \ln(-1). \tag{7}$$

Hence, if we write

$$C = C' - \ln(-1), \tag{8}$$

we have

$$y = \ln x + \ln(-1) + C' - \ln(-1), \tag{9}$$

or *

$$y = \ln x + C'. \tag{10}$$

Equation (3) is changed into (10), i.e., is unchanged except for C, by changing x to $-x$, and therefore represents the system of curves, symmetric to the y-axis, shown in Fig. 9.

The solution of the problem may be written in any of the forms (3), (4), or (5), each of which, under the proper specification of the range of C, is represented by the curves of Fig. 9.

Example 3. Find the family of curves having the property that the tangent is of constant length k, and determine the particular curve of the family which passes through the point $(0, k)$.

The length of AD in Fig. 6 is $y \cot \alpha = y(dx/dy)$. For P in any of the four quadrants and α either acute or obtuse, the length of AD is either $y(dx/dy)$ or $-y(dx/dy)$. Then, the length of the tangent

* An objection might be raised to saying that $\ln(-1) - \ln(-1) = 0$, since $\ln(-1)$ is multiple-valued. But we can *assume* that $\ln(-1)$ in (8) is the same as in (7).

AP being $\sqrt{y^2 + \overline{AD}^2}$, the differential equation of the family of curves is

$$\sqrt{y^2 + y^2\left(\frac{dx}{dy}\right)^2} = k. \tag{11}$$

Hence

$$y^2\left[1 + \left(\frac{dx}{dy}\right)^2\right] = k^2,$$

$$\left(\frac{dx}{dy}\right)^2 = \frac{k^2}{y^2} - 1,$$

$$dx = \pm\frac{\sqrt{k^2 - y^2}}{y}\,dy. \tag{12}$$

Integration (Peirce, 130) gives

$$\pm(x + C) = \sqrt{k^2 - y^2} - k\ln\frac{k + \sqrt{k^2 - y^2}}{y}, \tag{13}$$

or

$$\pm(x + C) = \sqrt{k^2 - y^2} - k\cosh^{-1}\frac{k}{y}. \tag{14}$$

If $y = k$ when $x = 0$, then $C = 0$, and the particular curve passing through $(0, k)$ is

$$\pm x = \sqrt{k^2 - y^2} - k\cosh^{-1}\frac{k}{y}. \tag{15}$$

The curve (15) is a tractrix (Fig. 10), the + or − sign on the left being used respectively for the left and right branches of the curve;

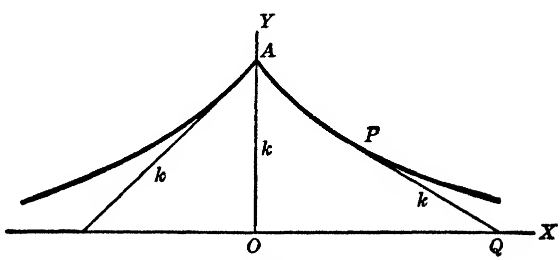

Fig. 10

these signs denote respectively positive and negative values for the slope dy/dx, as shown by equation (12).

Equation (14) includes all the curves obtained by shifting the tractrix of Fig. 10 to the right and left; furthermore, it includes the reflection of these curves in the x-axis, since equation (13), and hence (14), is unchanged except for C when y is changed to $-y$. (Cf. Ex. 2.) The complete family of tractrices represented by equation (14) is symmetrical to the x-axis, and each curve of the family has all its tangents of length k.

PROBLEMS

1. At any point P of a curve, the projection of the normal on the x-axis and the abscissa of P are equal in length. Find the curve if it passes through the point (2, 3).

2. Find the curve through the point (0, 2) such that the projection of the tangent on the x-axis is always of length 2.

3. Find the curve having both the following properties: (1) its ordinate equals the logarithm of its slope; (2) it crosses the line $x = 1$ at an angle of 60°.

4. Find the curve through the point (1, 2) whose normal at any point (except where $x = 0$) is bisected by the y-axis.

5. Find the family of curves having the following property: the perpendicular from the origin to the tangent line and the abscissa of the point of tangency are of equal length.

6. Find the curve through the point (2, 1) such that the x-intercept of the tangent is twice the ordinate of the point of tangency. Draw the curve.

7. Find the curves for which each normal and its x-intercept have the same length.

8. Find the curve through the point (4, 2) such that the line through each point of the curve and the origin bisects (a) the angle between the corresponding ordinate and tangent; (b) the angle between the corresponding ordinate and normal.

9. Work Prob. 8, using polar coordinates, if the curve passes through the point (2, 60°).

10. (a) Find the family of curves having the following property: at any point the slope of the normal is obtained by subtracting unity from the reciprocal of the abscissa.

(b) Find the particular curve of the above family passing through the origin, and the particular curve passing through the point (2, 1).

11. The normal at a point P of a curve meets the x-axis at Q. Find the equation of the curve if it passes through (0, 5) and if the locus of the midpoint of PQ is $y = x/3$. Check the result for the normals drawn at (0, 5) and at the vertex of the curve.

12. The normal at a point P of a curve meets the x-axis at Q. Find the equation of the curve if it passes through the point $(0, b)$ and if the locus of the midpoint of PQ is $y^2 = kx$.

13. Show that in Prob. 12 the curve is a parabola if $b = 2k$, and draw a figure illustrating the case $b = 2k = 4$.

14. Write the equation of the curve of Prob. 12 in each of the cases (a) $b = 3, k = 1$; (b) $b = 1, k = 2$. Show that curve (a) is an open curve crossing the y-axis at $y = \pm 3$ and the x-axis at $x = -1.33$. Show that curve (b) is a closed oval with $-1.61 < x < 3.77$ and $-3.64 < y < 3.64$.

15. The tractrix (Fig. 10) may be thought of as the path of a heavy particle P being dragged along a rough horizontal plane by a string PQ of length k. The end of the string, Q, is pulled along the x-axis, the initial positions of P and Q being at A and O respectively. If $k = 10$ in., how far have P and Q traveled from their initial positions when P is 5 in. from the x-axis?

16. If the length of the tangent to a tractrix is 2 ft, find the length of the line joining the two points on the curve where the slope is ± 1.

17. (a) Show that the family of catenaries obtained in Ex. 2, Art. 26, may also be derived from the following property: if at any point on a curve $(y > 0)$ an ordinate and a tangent are drawn, the perpendicular from the foot of the ordinate to the tangent is of unit length.

(b) Obtain the equation of the family of curves characterized by the above property if the perpendicular is of length k, instead of unit length. State the corresponding characteristic property involving arc length and area under the arc.

18. Find the curve for which the length of any arc equals half the area under the arc and above the x-axis, if the curve has slope 1 when $x = 0$.

19. An arc of a curve extends from a point $P\ (a, b)$ to a point Q upward and to the right of P. If for all positions of Q the area under the arc and above the x-axis is k times the vertical distance of Q above P, find the equation of the curve.

20. An arc of a curve is drawn upward and to the right, starting at the point $(a, 0)$. The area between the arc and the x-axis is equal to the nth power $(n > 1)$ of the ordinate which forms the right boundary of the area. Find the equation of the curve. Is the restriction $n > 1$ necessary?

21. On a curve which passes through the origin, take an arc from $(0, 0)$ to any point (x, y) of the curve. The area between this arc and the x-axis, when rotated about the x-axis, produces a volume V_x. The area between the same arc and the y-axis, when rotated about the y-axis, produces a volume V_y. Find the curves characterized by the property $V_x = V_y$.

22. A curve has the property that any arc of it, when rotated about the x-axis, produces an area equal to the difference of the ordinates at the endpoints of the arc. Show that the curve is one of the branches of a tractrix whose tangent is of length $1/2\pi$.

23. Find the curves for which any arc, rotated about the x-axis, produces a surface of revolution whose area is $1/n$ the volume enclosed by it.

24. Find the curve characterized by the following property: an arc of the curve in the first quadrant, from $(0, k)$ to any point (x, y), when rotated about the x-axis, generates a surface of revolution whose area is equal to the area of the cylinder formed by rotating the straight line from $(0, y)$ to (x, y) about the x-axis. Draw the curve.

25. At any point P of a curve the normal bisects the angle between the ordinate and the radius vector (or the radius vector extended through P). Find the curves characterized by this property, (*a*) using rectangular coordinates, (*b*) using polar coordinates.

26. Find the family of curves for which the vectorial angle is n times the angle between radius vector and tangent.

27. Find the family of curves for which the radius vector through any point trisects the angle between the corresponding tangent and ordinate.

28. A reflector is to be constructed so that light emanating from a point source will be reflected in a beam of parallel rays. Taking the origin as the point source and the x-axis as the direction of the reflected beam, the reflector will be a surface of revolution generated by rotating a certain curve about the x-axis. Find the equation of the curve, (*a*) using rectangular coordinates, (*b*) using polar coordinates. Note the physical law that the angle of incidence (the angle between the incident ray and the normal to the surface) equals the angle of reflection (the angle between the reflected ray and the normal to the surface).

29. Find the equation of an axial section of the convex surface of a lens, whose concave surface is spherical, if a ray of light parallel to the axis of the lens is focused at the center of the sphere, (*a*) using rectangular coordinates, (*b*) using polar coordinates. Take the origin at the focus, and the x-axis along the axis of the lens. Employ the physical law $\sin \alpha/\sin \beta = k$, where k is the constant index of refraction, α is the angle of incidence, and β is the angle of refraction. (*c*) What is the equation of the axial section if the index of refraction is 1.5 and the distance along the axis of the lens, from the convex surface to the point where the light is focused, is 3 ft?

28. Orthogonal trajectories. Suppose that a singly infinite family of curves is given. Consider a second family composed of all the curves which intersect all the curves of the given family at right angles. When two families of curves are so related each family is called the orthogonal trajectories of the other.

As an illustration, the equation $x^2 + y^2 = k$ represents a singly infinite family of concentric circles. Here k is a parameter, constant for any particular circle but varying from one circle to another. The orthogonal trajectories of this family of circles are represented by $y = cx$, the family of straight lines through the common center of the circles. The circles are also the orthogonal trajectories of the family of straight lines. The relation is geometrically obvious in this case, but in general we need an analytical method for finding orthogonal trajectories.

If we differentiate the equation of the given family of curves and eliminate the parameter, we obtain the differential equation of the given family—an equation which gives the slope dy/dx of any one of the curves in terms of the coordinates x, y of a point on the curve. Now the slope of the orthogonal trajectory at the point (x, y) must be the negative reciprocal of the slope of the given curve in order for the condition of perpendicularity to hold. The differential equation of the orthogonal trajectories is therefore obtained by writing dy/dx equal to the negative reciprocal of the value it has in the differential equation of the given family. Integration of the differential equation so obtained then yields the equation of the orthogonal trajectories. Let us apply this method to the family of concentric circles used in the above illustration.

Example 1. Find the orthogonal trajectories of the family of circles concentric about the origin.

The equation of the given family of circles is

$$x^2 + y^2 = k. \tag{1}$$

Differentiation with respect to x gives

$$2x + 2y\frac{dy}{dx} = 0,$$

or

$$\frac{dy}{dx} = -\frac{x}{y}. \tag{2}$$

In this case the parameter k was eliminated automatically by differentiation; equation (2) is the differential equation of the original family of circles (1). The differential equation of the orthogonal trajectories is therefore

$$\frac{dy}{dx} = \frac{y}{x},$$

or

$$\frac{dy}{y} = \frac{dx}{x}. \tag{3}$$

Integrating (3), we have

$$\ln y = \ln x + \ln c,$$

or

$$y = cx, \tag{4}$$

the equation of the orthogonal trajectories.

When the parameter does not disappear automatically by differentiation, it must be eliminated by combining the original equation with the one resulting from differentiation, in order to obtain the differential equation of the original family.

EXAMPLE 2. Find the orthogonal trajectories of the family of circles with centers on the x-axis and passing through the origin.

The equation of the given family is

$$x^2 + y^2 = kx. \tag{5}$$

Differentiating,

$$2x + 2y\frac{dy}{dx} = k. \tag{6}$$

The elimination of k between (5) and (6) gives

$$2x^2 + 2xy\frac{dy}{dx} = x^2 + y^2,$$

or

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}, \tag{7}$$

the differential equation of the original family (5).

The differential equation of the orthogonal trajectories is therefore

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}, \tag{8}$$

which can be written in the following form involving an integrable combination,

$$\frac{2xy\,dx - x^2\,dy}{y^2} + dy = 0.$$

Integration then yields

$$\frac{x^2}{y} + y = C,$$

or

$$x^2 + y^2 = Cy, \tag{9}$$

the equation of the orthogonal trajectories, a family of circles also passing through the origin, but with centers on the y-axis. (See Fig. 11.)

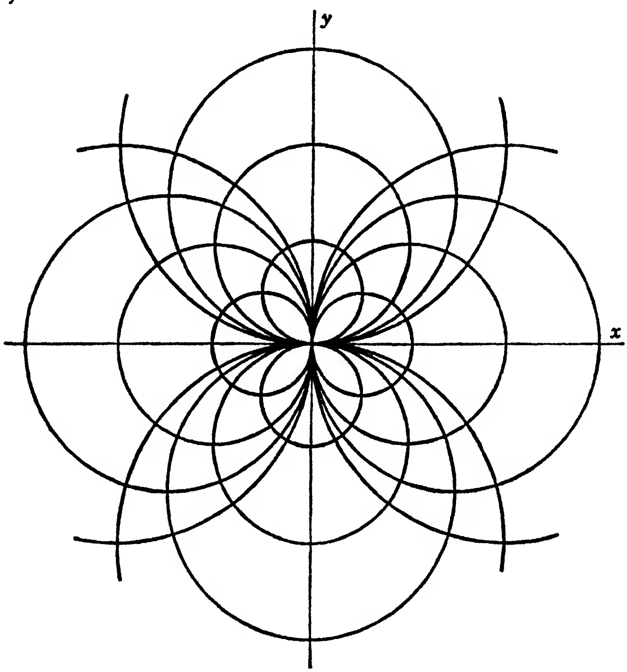

FIG. 11

If the equation of a singly infinite family of curves is given in terms of the polar coordinates ρ, θ, and a parameter k, we differentiate the equation and eliminate the parameter, thus obtaining the differential equation of the given family in polar coordinates. This equation enables us to express the value of $\rho\, d\theta/d\rho$, that is, by equation (2), Art. 26, the value of $\tan \psi$, for any one of the curves in terms of the coordinates ρ, θ of a point on the curve. Now the ψ of the orthogonal trajectory at the point (ρ, θ) differs by $90°$ from the ψ of the given curve, hence the $\tan \psi$ of the orthogonal trajectory equals the $-\cot \psi$ or $-1/\tan \psi$ of the given curve. The differential equation of the orthogonal trajectories is therefore obtained by writing $\rho\, d\theta/d\rho$ equal to the negative reciprocal of the value it has in the differential equation of the given family. Integration of the differential equation so obtained then yields the equation of the orthogonal trajectories.

EXAMPLE 3. Work Example 2, using polar coordinates.

In polar coordinates the equation of the given family of circles is

$$\rho = k \cos \theta. \tag{10}$$

Differentiating with respect to θ,

$$\frac{d\rho}{d\theta} = -k \sin \theta. \tag{11}$$

The parameter k may be eliminated by dividing (10) by (11):

$$\frac{\rho\, d\theta}{d\rho} = -\cot \theta, \tag{12}$$

the differential equation of the given family (10).

To obtain the differential equation of the orthogonal trajectories, we write $\rho\, d\theta/d\rho$ equal to the negative reciprocal of its values in (12):

$$\frac{\rho\, d\theta}{d\rho} = \tan \theta,$$

or

$$\frac{d\rho}{\rho} = \cot \theta\, d\theta. \tag{13}$$

Integration then yields

$$\ln \rho = \ln \sin \theta + \ln C,$$

or

$$\rho = C \sin \theta,$$

the equation of the orthogonal trajectories.

EXAMPLE 4. Find the orthogonal trajectories of the family of curves.

$$\rho = \sin \theta + k.$$

The differential equation of the given family is

$$\frac{d\rho}{d\theta} = \cos \theta,$$

or

$$\frac{\rho \, d\theta}{d\rho} = \frac{\rho}{\cos \theta}.$$

Hence the differential equation of the orthogonal trajectories is

$$\frac{\rho \, d\theta}{d\rho} = -\frac{\cos \theta}{\rho},$$

or

$$-\frac{d\rho}{\rho^2} = \sec \theta \, d\theta.$$

Integration then yields

$$\frac{1}{\rho} = \ln (\sec \theta + \tan \theta) + C,$$

the equation of the orthogonal trajectories.

PROBLEMS

Find the orthogonal trajectories of the following families of curves.

1. The family of curves of Ex. 1, Art. 26.
2. The family of curves of Ex. 3, Art. 26.

3. $(x - 1)^2 + y^2 + kx = 0.$

4. $x^2 = y^2 + ky^3.$

5. $e^x \cos y = k.$

6. $\sinh y = k \sec x.$

7. $y^2 = 4k(x + k).$

8. $x^2 + 3y^2 = ky.$

9. $y = x \tan \frac{1}{2}(y + k).$

10. $y = \sin x \sinh y + k.$

11. $y = \ln \tan (x + \sin x + k).$

12. $y = k \sin 2x.$

13. $\rho = k \sin 2\theta.$

14. $e^{2\rho} = k \cot \theta.$

15. $\rho^2 = k(\rho \sin \theta - 1).$

16. $\left(\rho + \frac{1}{\rho}\right) \sin \theta = k.$

Find the orthogonal trajectory, through the point specified, of each of the following families of curves.

17. $y^2 = kx \qquad (-2, 3).$

18. $y^2 = x^2 + ky \qquad (1, -2).$

19. $\rho = k \sin (\theta/4) \qquad (\frac{1}{4}, 180°).$

20. $\rho = k(1 + \sin \theta) \qquad (2, \pi/6).$

21. $y^2 = 2x + 1 + ke^{2x} \qquad (0, e).$

22. Find the curve through the point $(\pi/4, 0)$ belonging to the family

$$\sin x + \sinh y = k \cosh y,$$

and the curve through the same point belonging to the orthogonal family; verify that the two curves intersect at right angles.

23. Find the orthogonal trajectories of

$$ax^2 + y^2 = kx,$$

where a is a fixed constant and k is a parameter.

24. Find the orthogonal trajectories of

$$\cos y - a \cosh x = k \sinh x,$$

where a is a fixed constant and k is a parameter.

25. Find the orthogonal trajectories of

$$x^2 + ay^2 = b$$

(a) if a is a parameter and b is a fixed constant; (b) if b is a parameter and a is a fixed constant.

26. Find the orthogonal trajectories of

$$\rho = k \tan (\theta + a)$$

(a) if a is a fixed constant and k is a parameter; (b) if k is a fixed constant and a is a parameter.

27. Find the orthogonal trajectories of

$$\rho^m = k \sec n\theta,$$

where m and n are fixed constants and k is a parameter.

28. Show that the family of tractrices represented by equation (14), Art. 27, with $k = r$, are orthogonal trajectories of the family of circles in Prob. 2, Art. 10.

29. Given the family of curves

$$(x - y)(2x + y)^2 = kx^6,$$

find the family of orthogonal trajectories and show that one family can be transformed into the other by a rotation of 90° about the origin.

30. Given the family of curves

$$y = kx^2e^y,$$

find (*a*) a curve orthogonal to the family and passing through the point (2, 2); (*b*) a curve orthogonal to the family and passing through the point (2, −1). Show that curve (*a*) has a single branch with minimum point at (0, 1.57) and that curve (*b*) is an oval with x-intercepts ±2.29 and y-intercepts 0.88 and −2.58.

31. Consider a plane sheet of conducting material into which a current is fed at a point A (1, 0) and out of which it flows at a point B (−1, 0). It can be shown that the streamlines are represented by the family of circles passing through the points A and B. Find the orthogonal trajectories, i.e., the equipotentials. Draw both families of curves. Show that the equipotentials can be characterized by the property that at any point P on an equipotential the ratio PA/PB is constant.

32. In Prob. 31 the polar equation of the streamlines, taking the origin at B and the initial line along BA, is

$$\rho = 2 \cos \theta + k \sin \theta.$$

Find the polar equation of the equipotentials.

33. Given the fixed points A (1, 0) and B (−1, 0). A curve has the property that, for any point P (x, y) on it, the difference between the angles PAB and PBA is constant. Show that the curves characterized by this property are a family of hyperbolas, and find their orthogonal trajectories. Show that the orthogonal trajectories can be characterized by the property that the product $PA \cdot PB$ is constant and that they are therefore Cassinian ovals.

Chapter 4

LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

29. The linear equation. In Art. 21 of Chapter 3 we discussed the linear equation of first order. We shall now deal with linear equations of nth order, the word "linear" signifying that the equation is of first degree in the dependent variable and its derivatives. It is assumed in this chapter that the coefficients of the dependent variable and of its derivatives are constants; in the succeeding chapters linear equations with variable coefficients will appear. The type-form of linear equation of nth order with constant coefficients is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = R,$$

where the a's are constants, $a_0 \neq 0$, n is a positive integer, and the right member R is a function of x (or a constant).

30. The differential operator. Throughout this chapter we shall use symbolic methods in which the symbol D, called *the differential operator*, stands for $\frac{d}{dx}$ and symbolizes the operation of taking the derivative with respect to x of what follows it. Thus the equation $\frac{d}{dx} x^2 = 2x$ would be written, in operator notation, $\mathrm{D}x^2 = 2x$. Similarly D^2 stands for $\frac{d^2}{dx^2}$, and D^n for $\frac{d^n}{dx^n}$; thus $\mathrm{D}^2 x^2 = 2$, and $\mathrm{D}^n e^{ax} = a^n e^{ax}$.

In operator notation the equation of the preceding article is

$$a_0 \mathrm{D}^n y + a_1 \mathrm{D}^{n-1} y + \cdots + a_{n-1} \mathrm{D} y + a_n y = R,$$

or

$$(a_0 \mathrm{D}^n + a_1 \mathrm{D}^{n-1} + \cdots + a_{n-1} \mathrm{D} + a_n) y = R.$$

The expression in parentheses, a polynomial in D, may be denoted as a function of D by the symbol $f(\mathrm{D})$. The type-form linear differential equation of nth order will then be written

$$f(\mathrm{D})y = R,$$

where

$$f(\mathrm{D}) = a_0\mathrm{D}^n + a_1\mathrm{D}^{n-1} + \cdots + a_{n-1}\mathrm{D} + a_n$$

is a polynomial differential operator which, operating on y, produces R.

We shall now examine an important property of the operator $f(\mathrm{D})$. Take, for example, the case where $n = 2$, $a_0 = 1$, $a_1 = -1$, $a_2 = -2$, and let the operator be applied to the function $x^2 + x$; that is, consider the expression

$$(\mathrm{D}^2 - \mathrm{D} - 2)\,(x^2 + x). \tag{1}$$

This means: Take the second derivative of $x^2 + x$, minus the first derivative of $x^2 + x$, minus twice $x^2 + x$. Thus

$$(\mathrm{D}^2 - \mathrm{D} - 2)\,(x^2 + x) = 2 - (2x + 1) - 2(x^2 + x) = 1 - 4x - 2x^2.$$

Now, if $\mathrm{D}^2 - \mathrm{D} - 2$ is regarded as a quadratic expression in the letter D, without regard to the meaning of D, the expression can be factored by algebra into $(\mathrm{D} - 2)\,(\mathrm{D} + 1)$. Let us inquire whether, when D is regarded as a differential operator, the operator $(\mathrm{D} - 2)\,(\mathrm{D} + 1)$ has the same effect on $x^2 + x$ as the operator $\mathrm{D}^2 - \mathrm{D} - 2$. To evaluate the expression

$$(\mathrm{D} - 2)\,(\mathrm{D} + 1)\,(x^2 + x), \tag{2}$$

we must first operate on $x^2 + x$ with the operator $\mathrm{D} + 1$, then operate on the result with $\mathrm{D} - 2$:

$$(\mathrm{D} - 2)\,(\mathrm{D} + 1)\,(x^2 + x) = (\mathrm{D} - 2)\,(x^2 + 3x + 1) = 1 - 4x - 2x^2,$$

which is the same value as that obtained for the expression (1).

Finally, we may factor $D^2 - D - 2$ in the other order and obtain

$$(D + 1)(D - 2)(x^2 + x). \tag{3}$$

Evaluating this expression, we have

$$(D + 1)(D - 2)(x^2 + x) = (D + 1)(1 - 2x^2) = 1 - 4x - 2x^2.$$

We see that the expressions (1), (2), and (3) are equal. This illustrates an important property of an operator which is a polynomial in D with constant coefficients. D may be regarded as playing a dual role. The polynomial $f(D)$ may be factored with D playing its algebraic role, then, with the factors taken in any order, D may assume its role as operator. This may not be true, however, if the coefficients of $f(D)$ are not all constants, as the following example shows. We have

$$(D + x)Dy = (D^2 + xD)y,$$

but

$$D(D + x)y = D^2y + Dxy = D^2y + xDy + yDx = (D^2 + xD + 1)y;$$

so that

$$(D + x)Dy \neq D(D + x)y.$$

An operator, however, even with variable coefficients, is commutative with a constant; for example,

$$(D^2 + xD + 3)2y = 2(D^2 + xD + 3)y.$$

Consider next the effect of operating with $f(D)$ on the product of an exponential function e^{ax} and y, a function of x. We have

$$De^{ax}y = e^{ax}Dy + ae^{ax}y = e^{ax}(D + a)y,$$

$$D^2e^{ax}y = e^{ax}(D^2 + aD)y + ae^{ax}(D + a)y = e^{ax}(D + a)^2y,$$

and, in general,

$$D^ne^{ax}y = e^{ax}(D + a)^ny, \tag{4}$$

a formula which may be proved by mathematical induction [Problem 1, Art. 31(*b*)]. Hence, substituting in

$$f(D)e^{ax}y = (a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)e^{ax}y$$

the values of the derivatives of $e^{ax}y$ as given by (4), we have

$$f(D)e^{ax}y = e^{ax}[a_0(D + a)^n + a_1(D + a)^{n-1} + \cdots + a_{n-1}(D + a) + a_n]y,$$

or

$$f(D)e^{ax}y = e^{ax}f(D + a)y. \quad (5)$$

This formula states that when the operator $f(D)$ operates on a product $e^{ax}y$, the exponential e^{ax} may be shifted from right to left across the operator provided that the D in the operator is changed to $D + a$. We may refer to this process as the *exponential shift.* For example,

$$(D^2 - 4D + 4)e^{2x}x^3 = (D - 2)^2e^{2x}x^3 = e^{2x}D^2x^3 = 6xe^{2x}.$$

Reversing formula (5) and changing D to $D - a$, we have

$$e^{ax}f(D)y = f(D - a)e^{ax}y; \quad (6)$$

that is, we may shift the exponential e^{ax} from left to right across the operator $f(D)$ provided that the D in the operator is changed to $D - a$. This process may be called the *reverse exponential shift.* For example,

$$e^{-x}(D^2 - 2D + 5)y = e^{-x}[(D - 1)^2 + 4]y = (D^2 + 4)e^{-x}y.$$

We shall find considerable use for formulas (5) and (6) in the work which follows.

31. The linear equation with $R = 0$. Before developing methods for solving the general linear differential equation

$$f(D)y = (a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = R, \quad (1)$$

we consider first the case where $R = 0$, which is a step toward the solution of the more general case; that is, we shall now be concerned with solving the equation

$$f(D)y = (a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = 0. \quad (2)$$

Take first the case where all the a's, except a_0, are zero:

$$D^ny = 0. \quad (3)$$

Integrating n times in succession, we obtain

$$D^{n-1}y = c_1,$$

$$D^{n-2}y = c_1x + c_2,$$

$$D^{n-3}y = c_1\frac{x^2}{2} + c_2x + c_3,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$Dy = c_1\frac{x^{n-2}}{(n-2)!} + c_2\frac{x^{n-3}}{(n-3)!} + \cdots + c_{n-2}x + c_{n-1},$$

$$y = c_1\frac{x^{n-1}}{(n-1)!} + c_2\frac{x^{n-2}}{(n-2)!} + \cdots + c_{n-1}x + c_n.$$

We shall find it more convenient to write this value of y in a different form by letting $c_n = C_1, c_{n-1} = C_2, \cdots, c_2 = (n-2)!\,C_{n-1}$, $c_1 = (n-1)!\,C_n$; hence the general solution of the differential equation (3) is

$$y = C_1 + C_2x + C_3x^2 + \cdots + C_nx^{n-1}. \tag{4}$$

We are now almost ready to solve the differential equation (2). We need to notice one rather obvious fact, however. Suppose that y_1 and y_2 are two particular values of y that satisfy (2); that is, $f(D)y_1 = 0$ and $f(D)y_2 = 0$. It follows that the sum of y_1 and y_2 will satisfy (2), since $f(D)(y_1 + y_2) = f(D)y_1 + f(D)y_2 = 0$. Furthermore, if $y_1, y_2, \cdots, y_n$ are particular solutions, then $y_1 + y_2 + \cdots + y_n$ is a solution.

(*a*) *Solution of (2) when the roots of the polynomial equation* $f(D) = 0$ *are real and distinct.* Suppose that the polynomial equation

$$a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n = 0 \tag{5}$$

has all its n roots real and distinct: $r_1, r_2, \cdots, r_n$. When we say that r_1 is a root of equation (5) we are not saying that a value of the *operator* D is r_1. Equation (5) is not a differential equation; it is an algebraic equation of the nth degree in the letter D. Here D is playing its algebraic role as explained in Art. 30. When we have factored the polynomial in (5), we shall rewrite equation (2) in factored form; it will be a differential equation in which D is an operator. There need be no confusion regarding the dual role which D plays, and we shall not dwell further on this point. Since $D - r_1$, $D - r_2$, $\cdots$, $D - r_n$ are factors of the polynomial in (5), the differential equation (2) may be written

$$(D - r_1)(D - r_2)\cdots(D - r_{n-1})(D - r_n)y = 0. \quad (6)$$

Now the equation (6) is evidently satisfied if $(D - r_n)y = 0$, since

$$(D - r_1)(D - r_2)\cdots(D - r_{n-1})0 = 0.$$

But, if

$$(D - r_n)y = 0,$$

then

$$\frac{dy}{dx} = r_n y,$$

$$\frac{dy}{y} = r_n dx,$$

$$\ln y = r_n x + \ln C_n,$$

$$y = C_n e^{r_n x}. \quad (7)$$

Hence (7) is a particular solution of (6).

However, we saw in Art. 30 that the factors in (6) may be taken in any order, so that any one of them could be placed last, just before the y. Therefore (7) is a particular solution for each integral value of the subscripts from 1 to n and, by the principle enunciated earlier in this article,

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \cdots + C_n e^{r_n x} \quad (8)$$

is a solution of (6) and hence of (2). Furthermore, it is the general solution since the C's are all essential—that is, the solution (8) cannot be expressed with fewer than n arbitrary constants.

The problem of solving the differential equation (2) is merely a problem in algebra—finding the roots r_1, r_2, $\cdots$, r_n of the polynomial equation $f(D) = 0$, then, when these roots are distinct, substituting in (8) to obtain the general solution of the differential equation. Four methods of finding the roots of a polynomial equation are illustrated in the four following examples.

EXAMPLE 1. Solve

$$2\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0.$$

In operator notation, the differential equation is

$$(2D^2 + D - 1)y = 0.$$

By inspection, the roots of the quadratic equation $2D^2 + D - 1 = (2D - 1)(D + 1) = 0$ are $\frac{1}{2}$ and -1. Hence the general solution of the differential equation is

$$y = C_1e^{(1/2)x} + C_2e^{-x}.$$

EXAMPLE 2. Solve

$$(D^4 + 2D^3 - D^2 - 2D)y = 0.$$

Here we factor by grouping terms:

$$D^4 + 2D^3 - D^2 - 2D = D^3(D + 2) - D(D + 2) =$$
$$(D^3 - D)(D + 2) = D(D - 1)(D + 1)(D + 2) = 0.$$

Roots: $0, 1, -1, -2$.

General solution:

$$y = C_1 + C_2e^x + C_3e^{-x} + C_4e^{-2x}.$$

EXAMPLE 3. Solve

$$(D^3 - 8D + 3)y = 0.$$

Here we notice that the cubic $D^3 - 8D + 3$ will vanish when $D = -3$, so that $D + 3$ is a factor; we may find the other factor by synthetic division:

$$\begin{array}{rrrr|l} 1 & 0 & -8 & 3 & \underline{-3} \\ & -3 & 9 & -3 & \\ \hline 1 & -3 & 1 & 0 & \end{array}$$

The other factor is $D^2 - 3D + 1$ and the other roots are those of the quadratic equation $D^2 - 3D + 1 = 0$, namely, $(3 \pm \sqrt{5})/2$.

General solution:

$$y = C_1e^{-3x} + C_2e^{(1/2)(3+\sqrt{5})x} + C_3e^{(1/2)(3-\sqrt{5})x}.$$

EXAMPLE 4. Solve

$$(D^4 - 6D^2 + 1)y = 0.$$

In this case the quartic can be factored by arranging it as the difference of two squares:

$$D^4 - 6D^2 + 1 = (D^2 - 1)^2 - 4D^2 = (D^2 - 2D - 1)(D^2 + 2D - 1) = 0.$$

$$\text{Roots:} \quad \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}, \quad \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}.$$

General solution:

$$y = C_1e^{(1+\sqrt{2})x} + C_2e^{(1-\sqrt{2})x} + C_3e^{(-1+\sqrt{2})x} + C_4e^{(-1-\sqrt{2})x}.$$

(b) *Solution of (2) when the roots of the polynomial equation* f(D) = 0 *are real but not all distinct.* Suppose that the polynomial equation

$$a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n = 0$$

has all its n roots equal to r. The general solution of the differential equation (2) cannot then be found from the formula (8), for this formula would now yield

$$y = C_1e^{rx} + C_2e^{rx} + \cdots + C_ne^{rx} = (C_1 + C_2 + \cdots + C_n)e^{rx},$$

and, since $C_1 + C_2 + \cdots + C_n$ can be replaced by a single constant C, the result is

$$y = Ce^{rx},$$

which is only a particular solution.

Since $D - r$ is an n-fold factor of $f(D)$, equation (2) may be written

$$(D - r)^n y = 0.$$

To solve this equation, multiply by e^{-rx}:

$$e^{-rx}(D - r)^n y = 0.$$

Now, performing the reverse exponential shift [formula (6), Art. 30], we get

$$D^n e^{-rx} y = 0,$$

which is the same as equation (3) with y replaced by $e^{-rx}y$. Hence, from (4), the solution is

$$e^{-rx}y = C_1 + C_2x + C_3x^2 + \cdots + C_nx^{n-1},$$

or

$$y = (C_1 + C_2x + C_3x^2 + \cdots + C_nx^{n-1})e^{rx}. \qquad (9)$$

If the polynomial equation $f(D) = 0$ has some single roots and other multiple roots, the solution of the differential equation $f(D)y = 0$ is obtained by adding the terms corresponding to the single roots and the terms corresponding to the multiple roots.

EXAMPLE 5. Solve

$$D^2(D-1)^3(D+2)(D-3)y = 0.$$

The roots of $f(D) = 0$ are 0, 0, 1, 1, 1, -2, 3. The part of the solution corresponding to the double root 0 is, by (9), $(C_1 + C_2x)e^0$ or $C_1 + C_2x$; the part corresponding to the triple root 1 is $(C_3 + C_4x + C_5x^2)e^x$; and the parts corresponding to the single roots -2, 3 are respectively C_6e^{-2x} and C_7e^{3x}. Hence the general solution is

$$y = C_1 + C_2x + (C_3 + C_4x + C_5x^2)e^x + C_6e^{-2x} + C_7e^{3x}.$$

PROBLEMS

1. Prove formula (4) of Art. 30 by mathematical induction.
2. Find the results of the indicated operations:

(a) $(D^3 - 3D^2 + 3D - 1)x^3e^x$,
(b) $(D^3 + 4D^2 + 4D)x^2e^{-2x}$,
(c) $(D^4 - 8D^3 + 24D^2 - 32D)e^{2x}\sin 2x$,
(d) $(D - a)^n a^x$.

3. Show that the general solution of Ex. 4, Art. 31, may be written in the form

$$y = Ae^x \sinh(\sqrt{2}x + \alpha) + Be^{-x}\sinh(\sqrt{2}x + \beta).$$

4. Write the general solution of the differential equation

$$\frac{d^2y}{dx^2} = k^2y$$

in terms of exponential functions and in terms of hyperbolic functions.

Solve the following differential equations.

5. $(D^2 - 5D - 6)y = 0.$
6. $(D^2 + 6D - 5)y = 0.$
7. $\frac{d^4y}{dx^4} - 8\frac{d^2y}{dx^2} + 16y = 0.$
8. $(2D^3 + 5aD^2 + 2a^2D)y = 0.$
9. $(2D^3 - D^2 - 2D + 1)y = 0.$
10. $(D^4 - 7D^2 + 1)y = 0.$
11. $\frac{d^5y}{dx^5} = 2\frac{d^3y}{dx^3}.$
12. $(D^4 - 6D^2 + 8D - 3)y = 0.$
13. $[abD^2 - (a^2 + b^2)D + ab]y = 0.$
14. $(10D^4 - 9D^3 - 23D^2 + 4)y = 0.$
15. $(36D^4 - 37D^2 + 4D + 5)y = 0.$
16. Given the differential equation

$$\frac{d^4y}{dx^4} + \frac{d^3y}{dx^3} = 0$$

subject to the conditions

$$y = 0,\ dy/dx = 0, \text{ when } x = 0,$$

$$y = 1,\ dy/dx = 1, \text{ when } x = 1.$$

Obtain a formula for $y]_{x=-1}$ in terms of e, and use this formula, with $e = 2.718$, to compute the value of $y]_{x=-1}$.

17. Find a curve having slope $\frac{1}{4}$ at the origin and satisfying the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = y.$$

(*c*) *Solution of (2) when the polynomial equation* $f(D) = 0$ *has imaginary roots.* Suppose that the polynomial equation

$$a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n = 0,$$

where the a's are real constants, has an imaginary root. We know that such roots must occur in conjugate pairs: if $2 + 3i$ is a root, $2 - 3i$ must also be a root, and, in general, if $\alpha + \beta i$ is a root, $\alpha - \beta i$ must also be a root, α and β being real constants.

Suppose first that $n = 2$ and the equation has a pair of imaginary roots: $r_1 = \alpha + \beta i$, $r_2 = \alpha - \beta i$. Then formula (8) would give for the solution of (2),

$$y = C_1e^{(\alpha+\beta i)x} + C_2e^{(\alpha-\beta i)x}.$$

This is a correct solution but it is not customary to leave it in this form with imaginary exponents. We transform it as follows. Factoring the right member, we have

$$y = e^{\alpha x}(C_1e^{i\beta x} + C_2e^{-i\beta x}).$$

For the exponentials $e^{i\beta x}$ and $e^{-i\beta x}$ we may substitute the values obtained from *Euler's relation*,

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

by first putting $\theta = \beta x$, then $\theta = -\beta x$, namely,

$$e^{i\beta x} = \cos\beta x + i\sin\beta x,$$

$$e^{-i\beta x} = \cos\beta x - i\sin\beta x.$$

This gives

$$y = e^{\alpha x}(C_1\cos\beta x + iC_1\sin\beta x + C_2\cos\beta x - iC_2\sin\beta x),$$

or, changing $C_1 + C_2$ to A, $iC_1 - iC_2$ to B, which is merely a change of notation on the two arbitrary constants, we have

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x). \tag{10}$$

In Chapter 1, Art. 2, we saw that the expression in parentheses may be written in either of two equivalent forms, $K\sin(\beta x + \delta)$ or $K\cos(\beta x + \epsilon)$; hence alternative forms of (10) are

$$y = Ke^{\alpha x}\sin(\beta x + \delta), \tag{10'}$$

$$y = Ke^{\alpha x}\cos(\beta x + \epsilon). \tag{10''}$$

Formula (10) gives the general solution of (2) when $f(D) = 0$ is a quadratic equation with a pair of imaginary roots. If $f(D) = 0$ is a quartic equation having, besides $\alpha \pm \beta i$, another distinct pair of imaginary roots, $\gamma \pm \delta i$, the general solution of (2) is obtained by adding to the right member of (10) the expression $e^{\gamma x}(E \cos \delta x + F \sin \delta x)$ corresponding to the pair $\gamma \pm \delta i$. If $f(D) = 0$ has both imaginary and real roots, the solution of (2) is obtained by adding the terms corresponding to the imaginary roots and the terms corresponding to the real roots.

Finally, if the polynomial equation $f(D) = 0$ possesses a double pair of imaginary roots, $\alpha \pm \beta i$, $\alpha \pm \beta i$, the corresponding part of the solution of (2) is obtained by writing the expression for a single pair, namely, $e^{\alpha x}(A \cos \beta x + B \sin \beta x)$, with A replaced by $C_1 + C_2x$ and B replaced by $C_3 + C_4x$, a process analogous to that used for double real roots. For a triple pair of imaginary roots A would be replaced by $C_1 + C_2x + C_3x^2$, and B by $C_4 + C_5x + C_6x^2$, etc.

EXAMPLE 6. Solve

$$(D^3 + D + 10)y = 0.$$

Since the cubic $D^3 + D + 10$ vanishes when $D = -2$, we may find the other roots of the cubic equation $D^3 + D + 10 = 0$ by synthetic division:

1	0	1	10	$\lfloor -2$
	-2	4	-10	
1	-2	5	0	

The other roots are those of the quadratic equation $D^2 - 2D + 5 = 0$, namely, $(2 \pm \sqrt{4 - 20})/2 = 1 \pm 2i$. Hence $\alpha = 1$, $\beta = 2$, in (10), and the general solution of the differential equation is

$$y = C_1e^{-2x} + e^x(C_2 \cos 2x + C_3 \sin 2x).$$

EXAMPLE 7. Solve

$$(D^4 + 2D^2 + 1)y = 0.$$

The roots of $D^4 + 2D^2 + 1 = (D^2 + 1)^2 = 0$ are $\pm i$, $\pm i$. Here the pair $\pm i$, with $\alpha = 0$, $\beta = 1$, is repeated; hence we use (10) with A replaced by $C_1 + C_2x$ and B replaced by $C_3 + C_4x$.

General solution:

$$y = (C_1 + C_2x) \cos x + (C_3 + C_4x) \sin x.$$

PROBLEMS

Solve the following differential equations.

1. $\dfrac{d^2y}{dx^2} + k^2y = 0.$

2. $(D^2 - 2D + 4)y = 0.$

3. $(36D^2 + 36D + 13)y = 0.$

4. $\dfrac{d^2s}{dt^2} - 4\dfrac{ds}{dt} + 13s = 0.$

5. $\dfrac{d^2x}{dt^2} + 2a\dfrac{dx}{dt} + k^2x = 0$, where $0 < a < k$.

6. $(4D^4 + 5D^2 - 9D)y = 0.$

7. $\dfrac{d^6y}{dx^6} - y = 0.$

8. $(4D^3 - 107D - 222)y = 0.$

9. $(D^6 - 3D^2 + 2)y = 0.$

10. $\dfrac{d^4s}{dt^4} - 64s = 0.$

11. $\dfrac{d^4s}{dt^4} + 64s = 0.$

12. $(D^6 - 12D^2 - 16)y = 0.$

13. $(D^4 - D^2 + 2D + 2)y = 0.$

14. $(D^6 + 12D^4 + 48D^2 + 64)y = 0.$

15. $(D^5 + 2D^3 + 10D^2 + D + 10)y = 0.$

16. $(D^5 - 4D^4 + 2D^3 - 8D^2 + D - 4)y = 0.$

17. According to the preceding theory, if $f(D)y = 0$ is a fourth order differential equation with a repeated pair of imaginary roots, $\alpha \pm \beta i$, $\alpha \pm \beta i$, the general solution is

$$y = e^{\alpha x}[(C_1 + C_2x)\sin \beta x + (C_3 + C_4x)\cos \beta x].$$

Are the following also correct forms for the general solution:

$$(a)\quad y = (E + Fx)e^{\alpha x}(A \sin \beta x + B \cos \beta x),$$

$$(b)\quad y = e^{\alpha x}[A \sin (\beta x + E) + Bx \sin (\beta x + F)],$$

where A, B, E, and F are arbitrary constants?

32. Rectilinear motion. Before developing the theory for solving the linear equation $f(\mathrm{D})y = R$ with $R \neq 0$, we shall consider some applications of the equation $f(\mathrm{D})y = 0$ to problems in rectilinear motion. These problems will involve differential equations of second order, that is, the case where $f(\mathrm{D})$ is a quadratic in D; certain cases which could be treated by means of first order equations have already been studied in Chapter 3.

(*a*) *Attractive force proportional to displacement; resistance negligible.* Consider a particle moving without resistance along a straight line, which we may take as the x-axis, under the action of an attractive force located at the origin O. Suppose that the force is proportional to the displacement x of the particle from O at time t, x being positive to the right and negative to the left of O. Since force is proportional to acceleration (*cf.* Art. 14), the acceleration d^2x/dt^2 will be proportional to the displacement x. If the particle is to the right of O, the displacement is positive and the force (acting toward the left) and acceleration are negative; if the particle is to the left of O, the displacement is negative and the force and acceleration are positive; hence the acceleration and displacement are oppositely signed and the differential equation of the motion is

$$\frac{d^2x}{dt^2} = -k^2x, \tag{1}$$

where k^2 is used for convenience, instead of k, as the constant of proportionality.

Equation (1), written in operator notation, with $\mathrm{D} = d/dt$, is

$$(\mathrm{D}^2 + k^2)x = 0. \tag{1'}$$

Its general solution may be written down by the method of Art. 31(*c*) and is the same as that of Problem 1, Art. 31(*c*), with x and t taking the places of y and x, namely,

$$x = C_1 \sin kt + C_2 \cos kt. \tag{2}$$

A form equivalent to (2) is (Art. 2)

$$x = A \sin (kt + \alpha), \tag{3}$$

where

$$A = \sqrt{C_1^2 + C_2^2}, \qquad \alpha = \tan^{-1} \frac{C_2}{C_1}.$$

Differentiation of (2) and (3) gives two forms for the velocity $v = dx/dt$:

$$v = kC_1 \cos kt - kC_2 \sin kt, \tag{4}$$

$$v = kA \cos (kt + \alpha). \tag{5}$$

The motion defined by (1), (2), or (3) is called *simple harmonic motion.* Equation (3) shows that the particle vibrates between the extreme positions $x = \pm A$. A is called the *amplitude* of the motion.

The motion is periodic. If, at any time t, the displacement and (signed) velocity of the particle are noted, then after a certain time T, called the *period,* the particle will again have the same displacement and velocity. To find T we observe from equations (3) and (5) that x and v will both return to their original values if the angle $kt + \alpha$ increases by 2π, that is, if t increases by $2\pi/k$; hence

$$T = \frac{2\pi}{k}. \tag{6}$$

Any integral multiple of $2\pi/k$ would also be a period, but by the period T we mean the smallest period. In particular, T is the time of a complete vibration from one extreme position to the other and back again.

(*b*) *Repulsive force proportional to displacement; resistance negligible.* In the case of a repulsive force directed away from O, the force and acceleration have the same sign as the displacement; the differential equation of the motion is then

$$\frac{d^2x}{dt^2} = k^2x. \tag{7}$$

The general solution of (7) is [Problem 4, Art. 31(*b*)]

$$x = c_1 e^{kt} + c_2 e^{-kt}, \tag{8}$$

or

$$x = C_1 \sinh kt + C_2 \cosh kt. \tag{9}$$

The analogy between the solutions of equations (1) and (7) may be exhibited by considering two particles, each starting from rest at a distance a from O, the one under the action of an attractive force and the other under the action of a repulsive force emanating from O. We assume that the initial accelerations are numerically equal; that is, k will be the same in both equations. The initial conditions, $x = a$ and $v = 0$ when $t = 0$, serve to determine the arbitrary constants in the solutions in both cases.

Attraction	Repulsion
$\dfrac{d^2x}{dt^2} = -k^2x$	$\dfrac{d^2x}{dt^2} = k^2x$
$x = C_1 \sin kt + C_2 \cos kt$	$x = C_1' \sinh kt + C_2' \cosh kt$
$a = C_2 \quad (x = a,\ t = 0)$	$a = C_2' \quad (x = a,\ t = 0)$
$v = kC_1 \cos kt - ak \sin kt$	$v = kC_1' \cosh kt + ak \sinh kt$
$0 = C_1 \quad (v = 0,\ t = 0)$	$0 = C_1' \quad (v = 0,\ t = 0)$
$x = a \cos kt$	$x = a \cosh kt$

The displacement is a circular function of the time in the case of the attractive force, and the corresponding hyperbolic function of the time in the case of the repulsive force. This x, t relation is called the *equation of motion.*

Example 1. In the case of the two particles just discussed, let t_1 and t_2 be respectively the times required to travel the first $a/2$ units of distance; find the ratio t_1/t_2.

When each particle has traveled $a/2$ units, starting at $x = a$, the first will be at $x = a/2$ and the second at $x = 3a/2$; hence

$$\frac{kt_1}{kt_2} = \frac{\cos^{-1} 0.5}{\cosh^{-1} 1.5}, \qquad \frac{t_1}{t_2} = \frac{1.0472}{0.9625} = 1.09.$$

EXAMPLE 2. A particle moves with simple harmonic motion in a straight line. When $t = 0$, the acceleration is 9 ft/sec^2, the velocity is 4.5 ft/sec, and the displacement $x = -4$ ft. Find (*a*) the displacement when the time equals half a period; (*b*) the first time when the displacement is zero and when it is 1 ft; (*c*) the maximum velocity.

The differential equation is

$$\frac{d^2x}{dt^2} = -k^2x.$$

Since $d^2x/dt^2 = 9$ when $x = -4$,

$$9 = 4k^2, \qquad k = \tfrac{3}{2}.$$

Then x and v [equations (2) and (4)], together with the conditions for determining the arbitrary constants, are

$$x = C_1 \sin \tfrac{3}{2}t + C_2 \cos \tfrac{3}{2}t \qquad (t = 0, \quad x = -4),$$

$$v = \tfrac{3}{2}C_1 \cos \tfrac{3}{2}t - \tfrac{3}{2}C_2 \sin \tfrac{3}{2}t \quad (t = 0, \quad v = 4.5).$$

Substitution of the conditions in these equations gives $C_2 = -4$, $C_1 = 3$; hence the relation between displacement and time is

$$x = 3 \sin \tfrac{3}{2}t - 4 \cos \tfrac{3}{2}t. \tag{10}$$

The amplitude of the motion is $A = \sqrt{3^2 + (-4)^2} = 5$ ft, and the period is $T = 2\pi/(\frac{3}{2}) = 4\pi/3$ sec.

(*a*) To find the displacement when the time equals half a period, substitute $t = 2\pi/3$ in equation (10):

$$x\Big]_{t=2\pi/3} = 3 \sin \pi - 4 \cos \pi = 4 \text{ ft}.$$

(*b*) To find the time when $x = 0$ or 1 it is more convenient to use the alternative form for x [equation (3)],

$$x = 5 \sin (\tfrac{3}{2}t + \alpha), \tag{11}$$

where $\alpha = \tan^{-1} - \frac{4}{3}$. But care must be exercised in choosing the appropriate value for α, which could be an angle in either the second or fourth quadrant if there were no restrictions. Here, however, x is given negative when $t = 0$, hence $\sin \alpha$ is negative and α must be in the fourth quadrant. We write $\alpha = -\tan^{-1} \frac{4}{3} = -0.9273$

radian; then equation (11) gives, for the first positive value of t that makes $x = 0$,

$$\tfrac{3}{2}t - 0.9273 = 0, \qquad t = 0.618 \text{ sec.}$$

When $x = 1$, we solve for the smallest positive t that satisfies

$$1 = 5 \sin\left(\tfrac{3}{2}t - 0.9273\right).$$

$$\tfrac{3}{2}t - 0.9273 = \sin^{-1} 0.2 = 0.2013,$$

$$\tfrac{3}{2}t = 1.1286,$$

$$t = 0.752 \text{ sec.}$$

(c) Differentiating equation (11), we have

$$v = \tfrac{15}{2} \cos\left(\tfrac{3}{2}t + \alpha\right).$$

Hence the maximum velocity is 7.5 ft/sec; it occurs when $x = 0$.

The maximum value of v could also be found from

$$v = \tfrac{9}{2} \cos \tfrac{3}{2}t + 6 \sin \tfrac{3}{2}t$$

by taking the square root of the sum of the squares of the coefficients of the cos and sin functions:

$$v_{\max} = \sqrt{\tfrac{81}{4} + 36} = \sqrt{\tfrac{225}{4}} = 7.5 \text{ ft/sec.}$$

Let us now obtain the differential equation that represents the vibratory motion of a weight hanging on a spring, under the assumption that there is no resistance to the motion.

Given a spring fixed at its upper end and hanging vertically [Fig. 12(a)]. Suppose that a weight of w lb is attached to the lower end of the spring, stretching the spring a ft. The weight now hangs at rest in its equilibrium position [Fig. 12(b)]. Assume that *Hooke's law* holds; i.e., the tension in the spring is proportional to the elongation. The constant of proportionality is called the *spring constant.* In the equilibrium position the tension in the spring is w lb, just balancing the weight, and the elongation is a ft; hence $w = ca$, and the spring constant is $c = w/a$ lb/ft.

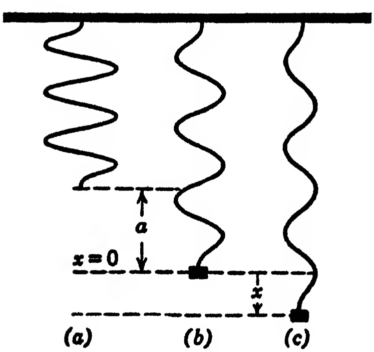

FIG. 12

We now set the weight in motion. This can be done by giving it an initial vertical velocity. Or, we may first displace the weight vertically from its equilibrium position and then either release it from rest or give it an initial vertical velocity. We take the equilibrium position as the origin and let x (ft), measured positive downward and negative upward, represent the displacement of the weight from its equilibrium position at time t (sec). The amount the spring is stretched at time t is therefore $a + x$ ft, no matter whether the weight is below or above the equilibrium position, and the tension in the spring is $c(a + x)$ lb [Fig. 12(c)]. The spring then pulls the weight upward with a force of $-c(a + x)$ lb, and gravity pulls it downward with a force of w lb. The resultant force acting on the weight at any time t is therefore $w - c(a + x)$ lb, which is equal to $-cx$ lb since $w = ca$. The force acting on the moving weight is also $(w/g)(d^2x/dt^2)$ lb. Equating these two expressions for pounds force, we obtain the differential equation

$$\frac{w}{g}\frac{d^2x}{dt^2} = -cx, \tag{12}$$

which shows that the resultant force acting on the weight is proportional and opposite in sign to the displacement. The weight will therefore execute a simple harmonic motion about its equilibrium position, $x = 0$.

EXAMPLE 3. A spring, fixed at its upper end, supports a weight at its lower end which stretches the spring 6 in. (*a*) If the weight is drawn down 3 in. below its equilibrium position and released, find the period of vibration and the equation of motion of the weight. (*b*) If the weight is drawn down 3 in. below its equilibrium position and given a downward velocity of 1 ft/sec, find the distance, below the equilibrium position, of the lowest point reached by the weight, the maximum velocity of the weight, and the time required by the weight to reach its equilibrium position.

The weight, say w lb, produces an elongation of $\frac{1}{2}$ ft; hence $w = c \cdot \frac{1}{2}$, and the spring constant is $c = 2w$ lb/ft. The differential equation is therefore [equation (12)]

$$\frac{w}{g}\frac{d^2x}{dt^2} = -2wx, \tag{13}$$

or

$$(D^2 + 2g)x = 0. \tag{14}$$

(*a*) The period of the motion is

$$T = \frac{2\pi}{\sqrt{2g}} = 0.783 \text{ sec.}$$

It is independent of the initial conditions and is obtained directly from the differential equation.

From (14) we have

$$\begin{aligned} x &= C_1 \sin \sqrt{2g}\, t + C_2 \cos \sqrt{2g}\, t, \\ v &= \sqrt{2g}\,(C_1 \cos \sqrt{2g}\, t - C_2 \sin \sqrt{2g}\, t). \end{aligned} \tag{15}$$

The initial conditions $x = \frac{1}{4}$ (ft), $v = 0$, $t = 0$, substituted in equations (15), give $C_2 = \frac{1}{4}$, $C_1 = 0$; hence the equation of motion is

$$x = \tfrac{1}{4} \cos \sqrt{2g}\, t \qquad \text{or} \qquad x = \tfrac{1}{4} \cos 8.02t.$$

(*b*) Substituting the initial conditions $x = \frac{1}{4}$, $v = 1$, $t = 0$ in equations (15), we obtain $C_2 = \frac{1}{4}$, $C_1 = 1/\sqrt{2g}$. The distance, below the equilibrium position, of the lowest point reached by the weight is the amplitude of the motion; that is

$$\sqrt{C_1^2 + C_2^2} = \sqrt{\frac{1}{16} + \frac{1}{2g}} \text{ ft} = 12\sqrt{\frac{1}{16} + \frac{1}{64.34}} \text{ in.} = 3.35 \text{ in.}$$

The maximum velocity of the weight is the maximum value of v in the second of equations (15), namely,

$$\sqrt{2g(C_1^2 + C_2^2)} = \sqrt{1 + \frac{g}{8}} = 2.24 \text{ ft/sec.}$$

It is attained when the weight passes through its equilibrium position, $x = 0$.

The time required by the weight to reach its equilibrium position is obtained by setting $x = 0$ in the first of equations (15) and solving for the smallest positive value of t:

$$0 = \frac{1}{\sqrt{2g}} \sin \sqrt{2g}\, t + \frac{1}{4} \cos \sqrt{2g}\, t,$$

$$\tan \sqrt{2g}\, t = -\frac{\sqrt{2g}}{4} = -2.0053,$$

$$\sqrt{2g}\, t = \pi - \tan^{-1} 2.0053 = 2.0334,$$

$$t = \frac{2.0334}{\sqrt{2g}} = 0.253 \text{ sec.}$$

The next example illustrates the vibratory motion of a body floating in water. The buoying force of the water takes the place of the force exerted by the spring in the preceding example.

EXAMPLE 4. A cubical block of wood 6 in. on an edge and weighing 4 lb floats in water (62.4 lb/ft^3). If it is depressed slightly and released, find the period of vibration, assuming that resistance is negligible and that the top remains horizontal. If the initial depression below the floating position is 2 in., find the equation of motion and the distance of the top of the block above water at the end of $\frac{5}{8}$ of a period.

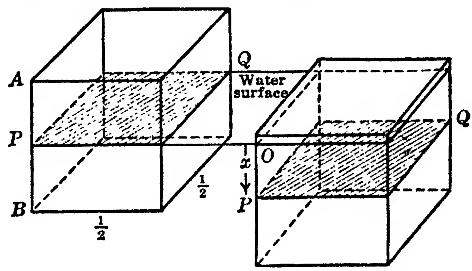

FIG. 13

The first position of the cube in Fig. 13 is the equilibrium position in which the cube floats without vertical motion. The fixed plane PQ which coincides with the plane of the water surface may be called the equilibrium plane. If a $(= PB)$ ft represents the depth of the submerged portion of the cube, the weight of the water it displaces is $(\frac{1}{2})^2 \times a \times 62.4$ lb $= 15.6a$ lb. According to the *Principle of Archimedes*, the buoying force of the water when the cube is in equilibrium is $15.6a$ lb, and this is just balanced by the weight of the cube, 4 lb, so that $15.6a = 4$ and $a = 1/3.9$ ft $= 3.08$ in. The cube floats with 3.08 in. submerged and 2.92 in. above water.

Now suppose that the cube is set in motion by drawing it down and then releasing it. We first determine the differential equation of the motion. Take the origin in the water surface, and let x (ft), measured positive downward and negative upward, represent the displacement of the equilibrium plane PQ from the water surface at any time t (sec). In the second position in Fig. 13, PQ is x ft below the water surface. But, no matter whether PQ is below or above the water surface, the depth of the submerged portion of the cube is $a + x$ ft and the buoying force of the water is

$$(\tfrac{1}{2})^2(a + x)(62.4) \text{ lb} = 15.6(a + x) \text{ lb},$$

acting upward, while gravity pulls the cube downward with a force of 4 lb. The resultant force acting on the cube is therefore $4 - 15.6(a + x)$ lb, which is equal to $-15.6x$ lb since $15.6a = 4$. Hence we have the differential equation

$$\frac{4}{g}\frac{d^2x}{dt^2} = -15.6x,$$

or

$$\frac{d^2x}{dt^2} = -3.9gx, \tag{16}$$

representing the simple harmonic motion of the plane PQ vibrating up and down through its equilibrium position, $x = 0$.

From equation (16) the period of vibration is

$$T = \frac{2\pi}{\sqrt{3.9g}} = 0.561 \text{ sec.}$$

Now suppose that the initial depression below the equilibrium position is 2 in. $= \frac{1}{6}$ ft. In order to find the equation of motion we must solve equation (16). Writing the general values of x and v [equations (2) and (4)], together with the initial conditions, we have

$$x = C_1 \sin \sqrt{3.9g}\, t + C_2 \cos \sqrt{3.9g}\, t \quad (t = 0, x = \tfrac{1}{6}),$$

$$v = \sqrt{3.9g}(C_1 \cos \sqrt{3.9g}\, t - C_2 \sin \sqrt{3.9g}\, t) \quad (t = 0, v = 0).$$

Substituting the conditions, we find $C_2 = \frac{1}{6}$ and $C_1 = 0$; hence the equation of motion is

$$x = \tfrac{1}{6} \cos \sqrt{3.9g}\, t.$$

At the end of $\frac{5}{8}$ of a period, $t = \frac{5}{8}T = \frac{5}{4}(\pi/\sqrt{3.9g}\,)$, and

$$x\Big]_{t=5T/8} = \frac{1}{6}\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{12} \text{ ft} = -1.41 \text{ in.}$$

Hence the equilibrium plane PQ is 1.41 in. above the water surface, and the top of the block is $1.414 + AP = 1.414 + 2.923 = 4.34$ in. above water at the end of $\frac{5}{8}$ of a period.

In all the problems of the following group resistance is assumed to be negligible. They are all problems in rectilinear motion except those concerned with the motion of a pendulum, but here the angular displacement θ takes the place of linear displacement x. Whenever the numerical value of g is needed, the value 32.17 ft/sec^2 should be used.

PROBLEMS

1. A particle P starts from rest at $x = a$ in accordance with the law $d^2x/dt^2 = -k^2x$. At the same time a second particle Q starts from rest at $x = a$ in accordance with the law $d^2x/dt^2 = k^2x$. Find (*a*) the distance traveled by P while Q moves its first $a/3$ units of distance; (*b*) the distance traveled by Q while P moves its first $a/3$ units of distance.

2. For the motion described in Ex. 2 of Art. 32, find (*a*) the relation between x and v; (*b*) the values of x and v when t equals a quarter-period; (*c*) the time when the particle first reaches its extreme left position.

3. A particle moves on a straight line under the action of a repulsive force, according to the law $d^2x/dt^2 = x/4$. If it is projected from the point $x = 0$ with a certain initial velocity, how much time will elapse before its velocity is double the initial velocity?

4. A weight stretches a spring 6 in. It is started vibrating at a point 2 in. below its equilibrium position with a downward velocity of 2 in./sec. (*a*) When will it first return to its starting position? (*b*) When will it reach its highest point? (*c*) Show that its maximum velocity is $2\sqrt{2g + 1}$ in./sec.

5. A weight which stretches a spring a ft is drawn down b ft below its equilibrium position and released. What initial velocity imparted to the weight would have doubled the amplitude of the vibration?

6. A weight of 10 lb stretches a spring 10 in. The weight is drawn down 2 in. below its equilibrium position and given an initial velocity of 4 in./sec. Another, identical spring has a different weight attached to it. This second weight is drawn down from its equilibrium position a distance equal to the amplitude of the first motion, then given an initial velocity of 2 ft/sec. If the amplitude of the second motion is twice that of the first, what weight was attached to the second spring?

7. A 20-lb weight is hung on an 18-in. spring, stretching it 8 in. The weight is pulled down 5 in., 5 lb are added to the weight, and it is given a vertical push. During vibration the weight rises to a point 6 in. above the point where the 20-lb weight originally hung. Find the initial velocity.

8. A weight hanging on a spring is pulled down 4 in. below its equilibrium position and then released. At the end of 1 sec it is passing upward for the second time through a point 2 in. below its equilibrium position. Find the period of the motion and the distance of the equilibrium position below the end of the unstretched spring.

9. A weight stretches a spring 2 in., and another weight stretches another spring 8 in. If both weights are pulled down 1 in. below their respective equilibrium positions and then released, find the first two times (after $t = 0$) when their velocities will be equal.

10. Two springs with weights attached are hung from a horizontal beam. If the weights are pulled down, and released at different times, show that when the distance between the weights is a maximum the velocities will be equal (in magnitude and direction).

11. In Prob. 10 suppose that the springs are identical, that each weight stretches the spring 1 ft, and that the weights are released $\frac{1}{8}$ sec apart at the same distance below their equilibrium positions. Find the time, after the release of the first weight, when the distance between the weights is first a maximum, and show that then the displacements of the two weights are equal in magnitude and opposite in sign.

12. A spring is 1.5 ft long. A weight is hung on it, stretching it 6 in. The weight is then drawn down 4 in. and given a downward velocity of 4 in./sec. (*a*) Find the distance of the weight below the top of the spring when the tension is a maximum. (*b*) If the maximum tension is 10 lb, what weight was used?

13. A weight stretches a spring 6 in. The weight is drawn down below its equilibrium position, then pushed upward, and reaches its uppermost position 4 in. above the equilibrium position in 0.3 sec. (*a*) How far was the weight drawn down below the equilibrium position? (*b*) What was the initial acceleration? (*c*) If during vibration the minimum tension is 5 lb, what weight was used?

14. A rubber band of natural length AB is suspended vertically from A and a weight is attached to it at B. The weight stretches it 18 in. If the weight is drawn up to the point B and then projected downward with a velocity of 5.67 ($= \sqrt{g}$) ft/sec, find the greatest extension of the band.

15. The resultant attraction of a spherical mass on a particle within the mass is directed toward the center of the sphere and is proportional to the distance from the center. Suppose that a straight tube is bored through the center of the earth and a particle weighing w lb is dropped into the tube. If the radius of the earth is 3960 miles, find how long it will take the particle (*a*) to pass through the tube; (*b*) to drop halfway to the center.

16. In Ex. 4 of Art. 32, assume that the block of wood is depressed until its upper face lies in the water surface and is then released. At the end of 2 sec, (*a*) how much of the block is submerged? (*b*) how fast is it moving, and in which direction?

17. A cylinder $\frac{1}{2}$ ft in diameter and weighing 10π lb floats half submerged in water. It is pressed down so that its top is in the water surface; then it is released, and vibrates with axis vertical. Find (*a*) the period of vibration; (*b*) the location of the midpoint of the cylinder after 1 sec.

18. A cylinder weighing 40 lb floats in water with its axis vertical. It is pressed down slightly and released, whereupon it vibrates with a period of 1.8 sec. Find the diameter of the cylinder.

18. A cubical block of wood 28 in. on an edge floats in water. It is pressed down slightly and released, whereupon it vibrates with a period of 1.35 sec, its top remaining horizontal. Find the specific gravity of the wood.

20. A rod of length L (ft) is pivoted at one end and has a weight w (lb) attached to the other end, forming a simple pendulum. If θ is the angular displacement of the rod from the vertical at time t (sec), show that

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta.$$

If θ is so small that $\sin\theta$ may be replaced by θ, the equation takes the form of equation (1), Art. 32. In this case, with α equal to the amplitude, or maximum angular displacement, find the period of vibration and the equation of motion when the initial conditions are $\theta = \alpha$, $\omega = d\theta/dt = 0$, when $t = 0$.

21. Find the equation of motion of a pendulum of length L (ft) if the amplitude of the vibration is small and the pendulum is started from its lowest position by giving it an angular velocity ω_0 rad/sec.

22. A seconds pendulum is one which swings through its complete arc in 1 sec, that is, one whose period is 2 sec. (*a*) Find the length of a seconds pendulum with small amplitude. (*b*) If the pendulum of Prob. 21 is a seconds pendulum with amplitude 4°, find ω_0.

23. A pendulum swings through an arc of 6°. How much of its period is consumed in traveling from its extreme position halfway to its lowest position?

24. A pendulum whose period is 1.5 sec is drawn aside 3° from its vertical position and released. With what angular velocity does it pass through its vertical position?

(*c*) *Attractive force proportional to displacement; resistance proportional to velocity.* The vibratory motion with constant amplitude, which we have just studied, resulted from a force proportional to the displacement, acting without resistance. If friction or other resisting forces are taken into account, the particle may execute damped vibrations in which the oscillations continually decrease in size, or, if the resistance is sufficiently large, the particle may gradually come to rest at its equilibrium position without vibrating.

We consider here only a resistance which is proportional to the velocity. Suppose that a particle moving along the x-axis suffers such a resistance. If the particle is moving toward the right, the velocity dx/dt is positive, and $-K\,dx/dt$ represents the negative resisting force, K being a positive constant. If the particle is moving toward the left, the velocity dx/dt is negative, and $-K\,dx/dt$ represents the positive resisting force. If, therefore, for either direction of motion, we add the resisting force $-K\,dx/dt$ to the force $-k^2x$, proportional to the displacement, we obtain the resultant force acting on the particle. Equating this resultant force, measured in pounds, to the symbol for pounds force, $(w/g)(d^2x/dt^2)$, where w (lb) is the weight of the particle, we have the differential equation

$$\frac{w}{g}\frac{d^2x}{dt^2} = -k^2x - K\frac{dx}{dt}.$$

Now multiply by g/w, write for convenience $k^2g/w = b^2$, $Kg/w = 2a$, and use the operator notation $\mathrm{D} = d/dt$, thus obtaining the *standard form* of the differential equation of motion,

$$(\mathrm{D}^2 + 2a\mathrm{D} + b^2)x = 0. \tag{17}$$

The roots of the quadratic $\mathrm{D}^2 + 2a\mathrm{D} + b^2 = 0$ are $-a \pm \sqrt{a^2 - b^2}$, so that the form of the general solution depends on the relative values of a and b.

If $a < b$, that is, if the resistance is sufficiently small, the roots will be imaginary, and the solution will contain trigonometric functions and represent vibratory motion. In this case we write the roots in the form $-a \pm \sqrt{b^2 - a^2}\,i$, and the general solution is

$$x = e^{-at}(C_1 \sin \sqrt{b^2 - a^2}\, t + C_2 \cos \sqrt{b^2 - a^2}\, t), \quad (18)$$

or

$$x = Ae^{-at} \sin (\sqrt{b^2 - a^2}\, t + \alpha). \quad (19)$$

If a were equal to zero, that is, if there were no resistance, we would have the case of simple harmonic motion discussed in Art. 32(a), where the x, t relation, if plotted on rectangular t, x axes, would represent a sine curve oscillating between the straight lines $x = \pm A$. Here the x, t relation represents a distorted sine curve oscillating between the *damping curves* $x = \pm Ae^{-at}$. The oscillations are damped down and gradually die out with increasing t; equation (18), or (19), represents *damped oscillatory motion.* The factor e^{-at} is called the *damping factor* (d.f.). It should be remembered, however, that the x, t curve is not the path of the particle. The particle oscillates back and forth through the origin in a straight line, the x-axis; the x, t curve pulls this motion out, so to speak, in the t direction.

We call $T = 2\pi/\beta$ the *period,* where $\beta = \sqrt{b^2 - a^2}$. If $a = 0$, T reduces to the period for simple harmonic motion, $T = 2\pi/b$. The period represents the time interval between two successive passages of the particle through the origin in the same direction.

If $a > b$, both roots $-a \pm \sqrt{a^2 - b^2}$ are real and negative. Writing $r_1 = -a + \sqrt{a^2 - b^2}$ and $r_2 = -a - \sqrt{a^2 - b^2}$, we obtain the general solution of the differential equation (17):

$$x = C_1 e^{r_1 t} + C_2 e^{r_2 t}, \qquad r_2 < r_1 < 0. \quad (20)$$

Then, differentiating equation (20),

$$v = r_1 C_1 e^{r_1 t} + r_2 C_2 e^{r_2 t}. \quad (21)$$

Suppose that at time $t = 0$ the particle is started at the point $x = x_0 > 0$ with initial velocity $v = v_0$. Substitution of these conditions in equations (20) and (21) determines the constants C_1 and C_2:

$$\left.\begin{aligned} x_0 &= C_1 + C_2 \\ v_0 &= r_1 C_1 + r_2 C_2 \end{aligned}\right\} \qquad C_1 = \frac{v_0 - r_2 x_0}{r_1 - r_2}, \qquad C_2 = \frac{-v_0 + r_1 x_0}{r_1 - r_2}.$$

By inserting these values of C_1 and C_2 in equation (20), it becomes

$$x = \frac{1}{r_1 - r_2}\left[(v_0 - r_2 x_0)e^{r_1 t} - (v_0 - r_1 x_0)e^{r_2 t}\right]. \qquad (22)$$

Since r_1 and r_2 are both negative, equation (22) shows that the particle will settle down to $x = 0$ as t increases indefinitely; the way it moves, however, depends on the value of the initial

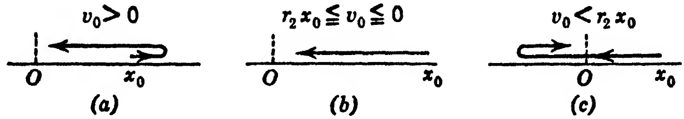

FIG. 14

velocity v_0. Since $r_1 - r_2$ is positive, x will be positive provided that

$$(v_0 - r_2 x_0)e^{(r_1 - r_2)t} > v_0 - r_1 x_0. \qquad (23)$$

For $t > 0$ the exponential in (23) is greater than 1; also, for $x_0 > 0$, $v_0 - r_2 x_0 > v_0 - r_1 x_0$. Therefore if $v_0 \geqq r_2 x_0$ the inequality (23) is satisfied and the particle remains to the right of $x = 0$; for $v_0 > 0$ the particle starts at $x = x_0$, moves toward the right, then reverses its direction of motion, and approaches $x = 0$ [Fig. 14(*a*)]; for $r_2 x_0 \leqq v_0 \leqq 0$ the particle starts toward the left and approaches $x = 0$ [Fig. 14(*b*)]. If $v_0 < r_2 x_0$ the inequality (23) is reversed for sufficiently large t; the particle starts toward the left at $x = x_0$, passes through $x = 0$, reverses its direction of motion, and approaches $x = 0$ [Fig. 14(*c*)].

EXAMPLE 5. A particle starts from rest at time $t = 0$ (sec) with a displacement $x = 5$ (ft) to the right of the origin, and moves along the x-axis according to the law

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + 1.25x = 0.$$

Find (*a*) the time required for the damping factor to decrease 50 per cent, (*b*) the percentage decrease in the damping factor after one period, (*c*) the location of the particle after one period.

The differential equation is given in standard form [equation (17)], with $a = \frac{1}{2}$, $b^2 = 1.25$, $a < b$.

(*a*) The d.f. is $e^{-at} = e^{-t/2}$. Its original value, for $t = 0$, is 1; it reduces to $\frac{1}{2}$ when

$$e^{-t/2} = \tfrac{1}{2},$$

$$-\frac{t}{2} = -\ln 2,$$

$$t = 2 \ln 2 = 1.39 \text{ sec.}$$

(*b*) The period is $2\pi/\sqrt{b^2 - a^2} = 2\pi$ sec. At this time the d.f. has the value $e^{-\pi} = 0.0432$. Hence the d.f. has decreased 95.7 per cent.

(*c*) Only in part (*c*) do we need to use the initial conditions and solve the differential equation

$$(D^2 + D + 1.25)x = 0. \tag{24}$$

The roots of the quadratic $D^2 + D + 1.25 = 0$ are $-\frac{1}{2} \pm i$; hence the general solution of (24) is

$$x = e^{-t/2}(C_1 \sin t + C_2 \cos t). \tag{25}$$

Differentiation of (25) gives

$$v = e^{-t/2}\left[\left(C_1 - \frac{C_2}{2}\right)\cos t - \left(\frac{C_1}{2} + C_2\right)\sin t\right]. \tag{26}$$

Substituting the initial conditions $x = 5$, $v = 0$, $t = 0$ in (25) and (26), we obtain

$$5 = C_2, \qquad 0 = C_1 - \frac{C_2}{2}, \qquad C_1 = \frac{5}{2}.$$

Hence the equation of motion is

$$x = e^{-t/2}(\tfrac{5}{2} \sin t + 5 \cos t).$$

Then

$$x]_{t=2\pi} = e^{-\pi}(5) = 0.216 \text{ ft},$$

and the particle is located 0.216 ft to the right of the origin.

Example 6. A particle moves along the x-axis in accordance with the law

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 16x = 0.$$

From a point 1 ft to the right of the origin the particle at time $t = 0$ (sec) is projected toward the left at the rate of 9 ft/sec. Find (a) the time when the particle passes through the origin; (b) the numerically greatest negative displacement; (c) the maximum (positive) velocity.

The differential equation is in standard form (17), with $a = 5$, $b = 4$, $a > b$, $r_1 = -2$, $r_2 = -8$. Since $x_0 = 1$ and $v_0 = -9$, the condition $v_0 < r_2x_0$ is satisfied and we have the type of motion in which the particle passes toward the left through the origin, reverses its direction of motion, and approaches $x = 0$.

For x and v [equations (20) and (21)] we have

$$x = C_1e^{-2t} + C_2e^{-8t},$$

$$v = -2C_1e^{-2t} - 8C_2e^{-8t}.$$

Substitution of the initial conditions $x = 1$, $v = -9$, $t = 0$, gives

$$\left.\begin{aligned} 1 &= C_1 + C_2 \\ -9 &= -2C_1 - 8C_2 \end{aligned}\right\} \qquad C_1 = -\tfrac{1}{6}, \qquad C_2 = \tfrac{7}{6}.$$

Hence

$$x = -\tfrac{1}{6}e^{-2t} + \tfrac{7}{6}e^{-8t}, \tag{27}$$

$$v = \tfrac{1}{3}e^{-2t} - \tfrac{28}{3}e^{-8t}. \tag{28}$$

(a) When $x = 0$, equation (27) gives

$$e^{6t} = 7, \qquad t = \tfrac{1}{6}\ln 7 = 0.324 \text{ sec}.$$

(b) The numerically greatest negative displacement occurs when $v = 0$, that is, from equation (28), when $e^{6t} = 28$, from which

$$e^{-2t} = 28^{-1/3}, \qquad e^{-8t} = 28^{-4/3}.$$

Substituting these values in equation (27), we obtain

$$x = \tfrac{1}{6}(-28 + 7)28^{-4/3} = -3.5(28)^{-1/3} = -0.0412 \text{ ft}.$$

(c) When the velocity is a maximum,

$$\frac{dv}{dt} = -\frac{2}{3}e^{-2t} + \frac{224}{3}e^{-8t} = 0.$$

Then $e^{6t} = 112$, from which $e^{-2t} = 112^{-1/3}$, $e^{-8t} = 112^{-4/3}$. Substituting these values in equation (28), we find

$$v_{\text{max}} = \tfrac{1}{3}(112 - 28)112^{-4/3} = 28(112)^{-4/3} = 0.0519 \text{ ft/sec.}$$

Example 7. A spring, fixed at its upper end, supports a weight of 10 lb at its lower end, which stretches the spring 6 in. If the weight is drawn down 3 in. below its equilibrium position and released, find the period of vibration and the equation of motion of the weight, assuming a resistance in pounds numerically equal to $\frac{1}{2}$ the speed in feet per second.

We obtain the differential equation for this example by putting $w = 10$ in equation (13) of Ex. 3, and adding the term $-\frac{1}{2}dx/dt$ which represents the resisting force with sign opposite that of the velocity. The differential equation is therefore

$$\frac{10}{g}\frac{d^2x}{dt^2} = -20x - \frac{1}{2}\frac{dx}{dt},$$

or, in standard form,

$$(D^2 + 2aD + b^2)x = 0,$$

where

$$a = \frac{g}{40} = 0.804, \qquad b = \sqrt{2g},$$

$$\beta = \sqrt{b^2 - a^2} = \sqrt{64.34 - .65} = 7.98.$$

The period is therefore $2\pi/\beta = 0.787$ sec. The general solution is

$$x = e^{-at}(C_1 \sin \beta t + C_2 \cos \beta t). \tag{29}$$

Differentiating (29), we get

$$v = e^{-at}[(\beta C_1 - aC_2) \cos \beta t - (aC_1 + \beta C_2) \sin \beta t]. \tag{30}$$

Substitution of the initial conditions $x = \frac{1}{4}$, $v = 0$, $t = 0$, in (29) and (30) gives

$$C_2 = \tfrac{1}{4}, \qquad 0 = \beta C_1 - aC_2, \qquad C_1 = \frac{a}{4\beta} = 0.0252.$$

Hence the equation of motion is

$$x = e^{-0.804t}(0.0252 \sin 7.98t + 0.250 \cos 7.98t).$$

PROBLEMS

1. For the motion represented by the differential equation [(17), Art. 32(*c*)]

$$(D^2 + 2aD + b^2)x = 0,$$

where $a < b$, show that, if v is the velocity at any time, the velocity one period later is $e^{-aT}v$, where T is the period.

2. Discuss the motion represented by the differential equation of Prob. 1 if $a = b$, with initial conditions $x = x_0 > 0$, $v = v_0$, when $t = 0$.

3. When $a > b$ in the differential equation of Prob. 1, there are two cases when the particle reverses its direction of motion, represented by Fig. 14(*a*) and (*c*). In each of these cases obtain a formula for the time elapsed until the reversal takes place.

4. The motion of a particle is represented by the differential equation of Prob. 1 with $a > b$ and initial conditions $x = x_0 > 0$, $v = 0$, when $t = 0$ [Fig. 14(*b*)]; at what time will the particle be moving fastest?

5. If a particle moves along the x-axis under the action of a repulsive force proportional to the displacement and resistance proportional to the velocity, the differential equation of the motion takes the form [cf. equation (17), Art. 32(*c*)]

$$(D^2 + 2ab - b^2)x = 0.$$

Discuss the motion under initial conditions $x = x_0 > 0$, $v = v_0$, when $t = 0$, and draw a diagram similar to Fig. 14 illustrating the motion.

6. A particle executes damped vibrations of period 1.35 sec. If the damping factor decreases by 25 per cent in 16.3 sec, find the differential equation, with numerical coefficients, representing the motion.

7. A weight hung on a spring and vibrating in air with negligible damping has a period of 1 sec. It is set vibrating with a practically weightless damping vane attached to it, causing a resistance proportional to the speed, and the period is found to be 1.5 sec. Find the damping factor, and write down the differential equation, with numerical coefficients, corresponding to the damped vibrations.

8. A weight suspended from a spring executes damped vibrations of period 2 sec. If the damping factor decreases by 90 per cent in 10 sec, find the acceleration of the weight when it is 3 in. below its equilibrium position and is moving upward with a speed of 2 ft/sec.

9. A weight of 4 lb is hung on a spring causing an elongation of 2 in. It is set vibrating, and its period is $\pi/6$ sec. Find the time required for the damping factor to decrease 75 per cent if resistance is proportional to velocity.

10. In Ex. 4, Art. 32(*b*), suppose that the water offers a resistance (lb) numerically equal to K times the velocity (ft/sec), thus increasing the period of vibration by 0.04 sec. Find the value of K and the time required for the damping factor to decrease 50 per cent.

11. A particle moves along the x-axis in accordance with the law

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 9x = 0.$$

From a point 2 ft to the right of the origin the particle at time $t = 0$ (sec) is projected toward the left at the rate of 20 ft/sec. Find (*a*) the time when the particle reaches its leftmost position; (*b*) the distance traveled and the velocity at the end of 1 sec.

12. If the particle of Prob. 11 is projected toward the right at the rate of 20 ft/sec, find (*a*) the time when it reaches its rightmost position; (*b*) the distance traveled and the velocity at the end of 1 sec.

13. If the particle of Prob. 11 is projected toward the left at the rate of 10 ft/sec, find the distance traveled and the velocity at the end of 1 sec.

14. A particle moves along the x-axis in accordance with the law

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} - 16x = 0.$$

From a point 2 ft to the right of the origin the particle at time $t = 0$ (sec) is projected toward the left at the rate of 10 ft/sec. Find (*a*) the time when the particle reaches its leftmost position; (*b*) the distance traveled and the velocity at the end of 1 sec.

15. A 10-lb weight is suspended by a spring which is stretched 2 in. by the weight. Assume a resistance whose magnitude (lb) is $40/\sqrt{g}$ times the speed (ft/sec) at any instant. If the weight is drawn down 3 in. below its equilibrium position and released, find (*a*) the displacement of the weight from its equilibrium position after $1/(2\sqrt{g})$ sec; (*b*) the time required to reach the equilibrium position.

16. A 10-lb weight hanging on a spring stretches it 2 ft. If the weight is drawn down 6 in. below its equilibrium position and released, find (*a*) the highest point reached by the weight; (*b*) the location of the weight after $1/\sqrt{g}$ sec. Assume a resistance whose magnitude (lb) is $10/\sqrt{g}$ times the speed (ft/sec).

17. In Prob. 16 draw the x, t curve and the damping curves. Show that throughout the motion the time required for the speed to increase from zero to a maximum is $\frac{1}{8}$ of a period, and that at maximum speed the x, t curve is tangent to a damping curve.

18. A 10-lb weight stretches a certain spring 1 ft. The weight is drawn down 6 in. below its equilibrium position and then released. Assuming a resistance whose magnitude (lb) is equal to the speed (ft/sec) at any point, find (*a*) the period; (*b*) the percentage decrease in the damping factor when the weight is at its highest point; (*c*) the distance (in.) of the weight from its starting position when $t = 0.6$ sec.

19. A weight of 16 lb hanging on a spring, which it stretches 6 in., is given a downward velocity of 10 ft/sec. If the resistance (lb) of the medium is numerically equal to $8/\sqrt{g}$ times the velocity (ft/sec), is the weight above or below the starting point at the end of 0.5 sec, and how much?

20. A simple pendulum makes small oscillations with a period of 2 sec in a resisting medium. Assume an angular retardation whose magnitude (rad/sec²) is 1/25 the magnitude of the angular velocity (rad/sec). If the pendulum is released from rest at an angular displacement of 1°, find the displacement at the end of 10 periods.

21. A 6-lb weight of specific gravity 3 stretches a spring 4 in. when immersed in water. If the period of vibration in water is $\frac{5}{6}$ sec, by what percentage will the damping factor decrease in 1 sec? Assume resistance proportional to velocity.

22. A weight having specific gravity ρ is immersed in water and supported by a spring which it stretches a in. It is set vibrating, and the resistance of the water to the motion of the weight is proportional to the velocity. If at the end of r periods the damping factor is $1/n$ its initial value, obtain a formula for the period T in terms of ρ, a, n, r, and g. Use the formula to compute the value of T if $\rho = 3$, $a = 8$ in., $n = 6$, $r = 1$, $g = 32.17$ ft/sec².

33. The linear equation with $R \neq 0$; simplest case. We consider now the simplest case of the differential equation

$$f(\mathrm{D})y = (a_0\mathrm{D}^n + a_1\mathrm{D}^{n-1} + \cdots + a_{n-1}\mathrm{D} + a_n)y = R,$$

with $R \neq 0$, namely, the case in which a_n and all the other a's except one, say a_{n-r}, are equal to zero; the differential equation then has the form

$$a_{n-r}\frac{d^ry}{dx^r} = R,$$

where R is a function of x, or

$$\frac{d^ry}{dx^r} = F(x). \tag{1}$$

The solution of the differential equation is effected merely by integrating r times, adding each time a constant of integration.

EXAMPLE 1. Solve

$$\frac{d^3y}{dx^3} = 6x + 2\cos 2x$$

Integrating three times, adding each time a constant of integration, we have

$$\frac{d^2y}{dx^2} = 3x^2 + \sin 2x + c_1,$$

$$\frac{dy}{dx} = x^3 - \frac{\cos 2x}{2} + c_1x + c_2,$$

$$y = \frac{x^4}{4} - \frac{\sin 2x}{4} + \frac{c_1x^2}{2} + c_2x + c_3.$$

General solution:

$$4y = x^4 - \sin 2x + C_1x^2 + C_2x + C_3.$$

34. Deflection of beams. As an application of equation (1), Art. 33, we consider some problems on the deflection of beams. Figure 15 shows a piece of a bent beam. It can be regarded as made up of fibers such as $F'F$, all originally of length s (ft).

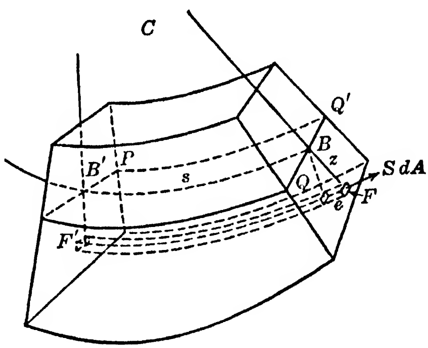

FIG. 15

The surface PQ, containing fibers whose lengths are unaltered when the beam is bent, is called the neutral surface. The curve of one of these fibers $B'B$ is called the elastic curve of the beam. Fibers below the neutral surface are stretched, and those above are compressed, when the beam is bent. Suppose that the fiber $F'F$ at a distance z below the neutral surface has been stretched an amount e by the force $S\,dA$, where S (lb/ft^2) is the stress per unit cross-section area and dA (ft^2) is the cross-section area of the fiber. Also let R (ft) be the length of BC, the radius of curvature at B of the fiber $B'B$, and let QQ' be perpendicular to $B'B$ and BC.

Now, by Hooke's law, the stress S per unit area is proportional to the stretch e/s per unit length in the fiber $F'F$; that is,

$$S = E\frac{e}{s}, \tag{1}$$

where the constant of proportionality E (lb/ft^2) is Young's modulus, or the modulus of elasticity. Furthermore, from the figure we have the proportion

$$\frac{e}{s} = \frac{z}{R}. \tag{2}$$

Then, from (1) and (2),

$$S = \frac{Ez}{R}.$$

The moment of the force $S\,dA$ about the axis QQ' is

$$zS\,dA = \frac{E}{R}z^2\,dA.$$

Integrating this over the cross section of the beam we get the bending moment M:

$$M = \frac{E}{R}\int z^2\,dA.$$

But

$$\int z^2\,dA = I,$$

where I (ft^4) is the moment of inertia of the cross-section area of the beam with respect to the axis QQ'; hence

$$M = \frac{EI}{R}. \tag{3}$$

Now take the x-axis horizontal through some point of the fiber $B'B$ chosen as the origin and the y-axis positive upward.

The formula for radius of curvature is $R = (1 + y'^2)^{3/2}/y''$, but for small bending y' is small and y'^2 may be neglected in comparison with unity, so that a close approximation to the radius of curvature is $R = 1/y''$. By use of this approximation, (3) becomes

$$M = EIy'', \tag{4}$$

the expression for the bending moment at any section of the beam. Equating this expression to the algebraic sum of the moments, with respect to the axis QQ' of the section, of all forces on one side of the section which tend to bend the beam about this axis, we obtain the differential equation of the elastic curve of the beam. When the moments of all these forces are expressible in terms of x the differential equation is of the form (1), Art. 33, with $r = 2$.

EXAMPLE 1. Find the elastic curve and the maximum deflection of a beam L ft long resting on supports at the ends and slightly bent under a uniform load of q lb/ft.

Take the origin O at the left support and the y-axis positive upward (Fig. 16). Then take a section of the beam through a point P at a

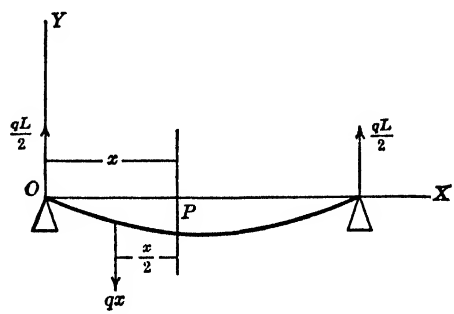

FIG. 16

distance x from O, and equate the expression for the bending moment at P, EIy'' [equation (4)], to the sum of the moments of all the forces on one side of P, say to the left, that tend to bend the beam at P. Since y'' is positive on a curve when the curve is concave toward the positive direction of the y-axis, we write a moment positive when it tends to produce concavity in the direction of the positive y-axis.

Since y'' is negative on a curve when the curve is concave toward the negative y-axis, we write a moment negative when it tends to produce concavity in the direction of the negative y-axis. The load qL is equally distributed on the supports, causing an upward force of $qL/2$ at the left support and a moment $qLx/2$ with respect to P. The load qx on OP acting downward at the middle point of OP produces a moment $-qx^2/2$ with respect to P. The differential equation of the elastic curve of the beam is therefore

$$EIy'' = \frac{qLx}{2} - \frac{qx^2}{2}. \tag{5}$$

In this equation the units of the various quantities involved are as follows: E (lb/ft^2), I (ft^4), y'' (ft^{-1}), q (lb/ft), L (ft), x (ft).

The differential equation (5) is solved merely by integrating twice and determining the two constants of integration from two conditions. Integrating once, we get

$$EIy' = \frac{qLx^2}{4} - \frac{qx^3}{6} + C_1. \tag{6}$$

The condition $y' = 0$, $x = L/2$, gives

$$0 = \frac{qL^3}{16} - \frac{qL^3}{48} + C_1; \qquad C_1 = -\frac{qL^3}{24}.$$

Inserting the value of C_1 in (6) and integrating again, we find

$$EIy = \frac{qLx^3}{12} - \frac{qx^4}{24} - \frac{qL^3x}{24} + C_2. \tag{7}$$

The condition $y = 0$, $x = 0$, gives $C_2 = 0$. Hence the elastic curve of the beam is

$$y = \frac{-q}{24EI}(x^4 - 2Lx^3 + L^3x). \tag{8}$$

There were just enough independent conditions to determine the two arbitrary constants. It may be noticed that a third known condition $y = 0$, $x = L$, is not independent; it satisfies (8) automatically, or, if substituted in (7), yields again $C_2 = 0$. If we had not seen at the beginning that the reaction at each support is $qL/2$, we could have called one reaction R and the other $qL - R$; then all three conditions would have been needed to determine R, C_1, and C_2.

It may be noticed further that, if we had taken moments to the right of P, instead of to the left, it would have been more complicated but the same differential equation would have been obtained, for in that case

$$EIy'' = \frac{qL(L - x)}{2} - \frac{q(L - x)^2}{2},$$

which reduces to equation (5).

The maximum deflection of the beam is the value of $-y$ when $x = L/2$, that is, from equation (8),

$$-y\Big]_{x=L/2} = \frac{q}{24EI}\left(\frac{L^4}{16} - \frac{L^4}{4} + \frac{L^4}{2}\right) = \frac{5qL^4}{384EI} \text{ ft.}$$

EXAMPLE 2. Solve Example 1 if the beam, instead of being simply supported at its ends, is embedded horizontally in masonry at its ends.

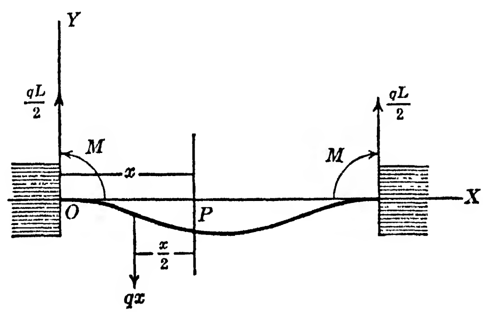

FIG. 17

The differential equation is the same as in Example 1, except that there is an additional unknown moment at each end of the beam, exerted by the masonry, which keeps the beam horizontal at the ends (Fig. 17). Since this moment tends to produce concavity downward it is negative; we denote it by $-M$. The differential equation is

$$EIy'' = \frac{qLx}{2} - \frac{qx^2}{2} - M.$$

Integrating once,

$$EIy' = \frac{qLx^2}{4} - \frac{qx^3}{6} - Mx + C_1.$$

The condition $y' = 0$, $x = 0$, gives $C_1 = 0$. Then the condition $y' = 0$, $x = L/2$, gives

$$0 = \frac{qL^3}{16} - \frac{qL^3}{48} - \frac{ML}{2}; \qquad M = \frac{qL^2}{12}.$$

Integrating again,

$$EIy = \frac{qLx^3}{12} - \frac{qx^4}{24} - \frac{qL^2x^2}{24} + C_2.$$

The condition $y = 0$, $x = 0$, gives $C_2 = 0$. Hence the equation of the elastic curve of the beam is

$$y = \frac{-q}{24EI}(x^4 - 2Lx^3 + L^2x^2),$$

and the maximum deflection is

$$-y\Big]_{x=L/2} = \frac{qL^4}{24EI}\left(\frac{1}{16} - \frac{1}{4} + \frac{1}{4}\right) = \frac{qL^4}{384EI}\text{ ft.}$$

Two other conditions $y = 0$, $x = L$ and $y' = 0$, $x = L$ are not independent of the ones used and are satisfied automatically.

Example 3. In Example 2, if there is an additional concentrated load of F lb at the midpoint of the beam, find the elastic curve of the left half of the beam and the maximum deflection.

The effect of the load F at the midpoint is to increase the reaction at each support to $(qL + F)/2$. But there is another important difference between this example and the preceding one. In Example 2 the point P can be taken anywhere along the beam and the moments to the left of P are given by the same expression; the differential equation obtained holds for the whole beam. In this example, however, if P is moved toward the right across the midpoint of the beam, an additional moment to the left of P is introduced, due to the concentrated load, so that the differential equation of the right half of the elastic curve is different from that of the left half; the two halves of the elastic curve have different equations. This may seem strange, since the elastic curve "looks like" a curve with a single equation, but, if the student will draw for illustration the curve $y = -x^3$ from -1 to 0 and the curve $y = x^3$ from 0 to 1, the result will be a curve whose two halves have different equations.

Taking P to the left of the midpoint of the beam and taking moments to the left of P, the differential equation representing the left half of the elastic curve is

$$EIy'' = \frac{(qL + F)x}{2} - \frac{qx^2}{2} - M.$$

Following exactly the same steps and using the same conditions as in Example 2 (the student should do this), the elastic curve of the left half of the beam is found to be

$$y = \frac{-q}{24EI}(x^4 - 2Lx^3 + L^2x^2) + \frac{F}{48EI}(4x^3 - 3Lx^2).$$

Notice that the two conditions $y = 0$, $x = L$ and $y' = 0$, $x = L$, which were satisfied automatically in Example 2, are not satisfied here; these conditions are true only for the right half of the beam.

For the maximum deflection we have

$$-y\Big]_{x=L/2} = \frac{qL^4}{384EI} - \frac{FL^3}{48EI}\left(\frac{1}{2} - \frac{3}{4}\right) = \frac{qL^4 + 2FL^3}{384EI} \text{ ft.}$$

This result could have been obtained by finding the deflection due to the concentrated load (with $q = 0$) and adding it to the result of Example 2.

PROBLEMS

1. Solve Ex. 1, Art. 33, if $y = y'' = 0$, $y' = -\frac{1}{2}$, when $x = \pi$.

2. Solve

$$y'' = 12x^2 + \frac{3}{x^2}$$

if $y' = y = 0$ when $x = 1$.

3. Solve

$$y'' = 4(e^{-x} - e^{2x})$$

if $y = 5$ when $x = 0$ and $y = \ln 2$ when $x = \ln 2$.

4. A beam of length 20 ft is simply supported at the ends and carries a weight of 240 lb at its midpoint. Taking the origin at the left end and neglecting the weight of the beam, find (*a*) the equation of the elastic curve of the left half of the beam; (*b*) the equation of the elastic curve of the right half of the beam; (*c*) the maximum deflection.

5. If the beam of Prob. 4 carries in addition a uniform load of 30 lb/ft find the maximum deflection.

6. A 2 by 6 in. board 20 ft long stands on edge, is simply supported at its ends, and carries a 240-lb weight at its midpoint. If the board weighs 40 lb/ft^3, and $E = 15 \times 10^5$ lb/in.^2, find the maximum deflection.

7. A cantilever beam (one end free and the other fixed horizontal) of length L ft weighs q lb/ft and carries a load of W lb at its free end. (*a*) Taking the origin at the free end, find the equation of the elastic curve of the beam. (*b*) What load, distributed uniformly along the beam, would produce the same maximum deflection as the load W at the free end?

8. A cantilever beam of length L ft weighs q lb/ft and carries a load of P lb at its midpoint. Find the maximum deflection of the beam.

9. A beam L ft long, carrying a uniform load of w lb/ft, is fixed horizontally at one end and is simply supported at the other. Find (*a*) the deflection of the midpoint of the beam; (*b*) the maximum deflection of the beam.

10. A beam 10 ft long, fixed at one end and simply supported at the other carries a uniform load. Find the distance from the supported end to the point where the maximum deflection occurs.

11. A beam 6 ft long is simply supported at its ends. Two concentrated loads, each equal to P lb, are supported at the points of trisection of the beam. Neglecting the weight of the beam, find the points at which the deflection has half its maximum value.

12. A cantilever beam of length L ft, vertical dimension h ft, and modulus of elasticity E lb/ft^2 carries a load of material whose width equals the width of the beam, whose density is w lb/ft^3, and whose depth at distance x ft from the free end is kx^2 ft. Neglecting the weight of the beam, find its maximum deflection.

13. A beam simply supported at its ends, of length L ft, vertical dimension h ft, and modulus of elasticity E lb/ft^2, carries a load of material whose width equals the width of the beam, whose density is w lb/ft^3, and whose depth at distance x ft from one end is kx^2 ft. Neglecting the weight of the beam, find the deflection of its midpoint.

14. Solve Prob. 13 if the depth of the load is proportional to the square root of the distance from one end and equal to H ft at the other end.

15. A beam of length $2L$ ft carrying a uniform load of q lb/ft is supported at its ends and at its middle point. (*a*) Taking the origin at the middle point, find the elastic curve of the right half of the beam. (*b*) Show that the maximum deflection is

$$(39 + 55\sqrt{33})\frac{qL^4}{2^{16}EI},$$

and that this is 42 per cent of the value it would have if the beam were cut in two at its middle point.

16. A simply supported beam L ft long carries a concentrated load of W lb at a distance of c ft from the left end. (*a*) Taking the origin at the left end, find the equation of the elastic curve of the portion of the beam to the left of W and of the portion of the beam to the right of W. (*b*) If $c < L/2$, show that the distance from the left end to the point of maximum deflection is greater than c and less than $L/2$, and find the maximum deflection.

✓**35. The linear equation with $R \neq 0$.** We now rewrite equations (1) and (2) of Art. 31:

$$f(\mathrm{D})y = R, \tag{1}$$
$$f(\mathrm{D})y = 0, \tag{2}$$

where

$$f(\mathrm{D}) = a_0\mathrm{D}^n + a_1\mathrm{D}^{n-1} + \cdots + a_{n-1}\mathrm{D} + a_n.$$

Equation (2) has been solved in Art. 31; we now seek the general solution of equation (1), in which R is a function of x.

The general solution of equation (2) is a value of y containing n essential arbitrary constants, n being the order of the differential equation; we call this the *complementary function* (c.f.) of equation (1) and denote it by y_c. Then $f(\mathrm{D})$ operating on y_c produces 0:

$$f(\mathrm{D})y_c = 0. \tag{3}$$

Now suppose that in some way we are able to find a particular value of y, free of arbitrary constants, which satisfies equation (1); we call this a *particular integral* (p.i.) of equation (1) and denote it by y_p. Then $f(\mathrm{D})$ operating on y_p produces R;

$$f(\mathrm{D})y_p = R. \tag{4}$$

We have seen in Arts. 2 and 13 that a particular integral of a differential equation can be obtained from the general solution by assigning particular values to the arbitrary constants which enter into the general solution; but here we are going to find a particular integral before the general solution is known—the finding of a particular integral is now to be a step in the process of finding the general solution.

Adding equations (3) and (4), we find

$$f(D)(y_c + y_p) = R;$$

hence

$$y = y_c + y_p \tag{5}$$

satisfies the differential equation (1), and since it contains n essential arbitrary constants it is the required general solution. Thus the general solution of (1) consists of two parts, the c.f. and a p.i. The operator $f(D)$, operating on the c.f., produces nothing—the c.f. merely acts as a carrier of the required arbitrary constants. The p.i., however, when operated on by $f(D)$, yields the right member R.

Example 1. Solve

$$(D^2 + 2D + 1)y = 4e^x.$$

The c.f. is found by solving $(D^2 + 2D + 1)y = 0$, or $(D + 1)^2y = 0$; it is

$$y_c = (C_1 + C_2x)e^{-x}.$$

A p.i., found by inspection, is $y_p = e^x$, since

$$(D^2 + 2D + 1)e^x = e^x + 2e^x + e^x = 4e^x.$$

Therefore the general solution of the given equation is

$$y = (C_1 + C_2x)e^{-x} + e^x.$$

This example illustrates the plan of solving an equation of form (1). We have but to find the c.f. by the method of Art. 31, then find a p.i., and add them together. However, we cannot rely on inspection to reveal a p.i. except in very simple cases. We shall explain in the next two articles two methods for finding a p.i. The first method, *undetermined coefficients,* will apply when R is of certain special form; the second, *variation of parameters,* is general and can be used theoretically for any form of R, although it is impracticable to carry it through for complicated forms of R.

36. Undetermined coefficients. We shall find it convenient in this article to call the part of a term which is multiplied by a constant coefficient the *variable part* (v.p.) of the term. Thus the v.p. of $7xe^{2x}$ is xe^{2x}, the v.p. of $6 \cos 2x$ is $\cos 2x$, and the v.p. of $3x$ is x. If a term is a constant, say 5, it can be thought of as $5x^0$, where 5 is the constant coefficient and x^0 or 1 is the v.p. Now a function may be of such kind that successive differentiation of it beyond a certain point ceases to yield terms with new variable parts.

For example, take the function x^2e^{3x} and its two successive derivatives:

$$x^2e^{3x}, \qquad 3x^2e^{3x} + 2xe^{3x}, \qquad 9x^2e^{3x} + 12xe^{3x} + 2e^{3x}.$$

Further differentiation would yield only terms with variable parts, x^2e^{3x}, xe^{3x}, e^{3x}.

As another example, take the function $3x^2 + \sin 2x$ and its two successive derivatives:

$$3x^2 + \sin 2x, \qquad 6x + 2\cos 2x, \qquad 6 - 4\sin 2x.$$

These and further derivatives contain only terms with v.p.'s x^2, x, 1, $\sin 2x$, $\cos 2x$.

As an example of a function which does not possess this property, take $\tan x$ and its successive derivatives:

$$\tan x, \qquad \sec^2 x, \qquad 2\sec^2 x \tan x, \qquad \cdots$$

Further differentiation continues to produce functions with new variable parts.

It can be shown * that functions possessing the above property are expressible as sums of terms of the form

$$Cx^pe^{\alpha x}\cos \beta x \qquad \text{or} \qquad Cx^pe^{\alpha x}\sin \beta x,$$

where p is a positive integer or zero, α and β are any real constants (including zero), and C is any constant.

* See Reddick and Miller, *Advanced Mathematics for Engineers*, 2nd Ed., Art 11, for proof of this and the rule following.

Thus the function x^2e^{3x} in the first example above is obtained from the first form by putting $C = 1$, $p = 2$, $\alpha = 3$, $\beta = 0$; and the function $3x^2 + \sin 2x$ in the second example is obtained by summing the first form with $C = 3$, $p = 2$, $\alpha = 0$, $\beta = 0$, and the second form with $C = 1$, $p = 0$, $\alpha = 0$, $\beta = 2$.

Suppose now that R is a function possessing the above property; that is, R is a function such that successive differentiation of it beyond a certain point ceases to yield terms with new variable parts. Otherwise stated, R is a function expressible as sums of terms of the form $Cx^pe^{\alpha x} \cos \beta x$ or $Cx^pe^{\alpha x} \sin \beta x$, where p is a positive integer or zero, α and β are any real constants (including zero), and C is any constant; then it is shown (*loc. cit.*) that the following rule will yield a particular integral of equation (1), Art. 35.

Write the variable parts of the terms in the right member R *and the variable parts of any other terms obtainable by successively differentiating* R. *Arrange the v.p.'s so found in groups such that all v.p.'s obtainable from a single term of* R *appear in only one group. Any group consisting of v.p.'s none of which is a v.p. of a term of the complementary function* y_c *is left intact, but if any member of a group is a v.p. of a term of* y_c *all the members of this group are to be multiplied by the lowest positive integral power of* x *that will make them all different from the v.p. of any term in* y_c. *Now multiply each member of all the groups by a general constant (undetermined coefficient), and take the sum of the expressions so obtained as* y_p. *Finally, operate on* y_p *with* f(D) *and equate the complete coefficients of the various v.p.'s to the coefficients in the corresponding terms of* R *in order to evaluate the undetermined coefficients in* y_p.

EXAMPLE 1. Solve

$$(D^2 - D)y = 3x^2 - 4x + 5 + 2e^x + \sin x. \qquad (1)$$

We give all the work necessary, followed by an explanation of the steps taken in finding y_p.

$$\text{Roots of } f(D) = (D^2 - D) = D(D - 1) = 0: \qquad 0, 1.$$

$$y_c = C_1 + C_2e^x.$$

$$y_p = Ax^3 + Bx^2 + Cx + Exe^x + F\sin x + G\cos x \quad (2)$$

$-1)$

$$Dy_p = 3Ax^2 + 2Bx + C + Exe^x + Ee^x - G\sin x + F\cos x \quad (3)$$

$1)$

$$D^2y_p = 6Ax + 2B + Exe^x + 2Ee^x - F\sin x - G\cos x \quad (4)$$

$-3A = 3$	$6A - 2B = -4$	$2B - C = 5$	$2E - E = 2$	$-F + G = 1, \quad -F - G = 0$
$A = -1$	$B = -1$	$C = -7$	$E = 2$	$F = -\frac{1}{2}, \quad G = \frac{1}{2}$

$$y_p = -x^3 - x^2 - 7x + 2xe^x - \tfrac{1}{2}\sin x + \tfrac{1}{2}\cos x.$$

The general solution, $y = y_c + y_p$, is

$$y = C_1 - 7x - x^2 - x^3 + (C_2 + 2x)e^x - \tfrac{1}{2}\sin x + \tfrac{1}{2}\cos x.$$

The v.p.'s of $3x^2$ and its two successive derivatives are x^2, x, 1. Corresponding to the term $-4x$ we would get x, 1, and corresponding to the term 5 merely 1, but these are already included in the group x^2, x, 1, so that this group of v.p.'s corresponds to the terms $3x^2 - 4x + 5$ of R. Now one of these v.p.'s, namely 1, is the v.p. of C_1, a term of the c.f.; hence we multiply each member of the group x^2, x, 1, by x, obtaining x^3, x^2, x, none of which is a v.p. of a term of y_c. By multiplying by A, B, C respectively we get the first three terms of y_p, namely, $Ax^3 + Bx^2 + Cx$.

The only v.p. obtainable from $2e^x$ is e^x, which itself forms the second group. Since e^x is the v.p. of the term C_2e^x in y_c, we multiply it by x and prefix a general constant E to obtain the fourth term, Exe^x, of y_p.

From $\sin x$ the only new v.p. obtainable by successive differentiation is $\cos x$, so that the third group consists of $\sin x$, $\cos x$. Since neither of these is a v.p. of a term of y_c, we multiply by F and G respectively, obtaining, for the last two terms of y_p, $F\sin x + G\cos x$.

Now write down y_p, and under it write Dy_p and D^2y_p, arranging terms with the same v.p.'s in the same column. Since y_p satisfies equation (1), the combination $D^2y_p - Dy_p$ must be identical with the right member of (1). To produce this combination we multiply equation (4) by 1, equation (3) by -1, and add, placing the multipliers 1 and -1 at the left of the respective equations. We do not use a multiplier (other than 0) for equation (2) since y, unaffected by a D, does not appear in the left member of (1). By equating the resulting coefficients of the various v.p.'s column by column to the coefficients of the corresponding terms in the right member of (1), we obtain just enough equations to solve for the values of the undetermined coefficients; any additional equations, although superfluous, must not be contradictory. In this example there are six equations, obtained from the 2nd, 3rd, 4th, 6th, 7th, and 8th columns, for determining the six undetermined coefficients, A, B, C, E, F, G. Note that the first column yields nothing, and the fifth gives $E - E = 0$, or merely $0 = 0$; if the fifth column had yielded a contradictory result, a mistake would have been indicated.

A short cut. In the case where R is the product of an exponential function e^{ax} and some other function of x, it is usually shorter to multiply the equation through by e^{-ax} and then make use of the reverse exponential shift [equation (6), Art. 30] in finding the p.i. As an illustration we shall work the following example first by the regular method just explained, then by use of the short cut.

EXAMPLE 2. Solve

$$(D^2 - 2D + 1)y = 2xe^x.$$

First solution. The roots of the quadratic

$$D^2 - 2D + 1 = (D - 1)^2 = 0$$

are 1, 1; hence

$$y_c = (C_1 + C_2x)e^x.$$

Writing the v.p.'s of $2xe^x$ and of the terms obtainable by differentiation, we have the group xe^x, e^x. Multiply the members of this group by x^2 in order to obtain the group x^3e^x, x^2e^x, both of whose members are different from the v.p. of any term in y_c. Now multiply these members by A, B, respectively, to obtain y_p.

$$\begin{aligned}
1)\quad y_p &= Ax^3e^x + Bx^2e^x \\
-2)\quad Dy_p &= Ax^3e^x + (3A + B)x^2e^x + 2Bxe^x \\
1)\quad D^2y_p &= Ax^3e^x + (6A + B)x^2e^x + (6A + 4B)xe^x + 2Be^x
\end{aligned}$$

	$6A + 4B - 4B = 2$	$2B = 0$
$y_p = \frac{1}{3}x^3e^x$	$A = \frac{1}{3}$	$B = 0$

(Note that the first two columns give $0 = 0$.)

33. Solve

$$\frac{d^2Q}{dt^2} + 10^6Q = 100 \sin 500t$$

if $Q = 0$, $dQ/dt = 0$ when $t = 0$; find the value of dQ/dt when $t = 1/100$ (i.e., when $500t = 5$ rad).

37. Variation of parameters. There was developed in the preceding article the method of undetermined coefficients which enables us to find a p.i. of the differential equation $f(D)y = R$, where R is of a certain special form. It happens that R is of this special form in most of the linear differential equations which arise in practice, but it is well to have a method for finding a p.i. in case an equation is encountered in which the method of undetermined coefficients does not apply. Such is the method of variation of parameters, due to Lagrange.

The method is general and theoretically can be used for any form of R, and even for a linear differential equation with variable coefficients * provided the complementary function is known, although it is impracticable to carry it through except for fairly simple forms of R. The difficulty in carrying it through increases not only with the complexity of R but also with the complexity of the complementary function. Even for Problem 22, Art. 36, where the complementary function is rather simple, the method is hardly practicable; in Problem 27 of the same article, however, although R is more complicated, the c.f. is simpler, and, as we shall see in Example 2, this problem can be solved by the method of variation of parameters with little more work than by the method of undetermined coefficients. The chief value of the method from the practical point of view is in solving an equation like that of the following Example 1, where R is a function whose successive derivatives continue to yield terms with different variable parts, so that the method of undetermined coefficients would involve a y_p having an infinite number of terms. The method of variation of parameters is powerful for use in theoretical considerations and is justly described by mathematicians as "elegant."

* For example, Prob. 12 at the end of this article.

We shall explain the theory for the differential equation of second order with constant coefficients,

$$f(\mathrm{D})y = (a_0\mathrm{D}^2 + a_1\mathrm{D} + a_2)y = R, \tag{1}$$

then indicate how it applies to equations of higher order.

First find the c.f. by the familiar method of Art. 31. We use, for convenience, A and B instead of the arbitrary constants C_1 and C_2, and write the c.f. in the form

$$y_c = Au + Bv,$$

where u and v are certain known functions of x.

Our problem is now to find a p.i. of equation (1), that is, a function such that $f(\mathrm{D})$ operating on it produces R. We know that $f(\mathrm{D})$ operating on the function $Au + Bv$ produces 0 if A and B are constants; it is reasonable to surmise that, *if A and B were functions of x* (which can be called parameters), it might be possible to determine them (vary the parameters, so to speak) so that $f(\mathrm{D})$ operating on $Au + Bv$ would produce R instead of 0.

Accordingly we write

$$y = Au + Bv, \tag{2}$$

where A and B now are undetermined functions of x. If we can determine A and B (free of arbitrary constants) so that (2) satisfies (1), the corresponding value of y will be the y_p that we are seeking.

We need two conditions on A and B in order to determine them. We may choose the first condition in some way that will simplify the problem. But the second condition is forced on us; it is that $y = Au + Bv$ must satisfy equation (1) after the first condition has been imposed. The form in which these two conditions appear is made clear by the following compact arrangement of the problem. Subscripts denote differentiation with respect to x.

$$\begin{array}{rl|l}
y &= Au + Bv & \\
\mathrm{D}y &= Au_1 + Bv_1 + & A_1u + B_1v = 0 \\
\mathrm{D}^2y &= Au_2 + Bv_2 + & A_1u_1 + B_1v_1 = R/a_0
\end{array}$$

Arrange the four terms obtained by differentiating $Au + Bv$ so that the first group of two terms contains A and B, and the second group contains A_1 and B_1. Now draw a vertical line separating the two groups, and set the second group equal to zero. This is the first condition, mentioned above, which we arbitrarily impose on the two functions A and B, namely, that the sum of their derivatives multiplied by u and v respectively shall vanish.

Next differentiate the part of Dy that remains on the left of the vertical line, obtaining four more terms for D^2y arranged as before. If now we imagine that the values of y, Dy, and D^2y are substituted in (1), the parts to the left of the vertical line yield nothing, since these would be the values of y and its derivatives if A and B were constant. There remains only $a_0(A_1u_1 + B_1v_1)$, which must equal R if the second condition mentioned above is satisfied. To say that $y = Au + Bv$ must satisfy (1) when $A_1u + B_1v = 0$ is the same as to say that $A_1u_1 + B_1v_1 = R/a_0$, which is the second condition that the functions A and B must satisfy.

In order to determine A and B, the equations

$$\begin{aligned} A_1u + B_1v &= 0, \\ A_1u_1 + B_1v_1 &= \frac{R}{a_0}, \end{aligned} \tag{3}$$

must be solved for A_1 and B_1; these expressions are then integrated (omitting constants of integration) to determine A and B. Substitution of these values of A and B in (2) yields a p.i. y_p of equation (1). The general solution of (1) is then

$$y = y_c + y_p.$$

If, when A_1 and B_1 were integrated to obtain A and B, constants of integration had been added and the resulting values of A and B substituted in (2), the general solution of (1) would have been obtained. Since y_c must be found at the beginning, however, it is expedient to use the method of variation of parameters merely to find y_p, then add y_c and y_p to obtain the general solution.

EXAMPLE 1. Solve ✓

$$(D^2 + 1)y = \tan x.$$

The c.f. is $y_c = C_1 \sin x + C_2 \cos x$.

$y = A \sin x + B \cos x$	
$Dy = A \cos x - B \sin x +$	$A_1 \sin x + B_1 \cos x = 0$
$D^2y = -A \sin x - B \cos x +$	$A_1 \cos x - B_1 \sin x = \tan x$
$A = -\cos x$	$A_1 = \sin x$
$B = \sin x - \ln(\sec x + \tan x)$	$B_1 = -\dfrac{\sin^2 x}{\cos x} = \cos x - \sec x$

$$y_p = -\cos x \ln(\sec x + \tan x)$$

$$y = C_1 \sin x + [C_2 - \ln(\sec x + \tan x)] \cos x$$

In solving the two equations to the right of the vertical line for A_1 and B_1 we multiplied the first of these equations by $\sin x$, the second by $\cos x$, then added to obtain A_1; we then multiplied the first equation by $\cos x$, the second by $\sin x$, and subtracted to obtain B_1. The expressions for A_1 and B_1 were then integrated (omitting constants of integration) to find A and B. When these values of A and B were substituted in the equation $y = A \sin x + B \cos x$, the terms $-\cos x \sin x$ and $\sin x \cos x$ cancelled and the expression for y_p was obtained. Finally the general solution was written down by adding y_c and y_p.

In studying this method the student may have been curious about the statement that we may choose a condition on A and B that will simplify the problem. In the above example we arbitrarily imposed on A and B the condition that $A_1 \sin x + B_1 \cos x = 0$. Was it necessary to impose this particular condition? Suppose we had written some other constant or a function of x instead of 0, would the result have been the same? The answer is that this particular condition was not necessary; it was convenient—it simplified the problem. If the student has sufficient curiosity, he should try working this example by imposing the condition $A_1 \sin x + B_1 \cos x = x$; the final result will be the same but the work will be longer.

We shall now solve Problem 27, Art. 36, by the method of variation of parameters, omitting explanations.

EXAMPLE 2. Solve

$$(4D^2 - 1)y = e^{-x/2}\left(x - \cos\frac{x}{2}\right).$$

The c.f. is $y_c = C_1e^{x/2} + C_2e^{-x/2}$.

$$\begin{array}{l|l}
y = Ae^{x/2} + Be^{-x/2} & \\
Dy = \frac{1}{2}Ae^{x/2} - \frac{1}{2}Be^{-x/2} + & A_1e^{x/2} + B_1e^{-x/2} = 0 \\
D^2y = \frac{1}{4}Ae^{x/2} + \frac{1}{4}Be^{-x/2} + & \frac{1}{2}A_1e^{x/2} - \frac{1}{2}B_1e^{-x/2} = \frac{1}{4}e^{-x/2}[x - \cos(x/2)]
\end{array}$$

$$\begin{array}{l|l}
B = -\dfrac{x^2}{8} + \dfrac{1}{2}\sin\dfrac{x}{2} & B_1 = -\dfrac{1}{4}\left(x - \cos\dfrac{x}{2}\right) \\
A = \dfrac{1}{4}e^{-x}(-x-1) - \dfrac{1}{5}e^{-x}\left(-\cos\dfrac{x}{2} + \dfrac{1}{2}\sin\dfrac{x}{2}\right) & A_1 = \dfrac{1}{4}e^{-x}\left(x - \cos\dfrac{x}{2}\right)
\end{array}$$

$$= e^{-x}\left(-\frac{x}{4} - \frac{1}{4} + \frac{1}{5}\cos\frac{x}{2} - \frac{1}{10}\sin\frac{x}{2}\right)$$

$$y_p = e^{-x/2}\left(-\frac{x}{4} - \frac{1}{4} + \frac{1}{5}\cos\frac{x}{2} - \frac{1}{10}\sin\frac{x}{2} - \frac{x^2}{8} + \frac{1}{2}\sin\frac{x}{2}\right)$$

$$y = C_1e^{x/2} + \left(C_2' - \frac{x}{4} - \frac{x^2}{8} + \frac{2}{5}\sin\frac{x}{2} + \frac{1}{5}\cos\frac{x}{2}\right)e^{-x/2} \qquad \left(C_2' = C_2 - \frac{1}{4}\right)$$

The method of variation of parameters is applicable theoretically to differential equations of any order, but practically it is seldom used for equations of higher order than the second. The amount of work necessary to carry it through increases rapidly with the order of the equation, owing to the necessity of solving a system of simultaneous equations for $A_1, B_1, \cdots$, the number of equations in the system being equal to the order of the differential equation.

In order to illustrate the method for a third order equation, we give the following example, although it could be solved much more readily by the method of undetermined coefficients. Note that here we need three conditions for determining A_1, B_1, C_1, so we arbitrarily set the tails of Dy and D^2y equal to zero for the first two conditions.

EXAMPLE 3. Solve

$$(D^3 - 6D^2 + 11D - 6)y = e^{-x}.$$

$f(D) = (D-1)(D-2)(D-3) \qquad y_c = C_1e^x + C_2e^{2x} + C_3e^{3x}$

$y = Ae^x + Be^{2x} + Ce^{3x}$	
$Dy = Ae^x + 2Be^{2x} + 3Ce^{3x} +$	$A_1e^x + B_1e^{2x} + C_1e^{3x} = 0$
$D^2y = Ae^x + 4Be^{2x} + 9Ce^{3x} +$	$A_1e^x + 2B_1e^{2x} + 3C_1e^{3x} = 0$
$D^3y = Ae^x + 8Be^{2x} + 27Ce^{3x} +$	$A_1e^x + 4B_1e^{2x} + 9C_1e^{3x} = e^{-x}$

Solving the system of three simultaneous equations to the right of the vertical line for A_1, B_1, C_1, we obtain

$$A_1 = \tfrac{1}{2}e^{-2x} \qquad A = -\tfrac{1}{4}e^{-2x}$$

$$B_1 = -e^{-3x} \qquad B = \tfrac{1}{3}e^{-3x}$$

$$C_1 = \tfrac{1}{2}e^{-4x} \qquad C = -\tfrac{1}{8}e^{-4x}$$

$$y_p = -\tfrac{1}{4}e^{-x} + \tfrac{1}{3}e^{-x} - \tfrac{1}{8}e^{-x} = -\tfrac{1}{24}e^{-x}$$

$$y = C_1e^x + C_2e^{2x} + C_3e^{3x} - \tfrac{1}{24}e^{-x}.$$

PROBLEMS

Solve the following differential equations, working 5, 9, and 10 by the method of Art. 36 and also by the method of Art. 37.

1. $(D^2 + 1)y = \csc x$.
2. $(D^2 + 4)y = \cot 2x$.
3. $(9D^2 + 1)y = \sec (x/3)$.
4. $(D^2 + 1)y = \tan^2 x$.
5. $(D^2 - 1)y = \sin^2 x$.
6. $(4D^2 + 4D + 5)y = 4e^{-x/2} \sec x + 4e^{x/2}$.
7. $(D^2 - 1)y = x^2e^{x^2/2}$.

8. $(D^2 - 2D + 2)y = 3x + e^x \tan x.$
9. $(D^2 + 1)y = x \cos x.$
10. $(D^3 - 7D - 6)y = 26e^{-2x} \cos x.$
11. (*a*) Solve by the method of variation of parameters:

$$(D^2 + 3D + 2)y = \sin e^x.$$

(*b*) Show that, although successive differentiation of the function $\sin e^x$ continues to yield terms with different v.p.'s, nevertheless the differential equation may be solved without using the method of variation of parameters.

12. Solve by the method of variation of parameters:

$$(D + P)y = Q,$$

where P and Q are functions of x, obtaining the solution (8), Art. 21.

38. Forced vibrations. In Example 7, Art. 32, we saw that the differential equation

$$\frac{10}{g}\frac{d^2x}{dt^2} = -20x - \frac{1}{2}\frac{dx}{dt} \tag{1}$$

represents the vibrations of a 10-lb weight, hung on a spring with spring constant 20, in a medium offering resistance numerically equal to half the velocity. If resistance is negligible, the last term is absent and the differential equation

$$\frac{10}{g}\frac{d^2x}{dt^2} = -20x \tag{2}$$

represents the special case of simple harmonic motion in a nonresisting medium.

These differential equations, of standard form

$$(D^2 + 2aD + b^2)x = 0, \qquad (a < b),$$

$$(D^2 + b^2)x = 0,$$

respectively, represent *free* or *natural vibrations*, that is, vibrations due to the inherent forces in the system. The system involved in (1) consists of the spring and the weight, together with the resisting medium in which they are immersed; in (2) the system is merely that of the spring and the weight.

Suppose now that there is applied to the weight, vibrating in accordance with equation (1), a periodic force $A \sin \omega t$, external to the system, of amplitude A and period $2\pi/\omega$. The differential equation of the motion will then be

$$\frac{10}{g}\frac{d^2x}{dt^2} = -20x - \frac{1}{2}\frac{dx}{dt} + A \sin \omega t,$$

of standard form

$$(\mathrm{D}^2 + 2a\mathrm{D} + b^2)x = C \sin \omega t, \qquad (a < b). \tag{3}$$

Neglecting the resistance due to the medium, the standard form of the differential equation is

$$(\mathrm{D}^2 + b^2)x = C \sin \omega t. \tag{4}$$

Equations (3) and (4) represent *forced vibrations* respectively with and without resistance proportional to velocity and due to the medium. The solutions of these equations consist of a complementary function x_c plus a particular integral x_p.

The form of the solution of equation (4) will depend on whether $\omega = b$ or $\omega \neq b$, that is, whether the period $2\pi/\omega$ of the impressed external force is or is not equal to the period $2\pi/b$ of the natural vibrations. If $\omega = b$, we have the case of *resonance*, where the vibrations get larger and larger. This condition would cause dangerous stresses in some vibration problems; on the other hand a resonant condition is desirable in certain acoustical and radio-circuit problems.

EXAMPLE 1. A 12-lb weight hangs at rest on a spring which is stretched 2 ft by the weight. The upper end of the spring is given the motion $y = \sin \sqrt{2g}\, t$, where y (ft) is the displacement, measured positive upward, of the upper end of the spring from its original position, at time t (sec). Find the equation of motion of the weight and the position of the weight $1.5\pi/\sqrt{2g}$ sec after the motion starts. Draw a figure representing the motion for the interval $0 \leqq \sqrt{g/2}\, t \leqq 2\pi$.

Since a force of 12 lb stretches the spring 2 ft, $12 = c \cdot 2$, and the spring constant is $c = 6$ lb/ft.

Take the origin at the equilibrium position of the weight, and let x (ft), measured positive upward, represent the displacement of the weight from its equilibrium position at time t (sec). The difference, $y - x$, represents the change in length of the spring caused by the displacement of its upper end. Combining this with the original stretch of 2 ft caused by the weight, we have $2 + y - x$ ft, the total stretch in the spring at time t, no matter which direction the weight or the upper end of the spring is moving, and no matter whether x or y is positive or negative. The spring pulls the weight upward with a force of $6(2 + y - x)$ lb while gravity pulls the weight downward with a force of -12 lb. The resultant force acting on the weight is therefore $6(2 + y - x) - 12 = 6(y - x)$ lb. Equating this to the expression for pounds force, we have the differential equation

$$\frac{12}{g}\frac{d^2x}{dt^2} = 6(y - x),$$

or, replacing y by $\sin\sqrt{2g}\,t$, multiplying by $g/12$, and using operator notation, we have

$$\left(D^2 + \frac{g}{2}\right)x = \frac{g}{2}\sin\sqrt{2g}\,t.$$

This equation is of standard form (4), with $\omega \neq b$. The complementary function is

$$x_c = C_1 \sin\sqrt{\frac{g}{2}}\,t + C_2 \cos\sqrt{\frac{g}{2}}\,t.$$

For the particular integral we have

$g/2)$ $$x_p = A\sin\sqrt{2g}\,t + B\cos\sqrt{2g}\,t$$

$1)$ $$D^2x_p = -2gA\sin\sqrt{2g}\,t - 2gB\cos\sqrt{2g}\,t$$

$$-\frac{3}{2}gA = \frac{g}{2} \qquad\Bigg|\qquad -\frac{3}{2}gB = 0$$

$$x_p = -\tfrac{1}{3}\sin\sqrt{2g}\,t \qquad A = -\tfrac{1}{3} \qquad\Bigg|\qquad B = 0$$

Hence the general solution is

$$x = C_1\sin\sqrt{\frac{g}{2}}\,t + C_2\cos\sqrt{\frac{g}{2}}\,t - \frac{1}{3}\sin\sqrt{2g}\,t.$$

The condition $x = 0$, $t = 0$, gives $C_2 = 0$. Then differentiation yields

$$\frac{dx}{dt} = \sqrt{\frac{g}{2}}\, C_1 \cos \sqrt{\frac{g}{2}}\, t - \frac{1}{3}\sqrt{2g} \cos \sqrt{2g}\, t,$$

and the condition $dx/dt = 0$, $t = 0$, gives

$$0 = \sqrt{\frac{g}{2}}\, C_1 - \frac{1}{3}\sqrt{2g}, \qquad C_1 = \frac{2}{3}.$$

The equation of motion is therefore

$$x = \frac{2}{3} \sin \sqrt{\frac{g}{2}}\, t - \frac{1}{3} \sin \sqrt{2g}\, t.$$

When $t = 1.5\pi/\sqrt{2g}$, we find

$$x = \frac{2}{3} \sin \frac{3\pi}{4} - \frac{1}{3} \sin \frac{3\pi}{2} = \frac{\sqrt{2}}{3} + \frac{1}{3} = \frac{2.4142}{3} = 0.805.$$

Hence, when $t = 1.5\pi/\sqrt{2g}$ sec, the weight is 0.805 ft above its original position of equilibrium.

In order to represent the motion graphically we can plot x and y against $\sqrt{g/2}\, t$ (instead of t) for convenience, since the motion of the upper end of the spring and of the weight are represented respectively by

$$y = \sin 2 \sqrt{\frac{g}{2}}\, t \tag{5}$$

and

$$x = \frac{2}{3} \sin \sqrt{\frac{g}{2}}\, t - \frac{1}{3} \sin 2 \sqrt{\frac{g}{2}}\, t. \tag{6}$$

Let O' (Fig. 18) denote the original position of the upper end of the spring, and draw the sine curve (5) representing the motion of the upper end of the spring. The upper end of the spring, of course, actually oscillates in the vertical line through O' with simple harmonic motion which is the projection of the motion of a point along the curve (5). Different scales are used horizontally and vertically.

Let A be the original position of the lower end of the spring. ($O'A$ is not drawn to scale—the length of the spring is not given.) When the weight is attached to the spring at A it will hang in equilibrium at O. With O as origin we draw the curve (6) representing the motion of the weight. This curve can best be drawn by compounding the two sine curves

$$x = \frac{2}{3}\sin\sqrt{\frac{g}{2}}\,t, \qquad x = \frac{1}{3}\sin 2\sqrt{\frac{g}{2}}\,t,$$

shown as broken lines.

The weight oscillates a distance of 0.866 ft above and below its equilibrium position (a fact to be established in Prob. 1 of the follow-

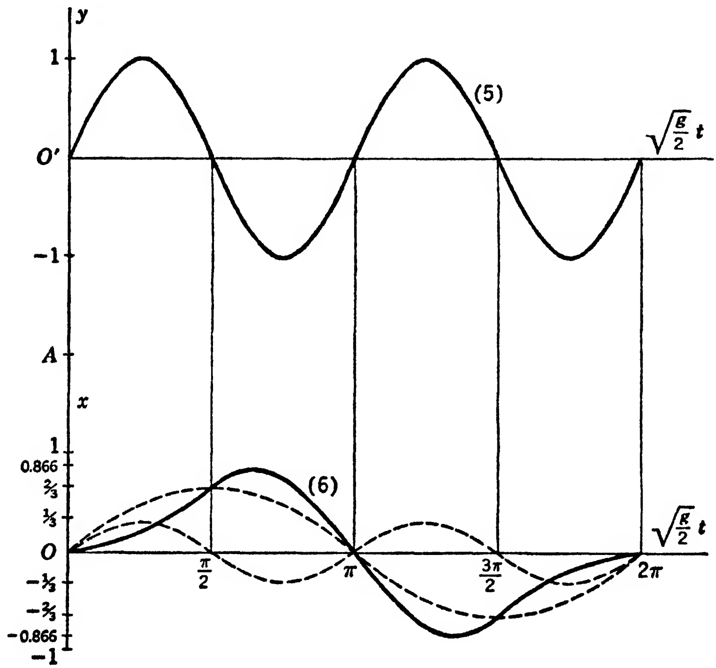

FIG. 18

ing list). The x, t curve repeats in intervals of $\sqrt{g/2}\,t = 2\pi$; that is, the motion of the weight is periodic, with period $2\pi/\sqrt{g/2} = 4\pi/\sqrt{2g}$ sec. But the motion of the weight is not simple harmonic. The x of equation (6) is a compound of two sine functions of different periods and hence is not expressible as a single sine function.

39. Electric circuits. In Art. 23 electric circuits leading to linear differential equations of first order were discussed. Using the same notation as in that article, the differential equation for a circuit containing resistance, inductance, and capacitance, is *

$$\left(LD^2 + RD + \frac{1}{C}\right) q = e, \qquad i = Dq. \tag{1}$$

Here $D = d/dt$, and, as in Art. 23, the capital letters L, R, and C represent constants of the circuit whereas the small letters are, in general, variables.

Example 1. An inductance of 1 henry and a capacitance of 10^{-6} farad are connected in series with an emf $e = 100 \sin 500t$ volts. If the charge and current are both initially zero, find (*a*) the charge and current at time t sec, (*b*) the value of the current when $t = 0.001$ sec, (*c*) the maximum magnitude of the current.

Substituting $L = 1$, $R = 0$, $C = 10^{-6}$ and $e = 100 \sin 500t$ in (1) we have the differential equation

$$(D^2 + 10^6)q = 100 \sin 500t.$$

(*a*) The solution—finding the complementary function and particular integral, and evaluating the constants of integration—proceeds as follows:

$$q_c = C_1 \sin 1000t + C_2 \cos 1000t$$

$$\begin{array}{lrcl} 10^6) & q_p & = & A \sin 500t + B \cos 500t \\ 1) & D^2 q_p & = & -500^2 A \sin 500t - 500^2 B \cos 500t \end{array}$$

$(10^6 - 500^2)A = 100$	$(10^6 - 500^2)B = 0$
$A = \frac{1}{7500}$	$B = 0$

$$q = q_c + q_p = C_1 \sin 1000t + C_2 \cos 1000t + \tfrac{1}{7500} \sin 500t$$

$$q = 0, \qquad t = 0; \qquad C_2 = 0$$

$$i = Dq = 1000C_1 \cos 1000t + \tfrac{1}{15} \cos 500t$$

$$i = 0, \qquad t = 0; \qquad 0 = 1000C_1 + \frac{1}{15}, \qquad C_1 = -\frac{1}{15{,}000}$$

$$q = \frac{1}{15{,}000}(2 \sin 500t - \sin 1000t).$$

$$i = \tfrac{1}{15}(\cos 500t - \cos 1000t).$$

* For references to the derivation of this equation see footnote, Art. 23.

(*b*) When $t = 0.001$ sec,

$$i\Big]_{t=0.001} = \tfrac{1}{15}(\cos 0.5 - \cos 1) = \tfrac{1}{15}(0.8776 - 0.5403)$$

$$= \frac{0.3373}{15} = 0.0225 \text{ amp.}$$

(*c*). The maximum (and the minimum) value of the current occurs when

$$Di = -\tfrac{500}{15}\sin 500t + \tfrac{1000}{15}\sin 1000t = 0,$$

or, if $500t = T$, when

$$2\sin 2T - \sin T = 4\sin T\cos T - \sin T = \sin T(4\cos T - 1) = 0.$$

For $\sin T = 0$:

$$T = 0, \qquad i = 0.$$

$$T = \pi, \qquad i = \tfrac{1}{15}(\cos\pi - \cos 2\pi) = -\tfrac{2}{15}.$$

For $\cos T = \frac{1}{4}$:

$$\cos 2T = 2\cos^2 T - 1 = \tfrac{1}{8} - 1 = -\tfrac{7}{8}; \qquad i = \tfrac{1}{15}(\tfrac{1}{4} + \tfrac{7}{8}) = \tfrac{3}{40}.$$

Hence the current varies between the extreme values of $-2/15$ and $3/40$ amp. Its maximum magnitude is $2/15$ amp.

PROBLEMS

1. In Ex. 1, Art. 38, find the amplitude of the oscillation of the weight.

2. In Ex. 1, Art. 38, find the smallest value that the tension in the spring can reach, and the time when this value first occurs.

3. Solve Ex. 1, Art. 38, if the upper end of the spring is given the motion $y = \sin\sqrt{g/2}\,t$.

4. A weight of w lb hangs on a spring. The upper end of the spring is given a simple harmonic motion $y = \sin\omega t$, where y (ft) is the displacement, measured positive upward, of the upper end of the spring from its original position, at time t (sec). If the period of the motion impressed on the upper end of the spring is $1/n$ the period of the natural vibration of the weight ($n \neq 1$), find the displacement of the weight after one period of the impressed motion.

5. A weight of w lb hangs on a spring, stretching it 18 in. The upper end of the spring is given simple harmonic motion $y = \sin\sqrt{6g}\,t$, where y (ft) is the displacement, measured positive upward, of the upper end of the spring from its original position, at time t (sec). (*a*) Find the location of the weight at the end of one period of the impressed motion. (*b*) Draw a figure similar to Fig. 18 representing the motion for the interval $0 \leq \sqrt{\frac{2}{3}g}\,t \leq 2\pi$, and show that the weight will oscillate a distance of $\frac{1}{2}$ ft above and below its original position of equilibrium with periodic but not simple harmonic motion. (*c*) Show that the maximum velocity is $\sqrt{2g}/3$ ft/sec, and find the time when it is first attained.

6. An 8-lb weight is hanging at rest on a spring which is stretched 2.25 ft by the weight. The upper end of the spring is given the motion $y = \sin 2\sqrt{g}\,t$, where y (ft) is the displacement, measured positive upward, of the upper end of the spring from its original position, at time t (sec). Find (*a*) the position of the weight $\pi/4\sqrt{g}$ sec after the motion starts; (*b*) the extreme positions attained by the weight.

7. If the upper end of the spring in Prob. 6 is given the motion $y = \sin \frac{2}{3}\sqrt{g}\,t$, find (*a*) the position of the weight when $t = \pi/2\sqrt{g}$ sec; (*b*) the time when the weight first passes through the equilibrium position.

8. A 10-lb weight is hanging at rest on a spring which is stretched 4 in. by the weight. The upper end of the spring is given the motion $y = \frac{1}{5} \sin \sqrt{2g}\,t$, where y (ft) is the displacement, measured positive upward, of the upper end of the spring from its original position, at time t (sec). Find the equation of motion of the weight, and the position of the weight when $t = \pi/\sqrt{3g}$ sec.

9. Solve Prob. 8 if the upper end of the spring is given the motion $y = \frac{1}{5} \sin \sqrt{3g}\,t$.

10. A 10-lb weight is hanging at rest on a spring which is stretched 4 in. by the weight. The upper end of the spring is given the motion $y = \frac{1}{5} \sin \sqrt{2g}\,t$, where y (ft) is the displacement, measured positive upward, of the upper end of the spring from its original position, at time t (sec). If the resistance (lb) of the medium in which the weight vibrates is equal to $20/\sqrt{g}$ times the speed (ft/sec), find the position of the weight when $t = \pi/\sqrt{2g}$ sec.

11. Draw a graph showing i from $t = 0$ to $t = \pi/250$ for Ex. 1, Art. 39.

12. In the circuit of Ex. 1, Art. 39, if $e = 100 \sin 1000t$ volts, find the current when $t = 0.01$ sec.

13. An inductance of 1 henry is connected in series with a capacitance of 10^{-4} farad and an emf $e = 100 \sin 50t$ volts. If $q = i = 0$ for $t = 0$, find (*a*) the current when $t = 0.02$ sec; (*b*) the maximum magnitude of the current; (*c*) the extreme values between which the current varies.

14. Solve Prob. 13 if $e = 100$ volts.

15. A circuit consists of an impedance coil of inductance L and negligible resistance connected in series with a condenser of capacitance C and an emf $e = 120 \sin 250t$ volts. Assume $i = 0$, $q = 0$, when $t = 0$. If $L = 2$ henries and $C = 2 \times 10^{-6}$ farad, find (*a*) the current when $t = 0.01$ sec; (*b*) the values between which the current varies.

16. A condenser of capacitance 2×10^{-4} farad, having a charge of 0.05

coulomb, is placed in series with a coil of inductance 2 henries and a resistance of 28 ohms, and discharges by sending a current through the circuit. The current being initially zero, find the current and the charge on the condenser when $t = 0.1$ sec.

17. An inductance of 1 henry, a resistance of 400 ohms, and a condenser of capacitance 1.6×10^{-5} farad are connected in series with an emf of $40 \cos 250t$ volts. If the charge and current are both zero when $t = 0$, find the current when $t = 10^{-3}$ sec.

18. An inductance of 1 henry, a resistance of 100 ohms, and a capacitance of 10^{-4} farad are connected in series with an emf of 100 volts. If no charge is present and no current is flowing at time $t = 0$, find the maximum value of the current.

19. An inductance of 2 henries, a resistance of 100 ohms, and a capacitance of 2×10^{-4} farad are connected in series with an emf $e = 100 \sin 50t$ volts. Initially the current is zero and the charge is 0.05 coulomb. Find the charge and current when $t = 0.02$ sec.

20. In a circuit containing resistance, inductance, and capacitance, a constant emf E is applied by closing a switch [Eq. (1), Art. 39, with $e = E$]. When $t = 0$ the current $i = 0$ and the charge on the condenser is $q = q_0$. Find the time after the switch is closed when the current is a maximum (*a*) if $4L > R^2C$; (*b*) if $4L = R^2C$; (*c*) if $4L < R^2C$. The result shows that the time of maximum current is independent of the applied voltage and of the initial charge on the condenser.

21. A particle slides freely in a tube which rotates in a vertical plane about its midpoint with constant angular velocity ω. If x is the distance of the particle from the midpoint of the tube at time t, and if the tube is horizontal when $t = 0$, show that the motion of the particle along the tube is given by

$$\frac{d^2x}{dt^2} - \omega^2 x = -g \sin \omega t.$$

Solve this equation if $x = x_0$, $dx/dt = v_0$, when $t = 0$. For what values of x_0 and v_0 is the motion simple harmonic?

22. (*a*) A cantilever beam of length L in. and weighing w lb/in. is subjected to a horizontal compressive force of P lb applied at the free end. Taking the origin at the free end and the y-axis positive upward, show that the differential equation of the elastic curve of the beam and the maximum deflection are, respectively,

$$\left(D^2 + \frac{P}{EI}\right) y = -\frac{wx^2}{2EI},$$

$$\text{Maximum deflection} = \frac{wEI}{P^2}\left(1 - \frac{\theta^2}{2} - \sec\theta + \theta\tan\theta\right), \qquad \theta = \sqrt{\frac{P}{EI}}\,L$$

(*b*) Find the maximum deflection of a wrought-iron cantilever beam 2 in. by 4 in. by 12 ft with $E = 15 \times 10^6$ lb/in.^{2} and weighing 490 lb/ft^3, if the 2-in. side is horizontal and the compressive force is $P = 6000$ lb.

23. (*a*) A cantilever beam of length L in. and weighing w lb/in. is subjected to a horizontal tensile force of P lb applied at the free end. Taking the origin at the free end and the y-axis positive upward, show that the differential equation of the elastic curve of the beam and the maximum deflection are, respectively,

$$\left(D^2 - \frac{P}{EI}\right) y = -\frac{wx^2}{2EI},$$

$$\text{Maximum deflection} = \frac{wEI}{P^2}\left(1 + \frac{\theta^2}{2} - \operatorname{sech}\theta - \theta\tanh\theta\right), \qquad \theta = \sqrt{\frac{P}{EI}}\,L.$$

(*b*) Find the maximum deflection of a wrought-iron cantilever beam 2 in. by 4 in. by 12 ft with $E = 15 \times 10^6$ lb/in.^{2} and weighing 490 lb/ft^3, if the 4-in. side is horizontal and the tensile force is $P = 6000$ lb.

24. If, in Probs. 22(*a*) and 23(*a*), a weight of W lb is added at the free end, and the weight of the beam may be neglected, show that the formulas for maximum deflection are, respectively,

$$\text{Compressive force } P: \quad \text{Maximum deflection} = \frac{WL}{P}\left(\frac{\tan\theta}{\theta} - 1\right),$$

$$\text{Tensile force } P: \quad \text{Maximum deflection} = \frac{WL}{P}\left(1 - \frac{\tanh\theta}{\theta}\right).$$

Chapter 5

SOME SPECIAL HIGHER ORDER EQUATIONS

40. Equations reducible to linear with constant coefficients. In this chapter a few of the common types of differential equations of order higher than the first will be discussed, beginning with a special kind of linear equation of nth order,

$$a_0x^n \frac{d^ny}{dx^n} + a_1x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}x \frac{dy}{dx} + a_ny = R. \quad (1)$$

Here the a's are constants, R is a function of x (or a constant), and n is an integer greater than 1; if n were equal to 1 the equation would be a special case of the linear equation of first order treated in Art. 21. It will be noticed that this equation differs from that studied in Chapter 4 by having variable coefficients; the exponent of x, however, in each coefficient must be the same as the order of the derivative which it multiplies. Such an equation can be solved by reducing it to the type treated in Chapter 4, that is, by reducing it to a linear equation with constant coefficients.

First write the equation in operator notation:

$$(a_0x^n\mathrm{D}^n + a_1x^{n-1}\mathrm{D}^{n-1} + \cdots + a_{n-2}x^2\mathrm{D}^2 + a_{n-1}x\mathrm{D} + a_n)y = R. \quad (2)$$

It will now be shown that a change of independent variable from x to z by means of the transformation $x = e^z$ will reduce equation (1) or (2) to a linear differential equation with constant coefficients. Use will be made of two operators—that of equation (2) indicating differentiation with respect to x, $\mathrm{D} = d/dx$, and a script $\mathscr{D}$ indicating differentiation with respect to z, $\mathscr{D} = d/dz$.

We shall express $x\mathrm{D}$, $x^2\mathrm{D}^2$, etc., in terms of $\mathscr{D}$.

$$x = e^z, \qquad \text{or} \qquad z = \ln x.$$

$$x\mathrm{D}y = x\frac{dy}{dx} = x\frac{dy}{dz}\frac{dz}{dx} = x\frac{dy}{dz}\frac{1}{x} = \mathscr{D}y.$$

Hence

$$x\mathrm{D}y = \mathscr{D}y, \tag{3}$$

and

$$x\mathrm{D} = \mathscr{D}. \tag{4}$$

Operating on equation (3) with D,

$$x\mathrm{D}^2y + \mathrm{D}y = \mathrm{D}\mathscr{D}y.$$

Transposing the term $\mathrm{D}y$, multiplying by x, and making use of (4),

$$x^2\mathrm{D}^2y = x\mathrm{D}(\mathscr{D}y - y) = \mathscr{D}(\mathscr{D} - 1)y.$$

Hence

$$x^2\mathrm{D}^2y = \mathscr{D}(\mathscr{D} - 1)y, \tag{5}$$

and

$$x^2\mathrm{D}^2 = \mathscr{D}(\mathscr{D} - 1). \tag{6}$$

Operating on equation (5) with D,

$$x^2\mathrm{D}^3y + 2x\mathrm{D}^2y = \mathrm{D}\mathscr{D}(\mathscr{D} - 1)y.$$

Transposing the term $2x\mathrm{D}^2y$, multiplying by x, and making use of (4) and (6),

$$x^3\mathrm{D}^3y = x\mathrm{D}\mathscr{D}(\mathscr{D} - 1)y - 2x^2\mathrm{D}^2y =$$
$$[\mathscr{D}^2(\mathscr{D} - 1) - 2\mathscr{D}(\mathscr{D} - 1)]y = \mathscr{D}(\mathscr{D} - 1)(\mathscr{D} - 2)y.$$

Hence

$$x^3\mathrm{D}^3y = \mathscr{D}(\mathscr{D} - 1)(\mathscr{D} - 2)y, \tag{7}$$

and

$$x^3\mathrm{D}^3 = \mathscr{D}(\mathscr{D} - 1)(\mathscr{D} - 2). \tag{8}$$

Looking at (4), (6), and (8), we have a strong suspicion that

$$x^n\mathrm{D}^n = \mathscr{D}(\mathscr{D} - 1)(\mathscr{D} - 2)\cdots(\mathscr{D} - n + 1). \tag{9}$$

This formula may be established by mathematical induction.

The solution of equation (1) or (2) now proceeds as follows:

$$(a_0x^n\mathrm{D}^n + a_1x^{n-1}\mathrm{D}^{n-1} + \cdots + a_{n-2}x^2\mathrm{D}^2 + a_{n-1}x\mathrm{D} + a_n)y = R.$$

On the left, substitute	On the right, substitute
$xD = \mathcal{D}$	$x = e^z$
$x^2D^2 = \mathcal{D}(\mathcal{D} - 1)$	$\ln x = z$
$x^3D^3 = \mathcal{D}(\mathcal{D} - 1)(\mathcal{D} - 2)$	
. . . .	

Solve the resulting equation, by the method of Chapter 4, for y in terms of z. Then substitute $z = \ln x$, or $e^z = x$, to obtain y in terms of x.

Example 1. Solve

$$x^3 \frac{d^3y}{dx^3} + x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} = 3x^2 + \ln x + 2.$$

We write the steps in the solution, following the preceding theory without further explanation.

$$(x^3D^3 + x^2D^2 - 4xD)y = 3x^2 + \ln x + 2$$

$$[\mathcal{D}(\mathcal{D} - 1)(\mathcal{D} - 2) + \mathcal{D}(\mathcal{D} - 1) - 4\mathcal{D}]y = 3e^{2z} + z + 2$$

$$(\mathcal{D}^3 - 2\mathcal{D}^2 - 3\mathcal{D})y = \mathcal{D}(\mathcal{D} + 1)(\mathcal{D} - 3)y = 3e^{2z} + z + 2$$

$$\begin{array}{rrlll} & y_c = & c_1 & + c_2e^{-z} + c_3e^{3z} & \\ & y_p = & Ae^{2z} + Bz^2 & + Cz & \\ -3) & \mathcal{D}y_p = & 2Ae^{2z} & + 2Bz & + C \\ -2) & \mathcal{D}^2y_p = & 4Ae^{2z} & & + 2B \\ 1) & \mathcal{D}^3y_p = & 8Ae^{2z} & & \end{array}$$

$8A - 8A - 6A = 3$	$-6B = 1$	$-4B - 3C = 2$
$A = -\frac{1}{2}$	$B = -\frac{1}{6}$	$C = -\frac{4}{9}$

$$y_p = -\tfrac{1}{2}e^{2z} - \tfrac{1}{6}z^2 - \tfrac{4}{9}z$$

$$y = c_1 + c_2e^{-z} + c_3e^{3z} - \tfrac{1}{2}e^{2z} - \tfrac{1}{6}z^2 - \tfrac{4}{9}z$$

$$y = c_1 + \frac{c_2}{x} + c_3x^3 - \tfrac{1}{2}x^2 - \tfrac{1}{6}\ln^2 x - \tfrac{4}{9}\ln x$$

A generalization of equation (1) is obtained by replacing the x's in the coefficients of the derivatives by $a + bx$. The transformation $a + bx = e^z$ then reduces the equation to an equation with constant coefficients. If the preceding theory is worked out for this transformation, we have the following process for solving the differential equation.

$$[a_0(a + bx)^n\mathrm{D}^n + \cdots + a_{n-2}(a + bx)^2\mathrm{D}^2 + a_{n-1}(a + bx)\mathrm{D} + a_n]y = R.$$

On the left, substitute	On the right, substitute
$(a + bx)\mathrm{D} = b\mathscr{D}$	$a + bx = e^z$
$(a + bx)^2\mathrm{D}^2 = b^2\mathscr{D}(\mathscr{D} - 1)$	$x = \dfrac{e^z - a}{b}$
$(a + bx)^3\mathrm{D}^3 = b^3\mathscr{D}(\mathscr{D} - 1)(\mathscr{D} - 2)$	$\ln(a + bx) = z$
. . . .	

Solve the resulting equation for y in terms of z, then substitute $z = \ln(a + bx)$ to obtain y in terms of x.

Example 2. Solve

$$[(3x + 1)^2\mathrm{D}^2 + (3x + 1)\mathrm{D} - 3]y = 9x.$$

Solution:

$$[9\mathscr{D}(\mathscr{D} - 1) + 3\mathscr{D} - 3]y = 3(e^z - 1)$$

$$(3\mathscr{D}^2 - 2\mathscr{D} - 1)y = (\mathscr{D} - 1)(3\mathscr{D} + 1)y = e^z - 1$$

$$y_c = C_1e^z + C_2e^{-z/3}$$

$$\begin{array}{ll} -1) & y_p = Aze^z \qquad\qquad + B \\ -2) & \mathscr{D}y_p = Aze^z + Ae^z \\ 3) & \mathscr{D}^2y_p = Aze^z + 2Ae^z \end{array}$$

$y_p = \frac{1}{4}ze^z + 1$	$6A - 2A = 1$	$-B = -1$
	$A = \frac{1}{4}$	$B = 1$

$$y = C_1e^z + C_2e^{-z/3} + \tfrac{1}{4}ze^z + 1$$

$$y = C_1(3x + 1) + C_2(3x + 1)^{-1/3} + \tfrac{1}{4}(3x + 1)\ln(3x + 1) + 1$$

PROBLEMS

1. If $\mathrm{D} = d/dx$, $\mathscr{D} = d/dz$, $x = e^z$, prove by mathematical induction that

$$x^n\mathrm{D}^n = \mathscr{D}(\mathscr{D} - 1)(\mathscr{D} - 2) \cdots (\mathscr{D} - n + 1).$$

2. If $\mathrm{D} = d/dx$, $\mathscr{D} = d/dz$, $a + bx = e^z$, show that

$$(a + bx)\mathrm{D} = b\mathscr{D}, \qquad (a + bx)^2\mathrm{D}^2 = b^2\mathscr{D}(\mathscr{D} - 1).$$

Solve the following differential equations.

✓**3.** $x^3 \dfrac{d^3y}{dx^3} - 2x\dfrac{dy}{dx} + 2y = 4x^3$. ✓ **4.** $x^2\dfrac{d^2y}{dx^2} + 2y = x\dfrac{dy}{dx} + x \ln x$. ✓

5. $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 4x^2 + \sin \ln x$.

6. $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 3y = 8x^3 \ln x + 6$.

✓**7.** $(x^3\mathrm{D}^3 + 6x^2\mathrm{D}^2 + 8x\mathrm{D} + 2)y = x^2 + 3x - 4$. ✓

✓**8.** $(x^4\mathrm{D}^4 - 11x^2\mathrm{D}^2)y = x + \ln x$.

9. $x^4\dfrac{d^3y}{dx^3} + 2x^3\dfrac{d^2y}{dx^2} + 2xy - 10 = 0$.

10. $(x^3\mathrm{D}^4 + 6\mathrm{D})y = 11\dfrac{\ln x}{x}$.

11. $x^2\dfrac{d^3y}{dx^3} + \dfrac{dy}{dx} = 7\ln^5 x + \dfrac{y}{x}$. **12.** $4x^2\dfrac{d^2y}{dx^2} + y = 3\sqrt{x} - 2x^3$.

13. $(4x^3\mathrm{D}^3 - 8x^2\mathrm{D}^2 - x\mathrm{D} + 1)y = x + \ln x + \ln^2 x$.

14. $(x^2\mathrm{D}^2 + 4x\mathrm{D} + 2)y = x + \sin x$. (*Cf.* Prob. 11, Art. 37.)

15. $(2x - 3)^2\dfrac{d^2y}{dx^2} = 2x - y$.

16. $(1 + 3x)^3\dfrac{d^3y}{dx^3} - 18(1 + 3x)\dfrac{dy}{dx} = 81(1 + x)$. ✓

17. $[(2x - 1)^3\mathrm{D}^3 + (2x - 1)\mathrm{D} - 2]y = 4x$.

18. Find the solution of

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 3x - x^2$$

if $y = 3$ and $dy/dx = 2$ when $x = 1$.

19. Find the equation of a curve which satisfies the differential equation

$$(4x^2\mathrm{D}^2 + 4x\mathrm{D} - 1)y = 0$$

and crosses the x-axis at an angle of 45° at $x = 1$.

20. Solve

$$\left(D^3 + \frac{1}{x} D^2\right) y = \frac{2y}{x^3}$$

if $y = 2$, $Dy = D^2y = -1$, when $x = 1$.

21. A steam pipe has inner and outer radii r_1 and r_2 respectively, and the temperatures at its inner and outer surfaces are, respectively, u_1 and u_2. Show that the temperature u at radial distance $r(r_1 < r < r_2)$ is given by the differential equation

$$r\frac{d^2u}{dr^2} + \frac{du}{dr} = 0.$$

Solve this equation under the conditions $u = u_1$ when $r = r_1$ and $u = u_2$ when $r = r_2$. (*Cf.* Prob. 27, Art. 17.)

22. A hollow spherical shell has inner and outer radii r_1 and r_2 respectively, and the temperatures at its inner and outer surfaces are, respectively, u_1 and u_2. Show that the temperature u at radial distance $r(r_1 < r < r_2)$ is given by the differential equation

$$r\frac{d^2u}{dr^2} + 2\frac{du}{dr} = 0.$$

Solve this equation under the conditions $u = u_1$ when $r = r_1$ and $u = u_2$ when $r = r_2$. (*Cf.* Prob. 28, Art. 17.)

41. Dependent variable absent. Consider a differential equation involving an independent variable x and a dependent variable y, but one in which y is absent except in the derivatives. If in such an equation we substitute for dy/dx a single letter p,* then substitute for the higher derivatives the corresponding expressions in terms of p and x, the result is an equation of lower order in p and x. If the latter equation can be solved in terms of p and x, we then substitute dy/dx for p and try to solve the resulting equation for the relation between y and x. The substitutions are

$$\frac{dy}{dx} = p, \qquad \frac{d^2y}{dx^2} = \frac{dp}{dx}, \qquad \frac{d^3y}{dx^3} = \frac{d^2p}{dx^2}, \qquad \text{etc.}$$

* If dy/dx also is absent, but two or more higher derivatives are present, we set the derivative of lowest order present equal to p and make corresponding substitutions for the higher derivatives.

EXAMPLE 1. Solve

$$x\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = x^3 + x.$$

Solution:

$$x\frac{dp}{dx} - 2p = x^3 + x,$$

$$\frac{dp}{dx} - \frac{2}{x}p = x^2 + 1.$$

The integrating factor of this equation of first order, linear in p and dp/dx, is $e^{\int(-2/x)\,dx} = e^{-2\ln x} = 1/x^2$. Hence

$$\frac{1}{x^2}\cdot p = \int \frac{1}{x^2}(x^2+1)\,dx = x - \frac{1}{x} + C_1,$$

$$\frac{dy}{dx} = x^3 - x + C_1x^2,$$

$$y = \frac{x^4}{4} - \frac{x^2}{2} + C_1'x^3 + C_2.$$

42. Independent variable absent. If, in a differential equation involving the independent variable x and the dependent variable y, x is absent except in the derivatives, we substitute for dy/dx a single letter p,* then substitute for the higher derivatives the corresponding expressions in terms of p and y. The result is an equation of lower order in p and y; if it can be solved in terms of p and y, we then substitute dy/dx for p and try to solve the resulting equation for the relation between y and x. The substitutions are

$$\frac{dy}{dx} = p, \qquad \frac{d^2y}{dx^2} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy},$$

$$\frac{d^3y}{dx^3} = \frac{d}{dy}\left(p\frac{dp}{dy}\right)\frac{dy}{dx} = \left[p\frac{d^2p}{dy^2} + \left(\frac{dp}{dy}\right)^2\right]p = p^2\frac{d^2p}{dy^2} + p\left(\frac{dp}{dy}\right)^2,$$

etc.

* See footnote, Art. 41.

If the dependent and independent variables are both absent, as in Problem 12 of the following group, then either the method of Art. 41 or that of Art. 42 may be used, but the former is usually simpler.

EXAMPLE 1. Solve

$$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx},$$

and find the value of y when $x = -\pi/12$ under (*a*) the conditions $y = 1$, $dy/dx = 2$ when $x = 0$; (*b*) the conditions $y = 0$, $dy/dx = -1$ when $x = 0$.

Substituting p for dy/dx and $p\,dp/dy$ for d^2y/dx^2 in the differential equation

$$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}, \tag{1}$$

it becomes

$$p\frac{dp}{dy} = 2yp.$$

Dividing by p [$p = 0$ would give the trivial solution, $y = C$, of equation (1) which would not satisfy the given conditions],

$$dp = 2y\,dy.$$

Integrating,

$$p = y^2 + C_1. \tag{2}$$

In problems of this kind the constants of integration should be evaluated as soon as they appear.

(*a*) Under the first set of conditions we have, from equation (2), $2 = 1 + C_1$, $C_1 = 1$, so that

$$\frac{dy}{y^2 + 1} = dx.$$

The next integral is

$$\tan^{-1} y = x + C_2,$$

and, under the given conditions, $\tan^{-1} 1 = 0 + C_2$, $C_2 = \pi/4$; hence

$$\tan^{-1} y = x + \frac{\pi}{4},$$

$$y = \tan\left(x + \frac{\pi}{4}\right),$$

which is the solution of equation (1) satisfying the first set of conditions. For the value of y when $x = -\pi/12$, we have

$$y\Big]_{x=-\pi/12} = \tan\frac{\pi}{6} = 0.577.$$

(*b*) Under the second set of conditions we have from equation (2), $-1 = 0 + C_1$, $C_1 = -1$, so that

$$\frac{dy}{y^2 - 1} = dx.$$

The next integral is

$$-\tanh^{-1} y = x + C_2,$$

and, under the given conditions, $-\tanh^{-1} 0 = 0 + C_2$, $C_2 = 0$; hence

$$\tanh^{-1} y = -x,$$

$$y = -\tanh x,$$

which is the solution of equation (1) satisfying the second set of conditions. For the value of y when $x = -\pi/12$, we have

$$y\Big]_{x=-\pi/12} = \tanh\frac{\pi}{12} = \tanh 0.2618 = 0.256.$$

PROBLEMS

Solve the following differential equations.

1. $x\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx} - x\left(\dfrac{dy}{dx}\right)^2.$
2. $xy'' + (x^2 - 1)y' + 1 = x^2.$
3. $xy'' = y' + y'^3$ (*cf.* Prob. 1, Art. 10).
4. $x^2y'' + y'^2 = 3xy'.$
5. $x\dfrac{d^4y}{dx^4} + \dfrac{d^3y}{dx^3} = 6.$
6. $(y + 1)\dfrac{d^2y}{dx^2} = 2\left(\dfrac{dy}{dx}\right)^2.$
7. $2yy'' - y'^2 = 0.$
8. $2yy'' + y'^2 = 0.$
9. $2yy'' - y'^2 = 4y^2.$
10. $3\left(\dfrac{d^2y}{dx^2}\right)^2 - 2\dfrac{dy}{dx}\dfrac{d^3y}{dx^3} = 0.$

11. Show that the differential equation of Prob. 7 can be changed into the differential equation of Prob. 8 by changing y to y^3.

12. Solve $y'' + y'^2 + 4 = 0$ by (*a*) the method of Art. 41; (*b*) the method of Art. 42.

13. Solve Prob. 18 of Art. 40 by the method of Art. 41.

14. Find a curve which satisfies the differential equation

$$yy'' = 2y'^2 + y^2$$

and has slope $\sqrt{3}$ at the point (0, 1).

15. Find a curve having slope 1 at the origin and satisfying the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{x}{(1 + x)^2}.$$

16. A curve is tangent to the x-axis at the origin and satisfies the differential equation

$$y'' = \sec y'.$$

Find the area between the curve and the x-axis.

Find a solution of each of the following differential equations satisfying the given conditions.

17. $\dfrac{d^2y}{dx^2} = \dfrac{3}{2}y^2$ $\quad\left(\dfrac{dy}{dx} = 1, y = 1, x = 1\right)$.

18. $(1 - x^2)y'' = xy'$ $\quad(y' = 2, y = 1, x = 0)$.

19. $2\dfrac{d^2y}{dx^2} = e^y$ $\quad(y' = -1, y = 0, x = 0)$.

20. $(1 - x)(y'' - y') + y' = 0$ $\quad(y' = -1, y = 0, x = 0)$.

21. $\dfrac{d^2y}{dx^2} = \sec^2 y \tan y$ $\quad\left(\dfrac{dy}{dx} = -1, x = \ln 2, y = \dfrac{\pi}{4}\right)$.

22. $\cos^3 y \dfrac{d^2y}{dx^2} = \sin y$ $\quad\left(\dfrac{dy}{dx} = \sqrt{2}, y = 0, x = 0\right)$.

23. $(1 - e^x)y'' = e^x y'$ $\quad(y' = x = 1, y = 0)$.

24. $y'' + y'^2 = \dfrac{yy'}{(y + 1)^2}$ $\quad(y' = 1, x = \frac{1}{2}, y = 0)$.

Find the particular value of the variable specified for each of the following differential equations with given conditions.

25. $\dfrac{d^2y}{dx^2} = 2y\dfrac{dy}{dx}$ $\quad\left(y = 1, \dfrac{dy}{dx} = 2, x = \dfrac{3\pi}{4}\right)$. Find $y]_{x=\pi/6}$.

26. $3\dfrac{dy}{dx}\dfrac{d^2y}{dx^2} = 2y$ $\quad\left(y = 8, \dfrac{dy}{dx} = 4, x = 1\right)$. Find $y]_{x=4}$.

27. $\dfrac{d^2y}{dx^2} = e^{2y}$ $\quad\left(y = 0, \dfrac{dy}{dx} = -1, x = 0\right)$. Find $y]_{x=\pi}$.

28. $\dfrac{d^2y}{dx^2} = \dfrac{1}{4\sqrt{y}}$ $\quad\left(\dfrac{dy}{dx} = 1, x = -\frac{8}{3}, y = 0\right)$. Find $x]_{y=9}$.

29. $y^2 \dfrac{d^2y}{dx^2} = 2\dfrac{dy}{dx}$ $\left(y = 2, \dfrac{dy}{dx} = 1, x = 1\right)$. Find $y]_{x=2}$.

30. $y'' = y'^2 \tanh y$ $(y' = 1, y = 0, x = 1)$. Find $y]_{x=0}$.

31. Given $y'' + yy' = 0$, with $y = -1$ and $y' = \frac{3}{2}$ when $x = 0$; find the values of y and y' when $x = -1$.

32. (*a*) Find the equation of a curve which is a solution of the differential equation

$$y'' = y' \sinh y$$

and passes through the origin with slope 1.

(*b*) Find the equation of a curve which is a solution of the differential equation

$$y'' = y' \tan x$$

and passes through the origin with slope 1.

Show that the curves of (*a*) and (*b*) are the same, namely,

$$\sinh y = \tan x.$$

33. Show that there are two curves satisfying the differential equation

$$y''^2 + y'^2 = 1$$

and conditions (*a*) $y' = \frac{1}{2}$, $y = \sqrt{3}/2$, $x = \pi/2$, but that there is only one curve satisfying the differential equation and conditions (*b*) $y' = y = 1$, $x = \pi/2$. Find the equations of the curves.

In the two following problems the differential equation is [*cf.* Prob. 20, Art. 32(*b*)]

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

34. At what angle must a pendulum be started from rest in order to attain half its maximum angular velocity when it is in a horizontal position?

35. If a pendulum is started from rest at an angle of 120° from its lowest position, at what angle will it attain $\frac{1}{3}$ its maximum velocity?

43. The inverse square law. If a particle moves from rest under the action of a force varying inversely as the square of the distance x from a point O, the differential equation of motion, taking the x-axis as the line of motion, is

$$\frac{d^2x}{dt^2} = \mp\frac{k^2}{x^2}, \tag{1}$$

the negative or positive sign being used according as the force is attractive or repulsive.

To solve these equations we use the method of Art. 42, letting $dx/dt = v$ (velocity) and $d^2x/dt^2 = v\,dv/dx$; then equations (1) become

$$v\,dv = \mp \frac{k^2\,dx}{x^2}. \tag{2}$$

Assuming that the particle starts from rest at $x = a$, the initial conditions are $v = 0$, $x = a$, when $t = 0$. We continue the solution of equations (2) in parallel columns:

For an attractive force

$$\frac{v^2}{2} = \frac{k^2}{x} + C_1$$

$$v = 0,\ x = a;\ C_1 = -\frac{k^2}{a}$$

$$v = -\sqrt{2}k\sqrt{\frac{1}{x} - \frac{1}{a}}$$

$$\frac{dx}{dt} = -k\sqrt{\frac{2}{a}}\sqrt{\frac{a-x}{x}}$$

$$t = -\frac{1}{k}\sqrt{\frac{a}{2}}\int\frac{\sqrt{x}\,dx}{\sqrt{a-x}}$$

Let $x = a\cos^2\theta$

$$\sqrt{a-x} = \sqrt{a}\sin\theta$$

$$dx = -2a\cos\theta\sin\theta\,d\theta$$

$$t = \frac{1}{k}\sqrt{\frac{a}{2}}\int 2a\cos^2\theta\,d\theta$$

$$t = \frac{a}{k}\sqrt{\frac{a}{2}}(\cos\theta\sin\theta + \theta) + C_2$$

$$t = 0,\ x = a,\ \theta = 0;\ C_2 = 0$$

$$t = \frac{a}{k}\sqrt{\frac{a}{2}}\left(\sqrt{\frac{x}{a}}\sqrt{1-\frac{x}{a}} + \cos^{-1}\sqrt{\frac{x}{a}}\right)$$

$$t = \frac{1}{k}\sqrt{\frac{a}{2}}\left(\sqrt{x(a-x)} + a\cos^{-1}\sqrt{\frac{x}{a}}\right)$$

For a repulsive force

$$\frac{v^2}{2} = -\frac{k^2}{x} + C_1$$

$$v = 0,\ x = a;\ C_1 = \frac{k^2}{a}$$

$$v = \sqrt{2}k\sqrt{\frac{1}{a} - \frac{1}{x}} \tag{3}$$

$$\frac{dx}{dt} = k\sqrt{\frac{2}{a}}\sqrt{\frac{x-a}{x}}$$

$$t = \frac{1}{k}\sqrt{\frac{a}{2}}\int\frac{\sqrt{x}\,dx}{\sqrt{x-a}} \tag{4}$$

Let $x = a\cosh^2\theta$

$$\sqrt{x-a} = \sqrt{a}\sinh\theta$$

$$dx = 2a\cosh\theta\sinh\theta\,d\theta$$

$$t = \frac{1}{k}\sqrt{\frac{a}{2}}\int 2a\cosh^2\theta\,d\theta$$

$$t = \frac{a}{k}\sqrt{\frac{a}{2}}(\cosh\theta\sinh\theta + \theta) + C_2$$

$$t = 0,\ x = a,\ \theta = 0;\ C_2 = 0$$

$$t = \frac{a}{k}\sqrt{\frac{a}{2}}\left(\sqrt{\frac{x}{a}}\sqrt{\frac{x}{a}-1} + \cosh^{-1}\sqrt{\frac{x}{a}}\right)$$

$$t = \frac{1}{k}\sqrt{\frac{a}{2}}\left(\sqrt{x(x-a)} + a\cosh^{-1}\sqrt{\frac{x}{a}}\right) \tag{5}$$

The v, x and t, x relations, equations (3) and (5) respectively, give the velocity and the time when the particle is at distance x from O.

If the force of attraction is that of the earth's gravitation, then, in equation (1), when $x = R$ (the radius of the earth), the acceleration due to gravity at the earth's surface is $d^2x/dt^2 = -g = -32.17$ ft/sec², so that $-g = -k^2/R^2$ and $k = R\sqrt{g}$.

44. The suspended cable. We shall now find the equation of the curve in which a uniform flexible cable of weight w (lb/ft) will hang if suspended from two points (Fig. 19). Let A be the lowest point of the cable, and consider a portion AP of length s (ft) and weight ws (lb). The three forces, measured in pounds, under which this portion of the cable is in equilibrium are T, the tension acting tangentially at P at an angle θ with the horizontal; H, the horizontal tension at A; and ws, the weight. Then, resolving vertically and horizontally,

$$T \sin \theta = ws, \tag{1}$$

$$T \cos \theta = H. \tag{2}$$

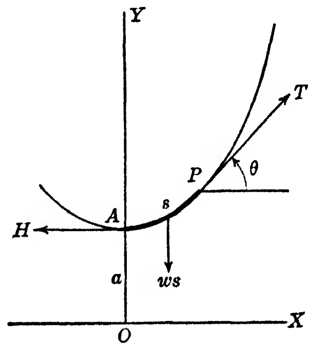

FIG. 19

Take the y-axis vertical through A and the x-axis horizontal at a distance a (ft) below A; a value for a will be chosen later which will make the equation of the curve come out in simplest form. Dividing (1) by (2) and writing dy/dx for $\tan \theta$, we have the differential equation of the curve,

$$\frac{dy}{dx} = \frac{ws}{H}. \tag{3}$$

Equation (3) contains three variables x, y, s. The s may be eliminated by differentiating with respect to x and replacing ds/dx by $\sqrt{1 + (dy/dx)^2}$:

$$\frac{d^2y}{dx^2} = \frac{w}{H}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{4}$$

Equation (4) can be solved by the method of Art. 41, letting $dy/dx = p$, $d^2y/dx^2 = dp/dx$; then

$$\frac{dp}{\sqrt{1+p^2}} = \frac{w}{H}\,dx.$$

Integrating,

$$\sinh^{-1} p = \frac{w}{H}x + C_1.$$

Since $p = dy/dx = 0$ when $x = 0$, $C_1 = 0$ and

$$\frac{dy}{dx} = \sinh\frac{w}{H}x.$$

Integrating again,

$$y = \frac{H}{w}\cosh\frac{w}{H}x + C_2.$$

But $y = a$ when $x = 0$. In order to make $C_2 = 0$, we choose $a = H/w$, and the equation of the curve is

$$y = a\cosh\frac{x}{a}, \tag{5}$$

the standard equation of a *catenary;* x and y are the coordinates of any point P, referred to the origin O which is at a distance $a = H/w$ ft below A.

The length of the arc AP is

$$s = \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^x \sqrt{1+\sinh^2\frac{x}{a}}\,dx$$

$$= \int_0^x \cosh\frac{x}{a}\,dx = a\sinh\frac{x}{a}. \tag{6}$$

Suppose (Fig. 20) that a cable of length S ft dips d ft when suspended from two points at the same level and L ft apart. We shall find relations connecting a with d and L, with S and L, and with S and d.

From equation (5),

$$d = y]_{x=L/2} - a = a\left(\cosh\frac{L}{2a} - 1\right),$$

a relation connecting a, d, and L.

From equation (6),

$$S = 2s]_{x=L/2} = 2a \sinh \frac{L}{2a},$$

a relation connecting a, S, and L.

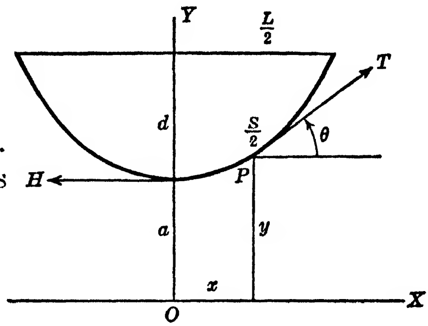

FIG. 20

From the last two relations we get

$$\cosh \frac{L}{2a} = \frac{d}{a} + 1,$$

$$\sinh \frac{L}{2a} = \frac{S}{2a},$$

$$\cosh^2 \frac{L}{2a} - \sinh^2 \frac{L}{2a} = 1 = \frac{d^2}{a^2} + \frac{2d}{a} + 1 - \frac{S^2}{4a^2},$$

$$4d^2 + 8ad - S^2 = 0;$$

hence

$$a = \frac{S^2}{8d} - \frac{d}{2},$$

a relation connecting a, S, and d.

The tension at P is

$$T = H \sec \theta = H \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = H \cosh \frac{x}{a} = wy,$$

and T_0, the tension at a point of support, is

$$T_0 = w(a + d).$$

Collecting the above results, we have the following working formulas:

$$y = a \cosh \frac{x}{a}, \qquad a = \frac{H}{w}, \tag{7}$$

$$d = a\left(\cosh \frac{L}{2a} - 1\right), \tag{8}$$

$$S = 2a \sinh \frac{L}{2a}, \tag{9}$$

$$a = \frac{S^2}{8d} - \frac{d}{2}, \tag{10}$$

$$T = wy, \qquad T_0 = w(a + d). \tag{11}$$

If d and L are given, we can find a from equation (8) by trial and error; then S can be found from equation (9). If S and L are given, we can find a from equation (9) by trial and error; then d can be found from equation (8). If S and d are given, equation (10) gives a; then L can be found from equation (9).

Example 1. A wire, fastened at the same level to two poles 120 ft apart, dips 30 ft. Find (a) the length of the wire; (b) the tension at the lowest point and at a point of support if the weight of the wire is 0.10 lb/ft.

Substituting $d = 30$, $L = 120$, in equation (8),

$$30 = a\left(\cosh\frac{60}{a} - 1\right),$$

from which the value of a is to be found. It is simpler to let $60/a = \lambda$, then solve for λ the equation

$$\frac{\lambda}{2} + 1 = \cosh\lambda.$$

Using the method of trial and error, we find, from Peirce's "Tables,"

λ	$\frac{\lambda}{2} + 1$	$\cosh\lambda$
1.0	1.500	1.543
0.9	1.450	1.433
0.930	1.465	1.465

Hence $\lambda = 0.930$, $a = 60/0.930$.

(a) Substituting in equation (9), the length of the wire is

$$S = \frac{120}{0.930}\sinh 0.930 = \frac{128.4}{0.930} = 138 \text{ ft.}$$

(*b*) The tension at the lowest point is, from (7),

$$H = aw = 6/0.930 = 6.45 \text{ lb.}$$

The tension at a point of support is, from (11),

$$T_0 = wa + wd = 6.45 + 3 = 9.45 \text{ lb.}$$

PROBLEMS

1. Evaluate the integrals in equations (4) of Art. 43 by use of the substitution $x = az^2$ and formulas 150, 151 of Peirce's "Tables."

2. In the motion described in Art. 43, assuming that k is the same for both attraction and repulsion, i.e., that the two forces have the same magnitude at a given distance from O, find the ratio of the times required to travel the first $a/2$ units of distance in the two respective cases.

3. Work Prob. 2 if the first particle travels from $x = a$ to $x = a/2$ and the second travels from $x = a/2$ to $x = a$, each starting from rest.

4. In the motion discussed in Art. 43, assuming that $k = 8 \text{ ft}^{3/2}/\text{sec}$ and $a = 8$ ft, find the distance traveled during the first second in the two respective cases.

5. A particle is repelled from O by a force obeying the inverse square law. If the particle starts 1 ft from O with an acceleration of 2 ft/sec^2, find the time required to travel 3 ft.

6. Find the number of hours required for a particle to fall from the distance of the moon to the surface of the earth. Assume that the radius of the earth is $R = 4000$ miles and that the distance from the center of the earth to the moon is $60R = 240{,}000$ miles.

7. Suppose that the earth's attractive force should suddenly become repulsive, remaining numerically the same. How many hours would be required for a particle to shoot out from the earth's surface to the distance of the moon? Assume distances as in Prob. 6.

8. Find the velocity with which a particle would strike the earth's surface if it started from rest at a very great (practically infinite) distance and moved subject only to the earth's attraction. Take R (radius of earth) = 3960 miles. (This is the "velocity of escape," i.e., the velocity with which a particle would have to be shot from the surface of the earth in order not to return.)

9. In Prob. 8 find the time consumed by the particle in traveling the last R miles.

10. Suppose that a particle, starting from rest at a distance $R = 3960$ miles from the surface of the earth, falls into a straight tube bored through the center of the earth. If the only force acting is that of the earth's gravitation, find the time required to reach the center. [*Cf.* Prob. 15, Art. 32(*b*).]

11. Solve Probs. 2, 6, 7, and 8 under the assumption that the earth attracts according to the inverse cube law, but that the acceleration due to gravity at the surface of the earth remains numerically the same (32.17 ft/sec²).

12. A particle moves under the action of a force which varies inversely as the nth power of the distance from a point O. If x is the distance from O at time t, and if the particle starts from rest at $x = a$, show that the t, x relations are

For an attractive force

$$t = \frac{1}{k}\sqrt{\frac{1-n}{2}}\int_x^a \frac{dx}{\sqrt{a^{1-n} - x^{1-n}}} \quad (n < 1)$$

$$t = \frac{1}{k}\sqrt{\frac{n-1}{2}}\int_x^a \frac{dx}{\sqrt{x^{1-n} - a^{1-n}}} \quad (n > 1)$$

For a repulsive force

$$t = \frac{1}{k}\sqrt{\frac{1-n}{2}}\int_a^x \frac{dx}{\sqrt{x^{1-n} - a^{1-n}}} \quad (n < 1)$$

$$t = \frac{1}{k}\sqrt{\frac{n-1}{2}}\int_a^x \frac{dx}{\sqrt{a^{1-n} - x^{1-n}}} \quad (n > 1)$$

13. Using the formulas of Prob. 12, with equal k's, find the ratio of the times required to travel a units of distance starting from rest at $x = a$, in the two cases of an attractive and of a repulsive force varying inversely as the square root of the distance from a point O.

14. Subject to an attractive force which varies inversely as the $\frac{3}{2}$ power of the distance from a point O, a particle starts from rest 16 ft from O with an acceleration numerically equal to 4 ft/sec. Find the time required for the particle to reach O.

15. A chain has its ends fastened at the same level to two poles 60 ft apart. (*a*) Find the dip in the chain if it is 70 ft long. (*b*) How long is the chain if the dip is 5 ft? (*c*) Find the tension at the lowest point and at a point of suspension of the chain of part (*b*) if the weight is 1.5 lb/ft.

16. A cable 60 ft long hangs from two supports at the same level and dips 6 ft. Find the distance between supports.

17. Two poles 80 ft high are 100 ft apart. A wire weighing 0.1 lb/ft is fastened to the top of one pole and drawn tight through a pulley at the top of the other pole. The wire, assumed flexible, is then let down through the pulley until the tension at the fixed end is 10 lb. Find the dip in the wire and its length between the poles.

18. If the wire in Prob. 17 is let down further, show that the tension at the fixed end will decrease to a minimum and then increase. Find (*a*) the minimum value of the tension at the fixed end; (*b*) the dip in the wire and its length between the poles when the tension at the fixed end is again 10 lb.

19. The ends of a cable 80 ft long are fastened at the same level to two poles. If the cable dips 8 ft, show that the slope of the cable at $x = L/4$ (one-quarter span) is $\sqrt{6}/12$.

20. A flexible cable of length b ft is hung over two smooth pegs at the same level L ft apart. Show that there are two, one, or no positions in which the cable will hang in equilibrium according as $b/L \gtreqless e$.

21. Find the equations of the two curves in which the cable of Prob. 20 will hang in equilibrium if $b = 30$ ft and $L = 10$ ft.

22. A flexible cable of length b ft hangs in equilibrium over two smooth pegs at the same level 100 ft apart. Find b so that the tension in the cable at the pegs is a minimum. What is the corresponding dip in the cable?

23. If the suspended cable (Art. 44) is tightly stretched, a will be large. (*a*) Show that, by replacing $a \cosh (x/a)$ by its series expansion and neglecting terms in $1/a$ of higher power than the first, the equation of the curve becomes

$$y = a + \frac{x^2}{2a},$$

a parabola which approximates the catenary for large a.

(*b*) Using the above approximating parabola obtain the formulas

$$d = \frac{L^2}{8a}, \qquad H = \frac{L^2 w}{8d}.$$

When the tension is large it varies little along the cable, so that H in the above formula can be taken as the tension in the cable.

(*c*) A wire weighing 0.1 lb/ft is stretched between two poles 120 ft apart. If the tension in the wire is 240 lb, find the dip.

24. A flexible chain L ft long is hung over the upper end of a smooth inclined plane which makes an angle θ with the horizontal. The end of the chain which hangs down is a ft below the end which rests on the plane. Show that the time required for the chain to slide off is

$$t = \sqrt{\frac{L}{g(1 + \sin\theta)}} \cosh^{-1} \frac{L}{a} \text{ (sec)},$$

where $g = 32.17$ ft/sec². Assume a space of at least L ft below the upper end of the plane.

25. Using $\theta = 0°$ and $90°$ respectively in the formula of Prob. 24, find the time required for a chain 13 ft long, with $a = 5$ ft, to slide (*a*) off a smooth horizontal table; (*b*) off a smooth peg.

26. If "upper" is replaced by "lower" in Prob. 24, obtain a formula for the time required for the chain to slide off the plane.

27. If the lower end of the chain in Prob. 26 is initially at the lower end of the plane, show that the time required for the chain to slide off is

$$t = \sqrt{\frac{L}{g(1 - \sin\theta)}} \cosh^{-1} \csc\theta;$$

also show that, as θ approaches $\pi/2$, t approaches the time required for a freely falling body to fall L ft. Assume that the plane is at least L ft long.

28. If the length of the radius of curvature at any point P of a curve is proportional to the length of the normal drawn from P to the x-axis,

$$\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = ky\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Find and identify the family of curves possessing this property in the four cases: (*a*) $k = 1$; (*b*) $k = -1$; (*c*) $k = 2$; (*d*) $k = -2$. Explain the significance of positive and negative values for k.

29. Solve the differential equation formed by setting the expression for radius of curvature equal to a constant r, and thus obtain the general equation of a circle of radius r.

30. At any point of a curve the product of the lengths of the ordinate and radius of curvature is equal to the length of the normal. Find the equation of the curve if it passes through the origin with slope 1 and is concave upward. What is the length of the radius of curvature at the points $x = 0$, $-\pi/4$, $-\pi/2$? Sketch the curve for $-3\pi/4 < x < \pi/4$.

31. Find the curve satisfying the conditions of Prob. 30 except that it is concave downward. Show that this curve is symmetrical with respect to the origin to the curve in Prob. 30.

32. Find and identify the family of curves such that the length of an arc of any curve of the family is proportional to the difference of the slopes at the ends of the arc.

33. A curve passes through the point P (1, 0) with slope 0 and has the property that the slope at any point is twice the length of the arc from P to that point. Find the ordinate and the slope of the curve at the point whose abscissa is 2.

34. The length of an arc of a curve is equal to the distance intercepted on the y-axis between the tangents at the ends of the arc. Find the equation of the curve having this property and passing through the point (1, 0) with slope 0.

35. The length of an arc of a curve is equal to the distance intercepted on the x-axis between the tangents at the end of the arc. Find the equation of the curve having this property and passing through the point (0, 1) with slope 4/3.

36. A particle Q starts from the origin and moves uniformly along the x-axis. It is pursued by a particle P which starts at the same time from a point on the y-axis at a distance a from the origin. The velocity of Q is k times that of P. Find the *curve of pursuit*, i.e., the path of P, (a) if $k \neq 1$; (b) if $k = 1$. Show that capture will occur if $k < 1$ after a time $ak/(1 - k^2)r$ sec, where a is measured in feet and r (ft/sec) is the rate of Q. If $k = 1$ show that the distance between P and Q approaches the limiting value $a/2$.

37. A wooden prism 4 ft long, whose cross section is an equilateral triangle, weighs 15 lb and floats in water with the vertex of the triangle downward. (a) If the prism is lifted up until the lower edge just touches the water, and is then released, show that it will pass through its equilibrium position with a velocity of $\sqrt{(4/3)ga}$ ft/sec, where a ft is the depth to which it submerges when floating. (b) Compute this velocity using w (density of water) $= 62.4$ lb/ft^3 and $g = 32.2$ ft/sec^2. Assume that the top face of the prism remains horizontal during the vibration.

Chapter 6

SIMULTANEOUS EQUATIONS

45. System of two first order equations. The simplest system of simultaneous differential equations contains two equations of first order in three variables—one independent variable and two dependent variables. Taking x as independent variable and y and z as dependent variables, we now consider a system of two first order equations which can be written in the form

$$\frac{dy}{dx} = M, \qquad \frac{dz}{dx} = N, \tag{1}$$

where M and N are functions of x, y, and z, or in the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}, \tag{2}$$

where P, Q, and R are functions of x, y, and z. Any or all of the functions M, N, P, Q, and R may actually contain less than three variables; they may even all be constant.

Solving such a system as (1) or (2) will be understood to mean finding two relations, free of derivatives, which together involve the three variables and two arbitrary constants, and which satisfy the equations. We take first an example of the simplest case where the capital letters are all constants, then an example in which M and N are not constant, but one at least of equations (1) contains only two variables.

Example 1. Solve

$$\frac{dy}{dx} = 1, \qquad \frac{dz}{dx} = 2. \tag{3}$$

Here each of the two equations may be integrated immediately, giving the solution

$$y = x + C_1,$$
$$z = 2x + C_2. \tag{4}$$

Example 2. Solve

$$\frac{dy}{dx} = 2x, \qquad \frac{dz}{dx} = \frac{y + z}{x}. \tag{5}$$

Integration of the first equation gives

$$y = x^2 + C_1. \tag{6}$$

Substituting the value of y from (6) into the second of equations (5) and rearranging the terms, we find

$$\frac{dz}{dx} - \frac{z}{x} = x + \frac{C_1}{x}.$$

This is a linear equation which may be solved by the method of Art. 21. It may be solved also by use of an integrable combination, as follows:

$$\frac{x\,dz - z\,dx}{x^2} = \left(1 + \frac{C_1}{x^2}\right) dx,$$

$$\frac{z}{x} = x - \frac{C_1}{x} + C_2,$$

$$z = x^2 + C_2 x - C_1. \tag{7}$$

Equations (6) and (7) constitute the required solution of the system (5).

Geometric interpretation. The term *space* will be used in this chapter to denote three-dimensional space in which a point is determined by its three rectangular coordinates, x, y, and z. A single equation (not a differential equation) in the three variables represents a surface. The surface is a plane if the equation is of first degree. If the equation contains only two variables it represents a cylindrical surface with elements parallel to the axis denoted by the missing variable; if the equation is linear in the two variables the surface is a plane parallel to the axis denoted by the missing variable. Two simultaneous equations in three variables represent the intersection of the two surfaces represented by the single equations, i.e., a curve in space; if the two surfaces are planes, the curve is a straight line.

In Example 1, the first of equations (4), for a fixed C_1, represents a plane parallel to the z-axis; the second, for a fixed C_2, represents a plane parallel to the y-axis; hence the two equations taken simultaneously, for fixed C_1 and C_2, represent the straight line intersection of the two planes. As C_1 and C_2 independently take on an infinity of values we have ∞^1 parallel planes intersecting another ∞^1 parallel planes in a family of ∞^2 parallel straight lines, one through each point in space. This doubly infinite family of straight lines is the geometric picture of the solution of the system of differential equations (3).

In Example 2, for a fixed C_1, the parabolic cylinder (6), with elements parallel to the z-axis, intersects the singly infinite family of parabolic cylinders (7), with elements parallel to the y-axis, in a family of ∞^1 curves. Thus, for each value of C_1 we have ∞^1 curves of intersection; for C_1 arbitrary, there are ∞^2 curves of intersection forming the geometric picture of the solution of the system of differential equations (5).

What does the system of differential equations tell us about the direction of the curves representing the solution? Let us look at the equations of the system written in the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{2}$$

The functions $P\ (x, y, z)$, $Q\ (x, y, z)$, and $R\ (x, y, z)$ determine at any point (x, y, z) of space three numbers, and these three numbers, regarded as direction components, fix a direction at that point. Thus if $P = x + y$, $Q = -2y$, $R = x + y + z$, then, at the point A (3, 2, 1), $P = 5$, $Q = -4$, $R = 6$. If from A we travel 5 units in the positive direction of the x-axis, then 4 units in the negative direction of the y-axis, then 6 units in the positive direction of the z-axis, we arrive at a point B, and the line joining A to B fixes a direction at the point A; we denote this direction by the notation [5, −4, 6]. Furthermore, the three differentials dx, dy, and dz are, at the point (x, y, z), the direction components of the tangent line to the curve through this point and satisfying the differential equations. Relations (2) state that dx, dy, and dz are proportional to P, Q, and R, and hence that the directions determined by these two sets of direction components are the same. Therefore the system of simultaneous differential equations (2) defines a family of curves, one passing through each point of space in a direction $[P, Q, R]$.

In Example 1 the differential equations can be written in the form

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{2}.$$

The solution

$$y = x + C_1, \qquad z = 2x + C_2, \tag{4}$$

represents a family of parallel straight lines in the direction [1, 1, 2].

If $y = -2$ and $z = 3$ when $x = 1$, then $C_1 = -3$, $C_2 = 1$, and the particular solution through the point (1, −2, 3) is

$$y = x - 3, \qquad z = 2x + 1. \tag{8}$$

This particular solution could be written down directly from the differential equations, since the equations of a line through (1, −2, 3) in the direction [1, 1, 2] are

$$\frac{x-1}{1} = \frac{y+2}{1} = \frac{z-3}{2}, \tag{8'}$$

the equivalent of (8).

In Example 2 let us find the particular solution through the point (1, 2, 3) and its direction at this point. Substituting $x = 1, y = 2, z = 3$ in equations (6) and (7),

$$2 = 1 + C_1, \qquad 3 = 1 + C_2 - C_1, \qquad C_1 = 1, \qquad C_2 = 3,$$

and the particular solution is

$$y = x^2 + 1, \qquad z = x^2 + 3x - 1. \tag{9}$$

Writing the differential equations in the form

$$\frac{dx}{x} = \frac{dy}{2x^2} = \frac{dz}{y + z},$$

we have for the direction of the curve (9) at the point (1, 2, 3),

$$[P, Q, R] = [1, 2, 5].$$

Systems in which both equations (1) contain all three variables. Certain systems in which both equations (1) contain all three variables may be solved by proceeding as in the following example which we shall solve by two different methods.

EXAMPLE 3. Solve

$$\frac{dz}{dx} = x + y, \qquad \frac{dy}{dx} = x + z. \tag{10}$$

First solution. By using a system of multipliers l,m,n, which may be either constant or variable, any of the fractions in equations (2) may be set equal to $\dfrac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$. Writing equations (10) in form (2), we have

$$\frac{dx}{1} = \frac{dy}{x + z} = \frac{dz}{x + y}.$$

Then

$$\frac{dx}{1} = \frac{dy - dz}{z - y} = \frac{dy + dz + 2dx}{y + z + 2x + 2}.$$

Integrating the first two fractions, then the first and third, we obtain

$$\ln (y - z) = -x + \ln C_1,$$

$$y - z = C_1e^{-x}, \tag{11}$$

$$\ln (y + z + 2x + 2) = x + \ln C_2,$$

$$y + z + 2x + 2 = C_2e^{x}. \tag{12}$$

By adding and subtracting (11) and (12),

$$y = Ae^{x} + Be^{-x} - x - 1, \tag{13}$$

$$z = Ae^{x} - Be^{-x} - x - 1. \tag{14}$$

Equations (11) and (12), or equations (13) and (14), constitute the solution of the system (10).

Second solution. Subtracting the first of equations (10) from the second, we have

$$\frac{d}{dx}(y - z) = z - y,$$

$$(\mathrm{D} + 1)(y - z) = 0,$$

$$y - z = C_1e^{-x}. \tag{15}$$

Substituting the value of z from equation (15) into the second of equations (10), we obtain

$$\frac{dy}{dx} - y = x - C_1e^{-x},$$

a linear equation, whose solution is

$$e^{-x}y = \int (xe^{-x} - C_1e^{-2x})\,dx + C_2,$$

$$= e^{-x}(-x - 1) + \frac{C_1}{2}e^{-2x} + C_2,$$

or

$$y = C_2e^{x} + \frac{C_1}{2}e^{-x} - x - 1. \tag{16}$$

Equations (15) and (16) constitute the solution of the system (10); if the value of y from (16) is substituted in (15), the equivalent solution consisting of equations (13) and (14) is obtained.

PROBLEMS

1. Find the equation of the cylinder formed by the straight lines which satisfy the equations

$$\frac{dy}{dx} = -1, \qquad \frac{dz}{dx} = 3,$$

and (*a*) pass through the circle $y^2 + z^2 = 1, x = 0$; (*b*) pass through the parabola $3y^2 = x, z = 0$.

2. Find the particular solution of

$$\frac{dy}{dx} = \frac{y}{x} + 1, \qquad \frac{dz}{dx} = \frac{y}{x} - 1,$$

through the point (1, 2, 3), and its direction at this point.

3. Find a curve through the point (1, 1, 1) satisfying the equations

$$\frac{dy}{dx} = -2xz, \qquad \frac{dz}{dx} = 2xy.$$

4. Find the two most general functions of x such that the derivative of each one equals the other one.

Solve the following systems of differential equations.

5. $$\frac{dy}{dt} - \frac{y}{x} = 0, \qquad \frac{dx}{dt} - \frac{t}{x} = 0.$$

6. $$\frac{dx}{1+y} = \frac{dy}{1+x} = \frac{dz}{z}.$$

7. $$\frac{dy}{dx} + z = \sin x, \qquad \frac{dz}{dx} + y = \cos x.$$

8. In a certain type of chemical reaction, if a substance A forms an intermediate substance B, which in turn changes into a third substance C, the respective concentrations, x, y, z, of the three substances obey the relations

$$\frac{dx}{dt} + ax = 0, \qquad \frac{dz}{dt} = by, \qquad x + y + z = c,$$

where a, b, and c are constants. Solve for x, y, and z, using the conditions $y = z = 0$ when $t = 0$, (*a*) if $a \neq b$; (*b*) if $a = b$.

9. There are two tanks, each of 100 gal capacity, the first being full of brine holding 50 lb of salt in solution and the second being full of water. If water runs into the first tank at the rate of 3 gal/min and the mixture, kept thoroughly stirred, passes out at the same rate through the second tank, when will the first tank contain twice as much salt as the second, and how much salt will have passed through the second tank at that time?

10. There are two tanks, each of 100 gal capacity, the first being full of brine holding 50 lb of salt in solution and the second being full of water. If the brine runs out of the first tank into the second at the rate of 3 gal/min and the mixture, kept thoroughly stirred, runs at the same rate out of the second tank into the first, when will the first tank contain twice as much salt as the second?

11. Tank A initially contains 100 gal of brine in which 20 lb of salt are dissolved. Two gallons of fresh water enter A per minute and the mixture, assumed uniform, passes at the same rate from A into a second tank B initially containing 50 gal of fresh water. The resulting mixture, also kept uniform, leaves B at the rate of 1 gal/min. Find the amount of salt in tank B at the end of 1 hr.

46. Systems of two linear equations.

A differential equation is linear if it is of first degree in the dependent variables and their derivatives. All the differential equations encountered so far in this chapter, except those of Problems 5 and 6, are linear but of the simplest type, each being of first order and containing only one derivative. We now develop a method for solving a system of two linear equations of any order, with constant coefficients, involving one independent and two dependent variables. The method may be used also for systems of three or more equations, and for systems of equations with variable coefficients, of the type discussed in Art. 40, which can be reduced to linear equations with constant coefficients. Two examples follow which illustrate the method.

EXAMPLE 1. Solve

$$\frac{dy}{dx} + \frac{dz}{dx} = 4y + 1,$$

$$\frac{dy}{dx} + z = 3y + x^2.$$

Using operator notation, we place all terms containing the dependent variables y and z on the left, those containing y in the first column and those containing z in the second:

$$(D - 4)y + Dz = 1, \tag{1}$$

$$(D - 3)y + z = x^2. \tag{2}$$

It can be shown * that for a system of differential equations written in this form the number of arbitrary constants appearing in the general solution is equal to the exponent of the highest power of D in the expansion of the determinant of the (operator) coefficients of the dependent variables. The highest power of D in the expansion of the determinant

$$\begin{vmatrix} D - 4 & D \\ D - 3 & 1 \end{vmatrix}$$

is D^2; hence two arbitrary constants will appear in the solution of the system of equations (1) and (2). If more than two arbitrary constants appear in the process of solution, the extra ones must be evaluated, or expressed in terms of only two.

Three methods of procedure are possible. We may eliminate z from equations (1) and (2), solve the resulting equation for y, then substitute this value of y in (2) to obtain z. If the value of y, containing two arbitrary constants, were substituted in (1) to obtain z, another constant of integration would appear and would have to be evaluated by substituting the values of y and z in (2); hence it is better to obtain z from equation (2).

Another method, obviously more complicated in this case, is first to eliminate y from equations (1) and (2), solve the resulting equation for z, then substitute the value found for z in (1) or (2) to obtain y; this method introduces a third arbitrary constant which must be evaluated.

A third method would be to eliminate z from equations (1) and (2), then solve the resulting equation, obtaining y in terms of x and two arbitrary constants, say C_1 and C_2. Next eliminate y from equations (1) and (2), solve the resulting equation and obtain z in terms of x and two arbitrary constants, say C_1' and C_2'. Finally C_1' and C_2' would be evaluated or expressed in terms of C_1 and C_2 by substituting the values of y and z in (1) or (2).

* See Forsyth's *A Treatise on Differential Equations*, 5th Ed., Art. 171.

We proceed to solve Example 1 by each of the three methods.

First solution. Multiplying equation (2) by D,

$$(D^2 - 3D)y + Dz = 2x. \tag{3}$$

Subtracting (1) from (3),

$$(D^2 - 4D + 4)y = (D - 2)^2\, y = 2x - 1. \tag{4}$$

Solving (4) by the method of Art. 36, we find

$$y_c = (C_1 + C_2x)e^{2x}.$$

$$\begin{array}{rrcl} 4) & y_p & = & Ax + B \\ -4) & Dy_p & = & A \\ 1) & D^2y_p & = & 0 \\ \hline \end{array}$$

$$\begin{array}{r|l} 4A = 2 & 4B - 4A = -1 \\ A = \frac{1}{2} & B \qquad = \frac{1}{4} \end{array}$$

$$y = (C_1 + C_2x)e^{2x} + \tfrac{1}{2}x + \tfrac{1}{4}. \tag{5}$$

Differentiating (5),

$$Dy = (2C_1 + C_2 + 2C_2x)e^{2x} + \tfrac{1}{2}. \tag{6}$$

From (5) and (6),

$$(D - 3)y = (-C_1 + C_2 - C_2x)e^{2x} - \tfrac{3}{2}x - \tfrac{1}{4}. \tag{7}$$

Then. from (2),

$$z = (C_1 - C_2 + C_2x)e^{2x} + x^2 + \tfrac{3}{2}x + \tfrac{1}{4}. \tag{8}$$

Equations (5) and (8) comprise the general solution of the system of equations (1) and (2).

Second solution. Multiplying equation (1) by D − 3 and (2) by D − 4, we have

$$(D - 3)(D - 4)y + (D^2 - 3D)z = -3, \tag{9}$$

$$(D - 3)(D - 4)y + (D - 4)z = 2x - 4x^2. \tag{10}$$

Subtracting (10) from (9),

$$(D^2 - 4D + 4)z = (D - 2)^2z = 4x^2 - 2x - 3. \tag{11}$$

The solution of (11) proceeds as follows:

$$z_c = (C_1' + C_2'x)e^{2x}.$$

$$\begin{array}{rrl} 4) & z_p = & Ax^2 + Bx + C \\ -4) & Dz_p = & \qquad 2Ax + B \\ 1) & D^2z_p = & \qquad\qquad 2A \end{array}$$

$4A = 4$	$4B - 8A = -2$	$4C - 4B + 2A = -3$
$A = 1$	$B = \frac{3}{2}$	$C = \frac{1}{4}$

$$z = (C_1' + C_2'x)e^{2x} + x^2 + \tfrac{3}{2}x + \tfrac{1}{4}. \tag{12}$$

Substituting the value of z from (12) into (2), we obtain the linear equation

$$(\mathrm{D} - 3)y = -(C_1' + C_2'x)e^{2x} - \tfrac{3}{2}x - \tfrac{1}{4},$$

whose solution is found by the method of Art. 21:

$$\begin{aligned} e^{-3x}y &= -\int [(C_1' + C_2'x)e^{-x} + (\tfrac{3}{2}x + \tfrac{1}{4})e^{-3x}]\,dx + C_3', \\ &= C_1'e^{-x} - C_2'e^{-x}(-x - 1) - \frac{3}{2}\frac{e^{-3x}}{9}(-3x - 1) + \frac{e^{-3x}}{12} + C_3', \end{aligned}$$

$$y = (C_1' + C_2' + C_2'x)e^{2x} + \tfrac{1}{2}x + \tfrac{1}{4} + C_3'e^{3x}. \tag{13}$$

To evaluate C_3' we multiply (12) by D and (13) by D − 4, giving

$$\mathrm{D}z = (2C_1' + C_2' + 2C_2'x)e^{2x} + 2x + \tfrac{3}{2},$$

$$(\mathrm{D} - 4)y = (-2C_1' - C_2' - 2C_2'x)e^{2x} - 2x - \tfrac{1}{2} - C_3'e^{3x},$$

which, substituted in (1), yield

$$1 - C_3'e^{3x} = 1.$$

Hence $C_3' = 0$, and equation (13) becomes

$$y = (C_1' + C_2' + C_2'x)e^{2x} + \tfrac{1}{2}x + \tfrac{1}{4}. \tag{14}$$

Equations (14) and (12) comprise the general solution of (1) and (2). If $C_2' = C_2$ and $C_1' = C_1 - C_2$, equations (14) and (12) reduce to (5) and (8).

Third solution. First eliminating z and then eliminating y from equations (1) and (2), and solving the resulting equations, as above,

$$y = (C_1 + C_2x)e^{2x} + \tfrac{1}{2}x + \tfrac{1}{4}, \tag{5}$$

$$z = (C_1' + C_2'x)e^{2x} + x^2 + \tfrac{3}{2}x + \tfrac{1}{4}. \tag{12}$$

In order to find the relations connecting the constants C_1, C_2, C_1', C_2', substitute (5) and (12) in (2), making use of the value of $(D - 3)y$ given by (7):

$$(-C_1 + C_2 - C_2x)e^{2x} - \tfrac{3}{2}x - \tfrac{1}{4} + (C_1' + C_2'x)e^{2x} + x^2 + \tfrac{3}{2}x + \tfrac{1}{4} = x^2,$$

$$(C_1 - C_2 + C_2x)e^{2x} = (C_1' + C_2'x)e^{2x},$$

$$C_1' = C_1 - C_2, \qquad C_2' = C_2.$$

Hence (12) reduces to

$$z = (C_1 - C_2 + C_2x)e^{2x} + x^2 + \tfrac{3}{2}x + \tfrac{1}{4}; \tag{8}$$

equations (5) and (8), as before, comprise the general solution of the given system of equations.

EXAMPLE 2. Solve

$$\frac{d^2x}{dt^2} + 2\left(\frac{dy}{dt} - a\right) = 0,$$

$$\frac{d^2y}{dt^2} - 2\left(\frac{dx}{dt} - b\right) = 0.$$

First write the equations in the form

$$D^2x + 2Dy = 2a, \tag{15}$$

$$2Dx - D^2y = 2b, \tag{16}$$

where D now denotes d/dt. The determinant of the coefficients of the dependent variables x and y yields D^4 as the highest power of D; hence the general solution will contain four arbitrary constants. Adding $D \times (15)$ and $2 \times (16)$, we get

$$(D^3 + 4D)x = 4b.$$

Then

$$x_c = C_1 + C_2 \sin 2t + C_3 \cos 2t, \qquad x_p = bt,$$

$$x = C_1 + C_2 \sin 2t + C_3 \cos 2t + bt. \tag{17}$$

Substituting in (15) the value of

$$D^2x = -4C_2 \sin 2t - 4C_3 \cos 2t,$$

we find

$$Dy = 2C_2 \sin 2t + 2C_3 \cos 2t + a. \tag{18}$$

Integration of (18) gives

$$y = -C_2 \cos 2t + C_3 \sin 2t + at + C_4. \tag{19}$$

Equations (17) and (19) contain the proper number of arbitrary constants and constitute the general solution of the given system. This solution may also be written in the form

$$x = A \cos (2t + \alpha) + bt + B,$$

$$y = A \sin (2t + \alpha) + at + C,$$

where A, B, C, and α are arbitrary constants.

PROBLEMS

Solve the following systems of differential equations.

1. $\dfrac{dy}{dx} - y - z = 2 \cos 2x, \qquad \dfrac{dz}{dx} - 3y + z = 0.$

2. $\dfrac{dy}{dx} + \dfrac{dz}{dx} = \cos x - 3y, \qquad \dfrac{dy}{dx} = 11 \sin x + y - z.$

3. $\dfrac{dx}{dt} + 2x = t - y, \qquad \dfrac{dx}{dt} + \dfrac{dy}{dt} = 1 - y.$

4. $\dfrac{dx}{dt} - y = \cos 2t, \qquad \dfrac{dx}{dt} - \dfrac{dy}{dt} - x - y = \sin 2t + \cos 2t.$

5. $\dfrac{dy}{dx} = e^x - y - z, \qquad 2\dfrac{dy}{dx} + \dfrac{dz}{dx} = \cos x - z.$

6. $\dfrac{dy}{dx} - \dfrac{dz}{dx} + y + z = 4 \sin x, \qquad 2\dfrac{dy}{dx} - y + 3z = 2 \cos x.$

7. $\dfrac{dy}{dx} + \dfrac{dz}{dx} + y = 2 \sin 2x, \qquad \dfrac{dy}{dx} + \dfrac{dz}{dx} + 2y + 2z = \cos 2x.$

8. $\dfrac{dx}{dt} = y - x + 3 \sin t, \qquad \dfrac{dx}{dt} + \dfrac{dy}{dt} = 2y - 3x + 2e^t + 6 \sin t - 2 \cos t$

9. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 1 - z, \qquad \dfrac{dy}{dx} + \dfrac{dz}{dx} = 4e^{-x} + y.$

10. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + z = 10e^{-2x}, \qquad \dfrac{dy}{dx} + \dfrac{dz}{dx} - y = 1 + 7e^{-2x}.$

11. $t\dfrac{dx}{dt} + y = t + x, \qquad t\dfrac{dy}{dt} + x = t + y.$

12. $t^2\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + x = 1 + \ln t, \qquad t\dfrac{dx}{dt} + t\dfrac{dy}{dt} - x - y = \ln t.$

13. Show that the system of differential equations

$$\mathrm{D}y + (\mathrm{D} + 1)z = x,$$
$$(\mathrm{D} - 1)y + [(n^2 - n + 1)\mathrm{D} + n]z = e^x,$$

where $\mathrm{D} = d/dx$, has none, one, or two arbitrary constants in its general solution when $n = 0, 1$, and $\frac{1}{3}$ respectively. Find the general solution in each of these cases.

14. Find a solution of the system

$$(7\mathrm{D}^2 + 23)x - 8y = 0, \qquad (3\mathrm{D}^2 - 13)x + 2(\mathrm{D}^2 + 5)y = 0,$$

where $\mathrm{D} = d/dt$, subject to the conditions: $x = 0$, $y = 0$, $\mathrm{D}x = 1$, and $\mathrm{D}y = 3$, when $t = 0$. Find the values of x and y when $t = 1$.

15. Solve the system

$$\frac{d^2x}{dt^2} - 8\frac{dy}{dt} + 9x = at + h,$$

$$\frac{d^2y}{dt^2} + 8\frac{dx}{dt} + 9y = bt + k.$$

16. Solve the three simultaneous equations

$$\frac{dx}{dt} = x + y, \qquad \frac{dy}{dt} = z - x, \qquad \frac{dz}{dt} = y + z.$$

47. Motion of a projectile. The theory of the motion of a projectile in the atmosphere is quite difficult, owing partly to complications connected with the law for variation of resistance with velocity. We shall treat here only the case in which resistance is assumed to be proportional to velocity, and the path is considered to be a plane curve traced out by the center of gravity $P\ (x, y)$ of the projectile.

If t is the time, dx/dt and dy/dt are the components of velocity in the x- and y-directions, and d^2x/dt^2 and d^2y/dt^2 represent the corresponding components of acceleration. Let x and y be measured in feet and t in seconds; also let w (lb) be the weight of the projectile and g (ft/sec^2), as usual, the gravity constant. We form a system of two simultaneous differential equations by equating $(w/g)(d^2x/dt^2)$ and $(w/g)(d^2y/dt^2)$ respectively to the components of force in the x- and y-directions:

$$\frac{w}{g}\frac{d^2x}{dt^2} = -k\frac{dx}{dt}, \qquad \frac{w}{g}\frac{d^2y}{dt^2} = -w - k\frac{dy}{dt}. \tag{1}$$

Here each of the equations contains only two variables and hence can be solved separately. The solution will consist of two equations giving the parametric equations of the path, x and y in terms of t. Four arbitrary constants will appear in the course of solution, for whose determination four conditions are necessary. We shall solve equations (1) under the assumption that the projectile starts at the origin with an initial velocity v_0 (ft/sec) in a direction making an angle α with the horizontal; that is, the four conditions are $x = 0$, $y = 0$, $dx/dt = v_0 \cos \alpha$, $dy/dt = v_0 \sin \alpha$, when $t = 0$.

After writing equations (1) in operator notation, with $\mathrm{D} = d/dt$, the solution proceeds as follows:

$$\left(\mathrm{D}^2 + \frac{kg}{w}\mathrm{D}\right)x = 0 \qquad \left(\mathrm{D}^2 + \frac{kg}{w}\mathrm{D}\right)y = -g$$

$$x = C_1 + C_2e^{-(kg/w)t} \qquad y = C_3 + C_4e^{-(kg/w)t} - \frac{w}{k}t$$

$$\mathrm{D}x = -\frac{kg}{w}C_2e^{-(kg/w)t} \qquad \mathrm{D}y = -\frac{kg}{w}C_4e^{-(kg/w)t} - \frac{w}{k}$$

$$0 = C_1 + C_2 \qquad 0 = C_3 + C_4$$

$$v_0 \cos \alpha = -\frac{kg}{w}C_2 \qquad v_0 \sin \alpha = -\frac{kg}{w}C_4 - \frac{w}{k}$$

$$C_1 = -C_2 = \frac{w}{kg}v_0 \cos \alpha \qquad C_3 = -C_4 = \frac{w}{kg}\left(v_0 \sin \alpha + \frac{w}{k}\right)$$

$$\begin{aligned} x &= \frac{w}{kg}v_0 \cos \alpha\,(1 - e^{-(kg/w)t}) \\ y &= \frac{w}{kg}\left(v_0 \sin \alpha + \frac{w}{k}\right)(1 - e^{-(kg/w)t}) - \frac{w}{k}t. \end{aligned} \tag{2}$$

Equations (2) are the parametric equations of the path and locate the position of the projectile at any time t. The x, y equation of the path could be obtained by eliminating t between the two equations.

PROBLEMS

1. Assuming that there is no resistance to the motion discussed in Art. 47, find (*a*) the x, y equation of the path; (*b*) the maximum height of the projectile and the maximum range.

2. (*a*) Find the maximum height of the projectile in Art. 47. (*b*) Show that this result approaches the value found for the maximum height of the projectile in Prob. 1(*b*) as $k \to 0$.

3. The small oscillations of a certain system with two degrees of freedom are given by the equations

$$\frac{d^2x}{dt^2} + 3x - 2y = 0, \qquad \frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} - 3x + 5y = 0.$$

If $x = 0$, $y = 0$, $dx/dt = 1$, $dy/dt = 3$, when $t = 0$, find (*a*) the values of x and y when $t = 1$; (*b*) the maximum and minimum values of x and y.

4. A particle of unit mass moves in accordance with the law

$$\frac{d^2x}{dt^2} = y, \qquad \frac{d^2y}{dt^2} = -x.$$

If it starts from rest at the point $(-1, 0)$, find the parametric equations of its path. Plot the path up to the point where it crosses the x-axis, and find the coordinates of the maximum point on this portion of the path curve.

5. The primary of a transformer has inductance L_1 henries and resistance R_1 ohms, the secondary has inductance L_2 henries and resistance R_2 ohms, and the mutual inductance is M henries, where $L_1L_2 > M^2$. The free oscillations in the two circuits are given by

$$L_1\frac{di_1}{dt} + M\frac{di_2}{dt} + R_1i_1 = 0, \qquad M\frac{di_1}{dt} + L_2\frac{di_2}{dt} + R_2i_2 = 0,$$

where i_1 and i_2 (amp) are the respective currents in the primary and secondary at time t (sec); show that i_1 and i_2 approach zero as t becomes infinite.

Suppose that the values of the circuit constants are $L_1 = 3$, $L_2 = 6$, $R_1 = 7$, $R_2 = 10$, $M = 4$; also that $i_1 = 2$, $i_2 = 3$, when $t = 0$. (*a*) Show that i_2 continually decreases but that i_1 increases to a maximum and then decreases indefinitely as t becomes infinite. (*b*) Find the maximum value of i_1 and the time required for it to reach the maximum. (*c*) Find the values of i_1 and i_2 when $t = 0.01$ sec.

6. If a particle of unit mass at $P(x, y)$ is attracted toward the origin O by a force F, the x- and y-components of the force are $-F \cos \theta$ and $-F \sin \theta$, where θ is the vectorial angle of P. The differential equations of motion are then

$$\frac{d^2x}{dt_2} = -\frac{Fx}{r}, \qquad \frac{d^2y}{dt^2} = -\frac{Fy}{r},$$

where r is the radius vector of P.

(*a*) Multiply the second equation by x, the first by y, and subtract; then, making use of the formula for differential sector area, Art. 3, show that the areal velocity (time derivative of sector area) is constant.

(*b*) Multiply the first equation by 2 dx, the second by 2 dy, and add, then, letting $r = 1/R$, show that

$$\frac{d^2R}{d\theta^2} + R = \frac{F}{h^2R^2},$$

where h is twice the areal velocity.

(*c*) Solve the differential equation in (*b*) if $F = kR^2$ (inverse square law), subject to the conditions $R = 1/r_0$, $dR/d\theta = 0$, when $\theta = 0$, and show that the path is a conic.

7. In Thomson's experimental determination of the ratio m/e of the mass to the charge of an electron, in which the electrons were subjected to an electric field of intensity E and a magnetic field of intensity H, the equations

$$m\frac{d^2x}{dt^2} + He\frac{dy}{dt} = Ee, \qquad m\frac{d^2y}{dt^2} - He\frac{dx}{dt} = 0,$$

were employed.* If $x = y = dx/dt = dy/dt = 0$ for $t = 0$, show that the path is a cycloid whose parametric equations are

$$x = \frac{Em}{H^2e}\left(1 - \cos\frac{He}{m}t\right), \qquad y = \frac{Em}{H^2e}\left(\frac{He}{m}t - \sin\frac{He}{m}t\right).$$

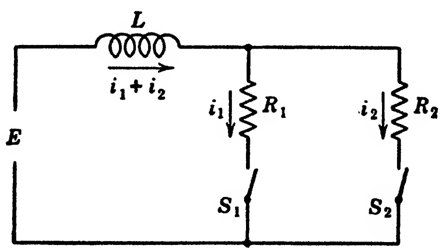

FIG. 21

8. In the network of Fig. 21, when the switches S_1 and S_2 are closed, the following differential equations (using the notation of Art. 23 with $D = d/dt$) hold:

* See *Phil. Mag.*, Vol. 48 (1899), p. 547.

$$LD(i_1 + i_2) + R_1 i_1 = E,$$

$$LD(i_1 + i_2) + R_2 i_2 = E.$$

(*a*) Find the currents i_1 and i_2 at time T sec after S_1 and S_2 are closed simultaneously. Given the circuit constants $R_1 = 10$ ohms, $R_2 = 20$ ohms, $L = 2$ henries, $E = 32$ volts, find the currents when $T = 0.03$ sec.

(*b*) If S_1 is closed and i_1 reaches a steady state while S_2 is open, find i_1 and i_2 at time T sec after S_2 is closed. Using the same circuit constants as in part (*a*), find the currents when $T = 0.06$ sec.

(*c*) If S_2 is closed T sec after S_1 is closed, find i_1 and i_2 at time T sec after S_2 is closed. Using the same circuit constants as in part (*a*), find the currents when $T = 0.06$ sec.

(*d*) Suppose that S_2 is closed T sec after S_1 is closed, and that T sec after S_2 is closed the transient currents in the two circuits are identical. Find T.

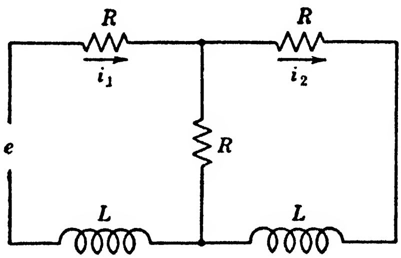

FIG. 22

9. In the network of Fig. 22 suppose that each R is equal to 10 ohms and that each L is equal to 10 henries; then the following differential equations hold:

$$(10D + 20)i_1 - 10i_2 = e,$$

$$(10D + 20)i_2 - 10i_1 = 0,$$

where $D = d/dt$. Assuming that $i_1 = i_2 = 0$ when $t = 0$, find i_1 and i_2 at time t if $e = 100$ volts.

10. Solve Prob. 9 if $e = 10 \sin t$ volts.

48. The roots of unity. The n roots of the equation

$$x^n = 1, \tag{1}$$

where n is a positive integer, are called the nth roots of unity. De Moivre's theorem gives the n values of x which satisfy equation (1) in the form

$$x = \cos\frac{2\pi k}{n} + i \sin\frac{2\pi k}{n}, \qquad (k = 0, 1, 2, \cdots, n - 1), \tag{2}$$

where $i = \sqrt{-1}$. These values of x are the nth roots of unity. They may be represented graphically by n points equally spaced around a unit circle centered at the origin, the first point being at $x = 1$, corresponding to $k = 0$. If r denotes the next of these points in the counterclockwise direction, corresponding to $k = 1$,

$$r = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}, \tag{3}$$

and we may call r the basic nth root of unity; all the nth roots of unity are obtained by raising r to the powers $0, 1, 2, \cdots, n - 1$. Thus instead of (2) we may write

$$x = 1, r, r^2, r^3, \cdots, r^{n-1}; \qquad (r^n = 1). \tag{4}$$

The relation

$$1 + r + r^2 + \cdots + r^{n-1} = 0, \qquad (n > 1), \tag{5}$$

follows from the symmetrical arrangement of the points. Furthermore

$$1 + r^k + r^{2k} + \cdots + r^{(n-1)k} = 0, \quad (k = 1, 2, 3, \cdots, n - 1), \tag{6}$$

and

$$r^{nk} = 1. \tag{7}$$

When $n = 2$, $r = -1$; the square roots of unity, $1, r$, are $1, -1$.

When $n = 3$, $r = -\frac{1}{2} + i\sqrt{3}/2$ and $r^2 = -\frac{1}{2} - i\sqrt{3}/2$. It is customary to denote r by the Greek letter ω when $n = 3$. Thus $\omega = -\frac{1}{2} + i\sqrt{3}/2$, $\omega^2 = -\frac{1}{2} - i\sqrt{3}/2$, and the cube roots of unity are $1, \omega, \omega^2$.

When $n = 4$, $r = i$; the fourth roots of unity, $1, r, r^2, r^3$, are $1, i, -1, -i$.

49. Cyclic systems of differential equations. By a cyclic interchange of the letters $x_1, x_2, x_3, \cdots, x_{n-1}, x_n$, we mean a change of x_1 to x_2, x_2 to x_3, $\cdots$, x_{n-1} to x_n, x_n to x_1. We call a system of differential equations cyclic if it is unchanged by a cyclic interchange of the dependent variables; for example, the following system in which the x's are the dependent variables and $\mathrm{D} = d/dt$:

$$\begin{aligned} \mathrm{D}x_1 &= x_2, \\ \mathrm{D}x_2 &= x_3, \\ &\cdots\cdots \\ \mathrm{D}x_{n-1} &= x_n, \\ \mathrm{D}x_n &= x_1. \end{aligned} \tag{1}$$

Certain cyclic systems of n equations in the dependent variables $x_1, x_2, \cdots, x_n$, may be solved by changing to a new set of dependent variables, $u_1, u_2, \cdots, u_n$, connected with the old set by means of the substitutions

$$u_k = x_1 + r^{k-1}x_2 + (r^{k-1})^2x_3 + \cdots + (r^{k-1})^{n-1}x_n, \tag{2}$$

where $k = 1, 2, 3, \cdots, n$, and r is the basic nth root of unity.

The original dependent variables $x_1, x_2, \cdots, x_n$ and the independent variable t are assumed to be real. Usually we wish to express the x's in a form involving t and n arbitrary constants but involving no fixed imaginary constants, that is, to obtain a solution in real form, although imaginary quantities are used to effect the solution.

Example 1. Solve the system

$$\begin{aligned} \mathrm{D}x &= ax + by + cz, \\ \mathrm{D}y &= ay + bz + cx, \\ \mathrm{D}z &= az + bx + cy, \end{aligned} \tag{3}$$

where $\mathrm{D} = d/dt$, and a, b, c are real constants.

In the substitutions (2) let $n = 3$, $r = \omega$, and replace x_1, x_2, x_3 by x, y, z, and u_1, u_2, u_3 by u, v, w. Then

$$\begin{aligned} u &= x + y + z, \\ v &= x + \omega y + \omega^2 z, \\ w &= x + \omega^2 y + \omega z. \end{aligned} \tag{4}$$

The last term of the third equation is $\omega^4 z$, but it is written ωz since $\omega^3 = 1$.

Adding equations (4), we find

$$u + v + w = 3x,$$

since $1 + \omega + \omega^2 = 0$ [equation (5), Art. 48]. Multiplying equations (4) by 1, ω^2, ω respectively and adding, then by 1, ω, ω^2 and adding, we find similarly

$$u + \omega^2 v + \omega w = 3y,$$

$$u + \omega v + \omega^2 w = 3z,$$

so that the solution of equations (4) for x, y, z gives

$$\begin{aligned} x &= \tfrac{1}{3}(u + v + w), \\ y &= \tfrac{1}{3}(u + \omega^2 v + \omega w), \\ z &= \tfrac{1}{3}(u + \omega v + \omega^2 w). \end{aligned} \tag{5}$$

Now, adding equations (3), we obtain

$$Du = (a + b + c)u,$$

whose solution is

$$u = 3C_1 e^{(a+b+c)t}. \tag{6}$$

Multiplying equations (3) by 1, ω, ω^2 respectively and adding, we find

$$Dv = (a + b\omega^2 + c\omega)v,$$

whose solution is

$$v = 3C_2 e^{(a+b\omega^2+c\omega)t}. \tag{7}$$

Finally, multiplying equations (3) by 1, ω^2, ω respectively and adding, we have

$$Dw = (a + b\omega + c\omega^2)w,$$

$$w = 3C_3 e^{(a+b\omega+c\omega^2)t}. \tag{8}$$

Substituting (6), (7), and (8) in (5) gives

$$\begin{aligned} x &= C_1 e^{(a+b+c)t} + C_2 e^{(a+b\omega^2+c\omega)t} + C_3 e^{(a+b\omega+c\omega^2)t}, \\ y &= C_1 e^{(a+b+c)t} + \omega^2 C_2 e^{(a+b\omega^2+c\omega)t} + \omega C_3 e^{(a+b\omega+c\omega^2)t}, \\ z &= C_1 e^{(a+b+c)t} + \omega C_2 e^{(a+b\omega^2+c\omega)t} + \omega^2 C_3 e^{(a+b\omega+c\omega^2)t}. \end{aligned} \tag{9}$$

Equations (9), where the C's are arbitrary constants, represent the general solution of the system (3), but the solution is not in real form since it involves the fixed imaginary constant ω. This solution, however, may be transformed into the following form (see Problem 1 at the end of this chapter):

$$\begin{aligned} x &= Ce^{(a+b+c)t} + Ke^{\lambda t}\sin(\mu t + \alpha), \\ y &= Ce^{(a+b+c)t} + Ke^{\lambda t}\sin\left(\mu t + \alpha - \frac{2\pi}{3}\right), \\ z &= Ce^{(a+b+c)t} + Ke^{\lambda t}\sin\left(\mu t + \alpha - \frac{4\pi}{3}\right), \end{aligned} \tag{10}$$

where

$$\lambda = a - \frac{b}{2} - \frac{c}{2}, \qquad \mu = \frac{\sqrt{3}}{2}(c - b),$$

and C, K, and α are arbitrary constants. Equations (10) represent the general solution of the system (3) in real form; for any real values of the arbitrary constants C, K, and α, the values of x, y, and z are real.

Example 2. Solve the system (1).

This system may be solved by use of the substitutions (2), but here it is easy to eliminate all but one of the dependent variables, say x_1, then solve the resulting equation for x_1 and find the other x's by successive differentiation. Substituting the value of x_2 from the first equation into the second, then the value of x_3 from the second equation into the third, $\cdots$, then the value of x_n from the $(n-1)$th equation into the nth, we have

$$D^n x_1 = x_1,$$

or

$$(D^n - 1)x_1 = 0,$$

from which we write down the value of x_1, then obtain x_2 from $Dx_1 = x_2$, $\cdots$, x_n from $Dx_{n-1} = x_n$.

The general solution of system (1) is therefore

$$\begin{aligned} x_1 &= C_1e^t + C_2e^{rt} + C_3e^{r^2t} + \cdots + C_ne^{r^{n-1}t}, \\ x_2 &= C_1e^t + rC_2e^{rt} + r^2C_3e^{r^2t} + \cdots + r^{n-1}C_ne^{r^{n-1}t}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n &= C_1e^t + r^{n-1}C_2e^{rt} + (r^2)^{n-1}C_3e^{r^2t} + \cdots + (r^{n-1})^{n-1}C_ne^{r^{n-1}t}, \end{aligned} \tag{11}$$

in which the derivative of each equation gives the following one and the derivative of the last gives the first.

As in the case of equations (9), the equations (11) represent a solution in imaginary form for $n > 2$, but in any particular case the solution can be exhibited in real form, for example in the following special case of (1).

EXAMPLE 3. Write in real form the solution of the system

$$\begin{aligned} Dx_1 &= x_2, \\ Dx_2 &= x_3, \\ Dx_3 &= x_4, \\ Dx_4 &= x_1. \end{aligned} \tag{12}$$

The first of equations (11) for $n = 4$, $r = i$, is

$$x_1 = C_1e^t + C_2e^{it} + C_3e^{-t} + C_4e^{-it}.$$

By means of Euler's relation we may replace $C_2e^{it} + C_4e^{-it}$ by $K \sin (t + \alpha)$ as in Art. 31(*c*). Also let $C_1 = A$, $C_3 = B$. We thus write x_1 in real form and obtain the other x's by successive differentiation:

$$\begin{aligned} x_1 &= Ae^t + Be^{-t} + K \sin (t + \alpha), \\ x_2 &= Ae^t - Be^{-t} + K \cos (t + \alpha), \\ x_3 &= Ae^t + Be^{-t} - K \sin (t + \alpha), \\ x_4 &= Ae^t - Be^{-t} - K \cos (t + \alpha). \end{aligned} \tag{13}$$

Equations (13), in which A, B, K, and α are arbitrary constants, form the required solution of the system (12).

50. A special case of Einstein's equations. Einstein's law of gravitation is expressed by a system of ten partial differential equations of second order. It has been shown by Kasner *

* "Solutions of the Einstein equations involving functions of only one variable," by Edward Kasner, *Transactions of the American Mathematical Society*, Vol. 27 (1925).

that in a special case these equations reduce to the following system of three ordinary differential equations of first order:

$$\begin{aligned} Dx &= yz - x^2, \\ Dy &= zx - y^2, \\ Dz &= xy - z^2, \end{aligned} \tag{1}$$

where $D = d/dt$.

We notice that this system of equations is cyclic in x, y, z, but not linear as in the case of the system in Example 1, Art. 49; however, the method used in that example will be followed in solving system (1). Letting

$$\begin{aligned} u &= x + y + z, \\ v &= x + \omega y + \omega^2 z, \\ w &= x + \omega^2 y + \omega z, \end{aligned} \tag{2}$$

we find

$$\begin{aligned} uv &= x^2 + \omega y^2 + \omega^2 z^2 - \omega^2 xy - yz - \omega zx, \\ vw &= x^2 + y^2 + z^2 - xy - yz - zx, \\ wu &= x^2 + \omega^2 y^2 + \omega z^2 - \omega xy - yz - \omega^2 zx. \end{aligned} \tag{3}$$

Now, adding equations (1), multiplying them by $1, \omega, \omega^2$ respectively and adding, then multiplying by $1, \omega^2, \omega$ respectively and adding, we reduce system (1) to the system

$$\begin{aligned} Du &= -vw, \\ Dv &= -uv, \\ Dw &= -wu. \end{aligned} \tag{4}$$

From the last two of these equations $dv/v = dw/w$, hence $v = k^2 w$, where k^2 is an arbitrary constant written in this form for convenience, and the system (4) is equivalent to

$$\begin{aligned} Du &= -k^2 w^2, \\ Dw &= - wu, \\ v &= k^2 w. \end{aligned} \tag{5}$$

Eliminating u from the first two of equations (5), we get

$$-k^2w^2 = \frac{-wD^2w + (Dw)^2}{w^2},$$

or

$$wD^2w - (Dw)^2 = k^2w^4.$$

Solving this equation by the method of Art. 42, letting $Dw = p$, $D^2w = p\,dp/dw$, we have

$$wp\frac{dp}{dw} - p^2 = k^2w^4,$$

then, as in Art. 25(a),

$$2p\frac{dp}{dw} - \frac{2}{w}p^2 = 2k^2w^3,$$

$$\frac{p^2}{w^2} = k^2w^2 - c^2,$$

where the constant of integration is taken as $-c^2$; if the + sign is used the resulting solution will involve hyperbolic instead of circular functions. Solving for $p(= dw/dt)$, separating the variables, and integrating again, we find

$$\frac{dw}{w\sqrt{k^2w^2 - c^2}} = \pm\,dt,$$

$$\frac{1}{c}\sec^{-1}\frac{kw}{c} = \pm(t + b),$$

$$w = \frac{c}{k}\sec c(t + b).$$

The last two of equations (5) then give v and u:

$$v = ck\sec c(t + b),$$

$$u = -c\tan c(t + b).$$

The relations (5) of Art. 49 yield the values of x, y, z:

$$
\begin{aligned}
x &= \frac{k^2+1}{3k}\, c \sec c(t+b) - \frac{c}{3} \tan c(t+b),\\
y &= \frac{\omega^2 k^2+\omega}{3k}\, c \sec c(t+b) - \frac{c}{3} \tan c(t+b),\\
z &= \frac{\omega k^2+\omega^2}{3k}\, c \sec c(t+b) - \frac{c}{3} \tan c(t+b).
\end{aligned}
\tag{6}
$$

Equations (6), in which k, b, and c are arbitrary constants, represent the general solution of the system (1).

It may be noticed that if we write

$$a_1 = \frac{k^2+1}{3k}, \qquad a_2 = \frac{\omega^2 k^2+\omega}{3k}, \qquad a_3 = \frac{\omega k^2+\omega^2}{3k},$$

then a_1, a_2, and a_3 satisfy the relations

$$
\begin{aligned}
a_1 + a_2 + a_3 &= 0,\\
a_1a_2 + a_2a_3 + a_3a_1 &= -\tfrac{1}{3},\\
a_1a_2a_3 &= \frac{k^6+1}{27k^3};
\end{aligned}
$$

hence a_1, a_2, a_3 are the roots of the cubic

$$a^3 - \tfrac{1}{3}a - \frac{k^6+1}{27k^3} = 0. \tag{7}$$

The solution (6) of the system (1) may therefore be written in the form

$$x, y, z = a_i c \sec c(t+b) - \frac{c}{3} \tan c(t+b), \qquad (i = 1,2,3), \tag{8}$$

where the a's are the roots of the cubic (7). This is the solution obtained by another method in the paper by Kasner previously cited.

An integral of a system of differential equations expressed as a relation free of derivatives, involving some or all of the dependent variables and an arbitrary constant, is of importance in the discussion of certain physical problems. From the third of equations (5) we see that

$$\frac{x + \omega y + \omega^2 z}{x + \omega^2 y + \omega z} = C \tag{9}$$

is an integral of the system (1).

PROBLEMS

1. Transform equations (9), Art. 49, into equations (10), making use of Euler's relation, $e^{i\theta} = \cos\theta + i\sin\theta$.

2. Obtain the solution of the system

$$\frac{dx}{dt} = y + z - x,$$

$$\frac{dy}{dt} = z + x - y,$$

$$\frac{dz}{dt} = x + y - z,$$

in the real form

$$x = Ae^t + Be^{-2t},$$

$$y = Ae^t + Ce^{-2t},$$

$$z = Ae^t - (B + C)e^{-2t},$$

(*a*) by putting $a = -1$, $b = c = 1$ in equations (9), Art. 49; (*b*) by putting $a = -1$, $b = c = 1$, hence $\lambda = -2$, $\mu = 0$ in equations (10), Art. 49.

Solve in real form the following systems, where $\mathrm{D} = d/dt$:

3. $\mathrm{D}x = ay,$
$\mathrm{D}y = az,$
$\mathrm{D}z = ax.$

4. $\mathrm{D}x = \mathrm{D}y = \mathrm{D}z = x + y + z.$

5. $\mathrm{D}x + \mathrm{D}y = z,$
$\mathrm{D}y + \mathrm{D}z = x,$
$\mathrm{D}z + \mathrm{D}x = y.$

6. $\mathrm{D}x = \frac{1}{2}(y + z + t),$
$\mathrm{D}y = \frac{1}{2}(z + x + t),$
$\mathrm{D}z = \frac{1}{2}(x + y + t).$

7. $t\,\mathrm{D}x = y + z,$
$t\,\mathrm{D}y = z + x,$
$t\,\mathrm{D}z = x + y.$

8. $t\,\mathrm{D}x_1 = ax_2,$
$t\,\mathrm{D}x_2 = ax_3,$
$t\,\mathrm{D}x_3 = ax_4,$
$t\,\mathrm{D}x_4 = ax_1.$

9. Show that the substitutions

$$u = x + y + z,$$

$$v = xy + yz + zx,$$

$$w = xyz,$$

reduce system (1) of Art. 50 to the system

$$\mathrm{D}u + u^2 = 3v,$$

$$\mathrm{D}v = 0,$$

$$\mathrm{D}w + 3uw = v^2.$$

10. Obtain the integrals

$$\frac{z - y}{z - x} = C \qquad \text{and} \qquad xy + yz + zx = C$$

of system (1), Art. 50.

11. Prove that the three integrals of system (1), Art. 50, given in equation (9) and in Prob. 10, namely,

$$f(x, y, z) = \frac{x + \omega y + \omega^2 z}{x + \omega^2 y + \omega z} = C_1,$$

$$g(x, y, z) = \frac{z - y}{z - x} = C_2,$$

$$h(x, y, z) = xy + yz + zx = C_3,$$

are not independent by showing that the Jacobian of f, g, h with respect to x, y, z vanishes. (*Cf.* Art. 20.)

12. Letting $x/z = \alpha$, $y/z = \beta$, the first two integrals in Prob. 11 may be written

$$F(\alpha, \beta) = \frac{\alpha + \omega\beta + \omega^2}{\alpha + \omega^2\beta + \omega} = C_1.$$

$$G(\alpha, \beta) = \frac{1 - \beta}{1 - \alpha} = C_2.$$

Show that F and G are not independent and find the relation connecting them.

Chapter 7

THE LINEAR EQUATION OF SECOND ORDER

51. Introduction. In Chapter 4 we examined the linear equation of second order, but with constant coefficients. Some special cases of linear equations with variable coefficients were considered in Chapter 5. We now turn our attention to the general linear equation of second order,

$$f_1 y'' + f_2 y' + f_3 y = f_4, \tag{1}$$

where primes denote differentiation with respect to x, and the f's are functions of x (or constants), $f_1 \neq 0$. But we exclude the cases previously treated, namely,

f_1, f_2, f_3 constant: linear equation with constant coefficients, Chapter 4;
$f_1 = a_0x^2$, $f_2 = a_1x$, $f_3 = a_2$: reducible to preceding case, Art. 40;
$f_3 = 0$: dependent variable absent, Art. 41.

When $f_4 = 0$ we have

$$f_1 y'' + f_2 y' + f_3 y = 0. \tag{2}$$

Using the notation of Art. 35, we may call y_c the complementary function of equation (1). It is the most general value of y satisfying equation (2) and contains two arbitrary constants. Also we call y_p a particular integral of equation (1). It is a particular value of y satisfying equation (1) and containing no arbitrary constants. Then

$$y = y_c + y_p \tag{3}$$

is the general solution of equation (1). This is a property of linear equations; the verification (Art. 35) holds whether the differential equation has constant or variable coefficients.

We shall consider in this chapter some cases where equation (2) can be solved without the use of infinite series. Methods involving the use of series will be taken up in the next chapter. With y_c known, a method is given for finding y_p and hence the general solution of equation (1). We shall see also that sometimes the general solution of equation (1) may be obtained without first finding y_c.

52. Exact equations. An equation of the form

$$f_1y'' + f_2y' + f_3y = f_4 \tag{1}$$

is *exact* if the expression forming the left member of the equation is the exact derivative of some function of x and y. This function will then be the integral of the given expression. We now illustrate a process that will not only tell when an expression is exact but will also, if the expression is exact, produce its integral. We carry out the process for the expression $x^2y'' + (x + 1)y' - y$, and follow up with an explanation of the steps.

$$\begin{array}{l|l} x^2y'' + (x+1)y' - y & \underline{x^2y' - (x-1)y} \\ \underline{x^2y'' + 2xy'} & \\ \quad -(x-1)y' - y & \\ \quad \underline{-(x-1)y' - y} & \end{array}$$

The first term x^2y'' would come from differentiating x^2y', and so we write this as the first term of the integral. Now differentiate x^2y', obtaining $x^2y'' + 2xy'$. Subtracting this from the given expression, we obtain $-(x - 1)y' - y$, whose first term comes from differentiating $-(x - 1)y$. Writing this as the second term of the integral and differentiating it, we get $-(x - 1)y' - y$. The process yields no remainder, hence the given expression is exact and its integral is $x^2y' - (x - 1)y$.

If we apply this process to the left member of equation (1), we have

$$\begin{array}{l|l}
f_1y'' + f_2y' + f_3y & f_1y' + (f_2 - f_1')y \\
\underline{f_1y'' + f_1'y'} & \\
(f_2 - f_1')y' + f_3y & \\
\underline{(f_2 - f_1')y' + (f_2' - f_1'')y} & \\
\qquad (f_3 - f_2' + f_1'')y &
\end{array}$$

The given expression will be exact if the remainder is 0. The test for exactness is therefore

$$f_3 - f_2' + f_1'' \equiv 0;$$

when this identity holds, the given expression is exact, and its integral is

$$f_1y' + (f_2 - f_1')y.$$

Hence equation (1), if it is exact, can be reduced at once by integration to a linear equation of first order whose solution will yield the general solution of equation (1). We have the result:

The differential equation

$$f_1y'' + f_2y' + f_3y = f_4 \tag{1}$$

is exact if

$$f_3 - f_2' + f_1'' \equiv 0, \tag{2}$$

in which case it integrates into

$$f_1y' + (f_2 - f_1')y = \int f_4\,dx + C_1. \tag{3}$$

EXAMPLE 1. Solve

$$(x^2 + x)y'' + (x - 1)y' - y = 2x. \tag{4}$$

Here

$$f_1 = x^2 + x, \quad f_1' = 2x + 1, \quad f_1'' = 2, \quad f_2 = x - 1,$$
$$f_2' = 1, \quad f_3 = -1,$$

and the test for exactness, equation (2), gives

$$f_3 - f_2' + f_1'' = -1 - 1 + 2 = 0;$$

hence the equation is exact, and the first integral, by equation (3), is

$$(x^2 + x)y' + [x - 1 - (2x + 1)]y = \int 2x\,dx + C_1,$$

or

$$x(x + 1)y' - (x + 2)y = x^2 + C_1. \tag{5}$$

Dividing through by $x(x + 1)$, we get

$$y' - \frac{x + 2}{x(x + 1)}y = \frac{x}{x + 1} + \frac{C_1}{x(x + 1)},$$

a linear equation of first order which we solve by the method of Art. 21.

$$e^{\int \frac{-(x+2)}{x(x+1)}dx} = e^{\int \left(\frac{1}{x+1} - \frac{2}{x}\right)dx} = e^{\ln(x+1) - 2\ln x} = \frac{x + 1}{x^2}.$$

Hence

$$\frac{x + 1}{x^2}y = \int \left(\frac{1}{x} + \frac{C_1}{x^3}\right)dx + C_2 = \ln x - \frac{C_1}{2x^2} + C_2.$$

Multiplying by x^2, and replacing $-C_1/2$ by A and C_2 by B, we have the general solution in the form

$$(x + 1)y = x^2 \ln x + A + Bx^2, \tag{6}$$

where A and B are arbitrary constants.

The above procedure produced the two parts of the general solution, y_c and y_p, simultaneously: $y_c = (A + Bx^2)/(x + 1)$ and $y_p = (x^2 \ln x)/(x + 1)$. If in any problem y_p were obtainable by inspection, the above procedure could be used merely to obtain y_c, then the addition of y_p would give the general solution. Such a case occurs when f_3 and f_4 are constant since then equation (1) is satisfied by $y = f_4/f_3$, $y' = 0$, $y'' = 0$, and we have $y_p = f_4/f_3$. Consider, for example, the equation obtained by changing the right member of equation (4) in the preceding example from $2x$ to 2:

EXAMPLE 2. Solve

$$(x^2 + x)y'' + (x - 1)y' - y = 2. \tag{7}$$

Equation (7) is satisfied if $y = -2$, so that we have, by inspection, $y_p = -2$ and we solve for y_c the equation

$$(x^2 + x)y'' + (x - 1)y' - y = 0. \tag{8}$$

The solution proceeds as in the previous example except that we omit the first term on the right; equation (5) becomes

$$x(x + 1)y' - (x + 2)y = C_1,$$

and finally, instead of equation (6), we have

$$(x + 1)y = A + Bx^2.$$

Hence

$$y_c = \frac{A + Bx^2}{x + 1},$$

and

$$y = y_c + y_p = \frac{A + Bx^2}{x + 1} - 2,$$

or, multiplying by $x + 1$ and replacing $A - 2$ by c_1 and B by c_2, we obtain

$$(x + 1)y = c_1 + c_2x^2 - 2x, \tag{9}$$

the general solution of equation (7).

If we put $c_1 = c_2 = 0$ in equation (9), we see that $y_p = -2x/(x + 1)$ is a particular integral of equation (7), but not as simple as the one we used, $y_p = -2$.

EXAMPLE 3. Solve

$$xy'' + (1 + x - 2x^2)y' + (1 - 4x)y = 0.$$

Applying the test for exactness, we get

$$1 - 4x - (1 - 4x) + 0 \equiv 0;$$

hence the equation is exact. The first integral is

$$xy' + (1 + x - 2x^2 - 1)y = C_1,$$

or

$$y' + (1 - 2x)y = \frac{C_1}{x}.$$

Then

$$e^{\int(1-2x)\,dx}y = e^{x-x^2}y = \int \frac{C_1}{x}\, e^{x-x^2}\, dx + C_2,$$

or

$$y = e^{x^2-x}\left[C_1 \int \frac{e^{x-x^2}}{x}\, dx + C_2\right].$$

Here we leave the solution in a form involving an integral which is not integrable in terms of elementary functions except by means of infinite series.

By the method of this article a solution is obtained whether the right member is zero or not, but only when the equation is exact. For non-exact equations some other procedures will now be developed, first for the equation with right member zero.

53. Right member zero; particular solution known. The linear equation with right member zero can be written in either of the forms

$$f_1y'' + f_2y' + f_3y = 0, \tag{1}$$

or

$$y'' + Py' + Qy = 0, \tag{2}$$

where $P = f_2/f_1$ and $Q = f_3/f_1$. It is understood that P and Q are not both constants and $Q \neq 0$. Equations (1) and (2) are equivalent; they have the same general solution of the form

$$y = C_1y_1 + C_2y_2, \tag{3}$$

where y_1 and y_2 are functions of x(y_1 not merely a constant times y_2) and C_1, C_2 are arbitrary constants; $y = y_1$ and $y = y_2$ are distinct particular solutions.

For convenience we write u for y_1, and v for y_2, and derive a formula for obtaining v in terms of u, so that if one particular solution u is known we can find another one, v, and hence the general solution (3).

Assuming that $y = u$ and $y = v$ are particular solutions of equation (2), we have

$$u'' + Pu' + Qu = 0, \tag{4}$$

$$v'' + Pv' + Qv = 0, \tag{5}$$

Subtracting $v \times$ (4) from $u \times$ (5), we find

$$uv'' - vu'' + P(uv' - vu') = 0,$$

or

$$\frac{uv'' - vu''}{uv' - vu'} = -P. \tag{6}$$

Now, since $uv'' - vu''$ is the exact derivative of $uv' - vu'$, we have, upon integrating equation (6),

$$\ln (uv' - vu') = -\int P\,dx,$$

or

$$uv' - vu' = e^{-\int P\,dx}. \tag{7}$$

We omit the constant of integration or take it equal to zero since we are seeking the simplest expression for v in terms of u.

Dividing equation (7) by u, we get

$$v' - \frac{u'}{u} v = \frac{1}{u} e^{-\int P\,dx},$$

a linear equation to be solved for v. Since $e^{-\int (u'/u)\,dx} = e^{-\ln u} = 1/u$, we obtain

$$\frac{1}{u} v = \int \frac{1}{u^2} e^{-\int P\,dx}\,dx,$$

again omitting the constant of integration, or

$$v = u \int \frac{e^{-\int P\,dx}}{u^2}\,dx. \tag{8}$$

When a particular solution u of equation (1) or (2) is known or can be found, formula (8) gives another particular solution v; then the general solution is

$$y = C_1 u + C_2 v. \tag{9}$$

In some cases a particular solution of (1) or (2) can be found by inspection or by a simple test. For example, substituting $y = e^{\lambda x}$ in equation (1), we obtain $(\lambda^2 f_1 + \lambda f_2 + f_3)e^{\lambda x} = 0$, hence the test: $y = e^{\lambda x}$ is a particular solution of (1) if

$$\lambda^2 f_1 + \lambda f_2 + f_3 \equiv 0. \tag{10}$$

Special cases for $\lambda = \pm 1$ are: e^x is a particular solution if $f_1 + f_2 + f_3 \equiv 0$; e^{-x} is a particular solution if $f_1 - f_2 + f_3 \equiv 0$.

Also $y = x + a$ is a particular solution of (1) if

$$f_2 + (x + a)f_3 \equiv 0; \tag{11}$$

in the special case where $a = 0$, x is a particular solution if $f_2 + xf_3 \equiv 0$.

EXAMPLE 1. Solve

$$(1 - x^2)y'' - xy' + y = 0.$$

Since $f_2 + xf_3 = 0$, $y = x$ is a particular solution, and a second particular solution is obtained by substituting $u = x$ and $P = -x/(1 - x^2)$ in formula (8):

$$v = x\int \frac{e^{\int x/(1-x^2)\,dx}}{x^2}\,dx = x\int \frac{e^{-\frac{1}{2}\ln(1-x^2)}}{x^2}\,dx$$

$$= x\int \frac{dx}{x^2\sqrt{1 - x^2}} = \text{[by Peirce, 153]} - \sqrt{1 - x^2}.$$

With these values of u and v, equation (9) gives the general solution

$$y = C_1x + C_2\sqrt{1 - x^2}.$$

Since $f_3 - f_2' + f_1'' = 0$, the differential equation is exact and can be solved also by the method of Art. 52.

EXAMPLE 2. Solve

$$xy'' + (x + 2)y' + 2y = 0.$$

Since $f_1 - f_2 + f_3 = 0$, $y = e^{-x}$ is a particular solution, and a second particular solution is obtained by substituting $u = e^{-x}$ and $P = 1 + 2/x$ in formula (8):

$$v = e^{-x}\int \frac{e^{-\int[1+(2/x)]\,dx}}{e^{-2x}}\,dx = e^{-x}\int \frac{e^{-x-2\ln x}}{e^{-2x}}\,dx = e^{-x}\int \frac{e^x\,dx}{x^2}.$$

Substituting these values of u and v in equation (9), we have the general solution

$$y = e^{-x}\left[C_1 + C_2\int \frac{e^x}{x^2}\,dx\right].$$

PROBLEMS

1. Solve

$$(1 - 4x^2)y'' - 4xy' + 4y = 0$$

(*a*) by the method of Art. 53; (*b*) by the method of Art. 52.

2. Solve

$$(2x^2 + 3x + 1)y'' + 2xy' - 2y = 0$$

(*a*) by the method of Art. 53; (*b*) by the method of Art. 52.

Solve the following differential equations.

3. $xy'' + (2 - x)y' - y = 2\cos x$.
4. $(1 - \ln x)xy'' + y' - y/x = 0$.
5. $xy'' + (2 - x)y' + y = 0$.
6. $xy'' + 2(x + 1)y' + 2y = 4xe^{-2x}$.
7. $y'' + (1 - x^2)y' - 2xy = 0$.
8. $(x + 1)y'' + (x + 1)y' + y = 0$.
9. $(x + 1)y'' + (x + 1)y' - y = 0$.
10. $y'' - xy' - y = xe^{x^2/2}$.
11. $(ax^2 + x)y'' + (4ax + 2)y' + 2ay = 0$.
12. $xy'' - y' + (1 - x)y = 0$.
13. $(1 - x^2)y'' - 2xy' + 2y = 0$.
14. $xy'' + (2 - x)y' - 2y = 0$.
15. $x^2y'' + (x + 1)y' - y = a$.
16. $x^2y'' + (x + 1)y' - y = 2x$.
17. $y'' + (x + 1)y' + xy = 0$.
18. $x(x - 1)y'' - (2x - 1)y' + 2y = 0$.
19. $x(x + 1)y'' + (x + 2)y' - y = 5$.
20. $x(x + 2)y'' - 2(x + 1)y' + 2y = 0$.
21. $xy'' - 2xy' - 2y = 2xe^{2x}$.

22. Find two particular solutions, and hence the general solution, of the differential equation

$$xy'' - (1 - 2x)y' - 2y = 0.$$

Apply tests (10) and (11), determining the values of λ and a.

23. Solve

$$(x + 1)y'' - (x + 2)y' - 2xy = 0,$$

first finding a particular solution by test (10).

24. Show that the differential equation

$$(a_1x + b_1)y'' + (a_2x + b_2)y' + (a_3x + b_3)y = 0$$

has a particular solution $y = e^{\lambda x}$ if

$$\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \cdot \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \neq 0,$$

where

$$\lambda = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \div \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

25. Apply the result of Prob. 24 in finding a particular solution of the differential equation of Prob. 23.

26. Find a particular solution of the differential equation

$$(x - 3)y'' - (x - 2)y' - 2(x - 4)y = 0$$

(*a*) by using the result of Prob. 24; (*b*) by applying test (10). Hence obtain the general solution.

27. Solve the differential equation of Prob. 14 by writing it in operator notation and factoring the operator.

28. Solve the differential equation of Prob. 23 by writing it in operator notation and factoring the operator.

54. The equivalent Riccati equation. If we change the dependent variable from y to v in the equation

$$y'' + Py' + Qy = 0 \tag{1}$$

by means of the substitution

$$y = e^{-\int v\,dx}, \qquad y' = -ve^{-\int v\,dx}, \qquad y'' = (v^2 - v')e^{-\int v\,dx},$$

we obtain

$$v^2 - v' - Pv + Q = 0,$$

or

$$v' = Q - Pv + v^2, \tag{2}$$

a Riccati equation * equivalent to (1).

Now, if we can find a particular solution of (2), say v_1, it follows that

$$y_1 = e^{-\int v_1\,dx} \tag{3}$$

* The general form of Riccati's equation is $dy/dx = P + Qy + Ry^2$, where P, Q, R are functions of x.

is a particular solution of (1). Then the substitution of this value for u in equation (8), Art. 53, gives another particular solution of (1) and hence the general solution. As we had no general rule for finding y_1 in Art. 53, so here we have no general rule for finding v_1.* But it may happen that it is easy to find v_1 although it is difficult to find y_1 directly; in such cases equation (3) is useful.

Carrying out the substitution of y_1 for u in equation (8), Art. 53, we obtain a second particular solution y_2:

$$y_2 = e^{-\int v_1\,dx} \int \frac{e^{-\int P\,dx}}{e^{-2\int v_1\,dx}}\,dx = e^{-\int v_1\,dx} \int e^{\int (2v_1 - P)\,dx}\,dx.$$

Then the general solution of equation (1) is

$$y = C_1 y_1 + C_2 y_2,$$

that is

$$y = e^{-\int v_1\,dx}\left[C_1 + C_2 \int e^{\int (2v_1 - P)\,dx}\,dx\right]. \tag{4}$$

Equation (4) can be used as a formula for solving equation (1) when v_1 is known. If two distinct particular solutions v_1 and v_2 of equation (2) are known, then

$$y_1 = e^{-\int v_1\,dx} \quad \text{and} \quad y_2 = e^{-\int v_2\,dx}$$

are particular solutions of equation (1), and the general solution of equation (1) is

$$y = C_1 e^{-\int v_1\,dx} + C_2 e^{-\int v_2\,dx}, \tag{5}$$

EXAMPLE 1. Solve

$$xy'' - (1 - 2x)y' - 2y = 0. \tag{6}$$

Dividing by x we have

$$y'' - \left(\frac{1}{x} - 2\right)y' - \frac{2}{x}y = 0;$$

* The condition that a polynomial can be a solution of equation (2) *if* P *and* Q *are polynomials,* is given in Rainville, *Intermediate Differential Equations,* John Wiley & Sons, Art. 13.

hence $P = -[(1/x) - 2]$, $Q = -2/x$, and the equivalent Riccati equation (2) is

$$v' = -\frac{2}{x} + \left(\frac{1}{x} - 2\right)v + v^2. \tag{7}$$

It might be observed that $v = 2$ is a particular solution of equation (7). Or, if this is not obvious, we might try

$$v = ax^n, \qquad v' = nax^{n-1},$$

and see if we can determine n and a so that the identity

$$nax^{n-1} \equiv -\frac{2}{x} + ax^{n-1} - 2ax^n + a^2x^{2n}$$

holds. We can make the terms $-2/x$ and ax^{n-1} cancel by choosing $n = 0$, $a = 2$. This gives

$$0 \equiv -\frac{2}{x} + \frac{2}{x} - 4 + 4;$$

hence $v = 2$ is a particular solution.

Substituting $v_1 = 2$ in equation (4), we have

$$\begin{aligned} y &= e^{-2x}[C_1 + C_2 \int e^{\int (2 + 1/x)\,dx}\,dx] \\ &= e^{-2x}[C_1 + C_2 \int xe^{2x}\,dx] \\ &= e^{-2x}\left[C_1 + C_2 \frac{e^{2x}}{4}(2x - 1)\right]. \end{aligned}$$

Hence, writing $C_1 = A$, $C_2/4 = B$, we have the general solution of equation (6):

$$y = Ae^{-2x} + B(2x - 1). \tag{8}$$

Another method. Although equation (6) is not exact, one might see that by rearranging the terms and dividing by x^2 it can be arranged in integrable combinations as follows:

$$\frac{xy'' - y'}{x^2} + \frac{2(xy' - y)}{x^2} = 0.$$

Integrating these combinations, we have

$$\frac{y'}{x} + 2\frac{y}{x} = C_1,$$

$$y' + 2y = C_1 x,$$

a linear equation of first order; hence

$$e^{2x}y = C_1 \int xe^{2x}\,dx + C_2 = C_1 \frac{e^{2x}}{4}(2x - 1) + C_2,$$

or

$$y = Ae^{-2x} + B(2x - 1).$$

This example can be solved also by the method of Art. 53. (See Problem 22, Art. 53.)

EXAMPLE 2. Solve

$$y'' - y' + e^{2x}y = 0. \tag{9}$$

The equivalent Riccati equation is

$$v' = e^{2x} + v + v^2. \tag{10}$$

We try an exponential function for v (although there is no guarantee beforehand that an exponential solution exists). Put

$$v = be^{ax}, \qquad v' = abe^{ax},$$

and see if a and b can be chosen so that the identity

$$abe^{ax} \equiv e^{2x} + be^{ax} + b^2e^{2ax}$$

holds. In order to get a term that will cancel e^{2x} we must have $a = 1$ or 2, but if $a = 2$ the last term would become b^2e^{4x}, forcing $b = 0$, so we take $a = 1$ and obtain

$$be^x \equiv e^{2x} + be^x + b^2e^{2x}.$$

This identity is satisfied if

$$b^2 + 1 = 0, \qquad b = \pm i.$$

Hence there are two particular solutions of equation (10) of the form be^{ax}:

$$v_1 = ie^x \qquad \text{and} \qquad v_2 = -ie^x.$$

The fact that a particular solution of (10) is imaginary does not mean that the corresponding solution of (9) need be imaginary.

Substitution of these values for v_1 and v_2 in equation (5) gives

$$y = C_1e^{-ie^x} + C_2e^{ie^x},$$

which, by use of Euler's relation as in Art. 31(*c*), reduces to

$$y = A \cos e^x + B \sin e^x, \tag{11}$$

the general solution of equation (9).

PROBLEMS

1. Solve

$$xy'' + (1 + x^2)y' + 2xy = 0$$

(*a*) by the method of Art. 54; (*b*) by the method of Art. 52.

2. Solve

$$y'' - x^2y' + xy = 0$$

(*a*) by the method of Art. 54; (*b*) by the method of Art. 53.

3. Solve

$$xy'' - (2x + 1)y' + 2y = 0$$

(*a*) by the method of Art. 54; (*b*) by finding two particular solutions as in Prob. 22, Art. 53.

4. Solve

$$2xy'' + (3x + 2)y' + (x + 1)y = 0$$

(*a*) by the method of Art. 54; (*b*) by the method of Art. 53, using the result of Prob. 24, Art. 53.

Solve the following differential equations.

5. $xy'' - (2x^2 + 3)y' + 4xy = 0.$
6. $xy'' + (2x^4 - 1)y' + 4x^3y = 0.$
7. $x^2y'' + x(1 - 2x^2)y' - (1 + 2x^2)y = 0.$
8. $4x^2y'' - 4x^2y' + (1 + 2x)y = 0.$
9. $xy'' - y' + x^3y = 0.$
10. $4xy'' + 2y' - y = 0.$
11. $xy'' - y' + 4x^3y = 0.$
12. $y'' + (4x - 1)y' + 2(2x^2 - x + 1)y = 0.$
13. $y'' + (e^{2x} - 1)y' + e^{2x}y = 0.$
14. $y'' + (e^{2x} - 1)y' - e^{2x}y = 0.$
15. $y'' - (\cot x)y' + (\sin^2 x)y = 0.$
16. $x^3y'' + x(3x - 1)y' - y = 0.$
17. Solve the differential equation

$$xy'' - (1 + x^2)y' + x^3y = 0$$

by obtaining the particular solutions $v_1 = \omega x$, $v_2 = \omega^2 x$, of the equivalent Riccati equation, where $1, \omega, \omega^2$, are the cube roots of unity.

55. Right member not zero. Consider now the general equation (1) of Art. 51:

$$f_1 y'' + f_2 y' + f_3 y = f_4, \tag{1}$$

with $f_4 \neq 0$, excluding the case where (1) is exact, which has been treated in Art. 52. We know that the general solution will be of the form

$$y = y_c + y_p, \tag{2}$$

where

$$y_c = C_1 y_1 + C_2 y_2 \tag{3}$$

is the complementary function representing the general solution of the equation

$$f_1 y'' + f_2 y' + f_3 y = 0, \tag{4}$$

and y_p is a particular function, free of arbitrary constants, satisfying equation (1).

We develop two methods which can be used in certain cases to solve equation (1). By the first method we obtain the solution (2) without finding y_c and y_p separately; by the second we obtain y_p after y_c is known.

(*a*) *A substitution method.* Divide equation (1) by f_1 and write it in the form

$$y'' + Py' + Qy = R. \tag{5}$$

Now, in an attempt to solve equation (5), suppose that we try the effect of substituting $y = uv$, where u and v are unknown functions of x. We have

$$y = uv, \qquad y' = uv' + u'v, \qquad y'' = uv'' + 2u'v' + u''v,$$

and equation (5) becomes

$$uv'' + 2u'v' + u''v + P(uv' + u'v) + Quv = R,$$

or

$$uv'' + (2u' + Pu)v' + (u'' + Pu' + Qu)v = R. \tag{6}$$

One way to simplify equation (6) is to choose u so that $u'' + Pu' + Qu = 0$, thus causing the terms in v to disappear from equation (6). This means that $y = u$ is a particular solution of the equation

$$y'' + Py' + Qy = 0. \tag{7}$$

If then we have given, or can find in some manner, a value u of y which satisfies equation (7), the substitution of $y = uv$ in (5) reduces it to (6) with the v-terms absent:

$$\frac{dv'}{dx} + \left(\frac{2u'}{u} + P\right)v' = \frac{R}{u}. \tag{8}$$

This equation is linear in v' and dv'/dx, and may be solved in terms of v' and x by the method of Art. 21. Then, writing $v' = dv/dx$, a second integration yields v. Having u and v, we multiply them together to obtain the general solution, $y = uv$, of equation (5).

This method depends on knowing a particular solution of equation (7), which may be obtainable as in Art. 53 or Art. 54. The method does not assume a knowledge of the general solution (3) of equation (7), although, if (3) is known, either y_1 or y_2 can be taken as u in equation (8).

EXAMPLE 1. Solve

$$(1 - x^2)y'' - xy' + y = 3x^2. \tag{9}$$

As in Example 1, Art. 53, we have the particular solution $y = x$, of the equation

$$(1 - x^2)y'' - xy' + y = 0, \tag{10}$$

or if we assume as known the general solution of (10), namely, $y = C_1x + C_2\sqrt{1 - x^2}$, the particular solution $y = x$ is obtained from it by putting $C_1 = 1$, $C_2 = 0$. Substituting

$$u = x, \qquad u' = 1, \qquad P = -\frac{x}{1 - x^2}, \qquad R = \frac{3x^2}{1 - x^2},$$

in equation (8), it becomes

$$\frac{dv'}{dx} + \left(\frac{2}{x} - \frac{x}{1 - x^2}\right)v' = \frac{3x}{1 - x^2}.$$

By the method of Art. 21,

$$e^{\int [(2/x) - x/(1-x^2)]\,dx} = e^{2 \ln x + \frac{1}{2} \ln (1-x^2)} = x^2\sqrt{1 - x^2}.$$

Hence

$$x^2\sqrt{1 - x^2}v' = \int \frac{3x^3}{\sqrt{1 - x^2}}\,dx + C_1' = -(2 + x^2)\sqrt{1 - x^2} + C_1',$$

$$v' = \frac{dv}{dx} = -\frac{2}{x^2} - 1 + \frac{C_1'}{x^2\sqrt{1 - x^2}},$$

$$v = \frac{2}{x} - x - \frac{C_1'\sqrt{1 - x^2}}{x} + C_2,$$

and the general solution, $y = xv$, is

$$y = 2 - x^2 + C_1\sqrt{1 - x^2} + C_2x.$$

(*b*) *Variation of parameters.* We now assume that the complementary function, $y_c = C_1u + C_2v$, of equation (1) is known. Then the theory explained in Art. 37 shows that y_p is given by the formula

$$y_p = Au + Bv, \tag{11}$$

where A and B are functions of x determined by the equations (3), Art. 37:

$$\begin{aligned} A_1u + B_1v &= 0, \\ A_1u_1 + B_1v_1 &= R. \end{aligned} \tag{12}$$

The subscripts indicate differentiation with respect to x, and R is f_4/f_1. Eliminating first B_1, then A_1, from equations (12), we have

$$A_1(uv_1 - u_1v) = -vR,$$

$$B_1(uv_1 - u_1v) = uR.$$

Solving for A and B, taking the constant of integration equal to zero, we get

$$A = -\int \frac{vR}{uv_1 - u_1v}\,dx, \qquad B = \int \frac{uR}{uv_1 - u_1v}\,dx. \tag{13}$$

These values of A and B substituted in (11) give y_p; then the general solution of equation (1) is $y = y_c + y_p$. For illustration we solve the above Example 1 by this method.

EXAMPLE 2. (Second solution of Example 1.) Solve

$$(1 - x^2)y'' - xy' + y = 3x^2.$$

The complementary function is (Example 1, Art. 53)

$$y_c = C_1x + C_2\sqrt{1 - x^2}.$$

Substituting

$$u = x, \qquad v = \sqrt{1 - x^2}, \qquad u_1 = 1,$$

$$v_1 = \frac{-x}{\sqrt{1 - x^2}}, \qquad R = \frac{3x^2}{1 - x^2},$$

$$uv_1 - u_1v = \frac{-x^2}{\sqrt{1 - x^2}} - \sqrt{1 - x^2} = -\frac{1}{\sqrt{1 - x^2}},$$

in equations (13), we find

$$A = \int 3x^2\,dx, \qquad B = -\int \frac{3x^3\,dx}{\sqrt{1 - x^2}},$$

or

$$A = x^3, \qquad B = (2 + x^2)\sqrt{1 - x^2}.$$

Hence

$$y_p = Au + Bv = x^4 + (2 + x^2)(1 - x^2) = 2 - x^2$$

and the general solution is

$$y = C_1x + C_2\sqrt{1 - x^2} + 2 - x^2.$$

PROBLEMS

1. Solve

$$xy'' - (2x + 1)y' + 2y = x^2e^{2x}$$

(*a*) by method (*a*), Art. 55; (*b*) by method (*b*), Art. 55.

2. Solve

$$y'' + x^2y' - xy = x^4$$

by method (*a*), Art. 55.

3. Solve

$$xy'' - y' - x^3y = xe^{x^2/2}$$

by method (*b*), Art. 55.

Solve the following differential equations.

4. $x^2y'' - (x^2 + 2x)y' + (x + 2)y = 2x^4$.

5. $xy'' - (2 + x)y' + 2y = 2x^3e^x$.

6. $xy'' + (2x + 1)y' + (x + 1)y = (2x + 1)e^x$.

7. $x^2y'' + 2x(1 - x)y' - x(2 - x)y = \sin x$.

8. $4xy'' - 4y' + (2 - x)y = x^2e^{-x}$.

9. $xy'' - y' - x^3y = x^5$.

10. $(x + 1)y'' - (x + 2)y' - 2xy = x^2 + 1$.

11. $y'' + 2xy' + (x^2 + 1)y = R$, where R is a function of x.

12. Another way to simplify equation (6), Art. 55, is to choose u so that the coefficient of v' will vanish, that is,

$$u = e^{-\frac{1}{2}\int P\,dx}.$$

Show that the resulting transformation

$$y = uv = ve^{-\frac{1}{2}\int P\,dx}$$

transforms the equation

$$y'' + Py' + Qy = R$$

into

$$v'' + (Q - \tfrac{1}{2}P' - \tfrac{1}{4}P^2)v = Re^{\frac{1}{2}\int P\,dx}$$

Hence, if

$$Q - \frac{1}{2}P' - \frac{1}{4}P^2 = c \qquad \text{or} \qquad \frac{c}{x^2},$$

where c is a constant, this differential equation in v and x is of the type considered in Chapter IV or Chapter V (Art. 40).

Solve the following differential equations using the result of Prob. 12.

13. $x^2y'' + xy' + (x^2 - \frac{1}{4})y = ax\sqrt{x}$.

14. $y'' + 2\tanh x \cdot y' + \left(1 - \dfrac{2}{x^2}\right)y = \operatorname{sech} x$.

15. $4x^2(y'' - y') + (x^2 + 1)y = e^{x/2}$.

16. $y'' + 2xy' + x^2y = 2$.

17. $y'' + 2xy' + (x^2 + 1)y = x^3$.

18. Solve the differential equation of Prob. 11 using the result of Prob. 12.

Chapter 8

SERIES SOLUTIONS

56. Power series. There are many simple-looking differential equations, for example, $d^2y/dx^2 + xy = 0$, for which the methods discussed up to this point are inadequate to yield a general solution. When such an equation is encountered it may be possible to solve it by use of an infinite series. We shall consider in this chapter only certain * equations of the form

$$f_1y'' + f_2y' + f_3y = 0, \tag{1}$$

where the f's are functions of x and primes denote differentiation with respect to x.

Assume that a value of y, expressible in the form of a power series in x, satisfies equation (1), and write down the expressions for y and its first two derivatives:

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots, \tag{2}$$

$$y' = a_1 + 2a_2x + 3a_3x^2 + \cdots + (n+1)a_{n+1}x^n + \cdots, \tag{3}$$

$$y'' = 2a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + \cdots + (n+2)(n+1)a_{n+2}x^n + \cdots. \tag{4}$$

* Not all differential equations of form (1) can be solved by the methods of this chapter. For example, the equation $4x^3y'' + 6x^2y' - y = 0$ (*cf.* Prob. 16, Art. 9) has a general solution which is not expressible by series of type (2), Art. 56, or of type (1), Art. 57. In this case the assumption of a solution of type (2), Art. 56, leads merely to the trivial solution $y = 0$, and the assumption of a solution of type (1), Art. 57, leads to a contradiction $a_0 = 0$. Furthermore, in other cases these assumptions may lead to series which do not converge except possibly for $x = 0$. It is understood that a solution expressed in series form is valid only for values of x for which the series is convergent. A fuller discussion may be found in Ince's *Ordinary Differential Equations*, Chapter VII, or in Cohen's *Differential Equations*, Chapter IX. See also Jeffrey's *Mathematical Physics*, Chapter 16.

Now substitute the values of y, y', and y'' from (2), (3), and (4) in equation (1), then equate to zero the complete coefficient of each power of x in order that (1) shall be satisfied identically. It may happen that two of the a's thus remain arbitrary but that the others are determined or expressible in terms of the two arbitrary ones. In this case equation (2) expresses y as a power series in x involving the two a's as arbitrary constants. If the series converges, equation (2) furnishes the general solution of (1), valid for values of x for which the series is convergent.

Some of the following examples illustrating the series method could be solved without the use of series; in such cases it is interesting to notice that the results obtained by the use of series are equivalent to those obtainable by previous methods.

EXAMPLE 1. Solve

$$(1 + x^2)y'' + xy' - y = 0. \tag{5}$$

Substitute the values of y, y', and y'' from (2), (3), and (4) in (5) and arrange the result in tabular form. At the left of the table are the terms of equation (5), at the top the various powers of x which occur, with their coefficients in the body of the table.

	con.	x	x^2	x^3	$\cdots$	x^n	$\cdots$
$-y$	$-a_0$	$-a_1$	$-a_2$	$-a_3$	$\cdots$	$-a_n$	$\cdots$
xy'		a_1	$2a_2$	$3a_3$	$\cdots$	na_n	$\cdots$
y''	$2a_2$	$3\cdot 2a_3$	$4\cdot 3a_4$	$5\cdot 4a_5$	$\cdots$	$(n+2)(n+1)a_{n+2}$	$\cdots$
x^2y''			$2a_2$	$3\cdot 2a_3$	$\cdots$	$n(n-1)a_n$	$\cdots$

The sum of each column of coefficients must vanish. The first two columns give

$$2a_2 - a_0 = 0, \qquad a_2 = \tfrac{1}{2}a_0, \qquad a_0 \text{ arbitrary,}$$

$$3\cdot 2a_3 = 0, \qquad a_3 = 0, \qquad a_1 \text{ arbitrary.}$$

The next two columns give

$$4\cdot 3a_4 + 3a_2 = 0, \qquad a_4 = -\frac{a_2}{4} = -\frac{a_0}{2\cdot 4},$$

$$5\cdot 4a_5 + 8a_3 = 0, \qquad a_5 = 0,$$

and so we see that all a's with odd subscripts vanish, except a_1 which is arbitrary, and that each a with even subscript can be expressed in terms of a_0 which is arbitrary.

Perhaps the best way to compute the a's is to obtain a general formula by equating to zero the column of coefficients of x^n:

$$(n+2)(n+1)a_{n+2} + (n^2 - 1)a_n = 0,$$

$$a_{n+2} = -\frac{n-1}{n+2}a_n.$$

Then, letting $n = 0, 1, 2, 3, 4, \cdots$, we find

$$n = 0, \qquad a_2 = \tfrac{1}{2}a_0,$$

$$n = 1, \qquad a_3 = 0,$$

$$n = 2, \qquad a_4 = -\tfrac{1}{4}a_2 = -\frac{1}{2\cdot 4}a_0,$$

$$n = 3, \qquad a_5 = -\tfrac{2}{5}a_3 = 0,$$

$$n = 4, \qquad a_6 = -\tfrac{3}{6}a_4 = \frac{1\cdot 3}{2\cdot 4\cdot 6}a_0,$$

. . . .

Substituting these values of the a's in equation (2) and remembering that a_1 is arbitrary, we have

$$y = a_0\left(1 + \frac{1}{2}x^2 - \frac{1}{2\cdot 4}x^4 + \frac{1\cdot 3}{2\cdot 4\cdot 6}x^6 - \cdots\right) + a_1x, \tag{6}$$

which, with a_0 and a_1 arbitrary constants, is the general solution of equation (5).

We might recognize the series in parentheses as the expansion of $(1 + x^2)^{1/2}$ and write the solution (6) in the finite form

$$y = a_0\sqrt{1 + x^2} + a_1x. \tag{7}$$

However, if the series in parentheses were not recognized, we could obtain the finite form of solution as follows. Putting $a_0 = 0$, $a_1 = 1$ in (6) yields the simple particular solution $y = x$. Then a second particular solution is obtained by substituting $u = x$ and $P = x/(1 + x^2)$ in formula (8), Art. 53:

$$x \int \frac{e^{-\int x/(1+x^2)\,dx}}{x^2}\,dx = x \int \frac{e^{-\frac{1}{2}\ln(1+x^2)}}{x^2}\,dx$$

$$= x \int \frac{dx}{x^2\sqrt{1+x^2}} = \text{[by Peirce, 152]} - \sqrt{1+x^2}.$$

Hence

$$y = Ax + B\sqrt{1+x^2},$$

where A and B are arbitrary constants, which is form (7). Notice that the series solution (6) is valid only for $-1 \leqq x \leqq 1$, whereas there is no such restriction on x in solution (7).

The solution may be verified by eliminating the arbitrary constants from the solution in finite form and thus obtaining the original differential equation (Problem 17, Art. 9). The differential equation can be solved without the use of series by the method of Art. 52 or the method of Art. 53.

It may be noticed also that the above general solution is obtainable from that of Example 1, Art. 53, by changing x to ix, since this substitution, together with the corresponding substitutions for y' and y'', transforms one differential equation into the other.

EXAMPLE 2. Given the differential equations

$$x^2y'' - 2xy' + 2y = 0, \qquad (a)$$

$$4x^2y'' + 4xy' - y = 0, \qquad (b)$$

obtain the general solution of (a) by the power series method and show that this method is inadequate to yield the general solution of (b).

Substitution of the values of y, y', and y'' from equations (2), (3), and (4) in equation (a) results in the following table:

	con.	x	x^2	x^3	$\cdots$	x^n	$\cdots$
$2y$	$2a_0$	$2a_1$	$2a_2$	$2a_3$	$\cdots$	$2a_n$	$\cdots$
$-2xy'$		$-2a_1$	$-2\cdot 2a_2$	$-2\cdot 3a_3$	$\cdots$	$-2na_n$	$\cdots$
x^2y''			$2a_2$	$3\cdot 2a_3$	$\cdots$	$n(n-1)a_n$	$\cdots$

Equating to zero the sum of each column of coefficients, we see that all the a's are zero except a_1 and a_2 which are arbitrary since the columns in which they occur vanish identically. This could also be seen by summing the coefficients of x^n:

$$(n^2 - 3n + 2)a_n = (n - 1)(n - 2)a_n = 0;$$

hence $a_n = 0$ unless $n = 1$ or 2, a_1 and a_2 being arbitrary.

Equation (2) then reduces to

$$y = a_1x + a_2x^2,$$

the general solution of equation (a) in finite form.

Applying the same method to equation (b) we have the table:

	con.	x	x^2	x^3	$\cdots$	x^n	$\cdots$
$-y$	$-a_0$	$-a_1$	$-a_2$	$-a_3$	$\cdots$	$-a_n$	$\cdots$
$4xy'$		$4a_1$	$4\cdot 2a_2$	$4\cdot 3a_3$	$\cdots$	$4na_n$	$\cdots$
$4x^2y''$			$4\cdot 2a_2$	$4\cdot 3\cdot 2a_3$	$\cdots$	$4n(n-1)a_n$	$\cdots$

Equating to zero the sum of each column we find that all the a's are zero; equation (2) gives only the trivial solution $y = 0$. There is no power series in x which satisfies equation (b). The power series method is inadequate for the solution of (b) but we shall obtain the solution by a more general series method in the next article. Both equations (a) and (b) can be solved by the method of Art. 40.

Example 3. Solve by the power series method

$$xy'' - (x + 2)y' + 2y = 0, \tag{8}$$

and express the solution in finite form.

Substituting the values of y, y', and y'' from (2), (3), and (4) in (8), we have the following table:

	con.	x	x^2	x^3	$\cdots$	x^n	$\cdots$
$2y$	$2a_0$	$2a_1$	$2a_2$	$2a_3$	$\cdots$	$2a_n$	$\cdots$
$-2y'$	$-2a_1$	$-2\cdot 2a_2$	$-2\cdot 3a_3$	$-2\cdot 4a_4$	$\cdots$	$-2(n+1)a_{n+1}$	$\cdots$
$-xy'$		$-a_1$	$-2a_2$	$-3a_3$	$\cdots$	$-na_n$	$\cdots$
xy''		$2a_2$	$3\cdot 2a_3$	$4\cdot 3a_4$	$\cdots$	$(n+1)na_{n+1}$	$\cdots$

Equating to zero the sum of each column, the first column gives $a_1 = a_0$, with a_0 arbitrary; the next gives $-2a_2 + a_1 = 0$, $a_2 = a_1/2$; the x^2 column vanishes identically, making a_3 arbitrary; the remaining columns determine $a_4, a_5, \cdots$ in terms of a_3. The general formula, obtained by summing the coefficients in the x^n column, is

$$(n + 1)(n - 2)a_{n+1} = (n - 2)a_n, \tag{9}$$

or

$$a_{n+1} = \frac{a_n}{n+1}, \qquad (n \neq 2),$$

from which we find

$$n = 0, \qquad a_1 = a_0,$$

$$n = 1, \qquad a_2 = \frac{a_1}{2} = \frac{a_0}{2},$$

$$n = 2, \qquad a_3 \text{ arbitrary}, \qquad \text{[from equation (9)]},$$

$$n = 3, \qquad a_4 = \frac{a_3}{4},$$

$$n = 4, \qquad a_5 = \frac{a_4}{5} = \frac{a_3}{4\cdot 5},$$

. . . .

Substitution of these values of the a's in (2) gives

$$y = a_0\left(1 + x + \frac{x^2}{2}\right) + a_3\left(x^3 + \frac{x^4}{4} + \frac{x^5}{4 \cdot 5} + \cdots\right), \qquad (10)$$

the general solution of equation (8) in series form, a_0 and a_3 being arbitrary constants.

In order to express the solution in finite form we might use the method of Example 1, but it is simpler to notice that the second part of the solution may be written

$$3\,!a_3\left(\frac{x^3}{3\,!} + \frac{x^4}{4\,!} + \frac{x^5}{5\,!} + \cdots\right),$$

and that the series in parentheses is now the series for e^x minus its first three terms, $1 + x + x^2/2$. Hence

$$y = a_0\left(1 + x + \frac{x^2}{2}\right) + 3\,!a_3\left[e^x - \left(1 + x + \frac{x^2}{2}\right)\right],$$

or, letting $a_0 - 3\,!a_3 = A$, $3\,!a_3 = B$,

$$y = A\left(1 + x + \frac{x^2}{2}\right) + Be^x, \qquad (11)$$

where A and B are arbitrary constants. Equations (10) and (11) are equivalent, (11) being the solution of (8) in finite form. This example can be solved without using series, by the method of Art. 53 or the method of Art. 54. Verification of solution (11) is found in Problem 18 of Art. 9.

EXAMPLE 4. Solve

$$y'' + xy = 0. \qquad (12)$$

Substituting the values of y, y', and y'' from (2), (3), and (4) in (12), we have the following table:

	con.	x	x^2	x^3	$\cdots$	x^n	$\cdots$
xy		a_0	a_1	a_2	$\cdots$	a_{n-1}	$\cdots$
y''	$2a_2$	$3 \cdot 2a_3$	$4 \cdot 3a_4$	$5 \cdot 4a_5$	$\cdots$	$(n+2)(n+1)a_{n+2}$	$\cdots$

Equating to zero the sum of each column, we see that $a_2 = 0$; a_0 and a_1 are arbitrary; $a_3, a_6, a_9, \cdots$ are expressible in terms of a_0; $a_4, a_7, a_{10}, \cdots$ are expressible in terms of a_1; $a_5, a_8, a_{11}, \cdots$ are expressible in terms of a_2 and are all zero. The general formula is

$$a_{n+2} = -\frac{a_{n-1}}{(n+1)(n+2)}.$$

Then, letting $n = 1, 2, 3, \cdots$, we have

$$a_3 = -\frac{a_0}{2\cdot 3}, \qquad a_4 = -\frac{a_1}{3\cdot 4}, \qquad a_5 = 0,$$

$$a_6 = -\frac{a_3}{5\cdot 6} = \frac{4a_0}{6!}, \qquad a_7 = -\frac{a_4}{6\cdot 7} = \frac{2\cdot 5a_1}{7!}, \qquad a_8 = 0,$$

$$a_9 = -\frac{a_6}{8\cdot 9} = -\frac{1\cdot 4\cdot 7a_0}{9!}, \quad a_{10} = -\frac{a_7}{9\cdot 10} = -\frac{2\cdot 5\cdot 8a_1}{10!}, \quad a_{11} = 0,$$

. . . .

$$y = a_0\left(1 - \frac{x^3}{3!} + \frac{1\cdot 4x^6}{6!} - \frac{1\cdot 4\cdot 7x^9}{9!} + \cdots\right) +$$
$$a_1\left(x - \frac{2x^4}{4!} + \frac{2\cdot 5x^7}{7!} - \frac{2\cdot 5\cdot 8x^{10}}{10!} + \cdots\right), \qquad (13)$$

the general solution of equation (12), where a_0 and a_1 represent arbitrary constants. Here we leave the solution expressed in terms of two infinite series which do not represent expansions of any elementary functions so far encountered. The solution may be expressed in terms of Bessel functions.*

PROBLEMS

Solve the following differential equations by the power series method; also, in case the result involves an infinite series, express the solution in finite form if possible.

1. $(x^2 - 1)y'' + 4xy' + 2y = 0$.
2. $x(3 - 2x)y'' - 6(1 - x)y' - 6y = 0$.
3. $x^2y'' - 2xy' + (2 - x^2)y = 0$.

* See Reddick and Miller's *Advanced Mathematics for Engineers,* Chapter VI. Following the theory of this chapter it may be shown that (13) is equivalent to

$$y = Ax^{1/2}J_{-1/3}(\tfrac{2}{3}x^{3/2}) + Bx^{1/2}J_{1/3}(\tfrac{2}{3}x^{3/2}).$$

4. $y'' + x^2y' + xy = 0.$
5. $(1 - x^2)y'' - xy' + 4y = 0.$
6. $y'' - xy' + 2y = 0.$
7. $xy'' - 2y' + xy = 0.$ (*Cf.* Prob. 19, Art. 9.)
8. $2(x^2 + 8)y'' + 2xy' + (x + 2)y = 0.$
9. (*a*) Show that the power series method gives only a particular integral of the differential equation

$$xy'' + 2y' - xy = 0;$$

find the particular integral, and express it in finite form.
(*b*) By making use of the particular integral found in (*a*), obtain the general solution of the differential equation.
(*c*) Eliminate the arbitrary constants from the general solution found in (*b*), thus obtaining the original differential equation. (*Cf.* Prob. 20, Art. 9.)
10. By the power series method find a particular integral of the differentia equation

$$xy'' + (2 - x)y' - y = 0;$$

whence obtain the general solution.
11. Find a particular integral of the differential equation

$$4x(1 - x)y'' + 2(1 - 3x)y' + 2y = 0$$

in the form of a power series. The general solution can be obtained by the method of Art. 57 (Prob. 9, Art. 58).
12. Solve by the use of a power series

$$y'' - x^2y' + xy = 0,$$

and show that the result agrees with the answer to Prob. 2, Art. 54.

57. The series of Frobenius. In Art. 56 we found that the differential equation (*b*) of Example 2 could not be solved by the method of that article since it has no solution in the form of a power series in x. We shall solve this differential equation as the next illustrative Example 1 by employing the more general Frobenius series.

The series of Frobenius is obtained from the power series (2) of Art. 56 by multiplying it by x^c:

$$y = x^c(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots), \quad a_0 \neq 0. \quad (1)$$

Here c is a constant to be determined, as well as the a's, by substituting the series in the equation to be solved and equating coefficients as in Art. 56. For zero or positive integral values of c, (1) is merely a power series, but for negative or non-integral positive values of c the series represents a new type. The Frobenius series (1) is therefore a generalization of a power series, including it as a special case. The assumption $a_0 \neq 0$ means that x^c is the lowest power of x appearing in the series.

The method of solving a differential equation by use of series (1) is similar to that of Art. 56. Assume that a value of y expressible in the form of series (1) satisfies the differential equation of form (1), Art. 56, and write down the expressions for y and its first two derivatives:

$$y = a_0x^c + a_1x^{c+1} + a_2x^{c+2} + a_3x^{c+3} + \cdots, \tag{2}$$

$$y' = ca_0x^{c-1} + (c+1)a_1x^c + (c+2)a_2x^{c+1} + \cdots, \tag{3}$$

$$y'' = c(c-1)a_0x^{c-2} + (c+1)ca_1x^{c-1} + (c+2)(c+1)a_2x^c + \cdots. \tag{4}$$

Substitute the values of y, y', and y'' from (2), (3), and (4) in the differential equation, form a table like those in Art. 56, starting with the lowest power of x which occurs after the substitutions are made, then equate to zero the complete coefficient of each power of x.

Suppose that two values of c are thus determined. If, for one value of c, we are able to determine all the a's in a manner involving only one of them which is arbitrary, we substitute these values in (2) and obtain a value of y, say y_1, which is a particular solution involving one arbitrary constant.* If, for the other value of c, we can determine the set of a's, which we now call a''s, involving one of them which is arbitrary, we have another particular solution, say y_2. The sum

$$y = y_1 + y_2,$$

involving two arbitrary constants, is the general solution of the differential equation.

It may happen that for one value of c the a's are determined in terms of two of them which are arbitrary; the corresponding

* It is understood here as in Art. 56 that the series represents a solution only for values of x for which it converges.

value of y, involving two arbitrary constants, then represents the general solution. Solutions in series are valid for values of x for which the series converge. There follow some illustrative examples, the first being the differential equation (b) of Example 2, Art. 56.

EXAMPLE 1. Solve

$$4x^2y'' + 4xy' - y = 0. \tag{5}$$

Substitute the values of y, y', and y'' from (2), (3), and (4) in (5), and arrange the result in tabular form, starting with the lowest power of x present after the substitutions, namely x^c:

	x^c	x^{c+1}	x^{c+2}	$\cdots$	x^{c+r}	$\cdots$
$-y$	$-a_0$	$-a_1$	$-a_2$	$\cdots$	$-a_r$	$\cdots$
$4xy'$	$4ca_0$	$4(c+1)a_1$	$4(c+2)a_2$	$\cdots$	$4(c+r)a_r$	$\cdots$
$4x^2y''$	$4c(c-1)a_0$	$4(c+1)ca_1$	$4(c+2)(c+1)a_2$	$\cdots$	$4(c+r)(c+r-1)a_r$	$\cdots$

Equating to zero the sum of the coefficients of x^c, we find

$$(4c^2 - 1)a_0 = 0, \qquad c = \tfrac{1}{2} \text{ or } -\tfrac{1}{2}, \qquad a_0 \text{ arbitrary}.$$

The set of a's corresponding to each of the two values of c will now be determined.

We obtain a general formula by equating to zero the sum of the coefficients of x^{c+r}, first for $c = \frac{1}{2}$, then for $c = -\frac{1}{2}$.

For $c = \frac{1}{2}$:

$$[4(\tfrac{1}{2} + r)^2 - 1]a_r = (4r^2 + 4r)a_r = 4r(r + 1)a_r = 0;$$

hence $a_r = 0$ unless $r = 0$, with a_0 arbitrary.

For $c = -\frac{1}{2}$ (using primes on the a's for the second value of c):

$$[4(-\tfrac{1}{2} + r)^2 - 1]a_r' = (4r^2 - 4r)a_r' = 4r(r - 1)a_r' = 0;$$

hence $a' = 0$ unless $r = 0, 1$, with a_0' and a_1' arbitrary.

Then equation (2) gives, for $c = \frac{1}{2}$ and $c = -\frac{1}{2}$ respectively,

$$y_1 = a_0x^{1/2}, \qquad y_2 = a_0'x^{-1/2} + a_1'x^{1/2}.$$

The first of these results is superfluous since the second contains two arbitrary constants and represents the general solution. Of course we would get the same general solution by setting $y = y_1 + y_2$ and replacing $a_0 + a_1'$ by a new arbitrary constant. The general solution of equation (5) is therefore

$$y = Ax^{1/2} + Bx^{-1/2},$$

where A and B are arbitrary constants. This solution can be found without the use of series by the method of Art. 40.

If we had chosen the value $c = -\frac{1}{2}$ first, we would have obtained the general solution without using the value $c = \frac{1}{2}$. When, as in this example, the values of c differ by an integer, it is better to try the smaller value first as this value may yield the general solution and make unnecessary the use of the other value of c. It can happen, however, as we shall see in Example 3, that there is no solution corresponding to the smaller value of c.

As a second example we solve the differential equation of Problem 9, Art. 56, by use of a Frobenius series.

EXAMPLE 2. Solve

$$xy'' + 2y' - xy = 0, \tag{6}$$

and express the result in finite form.

Substituting the values of y, y', and y'' from (2), (3), and (4) in (6), we have the following table:

	x^{c-1}	x^c	x^{c+1}	x^{c+2}	$\cdots$	x^{c+r}	$\cdots$
$-xy$			$-a_0$	$-a_1$	$\cdots$	$-a_{r-1}$	$\cdots$
$2y'$	$2ca_0$	$2(c+1)a_1$	$2(c+2)a_2$	$2(c+3)a_3$	$\cdots$	$2(c+r+1)a_{r+1}$	$\cdots$
xy''	$c(c-1)a_0$	$(c+1)ca_1$	$(c+2)(c+1)a_2$	$(c+3)(c+2)a_3$	$\cdots$	$(c+r+1)(c+r)a_{r+1}$	$\cdots$

Adding the first column and equating the sum to zero, we have

$$c(c+1)a_0 = 0.$$

Hence $c = 0$ or -1 with a_0 arbitrary. The second column gives

$$(c+1)(c+2)a_1 = 0.$$

Using the smaller value of c, namely $c = -1$, the above equation makes a_1 also arbitrary.

The general column gives, for $c = -1$,

$$r(r+1)a_{r+1} = a_{r-1}$$

$$a_2 = \frac{a_0}{1\cdot 2} = \frac{a_0}{2!} \qquad a_3 = \frac{a_1}{2\cdot 3} = \frac{a_1}{3!}$$

$$a_4 = \frac{a_2}{3\cdot 4} = \frac{a_0}{4!} \qquad a_5 = \frac{a_3}{4\cdot 5} = \frac{a_1}{5!}$$

$$a_6 = \frac{a_4}{5\cdot 6} = \frac{a_0}{6!} \qquad a_7 = \frac{a_5}{6\cdot 7} = \frac{a_1}{7!}$$

$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots$$

Substituting these values of the a's and $c = -1$ in equation (2), we have the general solution:

$$y = a_0\left(\frac{1}{x} + \frac{x}{2!} + \frac{x^3}{4!} + \frac{x^5}{6!} + \cdots\right) + a_1\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \cdots\right). \tag{7}$$

If we had used the value $c = 0$, we would have obtained corresponding to it merely the second series of (7).

To express (7) in finite form, we make use of the expansions:

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots,$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots.$$

Then (7) reduces to

$$y = a_0\frac{\cosh x}{x} + a_1\frac{\sinh x}{x}. \tag{8}$$

A trick method. If we notice that

$$D(xy) = xy' + y, \qquad D^2(xy) = xy'' + 2y',$$

where $D = d/dx$, then equation (6) can be written

$$(D^2 - 1)xy = 0.$$

Hence we have at once by the method of Art. 31,

$$xy = C_1e^x + C_2e^{-x},$$

or

$$y = C_1\frac{e^x}{x} + C_2\frac{e^{-x}}{x}, \tag{9}$$

which is the general solution of equation (6) and is equivalent to (8).

EXAMPLE 3. Show that the differential equation

$$xy'' - xy' + y = 0 \tag{10}$$

has only one particular solution of type (2). Making use of this particular solution, find the general solution in series form.

Substitute the values of y, y', and y'' from (2), (3), and (4) in (10), and form the table:

	x^{c-1}	x^c	x^{c+1}	$\cdots$	x^{c+r}	$\cdots$
y		a_0	a_1	$\cdots$	a_r	$\cdots$
$-xy'$		$-ca_0$	$-(c+1)a_1$	$\cdots$	$-(c+r)a_r$	$\cdots$
xy''	$c(c-1)a_0$	$(c+1)ca_1$	$(c+2)(c+1)a_2$	$\cdots$	$(c+r+1)(c+r)a_{r+1}$	$\cdots$

The first column gives

$$c(c-1)a_0 = 0,$$

from which $c = 0$ or $c = 1$ with a_0 arbitrary.

The second column gives

$$(c+1)ca_1 = (c-1)a_0,$$

which cannot be satisfied by $c = 0$, since $a_0 > 0$; but for $c = 1$, $a_1 = 0$. Furthermore all the successive a's are 0 when $c = 1$, as can be seen from the formula

$$(r+2)(r+1)a_{r+1} = ra_r.$$

Therefore the only solution of type (2) is for $c = 1$, a_0 arbitrary, $a_1 = a_2 = \cdots = 0$, namely, the particular solution

$$y = a_0x. \tag{11}$$

Using $u = x$ and $P = -1$ in formula (8), Art. 53, we obtain a second particular solution $x \int e^x \, dx/x^2$, and hence the general solution:

$$y = C_1 x + C_2 x \int \frac{e^x}{x^2} \, dx. \tag{12}$$

To express the result in series form we substitute for e^x its power series expansion and find

$$y = C_1 x + C_2 x \int \left(\frac{1}{x^2} + \frac{1}{x} + \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots \right) dx,$$

or

$$y = C_1 x + C_2 \left(-1 + x \ln x + \frac{x^2}{2!} + \frac{x^3}{2 \cdot 3!} + \frac{x^4}{3 \cdot 4!} + \cdots \right).$$

The solution in form (12) can be obtained by the method of Art. 53.

58. Bessel functions of zero or positive integral order. Bessel functions are named after the German mathematician and astronomer, Friedrich Wilhelm Bessel, who was director of the observatory at Königsberg. He obtained them in solving a differential equation connected with a problem in planetary motion, but they occur in various other problems in applied mathematics.

Bessel functions are particular solutions of the differential equation

$$x^2 y'' + x y' + (x^2 - n^2) y = 0. \tag{1}$$

The number n may have any real or imaginary value, but we limit this discussion * to the case where n is zero or a positive integer: $n = 0, 1, 2, 3, \cdots$. With this restriction on n let us make use of a Frobenius series to find a particular solution of equation (1).

Substituting in (1) the values of y, y', and y'' from equations (2), (3), and (4) of Art. 57, we form the following table of coefficients:

* See footnote to Ex. 4, Art. 56, for reference to a more extensive discussion of Bessel functions.

	x^c	x^{c+1}	x^{c+2}	x^{c+3}	$\cdots$	x^{c+r}	$\cdots$
$-n^2y$	$-n^2a_0$	$-n^2a_1$	$-n^2a_2$	$-n^2a_3$	$\cdots$	$-n^2a_r$	$\cdots$
x^2y			a_0	a_1	$\cdots$	a_{r-2}	$\cdots$
xy'	ca_0	$(c+1)a_1$	$(c+2)a_2$	$(c+3)a_3$	$\cdots$	$(c+r)a_r$	$\cdots$
x^2y''	$c(c-1)a_0$	$(c+1)ca_1$	$(c+2)(c+1)a_2$	$(c+3)(c+2)a_3$	$\cdots$	$(c+r)(c+r-1)a_r$	$\cdots$

Setting the sum of the first column equal to zero, we find $(c^2 - n^2)a_0 = 0$, which with a_0 arbitrary is satisfied if $c = n$. We now obtain the particular solution of equation (1) corresponding to $c = n$. The second column gives $a_1 = 0$, then the fourth column gives $a_3 = 0$, etc.; all a's with odd subscripts vanish.

We compute the a's with even subscripts from the formula obtained by equating to zero the sum of the coefficients in the x^{c+r} column with $c = n$:

$$[(n+r)^2 - n^2]a_r + a_{r-2} = 0,$$

$$a_r = -\frac{a_{r-2}}{r(2n+r)}.$$

Hence

$$a_2 = -\frac{a_0}{2(2n+2)},$$

$$a_4 = -\frac{a_2}{4(2n+4)} = \frac{a_0}{2\cdot 4(2n+2)(2n+4)},$$

. . . .

$$a_{2k} = (-1)^k \frac{a_0}{2\cdot 4 \cdots 2k(2n+2)(2n+4)\cdots(2n+2k)}.$$

Now since

$$2\cdot 4 \cdots 2k = 2^k(1\cdot 2 \cdots k) = 2^k k!$$

and

$$(2n+2)(2n+4)\cdots(2n+2k) = 2^k(n+1)(n+2)\cdots(n+k),$$

we may write

$$a_{2k} = (-1)^k \frac{a_0}{2^{2k}k!(n+1)(n+2)\cdots(n+k)}.$$

Multiplying both numerator and denominator by $2^n n!$,

$$a_{2k} = (-1)^k \frac{2^n n! a_0}{2^{n+2k} k!(n+k)!},$$

or, setting $2^n n! a_0$ equal to a_0', another arbitrary constant,

$$a_{2k} = (-1)^k \frac{a_0'}{2^{n+2k} k!(n+k)!}.$$

In this formula let $k = 0, 1, 2, 3, \cdots$ and substitute the resulting a's, together with $c = n$, in the Frobenius series (2) of Art. 57 (remembering that all a's with odd subscripts vanish); then

$$y = a_0' \left[\frac{x^n}{2^n n!} - \frac{x^{n+2}}{2^{n+2} 1!(n+1)!} + \frac{x^{n+4}}{2^{n+4} 2!(n+2)!} - \cdots \right].$$

The series in brackets is known as the Bessel function $J_n(x)$ of order $n (n = 0, 1, 2, 3, \cdots)$, which, written in Σ notation, is

$$J_n(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{n+2k}}{2^{n+2k} k!(n+k)!}. \tag{2}$$

We therefore have the following particular solution of the differential equation (1):

$$y = a_0' J_n(x), \tag{3}$$

where a_0' is an arbitrary constant and $J_n(x)$ is the Bessel function of zero or positive integral order given by (2).

The power series (2) is convergent for any finite value of x, and the derivative or integral of $J_n(x)$ may be found by differentiating or integrating the corresponding series term by term. Writing out the expansion of $J_0(x)$ and $J_1(x)$ from (2) we have

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4 (2!)^2} - \frac{x^6}{2^6 (3!)^2} + \cdots + (-1)^k \frac{x^{2k}}{2^{2k} (k!)^2} + \cdots, \tag{4}$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^3 2!} + \frac{x^5}{2^5 2!3!} - \frac{x^7}{2^7 3!4!} + \cdots + (-1)^k \frac{x^{2k+1}}{2^{2k+1}k!(k+1)!} + \cdots. \quad (5)$$

We notice that the derivative of the second term in $J_0(x)$ is the negative of the first term of $J_1(x)$, the derivative of the third term in $J_0(x)$ is the negative of the second term in $J_1(x)$, etc. In general the derivative of the term containing x^{2k+2} in $J_0(x)$ is

$$\frac{d}{dx}(-1)^{k+1}\frac{x^{2k+2}}{2^{2k+2}(k+1)!^2} = -(-1)^k \frac{(2k+2)x^{2k+1}}{2^{2k+2}(k+1)!^2}$$
$$= -(-1)^k \frac{x^{2k+1}}{2^{2k+1}k!(k+1)!},$$

which is the negative of the term containing x^{2k+1} in $J_1(x)$; also the derivative of the first term of $J_0(x)$ is zero. Hence

$$\frac{d}{dx}J_0(x) = -J_1(x), \qquad \int J_1(x)\,dx = -J_0(x). \quad (6)$$

PROBLEMS

1. Solve the differential equation

$$x^2y'' + 4xy' + 2y = 0$$

by use of a Frobenius series, and check the result by the method of Art. 40

2. Show that the differential equation

$$xy'' + (2 - x)y' - 2y = 0$$

has only one particular solution of Frobenius type. Making use of this particular solution, find the general solution in series form. (*Cf.* Prob. 14, Art. 53.)

3. Solve by use of a Frobenius series

$$x^2y'' + (x - 2x^3)y' - (1 + 2x^2)y = 0,$$

and show that the result agrees with the answer to Prob. 7, Art. 54.

4. Show that the differential equation

$$4x^2y'' - 4x^2y' + (1 + 2x)y = 0$$

has only one particular solution of Frobenius type. Using this particular solution find the general solution in series form. (*Cf.* Prob. 8, Art. 54.)

Solve each of the following equations by use of a Frobenius series, and express the result in finite form if possible.

5. $(1 + 3x^2)y'' + \frac{2}{x}y' - 6y = 0.$
6. $(2x - 1)x^2y'' + 2x(3x - 2)y' - 2y = 0.$
7. $xy'' + 3y' - x^3y = 0.$
8. $(3x^2 - x^4)y'' + 2x^3y' - (2x^2 + 6)y = 0.$
9. $4x(1 - x)y'' + 2(1 - 3x)y' + 2y = 0.$
10. $(2x^3 + 6x^2)y'' + (x^2 + 9x)y' - 3y = 0.$
11. $4x^2(1 - x)y'' + 2x(x - 2)y' + (3 - 2x)y = 0.$
12. $4x(1 - x)y'' + 2(1 - 2x)y' + y = 0.$
13. $2x(1 + x)y'' + y' - 4y = 0.$
14. $2xy'' + 3y' + xy = 0.$
15. Show that the method of Art. 57 yields only a particular solution of

$$x(x - 1)y'' - 2(2x - 1)y' + 2y = 0,$$

and find the particular solution in series form.
16. Compute to four decimal places the values of $J_0(3)$ and $J_1(2)$.
17. Prove that

$$\frac{d}{dx}[xJ_1(x)] = xJ_0(x), \qquad \int xJ_0(x)\,dx = xJ_1(x).$$

18. Prove that

$$J_1'(x) = J_0(x) - \frac{1}{x}J_1(x).$$

Find a particular solution of each of the four following differential equations, and express the result in terms of a Bessel function. Prove in each case, by the use of formula (6) and Probs. 17 and 18, that the particular solution found satisfies the differential equation.

19. $xy'' - y' + xy = 0.$
20. $xy'' + 3y' + xy = 0.$
21. $xy'' + y' + x^3y = 0.$
22. $\frac{d^2y}{dx^2} = \frac{y}{x}.$

23. Find a solution of

$$4x^2y'' + (x + 1)y = 0$$

such that $y = 1$ when $x = 4$.

24. Find the equation of a curve through the point $(3, -6)$ representing a particular solution of the differential equation

$$x^2y'' - xy' + (x^2 + 1)y = 0.$$

Also find the area from $x = 0$ to $x = 2$ between this curve and the x-axis.

Answers

Art. 5

1. (a) $y = \sin x + C$; (b) $y = x^4 + C_1x^2 + C_2x + C_3$.
2. $y - x = C(1 + xy)$. **3.** (a) 5; (b) 1; (c) 3. **4.** $(ax + b)(ay + b) = C$.
5. $C_1(y^2 - C_1) = (C_2 + x)^2$. **8.** 1. **9.** 0.533. **10.** 0.693. **13.** 0.0676.
14. $\frac{1}{2}\ln\frac{x+1}{x-1}$, -0.549, $-\frac{1}{3}$. **15.** $(0.707a, 0.881a)$.

Art. 9

5. $xy' = x + 2y$. **6.** $(1 + y'^2)y''' = 3y'y''^2$.
8. $(x - y)y'' + 2y' + 2y'^2 = 0$. **11.** $x^3y' = y^3$.
12. $(y - 1)y'' + y = (y - 2)y'^2$. **13.** $y''' - 2y'' - 5y' + 6y = 0$.
14. $x + y + 1 + 2(3x + 5y + 7)y' = 0$. **15.** $y' = \cosh[(x/2) + y]$.
16. $4x^3y'' + 6x^2y' - y = 0$. **17.** $(1 + x^2)y'' + xy' - y = 0$.
18. $xy'' - (x + 2)y' + 2y = 0$. **19.** $xy'' - 2y' + xy = 0$.
20. $xy'' + 2y' - xy = 0$. **21.** $y^{\text{iv}} + 64y = 0$. **22.** $y'' - 2xy' - 2y = 0$.
23. $y'' - x^2y' + xy = 0$. **24.** (a) $x^2y'' - n(n - 1)y = 0$;
(b) $xyy''' + 2yy'' - xy'y'' = 0$.

Art. 10

1. $xy'' = y' + y'^3$. **2.** $y^2(1 + y'^2) = r^2$.
3. $(x - y)y'' = (1 + y')(1 + y'^2)$. **4.** $(x^2 - 2xy - y^2)y' = x^2 + 2xy - y^2$.
5. $2xyy' = y^2 - x^2$. **6.** $x^2y'^2 + y^2 = 2xy(1 + y' + y'^2)$.
7. $y^2y'^2 - 2xyy' + 2y^2 - x^2 = 0$. **8.** $(2x^2 - y^2)y'^2 - 2xyy' + x^2 = 0$.
9. $yy'' + y'^2 = 0$. **10.** $2xy' = y(1 - y'^2)$. **11.** $2xy'' + y' = 0$.
12. $(x - a)^2y'' = 2(x - a)y' - 2(y - b)$. **13.** $y'e^{xy'/y} + 2\pi xy^3 = 0$.
14. $y^2[1 + (dx/dy)^2] = k^2$. **15.** (a) $yy'' = 1 + y'^2$; (b) $(1 + y'^2)y''' = y'y''^2$.
16. $2x^2y'' - xy' + y = 0$. **17.** (a) $2(xy' - y)(1 + y'^2) = (x^2 + y^2)y''$;
(b) $(d^2\rho/d\theta^2) + \rho = 0$. **18.** (a) $2y'y''' = 3y''^2$;
(b) $(x - y)y'' + 2y'^2 + 2y' = 0$. **19.** $9y''^2y^{\text{v}} - 45y''y'''y^{\text{iv}} + 40y'''^3 = 0$.
20. $3y''y^{\text{iv}} - 5y'''^2 = 0$.

Art. 13

1. $y = Cx^3$. **2.** $\sin^2 y = C \sin x$. **3.** $x\sqrt{1-y^2} + y\sqrt{1-x^2} = C$.
4. $e^x + e^{-y} = C$. **5.** $x - 2xy + 2y^2 = Cxy^2$.
6. $x^2 = 2[2 - y - \ln(1-y)]$. **7.** $y = 2xe^{1-x}$. **8.** $x = (1/c)(ae^{ct/a} - b)$.
9. $x = a \sin y$. **10.** $x = 3(2^{t/10} - 1)(2^{t/10} - \frac{1}{2})$. **11.** 1.90. **12.** $\frac{3}{5}$.
13. (*a*) 0.255; (*b*) 0.720. **14.** $\frac{18}{7}$. **15.** 0.917. **16.** $xy + 2(x + y) = 0$.
17. $\rho^2 - 1 = \frac{4}{3}\rho \sec \theta$. **18.** $(4 + x^2)(1 - y^2)^2 = C$. **19.** $y = e^{-x} \tan x + C$.
20. $(1 - x - y + xy)e^{(x+y)/2} = C$. **21.** $x(1 + y) = Ce^{1/y}$. **23.** 3.81×10^6.
24. (*a*) $y = (x^{3/2} + C)^2$; (*b*) $y = x^3$, $y = (x^{3/2} - 1)^2$.
25. $x^2 + 3y = 7$, $x^2 + 3y = 3 + 4x^{3/2}$.

Art. 17

1. 2.88 ft/sec; 1.05 ft. **2.** −0.557. **3.** 1.22. **4.** (*a*) 146 ft/sec; (*b*) 945 ft; (*c*) 3.73 sec; (*d*) 3.65 sec. **5.** (*a*) $6.83V$ sec; (*b*) $5.70V$ sec.
6. (*a*) 2.16 sec; (*b*) 67.0 ft. **7.** $t = [wV/g(w - E)] \ln [100/(100 - p)]$ sec.
8. (*a*) 8.83 ft/sec; (*b*) 1.01 sec; (*c*) 5.88 ft. **9.** 126 ft/sec; 3.10.
10. 147 ft/sec; 2.67. **11.** $16A/7$. **12.** (*a*) 29.3 per cent; (*b*) $0.415n$ yr.
13. (*a* and *b*) 2.41 min; (*c*) 27.3°F. **14.** (*a*) 2.36; (*b*) 2.57. **15.** 10.3.
16. $2a/(n + 1)$. **17.** 1460 lb/ft². **18.** 17,900 ft. **19.** 8.85 lb/in.²
20. 344×10^{-7} lb/in.³ **21.** 15,800 ft. **22.** 1.12 in. **23.** 64.9 lb/ft³; 680,000 lb/ft². **24.** 3.11×10^8 cal/day; 63.4°C. **25.** 117 cal/sec; 69.1°C. **26.** 1.55 in. **27.** (*a*) $u = u_1 - (u_1 - u_2) \ln (r/r_1)/\ln (r_2/r_1)$; (*b*) 86.4°C. **28.** (*a*) $u = u_1 - [(u_1 - u_2)(r - r_1)/(r_2 - r_1)](r_2/r)$; (*b*) 76°C. **29.** 5.47 cm. **30.** $k_1k_2A(u_1 - u_2)/(k_2t_1 + k_1t_2)$ cal/sec.
31. $2\pi L(u_0 - u_n)/\sum_{r=1}^{n} (1/k_r) \ln (x_r/x_{r-1})$ cal/sec. **33.** 10.2 min.
34. 3.93 min. **35.** $2.28t$. **37.** 0.456. **38.** (*a*) 45.3 min; (*b*) 54.3 min.
39. (*a*) 19.2 min; (*b*) 1.31 ft. **41.** 81 min. **42.** $\frac{5}{7}$. **43.** 56.9 hr.

Art. 20

2. $x = Cy - y^3$. **3.** $y = x(\ln x + C)$. **4.** $y = x \tan (x + C)$.
5. $y^2 = xe^y + C$. **6.** $x^2 - y^2 = Cx$. **7.** $xe^{x/y} = C$. **8.** $3x^2 + 2y^2 = Cx^4$.
9. $(x - y)e^{x/y} = C$. **10.** $e^{xy}(C - x^2) = 2$. **11.** $\ln (y/x) = 1 + Cx$.
12. $(x + y)e^{x+y} = C(x - y)$. **13.** $x^2 - 2xy - y^2 = C$.
14. $6x^2y^2 + y^4 = C$. **15.** $x^2 + 2y = Cy^2$. **16.** $y^3 = Cx^2 + x^3$.
17. $x^2 = Cy^3e^{\sin x}$. **18.** $y = C[1 + \ln (x/y)]$. **19.** $x^2 = C \sin (y/x)$.
20. $(x^2 + y^2)^2 = Cxy$. **21.** $7x^2 + 6xy + 11y^2 + 10x + 34y = C$.
22. $x^3 + y^3 = xy(x + y + C)$. **23.** $x + y = Ce^{x/y}$.

24. $y^2 + y\sqrt{x^2 + y^2} = Cx$. **25.** $xy + \sqrt{x^2 - 1}\sqrt{y^2 - 1} = \cosh(xy + C)$.
26. $\sqrt{x - y}(\sqrt{x} - \sqrt{y}) = Ce^{\sqrt{y}/(\sqrt{x}-\sqrt{y})}$.
27. $y + \sqrt{x^2 + y^2} = Cx^2e^{\sqrt{x^2+y^2}/y}$. **28.** $e^x = y(1 + 2\ln y)$.
29. $(1 - x^2)y^2 - 2xy = 4$. **30.** $(2x + y)y^3 = 2e^{2x/y-1}$; ± 1.16.
32. $xy = -\frac{1}{3}(2 - x^2)^{3/2} + \frac{5}{6}$, $xy = \frac{1}{3}(2 - x^2)^{3/2} - \frac{1031}{750}$.
33. $C_1 = C_2 + \pi/3$. **34.** $C_1 = C_2 - \pi/3$. **35.** $\sqrt{9 + e^{2x}}\,dy = 3dx$; $C_1C_2 = 1$.

Art. 21

1. $y = (x + C)e^x$. **2.** $15x^3y = 3x^5 + 5x^3 + C$. **3.** $y = (2/x) + C\sqrt{x}$.
4. $x^2y = 3 + C\sqrt{x}$. **5.** $3y = 1 + x^2 + C(1 + x^2)^{-1/2}$.
6. $y\sqrt{1 - x^2} = \sin^{-1} x + C$, or $y\sqrt{x^2 - 1} + \cosh^{-1} x = C$.
7. $7x + 2y^4 = C\sqrt{y}$. **8.** $xy = x - 1 + Ce^{-x}$.
9. $9y + 6x^2 \ln x - 4x^2 + C\sqrt{x}$. **10.** $y = Cx - x^2$.
11. $y = e^x + C(x - 1)$. **12.** $x + y + 2 = Ce^x$. **13.** $y = x^2(1 + Ce^{1/x})$.
14. $2x + y^2 + 1 = Ce^{y^2}$. **15.** $y = \frac{1}{2}(50 + x) + C(50 + x)^{-1/3}$.
16. $x + y = Cx(x - 1)$. **17.** $y = Ce^{x+x^2/2} - x - 3$. **18.** $y^2 = Ce^{x/y}$.
19. $y = -\sin x$. **20.** $y = 2\sin^2 x - 2\sin x$; 4, $-\frac{1}{2}$. **21.** 4.
22. 7.44. **23.** $-8/5$. **24.** 9.39. **25.** 0.103. **26.** $\frac{31}{160}$ or $-\frac{33}{160}$.

Art. 24

1. 1.32 lb/gal. **2.** (*a*) 45 lb; (*b*) $\frac{39}{64}$ lb/gal. **3.** (*a*) 124 lb; (*b*) 54 min.
4. (*a*) 4 hr 25 min; (*b*) 2 hr 27 min. **5.** 287 lb; 14.9 min. **7.** 14.8 lb; 27.8 min. **8.** $crV(1 - r)^{(1-r)/r}$ lb; $V(1 - r)^{(1-r)/r}$ gal.
9. (*a*) 0.416 amp; (*b*) -0.413 amp; (*c*) 0.432. **10.** (*a*) 0.476 amp; (*b*) 0.299 amp. **11.** (*a*) $q = EC + K\epsilon^{-t/RC}$;
(*b*) $q = [EC/(1 + \omega^2R^2C^2)](\sin\omega t - \omega RC\cos\omega t) + K\epsilon^{-t/RC}$.
12. $i = [\omega EC/(1 + \omega^2R^2C^2)](\cos\omega t + \omega RC\sin\omega t) + K\epsilon^{-t/RC}$.
13. (*a*) 1.35 amp, 0.0432 coulomb; (*b*) -8.50 amp, 0.0131 coulomb.
14. -1.35 amp; 0.00677 coulomb. **15.** 0.181 amp. **16.** 79.1 volts.
17. 0.721 henry. **18.** (*a*) $E = Q/C(1 - \epsilon^{-t/RC})$; (*b*) 120 volts.
19. (*a*) 6.93 ohms; (*b*) 8.66 amp. **20.** 15.9 ohms.
21. $t = RC\ln(RC + 1)/RC$ sec. **23.** 0.0487 coulomb; 2.94 ohms.
24. (*a*) $i = \frac{11}{6}[1 + (1/\sqrt{37})\sin(120\pi t - \tan^{-1} 6) - (31/37)\epsilon^{-20\pi t}]$;
(*b*) 0.969 amp; (*c*) $\frac{11}{6}[1 + (1/\sqrt{37})\sin(120\pi t - \tan^{-1} 6)]$;
(*d*) 42.3 volts; 116 volts. **26.** $L = [2a\mu/(1 + \mu^2)](1 + e^{\mu\pi})$.
27. (*a*) $2\mu = (1 - \mu^2)e^{\mu\pi/2}$; (*b*) 0.732. **28.** 2.01 lb. **29.** 0.630 ft.
30. 8.38 ft. **31.** 19.6 lb; 2.22. **32.** (*a*) 258 lb; (*b*) 241 lb; (*c*) 50.9 lb.
33. 73°44′ from the top of the cylinder.

34. $1 - 10\mu + \mu^2 = 5[\mu - (1-\mu^2)(\sqrt{3}/2)]e^{2\mu\pi/3}$. **35.** 41°44′.
36. 5.85 ft; 0.13 ft. **37.** 13,900.

Art. 25

1. $xe^{1/xy} = C$. **2.** $x^4 = Cy + y^2$. **3.** $3y^2 = 1 + Cx^2y^2$. **4.** $ye^{x^2/2y^2} = C$.
5. $xy = C(xy+1)e^{y^2/2}$. **6.** $2y^2 = 2x - 1 + Ce^{-2x}$. **7.** $xy^3(x + C) = 1$.
8. $x^3 = \sin y + \cos y + Ce^{-y}$. **9.** $3y^4 = x^2(y^6 + C)$.
10. $1/y = \sin x + C\cos x$. **11.** $x^2 = (C - 3x)y^2$.
12. $y = 2\cos x/(C\cos^2 x - 1)$. **13.** $x^4(C - \ln\tan y) = \tan y$.
14. $y^2 = e^{1/x}/(4e^{1/x} + C)$. **15.** $(y - 2x + 4)^4 = C(y - 3x + 6)^3$.
16. $2(x - y) + \ln(2x + 2y + 1) = C$.
17. $(x-1)^2 + y^2 = Ce^{2\tan^{-1}[y/(x-1)]}$. **18.** $(x + 3y - 5)^2 = C(x + 2y - 2)$.
19. $(x + y + 1)e^{(x+2)/(x+y+1)} = C$. **20.** $y^2 + 2xy - 2x^2 - 18x = C$.
21. $\ln[9(y - x + 3)^2 + (3x - 5)^2] + 6\tan^{-1}[3(y - x + 3)/(3x - 5)] = C$.
22. $(x + y)^2 = C(x^2y^2 - 1)$. **23.** $3y = 5x - 2\tan^{-1}[\frac{1}{2}\tan(2x + C)]$.
24. $y^2 = (C - 2\ln y)e^{2x}$. **25.** $2(x + 4y + 1) = 3\tan(6x + C)$.
26. $x^2 = (1 + Ce^{x^4})y^2$. **27.** $y^2 = (x-2)^2 + e^{1-(x/2)}$.
28. $xy^{1-n} = 3(1-n)/(2n-3) + Cx^{3-2n}$ $(n \neq 1, \frac{3}{2})$; $y = Cx^2e^{-3/x}$ $(n = 1)$; $2x = (C - 3\ln x)\sqrt{y}$ $(n = \frac{3}{2})$.
29. $(x + y - 4)\ln(x + y - 4) + 4x + 3y = 19$. **30.** 2.20.
31. $x\ln(y/x) = \sin x - x\cos x + C$. **32.** (a) $x^2(x^2y^2 + 2) = Cy^2$; (b) $y = Cxe^{x^2y^2/4}$. **33.** (a) $x = 1 + Ce^{(4x-4)/(x-4y-2)}$; (b) $6(y - x) + C = \ln(2x + 2y + 1)(x^2 + 2xy + y^2 + x + y + 1)^4$.
34. (a) $y = x^2\tan[(x^2/2) + C]$; (b) $y(\sin x + C) = \sec x + C\tan x$.

Art. 27

1. $y^2 - x^2 = 5$ or $x^2 + y^2 = 13$. **2.** $y^2 = 4e^{\pm x}$. **3.** $x = 1 + \sqrt{3} - e^{-y}$.
4. $2x^2 + y^2 = 6$. **5.** $x^2 + y^2 = Cx$. **6.** $y^2 = e^{2-(x/y)}$. **7.** $x^2 + y^2 = Cx$.
8. (a) $x^2 + y^2 = 5x$; (b) $x^2 + y^2 = 10y$. **9.** (a) $\rho = 4\cos\theta$; (b) $\rho = (4\sqrt{3}/3)\sin\theta$. **10.** (a) $y = x + \ln(x - 1) + C$; (b) $y = x + \ln(1 - x)$, $y = x - 1 + \ln(x - 1)$.
11. $(y - 2x)^2 = 5(y - x)$. **12.** $y^2 = 4k(x + k) + (b^2 - 4k^2)e^{x/k}$.
14. (a) $y^2 = 4(x + 1) + 5e^x$; (b) $y^2 = 8(x + 2) - 7e^{x/2}$. **15.** P, 6.93 in.; Q, 13.2 in. **16.** 0.697 ft. **17.** (b) $y = k\cosh[(x/k) + C]$; area under arc = $k \times$ arc length. **18.** $y = 2\cosh[x/2 + 0.881]$.
19. $y = be^{(x-a)/k}$. **20.** $ny^{n-1} = (n - 1)(x - a)$. **21.** $x - y = Cxy$.
23. $y = 2n\cosh[(x/2n) + C]$ and $y = 2n$. **24.** $y^3 - 3x^2y = k^3$ and $y = k$.
25. Confocal parabolas: (a) $x^2 = 2Cy + C^2$; (b) $\rho(1 + \sin\theta) = C$.
26. $\rho = C\sin^n(\theta/n)$ or $\rho\sin^n(\theta/n) = C$. **27.** $\rho^2 = C\sin 2\theta$ or $\rho = C(1 - \sin\theta)$. **28.** (a) $y^2 = 2Cx + C^2$; (b) $\rho(1 - \cos\theta) = C$.

29. (*a*) $(k^2 - 1)x^2 + k^2y^2 = 2Cx + C^2$; (*b*) $\rho = C/(k - \cos\theta)$; (*c*) $\rho = 3/(3 - 2\cos\theta)$.

Art. 28

1. $y = Ce^{\mp x/k}$. **2.** $\rho = Ce^{\mp\theta/k}$. **3.** $x^2 + y^2 - 1 = Cy$. **4.** $3x^2 + y^2 = Cx$.
5. $e^x \sin y = C$. **6.** $\sin x = C \operatorname{sech} y$. **7.** $y^2 = 4C(x + C)$.
8. $y^2 - x^2 = Cx^3$. **9.** $x^2 + y^2 = Ce^x$. **10.** $x + \cos x \cosh y = C$.
11. $2\sinh y + \tan(x/2) = C$. **12.** $y^2 = \frac{1}{2}\ln\cos 2x + C$.
13. $\rho^4 = C\cos 2\theta$. **14.** $\rho = 1/(C - \sin^2\theta)$. **15.** $\rho = 2\sin\theta + C\cos\theta$.
16. $\rho^2 - 1 = C\rho\sec\theta$. **17.** $2x^2 + y^2 = 17$. **18.** $x^3 + 3xy^2 = 13$.
19. $\rho = 64\cos^{16}(\theta/4)$. **20.** $\rho = 4(1 - \sin\theta)$. **21.** $x = y^2(1 - \ln y)$.
22. $\sqrt{2}(\sin x + \sinh y) = \cosh y$; $\sinh y = \sin x - \cos x$.
23. $y^a = C[(a - 2)x^2 - y^2]$, $(a \neq 2)$; $ye^{x^2/y^2} = C$, $(a = 2)$.
24. $\cosh x - a\cos y = C\sin y$. **25.** (*a*) $x^2 + y^2 = 2b\ln x + C$; (*b*) $y = Cx^a$. **26.** (*a*) $\ln\rho = \frac{1}{4}\cos 2(\theta + a) + C$; (*b*) $(k/\rho) - (\rho/k) = \theta + C$. **27.** $\rho(\sin n\theta)^{m/n^2} = C$.
29. $(x + y)(x - 2y)^2 = Cy^6$. **30.** (*a*) $x^2 = 4[y - 1 + \ln(y - 1)]$; (*b*) $x^2 = 4\left[y + 2 + \ln\dfrac{1 - y}{2}\right]$. **31.** $x^2 + y^2 + 1 = Cx$.
32. $\rho^2 = C(\rho\cos\theta - 1)$. **33.** $(x^2 + y^2 + 1)^2 - 4x^2 = C$.

Art. 31(*b*)

2. (*a*) $6e^x$; (*b*) $-4e^{-2x}$; (*c*) 0; (*d*) $(\ln a - a)^n a^x$. **4.** $y = C_1e^{kx} + C_2e^{-kx}$; $y = A\sinh kx + B\cosh kx$. **5.** $y = C_1e^{6x} + C_2e^{-x}$.
6. $y = C_1e^{(\sqrt{14}-3)x} + C_2e^{-(\sqrt{14}+3)x}$. **7.** $y = (C_1 + C_2x)e^{2x} + (C_3 + C_4x)e^{-2x}$.
8. $y = C_1 + C_2e^{-(a/2)x} + C_3e^{-2ax}$. **9.** $y = C_1e^x + C_2e^{-x} + C_3e^{(1/2)x}$.
10. $y = Ae^{(3/2)x}\sinh[(\sqrt{5}/2)x + \alpha] + Be^{-(3/2)x}\sinh[(\sqrt{5}/2)x + \beta]$.
11. $y = C_1 + C_2x + C_3x^2 + C_4e^{\sqrt{2}x} + C_5e^{-\sqrt{2}x}$.
12. $y = (C_1 + C_2x + C_3x^2)e^x + C_4e^{-3x}$. **13.** $y = C_1e^{(a/b)x} + C_2e^{(b/a)x}$.
14. $y = C_1e^{-x} + C_2e^{2x} + C_3e^{(2/5)x} + C_4e^{-(1/2)x}$.
15. $y = C_1e^{-x} + C_2e^{(1/2)x} + C_3e^{(5/6)x} + C_4e^{-(1/3)x}$.
16. $y]_{x=-1} = (1 - e)(2 - e)/(3 - e)$; 4.37.
17. $y = (\sqrt{2}/8)e^{-x}\sinh\sqrt{2}x$.

Art. 31(*c*)

1. $y = A\cos kx + B\sin kx$. **2.** $y = e^x(A\cos\sqrt{3}x + B\sin\sqrt{3}x)$.
3. $y = e^{-x/2}[A\cos(x/3) + B\sin(x/3)]$. **4.** $s = e^{2t}(A\cos 3t + B\sin 3t)$.
5. $x = e^{-at}(A\cos\sqrt{k^2 - a^2}t + B\sin\sqrt{k^2 - a^2}t)$.
6. $y = C_1 + C_2e^x + e^{-x/2}(C_3\cos\sqrt{2}x + C_4\sin\sqrt{2}x)$.

7. $y = A\sinh(x+\alpha) + Be^{x/2}\sin[(\sqrt{3}/2)x+\beta] + Ce^{-x/2}\sin[(\sqrt{3}/2)x+\gamma]$.
8. $y = C_1e^{6x} + e^{-3x}[C_2\cos(x/2) + C_3\sin(x/2)]$.
9. $y = (C_1 + C_2x)e^x + (C_3 + C_4x)e^{-x} + C_5\cos\sqrt{2}x + C_6\sin\sqrt{2}x$.
10. $s = A\sinh(2\sqrt{2}t + \alpha) + B\sin(2\sqrt{2}t + \beta)$.
11. $s = Ae^{2t}\sin(2t + \alpha) + Be^{-2t}\sin(2t + \beta)$.
12. $y = C_1e^{2x} + C_2e^{-2x} + (C_3 + C_4x)\cos\sqrt{2}x + (C_5 + C_6x)\sin\sqrt{2}x$.
13. $y = (C_1 + C_2x)e^{-x} + e^x(C_3\cos x + C_4\sin x)$.
14. $y = (C_1 + C_2x + C_3x^2)\cos 2x + (C_4 + C_5x + C_6x^2)\sin 2x$.
15. $y = C_1e^{-2x} + C_2\cos x + C_3\sin x + e^x(C_4\cos 2x + C_5\sin 2x)$.
16. $y = C_1e^{4x} + (C_2 + C_3x)\cos x + (C_4 + C_5x)\sin x$. **17.** (*a*) No; (*b*) yes.

Art. 32(*b*)

1. (*a*) $0.300a$; (*b*) $0.375a$. **2.** (*a*) $9x^2 + 4v^2 = 225$; (*b*) 3 ft, 6 ft/sec; (*c*) 3.76 sec. **3.** 2.63 units. **4.** (*a*) 0.031 sec; (*b*) 0.407 sec.
5. $\pm b\sqrt{3g/a}$ ft/sec. **6.** 8.88 lb. **7.** ±3.84 ft/sec. **8.** $\frac{6}{7}$ sec; 7.18 in.
9. 0.190 sec; 0.452 sec. **11.** 0.377 sec. **12.** (*a*) 2.34 ft; (*b*) 5.98 lb.
13. (*a*) 0.247 ft; (*b*) -15.9 ft/sec²; (*c*) 15 lb. **14.** 3.44 ft. **15.** (*a*) 42.2 min; (*b*) 14.1 min. **16.** (*a*) 0.40 in.; (*b*) 13.0 in./sec downward.
17. (*a*) 1.77 sec; (*b*) 2.36 ft above water. **18.** 6.67 in. **19.** 0.636.
20. $2\pi\sqrt{L/g}$; $\theta = \alpha\cos\sqrt{g/L}t$. **21.** $\theta = \sqrt{L/g}\omega_0\sin\sqrt{g/L}t$.
22. (*a*) 3.26 ft; (*b*) 0.219 rad/sec. **23.** $\frac{1}{6}$. **24.** ±0.219 rad/sec.

Art. 32(*c*)

3. $t = [1/(r_1 - r_2)]\ln[(r_2v_0 - b^2x_0)/(r_1v_0 - b^2x_0)]$.
4. $t = [\ln(r_2/r_1)]/(r_1 - r_2)$. **6.** $(D^2 + 0.0353D + 21.7)x = 0$.
7. $e^{-4.68t}$; $(D^2 + 9.37D + 39.5)x = 0$. **8.** -1.56 ft/sec². **9.** 0.198 sec.
10. $K = 1$; 0.172 sec. **11.** (*a*) 0.549 sec; (*b*) 2.16 ft; 0.0895 ft/sec.
12. (*a*) 0.206 sec; (*b*) 3.12 ft; -1.74 ft/sec. **13.** 1.63 ft; -0.369 ft/sec.
14. (*a*) 0.223 sec; (*b*) 4.09 ft; 8.86 ft/sec. **15.** (*a*) 0.154 ft; (*b*) 0.315 sec.
16. (*a*) 0.259 in. above equilibrium; (*b*) 4.94 in. below equilibrium.
18. (*a*) 1.16 sec; (*b*) 60.5 per cent; (*c*) 8.35 in. **19.** 5.40 in. above.
20. 40.2′. **21.** 93.5 per cent.
22. $T = \sqrt{a\rho(4\pi^2r^2 + \ln^2 n)}/2r\sqrt{3g(\rho - 1)}$; 1.15 sec.

Art. 34

1. $4y = x^4 - \sin 2x - 6\pi^2x^2 + 8\pi^3x - 3\pi^4$. **2.** $y = x^4 - x - 3\ln x$.
3. $y = 4e^{-x} - e^{2x} + x + 2$. **4.** (*a*) $EIy = 20x(x^2 - 300)$; (*b*) $EIy = 20(20 - x)(100 - 40x + x^2)$; (*c*) $(4 \times 10^4)/EI$ ft.

Art. 40

3. $y = C_1x + C_2x^{1+\sqrt{3}} + C_3x^{1-\sqrt{3}} + 2x^3.$
4. $y = x(C_1 \sin \ln x + C_2 \cos \ln x + \ln x).$
5. $y = C_1x + C_2x^2 + 4x^2 \ln x + \frac{1}{10} \sin \ln x + \frac{3}{10} \cos \ln x.$
6. $y = (C_1 + \ln^2 x - \frac{1}{2} \ln x)x^3 + (C_2/x) - 2.$
7. $y = (C_1 + C_2 \sin \ln x + C_3 \cos \ln x)(1/x) + (x^2/30) + (3x/10) - 2.$
8. $y = C_1 + C_2x + C_3x^{(5+3\sqrt{5})/2} + C_4x^{(5-3\sqrt{5})/2} - \frac{1}{9}x \ln x + \frac{1}{10} \ln^2 x.$
9. $y = (C_1 + 2 \ln x)(1/x) + (C_2 \sin \ln x + C_3 \cos \ln x)x.$
10. $y = C_1 + C_2 \ln x + [C_3 \sin (\sqrt{2} \ln x) + C_4 \cos (\sqrt{2} \ln x)]x^3 + \frac{1}{6} \ln^3 x + \frac{3}{11} \ln^2 x.$ **11.** $y = [C_1 + C_2 \ln x + C_3 \ln^2 x + (\ln^8 x)/48]x.$
12. $y = (C_1 + C_2 \ln x + \frac{3}{8} \ln^2 x)\sqrt{x} - \frac{2}{25}x^3.$
13. $y = C_1x + C_2x^{2+\sqrt{4.25}} + C_3x^{2-\sqrt{4.25}} - \frac{1}{13}x \ln x + \ln^2 x - 29 \ln x + 475.$
14. $y = (C_1/x) + (C_2/x^2) + (x/6) - (\sin x)/x^2.$
15. $y = [C_1 + C_2 \ln (2x - 3)]\sqrt{2x - 3} + 2x.$
16. $y = C_1 + C_2 \ln (1 + 3x) + C_3(1 + 3x)^3 - \frac{1}{3} \ln^2 (1 + 3x) - 3x/2.$
17. $y = C_1(2x - 1) + C_2(2x - 1)^{1+(\sqrt{3}/2)} + C_3(2x - 1)^{1-(\sqrt{3}/2)} - \frac{1}{3}(2x - 1) \ln (2x - 1) - 1.$ **18.** $y = \frac{13}{4} - 3x + \frac{11}{4}x^2 - \frac{1}{2}x^2 \ln x.$
19. $y = \sqrt{x} - 1/\sqrt{x}.$ **20.** $y = 2 \cos \ln x - \sin \ln x.$
21. $u = u_1 - [(u_1 - u_2) \ln (r/r_1)]/\ln (r_2/r_1).$
22. $u = u_1 - [(u_1 - u_2)(r - r_1)/(r_2 - r_1)](r_2/r).$

Art. 42

1. $y = \ln (x^2 + C_1) + C_2.$ **2.** $y = x + C_1e^{-x^2/2} + C_2.$
3. $x^2 + (y + C_2)^2 = C_1.$ **4.** $y = x^2 + C_1 \ln (x^2 - C_1) + C_2.$
5. $y = x^3 + C_1x^2 \ln x + C_2x^2 + C_3x + C_4.$ **6.** $y = (C_1 + x)/(C_2 - x).$
7. $y = (C_1x + C_2)^2.$ **8.** $y^3 = (C_1x + C_2)^2.$ **9.** $y = C_1 \sinh^2 (x + C_2).$
10. $xy + C_1x + C_2y = C_3.$ **12.** $y = \ln \cos (2x + C_1) + C_2.$
14. $2y = \sec [x + (\pi/3)].$ **15.** $y = \ln (1 + x).$ **16.** 0.356.
17. $x = 3 - (2/\sqrt{y}).$ **18.** $y = 2 \sin^{-1} x + 1.$ **19.** $x = 2(e^{-y/2} - 1).$
20. $y = (x - 2)e^x + 2.$ **21.** $x = \frac{1}{2} \ln 2 - \ln \sin y.$ **22.** $\sin y = \sqrt{2} \sin x.$
23. $y = (e - 1) \ln [(1 - e^{-x})/(1 - e^{-1})].$ **24.** $(y + 1)^2 = 2x.$ **25.** $-1.73.$
26. 27. **27.** $-1.42.$ **28.** $\frac{8}{3}.$ **29.** 3.21. **30.** $-1.23.$ **31.** -1.83; 0.330.
33. $y = \sin\left(x - \frac{\pi}{6}\right), y = \sqrt{3} - \cos\left(x - \frac{\pi}{3}\right); y = 1 - \cos x.$
34. 109° 28′. **35.** 109° 28′.

Art. 44

2. 0.843. **3.** 1.58. **4.** 0.51 ft; 0.49 ft. **5.** 2.39 sec. **6.** 116 hr.
7. 9.90 hr. **8.** 6.95 mi/sec. **9.** 11.6 min. **10.** 45.1 min. **11.** (2) 0.775; (6) 810 hr; (7) 13.5 hr; (8) 4.91 mi/sec. **13.** 0.910. **14.** 2.36 sec.

15. (*a*) 15.9 ft; (*b*) 61.1 ft; (*c*) 136 lb, 144 lb. **16.** 58.4 ft. **17.** 15.2 ft, 106 ft. **18.** (*a*) 7.54 lb; (*b*) 76.5 ft, 194 ft. **21.** $y = 8.08 \cosh 0.124x$, $y = 3.31 \cosh 0.302x$. **22.** 277 ft; 33.8 ft. **23.** (*c*) 9 in.

25. (*a*) 1.02 sec; (*b*) 0.723 sec. **26.** $t = \sqrt{L/g(1 - \sin\theta)} \cosh^{-1}(L/a)$.

28. (*a*) catenaries; (*b*) circles; (*c*) parabolas; (*d*) cycloids.

30. $y = -\ln(\cos x - \sin x)$; $\sqrt{2}$, 1, $\sqrt{2}$. **31.** $y = \ln(\cos x + \sin x)$.

32. The family of catenaries: $y - b = k \cosh[(x - a)/k]$. **33.** 1.38; 3.63.

34. $y = \pm[\frac{1}{4}(x^2 - 1) - \frac{1}{2}\ln x]$. **35.** $x = \frac{1}{8}(1 - y^2) + \ln y$ and $x = \frac{1}{2}(y^2 - 1) - \frac{1}{4}\ln y$.

36. (*a*) $x = [a^{-k}y^{1+k}/2(1 + k)] - [a^k y^{1-k}/2(1 - k)] + ak/(1 - k^2)$; (*b*) $x = \frac{1}{2}[(y^2/2a) - (a/2) + a\ln(a/y)]$. **37.** (*b*) 3.72 ft/sec.

Art. 45

1. (*a*) $(y + x)^2 + (z - 3x)^2 = 1$; (*b*) $(3y + z)^2 = 3x - z$.

2. $y = x\ln x + 2x$, $z = x\ln x + 3$; [1, 3, 1].

3. $y = \sqrt{2}\cos[x^2 - 1 + (\pi/4)]$, $z = \sqrt{2}\sin[x^2 - 1 + (\pi/4)]$.

4. $C_1e^x + C_2e^{-x}$ and $C_1e^x - C_2e^{-x}$. **5.** $x^2 = t^2 + C_1$, $y = C_2(x + t)$.

6. $(1 + y)^2 = (1 + x)^2 + C_1$, $z = C_2(x + y + 2)$. **7.** $y = C_1e^{-x} + C_2e^x$, $z = C_1e^{-x} - C_2e^x + \sin x$. **8.** (*a*) $x = ce^{-at}$, $y = [ac/(a - b)](e^{-bt} - e^{-at})$, $z = c + [c/(a - b)](be^{-at} - ae^{-bt})$; (*b*) $x = ce^{-at}$, $y = acte^{-at}$, $z = c[1 - (1 + at)e^{-at}]$. **9.** $16\frac{2}{3}$ min, 4.51 lb. **10.** 18.3 min. **11.** 9.43 lb.

Art. 46

1. $y = C_1e^{2x} + C_2e^{-2x} + \frac{1}{2}\sin 2x - \frac{1}{4}\cos 2x$,
$z = C_1e^{2x} - 3C_2e^{-2x} - \frac{3}{4}\cos 2x$.

2. $y = C_1e^{3x} + C_2e^{-x} - 2\cos x - \sin x$,
$z = -2C_1e^{3x} + 2C_2e^{-x} - \cos x + 8\sin x$.

3. $x = e^{-t}(C_1\sin t + C_2\cos t) + (t/2) - \frac{1}{2}$,
$y = e^{-t}[(C_2 - C_1)\sin t - (C_2 + C_1)\cos t] + \frac{1}{2}$.

4. $x = C_1\sin t + C_2\cos t + \sin 2t$, $y = C_1\cos t - C_2\sin t + \cos 2t$.

5. $y = C_1\sin x + C_2\cos x + e^x - \frac{1}{2}x\sin x$,
$z = [C_2 - C_1 + \frac{1}{2} + (x/2)]\sin x - [C_2 + C_1 - (x/2)]\cos x - e^x$.

6. $y = C_1\sin\sqrt{2}x + C_2\cos\sqrt{2}x + 5\sin x - \cos x$,
$z = [(C_1 + 2\sqrt{2}C_2)/3]\sin\sqrt{2}x + [(C_2 - 2\sqrt{2}C_1)/3]\cos\sqrt{2}x + \sin x - 3\cos x$.

7. $y = C_1e^{-2x} + \frac{5}{2}\sin 2x - \frac{1}{2}\cos 2x$,
$z = -(C_1/2)e^{-2x} + \frac{3}{4}\cos 2x - \frac{9}{4}\sin 2x$.

8. $x = C_1\sin t + C_2\cos t + e^t + (t/2)\sin t$,
$y = [C_1 - C_2 - \frac{5}{2} + (t/2)]\sin t + [C_1 + C_2 + (t/2)]\cos t + 2e^t$.

9. $y = (C_1 + C_2x)e^x + (C_3 - x)e^{-x},\ z = 1 - C_2e^x - (2C_3 + 3 - 2x)e^{-x}$.
10. $y = (C_1 + C_2x)e^x + C_3e^{-x} + 3e^{-2x} - 1,\ z = -C_2e^x - 2C_3e^{-x} - 8e^{-2x}$.
11. $x = C_1 + C_2t^2 + t,\ y = C_1 - C_2t^2 + t$.
12. $x = [C_1 - C_2 + (C_1/2)\ln t]t - (C_3/t) + \ln t + 1$,
$y = [C_2 - (C_1/2)\ln t]t + (C_3/t) - 2\ln t - 2$.
13. $n = 0: y = 1 - 2e^x,\quad z = e^x + x - 1$.
$n = 1: y = x - e^x,\quad z = Ce^{-x} + \frac{1}{2}e^x + x - 2$.
$n = \frac{1}{3}: z = C_1e^{3x} + C_2e^{-(3/2)x} + \frac{9}{10}e^x + x - \frac{4}{3}$,
$y = -\frac{4}{3}C_1e^{3x} - \frac{1}{3}C_2e^{-(3/2)x} - \frac{9}{5}e^x + (x/3) + \frac{2}{3}$.
14. $x = \frac{8}{7}\sin t - \frac{1}{21}\sin 3t,\ y = \frac{16}{7}\sin t + \frac{5}{21}\sin 3t$; 0.955, 1.96.
15. $x = A\sin(t + \alpha) + B\sin(9t + \beta) + (at/9) + (h/9) + (8b/81)$,
$y = -A\cos(t + \alpha) + B\cos(9t + \beta) + (bt/9) + (k/9) - (8a/81)$.
16. $y = C_1 + C_2e^t,\ x = -C_1 + (C_2t + C_3)e^t,\ z = -C_1 + (C_2t + C_2 + C_3)e^t$.

Art. 47

1. (*a*) $y = -[gx^2/(2v_0^2\cos^2\alpha)] + x\tan\alpha$;
(*b*) $(v_0^2\sin^2\alpha)/2g$ ft, $(v_0^2\sin 2\alpha)/g$ ft.
2. (*a*) $[(wv_0\sin\alpha)/kg] - (w^2/k^2g)\ln[(kv_0\sin\alpha + w)/w]$ ft.
3. (*a*) 1.24, 1.33; (*b*) $-\frac{5}{3} \leqq x \leqq \frac{5}{3},\ -\sqrt{2} \leqq y \leqq \sqrt{2}$.
4. $x = -\cos(t/\sqrt{2})\cosh(t/\sqrt{2}),\ y = \sin(t/\sqrt{2})\sinh(t/\sqrt{2})$;
(3.83, 3.70). **5.** (*b*) 2.37 amp, 0.061 sec; (*c*) 2.15 amp, 2.85 amp.
6. (*c*) $R = [(1/r_0) - (k/h^2)]\cos\theta + (k/h^2)$.
8. (*a*) $i_1 = (E/R_1)[1 - \epsilon^{-R_1R_2T/L(R_1+R_2)}]$,
$i_2 = (E/R_2)[1 - \epsilon^{-R_1R_2T/L(R_1+R_2)}]$; 0.305 amp, 0.152 amp.
(*b*) $i_1 = E/R_1,\ i_2 = (E/R_2)[1 - \epsilon^{-R_1R_2T/L(R_1+R_2)}]$; 3.2 amp, 0.290 amp.
(*c*) $i_1 = (E/R_1)[1 - \epsilon^{-R_1(R_1+2R_2)T/L(R_1+R_2)}]$,
$i_2 = (E/R_2)[1 - \epsilon^{-R_1R_2T/L(R_1+R_2)}]$; 1.26 amp, 0.290 amp. (*d*) 0.139 sec.
9. $i_1 = \frac{20}{3} - 5\epsilon^{-t} - \frac{5}{3}\epsilon^{-3t},\ i_2 = \frac{10}{3} - 5\epsilon^{-t} + \frac{5}{3}\epsilon^{-3t}$.
10. $i_1 = \frac{1}{4}\epsilon^{-t} + \frac{1}{20}\epsilon^{-3t} - \frac{3}{10}\cos t + \frac{2}{5}\sin t$,
$i_2 = \frac{1}{4}\epsilon^{-t} - \frac{1}{20}\epsilon^{-3t} - \frac{1}{5}\cos t + \frac{1}{10}\sin t$.

Art. 50

3. $x = Ae^{at} + Be^{-at/2}\sin[(\sqrt{3}/2)at + \alpha]$,
$y = Ae^{at} + Be^{-at/2}\sin[(\sqrt{3}/2)at + \alpha + (2\pi/3)]$,
$z = Ae^{at} + Be^{-at/2}\sin[(\sqrt{3}/2)at + \alpha + (4\pi/3)]$.
4. $x = Ae^{3t} + B,\ y = Ae^{3t} + C,\ z = Ae^{3t} - (B + C)$.
5. $x = Ae^{t/2} + Be^{-t},\ y = Ae^{t/2} + Ce^{-t},\ z = Ae^{t/2} - (B + C)e^{-t}$.
6. $x = Ae^t + Be^{-t/2} - t/2 - \frac{1}{2},\ y = Ae^t + Ce^{-t/2} - t/2 - \frac{1}{2}$,
$z = Ae^t - (B + C)e^{-t/2} - t/2 - \frac{1}{2}$.
7. $x = At^2 + Bt^{-1},\ y = At^2 + Ct^{-1},\ z = At^2 - (B + C)t^{-1}$.

8. $x_1 = At^a + Bt^{-a} + K\sin(a\ln t + \alpha)$,
$x_2 = At^a - Bt^{-a} + K\cos(a\ln t + \alpha)$,
$x_3 = At^a + Bt^{-a} - K\sin(a\ln t + \alpha)$,
$x_4 = At^a - Bt^{-a} - K\cos(a\ln t + \alpha)$.
12. $F = (1 + \omega G)/(1 + \omega^2 G)$.

Art. 53

1. $y = C_1x + C_2\sqrt{1 - 4x^2}$. **2.** $y = C_1x + C_2/(x + 1)$.
3. $xy = C_1 + C_2e^x - \sin x - \cos x$. **4.** $y = C_1x + C_2\ln x$.
5. $y = (x - 2)[C_1 + C_2\int e^x\,dx/x^2(x - 2)^2]$.
6. $y = (C_1 + C_2e^{-2x})/x - (x + 1)e^{-2x}$.
7. $y = e^{-x+x^3/3}[C_1\int e^{x-x^3/3}\,dx + C_2]$.
8. $y = (x + 1)e^{-x}[C_1 + C_2\int e^x\,dx/(x + 1)^2]$.
9. $y = (x + 1)[C_1 + C_2\int e^{-x}\,dx/(x + 1)^2]$.
10. $y = e^{x^2/2}[x + C_1\int e^{-x^2/2}\,dx + C_2]$. **11.** $y = C_1/x + C_2/(ax + 1)$.
12. $y = C_1e^x + C_2(2x + 1)e^{-x}$. **13.** $y = C_1x + C_2(x\tanh^{-1}x - 1)$.
14. $y = e^x[C_1 + C_2\int e^{-x}\,dx/x^2]$. **15.** $y = C_1(x + 1) + C_2xe^{1/x} + ax$ or $A(x + 1) + Bxe^{1/x} - a$. **16.** $y = C_1(x + 1) + xe^{1/x}[C_2 + \int e^{-1/x}\,dx/x]$.
17. $y = e^{-x}[C_1 + C_2\int e^{x-x^2/2}\,dx]$. **18.** $y = C_1(2x - 1) + C_2x^2$.
19. $y = C_1/x + C_2(x + 2) - 5$. **20.** $y = C_1(x + 1) + C_2x^2$.
21. $y = xe^{2x}[\ln x + 1/(2x) + C_1\int e^{-2x}\,dx/x^2 + C_2]$.
22. $y = Ae^{-2x} + B(2x - 1)$. **23.** $y = Ae^{2x} + B(3x + 4)e^{-x}$.
26. $y = Ae^{2x} + B(3x - 8)e^{-x}$.

Art. 54

1. $y = e^{-x^2/2}(C_1 + C_2\int e^{x^2/2}\,dx/x)$. **2.** $y = x(C_1 + C_2\int e^{x^3/3}\,dx/x^2)$.
3. $y = C_1e^{2x} + C_2(2x + 1)$. **4.** $y = e^{-x/2}(C_1 + C_2\int e^{-x/2}\,dx/x)$.
5. $y = C_1e^{x^2} + C_2(x^2 + 1)$. **6.** $y = e^{-x^4/2}(C_1 + C_2\int xe^{x^4/2}\,dx)$.
7. $xy = C_1 + C_2e^{x^2}$. **8.** $y = \sqrt{x}(C_1 + C_2\int e^x\,dx/x)$.
9. $y = A\cos(x^2/2) + B\sin(x^2/2)$. **10.** $y = C_1e^{\sqrt{x}} + C_2e^{-\sqrt{x}}$ or $A\cosh\sqrt{x} + B\sinh\sqrt{x}$. **11.** $y = A\cos x^2 + B\sin x^2$.
12. $y = e^{-x^2}(C_1 + C_2e^x)$. **13.** $y = e^{-\frac{1}{2}e^{2x}}(C_1 + C_2\int e^{x+\frac{1}{2}e^{2x}}\,dx)$.
14. $y = e^x(C_1 + C_2\int e^{-x-\frac{1}{2}e^{2x}}\,dx)$. **15.** $y = A\cos(\cos x) + B\sin(\cos x)$.

16. $y = e^{-1/x}(C_1 + C_2 \int e^{1/x}\, dx/x^3)$.

17. $y = e^{x^2/4}[A \cos(\sqrt{3}x^2/4) + B \sin(\sqrt{3}x^2/4)]$.

Art. 55

1. $y = C_1(2x + 1) + (C_2 + \frac{1}{4}x^2)e^{2x}$.
2. $y = \frac{1}{2}x^3 + 3 + C_1 x + C_2 x \int e^{-x^3/3}\, dx/x^2$.
3. $2y = (C_1 + \ln x)e^{\frac{1}{2}x^2} + (C_2 - \int e^{x^2}\, dx/x)e^{-\frac{1}{2}x^2}$.
4. $y = x(C_1 e^x + C_2 - 2x - x^2)$. 5. $y = C_1(x^2 + 2x + 2) + (\frac{2}{3}x^3 + C_2)e^x$.
6. $y = \frac{1}{2}e^x + (C_1 \ln x + C_2)e^{-x}$.
7. $y = e^x\left[\dfrac{A}{x} + B - \frac{1}{2}\int e^{-x}(\sin x + \cos x)\, dx/x^2\right]$.
8. $y = Ae^{x/2} + B(x + 1)e^{-x/2} + \left(\dfrac{x}{3} + \dfrac{2}{9}\right)e^{-x}$.
9. $y = Ae^{\frac{1}{2}x^2} + Be^{-\frac{1}{2}x^2} - x^2$. 10. $y = Ae^{2x} + B(3x + 4)e^{-x} - \frac{1}{2}x + \frac{1}{4}$.
11. $y = e^{-\frac{1}{2}x^2}[C_1 x + C_2 + \iint e^{\frac{1}{2}x^2} R\, dx^2]$.
13. $y\sqrt{x} = C_1 \sin x + C_2 \cos x + a$. 14. $y = \left(C_1 x^2 + \dfrac{C_2}{x} + \dfrac{x^2}{3}\ln x\right)\operatorname{sech} x$.
15. $y = (C_1 + C_2 \ln x)\sqrt{xe^x} + e^{x/2}$.
16. $y = e^{x-\frac{1}{2}x^2}(C_1 + \int e^{-x+\frac{1}{2}x^2}\, dx) + e^{-x-\frac{1}{2}x^2}(C_2 - \int e^{x+\frac{1}{2}x^2}\, dx)$.
17. $y = e^{-\frac{1}{2}x^2}\left[\int e^{\frac{1}{2}x^2}(x^2 - 2)\, dx + C_1 x + C_2\right]$.

Art. 56

1. $y = A/(1 - x) + B/(1 + x)$. 2. $y = a_0(1 - x) + a_3 x^3$.
3. $y = x(a_1 \cosh x + a_2 \sinh x)$ or $x(Ae^x + Be^{-x})$.
4. $y = a_0\left(1 - \dfrac{x^3}{3!} + \dfrac{4^2x^6}{6!} - \dfrac{4^2 7^2 x^9}{9!} + \dfrac{4^2 7^2 10^2 x^{12}}{12!} - \cdots\right)$
$+ a_1\left(x - \dfrac{2^2x^4}{4!} + \dfrac{2^2 5^2 x^7}{7!} - \dfrac{2^2 5^2 8^2 x^{10}}{10!} + \cdots\right)$.
5. $y = a_0(1 - 2x^2) + a_1[x - \frac{1}{2}x^3 - (1/2\cdot 4)x^5 - (1\cdot 3/2\cdot 4\cdot 6)x^7 - \cdots]$
$= a_0(1 - 2x^2) + a_1 x\sqrt{1 - x^2}$.
6. $y = a_0(1 - x^2) + a_1[x - (x^3/3!) - (x^5/5!) - (3x^7/7!) - (3\cdot 5x^9/9!)$
$- (3\cdot 5\cdot 7x^{11}/11!) - \cdots] = (1 - x^2)\{a_0 + a_1 \int [e^{x^2/2}/(1 - x^2)^2]\, dx\}$.
7. $y = a_0[1 + (x^2/2!) - (3x^4/4!) + (5x^6/6!) - (7x^8/8!) + \cdots]$
$+ a_3'[(2x^3/3!) - (4x^5/5!) + (6x^7/7!) - (8x^9/9!) + \cdots]$
$= A(\cos x + x \sin x) + B(\sin x - x \cos x)$.

8. $y = a_0[1 - (x^2/16) - (x^3/96) + (5x^4/1536) + \cdots]$
$+ a_1[x - (x^3/24) - (x^4/192) + (x^5/384) + \cdots].$

9. (*a*) $y = a_0[1 + (x^2/3!) + (x^4/5!) + \cdots] = a_0(\sinh x)/x;$
(*b*) $y = C_1(\cosh x)/x + C_2(\sinh x)/x.$ **10.** $y = (Ae^x + B)/x.$

11. $y = a_0\left(1 - x - \dfrac{x^2}{3} - \dfrac{x^3}{5} - \dfrac{x^4}{7} - \cdots\right).$

12. $y = a_0\left(1 - \dfrac{x^3}{2\cdot3\cdot1!} - \dfrac{x^6}{5\cdot3^2\cdot2!} - \dfrac{x^9}{8\cdot3^3\cdot3!} - \dfrac{x^{12}}{11\cdot3^4\cdot4!} - \cdots\right) + a_1x.$

Art. 58

1. $y = (A/x) + (B/x^2).$

2. $y = e^x\{C_1 + C_2[(1/x) + \ln x - (x/2!) + (x^2/2\cdot3!) - (x^3/3\cdot4!) + \cdots]\}.$

3. $y = (a_0/x) + a_2[x + (x^3/2!) + (x^5/3!) + (x^7/4!) + \cdots] = (A + Be^{x^2})/x.$

4. $y = \sqrt{x}\{C_1 + C_2[\ln x + x + (x^2/2\cdot2!) + (x^3/3\cdot3!) + (x^4/4\cdot4!) + \cdots]\}.$

5. $y = a_0/x + a_1(1 + x^2).$ **6.** $y = a_0/x^2 + a_1[(1/x) - 1].$

7. $y = [A\cosh(x^2/2) + B\sinh(x^2/2)]/x^2.$ **8.** $y = a_0[(1/x) - x] + a_3x^2.$

9. $y = a_0\left(1 - x - \dfrac{x^2}{3} - \dfrac{x^3}{5} - \dfrac{x^4}{7} - \cdots\right) + a_0'\sqrt{x}$
$= A\sqrt{x} + B(1 - \sqrt{x}\tanh^{-1}\sqrt{x}).$

10. $y = a_0[(1/x) + 1] + a_0'\sqrt{x}.$ **11.** $y = (A + B\sqrt{1 - x})\sqrt{x}.$

12. $y = a_0[1 - (x/2) - (x^2/2\cdot4) - (1\cdot3x^3/2\cdot4\cdot6) - (1\cdot3\cdot5x^4/2\cdot4\cdot6\cdot8)$
$- \cdots] + a_0'x^{1/2} = a_0\sqrt{1 - x} + a_0'\sqrt{x}.$

13. $y = a_0(1 + 4x + \frac{8}{3}x^2) + a_0'x^{1/2}[1 + \frac{3}{2}x + (3\cdot1/2\cdot4)x^2$
$- (3\cdot1\cdot1/2\cdot4\cdot6)x^3 + (3\cdot1\cdot1\cdot3/2\cdot4\cdot6\cdot8)x^4 - \cdots] = a_0(1 + 4x + \frac{8}{3}x^2)$
$+ a_0'x^{1/2}(1 + x)^{3/2}.$

14. $y = a_0[1 - (x^2/2\cdot5) + (x^4/2\cdot4\cdot5\cdot9) - (x^6/2\cdot4\cdot6\cdot5\cdot9\cdot13) + \cdots]$
$+ a_0'x^{-1/2}[1 - (x^2/2\cdot3) + (x^4/2\cdot4\cdot3\cdot7) - (x^6/2\cdot4\cdot6\cdot3\cdot7\cdot11) + \cdots].$

15. $y = a_0\left(x^3 - x^4 + \dfrac{x^5}{5} + \dfrac{x^6}{45} + \dfrac{2x^7}{315} + \cdots\right).$ **16.** $-0.2601, 0.5767.$

19. $y = a_0x^2[1 - (x^2/2\cdot4) + (x^4/2\cdot4^2\cdot6) - (x^6/2\cdot4^2\cdot6^2\cdot8) + \cdots] = AxJ_1(x).$

20. $y = (A/x)J_1(x).$ **21.** $y = a_0J_0(x^2/2).$

22. $y = a_0\left(x + \dfrac{x^2}{1!2!} + \dfrac{x^3}{2!3!} + \dfrac{x^4}{3!4!} + \cdots\right) = A\sqrt{x}J_1(2i\sqrt{x}).$

23. $y = 2.23\sqrt{x}J_0\sqrt{x}.$ **24.** $y = 7.69xJ_0(x); 8.87.$

Index

(The numbers refer to pages.)