THEORY OF STATICALLY INDETERMINATE STRUCTURES

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PREFACE

This book is an attempt to present in a thorough and cohesive manner the principles which underlie the analysis of statically indeterminate structures. In the past few years there has been some criticism of the so-called classic methods of analysis on the ground that the methods are abstruse and that excessive labor is involved in using them to analyze statically indeterminate structures. The authors are convinced, however, that the principles involved form the best foundation for a knowledge of structural theory and that familiarity with them is essential to an understanding of structural behavior. To hold to this point of view is not to belittle the importance of some of the methods presented more recently, for the authors are of the opinion that the ease of solution of certain problems made possible by these later methods makes such methods an essential part of the equipment of the structural analyst.

The authors wish to acknowledge their debt to Professor Charles M. Spofford who has been responsible for much of their training. They have been greatly influenced also by Dr. Heinrich Müller-Breslau's comprehensive treatment of this subject.

W. M. Fife.
J. B. Wilbur.

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CONTENTS

SYMBOLS

A = area.
A' = shear area.
b = width of a member.
E = modulus of elasticity.
e = strain (linear).
F = total axial stress in a member.
f = normal stress intensity.
G = shear modulus.
g = shear strain.
I = moment of inertia.
L = length.
M = moment.
M_s = static moment.
P = external load.
p = external load intensity.
Q = any external force.
R = reaction.
S = total transverse shear.
t = temperature.
s = shear intensity.
W = work.
δ = deflection.
Δ = change, *e.g.*, ΔL = change in L.
ϵ = coefficient of thermal expansion.
θ = change in slope of a member referred to its original position.
τ = change in slope referred to a chord.

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CHAPTER I

BASIC CONCEPTS AND THEOREMS

1. Introduction.—A structure is built to perform a certain function. To perform this function satisfactorily, it must have sufficient strength and rigidity. Economy and good appearance are further objectives of major importance in structural design.

Stress analysis serves as an important guide in so proportioning the members of a structure that the requisite strength, rigidity, and economy will be attained. However rigorous a stress analysis may be, the stresses which will actually occur cannot be predicted with absolute certainty. As actually built, a structure and its loading will always differ somewhat, and may differ appreciably, from the hypothetical structure which serves as a basis for stress analysis. Recognition of this fact, together with considerations of economy in time and design expense, often leads the designer to make assumptions known to be incorrect, and approximations known to introduce errors. Only a knowledge of the basic principles of stress analysis can serve as a safe basis for departure from methods more rigorously correct. The assumptions underlying the basic theory and the limitations imposed in the development of basic theorems must be fully understood and constantly kept in mind. If approximate methods are to be introduced, they must be carefully studied in order to ascertain that they may be safely and properly applied to a given problem.

2. Elasticity.—Most of the materials used in building structures follow Hooke's law: Within a certain range dependent upon the material considered, stress and strain are directly proportional to each other. If stresses do not exceed a certain value termed the *elastic limit* for the material considered, the strains will vanish if the stress is removed. Materials behaving in this manner are called elastic. Steel, wrought iron, and wood are elastic, while it is permissible to consider reinforced concrete as elastic, provided the stresses are not too great.

When structures which are composed of elastic members and which rest on elastic supports are subjected to the action of forces or to imposed deformations such as the changing of the length of a bar by taking up a turnbuckle, there are corresponding changes of shape. The movements of points on the structure during such a change are called deflections. Distortion may or may not be accompanied by stresses in the members of a structure.

3. Assumptions and Limitations.—For the treatment to come, the following assumptions are made:

In general, it is assumed that deflections in a structure are so small that the changes in the dimensions of the structure may be considered as infinitesimals, *i.e.*, the errors in the results obtained by solving equations of static equilibrium in which these changes of dimension have been neglected are so small that they are of no importance. It follows that these movements are so small that forces applied to the joints of a structure have the same lines of action after distortion as before distortion. With respect to the inner forces, this may be interpreted to mean that their lines of action do not change relative to the original positions of the members in which they act.

It will be assumed that applied forces and imposed distortions increase gradually, though not necessarily uniformly, from zero to their final magnitudes, and that a condition of equilibrium is ultimately reached.

For the purpose of computing primary stresses in trussed structures, it will be assumed that, even if the joints are capable of carrying bending, the moments at the ends of members are zero; that members of a truss intersect at their gravity axes; that the weights of the members in a trussed structure are concentrated at the ends of the members, so that the bending which may be present owing to weight distribution is neglected. It is recognized that the primary stresses thus obtained are, in general, approximate: a consideration of secondary stresses leads to corrections which should be applied to the primary stresses to compensate for these approximations.

Most of the illustrations used will be of planar structures. This involves the assumption that, even though such a structure forms part of a three-dimensional frame, the part of the structure lying in one plane is not affected by parts lying in other planes. This amounts to neglecting what are called *participating stresses.*

In analyzing trussed structures it will be assumed that the external forces are applied at the joints only. From this it follows that, if the joints are incapable of resisting bending, the stresses in the members will be axial only, *i.e.*, direct tension or direct compression, without either shear or bending moment.

Lastly, it will be assumed that the material of which the structure is composed has an elastic limit, and that stress intensities in members due to all causes combined are less than this elastic limit.

These assumptions having been made, the application of theorems to be developed is limited to problems where conditions correspond closely to those assumed.

The theorems developed will be applicable to both three-dimensional and planar structures, so long as any individual member is in a condition of planar stress. Problems involving three-dimensional stress in any one member are discussed in works on the theory of elasticity and lie beyond the scope of this book.

4. Geometric Relations.—Let the positions of the ends of a member of a three-dimensional frame, the joints of which are capable of resisting bending, be defined by their coordinates referred to the rectangular axes OX, OY, and OZ, so that the coordinates of joint i are x_i, y_i, and z_i, while those of joint k are x_k, y_k, and z_k. Let the angles which the axis ik of the member makes with lines parallel to the OX, OY, and OZ axes be α_{ik}, β_{ik}, and γ_{ik}, respectively.

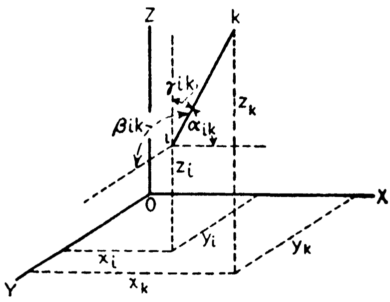


FIG. 1.

Let the coordinates of point k increase by small amounts Δx_k, Δy_k, and Δz_k, and the coordinates of point i increase by small amounts Δx_i, Δy_i, and Δz_i. Let the angles which the axis of the member at k makes with lines parallel to the OX and OY

axes increase by small amounts $\Delta\alpha_k$ and $\Delta\beta_k$, respectively. The angular change of this portion of the axis with respect to a line parallel to OZ is thereby fixed. Let the angles which the axis of the member at i makes with lines parallel to the OX and OY axes increase by small amounts $\Delta\alpha_i$ and $\Delta\beta_i$, respectively. The angular change of this portion of the axis with respect to OZ is thereby fixed. Finally let the end k of the member be given a torsional rotation about the axis ik of the member through a small angle $\Delta\phi_k$, and the i end of the member be rotated about the axis ik of the member through a small angle $\Delta\phi_i$.

These six movements introduced at each end of the member represent all of the independent movements which may occur at a point. From a consideration of the six movements at each end, six independent geometric relations may be written. These may be summarized as follows:

1. The change of length of a member may be expressed in terms of the changes in the end coordinates of the member.

2. The difference of the changes of slope of the ends of a member with respect to lines parallel to the OX axis must equal the sum of the changes of slope with respect to lines parallel to the OX axis occurring at points along the member.

3. The differenee of the changes of slope of the ends of a member with respect to lines parallel to the OY axis must equal the sum of the changes of slope with respect to lines parallel to the OY axis occurring at points along the member.

4. The change of slope with respect to a line parallel to the OX axis of a line joining the ends of a member, as determined by the changes in coordinates at the ends of the member, must be consistent with the changes of slope with respect to lines parallel to the OX axis occurring at points along the member.

5. The change of slope with respect to a line parallel to the OY axis of a line joining the ends of a member, as determined by the changes in coordinates at the ends of the member, must be consistent with the changes of slope with respect to lines parallel to the OY axis occurring at points along the member.

6. The difference in torsional rotations about the axis of a member occurring at the two ends of the member equals the sum of the torsional movements about the axis of the member occurring at points along the member.

The first of these geometric relations may be derived as follows: Let the length of the member joining joints i and k before any movement of the ends of the member occurs be L_{ik}, and the increase in the length of this member corresponding to the end movements be ΔL_{ik}. From the geometry of the figure we may write

$$L_{ik}^2 = (x_k - x_i)^2 + (y_k - y_i)^2 + (z_k - z_i)^2 \qquad (1)$$

After distortion has occurred, the corresponding equation is

$$(L_{ik} + \Delta L_{ik})^2 = [(x_k + \Delta x_k) - (x_i + \Delta x_i)]^2 + [(y_k + \Delta y_k) - (y_i + \Delta y_i)]^2 + [(z_k + \Delta z_k) - (z_i + \Delta z_i)]^2 \qquad (2)$$

If the two sides of Eq. (1) be subtracted from the two sides of Eq. (2), we have

$$\begin{aligned} 2L_{ik}(\Delta L_{ik}) + (\Delta L_{ik})^2 = {} & 2(x_k - x_i)(\Delta x_k - \Delta x_i) + (\Delta x_k - \Delta x_i)^2 \\ & + 2(y_k - y_i)(\Delta y_k - \Delta y_i) + (\Delta y_k - \Delta y_i)^2 \\ & + 2(z_k - z_i)(\Delta z_k - \Delta z_i) + (\Delta z_k - \Delta z_i)^2 \qquad (3) \end{aligned}$$

If the distortion is so small that the increments in the ordinates may be treated as infinitesimals, *i.e.*, if the second powers of the increments may be considered as negligible compared with the first powers, Eq. (3) may be written as follows:

$$L_{ik}(\Delta L_{ik}) = (x_k - x_i)(\Delta x_k - \Delta x_i) + (y_k - y_i)(\Delta y_k - \Delta y_i) + (z_k - z_i)(\Delta z_k - \Delta z_i) \qquad (4)$$

But

$$\begin{aligned} x_k - x_i &= L_{ik}(\cos \alpha_{ik}) \\ y_k - y_i &= L_{ik}(\cos \beta_{ik}) \\ z_k - z_i &= L_{ik}(\cos \gamma_{ik}) \end{aligned}$$

and Eq. (4) may be written in the form

$$\Delta L_{ik} = (\Delta x_k - \Delta x_i) \cos \alpha_{ik} + (\Delta y_k - \Delta y_i) \cos \beta_{ik} + (\Delta z_k - \Delta z_i) \cos \gamma_{ik} \qquad (5)$$

The second, third, fourth, and fifth geometric relations are important. They are considered in detail in other chapters of this book.

The sixth geometric relation, dealing with torsional distortion, is difficult to analyze mathematically. Such research evidence as is available indicates that its effect may usually be ignored without appreciable error.

5. Theory of Linear Simultaneous Equations.—In order that a group of linear simultaneous equations may be solved, the following conditions must be satisfied:

1. There must be as many equations as there are unknowns.
2. The equations must be consistent, *i.e.*, they must represent possible conditions.
3. The equations must be independent, *i.e.*, not obtainable from each other.

The equations $x + y = 1$ and $x - y = 2$ are solvable for x and y because they fulfill the foregoing conditions. The equations $x + y = 1$ and $2x + 2y = 3$ cannot be solved for x and y because they are not consistent. A functional relation exists between the left sides of the two equations—in this case the left side of the second equation equals twice the left side of the first equation. The same functional relation does not exist, however, between the right sides of the equations. Hence the equations represent an impossible condition. The equations $x + y = 1$ and $2x + 2y = 2$ cannot be solved for x and y because the same functional relation exists on both sides of the equations, *i.e.*, the equations are not independent.

If a determinant is evaluated and found to equal zero, a functional relation exists between the coefficients which compose the determinant. Suppose the following equations be solved by the method of determinants

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= C_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= C_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= C_3 \end{aligned}$$

Then

$$x_1 = \frac{\begin{vmatrix} C_1 & a_{12} & a_{13} \\ C_2 & a_{22} & a_{23} \\ C_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} = \frac{D_1}{D_2}, \text{ etc.}$$

If D_2 equals zero, a functional relation exists between the left sides of the equations to be solved. If D_1 also equals zero, the same functional relation exists between the constants C_1, C_2, and C_3. Under these conditions the equations are not all independent of each other, hence by themselves they are indeterminate. If, however, D_2 equals zero and D_1 has a finite value, the functional relation existing between the left sides of the equations does not exist between the constants of the right sides of the equations. The equations are therefore inconsistent and cannot be satisfied.

If there are more independent equations than there are unknowns, the equations cannot be consistent; hence no solution is available which will satisfy all the equations simultaneously. If there are fewer independent equations than unknowns, an infinite number of solutions are available, although there is a definite relation between the different solutions.

Even though determinants may not be used as an expedient for the solution of simultaneous equations, their use makes possible an investigation of the consistency and independence of the equations.

6. Structural Analysis.—The general problem of analysis of a structure involves the determination of the reactions, the stress intensity at any point in the structure, and the shape of the structure after distortion. For a three-dimensional frame, the joints of which are capable of resisting bending, the stress intensity at any point in a member may be determined when the complete state of stress is known at any one section through the member. Six independent elements of stress occur at a given section: The total direct stress in the member, assumed to act at the centroid of the cross section of the member; the transverse shears, acting in the plane of the cross section in any two directions at right angles to each other; the bending moments acting about two axes in the plane of the cross section, normal to each other; lastly, a torsional moment lying in the plane of the cross section. For structures of this type, the movement of any point in a member may be determined when the six independent components of deflection which may occur at each end of the member are known.

Consider a structure of this type, under the action of known external loads in equilibrium, or any other known cause of distortion. In the general case, none of the external forces will be taken as reactions in the sense that the movements of their points of application are known. Let j equal the number of joints in the structure, and m the number of members. There will then be present as independent unknowns in the complete solution six components of deflection at each joint and six components of stress in each member, or a total number of independent unknowns equal to $6(j + m)$. At each joint of the structure, six equations of static equilibrium may be written as follows: Let OX, OY, and OZ be rectangular axes through a joint: the sum of the components parallel to OX of all the forces acting on the joint must equal zero; the sum of the components parallel to OY of all the forces acting on the joint must equal zero; the sum of the components parallel to OZ of all the forces acting on the joint must equal zero; the sum of the components about the OX axis of all the moments acting on the joint must equal zero; the sum of the components about the OY axis of all the moments acting on the joint must equal zero; and the sum of the components about the OZ axis of all the moments acting on the joint must equal zero. For each member of the structure, six geometric equations relating to the conditions of end deflection may be written. There are, then, available for the complete solution, six equations of static equilibrium for each joint and six geometric equations for each member, or a total number of available equations equal to $6(j + m)$, this equaling the number of independent unknowns in the structure.

At any joint in the structure, six independent reactions may be introduced, these being forces along, or moments about, each of three coordinate axes. Each independent reaction introduced offers restraint to a corresponding type of movement at the point of application of the reaction. The restraint may be complete, in which case the movement of the point, of the type resisted by the reaction, is zero, or the restraint may be partial, in which case the point of application of the reaction is permitted a limited amount of movement of the type resisted by the reaction. In either case, the movement of the point of application of the reaction enters the solution as a known quantity, even though this amount must usually be estimated in event of partial restraint. The introduction of each independent reaction therefore removes an unknown component of deflection from those present in the solution of the general problem of structural analysis. The magnitude of the reaction, however, enters as a new unknown, so that equality between the number of available equations and the number of independent unknowns is independent of the number of complete or partial restraints introduced against movements by the reactions.

7. Statically Determinate Structures.—The equations of static equilibrium of joints involve as unknowns only stress components of members and reactions. If the number of equations of static equilibrium available in a structure equals the number of independent stress components, and if the equations are independent and consistent, the equations of equilibrium may be solved by themselves, yielding values of stresses and reactions for the structure. A structure where such a solution is possible is called statically determinate. In such a structure, the equations of equilibrium, although simultaneous, may usually be easily solved, the character of the equations being such that the unknowns may be determined successively. Geometric relations then provide the necessary equations for computing the change of shape of the structure due to stresses or to any other cause.

Four important characteristics of statically determinate structures should be observed:

1. Each member and reaction of the structure is necessary for stability; if any one is removed, the structure will collapse without stress under some possible arrangement of loads.

2. The total load taken by any member, or the value of any reaction, depends only on the external loads and the layout of the structure, and is not a function of the stiffnesses of the members of the structure.

3. If a distortion is introduced internally into any member, such as a change of length due to temperature, or if any point of support is moved with a movement of the type resisted by a reaction at that point, no stresses will occur in the structure.

4. If an external load is applied to the structure, it will immediately encounter an elastic restraint.

8. Stability and Instability.—If it is possible to apply an external load to a structure in such a way that the load encounters no elastic restraint immediately upon its application, the structure is unstable. It is often possible that an unstable structure may be in a condition of unstable equilibrium, in which event elastic restraint will be immediately encountered by a given set of applied loads, although under some other condition of loading no elastic restraint would be offered by the structure to the loads upon their immediate application.

An equation of statics is a statement of the equilibrium of forces which, acting on a body, will prevent nonelastic motion. If n independent equations of statics may be written for a body, it follows that n independent motions of the body are possible, these motions being of the types resisted by the forces for which the equations of statics are written. If no nonelastic movement is to occur, each type of nonelastic movement must meet a corresponding restraint—an independent reaction resulting from the action of adjacent bodies upon the body under consideration. There must then be n restraints available, these restraints being independent components of stress in adjacent supports or structural members.

If a three-dimensional truss, the joints of which are capable of resisting bending, is to be stable, each joint must be restrained against all types of nonelastic movement. A given joint, if unrestrained, may perform six nonelastic movements: translation along, and rotation about, each of three coordinate axes. Six corresponding equations of statics may be written. If there are j joints, $6j$ nonelastic joint movements are possible: for these conditions $6j$ equations of statics may be written. For stability, it is necessary that at least $6j$ independent components of stress in members and reactions be present.

A certain number of the independent stress components must necessarily be provided by external reactions on the structure. The minimum number of independent components of reactions must correspond to the number of independent equations of statics which may be written for the external forces on the structure, this number corresponding to the number of independent types of nonelastic motion which could otherwise occur, considering the structure as a whole. For a three-dimensional structure, the minimum required number of independent reaction components is six, but this number will be increased for each condition of construction, such as the introduction of an internal hinge in the structure, which provides an extra equation of statics for the external forces.

It follows, then, that a necessary but not sufficient requirement for stability is that there must be at least as many independent components of member stresses and reactions as there are independent equations of statics for the structure, and that of these there must be at least as many independent reaction components as there are independent equations of statics for the external forces acting on the structure.

Three types of instability may be considered:

1. When there are fewer independent stress components than there are equations of statics, a structure is unstable. The equations of statics which would be written for this case represent impossible conditions and are inconsistent. A structure of this type is illustrated in Fig. 2, where for the structure shown six equations of statics may be written:

FIG. 2.

$$\Sigma X_a = 0; \quad \Sigma Y_a = 0; \quad \Sigma Z_a = 0; \quad \Sigma X_b = 0; \quad \Sigma Y_b = 0; \quad \Sigma Z_b = 0.$$

There are, however, only four possible stress components: the bar stress F_{ab}, and the reaction components X_b, Y_b, and Z_b.

2. When there are as many independent stress components as there are equations of statics, but when the structure is in certain respects statically indeterminate, it is unstable. The fact that more restraints exist than are necessary to prevent nonelastic movements under one loading makes it unavoidable that under some other loading fewer restraints than are required are available. The planar structure shown in Fig. 3 illustrates instability of this type. Considering only the external forces acting on this structure, three equations of statics may be written: $\Sigma H = 0$; $\Sigma V = 0$; $\Sigma M = 0$. There are three independent reaction components: V_a, V_b, and V_c. Under the load P acting as shown, the structure acts as a continuous beam and is statically indeterminate: if, however, a horizontal load is applied to the structure, it becomes unstable. Even under the vertical load, then, the structure is in unstable equilibrium, so that it belongs to the unstable class of structures.

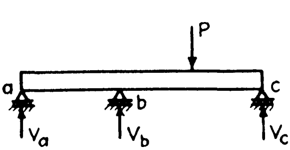


FIG. 3.

3. When there are as many independent stress components as there are equations of statics, and when the members and reactions are so arranged that with certain dimensions the structure is statically determinate, there may be specific relations between the dimensions which would render the structure unstable. Such a condition will be called geometric instability.

For geometric instability, the dimensions are such that different portions of the structure have instantaneous centers of rotation, about which simultaneous rotations without elastic restraint may begin. Usually after a small amount of nonelastic movement the structure has changed its shape sufficiently to act as a new structure which is geometrically stable. The condition of equilibrium which will be reached can be determined only by a consideration of the elastic properties of the structure. The three-hinged arch shown in Fig. 4*a* is statically determinate. If, however, the rise at mid-span is reduced to zero, as shown in Fig. 4*b*, the structure is geometrically unstable. Each half

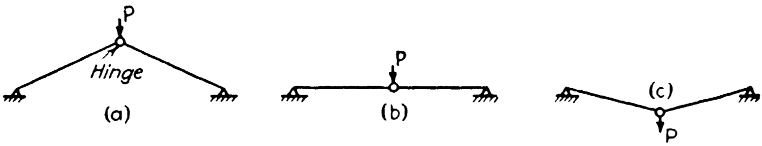


FIG. 4.

of the arch may rotate about its end hinge a small amount without encountering elastic restraint. After a certain amount of movement of the center hinge, a condition of equilibrium is reached, as shown in Fig. 4*c*. The structure is now essentially a different one from that shown in Fig. 4*b*, the deflection of the center hinge during the time the structure is coming into a state of equilibrium depending upon the elastic properties of the structure. If the resultant deflection of this hinge is once determined, the stresses could then be computed by the equations of statics.

For a simple case, geometric instability may be recognized by inspection, or by inconsistent results in a stress analysis by the equations of statics. For more complex cases, a simple model may be constructed and tested, or the following mathematical investigation may be made. Referring to Art. 5, suppose the value of an unknown stress component be expressed as the ratio of two determinants, such as D_1/D_2. In this ratio, the value of D_2 is the same for all stresses. If D_2 equals zero, a functional relation exists between the coefficients of the unknown stresses in the different equations of statics which may be written. If, also, D_1 equals a finite number, the same functional relation does not exist between the applied loads; the equations are inconsistent and impossible, and the structure is geometrically unstable. Should D_1 equal zero, the equations are not independent of each other, and the structure is statically indeterminate under the loading considered. Such a structure will therefore be unstable under some other loading.

As an illustration of geometric instability, consider the structure shown in Fig. 5. In finding the reactions, the available equations are, $\Sigma V = 0$, $\Sigma H = 0$, and $\Sigma M = 0$, together with three equations of condition, namely, the bending moment at joint 9 is zero, the bending moment at joint 27 is zero and the shear in panel 17-19 is zero; in all, six equations. The number

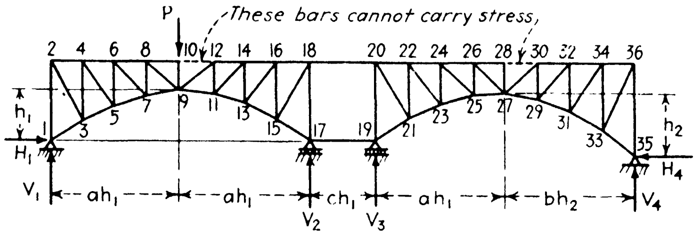


FIG. 5.

of unknown reaction components is six, so that, if the equations have a finite solution, the structure is statically determinate with respect to the outer forces. These equations are:

$$V_1 + V_2 + V_3 + V_4 - P = 0$$
$$H_1 - H_4 = 0$$
$$V_3 + V_4 = 0$$
$$V_1(3ah_1 + ch_1 + bh_2) + V_2(ah_1 + ch_1 + bh_2) + V_3(ah_1 + bh_2) + H_1(h_2 - h_1) - P(2ah_1 + ch_1 + bh_2) = 0$$
$$V_1ah_1 - H_1h_1 = 0$$
$$V_4bh_2 - H_4h_2 = 0$$

If the last two equations be divided by h_1 and h_2, respectively, the denominator determinant D_2 which appears in the solution of the equations is

$$\begin{vmatrix} +1 & +1 & +1 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & +1 & +1 & 0 & 0 \\ 3ah_1+ch_1+bh_2, & ah_2+ch_1+bh_2, & ah_1+bh_2, & 0, & h_2-h_1, & 0 \\ +a & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +b & 0 & -1 \end{vmatrix}$$

which reduces to $ah_1(a - b)$. Hence $D_2 = 0$ when $a = b$, and under these circumstances the reactions will be either infinite or indeterminate. The structure will then be geometrically unstable.

As a summary of the foregoing discussion of instability, one may state: In order that a structure may be stable, there must be at least as many independent components of member and reaction stresses as there are available independent equations of statics for the structure; and of these independent components of member and reaction stresses, there must be at least as many independent reaction components as there are independent equations of statics for the external forces acting on the structure; the equations of statics must be not only independent but consistent.

9. Statically Indeterminate Structures.—It has been shown that the total number of independent equations of statics plus independent equations of geometry is equal to the number of independent components of stress plus independent components of deflection, and that for stability there must be at least as many unknown stress components as there are equations of statics. In addition to the unknown stress components required for stability, there remain, then, as many additional unknowns as there are equations of geometry for the structure. The division of these additional unknowns between stress and deflection components depends upon the structure under consideration and may range from a condition where they are all stress components to the other extreme where they are all deflection components. For this latter condition a structure may be statically determinate with respect to its stresses. When, however, the number of unknown deflection components is less than the number of available equations of geometry, it follows that the total number of unknown stress components exceeds the number of available equations of statics. Solution for the stresses cannot be made for this condition from the equations of statics alone; such a structure is statically indeterminate with respect to its stresses.

Since in a statically indeterminate structure the number of independent stress components exceeds the number of independent equations of statics, this is analogous to stating that the number of restraining influences exceeds the number of independent movements of the structure which could take place were the restraining influences not present. Under these conditions, nonelastic motion meets more than one restraint. The part which the available restraints against a motion play in resisting the motion depends upon the effectiveness of the restraints involved, which in turn is a function of the elastic properties of the members of the structure and its foundation. The dependence of stresses upon elastic properties of the structure constitutes an important characteristic of the statically indeterminate structure.

The excess of independent stress components in a statically indeterminate structure, over those required for stability, makes possible the removal of certain restraints, still leaving the structure stable. If the maximum number of restraints which can be removed without making the structure unstable are imagined as inoperative, the resulting structure is called the primary structure. Restraints are removed by removing independent stress components of reaction and member stresses. The number of restraints removed to give the primary structure is the degree to which a structure is statically indeterminate. The independent components of stress which may be so removed are called redundant stresses. The degree to which a structure is statically indeterminate is numerically equal to the excess in the number of independent stress components over the number of independent equations of statics; it is also numerically equal to the number of independent equations of geometry in excess of the number of independent deflection components.

A structure may be statically indeterminate in whole or in part. If the reactions can be determined by statics but the member stresses depend upon the elastic properties of the structure, the structure is statically indeterminate internally. Often a portion of the member stresses along with the reactions can be determined by statics, but the remaining member stresses depend upon the elastic properties of the members involved. If the number of independent reaction components exceeds the number of independent equations of statics for the external forces, but the structure itself would be statically determinate were the reactions known, the structure is both internally and externally indeterminate, since the actual magnitude of the reactions depends upon the elastic properties of the members and foundations. An important characteristic of a statically indeterminate stress component is as follows: If a deformation is imposed on a structure of the type resisted by a redundant stress, the deformation meets elastic restraint.

In general there are two approaches which may be made in the solution of statically indeterminate structures: The stresses may be first determined and the deformations computed to correspond; or the deflection components may be evaluated directly in order that they may serve as a basis for stress analysis. In the first approach, there may be added to the equations of statics as many equations dealing with the elastic properties of the structure (these corresponding to the equations of geometry) as there are redundant stress components in the structure. The solution of these equations gives the stresses directly. The equations of statics can usually be solved progressively and with comparatively little labor; the equations of elasticity are usually interrelated in such a way that they must be solved simultaneously. The second approach consists of expressing the independent stress components in terms of the independent components of deflection and solving the equations of statics in which the stresses have been so expressed. The solution of these equations yields the components of deflection from which the stresses may be determined. In this procedure, the resulting equations of statics are interrelated in such a way that they must be solved simultaneously. In general, the necessary equations in either approach are easily written, and the labor involved in the solution of the equations is a major criterion in determining which is the better approach. Since this usually depends upon the number of equations which must be solved simultaneously, we may conclude that if the number of unknown deflection components exceeds the number of unknown stress components in excess of the number of independent equations of statics, the first approach is superior; but that if the number of unknown deflection components is less than this, the second approach will be the better.

10. The Law of Virtual Work.—One of the problems arising in structural analysis is the determination of the distortion of a structure when it is subjected to external forces, to change of temperature or to any cause whatsoever. The solution of this problem is important not only because it is sometimes necessary to know the positions of points in the distorted structure, but also because one of the general methods of attacking the problem of stress analysis for statically indeterminate structures is based on setting up expressions for the movements of points in the structure in certain conditions of distortion. The *law of virtual work* may be used as a basis for determining the relation between the condition of distortion as defined by the internal strains and the movements of certain definite points in the structure. Its validity may be demonstrated in the following manner:

Consider a body formed of isotropic material which is in equilibrium under a group of external forces Q. Assume that a small change in the shape of the body occurs, this change being measured from that which exists in the condition of equilibrium and that it is independent of the forces Q. Such a distortion is called a virtual distortion, the particular significance of the term virtual being that the distortion is independent of the forces Q. In the condition described any small particle within the body is in equilibrium under the forces applied to its surfaces by the adjacent particles and any forces which may act on it owing to the fact that it has mass; these latter forces will be called inertia forces and in most bodies the only one which exists is the weight of the particle. During the virtual distortion each particle may be translated, rotated, and distorted: if these changes occur, the forces applied to the particle will perform certain amounts of work which are designated as virtual work because the displacements of the points of application of the forces are not dependent on the magnitudes of the forces. Let the work done by the forces acting on the surfaces of the particle be designated by dW_s and the work done by the inertia forces be called dW_i; further, let the work done by the surface forces be divided into two parts, first, that part which is done during the distortion only of the particle, and, second, that part which is the result of the translation and rotation. If the first of these two parts is designated by dW_d, the second will be

$$dW_s - dW_d.$$

Since the surface forces and inertia forces form a system in equilibrium, their resultant is zero and the amount of virtual work done by them during the translation and rotation of the particle must be zero also; therefore,

$$dW_i + dW_s - dW_d = 0 \tag{6}$$

which may be written

$$dW_d = dW_i + dW_s \tag{7}$$

If the virtual work be integrated over the whole of the body, the equation becomes

$$W_d = W_i + W_s \tag{8}$$

On each surface which lies inside the body and is, therefore, a boundary surface between two adjacent particles, two sets of forces operate, these being the reciprocal effects of the two particles on each other. These two sets of forces must be equal in magnitude but opposite in direction; consequently, the amounts of virtual work done by the two sets in any translation and rotation of the boundary surface must have an algebraic sum equal to zero. This being so, that part of the virtual work indicated by the term W_s in Eq. (8) which refers to the work done by the forces acting on interior surfaces, vanishes, and the term must be interpreted as the virtual work done by the forces applied to the outer surface of the whole body. Under these circumstances, Eq. (8) may be stated as follows: *If a body is in equilibrium and remains in equilibrium while it is subjected to a small virtual distortion, the virtual work done by the external forces acting on the body plus the virtual work done by the inertia forces is equal to the virtual work of distortion.* This statement is known as the law of virtual work.

11. Stresses Acting on a Particle.—Before the law of virtual work can be used in the problem of determining the relation between the movements of certain points in a structure and the condition of internal strain, it is necessary to develop expressions for the virtual work of distortion. With this purpose in view consider the distortion a small parallelepiped isolated from the surrounding material as shown in Fig. 6. Let the position of the particle be referred to a set of perpendicular coordinate axes OX, OY, and OZ and let the lengths of the edges be dx, dy, and dz, respectively. Let the stress intensities acting on the face which is perpendicular to the X axis and passes through the origin O be

f_x, parallel to the X axis and positive when it is tension,

s_{xy}, parallel to the Y axis and positive when it acts in the direction $(-y)$,

s_{xz}, parallel to the Z axis and positive when it acts in the direction $(-z)$.

Similarly, let the stress intensities acting on the face perpendicular to the Y axis and passing through the origin be

f_y, parallel to the Y axis and positive when it is tension,

s_{yz}, parallel to the Z axis and positive when it is in the direction $(-z)$,

s_{yx}, parallel to the X axis and positive when it is in the direction $(-x)$.

There will be similar notation for the stress intensities acting on the face passing through the origin and perpendicular to

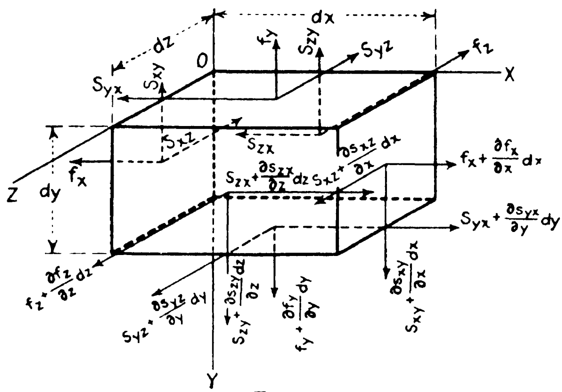


FIG. 6.

the Z axis. The stress intensities on the faces opposite to these three may differ from them by increments. For example, the stress intensities acting on the face perpendicular to the X axis but not passing through the origin are $f_x + \frac{\partial f_x}{\partial x}dx$, $s_{xy} + \frac{\partial s_{xy}}{\partial x}dx$ and $s_{xz} + \frac{\partial s_{xz}}{\partial x}dx$, and are to be taken as positive as shown in the figure. As the lengths of the edges of the parallelepiped decrease, approaching zero as a limit, it is permissible to neglect any variation of stress intensity over a face so that the resultant stress on any face may be considered as acting through the geometric center of that face.

Since the particle is in equilibrium under the surface forces and inertia forces three equations of the form $\Sigma M = 0$ may be written, taking, in turn, as axes of moments, three axes passing through the center of gravity of the particle and parallel to the X, Y, and Z axes, respectively. If the first equation be that about the axis parallel to the X axis, the resultants of the stress intensities f_x, f_y, f_z, $f_x + \frac{\partial f_x}{\partial x}dx$, $f_y + \frac{\partial f_y}{\partial y}dy$, $f_z + \frac{\partial f_z}{\partial z}dz$, and the resultant inertia force will have lines of action intersecting the axis of moments; this is true also of the lines of action of the resultants of the stress intensities s_{xy}, s_{xz}, $s_{xy} + \frac{\partial s_{xy}}{\partial x}dx$, and $s_{xz} + \frac{\partial s_{xz}}{\partial x}dx$, while the lines of action of the resultants of s_{yx}, s_{zx}, $s_{yx} + \frac{\partial s_{yx}}{\partial y}dy$ and $s_{zx} + \frac{\partial s_{zx}}{\partial z}dz$ are parallel to the axis of moments. The moments of the resultants of all these stress intensities will, therefore, vanish from this equation $\Sigma M = 0$, leaving

$$s_{yz}\,dx\,dz\frac{dy}{2} + \left(s_{yz} + \frac{\partial s_{yz}}{\partial y}dy\right)dx\,dz\frac{dy}{2} = s_{zy}\,dx\,dy\frac{dz}{2} + \left(s_{zy} + \frac{\partial s_{zy}}{\partial z}dz\right)dx\,dy\frac{dz}{2}$$

which may be written

$$s_{yz}\,dx\,dy\,dz + \frac{\partial s_{yz}}{\partial y}\frac{1}{2}dx(dy)^2dz = s_{zy}\,dx\,dy\,dz + \frac{1}{2}\frac{\partial s_{zy}}{\partial z}dx\,dy(dz)^2$$

If both sides of this equation be divided by $dx\,dy\,dz$, it becomes

$$s_{yz} + \frac{1}{2}\frac{\partial s_{yz}}{\partial y}dy = s_{zy} + \frac{1}{2}\frac{\partial s_{zy}}{\partial z}dz$$

As dx, dy, and dz approach zero, the infinitesimals in this equation approach zero, also, and in the limit vanish, therefore, at any point,

$$s_{yz} = s_{zy} \qquad (9a)$$

The other two equations $\Sigma M = 0$ lead in a similar fashion to

$$s_{zx} = s_{xz}, \qquad s_{xy} = s_{yx} \qquad (9b)$$

This is merely an extension to the condition of three-dimensional stress of the familiar statement that in beams, which are considered to be in a condition of planar stress, the intensity of transverse shear at any point is equal to the intensity of longitudinal shear. For the purpose of brevity there will, in future, be no distinction made between s_{yz} and s_{zy}, between s_{zx} and s_{xz}, or between s_{xy} and s_{yx}. They will be designated by s_x, s_y, and s_z, respectively, in which it should be noted that s_x is perpendicular to the X axis, s_y is perpendicular to the Y axis, and s_z is perpendicular to the Z axis.

12. The Virtual-work Equations.—During the virtual distortion of a particle let the system of forces acting on the body, including the inertia forces, be designated by Q' and let the stress intensities due to these forces be f'_x, f'_y, f'_z, s'_x, s'_y, and s'_z. Suppose that during the distortion the surface YOZ moves away from the center of gravity of the particle a distance $\Delta'(dx/2)$ and that the opposite face moves away from the center of gravity a distance $\Delta''(dx/2)$. Owing to this part of the distortion the stress intensities f'_x and $f'_x + \frac{\partial f'_x}{\partial x}dx$ will perform the virtual work

$$(f'_x\,dy\,dz)\Delta'\left(\frac{dx}{2}\right) + \left[\left(f'_x + \frac{\partial f'_x}{\partial x}dx\right)dy\,dz\right]\Delta''\left(\frac{dx}{2}\right)$$
$$= (f'_x dy\,dz)\Delta(dx) + \left(\frac{\partial f'_x}{\partial x}dx\,dy\,dz\right)\Delta''\left(\frac{dx}{2}\right)$$
$$= f'_x\,dx\,dy\,dz\frac{\Delta(dx)}{dx} + \frac{\partial f'_x}{\partial x}(dx)^2 dy\,dz\frac{\Delta''(dx/2)}{dx}$$

The first term of this expression is an infinitesimal of the third order and the second is an infinitesimal of the fourth order. Since it is intended to find the total virtual work of distortion by integrating over the whole body and since, in such an integration, the integrals of the infinitesimals of the fourth order will vanish from the definite integral obtained, it is unnecessary to consider the second term further and one may write, for this part of the virtual work, the expression

$$f'_x\frac{\Delta(dx)}{dx}dV$$

The ratio $\Delta(dx)/dx$ is the distance strain parallel to the X axis, and this will be designated by the notation e_x so that the expression above becomes

$$f'_x e_x\,dV$$

Similar treatment with respect to the stress intensities and virtual strains parallel to the Y and Z axes leads to the expressions $f'_y e_y \, dV$ and $f'_z e_z \, dV$, respectively, for the virtual work done by the normal stress intensities parallel to the other two axes.

In addition to the changes in the lengths of the edges of the parallelepiped, the condition of distortion may involve changes in the angles YOZ, ZOX, and XOY. If such changes occur, the shearing stresses will perform work. Consider first the effect of a change in the angle YOZ, this change being taken as positive when the angle decreases. In this change the only stress intensities which do work which does not vanish when added to the work done by other shearing stress intensities are s'_x, $s'_x + \frac{\partial s'_x}{\partial y}dy$ and $s'_x + \frac{\partial s'_x}{\partial z}dz$. To evaluate this amount of work consider Fig. 7.

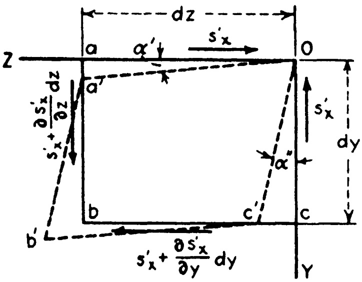


FIG. 7.

Let the original cross section of the parallelepiped be $Oabc$ and after distortion let the shape be $Oa'b'c'$. Let the decrease in the angle YOZ be $\alpha' + \alpha'' = g_x$, which is the shear strain about the X axis. The virtual work done in this part of the virtual distortion is

$$\left(s'_x + \frac{\partial s'_x}{\partial z}dz\right)dx\,dy\,\alpha'\,dz + \left(s'_x + \frac{\partial s'_x}{\partial y}dy\right)dx\,dz\,\alpha''\,dy$$
$$= s_x\,dx\,dy\,dz(\alpha' + \alpha'') + \alpha'\frac{\partial s'_x}{\partial z}dx\,dy(dz)^2 + \alpha''\frac{\partial s'_x}{\partial y}dx(dy)^2dz$$

As in the expressions having to do with the virtual work done by the normal stress intensities, the infinitesimals of the fourth order will vanish when the virtual work is integrated over the body and need not be considered further. This part of the virtual work becomes, therefore, $s'_x g_x \, dV$ and, similarly, the virtual work done when the angles ZOX and XOY change is $s'_y g_y \, dV$ plus $s'_z g_z \, dV$.

The total virtual work of distortion done by the stress intensities acting on the particle is

$$dW_d = [f'_x e_x + f'_y e_y + f'_z e_z + s'_x g_x + s'_y g_y + s'_z g_z]dV \qquad (10)$$

and the virtual work done during the distortion of the whole body is

$$W_d = \int (f'_x e_x + f'_y e_y + f'_z e_z + s'_x g_x + s'_y g_y + s'_z g_z) dV \quad (11)$$

The law of virtual work states that, in the kind of distortion under consideration in this article, the virtual work of distortion is equal to the virtual work done by the external forces plus the virtual work done by the inertia forces. If all these forces be included in a system called the Q' system and if the displacement of the point of application of any one of these forces along its line of action be designated by δ; also, if the internal stress intensities set up by the Q' system and, consequently, forming with the Q' system a system in equilibrium, are those designated by f'_x, f'_y, f'_z, s'_x, s'_y and s'_z, while the virtual distortion is defined by the strains e_x, e_y, e_z, g_x, g_y and g_z, Eq. (11) becomes

$$\Sigma Q' \delta = \int (f'_x e_x + f'_y e_y + f'_z e_z + s'_x g_x + s'_y g_y + s'_z g_z) dV \quad (12)$$

The limitations on the validity of this equation are those defined in the previous statement and should be carefully kept in mind.

The condition of distortion encountered most frequently in structural analysis is that due to the imposition of loads or to a change of temperature. In order to use Eq. (12) as an expedient to determine the relation between the movement of a point in the structure and the internal strains, an arbitrarily chosen Q' system is adopted, this being such that the left-hand side of the equation may be interpreted as numerically equal to a simple multiple of the deflection component desired while the virtual distortion is defined by the strains due to the applied loads and the change of temperature. Usually it is convenient to express these strains, as far as possible, in terms of the corresponding stress intensities. If the stress intensities due to the loads and temperature changes be designated by f_x, f_y, f_z, s_x, s_y, and s_z, and the change of temperature by Δt, the strains may be expressed in the form

$$\begin{aligned} e_x &= \frac{1}{E}[f_x - \nu(f_y + f_z)] + \epsilon \, \Delta t \qquad & g_x &= \frac{s_x}{G} \\ e_y &= \frac{1}{E}[f_y - \nu(f_z + f_x)] + \epsilon \, \Delta t \qquad & g_y &= \frac{s_y}{G} \\ e_z &= \frac{1}{E}[f_z - \nu(f_x + f_y)] + \epsilon \, \Delta t \qquad & g_z &= \frac{s_z}{G} \end{aligned} \quad (13)$$

in which ϵ is the coefficient of thermal expansion for the material of which the body is composed, and ν is Poisson's ratio.

It is still more convenient to have the strains expressed in terms of the bending moments, total axial stresses, and total shears. This can be done without trouble if the stress condition in each member of the structure is one of planar stress; this condition is usually assumed to exist even in a three-dimensional structure. If a member is in a condition of planar stress and if the X axis for the member is taken as coinciding with the gravity axis of the member

$$f_x = \frac{My}{I} + \frac{F}{A}; \quad f_y = 0; \quad f_z = 0; \quad s_x = 0; \quad s_y = 0; \quad s_z = \frac{SM_s}{bI}$$

in which M = the bending moment at any section of a member,
y = the distance of a point in the cross section from the gravity axis and is to be taken as positive on the side in which the bending moment causes tension,
I = the moment of inertia of the cross-sectional area about its gravity axis,
F = total axial stress,
A = the area of a cross section,
S = total transverse shear at any section of a member,
b = the width of the cross section at a point distance y from the gravity axis,
M_s = the static moment about the gravity axis of that part of the cross-sectional area which lies outside of the point at which the shear intensity is to be computed.

Also

$$f'_x = \frac{M_q y}{I} + \frac{F_q}{A}, \qquad f'_y = f'_z = 0, \qquad s'_x = s'_y = 0, \qquad s'_z = \frac{S_q M_s}{bI}$$

in which M_q = the bending moment at a cross section due to the Q' system,
F_q = the total axial stress at a cross section due to the Q' system,
S_q = the total shear at a cross section due to the Q' system.

Using this notation, Eq. (12) may be written as

$$\sum Q'\,\delta = \sum \int \left\{ \left(\frac{M_q y}{I} + \frac{F_q}{A} \right) \left(\frac{My}{EI} + \frac{F}{AE} + \epsilon\,\Delta t \right) + \frac{S_q M_s}{bI} \frac{SM_s}{bIG} \right\} dV$$

$$= \sum \int \left(\frac{M_q M y^2}{EI^2} + \frac{F_q M y}{EAI} + \frac{M_q F y}{EAI} + \frac{F_q F}{A^2 E} + \epsilon \frac{M_q \Delta t\, y}{I} + \frac{F_q}{A}\epsilon\,\Delta t + \frac{S_q S M_s^2}{b^2 I^2 G} \right) dV$$

$$= \sum \left[\int_{x=0}^{x=L} \frac{M_q M}{EI^2} dx \int\int y^2\, dy\, dz + \int_{x=0}^{x=L} \frac{F_q M}{EAI} dx \int\int y\, dy\, dz + \int_{x=0}^{x=L} \frac{M_q F}{EAI} dx \int\int y\, dy\, dz \right.$$
$$+ \int_{x=0}^{x=L} \frac{F_q F}{A^2 E} dx \int\int dy\, dz + \epsilon \int_{x=0}^{x=L} \frac{M_q}{I} dx \int\int \Delta t\, y\, dy\, dz + \epsilon \int_{x=0}^{x=L} \frac{F_q}{A} dx \int\int \Delta t\, dy\, dz$$
$$\left. + \int_{x=0}^{x=L} \frac{S_q S}{I^2 G} dx \int\int \frac{M_s^2}{b^2} dy\, dz \right] \qquad (14)$$

In this expression the double integrals (over the cross section) have the following meanings:

$$\int\int y^2\, dy\, dz = I; \qquad \int\int y\, dy\, dz = 0; \qquad \int\int dy\, dz = A$$

and Eq. (14) becomes

$$\sum Q'\,\delta = \sum \left[\int \frac{M_q M}{EI} dx + \int \frac{F_q F}{AE} dx + \epsilon \int \frac{M_q}{I} dx \int\int \Delta t\, y\, dy\, dz + \epsilon \int \frac{F_q}{A} dx \int\int \Delta t\, dy\, dz + \int \frac{S_q S}{I^2 G} dx \int\int \frac{M_s^2}{b^2} dy\, dz \right] \qquad (15)$$

the single integrals being from $x = 0$ to $x = L$.

The third and fourth terms of the right-hand side of Eq. (15) cannot be simplified further without defining the variation of the change of temperature over the member and the last term cannot be simplified without defining the shape of the cross section. Further simplification may be illustrated, however, by carrying out the process for a particular set of conditions. For example, assume a temperature change which is constant across the width of a member but varies linearly from the upper to the lower side, the plane of stress being vertical (see Fig. 8).

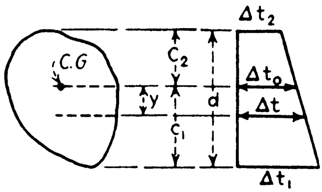


FIG. 8.

$$\Delta t = \Delta t_2 + (\Delta t_1 - \Delta t_2)\frac{c_2 + y}{d}$$
$$= \Delta t_1\frac{c_2 + y}{d} + \Delta t_2\left(1 - \frac{c_2 + y}{d}\right)$$
$$= \frac{1}{d}[\Delta t_1(c_2 + y) + \Delta t_2(c_1 - y)]$$

At the C.G., $\Delta t_0 = \frac{1}{d}(c_2\,\Delta t_1 + c_1\Delta t_2)$

$$\int\int \Delta t\, y\, dy\, dz = \frac{1}{d}\int\int \{\Delta t_1(c_2 + y) + \Delta t_2(c_1 - y)\}y\, dy\, dz$$
$$= \frac{\Delta t_1 c_2 + \Delta t_2 c_1}{d}\int\int y\, dy\, dz + \frac{\Delta t_1 - \Delta t_2}{d}\int\int y^2\, dy\, dz$$
$$= \frac{I}{d}\,\Delta(\Delta t) \tag{16}$$

where

$$\Delta(\Delta t) = \Delta t_1 - \Delta t_2$$

$$\int\int \Delta t\, dy\, dz = \frac{1}{d}\int\int \{\Delta t_1(c_2 + y) + \Delta t_2(c_1 - y)\}dy\, dz$$
$$= \frac{\Delta t_1 c_2 + \Delta t_2 c_1}{d}\int\int dy\, dz + \frac{\Delta(\Delta t)}{d}\int\int y\, dy\, dz$$
$$= A\,\Delta t_0 \tag{17}$$

The third and fourth terms become, for this set of conditions,

$$\epsilon\int\frac{M_q}{d}\Delta(\Delta t)dx + \epsilon\int F_q\Delta t_0\, dx$$

which, when the member is prismatic and $\Delta(\Delta t)$ is the same for all cross sections of the member, may be written

$$\epsilon \frac{\Delta(\Delta t)}{d} \int M_q \, dx + F_q \, \epsilon \, \Delta t_0 \, L$$

The simplification of the shear term may be illustrated by carrying it through for a member of rectangular cross section.

$$M_s = b\left(\frac{d}{2} - y\right)\frac{1}{2}\left(\frac{d}{2} + y\right) = \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

$$\int\int \frac{M_s^2}{b^2} dy\, dz = \frac{1}{b^2}\int_{-\frac{d}{2}}^{+\frac{d}{2}} \frac{b^2}{4}\left(\frac{d^4}{16} - \frac{d^2y^2}{2} + y^4\right) b\, dy$$

$$= \frac{b}{4}\left\{\frac{d^4}{16}\left(\frac{d}{2} + \frac{d}{2}\right) - \frac{d^2}{6}\left(\frac{d^3}{8} + \frac{d^3}{8}\right) + \frac{1}{5}\left(\frac{d^5}{32} + \frac{d^5}{32}\right)\right\}$$

$$= \frac{bd^5}{4}\left(\frac{1}{16} - \frac{1}{24} + \frac{1}{80}\right) = \frac{bd^5}{120} = \frac{d^2 I}{10} \qquad (18)$$

Therefore, for a member of rectangular cross section, the shear term becomes

$$\int \frac{S_q S\, d^2}{10GI} dx = \frac{12}{10}\int \frac{S_q S}{AG} dx$$

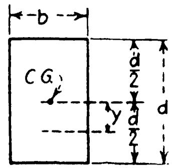


FIG. 9.

The evaluation of the shear term for members such as I beams and channels is more complicated but the evaluation for a few sections lead to the following results:

For an 18-in. C.B. 70 lb., $0.994 \int \frac{S_q S}{A'G} dx$

For an 18-in. C.B. 124 lb., $1.000 \int \frac{S_q S}{A'G} dx$

For a 36-in. C.B. 360 lb., $0.997 \int \frac{S_q S}{A'G} dx$

For a 24-in. I beam 79.9 lb., $1.018 \int \frac{S_q S}{A'G} dx$

For an 18-in. I beam 42.9 lb., $0.997 \int \frac{S_q S}{A'G} dx$

where A' is the product of the web thickness and the depth of the beam. It is evident, therefore, that if the shear were considered as uniformly distributed over the area of a beam of rectangular cross section the contribution made by the shear term to the total deflection would be but 83 per cent of the correct magnitude, but that in an I beam the assumption that the shear is uniformly distributed over the area of the web (beam depth times web thickness) leads to an error in the evaluation of this term of but 2 per cent or less if these values may be taken as typical. Since, in general, the shear term is a comparatively small part of the total, the error involved in using this approximate method is negligible, particularly when the members are beams of the shape used for rolled steel beams. With this in mind, Eq. (15) may be written as

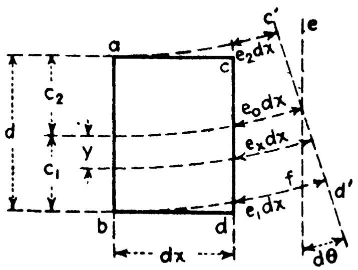


FIG. 10.

$$\sum Q'\delta = \sum\left[\int \frac{M_qM}{EI}dx + \int \frac{F_qF}{AE}dx + \epsilon\int M_q\frac{\Delta(\Delta t)}{d}dx + \epsilon\int F_q\Delta t_0\,dx + K\int \frac{S_qS}{A'G}dx\right] \quad (19)$$

where K is a numerical coefficient equal to 1.2 for members of rectangular cross section and 1.0 for members such as I beams, and A' is an area over which the shear is assumed to be uniformly distributed. If the member be part of an ideal truss, in which the members are subjected to axial stress only, the equation becomes

$$\sum Q'\delta = \sum \frac{F_qFL}{AE} + \sum F_q\epsilon\,\Delta t_0 L = \Sigma F_q\Delta L \quad (20)$$

This discussion might be made somewhat broader in its application if the cause of distortion is not limited to the application of loads and to change of temperature. The assumption that each member is in a condition of planar stress is retained and the assumption that transverse sections, planar before distortion, are planar after distortion, which was a prerequisite to the equations

$$f'_x = \frac{M_q y}{I} + \frac{F_q}{A}, \qquad s'_x = \frac{S_q M_s}{bI}$$

is again pointed out. For a condition of distortion which fits these assumptions, an element of a member such as that shown by *abcd* in Fig. 10 has a shape after distortion as shown by *abc'd'*. The section *ef* is parallel to the original position of the section *cd*; e_0 is the distance strain at the gravity axis of the member.

$$\begin{aligned} e_x\,dx &= e_0\,dx + \frac{y}{c_1}(e_1\,dx - e_0\,dx) \\ &= e_0\,dx + y\,d\theta \end{aligned}$$

In such circumstances Eq. (12) becomes

$$\begin{aligned} \sum Q'\,\delta &= \sum\left\{\int\left[\left(\frac{M_q y}{I} + \frac{F_q}{A}\right)\left(e_0 + y\frac{d\theta}{dx}\right) + s_q g_s\right]dV\right\} \\ &= \sum\left[\int\left(\frac{M_q}{I}ye_0 + \frac{F_q}{A}e_0 + \frac{M_q}{I}y^2\frac{d\theta}{dx} + \frac{F_q}{A}y\frac{d\theta}{dx} + s_q g_s\right)dV\right] \\ &= \sum\left[\int\frac{M_q}{I}e_0\,dx\int\int y\,dy\,dz + \int\frac{F_q}{A}e_0\,dx\int\int dy\,dz\right. \\ &\left. + \int\frac{M_q}{I}\frac{d\theta}{dx}dx\int\int y^2\,dy\,dx + \int\frac{F_q}{A}\frac{d\theta}{dx}dx\int\int y\,dy\,dz + \int s_q g_s\,dV\right] \end{aligned}$$

and, since $\int\int y\ dy\ dz = 0$, $\int\int dy\ dz = A$, $\int\int y^2\ dy\ dz = I$, this becomes

$$\sum Q'\,\delta = \sum\left[\int M_q\frac{d\theta}{dx}dx + \int F_q e_0\,dx + \int s_q g_s\,dV\right] \quad (21)$$

and if F_q is constant over the length of each member, this becomes

$$\sum Q'\,\delta = \sum\int M_q\frac{d\theta}{dx}dx + \sum F_q\Delta L + \int s_q g_s\,dV. \quad (22)$$

When the causes of distortion are limited to the application of loads and to change of temperature,

$$\begin{aligned} e_1 &= \frac{1}{E}\left(\frac{Mc_1}{I} + \frac{F}{A}\right) + \epsilon\,\Delta t_1 \\ e_2 &= \frac{1}{E}\left(\frac{Mc_2}{I} + \frac{F}{A}\right) + \epsilon\,\Delta t_2 \end{aligned}$$

where c_2 is negative;

$$d\theta = (e_1\,dx - e_2\,dx)\frac{1}{d}$$
$$= \left[\frac{1}{E}\left(\frac{Mc_1}{I} + \frac{F}{A} - \frac{Mc_2}{I} - \frac{F}{A}\right)dx + \epsilon(\Delta t_1 - \Delta t_2)dx\right]\frac{1}{d}$$
$$= \frac{M}{EI}dx + \epsilon\frac{\Delta(\Delta t)}{d}dx$$

Also

$$e_0 = \frac{F}{AE} + \epsilon\,\Delta t_0$$

It is evident that if these values are substituted in Eq. (22) the resulting equation is the same as Eq. (19). It should be remembered that the $d\theta$ used here is the change in the slope of a section originally perpendicular to the axis of the member. If the distortion includes the effect of shear, the transverse sections do not remain perpendicular to the axis of the member and the change of slope of the axis is not the same as the change in the slope of the transverse sections.

13. Alternate Proof of Law of Virtual Work for Trussed Structures.—An alternative method of demonstrating the law of virtual work for trussed structures may be based on the geometrical relation stated in Eq. (5):

$$\Delta L_{ik} = (\Delta x_k - \Delta x_i)\cos\alpha_{ik} + (\Delta y_k - \Delta y_i)\cos\beta_{ik} + (\Delta z_k - \Delta z_i)\cos\gamma_{ik}$$

Suppose that some system of external forces Q' applied to the joints of the truss causes a stress F'_{ik} in member ik. Multiply both sides of Eq. (5) by this stress. Then

$$F'_{ik}\Delta L_{ik} = (\Delta x_k - \Delta x_i)F'_{ik}\cos\alpha_{ik} + (\Delta y_k - \Delta y_i)F'_{ik}\cos\beta_{ik} + (\Delta z_k - \Delta z_i)F'_{ik}\cos\gamma_{ik} \quad (23)$$

If an equation of this sort be written for each member of the truss and all these equations be added together, the result is

$$\Sigma F'_{ik}\Delta L_{ik} = \Sigma[(\Delta x_k - \Delta x_i)F'_{ik}\cos\alpha_{ik} + (\Delta y_k - \Delta y_i)F'_{ik}\cos\beta_{ik} + (\Delta z_k - \Delta z_i)F'_{ik}\cos\gamma_{ik}] \quad (24)$$

If it be remembered that the angles α, β, and γ are measured from the positive directions of the coördinate axes, it is evident that

$$\alpha_{ki} = 180° + \alpha_{ik}, \qquad \beta_{ki} = 180° + \beta_{ik}, \qquad \gamma_{ki} = 180° + \gamma_{ik}$$

therefore,

$$\cos \alpha_{ki} = -\cos \alpha_{ik}, \qquad \cos \beta_{ki} = -\cos \beta_{ik}, \qquad \cos \gamma_{ki} = -\cos \gamma_{ik}$$

and Eq. (24) may be written as

$$\Sigma F'_{ik} \Delta L_{ik} = -\Sigma[\Delta x_i F'_{ik} \cos \alpha_{ik} + \Delta y_i F'_{ik} \cos \beta_{ik} + \Delta z_i F'_{ik} \cos \gamma_{ik} + \Delta x_k F'_{ik} \cos \alpha_{ki} + \Delta y_k F'_{ik} \cos \beta_{ki} + \Delta z_k F'_{ik} \cos \gamma_{ki}] \quad (25)$$

If the terms in the right-hand side of this equation are regrouped, one group for each joint, and the stress in any member is considered as a force applied to each of the joints at the ends of the member, Eq. (25) may be written as

$$\Sigma F'_{ik} \Delta L_{ik} = -\Sigma[\Delta x_i \Sigma F'_{ik} \cos \alpha_{ik} + \Delta y_{ik} \Sigma F'_{ik} \cos \beta_{ik} + \Delta z_i \Sigma F'_{ik} \cos \gamma_{ik}] \quad (26)$$

where i designates any joint of the truss, F'_{ik} is the stress in any member due to a system of external forces Q' applied to the joints, and in which each summation within the brackets includes all the members which are connected to joint i while the summation outside the brackets includes all the joints of the structure. If the resultant external force applied at joint i be Q'_i and the angles of inclination of this resultant to the X, Y, and Z axes be ξ_i, η_i, and ζ_i, respectively, the equations of equilibrium for the joint are

$$\begin{aligned} Q'_i \cos \xi_i + \Sigma F'_{ik} \cos \alpha_{ik} &= 0 \\ Q'_i \cos \eta_i + \Sigma F'_{ik} \cos \beta_{ik} &= 0 \\ Q'_i \cos \zeta_i + \Sigma F'_{ik} \cos \gamma_{ik} &= 0 \end{aligned} \quad (27)$$

Using these relations, Eq. (26) may be written in the form

$$\begin{aligned} \Sigma F'_{ik} \Delta L_{ik} &= \Sigma[\Delta x_i Q'_i \cos \xi_i + \Delta y_i Q'_i \cos \eta_i + \Delta z_i Q'_i \cos \zeta_i] \\ &= \Sigma Q'_i(\Delta x_i \cos \xi_i + \Delta y_i \cos \eta_i + \Delta z_i \cos \zeta_i) \end{aligned} \quad (28)$$

The quantity within the parentheses in the right-hand side of this equation is the sum of the projections, on the line of action of Q'_i, of the components of the displacement of joint i and might be written as the projection, on this line of action, of the displacement. If this projection be designated by δ_i Eq. (28) becomes

$$\Sigma F'_{ik}\,\Delta L_{ik} = \Sigma Q'_i\delta_i \tag{29}$$

which is an algebraic statement of the law of virtual work. The only limitations on the validity of this equation are that the stresses F'_{ik} must be those due to the external forces Q'_i with the truss in its distorted condition, that the changes of length ΔL and the joint displacements δ_i are those existing in the condition of distortion under investigation and that the distortion be such that Eq. (5) is satisfied.

14. Clapeyron's Law.—Consider a structure that is subjected to external forces which vary continuously from zero to their final magnitudes and which ultimately attains a condition of equilibrium. Let the external forces and internal stresses form a system in equilibrium at every instant of the interval during which the forces are increasing and the distortion of the structure is progressing. Assume that there is no change of temperature. At some instant let the simultaneous magnitudes of the external forces and internal stress intensities be Q', f'_x, f'_y, f'_z, s'_x, s'_y, and s'_z, respectively. At the same instant let the displacement of the point of application of any force Q' be δ. During a small interval immediately following the instant at which these stress intensity magnitudes exist, let the displacements of the points of application of the forces Q' increase by increments $d\delta$ while the strains in the structure increase by the increments de'_x, de'_y, de'_z, dg'_x, dg'_y, and dg'_z. Since the law of virtual work is valid for any system of forces Q, it is valid for that particular case where the forces Q have the same magnitudes as a set of forces which are causing the distortion; consequently one may write,

$$\Sigma Q'\,d\delta = \Sigma\int(f'_x\,de'_x + f'_y\,de'_y + f'_z\,de'_z + s'_x\,dg'_x + s'_y\,dg'_y + s'_z\,dg'_z)dV \tag{30}$$

Moreover, since these are the real stresses and real strains occurring at the particular instant under consideration, the amounts of work appearing in Eq. (30) are the amounts of real work that would be done if the external forces and internal stresses remained constant during the small interval. The forces and stress intensities do not remain constant in the small interval, however, and the amounts of real work done by the increments to the forces and stress intensities may be stated as $K_1 \Sigma dQ' \, d\delta$ and

$$K_2 \int (df'_x \, de'_x + df'_y \, de'_y + df'_z \, de'_z + ds'_x \, dg'_x + ds'_y \, dg'_y + ds'_z \, dg'_z) dV$$

where K_1 and K_2 are factors which are less than unity. The terms in this expression are infinitesimals of an order one degree higher than those in Eq. (30) and will vanish in the definite integral obtained by integrating the work over the whole interval. They need not be considered further and Eq. (30) may be taken as a sufficient statement of the real work done in the small interval.. Since the strains are due to the forces Q' they may be expressed as

$$\begin{aligned} e'_x &= \frac{1}{E}[f'_x - \nu(f'_y + f'_z)] \qquad & g'_x &= \frac{s'_x}{G} \\ e'_y &= \frac{1}{E}[f'_y - \nu(f'_z + f'_x)] \qquad & g'_y &= \frac{s'_y}{G} \\ e'_z &= \frac{1}{E}[f'_z - \nu(f'_x + f'_y)] \qquad & g'_z &= \frac{s'_z}{G} \end{aligned}$$

and, differentiating,

$$\begin{aligned} de'_x &= \frac{1}{E}[df'_x - \nu(df'_y + df'_z)] \qquad & dg'_x &= \frac{ds'_x}{G} \\ de'_y &= \frac{1}{E}[df'_y - \nu(df'_z + df'_x)] \qquad & dg'_y &= \frac{ds'_y}{G} \\ de'_z &= \frac{1}{E}[df'_z - \nu(df'_x + df'_y)] \qquad & dg'_z &= \frac{ds'_z}{G} \end{aligned}$$

If these values are substituted in Eq. (30), it becomes

$$\begin{aligned} \sum Q' \, d\delta = \sum \int \Big[\frac{1}{E} \{ f'_x \, df'_x - \nu(f'_x \, df'_y + f'_x \, df'_z) + f'_y \, df'_y \\ - \nu(f'_y \, df'_z + f'_y \, df'_x) + f'_z \, df'_z - \nu(f'_z \, df'_x + f'_z \, df'_y) \} \\ + \frac{1}{G}(s'_x \, ds'_x + s'_y \, ds'_y + s'_z \, ds'_z) \Big] dV \end{aligned}$$

If this be integrated over the whole interval, *i.e.*, between the limits zero and $Q' = Q$, $f'_x = f_x$, $f'_y = f_y$, $f'_z = f_z$, $s'_x = s_x$, $s'_y = s_y$, and $s'_z = s_z$, the equation becomes

$$\sum \int Q' \, d\delta = \sum \int \left[\frac{1}{2E}(f_x^2 + f_y^2 + f_z^2) - \frac{\nu}{E}(f_x f_y + f_y f_z + f_z f_x) + \frac{1}{2G}(s_x^2 + s_y^2 + s_z^2) \right] dV \quad (31)$$

From Eq. (12)

$$\begin{aligned} \Sigma Q\,\delta &= \Sigma \textstyle\int (f_x e_x + f_y e_y + f_z e_z + s_x g_x + s_y g_y + s_z g_z) dV \\ &= \sum \int \left\{ f_x \frac{1}{E}[f_x - \nu(f_y + f_z)] + f_y \frac{1}{E}[f_y - \nu(f_z + f_x)] + f_z \frac{1}{E}[f_z - \nu(f_x + f_y)] + s_x \frac{s_x}{G} + s_y \frac{s_y}{G} + s_z \frac{s_z}{G} \right\} dV \\ &= \sum \int \left[\frac{1}{E}(f_x^2 + f_y^2 + f_z^2) - \frac{2\nu}{E}(f_x f_y + f_y f_z + f_z f_x) + \frac{1}{G}(s_x^2 + s_y^2 + s_z^2) \right] dV \quad (32) \end{aligned}$$

The summation in the right-hand side of Eq. (32) is just double the summation in the right-hand side of Eq. (31), therefore

$$\sum \int Q' \, d\delta = \frac{1}{2} \sum Q\delta \quad (33)$$

In the development of Eq. (31) no limitation was imposed on the rate at which the forces Q' increased from zero to their final magnitudes nor on the order in which they were applied to the structure. The development does depend, however, on the validity of the law of virtual work; therefore, Eq. (31) can be true only when the distortions are so small that the law of virtual work is applicable. Within these limitations, a consideration of Eqs. (31), (32), and (33) leads to the following conclusion:

If a structure is initially without stress and is not subjected to a change of temperature, but is operated on by external forces which vary continuously from zero to their final magnitudes, and which produce a small distortion of the structure, the mechanical work done by the outer forces while the structure is arriving at a condition of equilibrium is equal to the mechanical work done by the internal stresses. Each of these amounts is independent of the rate at which the forces increase and of the order in which the external forces are applied to the structure; each is half as great as it would have been if the forces, or internal stresses, had had their final magnitudes throughout the whole of the distortion.

This statement was formulated first by Clapeyron and is known as Clapeyron's law.

It is often convenient to express the mechanical work of distortion (strain energy) as a function of the bending moments, shears and total axial stresses. Consideration will be limited to structures in which each member is in a condition of planar stress. As in Art. 12,

$$f_x = \frac{My}{I} + \frac{F}{A}, \qquad f_y = f_z = 0, \qquad s_x = s_y = 0; \qquad s_z = \frac{SM_s}{bI}$$

$$W_d = \frac{1}{2}\sum Q\,\delta = \sum \int \left[\frac{1}{2E}\left(\frac{My}{I} + \frac{F}{A}\right)^2 + \frac{1}{2G}\left(\frac{SM_s}{bI}\right)^2\right] dV$$

$$= \sum \int \left[\frac{1}{2E}\left(\frac{M^2y^2}{I^2} + 2\frac{MFy}{AI} + \frac{F^2}{A^2}\right) + \frac{S^2M_s^2}{2Gb^2I^2}\right] dV$$

In trying to develop a general expression for the last term which will be simpler than the form just obtained there is the same difficulty as was encountered in the expressions for virtual work, and, since no great error is involved in the approximation based on the assumption that the shear is uniformly distributed over an area A', $(K = 1)$, the further simplification will be based on this assumption.

$$W_d = \sum \left[\int \frac{M^2}{2EI^2} dx \int \int y^2\, dy\, dz + \int \frac{MF}{EAI} dx \int \int y\, dy\, dz + \int \frac{F^2}{2A^2E} dx \int \int dy\, dz + \int \frac{S^2}{2GA'^2} dx \int \int dy\, dz \right]$$

$$= \sum \left[\int \frac{M^2}{2EI} dx + \int \frac{F^2}{2AE} dx + \int \frac{S^2}{2GA'} dx \right] \tag{34}$$

If the members of the structure are prismatic and if the loads are applied at the joints only, the second term becomes

$$\sum \frac{F^2L}{2AE}$$

If the structure is an ideal truss, the shear and moment terms vanish and Eq. (34) becomes

$$W_d = \sum \frac{F^2L}{2AE} \tag{35}$$

15. Castigliano's Law.—Let the external forces Q acting on a structure be separated into two groups: First, those forces which are independent of each other such as loads and which will be designated by the letter P; and, second, a group, such as reactions, which is dependent on the first group. Equation (19) may then be written as

$$\sum P\,\delta + W_r = \sum\left[\int \frac{M_q M}{EI}dx + \int \frac{F_q F}{AE}dx + K\int \frac{S_q S}{A'G}dx + \epsilon\int \frac{M_q \Delta(\Delta t)}{d}dx + \epsilon\int F_q \Delta t_0\, dx\right] \quad (36)$$

in which W_r is the virtual work done by those external forces which are not independent of the forces P. If the only forces in this last group are reactions, the term W_r will vanish unless the condition of distortion involves yielding of the supports. Since the external forces P and the resulting internal stresses M_q, F_q, and S_q are independent of the condition of distortion as defined by the internal stresses M, F, and S, and of the changes of temperature $\Delta(\Delta t)$ and Δt_0, if the partial derivative of the two sides of Eq. (36), with respect to one of the external forces P_m, be written, Eq. (36) becomes

$$1\,\delta_m + \frac{\partial W_r}{\partial P_m} = \sum\left[\int \frac{\partial M_q}{\partial P_m}\frac{M}{EI}dx + \int \frac{\partial F_q}{\partial P_m}\frac{F}{AE}dx + K\int \frac{\partial S_q}{\partial P_m}\frac{S}{A'G}dx + \epsilon\int \frac{\partial M_q}{\partial P_m}\frac{\Delta(\Delta t)}{d}dx + \epsilon\int \frac{\partial F_q}{\partial P_m}\Delta t_0\, dx\right] \quad (37)$$

Since Eqs. (36) and (37) are valid for all Q systems, they are valid for that particular Q system which is the same as that which is operative in causing the distortion and which corresponds to the moments M, the axial stresses F, and the shearing stresses S. Therefore one may write

$$\begin{aligned} 1\,\delta_m + \frac{\partial W_r}{\partial P_m} &= \sum\left[\int \frac{M}{EI}\frac{\partial M}{\partial P_m}dx + \int \frac{F}{AE}\frac{\partial F}{\partial P_m}dx + K\int \frac{S}{A'G}\frac{\partial S}{\partial P_m} + \epsilon\int \frac{\partial M}{\partial P_m}\frac{\Delta(\Delta t)}{d}dx + \epsilon\int \frac{\partial F}{\partial P_m}\Delta t_0\, dx\right] \\ &= \frac{\partial}{\partial P_m}\sum\left[\int \frac{M^2}{2EI}dx + \int \frac{F^2}{2AE}dx + K\int \frac{S^2}{2A'G}dx\right] \\ &\quad + \sum \epsilon\int \frac{\partial M}{\partial P_m}\frac{\Delta(\Delta t)}{d}dx + \sum \epsilon\int \frac{\partial F}{\partial P_m}\Delta t_0\, dx \quad (38) \end{aligned}$$

By referring to Eq. (34) it may be seen that the expression within the brackets is the strain energy W_d due to the external forces; therefore

$$1\ \delta_m + \frac{\partial W_r}{\partial P_m} = \frac{\partial W_d}{\partial P_m} + \epsilon \int \frac{\partial F}{\partial P_m} \Delta t_0 \, dx + \epsilon \int \frac{\partial M}{\partial P_m} \frac{\Delta(\Delta t)}{d} dx \quad (39)$$

If the virtual work W_r done by those forces which are not independent of the loads P is zero, which usually means that the points of support do not yield, and if there is no change of temperature, this equation becomes

$$1\ \delta_m = \frac{\partial W_d}{\partial P_m} \quad (40)$$

which may be stated as follows:

If a structure be in equilibrium under the action of external forces, if the points of supports do not yield and there is no change of temperature, the displacement of the point of application of any one of the external forces in the direction of that force due to the application of the external forces mentioned is equal to the partial derivative, with respect to that force, of the strain energy of the structure due to these external forces.

This is known as Castigliano's second law. The demonstration just given is limited to structures in which each member is in a condition of planar stress. It might have been carried out by starting with Eq. (12) as a basis, in which case no limitation would have been placed on the character of stress existing in the members, and the conclusion stated at the end would still be valid. Thus, Castigliano's law is valid for structures of all types.

16. Betti's Law and Maxwell's Law.—Let a structure be subjected to the action of a set of external forces Q_n which cause a condition of distortion which may be defined by the deflections of certain points m in the directions mo and let the deflection of any point m, in the direction mo due to the action of a unit value of one of the forces Q_n be designated by δ_{mn}. The deflection of a point m due to the group of forces Q_n is, therefore, $\Sigma Q_n \delta_{mn}$. If Eq. (12) is written for this condition of distortion, using as a Q' system a set of forces Q_m applied at joints m and acting in the directions mo, it becomes

$$\Sigma Q_m Q_n \delta_{mn} = \sum \int (f_{mx} e_{nx} + f_{my} e_{ny} + f_{mz} e_{nz} + s_{mx} g_{nx} + s_{my} g_{ny} + s_{mz} g_{nz}) dV$$

$$= \sum \int \Big\{ f_{mx} \frac{1}{E} [f_{nx} - \nu(f_{ny} + f_{nz})] + f_{my} \frac{1}{E} [f_{ny} - \nu(f_{nz} + f_{nx})] + f_{mz} \frac{1}{E} [f_{nz} - \nu(f_{nx} + f_{ny})] + \frac{1}{G}(s_{mx} s_{nx} + s_{my} s_{ny} + s_{mz} s_{nz}) \Big\} dV$$

$$= \sum \int \Big\{ \frac{1}{E}(f_{mx} f_{nx} + f_{my} f_{ny} + f_{mz} f_{nz}) - \frac{\nu}{E}[f_{mx}(f_{ny} + f_{nz}) + f_{my}(f_{nz} + f_{nx}) + f_{mz}(f_{nx} + f_{ny})] + \frac{1}{G}(s_{mx} s_{nx} + s_{my} s_{ny} + s_{mz} s_{nz}) \Big\} dV \quad (41)$$

Now consider the same structure in the condition of distortion which would be caused by the external forces Q_m and apply the law of virtual work using the Q_n system as the forces Q' in Eq. (12).

$$\Sigma Q_n Q_m \delta_{nm} = \sum \int (f_{nx} e_{mx} + f_{ny} e_{my} + f_{nz} e_{mz} + s_{nx} g_{mx} + s_{ny} g_{my} + s_{nz} g_{mz}) dV$$

$$= \sum \int \Big\{ \frac{f_{nx}}{E} [f_{mx} - \nu(f_{my} + f_{mz})] + \frac{f_{ny}}{E} [f_{my} - \nu(f_{mz} + f_{mx})] + \frac{f_{nz}}{E} [f_{mz} - \nu(f_{mx} + f_{my})] + \frac{1}{G}(s_{nx} s_{mx} + s_{ny} s_{my} + s_{nz} s_{mz}) \Big\} dV$$

$$= \sum \int \Big\{ \frac{1}{E}(f_{nx} f_{mx} + f_{ny} f_{my} + f_{nz} f_{mz}) - \frac{\nu}{E}[f_{nx}(f_{my} + f_{mz}) + f_{ny}(f_{mx} + f_{mz}) + f_{nz}(f_{mx} + f_{my})] + \frac{1}{G}(s_{nx} s_{mx} + s_{ny} s_{my} + s_{nz} s_{mz}) \Big\} dV \quad (42)$$

The expressions in the right-hand sides of Eqs. (41) and (42) may be seen to be exactly alike though the coefficients of ν/E

are not arranged in exactly the same order; therefore,

$$\Sigma Q_m Q_n \delta_{mn} = \Sigma Q_n Q_m \delta_{nm} \tag{43}$$

This equation is an algebraic presentation of what is known as Betti's law, which may be stated as follows:

The virtual work done by a group of external forces Q_m during a distortion due to a group of external forces Q_n is equal to the virtual work done by the forces Q_n in the distortion due to the forces Q_m.

In the particular case where each of the groups Q_m and Q_n is reduced to a single unit force, together with the reactions they cause, and in which the points of support do not move, Eq. (43) becomes

$$\delta_{mn} = \delta_{nm} \tag{44}$$

which may be stated as follows:

In a body supported on unyielding supports the deflection of a point $\underline{m}$ in the direction $\underline{mo}$ due to the action of a unit force applied at point $\underline{n}$ and acting in the direction $\underline{np}$ is equal to the deflection of point $\underline{n}$ in the direction $\underline{np}$ due to the action of a unit force applied at joint $\underline{m}$ and acting in the direction $\underline{mo}$.

This is known as Maxwell's law of reciprocal deflections. It will be used frequently in the analysis of indeterminate structures.

CHAPTER II

DEFLECTIONS

17. Introductory.—It has been suggested already that the problem of determining the distortion of a structure is of much interest, both on its own account and because of its use as a step in the stress analysis of indeterminate structures. It is, consequently, important that the structural analyst should be familiar with various methods of solving this problem and this chapter will be devoted to discussion of some of these methods.

18. Computing Deflections by the Law of Virtual Work.—If the law of virtual work as expressed in Eq. (21) is applied to a structure, any element of which may be in a condition of distortion as defined by angular strains $\frac{d\theta}{dx}dx$, linear strains $e_0\,dx$ and shear strains $g_s\,dx$, and if the Q' system used in the equation consists of a unit load applied at joint m in the direction mo together with the reactions set up by this unit load, the left-hand side of the equation becomes $1\,\delta_m + W_r$ where δ_m is the movement of m in the direction mo and W_r is the virtual work done by the reactions due to the unit load if the condition of distortion involves movement of the points of support. When the points of support are unyielding, the term W_r vanishes and the left-hand side of the equation is numerically equal to δ_m. If there is some yielding of the supports, such yielding must be known before δ_m can be evaluated. Designating the bending moments, axial stresses, and shearing stresses due to this particular Q' system by M_m, F_m, and S_m, respectively, Eq. (21) becomes

$$1\,\delta_m + W_r = \sum\left[\int M_m\frac{d\theta}{dx}dx + \int F_m\frac{d(\Delta L)}{dx}dx + \int S_m g_s\,dx\right] \tag{45}$$

Since $\frac{M}{EI} + \epsilon\frac{\Delta(\Delta t)}{d}$ is the rate of change of slope of a member, $\frac{F}{AE} + \epsilon\,\Delta t_0$ is the rate of change of length and $K\frac{S}{A'G}$ is the rate of shear distortion, Eq. (45), which is general in that it includes the effects of all types of distortion, may be written in the form

$$1\ \delta_m + W_r = \sum \left[\int \frac{M_m M}{EI} dx + \int \frac{F_m F}{AE} dx + K \int \frac{S_m S}{A'G} dx + \epsilon \int M_m \frac{\Delta(\Delta t)}{d} dx + \epsilon \int F_m \Delta t_0 \, dx \right] \tag{46}$$

which is directly applicable when deformations are due to stress and temperature changes.

It is evident, therefore, that, provided there is knowledge as to any possible yielding of the supports, Eqs. (45) and (46) may be used as an expedient to determine the movement of a particular point in a particular direction in any condition of distortion of a structure if the stresses M_m, F_m, and S_m can be computed and the strains, expressed either as in Eq. (45) or as in Eq. (46) can be found. When the structure under consideration is a truss, either planar or three-dimensional, those terms which express the result of changes of slope and of shear deformations vanish, and Eq. (46) becomes

$$1\ \delta_m + W_r = \sum \left[\int \frac{F_m F}{AE} dx + \epsilon \int F_m \Delta t_0 \, dx \right] \tag{47}$$

and if, in addition, the members are prismatic and Δt_0 is constant over the length of a member, the equation may be written as

$$1\ \delta_m + W_r = \sum \frac{F_m F L}{AE} + \sum \epsilon F_m \Delta t_0 L \tag{48}$$

$$= \Sigma F_m \, \Delta L \tag{49}$$

As a simple illustration of the use of this expedient consider

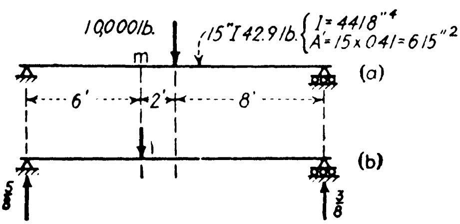


FIG. 11.

the problem of determining the vertical deflection of point *m* of the beam shown in Fig. 11*a* due to the loading given there.

Assume that the points of support do not yield. In the condition of distortion to be investigated here, $F = 0$, $\Delta t_0 = 0$, $\Delta(\Delta t) = 0$, and the functions expressing M and S vary for different parts of the span. The Q' system is shown in Fig. 11*b*. Measuring x from the left reaction, for

$0 < x < 6$	$6 < x < 8$	$8 < x < 16$
$M = 5x$	$M = 5x$	$M = -5x + 80$
$M_m = \frac{5}{8}x$	$M_m = -\frac{3}{8}x + 6$	$M_m = -\frac{3}{8}x + 6$
$S = +5$	$S = +5$	$S = -5$
$S_m = +\frac{5}{8}$	$S_m = -\frac{3}{8}$	$S_m = -\frac{3}{8}$

in which the stresses for the distorting condition are stated in units of 1,000 lb. Equation (46) becomes

$$1\ \delta_m = \int \frac{M_m M}{EI} dx + K \int \frac{S_m S}{A'G} dx \qquad \text{(take } K = 1\text{)}$$

$$= \frac{1}{EI}\left[\int_0^6 \left(\frac{5x}{8}\right)(5x)dx + \int_6^8 \left(-\frac{3x}{8} + 6\right)(5x)dx \right.$$

$$\left. + \int_8^{16} \left(-\frac{3x}{8} + 6\right)(-5x + 80)dx\right] + \frac{1}{A'G}\left[\int_0^6 \left(+\frac{5}{8}\right)(+5)dx \right.$$

$$\left. + \int_6^8 (+5)\left(-\frac{3}{8}\right)dx + \int_8^{16} \left(-\frac{3}{8}\right)(-5)dx\right]$$

$$= \frac{1}{EI}\left[\frac{25}{8} \times \frac{6^3}{3} - \frac{15}{8} \times \frac{8^3 - 6^3}{3} + 30\frac{8^2 - 6^2}{2} \right.$$

$$\left. + \frac{15}{8} \times \frac{16^3 - 8^3}{3} - 60 \times \frac{16^2 - 8^2}{2} + 480(16 - 8)\right]$$

$$+ \frac{1}{A'G}\left[+\frac{25}{8} \times 6 - \frac{15}{8}(8 - 6) + \frac{15}{8}(16 - 8)\right]$$

$$= +\frac{780}{EI} + \frac{30}{A'G};$$

taking $E = 3 \times 10^7$ lb. per sq. in., and $G = 1.15 \times 10^7$ lb. per sq. in.,

$$\delta_m = +\frac{780}{3 \times 10^4 \times 12^2 \times 441.8 \times 12^{-4}} + \frac{30}{6.15 \times 12^{-2} \times 11.5 \times 10^3 \times 12^2}$$

$$= +0.00848 + 0.00042 \text{ (ft.)}$$

$$= +0.00890 \text{ ft.}$$

It might be noted that the deflection due to the shear distortion is but 5 per cent of the total deflection in this particular example. In most practical cases the shear deflection is of the same general proportional magnitude as this and need not be computed. The relative importance increases as the ratio of the length of the span to the depth of the beam decreases.

As a further illustration consider the deflection of point m in the condition of distortion caused by a change of temperature which is the same for all cross sections of the beam but varies uniformly from an increase of 50°F. at the top of the beam to an increase of 10°F. at the bottom. Since $\Delta(\Delta t) = \Delta t_1 - \Delta t_2$ and Δt_1 is the increase in temperature at the edge for which the ordinate y is positive (see Fig. 8) and, therefore, is for the side on which the bending moment causes tension $\left(f = \frac{My}{I} + \frac{F}{A}\right)$, also, since the bending moment here has been taken as positive when it causes tension in the lower part of the beam,

$$\Delta(\Delta t) = 10° - 50° = -40°;$$

$\Delta t_0 = +30°$. Equation (46) becomes

$$\begin{aligned} 1\ \delta_m &= \epsilon\Delta(\Delta t)\frac{1}{1.25}\int M_m\,dx; \qquad F_m = 0 \\ &= -\frac{40}{1.25}\epsilon\left[\int_0^6 \frac{5x}{8}dx + \int_6^{16}\left(-\frac{3x}{8} + 6\right)dx\right] \\ &= -32\epsilon\left[\frac{5}{8}\times\frac{6^2}{2} - \frac{3}{8}\left(\frac{16^2 - 6^2}{2}\right) + 6(16 - 6)\right] \\ &= -960 \times 6.5 \times 10^{-6} \\ &= -0.00624 \text{ ft.} \end{aligned}$$

The negative sign indicates that the deflection is in a direction opposite to that of the unit load.

The law of virtual work also provides an expedient which will enable the analyst to determine the change of slope at a point in a structural member. In such a problem the Q system to be adopted consists of a unit couple to be applied at the point where the change of slope is desired, together with the reactions set up by this couple. An examination of Fig. 12 shows that when such a Q system is used the left-hand side of Eq. (21) becomes

$$\frac{1}{a}\,\delta_{m1} + \frac{1}{a}\,\delta_{m2} + W_r = \frac{1}{a}(a \tan \theta) + W$$
$$= \tan \theta + W_r$$

The rotation encountered in the usual structure is so small that no distinction need be made between tan θ and the angle θ expressed in radians; therefore, if there is no yielding of the supports, the left-hand side of Eq. (21) may be interpreted as $1(\theta)$. As an illustration, consider the problem of determining

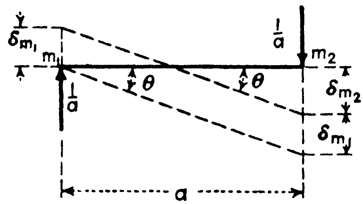


FIG. 12.

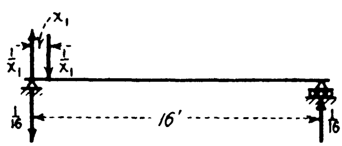


FIG. 13.

the change of slope at the left end of the beam of Fig. 11*a*. The Q system to be adopted is shown in Fig. 13. For the condition of distortion defined by the loading and temperature change previously used, Eq. (46) becomes

$$1\,\theta = \frac{1}{EI}\left[\int_0^8\left(1 - \frac{x}{16}\right)5x\,dx + \int_8^{16}\left(1 - \frac{x}{16}\right)(-5x + 80)dx\right]$$
$$+ \frac{-40}{1.25}\epsilon\int_0^{16}\left(1 - \frac{x}{16}\right)dx + \frac{1}{A'G}\left[\int_0^{x_1}\left(\frac{1}{x_1} - \frac{1}{16}\right)5dx + \right.$$
$$\left.\int_{x_1}^{8}\left(-\frac{1}{16}\right)(5)dx + \int_8^{16}\left(-\frac{1}{16}\right)(-5)dx\right]$$
$$= \frac{1}{EI}\left[5 \times \frac{8^2}{2} - \frac{5}{16} \times \frac{8^3}{3} - 5\frac{(16^2 - 8^2)}{2} + \frac{5}{16}\,\frac{(16^3 - 8^3)}{3}\right.$$
$$\left. + 640 - \frac{80}{16}\,\frac{(16^2 - 8^2)}{2}\right] - 32\epsilon\left(16 - \frac{16^2}{2 \times 16}\right)$$
$$+ \frac{1}{A'G}\left[\frac{5}{x_1}x_1 - \frac{5}{16}x_1 - \frac{5}{16}(8 - x_1) + \frac{5}{16}8\right]$$

$$1\,\theta = \frac{160}{EI} - 256\epsilon + \frac{5}{A'G}$$
$$= +\frac{160}{3 \times 10^4 \times 12^2 \times 441.8 \times 12^{-4}} - 256 \times 6.5 \times 10^{-6}$$
$$+ \frac{5}{6.15 \times 12^{-2} \times 11.5 \times 10^3 \times 12^2}$$
$$= +0.00173 - 0.00166 + 0.00007$$
$$= +0.00014 \text{ rad.}$$

In the second last line of the foregoing computation, the first term shows the effect of the bending moment, the second term is the change of slope due to the change of temperature, while the third is the change of slope due to the shear. As in computing the vertical deflection, the effect of the shear is a comparatively small fraction of the effect of bending moment or of the effect of change of temperature.

The use of the law of virtual work as an expedient to enable one to determine the movement of a joint of a truss in some particular condition of distortion is very similar to what has been demonstrated in the previous illustrations. For trusses, however, Eq. (46) is expressed in the form given in Eq. (49), *i.e.*,

$$1\,\delta_m + W_r = \Sigma F_m\,\Delta L$$

To illustrate, consider the problem of determining the vertical deflection of joint 5 of the truss shown in Fig. 14*a* when the condition of distortion is that caused by the loading given in the

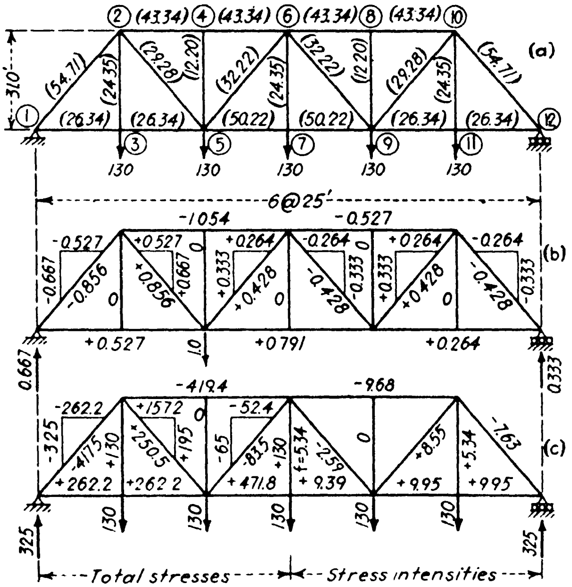


FIG. 14.

figure. The loads are stated in units of 1,000 lb.; the numbers written in parentheses on the members are their cross-sectional areas in square inches.

The Q system to be used consists of a unit vertical load applied at joint 5 together with the reactions it causes. The first step in the solution is to make two stress analyses, one to find the stress intensities due to the distorting loads and, through them, the changes of length $\Delta L = \frac{f}{E}L$ and the other to determine the stresses F_m. The first of these is carried out in Fig. 14c and the second in Fig. 14b. If the points of support do not yield, the term W_r vanishes and Eq. (49) becomes

$$1\ \delta_5 = \sum F_5\ \Delta L = \sum F_5 \frac{f}{E} L$$

The summation may be arranged in tabular form as follows:

Bar	L (ft.)	$f \cdot 10^{-3}$	F_5	$F_5 E\ \Delta L\ 10^{-3}$
1- 3- 5	50	+9.95	+0.527	+ 262.1
5- 7- 9	50	+9.39	+0.791	+ 371.2
9-11-12	50	+9.95	+0.264	+ 131.0
2- 4- 6	50	−9.68	−1.054	+ 510.0
6- 8-10	50	−9.68	−0.527	+ 255.0
1- 2	39.8	−7.63	−0.856	+ 260.2
2- 5	39.8	+8.55	+0.856	+ 291.6
5- 6	39.8	−2.59	+0.428	− 44.1
6- 9	39.8	−2.59	−0.428	+ 44.1
9-10	39.8	+8.55	+0.428	+ 145.7
10-12	39.8	−7.63	−0.428	+ 130.0
2- 3	31.0	+5.34	0	0
4- 5	31.0	0.0	0	0
6- 7	31.0	+5.34	0	0
8- 9	31.0	0.0	0	0
10-11	31.0	+5.34	0	0
				+2356.8

$$1\ \delta_5 = \frac{2357}{3 \times 10^4} = 0.0786 \text{ ft.}$$

If this condition of distortion had involved a yielding of the points of support, for example, a movement downward of joint 1 amounting to ½ in. and a movement of ¼ in. downward at joint 12, the term W_r would not have vanished and Eq. (49) would have been

$$-0.667 \times 0.5 \times \frac{1}{12} - 0.333 \times 0.25 \times \frac{1}{12} + 1\ \delta_m = +\frac{2357}{3 \times 10^4}$$

$$\delta_m = 0.0786 + 0.0277 + 0.0069 = 0.1132 \text{ ft.}$$

If the deflection desired is the relative deflection of joints 3 and 8 along the line joining them, the Q system to be adopted consists of a pair of unit forces, one applied at joint 3 and acting along the line 3-8 and the other applied at joint 8 and acting in a direction opposite to that of the unit load applied at joint 3, together with any reactions which may be caused by this pair of unit forces. In this problem these reactions are zero. If the condition of distortion is the same as that in the illustration

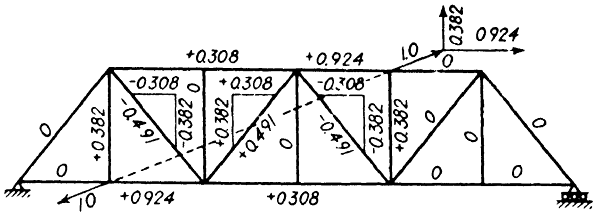


FIG. 15.

just completed, the stress analysis for the effect of the distorting loads need not be repeated but a stress analysis must be made to compute the stresses F_m, which, in this example, become the stresses $F_{3\text{-}8}$. The left-hand side of Eq. (49) becomes

$$1\ \delta_3 + 1\ \delta_8 = 1\ \delta_{3\text{-}8}$$

When the unit forces act outward, a positive value for $\delta_{3\text{-}8}$ indicates that joints 3 and 8 have moved apart; while if the forces act inward, a positive value means that the joints have approached each other. The numerical solution may be completed as follows:

It is not necessary to include in the tabulation any members in which either $F = 0$ or $F_{3\text{-}8} = 0$.

Bar	L (ft.)	$f \cdot 10^{-3}$	$F_{3\text{-}8}$	$F_{3\text{-}8}\, E\, \Delta L\ 10^{-3}$
3-5	25	+9.95	+0.924	+229.9
5-7-9	50	+9.39	+0.308	+144.6
2-4-6	50	−9.68	+0.308	−149.1
6-8	25	−9.68	+0.924	−223.6
2-5	39.8	+8.55	−0.491	−167.1
5-6	39.8	−2.59	+0.491	− 50.6
6-9	39.8	−2.59	−0.491	+ 50.6
2-3	31.0	+5.34	+0.382	+ 63.2
				−102.1

$$\delta_{3\text{-}8} = -\frac{102.1}{3 \times 10^4} = -0.0034 \text{ ft.}$$

The minus sign indicates that, in this example, points 3 and 8 have moved toward each other.

19. Deflections by Castigliano's Law.—When the distortions of a structure are due to the application of external loads on the structure, Castigliano's law provides an expedient for the computation of deflections, provided the limitations imposed in the statement of the law are fulfilled. In accordance with this law, if the internal work in a structure is differentiated with respect to any particular load acting on the structure, the deflection of the point of application of the load considered, in the direction of the load, is obtained. As a general procedure, a load X, which corresponds in point of application and line of action to the desired deflection, is applied to the structure. If the load X corresponds to an actual load on the structure, it may be considered as replacing the actual load; if, however, load X does not correspond to an actual applied load, it is considered as acting in addition to the other loads on the structure. In either event, the application of Castigliano's law to the structure, with respect to the load X, gives the required deflection in terms of X; and, if, in the resultant expression for deflection, X is given its true value, the required deflection is obtained. If X has replaced an actual load acting on the structure, it is given the value of this load; otherwise its true value is zero.

This method may also be used to obtain the sum of the deflections in given directions of different points on a structure, by applying simultaneously loads X_a, X_b, . . . at points a, b, ... In accordance with Eq. (38), one may then write

$$\delta_a = \frac{\partial W_d}{\partial X_a} = \sum \int \frac{M}{EI}\frac{\partial M}{\partial X_a}dx + \sum \int \frac{F}{AE}\frac{\partial F}{\partial X_a}dx + \sum K \int \frac{S}{A'G}\frac{\partial S}{\partial X_a}dx$$

$$\delta_b = \frac{\partial W_d}{\partial X_b} = \sum \frac{M}{EI}\frac{\partial M}{\partial X_b}dx + \sum \int \frac{F}{AE}\frac{\partial F}{\partial X_b}dx + \sum K \int \frac{S}{A'G}\frac{\partial S}{\partial X_b}dx$$

from which, adding,

$$\delta_a + \delta_b + \cdots = \sum \int \frac{M}{EI}\left(\frac{\partial M}{\partial X_a} + \frac{\partial M}{\partial X_b} + \cdots\right)dx + \sum \int \frac{F}{AE}\left(\frac{\partial F}{\partial X_a} + \frac{\partial F}{\partial X_b} + \cdots\right)dx \sum K \int \frac{S}{A'G}\left(\frac{\partial S}{\partial X_a} + \frac{\partial S}{\partial X_b} + \cdots\right)dx$$

If it is possible to replace the forces X_a, X_b, . . . , by an equivalent function X_e, such that a unit change in X_e would have the same effect upon moments, direct stresses, and shears occurring throughout a structure as would unit changes occurring simultaneously in each of the forces X_a, X_b, . . . , one may write

$$\delta_a + \delta_b + \cdots = \sum \int \frac{M}{EI}\frac{\partial M}{\partial X_e}dx + \sum \int \frac{F}{AE}\frac{\partial F}{\partial X_e}dx + \sum K \int \frac{S}{A'G}\frac{\partial S}{\partial X_e}dx$$
$$= \frac{\partial W_d}{\partial X_e}$$

The replacement of two equal and opposite forces X, one force X acting on either side of an assumed cut at a cross section of a member, by the stress X in the member, furnishes an example of such equivalent systems.

As illustrations of the application of Castigliano's law to the determination of deflections, the following examples are given:

1. Determine the deflection at mid-span of an end-supported beam with a span of 10.00 ft., $E = 3 \times 10^7$ lb. per sq. in., $I = 100$ in.^{4}, due to a single concentrated load of 10,000 lb. acting downward at the center of the beam, neglecting the effect of shear.

Solution.—Replace the force of 10,000 lb. by load X. The moment at distance x from either support is given by $Xx/2$, from which, using symmetry,

$$\delta = \frac{2}{EI}\int_0^{\frac{L}{2}}\left(\frac{X}{2}x\right)\left(\frac{x}{2}\right)dx = \frac{X}{2EI}\int_0^{\frac{L}{2}} x^2\,dx = \frac{X}{2EI}\frac{1}{3}\frac{L^3}{8} = \frac{XL^3}{48EI}$$
$$= \frac{10{,}000 \times 1{,}000 \times 144 \times 144}{48 \times 3 \times 10^7 \times 144 \times 100} = \frac{1}{100}\text{ ft.}$$

2. Determine the vertical deflection of point a on the truss shown in Fig. 16 due to the load acting as shown. The area of each member of the truss is 6 sq. in.; $E = 3 \times 10^7$ lb. per sq. in.

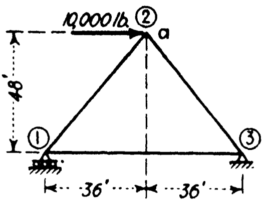


FIG. 16.

Solution.—Apply the vertical load X acting downward at point a. Note that, considering this structure to act as an ideal truss, no moments or shears are present, so that

$$\delta_a = \int \frac{F}{AE} \frac{\partial F}{\partial X} dx.$$

Since the members of the truss are prismatic, this reduces to $\delta_a = \sum \frac{FL}{AE} \frac{\partial F}{\partial X}$. The following tabular arrangement gives the steps of the solution:

Bar	F	$\frac{L}{A}$ (ft.)/(in.)	$\frac{\partial F}{\partial X}$	$\frac{FL}{A} \frac{\partial F}{\partial X}$
1-2	$+\frac{25}{3} - \frac{5}{8}X$	10	$-\frac{5}{8}$	$-\frac{1250}{24} + \frac{250}{64}X$
2-3	$-\frac{25}{3} - \frac{5}{8}X$	10	$-\frac{5}{8}$	$+\frac{1250}{24} + \frac{250}{64}X$
3-1	$-\frac{15}{3} + \frac{3}{8}X$	12	$+\frac{3}{8}$	$-\frac{540}{24} + \frac{108}{64}X$

$$E\delta_a = \sum = -\frac{540}{24} + \frac{608}{64}X = -\frac{540}{24} \qquad \text{since} \qquad X = 0$$

$$\delta_a = -\frac{540}{24 \times 3 \times 10^7} = -7.5 \times 10^{-7} \text{ ft.}$$

Since the result has a negative sign, the deflection of point a is in a direction opposite to that of the assumed load X and is therefore upward.

3. Show that if a portion of a structure is acted upon by a uniform load of p lb. per ft., the derivative of the internal work with respect to p equals the area under the deflection curve for that portion of the structure so loaded.

Solution.—Let the portion of the structure loaded be divided into equal distances dx, and at the center of each distance dx apply forces P_1, P_2, . . . , at points 1, 2, ... Then

$$\delta_1 + \delta_2 + \cdot \cdot \cdot = \frac{\partial W_d}{\partial P_1} + \frac{\partial W_d}{\partial P_2} + \cdot \cdot \cdot$$

Multiplying each side of the equation by dx,

$$\delta_1\, dx + \delta_2\, dx + \cdot \cdot \cdot = \frac{\partial W_d}{\partial P_1}dx + \frac{\partial W_d}{\partial P_2}dx + \cdot \cdot \cdot$$

But $\delta_1\, dx + \delta_2\, dx + \cdot \cdot \cdot = A =$ the area under the deflection curve for the portion of the structure loaded with p lb. per ft., and $P_1 = p_1\, dx$, $P_2 = p_2\, dx$, $\cdot \cdot \cdot$, where p_1, p_2, $\cdot \cdot \cdot$ are intensities of loading through the distances dx_1, dx_2, . . . , respectively. We may therefore write

$$A = \frac{\partial W_d}{\partial p_1} + \frac{\partial W_d}{\partial p_2} + \cdot \cdot \cdot$$

But $p_1 = p_2 = \cdot \cdot \cdot = p$, so that the series of load intensities p_1, p_2, . . . , each acting over distance dx, is a loading which is equivalent to a load p lb. per ft. acting over the loaded portion of the structure, or

$$\frac{\partial W_d}{\partial p_1} + \frac{\partial W_d}{\partial p_2} + \cdot \cdot \cdot = \frac{\partial W_d}{\partial p}$$

so that,

$$\frac{\partial W_d}{\partial p} = A$$

A comparison of the computation of deflections by Castigliano's law to their computation by the method of virtual work discloses the fact that the terms $\partial M/\partial X$, $\partial F/\partial X$, and $\partial S/\partial X$ are equivalent to the functions M_Q, F_Q, and S_Q, respectively, provided the Q system consists of a unit load applied at the point of, and in the direction of, the desired deflection together with the reactions due to the unit load and if the limitations imposed in the statement of Castigliano's law are fulfilled. Under these circumstances the stress in any member due to a unit load is a measure of the rate of change in the stress of the member with respect to an actual applied load corresponding in direction and point of application to the unit load.

20. The Williot-Mohr Method.—For planar trussed structures this method may be illustrated by a very simple problem, which, however, demonstrates all the ideas necessary for the solution of any problem having to do with the deformation of a trussed structure so long as primary stresses alone are considered. Consider a frame such as that in Fig. 17*a* in which each joint is fixed in its position relative to two other joints by its connection to them by means of a pair of members. In this structure joint *c* is fixed in position by being connected to *a* and *b* by members *ac* and *bc*, respectively. Joint *d* is fixed in position through its connection to joints *a* and *b* by members *ad* and *bd*, respectively. Similarly the positions of joints *e* and *f* are determined by the connection of each of them to two other joints by a pair of members. It is assumed that each of these members is subjected to a change of length and it is desired to

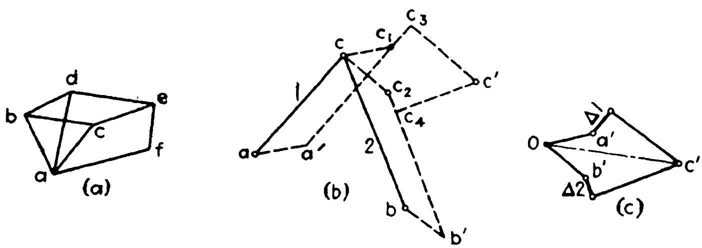


FIG. 17.

determine the changes in the positions of the joints corresponding to these changes in length.

The problem is solved in a series of steps, each step leading to the determination of the movement of one of the joints. Referring to Fig. 17*b*, assume that joint *a* moves from *a* to *a′* and that joint *b* moves from *b* to *b′* and that these movements are known in direction and magnitude. Imagine that the connection of the members *ac* and *bc*, hereafter called members 1 and 2, is broken and that when joints *a* and *b* move, the members move parallel to themselves into the positions $a'c_1$ and $b'c_2$, respectively. Now suppose that member 1 has an increase in its length amounting to Δ1 so that joint *c*, considered as the end of member 1, moves from c_1 to c_3 and that member 2 is subjected to a decrease, Δ2, in its length, so that joint *c*, considered as the end of member 2, moves from c_2 to c_4. Now consider the connection between the two members reestablished; the only way in which this can occur is by means of rotations of the two members about joints *a* and *b*, respectively, these joints being in their new positions *a′* and *b′*. In the distortions which occur in actual structures, the movements of the joints are so small compared with the lengths of the members that no appreciable error is made if the movements of points c_3 and c_4 during the rotations of members 1 and 2 are represented by straight lines perpendicuiar to the original directions of the members. These rotations must continue till points c_3 and c_4 move into a common position c'.

Since the movements of the joints are very small compared to the dimensions of the structure, if one tried to represent them to the same scale as a sketch of the structure, either the line sketch would be very large or the lines representing the joint movements would be so small that it would be impossible to measure them with sufficient precision to be of value. To overcome this difficulty, it is customary to draw only that part of Fig. 17*b* which is shown in the polygon $cc_1c_3c'c_4c_2$ and this may be done as in Fig. 17*c* to any scale desired. The procedure is as follows: From an origin O draw the vectors Oa' and Ob' to represent the motions of the c ends of the two members while the members are being moved parallel to themselves; then draw from a' a line representing the movement of the c end of member 1 due to the change in the length of that member and from b' draw a line representing the movement of the c end of member 2 due to the change in length of member 2; these two vectors are shown by $\Delta 1$ and $\Delta 2$ in Fig. 17*c*; from the outer end of the vector $\Delta 1$ draw a line perpendicular to the original direction of member 1 to represent the movement of the c end of that member during its rotation about the a end and similarly from the outer end of the vector $\Delta 2$ draw a line perpendicular to the original position of member 2 to represent the movement of the c end of that member due to its rotation about the b end; produce these two perpendiculars till they intersect at c'; the vector Oc' represents the movement of joint c due to the movement of joints a and b and the changes in the lengths of members 1 and 2.

A repetition of this procedure applied to members ad and bd would lead to the determination of a vector Od' which represents the movement of joint d; this diagram might be drawn from the same pole O as was used in finding the movement of joint c. If the procedure is repeated once more with reference to the members de and ce, a vector Oe' will be found which represents the movement of joint *e* and it may be noted that if the same pole *O* is used for all three diagrams the vectors *Oc′* and *Od′* are already drawn when one starts the third operation. The procedure is repeated as many times as is necessary to find the movements of all the joints of the truss. Usually it must be assumed that one of the joints is fixed in position and that the axis of one of the members connected to this joint is fixed in direction. If it is not true that there is such a member, the solution as described above will serve to find the movements of the joints referred to the fixed joint and to the direction of the axis of the member assumed to be fixed. Under such circumstances a further correction must be made to take account of the error made in assuming one of the members to have a fixed direction.

The method may be illustrated by a consideration of the truss shown in Fig. 18*a* in which it will be assumed to begin with that joint *a* is fixed in position and that the axis of member 1 does not change direction. The members marked (+) are assumed to be in tension and, consequently, to have increases in length while those marked (−) are assumed to be in compression and to be subjected to decreases in length. It is assumed that these changes in length are known. If the changes in the lengths of the members are not given, they must be found from a consideration of the causes of distortion. Usually this is the application of a group of loads or a change in temperature for some or all of the members. If the stress in any member due to a group of applied loads be F the stress intensity in the member will be $f = F/A$ and the corresponding change in the length of the member will be

$$\Delta L = \frac{f}{E}L = \frac{FL}{EA} = F\rho$$

where $\rho = L/AE$. If the cause of distortion be a change $\Delta t°$ in the temperature of a member and if the coefficient of thermal expansion be ϵ, the change in the length of the member will be $\Delta L = \epsilon\, \Delta t\, L$.

Since, in the problem under consideration, joint *a* has been assumed to be fixed in position and member 1 fixed in direction, the vector *Oa′* representing the movement of joint *a* is zero as shown in Fig. 18*b*. From *O* draw a line of length Δ1 parallel to the direction of member 1, using any scale deemed suitable; this vector Ob' represents the movement of joint b. Now use the procedure previously described to determine the movement of joint c. The details are as follows: from a' draw a vector $\Delta 2$ to represent the change in length of member 2 and at its end erect a perpendicular; from b' draw a vector to represent the change in the length of member 3 and at its outer end erect a perpendicular which is to be produced till it intersects the perpendicular drawn from the end of the vector $\Delta 2$; this intersection is the position of c' and the vector Oc' represents the movement of joint c. It is to be noted that the direction of a vector

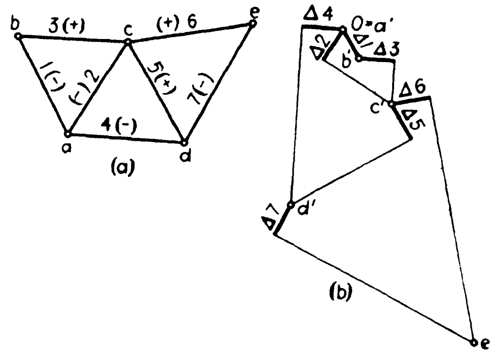


FIG. 18.

representing the change in the length of a member depends not only on the direction of the axis of the member but on the character of the change in length also. For example, in drawing $\Delta 3$, since this member is in tension the length of the member increases and joint c moves to the right with respect to joint b, hence the vector $\Delta 3$ must be drawn to the right. Next, repeat the operation to find the point d' from the points c' and a', then repeat again to find the position of e' from the positions of points c' and d'.

If the conditions of support are such that the initial assumptions with respect to the position of joint a and to the direction of the axis of member 1 are actually true, the actual movements of the joints are shown by the vectors Ob', Oc', Od', and Oe'. If, on the other hand, the conditions of support are such that member ab rotates during the distortion a correction is necessary. For example, suppose that this structure was supported at joints *a* and *d* so that joint *a* was immovable while joint *d* could move in a horizontal direction only. Figure 18*b* shows joint *d* to have moved to the left and downward; consequently the assumption that the axis of member 1 was fixed in direction was incorrect, and there is actually a rotation of this member and of the rest of the structure about joint *a*. The correction to be made, therefore, is to superimpose on the joint movements already found movements which are the results of a rotation of the truss about joint *a* sufficient to bring joint *d* back to the same elevation as joint *a*. A convenient method of finding the

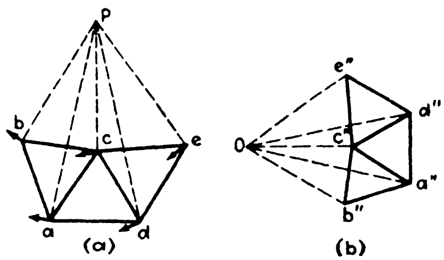


FIG. 19.

movements of the joints during such a rotation was suggested by Mohr and is explained in the following paragraph.

Let the figure *abced* shown in Fig. 19*a* rotate through a very small angle α about the pole *P*. During this rotation each point moves along a path which, for very small angles, may be taken as perpendicular to the line joining the original position of the point to the center of rotation. The distance each point moves is equal to the angle α multiplied by the distance of the point from the center of rotation. Now as in Fig. 19*b*, draw from (or to) a pole *O* a vector representing the movement, during this small rotation, of each point so that the movement of joint *a* is represented by the vector $a''O$, the movement of joint *b* is represented by the vector $b''O$, with similar vectors for the other joints. Connect the points $a''b''c''e''d''$. Since the vector $a''O$ is perpendicular to the line *aP* and the vector $b''O$ is perpendicular to the line *bP*, the angle $a''Ob''$ is equal to the angle *aPb*. Since, also $a''O = \alpha\, aP$ and $b''O = \alpha\, bP$,

$$\frac{a''O}{b''O} = \frac{\alpha}{\alpha}\frac{aP}{bP} = \frac{aP}{bP}$$

therefore the triangles abP and $a''b''O$ are geometrically similar. In like manner one may show that the triangle acP is geometrically similar to the triangle $a''c''O$, the triangle cdP is geometrically similar to the triangle $c''d''O$ and the triangle deP is similar to the triangle $d''e''O$. It is possible, therefore, to state the following conclusion:

If a rigid body rotates about a point through an angle so small that the movement of any point in the body may be considered as a linear, and if the movements of two points i and k in the body are plotted as vectors, Oi'' and Ok'', from (or to) a pole O, and if the shape of the body is plotted to scale with the points i'' and k'' as a base, each line in this scale sketch will be perpendicular to the corresponding line in the actual body and the line joining any point in the scale sketch to the pole O will represent, as a vector, the movement of the corresponding point in the real body during the rotation.

If this theorem be applied to the structure in Fig. 18 in order to find the vectors which represent the movements of the joints during the rotation which is necessary to correct the apparent vertical movement of joint d, the procedure is as follows: The vertical movement of joint d is shown in Fig. 20 by the vector $d''O$ and the vector showing the movement of joint a is zero so that a'' falls on O, the original pole for the distortion diagram. Construct, with the points d'' and a'' as a base a figure which is geometrically similar to the sketch $abcde$ of the truss but in which every line is perpendicular to the corresponding line in the sketch of the truss. This figure is $a''b''c''d''e''$. The movement of joint b during the rotation is shown by the vector $b''O$ and when this vector is added to the vector Ob' which represents the movement of joint b due to distortion only, the sum is the vector $b''b'$ which, therefore, represents the total movement of joint b. Similarly, the movement of joint c is represented by the vector $c''c'$, the movement of joint d by the vector $d''d'$, and the movement of joint e by the vector $e''e'$.

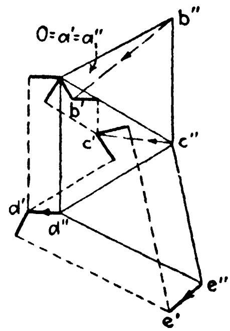


FIG. 20.

The solution, by this method, of a typical problem is shown in Fig. 21. In this solution it was assumed that joint a was fixed in position and that member *ab* did not change direction. The number written on each member in the sketch of the truss is E times its change in length. The vector diagram $a'b'c'd'e'f'g'h'$ was drawn by the method illustrated in Fig. 18*b*; consequently

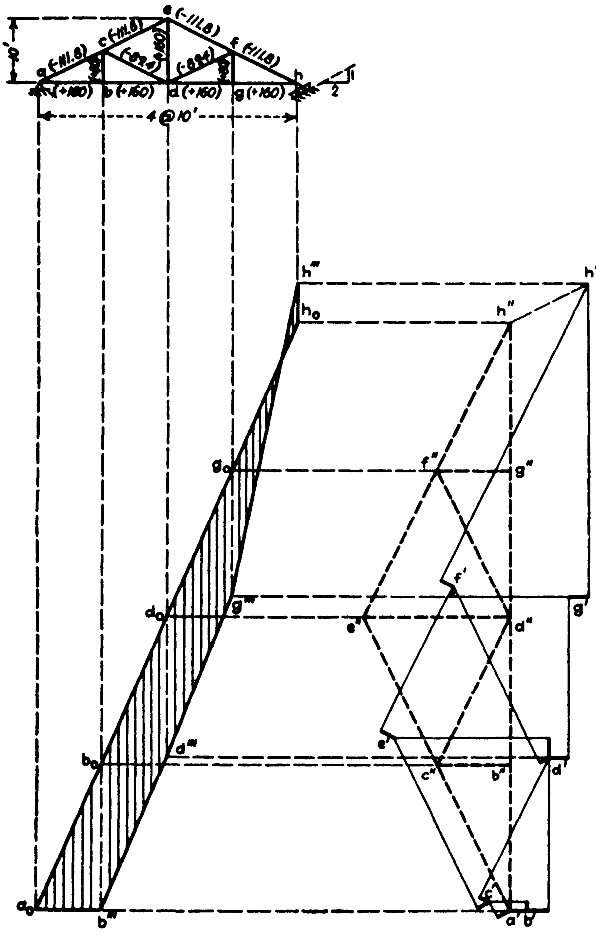


FIG. 21.

the movement of each joint relative to *a* and the direction of the member *ab* are shown by the vector drawn from a' to the point in the vector diagram corresponding to the joint in question. Thus the relative movement of joint *h* is shown by the vector $a'h'$. An examination of the truss, however, shows that the actual movement of joint h must be parallel to the surface supporting the right-hand shoe; consequently the assumption that the axis of member ab did not change direction was incorrect and a correction diagram must be added to the Williot diagram. This correction diagram was drawn as follows. The rotation of the truss required to correct the error due to the incorrect assumption is about joint a and during this rotation joint h moves in an arc with center at a. The angular movement is so small that no appreciable error is made if the arc is replaced by a straight line perpendicular to the line ah, in this problem a vertical line. This movement of joint h is represented in the vector diagram by a vertical vector drawn through a' and is such that when added to the vector $a'h'$ representing the movement of joint h during the distortion of the truss the sum will be a vector parallel to the actual movement of joint h. This idea serves to locate the point h''. A line is drawn through h' parallel to the actual movement of joint h and produced till it intersects the vertical vector through a'; this point of intersection is h''. The vector $h''a'$ represents the movement of joint h during the corrective rotation and the vector $a'h'$ represents the movement of joint h during distortion only; their sum is the vector $h''h'$ which satisfies the conditions of support. The correction diagram $a'b''c''d''e''f''g''h''$ is now drawn using the points a' and h'' as a base. It is geometrically similar to the sketch of the truss but every line in it is perpendicular to the corresponding line in the sketch of the truss. According to the theorem previously developed, the movement of any joint during the rotation is represented by the vector drawn from the corresponding point in the correction diagram to a', and this vector, when added to the vector drawn from a' to the corresponding point in the Williot diagram, gives a vector sum which represents the total movement of the joint. For example, the total movement of joint d is represented by the vector $d''d'$, which is the sum of the vectors $d''a'$ and $a'd'$. The polygon $a_0b'''d'''g'''h'''h_0$ is the deflection diagram for the bottom chord of the truss, the ordinates being measured vertically from the base line a_0h_0.

Usually it is advantageous to select as the joint which is assumed to be fixed in position one that is near the middle of the truss and to choose as the member which is assumed to be fixed in direction one of those connected to this joint, the middle vertical, for example, if there be one. Such a selection would lead to a Williot diagram which, if drawn to the same scale as might be used if the procedure in Fig. 21 were carried out, would

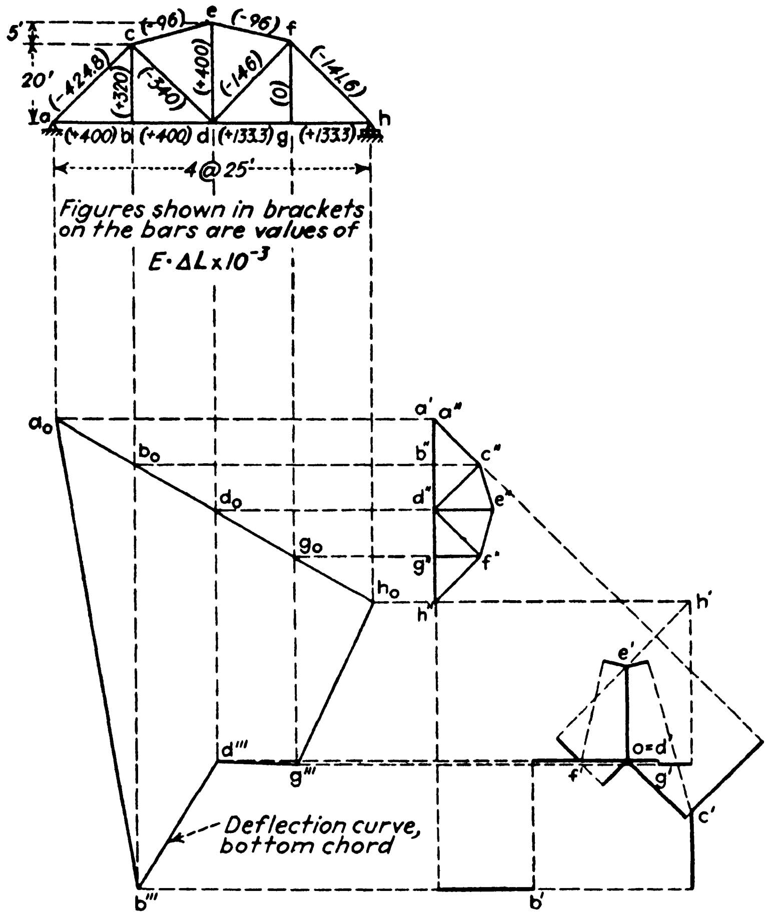
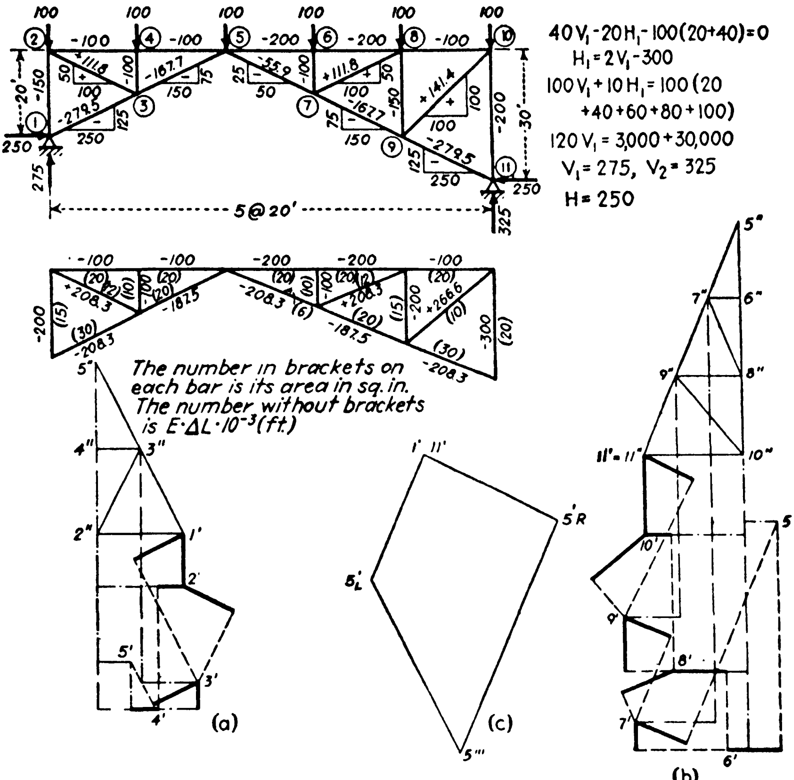


FIG. 22.

occupy an area much less extensive than that in Fig. 21. It is possible, then, to draw the diagram to a larger scale than was used in Fig. 21 without drawing paper whose size is unwieldy or to produce a more compact Williot diagram by using the same scale. In addition, if the truss and loading are symmetrical and the member assumed to be fixed in direction lies on the axis of symmetry, it is necessary to draw only half the Williot diagram and no Mohr correction diagram is needed. This variation of the procedure, applied to a truss which is symmetrical about a middle vertical but which is loaded asymmetrically, is shown in Fig. 22. Joint *d* was assumed to be fixed in position and member *de* was assumed to be fixed in direction. When the Williot diagram was completed, it showed that if these assumptions had been correct joint *h* would have moved downward relative to joint *a*. The conditions of support, however, show that the only possible movement of joint *h* relative to joint *a* is horizontal; therefore, the assumptions were incorrect and a Mohr correction diagram is necessary. This is carried out just as in Fig. 21. If the loading had been symmetrical, the Williot diagram would have been symmetrical about *d′e′* and the movement of joint *h* relative to joint *a* would have been shown as horizontal. This satisfies the conditions of support; no correction diagram would have been necessary and the Williot diagram would have shown true deflections.

21. The Williot-Mohr Diagram for a Three-hinged Arch.—In applying this procedure to a three-hinged arch the crown connection between the two sections of the arch is assumed to be broken and in each section one joint, usually the point of support, is assumed to be fixed in position; also one of the members connected to that joint is assumed to be fixed in direction. A Williot diagram is drawn for each section. In general the two Williot diagrams will show movements for the crown hinge which are not alike, and corrective rotations of the two sections are necessary. Each section is rotated about the fixed joint through an angle which is such as to produce, together with the relative deflections previously found, final deflections for the crown joint which are alike for the two sections.

The procedure described above has been applied to the arch shown in Fig. 23. In part (*a*) the Williot diagram, 1′-2′-3′-4′-5′ for the left section of the arch is shown; this was drawn on the assumption that joint 1 was fixed in position and that member 1-2 did not change direction. Similarly the Williot diagram 11′-10′-9′-8′-7′-6′-5′ for the right section of the arch was drawn on the assumption that joint 11 was fixed in position and that member 11-10 was fixed in direction. This is shown in (*b*). In these two diagrams the vectors 1′-5′ and 11′-5′ which represent movements of joint 5 based on two separate assumptions are not alike, consequently the assumptions were incorrect. There must be a further movement of each section; this movement is a rotation about the fixed joint in each section, such



Joint	$\delta_v \cdot E \cdot 10^{-3}$(ft)	$\delta_h \cdot E \cdot 10^{-3}$(ft)	Joint	$\delta_v \cdot E \cdot 10^{-3}$(ft)	$\delta_h \cdot E \cdot 10^{-3}$(ft)
2	203 ↓	322 →	7	1608 ↓	290 ←
3	892 ↓	210 →	8	1120 ↓	270 ←
4	996 ↓	223 →	9	915 ↓	203 ←
5	1140 ↓	123 →	10	302 ↓	375 ←
6	1710 ↓	71 ←			

Fig. 23.

that the movement of joint 5 during the rotation when added to the movement shown in the Williot diagram will produce identical total deflections of joint 5 in the two sections. The amount of this additional movement of joint 5 was found in (*c*). From point 1′, 11′ are drawn 1′-$5'_L$ which is a reproduction of vector 1′-5′ in part (*a*) and 11′-$5'_R$ which is a reproduction of 11′-5′ in part (*b*). To 1′-$5'_L$ must be added a movement of joint 5 due to the rotation of the left section about joint 1 which is a movement perpendicular to the line joining joints 1 and 5. Similarly one must add to 11′-$5'_R$ a movement of joint 5 about joint 11 which is represented by a line perpendicular to the line joining joints 11 and 5. These two lines are produced till they meet at point 5‴. The total movement of joint 5 is represented in direction and magnitude by the line 1′-5‴ in (*c*).

It still remains to find the total movements of the other joints of the structure by adding to the movements shown in parts (*a*) and (*b*) the movements occurring during the rotations of the two sections. This is done by drawing a Mohr correction diagram for each section of the arch. In part (*a*) the correction diagram is determined by the points 1′ and 5″. Of these two, point 1′ is already known and point 5″ is found from the fact that the final movement must be shown by the vector 5″-5′, which, therefore, must be equal and parallel to the vector 1′-5‴ in part (*c*). Once point 5″ has been found, the rest of the correction diagram is found by drawing a figure geometrically similar to the left section of the arch but turned through an angle of 90 deg. A similar procedure will serve to find the correction diagram for the right-hand section.

22. The Elastic Curve as a Funicular Polygon; Elastic Loads.—The elastic curve for a series of members in a truss may be treated as a funicular polygon, drawn for certain forces, using a pole distance equal to unity. These forces are called the elastic loads for the series of members and the condition of distortion under investigation. In determining these elastic loads it is necessary to consider two separate problems: First, what forces must be used to produce a particular polygon and, second, what are the relations between these forces and the condition of distortion of the structure? In many structures it is necessary to investigate the deflections of certain joints only, and in finding these deflections it is not necessary to take into account, directly, any members other than those forming a pin-jointed chain which has, as connections, the joints whose deflections are desired. This method of investigation was developed by Müller-Breslau and is called "Müller-Breslau's bar-chain method."

This method may be explained by beginning with the first of the two problems mentioned in the preceding paragraph and then considering the second. Consider Fig. 24 in which is shown the funicular polygon for certain forces P_{m-1}, P_m, P_{m+1}, ... Lines *ab* and *me* are drawn parallel to the base line *AA* and *md* is *am* produced. The lines marked with corresponding Roman numerals are parallel.

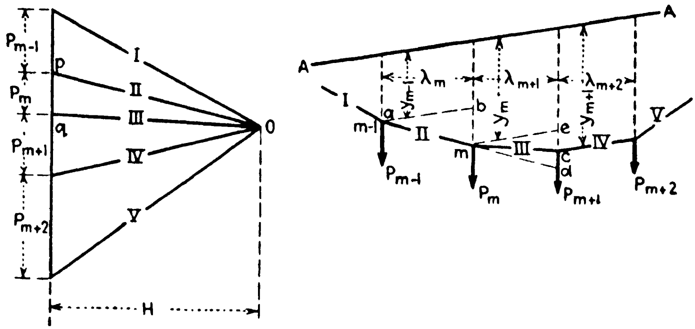


FIG. 24.

In the triangles *amb* and *mde*, *ab* is parallel to *me*, *am* and *md* are in the same straight line, while *bm* and *ed* are both vertical; therefore the two triangles are geometrically similar and one may write

$$\frac{bm}{\lambda_m} = \frac{ed}{\lambda_{m+1}}$$

Now, $bm = y_m - y_{m-1}$ and $ed = cd + (y_{m+1} - y_m)$, therefore

$$\frac{y_m - y_{m-1}}{\lambda_m} = \frac{cd}{\lambda_{m+1}} + \frac{(y_{m+1} - y_m)}{\lambda_{m+1}}$$

It may be seen also that the triangles *mcd* and *pqO* are geometrically similar, so that

$$\frac{cd}{\lambda_{m+1}} = \frac{P_m}{H}$$

therefore,

$$\frac{P_m}{H} = +\frac{y_m - y_{m-1}}{\lambda_m} - \frac{y_{m+1} - y_m}{\lambda_{m+1}} \tag{50}$$

If the funicular polygon under consideration is such that it is to be the elastic curve for a structure, the pole distance *H* becomes unity and the force P_m is the elastic load to be used at joint m and will be designated by w_m. Also, if the base line AA is the line from which deflections are to be measured, the ordinates y are the deflections of the joints which, in the previous articles, have been designated by δ. Using this notation, Eq. (50) becomes

$$w_m = +\frac{\delta_m - \delta_{m-1}}{\lambda_m} - \frac{\delta_{m+1} - \delta_m}{\lambda_{m+1}} \tag{51}$$

This is the general equation expressing the elastic load for any joint in terms of the deflections at that joint and the two adjacent joints.

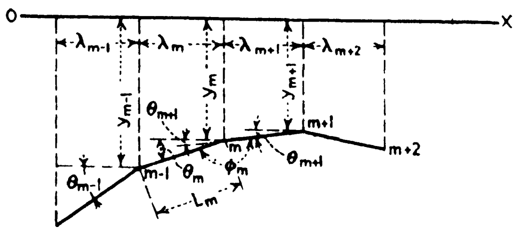


FIG. 25.

To find the relation between such an elastic load and changes in the lengths of the members of a trussed structure consider the series of members $(m - 1)$-(m)-$(m + 1)$-$(m + 2)$ in Fig. 25. The connections between these members are assumed to be frictionless pins; the members are subjected to changes of length with consequent changes in their slopes and in the positions of the joints. The original position of the bar chain is fixed with reference to the axis OX by means of the ordinates y_{m-1}, y_m, y_{m+1}, etc.

The notation used is as follows:

L_m = length of the member to the left of joint m,

θ_m = slope of that member to be taken as positive when it is counterclockwise from the horizontal,

ϕ_m = angle at joint m between members $(m - 1)$-(m) and (m)-$(m + 1)$ and is measured on the lower side of the two members.

It is evident that

$$y_{m-1} - y_m = L_m \sin \theta_m$$

If there is a displacement of the bar chain so small that the changes in the lengths of the members and the movements of the joints may be written as infinitesimals,

$$dy_{m-1} - dy_m = dL_m \sin \theta_m + L_m \cos \theta_m \, d\theta_m$$

If both sides of this equation are divided by $\lambda_m = L_m \cos \theta_m$ it becomes

$$\frac{dy_{m-1} - dy_m}{\lambda_m} = \frac{dL_m}{L_m} \tan \theta_m + d\theta_m$$

The infinitesimals dy are the vertical components of the displacements of the joints which previously were designated by δ. Hence,

$$\frac{\delta_{m-1} - \delta_m}{\lambda_m} = \frac{dL_m}{L_m} \tan \theta_m + d\theta_m$$

A similar relation may be written for the member connecting joints m and $m + 1$, thus

$$\frac{\delta_m - \delta_{m+1}}{\lambda_{m+1}} = \frac{dL_{m+1}}{L_{m+1}} \tan \theta_{m+1} + d\theta_{m+1}$$

When the first of these equations is subtracted from the second, the result is

$$+\frac{\delta_m - \delta_{m-1}}{\lambda_m} - \frac{\delta_{m+1} - \delta_m}{\lambda_{m+1}} = -\frac{dL_m}{L_m} \tan \theta_m + \frac{dL_{m+1}}{L_{m+1}} \tan \theta_{m+1} - d\theta_m + d\theta_{m+1}$$

The left-hand side of this equation is the same as the right-hand side of Eq. (51); therefore,

$$w_m = -\frac{dL_m}{L_m} \tan \theta_m + \frac{dL_{m+1}}{L_{m+1}} \tan \theta_{m+1} - d\theta_m + d\theta_{m+1}$$

From Fig. 25,

$$\phi_m = 180° - (\theta_m - \theta_{m+1})$$

and, differentiating,

$$d\phi_m = -d\theta_m + d\theta_{m+1}$$

therefore

$$w_m = -\frac{dL_m}{L_m} \tan \theta_m + \frac{dL_{m+1}}{L_{m+1}} \tan \theta_{m+1} + d\phi_m \tag{52}$$

In using this equation it is to be remembered that the angles θ are to be taken as positive when they are measured counterclockwise from the horizontal and that the fractions dL_m/L_m and dL_{m+1}/L_{m+1} are the strains (unit) for the two members of the chain adjacent to joint m; these strains are to be taken as positive when they are increases in the lengths of the members. It should be pointed out also that these strains may be due to any cause whatever. If the strains correspond to a stress intensity f and an increase of $\Delta t°$ in the temperature

$$\frac{dL_m}{L_m} = \frac{f_m}{E} + \epsilon\,\Delta t$$

One of the properties of the funicular polygon is that if the pole distance is unity and if the outer strings of the polygon are produced to intersect the lines of action of the reactions for a simple end-supported beam which supports the loads for which the polygon is drawn and if a straight line is drawn connecting these two points of intersection, the ordinates from this line to the funicular polygon represent, to a scale which is numerically the same as the distance scale to which the beam was drawn, the bending moments for that beam. This idea makes it possible to obtain the funicular polygon for the elastic loads, *i.e.*, the elastic curve for the bar chain, by imagining the elastic loads to be applied to a simple end-supported beam and drawing the bending-moment curve for that beam. It should be pointed out also that this procedure merely serves to determine the shape of the elastic curve and that the base line from which the bending moments due to the elastic loads are measured is not necessarily the line from which the deflections are to be measured: This line of zero deflections is found by drawing through the points on the funicular polygon at which the deflections are zero a line which will be the base line from which the deflections are to be measured. If the imaginary end-supported beam to which the elastic loads are applied is in the same position relative to the elastic loads as the real structure and if the latter is end-supported, the line of no deflections coincides with the base line from which the bending moments were measured and no adjustment is necessary.

The first two terms in the right-hand side of Eq. (52) may be taken directly from the condition of strain which defines the condition of distortion of the structure but a little further explanation is necessary before one can evaluate the terms $d\phi_m$. It is evident that the angle ϕ_m is equal to 360 deg. minus the sum of the angles of the triangles of the truss having vertices at joint m. If it is possible to compute the changes in the angles of these triangles, $d\phi_m$ can be determined; it is, in fact, minus the sum of the changes in these angles. These changes can be computed as follows: Consider the triangle shown in Fig. 26 and suppose that the lengths of the sides of this triangle change by small increments.

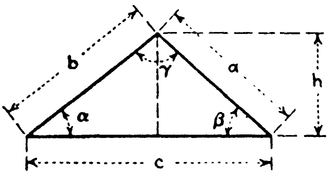


FIG. 26.

$$c = b \cos \alpha + a \cos \beta$$

Differentiating,

$$dc = db \cos \alpha - b \sin \alpha \, d\alpha + da \cos \beta - a \sin \beta \, d\beta$$

Also

$$h = a \sin \beta = b \sin \alpha$$

therefore

$$dc = db \cos \alpha + da \cos \beta - h(d\alpha + d\beta)$$

Since

$$\alpha + \beta + \gamma = 180°$$
$$d\alpha + d\beta = -d\gamma$$

Therefore

$$h \, d\gamma = dc - db \cos \alpha - da \cos \beta$$

Dividing both sides of this equation by $h = a \sin \beta = b \sin \alpha$

$$d\gamma = \frac{dc}{c}\frac{c}{h} - \frac{db \cos \alpha}{b \sin \alpha} - \frac{da \cos \beta}{a \sin \beta}$$
$$= \frac{dc}{c}\frac{b \cos \alpha + a \cos \beta}{b \sin \alpha = a \sin \beta} - \frac{db \cos \alpha}{b \sin \alpha} - \frac{da \cos \beta}{a \sin \beta}$$
$$= \left(\frac{dc}{c} - \frac{db}{b}\right) \cot \alpha + \left(\frac{dc}{c} - \frac{da}{a}\right) \cot \beta \qquad (53)$$

Similarly the changes in the other two angles of the triangle are

$$\left.\begin{aligned} d\alpha &= \left(\frac{da}{a} - \frac{dc}{c}\right)\cot\beta + \left(\frac{da}{a} - \frac{db}{b}\right)\cot\gamma \\ d\beta &= \left(\frac{db}{b} - \frac{da}{a}\right)\cot\gamma + \left(\frac{db}{b} - \frac{dc}{c}\right)\cot\alpha \end{aligned}\right\} \qquad (53a)$$

If the triangle is one of the triangles formed by the members of a truss and the changes in the lengths of the sides are due to stress intensities and changes in temperature

$$\frac{da}{a} = \frac{f_a}{E} + \epsilon\,\Delta t_a, \qquad \frac{db}{b} = \frac{f_b}{E} + \epsilon\,\Delta t_b, \qquad \frac{dc}{c} = \frac{f_c}{E} + \epsilon\,\Delta t_c$$

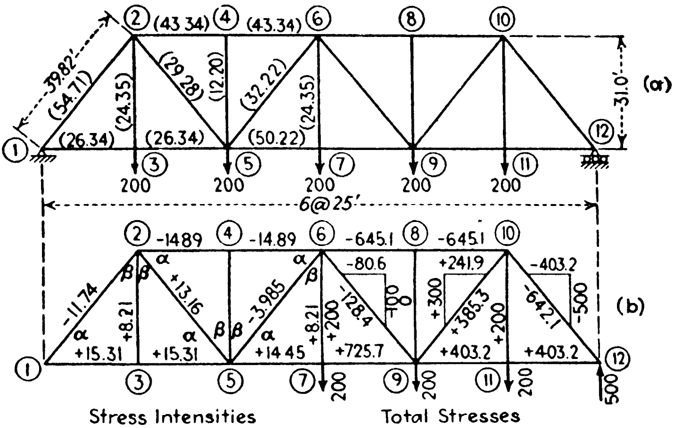


FIG. 27.

As a numerical illustration consider the truss shown in Fig. 27*a*. It is desired to find the vertical components of the deflections of the joints of the bottom chord. The truss is symmetrical about member 6-7, the loads are stated in units of 1,000 lb. and the numbers in parenthesis are the cross-sectional areas in square inches.

Since it is desired to find the vertical deflection components for the joints of the bottom chord, the bar chain to be used consists of the members of the bottom chord. Since all of these members are horizontal each one has a slope $\theta = 0$ and the first and second terms of the right-hand side of Eq. (52) vanish leaving

$$w_m = d\phi_m$$

All that is necessary in computing the elastic loads is to compute the changes in the angles of the triangles of the truss which have vertices at the joints of the bottom chord. This is carried out in tabular form basing the computations on Eqs. (53).

Angle	Coefficient of cot α cot $\alpha = \frac{25}{31}$	Coefficient of cot β cot $\beta = \frac{31}{25}$	1st term	2d term	$E\,d\phi\,10^{-}$
1-3-2	−11.74−15.31 = −27.05	−11.74− 8 21 = −19 95	−21 82	−24.74	+42.15
2-3-5	+13.16−15.31 = − 2.15	+13.16− 8.21 = + 4 95	− 1 73	+ 6.14	
3-5-2		+ 8 21−13 16 = − 4 95		− 6.14	
2-5-4	−14 89−13 16 = −28.05		−22 62		+22.42
4-5-6	−14.89+ 3 99 = −10.90		− 8 79		
6-5-7		+ 8.21+ 3.99 = +12 20		+15 13	
5-7-6	− 3.99−14.45 = −18.44	− 3 99− 8 21 = −12 20	−14 87	−15 13	+60.00
6-7-9			−14 87	−15 13	

Since the truss is symmetrical, $d\phi_9 = d\phi_5$ and $d\phi_{11} = d\phi_3$.

The most convenient imaginary beam is one which has the same span as the truss and has the same position relative to the elastic loads as the bottom chord of the truss. If this choice is made the line from which the bending moments due to the elastic loads are measured is also the base line from which deflections are to be measured. The deflections are computed as shown in Fig. 28.

	$E\,\delta\,10^{-3}$	δ(in.)
+94.57 × 25 =	+2364	+0.946
−42.15		
+52.42 × 25 =	+1310	
−22.42	+3674	+1.480
+30.00 × 25 =	+ 750	
−60.00	+4424	+1.770
−30.00 × 25 =	− 750	
−22.42	+3674	+1.480
−52.42 × 25 =	−1310	
−42.15	+2364	+0.946
−94.57 × 25 =	−2364	
	0	

If the vertical components of the deflections of the joints of the top chord are desired, they may be found from the fact that the difference between the deflections of top and bottom chord joints at the upper and lower ends of any vertical must differ by the change in the length of that vertical. For example,

$$\delta_2 = \delta_3 - \Delta L_{2-3} = 0.946 \text{ in.} - \frac{8.21}{3 \times 10^4} \times 31 \times 12 \text{ in.}$$
$$= 0.946 \text{ in.} - 0.102 \text{ in.} = 0.844 \text{ in.}$$

As a second illustration compute the vertical components of the deflections of the joints of the bottom chord of the truss shown in Fig. 29.

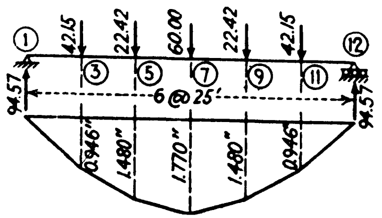
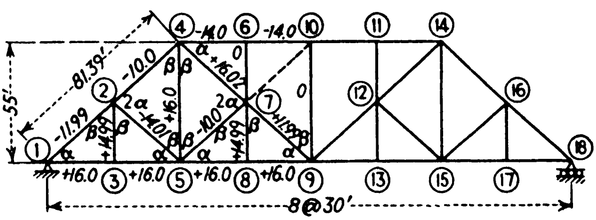


FIG. 28.

The solution of this problem is carried out in the same way as the solution of the previous illustration; there is a difference in one detail only. This difference arises from the fact that the angle 7-9-10 is an angle of a quadrilateral formed by members of the truss instead of being an angle of a triangle. Consequently it is necessary to find the change in this angle by a method which differs slightly from that used for the other angles. Two methods are available. Since the changes of the angles of a triangle may be computed when the changes in the lengths of its sides are known, it is possible to compute the change in the angle 7-9-10 if the change in the distance 7-10 has been found, treating



The number written on each member is its stress intensity in units of 1,000 lb. per sq. in. The truss and loading are symmetrical about member 9-10

FIG. 29.

the angle as one of the angles of the triangle 7-9-10. The two methods mentioned differ in the means adopted to find the change in the distance. One way is to use the law of virtual work to find the relative deflection, along the line 7-10, of joints 7 and 10. This is a problem which has been explained in Art. 18

and needs no further explanation here. A second means is based on the idea that the sum of the angles 9-7-8, 8-7-5, 5-7-4, 4-7-6, 6-7-10, and 10-7-9 before distortion is 360 deg. and must be 360 deg. after distortion also; in other words, the sum of the changes in these angles is zero. If the changes in these angles be expressed in terms of the strains in the sides of the triangles of which they are parts, the only unknown which will appear is the strain in the distance 7-10: this strain may, therefore, be computed. The equation may be set up as follows:

$$\cot \alpha = \frac{12}{11} = 1.0909; \qquad \cot \beta = \frac{11}{12} = 0.9167;$$

$$\cot 2\alpha = 0.08712$$

$$\begin{aligned}(+16.0 - 11.99)1.0909 + (+16.0 + 10.0)1.0909 \\ + (+16.0 + 10.0)0.9167 + (+16.0 - 16.02)0.9167 + \\ (-14.0 - 16.02)1.0909 \\ + \left(-14.0 - \frac{\Delta L_{7\text{-}10}}{L_{7\text{-}10}}E\right)1.0909 + \left(0 - E\frac{\Delta L_{7\text{-}10}}{L_{7\text{-}10}}\right)0.9167 \\ + (0 - 11.99)0.9167 = 0\end{aligned}$$

This becomes

$$2.0076E\frac{\Delta L_{7\text{-}10}}{L_{7\text{-}10}} = -2.460$$

$$E\left(\frac{\Delta L}{L}\right)_{7-10} = -1.226$$

The computation of the changes in the angles of the triangles and of $d\phi$ is carried out in the table shown on page 73.

For each member of the bar chain, $\theta = 0$; therefore, $w_m = d\phi_m$.

	$E\,\delta\,10^{-3}$	δ (in.)
$+144.21 \times 30 =$	$+4326$	$+1.730$
-114.58		
$+\ 29.63 \times 30 =$	$+\ 889$	
$+\ 28.28$	$+5215$	$+2.086$
$+\ 57.91 \times 30 =$	$+1737$	
$-\ 58.38$	$+6952$	$+2.781$
$-\ 0.47 \times 30 =$	$-\ 14$	
	$+6938$	$+2.775$

Angle	Coef. of cot α cot $\alpha = 1.0909$	Coef. of cot β cot $\beta = 0.9167$	Coef. of cot 2α cot $2\alpha = 0.0871$	1st term	2d term	3d term	$E\,d\phi\,10^{-3}$
1-3- 2	−11.99−16.00 = −27.99	−11.99−14.99 = −26.98		−30.53	−24.73		+114.58
2-3- 5	−14.01−16.00 = −30.01	−14.01−14.99 = −29.00		−32.74	−26.58		
3-5- 2		+14.99+14.01 = +29.00			+26.58		
2-5- 4		−10.00−16.00 = −26.00	−10.00+14.01 = + 4.01		−23.83	+0.35	− 28.28
4-5- 7		+16.02−16.00 = + 0.02	+16.02+10.00 = +26.02		+ 0.02	+2.26	
7-5- 8		+14.99+10.00 = +24.99			+22.90		
5-8- 7	−10.00−16.00 = −26.00	−10.00−14.99 = −24.99		−28.36	−22.90		+ 58.38
7-8- 9	+11.99−16.00 = − 4.01	+11.99−14.99 = − 3.00		− 4.37	− 2.75		
8-9- 7		+14.99−11.99 = + 3.00			+ 2.75		
7-9-10		− 1.23− 0.0 = − 1.23	− 1 23−11 99 = −13.22		1.13	−1.15	− 0.94
10-9-12					− 1.13	−1.15	
12-9-13					+ 2.75		

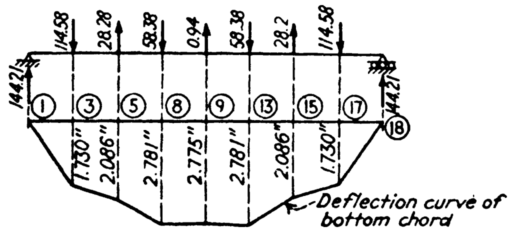


FIG. 30.

An interesting variation in the details of the method is illustrated in the determination of the vertical components of the deflections of the bottom chord of the cantilever truss shown in Fig. 31. The truss is symmetrical about member 6-7 and is

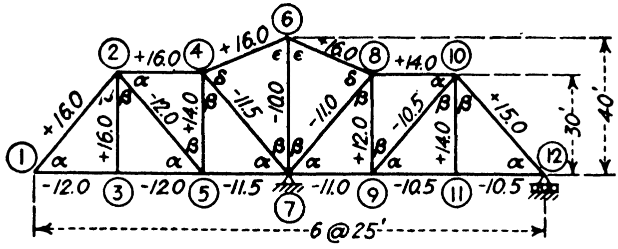


FIG. 31.

loaded so as to produce in the members the stress intensities written thereon in units of 1,000 lb. per sq. in. Each member of the bottom chord being horizontal, the expression for the elastic load to be used at each joint of the bottom chord, used as a bar chain, is

$$w_m = d\phi_m$$

and the computation of these quantities is carried out as in the preceding illustrations. The computation is shown in the table at the top of page 75.

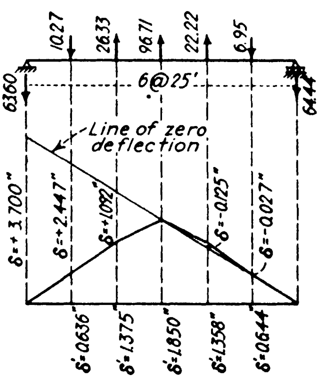


FIG. 32.

In this problem, as in the preceding illustrations, the shape of the elastic curve for the bar chain selected may be found by

Angle	Coef. of cot α $\cot\alpha = \frac{5}{6}$	Coef. of cot β $\cot\beta = \frac{6}{5}$	1st term	2d term	$E\,d\phi\;10^{-3}$
1-3-2	+16.0 + 12.0 = +28.0	+16.0 − 16.0 = 0.0	+23.33	0.0	+10.27
2-3-5	−12.0 + 12.0 = 0.0	−12.0 − 16.0 = −28.0	0.0	−33.60	
3-5-2		+16.0 + 12.0 = +28.0		+33.60	
2-5-4	+16.0 + 12.0 = +28.0		+23.33		−26.33
4-5-7	−11.5 + 11.5 = 0.0	−11.5 − 14.0 = −25.5	0.0	−30.60	
5-7-4		+14.0 + 11.5 = +25.5		+30.60	
8-7-9		+12.0 + 11.0 = +23.0		+27.60	
	Coef. of cot δ $\cot\delta = 0.3250$	Coef. of cot ε $\cot\epsilon = \frac{2}{5}$			−96.71
4-7-6	+16.0 + 11.5 = +27.5	+16.0 + 10.0 = +26.0	+ 8.94	+10.40	
6-7-8	+16.0 + 11.0 = +27.0	+16.0 + 10.0 = +26.0	+ 8.77	+10.40	
	Coef. of cot α $\cot\alpha = \frac{5}{6}$	Coef. of cot β $\cot\beta = \frac{6}{5}$			
7-9-8	−11.0 + 11.0 = 0.0	−11.0 − 12.0 = −23.0	0.0	−27.60	
8-9-10	+14.0 + 10.5 = +24.5		+20.42		−22.22
10-9-11		+14.0 + 10.5 = +24.5		+29.40	
9-11-10	−10.5 + 10.5 = 0.0	−10.5 − 14.0 = −24.5	0.0	−29.40	+ 6.95
10-11-12	+15.0 + 10.5 = +25.5	+15.0 − 14.0 = + 1.0	+21.25	+ 1.20	

considering the elastic loads applied to an imaginary beam freely supported at the ends and drawing the curve of bending moments for this beam. This, however, is merely a means of determining the shape of the elastic curve and the base line for bending moments is not necessarily the base line from which deflections are to be measured. In this problem the base line from which deflections are to be measured is a straight line passing through the points on the elastic curve corresponding to the points of support of the truss. The determination of the deflections is shown in Fig. 32.

$$
\begin{array}{rcrcr}
-\;22.22 & \times 2 & = & -\;44.44 \\
-\;96.71 & \times 3 & = & -290.13 \\
-\;26.33 & \times 4 & = & -105.32 \\
\hline
-145.26 & & & -339.89
\end{array}
$$

+ 6.95	× 1	=	+ 6.95
+ 10.27	× 5	=	+ 51.35
−128.04			6) −381.59
− 63.598			− 63.598
− 64.442			

				δ', in.
− 63.60	× 25	=	−1590	−0.636
− 10.27				
− 73.87	× 25	=	−1847	
+ 26.33			−3437	−1.3748
− 47.54	× 25	=	−1188	
+ 96.71			−4625	−1.8500
+ 49.17	× 25	=	+1229	
+ 22.22			−3396	−1.3584
+ 71.39	× 25	=	+1785	
− 6.95			−1611	−0.6444
+ 64.44	× 25	=	+1611	
			000	

23. Deflections of the Joints of a Three-hinged Arch by the Use of Elastic Loads.—In the illustrations in Art. 22 the trusses were such that changes in the angles between adjacent members of the bar chain could be computed from the strains in the members of the truss. At the crown hinge of a three-hinged arch the relative rotation of any two members connected to the crown hinge is not only a function of the strains in the members forming the boundaries of triangles but is a function of the relative rotation of the two sections of the arch when they are considered as rigid bodies. Some addition to the theory already developed is necessary, therefore, if one is to be able to use the method of elastic loads in computing the deflections of the joints of a three-hinged arch. This addition is a method of computing the change $d\phi$ in the angle ϕ between two members of the bar chain adjacent to the crown hinge and is based on the fact that the change in the length of the chord of the bar chain joining the points of support must be equal to whatever relative yielding of the supports there may be along the line joining them. Usually this yielding is assumed to be zero. If it is possible to set up an expression for the change in the length of this chord in which the only unknown is the change $d\phi$ mentioned above, there is an equation available from which this change may be

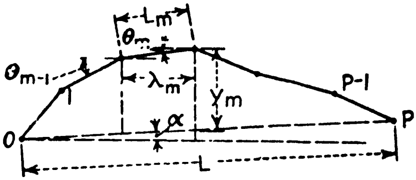


FIG. 33.

computed. In order to develop such an equation consider the bar chain shown in Fig. 33.

$$L = \Sigma L_m \cos(\theta_m - \alpha)$$

Differentiating,

$$dL = \Sigma dL_m \cos(\theta_m - \alpha) - \Sigma L_m \sin(\theta_m - \alpha) d\theta_m$$

Also

$$y_m + L_m \cos\theta_m \tan\alpha - y_{m-1} = L_m \sin\theta_m,$$

or

$$y_m - y_{m-1} = L_m \sin\theta_m - L_m \cos\theta_m \tan\alpha = L_m \frac{\sin(\theta_m - \alpha)}{\cos\alpha}$$

therefore

$$dL = \Sigma dL_m \cos(\theta_m - \alpha) - \Sigma d\theta_m \cos\alpha (y_m - y_{m-1})$$

For each ordinate y in the second term of the right-hand side of this equation there are two multipliers; for example, there is a product $y_m \, d\theta_m$ and a product $-y_m \, d\theta_{m+1}$. Regrouping the products in this second term one obtains

$$dL = \Sigma dL_m \cos(\theta_m - \alpha) - \cos\alpha \Sigma y_m (d\theta_m - d\theta_{m+1})$$

Using the relation

$$\phi_m = 180° - (\theta_m - \theta_{m+1})$$

or

$$d\phi_m = d\theta_{m+1} - d\theta_m$$

in the last expression for dL, it becomes

$$dL = \Sigma dL_m \cos(\theta_m - \alpha) + \cos\alpha \Sigma y_m \, d\phi_m \qquad (54)$$

and this equation would serve as a basis for the evaluation of the change $d\phi$ in the angle ϕ at the crown hinge of the arch. A more convenient procedure, however, is to continue this development until dL is expressed in terms of the elastic loads and to compute the elastic load to be used at the crown hinge directly. From Eq. (52)

$$d\phi_m = w_m + \frac{dL_m}{L_m}\tan\theta_m - \frac{dL_{m+1}}{L_{m+1}}\tan\theta_{m+1}$$

Substituting this expression for $d\phi_m$ in Eq. (54),

$$dL = \sum dL_m \cos(\theta_m - \alpha) + \cos\alpha \sum \left\{ y_m w_m + \frac{dL_m}{L_m} y_m \tan\theta_m - \frac{dL_{m+1}}{L_{m+1}} y_m \tan\theta_{m+1} \right\}$$

$$= \cos\alpha \sum y_m w_m + \sum dL_m \cos(\theta_m - \alpha) + \sum \frac{dL_m}{L_m}\tan\theta_m (y_m - y_{m-1})\cos\alpha$$

this transformation being based on a regrouping of the members of the last two terms of the summation. Substituting for $y_m - y_{m-1}$

$$dL = \cos\alpha \sum y_m w_m + \sum dL_m \cos(\theta_m - \alpha) + \sum \frac{dL_m}{L_m}\tan\theta_m \, L_m \frac{\sin(\theta_m - \alpha)}{\cos\alpha}\cos\alpha$$

$$= \cos\alpha \sum y_m w_m + \sum dL_m[\cos(\theta_m - \alpha) + \tan\theta_m \sin(\theta_m - \alpha)]$$

$$= \cos\alpha \sum y_m w_m + \sum dL_m \left[\cos(\theta_m - \alpha) + \frac{\sin^2\theta_m \cos\alpha - \sin\theta_m \cos\theta_m \sin\alpha}{\cos\theta_m}\right]$$

$$= \cos\alpha \sum y_m w_m + \sum dL_m \frac{\cos^2\theta_m \cos\alpha + \sin\theta_m\cos\theta_m\sin\alpha + \sin^2\theta_m\cos\alpha - \sin\theta_m\cos\theta_m\sin\alpha}{\cos\theta_m}$$

$$dL = \cos\alpha\left[\sum y_m w_m + \sum dL_m \sec\theta_m\right] \qquad (55)$$

To illustrate the application of these ideas consider the arch shown in Fig. 34*a*. The condition of distortion is defined by the stress intensities which are written (in units of 1,000 lb. per sq. in.) on the members.

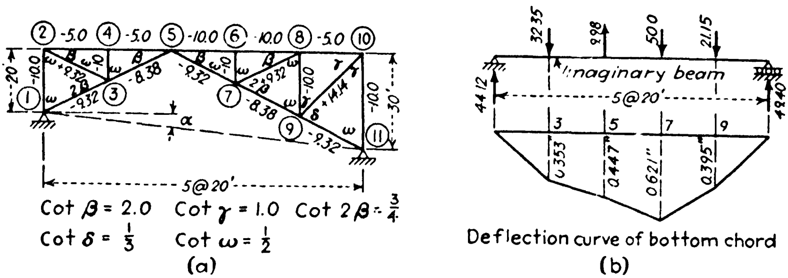


Fig. 34.

Changes in angles

$$E\,d\alpha = (f_a - f_b)\cot\gamma + (f_a - f_c)\cot\beta$$

α	$f_a - f_b$	cot δ	$f_a - f_c$	cot β	1st term	2d term	$E\,d\phi\ 10^{-4}$
1-3-2	−10.00+ 9.32= − 0.68	0.5	−10.00− 9.32= −19.32	0.5	− 0.34	− 9.66	
2-3-4	− 5.00− 9.32= −14.32	2.0		0	−28.64	0	+31.88
4-3-5		0	− 5.00+ 8.38= + 3.38	2.0	0	+ 6.76	
5-7-6	−10.00+ 9.32= − 0.68	2.0		0	− 1.36	0	
6-7-8		0	−10.00− 9.32= −19.32	2.0	0	−38.64	+50.47
8-7-9	−10.00− 9.32= −19.32	0.5	−10.00+ 8.38= − 1.62	0.5	− 9.66	− 0.81	
7-9-8	+ 9.32+ 8.38= +17.70	0.75	+ 9.32+10.00= +19.32	0.5	+13.28	+ 9.66	
8-9-10		0	− 5.00−14.14= −19.14	1.0	0	−19.14	+20.68
10-9-11	−10.00−14.14= −24.14	1.0	−10.00+ 9.32= − 0.68	0.5	−24.14	− 0.34	

$$Ew_m = E\,d\phi_m - f_m \tan\theta_m + f_{m+1}\tan\theta_{m+1}$$

m	f_m	$\tan\theta_m$	f_{m+1}	$\tan\theta_{m+1}$	$E\,d\phi_m$	$-f_m\tan\theta_m$	$+f_{m+1}\tan\theta_{m+1}$	$E\,w_m$
3	−9.32	+0.5	−8.38	+0.5	+31.88	+4.66	−4.19	+32.35
7	−9.32	−0.5	−8.38	−0.5	+50.47	−4.66	+4.19	+50.00
9	−8.38	−0.5	−9.32	−0.5	+20.68	−4.19	+4.66	+21.15

$$E\,dL = \cos\alpha[\Sigma y_m\,Ew_m + \Sigma f_m L_m \sec\theta_m]$$

m	y_m	$E\,w_m$	$y_m\,E\,w_m$	f_m	L_m	$\sec\theta_m$	$f_m\,L_m\sec\theta_m$
1	0.0	0.0					
3	12.0	+32.35	+388.2	−9.32	$10\sqrt{5}$	$\frac{1}{2}\sqrt{5}$	− 233.0
5	24.0	Ew_5	$24Ew_5$	−8.38	$10\sqrt{5}$	$\frac{1}{2}\sqrt{5}$	− 209.5
7	16.0	+50.00	+800.0	−9.32	$10\sqrt{5}$	$\frac{1}{2}\sqrt{5}$	− 233.0
9	8.0	+21.15	+169.2	−8.38	$10\sqrt{5}$	$\frac{1}{2}\sqrt{5}$	− 209.5
11	0.0	0.0	0.0	−9.32	$10\sqrt{5}$	$\frac{1}{2}\sqrt{5}$	− 233.0
			$+1357.4 + 24Ew_5$				−1118.0

Therefore

$$\frac{10}{\sqrt{101}}[1357.4 + 24Ew_5 - 1118.0] = 0$$

$$Ew_5 = -\frac{239.4}{24} = -9.975$$

+ 21.15 × 1 =	+ 21.15
+ 50.00 × 2 =	+100.00
+ 32.35 × 4 =	+129.40
+103.50	+250.55
− 9.98 × 3 =	− 29.94
+ 93.52	5) +220.61
− 44.12	44.12
49.40	

	$E\,\delta\,10^{-3}$	δ, in.
+44.12 × 20 =	+ 882.4	+0.353
−32.35		
+11.77 × 20 =	+ 235.4	
+ 9.98	+1117.8	+0.447
+21.75 × 20 =	+ 435.0	
−50.00	+1552.8	+0.621
−28.25 × 20 =	− 565.0	
−21.15	+ 987.8	+0.395
−49.40 × 20 =	− 988.0	

24. The Moment-area Theorems.—Two methods of determining the deflection of a point on the axis of a member which is subjected to bending moment, axial stress, and shear have been discussed already. A method of determining such deflections which is often more convenient is based on a consideration of the geometry of the elastic curve and applying the relation between the rate of change of slope at a point and the causes of that change. The basic ideas may be developed as follows: Let ACB in Fig. 35 be the elastic curve of part of any member which is bent due to any cause whatever and let AE be the tangent to the elastic curve at A. Suppose also that tangents to the elastic curve are drawn at the ends of an element of the elastic curve which has a horizontal projection dx. Let the curve MNO be such that its ordinate at any point is $d\theta/dx$, which is the rate of change of slope of the elastic curve at that point. It is evident that the total change of slope along the section ACB is

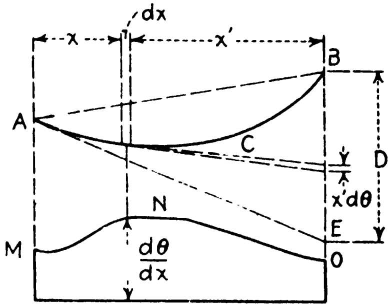


FIG. 35.

$$\int_a^b d\theta = \int_a^b \frac{d\theta}{dx} dx$$

which may be interpreted as the area under the rate of change of slope curve.

If the causes of distortion be limited to the application of loads and to change of temperature, the rate of change of slope at any point is

$$\frac{d\theta}{dx} = \frac{M}{EI} + \epsilon \frac{\Delta(\Delta t)}{d} - \frac{dg_s}{dx} \tag{56}$$

where g_s is the shear strain. The shear strain causes a change of slope of the axis of the member which occurs only when the intensity of shearing stress changes. The change of slope is clockwise when the change in the shear intensity is positive and, therefore, is in the opposite direction to the change measured by M/EI. Consequently, if x is measured to the right, the rate of change of slope corresponding to change of shear intensity must be written as negative. If there is no difference between the changes of temperature at the two faces of the member, $\Delta(\Delta t) = 0$, and if the shear distortion be neglected, Eq. (56) becomes

$$\frac{d\theta}{dx} = \frac{M}{EI}$$

and the statement with respect to the change of slope between two points becomes:

I. The change of slope of the elastic curve between two points A and B is equal to the area under the M/EI curve between the two points.

This is known as the first "moment-area theorem."

When it is remembered that the distortion occurring in a structure is very small, it is evident that the intercept on the line BE between the tangents at the ends of the element dx may be written as $x'\,d\theta$ and, consequently, that the distance BE is equal to $\int_a^b x'\,d\theta$ and, since

$$\int_a^b x'\,d\theta = \int_a^b x'\,\frac{d\theta}{dx}dx$$

it is evident that the distance BE may be interpreted as the static moment, about an axis through B, of the area under the rate of change of slope curve between points A and B. When there is no change in temperature and when the shear distortion is neglected, this becomes what is known as the "second moment-area theorem" which may be stated as follows:

II. The distance of a point B on the elastic curve of a member from the tangent at A to this curve is equal to the static moment about an axis through B of the area under the M/EI curve between points A and B.

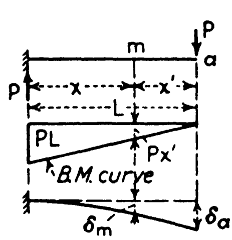


FIG. 36.

A very simple illustration of the application of these theorems is the computation of the deflection at a point on the axis of a simple cantilever beam carrying a concentrated load at the free end (see Fig. 36). If only the distortion corresponding to bending moment is considered, the elastic curve is tangential to the original position of the beam, the point of tangency being at the left end. Consequently, the deflection at any point m is equal to the static moment about an axis through m of that part of the M/EI curve which lies between the left end and the point m. Therefore

$$EI\,\delta_m = Px'\frac{x}{2}\frac{1}{3}x + PL\frac{x}{2}\frac{2}{3}x$$

$$\delta_m = \frac{Px^2}{6EI}(2L + x')$$

$$\delta_a = \frac{PL^3}{3EI}$$

and the slope at any point m is

$$EI\ \theta_m = -Px'\frac{x}{2} - PL\frac{x}{2}$$

$$\theta_m = -\frac{Px}{2EI}(L + x')$$

If it is desired to consider the deflection corresponding to shear distortion, the distance D of point B on the elastic curve from the tangent at A to the elastic curve must be written as

$$\begin{aligned} D &= \int_a^b x'\frac{d\theta}{dx}dx \\ &= \int_a^b x'\left(\frac{M}{EI} - \frac{dg_s}{dx}\right)dx \\ &= \int_a^b \frac{M}{EI}x'\,dx - \int_a^b \frac{d}{dx}\left(K\frac{S}{GA'}\right)x'\,dx \end{aligned}$$

The first term of this expression is the static moment, about an axis through B, of the M/EI area as defined under the moment-area theorems. The second may be interpreted in a similar fashion.

$$\begin{aligned} -\int_a^b \frac{d}{dx}\left(K\frac{S}{GA'}\right)x'\,dx &= -K\int_a^b \frac{d}{dx}\left(\frac{S}{GA'}\right)(x_b - x)dx \\ &= -Kx_b\int_a^b \frac{d}{dx}\left(\frac{S}{GA'}\right)dx + K\int_a^b x\frac{d}{dx}\left(\frac{S}{GA'}\right)dx \\ &= -Kx_b\left|\frac{S}{GA'}\right|_a^b + K\left[\left|\frac{S}{GA'}x\right|_a^b - \int_a^b \frac{S}{GA'}dx\right] \\ &= -Kx_b\left(\frac{S_b}{GA'_b} - \frac{S_a}{GA'_a}\right) + K\left[\frac{S_b}{GA'_b}x_b - \frac{S_a}{GA'_a}x_a - (A)_{\frac{S}{GA'}}\right] \end{aligned}$$

where $(A)_{\frac{S}{GA'}}$ is the area under the S/GA' curve between the points A and B. In this expression $x_a = 0$, therefore

$$-\int_a^b \frac{d}{dx}\left(K\frac{S}{GA'}\right)x'\,dx = +K\frac{S_a}{GA'_a}x_b - K(A)_{\frac{S}{GA'}}$$

which may be interpreted as K times the difference between the area of a rectangle bounded by the distance x_b and the ordinate to the S/GA' curve just to the left of A, and the area under the S/GA' curve between points A and B. If this difference is positive point B has moved up with reference to the tangent at A. It should be pointed out that S_a and the point of tangency should be immediately to the left of A, a suggestion which is of importance when the shear changes suddenly at point A. For example, if it is desired to find the deflection due to shear at point m of the beam in Fig. 36, since the deflection from the original position is also the deflection measured from the tangent to the elastic curve at a point just to the left of the support, and since the shear at such a point is taken as zero in the customary analysis,

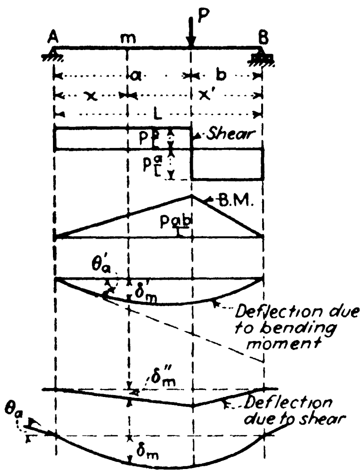


FIG. 37.

$$\delta_m = -\frac{Px}{GA'}, \qquad \delta_a = -\frac{PL}{GA'}$$

where the minus sign indicates that point a has moved downward with respect to the tangent.

The computation of the deflections for points on the axis of an end-supported beam is not so simple since, in order to use the second of the theorems relative to the $d\theta/dx$ curve, it is necessary, as a first step, to determine the position of the tangent to the elastic curve, the point of tangency being just outside of the support. The method may be illustrated by a consideration of the beam in Fig. 37. The tangent to the elastic curve at a point just outside of the support at A may be located by equating two expressions for the intercept, on the vertical through support B, between the tangent and the elastic curve; one expression is $L\,\theta_a$ and the other is computed by applying the second theorem with respect to the $d\theta/dx$ curve. Thus,

$$L\,\theta_a = \int_a^b \frac{M}{EI}x'\,dx + K\left[\frac{S_a}{GA'}L - (A)_{\frac{S}{GA'}}\right]$$

in which S_a is zero since it is the shear just outside of the point of support. If I and A' are constant over the length of the span,

$$L\,\theta_a = \frac{Pab}{EIL}\left[\frac{b}{2}\,\frac{2}{3}b + \frac{a}{2}\left(b + \frac{a}{3}\right)\right] - \frac{1}{GA'}K\left(P\frac{b}{L}\,a - P\frac{a}{L}\,b\right)$$
$$= \frac{Pab}{6EIL}(2b^2 + 3ab + a^2) - 0$$
$$= \frac{Pab}{6EIL}(2b + a)(b + a)$$
$$\theta_a = \frac{Pab}{6EIL}(2b + a)$$

Due to bending moment only,

$$\delta'_m = \theta'_a x - \frac{1}{EI}\,P\frac{b}{L}x\,\frac{x}{2}\,\frac{x}{3} = \frac{Pb}{6EIL}[ax(2b + a) - x^3]$$

Due to shear only,

$$\delta''_m = \frac{K}{GA'}\,\frac{Pbx}{L}$$

In both of the above expressions δ_m is taken as positive downward. Computing the deflections due to bending moment and shear together,

$$\delta_m = \theta_a x - \left[\frac{1}{EI}\left(\frac{Pb}{L}x\,\frac{x}{2}\,\frac{x}{3}\right) - K\frac{Pb}{GA'L}x\right]$$
$$= \frac{Pb}{6EIL}[ax(2b + a) - x^3] + K\frac{Pbx}{GA'L}$$

The point at which the greatest deflection occurs is the point at which the slope of the elastic curve is zero, *i.e.*, the change of slope between the end of the beam and the point of maximum deflection must be equal to the slope of the tangent to the elastic curve just outside of the support. This change must include the sudden change of direction which occurs just over the support due to the sudden increase of shear at that point.

25. Elastic Loads for Straight Members Subjected to Bending. See Fig. 38. Let ACB be the elastic curve of a section AB of a member which was initially straight and has been bent owing to any cause and let MNO be the corresponding rate-of-change-of-slope curve. Since

$$L\tau_a = \int_a^b x'\frac{d\theta}{dx'}dx'$$

τ_a is equal to $1/L$ times the static moment, about an axis through B, of the area under the rate-of-change-of-slope curve between points A and B. The computation of τ_a is exactly the same as that which would be done if it had been desired to find the left-hand reaction for an imaginary end-supported beam AB subjected to a distributed load whose intensity at any point is $d\theta/dx$. Hence one may state the following:

If a section AB of a member which is initially straight is bent due to any cause, and if the rate of change of slope at any point is $d\theta/dx$, the slope at one end of the section, referred to the chord AB of the elastic curve, is equal to the reaction at that end of an imaginary beam AB which is carrying a distributed load of intensity $d\theta/dx$.

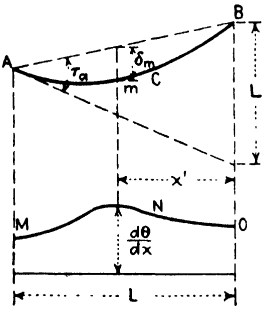


FIG. 38.

Consequently, $d\theta/dx$ may be used as an elastic load for such members.

In the particular case where one approximates by considering only the distortion corresponding to bending moment, the elastic load intensity is M/EI. If it is desired to include the effect of shear distortion, the elastic load intensity is

$$\frac{M}{EI} - \frac{dg_s}{dx} = \frac{M}{EI} - \frac{d}{dx}\left(K\frac{S}{A'G}\right)$$

Further, since the slope of the elastic curve at any point, this slope being measured with reference to the chord AB of the elastic curve, is equal to the slope at the end minus the area under the $d\theta/dx$ curve between that end and the point under consideration, if the $d\theta/dx$ curve be considered as defining an elastic load on an imaginary beam AB, one may state that

For a section AB of a member initially straight, but bent owing to any cause whatever, the slope at a point m of the elastic curve, referred to the chord AB of that curve, is equal to the shear at point m of an imaginary end-supported beam AB which carries a distributed load of intensity $d\theta/dx$.

Carrying the analogy further, since the deflection δ_m, measured from the chord AB of the elastic curve to point m on the elastic curve, is equal to $\tau_a\, x$ minus the moment about point m of the area under the $d\theta/dx$ curve between points A and m, the computation of the deflection δ_m is exactly the same as the operation of computing the bending moment at point m of the imaginary beam AB described above, so that one may state that

At any point m of a member which was initially straight but has been bent due to any cause, the deflection of the elastic curve, measured from the chord AB of that curve, is equal to the bending moment at point m of an imaginary end-supported beam of span AB which is subjected to a distributed load whose intensity is $d\theta/dx$.

If the bending is due to the action of transverse loads and to a change of temperature which varies from one side of the member to the other,

$$\frac{d\theta}{dx} = \frac{M}{EI} + \epsilon\frac{\Delta(\Delta t)}{d} - K\frac{d}{dx}\left(\frac{S}{GA'}\right)$$

The last term of this expression may have magnitude either because of variation in the shear S or because of variation in A'. If the transverse load is continuous and has an intensity p at any point, the last term may be written as

$$-\frac{K}{G}\frac{d}{dx}\left(\frac{S}{A'}\right) = +\frac{K}{G}\frac{p}{A'}.$$

Where the transverse forces are concentrated, that part of the elastic load which corresponds to shear distortion must consist of concentrated loads, one at each point of application of a concentrated force on the real beam and equal to $+\frac{K}{G}\frac{P}{A'}$.

A distributed elastic load of intensity $-\frac{K}{G}\frac{d}{dx}\left(\frac{S}{A'}\right)$ should be applied over portions of the beam where the cross-sectional area is varying continuously. In evaluating this term the only variable is A' and the elastic load intensity becomes $+\frac{KS}{GA'^2}\frac{dA'}{dx}$

Should the cross-sectional area vary suddenly, at such points concentrated elastic loads should be applied, of magnitude

$$-\frac{K}{G}\left(\frac{S}{A'_R} - \frac{S}{A'_L}\right)$$

or $+\frac{K}{G}\left(\frac{S}{A'_L} - \frac{S}{A'_R}\right)$, where A'_L and A'_R refer to conditions just to the left and just to the right of the section under consideration.

As an illustration consider the beam in Fig. 39*a*, the problem being to determine the curve of deflection. The curves of shear and bending moment for the beam are shown in Fig. 39*b* and *c*, respectively. At any point the intensity p_e of the elastic load is

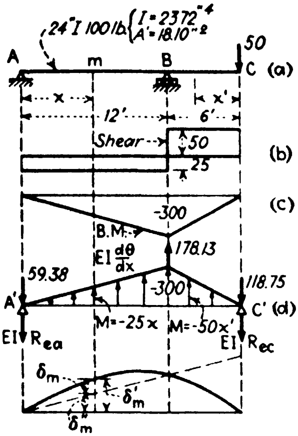


FIG. 39.

$$p_e = \frac{M}{EI} - \frac{K}{G}\frac{d}{dx}\left(\frac{S}{A'}\right)$$

Since I and A' are constant and since $K = 1$, this may be written

$$EIp_e = M - \frac{EI}{GA'}\frac{dS}{dx}$$

The first term of this expression is defined by the curve of bending moment and is negative because M is negative. The second term is zero for all points except at the two ends and at point B where the shear changes suddenly. At these points there must be concentrated elastic loads of

$$-\frac{EI}{GA'}(-25) = +59.375, \qquad -\frac{EI}{GA'}(+75) = -178.125,$$

and

$$-\frac{EI}{GA'}(-50) = +118.75, \text{ respectively.}$$

The only difference between the rest of the solution of this problem and the solution for the corresponding part of the illustration based on Fig. 31 is that due to the fact that in this case the elastic loading consists of a distributed load and a number of concentrated loads while in the former illustration there were concentrated loads only. The numerical solution is

$$EI \times 18R_{ea} = 178.13 \times 6 - 59.38 \times 18 + 300\left[\frac{6}{2} \times \frac{2}{3} \times 6 + \frac{12}{2}(6 + 4)\right]$$

$$EIR_{ea} = +1200$$

$$EIR_{ec} = 178.13 + 300 \times \frac{18}{2} - 59.38 - 1200 - 118.75 = +1500$$

For $0 < x < 12$ ft.

$$EI\delta_m' = -1259.38x + 25x\frac{x}{2}\frac{x}{3} = -1259.38x + \frac{25}{6}x^3$$

For $0 < x' < 6$ ft.,

$$EI\delta_m' = -1618.75x' + 50x'\frac{x'}{2}\frac{x'}{3} = -1618.75x' + \frac{50}{6}(x')^3$$

x	$-1259.4x$	$+\frac{25}{6}x^3$	$EI\delta_m'$	$EI\delta_m''$	$EI\delta_m$
2	− 2518.7	+ 33.3	−2485.0	− 1318.7	− 1166.7
4	− 5037.5	+ 266.7	−4770.8	− 2637.5	− 2133.3
6	− 7556.2	+ 900.0	−6556.2	− 3956.2	− 2700.0
8	−10075.0	+2133.3	−7941.7	− 5275.0	− 2666.7
10	−12593.8	+4166.7	−8427.1	− 6593.5	− 1833.6
12	−15112.5	+7200.0	−7912.5	− 7912.5	0.0
x'	$-1618.75x'$	$+\frac{50}{6}(x')^3$	$EI\delta_m'$	$EI\delta_m''$	$EI\delta_m$
6	− 9712.5	+1800.0	−7912.5	− 7912.5	0.0
4	− 6475.0	+ 533.3	−5941.7	− 9231.2	+ 3289.5
2	− 3237.5	+ 66.7	−3170.8	−10549.9	+ 7379.1
0	0.0	0.0	0.0	−11867.7	+11867.6

If the shear distortion had been neglected, the solution would have been

$$\frac{d\theta}{dx} = \frac{M}{EI}; \qquad EI\ R_{ea} = 1200; \qquad EI\ R_{ec} = 1500$$

For $0 < x < 12$ ft.,

$$EI\delta_m' = -1200x + \frac{25}{6}x^3$$

For $0 < x' < 6$ ft.,

$$EI\delta_m' = -1500x' + \frac{50}{6}(x')^3$$

x	$-1200x$	$+\frac{25}{6}x^3$	$EI\delta'_m$	$EI\delta''_m$	$EI\delta_m$	Error
2	− 2400	+ 33.3	−2366.6	−1200	−1166.6	0
4	− 4800	+ 266.7	−4533.3	−2400	−2133.3	0
6	− 7200	+ 900.0	−6300.0	−3600	−2700.0	0
8	− 9600	+2133.3	−7466.7	−4800	−2666.7	0
10	−12000	+4166.7	−7833.3	−6000	−1833.3	0
12	−14400	+7200.0	−7200.0	−7200	0.0	0

For $0 < x' < 6$ ft.,

x'	$-1500x'$	$+\frac{50}{6}x'^3$	$EI\delta'_m$	$EI\delta''_m$	$EI\delta_m$	Error, per cent
6	−9000	+1800.0	−7200.0	− 7200	0.0	0.0
4	−6000	+ 533.3	−5466.7	− 8400	+ 2933.3	−15.9
2	−3000	+ 66.7	−2933.3	− 9600	+ 6666.7	−16.1
0	0	0.0	0.0	−10800	+10800.0	−16.2

26. The Conjugate-beam Method.—For the determination of slopes and deflections in a given beam by the moment-area theorems, it may be convenient to introduce as an expedient, the conjugate beam. The conjugate beam is a fictitious beam of the same length as the given beam, but one which is supported and loaded in such a way that its shear and moment diagrams become identical with the slope and deflection diagrams, respectively, for the given beam.

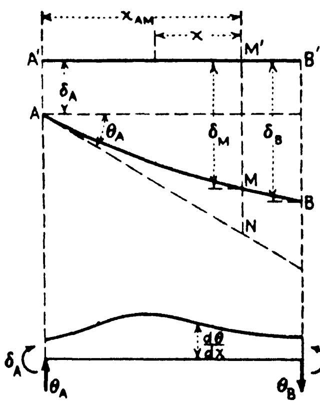


FIG. 40.

Let $A'M'B'$ in Fig. 40 be the unstrained position of a portion of any straight beam, which, owing to any cause whatsoever, deforms to the shape and position shown by the line AMB. Let δ_A and θ_A be the deflection and slope, respectively, of point A, measured relative to the original position of the beam as a base line. Using the moment-area relations, the slope at any point M

in the portion of the beam considered is given by

$$\theta_M = \theta_A - \int_A^M \frac{d\theta}{dx} dx,$$

while the deflection at point M is given by

$$\delta_M = \delta_A + \theta_A(x_{AM}) - \int_A^M x \frac{d\theta}{dx} dx$$

It is evident that if θ_A is considered as the shear in a conjugate beam at point A, and if δ_A is taken as the bending moment at the same point, and if $d\theta/dx$ is considered as a loading, the shear and moment diagrams for this conjugate beam will be identical with diagrams for slope and deflection, respectively, in the given beam, each referred to the unstrained position of the beam. The same relation holds regardless of which side of the section is considered in computing shear and moment. From this it follows that the conjugate beam is in static equilibrium under the elastic loading $d\theta/dx$ and the end slopes and deflections, considered as vertical force and moment-resisting reactions, respectively.

At points where the deflections or the slopes of the given beam are known, the moments and shears of the conjugate beam must be made to assume those definite values; correspondence between zero values of these functions is of particular importance. At special points, such as at an internal hinge in a given beam, the conjugate beam must be such that its shear and moment diagrams correspond to the conditions of slope and deflection known. The choice of proper types of support for the conjugate beam will ensure the fulfillment of these requirements, and at other points on the conjugate beam the desired relations will then exist.

The following procedure may be followed in the selection of supports and the placing of hinges and free ends on the conjugate beam:

1. When the deflection at a point in a given beam is zero, a hinge is introduced in the conjugate beam, so that the moment at that point will be zero.

2. At the end of a given beam, where the deflection may not be zero, a moment-resisting reaction is provided in the conjugate beam, so that moment may occur at this point.

3. When the slope at the end of a given beam is zero, no vertical reaction can occur at the corresponding end of the conjugate beam. The shear in the conjugate beam must be zero at that point.

4. At the end of a given beam where the slope may not be zero, a vertical reaction is provided for the conjugate beam, so that shear may occur at this point.

5. When an intermediate hinge occurs at a point in the given beam, a vertical reaction is introduced at the corresponding

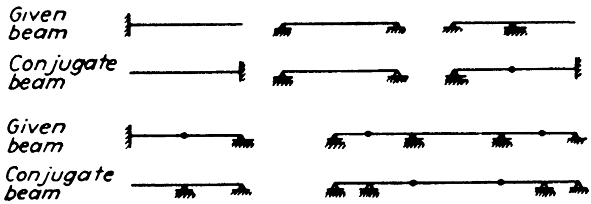


FIG. 41.

point in the conjugate beam. Thus equal deflections but different slopes on the two sides of the hinge in the given beam are shown by equal moments but different shears at corresponding points on the conjugate beam.

Thus a simple end support for a given beam, where slope may exist but deflection is zero, would be reproduced on the conjugate beam as a simple end support, since shear might then have a value, but moment would be zero. A fixed end on a given beam, where both slope and deflection are zero, would be reproduced on the conjugate beam by an end hinge and the absence of a vertical end reaction. This amounts to leaving such an end free, under which condition both shear and moment at the end are zero. A free end on the given beam, where both slope and deflection may occur, may be reproduced on the conjugate beam by a fixed end support, since both shear and moment will then be possible.

Figure 41 illustrates the type of support to be used on conjugate beams for a number of given beams. It will be noted that a reciprocity exists between given and conjugate beams.

Since the conjugate beam is in equilibrium under the elastic loads and the end slopes and deflections considered as reactions, once the condition of support and location of hinges in the conjugate beam have been determined, the reactions may be computed by statics.

As an illustration of the use of the conjugate-beam method, suppose it is desired to compute the slope in terms of EI, at point a on the beam given in Fig. 42, due to the load shown. Distortion due to shear will be neglected in this computation. The conjugate beam, together with the M/EI diagram, which in this case constitutes the elastic loading, is shown below the actual beam. On the conjugate beam, taking moments, about the hinge, of the forces to the left of the hinge,

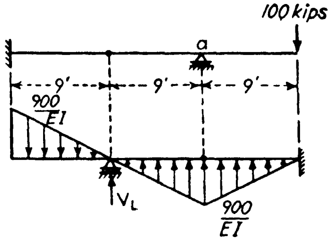


FIG. 42.

$$+9V_L + \frac{900}{EI} \times \frac{9}{2} \times 3 - \frac{900}{EI} \times \frac{9}{2} \times 15 = 0$$

$$V_L = +\frac{5400}{EI}$$

The slope at a equals the shear at a on the conjugate beam, which in this case is equal to

$$-\frac{900}{EI} \times \frac{9}{2} + \frac{5400}{EI} + \frac{900}{EI} \times \frac{9}{2} = +\frac{5400}{EI}$$

In a similar manner, the slope or deflection at any point in the given beam may be obtained by computing shear or moment, respectively, at the corresponding point on the conjugate beam.

CHAPTER III

STRESS ANALYSIS FOR STATICALLY INDETERMINATE STRUCTURES

27. Introduction.—It has been stated already that the methods of solution to be used to determine the stresses in statically indeterminate structures may be classified in two groups. In one group, the order of procedure is to use the redundant stresses or reaction components as the primary variables and to derive the equations which are needed in addition to the available equations of equilibrium as statements of certain necessary requirements with respect to the distortion of the structure. In the other group, a solution is obtained by expressing the stresses as functions of certain characteristics of the distortion, which latter are used as the independent variables, and writing the equations of equilibrium for the structure, thus providing a means of computing the distortion characteristics and, from them, the stresses. The discussion which follows will begin with solutions which fall in the first-mentioned group.

28. The Use of the Law of Virtual Work in the Analysis of Statically Indeterminate Trussed Structures.—It is always possible to solve the problem of stress analysis for a statically indeterminate structure by using the law of virtual work as an expedient to enable one to state certain conditions of distortion which the structure must satisfy. These statements are equations which, together with the available equations of equilibrium, are sufficient to define all the unknown stresses and reaction components. The procedure is best explained by consideration of a particular illustration such as the arch shown in Fig. 43.

It is assumed that this structure and its loads lie in one plane and that the loads are applied at the joints only. If this be so there are, as unknowns to be found, 6 reaction components and 69 bar stresses, 75 in all: the number of joints is 36, therefore there are 72 equations of equilibrium available. It is evident that there are three restraints more than are necessary for stability, or, to put it in another way, that there are three redundant stresses or reaction components. This is usually stated by saying that the structure is statically indeterminate to the third degree.

The 72 equations of equilibrium might be used to express 72 of the unknowns in terms of the remaining three; if, thereafter,

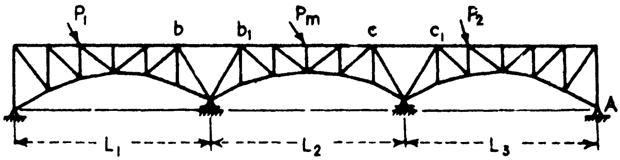


FIG. 43.

some way of computing the last three can be found, the problem is solved. Suppose that the last three unknowns are the horizontal component of the reaction at A, and the stresses in the members bb_1 and cc_1. Let these be designated by X_a, X_b, and X_c, respectively, X_a to be taken as positive when it acts toward the left and the other two to be taken as positive when they are tensions. A means of determining the magnitudes of these three redundants may be found by considering Fig. 44. This structure differs from the one shown in Fig. 43 in three details: The right-hand support has been made movable horizontally and the two members bb_1 and cc_1 have been cut; at the same time there have been added a force X_a applied at A, and acting to the left, two forces X_b aoplied on opposite sides of the section

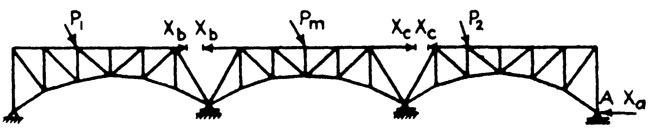


FIG. 44.—Primary structure

through member bb_1, and acting in directions such as to produce tension in the two parts of the member and two similar forces X_c applied on opposite sides of the section through member cc_1. This structure is stable and statically determinate. The redundants must be chosen so that the stresses in the primary structure, due to the known loads, or due to the redundants, can be computed; most frequently it is statically determinate. If the forces X_a, X_b, and X_c are equal, respectively, to the reaction component X_a and the stresses X_b and X_c which remain as the last three unknowns for the structure shown in Fig. 43, the condition of stress in the structure shown in Fig. 44 must be exactly the same as the condition of stress for the structure shown in Fig. 43; the conditions of distortion in the two structures must be alike also. It follows that these three redundants must be such as to cause the structure in Fig. 44 to satisfy the following conditions of distortion: The horizontal displacement of the point of application of the force X_a must be equal to whatever horizontal movement there may be at joint A of the structure in Fig. 43; the points of application of the two forces X_b can have no movement relative to each other nor can the points of application of the two forces X_c have any movement relative to each other. If it is possible to express these three deflections as functions of the known loads, the proportions of the structure and of X_a, X_b and X_c, three equations are available, having as unknowns only the three redundants. It is possible, therefore, to compute the three redundants and, thereafter, all the other unknowns.

In this illustration the law of virtual work will be used to express the deflection characteristics desired in terms of the known loads and the redundants. The stress in any member of the structure shown in Fig. 44, hereafter called the primary structure, may be expressed as the sum of the stresses caused by the loads and redundants separately, *i.e.*,

$$F = F_0 + X_aF_a + X_bF_b + X_cF_c + \text{etc.} \qquad (57)$$

in which F = the total stress in any member.

F_0 = the stress in that member due to the known loads only.

F_a = the stress in that member when the only forces applied to the structure are a unit force applied at A in the direction of X_a, and the reactions caused by this unit force.

F_b = the stress in that member of the structure caused by a pair of unit forces applied at the points of application of the forces X_b, each in the direction of the corresponding force X_b, together with the reactions caused by this pair of unit forces.

F_c = the stress in that member caused by a pair of unit forces applied at the points of application of the forces X_c, each in the direction of the corresponding force X_c, together with the reactions caused by this pair of unit forces.

With this understanding, the coefficients X_a, X_b, and X_c in Eq. (57) become the magnitudes of the forces numerically and are pure numbers.

In applying the law of virtual work to determine the horizontal movement of point A of the primary structure, the Q system to be adopted consists of the unit force applied at A and its reactions as described in the definition for F_a. This condition of loading will be called "condition $X_a = 1$." The stresses in the members due to this Q system are, therefore, the same as those defined by the notation F_a and the equation with respect to a movement of point A becomes

$$\begin{aligned} 1\,\delta_a + W_a &= \Sigma F_a\,\Delta L \\ &= \sum F_a\left(\frac{FL}{AE} + \epsilon\,\Delta t\,L\right) \\ &= \sum \frac{F_aL}{AE}(F_0 + X_aF_a + X_bF_b + X_cF_c) \\ &\qquad + \sum F_a\,\epsilon\,\Delta t\,L \\ &= \sum F_0F_a\frac{L}{AE} + X_a\sum F_a^2\frac{L}{AE} + X_b\sum F_aF_b\frac{L}{AE} \\ &\qquad + X_c\sum F_aF_c\frac{L}{AE} + \sum F_a\,\epsilon\,\Delta t\,L \qquad (58) \end{aligned}$$

in which W_a is the virtual work done by the reactions in condition $X_a = 1$ due to any possible movement of the points of support. No solution will be possible unless such movements are known or can be found by some means not included in this particular problem. It should be emphasized that this equation deals with the distortion of the primary structure.

Similarly, in expressing the relative deflection of the points of application of the two forces X_b, the Q system to be adopted consists of a pair of unit forces acting at the points of application of the forces X_b each in the same direction as the corresponding force X_b together with any reactions due to these unit loads. This condition is the same as that described in defining the stress F_b and will be designated as "condition $X_b = 1$." The bar stresses it causes are the stresses F_b. The relative deflection δ_b of the two points in question is stated by the equation

$$
\begin{aligned}
1\,\delta_b + W_b &= \Sigma F_b\,\Delta L \\
&= \sum F_b\left(\frac{FL}{AE} + \epsilon\,\Delta t\,L\right) \\
&= \sum F_b\frac{L}{AE}(F_0 + X_aF_a + X_bF_b + X_cF_c) \\
&\qquad\qquad + \sum F_b\,\epsilon\,\Delta t\,L \\
&= \sum F_bF_0\frac{L}{AE} + X_a\sum F_bF_a\frac{L}{AE} + X_b\sum F_b^2\frac{L}{AE} \\
&\qquad + X_c\sum F_bF_c\frac{L}{AE} + \sum F_b\,\epsilon\,\Delta t\,L \qquad (59)
\end{aligned}
$$

where W_b is the virtual work done by the reactions in condition $X_b = 1$ during any possible actual yielding of the supports.

In the same way the condition with respect to the relative deflection of the points of application of the forces X_c may be stated as

$$
\begin{aligned}
1\,\delta_c + W_c = \sum F_0F_c\frac{L}{AE} &+ X_a\sum F_cF_a\frac{L}{AE} + X_b\sum F_cF_b\frac{L}{AE} \\
&+ X_c\sum F_c^2\frac{L}{AE} + \sum F_c\,\epsilon\,\Delta t\,L \qquad (60)
\end{aligned}
$$

In these equations δ_a is usually zero; δ_b and δ_c, each being the relative deflection of two points on opposite sides of an imaginary section across a member, must be zero; and if the points of support of the structure do not yield, as is usually assumed to be the case, W_a, W_b, and W_c must be zero also.

There is a physical interpretation for each of the summations in the right-hand sides of Eqs. (58), (59), and (60), which is of considerable interest and is important because of ways in which it may be used. For example, having Eq. (49) in mind, it may be seen that $\sum F_0F_a\frac{L}{AE}$ is the horizontal movement inward of point A of the primary structure when the condition of distortion is that due to "condition $X = 0$." Similarly, $\sum F_a^2\frac{L}{AE}$

is the horizontal movement inward of point A when the condition of distortion is that induced by "condition $X_a = 1$," $\sum F_a F_b \frac{L}{AE}$ is the movement horizontally and inward of point A when the condition of distortion is that due to "condition $X_b = 1$" and might be also interpreted as the relative deflection of the points of application of the forces X_b when the condition of distortion is that due to "condition $X_a = 1$"; there are similar interpretations for the remaining summations. These interpretations suggest that the summations might be designated by δ_{a0}, δ_{aa}, δ_{ab}, etc., in which the first subscript names the point whose deflection is under consideration at the moment and the direction of that deflection, while the second names the cause of the distortion, *i.e.*, some particular unit force or stress acting in a particular direction. Using this notation, Eqs. (58), (59), and (60) may be written in the form

$$\begin{aligned} 1\ \delta_a + W_a &= \delta_{a0} + X_a\delta_{aa} + X_b\delta_{ab} + X_c\delta_{ac} + \delta_{at} \\ 1\ \delta_b + W_b &= \delta_{b0} + X_a\delta_{ba} + X_b\delta_{bb} + X_c\delta_{bc} + \delta_{bt} \\ 1\ \delta_c + W_c &= \delta_{c0} + X_a\delta_{ca} + X_b\delta_{cb} + X_c\delta_{cc} + \delta_{ct} \end{aligned} \qquad (61)$$

The quantities designated by δ in the equations above may be computed in several different ways. If the law of virtual work is used as the expedient, each one is computed as the summation corresponding to it in Eqs. (58), (59), or (60). This requires that, for the structure under consideration, four separate stress analyses of the statically determinate primary structure be made, one to determine the stresses F_0, one to find the stresses F_a, one to find the stresses F_b and one to find the stresses F_c. The analyst should note here that the labor is somewhat reduced because $\delta_{ab} = \delta_{ba}$, $\delta_{ac} = \delta_{ca}$, $\delta_{bc} = \delta_{cb}$; this is to be expected since Maxwell's law is applicable.

Numerical Illustration 1.—Compute the magnitude of the middle reaction for the truss shown in Fig. 45*a*. The numbers written in () on the members are their cross-sectional areas in square inches; the loads are stated in units of 1,000 pounds. There is no yielding of the supports.

Since the structure has 12 joints, there are 24 equations of equilibrium available: there are, as unknowns, 4 reaction components and 21 bar stresses, a total of 25. The structure is, therefore, indeterminate to the first degree. Let the middle reaction be chosen as the redundant. Since there is but one redundant, only one equation of the form given in Eqs. (61) is needed and this reduces to

$$\delta_{a0} + X_a\delta_{aa} = 0$$

which may be written as

$$X_a = -\frac{\delta_{a0}}{\delta_{aa}}$$

in which

$$\delta_{a0} = \sum F_a F_0 \frac{L}{AE}, \qquad \delta_{aa} = \sum F_a^2 \frac{L}{AE}.$$

The primary structure is shown in Fig. 45*b*, condition $X = 0$ and the stress analysis therefor in Fig. 45*c*, and condition $X_a = 1$ with its stress analysis in Fig. 45*d*.

Bar	L	A	$F_0\ 10^{-3}$	F_a	$F_0F_a\frac{L}{A}\ 10^{-3}$	$F_a^2\frac{L}{A}$
1-3-5	60	25	+30	−0.375	− 27.00	+ 0.338
5-7	30	25	+45	−1.125	− 60.75	+ 1.519
2-4-6	60	30	−45	+0.750	− 67.50	+ 1.125
1-2	50	25	−50	+0.625	− 62.50	+ 0.781
2-5	50	20	+25	−0.625	− 39.06	+ 0.976
5-6	50	15	0	+0.625	0.0	+ 1.302
					−256.81	+ 6.041
					2	2
					−513.62	+12.082
6-7	40	20	0	−1.000	0.0	+ 2.000
					−513.62	+14.082

$$X_a = -\frac{-513.62 \times 10^3}{14.082} = +36.47 \times 10^3$$

If under the loads shown the supports yield downward as follows: joint 1, 0.25 in.; joint 7, 0.5 in.; joint 12, 0.375 in.;

the correct equation would be

$$\delta_a + W_a = \delta_{a0} + X_a\delta_{aa}$$

which becomes, numerically

$$-1 \times \frac{0.5}{12} + 0.5 \times \frac{0.25}{12} + 0.5 \times \frac{0.375}{12} = -\frac{513.62}{E} + \frac{14.082}{E}X_a$$

in which E is in units of 1,000 lb. per sq. in. and X_a is in units of

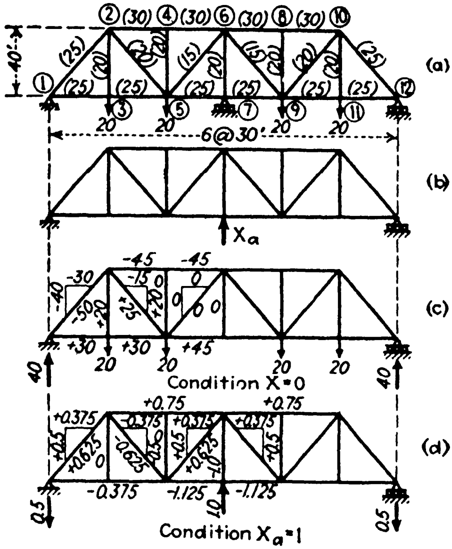


FIG. 45.

1,000 lb. From this

$$X_a = 3.18 \text{ in units of } 1{,}000 \text{ lb.}$$

Numerical Illustration 2.—Compute the stresses in the members of the arch shown in Fig. 46*a* due to the loads shown there and an increase in temperature $\Delta t = +40°$. $\epsilon = 6.5 \times 10^{-6}$;

$$E = 3 \times 10^7 \text{ lb. per sq. in.}$$

The loads are stated in units of 1,000 lb.; the numbers written in parentheses on the members are their cross-sectional areas in square inches.

In this structure there are 12 joints; hence there are 24 equations of equilibrium available. The unknowns are 4 reaction components and 22 bar stresses, 26 in all. Therefore the structure is statically indeterminate to the second degree. Let the horizontal component of the right-hand reaction and the stress in member 5-8 be chosen as the redundants X_a and X_b,

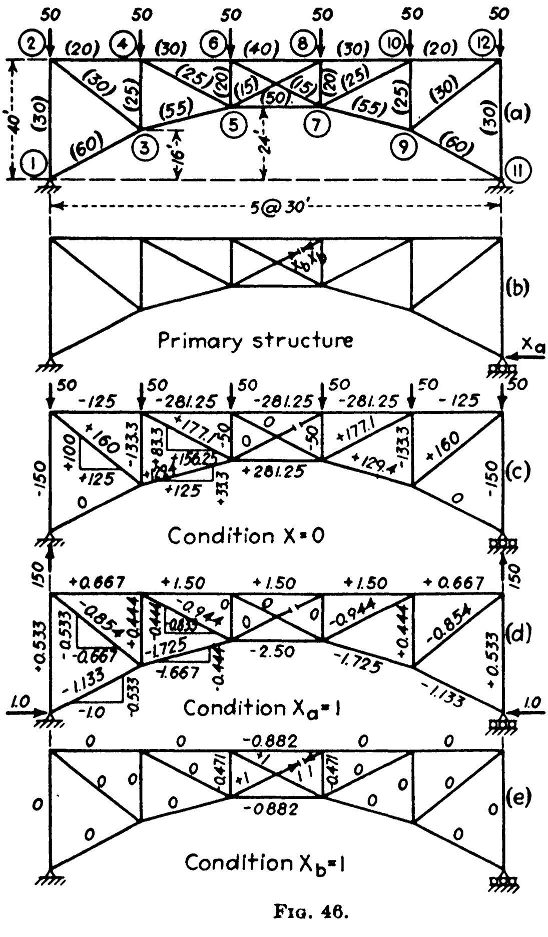


Fig. 46.

respectively. With this choice the primary structure is as shown in Fig. 46*b*. When there are but two redundants, Eqs. (61) become

$$\delta_a + W_a = \delta_{a0} + X_a\delta_{aa} + X_b\delta_{ab} + \delta_{at}$$
$$\delta_b + W_b = \delta_{b0} + X_a\delta_{ba} + X_b\delta_{bb} + \delta_{bt}$$

For this problem δ_b is zero and, if there is no yielding of the supports, as will be assumed here, δ_a, W_a, and W_b will be zero also. The equations may be written as

$$\begin{aligned} X_a\delta_{aa} + X_b\delta_{ab} &= -\delta_{a0} - \delta_{at} \\ X_a\delta_{ba} + X_b\delta_{bb} &= -\delta_{b0} - \delta_{bt} \end{aligned} \tag{62}$$

in which

$$\delta_{a0} = \sum F_a F_0 \frac{L}{AE}; \qquad \delta_{aa} = \sum F_a^2 \frac{L}{AE}; \qquad \delta_{ab} = \sum F_a F_b \frac{L}{AE};$$

$$\delta_{at} = \epsilon\,\Delta t \sum F_a L$$

$$\delta_{b0} = \sum F_b F_0 \frac{L}{AE}; \qquad \delta_{ba} = \sum F_b F_a \frac{L}{AE}; \qquad \delta_{bb} = \sum F_b^2 \frac{L}{AE};$$

$$\delta_{bt} = \epsilon\,\Delta t \sum F_b L$$

Three stress analyses of the primary structure are necessary, one to determine the stresses F_0, one to find the stresses F_a and one for the stresses F_b. These are shown in Fig. 46*c*, 46*d*, and 46*e*, respectively. The summations are carried out in the following table.

Bar	L	A	F_0	F_a	F_b	$F_aF_0\frac{L}{A}$	$F_a^2\frac{L}{A}$	$F_aF_b\frac{L}{A}$	F_aL	$F_bF_0\frac{L}{A}$	$F_b^2\frac{L}{A}$	F_bL
1-3	34 0	60.0	0 0	−1.133	0 0	0.0	+ 0.727	0 0	− 38 5	0.0	0.0	0.0
3-5	31.05	55 0	+129 4	−1 725	0 0	− 126 0	+ 1 680	0.0	− 53 6	0 0	0 0	0.0
2-4	30 0	20 0	− 125 0	+0 667	0.0	− 125.0	+ 0.667	0.0	+ 20 0	0 0	0 0	0.0
4-6	30 0	30 0	− 281 2	− 1 500	0 0	− 422 0	+ 2 250	0.0	+ 45 0	0.0	0 0	0 0
1-2	40.0	30 0	− 150 0	+0 533	0 0	− 106.6	+ 0.379	0.0	+ 21 3	0 0	0 0	0 0
3-4	24.0	25 0	− 133 3	+0 444	0 0	− 57.1	+ 0 190	0 0	+ 10 7	0.0	0.0	0 0
5-6	16 0	20 0	− 50 0	0 0	−0 471	0.0	0 0	0.0	0 0	+ 18 9	+0 177	− 7 54
2-3	38 41	30 0	+160.0	−0.854	0.0	− 174.9	+ 0 933	0.0	− 32 8	0.0	0 0	0 0
4-3	34.0	25.0	+177.1	−0.944	0.0	− 227 6	+ 1 213	0 0	− 32 1	0.0	0 0	0 0
						−1239.2	+ 8.039	0 0	− 60 0	+ 18 9	+0 177	− 7.54
						2	2		2	2	2	2
						−2478 4	+16 078	0 0	−120 0	+ 37.8	+0 354	−15.08
5-7	30.0	50 0	+281.2	−2.50	−0 882	− 422.0	+ 3 750	+1.325	− 75 0	−148.9	+0 467	−26 46
6-8	30 0	40 0	− 281.2	+1.50	−0 882	− 316.2	+ 1 688	−0.992	+ 45 0	+186.1	+0.584	−26 46
5-8	34 0	15.0	0 0	0 0	+1.000	0.0	0 0	0 0	0 0	0.0	+2.269	+34 00
6-7	34 0	15.0	0.0	0 0	+1 000	0 0	0 0	0.0	0 0	0 0	+2 269	+34 00
						−3216.6	+21 516	+0 333	−150.0	+ 75 0	+5 943	0.0

It should be noticed that in this particular structure there is no need of going through the detail of computing the summations $\epsilon\,\Delta t\Sigma F_aL$ and $\epsilon\,\Delta t\Sigma F_bL$. Since the change of temperature is the same for all parts of the structure and since ϵ is the same for all members, the primary structure after the temperature change will have a form which is geometrically similar to its shape before the change of temperature occurred and every dimension will have increased in the ratio $(1 + \epsilon\,\Delta t)$. Among these dimensions is the distance between the points of support which, therefore, will have become $150(1 + \epsilon\,\Delta t)$. Since δ_a is equal to the change in the length of the span of the primary structure and is positive when this length decreases,

$$\delta_{at} = -150\epsilon\,\Delta t$$

By using the same reasoning it may be seen directly that $\delta_{bt} = 0$.

If each side of each equation in Eq. (62) is multiplied by E and units of 1,000 lb. are used, the equations become

$$\begin{aligned} 21.516X_a + 0.333X_b &= +3216.6 \\ &\qquad + 3 \times 10^4 \times 6.5 \times 10^{-6} \times 40(-150) \\ &= +3216.6 + 1170 = +4386.6 \\ 0.333X_a + 5.943X_b &= -75.0 + 0 = -75.0 \end{aligned}$$

The solution of the equations shows that

$$X_a = +204.2 \qquad \text{and} \qquad X_b = -24.06$$

The stress in any member is stated by the expression

$$F = F_0 + X_aF_a + X_bF_b$$

and the evaluation of the stresses is carried out in the table below.

Bar	F_0	F_a	F_b	X_aF_a	X_bF_b	F
1-3	0.0	−1.133	0.0	−231.5	0.0	−231.5
3-5	+129.4	−1.725	0.0	−352.4	0.0	−223.0
5-7	+281.2	−2.500	−0.822	−510.7	+19.78	−209.7
2-4	−125.0	+0.667	0.0	+136.3	0.0	+ 11.3
4-6	−281.2	+1.500	0.0	+306.3	0.0	+ 25.1
6-8	−281.2	+1.500	−0.822	+306.3	+19.78	+ 44.9
1-2	−150.0	+0.533	0.0	+108.9	0.0	− 41.1
3-4	−133.3	+0.444	0.0	+ 90.7	0.0	− 42.6
5-6	− 50.0	0.0	−0.471	0.0	+11.33	− 38.7
2-3	+160.0	−0.854	0.0	−174.4	0.0	− 14.4
4-5	+177.1	−0.944	0.0	−192.8	0.0	− 15.8
6-7	0.0	0.0	+1.000	0.0	−24.06	− 24.1
5-8	0.0	0.0	+1.000	0.0	−24.06	− 24.1

29. The Application of the Law of Virtual Work in the Analysis of Statically Indeterminate Structures with Moment-resisting Joints.—In structures with moment-resisting joints the members are, in general, subjected to bending moment, shear, and axial stress. If such a structure is indeterminate, it is possible to use the same general type of solution as was used in the analysis of the trussed structure considered in the previous article, *i.e.*, to consider an equivalent statically determinate primary structure subjected to the same loads and to certain redundant forces and to set up equations which state that the condition of distortion of the primary structure is the same as that of the original structure. The procedure is best explained by considering a particular structure such as that shown in Fig. 47*a*. Adopt as a primary structure

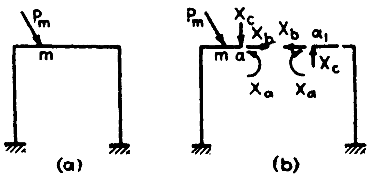


FIG. 47.

that shown in Fig. 47*b*. This is obtained by considering that the original structure is cut by a transverse section through the middle of the girder and that the reciprocal effects of the two parts on each other are replaced by the moments X_a, the axial stresses X_b, and the shearing forces X_c. If a and a_1 are two points on opposite sides of this transverse section, the conditions of distortion of the primary structure which must be satisfied if it is to stimulate exactly the original structure are:

a. There can be no relative rotation of the tangents, at a and a_1, to the elastic curves of the two parts of the girder.

b. There can be no relative horizontal movements of points a and a_1.

c. There can be no relative vertical movement of points a and a_1.

These conditions might be stated more briefly as $\delta_a = 0$, $\delta_b = 0$, $\delta_c = 0$.

The detail of the analysis consists of expressing these conditions in the form of equations and solving the equations. The equations of equilibrium are used to eliminate from the expressions all unknowns except the redundants. Since there are three equations expressing the conditions stated above and since there are three redundants, the latter can be computed and, thereafter, the bending moment, axial stress, and shear at any point in the bent can be computed in the usual manner. There are several ways of setting up the expressions for the relative deflections δ_a, δ_b, and δ_c; in this discussion the law of virtual work will be used. The notation adopted is as follows:

M_0 = the bending moment at any point of the primary structure due to the known loads only; this condition of loading will be called "condition $X = 0$."

F_0 = the axial stress at this point in condition $X = 0$.

S_0 = the transverse shear at this point in condition $X = 0$.

M_a = the bending moment at any point in the primary structure when the only forces acting on it are a pair of unit moments applied at the points of application of the moments X_a, each in the direction of the corresponding moment X_a, and the reactions set up by this pair of unit moments; this condition of loading will be called "condition $X_a = 1$."

F_a = the axial stress at any point in the primary structure in condition $X_a = 1$.

S_a = the shear at this point in condition $X_a = 1$.

M_b, F_b, and S_b are the bending moment, axial stress, and transverse shear, respectively, at any point in the primary structure when the only forces acting on it are a pair of unit forces applied at the points of application of the redundants X_b, each in the same direction as the corresponding force X_b, and the reactions set up by this pair of unit forces. This condition of loading will be called "condition $X_b = 1$."

M_c, F_c, and S_c are the bending moment, axial stress, and shear, respectively, when the only forces acting on the primary structure are a pair of unit forces applied at the points of application of the redundants X_c, each in the direction of the corresponding force X_c, and the reactions caused by this pair of unit forces; this condition of loading will be called "condition $X_c = 1$."

If M, F, and S are the bending moment, axial stress and transverse shear at any point of the primary structure, due to all causes, one may write:

$$\left.\begin{aligned} M &= M_0 + X_aM_a + X_bM_b + X_cM_c \\ F &= F_0 + X_aF_a + X_bF_b + X_cF_c \\ S &= S_0 + X_aS_a + X_bS_b + X_cS_c \end{aligned}\right\} \qquad (63)$$

in which X_a, X_b, and X_c are the numerical magnitudes of the redundants and are pure numbers.

In using the law of virtual work to express the deflection δ_a adopt a Q system which is the same as that which causes the stresses M_a, F_a, and S_a. Under these circumstances, Eq. (19) becomes

$$1\,\delta_a + W_a = \sum\int M_a M\frac{dx}{EI} + \sum\int F_a F\frac{dx}{AE} + \sum\int S_a S\frac{dx}{A'G} + \epsilon\sum\int M_a\frac{\Delta(\Delta t)}{d}dx + \epsilon\sum\int F'_a\Delta t_0\,dx \quad (64)$$

in which W_a is the virtual work done by the reactions in condition $X_a = 1$ if there is any yielding of the supports in the actual condition of distortion. Substituting in Eq. (64) the values of M, F, and S given in Eqs. (63),

$$\begin{aligned}1\,\delta_a + W_a &= \sum\int M_a(M_0 + X_aM_a + X_bM_b + X_cM_c)\frac{dx}{EI}\\ &\quad + \sum\int F_a(F_0 + X_aF_a + X_bF_b + X_cF_c)\frac{dx}{AE}\\ &\quad + \sum\int S_a(S_0 + X_aS_a + X_bS_b + X_cS_c)\frac{dx}{A'G}\\ &\quad + \epsilon\sum\int M_a\frac{\Delta(\Delta t)}{d}dx + \epsilon\sum\int F_a\Delta t_0\,dx\\ &= \sum\left[\int M_aM_0\frac{dx}{EI} + \int F_aF_0\frac{dx}{AE} + \int S_aS_0\frac{dx}{A'G}\right]\\ &\quad + X_a\sum\left[\int M_a^2\frac{dx}{EI} + \int F_a^2\frac{dx}{AE} + \int S_a^2\frac{dx}{A'G}\right]\\ &\quad + X_b\sum\left[\int M_aM_b\frac{dx}{EI} + \int F_aF_b\frac{dx}{AE} + \int S_aS_b\frac{dx}{A'G}\right]\\ &\quad + X_c\sum\left[\int M_aM_c\frac{dx}{EI} + \int F_aF_c\frac{dx}{AE} + \int S_aS_c\frac{dx}{A'G}\right]\\ &\quad + \epsilon\sum\int M_a\frac{\Delta(\Delta t)}{d}dx + \epsilon\sum\int F_a\Delta t_0\,dx \quad (65)\end{aligned}$$

The first term of the right-hand side of Eq. (65) may be seen to be the relative rotation of the tangents, at a and a_1, to the elastic curves of the two parts of the girder in the primary structure due to condition $X = 0$ and hence may be written as δ_{a0}. The subscripts have the same meaning as in Eqs. (61).

Similarly, the coefficient of X_a may be recognized as the relative rotative deflection of the tangents at a and a_1 to the two parts of the elastic curve in the primary structure due to condition $X_a = 1$; it may, therefore, be written as δ_{aa}. In the same way the coefficient of X_b may be written as δ_{ab} and the coefficient of X_c as δ_{ac}, while the last two terms become δ_{at}. Consequently, Eq. (65) may be written as

$$1\ \delta_a + W_a = \delta_{a0} + X_a\delta_{aa} + X_b\delta_{ab} + X_c\delta_{ac} + \delta_{at}$$

In applying the law of virtual work to develop an expression for the relative horizontal deflection of points a and a_1, the Q system to be adopted is the same as that which was defined in describing the stresses M_b, F_b, and S_b and the stresses this Q system causes are M_b, F_b, and S_b; therefore, Eq. (19) becomes

$$1\ \delta_b + W_b = \sum\int M_bM\frac{dx}{EI} + \sum\int F_bF\frac{dx}{AE} + \sum\int S_bS\frac{dx}{A'G} + \epsilon\sum\int M_b\frac{\Delta(\Delta t)}{d}dx + \epsilon\sum\int F_b\Delta t_0\,dx \qquad (66)$$

and if one follows through the same steps as were used in developing Eq. (65), the result is

$$1\ \delta_b + W_b = \delta_{b0} + X_a\delta_{ba} + X_b\delta_{bb} + X_c\delta_{bc} + \delta_{bt}$$

A similar treatment with respect to the relative vertical deflections of points a and a_1 of the primary structure leads to

$$1\ \delta_c + W_c = \delta_{c0} + X_a\delta_{ca} + X_b\delta_{cb} + X_c\delta_{cc} + \delta_{ct}$$

Thus there are three equations of the form

$$\begin{aligned} 1\ \delta_a + W_a &= \delta_{a0} + X_a\delta_{aa} + X_b\delta_{ab} + X_c\delta_{ac} + \delta_{at} \\ 1\ \delta_b + W_b &= \delta_{b0} + X_a\delta_{ba} + X_b\delta_{bb} + X_c\delta_{bc} + \delta_{bt} \\ 1\ \delta_c + W_c &= \delta_{c0} + X_a\delta_{ca} + X_b\delta_{cb} + X_c\delta_{cc} + \delta_{ct} \end{aligned} \qquad (67)$$

which are of exactly the same form as Eqs. (61). In this structure δ_a, δ_b, and δ_c are seen to be zero and if there is no yielding of the supports, W_a, W_b, and W_c must be zero also. If some yielding of the supports occurs and it is not possible to express this yielding as a function of the known loads and the redundants it is necessary to make arbitrary assumptions as to the possible magnitudes of the yieldings and to investigate the effect of these on the stresses in the structure. It might be noted that if the arrangement of the structure were such that no bending moment or shear could occur in any member, Eq. (65) and its companions would be exactly the same as Eqs. (58), (59), and (60) which were developed for trussed structures.

Numerical Example 1.—Compute the magnitude of the middle reaction for the beam in Fig. 48*a* due to the loads shown and to a change of temperature which is the same at all cross sections of the beam but varies linearly from $\Delta t = +40°$F. at the top to $\Delta t = +10°$F. at the bottom. The loads are stated in units of 1,000 lb. The beam is a 24-in. I 79.9 lb.; $I = 2{,}087$ in.^{4}. Assume that the shear is uniformly distributed over the area of the web ($A' = 12$ in.^{2}). $E = 3 \times 10^7$ lb. per sq. in. Compute separately the effects of the loads and the change in temperature.

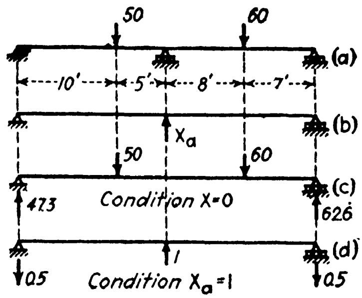


FIG. 48.

This structure is statically indeterminate to the first degree. Let the redundant chosen be the middle reaction. The primary structure is shown in Fig. 48*b*, condition $X = 0$ in Fig. 48*c* and condition $X_a = 1$ in Fig. 48*d*. Since there is but one redundant, Eqs. (67) reduce to

$$1\,\delta_a + W_a = \delta_{a0} + X_a\delta_{aa} + \delta_{at}$$

in which both δ_a and W_a are zero if there is no yielding of the supports. This condition will be assumed here. If the effects of the loads and temperature change be computed separately, X_a being the effect of the loads on the middle reaction while X_{at} is the effect of the change of temperature

$$X_a = -\frac{\delta_{a0}}{\delta_{aa}}, \qquad X_{at} = -\frac{\delta_{at}}{\delta_{aa}}$$

Since there is no axial stress,

$$\delta_{a0} = \sum \int M_a M_0 \frac{dx}{EI} + \sum \int S_a S_0 \frac{dx}{A'G}$$

$$= \frac{1}{EI}\left\{\int_0^{10} (47.3x)(-0.5x)dx\right.$$

$$+ \int_{10}^{15} [47.3x - 50(x - 10)]\,(-0.5x)dx + \int_0^{7} (62.67x)(-0.5x)dx +$$

$$\left.\int_7^{15} [62.67x - 60(x - 7)](-0.5x)dx\right\} + \frac{1}{A'G}\left[\int_0^{10} 47.3(-0.5)dx +\right.$$

$$\int_{10}^{15} (-2.67)(-0.5)dx + \int_0^{8} (-2.67)(+0.5)dx +$$

$$\left.\int_8^{15} (-62.6)(+0.5)dx\right]$$

$$EI\ \delta_{a0} = -45{,}868 + \frac{EI}{A'G}(-460)$$

$$\frac{EI}{A'G} = 2.6 \times \frac{2087 \times 12^{-4}}{12 \times 12^{-2}} = 3.14$$

$$EI\ \delta_{a0} = -45{,}868 - 1{,}444 = -47{,}312$$

$$\delta_{aa} = \sum \int M_a^2 \frac{dx}{EI} + \sum \int S_a^2 \frac{dx}{A'G}$$

$$= \frac{2}{EI}\int_0^{15} (-0.5x)^2 dx + \frac{2}{A'G}\int_0^{15} (-0.5)^2 dx$$

$$EI\ \delta_{aa} = 0.5 \times \frac{15^3}{3} + 0.5\frac{EI}{A'G} \times 15 = 562.5 + 23.6 = 586.1$$

$$X_a = -\frac{EI\ \delta_{a0}}{EI\ \delta_{aa}} = +\frac{47312}{586.1} = +80.70 \text{ in units of 1,000 lb.}$$

If the effect of shear distortion is neglected,

$$EI\ \delta_{a0} = -45868, \qquad EI\ \delta_{aa} = 562.5, \qquad X_a = +81.54$$

The error involved in this approximation is but 1 per cent. This is typical of the magnitude of the error due to omitting the effect of the shear strain in the solution of such problems and it is customary to make this approximation.

To find X_{at},

$$\delta_{at} = \epsilon\frac{\Delta(\Delta t)}{d}\int M_a\,dx$$
$$= \epsilon\frac{(10 - 40)}{2} \times 2\int_0^{15}(-0.5x)dx$$
$$= -30\epsilon(-0.5)112.5$$
$$= +0.010970$$

$$X_{at} = -\frac{EI\ \delta_{at}}{EI\ \delta_{aa}} = -\frac{0.010970 \times 3 \times 10^4 \times 12^2 \times 2087 \times 12^{-4}}{586.1}$$
$$= -8.14 \text{ in units of 1,000 lb.}$$

Neglecting shear distortion,

$$X_{at} = -8.48$$

in which the error due to neglecting shear distortion is four per cent.

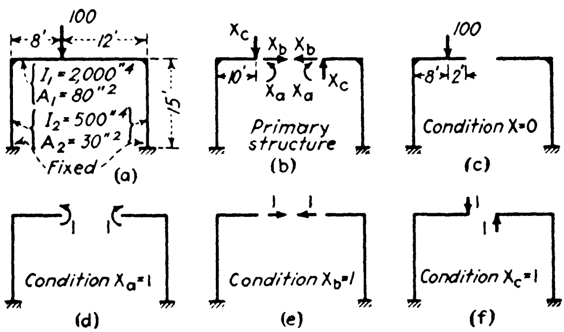


FIG. 49.

Numerical Example 2.—Compute the bending moment, axial stress, and shear at mid-span of the girder in the bent shown in Fig. 49*a* due to the load shown. Neglect the effect of shear distortion.

$$1\ \delta_a + W_a = \delta_{a0} + X_a\,\delta_{aa} + X_b\,\delta_{ab} + X_c\,\delta_{ac}$$
$$1\ \delta_b + W_b = \delta_{b0} + X_a\,\delta_{ba} + X_b\,\delta_{bb} + X_c\,\delta_{bc}$$
$$1\ \delta_c + W_c = \delta_{c0} + X_a\,\delta_{ca} + X_b\,\delta_{cb} + X_c\,\delta_{cc}$$

In this structure $\delta_a = \delta_b = \delta_c = 0$; assuming that the points of support do not yield, $W_a = W_b = W_c = 0$ also.

$$\delta_{a0} = \sum \int M_a M_0 \frac{dx}{EI} + \sum F_a F_0 \frac{L}{AE}$$

$$= \frac{1}{EI_1}\int_2^{10} (+1)[-100(x - 2)]dx + \frac{1}{EI_2}\int_0^{15} (+1)(-800)dx$$

$$= -\frac{100}{EI_1}\left[\frac{10^2 - 2^2}{2} - 16\right] - \frac{800}{EI_2} \times 15 = -\frac{3200}{EI_1} - \frac{12000}{EI_2}$$

$$\delta_{b0} = \sum \int M_b M_0 \frac{dx}{EI} + \sum F_b F_0 \frac{L}{AE} = \frac{1}{EI_2}\int_0^{15} (-x)(-800)dx$$

$$= +\frac{800}{EI_2} \times \frac{15^2}{2} = +\frac{90000}{EI_2}$$

$$\delta_{c0} = \sum \int M_c M_0 \frac{dx}{EI} + \sum F_c F_0 \frac{L}{AE}$$

$$= \frac{1}{EI_1}\int^{10} (-x)[-100(x - 2)]dx$$

$$+ \frac{1}{EI_2}\int_0^{15} (-10)(-800)dx + \frac{15}{A_2E}(-1)(-100)$$

$$= \frac{1}{EI_1}\left[+100\frac{10^3 - 2^3}{3} - 200\frac{10^2 - 2^2}{2}\right] + \frac{8000}{EI_2} \times 15 + \frac{1500}{A_2E}$$

$$= \frac{23467}{EI_1} + \frac{120000}{EI_2} + \frac{1500}{A_2E}$$

$$\delta_{aa} = \sum \int M_a^2 \frac{dx}{EI} + \sum F_a^2 \frac{L}{AE}$$

$$= \frac{2}{EI_1}\int_0^{10} (+1)^2 dx + \frac{2}{EI_2}\int_0^{15} (+1)^2 dx$$

$$= \frac{20}{EI_1} + \frac{30}{EI_2}$$

$$\delta_{ab} = \sum \int M_a M_b \frac{dx}{EI} + \sum F_a F_b \frac{L}{AE} = \frac{2}{EI_2}\int_0^{15} (+1)(-x)dx$$

$$= -\frac{225}{EI_2}$$

$$\delta_{ac} = \sum \int M_a M_c \frac{dx}{EI} + \sum F_a F_c \frac{L}{AE} = \frac{1}{EI_1}\int_0^{10} (+1)(-x)dx$$

$$+ \frac{1}{EI_1}\int_0^{10} (+1)(+x)dx + \frac{1}{EI_2}\int_0^{15} (+1)(-10)dx$$

$$+ \frac{1}{EI_2}\int_0^{15} (+1)(+10)dx = 0$$

$$\delta_{bb} = \sum \int M_b^2 \frac{dx}{EI} + \sum F_b^2 \frac{L}{AE}$$
$$= \frac{2}{EI_2}\int_0^{15} (-x)^2 dx + \frac{2 \times 10}{A_1E}(+1)^2$$
$$= +\frac{2250}{EI_2} + \frac{20}{A_1E}$$
$$\delta_{bc} = \sum \int M_b M_c \frac{dx}{EI} + \sum F_b F_c \frac{L}{AE}$$
$$= \frac{1}{EI_2}\int_0^{15} (-x)(-10)dx + \frac{1}{EI_2}\int_0^{15} (-x)(+10)dx = 0$$
$$\delta_{cc} = \sum \int M_c^2 \frac{dx}{EI} + \sum F_c^2 \frac{L}{AE} = \frac{2}{EI_1}\int_0^{10} x^2\,dx + \frac{2}{EI_2}\int_0^{15} 100dx$$
$$+ \frac{2 \times 15}{A_2E}(1^2) = \frac{2000}{3EI_1} + \frac{3000}{EI_2} + \frac{30}{A_2E}$$

The equations can be handled more conveniently if each side of each equation is multiplied by EI_1 or EI_2 than if they are left in the form given at the beginning of the solution. Multiplying both sides by EI_2 leads to

$$35X_a - 225X_b = +12800$$
$$-225X_a + X_b\left(2250 + 20\frac{500 \times 12^{-4}}{80 \times 12^{-2}}\right) = -90000$$
$$X_c\left(166.7 + 3000 + 30\frac{500 \times 12^{-4}}{30 \times 12^{-2}}\right) = -5866.7 - 120000 - 1500\frac{500 \times 12^{-4}}{30 \times 12^{-2}}$$

From these

$$X_c = -\frac{126040}{3170} = -39.76$$

X_a	X_b	
+ 35 −225	− 225 +2250.9	+12800 −90000
− 22.49	+ 225.0	− 8994
+ 12.51	. . .	+ 3806

$$+10650 - 225X_b = +12800$$
$$X_b = -9.555$$

$$X_a = +304.2$$

If the distortion due to axial stress is omitted in the solution, the result is

$$X_c = -\frac{125867}{3166.7} = -39.74$$

X_a	X_b	
$+35$ -225	-225 $+2250$	$+12800$ -90000
-22.5	$+225$	-9000
$+12.5$		$+3800$

$$+10640 - 225X_b = +12800$$
$$X_b = -9.599$$

$$X_a = +304.0$$

The difference between these results and those obtained before is negligible.

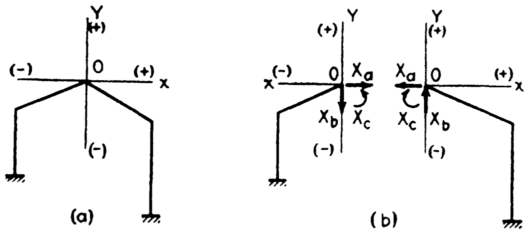


FIG. 50.

30. Use of the Elastic Center.—The labor involved in the solution of a structure such as the bent just considered may be decreased considerably by proper selection of the redundants, particularly if one makes an approximation by omitting from the computation the effects of distortion corresponding to the axial stress and shear. If the redundants could be chosen so that $\delta_{ab} = 0$, $\delta_{ac} = 0$, and $\delta_{bc} = 0$, not only would the necessity for evaluating them disappear, but the three equations which, in the last illustration, were solved simultaneously would become equations each of which contained but one unknown and which could, therefore, be solved independently. To demonstrate how this objective may be attained, consider the bent in Fig. 50*a* in which the location of any point on the axis of the bent is defined by its coördinates measured from a pair of axes OX and OY. As a first selection choose the redundants as shown in

Fig. 50*b*. Neglecting the distortion corresponding to axial stress and shear,

$$\delta_{aa} = \sum \int M_a^2 \frac{ds}{EI} = \sum \int (1\,y)^2 \frac{ds}{EI}$$

in which M_a is positive at all points for which y is positive and is negative at all points for which y is negative; ds is any element of length measured along the axis of a member. If ds/EI is called the elastic weight of the element whose length is ds, the quantity $\sum \int y^2 \frac{ds}{EI}$ may be interpreted as the moment of inertia, about the axis parallel to X_a, of the elastic weight of the structure. Similarly,

$$\delta_{bb} = \sum \int M_b^2 \frac{ds}{EI} = \sum \int x^2 \frac{ds}{EI}$$

which may be interpreted as the moment of inertia, about the axis parallel to X_b, of the elastic weight of the bent;

$$\delta_{cc} = \sum \int M_c^2 \frac{ds}{EI} = \sum \int (+1)^2 \frac{ds}{EI}$$

which may be interpreted as the total elastic weight of the bent;

$$\delta_{ab} = \sum \int M_a M_b \frac{ds}{EI} = \sum \int yx \frac{ds}{EI}$$

which may be interpreted as the product of inertia of the elastic weight of the bent referred to the axes parallel to X_a and X_b, respectively;

$$\delta_{ac} = \sum \int M_a M_c \frac{ds}{EI} = \sum \int y(+1) \frac{ds}{EI}$$

which may be interpreted as the static moment, about an axis parallel to X_a, of the elastic weight of the bent;

$$\delta_{bc} = \sum \int M_b M_c \frac{ds}{EI} = \sum \int x(+1) \frac{ds}{EI}$$

which may be interpreted as the static moment, about an axis parallel to X_b, of the elastic weight of the bent;

$$\delta_{a0} = \sum \int M_a M_0 \frac{ds}{EI} = \sum \int y M_0 \frac{ds}{EI}$$

which is the sum of the static moments, about an axis parallel to X_a, of the areas under the M_0/EI curves for the various members of the structure, each being assumed to be drawn so that the centers of its ordinates lie on the axis of the member;

$$\delta_{b0} = \sum \int M_b M_0 \frac{ds}{EI} = \sum \int x M_0 \frac{ds}{EI}$$

which is the sum of the static moments, about an axis parallel to X_b, of the M_0/EI areas for the members of the structure;

$$\delta_{c0} = \int M_c M_0 \frac{ds}{EI} = \sum \int (+1) M_0 \frac{ds}{EI}$$

which is the sum of the areas under the M_0/EI curves for the members of the structure.

It is evident that, if the origin of coordinates, which is also the point of application of the redundant forces, is placed at the center of gravity of the elastic weight of the structure, the static moment of the elastic weight of the structure about each of the coordinate axes must be zero; consequently, $\delta_{ac} = 0$ and $\delta_{bc} = 0$. If, in addition, the structure has an axis of symmetry, the product of inertia about the coordinate axes must be zero, so that $\delta_{ab} = 0$. When the redundants are selected in this way their effect will be transferred to the actual structure in the same manner which would occur if their points of application were connected to the structure by a pair of rigid arms. If the effect of the redundant forces on the bending moment at any point in the structure is designated by M_r,

$$\begin{aligned} M_r &= X_a y + X_b x + X_c \\ &= -\frac{\delta_{a0}}{\delta_{aa}} y - \frac{\delta_{b0}}{\delta_{bb}} x - \frac{\delta_{c0}}{\delta_{cc}} \\ &= -\frac{(M_s)_{0x} y}{I_x} - \frac{(M_s)_{0y} x}{I_y} - \frac{W_{0e}}{P_e} \end{aligned}$$

in which $(M_s)_{0x}$ is the static moment of the M_0/EI areas about the X axis, $(M_s)_{0y}$ is the static moment of these areas about the Y axis, and W_{0e} is the sum of these areas, while I_x and I_y are the moments of inertia of the elastic weight of the structure about the X axis and Y axis, respectively, and P_e is the total elastic weight. This expression is the same in form as the familiar expression

$$f = \frac{P}{A} + \frac{M_x y}{I_x} + \frac{M_y x}{I_y}$$

which states the stress intensity at any point in the cross section of a column due to the application of an eccentric load. It is evident, therefore, that if the axis of the bent represented the axis of the cross section of a column and the thickness of the column section at any point was 1 *EI* for the actual member of the bent at that point, also if the axial load applied to the

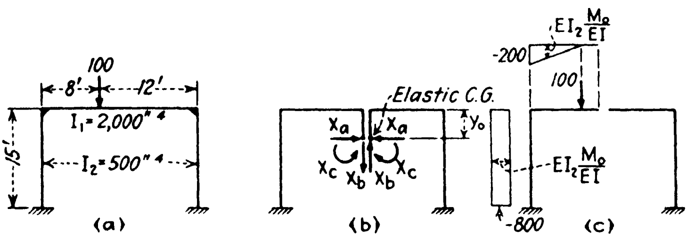


FIG. 51.

column is represented by the M_0/EI curves for the various members of the bent, that the bending moment at any point in the bent due to the redundant forces only can be computed as the stress intensity in the analogous column section just described. To find the total bending moment one must add to the bending moment so computed the value of M_0 at that point. This idea was suggested by Professor Hardy Cross and was called by him the *column analogy.**

As an illustration of the solution when the redundants are selected as suggested above, consider again the bent of Fig. 49, but let the redundants be applied at the center of gravity of the elastic weight of the bent. The primary structure is shown in Fig. 51*b*.

To find the elastic center of gravity

$$EI_2P_e = \frac{1}{4} \times 20 + 2 \times 15 = 35$$

$$35y_0 = 2 \times 15 \times 7.5 = 225; \qquad y_0 = \frac{225}{35} = 6.429'$$

* *Univ. Illinois Eng. Exp. Sta., Bull.* 215; Cross and Morgan's "Continuous Frames of Reinforced Concrete"; "Structural Theory and Design," by Sutherland and Bowman.

$$EI_2\,\delta_{aa} = I_x = \frac{20}{4} \times 6.429^2 + 2 \times 15\left(\frac{15^2}{12} + 1.071^2\right)$$
$$= 206.6 + 596.9 = 803.52$$
$$EI_2\,\delta_{bb} = I_y = \frac{20}{4} \times \frac{20^2}{12} + 2 \times 15 \times 10^2 = 166.6 + 3000 = 3166.7$$
$$EI_2\,\delta_{cc} = 35$$
$$EI_2\,\delta_{a0} = -200 \times \frac{8}{2} \times 6.429 + [-800 \times 15(-1.071)]$$
$$= -5143 + 12857 = +7714$$
$$EI_2\,\delta_{b0} = \left(-200 \times \frac{8}{2}\right)\left[-\left(2 + \frac{2}{3} \times 8\right)\right] + (-800 \times 15)(-10)$$
$$= +5866.7 + 120000 = +125867$$
$$EI_2\,\delta_{c0} = -200 \times \frac{8}{2} - 800 \times 15 = -12800$$
$$X_a = -\frac{EI_2\,\delta_{a0}}{EI_2\,\delta_{aa}} = -\frac{7714}{803.5} = -9.601$$
$$X_b = -\frac{EI_2\,\delta_{b0}}{EI_2\,\delta_{bb}} = -\frac{125867}{3166.7} = -39.74$$
$$X_c = -\frac{EI_2\,\delta_{c0}}{EI_2\,\delta_{cc}} = +\frac{12800}{35} = +365.71$$

At mid-span of the girder
$F = -9.601$
$S = -39.74$
$M = +365.71 - 9.601 \times 6.429 = +365.71 - 61.71 = +304.0$
These results are the same as those obtained by the previous analysis.

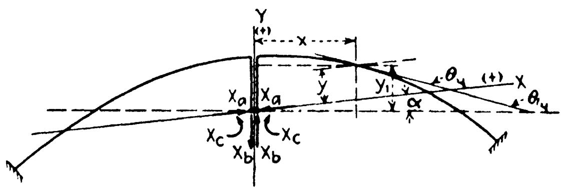


FIG. 52.

If the structure has no axis of symmetry the procedure just explained must, in general, be altered. In considering the necessary changes the effect of distortion corresponding to axial stress will be taken into account. Consider the primary structure shown in Fig. 52. Let the position of the axis of the structure be defined with reference to a pair of coordinate axes of which the Y axis is vertical and the X axis is inclined at an angle α to the horizontal. Let the position of any point (x,y) be defined by its perpendicular distances from the two axes, these distances being positive to the right and upward. Let the slope of the axis of the structure at any point be θ which is measured counter-clockwise from the X axis. The redundants are applied at the origin of coordinates. Since

$$\delta_{ac} = \sum \int M_a M_c \frac{ds}{EI} + \sum \int F_a F_c \frac{ds}{AE}$$

in which $F_c = 0$

$$\delta_{ac} = \sum \int y(+1)\frac{ds}{EI}$$

and will vanish if the X axis passes through the center of gravity of the elastic weight of the structure. Similarly,

$$\delta_{bc} = \sum \int M_b M_c \frac{ds}{EI} + \sum \int F_b F_c \frac{ds}{AE}$$

in which the second term is equal to zero, therefore

$$\delta_{bc} = \sum \int x(+1)\frac{ds}{EI}$$

which vanishes also if the Y axis passes through the elastic center of the structure.

For

$$\begin{aligned}\delta_{ab} &= \sum \int M_a M_b \frac{ds}{EI} + \sum \int F_a F_b \frac{ds}{AE} \\ &= \sum \int yx \frac{ds}{EI} + \sum \int \cos\,(180^\circ - \theta)\,\sin\,\{180 - (\theta + \alpha)\}\frac{ds}{AE} \\ &= \sum \int xy \frac{ds}{EI} + \sum \int -\cos\,\theta\,\sin\,(\theta + \alpha)\frac{ds}{AE}\end{aligned}$$

In order to determine the magnitude of the angle α which will make δ_{ab} vanish, express y and θ in terms of y_1 and θ_1 which are, respectively, the ordinate to the point (x,y) and the slope of the axis of the structure referred to a horizontal axis through the elastic center.

$$y_1 = y\sec\alpha + x\tan\alpha$$

or

$$y = y_1 \cos \alpha - x \sin \alpha$$
$$\theta = \theta_1 - \alpha$$

therefore, for $\delta_{ab} = 0$

$$\sum \int x(y_1 \cos \alpha - x \sin \alpha)\frac{ds}{EI} - \sum \int \cos (\theta_1 - \alpha) \sin \theta_1 \frac{ds}{AE} = 0$$

$$\cos \alpha \sum \int xy_1 \frac{ds}{EI} - \sin \alpha \sum \int x^2 \frac{ds}{EI}$$
$$- \cos \alpha \sum \int \cos \theta_1 \sin \theta_1 \frac{ds}{AE} - \sin \alpha \sum \int \sin^2 \theta_1 \frac{ds}{AE} = 0$$

$$\sum \int xy_1 \frac{ds}{EI} - \sum \int \cos \theta_1 \sin \theta_1 \frac{ds}{AE} = \tan \alpha \left[\sum \int x^2 \frac{ds}{EI} + \sum \int \sin^2 \theta_1 \frac{ds}{AE} \right]$$

$$\tan \alpha = \frac{\sum \int xy_1 \frac{ds}{EI} - \sum \int \sin \theta_1 \cos \theta_1 \frac{ds}{AE}}{\sum \int x^2 \frac{ds}{EI} + \sum \int \sin^2 \theta_1 \frac{ds}{AE}} \qquad (68)$$

If distortion due to axial stress is neglected, the second terms in both numerator and denominator of the right-hand side of Eq. (68) vanish.

It might be pointed out here that Eq. (68) is valid also if y_2 is substituted for y_1 and the ordinates y_2 are measured vertically from the X axis.

There is no advantage to be gained by the use of inclined axes in the solution of bents such as have been considered in the previous illustrations, but the selection of the elastic center as the origin of coordinates and point of application of the redundants will always eliminate some of the labor involved in the solution. In the analysis of hingeless arches, however, particularly when it is desirable to draw influence lines for stresses in the arch, it is expedient to choose the elastic center as the origin of coordinates and the principal axes of the elastic weight of the arch as the axes of coordinates. These principal axes will be perpendicular to each other if the arch is symmetrical but will be inclined to each other at some angle other than 90 deg. if the arch is not symmetrical.

31. Application of Castigliano's Law in the Analysis of Statically Indeterminate Trussed Structures. The Theorem of Least Work.—Articles 28 and 29 deal with a method of analyzing statically indeterminate structures which is based on the investigation of equivalent statically determinate primary structures subjected to the same loads as the original structures and to certain redundant forces, the latter being such as to ensure that the condition of strain and, consequently, the condition of stress in a primary structure should be the same as in the original structure to which it is equivalent. The equations which are needed in addition to the equations of statics were based on the necessity of satisfying certain conditions of distortion in the primary structure; in the articles mentioned the law of virtual work was used as an expedient to enable one to set up expressions for the characteristics used to define the condition of distortion. A second expedient which may be used as a basis for such expressions is provided by Castigliano's law. For example, in the structure of Fig. 43, using the primary structure shown in Fig. 44, the condition that the horizontal movement of point A of the primary structure must be equal to any horizontal yielding that may occur at point A of the original structure may be written as

$$\frac{\partial W_d}{\partial X_a} = \delta_a$$

in which W_d is the strain energy for the primary structure when it is subjected to the given loads and to the redundants, and δ_a is any possible movement inward of point A of the original structure. If such a movement should be outward, δ_a must be taken as negative. In a similar way, the condition that there can be no relative movement, along the line bb_1, of the points of application of the two forces X_b may be written as

$$\frac{\partial W_d}{\partial X_b} = 0$$

and the condition that there can be no relative movement of the points of application of the forces X_c can be written as

$$\frac{\partial W_d}{\partial X_c} = 0$$

If point A does not yield, each of the three partial derivatives is zero; in such circumstances one may make the following statement:

In a statically indeterminate structure, providing that there is no yielding of the supports and no change of temperature, the redundants must be such as to make the strain energy a minimum.

This is known as the theorem of least work.

If these ideas be used in finding the redundant reaction X_a of the truss in Fig. 45*a*, the yielding of the supports being zero, the solution is as follows: Since

$$\frac{\partial W_d}{\partial X_a} = 0 \qquad \text{and} \qquad W_d = \sum \frac{F^2L}{2AE}$$

$$\frac{\partial W_d}{\partial X_a} = \sum \frac{FL}{AE}\frac{\partial F}{\partial X_a} = 0$$

The solution requires only the evaluation of this summation and the solution resulting from equating this summation to zero. The details follow (see Fig. 53).

Bar	L	A	F	$\frac{\partial F}{\partial X_a}$	$\frac{FL}{A}\frac{\partial F}{\partial Xa}$
1-3-5	60	25	$+30 - 0.375X_a$	-0.375	$-27.00 + 0.338X_a$
5-7	30	25	$+45 - 1.125X_a$	-1.125	$-60.75 + 1.519X_a$
2-4-6	60	30	$-45 + 0.750X_a$	$+0.750$	$-67.50 + 1.125X_a$
1-2	50	25	$-50 + 0.625X_a$	$+0.625$	$-62.50 + 0.781X_a$
2-5	50	20	$+25 - 0.625X_a$	-0.625	$-39.06 + 0.976X_a$
5-6	50	15	$+0.625X_a$	$+0.625$	$+1.302X_a$
					$-256.81 + 6.041X_a$ 2
					$-513.62 + 12.082X_a$
6-7	40	20	$-1.000X_a$	-1.000	$+2.000X_a$
					$-513.62 + 14.082X_a$

$$14.082X_a - 513.62 = 0$$
$$X_a = +36.47$$

If the middle point of support yields $\frac{1}{8}$ in. under the loads given, the conditions are not those in which the theorem of least work is valid, but the application of Castigliano's law leads to

$$\sum \frac{FL}{AE} \frac{\partial F}{\partial X_a} = -\frac{1}{8} \text{ in.}$$

the negative sign indicating that the displacement of the point

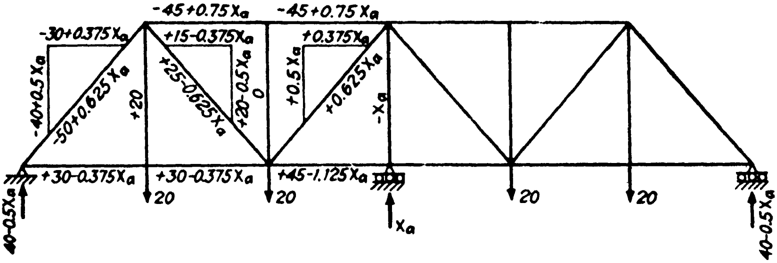


FIG. 53.

of application of X_a is in a direction opposite to that of X_a. For these circumstances,

$$-513.62 + 14.082X_a = -\frac{1}{96} \times E$$

in which E must be written in the same force units as were used for the loads and the same area units as were used for the cross-sectional areas of the members. In this illustration these units are 1,000 lb. per sq. in.; therefore, $E = 3 \times 10^4$ and

$$14.082X_a = +513.62 - \frac{1}{96} \times 30{,}000 = +201.12,$$

$$X_a = +14.22 \text{ in units of 1,000 lb.}$$

As an illustration of the application of Castigliano's law to a trussed structure which is indeterminate to the second degree, consider the structure shown in Fig. 46*a*. If there is no yielding of the supports and no change in temperature,

$$\frac{\partial W_d}{\partial X_a} = 0, \qquad \frac{\partial W_d}{\partial X_b} = 0$$

i.e., the theorem of least work is valid. These two equations become

$$\sum \frac{FL}{AE} \frac{\partial F}{\partial X_a} = 0, \qquad \sum \frac{FL}{AE} \frac{\partial F}{\partial X_b} = 0$$

Bar	L	A	F	$\frac{\partial F}{\partial X_a}$	$\frac{FL}{A}\frac{\partial F}{\partial X_a}$	$\frac{\partial F}{\partial X_b}$	$\frac{FL}{A}\frac{\partial F}{\partial X_b}$
1-3	34.0	60 0	$-1\ 133X_a$	-1.133	$+0.727X_a$		
3-5	31.05	55.0	$+129.4-1.725X_a$	-1.725	$-126.0+1.680X_a$		
2-4	30.0	20.0	$-125.0+0.667X_a$	$+0.667$	$-125.0+0.667X_a$		
4-6	30.0	30.0	$-281.2+1.500X_a$	$+1.500$	$-421.8+2.250X_a$		
1-2	40.0	30.0	$-150.0+0.533X_a$	$+0.533$	$-106.7+0.379X_a$		
3-4	24.0	25.0	$-133.3+0\ 444X_a$	$+0.444$	$-59.6+0.190X_a$		
5-6	16.0	20.0	$-50.0\quad-0.471X_b$			-0.471	$+18.93\quad+0.177X_b$
2-3	38.42	30.0	$+160.1-0.854X_a$	-0.854	$-175\ 0+0.933X_a$		
4-5	34.0	25.0	$+177.1-0\ 944X_a$	$-0\ 944$	$-227\ 5+1.213X_a$		
6-7	34.0	15.0	$+1\ 000X_b$			$+1.000$	$+2.267X_b$
					$-1238\ 9+8\ 039X_a$		$+18.93\quad+2.444X_b$
					2		2
					$-2477\ 8+16.078X_a$		$+37.86\quad+4.888X_b$
6-8	30.0	40 0	$-281.2+1\ 500X_a-0.882X_b$	$+1.500$	$-316.4+1.687X_a-0.993X_b$	-0.882	$+186.1-0.993X_a+0.584X_b$
5-7	30.0	50.0	$+281.2-2.500X_a-0.882X_b$	-2.500	$-421.8+3.750X_a+1.324X_b$	-0.882	$-148.9+1.324X_a+0.467X_b$
					$-3216.0+21.515X_a+0.331X_b$		$+75.1+0.331X_a+5.939X_b$

The stress analysis for the primary structure is shown in Fig. 54. Since the truss and the condition of stress are symmetrical about a vertical mid-axis, only half the structure need be shown.

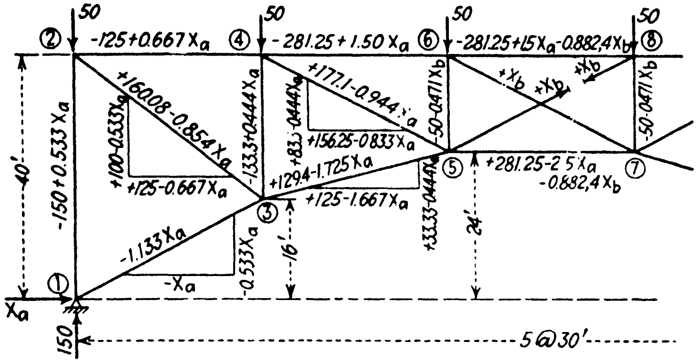


FIG. 54.

X_a	X_b	
21.515 0.331	0.331 5.939	−3216.0 + 75.1
0.331	0.005	− 49.49
	5.934	+ 124.59

$$21.515X_a - 6.9 - 3216.0 = 0$$
$$X_a = +149.8$$

$$X_b = -20.99$$

If it is desired to determine the effect of temperature change on the magnitude of the redundants, Castigliano's law as stated in Art. 15 is not applicable and one must revert to Eq. (39). When the structure is an ideal truss, as this is assumed to be, the last term in the right-hand side of the equation vanishes. If, also, there is no yielding of the supports, the term $\partial W_r/\partial P_m$ is zero and the equation becomes

$$\frac{\partial W_d}{\partial X_a} + \epsilon \sum \int \frac{\partial F}{\partial X_a} \Delta t_0 \, dx = 0$$
$$\frac{\partial W_d}{\partial X_b} + \epsilon \sum \int \frac{\partial F}{\partial X_b} \Delta t_0 \, dx = 0$$

which, for this particular case, become

$$\sum \frac{FL}{A}\frac{\partial F}{\partial X_a} + E\epsilon\, \Delta t_0 \sum \frac{\partial F}{\partial X_a} L = 0$$

$$\sum \frac{FL}{A}\frac{\partial F}{\partial X_b} + E\epsilon\, \Delta t_0 \sum \frac{\partial F}{\partial X_b} L = 0$$

The first term of the left-hand side of each of these equations is the same as the corresponding term in the previous problem except that the numerical term is omitted. The last terms in the left-hand sides are evaluated in the table which follows.

Bar	L	$\frac{\partial F}{\partial X_a}$	$\frac{\partial F}{\partial X_a}L$	$\frac{\partial F}{\partial X_b}$	$\frac{\partial F}{\partial X_b}L$
1-3	34.0	−1.133	− 38.53		
3-5	31.05	−1.725	− 53.56		
2-4	30.0	+0.667	+ 20.0		
4-6	30.0	+1.50	+ 45.0		
1-2	40.0	+0.533	+ 21.33		
3-4	24.0	+0 444	+ 10.67		
5-6	16.0	0.0	0.0	−0.471	− 7.53
2-3	38.42	−0.854	− 32.80		
4-5	34.0	−0.944	− 32.11		
6-7	34.0	0.0	0.0	+1.0	+34.0
			− 60.00		+26.47
			2		2
			−120.00		+52.94
6-8	30.0	+1.50	+ 45.00	−0.882	−26.46
5-7	30.0	−2.50	− 75.00	−0.882	−26.46
			−150.00		+ 0.02

For $\Delta t_0 = +40°F$.

$$E\epsilon\, \Delta t_0 \sum \frac{\partial F}{\partial X_a} L = 3 \times 10^4 \times 6.5 \times 10^{-6} \times 40(-150.0)$$
$$= -1,170 \text{ in units of } 1,000 \text{ lb.};$$
$$E\epsilon\, \Delta t_0 \sum \frac{\partial F}{\partial X_b} L = 0.0$$

Owing to temperature change alone the redundants are defined by the equations

$$21.515X_a + 0.331X_b = +1170$$
$$0.331X_a + 5.939X_b = 0$$

which lead to the solution

$$X_a = +54.41, \qquad X_b = -3.03$$

32. Application of Castigliano's Law to the Analysis of Statically Indeterminate Structures with Moment-resisting Joints.—The analysis of a statically indeterminate structure with moment-resisting joints by the use of Castigliano's law follows the same general procedure as that given in Art. 29, but differs in detail owing to the fact that the expressions for the relative deflections of points in the primary structure are based on Castigliano's law instead of the law of virtual work. The method is best explained by consideration of a particular problem; the illustration used will be the structure shown in Fig. 47*a*; the primary structure and redundants will be as shown in Fig. 47*b*. The conditions of distortion which must be satisfied are:

a. There can be no relative rotation of the tangents, at a and a_1, to the elastic curves of the two sections of the girder; using Castigliano's law as a basis, this condition may be written as

$$\frac{\partial W_d}{\partial X_a} = 0$$

in which

$$W_d = \sum\left[\int \frac{M^2}{2EI}dx + \int \frac{F^2}{2AE}dx + K\int \frac{S^2}{2GA'}dx\right];$$

b. There can be no relative horizontal movement of points a and a_1, which may be expressed as

$$\frac{\partial W_d}{\partial X_b} = 0$$

c. There can be no relative vertical movement of points a and a_1, which may be written as

$$\frac{\partial W_d}{\partial X_c} = 0$$

It is evident that the theorem of least work is applicable to this structure.

The analysis of this structure carried out in Art. 29 demonstrated that there was no appreciable error in the approximate solution in which the distortion corresponding to axial stress and the distortion corresponding to shear were neglected. Such an approximate solution will be used here. Article 30 demonstrated also that some advantage was to be gained by selecting the redundant stresses so that they were applied at the elastic center of the structure. In order to make use of these ideas the primary structure will be as shown in Fig. 55*b*:

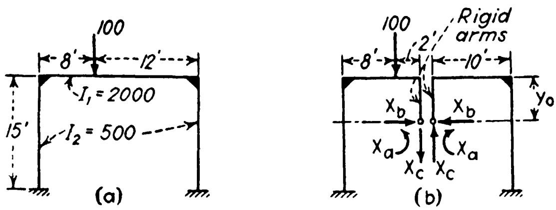


FIG. 55.

$$EI_2P_e = \frac{500}{2000} \times 20 + 2 \times 15 = 35$$

$$35y_0 = 2 \times 15 \times 7.5; \qquad y_0 = \frac{45}{7} = 6.4286 \text{ ft.}$$

The equations are:

$$\frac{\partial W_d}{\partial X_a} = \sum \int \frac{M}{EI} \frac{\partial M}{\partial X_a} dx = 0$$

$$\frac{\partial W_d}{\partial X_b} = \sum \int \frac{M}{EI} \frac{\partial M}{\partial X_b} dx = 0$$

$$\frac{\partial W_d}{\partial X_c} = \sum \int \frac{M}{EI} \frac{\partial M}{\partial X_c} dx = 0$$

If both sides of each equation be multiplied by EI_2, they are as follows:

$$\frac{I_2}{I_1}\left[\int_0^2 [X_a + X_by_0 - X_cx]dx + \int_2^{10} [X_a + X_by_0 - X_cx - 100(x-2)]dx + \int_0^{10} [X_a + X_by_0 + X_cx]dx\right]$$
$$+ \int_{(-15+y_0)}^{+y_0} [X_a + X_by - 10X_c - 800]dy + \int_{(-15+y_0)}^{+y_0} [X_a + X_by + 10X_c]dy = 0$$

$$X_a\left(2\frac{I_2}{I_1} \times 10 + 2 \times 15\right) + X_b\left[2\frac{I_2}{I_1}y_0 \times 10 + 2\frac{y_0^2 - (-15 + y_0)^2}{2}\right] + X_c(0 + 0) - 100\frac{I_2}{I_1}\left(\frac{10^2 - 2^2}{2} - 2 \times 8\right) - 800 \times 15 = 0$$

$$X_a(35) + X_b(5y_0 + 30y_0 - 225) = 25 \times 32 - 12{,}000$$

$$35X_a + X_b\left(35 \times \frac{45}{7} - 225\right) = 12{,}800$$

$$X_a = \frac{12800}{35} = 365.71$$

$$\frac{I_2}{I_1}\left\{\int_0^2 [X_a + X_by_0 - X_cx]y_0\,dx + \int_2^{10} [X_a + X_by_0 - X_cx - 100(x - 2)]y_0\,dx + \int_0^{10} [X_a + X_by_0 + X_cx]y_0\,dx\right\} + \int_{(-15+y_0)}^{y_0} [X_a + X_by - 10X_c - 800]y\,dy + \int_{(-15+y_0)}^{y_0} [X_a + X_by + 10X_c]y\,dy = 0$$

$$X_a\left[2\frac{I_2}{I_1}y_0 \times 10 + 2\frac{y_0^2 - (-15 + y_0)^2}{2}\right] + X_b\left[2\frac{I_2}{I_1}y_0^2 \times 10 + 2\frac{y_0^3 - (-15 + y_0)^3}{3}\right] - 100\frac{I_2}{I_1}\left[y_0\frac{10^2 - 2^2}{2} - 2y_0 \times 8\right] - 800\frac{y_0^2 - (-15 + y_0)^2}{2} = 0$$

$$X_a(5y_0 - 225 + 30y_0) + X_b\left(\frac{5625}{7}\right) = 25 \times 32y_0 + 400(-225 + 30y_0)$$

$$X_b = -\frac{54000}{7} \times \frac{7}{5625} = -9.60$$

$$\frac{I_2}{I_1}\left\{\int_0^2 [X_a + X_by_0 - X_cx](-x)dx + \int_2^{10} [X_a + X_by_0 - X_cx - 100(x - 2)](-x)dx + \int_0^{10} [X_a + X_by_0 + X_cx](+x)dx\right\} + \int_{(-15+y_0)}^{y_0} X_a + X_by - 10X_c - 800](-10)dy + \int_{(-15+y_0)}^{(+y_0} [X_a + X_by + 10X_c](+10)dy = 0$$

$$X_c\left[2\frac{I_2}{I_1} \times \frac{1000}{3} + 2 \times 100 \times 15\right]$$

$$+ 100\frac{I_2}{I_1}\left[\frac{10^3 - 2^3}{3} - 2\frac{10^2 - 2^2}{2}\right] + 8000 \times 15 = 0$$

$$\frac{9500}{3}X_c = -\frac{377600}{3}$$

$$X_c = -39.75$$

33. Members with Restrained Ends.—Let the member AB in Fig. 56 be subjected to transverse loads and to restraining moments M_a and M_b at A and B, respectively. $A'C'B'$ is the elastic curve. Let τ_a and τ_b be the slopes, referred to the chord $A'B'$, of the tangents to the elastic curve at A and B, respectively. The curve of bending moments is shown in Fig. 56*b*. Let any ordinate to the bending moment curve be divided into two parts, first, M_0, which is that part due to the transverse loads only and which would occur if the restraining moments were zero, and second, M', which is that part which would occur if there were no transverse loads and if the member were subjected to the end moments only. The usual conventions with respect to bending moment will be adopted. Shear distortion will be neglected. Using the second moment-area theorem and assuming that the cross section is constant between A and B,

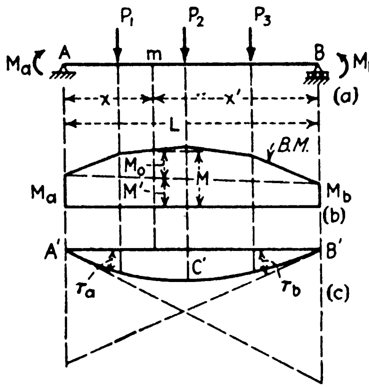


FIG. 56.

$$EIL\tau_a = \int_0^L Mx'\,dx$$

In greater detail

$$EIL\tau_a = M_a \times \frac{L}{2} \times \frac{2}{3}L + M_b \times \frac{L}{2} \times \frac{L}{3} + (M_s)_{0b}$$

$$EIL\tau_b = M_a \times \frac{L}{2} \times \frac{L}{3} + M_b \times \frac{L}{2} \times \frac{2}{3}L + (M_s)_{0a} \quad (69)$$

in which $(M_s)_{0b}$ is the static moment, about an axis through B, of the area under the M_0 curve and $(M_s)_{0a}$ is the corresponding static moment about an axis through A. These may be written as

$$\tau_a = \frac{L}{6EI}(2M_a + M_b) + \frac{(M_s)_{0b}}{EIL}$$
$$\tau_b = \frac{L}{6EI}(M_a + 2M_b) + \frac{(M_s)_{0a}}{EIL} \qquad (70)$$

If these equations are solved simultaneously for M_a and M_b, the result is

$$M_a = \frac{2EI}{L}(2\tau_a - \tau_b) + \frac{2}{L^2}[(M_s)_{0a} - 2(M_s)_{0b}]$$
$$M_b = \frac{2EI}{L}(-\tau_a + 2\tau_b) + \frac{2}{L^2}[-2(M_s)_{0a} + (M_s)_{0b}] \qquad (71)$$

So far, the condition of loading has not been defined and Eqs. (70) and (71) are valid for any condition of loading. It is worth while to evaluate the last terms of the right-hand sides of Eqs. (71) for some of the more common conditions. For example, if the member is subjected to a distributed transverse load of uniform intensity p continuous over the whole of the length, the last terms are as follows (see Fig. 57):

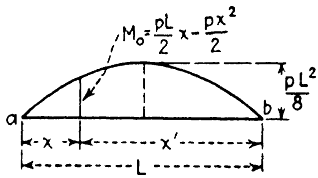


FIG. 57.

$$(M_s)_{0a} = (M_s)_{0b} = \frac{pL^2}{8} \times \frac{2}{3}L \times \frac{L}{2} = \frac{pL^4}{24}$$
$$\frac{2}{L^2}[(M_s)_{0a} - 2(M_s)_{0b}] = -\frac{pL^2}{12}$$
$$\frac{2}{L_a}[-2(M_s)_{0a} + (M_s)_{0b}] = -\frac{pL^2}{12}$$

For a single concentrated load (see Fig. 58),

$$(M_s)_{0b} = \frac{Pab}{L}\left[\frac{b}{2} \times \frac{2}{3}b + \frac{a}{2}\left(b + \frac{a}{3}\right)\right]$$
$$= \frac{Pab}{6L}(2b^2 + 3ab + a^2)$$
$$= \frac{Pab}{6L}(2b + a)(b + a)$$
$$= \frac{Pab}{6}(2b + a)$$

$$(M_s)_{0a} = \frac{Pab}{6}(2a + b)$$

$$\frac{2}{L^2}[(M_s)_{0a} - 2(M_s)_{0b}] = -\frac{Pab^2}{L^2};$$

$$\frac{2}{L^2}[-2(M_s)_{0a} + (M_s)_{0b}] = -\frac{Pa^2b}{L^2}.$$

If the member carries a number of concentrated loads P_1, P_2, P_3, whose distances from the left and right ends are a_1 and

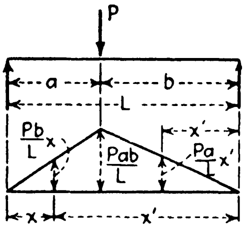


FIG. 58.

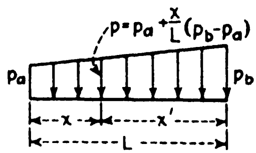


FIG. 59.

b_1, a_2 and b_2, a_3 and b_3, etc., the load terms are

$$\frac{2}{L^2}[(M_s)_{0a} - 2(M_s)_{0b}] = -\sum\frac{Pab^2}{L^2}$$

$$\frac{2}{L^2}[-2(M_s)_{0a} + (M_s)_{0b}] = -\sum\frac{Pa^2b}{L^2}.$$

If the load is continuous and varies uniformly from p_a at A to p_b at B (see Fig. 59),

$$R_a = \frac{1}{L}\left[p_a\frac{L}{2} \times \frac{2}{3}L + p_b\frac{L}{2} \times \frac{L}{3}\right] = \frac{L}{6}(2p_a + p_b)$$

$$M_0 = \frac{L}{6}(2p_a + p_b)x - p_a\frac{x}{2} \times \frac{2}{3}x - \left[p_a + \frac{x}{L}(p_b - p_a)\right]\frac{x^2}{6}$$

$$= p_a\left(\frac{Lx}{3} - \frac{x^2}{3} - \frac{x^2}{6} + \frac{x^3}{6L}\right) + p_b\left(\frac{Lx}{6} - \frac{x^3}{6L}\right)$$

$$(M_s)_{0a} = \int_0^L M_0 x\,dx$$

$$= \int_0^L\left[p_a\left(\frac{Lx^2}{3} - \frac{x^3}{2} + \frac{x^4}{6L}\right) + p_b\left(\frac{Lx^2}{6} - \frac{x^4}{6L}\right)\right]dx$$

$$= p_a\left(\frac{L^4}{9} - \frac{L^4}{8} + \frac{L^4}{30}\right) + p_b\left(\frac{L^4}{18} - \frac{L^4}{30}\right)$$

$$= \frac{L^4}{360}(7p_a + 8p_b)$$

$$(M_s)_{0b} = \frac{L^4}{360}(8p_a + 7p_b)$$

$$\frac{2}{L^2}[(M_s)_{0a} - 2(M_s)_{0b}] = -\frac{L^2}{180}(9p_a + 6p_b)$$
$$= -\frac{L^2}{60}(3p_a + 2p_b)$$

$$\frac{2}{L^2}[-2(M_s)_{0a} + (M_s)_{0b}] = -\frac{L^2}{60}(2p_a + 3p_b)$$

If there are no transverse loads but the ends of the member rotate owing to some cause or other, such as might occur if this member were connected by moment-resisting joints to other members which are loaded and are forced to rotate at the ends, Eqs. (70) become

$$\begin{aligned} \tau_a &= \frac{L}{6EI}(2M_a + M_b) \\ \tau_b &= \frac{L}{6EI}(M_a + 2M_b) \end{aligned} \tag{72}$$

and Eqs. (71) become

$$\begin{aligned} M_a &= \frac{2EI}{L}(2\tau_a - \tau_b) \\ M_b &= \frac{2EI}{L}(-\tau_a + 2\tau_b) \end{aligned} \tag{73}$$

These equations were used in the Winkler variation of the solution suggested by Manderla for the problem of secondary-stress analysis in trusses.* When used for such problems it is convenient to use conventions with respect to positive and negative moment and to positive and negative τ which are different from those used heretofore. If the conventions adopted are:

1. Moment applied to the end of a member is positive when counterclockwise;

2. The angle τ is positive when the tangent at the end of a member has rotated counterclockwise from the chord of the elastic curve;

* See *Trans.* A.S.C.E., vol. 88, 1925; also Johnson, Bryan, Tourneaure, "Modern Framed Structures," Part II.

Eqs. (73) become

$$M_a = \frac{2EI}{L}(2\tau_a + \tau_b)$$
$$M_b = \frac{2EI}{L}(\tau_a + 2\tau_b) \qquad (74)$$

In Eqs. (70) to (74) the end moments are defined in terms of the transverse loads and the rotations of the end tangents with respect to the chord of the elastic curve. The equations

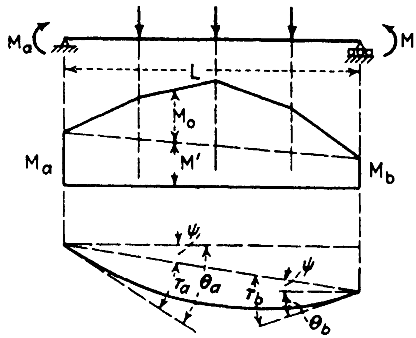


FIG. 60.

can be used to advantage in the stress analysis of structures if these chords do not rotate or, when they do rotate, if one can find some way to evaluate such rotation before using the equations. If this last is not possible, a more convenient pair of equations is available. In these equations the end moments are stated as functions of the transverse loads and of the rotations of the end tangents with respect to the original position of the member. Such equations may be developed by a consideration of Fig. 60. The member shown here is the same as that shown in Fig. 56 except that the chord of the elastic curve has rotated through the angle ψ. It is evident that

$$\tau_a = \theta_a - \psi$$
$$\tau_b = \theta_b + \psi$$

therefore Eqs. (71) may be written in the form

$$M_a = \frac{2EI}{L}(2\theta_a - 2\psi - \theta_b - \psi) + \frac{2}{L^2}[(M_s)_{0a} - 2(M_s)_{0b}]$$
$$M_b = \frac{2EI}{L}(-\theta_a + \psi + 2\theta_b + 2\psi) + \frac{2}{L^2}[-2(M_s)_{0a} + (M_s)_{0b}] \qquad (75)$$

These equations can be used more conveniently if the conventions adopted are, in general, similar to those adopted in Eqs. (74). Those usually adopted are:

1. Moments applied to the ends of members are positive when clockwise.

2. The angle θ is positive when the tangent to the elastic curve has rotated in a clockwise direction from its original position.

3. The angle ψ is positive when the chord of the elastic curve has rotated in a clockwise direction from its original position.

When Eqs. (75) are rewritten to suit these conventions, they become

$$\begin{aligned} M_a &= \frac{2EI}{L}(2\theta_a + \theta_b - 3\psi) + \frac{2}{L^2}[(M_s)_{0a} - 2(M_s)_{0b}] \\ M_b &= \frac{2EI}{L}(\theta_a + 2\theta_b - 3\psi) + \frac{2}{L^2}[2(M_s)_{0a} - (M_s)_{0b}] \end{aligned} \tag{76}$$

These are known as the "slope-deflection" equations.

Numerical Illustration.—Draw the curve of bending moments for the beam shown in Fig. 61. In computing the redundants neglect the distortion corresponding to axial stress and shear.

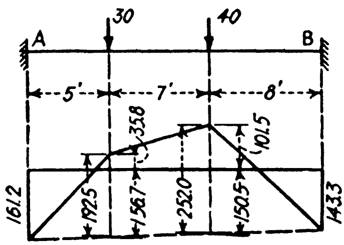


FIG. 61.

Equations (71) may be used to determine the fixing moments at A and B. Since the ends are fixed,

$$\tau_a = \tau_b = 0$$

therefore

$$M_a = \frac{2}{L^2}[(M_s)_{0a} - 2(M_s)_{0b}] = -\sum\frac{Pab^2}{L^2}$$

$$M_b = \frac{2}{L^2}[-2(M_s)_{0a} + (M_s)_{0b}] = -\sum\frac{Pa^2b}{L^2}$$

which, for this beam, become

$$M_a = -\frac{1}{400}(30 \times 5 \times 15^2 + 40 \times 12 \times 8^2) = -161.18$$

$$M_b = -\frac{1}{400}(30 \times 5^2 \times 15 + 40 \times 12^2 \times 8) = -143.32$$

The curve of bending moments may be obtained by superimposing the M_0 curve on the curve of bending moments due to M_a and M_b alone.

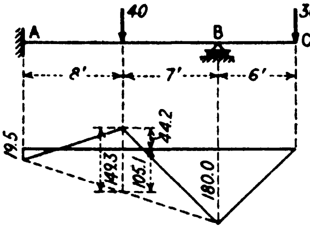


FIG. 62.

Numerical Illustration.—Draw the curve of bending moments for the beam shown in Fig. 62. Neglect the distortion corresponding to axial stress and shear. By statics, $M_B = -180;\ \tau_A = 0.$

Using Eqs. (71), for the section AB, therefore,

$$\frac{2EI}{L}(2\tau_B) - \frac{40 \times 8^2 \times 7}{15^2} = -180.0$$

$$\frac{2EI}{L}\tau_B = \frac{1}{2}(+79.64 - 180.0) = -50.18$$

$$M_A = \frac{2EI}{L}(-\tau_B) - \frac{40 \times 8 \times 7^2}{15^2}$$

$$= +50.18 - 69.69 = -19.51$$

An alternative solution for either of these problems could be based on the use of Eqs. (76). For example, in the second illustration, $\theta_A = \tau_A = 0$, $\theta_B = \tau_B$, $\psi = 0$, and the two sets of algebraic equations lead to numerical equations which are exactly alike.

A second alternative solution might be based on the use of the conjugate beam. For example, the conjugate beam corresponding to the beam of Fig. 62 is as shown in Fig. 63 and the elastic-load curve, being the M/EI curve for the beam, is completely defined except for the ordinate M_A/EI. Since the beam is fixed at point A the slope at that point is zero, therefore the elastic shear at the corresponding point of the conjugate beam is zero. This condition may be stated in the form of the equation

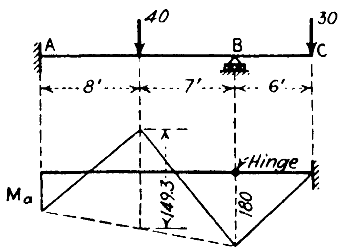


FIG. 63.

$$-M_A \times \frac{15}{2} \times \frac{2}{3} \times 15 - 180 \times \frac{15}{2} \times \frac{15}{3} + 149.3\left[\frac{8}{2}\left(7 + \frac{8}{3}\right) + \frac{7}{2} \times \frac{2}{3} \times 7\right] = 0$$

which leads to

$$75M_A = -6750 + 8213 = +1463$$
$$M_A = +19.51$$

the positive sign indicating that M_A has the character assumed in the sketch of the elastic-load diagram of Fig. 63.

34. Beams Continuous over More than One Span: The Equation of Three Moments.—Beams which are continuous over more than one span may, in most cases, be analyzed by using Eqs. (70) or (71). If these equations are used, the procedure is to select the bending moments at the points of support as the redundants and to use Eqs. (70) or (71) as a means of stating the conditions

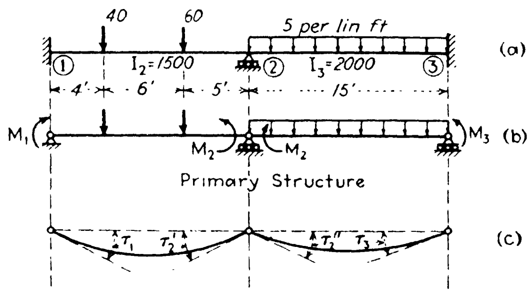


FIG. 64.

of distortion which must be satisfied if the primary structure is to behave in the same way as the original structure. To illustrate consider the beam in Fig. 64*a*. The loads are stated in units of 1,000 lb. The conditions of distortion which must be satisfied if the primary structure is to behave in the same way as the original structure are:

$$\tau_1 = 0, \qquad \tau_2' + \tau_2'' = 0, \qquad \tau_3 = 0$$

these conditions being true only if the points of support do not yield. Using Eqs. (70) these conditions lead to

$$\frac{15}{6EI_2}(2M_1 + M_2) + \frac{40 \times 4 \times 11}{6}(22 + 4)\frac{1}{15EI_2} + \frac{60 \times 10 \times 5}{6} \times (10 + 10)\frac{1}{15EI_2} = 0$$

$$\frac{15}{6EI_2}(M_1 + 2M_2) + \frac{40 \times 4 \times 11}{6}(8 + 11)\frac{1}{15EI_2} + \frac{60 \times 10 \times 5}{6}(20 + 5)\frac{1}{15EI_2} + \frac{15}{6EI_3}(2M_2 + M_3) + \frac{5 \times 15^4}{24 \times 15EI_3} = 0$$

$$\frac{15}{6EI_3}(M_2 + 2M_3) + \frac{5 \times 15^4}{24 \times 15EI_3} = 0$$

which may be written as

$$2M_1 + M_2 = -470.04$$
$$M_1 + 3.5M_2 + 0.75M_3 = -692.89$$
$$M_2 + 2M_3 = -281.25$$

The solution of these equations is

$$M_1 = -167.90; \qquad M_2 = -134.25; \qquad M_3 = -73.50$$

Another method of solving this problem is based on the equation obtained by combining the expressions found by

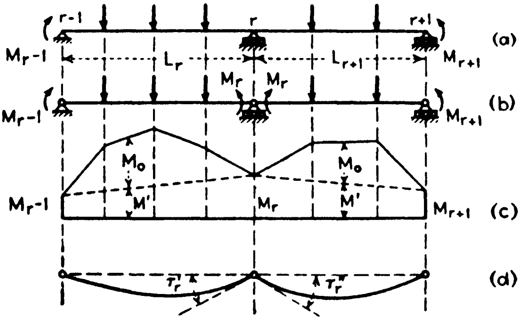


Fig. 65.

applying Eqs. (70) at a support common to two adjacent spans. Consider, for example, the two spans shown in Fig. 65*a*; these are any two adjacent spans selected from a longer series; the beam is continuous over the supports. The moments M_{r-1} and M_{r+1} are end moments which are applied by the spans adjacent to the pair shown or by supports at points $(r - 1)$ and $(r + 1)$ which supply restraint against rotation. Let a primary structure be selected as shown in Fig. 65*b*. The usual conventions for bending moment will be adopted. Let the end moments be positive when they tend to cause positive bending moment. If the end moments are assumed to be positive, the curves of bending moment for the two spans will be as shown in Fig. 65c. In these curves M' is that part of the bending moment which would occur if the primary structure were subjected to the end moments only, while M_0 is that part which would occur if nothing but the transverse loads acted on the primary structure. The condition of distortion which must be satisfied if the primary structure is to behave in the same way as the original structure is that the tangents at the inner ends of the two parts of the elastic curve must lie in the same straight line, or, algebraically,

$$\tau_r' + \tau_r'' = 0$$

If τ_r' and τ_r'' are expressed as functions of the end moments and transverse loads by using Eqs. (70), this equation becomes

$$\frac{L_r}{6EI_r}(M_{r-1} + 2M_r) + \frac{(M_s)_{0(r-1)}}{EI_rL_r} + \frac{L_{r+1}}{6EI_{r+1}}(2M_r + M_{r+1}) + \frac{(M_s)_{0(r+1)}}{EI_{r+1}L_{r+1}} = 0$$

which may be written as

$$M_{r-1}\frac{L_r}{I_r} + 2M_r\left(\frac{L_r}{I_r} + \frac{L_{r+1}}{I_{r+1}}\right) + M_{r+1}\frac{L_{r+1}}{I_{r+1}} = -\frac{6(M_s)_{0(r-1)}}{I_rL_r} - \frac{6(M_s)_{0(r+1)}}{I_{r+1}L_{r+1}} \quad (77)$$

This is known as the equation of three moments. In applying this equation it should be remembered that the equation

$$\tau_r' + \tau_r'' = 0$$

infers that there has been no change in the relative positions of the points $r - 1$, r, and $r + 1$. If a change in the relative positions of these points does occur, it is possible to modify Eq. (77) so that it will be valid under such circumstances. With this end in view consider Fig. 66. Let the points $r - 1$, r, and $r + 1$ move downward through the distances δ_{r-1}, δ_r, and δ_{r+1}, respectively. Since the tangents at r to the two parts of the elastic curve must be colinear,

$$\tau_r' + \tau_r'' - \frac{\delta_r - \delta_{r-1}}{L_r} - \frac{\delta_r - \delta_{r+1}}{L_{r+1}} = 0$$

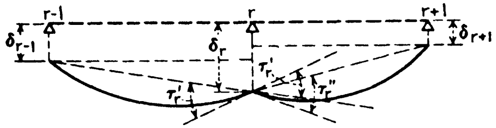


FIG. 66.

which may be written as

$$M_{r-1}\frac{L_r}{I_r} + 2M_r\left(\frac{L_r}{I_r} + \frac{L_{r+1}}{I_{r+1}}\right) + M_{r+1}\frac{L_{r+1}}{I_{r+1}} = -\frac{6(M_s)_{0(r-1)}}{I_rL_r}$$

$$-\frac{6(M_s)_{0(r+1)}}{I_{r+1}L_{r+1}} - \frac{6E\delta_{r-1}}{L_r} + 6E\delta_r\left(\frac{1}{L_r} + \frac{1}{L_{r+1}}\right) - \frac{6E\delta_{r+1}}{L_{r+1}} \quad (78)$$

To illustrate the application of these equations consider again the beam of Fig. 64, assuming first that there is no yielding of the supports. With this assumption, Eq. (77) is applicable. Since the beam is statically indeterminate to the third degree, one must find three equations in addition to the equations of statics. These three equations are found by writing the equation of three moments three times, once for $r = 1$, once for $r = 2$ and for $r = 3$. The fixed ends at points 1 and 3 are accounted for by considering that the part of the beam which extends into the support and through which the rotative restraint is applied is a span of zero length. If the supports of such a span do not move, the effect is the same as fixing the direction of the axis of the beam at the point of support. The detail of the computation is

$$2M_1\left(\frac{15}{I_2}\right) + M_2\left(\frac{15}{I_2}\right) = -\frac{40 \times 4 \times 11}{15I_2}(4 + 22)$$

$$\frac{60 \times 10 \times 5}{15I_2}(10 + 10)$$

$$M_1\left(\frac{15}{I_2}\right) + 2M_2\left(\frac{15}{I_2} + \frac{15}{I_3}\right) + M_3\left(\frac{15}{I_3}\right) = -\frac{40 \times 4 \times 11}{15I_2}(8 + 11)$$

$$-\frac{60 \times 10 \times 5}{15I_2}(20 + 5) - \frac{5 \times 15^4}{4 \times I_3 \times 15}$$

$$M_2\left(\frac{15}{I_3}\right) + 2M_3\left(\frac{15}{I_3}\right) = -\frac{5 \times 15^4}{4 \times I_3 \times 15}$$

M_1	M_2	M_3	
2.0	1.0		−470.04
1.0	3.5	0.75	−602.89
	1.0	2.0	−281.25

These equations are the same as those developed by using Eq. (70) and the solution need not be carried farther.

If, in the same continuous beam, the middle support should yield ¼ in. under these loads, Eq. (77) is not valid but Eq. (78) may be used in the same way as Eq. (77) was used in the previous illustration. The solution is

$$2M_1\left(\frac{15}{I_2}\right) + M_2\left(\frac{15}{I_2}\right) = -\frac{1760 \times 26 + 60000}{15I_2} - \frac{6E \times 1}{48 \times 15}$$

$$M_1\left(\frac{15}{I_2}\right) + 2M_2\left(\frac{15}{I_2} + \frac{15}{I_3}\right) + M_3\left(\frac{15}{I_3}\right) = -\frac{1760 \times 19 + 75000}{15I_2} - \frac{5 \times 15^3}{4I_3} + 6E \times \frac{1}{48}\left(\frac{1}{15} + \frac{1}{15}\right)$$

$$M_2\left(\frac{15}{I_3}\right) + 2M_3\left(\frac{15}{I_3}\right) = -\frac{5 \times 15^3}{4I_3} - \frac{6E}{48 \times 15}$$

which may be solved as follows:

M_1	M_2	M_3	
2.0	1.0		−643.65
1.0	3.5	0.75	−345.67
	1.0	2.0	−512.73
1.0	0.5		−321.82
	3.0	0.75	− 23.85
	1.0	2.0	−512.73
	1.0	0.25	− 7.95
		1.75	−504.78

$$2.0M_1 + 64.16 = -643.65$$

$$M_1 = -353.90$$

$$3.0M_2 - 216.33 = -23.85$$

$$M_2 = +64.16$$

$$M_3 = -288.45$$

35. Application of the Slope-deflection Equations.—It was pointed out in Art. 33 that the problems solved there could have been handled equally well by using Eqs. (75) or (76) instead of Eqs. (70) and (71), and that the details of the computation would not have been changed by a new choice of equations. There are structures, however, particularly those with moment-resisting joints, in which there are rotations of the chords of the elastic curves of some or all of the members while the structure is attaining equilibrium. The analysis of such structures can be carried out more conveniently by using the slope-deflection equations than by applying Eqs. (70) or (71), and may be illustrated by a consideration of the structure shown in Fig. 67. As demonstrated in Art. 29, there will be no appreciable error in the analysis of this structure if the distortion corresponding to axial stress and to shear is neglected; consequently, a solution based on the use of Eqs. (76) is a close approximation to a precise solution. A good procedure is to write the slope-deflection equations for each of the members, then to write certain of the equations of equilibrium for parts of the structure, and to use the slope-deflection equations to eliminate from these equations of equilibrium all unknowns except those measuring the distortion, *i.e.*, the unknown angles θ and ψ. These are determined by solving the equations of equilibrium simultaneously, after which the moments at the ends of the members are found by substituting in the slope-deflection equations the newly found values for the angles θ and ψ. For this particular illustration the slope-deflection equations are

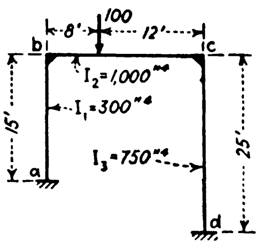


FIG. 67.

$$M_{ab} = 2EK_{ab}(2\theta_a + \theta_b - 3\psi_{ab}) \qquad \text{where} \qquad K = \frac{I}{L}$$

$$M_{ba} = 2EK_{ab}(\theta_a + 2\theta_b - 3\psi_{ab})$$

$$M_{bc} = 2EK_{bc}(2\theta_b + \theta_c - 3\psi_{bc}) - \frac{100 \times 8 \times 12^2}{20^2}$$

$$M_{cb} = 2EK_{bc}(\theta_b + 2\theta_c - 3\psi_{bc}) + \frac{100 \times 8^2 \times 12}{20^2}$$

$$M_{cd} = 2EK_{cd}(2\theta_c + \theta_d - 3\psi_{cd})$$

$$M_{dc} = 2EK_{cd}(\theta_c + 2\theta_d - 3\psi_{cd}) \qquad (a)$$

Since the columns are fixed at a and d, respectively, $\theta_a = \theta_d = 0$. If the changes in the lengths of the members are neglected, as suggested on p. 142, there can be no change in the relative elevations of joints b and c; consequently $\psi_{bc} = 0$. Further, if member bc is assumed to have no change in length, points b and c must have equal horizontal deflections; therefore

$$15\psi_{ab} = 25\psi_{cd}, \qquad \text{or} \qquad \psi_{ab} = \frac{5}{3}\psi_{cd}$$

This leaves, as independent unknowns, the moments at the ends of the members and θ_b, θ_c, ψ_{cd}. These angles may be found

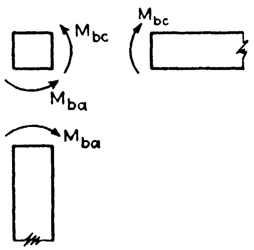


FIG. 68.

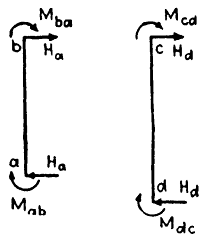


FIG. 69.

directly by writing three equations of the form $\Sigma M = 0$, each for some part of the structure and using the slope-deflection equations to eliminate from them all unknown moments. For joint b, $\Sigma M_b = 0$, therefore

$$M_{ba} + M_{bc} = 0 \qquad (b)$$

Similarly $\Sigma M = 0$ for joint c leads to

$$M_{cb} + M_{cd} = 0 \qquad (c)$$

A third equation may be obtained by writing $\Sigma M = 0$ for each of the columns (see Fig. 69).

$$M_{ab} + M_{ba} + 15H_a = 0$$
$$M_{cd} + M_{dc} + 25H_d = 0$$

$$\frac{1}{15}(M_{ab} + M_{ba}) + \frac{1}{25}(M_{cd} + M_{dc}) + H_a + H_d = 0$$

But, from $\Sigma H = 0$ for the whole structure,

$$H_a + H_d = 0$$

Therefore

$$5M_{ab} + 5M_{ba} + 3M_{cd} + 3M_{dc} = 0 \qquad (d)$$

Substituting in Eqs. (b), (c), and (d) the values of the moments in terms of the angles θ and ψ as given in Eqs. (a) leads to

$$2E\theta_b(2K_{ab} + 2K_{bc}) + 2E\theta_c(K_{bc}) - 6E \times \frac{5}{3}\psi_{cd}\, K_{ab} - 288 = 0$$

$$2E\theta_b(K_{bc}) + 2E\theta_c(2K_{bc} + 2K_{cd}) - 6E\psi_{cd}K_{cd} + 192 = 0$$

$$2E\theta_b(5K_{ab} + 10K_{ab}) - 6E\psi_{cd}\left(5 \times 2 \times \frac{5}{3}K_{ab}\right) + 2E\theta_c(9K_{cd})$$
$$- 6E\psi_{cd}(2K_{cd} \times 3) = 0$$

or, in tabular form,

$2E\theta_b$	$2E\theta_c$	$6E\psi_{cd}$	$= 0$
$2K_{ab} + 2K_{bc}$	K_{bc}	$-\frac{5}{3}K_{ab}$	-288
K_{bc}	$2K_{bc} + 2K_{cd}$	$-K_{cd}$	$+192$
$15K_{ab}$	$9K_{cd}$	$-\frac{50}{3}K_{ab} - 6K_{cd}$	0

$$K_{ab} \times 12^4 = \frac{300}{15} = 20; \qquad K_{bc} \times 12^4 = \frac{1000}{20} = 50;$$
$$K_{cd} \times 12^4 = \frac{750}{25} = 30$$

and the equations become

$2E\theta_b$	$2E\theta_c$	$6E\psi_{cd}$	Num $\times 12^{-4} = 0$
140	50	− 33.3	−288
50	160	− 30.0	+192
100	90	−171.1	0

The solution of these equations is

$2E\theta_b$	$2E\theta_c$	$6E\psi_{cd}$	
140	50	−33.3	−288.0
50	160	−30.0	+192.0
10	9	−17.1	0.0
50	17.8571	−11.9047	−102.8571
10	3.5714	− 2.3809	− 20.5714
	142.1429	−18.0953	+294.8571
	5.4286	−14.7302	+ 20.5714
	5.4286	− 0.6911	+ 11.2609
		−14.0391	+ 9.3105

$$6E\psi_{cd} = +0.66318(12^4)$$
$$142.1429(2E\theta_c) - 12.0001 + 294.8571 = 0$$
$$2E\theta_c = -1.9899(12^4)$$
$$140(2E\theta_b) - 99.496 - 22.106 - 288 = 0$$
$$2E\theta_b = +2.9257(12^4)$$

$$\begin{aligned}
M_{ab} &= 20(2.9257 - 1.1053) &&= + \quad 36.408 \\
M_{ba} &= 20(5.8514 - 1.1053) &&= + \quad 94.922 \\
M_{bc} &= 50(5.8514 - 1.9899) - 288.0 &&= - \quad 94.925 \\
M_{cb} &= 50(2.9257 - 3.9798) + 192.0 &&= + \quad 139.295 \\
M_{cd} &= 30(-3.9798 - 0.6632) &&= - \quad 139.290 \\
M_{dc} &= 30(-1.9899 - 0.6632) &&= - \quad 79.593
\end{aligned}$$

These moments, together with the shears and direct stresses which correspond, are shown in Fig. 70.

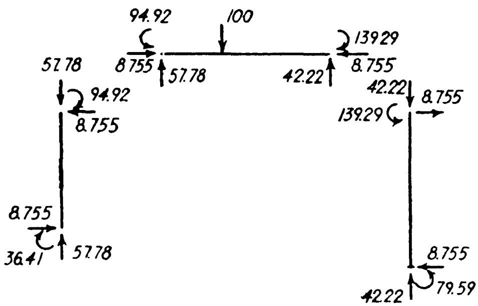


FIG. 70.

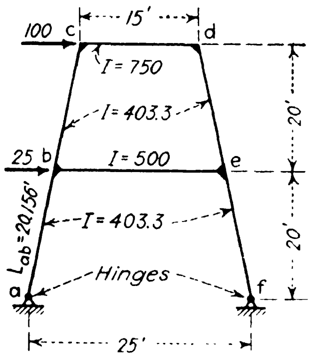


FIG. 71.

As a second illustration consider the bent in Fig. 71. The solution follows the same general procedure as in the previous illustration, though there are differences in certain details. The slope-deflection equations are

$$\begin{aligned}
M_{ab} &= 2EK_{ab}(2\theta_a + \theta_b - 3\psi_{ab}) \\
M_{ba} &= 2EK_{ab}(\theta_a + 2\theta_b - 3\psi_{ab}) \\
M_{be} &= 2EK_{be}(2\theta_b + \theta_e - 3\psi_{be}) \\
M_{bc} &= 2EK_{bc}(2\theta_b + \theta_c - 3\psi_{bc}) \\
M_{cb} &= 2EK_{bc}(\theta_b + 2\theta_c - 3\psi_{bc}) \\
M_{cd} &= 2EK_{cd}(2\theta_c + \theta_d - 3\psi_{cd}) \\
M_{dc} &= 2EK_{cd}(\theta_c + 2\theta_d - 3\psi_{cd}) \\
M_{de} &= 2EK_{de}(2\theta_d + \theta_e - 3\psi_{de}) \\
M_{ed} &= 2EK_{de}(\theta_d + 2\theta_e - 3\psi_{de})
\end{aligned}$$

$$M_{eb} = 2EK_{eb}(2\theta_e + \theta_b - 3\psi_{be})$$
$$M_{ef} = 2EK_{ef}(2\theta_e + \theta_f - 3\psi_{ef})$$
$$M_{fe} = 2EK_{ef}(\theta_e + 2\theta_f - 3\psi_{ef})$$

in which M_{ab} and M_{fe} are zero. If the changes in the lengths of the members be neglected, the relations between the various

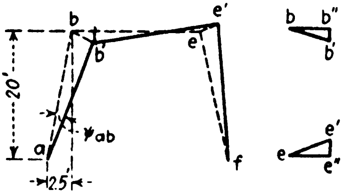


FIG. 72.

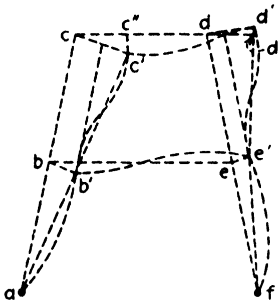


FIG. 73.

angles ψ are found to be as follows:

$$bb' = 20.16\psi_{ab}$$
$$bb'' = 20.16\psi_{ab} \times \frac{20}{20.16} = 20\psi_{ab}$$
$$b''b' = 20.16\psi_{ab} \times \frac{2.5}{20.16} = 2.5\psi_{ab}$$
$$ee'' = 20.16\psi_{fe} \times \frac{20}{20.16} = bb''$$

Therefore

$$20\psi_{fe} = 20\psi_{ab}$$
$$\psi_{fe} = \psi_{ab}$$
$$e''e' = 20.16\psi_{fe} \times \frac{2.5}{20.16} = 2.5\psi_{ab}$$
$$\psi_{be} = -\frac{1}{20}(b''b' + e''e') = -0.25\psi_{ab}$$
$$cc'' = bb'' + 20.16\psi_{bc} \times \frac{20}{20.16}$$
$$= 20(\psi_{ab} + \psi_{bc})$$
$$dd'' = cc'' = ee'' + 20.16\psi_{de} \times \frac{20}{20.16}$$
$$= 20(\psi_{ab} + \psi_{de})$$

Therefore

$$\psi_{de} = \psi_{bc}$$

$$\psi_{cd} = -\frac{1}{15}(c'c'' + d'd'')$$
$$= -\frac{1}{15}\left(b'b'' + 20.16\psi_{bc}\frac{2.5}{20.16} + e'e'' + 20.16\psi_{de} \times \frac{2.5}{20.16}\right)$$
$$= -\frac{1}{15}(2.5\psi_{ab} + 2.5\psi_{bc} + 2.5\psi_{ab} + 2.5\psi_{bc})$$
$$= -\frac{1}{3}\psi_{ab} - \frac{1}{3}\psi_{bc}$$

The unknown independent angles are θ_a, θ_b, θ_c, θ_d, θ_e, θ_f, ψ_{ab}, and ψ_{bc}. The equations of equilibrium which are available for the solution are $M_{ab} = 0$; $M_{fe} = 0$; $M_{ba} + M_{be} + M_{bc} = 0$;

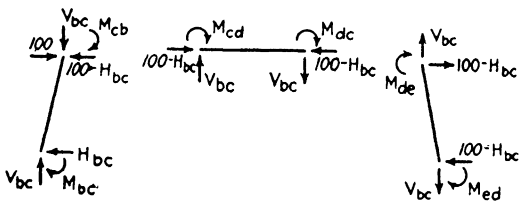


FIG. 74.

$M_{cb} + M_{cd} = 0$; $M_{dc} + M_{de} = 0$; $M_{ed} + M_{eb} + M_{cf} = 0$, with two more which may be derived from the conditions of equilibrium for the various members. For the legs of the top story of this bent (see Fig. 74)

$$2.5V_{bc} + 20H_{bc} + M_{bc} + M_{cb} = 0$$
$$2.5V_{bc} + 20(100 - H_{bc}) + M_{de} + M_{ed} = 0$$

$$M_{bc} + M_{cb} + M_{de} + M_{ed} + 5V_{bc} + 2000 = 0$$

also, for the girder *cd*

$$M_{cd} + M_{dc} + 15V_{bc} = 0$$

which may be written

$$-\frac{1}{3}M_{cb} - \frac{1}{3}M_{de} + 5V_{bc} = 0$$

therefore

$$M_{bc} + \frac{4}{3}M_{cb} + \frac{4}{3}M_{de} + M_{ed} + 2000 = 0 \qquad (a)$$

For the bottom story (see Fig. 75)

$$M_{ba} + 20H_{ab} + 2.5V_{ab} = 0$$
$$M_{ef} + 20(125 - H_{ab}) + 2.5V_{ab} = 0$$

$$M_{ba} + M_{ef} + 2500 + 5.0V_{ab} = 0$$

$$M_{be} + M_{eb} + (V_{ab} - V_{bc})20 = 0; \qquad M_{ba} + M_{bc} + M_{be} = 0$$
$$M_{ed} + M_{ef} + M_{eb} = 0$$

$$-\frac{1}{4}M_{ba} - \frac{1}{4}M_{bc} - \frac{1}{4}M_{ed} - \frac{1}{4}M_{ef} + 5.0V_{ab} - \frac{1}{3}M_{cb} - \frac{1}{3}M_{de} = 0$$

$$\frac{5}{4}M_{ba} + \frac{1}{4}M_{bc} + \frac{1}{4}M_{ed} + \frac{5}{4}M_{ef} + \frac{1}{3}M_{cb} + \frac{1}{3}M_{de} + 2500 = 0 \quad (b)$$

Expressing Eqs. (*a*) and (*b*) and the equations $\Sigma M = 0$ in terms of the distortion constants

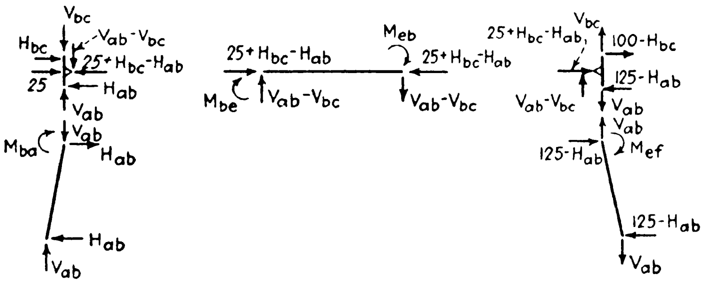


FIG. 75.

$$2EK_{ab}(2\theta_a + \theta_b - 3\psi_{ab}) = 0$$
$$2EK_{ef}(\theta_e + 2\theta_f - 3\psi_{ab}) = 0$$

$$2E\theta_a K_{ab} + 2E\theta_b(2K_{ab} + 2K_{be} + 2K_{bc}) + 2E\theta_c K_{bc} + 2E\theta_e K_{be} - 6E\psi_{ab}\left(K_{ab} - \frac{1}{4}K_{be}\right) - 6E\psi_{bc} K_{bc} = 0$$

$$2E\theta_b K_{bc} + 2E\theta_c(2K_{bc} + 2K_{cd}) + 2E\theta_d K_{cd} - 6E\psi_{bc}\left(K_{bc} - \frac{1}{3}K_{cd}\right) - 6E\psi_{ab}\left(-\frac{1}{3}K_{cd}\right) = 0$$

$$2E\theta_c K_{cd} + 2E\theta_d(2K_{cd} + 2K_{de}) + 2E\theta_e K_{de} - 6E\psi_{bc}\left(K_{de} - \frac{1}{3}K_{cd}\right) - 6E\psi_{ab}\left(-\frac{1}{3}K_{cd}\right) = 0$$

$$2E\theta_d K_{de} + 2E\theta_e(2K_{de} + 2K_{eb} + 2K_{ef}) + 2E\theta_b K_{eb} + 2E\theta_f K_{ef} - 6E\psi_{bc}(K_{de}) - 6E\psi_{ab}\left(K_{ef} - \frac{1}{4}K_{eb}\right) = 0$$

$$2E\theta_b\left(\frac{10}{3}K_{bc}\right) + 2E\theta_c\left(\frac{11}{3}K_{bc}\right) + 2E\theta_d\left(\frac{11}{3}K_{de}\right) + 2E\theta_e\left(\frac{10}{3}K_{de}\right)$$

$$-6E\psi_{bc}\left(\frac{7}{3}K_{bc} + \frac{7}{3}K_{de}\right) + 2000 = 0$$

$$2E\theta_a\left(\frac{5}{4}K_{ab}\right) + 2E\theta_b\left(\frac{10}{4}K_{ab} + \frac{1}{2}K_{bc} + \frac{1}{3}K_{bc}\right) + 2E\theta_c\left(\frac{1}{4}K_{bc}\right.$$

$$\left. + \frac{2}{3}K_{bc}\right) + 2E\theta_d\left(\frac{1}{4}K_{de} + \frac{2}{3}K_{de}\right) + 2E\theta_e\left(\frac{1}{2}K_{de} + \frac{10}{4}K_{ef} + \frac{1}{3}K_{de}\right)$$

$$-6E\psi_{ab}\left(\frac{5}{4}K_{ab} + \frac{5}{4}K_{ef}\right) + 2E\theta_f\left(\frac{5}{4}K_{ef}\right)$$

$$-6E\psi_{bc}\left(\frac{1}{4}K_{bc} + \frac{1}{4}K_{de} + \frac{1}{3}K_{bc} + \frac{1}{3}K_{de}\right) + 2500 = 0$$

$$K_{ab} = K_{ef} = K_{bc} = K_{de} = \frac{403.3}{20.165 \times 12^4} = 20 \times 12^{-4}$$

$$K_{be} = \frac{500}{20 \times 12^4} = 25 \times 12^{-4}$$

$$K_{cd} = \frac{750}{15 \times 12^4} = 50 \times 12^{-4}$$

The equations become

$2E\theta_a$	$2E\theta_b$	$2E\theta_c$	$2E\theta_d$	$2E\theta_e$	$2E\theta_f$	$6E\psi_{ab}$	$6E\psi_{bc}$	Num $\times 12^{-4}$
40.0	20.0					−20 0		0 0
20 0	130.0	20.0		25 0		−13 75	−20.0	0.0
25.0	$66.\dot{6}$	18.3	$18.\dot{3}$	$66.\dot{6}$	25	−50 0	$-23.\dot{3}$	+2500 0
	20.0	140.0	50.0			$+16.\dot{6}$	− 3.3	0.0
	25.0		20.0	130.0	20 0	−13.75	−20.0	0.0
	$66.\dot{6}$	$73.\dot{3}$	$73.\dot{3}$	$66.\dot{6}$			−93.3	+2000.0
		50.0	140.0	20 0		$+16.\dot{6}$	$-3.\dot{3}$	0 0
				20.0	40.0	−20.0		0 0
20.0	10.0					−10 0		0.0
25.0	12.5					−12.5		0.0
	120.0	20.0		25.0		− 3.75	−20.0	0 0
	$54.1\dot{6}$	$18.\dot{3}$	$18.\dot{3}$	$66.\dot{6}$	25.0	−37.5	$-23.\dot{3}$	+2500.0
	20.0	140.0	50.0			$+16.\dot{6}$	$-3\ \dot{3}$	0 0
	25.0		20.0	130.0	20.0	−13.75	−20.0	0.0
	$66\ \dot{6}$	$73.\dot{3}$	$73.\dot{3}$	$66\ \dot{6}$			−93 3	+2000 0
	$54.1\dot{6}$	9.028		11.285		− 1.693	− 9.028	0.0
	20.0	$3.33\dot{3}$		$4.16\dot{6}$		− 0.625	$-3.33\dot{3}$	0.0
	25.0	$4.16\dot{6}$		$5.208\dot{3}$		− 0.7812	$-4.16\dot{6}$	0.0
	$66.\dot{6}$	$11.11\dot{1}$		$13.888\dot{8}$		$-2.08\dot{3}$	$-11.11\dot{1}$	0.0
		9.305	$18.\dot{3}$	55.3816	25.0	−35.807	−14.305	+2500.0
		$136.66\dot{6}$	50.0	$-4.166\dot{6}$		$+17.291\dot{6}$	0.0	0.0
		−4.166	20.0	$124.791\dot{6}$	20.0	−12.9688	$-15.83\dot{3}$	0.0
		62.222	$73.\dot{3}$	$52.777\dot{7}$		$+2.08\dot{3}$	$-82.22\dot{2}$	+2000.0
		50.0	140.0	20.0		$+16.\dot{6}$	$-3.\dot{3}$	0.0
		9.305	3.4045	−0.2838		+ 1.1773	0.0	0.0
		4.16	1.525	−0.1271		+ 0.5271	0.0	0.0
		62.222	22.762	−1.8978		+ 7.873	0.0	0.0
		50.0	18.292	−1.5248		+ 6.326	0.0	0.0

$2E\theta_a$	$2E\theta_b$	$2E\theta_c$	$2E\theta_d$	$2E\theta_e$	$2E\theta_f$	$6E\psi_{ab}$	$6E\psi_{bc}$	Num × 12^{-4}
			14.9288	55.6654	25.0	−36.9843	−14.305	+2500.0
			21.525	124.6645	20.0	−12.4417	−15.833	0.0
			50.571	54.6755		− 5.790	$-82.22\dot{2}$	+2000.0
			121.708	21.5248		+10.340	$-3.\dot{3}$	0.0
			14.9288	2.6413		+ 1.2687	− 0.4089	0.0
			21.525	3.8075		+ 1.8292	− 0.5895	0.0
			50.571	8.9440		+ 4.296	− 1.3847	0.0
				53.0241	25.0	−38.253	−13.8961	+2500.0
				120.857	20.0	−14.2709	−15.2435	0.0
				45.7315		−10.086	−80.8375	+2000.0
				20.0	40.0	−20.0		0.0
				53 0241	25.0	−38.253	−13.8961	+2500.0
				120.857	20.0	−14.2709	−15.2435	0.0
				45.7315		−10.086	−80.8375	+2000.0
				20.0	40.0	−20.0		0.0
				53.0241	8.786	− 6.260	− 6.688	0.0
				45.7315	7.568	− 5.399	− 5.768	0.0
				20.0	3.309	− 2.361	− 2.522	0.0
					16 214	−31.993	− 7 208	+2500.0
					−7.568	− 4.687	−75.0695	+2000.0
					36 691	−17.639	+ 2.522	0.0
					16 214	− 7.797	+ 1.1146	0 0
					7 568	− 3.638	+ 0.5201	0.0
						−24.196	− 8 3226	+2500.0
						− 8.325	−74.5494	+2000.0
						− 8 325	− 2.864	+ 860.2
							−71 6854	+1139 8

$-24\ 196(6E\psi_{ab}) - 132.37 + 2500.0 = 0$
$36.691(2E\theta_f) - 1726.0 + 40.11 = 0$
$120.857(2E\theta_e) + 919.2 - 1396.2 - 242.35 = 0$
$121.708(2E\theta_d) + 128.10 + 1011.9 - 53.0 = 0$
$136.6(2E\theta_c) - 446.6 - 24.8 + 1691.8 = 0$
$120.0(2E\theta_b) - 178.64 + 148.75 - 366.9 - 318.04 = 0$
$40.0(2E\theta_a) + 119.14 - 1957.2 = 0$

$6E\psi_{bc} = +15.902$
$6E\psi_{ab} = +97.86$
$2E\theta_f = +45.96$
$2E\theta_e = +5.95$
$2E\theta_d = -8.932$
$2E\theta_c = -8.932$
$2E\theta_b = +5.957$
$2E\theta_a = +45.95$

$M_{ab} = 20(+91.90 + 5.96 - 97.86) = 0$
$M_{ba} = 20(+45.95 + 11.91 - 97.86) = -800.0$
$M_{be} = 25(+11.91 + 5.95 + 24.46) = +1058.3$
$M_{bc} = 20(+11.91 - 8.93 - 15.90) = -258.4$
$M_{cb} = 20(+5.95 - 17.864 - 15.902) = -556.2$
$M_{cd} = 50(-17.864 - 8.932 + 37.921) = +556.25$
$M_{dc} = 50(-8.932 - 17.864 + 37.921) = +556.25$
$M_{de} = 20(-17.864 + 5.95 - 15.902) = -556.2$
$M_{ed} = 20(-8.932 + 11.90 - 15.902) = -258.4$
$M_{eb} = 25(+11.90 + 5.95 + 24.465) = +1058.3.$
$M_{ef} = 20(+11.90 + 45.96 - 97.86) = -800.0$
$M_{fe} = 20(+91.92 + 5.95 - 97.86) = 0.0$

36. Stress Analysis by Distribution of End Moments.—An ingenious and convenient method of finding the stresses in structures with moment-resisting joints has been suggested by Professor Hardy Cross and is called the method of moment distribution. Each of the methods discussed in previous articles involves the solution of a number of simultaneous equations and, in general, is somewhat laborious. The method of moment distribution does not require equations, is often much shorter than

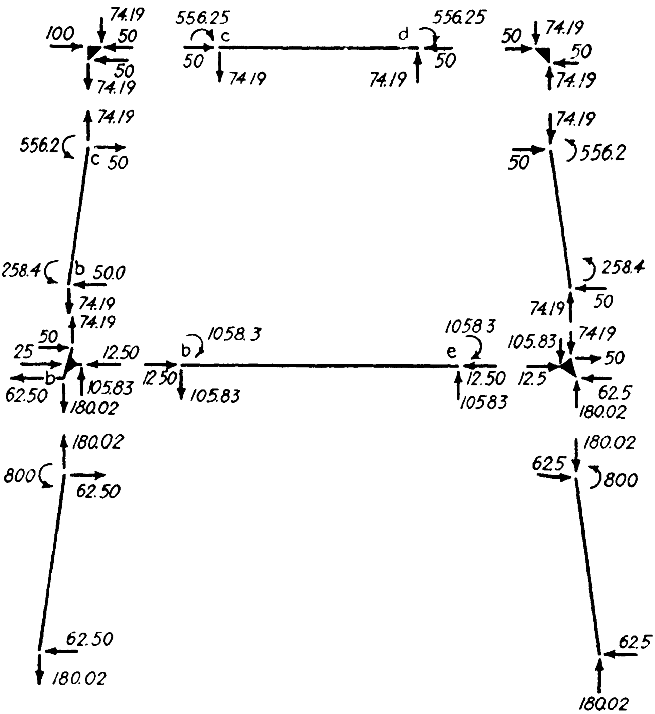


FIG. 76.—End moments, shears, and axial stresses for the bent of Fig. 71.

any of the methods already discussed, and may be carried only far enough to provide results of a degree of precision required by the particular problem in hand.

Consideration of the slope-deflection equations leads to the conclusion that the moment at the end of any member is determined by the following factors: The rotations of the tangents, at the ends of the member, to the elastic curve; the rotation of the chord joining the ends of the elastic curve and the applied loads. The method of moment distribution is a process of evaluating the effects of these factors separately and arriving at the final result by a series of approximations. It may be illustrated by the problem shown in Fig. 77. The slope-deflection equations for this beam are

$$
\begin{aligned}
M_{ab} &= 2EK_{ab}(2\theta_a + \theta_b - 3\psi_{ab}) - \frac{50 \times 4 \times 8^2}{12^2} \\
M_{ba} &= 2EK_{ab}(\theta_a + 2\theta_b - 3\psi_{ab}) + \frac{50 \times 4^2 \times 8}{12^2} \\
M_{bc} &= 2EK_{bc}(2\theta_b + \theta_c - 3\psi_{bc}) - \frac{5 \times 12^2}{12} \\
M_{cb} &= 2EK_{cb}(\theta_b + 2\theta_c - 3\psi_{bc}) + \frac{5 \times 12^2}{12}
\end{aligned}
\qquad (a)
$$

also, by statics,

$$M_{cd} = -5 \times \frac{6^2}{2} = -90$$

If the points of support do not yield, the angles ψ_{ab} and ψ_{cd}

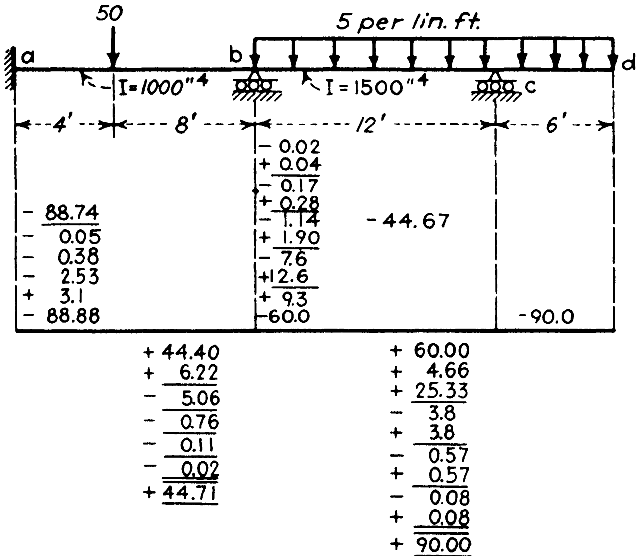


FIG. 77.

are zero. The next step is to write on the members the end moments which would occur if the angles θ were zero also, *i.e.*, the moments which would exist if each member were fixed at the ends. It is evident that these are the last terms of Eq. (*a*). At this point it is well to adopt some convention as to positive and negative moments and to adopt some orderly procedure with respect to the places in which end moments are written. The authors prefer to retain the slope-deflection conventions for moment and to write each moment on the side of the member first encountered in a clockwise movement around the joint. There is no particular advantage in these conventions for such problems as that in Fig. 77, but it is believed that there is con-

FIG. 78.

siderable advantage in more complicated problems, and that there is merit in adopting the same procedure for all problems.

If either joint b or joint c be considered as isolated in the condition just described (see Fig. 78), it is evident that neither joint is in equilibrium; consequently each joint must rotate till equilibrium is established. The effects of such rotations will be evaluated separately. The unbalanced moment at joint c is greater than that at joint b and in general its rotation would be considered first, but, for the purpose of explaining the method, joint b will be taken first. Assume that, for the moment, joint b rotates, all other joints remaining fixed. During this rotation the moments developed are stated by

$$M_{ab} = 2EK_{ab}(\theta_b) \cdot \qquad M_{ba} = 2EK_{ab}(2\theta_b); \qquad M_{bc} = 2EK_{bc}(2\theta_b); \qquad M_{cb} = 2EK_{bc}(\theta_b) \quad (b)$$

Since joint b rotates till equilibrium is established, the moments M_{ba} and M_{bc} developed during the rotation must be together equal to the unbalanced moment and be of opposite sign, *i.e.*, $+15.\dot{5}$, also, from Eqs. (b)

$$\frac{M_{ba}}{M_{bc}} = \frac{K_{ab}}{K_{bc}} = \frac{2}{3}$$

therefore the moments developed during the rotation are

$$M_{ba} = \frac{2}{5}(+15.\dot{5}) = +6.\dot{2}; \qquad M_{bc} = \frac{3}{5}(+15.\dot{5}) = +9.3$$

Each of these is written at the end of the member in which it occurs. This is called moment distribution. It is evident from Eqs. (b) that this rotation of joint b through an angle θ_b produces moments at joint a and joint c, further that the moment produced at joint *a* is just half as great as that produced in the *b* end of member *ba* and has the same sign, while the moment produced at point *c* is just half as great as that produced at the *b* end of member *bc* and has the same sign. These moments are written on the ends of the members in which they occur. This is called the carryover. An examination of joint *c* at this stage shows that there is an unbalanced moment of -25.3; therefore this joint must rotate till equilibrium is established. During this rotation there can be no change in the moment -90.0 in *cd* because the far end is free to move and this moment is determined by the conditions of equilibrium for the cantilever end *cd*; consequently, all the balancing moment must occur in member *cb*; it is so written. Since this rotation is assumed to occur with all other joints temporarily fixed, there will be a carryover moment of $+25.\dot{3} \times \frac{1}{2}$ developed at the *b* end of member *bc*, the reasoning leading to this conclusion being exactly the same as that which determined the carryover moments due to the rotation of joint *b*. A new examination of joint *b* shows that there is a new unbalance of moments at that point. Therefore there must be some additional rotation of that joint, this rotation proceeding till equilibrium of the joint is established. It is assumed that for the moment all other joints are fixed and a moment distribution and moment carryover are carried out just as in the previous rotation of joint *b*. This produces a new unbalance at joint *c*, which must, therefore, rotate till equilibrium is established and is assumed to do so while all other joints are fixed, resulting in a balancing moment at joint *c* and a new carryover moment at joint *b* and a new unbalance there. These steps are repeated till the unbalanced moments are so small that further investigation is deemed unnecessary, this particular decision resulting on the precision desired in the analysis. The total moment at the end of any member is the algebraic sum of the moments written there.

In the solution just completed it was assumed that there was no yielding of the supports and that, consequently, all angles ψ were zero. If the conditions were such that joint *b* should be expected to yield under the loading given, a further investigation must be made to determine the effect of this yielding and the end moments developed must be added to those already found. This further investigation is carried out in a manner very similar to that used in finding the effect of the loads, but starting with the end moments which would be developed if the settlement occurred without any rotation of the joints and then considering the effect of any possible rotation of the joints (moment distribution and carryover) just as in the previous problem. As an

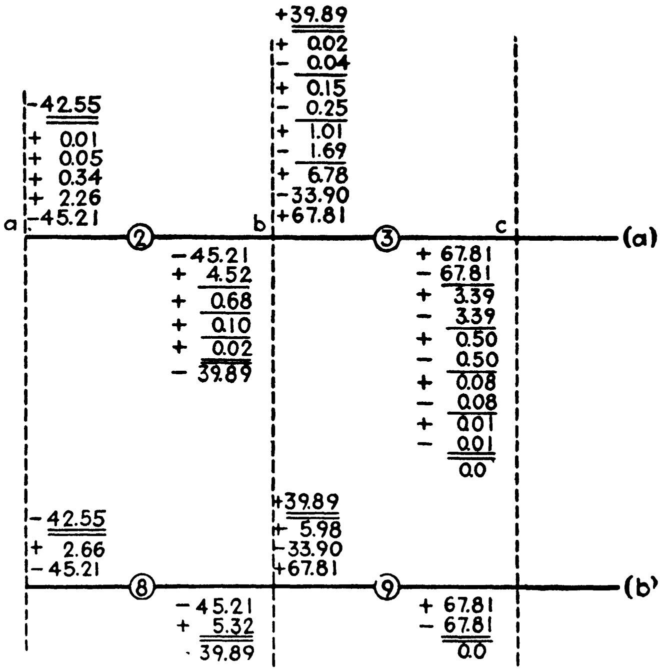


FIG. 79.

illustration assume that joint b settles $\frac{1}{16}$ in. under the loading shown. It follows that

$$\psi_{ab} = +\frac{1}{16} \times \frac{1}{12} \times \frac{1}{12} \qquad {}_{bc} = -\frac{1}{16} \times \frac{1}{12} \times \frac{1}{12}$$

and the fixed-end moments due to the settlement are

$$M_{ab} = M_{ba} = -6EK_{ab}\psi_{ab} = -6 \times \frac{3 \times 10^7 \times 12^2 \times 1000}{12^4 \times 12} \times \frac{1}{16 \times 12 \times 12}$$

$$= -45.21 \text{ (in 1,000-lb. units)}$$

$$M_{bc} = M_{cb} = -6EK_{cd}\psi_{cd} = +45.21 \times 1.5 = 67.81$$

The moment distribution is given in Fig. 79a.

In this beam joint c behaves as if it were pin-connected and some labor may be saved if this is kept in mind in carrying out the moment distribution at joint b. For adjacent members ab and bc, joint a being fixed and joint c free to rotate, the slope-deflection equations are

$$\begin{aligned} M_{ba} &= 2EK_{ab}(\theta_a + 2\theta_b - 3\psi_{ab}) \\ M_{bc} &= 2EK_{bc}(2\theta_b + \theta_c - 3\psi_{bc}) \\ M_{cb} &= 2EK_{cb}(\theta_b + 2\theta_c - 3\psi_{bc}) = 0 \end{aligned}$$

In evaluating the effect of a rotation of joint b, when joint a is fixed and joint c free to rotate, and the angles ψ being assumed to be zero, even if only temporarily, these equations become

$$\theta_c = -\frac{1}{2}\theta_b$$

$$M_{bc} = 2EK_{bc}\left(\frac{3}{2}\theta_b\right) = 3EK_{bc}\theta_b$$

$$M_{ba} = 2EK_{ba}(2\theta_b) = 4EK_{ba}\theta_b$$

It is evident that a rotation of joint b under these circumstances leads to the development of moments M_{ba} and M_{bc} which are not,

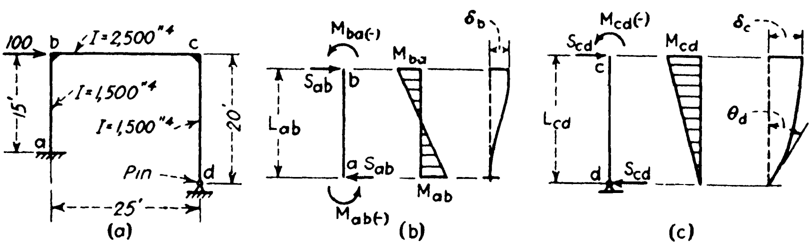


FIG. 80.

as before, proportional to the stiffness factors K_{ab} and K_{bc}, but are proportional to K_{ab} and $\frac{3}{4}K_{bc}$. If this idea is used, the computation in Fig. 79*a* becomes that shown in Fig. 79*b*.

As a further illustration consider the bent shown in Fig. 80*a*. In this condition of loading one cannot start with the fixed-end moments for the members because there are no transverse loads between the ends of any member. It is impossible, also, to begin with the end moments due to some definite angle ψ for each member, though a little consideration of the structure will show that $\psi_{bc} = 0$ and that neither ψ_{ab} nor ψ_{cd} is zero. A logical beginning would be to start with the effects of some definite value for ψ_{ab} or for ψ_{bc}, and, since the purpose of the moment distribution is the evaluation of the effect of rotation of the joints, it might be assumed that, temporarily, the angles ψ occur without rotation of the joints. Further, since the sum of the horizontal shears in the two columns must be equal to the load applied, it might be assumed that the structure moves sideways till this condition is satisfied. Since the two columns are of different lengths and have different conditions of support, though under the assumptions stated above the horizontal deflections of joints b and c are alike, the horizontal shears in the two columns will not be alike and the logical procedure is to determine the relation between the horizontal shears which will cause the two columns to have equal deflections at the upper ends, both being assumed fixed in direction at the top, column ab fixed at the base while column cd is pin-connected at the base. The comparison between the two conditions is as follows:

Since the column ab is fixed in direction at both ends, the slope-deflection equation gives (see Fig. 80b):

$$M_{ab} = M_{ba} = +\frac{2EI_{ab}}{L_{ab}}\left(-3\frac{\delta_b}{L_{ab}}\right),$$

from which

$$\delta_b = -\frac{L_{ab}^2 M_{ab}}{6EI_{ab}}$$

From the equation $\Sigma M = 0$ applied to the column,

$$S_{ab}L_{ab} = -M_{ab} - M_{ba} = -2M_{ab}.$$

Combining these relations:

$$\delta_b = +\frac{S_{ab}L_{ab}^3}{12EI_{ab}}$$

For the column cd, the slope-deflection equation gives (see Fig. 80c)

$$M_{dc} = 0 = 2E\frac{I_{cd}}{L_{cd}}\left(2\theta_d - 3\frac{\delta_c}{L_{cd}}\right),$$

from which

$$\theta_d = \frac{3}{2}\frac{\delta_c}{L_{cd}}$$

$$M_{cd} = +2E\frac{I_{cd}}{L_{cd}}\left(\theta_d - 3\frac{\delta_c}{L_{cd}}\right) = 2E\frac{I_{cd}}{L_{cd}}\left(-\frac{3}{2}\frac{\delta_c}{L_{cd}}\right),$$

from which

$$\delta_c = -\frac{M_{cd}L_{cd}^2}{3EI_{cd}}$$

By statics,

$$S_{cd}L_{cd} = -M_{cd}.$$

Combining these expressions,

$$\delta_c = \frac{S_{cd}L_{cd}^3}{3EI_{cd}}$$

Since $\delta_b = \delta_c$,

$$\frac{S_{ab}}{S_{cd}} = \frac{12EI_{ab}/L_{ab}^3}{3EI_{cd}/L_{cd}^3} = \frac{4K_{ab}/L_{ab}^2}{K_{cd}/L_{cd}^2}$$

If both columns had been fixed at the base, the relation would have been

$$\frac{S_{ab}}{S_{cd}} = \frac{12EI_{ab}/L_{ab}^3}{12EI_{cd}/L_{cd}^3} = \frac{K_{ab}/L_{ab}^2}{K_{cd}/L_{cd}^2}$$

For this particular problem the tentative shear distribution is

$$\frac{S_{ab}}{S_{cd}} = \frac{4\frac{1500}{15^3}}{1500/20^3} = \frac{4^4}{3^3} = \frac{256}{27}$$

$$S_{ab} = 100 \times \frac{256}{283} = 90.46; \qquad S_{bc} = 100 \times \frac{27}{283} = 9.54$$

and the corresponding fixed-end moments are

$$M_{ab} = M_{ba} = -\frac{1}{2} \times 90.46 \times 15 = -678.4$$

$$M_{cd} = -9.54 \times 20 = -190.8$$

In carrying out the moment distribution only the relative magnitudes of the K factors are of importance and since the column *cd* is hinged at the base the ratios are

$$\frac{1500}{15}:\frac{2500}{25}:\frac{3}{4} \times \frac{1500}{20} = 100:100:56.25 = 16:16:9$$

The moment distribution follows.

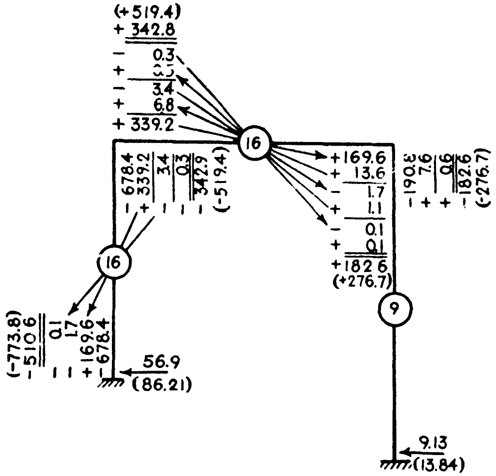


FIG. 81.

The equations of equilibrium for the two columns lead to

$$S_{ab} = \frac{342.9 + 510.6}{15} = 56.9; \qquad S_{cd} = \frac{182.6}{20} = 9.13$$

The sum of these is 66.0 and the total load is 100; therefore, all quantities should be multiplied by the ratio 100:66. The results are shown in parentheses in Fig. 81.

If the bent is of more than one story, the shear distribution is not quite so easy. There are several ways of carrying out the desired process, but what is perhaps the most convenient is that suggested by Professor Clyde T. Morris in his discussion of Professor Cross's paper entitled "Analysis of Continuous Frames by Distributing Fixed-end Moments," in the *Transactions of A.S.C.E.*, vol. 96, 1932. Consider the analysis of the bent shown in Fig. 82. The stiffness factors are in the ratios of the figures written on the members. If the changes in the lengths of the girders be neglected, the angles ψ for the columns of any one story are alike. Hence, assuming no joint rotation, the

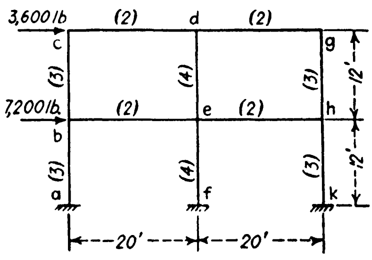


FIG. 82.

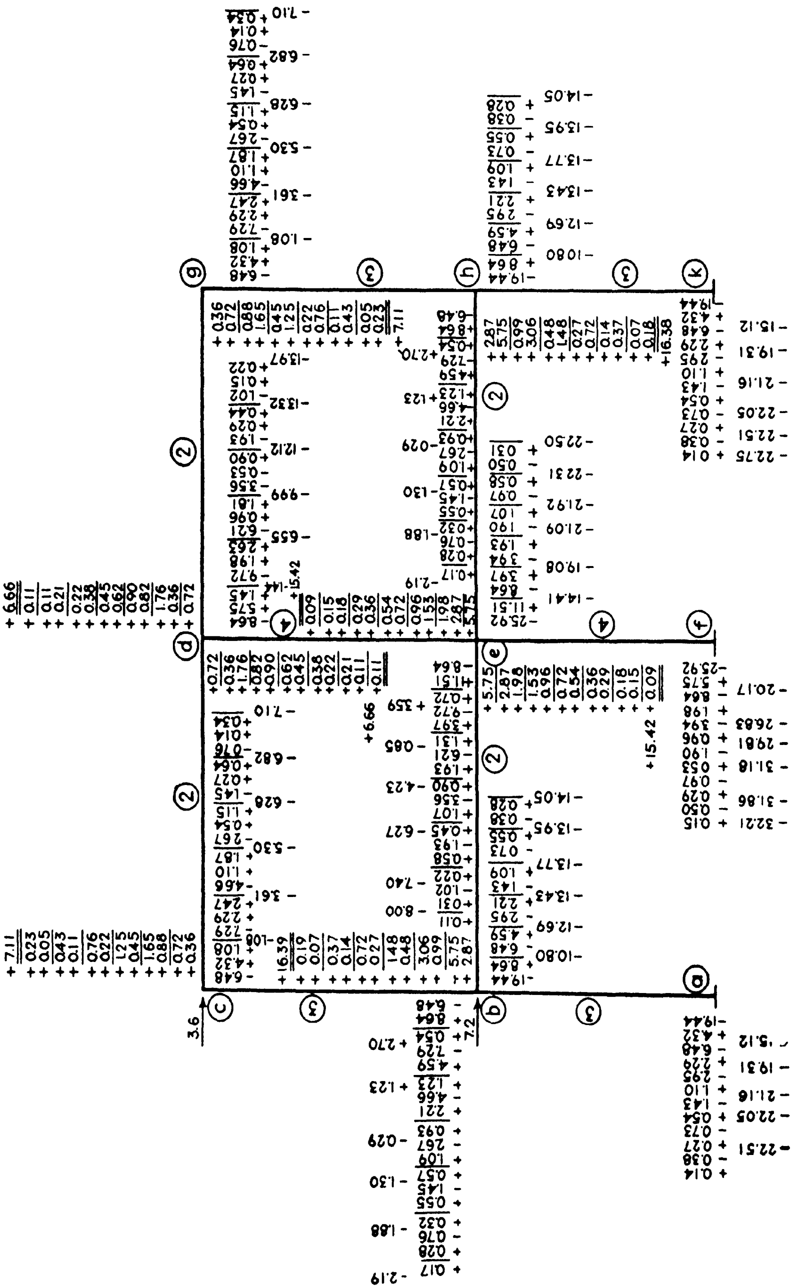


FIG. 83.

end moments for any column mn, being equal to $-6EK_{mn}\psi_{mn}$, are proportional to the stiffness coefficient for the column. Thus in the upper story of the bent, the total shear being 3.600 lb., the sum of the end moments for the columns is

$$3{,}600 \times 12 = 43{,}200 \text{ ft.-lb.}$$

This is divided among the columns in the proportion 3:4:3, *i.e.*, using 1,000-lb. units, 12.96:17.28:12.96, and the end moments for any column being equal, the end moments for the three columns are 6.48, 8.64, and 6.48, respectively. Similarly, in the lower story the end moments are 19.44, 29.52, and 19.44, respectively. These moments are all counterclockwise and hence are written as negative. When this is done the moments at any joint, in general, will not balance. Hence the joints will rotate. This is taken into account by distributing the moments at

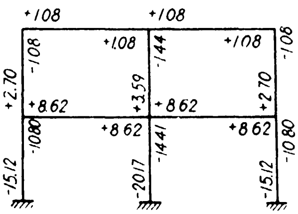


FIG. 84.

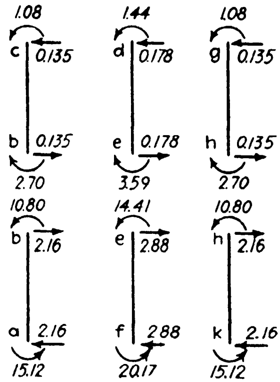


FIG. 85.

each joint and carrying over the "carryover moments." In the illustration in hand this is performed in Fig. 83, taking the joints in the order e, b, h, d, c, g, and leading to the results as shown in Fig. 84. The shears in the columns corresponding to these end moments are shown in Fig. 85.

In the top story the column shears have as a sum 0.448 to the right when it should be 3.6 to the left. The sum of the column shears in the bottom story is 7.20 when it should be 10.8. To take care of this, impose new moments on the top story columns whose total is $(3.6 + 0.448)12 = 48.576$ divided among the three columns in the ratios 3:4:3, *i.e.*,

$$14.57 : 19.43 : 14.57,$$

which means end moments of -7.29, -9.72, -7.29, in columns bc, ed, and hg, respectively. In the lower story impose new moments whose sum is (10.8 − 7.20)12 = 43.2, which, divided among the three columns in the proper proportions, leads to end moments of −6.48, −8.64, −6.48 in columns *ab*, *fe*, and *kh*, respectively. Now repeat the distribution and carrying over of unbalanced moments and check up the total shears in the two stories once more. Repeat this process until the unbalanced moments are so small that further computation is considered unnecessary. The process described was carried out six times in Fig. 83. The results obtained are listed below together with those obtained from a slope-deflection solution of the same problem.

Member	Moment distribution solution	Slope-deflection solution	Member	Moment distribution solution	Slope-deflection solution
ab	−22.75	−23.97	*ed*	− 8.00	− 8.64
ba	−14.05	−14.08	*de*	−13.97	−14.09
bc	− 2.19	− 2.68	*be*	+16.39	+16.76
cb	− 7.10	− 7.55	*eb*	+15.42	+15.74
fe	−32.21	−32.66	*cd*	+ 7.11	+ 7.55
ef	−22.50	−22.84	*dc*	+ 6.66	+ 7.05

CHAPTER IV

INFLUENCE LINES FOR STATICALLY INDETERMINATE STRUCTURES

37. Introduction.—In the preceding chapter, various methods used to determine stresses in structures with redundant members or reactions were demonstrated. The structures discussed were, however, subjected to a particular condition of loading. Very often one must investigate the effect on a structure of many possible conditions of loading, as, for example, when live load moves across a structure. In such circumstances it is necessary to determine what position of the live load will cause a maximum value of some particular function, such as stress or deflection. For this purpose the influence line serves as the most convenient means of obtaining the desired result.

The first step in finding the influence lines for the stresses in various members of a structure is to find the influence lines for the redundants. Once this is done, it is possible to find the stress in any member, for any position of the unit load, as a function of the unit load and the redundant stress or stresses.

38. Influence Lines by Successive Positions of Unit Load.—It is, of course, possible to place the unit load in each position possible and, for each of these positions, compute the values of the redundants by one of the methods given in Chap. III. While the amount of time involved in this procedure may be large, the work may be systematized in such a way that it is not unduly laborious.

For illustrative purposes, Castigliano's law will be used as a basis for stress analysis, and the ordinates to the influence lines for the stresses in members 5-6 and 6-9 of the doubly redundant truss shown in Fig. 86*a* will be computed. The numbers shown in parentheses on this figure are the member areas in square inches. The primary structure is shown in Fig. 86*b*, the tension in member 5-6, X_a, and the tension in member 6-9, X_b, having been taken as the redundants. For any position of the unit load, the stress in any member of the structure may be expressed by

$$F = F_0 + X_aF_a + X_bF_b$$

in which F = the total stress in any member.
F_0 = the stress in that member due to the unit load acting on the primary structure.
F_a = the stress in that member for "condition $X_a = 1$."
F_b = the stress in that member for "condition $X_b = 1$.'

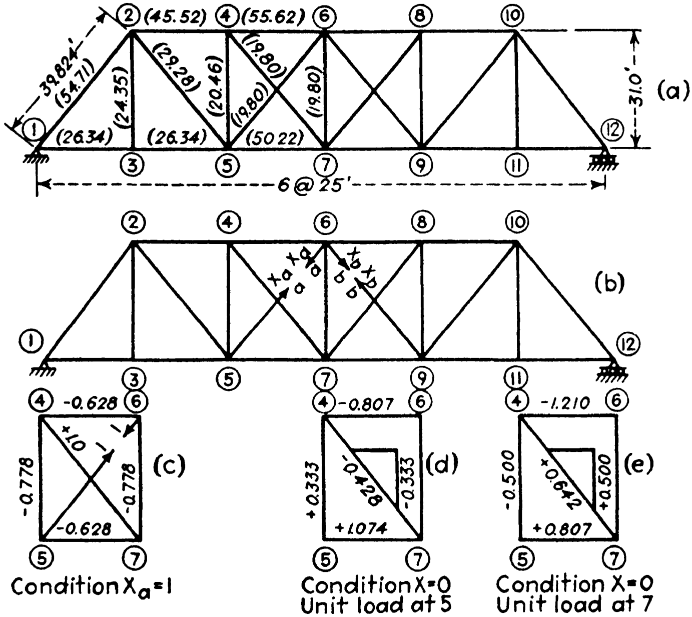


Fig. 86.

If one assumes no yielding of supports and no change of temperature, the equations $\partial W_d/\partial X_a = 0$ and $\partial W_d/\partial X_b = 0$, solved simultaneously, will yield the required solution, for any given position of the unit load.

These equations become

$$\frac{\partial W_d}{\partial X_a} = \sum \frac{FL}{AE}\frac{\partial F}{\partial X_a} = \sum (F_0 + X_aF_a + X_bF_b)\frac{L}{AE}F_a$$

$$= \sum \frac{F_0F_aL}{AE} + X_a\sum \frac{F_a^2L}{AE} + X_b\sum \frac{F_aF_bL}{AE} = 0$$

and

$$\frac{\partial W_d}{\partial X_b} = \sum \frac{FL}{AE}\frac{\partial F}{\partial X_b} = \sum (F_0 + X_aF_a + X_bF_b)\frac{L}{AE}F_b$$

$$= \sum \frac{F_0F_bL}{AE} + X_a\sum \frac{F_aF_bL}{AE} + X_b\sum \frac{F_b^2L}{AE} = 0$$

But

$$\sum \frac{F_0F_aL}{AE} = \delta_{a0}; \qquad \sum \frac{F_0F_bL}{AE} = \delta_{b0}; \qquad \sum \frac{F_a^2L}{AE} = \delta_{aa};$$

$$\sum \frac{F_cF_bL}{AE} = \delta_{ab} = \delta_{ba}$$

and

$$\sum \frac{F_b^2L}{AE} = \delta_{bb}$$

from which one may write

$$\begin{aligned} \delta_{a0} + X_a\delta_{aa} + X_b\delta_{ab} &= 0 \\ \delta_{b0} + X_a\delta_{ba} + X_b\delta_{bb} &= 0 \end{aligned} \qquad (a)$$

these equations being identical with those obtained by the method of virtual work. If Eqs. *a* are solved simultaneously, the following results are obtained:

$$X_a = \frac{\delta_{b0}\delta_{ab} - \delta_{a0}\delta_{bb}}{\delta_{aa}\delta_{bb} - \delta_{ab}^2}$$

$$X_b = \frac{\delta_{a0}\delta_{ab} - \delta_{b0}\delta_{aa}}{\delta_{aa}\delta_{bb} - \delta_{ab}^2}$$

In the foregoing equations, δ_{aa}, δ_{ab}, and δ_{bb} are independent of the position of the unit load, hence need be computed but once. Moreover, δ_{aa} equals δ_{bb}, owing to symmetry of the structure and symmetrical choice of redundants. δ_{a0} and δ_{b0} have different values for each position of the unit load. Since, however, the primary structure is symmetrical, if the various values of δ_{a0} are computed, values of δ_{b0} may be obtained by symmetry. As a further means of saving labor, it should be noted that for condition $X_a = 1$, only the members which compose the panel 4-6-5-7 are stressed, so that only the members in this panel will contribute to δ_{a0} and δ_{aa}; for the condition $X_b = 1$, only the members which compose the panel 6-8-7-9 are stressed, so that member 6-7 is the only member in the structure for which the product F_aF_b does not equal zero, and in consequence is the only member contributing to δ_{ab}. Moreover, for this member, $F_a = F_b$, owing to symmetry.

Bar	F_a	F_b	F_s due to unit load at					L, ft.	A (sq. in.)	$\frac{L}{A}\left(\frac{1}{\text{in.}}\right)$	$\frac{F_a^2L}{A}$	$\frac{F_aF_bL}{A}$	$\frac{F_aF_sL}{A}$ due to unit load at				
			3	5	7	9	11						3	5	7	9	11
4-6	−0.628		−0.403	−0.807	−1.210	−0.807	−0.403	25.0	55.62	5.39	+ 2.13		+1.37	+ 2.74	+ 4.09	+ 2.74	+1.37
5-7	−0.628		+0.537	+1.074	+0.807	+0.538	+0.269	25.0	50.22	5.97	+ 2.36		−2.02	− 4.02	− 3.02	− 2.02	−1.01
4-5	−0.778		+0.167	+0.333	−0.500	−0.333	−0.167	31.0	20.46	18.15	+11.00	+11.35	−2.36	− 4.70	+ 7.07	+ 4.70	+2.36
6-7	−0.778	−0.778	0	0	0	0	0	31.0	19.80	18.76	+11.35		0	0	0	0	0
4-7	+1.000		−0.214	−0.428	+0.642	+0.428	+0.214	39.824	19.80	24.10	+24.10		−5.16	−10.30	+15.45	+10.30	+5.16
5-6	+1.000		0	0	0	0	0	39.824	19.80	24.10	+24.10		0	0	0	0	0
											+75.04 ($E\delta_{aa}$)	+11.35 ($E\delta_{ab}$)	−8.17	−16.28	+23.59	+15.72	+7.88
													$E\delta_{a0}$				

$$E^2(\delta_{aa}\delta_{bb} - \delta_{ab}^2) = (75.04)^2 - (11.35)^2 = +5502$$

Load at	$E\delta_{b0}$	$E\delta_{a0}$	$E^2\delta_{b0}\delta_{ab}$	$E^2\delta_{a0}\delta_{bb}$	$E^2(\delta_{b0}\delta_{ab} - \delta_{a0}\delta_{bb})$	X_a	$E\delta_{a0}\delta_{ab}$	$E\delta_{b0}\delta_{aa}$	$E^2(\delta_{a0}\delta_{ab} - \delta_{b0}\delta_{aa})$	X_b
3	+ 7.88	− 8.17	+ 89.3	− 613	+ 702	+0.127	− 92.5	+ 592	− 685	−0.124
5	+15.72	−16.28	+178 5	−1220	−1399	+0.254	−184.5	+1180	−1365	−0.248
7	+23.59	+23.59	+267.5	+1770	−1502	−0.272	+267.5	+1770	−1502	−0.272
9	−16.28	+15.72	−184.5	+1180	−1365	−0.248	+178.5	−1220	+1399	+0.254
11	− 8.17	+ 7 88	− 92 5	+ 592	− 685	−0.124	+ 89.3	− 613	+ 702	+0.127

In Fig. 86*c* the stresses F_a for condition $X_a = 1$ are given; in Figs. 86*d* and 86*e* the stresses F_0 for condition $X = 0$ are shown, with the unit load at points 5 and 7, respectively. Consideration of the principles of statics permits an easy computation of the F_0 stresses for other positions of the unit load. With the unit load at 3, the F_0 stresses in panel 5-7 are equal to one-half those with the unit load at 5; with the unit load at 9 and 11, the F_0 stresses are equal to two-thirds and one-third, respectively, of those with the unit load at 7.

The computations arranged in tabular form are shown on p 166.

39. Influence Lines for Elastic Deflections.—If one is confronted with the problem of determining the maximum deflection due to moving loads, of a point on a structure, the influence line again serves as an expedient to determine the position of

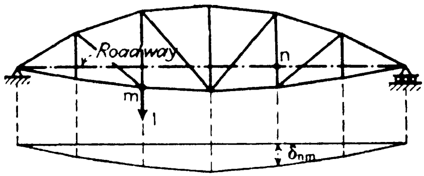


FIG. 87.

loads which will cause the maximum deflection and may be used as a basis for computing the value of the maximum deflection once this position of the loads has been determined. Influence lines for deflections may be constructed by investigating successive positions of the unit load in a manner similar to that discussed in the preceding paragraph, the effect of each loading being investigated by one of the methods demonstrated in Chap. II. A more convenient approach, however, may be found.

If a unit load is applied at m, as shown in Fig. 87, and the deflection curve is drawn for the points of application of the load system, the ordinate to such a curve at any point n is δ_{nm}. Now, by Maxwell's law, $\delta_{mn} = \delta_{nm}$, so that the ordinates to the deflection curve are also ordinates which show the various values of the deflection of joint m as the unit load moves across the span; *i.e.*, the deflection curve of the roadway due to a unit load at m is the influence line for δ_m.

To illustrate the application of this method, the maximum deflection of joint 7 of the truss in Fig. 88*a* due to a uniform live load of 4,000 lb. per ft. will be determined. A unit load of 1,000 lb. will be used in this computation, and the method of elastic loads will be employed to determine the deflection curve of the bottom chord due to the unit load at joint 7, which is

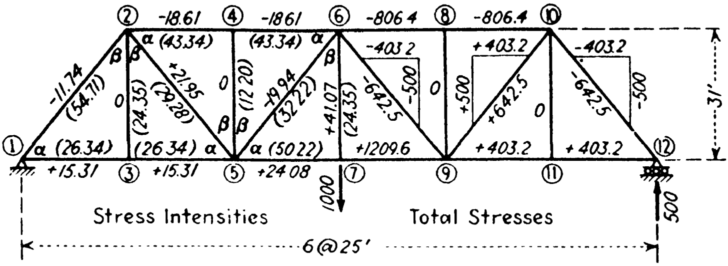


FIG. 88*a*.

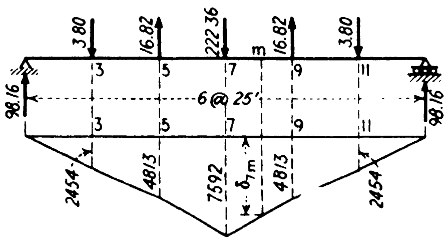


FIG. 88*b*.

shown in Fig. 88*b*. Since this elastic curve is the influence line for the vertical deflection at joint 7 and all the ordinates to the curve indicate downward deflection, the area under the curve multiplied by the intensity of live loading gives the required maximum deflection.

Angle	Coefficient of cot α $\cot\alpha = \frac{25}{31}$	Coefficient of cot β $\cot\beta = \frac{31}{25}$	1st term	2d term	$E\ d\theta$
1-3-2	−11.74−15 31 = −27.05	−11.74− 0.0 = −11.74	−21.81	−14.56	+ 3.80
2-3-5	+21.95−15.31 = + 6.64	+21.95− 0.0 = +21.95	+ 5.35	+27.22	
3-5-2		0.0 −21.95 = −21.95		−27.22	− 16.82
2-5-4	−18.61−21.95 = −40.56		−32.71		
4-5-6	−18.61+19.94 = + 1.33		+ 1.07		
6-5-7		+41.07+19.94 = +61.01		+75.68	
5-7-6	−19.94−24.08 = −44.02	−19.94−41.07 = −61.01	−35.50	−75.68	+222.36
6-7-9	−19.94−24.08 = −44.02	−19 94−41.07 = −61.01	−35.50	−75.68	

$E\,\delta_{7m}$ for 1,000-lb. load

$+\ 98.16 \times 25 =$	$+2454$
$-\ 3.80$	
$+\ 94.36 \times 25 =$	$+2359$
$+\ 16.82$	$+4813$
$+111.18 \times 25 =$	$+2779$
-222.36	$+7592$
$-111.18 \times 25 =$	-2779
$+16.82$	$+4813$
$-\ 94.36 \times 25 =$	-2359
$-\ 3.80$	$+2454$
$-\ 98.16 \times 25 =$	-2454

2454
4813
7592
4813
2454

$22126(25) = 553{,}000 = E \text{ (area)}$

Therefore

$$\text{Max. } \delta_n = \frac{553000(4)}{30 \times 10^6} = 0.0738 \text{ ft.}$$
$$= 0.885 \text{ in. due to live load only}$$

40. Influence Lines for Stress by Use of the Elastic Curve.—The use of the elastic curve and Maxwell's law in constructing influence lines for deflections, and the advantage gained by their use, suggest a similar approach in constructing influence lines for functions such as bar stresses or reactions, through the application of Eqs. (61).

Consider the application of these equations to the truss in Fig. 89*a*, which is symmetrical about the vertical center line. The cross-sectional areas of the members are given by the numbers in parentheses written on the left half of the sketch It will be assumed that there is no yielding of the supports. As the redundant force, the vertical component of the middle reaction, assumed acting up, will be chosen. Under these circumstances the statically determinate primary structure is as shown in Fig. 89*b*.

If one writes Eqs. (61) for this structure,

$$\delta_{a0} + X_a\delta_{aa} = 0$$

in which δ_{a0} is the deflection of point a of the primary structure in condition $X = 0$. But, when one is determining the influence line, this condition reduces to a unit concentrated load at one of the joints m of the loaded chord, for which

$$\delta_{a0} = \delta_{am}$$

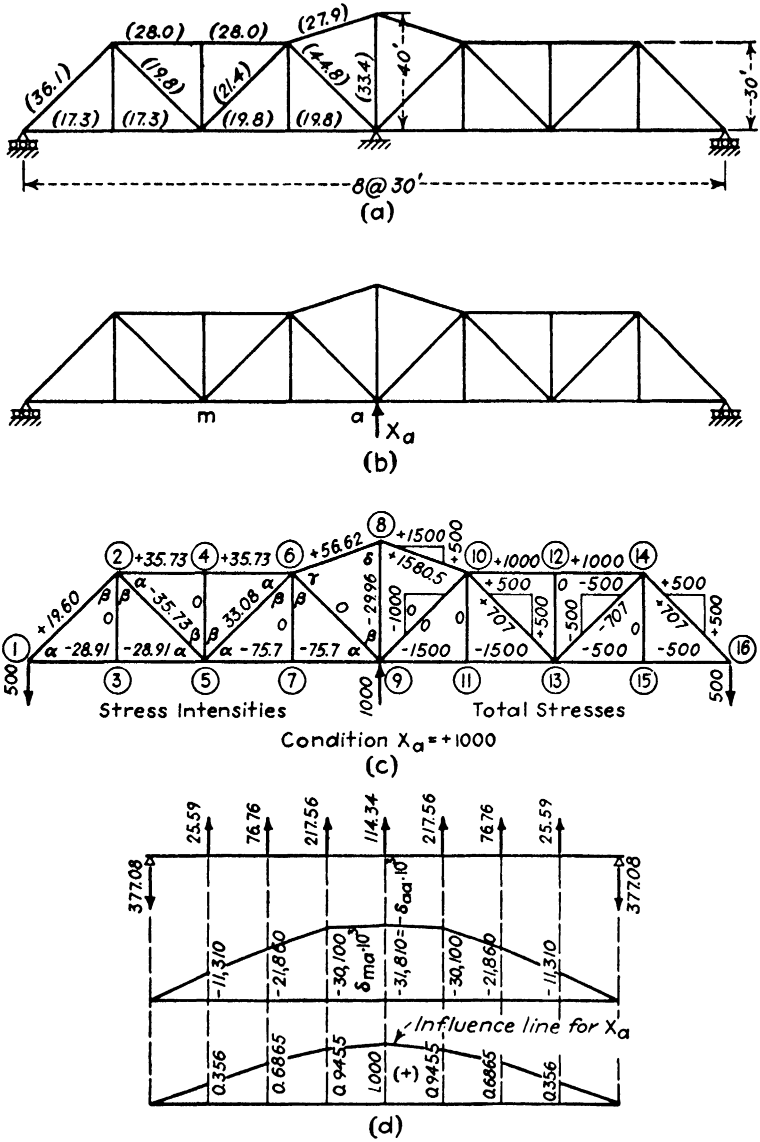

(28.0)
(28.0)
(27.9)
(36.1)
(19.8)
(21.4)
(44.8)
(33.4)
40'
30'
(17.3)
(17.3)
(19.8)
(19.8)
8@30'
(a)
m
a
X_a
(b)
Stress Intensities
Total Stresses
Condition X_a = +1000
(c)
Influence line for X_a
(d)

FIG. 89.

Therefore one may write, when finding the magnitude of X_a due to the unit load at a joint m,

$$X_a = -\frac{\delta_{am}}{\delta_{aa}} \tag{79}$$

All that must be done in order to draw the influence line for X_a is to draw the influence line for $-\delta_{am}$ and divide each ordinate by the quantity $+\delta_{aa}$.

If the deflection diagram of the loaded chord is drawn for condition $X_a = +1$, the ordinates δ_{ma} to this curve will, by Maxwell's law, be equal to δ_{am}. Since the live loads, hence the unit loads at m from which the deflections δ_{ma} would be computed by the method of virtual work are positive downward, upward deflections δ_{ma} are negative and indicate negative values of δ_{am}. δ_{aa} is, of course, positive.

The numerical solution follows. For convenience, use

$$X_a = +1{,}000,$$

and find $\delta_{am} \times 10^3$ and $\delta_{aa} \times 10^3$. In this particular problem, the absolute value of δ_{aa} is equal to one of the particular values of δ_{ma}, so that no special computation is needed for its evaluation.

Angle	Coefficient of cot α cot $\alpha = 1$	Coefficient of cot β cot $\beta = 1$	1st term	2d term	$E\,d\theta$
1-3-2	+19.60+28.91 = + 48.51	+19.60+ 0.0 = +19.60	+ 48.51	+19.60	− 25.59
2-3-5	−35.73+28.91 = − 6 82	−35.73+ 0 0 = −35 73	− 6 82	−35.73	
3-5-2		0.0 +35.73 = +35.73		+35.73	
2-5-4	+35.73+35.73 = + 71 46		+ 71.46		− 76.76
4-5-6	+35.73−33.08 = + 2 65		+ 2.65		
6-5-7		0.0 −33.08 = −33.08		−33.08	
5-7-6	+33.08+75 70 = +108.78	+33.08+ 0 0 = +33.08	+108.78	+33.08	−217.56
6-7-9	0.0 +75 70 = + 75.70	0.0	+ 75.70	0.0	
7-9-6			0.0	0.0	
	Coefficient of cot γ cot $\gamma = 0.50$	Coefficient of cot δ cot $\delta = \frac{1}{2}$			
6-9-8	+56.62+ 0.0 = + 56.62	+56.62+29.96 = +86.58	+ 28.31	+28.86	
8-9-10	+56.62+ 0.0 = + 56.62	+56.62+29.96 = +86.58	+ 28.31	+28.86	−114.34
10-9-11			0.0	0 0	

		$E\,\delta_{ma}\,10^3$	X_a
-377.08	$\times 30 =$	-11312.4	$+0.356$
$+\ 25.59$			
-351.49	$\times 30 =$	-10544.7	
$+\ 76.76$		-21857.1	$+0.6865$
-274.73	$\times 30 =$	$-\ 8241.9$	
$+217.56$		-30099.0	$+0.9455$
$-\ 57.17$	$\times 30 =$	$-\ 1715.1$	
$+114.34$		-31814.1	$+1.000$
$+\ 57.17$	$\times 30 =$	$+\ 1715.1$	
$+217.56$		-30099.0	$+0.9455$
$+274.73$	$\times 30 =$	$+\ 8241.9$	
$+\ 76.76$		-21857.1	$+0.6865$
$+351.49$	$\times 30 =$	$+10544.7$	
$+\ 25.59$		-11312.4	$+0.356$
$+377.08$	$\times 30 =$	$+11312.4$	
		0	

As a second illustration, the truss in Fig. 90*a*, in which there is one redundant member, will be investigated. The primary structure shown in Fig. 90*b* is formed by cutting the redundant member, for which bar 6-9 is selected, by a section; for convenience the section will be taken just inside joint 6. Proceeding as in the last illustration,

$$X_a = -\frac{\delta_{am}}{\delta_{aa}} = -\frac{\delta_{ma}}{\delta_{aa}}$$

The deflection curve of the loaded chord, the ordinates of which are $\delta_{am} = \delta_{ma}$, is drawn for condition $X_a = +1$. As in the preceding illustration, upward values of δ_{am} are negative. This is shown in Fig. 90*c*. The influence line for $F_{6\text{-}9}$ is given in Fig. 90*d*. The method of elastic loads is used in determining values of δ_{ma}. To find the elastic loads,

Angle	Coefficient of $\cot\alpha$ $\cot\alpha = \frac{25}{30}$	Coefficient of $\cot\beta$ $\cot\beta = \frac{30}{25}$	1st term	2d term	$E\,d\theta$
4-7- 6	0	0	0	0	
6-7- 8	$-0\ 00915 - 0.055 = -0.06415$	0	-0.05346	0	$+0.16602$
8-7- 9	0	$-0.0388 - 0.075 = -0.0938$	0	-0.11256	
7-9- 8	$+0\ 055 + 0.00935 = +0.06435$	$+0.055 + 0.0388 = +0.0938$	$+0.05363$	$+0.11256$	-0.16619
8-9-10	0	0	0	0	

In this example the easiest way to find δ_{aa} is by a direct application of the law of virtual work leading to

$$\delta_{aa} = \sum F_a^2 \frac{L}{AE} \qquad \text{or} \qquad E\delta_{aa} = \sum F_a f_a L$$

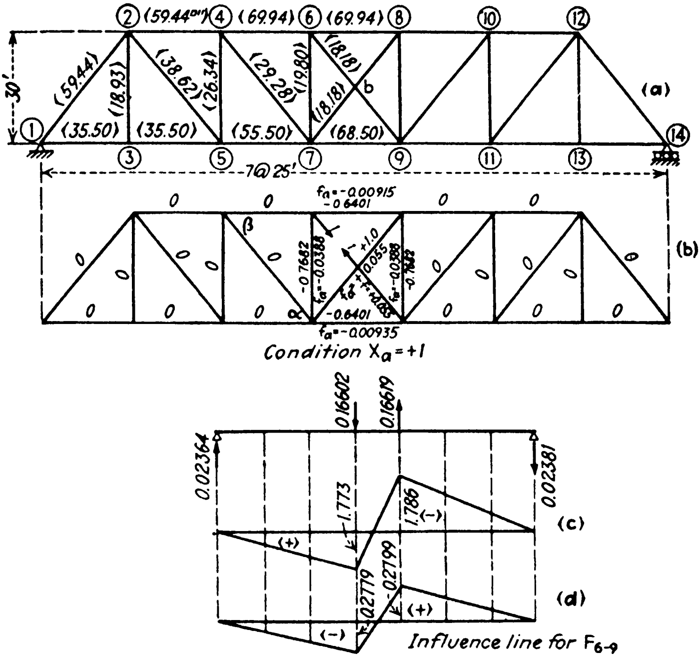


Fig. 90.

Bar	L	F_a	f_a	$F_a f_a L$
6-8	25	−0.6401	−0.00915	+0.1465
7-9	25	−0.6401	−0.00935	+0.1497
6-7	30	−0.7682	−0.0388	+0.8941
8-9	30	−0.7682	−0.0388	+0.8941
6-9	39.05	+1.0	+0.055	+2.148
7-8	39.05	+1.0	+0.055	+2.148
				+6.3804

$$\begin{array}{rcr} -0.16602 \times 4 & = & -0.66408 \\ +0.16619 \times 3 & = & +0.49857 \\ \hline +0.00017 & & 7)\,-0.16551 \\ 0.02364 & & 0.02364 \text{ up} \\ \hline 0.02381 & & \end{array}$$

	$E\,\delta_{ma}$	X_a
$+0.02364 \times 75 =$	$+1.773$	-0.2779
-0.16602		
$-0.14238 \times 25 =$	-3.559	
$+0.16619$	-1.786	$+0.2799$
$+0.02381 \times 75 =$	$+1.786$	
	0.0	

41. Influence Lines for Trusses with Two Redundant Members.—The construction of influence lines for structures with two redundant members, based on the use of the elastic curve and Maxwell's law, may be illustrated by an alternate solution to the problem solved in Art. 38 by investigating successive positions of the unit load. Referring to Fig. 86, one may write

$$\begin{aligned} \delta_{a0} + X_a\delta_{aa} + X_b\delta_{ab} &= 0 \\ \delta_{b0} + X_a\delta_{ba} + X_b\delta_{bb} &= 0 \end{aligned} \qquad (a)$$

If it is desired to draw influence lines, the deflection δ_{a0} becomes the relative deflection inward, δ_{am}, of the two points a of the primary structure, due to a unit load at joint m of the loaded chord, while δ_{b0} becomes the relative deflection inward, δ_{bm}, of the points b of the primary structure due to a unit load at joint m. By Maxwell's law

$$\delta_{am} = \delta_{ma} \qquad \text{and} \qquad \delta_{bm} = \delta_{mb}$$

and Eqs. (a) become

$$\begin{aligned} \delta_{ma} + X_a\delta_{aa} + X_b\delta_{ab} &= 0 \\ \delta_{mb} + X_a\delta_{ba} + X_b\delta_{bb} &= 0 \end{aligned}$$

Thus, the determination of influence lines for X_a and X_b resolves itself into the solution of a number of pairs of simultaneous equations, in which the coefficients of the unknowns are always the same, while the numerical terms δ_{ma} and δ_{mb} may be obtained from the deflection diagrams of the bottom chord; δ_{ma} from that corresponding to condition $X_a = +1$, and δ_{mb} from that corresponding to $X_b = +1$. What is probably the most convenient way of evaluating the coefficients δ_{aa}, δ_{ab}, and δ_{bb} is by evaluating the equations

$$\delta_{aa} = \sum \frac{F_a^2 L}{AE}, \qquad \delta_{ab} = \sum \frac{F_a F_b L}{AE}, \qquad \delta_{bb} = \sum \frac{F_b^2 L}{AE}$$

This is the same procedure which was followed in the solution of this problem in Art. 38, from which the following results were obtained: $E\delta_{aa} = +75.04$; $E\delta_{ab} = +11.35$; $E\delta_{bb} = +75.04$. In these equations E is in pounds per square inch, while δ_{aa}, δ_{ab}, and δ_{bb} are in inches. If the deflection units are changed to feet, the following values are obtained:

$$E\delta_{aa} = +6.2623; \qquad E\delta_{ab} = +0.9482; \qquad E\delta_{bb} = +6.2623$$

To find the elastic loads in condition $X_a = +1$, referring to Fig. 91*a*

Angle	Coefficient of cot α, $\cot\alpha = \frac{25}{31}$	Coefficient of cot β, $\cot\beta = \frac{31}{25}$	1st term	2d term	$E\,d\theta$
4-5-7	+0 0505 + 0.01249 = +0 06299	+0 0505 + 0.03804 = +0.08854	+0 0508	+0.1098	−0.1606
5-7-4		−0 03804 − 0 0505 = −0 08854		−0 1098	
4-7-6	−0 01128 − 0 0505 = −0 06178		−0 0498		+0.1596

$+0.1606 \times 4 =$	$+0.6424$	$-0.0272\dot{6} \times 50 =$	$-1.36\dot{3}$
$-0.1596 \times 3 =$	-0.4788	$+0.1606$	
$+0.0010$	$6)+0.1636$	$+0.133\dot{3} \times 25 =$	$+3.33\dot{3}$
$-0.0272\dot{6}$	$0.0272\dot{6}$ down	-0.1596	$+1.970$
$-0.0262\dot{6}$		$-0.0262\dot{6} \times 75 =$	-1.970
			0.0

The elastic loads together with the curve of $E\delta_{ma}$ are given in Fig. 91*b*. It may be seen that condition $X_b = +1$ is the complement in symmetry of $X_a = +1$. Hence no separate computation need be made for the elastic loads for condition $X_b = +1$, or, indeed, for the $E\delta_{bm}$ curve.

The solution of the five sets of simultaneous equations follows. It is to be noted that instead of using the coefficients δ_{aa}, δ_{ab},

X_a	X_b	Numerical term for unit load at joint				
		3	5	7	9	11
+6.262	+0.9482	−0.6816	−1.363	+1.970	+1.313	+0.656 = 0
+0.9482	+6.262	+0.656	+1 313	+1.970	−1.363	−0.6816 = 0
+0.9482	+0.1436	−0.1032	−0 2064	+0.2983	+0.1988	+0 0994 = 0
	+6.1184	+0.7598	+1.5197	+1.6717	−1.5621	−0.7810 = 0
	X_b	= −0.1242	−0 2484	−0.2732	+0.2558	+0.1176
	$0.9482X_b$	= −0.1178	−0.2355	−0.2591	+0.2421	+0 1210
6 262X_a		= +0 7994	+1.5988	−1.7109	−1.5555	−0 7777
X_a		= +0.1276	+0.2552	−0.2732	−0 2483	−0.1242

δ_{bb}, δ_{ma}, δ_{mb}, the quantities $E\ \delta_{aa}$, $E\ \delta_{ab}$, $E\ \delta_{bb}$, $E\ \delta_{ma}$, $E\ \delta_{mb}$, are used, which is the same as multiplying both sides of each equation by E.

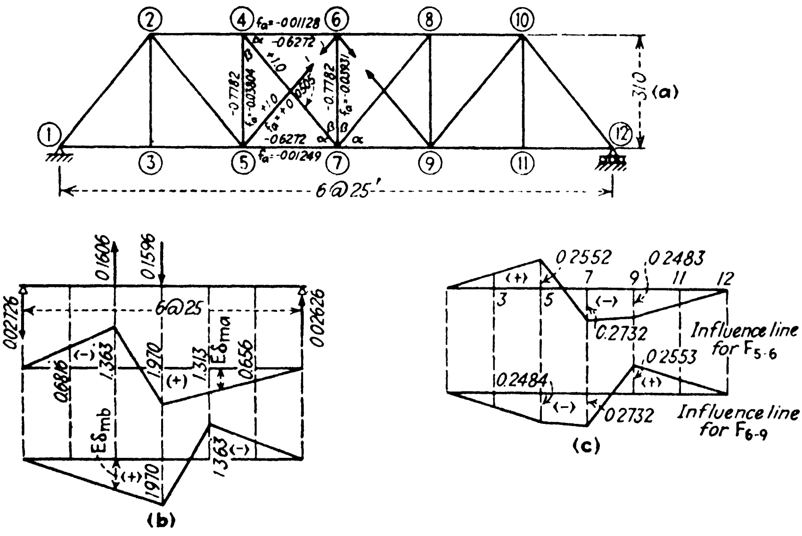


FIG. 91.

The influence lines for the members 5-6 and 6-9 are shown in Fig. 91*c*. It may be noticed they are complementary in symmetry, as might be expected from the symmetry of the truss.

Another interesting application of these ideas is the analysis of the arch shown in Fig. 92*a* for which conditions $X_a = +10^3$ and $X_b = +10^3$ are shown in Fig. 92*b* and 92*c*.

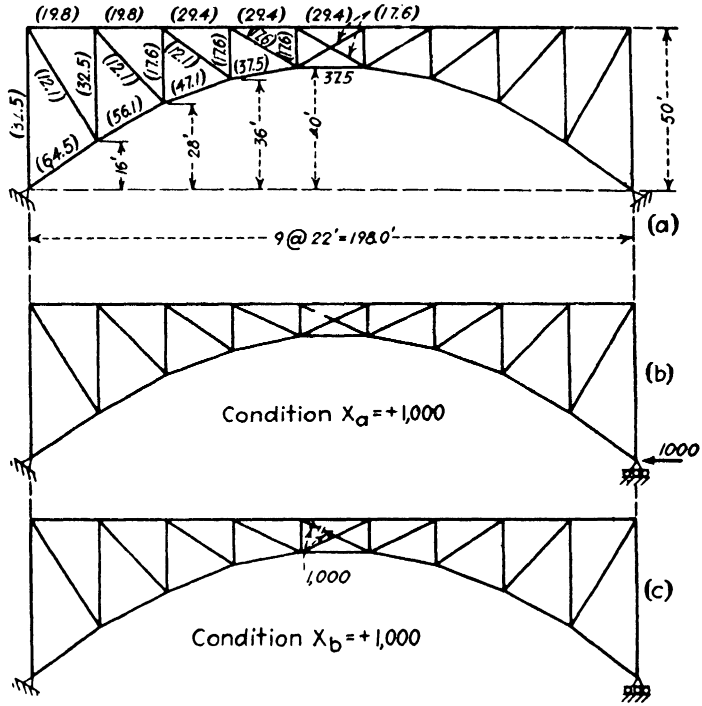


FIG. 92.

The dimensions of the structure are as shown in Fig. 93*a*, the stress analysis for condition $X_a = +1{,}000$ in Fig. 93*b* and for condition $X_b = +1{,}000$ in Fig. 93*c*.

The live load for this structure moves along the top chord. Therefore the values δ_{ma} required are the deflections of the joints of the top chord. These might be obtained by drawing a Williot diagram for the structure, by successive positions of the unit load and the application of Castigliano's law or of the theorem of virtual work, or by using the bar-chain procedure.

If the last mentioned is adopted, two methods are available: First, to use the bottom chord as the bar chain, find the deflections of its joints, and, from these, find the deflections of the joints of the top chord; and second, to use the top chord as the bar chain. The use of the top chord is the more convenient so far as finding the values of δ_{ma} is concerned, but, looking ahead

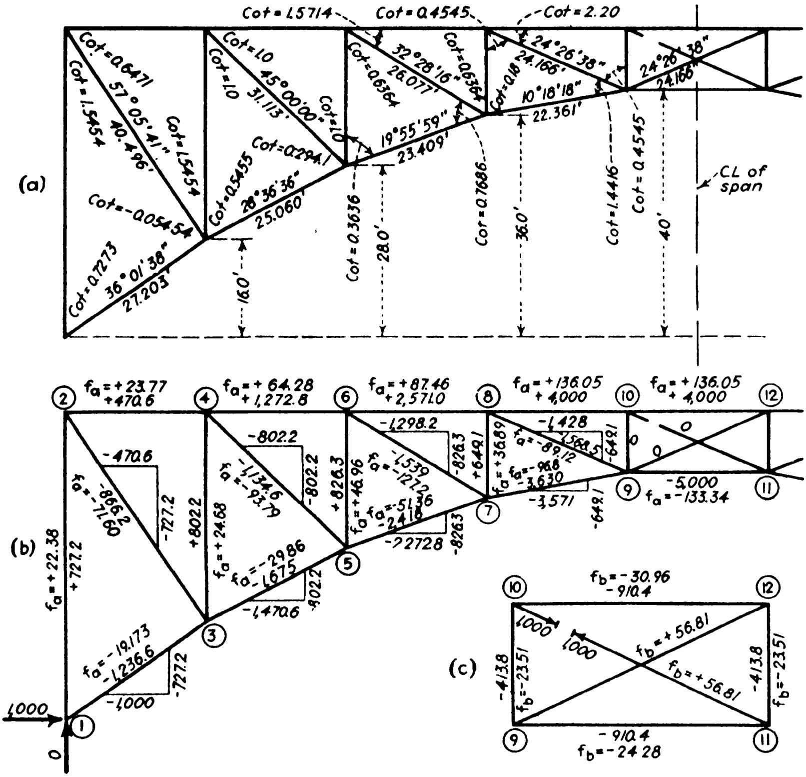


Fig. 93.

to the problem of finding δ_{aa}, the use of the bottom chord may offer advantages. This is because one may use Eq. (55) to find δ_{aa} without much labor if the bottom chord is used. Finding first the changes $d\phi$, the tabular form used is different from that used before because there is so much variation in the angles of the triangles forming the truss. For condition $X_a = +1{,}000$, $E\,d\alpha = (f_a - f_b)\cot\gamma + (f_a - f_c)\cot\beta$.

Angle	$f_a - f_b$	cot γ	$f_a - f_c$	cot β	1st term	2d term	$E\,d\theta$
1-3-2	+ 22.38 + 19.17 = + 41.55	0.7273	+ 22.38 + 71.60 = + 93.98	1.5454	+ 30.21	+145.23	
2-3-4	+ 23.77 + 71.60 = + 95.37	0.6471		0	+ 61.71	0.0	− 99.87
4-3-5	− 93.79 − 24.68 = −118.47	1.0	− 93.78 + 29.86 = − 63.93	0.2941	−118.47	− 18.81	
3-5-4	+ 24.68 + 29.86 = + 54.54	0.5454	+ 24.68 + 93.79 = +118.47	1.0	+ 29.74	+118.47	
4-5-6	+ 64.28 + 93.79 = +158.07	1.0		0	+158.07	0.0	−137.16
6-5-7	−127.2 − 46.96 = −174.16	0.6364	−127.2 + 51.36 = − 75.84	0.7686	−110.83	− 58.29	
5-7-6	+ 46.96 + 51.36 = + 98.32	0.3636	+ 46.96 + 127.2 = +174.16	0.6364	+ 35.75	+110.83	
6-7-8	+ 87.46 + 127.2 = +214.66	1.5714		0	+337.30	0	−437.67
8-7-9	− 89.12 − 36.89 = −126.01	0.4545	− 89.12 + 96.80 = + 7.68	1.4416	− 57.28	+ 11.07	
7-9-8	+ 36.89 + 96.80 = +133.69	0.1818	+ 36.89 + 89.12 = +126.01	0.4545	+ 24.30	+ 57.28	
8-9-10	+136.05 + 89.12 = +225.17	2.20		0	+495.25	0.0	−876.06
10-9-12		0	+136.05 + 0.0 = +136.05	2.20	0.0	+299.23	
12-9-11		0		0	0	0.0	
9-11-12	+ 0.0 + 133.34 = +133.34	2.20	0.0 − 0.0	0	+293.30	0.9	
12-11-14		0	+136.05 + 89.12 = +225.17	2.20	0.0	+495.30	−870.18
14-11-13	+ 36.89 + 89.12 = +126.01	0.4545	+ 36.89 + 96.80 = +133.69	0.1818	+ 57.28	+ 24.30	

$$Ew_{ma} = E\,d\varphi_{ma} - f_{ma}\tan\theta_m + f_{(m+1)a}\tan\theta_{m+1}$$

m	$-f_{ma}$	$\tan\theta_m$	$+f_{(m+1)a}$	$\tan\theta_{m+1}$	$E\,d\varphi_m$	$-f_{ma}\times\tan\theta_m$	$+f_{(m+1)a}\times\tan\theta_{m+1}$	Ew_{ma}
3	+ 19.17	+0.7273	− 29.86	+0.5454	− 99.87	+13.94	−16.28	−102.21
5	+ 29.86	+0.5454	− 51.36	+0.3636	−137.16	+16.28	−18.68	−139.56
7	+ 51.36	+0.3636	− 96.80	+0.1818	−437.67	+18.68	−17.60	−436.59
9	+ 96.80	+0.1818	−133.34	0.0	−876.06	+17.60	0.0	−858.46
11	+133.34	0.0	− 96.80	−0.1818	−870.18	0.0	+17.60	−852.58
13	+ 96.80	−0.1818	− 51.36	−0.3636	−437.67	−17.60	+18.68	−436.59
15	+ 51.36	−0.3636	− 29.86	−0.5454	−137.16	−18.68	+16.28	−139.56
17	+ 29.86	−0.5454	− 19.17	−0.7273	− 99 87	−16 28	+13.94	−102.21

For condition $X_b = +1$:

Changes of Angle

Angle	$f_a - f_b$	$\cot\gamma$	$f_a - f_c$	$\cot\beta$	First term	Second term	$E\,d\varphi$
10-9-12		0 0	−30.96 − 56.81 = −87.77	2.20		−193.05	
12-9-11	−23.51 − 56.81 = −80 32	0.4545			− 36.51		+229.56
9-11-12	+56.81 + 24.28 = +81.09	2 20	+56 81 + 23.51 = +80.32	0.4545	+178 38	+36 51	−214.89

In condition $X_b = +1$, for every joint, either $\tan\theta_m = 0$ or $f_{mb} = 0$; also either $\tan\theta_{m+1} = 0$ or $f_{(m+1)b} = 0$, so that $w_m = d\varphi_m$.

To find δ_{aa}, Eq. (55) will be used. Since $\alpha = 0$ this equation becomes

$$dL = \Sigma y_m w_{ma} + \Sigma\, dL_{ma}\sec\theta_m$$

The evaluation for condition $X_a = 1000$ is in the table below.

m	y_m	Ew_{ma}	Ey_mw_{ma}	f_{ma}	L_m	$\sec\theta_m$	$E\,dL_{ma}\sec\theta_m$
1	0	0	0	0	0	0	0
3	16	−102.21	− 1635	− 19.17	27.20	1.236	− 644
5	28	−139.56	− 3905	− 29.86	25.06	1.140	− 853
7	36	−436.59	−15700	− 51.36	23.41	1.063	−1277
9	40	−858.48	−34400	− 96.80	22.36	1.016	−2195
11	40	−852.58	−34075	−133.34	22.00	1.000	−2935
13	36	−436.59	−-15700	− 96.80	22.36	1.016	−2195
15	28	−139.56	− 3905	− 51.36	23.41	1.063	−1277
17	16	−102.21	− 1635	− 29.86	25.06	1.140	− 853
19	0	0	0	− 19.17	27.20	1.236	− 644

$$\Sigma\,E\,dL_{ma}\sec\theta_m = -\ 12873$$
$$\Sigma\,Ey_mw_{ma} = -110955$$
$$E\,dL = -123828$$

Therefore $E\delta_{aa} \times 10^3 = +123828$ (points move inward).

Similarly, $\delta_{ab} = -dL$ under condition $X_b = +1000$.

m	y_m	Ew_{mb}	$Ey_m w_{mb}$	f_{mb}	L_m	sec θ_m	$E\ dL_{mb} \times$ sec θ_m
9	40	+229.56	+9177	0	22.36	1.016	0
11	40	−214.89	−8590	−24.28	22.00	1.000	−534.2

$$\Sigma\ Ey_m w_{mb} = +587.0$$
$$\Sigma\ E\ dL_{mb} \sec \theta_m = -534.2$$
$$E\ dL = +\ 52.8$$
$$E\delta_{ab} \times 10^3 = -\ 52.8$$

To find δ_{bb}, use $E\delta_{bb} = \Sigma\ F_b^2 \frac{L}{A}$.

Bar	L	A	$F_b \times 10^3$	$\frac{F_b^2 L}{A} \times 10^3$
10–12	22	29.4	− 910.4	+ 620.30
9–11	22	37.5	− 910.4	+ 486.40
9–10	10	17.6	− 413.8	+ 97.29
11–12	10	17.6	− 413.8	+ 97.29
9–12	24.17	17.6	+1000	+1373.00
10–11	24.17	17.6	+1000	+1373 00
				+4047.28

$$E\delta_{bb} \times 10^3 = 4047.3$$

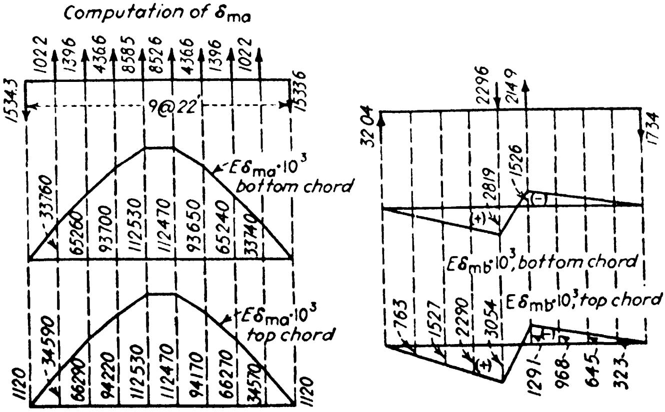


FIG. 94.

	102.2 × 1 =	102.2
	139.6 × 2 =	279.2
	436.6 × 3 =	1309.8
	852.6 × 4 =	3410.4
	858.5 × 5 =	4292.5
	436.6 × 6 =	2619.6
	139.6 × 7 =	977.2
	102.2 × 8 =	817.6
Σw_m =	3067.9	9)13808.5
V_{el} =	1534.3	1534.3 down
V_{er} =	1533.6 down	

	$E\,\delta_{ma} \times 10^3$ bottom chord	$E\,dL \times 10^3$ verticals	$E\,\delta_{ma} \times 10^3$ top chord
−1534.3 × 22 =	− 33755	+ 839	− 34594
+ 102.2			
−1432.1 × 22 =	− 31505		
+ 139.6	− 65260	+1033	− 66293
−1292.5 × 22 =	− 28440		
+ 436.6	− 93700	+ 516	− 94216
− 855.9 × 22 =	− 18828		
+ 858.5	−112528	0	−112528
+ 2.6 × 22 =	+ 57		
+ 852.6	−112471	0	−112471
+ 855.2 × 22 =	+ 18818		
+ 436.6	− 93653	+ 516	− 94169
+1291.8 × 22 =	+ 28418		
+ 139.6	− 65235	+1033	− 66268
+1431.4 × 22 =	+ 31500		
+ 102.2	− 33735	+ 839	− 34574
+1533.6 × 22 =	+ 33735		
	0	+1119	− 1119

+214.9 × 4 =	+ 859.6
−229.6 × 5 =	−1148.0
− 14.7	9)− 288.4
32.04	32.04 up
+ 17.34 up	

	$E\delta_{mb} \times 10^3$ bottom chord	$E\,dL \times 10^3$ verticals	$E\delta_{mb} \times 10^3$ top chord
$-\ 32.04 \times 88 =$	$+2819$	-235.1	$+3{,}054$
-229.6			
$-197.56 \times 22 =$	-4345		
$+214.9$	-1526	-235.1	$-1{,}291$
$+\ 17.34 \times 88 =$	$+1526$		

The equations available for the determination of the various values of the redundant forces, set up and solved in tabular form as shown on p. 184.

The influence lines for X_a and X_b are shown in Fig. 95.

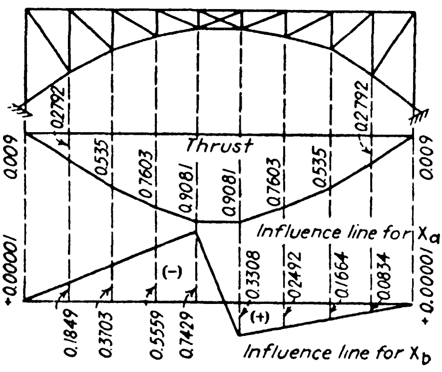


FIG. 95.

42. Influence Lines for Beams with Fixed Ends.—Influence lines for moment-resisting structures may be constructed by the same general methods available for trussed structures, although it may be convenient to employ some of the special methods for determining deflections in moment-resisting structures which were demonstrated in Chap. II. The variation in methods of computing deflections leads to variations in ways of drawing influence lines. Consideration of the beam with fixed ends shown in Fig. 96*a* will permit an illustration of these methods.

X_a	X_b	Unit load at joint									
		2	4	6	8	10	12	14	16	18	20
+123.83 − 0.0528	−0.0528 +4.0473	−1.12 0	−34.59 + 0.763	−66.29 + 1.527	−94.22 + 2.29	−112.53 + 3.054	−112.47 − 1.291	−94.17 − 0.968	−66.27 − 0.645	−34.57 − 0.323	−1.12 0
+ 0.0528	−0.0002	−0.00005	− 0.01475	− 0.02826	− 0.04018	− 0.04798	− 0.04795	− 0.04015	− 0.02826	− 0.01474	−0.00005
	+4.0471	−0.00005	+ 0.7482	+ 1.4987	+ 2.2498	+ 3.0060	− 1.3389	− 1.0082	− 0.6733	− 0.3377	−0.00005
	X_b =	+0.00001	− 0.1849	− 0 3703	− 0.5559	− 0.7429	+ 0.3308	+ 0.2492	+ 0.1664	+ 0.0834	+0.00001
	−0.0528X_b =	0	+ 0.0098	+ 0.0196	+ 0.0293	+ 0.0392	− 0.0175	− 0.0132	− 0.0088	− 0.0044	0
+123.83X_a =		+1.12	+34.58	+66.27	+94 19	+112 49	+112.49	+94.18	+66.28	+34.57	+1.12
X_a =		+0.0090	+ 0.2792	+ 0.5350	+ 0 7603	+ 0 9081	+ 0.9081	+ 0 7602	+ 0.5350	+ 0.2791	+0.0090

If axial stresses are neglected, as they may be when the loads have no axial components, we may treat the beam as though it were statically indeterminate to the second degree only. Suppose it is desired to draw the influence lines for the redundant reactions of the beam under consideration, basing the solution upon the law of virtual work. Assuming no abutment yielding or temperature change

$$\delta_{am} + X_a\delta_{aa} + X_b\delta_{ab} = 0$$
$$\delta_{bm} + X_a\delta_{ba} + X_b\delta_{bb} = 0$$

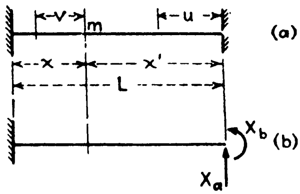


FIG. 96.

If the vertical-reaction component and the fixing moment at the right support are chosen as redundants, the statically determinate primary beam is that shown in Fig. 96*b*.

For condition $X_a = +1$

$$EI\delta_{aa} = \int_0^L u^2\,du = \left[\frac{u^3}{3}\right]_0^L = +\frac{L^3}{3}$$
$$EI\delta_{ba} = \int_0^L u(+1)du = \left[\frac{u^2}{2}\right]_0^L = +\frac{L^2}{2}$$

For condition $X_b = +1$

$$EI\delta_{bb} = \int_0^L (+1)(+1)du = \left[u\right]_0^L = +L$$

For condition $X = 0$

$$EI\delta_{am} = \int_x^0 (-v)(x' + v)dv = \int_0^x (-vx' - v^2)dv$$
$$= \left[-\frac{x'v^2}{2} - \frac{v^3}{3}\right]_0^x = -\frac{x'x^2}{2} - \frac{x^3}{3}$$
$$EI\delta_{bm} = \int_0^x (-v)(+1)dv = -\left[\frac{v^2}{2}\right]_0^x = -\frac{x^2}{2}$$

The equations to be solved, with both sides multiplied by EI, become

$$-\frac{x'x^2}{2} - \frac{x^3}{3} + X_a\frac{L^3}{3} + X_b\frac{L^2}{2} = 0$$
$$-\frac{x^2}{2} + X_a\frac{L^2}{2} + X_bL = 0$$

which solved simultaneously, lead to

$$X_a = \frac{x^2}{L^2} + 2\frac{x^2x'}{L^3}$$

$$X_b = -\frac{x^2x'}{L^2}$$

In the foregoing illustration, the deflections δ_{aa}, δ_{ba}, δ_{bb}, δ_{am}, and δ_{bm} might have been computed by the moment-area theorems. Referring to Fig. 97*b* where the moment curve for $X_a = +1$ is shown, one may write

$$EI\delta_{aa} = \frac{L}{2}L\frac{2}{3}L = +\frac{L^3}{3}$$

$$EI\delta_{ba} = \frac{L}{2}L = +\frac{L^2}{2}$$

$$EI\delta_{am} = -\frac{Lx}{2}\frac{2}{3}x - \frac{x'x}{2}\frac{x}{3}$$

$$= -\frac{Lx^2}{3} - \frac{x'x^2}{6} = -\frac{x'x^2}{2} - \frac{x^3}{3}$$

These deflections are shown in Fig. 97*c*. Referring to Fig. 97*d*, which gives the moment curve for $X_b = +1$,

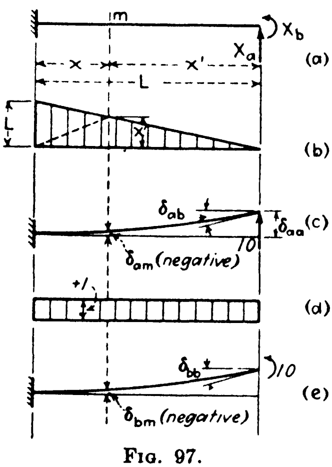


FIG. 97.

$$EI\delta_{bb} = L(+1) = +L$$

$$EI\delta_{bm} = -x(1)\left(\frac{x}{2}\right) = -\frac{x^2}{2}$$

These deflections are shown in Fig. 97*e*. They agree with the results obtained by the method of virtual work and lead, of course, to the same solution.

The same problem may be solved by using the slope-deflection equations (76). It is to be noted that angles θ_a, θ_b, and ψ, and end moments M_a and M_b are taken as positive when they are clockwise. The equations

$$M_a = \frac{2EI}{L}(2\theta_a + \theta_b - 3\psi) + \frac{2}{L^2}[(M_s)_{0a} - 2(M_s)_{0b}]$$
$$M_b = \frac{2EI}{L}(\theta_a + 2\theta_b - 3\psi) + \frac{2}{L^2}[2(M_s)_{0a} - (M_s)_{0b}]$$

become, since $\theta_a = \theta_b = \psi = 0$,

$$M_a = \frac{2}{L^2}[(M_s)_{0a} - 2(M_s)_{0b}]; \qquad M_b = \frac{2}{L^2}[2(M_s)_{0a} - (M_s)_{0b}$$

Referring to Art. 33 in evaluating $(M_s)_{0a}$ and $(M_s)_{0b}$,

$$M_a = -\frac{x(x')^2}{L^2}$$
$$M_b = +\frac{x^2x'}{L^2} = -X_b \qquad \text{(see Fig. 97)}$$

Taking moments about the left end of the beam,

$$+1(x) + M_b + M_a - X_aL = 0; \qquad X_a = \frac{x}{L} + \frac{x^2x'}{L^3} - \frac{x(x')^2}{L^3}$$
$$\frac{x^2}{L^2} + \frac{2x^2x'}{L^3}$$

The results agree with those obtained by the method of virtual work.

43. Influence Lines for Beams with Restrained Ends and for Continuous Beams. Use of Statically Indeterminate Primary Structures.—If one considers the equation

$$\delta_a = \delta_{a0} + X_a\delta_{aa} + X_b\delta_{ab} + X_c\delta_{ac} + \cdots + \delta_{at} \qquad (a)$$

where δ_{a0} is the deflection of point a of a statically determinate structure under the known loads only, δ_{aa} is the deflection of point a of the primary structure due to $X_a = +1$, δ_{ab} is the deflection of point a due to $X_b = +1$, and so on, it is evident that this is merely an application of the law of superposition of effects to the deflection of point a. If a structure statically indeterminate to the nth degree were under consideration, the same idea could be expressed by the equation

$$\delta_a = \delta'_{a0} + X_a\,\delta'_{aa} + \delta'_{at} \qquad (b)$$

in which δ'_{a0} means the deflection of point a of a primary structure which is just like the original structure except that one redundant restraint has been replaced by the force, or moment, designated by X_a. This primary structure is redundant to the degree

$(n - 1)$. If δ_a is known, as it must be before the structure can be analyzed, and if one can evaluate δ'_{a0} and δ'_{aa}, X_a can be found without solving simultaneously n equations of the form of Eq. (a). This idea may be used to advantage in drawing influence lines for restrained and continuous beams and, in fact, is the basis of the mechanical methods of analysis such as the Begg's method or the Gottschalk "Continostat."

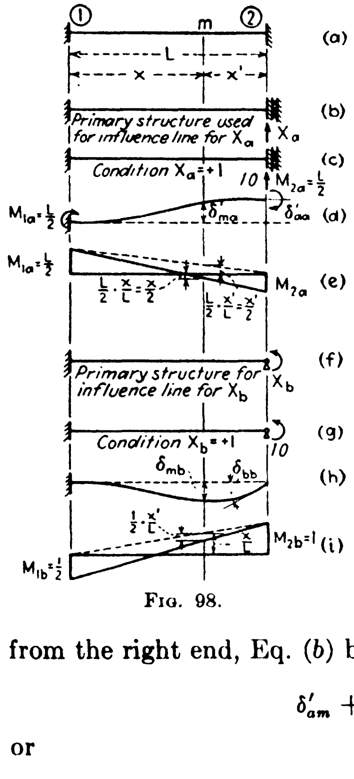


FIG. 98.

As a first illustration the beam with fixed ends as shown in Fig. 98a will be considered, drawing, first, the influence line for the vertical-reaction component X_a at the right end. The primary structure to be used is a beam fixed in direction at both ends and restrained against vertical movement at the left end, but having no restraint against vertical movement at the right end. If a vertical force X_a is applied to stop this vertical movement at the right end, it must be such that $\delta_a = 0$. In drawing the influence line, where the load system consists of a single unit load at point m, m being at distance x from the left end and x' from the right end, Eq. (b) becomes

$$\delta'_{am} + X_a \, \delta'_{aa} = 0$$

or

$$X_a = \frac{-\delta'_{am}}{\delta'_{aa}}$$

just as if the structure were statically indeterminate to the first degree only and the problem resolves itself into that of finding the value of δ'_{aa} and the various values of δ'_{am} as x varies, the only difference being that the primary structure is indeterminate to the first degree. There are several ways of analyzing condition $X = +1$; the moment-area theorems will be used. The change of slope between two points being equal to the area under the M/EI curve between the two points, and I being constant over the span,

$$+M_{2a}\frac{L}{2} - M_{1a}\frac{L}{2} = 0$$

Therefore $$M_{1a} = M_{2a}$$

Then, applying $\Sigma M = 0$ about point 1, we get

$$+M_{1a} - 1 \times L + M_{2a} = 0$$

Therefore $$M_{1a} = M_{2a} = +\frac{L}{2}$$

To compute the quantities δ'_{ma} and δ'_{aa}, the second moment-area theorem is used, which leads to

$$EI\ \delta'_{ma} = -\frac{L}{2}\frac{x}{2}\frac{2}{3}x - \frac{x'}{2}\frac{x}{2}\frac{x}{3} + \frac{x}{2}\frac{x}{2}\frac{x}{3}$$
$$= -\frac{Lx^2}{6} - \frac{x^2x'}{12} + \frac{x^3}{12}$$

$EI\ \delta'_{aa}$ is the minus value of $EI\ \delta'_{ma}$ for $x' = 0$, $x = L$, hence

$$EI\ \delta'_{aa} = +\frac{L^3}{6} - \frac{L^3}{12} = \frac{L^3}{12}$$

$EI\delta'_{ma}$ being positive downward and $EI\delta'_{aa}$ being positive upward.

$$X_a = -\frac{EI\delta'_{ma}}{EI\delta'_{aa}} = -\left[\frac{-2Lx^2 - x^2x' + x^3}{L^3}\right]$$
$$= \frac{x^2(2x + 2x' + x' - x)}{L^3}$$
$$= \frac{x^2(x + 3x')}{L^3}$$
$$= \frac{x^2L + 2x^2x'}{L^3}$$
$$= \frac{x^2}{L^2} + 2\frac{x^2x'}{L^3}$$

which agrees with the expression of Art. 42.

In finding the influence line for X_b, the primary structure to be used is as shown in Fig. 98*f*. In order to find M_{1b} one may use the second moment-area theorem to express the vertical deflection of point 2, which must be zero.

$$M_{1b}\frac{L}{2}\frac{2}{3}L - 1\frac{L}{2}\frac{L}{3} = 0$$

$$M_{1b} = \frac{1}{6} \times 3 = \frac{1}{2}$$

Then

$$\begin{aligned} EI\,\delta'_{mb} &= +\frac{1}{2}\frac{x}{2}\frac{2}{3}x + \frac{x'}{2L}\frac{x}{2}\frac{x}{3} - \frac{x}{L}\frac{x}{2}\frac{x}{3} \\ &= +\frac{x^2}{6} + \frac{x^2x'}{12L} - \frac{x^3}{6L} = +\frac{x^2}{12L}(2L + x' - 2x) = +\frac{x^2}{12L}(3x') \\ &= +\frac{x^2x'}{4L} \end{aligned}$$

In this case δ'_{bb} is the change of slope between points 1 and 2 and is positive counterclockwise. Using the first moment-area theorem,

$$EI\,\delta'_{bb} = -\frac{1}{2}\frac{L}{2} + 1\frac{L}{2} = +\frac{L}{4}$$

Therefore

$$X_b = -\frac{EI\,\delta'_{mb}}{EI\,\delta'_{bb}} = -\frac{x^2x'}{4L}\frac{4}{L} = -\frac{x^2x'}{L^2}$$

which also agrees with the results obtained previously.

It is not suggested that this method is any better in this particular problem than methods presented previously, but it is a good illustration of the use of the indeterminate primary structure.

It would be possible also to use, instead of $X_a = +1$, $X_a = +Q$, where Q is some force whose magnitude is chosen arbitrarily. In such circumstances one would determine, by a procedure exactly the same as that just used, the quantities $EI\,\delta'_{ma}\,Q$ and $EI\,\delta'_{aa}\,Q$ and could write

$$X_a = -\frac{EIQ\,\delta'_{ma}}{EIQ\,\delta'_{aa}}$$

It does not matter whether the deflections are computed as shown or whether one uses a model, producing the deflections

$Q\ \delta'_{ma}$ and $Q\ \delta'_{aa}$ by the application of an arbitrary force $X_a = +Q$ and measuring the deflections produced, or, what amounts to the same thing, introducing an arbitrary deflection $Q\ \delta'_{aa}$ and measuring the resulting deflections $Q\ \delta'_{ma}$, and obtaining the influence ordinates as the various values of

$$-\frac{Q\ \delta'_{ma}}{Q\ \delta'_{aa}}$$

Similarly, to obtain the influence line for X_b, one could introduce an arbitrary deflection $Q\ \delta'_{bb}$, measure the resulting deflections $Q\ \delta'_{mb}$, and obtain the influence ordinates as the various values of the ratio

$$-\frac{Q\ \delta'_{mb}}{Q\ \delta'_{bb}}$$

This procedure can be followed for any structure, no matter what the degree of indeterminacy, and is, in fact, the procedure followed in the Beggs method of mechanical analysis. It should be pointed out that the theory is applicable only when the distortion of the structure is small, so small, in fact, that it may be considered infinitesimal and that infinitesimals of the second order may be neglected in comparison with the first. This requirement is satisfied in the Beggs method, where the distortions used are so small that microscopes are used to measure them. In the use of the Gottschalk continostat, however, the distortions of the model are usually so great in comparison with the dimensions of the model that they cannot be considered infinitesimal and, consequently, the results obtained must be approximate only.

This same idea may be used to advantage in drawing influence lines for the redundant forces in continuous beams. Consider the beam shown in Fig. 99*a*. To draw the influence line for the moment over support (2), use the primary structure in Fig. 99*b*. Condition $X_a = +1$ is shown in Fig. 99*c*. In order to determine the deflections δ'_{ma}, draw the curve of bending moments for condition $X_a = +1$ and use the method of elastic loads, using as the elastic load the curve of bending moment just mentioned. The δ'_{ma} curve is shown in Fig. 99*e*. The ordinate to the moment curve for $X_a = +1$ at joint 2 is unity, and the curve is completely defined as soon as the ordinates at joints 0, 1, and 3 have been determined. To find these ordinates consider the equations arising from the application of the equation of three moments to condition $X_a = +1$; $I_1 = I_2 = I_3 = I_4$.

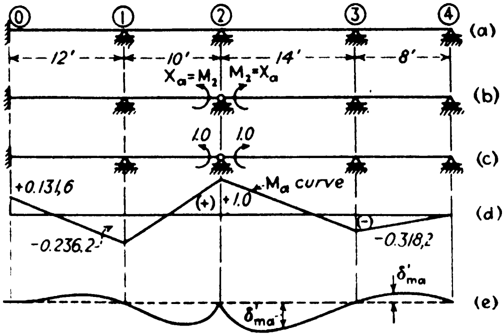


FIG. 99.

$$2M_0 \times 12 + 12M_1 = 0$$

$$M_0 = -\frac{1}{2}M_1$$

$$12M_0 + 2M_1(12 + 10) + 10M_2 = 0$$

in which $M_2 = +1$.

$$M_1(24 - 6 + 20) + 10 = 0$$

$$M_1 = -\frac{10}{38} = -0.2632$$

$$M_0 = +0.1316$$

Also
$$14M_2 + 2M_3(14 + 8) + 8M_4 = 0$$

in which $M_2 = +1$, $M_4 = 0$

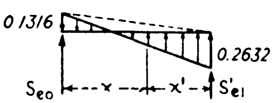


FIG. 100.

$$+14 + 44M_3 = 0$$

$$M_3 = -\frac{14}{44} = -0.3182$$

To compute the δ'_{ma} curve for span 0-1

$$12S_{e0} = +0.1316 \times \frac{12}{2} \times 8 - 0.2632 \times \frac{12}{2} \times 4$$

$$S_{e0} = +0.5264 - 0.5264 = 0$$

which is what should be expected, since the beam is fixed at point O and the reaction due to elastic loads is equal to the slope at that point.

$$S'_{e1} = -0.2632 \times \frac{12}{2} + 0.1316 \times \frac{12}{2} = -0.7896$$

$$EI\ \delta'_{ma} = -0.1316\frac{x}{2}\frac{2}{3}x - 0.1316\frac{x'}{12}\frac{x}{2}\frac{x}{3} + 0.2632\frac{x}{12}\frac{x}{2}\frac{x}{3}$$

$$= -0.0438\dot{6}x^2 - 0.001829x'x^2 + 0.003657x^3$$

x	x'	$-0.0438\dot{6}x^2$	$-0.001829x'x^2$	$+0.003657x^3$	$EI\ \delta'_{ma}$
0	12	−0.0	0.0	0.0	0.0
2	10	−0.1755	−0.0731	+0.0293	−0.2193
4	8	−0.7019	−0.2341	+0.2340	−0 7020
6	6	−1 5792	−0.3951	+0.7899	−1 1844
8	4	−2.8075	−0.4682	+1.8724	−1.4033
10	2	−4.3867	−0.3658	+3.6570	−1.0955
12	0	−6.3168	0.0	−6.3168	0 0

To compute $EI\ \delta'_{ma}$ for span 1-2,

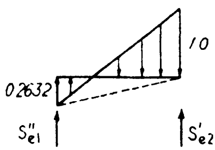


FIG. 101.

$$10S''_{e1} = -0.2632 \times \frac{10}{2} \times \frac{2}{3} \times 10 + 1.0 \times \frac{10}{2} \times \frac{10}{3}$$

$$S''_{e1} = -0.877\dot{3} + 1.\dot{6} = +0.789\dot{3}$$

$$10S'_{e2} = -0.2632 \times \frac{10}{2} \times \frac{10}{3} + 1.0 \times \frac{10}{2} \times \frac{2}{3} \times 10$$

$$= -4.38\dot{6} + 33.\dot{3} = +28.94\dot{6}$$

$$S'_{e2} = +2.894\dot{6}$$

$$EI\ \delta'_{ma} = +0.789\dot{3}x + 0.2632\frac{x}{2}\frac{2}{3}x + 0.2632\frac{x'}{10}\frac{x}{2}\frac{x}{3} - \frac{x}{10}\frac{x}{2}\frac{x}{3}$$

$$= +0.789\dot{3}x + 0.0877\dot{3}x^2 + 0.004386x'x^2 - 0.01\dot{6}x^3$$

x	x'	$+0.7893x$	$+0.0877\dot{3}x^2$	$+0.004386x'x^2$	$-0.0167x^3$	$EI\ \delta_{ma}$
0	10	0.0	0.0	0.0	0.0	0.0
2	8	+1.5786	+0.3509	+0.1404	$-\ 0.1\dot{3}$	$+1.93\dot{6}$
4	6	+3.1572	+1.4037	+0.4211	$-\ 1.0\dot{6}$	+3.915
6	4	+4.7358	+3.1584	+0.6317	− 3.60	+4.926
8	2	+6.3144	+5.6149	+0.5615	$-\ 8\ 5\dot{3}$	+3.958
10	0	+7.893	+8.7733	0.0	$-16.\dot{6}$	0.0

To compute $EI\ \delta'_{ma}$ for span 2-3,

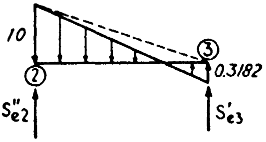


FIG. 102.

$$14S''_{e2} = +1.0 \times \frac{14}{2} \times \frac{2}{3} \times 14 + 0.3182 \times \frac{14}{2} \times \frac{14}{3} = \frac{14^2}{6}(+2.0 - 0.3182)$$

$$S''_{e2} = \frac{14}{6}(+1.6818) = +3.924$$

$$14S'_{e3} = +1.0 \times \frac{14}{2} \times \frac{14}{3} - 0.3182 \times \frac{14}{2} \times \frac{2}{3} \times 14$$

$$= \frac{14^2}{6}(+1.0 + 0.6364)$$

$$S'_{e3} = \frac{7}{3}(+0.3636) = +0.8484$$

$$EI\ \delta'_{ma} = +3.924x - 1\frac{x}{2}\frac{2}{3}x - 1\frac{x'}{14}\frac{x}{2}\frac{x}{3} + 0.3182\frac{x}{14}\frac{x}{2}\frac{x}{3}$$

$$= +3.924x - \frac{x^2}{3} - \frac{x'x^2}{84} + 0.003788x^3$$

x	x'	$+3.924x$	$-\frac{x^2}{3}$	$-\frac{x'x^2}{84}$	$+0.003788x^3$	$EI\ \delta'_{ma}$
0	14	0.0	0.0	0.0	0.0	0.0
2	12	+ 7.848	− 1.3	−0.5714	+ 0.0303	+5.974
4	10	+15.696	− 5.3	−1.9048	+ 0.2424	+8.700
6	8	+23.544	−12.0	−3.4286	+ 0.8182	+8.933
8	6	+31.392	−21 3	−4.5714	+ 1.9395	+7.427
10	4	+39.240	−33.3	−4.7619	+ 3.7880	+4.933
12	2	+47.088	−48.0	−3.4286	+ 6.5456	+2.205
14	0	+54.936	−65.3	0.0	+10.3943	0.0

To compute $EI\ \delta'_{ma}$ for span 3-4,

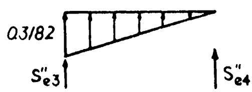


FIG. 103.

$$8S''_{e3} = -0.3182 \times \frac{8}{2} \times \frac{2}{3} \times 8$$

$$S''_{e3} = -\frac{2.5456}{3} = -0.8485$$

$$S_{e4} = -0.4242$$

$$EI\ \delta'_{ma} = -0.4242x' + 0.3182\frac{x'}{8}\frac{x'}{2}\frac{x'}{3}$$

$$= -0.4242x' + 0.006629x'^3$$

x	x'	$-0.4242x'$	$+0.006629x'^3$	$EI\ \delta'_{ma}$
0	8	−3.3940	+3.3940	0.0
2	6	−2.5455	+1.4319	−1.1136
4	4	−1.6970	+0.4243	−1.2727
6	2	−0.8485	+0.0530	−0.7955
8	0	−0.0	0.0	0.0

In this problem δ'_{aa} is the relative rotation of the two tangents to the elastic curve for condition $X_a = +1$, one just to the left of point of support 2, and the other just to the right of that point. The slopes of these two tangents are the reactions S'_{e2} and S''_{e2}, but as compared here both slopes have been computed including the factor EI. From the computation of $EI\delta'_{ma}$ for span 1-2, we find $EI\ S'_{e2} = +2.894\dot{6}$, the positive sign indicating that the tangent has rotated in a counterclockwise direction. Similarly, from the computation of $EI\ \delta'_{ma}$ for span 2-3, we find $EI\ S''_{e2} = +3.924$ which here indicates rotation in a clockwise direction. Both of these rotations, however, are in the same directions as the corresponding moments $X_a = +1$ and therefore contribute positive terms to the quantity δ'_{aa}. Therefore,

$$EI\ \delta'_{aa} = 2.895 + 3.924 = +6.819$$

The ordinates to the influence line for X_a being equal to $-\dfrac{EI\ \delta'_{ma}}{EI\ \delta'_{aa}}$ are found as below.

Span 0-1			Span 2-3		
x	$EI\ \delta'_{ma}$	X_a	x	$EI\ \delta'_{ma}$	X_a
0	0.0	0.0	2	+5.974	−0.8765
2	−0.2193	+0.0322	4	+8.700	−1.276
4	−0.7020	+0.1030	6	+8.933	−1.3105
6	−1.1844	+0.1738	8	+7.427	−1.0895
8	−1.4033	+0.2058	10	+4.933	−0.7236
10	−1.0955	+0.1607	12	+2.205	−0.3235
12	0.0	0.0	14	0.0	0.0
Span 1-2			Span 3-4		
2	+1.936	−0.2841	2	−1.1136	+0.1633
4	+3.915	−0.5742	4	−1.2727	+0.1867
6	+4.926	−0.7226	6	−0.7955	+0.1167
8	+3.958	−0.5806	8	0.0	0.0
10	0.0	0.0			

44. The Fixed-point Method.—In cases where there are a large number of spans, it may be convenient to have some of the labor of solving the sets of three-moment equations done in advance, as much, in fact, as can be done without taking into account the details of the particular problem in hand. Referring to Fig. 104, which illustrates the condition which may be expected between

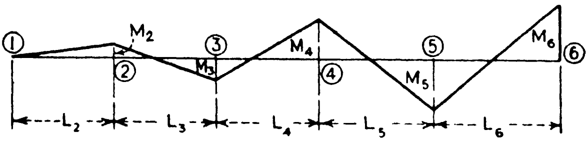


FIG. 104.

the point of application of a redundant moment and the left end of a beam supported over a number of spans, equations may be written as follows:

$$
\begin{aligned}
M_1L_1' + 2M_2(L_2' + L_3') + M_3L_3' &= 0\\
M_2L_3' + 2M_3(L_3' + L_4') + M_4L_4' &= 0\\
M_3L_4' + 2M_4(L_4' + L_5') + M_5L_5' &= 0\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
M_{r-1}L_r' + 2M_r(L_r' + L_{r+1}') + M_{r+1}L_{r+1}' &= 0
\end{aligned}
$$

where $L_r' = L_r/I_r$

If $M_1 = 0$, $M_2 = -\mu_2 M_3$, where $\mu_2 = \dfrac{L_3'}{2(L_2' + L_3')}$

The second equation becomes

$$M_3[-\mu_2L_3' + 2(L_3' + L_4')] + M_4L_4' = 0$$

from which

$$M_3 = -\mu_3M_4, \qquad \text{where} \qquad \mu_3 = \frac{L_4'}{-\mu_2L_3' + 2(L_3' + L_4')}$$

Similarly, the third equation becomes

$$M_4[-\mu_3L_4' + 2(L_4' + L_5')] + M_5L_5' = 0$$

from which

$$M_4 = -\mu_4M_5, \qquad \text{where} \qquad \mu_4 = \frac{L_5'}{-\mu_3L_4' + 2(L_4' + L_5')}$$

In general, one may write

$$M_{r-1} = -\mu_{r-1}M_r, \qquad \text{where} \qquad \mu_{r-1} = \frac{L_r'}{-\mu_{r-2}L_{r-1}' + 2(L_{r-1}' + L_r')}$$

Since μ_r is always less than 1 and is positive, the support moments M_{r-1}, M_r, M_{r+1}, etc., must alternate in sign. Hence the moment curve for such a condition is as shown in Fig. 104, and the ratio between the support moments $M_{r-1}/M_r = -\mu_{r-1}$. It is evident, also, that the ratio between the distances of the point of inflection in a span from the span's outer and inner ends, respectively, is the μ for that span. If the outer end of the continuous beam had been fixed, *i.e.*, at support r, $L'_r = 0$ and the coefficient for the end span becomes

$$\mu_r = \frac{L'_{r+1}}{-\mu_{r-1}(0) + 2(0 + L'_{r+1})} = \frac{1}{2}$$

For illustration, this idea will be applied in the determination of the influence line for the moment at support 3 of the beam shown in Fig. 105. I is constant throughout the 5 spans;

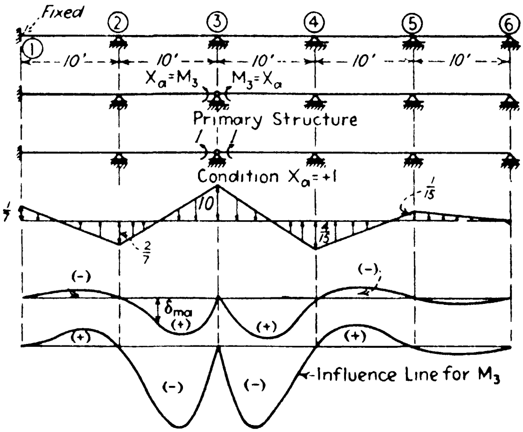


FIG. 105.

therefore one may use L_r in place of L'_r. To draw the curve of bending moment for condition $X_a = +1$,

$$\mu_1 = \frac{1}{2} \qquad\qquad M_3 = +1$$

$$\mu_2 = \frac{10}{-\frac{1}{2} \times 10 + 2(10 + 10)} = \frac{10}{35} = \frac{2}{7} \qquad M_2 = -\frac{2}{7}$$

$$M_1 = +\frac{1}{7}$$

Using μ'_r to indicate the coefficients for the right-hand part of the beam,

$$\mu'_r = \frac{L'_r}{-\mu_{r+1}L'_{r+1} - 2(L'_{r+1} + L'_r)}$$

$$\mu'_6 = 0 \qquad\qquad M_3 = +1$$

$$\mu'_5 = \frac{10}{2(10 + 10)} = \frac{1}{4} \qquad\qquad M_4 = -\frac{4}{15}(+1) = -\frac{4}{15}$$

$$\mu'_4 = \frac{10}{-\frac{1}{4} \times 10 + 2(10 + 10)} = \frac{10}{37.5} = \frac{4}{15}$$

$$M_5 = -\frac{1}{4}\left(-\frac{4}{15}\right) = +\frac{1}{15}; \qquad M_6 = 0$$

The moment curve being obtained in this way, values of δ'_{ma} and δ'_{aa} may be computed by methods already illustrated, and the influence line for M_3 easily obtained.

It might be well to point out that as a further alternative approach to the determination of the ordinates to the moment

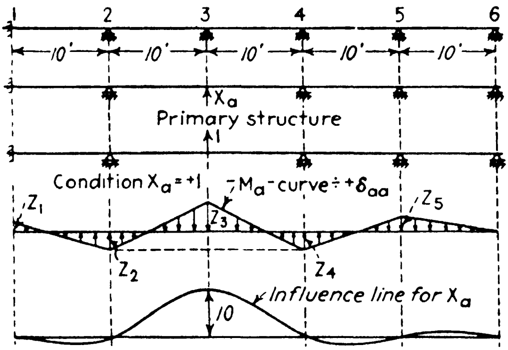


FIG. 106.

curve for condition $X_a = +1$ the method of moment distribution may be used.

The same general procedure may be followed when the redundant under investigation is a reaction, though in this case it will be convenient to change some of the details. For illustration, the influence line for the reaction at support 3 of the beam in Fig. 105 will be drawn (see Fig. 106). Since the influence line for X_a is obtained from the $-\delta_{ma}$ curve by dividing each of the $-\delta_{ma}$ ordinates by $+\delta_{aa}$, it is possible to think of the influence line as the deflection curve obtained by using an elastic loading whose ordinates are the ordinates $-M_a/EI$ divided by $+\delta_{aa}$, in which case the moment curve for the elastic load has an ordinate of one at support 3. Also, since the elastic reaction for any span represents the slope of the beam at that point, the elastic reaction at support 1 is zero. The sum of the elastic reactions at support 2 from span 1-2 and span 2-4 is zero. Similarly the total elastic reaction at support 4 is zero as is also the total elastic reaction at support 5. These facts, together with the coefficients μ as used in the previous problem, should serve to determine the ordinates z which define the elastic-load curve. Thus,

$$z_1 = -\mu_1 z_2 = \frac{10}{-\mu_0(0) + 2(0 + 10)}z_2 = -\frac{1}{2}z_2$$

$$z_6 = -\mu_6 z_5 = 0, \qquad \text{Therefore} \qquad \mu_6 = 0$$

$$\mu_5 = \frac{10}{0(10) + 2(10 + 10)} = \frac{1}{4}, \qquad z_5 = -\frac{1}{4}z_4$$

For span 2-4,

$$20S_2'' = z_2\frac{10}{2}\left(10 + \frac{2}{3} \times 10\right) + z_3\frac{10}{2}\left(10 + \frac{10}{3} + \frac{2}{3}10\right) + z_4\frac{10}{2} \times \frac{10}{3}$$

$$= z_2\frac{10^2}{6}(5) + z_3\frac{10^2}{6}(6) + z_4\frac{10^2}{6}$$

$$S_2'' = \frac{5}{6}(5z_2 + 6z_3 + z_4)$$

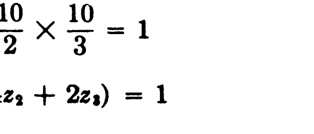


FIG. 107.

Since the deflection at joint 3 = 1,

$$10S_2'' - z_2\frac{10}{2} \times \frac{2}{3}10 - z_3\frac{10}{2} \times \frac{10}{3} = 1$$

$$\frac{50}{6}(5z_2 + 6z_3 + z_4) - \frac{50}{6}(4z_2 + 2z_3) = 1$$

$$\frac{50}{6}(z_2 + 4z_3 + z_4) = 1$$

For span 1-2,

$$10S_2' = z_1\frac{10}{2} \times \frac{10}{3} + z_2\frac{10}{2} \times \frac{2}{3} \times 10$$

$$10S_2' = \frac{50}{6}(2z_1 + 4z_2)$$

$$S_2' = \frac{10}{6}(z_1 + 2z_2)$$

in which $z_1 = -\frac{1}{2}z_2$

Therefore

$$S_2' = \frac{10}{6}\left(\frac{3}{2}z_2\right)$$

Similarly, $S_4'' = \frac{10}{6}(2z_4 + z_5)$, in which $z_5 = -\frac{1}{4}z_4$. hence

$$S_4'' = \frac{10}{6}\left(\frac{7}{4}z_4\right)$$

Therefore we have, from $S_2' + S_2'' = 0$,

$$\frac{5}{6}(5z_2 + 6z_3 + z_4) + \frac{5}{6}(3z_2) = 0$$

i.e.,

$$8z_2 + 6z_3 + z_4 = 0$$

and, from $S_4' + S_4'' = 0$,

$$\frac{5}{6}(z_2 + 6z_3 + 5z_4) + \frac{5}{6}\left(\frac{7}{2}z_4\right) = 0$$

$$z_2 + 6z_3 + \frac{17}{2}z_4 = 0$$

Solving these equations

z_2	z_3	z_4	Num.	
25	100	25	3	$8z_2 + 0.26652 - 0.027835 = 0$
8	6	1	0	$z_2 = -0.029836$
2	12	17	0	
8	32	8	0.96	
2	8	2	0.24	
	+ 26	+ 7	+0.96	$26z_3 - 0.194845 = +0.96$
	4	15	−0.24	$z_3 = +0.04442$
	4	+ 1.0771	+0.1477	
		+13.9229	−0.3877	$z_4 = -0.027835$

Therefore $z_1 = +0.014918$
$z_5 = +0.006959$

To find X_a, for span 1-2,

$$S_1'' = 0$$

Therefore

$$X_a = -z_1\frac{x}{2}\frac{2}{3}x - \left(z_1\frac{x'}{L_2} + z_2\frac{x}{L_2}\right)\frac{x}{2}\frac{x}{3}$$
$$= -\frac{x^2}{6}\left[+2z_1 + z_1\frac{x'}{L_2} + z_2\frac{x}{L_2}\right]$$

x	$\frac{x}{L_2}$	$\frac{x'}{L_2}$	$+2z_1$	$z_1\frac{x'}{L_2}$	$z_2\frac{x}{L_2}$	$\frac{6}{x^2}X_a$	X_a
0	0.0	1.0	+0.029836	+0.014918	−0.0	+0.044754	−0.0
1	0.1	0.9	+0.029836	+0.013426	−0.002984	+0.040278	−0.006713
2	0.2	0.8	+0.029836	+0.011934	−0.005967	+0.035803	−0.023869
3	0.3	0.7	+0.029836	+0.010443	−0.008951	+0.031328	−0.046992
4	0.4	0.6	+0.029836	+0.008951	−0.011934	+0.026853	−0.071608
5	0.5	0.5	+0.029836	+0.007459	−0.014918	+0.022377	−0.093237
6	0.6	0.4	+0.029836	+0.005967	−0.017902	+0.017901	−0.107406
7	0.7	0.3	+0.029836	+0.004475	−0.020885	+0.013426	−0.109646
8	0.8	0.2	+0.029836	+0.002984	−0.023869	+0.008951	−0.095476
9	0.9	0.1	+0.029836	+0.001492	−0.026852	+0.004476	−0.060426
10	1.0	0.0	+0.029836	+0.0	−0.029836	0.0	0.0

For other spans, the procedure is similar.

CHAPTER V

SECONDARY STRESSES IN TRUSSES

45. Introduction.—Stress analysis for trusses, as it has been discussed so far, has been based on the following assumptions: (1) The members of the structure are connected to each other by frictionless pins; (2) the external loads, assumed to include the weight of the structure, are applied at the joints; (3) the axes of the members are straight; (4) the gravity axes of the members meeting at a joint intersect in a point; (5) a truss is subjected to external forces in its own plane only. These assumptions are not in accord with the facts. Assumption (1) is never true, even in a pin-connected truss, because no member can turn on a pin without frictional resistance and in a truss with riveted joints the gusset plates exercise considerable restraint when the members tend to change their directions relative to each other as the truss is distorted under the loads. If the structure is well arranged, assumption (2) may be satisfied by the way in which the live loads and the weights of the floor system and bracing are applied, but each member of the truss must act as a beam in supporting its own weight. Assumption (3) is in error only to a small extent, this being due to the inability of manufacturers to fabricate members which are completely straight; such imperfections are important mainly in compression members and column formulas are designed to make allowance for them; they will not be discussed further here. Assumption (4) is satisfied at most of the joints in heavy trusses but is almost always untrue in the lighter roof trusses. These remarks apply to all trussed structures. It is to be remembered that very few trusses exist as complete structures; almost all are units of more complicated frameworks which are three-dimensional rather than planar. Except in three-dimensional analysis, it is assumed that a truss will be called on to carry loads in its own plane only and will not be affected by stresses in other parts of the structure. Often, however, the distortion of members lying in a plane other than that of the truss under consideration will cause a truss which was initially planar to distort in a nonplanar fashion, or, if one or more of the members of the truss act also as members of a truss lying in another plane, loads applied in this second plane may cause "participating stresses" in the members of the first truss. A typical example of such action occurs in the reciprocal effects of the stresses in the members of the vertical trusses and the stresses in the members of the lateral bracing trusses of a truss bridge.

These considerations lead to the conclusion that the customary stress analysis does not take into account many of the conditions existing in a truss and that the results obtained, the so-called "primary stresses," do not present a true picture of the condition of stress. The stresses due to conditions which have not been considered in the primary-stresses analysis are called "secondary stresses." Of these, the most important are those due to the fact that the members are not free to change their relative directions when the truss is distorted owing to the application of loads. There are several methods of making approximate analyses of the secondary stresses due to this cause. What is probably the most comprehensive presentation of these is to be found in a paper by Cecil Vivian von Abo published in the *Transactions of A.S.C.E.*, vol. 89, 1926. Of these methods, it is intended to present two and to add thereto the method of "moment distribution" suggested by Professor Hardy Cross. Each of the first two methods to be presented is based on writing a set of equations, each being the equation $\Sigma M = 0$ for one of the joints and expressing the moments as functions of certain of the characteristics of the distortion of the truss; when these have been found by simultaneous solution of the equations, one may evaluate the moments in the ends of the members and, thereafter, the stress intensities. The two methods differ primarily in the distortion characteristics selected to serve as the primary variables: In the first, the variation of the Manderla method suggested by Winkler, each moment applied to the end of a member is expressed in terms of the angles τ between the chord of the elastic curve of the member and the tangents to this curve at its ends; in the second, known in secondary-stress literature as the Mohr semigraphic method, one uses Eqs. (76) which are more commonly known as the slope-deflection equations.

46. The Manderla Method, Winkler's Variation.—When a truss distorts under load, the strains in the members are, in general, not alike and the triangles having as vertices the joint centers have shapes after distortion which are not geometrically similar to their shapes before distortion. The angles of these triangles have been changed by small increments but the members of the truss, being prevented by the resistance of the gusset plates in a riveted truss, or by friction on the pins in a pin-connected truss, from changing their relative directions must bend to accommodate themselves to the new relative positions of the joint centers. Consider, for example, a joint n, as shown in Fig. 108, and the members na, nb, nc, and nd connected

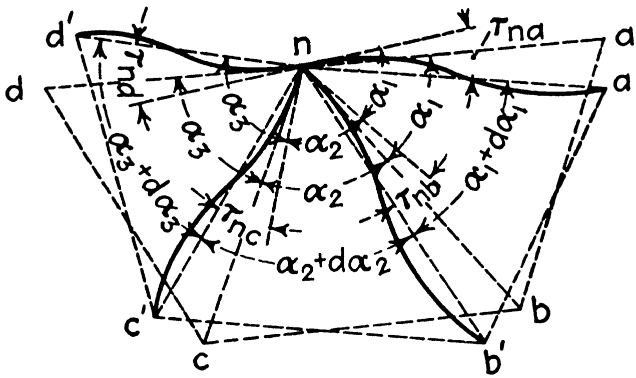


FIG. 108.

thereto. Suppose that the original positions of these joints are indicated by n, a, b, c, and d, while the relative positions after distortion of the truss are n, a', b', c', and d'. Since the joint is in equilibrium, $\Sigma M = 0$; therefore, if there are no eccentricities in the application of the total axial stresses and M_{na}, M_{nb}, M_{nc}, and M_{nd} are moments about the gravity axes at the ends of the members,

$$M_{na} + M_{nb} + M_{nc} + M_{nd} = 0$$

If there are eccentricities in the application of the axial stresses to the joint center, this equation may be written as

$$M_{na} + M_{nb} + M_{nc} + M_{nd} + M_e = 0$$

where M_e is the algebraic sum of the products of the axial stresses and their eccentricities measured from the joint center. Using

Eq. (74) and introducing the notation $K = I/L$,

$$M_{na} = 2EK_{na}(2\tau_{na} + \tau_{an})$$
$$M_{nb} = 2EK_{nb}(2\tau_{nb} + \tau_{bn})$$
$$M_{nc} = 2EK_{nc}(2\tau_{nc} + \tau_{cn})$$
$$M_{nd} = 2EK_{nd}(2\tau_{nd} + \tau_{dn})$$

Before writing the equation $\Sigma M = 0$ in terms of the angles τ, it is advisable to express each angle τ in terms of a particular τ and the changes in the angles at the joint. This particular τ is called the reference τ for the joint, and the one selected will be that for the first member encountered in moving clockwise around the joint through the internal triangles. For joint n it is τ_{na}. From Fig. 108,

$$\tau_{nb} + \alpha_1 = \tau_{na} + \alpha_1 + d\alpha_1$$

Therefore,

$$\tau_{nb} = \tau_{na} + d\alpha_1$$

Similarly,

$$\tau_{nc} + \alpha_2 + \alpha_1 = \tau_{na} + \alpha_1 + d\alpha_1 + \alpha_2 + d\alpha_2$$

or

$$\tau_{nc} = \tau_{na} + d\alpha_1 + d\alpha_2$$

and

$$\tau_{nd} = \tau_{na} + d\alpha_1 + d\alpha_2 + d\alpha_3$$

For convenience the reference τ at joint n will be called τ_n. The relations above may be summarized in the statement that the angle τ at the end of any member connected to joint n is equal to τ_n plus the sum of the changes in the angles between the reference bar at joint n and the bar under consideration. Thus the equation $\Sigma M = 0$ for joint n becomes

$$2EK_{na}\left(2\tau_n + \tau_a + \sum_a^{an} d\alpha\right) + 2EK_{nb}\left(2\tau_n + 2\sum_n^{nb} d\alpha + \tau_b + \sum_b^{bn} d\alpha\right) + 2EK_{nc}\left(2\tau_n + 2\sum_n^{nc} d\alpha + \tau_c + \sum_c^{cn} d\alpha\right) + 2EK_{nd}\left(2\tau_n + 2\sum_n^{nd} d\alpha + \tau_d + \sum_d^{dn} d\alpha\right) + M_e = 0$$

where the summation $\sum^{nc} d\alpha$ means the sum of the changes in the angles α at joint n from the reference member around to member nc. Since Eqs. (74) have been used in this development, the conventions used in those equations are applicable here also, *i.e.*, the angle τ is positive when the tangent to the elastic curve has rotated contraclockwise from the chord of the elastic curve; a moment applied to the end of a member is positive when it is counterclockwise or, conversely, a moment applied to a joint is positive when it is clockwise (see Fig. 108*a*).

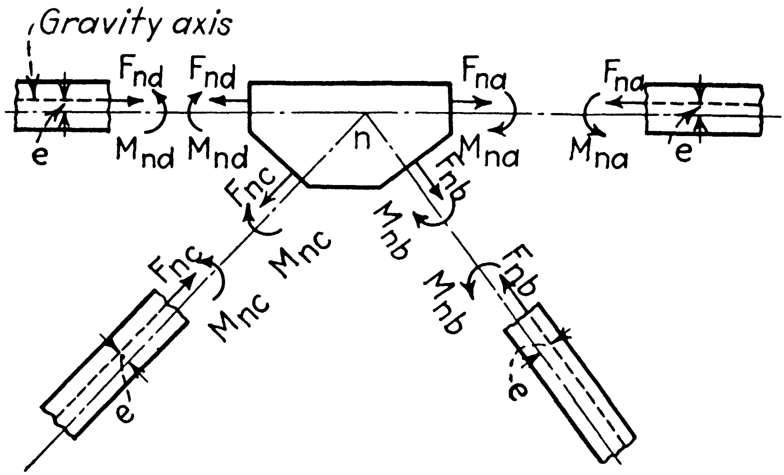


Fig. 108*a*.

If the terms in the last equation are collected and both sides of the equation are divided by two, it becomes

$$E\tau_n\, 2\sum_n K_{nm} + 2\sum_n \left\{ K_{nm}\sum_n^{nm} E\, d\alpha \right\} + K_{na}\, E\tau_a + K_{na}\sum_a^{an} E\, d\alpha$$
$$+ K_{nb}\, E\tau_b + K_{nb}\sum_b^{bn} E\, d\alpha + K_{nc}\, E\tau_c + K_{nc}\sum_c^{cn} E\, d\alpha$$
$$+ \frac{1}{2}M_e = 0 \quad (80)$$

Since the changes $d\alpha$ in the angles of the triangles outlined by the members may be computed as in Art. 22 (Eqs. 53) and since it is possible to write an equation of the form of (80) for each joint of the truss, there are, as unknowns, only the reference angles τ_n, one for each joint, and there are available just as many equations as there are unknowns. It is possible, therefore, to determine all the angles τ_n and, from them, all the angles τ such as τ_{na}, τ_{nb}, . . . Thereafter, using Eqs. (74), the moment at each end of each member may be computed and, from these, the stress intensities. Since

$$f_{nm} = \frac{M_{nm}c_{nm}}{I_{nm}} = \frac{c_{nm}}{I_{nm}}2E\frac{I_{nm}}{L_{nm}}(2\tau_{nm} + \tau_{mn})$$
$$= 2\frac{c_{nm}}{L_{nm}}(2E\tau_{nm} + E\tau_{mn}) \tag{81}$$

it is unnecessary to perform the intermediate step of computing the end moments.

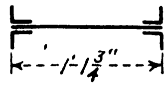
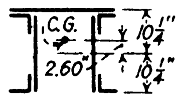
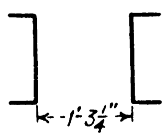
The procedure may be illustrated by applying it to a simple truss such as is shown in Fig. 109. After the characteristics of the members of the truss have been found the first step is to compute the changes $E\,d\alpha$ in the angles of the triangles of the truss for the condition of distortion under consideration. The procedure is the same as that used in Art. 22 as one of the steps in computing elastic loads. Following this, Table III is begun, the first five columns serving as a basis for the formulation of Eqs. (80) for the particular problem in hand. If there is eccentricity in the connection of any of the members, the moments M_e should be computed next and, following this, Eqs. (80) are set up and solved. The last step is to complete Table III.

In any truss the problem which is of interest is to determine the maximum stress intensity which can possibly occur in any given part of the structure. If the truss is subjected to moving loads, it would seem logical to compute the secondary stresses due to the dead load, to draw influence lines for secondary stresses in various parts of the structure, and to use these to determine the maximum magnitudes of the secondary stresses due to live load and impact. It is not generally true, however, that the position of live load which will cause the maximum secondary stress at any point is the same as that which will cause the maximum primary stress at that same point; consequently, since the live-load primary stress is almost always much greater than the maximum secondary stress, it is more likely that the worst combination of primary and secondary stresses will occur when the live load is placed so as to cause the maximum primary stress. A comparison between the ratios between secondary-stress intensities and primary-stress intensities computed, first, for dead load, live load, and impact, with the live load placed so as to cause maximum primary stress, and, second, for load distributed uniformly over the whole of the floor system, is shown in the following table.

Position in truss	Ratio (%) between secondary- and primary-stress intensity	
	D.L. + L.L. + I, L.L. in position for maximum primary stress	Load distributed uniformly over whole floor system
1-2 (top)	4.4	6.2
2-1 (bottom)	1 4	2.4
2-4 (bottom)	1 7	2.7
4-2 (top)	9 5	10 5
4-6 (top)	10.2	9.7
6-4 (top)	9.0	8.0
3-1	29.0	25.6
1-3	23.4	22 2
3-5	27.2	23 5
5-3	11.8	5.8
5-7	9.8	7.7
7-5	32.3	28.9
2-5	6.2	7.1
5-2	2.7	1.7
5-6	17.1	23.1
6-5	2.9	5.1
2-3	16.4	22.9
3-2	21.1	25.0
4-5	545.0	0
5-4	636.0	0
6-7	5.4	0
7-6	5.2	0

Since the designer of today is not usually interested in obtaining precise values for the secondary-stress intensities but desires approximate magnitudes only and since the theory suggested above is approximate in that it assumes that all members are prismatic from joint center to joint center and that the joints are reduced to surfaces of contact, the discrepancies between the two sets of values in the table above are of minor importance and the investigator may be satisfied with an analysis such as

TABLE I

Member	Length, in.	Area, sq. in.	I, in.^{4}	c, in.	K	$\frac{c}{L}$	Arrangement	
1-3 3-5	320.0	36.62	2089.2	10.75	6.524	0.0336	4 ⌊s 3½″ × 3½″ × ½″ 2 webs 21″ × 9/16″	
5-7	320.0	63.17	3219.0	10.75	10.059	0.0336	4 ⌊s 3½″ × 3½″ × ⅝″ 4 webs 21″ × 9/16″	
2-3, 4-5 6-7	348 0	19 32	119 4	6.188	0 3431	0 01777	4 ⌊s 6″ × 4″ × ⅜″ 1 web 13″ × ⅜″	
2-5	472 88	33 72	235 8	6 313	0 498	0 01335	4 ⌊s 6″ × 4″ × 11/16″ 1 web 13″ × ⅝″ × ⅝	Same as 2-3
1-2	472.88	60 92	3612 2	8 275 12 85	7 641	0 0175 0 02718	1 cov. pl. 24″ × ⅝″ 4 ⌊s 3½″ × 3½″ × ⅝″ 2 webs 20″ × ¾″ × ⅝	
2-4 4-6	320 0	56.92	3409.3	8 310 12 746	10 652	0 02596 0 03982	1 cov. pl. 24″ × 9/16″ 4 ⌊s 3½″ × 3½″ × ⅝″ 2 webs 20″ × 11/16″	Same as 1-2 except dist. to c.g. = 2.496″
5-6	472 88	29.28	802.8	7.5	1 698	0 01586	2 [s 15″ @ 50#	

that which led to the results in the second column. The illustrative problem will be carried out on this basis. Since it is the intention to compute ratios between secondary- and primary-stress intensities, it is not important that the loads to which the truss is assumed to be subjected should have any particular magnitude and a panel load of 1,000 lb. will be used. The layout of the truss and the characteristics of its members are

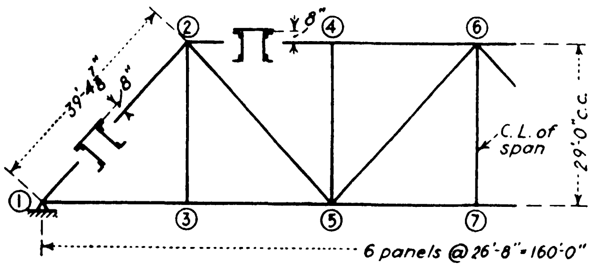


FIG. 109.

given in Fig. 109 and Table I. The stress analysis is carried out in Fig. 110 and the changes in the angles are computed in Table II.

TABLE II.—CHANGES IN ANGLES

Angle	Coeff. of cot α $\cot \alpha = \frac{26.67}{29.0}$	Coeff. of cot β $\cot \beta = \frac{29}{26.67}$	1st term	2d term	E dα
2-1-3		+51.76+55.75 = +107.51		+116.92	+116.92
3-2-1	+62.76+55.75 = +118.51		+108.97		+108.97
1-3-2	−55.75−62.76 = −118.51	−55.75−51.76 = −107.51	−108.97	−116.92	−225.89
5-2-3	+62.76−60.43 = + 2.33		+ 2.14		+ 2.14
2-3-5	+60.43−62.76 = − 2.33	+60.43−51.78 = + 8.65	− 2.14	+ 9.41	+ 7.27
3-5-2		+51.78−60.43 = − 8.65		− 9.41	− 9.41
4-2-5		0.0 −60.43 = − 60.43		− 65.72	− 65.72
5-4-2	+60.43+64.62 = +125.05	+60.43− 0.0 = + 60.43	+114.99	+ 65.72	+180.71
2-5-4	−64.62−60.43 = −125.05		−114.99		−114.99
6-4-5	−23.20+64.62 = + 41.42	−23.20− 0.0 = − 23.20	+ 38.09	− 25.23	+ 12.86
4-5-6	−64.62+23.20 = − 41.42		− 38.09		− 38.09
5-6-4		0.0 +23.20 = + 23.20		+ 25.23	+ 25.23
6-5-7		+51.76+23.20 = + 74.96		+ 81.52	+ 81.52
7-6-5	+65.50+23.20 = + 88.70		+ 81.56		+ 81.56
5-7-6	−23.20−65.50 = − 88.70	−23.20−51.76 = − 74.96	− 81.56	− 81.52	−163.08

The next step is to fill in the first five columns of Table III, which, together with the computations immediately following that table (see also Fig. 111), are used to supply the information

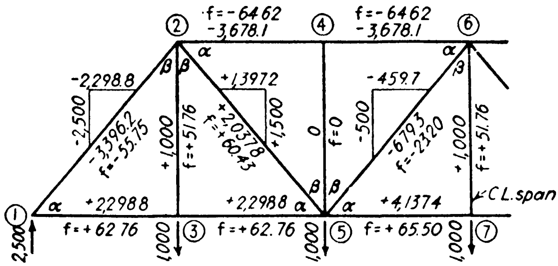


FIG. 110.

required in formulating Eqs. (80) as numerical equations. Since the structure and condition of loading are symmetrical about the vertical mid-axis, the truss in its distorted condition must be symmetrical about this axis also and, therefore, member 6-7 must remain straight. Consequently,

$$\tau_{7-6} = \tau_7 + d(5\text{-}7\text{-}6) = 0; \qquad E\tau_7 - 163.08 = 0;$$
$$E\tau_7 = +163.08$$

Similarly,

$$\tau_{6-7} = \tau_6 + d(8\text{-}6\text{-}9) + d(9\text{-}6\text{-}7) = 0;$$
$$E\tau_6 + 25.23 + 81.56 = 0$$
$$E\tau_6 = -106.89$$

Under such circumstances it is necessary to carry out the solution for half the truss only.

For joint 1:

$$M_e = -3396.2 \times 0.35$$
$$= -1188.67$$
$$\frac{1}{2}M_e = -\ 594.33$$

For joint 2:

$$+3396.2 \times 0.35 = +1188.67$$
$$-3678.1 \times 0.246 = -\ 904.80$$
$$M_e = +\ 283.87$$
$$\frac{1}{2}M_e = +\ 141.93$$

TABLE III

Bar	K	$E\,d\alpha$	$\Sigma E\,d\alpha$	$K\Sigma E\,d\alpha$	$E\tau$	$E(2\tau_{nm}+\tau_{nm})$	$\frac{1}{2}M$	$2\frac{c}{L}$	f_s	$\frac{f_s}{f_p}$, %
1-2	7 641				− 58.14	− 99.57	− 760 8	0 035	− 3 48	6.2
		+116.92						0 05436	+ 5 41	
1-3	6.524		+116.92	+ 762.77	+ 58.78	+207.69	+1355.1	0.0672	± 13 93	22.2
	14 165			+ 762 77						
2-4	10.652				− 28 68	+ 22 23	+ 236 8	0 05192	+ 1 15	
		− 65.72						0 07964	− 1 77	2 7
2-5	0.498		− 65.72	− 32.73	− 94 40	−161.18.	− 80 3	0 0267	∓ 4 30	7.1
		+ 2.14								
2-3	0.343		− 63 58	− 21.81	− 92 26	−320 28	− 109 9	0 03554	∓11 39	22 0
		+108 97								
2-1	7.641		+ 45 39	+ 346 81	+ 16 71	− 24 72	− 188 9	0 05436	− 1 34	2 4
								0 035	+ 0 86	
	19.134			+ 292 27						
3-1	6 524				+ 90.13	+239 04	+1559 8	0 0672	± 16 07	25 6
		−225.89								
3-2	0 343		−225.89	− 77 48	−135 76	−363 78	− 124 8	0.03554	∓12 91	25.0
		+ 7.27								
3-5	6 524		−218 62	−1426 27	−128 49	−219 95	−1435 0	0 0672	∓14 77	23 5
	13 391			−1503 75						
4-6	10.652				−113.98	−121 18	−1291 3	0.05192	− 6 29	9.7
		+ 12 86						0 07964	+ 9.66	
4-5	0.343		+ 12 86	+ 4 41	−101 12	−289 61	− 99 4	0 03554	∓10 28	—
		+180 71								
4-2	10 652		+193 57	+2061 89	+ 79 59	+130 50	+1390 3	0 07964	+10 40	
								0 05192	− 6.77	10.5
	21 647			+2066 30						
5-3	6 524				− 37 03	− 54 43	− 355 2	0 0672	∓ 3 66	5 8
		− 9.41								
5-2	0.498		− 9 41	− 4 69	+ 27.62	− 39 16	− 19 5	0.0267	∓ 1.04	1 7
		−114.99								
5-4	0.343		−124 40	− 42.67	− 87 37	−275 86	− 94 7	0 03554	∓ 0 79	—
		− 38.09								
5-6	1.698		−162.49	− 275 88	−125 46	−169 36	− 287 6	0.03172	∓ 5 38	23 2
		+ 81.52								
5-7	10.059		− 80.97	− 814 46	− 43 94	+ 75 20	+ 756.3	0 0672	± 5 05	7.8
	19.122			−1137.70						
6-8	10.652				−106 79		−1061 0			
		+ 25.23								
6-9	1.698		+ 25 23	+ 42.83	− 81 56		− 64 0			
		+ 81.56								
6-7	0.343		+106.79	+ 36 63	0.0	0.00	0 0	0.03554	0.0	0.0
		+ 81.56								
6-5	1.698		+188.35	+ 319.81	+ 81.56	+ 37.66	+ 64.0	0.03172	± 1.19	5.1
		+ 25.23								
6-4	10.652		+213.58	+2275.03	+106.78	+ 99.58	+1061.0	0 07964	+ 7.94	
								0.05192	− 5.16	8.0
	25.043			+2674 30						
7-5	10.059				+163.08	+282.22	+2838.5	0.0672	± 18.98	28.9
		−163.08								
7-6	0.343		−163.08	− 55.94	0.0	0 0	0 0	0.03554	0.0	0.0
		−163.08								
7-8	10.059		−326.16	−3280 83	−163.08	−282 22	−2838.5	0 0672	∓18.98	28.9
	20.461			−3336.77						

Formulation of Equations.—For joint 1:

$$2E\tau_1\sum_1 K_{1-m} + 2\sum_1\left[K_{1-m}\sum_1^{1-m} E\,d\alpha\right] = 28.330E\tau_1 + 1525.54$$

$$K_{1-2}E\tau_2 + K_{1-2}\sum_2^{2-1} E\,d\alpha =$$

$$7.641E\tau_2 + 346.81$$

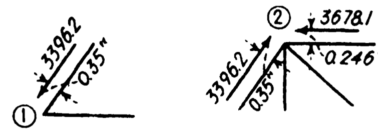


FIG. 111.

$$K_{1-3}E\tau_3 + K_{1-3}\sum_3^{3-1} E\,d\alpha = 6.524E\tau_3$$

$$\frac{1}{2}(M_s)_1 = \frac{-\ 594.33}{+1278.02}$$

For joint 2:

$$\begin{aligned}
38.268E\tau_2 &+ 584.54\\
10.652E\tau_4 &+ 2061.89\\
0.498E\tau_5 &- 4.69\\
0.343E\tau_3 &- 77.48\\
7.641E\tau_1 &\\
&+ 141.93\\
\hline
&+ 2706.19
\end{aligned}$$

For joint 3:

$$\begin{aligned}
26.782E\tau_3 &- 3007.50\\
6.524E\tau_1 &+ 762.77\\
0.343E\tau_2 &- 21.81\\
6.524E\tau_5 &\\
\hline
&- 2266.54
\end{aligned}$$

$$E\tau_6 = -106.89; \qquad E\tau_7 = +163.08$$

For joint 4:

$$\begin{aligned}
43.294E\tau_4 &+ 4132.60\\
&+ 1137.51 = 10.652E\tau_6 + 2275.03\\
0.343E\tau_5 &- 42.67\\
10.652E\tau_2 &\\
\hline
&+ 5227.44
\end{aligned}$$

For joint 5:

$$\begin{aligned}
38.244E\tau_5 &- 2275.40\\
6.524E\tau_3 &- 1426.27\\
0.498E\tau_2 &- 32.73\\
0.343E\tau_4 &+ 4.41\\
&+ 138.48 = 1.698E\tau_6 + 319.81\\
&+ 1640.41 = 10.059E\tau_7\\
\hline
&- 1951.10
\end{aligned}$$

The equations are, therefore,

	$E\tau_1$	$E\tau_2$	$E\tau_3$	$E\tau_4$	$E\tau_5$	Num. $\times\ 10^{-2}$	Check
1	28.330	7.641	6.524			+12.780	+ 55.275
2	7.641	38.268	0.3431	10.652	0.4981	+27.062	+ 84.464
3	6.524	0.3431	26.782		6.524	−22.665	+ 17.508
4		10.652		43.294	0.3431	+52.274	+106.563
5		0.4981	6.524	0.3431	38.244	−19.511	+ 26.098

After the equations have been solved, the values for the reference τ's are entered in column 6 of Table III and from these the value of τ for each end of each member is computed by adding to the reference τ for the joint the sum of the changes in the angles between the reference bar and the bar in question. After all τ's have been computed, the quantities in column 7 are computed; this needs no explanation, the heading of the column being sufficient. Column 8, headed $\frac{1}{2}M$, is not essential, but the values are useful in that they enable the computer to ascertain whether his computed results satisfy the equations $\Sigma M = 0$. The individual numbers in this column are the products

$$KE(2\tau_{nm} + \tau_{mn}).$$

Column 9 is a recapitulation of information found previously and entered here for convenience. The numbers in column 10 are the products $2\frac{c}{L}E(2\tau_{nm} + \tau_{mn})$, which, as demonstrated already, are values of secondary-stress intensities f_s, and the last column explains itself.

47. The Mohr Semigraphic Method.—This method of determining the secondary stresses in a truss is based on the use of the slope-deflection equations. As in the Manderla method and as in other problems solved by the use of these equations, the procedure to be followed consists of writing an equation $\Sigma M = 0$ for each of the joints of the structure; in this method the moments are expressed in terms of the distortion characteristics θ and ψ; enough additional equations are set up to enable the computer to solve simultaneously for these primary variables and, after these have been computed, he determines the moments applied to the ends of the members. In a secondary-stress analysis the angle ψ for each member can be determined before the equations are set

TABLE IV.—SOLUTION OF EQUATIONS

	E_{T1}	E_{T2}	E_{T3}	E_{T4}	E_{T5}	Num. × 10^{-2}	Check
1	28.330	7.641	6.524			+12 780	+ 55.275
2	7.641	38.268	0.3431	10.652	0.4981	+27.062	+ 84.464
3	6.524	0.3431	26.782		6.524	−22.665	+ 17.508
$1 \times \frac{7.641}{28.330} =$ 1′	7.641	2.0615	1.760			+ 3 447	+ 14.910
$1 \times \frac{6.524}{28.33} =$ 1″	6.524	1.760	1.503			+ 2.943	+ 12.730
2 − 1′ = 6		36.2065	− 1.4169	10.652	0.4981	+23.615	+ 69.554
3 − 1″ = 7		− 1.4169	25.279		6.524	−25.608	+ 4.778
4		10.652		43.294	0.3431	+52.274	+106.563
5		0.4981	6.524	0.3431	38.244	−19.511	+ 26 098
$6 \times \frac{1.4169}{36.2065} =$ 6′		1.4169	− 0.0554	0.4168	0.0195	+ 0.923	+ 2.721
$6 \times \frac{10.652}{36.2065} =$ 6″		10.652	− 0.4168	3.134	0.1465	+ 6.947	+ 20.463
$6 \times \frac{0.4981}{36.2065} =$ 6‴		0.4981	− 0.0195	0.1466	0.0069	+ 0.325	+ 0.957
7 + 6′ = 8			25.224	0.4168	6.5435	−24.685	+ 7.499
4 − 6″ = 9			+ 0.4168	40.160	0 1966	+45.327	+ 86.100
5 − 6‴ = 10			6.5435	0.1966	38.237	−19 836	+ 25 141
$8 \times \frac{0\ 4168}{25\ 224} =$ 8′			0.4168	0.0069	0.1081	− 0.408	+ 0.124
$8 \times \frac{6\ 5435}{25.224} =$ 8″			6.5435	0.1081	1.698	− 6.404	+ 1.945
9 − 8′ = 11				40.153	0.0885	+45.735	+ 85 976
10 − 8″ = 12				0.0885	36.539	−13 432	+ 23 196
$11 \times \frac{0\ 0884}{40.153} =$ 11′				0.0885	0.0002	+ 0 101	+ 0.190
12 − 11′ = 13					36.539	−13.533	+ 23 006
1	28 330	7 641 − 2.192	6.524 + 5.881			+12.780	$E_{T1} = -0\ 5814$
6		36 206	− 1.4169 − 1 277	10.652 −12.142	0 4981 + 0 1845	+23 615	$E_{T2} = -0\ 2868$
8			25.224	0.4168 − 0.475	6.5435 + 2 424	−24 685	$E_{T3} = +0.9013$
11				40 153	0.0585 + 0.0327	+45 735	$E_{T4} = -1.13983$
							$E_{T5} = +0.37037$

up, thus reducing the number of unknowns to the θ^s, one for each joint, and the only equations needed are the equations $\Sigma M = 0$.

The angles ψ can be found by either of two procedures, each of which is favored by a number of computers. In one, the angles ψ are found by assuming a value of the angle ψ for one of the members and using the changes in the angles of the triangles of the truss to compute the corresponding angles ψ for the other members. Since the secondary-stress intensities are dependent on the distortion of the truss and not on its position in space, it is immaterial, except as a matter of convenience, whether the assumed value for the angle ψ with which one starts is a correct value. The second method is based on the use of the Williot diagram.

Referring again to Fig. 108a, the equation $\Sigma M = 0$ may be written as

$$M_{na} + M_{nb} + M_{nc} + M_{nd} + M_e = 0$$

in which

$$\begin{aligned} M_{na} &= 2EK_{na}(2\theta_n + \theta_a - 3\psi_{na}) \\ M_{nb} &= 2EK_{nb}(2\theta_n + \theta_b - 3\psi_{nb}) \\ M_{nc} &= 2EK_{nc}(2\theta_n + \theta_c - 3\psi_{nc}) \\ M_{nd} &= 2EK_{nd}(2\theta_n + \theta_d - 3\psi_{nd}) \end{aligned} \qquad (82)$$

In these equations, since the tangents at joint n to the n-ends of the members connected to joint n do not change their directions relative to each other,

$$\theta_{na} = \theta_{nb} = \theta_{nc} = \theta_{nd} = \theta_n$$

with similar relations at the other joints of the truss. Consequently the equation $\Sigma M = 0$ may be rewritten, after dividing both sides by two, as

$$2\sum_n K_{nm}\, E\theta_n + K_{na}\, E\theta_a + K_{nb}\, E\theta_b + K_{nc}\, E\theta_c + K_{nd}\, E\theta_d - 3\sum_n \{K_{nm}\, E\psi_{nm}\} + \frac{1}{2}M_e = 0 \quad (83)$$

It will be convenient in deciding the character of secondary-stress intensities if the conventions adopted here are similar to those used in the Manderla solution. These are just the reverse of those used in the slope-deflection solutions previously illustrated and are:

The moment applied to the end of a member is positive when counterclockwise; this is the same as saying that a clockwise moment on a joint is positive.

Angles θ and ψ are positive when counterclockwise. The adoption of these conventions leaves the equations unchanged in form.

When the Eqs. (83) have been formulated and solved, the moments may be evaluated by using Eqs. (82) or one may proceed by using Eqs. (84):

$$f_{nm} = \frac{M_{nm}c_{nm}}{I_{nm}} = \frac{c_{nm}}{I_{nm}} 2E\frac{I_{nm}}{L_{nm}}(2\theta_n + \theta_m - 3\psi_{nm})$$
$$= 2\frac{c_{nm}}{L_{nm}}E(2\theta_n + \theta_m - 3\psi_{nm}) \qquad (84)$$

without computing the end moments.

To illustrate the procedure, the truss and loading of Figs. 109 and 110 will be used and the angles ψ will be computed by using the changes in angle computed in Table II. Since both truss and loading are symmetrical there can be no rotation for either joint 6 or joint 7, nor can there be any rotation of the chord of the elastic curve of member 6-7. In slope-deflection notation these conditions are

$$\theta_6 = 0; \qquad \theta_7 = 0; \qquad \psi_{6\text{-}7} = 0 \qquad (85)$$

It is necessary to formulate and solve only the equations $\Sigma M = 0$ for joints 1, 2, 3, 4, and 5. If the truss were not symmetrical or were not loaded symmetrically, the conditions (85) would not be true, but it would still be possible to start with any assumed value for $\psi_{6\text{-}7}$; under such circumstances it is necessary to formulate and solve the equations for all the joints of the truss.

For the problem in hand, since $\psi_{6\text{-}7} = 0$ and the angle 5-7-6 has decreased by $1/E \times 163.08$, the chord 7-5 must have rotated clockwise through this angular change, that is, $E\psi_{7\text{-}5} = -163.08$. Similarly, since the change in the angle 6-5-7 $= +\frac{1}{E} \times 81.52$, the chord 5-6 must have rotated through an angle $+\frac{1}{E} \times 81.52$ counterclockwise relative to member 5-7; therefore its angle

TABLE V

Member	K	$E\,d\alpha$	$E\psi$	$K_{nm} \times E\psi_{nm}$	$E\theta$	$E(2\theta_n + \theta_m - 3\psi_{nm})$	$\frac{1}{2}M$	$2\frac{c}{L}$	f_s	$\frac{f_s}{f_p}$, %
1-2	7 641	+116 92	−345 75	−2641.9	−403.89	− 99.57	− 760.8	0 035 0.05436	− 3.48 + 5.41	6.2
1-3	6 524		−462 67	−3018.5		+207 69	+1354.9	0 0672	± 13.93	22.2
	14 165			−5660 4						
2-4	10 652	− 65 72	−300 36	−3199.4		+ 22.23	+ 236 8	0 05192 0 07964	+ 1.15 − 1 77	2 7
2-5	0 4981	+ 2 14	−234 64	− 116 9	−329.04	−161.17	− 80 3	0 0267	∓ 4.31	7.1
2-3	0 3431	+108 97	−236 78	− 81 2		−320 28	− 109 9	0 03554	∓11 40	22.0
2-1	7 641		−345 75	−2641.9		− 24 72	− 188 9	0 05436 0 035	− 1 34 + 0.86	2.4
	19 134			−6039 4						
3-1	6 524	−225 89	−462 67	−3018 5		+239.04	+1559 5	0 0672	± 16 08	25.6
3-2	0 3431	+ 7 27	−236 78	− 81 2	−372 54	−363.78	− 124.8	0.03554	∓12.92	25 0
3-5	6 524		−244 05	−1592.2		−219 94	−1435 0	0 0672	∓14 77	23.5
	13 391			−4691 9						
4-6	10 652	+ 12 86	−106 79	−1137 5		−121.17	−1290 7	0.05192 0 07964	− 6 29 + 9 68	9 7
4-5	0 3431	+180 71	−119 65	− 41 0	−220.77	−289 60	− 99 4	0 03554	∓10 29	—
4-2	10 652		−300 36	−3199.4		+130 50	+1390 2	0 07964 0 05192	+10 41 − 6 77	10.5
	21 647			−4377 9						
5-3	6 524	− 9 41	−244 05	−1592 2		− 54 41	− 355 1	0 0672	∓ 3 65	5 8
5-2	0 4981	−114 99	−234 64	− 116.9		− 39 14	− 19 5	0 0267	∓ 1 05	1 7
5-4	0 3431	− 38.09	−119.65	− 41 0	−207.01	−275 84	− 94 6	0 03554	∓ 9.80	—
5-6	1 698	+ 81 52	− 81.56	− 138 5		−169.34	− 287 5	0 03172	∓ 5 37	23.1
5-7	10 059		−163 08	−1640 4		+ 75.22	+ 756 8	0 0672	± 5 05	7 7
	19 122			−3529 0						
6-8	10 652	+ 25 23	+106 79	+1137 5		− 99 60	−1060 9	0 05192 0 07964	− 5 17 + 7 95	8 0
6-9	1 698	+ 81 56	+ 81 56	+ 138.5		− 37 67	− 64 0	0 03172	∓ 1.19	5.1
6-7	0 3431	+ 81 56	0 0	0 0	0.0	0.0	0 0	0 03554	0 0	0.0
6-5	1 698	+ 25 23	− 81.56	− 138 5		+ 37.67	+ 64 0	0 03172	± 1.19	5 1
6-4	10 652		−106 79	−1137 5		+ 99.60	+1060 9	0 07964 0 05192	+ 7 95 − 5.17	8.0
	25 043									
7-5	10.059	−163 08	−163 08	−1640 4		+282.23	+2838 5	0.0672	∓18.96	28.9
7-6	0 3431	−163 08	0 0	0 0	0.0	0 0	0 0	0.03554	0.0	0.0
7-9	10 059		+163 08	+1640 4		−282 23	−2838.5	0 0672	± 18.96	28.9
	20 461									

$\psi_{5\text{-}6} = \frac{1}{E}(-163.08 + 81.52) = -\frac{1}{E} \times 81.56.$ The extension of this procedure can be carried out conveniently as part of Table V, which will be used to compile the information needed in formulating the equations and, after the equations have been solved, in computing the secondary-stress intensities f_s.

In formulating the equations, the moments due to eccentricity of connection are taken from the previous article.

at joint 1:

$$
\begin{aligned}
2\sum_1 K_{1m}E\theta_1 - 3\sum(K_{1m}\,E\psi_{1m}) &= +28.230E\theta_1 + 16983.9 \\
+K_{1-2}E\theta_2 &= +\ 7.641E\theta_2 \\
+K_{1-3}E\theta_3 &= +\ 6.524E\theta_3 \\
+\tfrac{1}{2}M_e &= \qquad\qquad -\ 594.3 \\
&\quad +42.395 \qquad + 16389.6
\end{aligned}
$$

At joint 2:

$$
\begin{aligned}
&+38.268E\theta_2 + 18118.8 \\
&+10.652E\theta_4 + \quad 141.9 \\
&+\ 0.498E\theta_5 \\
&+\ 0.343E\theta_3 \\
&+\ 7.641E\theta_1 \\
&+57.402 \qquad + 18260.7
\end{aligned}
$$

At joint 3:

$$
\begin{aligned}
&26.782E\theta_3 + 14079.9 \\
&6.524E\theta_1 \\
&0.343E\theta_2 \\
&6.524E\theta_5 \\
&40.173 \qquad + 14079.9
\end{aligned}
$$

At joint 4:

$$
\begin{aligned}
&43.294E\theta_4 + 13136.4 \\
&0.343E\theta_5 \\
&10.652E\theta_2 \\
&54.289 \qquad + 13136.4
\end{aligned}
$$

At joint 5:

$$
\begin{aligned}
&38.244E\theta_5 + 10588.8 \\
&6.524E\theta_3 \\
&0.498E\theta_2 \\
&0.343E\theta_4 \\
&45.609 \qquad + 10588.8
\end{aligned}
$$

	$E\theta_1$	$E\theta_2$	$E\theta_3$	$E\theta_4$	$E\theta_5$	Num. $\times 10^{-2}$	Check
1	28.330	7.641	6.524			+163.869	+206.364
2	7.641	38.268	0.3431	10.652	0.4981	+182.601	+240.003
3	6.524	0.3431	26.782		6.524	+140.757	+180.930
4		10.652		43.294	0.3431	+131.337	+185.626
5		0.4981	6.524	0.3431	38.244	+105.870	+151.479

TABLE VI

	$E\theta_1$	$E\theta_2$	$E\theta_3$	$E\theta_4$	$E\theta_5$	Num. × 10^{-2}	Check
1	28.330	7.641	6.524			+163.869	+206.364
2	7.641	38.268	0.3431	10.652	0.4981	+182.601	+240.003
3	6.524	0.3431	26.782		6.524	+140.757	+180.930
$1 \times \frac{7.641}{28.330} = 1'$	7.641	2.061	1.7602			+ 44.198	+ 55.659
$1 \times \frac{6.524}{28.33} = 1''$	6.524	1.7602	1.5032			+ 37.737	+ 47.523
$2 - 1' = 6$		36.207	− 1.4171	10.652	0.4981	+138.403	+184 344
$3 - 1'' = 7$		− 1.4171	+25.279		6.524	+103.020	+133.407
4		10.652		43.294	0.3431	+131.337	+185.626
5		0.4981	6.524	0 3431	38.244	+105.870	+151.479
		+ 1.4171	− 0.0554	0.4169	0.0195	+ 5 417	+ 7.214
		10.652	− 0.4169	3.134	0.1465	+ 40.717	+ 54.233
		0.4981	− 0 0195	0.1465	0.0069	+ 1.904	+ 2 536
			25 224	0 4169	6.5435	+108.437	+140.621
			0.4169	40 160	0.1966	+ 90 620	+131.393
			6.5435	0.1966	38.2371	+103.966	+148.943
			0.4169	0.0069	0.1081	+ 1.792	+ 2 324
			6.5435	0 1081	1.6978	+ 28.129	+ 36.478
				40.1531	0 0885	+ 88 828	+129.069
				0.0885	36.5393	+ 75.837	+112.465
				0.0885	0 0002	+ 0.196	+ 0.285
					36.539	+ 75.641	+112.180
	28.330	7.641	6.524			+163.869	$E\theta_1 = -4.0319$
		−25.142	−24 304				
		36.207	− 1 4171	10.652	0 4981	+138.403	$E\theta_2 = -3.2904$
			+ 5 279	−23.515	− 1.0312		
			25.224	0.4169	6 5435	+106.437	$E\theta_3 = -3.7254$
				− 0.9202	−13.546		
				40.153	0.0885	86.828	$E\theta_4 = -2.2077$
					− 0.1832		
							$E\theta_5 = -2.0701$

48. Secondary Stresses by Distributing End Moments.—In computing secondary stresses in trusses, one is considering a truss essentially as a structure with moment-resisting joints. Consequently, any of the methods used for the analysis of such structures is applicable in the problem of secondary-stress analysis. A truss so considered has so many redundants that a primary-stress analysis on such a basis is not practicable. This situation, however, does not introduce any real difficulty since the changes in the positions of the joints are dependent primarily on the changes in the lengths of the members. A fairly close approximation may be obtained by a method of successive steps, the first of which is the usual so-called primary-stress analysis; the second, the secondary-stress analysis, uses the results of the first step.

Since the primary-stress analysis enables one to compute the angles ψ, as was demonstrated in the previous article, the method of distributing end moments is applicable and can be started by computing the moments which would exist if the truss distorted so that the chords of the elastic curves of the members rotated through the angles ψ just mentioned, but that during this distortion the joints were not permitted to rotate. If one considers the slope-deflection equation

$$M_{nm} = 2EK_{nm}(2\theta_n + \theta_m - 3\psi_{nm})$$

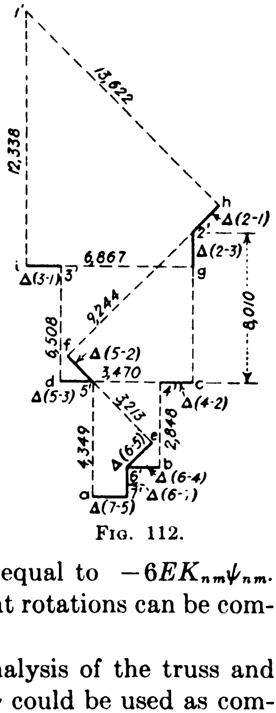


FIG. 112.

it is evident that the fixed-end moment in a member nm, as just described, is equal to $-6EK_{nm}\psi_{nm}$. It is evident also that the effect of the joint rotations can be computed just as in Art. 36.

In illustrating the procedure by the analysis of the truss and loading of Figs. 109 and 110, the angles ψ could be used as computed in the previous article, but, for the sake of presenting an alternative method, they will be computed here by using the Williot diagram. When the truss is symmetrical and loaded symmetrically, as in this illustration, it will be sufficient to draw the diagram for half the truss only. It will be drawn assuming that joint 7 is fixed in position and member 6-7 fixed in direction (see Fig. 112). In this diagram a-5′ represents the movement of joint 5 during rotation of member 5-7 about joint 7; therefore, $\psi_{7\text{-}5} = \frac{1}{L_{7\text{-}5}}(a\text{-}5')$ and is clockwise. Using the same conventions as in the previous article, this is negative. Similarly,

$$\psi_{6\text{-}5} = -\frac{e\text{-}5'}{L_{6\text{-}5}}; \quad \psi_{6\text{-}4} = -\frac{b\text{-}4'}{L_{6\text{-}4}}; \quad \psi_{5\text{-}4} = -\frac{5'\text{-}4'}{L_{5\text{-}4}};$$

$$\psi_{5\text{-}3} = -\frac{d\text{-}3'}{L_{5\text{-}3}}; \quad \psi_{4\text{-}2} = -\frac{c\text{-}2'}{L_{4\text{-}2}}; \quad {}_{2\text{-}3} = -\frac{g\text{-}3'}{L_{3\text{-}2}};$$

$$\psi_{5\text{-}2} = -\frac{f\text{-}2'}{L_{5\text{-}2}}; \quad \psi_{2\text{-}1} = -\frac{h\text{-}1'}{L_{2\text{-}1}}; \quad \psi_{3\text{-}1} = -\frac{i\text{-}1'}{L_{3\text{-}1}}.$$

The fixed-end moments are computed in Table VII. The moment distribution is carried out in Fig. 113.

TABLE VII

Bar	L	ψ	K	$-6EK\psi$
6-7	29.0	0.0	0.3431	+ 0
7-5	26.67	−163.08	10.059	+ 9,841
6-5	39.40	− 81.56	1.698	+ 831
6-4	26.67	−106.79	10.652	+ 6,825
5-4	29.0	−119.65	0.3431	+ 246.4
4-2	26.67	−300.36	10.652	+19,197
5-2	39.40	−234.64	0.4981	+ 701
5-3	26.67	−244.07	6.524	+ 9,552
2-3	29.0	−236.77	0.3431	+ 487.5
2-1	39.40	−345.76	7.641	+15,852
3-1	26.67	−462.68	6.524	+18,155

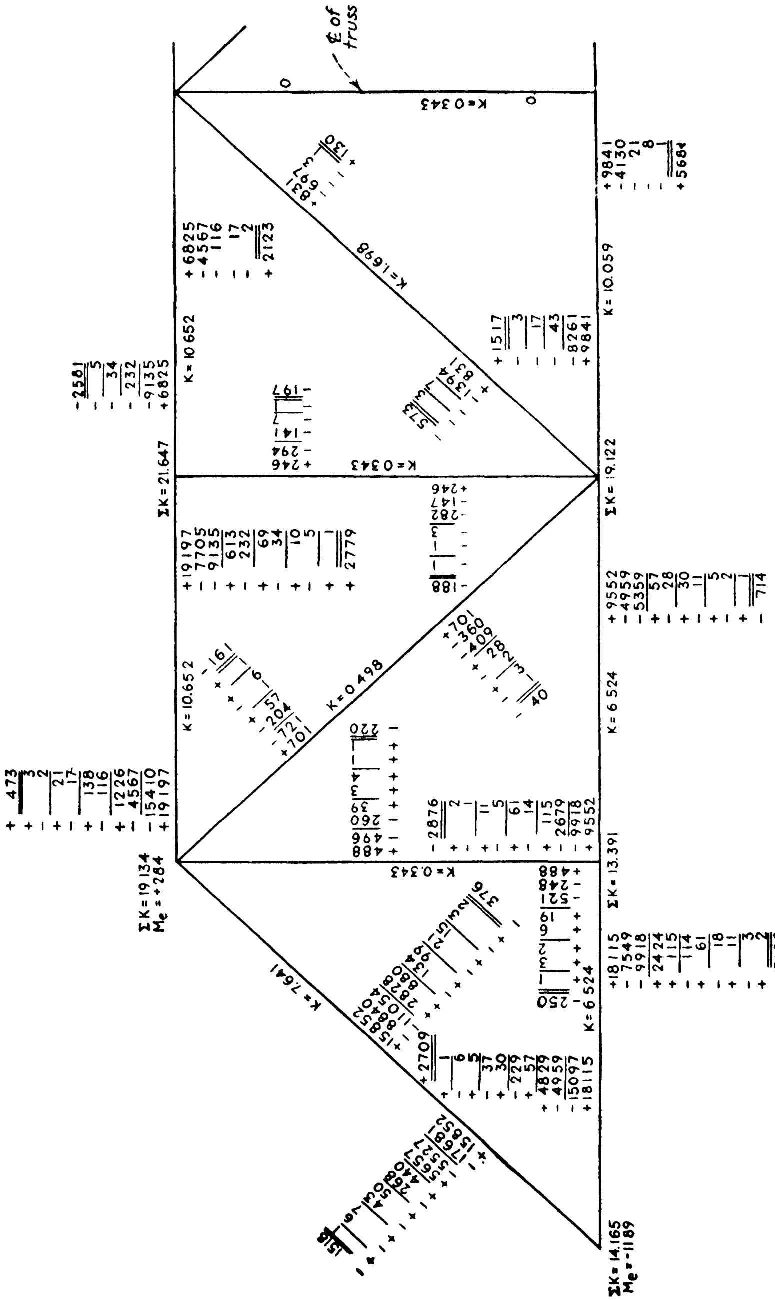
It is to be remembered that in making the moment distribution the moments due to eccentricity must be taken into account. The convention that moments applied to the ends of members are positive when counterclockwise implies that moments applied to joints are positive when clockwise. This means that moments due to eccentricity are to be taken as positive when they act in a clockwise direction on the joints. Starting with joint 1, the unbalance of moments is

$$+15{,}852 + 18{,}115 - 1{,}189 = +32{,}778.$$

This is distributed as shown and the carryover moments are written at the far ends of members 1-2 and 1-3. Taking joint 2 next, the unbalance of moments is now

$$+15{,}852 + 19{,}197 + 701 + 488 + 284 - 8{,}840 = +27{,}682.$$

This is distributed and carried over. The moment distribution is continued, taking joints 3, 4, and 5 in the order stated. In



℄ of truss
K = 0.343
K = 1.698
K = 10.652
K = 10.059
ΣK = 21.647
ΣK = 19.122
K = 10.652
K = 0.498
K = 6 524
ΣK = 19.134
M_e = +284
ΣK = 13.391
K = 7.641
K = 6 524
ΣK = 14.165
M_e = -1189

FIG. 113.

this problem, where truss and loading are symmetrical about member 6-7, if it is assumed that the same procedure is carried out taking joints 12, 10, 11, 8, and 9 in order, it will be found that joints 6 and 7 are balanced and no distribution is needed there. The procedure is repeated, joint by joint, until the remaining unbalance in the moment at each joint is small enough to be considered negligible. The total moments are found by summing the various terms written on the ends of the various members, and the secondary-stress intensities are computed by multiplying each moment by the proper value of c/I. This step is not carried out here since it is merely a repetition of what has been done in the previous articles.

49. Effect of Weight Distribution.—In all truss analyses considered so far, it has been assumed that the weights of the members were concentrated at the joints. In order to form an idea of the error involved in this assumption, an analysis of the truss shown in Fig. 109 will be carried out, using the moment-distribution procedure. This has been done already to the extent of determining the effect of the angles ψ and the joint rotations θ and the only step which remains is to determine the effect of the load distribution. The moments stated in the previous problem are stated in inch-pounds and the same unit will be used here so that the moments developed may be compared easily with those due to other causes. In computing the fixed-end moments it is to be remembered that the weight per unit length w' is that component of the weight which is perpendicular to the axis of the member. The fixed-end moments are tabulated below.

Bar	L, ft.	w, lb. per ft.	w'	$\frac{w'L^2}{12}$, ft.-lb.	$\frac{w'L^2}{12}$, in.-lb.
1-2	39.40	209.5	141.8	18,340	220,100
2-4, 4-6	26.67	195.7	195.7	11,597	139,180
1-3, 3-5	26.67	125.9	125.9	7,460	89,520
5-7	26.67	217.2	217.2	12,870	154,450
2-5	39.40	115.9	78.4	10,140	121,700
5-6	39.40	100.7	68.2	8,820	105,820
2-3, 4-5, 6-7	29.0	66.4	0.0	0	0.0

These moments are written at the ends of the members in the diagram of the truss shown in Fig. 114. In this step it must be

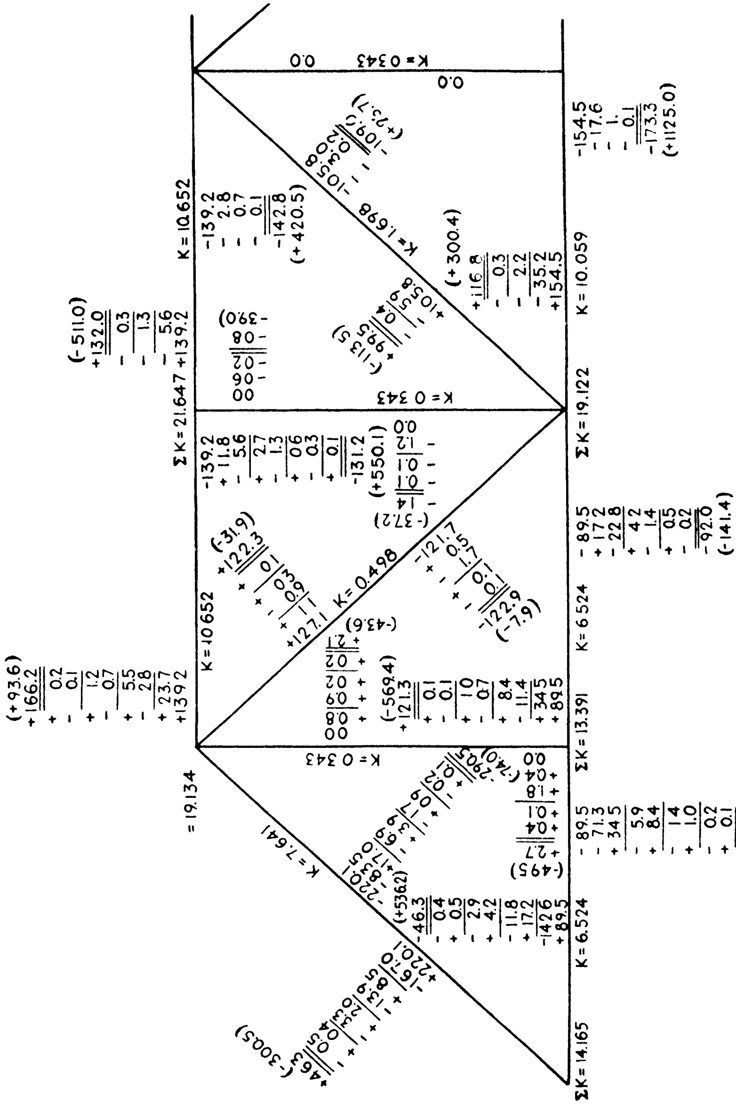

K = 10.652
ΣK = 21.647
K = 10.652
= 19.134
K = 10.059
ΣK = 19.122
K = 6.524
ΣK = 13.391
K = 6.524
ΣK = 14.165
K = 0.343
K = 0.343
K = 0.343
K = 1.698
K = 0.498
K = 7.641

FIG. 114.

remembered that the conventions previously adopted stated that moments acting counterclockwise on the ends of members were positive. The moments are written in units of 1,000 in.-lb. Following this step, the moment-distribution process is carried through. This is shown in Fig. 114, also.

TABLE VIII

Member	Weight distribution neglected $\left(\frac{f_s}{f_p}, \%\right)$	Weight distribution considered $\left(\frac{f_s}{f_p}, \%\right)$
1-2	6.2	5.3
1-3	22.2	20.3
2-4	2.7	7.6
2-5	7.1	20.3
2-3	22.0	20.9
2-1	2.4	11.8
3-1	25.6	20.4
3-2	25.0	23.6
3-5	23.5	18.5
4-6	9.7	7.2
4-2	10.5	8.0
5-3	5.8	9.6
5-2	1.7	29.3
5-4		
5-6	23.2	2.9
5-7	7.8	10.7
6-7	0.0	0.0
6-5	5.1	17.1
6-4	8.0	5.3
7-5	28.9	24.5
7-6	0.0	0.0

The effect of the weight distribution should be compared with the moments due to full loading of the truss. With this in view, consider a load of 198,000 lb. at each of the bottom-chord panel points. The secondary moments (in units of 1,000 in.-lb.) due to this loading may be found by multiplying the moments shown in Fig. 113 by 0.198. The results are shown in parentheses in Fig. 114. The moments due to the weight distribution are not negligible compared with those computed by taking into account only the changes in the lengths of the members and the eccentricities in the connections, but when the secondary-stress intensities are expressed as percentages of the primary-stress intensities the variation between the results of the two analyses is proportionally much smaller than the difference between the two sets of moments. A comparison is shown in Table VIII.

If it is desired to use the Manderla solution when the weight distribution is to be considered, the basic equations (74) must be modified to include the load terms

$$-\frac{2}{L_2}[(M_s)_{0a} - 2(M_s)_{0b}]$$
$$-\frac{2}{L_2}[2(M_s)_{0a} - (M_s)_{0b}]$$

respectively. Since the load is uniformly distributed over the full length of the member, the M_0 curve is a parabola with a middle ordinate equal to $w'L^2/8$, where w' is the component, normal to the axis of the member, of the weight per unit length and L is the length of the member.

$$(M_s)_{0a} = (M_s)_{0b} = \frac{2}{3}\frac{w'L^2}{8}L\frac{L}{2} = \frac{w'L^4}{24}$$

and Eqs. (74) become

$$M_{ab} = 2EK_{ab}(2\tau_a + \tau_b) + \frac{w'L^2}{12}$$
$$M_{ba} = 2EK_{ab}(\tau_a + 2\tau_b) - \frac{w'L^2}{12}$$

For this situation, Eq. (80) becomes

$$E\tau_n 2\sum_n K_{nm} + 2\sum_n\left[K_{nm}\sum_n^{nm} E\,d\alpha\right] + \sum_n\left[K_{nm}E\tau_m + K_{nm}\sum_m^{mn} E\,d\alpha\right] + \frac{1}{2}M_e + \sum\frac{w'_{nm}L^2_{nm}}{24} = 0 \quad (86)$$

where the terms in the last summation are to be taken as positive or negative according to whether the weight tends to rotate the member in a clockwise or counterclockwise direction about joint n. If the slope-deflection equations are to be used and the effect of weight distribution is to be taken into account, Eq. (83) must be rewritten in the form

$$2\sum_n K_{nm} E\theta_n + \sum_n K_{nm}E\theta_m - 3\sum_n K_{nm}E\psi_{nm} + \frac{1}{2}M_e + \sum \frac{w'_{nm}L^2_{nm}}{24} = 0 \quad (87)$$

50. Participating Stresses.—In the previous articles it has been assumed that the condition of stress in the truss under consideration was not affected by any members connecting it to other elements of the structure of which it forms a part. More often than not, such an assumption is only approximately true. For example, the truss of Fig. 109 is one of the main trusses of a single-track railway bridge in which the top chords are connected by top lateral bracing as shown in Fig. 115, while the bottom chords are connected not only by the bottom lateral bracing but by the floor system also. Each of the lateral bracing systems, and the floor system as well, offers resistance to movement of

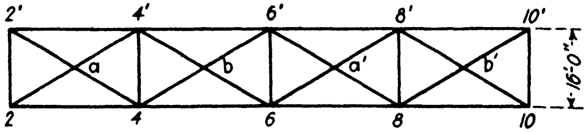


FIG. 115.

its joints. Consequently, when, under the action of vertical loads applied to the bridge, the top chords become shorter and the bottom chords extend, the movements of the joints are resisted by the lateral systems and the floor system, which, therefore, are subjected to axial stress and help to resist the distortion of the structure. Since they do so help, the stresses in the chord members are not as computed in the primary-stress analysis. Moreover, since the lengths of the members in the lateral systems change, there are changes in the angles of the triangles formed by these members with resultant bending of the members in the horizontal plane, so that, owing to the application of vertical loads to the structure, the bracing members are subjected to both axial and bending stresses, while the chord members are subjected not only to bending in the vertical plane but to bending in the horizontal plane as well.

Bar	A	L	I_h	c_h	K_h	$\frac{c_h}{L}$	
Chord	56.92	320.0	3570.2	11.938	11.160	0.0373	See Fig. 109
2′-a, 4′-a + other diagonals	9.92	186.59	30.63	3.875	0.1642	0.02076	1′-8″=d; 3/4″
4′-4 6′-6	9.92	192.0	30.63	3.875	0.1596	0.02018	Same as 2′-a except d = 2′-8″
2-2′	30.20	192.0	3045.	15.51	15.862	0.08081	

If the final condition of stress is determined by first computing the primary stress and then computing secondary effects, a chord system with its lateral bracing may be treated like any

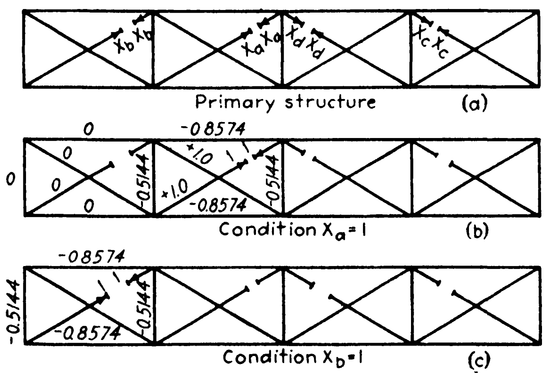


FIG. 116.

other indeterminate structure. The procedure will be illustrated by an analysis of the top-chord system shown in Fig. 115. The primary structure is shown in Fig. 116a and conditions $X_a = 1$ and $X_b = 1$ are shown in Fig. 116b and Fig. 116c, respectively. In condition $X = 0$ there are no stresses in the members of the bracing system and the stresses F_0 in the chord members are those found by the primary-stress analysis. The condition of loading is that shown in Fig. 117. The loads and stresses are

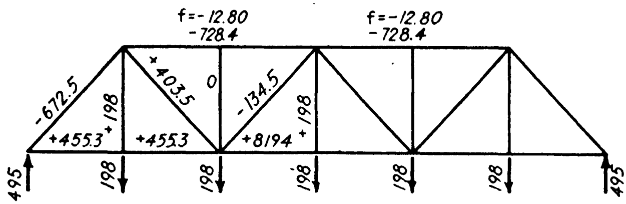


FIG. 117.

stated in units of 1,000 lb. Using the law of virtual work,

$$
\begin{aligned}
X_a\,\delta_{aa} + X_b\,\delta_{ab} + X_c\,\delta_{ac} + X_d\,\delta_{ad} &= -\delta_{a0}\\
X_a\,\delta_{ba} + X_b\,\delta_{bb} + X_c\,\delta_{bc} + X_d\,\delta_{bd} &= -\delta_{b0}\\
X_a\,\delta_{ca} + X_b\,\delta_{cb} + X_c\,\delta_{cc} + X_d\,\delta_{cd} &= -\delta_{c0}\\
X_a\,\delta_{da} + X_b\,\delta_{db} + X_c\,\delta_{dc} + X_d\,\delta_{dd} &= -\delta_{d0}
\end{aligned}
$$

The coefficients of the unknowns are computed in Table IX.

TABLE IX

Bar	L, ft.	A, in.	F_0	F_a	F_b	$\frac{F_aF_0L}{A}$	$\frac{F_bF_0L}{A}$	$\frac{F_a^2L}{A}$	$\frac{F_aF_bL}{A}$	$\frac{F_b^2L}{A}$
2-4	26 67	56.92	−728 4		−0.8574		+292 6			+0 3444
2′-4′	26 67	56.92	−728.4		−0.8574		+292.6			+0 3444
4-6	26.67	56.92	−728 4	−0 8574		+292.6		+0 3444		
4′-6′	26 67	56 92	−728.4	−0 8574		+292.6		+0 3444		
2-2′	16 0	30 20	0		−0.5144					+0 1402
4-4′	16 0	9 92	0	−0 5144	−0.5144			+0.4269	+0 4269	+0 4269
2-4′	31.10	9 92	0		+1.0					+3 135
2′-4	31 10	9 92	0		+1.0					+3 135
4-6′	31.10	9.92	0	+1 0				+3 135		
4′-6	31.10	9 92	0	+1 0				+3 135		
6-6′	16.0	9 92	0	−0.5144				+0 4269		
						+585.2	+585 2	+7 8126	+0 4269	+7 5259

$$
\begin{aligned}
E\delta_{d0} &= E\delta_{a0} = +585.2; \qquad E\delta_{c0} = E\delta_{b0} = +585.2\\
E\delta_{cd} &= E\delta_{dc} = E\delta_{ab} = E\delta_{ba} = +0.4269\\
E\delta_{cc} &= E\delta_{bb} = +7.5259\\
E\delta_{ad} &= \frac{F_aF_dL}{A} = \frac{16}{9.92} \times (-0.5144)^2 = +0.4269\\
E\delta_{aa} &= E\delta_{dd} = +7.8126
\end{aligned}
$$

The equations are

X_a	X_b	X_c	X_d	Num. × 10^{-2}	
7.8126	0.4269		0.4269	+5.852	
0.4269	7.5259			+5.852	
		7.5259	0.4269	+5.852	
0.4269		0.4269	7.8126	+5.852	
0.4269	0.0233		0.0233	+0.3198	
	7.5026		−0.0233	+5.532	
	−0.0233	0.4269	7.7893	+5.532	
	+0.0233		−0.0001	+0.017	
		0.4269	7.7892	+5.549	
		7.5259	0.4269	+5.852	
		0.4269	0.0242	+0.332	
			7.7650	+5 217	X_d = −0.6719
7.8126	0.4269		0.4269	+5.852	
	−0.3165		−0.2867		X_a = −0.6719
	7.5026		−0.0233	+5.532	
			+0.0157		X_b = −0.7394
		7.5259	0.4269	+5.852	
			−0.2867		X_c = −0.7396

The stresses are as shown in Fig. 118. The stress intensities in the top-chord members are reduced by approximately 8 per cent, while the lateral bracing members are subjected to stress

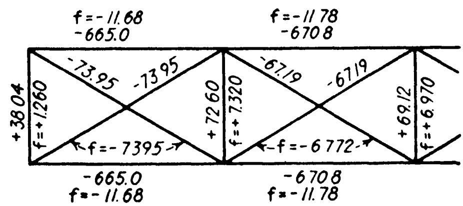


FIG. 118.

intensities of approximately 7000 lb. per sq. in. In addition, the members are subjected to bending of the sort discussed in

Art. 49 such that the stress condition in the top-chord lateral system due to uniformly distributed vertical loading sufficient to utilize what is usually considered to be the capacity of the truss is as shown in Fig. 119. The stress intensities are stated in thousands of pounds per square inch. The primary stresses are written in brackets. The effect of the bending of the members is small, being in no place as great as 1,200 lb. per sq. in.

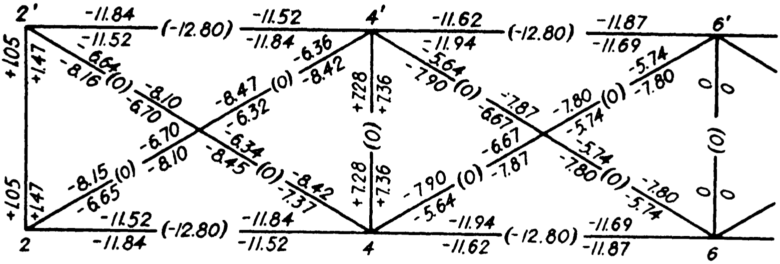


FIG. 119.

51. Participation Stresses Due to Floor Systems.—When a floor system is connected to its supporting truss at panel points, and the lengths of the truss members connecting those panel points change, the floor system resists these changes, and, consequently, there are participating stresses in the floor system and corresponding deviations in the axial stresses in the chord members from those computed in the primary analysis. As an illustration, consider the floor system for the bridge, one of whose trusses has been investigated in the previous articles. This floor system is as shown in Fig. 120.

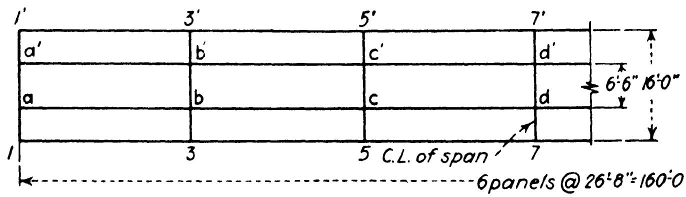
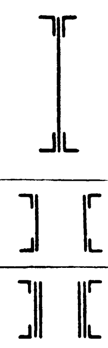


FIG. 120.

The problem involved is that of analysis of a structure which is statically indeterminate and is subjected to loads which in this case are the horizontal components of the stresses in the diagonals of the main trusses. If it is assumed that the joints cannot carry bending moment, the chord members of the trusses and the stringers are subjected to axial stresses only, while the floor beams are subjected to bending moment. The primary structure is as shown in Fig. 121. In condition $X = 0$ there are no stresses in the members of the floor system and the stresses in the chord members of the trusses are those found in the primary-stress analysis; these are taken as in Fig. 117. There are twelve redundant stresses but the structure and condition of stress are

Member	L, in.	A, sq. in.	I, in.[4]	Arrangement
3-3′, 5-5′ 7-7′	192	51.00	162.67	4 flange ∟s 6″ × 6″ × ½″ web 64″ × 7/16″
1-1′	192	46.72	38.26	4 flange ∟s 6″ × 4″ × ⅜″ web 84¾″ × ⅜″
ab, *bc*, *cd* *a′b′*, *b′c′*, *c′d′*	320	53.00	225.48	4 flange ∟s 6″ × 6″ × 11/16″ web 50″ × 7/16″
1-3, 1′-3′ 3-5, 3′-5′	320	36.62		4∟s 3½″ × 3½″ × ½″ 2 webs, 21″ × 9/16″
5-7, 5′-7′	320	63.17		4 ∟s 3½″ × 3½″ × ⅝″ 4 webs, 21″ × 9/16″



symmetrical about axis 7-7′ and an axis perpendicular to 7-7′. The equations which define the redundants are similar to Eqs.

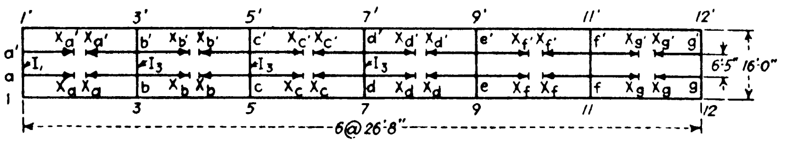


FIG. 121.

(67). The setting up of these equations and their solution involves no unusual procedure.

52. Correlation of Primary and Secondary Stresses.—The treatment of secondary-stress analysis as hitherto presented has been based on a method of successive approximations. The primary stresses were first computed. These were used as the basis for computation of the angles τ, or of the angles θ and ψ, dependent upon the method of attack utilized. With these angles computed, the corresponding bending moments were then determined and were assumed to be the true values of the secondary moments. Actually, the presence of secondary bending moments shows that the direct stresses in the various bars, as computed in the primary-stress analysis, are only approximately correct. If these values are now corrected, and a new secondary-stress analysis is carried through, more exact values of the secondary bending moments will be obtained. This procedure can, of course, be repeated to produce any desired degree of accuracy.

If the first cycle of this operation is to give results which are substantially correct, the following conditions must hold: The direct stresses as computed from the primary-stress analysis

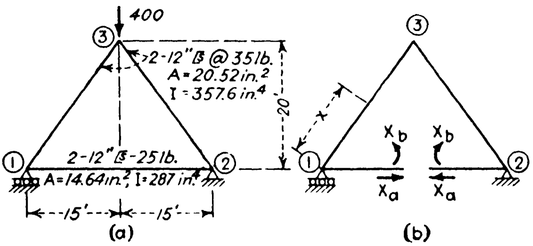


FIG. 122.

must be substantially correct, and the deflections of joints as computed from the primary structure must be equally precise.

The method of successive approximations is used in secondary-stress analysis as an expedient, rather than from necessity. A truss the joints of which are capable of resisting bending can be analyzed as a rigid frame. As such, it is a statically indeterminate structure. Although considerable labor may be involved, a direct solution, yielding exact values of direct stresses and bending moments, may be made. These results are equivalent to the combined values of the primary and secondary stresses. Either the law of virtual work or Castigliano's theorem may be used as the basis for solution. Even though such an approach is too laborious for the solution of practical problems, it offers the opportunity to investigate the validity of the assumptions upon which is based the acceptance of the methods which depend upon successive approximations.

For the purpose of making this investigation, the structure shown in Fig. 122*a* will be considered. The load shown is in units of 1,000 lb. The sections as given were designed for the load shown, on the basis of the primary stresses. Castigliano's theorem is used as a basis for analysis. The primary structure is shown in Fig. 122*b*, the bottom chord having been assumed cut at its center. Because of symmetry, no shear can exist at this point.

For bar 1-2:

$$F = +X_a$$
$$M = +X_b$$

For bar 1-3:

$$F = -160 - \frac{3X_a}{5}$$

$$M = +200\left(\frac{3}{5}x\right) - X_a\left(\frac{4}{5}x\right) - X_b = +120x - \frac{4}{5}X_ax - X_b$$

Taking advantage of the symmetry of both the structure and its loading,

$$E\frac{\partial W}{\partial X_a} = \frac{X_a(180)}{14.64}(+1) + \frac{(-160 - 0.6X_a)(300)}{20.52}\left(-\frac{3}{5}\right) + \int_0^{300}\frac{(+120x - 0.8X_ax - X_b)}{357.6}\left(-\frac{4}{5}x\right)dx = 0$$

which reduces to

$$+16{,}124.940X_a + 100.67115X_b = 2{,}414{,}703.9 \qquad (a)$$

$$E\frac{\partial W}{\partial X_b} = \int_0^{180}\frac{X_b}{287.0}dx + \int_0^{300}\frac{(+120x - 0.8X_ax - X_b)}{357.6}(-1)dx = 0$$

which reduces to

$$+1.46611X_b + 100.67115X_a = +15{,}100.671 \qquad (b)$$

The solution of Eqs. (*a*) and (*b*) leads to

$$X_a = +149.56179 \text{ kips}; \qquad X_b = +30.0895 \text{ in. kips}$$
$$F_{1\text{-}2} = X_a = +149.56179 \qquad \text{(primary stress} = +150.0)$$
$$F_{2\text{-}3} = -160 - \frac{3}{5}(+149.56179) = -249.73707$$
$$\text{(Primary stress} = -250.0)$$

The foregoing comparison shows that for the structure considered the direct stresses as computed from the primary-stress analysis are essentially correct. The deflections of joints in a truss are dependent upon the change of length of chords connecting the joints. The effect of stresses upon these changes of length is twofold: The neutral axis changes its length by an amount directly proportional to the total direct stress in the member, while, owing to bending, the neutral axis becomes, in general, curved. The true length of the chord connecting two joints is therefore always equal to or less than the length of the neutral axis connecting the same joints. The effect of this difference in length between the length of the chord and the neutral axis may be shown to be negligible in comparison with the effect of stress in changing the length of the neutral axis. It may therefore be concluded that deflections computed on the basis of primary stresses will be essentially correct. In order to illustrate this point, the vertical deflection of joint 3 of the truss shown in Fig. 122*a* will be computed, first by an exact consideration of the truss acting as a rigid frame and then, for the sake of comparison on the basis of the primary stresses. The following deflections of the primary structure due to unit loads are first computed:

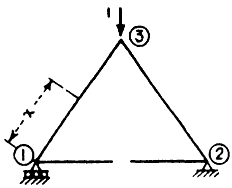


FIG. 123.

Let $\delta_{3\text{-}3}$ equal the vertical downward deflection of joint 3 due to the unit load acting as shown in Fig. 123. The method of virtual work will be used.

$$E\delta_{3\text{-}3} = 2\left[\frac{-0.4(300)}{20.52}\left(-\frac{4}{10}\right) + \int_0^{300}\frac{(0.3x)(0.3x)dx}{357.6}\right]$$
$$= +4534.8798$$

Let $\delta_{a\text{-}3}$ equal the relative inward horizontal movements of the cut ends of the bottom chord due to the unit load acting as shown in Fig. 123.

$$E\delta_{a\text{-}3} = 2\left[\frac{(-0.4)(300)}{20.52}\left(-\frac{3}{5}\right) + \int_0^{300}\frac{(+0.3x)(-0.8x)dx}{357.6}\right]$$
$$= -12073.5194$$

Let $\delta_{b\text{-}3}$ equal the relative rotation (in the direction of the redundants X_b) of tangents to the elastic curves of the cut ends of the bottom chord, due to the unit load acting as shown in Fig. 123.

$$E\delta_{b\text{-}3} = \frac{2}{357.6}\int_0^{300}\frac{3x}{10}(-1)dx = -75.50336$$

The true vertical deflection of point 3 due to the load shown in Fig. 122*a* may now be computed as follows:

$$\begin{aligned} E\delta_3 &= +400(E\delta_{3\text{-}3}) + 149.56179(E\delta_{3\text{-}a}) + 30.0895(E\delta_{3\text{-}b}) \\ &= +400(E\delta_{3\text{-}3}) + 149.56179(E\delta_{a\text{-}3}) + 30.0895(E\delta_{b\text{-}3}) \\ &= +400(+4534.8798) + 149.56179(-12073.5194) \\ &\qquad\qquad +30.0895(-75.50336) \\ &= +5942.95 \end{aligned}$$

If $E\delta_3$ be computed by the method of virtual work, taking the stresses as those resulting from the primary-stress analysis, the relation is

$$\begin{aligned} E\delta_3 &= 2\left[\frac{(250)(300)}{20.52}\left(\frac{5}{8}\right)\right] + \frac{(150)(360)}{14.64}\left(\frac{3}{8}\right) \\ &= +5951.91 \end{aligned}$$

The discrepancy between this value and the value

$$E\delta_3 = +5942.95$$

as obtained from a consideration of both primary and secondary stresses is small. The results of the investigation of this structure substantiate therefore the acceptability of the methods of secondary-stress analysis hitherto given.

In computing deflections in a statically indeterminate structure, the actual strains in the members of the structure must first be determined. This procedure involves a stress analysis of the indeterminate structure. Once this step has been carried out, the computation for the deflection of a point may be carried out by a consideration of any portion of the actual structure which in itself is stable, and which includes the point, the deflection of which is to be determined. The entire structure constitutes one such system, and has been used in the exact computation of $E\delta_3$ in the foregoing discussion.

In general, a more direct solution may be obtained by applying the law of virtual work to the primary structure. The Q system to be used consists of a unit load applied at the point whose deflection is desired and in the direction of the displacement component to be found. The stresses due to the Q system are thus those in a statically determinate structure. The condition of distortion is that defined by the stresses found in the analysis of the indeterminate structure. The procedure is valid because the stress analysis is based on the condition that the distortion of the primary structure must be the same as that of the original structure. To illustrate the advantages of this approach, the computation of $E\delta_3$ follows, based upon the primary structure shown in Fig. 123. Owing to the unit load acting as shown, the direct stress and bending in member 1-2 are zero so that members 1-3 and 2-3 only need be considered. Taking advantage of symmetry in the solution, for member 1-3,

$$F = -249.73707$$

$$M = +120x - \frac{4}{5}(+149.56179)x - 30.0895$$

$$= +0.35057x - 30.0895$$

$$F_q = -0.4$$

$$M_q = +0.3x$$

$$E\delta_3 = 2\left[\frac{(-0.4)(-249.73707)(300)}{20.52} + \int_0^{300} \frac{(+0.3x)(+0.35057x - 30.0895)dx}{357.6}\right]$$

$$= 5942.89$$

The small discrepancy between this value and that previously obtained ($E\delta_3 = +5942.95$) is due entirely to lack of arithmetic precision.

PROBLEMS FOR SOLUTION

Chapter I

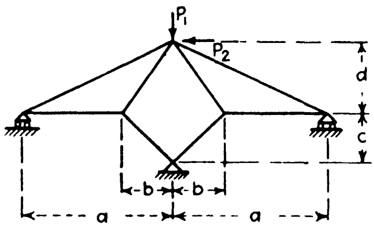


FIG. I-1.

1. What is the relation between *a*, *b*, *c*, and *d*, for which this structure becomes geometrically unstable?

Chapter II

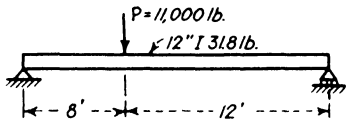


FIG. II-1.

1. The beam shown is distorted due to the load and to a change of temperature which is the same for all cross sections of the span, but varies linearly from an increase of 60°F. at the top to an increase of 10°F. at the bottom. Compute, for each cause, by the method of virtual work:

a. The vertical deflection at mid-span.

b. The change of slope at the left end.

c. The error occurring owing to neglecting the shear distortion.

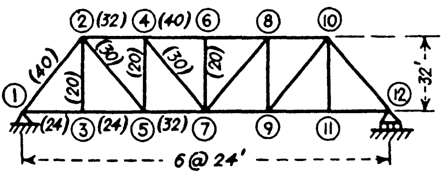


FIG. II-2.

2. The truss shown is symmetrical about its vertical mid-axis. Numbers in parentheses are cross-sectional areas in square inches. By the method of virtual work, compute:

a. The vertical deflection of joint (7) due to a vertical load of 72,000 lb. at each of the panel points of the bottom chord.

b. For the loading stated in part (*a*), the relative displacement of joints (3) and (6) along the line joining them.

c. For the loading stated in part (*a*), the rotation of the line joining joints (2) and (7).

d. The vertical deflection of joint (5) due to the following changes of temperature:

Top chords and end posts	Δt	$= +50°F.$
Web members	Δt	$= +20°F.$
Bottom chords	Δt	$= +10°F.$

e. Solve part (*a*) with the further data that the hinge support of the truss settles 1 in., while the roller support of the truss settles $\frac{1}{2}$ in.

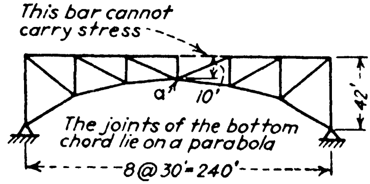


FIG. II-3.

3. Compute the vertical deflection of joint (*a*) due to an increase of 60°F in the temperature of this arch. This change is the same for all parts of the structure. Use the law of virtual work.

4. Compute the vertical deflection at mid-span of the beam shown in Fig. II-1 due to the load only, by using Castigliano's theorem.

5. Compute the vertical deflection of joint (7) of the truss described in Prob. 2*a*, Chap. II, by using Castigliano's theorem.

6. Draw the Williot-Mohr diagram for the truss of Prob. 2*a*, Chap. II, under the loading described there, and find the vertical and horizontal components of the deflections of all the joints.

7. Find the vertical components of the deflections of the joints of the three-hinged arch shown in Fig. II-3 due to the temperature change described in Prob. 3, Chap. II. Use the Williot-Mohr procedure.

8. Find the vertical components of the deflections of the panel points of the bottom chord of the truss of Prob. 2*a*, Chap. II, due to the loads described there. Use the elastic-load method.

9. Using elastic loads, find the vertical components of the deflections of the panel points of the top chord of the arch shown in Fig. II-3. The distortion is due to the temperature change stated in that problem.

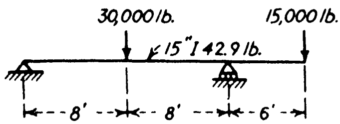


FIG. II-10

10. Compute the ordinates, at 2-ft. intervals, to the elastic curve of this beam, using the methods of Art. 24:

a. Corresponding to moment distortion only.

b. Corresponding to moment and shear distortion.

c. Due to a change in temperature varying from +40°F. at the top to +5°F. at the bottom, and the same for all cross sections of the span.

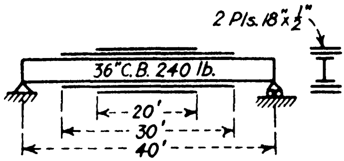


FIG. II-11.

11. This girder carries a uniformly distributed load of 5,000 lb. per lin. ft. extending over the whole span. Compute the ordinates to the elastic curve at intervals of 5 ft. Neglect shear distortion. Use the methods of Art. 24.

12. Solve Prob. 10, Chap. II, by the conjugate-beam method.

Chapter III

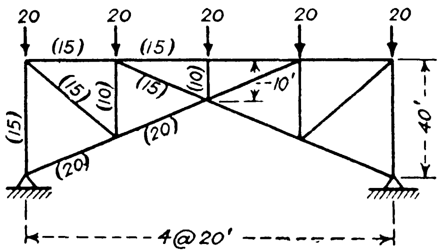


FIG. III-1.

1. This arch is symmetrical about a vertical mid-axis. The numbers in parentheses are the cross-sectional areas in square inches. The loads are in units of 1,000 lb. Compute the stresses in the members by the method of virtual work:

a. Assuming no yielding of supports.

b. Assuming that the right-hand support moves ¼ in. to the right but has no vertical movement.

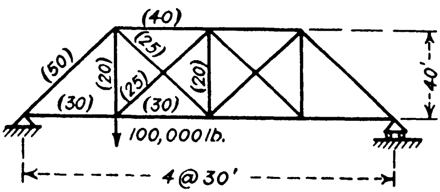


FIG. III-2.

2. This truss is symmetrical about the vertical mid-axis. The numbers in parentheses are cross-sectional areas in square inches. Compute the stresses in the members of the truss by the method of virtual work.

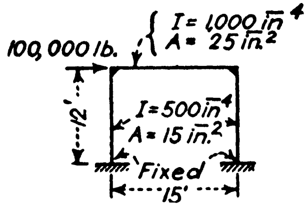


Fig. III-3.

3. Draw the curves of bending moment and shear for the members of this bent. Use the law of virtual work as a basis for analysis.

4. Draw the curves of bending moment and shear for the bent of Prob. 3, Chap. III, due to a temperature increase of 40°F. The bent is built of steel. Use the method of virtual work.

5. Compute the stresses in the arch of Prob. 1, Chap. III, due to a temperature increase of 50°F. uniform over the whole structure. Use the method of virtual work.

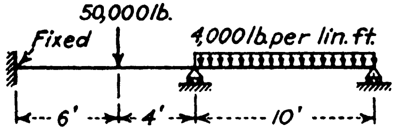


Fig. III-6.

6. Draw the curve of bending moments for this beam, using the virtual-work procedure throughout.

7. Draw the curve of bending moments for the beam of Prob. 6, Chap. III, using the virtual-work procedure, but computing deflections by using the moment-area theorems.

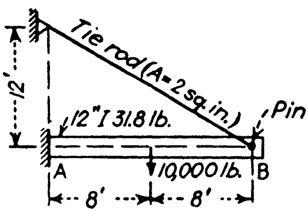


Fig. III-8.

8. Draw the curve of bending moments for the beam *A-B*. Use the virtual-work procedure for the analysis.

9. The temperature in the tie rod in Prob. 8, Chap. III, decreases by 30°F. while that in the beam does not change. Compute the stress in the tie rod and the maximum bending moment in the beam. Use the method of virtual work.

10. Assuming that the gusset plate is stiff enough to transmit bending, and using the method of virtual work, draw:

a. The curve of bending moments for the beam *ABC* due to the load shown.

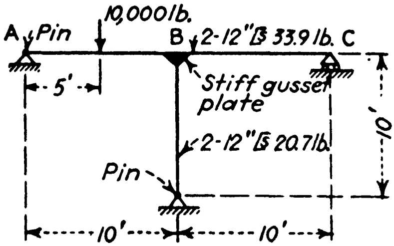


Fig. III-10.

b. The curve of bending moments for the beam due to a temperature increase of 50°F. uniform over the whole structure.

11. Solve Prob. 1*a*, Chap. III, by Castigliano's theorem.

12. Solve Prob. 2, Chap. III, by Castigliano's theorem.

13. Solve Prob. 3, Chap. III, by Castigliano's theorem.

14. Solve Prob. 8, Chap. III, by Castigliano's theorem.

15. Solve Prob. 6, Chap. III, by the slope-deflection procedure.

16. Solve Prob. 3, Chap. III, by the slope-deflection procedure.

17. Solve Prob. 10, Chap. III, by the slope-deflection procedure.

18. Solve Prob. 6, Chap. III, by the equation of three moments.

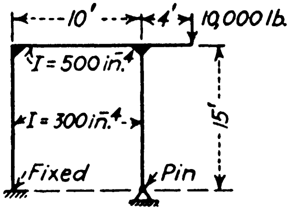


Fig. III-19.

19. Find the end moments for the members of this structure by using the slope-deflection procedure.

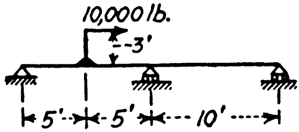


Fig. III-20.

20. Draw the curve of bending moments for this beam. Use the equation of three moments.

21. Solve Prob. 20 Chap. III, using the slope-deflection procedure.

22. Solve Prob. 3, Chap. III, by the method of moment distribution.

23. Solve Prob. 10, Chap. III, by the method of moment distribution.

24. Solve Prob. 19, Chap. III, by the method of moment distribution.

25. Solve Prob. 6, Chap. III, by the method of moment distribution.

Chapter IV

1. Draw the influence line for the horizontal component of the right-hand reaction of the arch shown in Fig. III-1. Use the method of virtual work.

2. Draw the influence line for the stress in the center vertical in the truss of Fig. III-2. Use the method of virtual work.

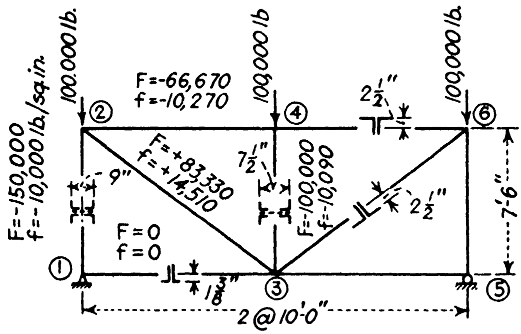
3. Draw the influence line for the bending moment at the middle support of the beam in Prob. 6, Chap. III. Use the method of virtual work.

4. Solve Prob. 1, Chap. IV, using elastic loads.

5. Solve Prob. 2, Chap. IV, using elastic loads.

6. Solve Prob. 3, Chap. IV, using the $d\theta/dx$ curve as an elastic load.

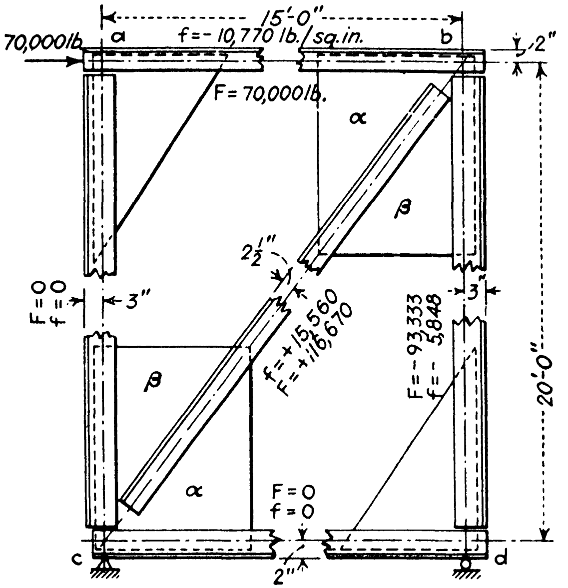
Chapter V



Bar	Length in.	Area	I	C	K	C/L	
3-4	90	9.92	85.96	3.75	0.9545	0.0416	4 Ls, 3½"×3½"×⅜"
1-2	90	15.00	187.64	4.50	2.083	0.050	4 Ls, 4"×4"×½"
2-4	120	6.50	10.10	1.33 2.67	0.08416	0.01108 0.02225	2 Ls, 4"×3"×½"
1-3	120	2.12	1.30	0.79 1.71	0.01083	0.00658 0.01425	2 Ls, 2½"×2"×¼"
2-3	150	5.74	9.04	1.30 2.70	0.0602	0.00867 0.0180	2 Ls, 4"×3"×7/16"

FIG. V-1.

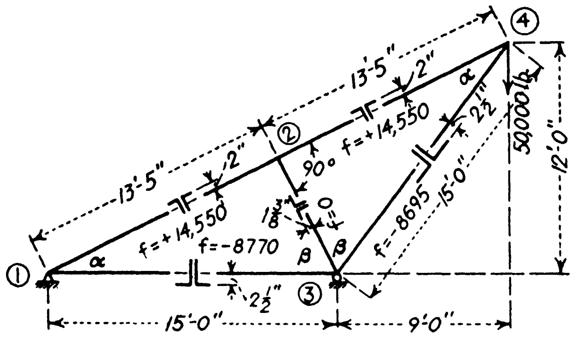
1. Compute for each end of each bar of one-half of this truss the ratio of secondary to primary stress. Use the Winkler variation of the Manderla solution.



Bar	A	L(in.)	I	C	K	$\frac{C}{L}$	Arrangement
ab cd	6.50	180	7.28	1.06 2.44	0.04044	0.00589 0.01355	2Ls $3\frac{1}{2}$"x$3\frac{1}{2}$"x$\frac{1}{2}$" c.g. 1.06"
ac bd	15.96	240	35.60	1.57 3.43	0.1484	0.00654 0.01429	2Ls 5"x5"x$\frac{7}{8}$" 1.57"
cb	7.50	300	11.12	1.18 2.82	0.03706	0.003933 0.00940	2Ls 4"x4"x$\frac{1}{2}$" 1.18"

Fig. V-2.

2. Compute the secondary-stress intensities at each end of each member of this truss. In the analysis include the effects of truss distortion and eccentricity. Use the Winkler variation of the Manderla solution.



Bar	Length	Area	I	C	K	$\frac{C}{L}$	Arrangement	Wt. per ft.
1-2 2-4	161	4.60	5.40	1.08 2.42	0.03354	0.006709 0.01503	2 Ls, $3\frac{1}{2}$" × 3" × $\frac{3}{8}$"	15.8 lb.
1-3	180	6.84	25.80	2.04 3.96	0.1433	0.01133 0.02200	2 Ls, 6" × $3\frac{1}{2}$" × $\frac{3}{8}$"	23.4 lb.
3-4	180	11.50	39.80	1.68 4.32	0.2211	0.00933 0.0240	2 Ls, 6" × 6" × $\frac{1}{2}$"	39.2 lb.
2-3	80.5	3.10	1.82	0.83 1.67	0.02261	0.01031 0.02075	2 Ls, $2\frac{1}{2}$" × 2" × $\frac{3}{8}$"	10.6 lb.

FIG. V-3.

3. For each end of each of the members of this structure, compute the ratio, secondary-stress intensity to primary-stress intensity. Include the effects of distortion, eccentricity and weights of the members, but do not attempt to compute these effects separately. Use the Winkler variation of the Manderla solution.

4. Solve Prob. 1, Chap. V, by the Mohr semigraphic method.

5. Solve Prob. 2, Chap. V, by the Mohr semigraphic method.

6. Solve Prob. 3, Chap. V, by the Mohr semigraphic method.

7. Solve Prob. 1, Chap. V, by the method of moment distribution.

8. Solve Prob. 2, Chap. V, by the method of moment distribution.

9. Solve Prob. 3, Chap. V, by the method of moment distribution.

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