Elementary Pile Theory

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FOREWORD

During the war there was assembled at the University of Chicago a group of scientists and engineers who undertook the many complex problems associated with the planning and construction of nuclear-chain reactors which could be employed to produce plutonium and many other radioactive materials and radiations. This task reached its climax in 1944 when the reactors at Hanford were placed in operation. After this date most members of the group at Chicago dispersed to undertake other tasks at other sites of the Manhattan Project, most notably at Los Alamos. There remained at Chicago, however, a small group which was concerned with the "stand-by" problems associated with the operating reactors and which devoted a fraction of its time to consideration of the longer-range aspects of reactor development. This group, which was centered for the most part on the fourth floor of Eckhart Hall of the University and looked to Professor Wigner for leadership, contained a number of men who intended to devote an appreciable fraction of their future life to the problem of reactor development. These individuals were interested in seeing established a plan whereby the science and technology of reactors could be projected continuously into the future once peacetime conditions would make it possible to broaden the basis of development to general as well as military problems.

It was agreed at this time that one of the first steps that should be taken after the end of hostilities would be the establishment of a training school at which the entire field of reactor science could be reviewed systematically and presented to younger men who would in their turn become the future leaders of the field. The group disbanded in the fall of 1945 with the hope that the time to establish such a training program would not lie too far in the future.

During the winter of 1945–46 the Monsanto Chemical Company accepted a contract to carry on the operation of Clinton Laboratories at Oak Ridge (now the Oak Ridge National Laboratory). Dr. C. A. Thomas of that company approached Wigner with the proposal that he spend a year at Oak Ridge on leave of absence from Princeton University and attempt to continue at Oak Ridge the work that had been started at Chicago during the latter period of the war. Part of this program was to be devoted to detailed planning of reactors, which would be constructed as soon as possible, and part would be devoted to a training program. Wigner accepted this proposal and was joined by a number of his wartime associates, most notably A. M. Weinberg, who is now Associate Director of Oak Ridge National Laboratory, and Gale Young who is now with Nuclear Development Associates in New York City. In addition a number of other scientists and engineers too numerous to mention who had had wartime experience agreed to participate in the program. The undersigned was granted leave from the Carnegie Institute of Technology and was appointed director of the training aspect of the proposal. Fortunately, he was able to procure the part-time services of many of the outstanding men who were assembled at Oak Ridge in planning and carrying out a fairly comprehensive series of lectures, which covered two academic semesters extending from October 1946 to June 1947. The course ranged over various topics extending from theoretical nuclear physics and health physics to the engineering problems of reactor design.

Students were recruited both from academic institutions and from industrial laboratories. Approximately forty mature individuals were obtained to constitute the core of the student body. These scientists and engineers had a median age of about thirty and were recognized experts in other fields of science or engineering. They spent about half their time taking formal courses and the remaining half in the research laboratories on practical problems of immediate interest to the atomic energy program.

A fraction of the lecture program was open to the regular employees of Clinton Laboratories who made excellent use of the opportunities. In addition a contingent of officers and civilians of the United States Navy, who were stationed in Oak Ridge to investigate problems of specific interest to the Navy, employed the training program to facilitate its own program.

The following pages of this book represent a declassified version of the lectures given by Dr. H. Soodak who was a member of the Physics Division of Clinton Laboratories and who kindly devoted a fraction of his time to the training program. These lectures were introductory in nature and were among the most popular and valuable presented. They were organized for publication by Dr. E. C. Campbell who left Princeton University to join the student body of the training program. Like many of the other trainees, Dr. Campbell elected to remain in the atomic energy program after completing the educational period.

It should be emphasized, of course, that these pages represent only a small, though very significant, part of the training program. The greatest part of the work is still classified and probably will remain so for some time to come.

On the purely personal side, I should like to take this opportunity to thank once again the large number of individuals from all walks of Clinton Laboratories for the understanding and cooperation which made the training program successful. I believe I speak for them in thanking both the Manhattan District and the Atomic Energy Commission for the opportunity to participate in this work.

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Pittsburgh, Pennsylvania
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CONTENTS

1. INTRODUCTION

It is the purpose of this course to consider the chain-reacting pile in which fast neutrons are produced by fission. Some of these neutrons, after being slowed down in a moderator, are captured by other fissionable nuclei, producing more neutrons and thus perpetuating the chain reaction. This is possible only in the case that not too many of the neutrons produced leak out of the pile structure or are unproductively absorbed before they can produce fission. A delicate balance must be reached in the neutron economy in which the rate of neutron loss is equal to the rate of neutron production if the pile is to operate at a steady level of neutron density.

This idea may be expressed in symbols in the following way: Let L be the number of neutrons that leak out of a pile per second and A the total number of neutrons absorbed per second in the pile. Of these a certain portion A_f will produce fissions, and in each fission ν neutrons will be emitted. The balance condition is then

$$\text{Leakage} + \text{absorption} = \text{production}$$

$$L + A = A_f\nu \tag{1-1}$$

We must investigate in detail what happens to the fast neutrons given off in the act of fission, how they move about in space, and how they are slowed down in matter. A fast neutron passing through a moderator substance, such as graphite, will collide with carbon nuclei. Its path will consist of many short zigzags of various lengths. The segments of the path will be oriented more or less at random, since the neutron may make either a glancing collision with slight deflection or a head-on collision with large deflection and maximum energy loss.

2. CROSS SECTIONS

It is customary to represent the probability of a particular process (scattering, absorption, fission) that occurs when nuclear projectiles pass through matter by giving the effective target area of the bombarded nucleus for that particular process. This quantity is known as the cross section of the nucleus for that process and is usually denoted by the symbol σ. If nuclei are pictured as tiny hard spheres of radius R, one might expect the effective target area to be of the order of πR^2. Since the radii of nuclei are known from other evidence to be of the order of 10^{-12} cm, it is not surprising that certain cross sections for scattering of neutrons are of the order of 10^{-24} cm^2. This quantity is known as a "barn," presumably for the reason that it denotes simply that area a poor marksman would have difficulty in hitting.

The exact meaning of the cross section is as follows: if a beam of I_0 neutrons per sec per cm^2 passes through a region in which there are N nuclei per cm^3, each with a scattering cross section σ_s, then the number of scattering processes that occurs per second per cubic centimeter is $I_0 N\sigma_s$. This can be visualized in the impractical case where $I_0 = 1$ and $N = 1$. If 1 neutron per sec per cm^2 enters a cubic centimeter in which there is just one nucleus, the probability that it will be scattered is the ratio of the target area of the nucleus σ_s to the whole area of the "beam," namely, 1 cm^2. This probability is therefore numerically equal to σ_s.

If the 1-neutron beam enters a slab of moderator, the probability of its being scattered in a thickness Δx we call $\Sigma_s\,\Delta x$. The probability that it will pass Δx without being scattered is $(1 - \Sigma_s\,\Delta x)$, and the probability that it will pass through n such thicknesses, $(1 - \Sigma_s\,\Delta x)^n$. Let x be the total thickness of the moderator traversed. Then, as n approaches infinity and $\Delta x = x/n$ approaches zero, the probability that the neutron will pass unscattered through a thickness x is just $e^{-\Sigma_s x}$, since

$$\lim_{y \to 0} \left(1 - y\right)^{a/y} = e^{-a}$$

From the previous results it is clear that Σ_s is related to the cross section by $\Sigma_s = N\sigma_s$. The fraction of the neutrons that will penetrate unscattered to distance x is $e^{-\Sigma_s x}$. The mean free path for scattering λ_s is defined as the average distance a neutron travels before being scattered. This is given by

$$\lambda_s = \int_0^\infty x e^{-\Sigma_s x} \cdot \Sigma_s\, dx = \frac{1}{\Sigma_s} \qquad (2\text{-}1)$$

In the integral $e^{-\Sigma_s x}$ is the probability that the neutron will penetrate to x without being scattered and $\Sigma_s\, dx$ is the probability that it will be scattered in the next interval dx. The product is therefore the probability of being scattered between x and $x + dx$. The integral then gives the mean free path for scattering, λ_s, which is just equal to the reciprocal of Σ_s, the probability of being scattered per centimeter of path.

The quantity $\Sigma_s = N\sigma_s$ is called the "macroscopic scattering cross section" or the "scattering cross section per cubic centimeter." In a similar way one can define analogous quantities which refer to other processes such as absorption, denoting them by λ_a, σ_a, and Σ_a. The number of atoms per cubic centimeter, N is equal to the number of moles of the substance per cubic centimeter times the number of atoms per mole. Thus

$$N = \frac{\text{density (g/cm}^3\text{)}}{\text{atomic weight (g/mole)}} \times \text{Avogadro's number (atoms/mole)} \qquad (2\text{–}2)$$

In the case of graphite, one obtains

$$N = \frac{1.6}{12} \times 6.02 \times 10^{23} = 8 \times 10^{22} \text{ atoms/cm}^3$$

Since the thermal-neutron scattering cross section σ_s is 4.8 barns for graphite, the macroscopic cross section,

$$\Sigma_s = N\sigma_s = 8 \times 10^{22} \times 4.8 \times 10^{-24} = 0.38 \text{ cm}^{-1}$$

The scattering mean free path λ_s is then $1/0.38 = 2.7$ cm. The circumstance that graphite is a useful moderator depends in part on its extremely small cross section for absorption, which is only one thousandth of its scattering cross section, that is, $\sigma_a = 0.0048$ barn. The mean free path for absorption of thermal neutrons by graphite is therefore about 2700 cm.

In general, both σ_s and σ_a depend on the neutron energy, and it is necessary to specify this energy in giving cross sections. The absorption cross section is a more sensitive function of the neutron energy than the scattering cross section. For many substances, for example boron, σ_a is inversely proportional to the neutron velocity v. Other substances show the phenomenon of "resonance" in which σ_a attains very large values for neutron energies close to a particular value.

3. SLOWING DOWN OF NEUTRONS

Consider a fast neutron ($v \sim 10^9$ cm/sec) in a large block of moderator, for example graphite. We must now examine the details of the elastic collision process by which the neutron is brought from high to low energies. When the energy becomes so low that it is comparable to the energy of the thermal motion of the graphite atoms, it is clear that the neutron will be as likely to gain as to lose energy. It will therefore retain an energy in the thermal region until it is finally captured.

A straightforward application of the laws of conservation of momentum and energy leads to a relation giving the fractional loss of energy of the neutron in terms of the angle of deflection of the neutron and the mass ratio of nucleus to neutron.

It is more convenient, however, to consider the collision from the point of view of an observer who rides with the center of mass of

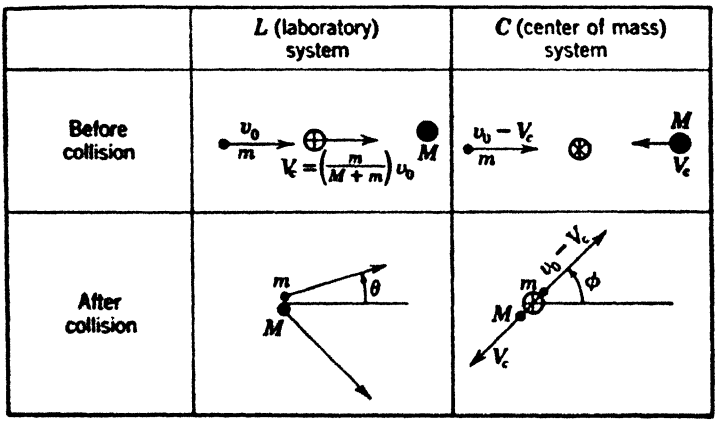


FIG. 1

the two particles. In this center of gravity system (denoted by C) the total momentum (vector sum) of the particles before collision is zero. From the law of conservation of momentum it follows that after the collision the total momentum is again zero. This means that the observer who rides with the center of gravity sees the two particles depart in precisely opposite directions. Moreover, if it is an elastic collision (kinetic energy conserved) their velocities are unchanged from what they were before collision, since such a change would mean a change in the total kinetic energy of the two particles. The total effect of the collision, as viewed in the C system, is to change the directions of the velocities but not their magnitudes. In the laboratory system (denoted by L), in which the nucleus was originally at rest, the magnitudes of the velocities are changed and their directions are not opposite. To determine the new velocity of the neutron in the L system, we must transform back from the C system to the L frame of reference.

Let the neutron move to the right with speed v_0, energy E_0, and mass m. The nucleus has zero velocity and mass M. The velocity of the center of mass

$$V_C = \left(\frac{m}{M + m}\right) v_0$$

In the C system the neutron moves to the right with a speed

$$v_0 - V_C = \left(\frac{M}{m + M}\right) v_0$$

and the nucleus moves to the left with a speed

$$V_C = \left(\frac{M}{m + M}\right) v_0$$

The total momentum is then zero, since

$$m \frac{M}{M + m} v_0 - M \frac{m}{M + m} v_0 = 0$$

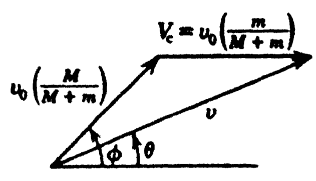


FIG. 2

After the collision the neutron flies off at an angle ϕ and the nucleus at an angle of $180° + \phi$ in the C system. In the L system the neutron leaves at an angle θ and has a velocity $\vec{v}$ which is the vector sum of the velocity of the neutron in the C system and the velocity of the center of mass. Two special cases are of particular interest. In the case of a glancing collision,

$$\phi = 0, \quad v = v_0 \quad \text{and} \quad E = E_0$$

In the case of a head-on collision,

$$\phi = 180°, \quad v = \left(\frac{M - m}{M + m}\right) v_0 \quad \text{and} \quad E = \left(\frac{M - m}{M + m}\right)^2 E_0$$

It is apparent that under this condition the neutron loses the maximum amount of energy in a collision. In the case of carbon,

$$E = \left(\frac{12 - 1}{12 + 1}\right)^2 E_0 = 0.72E_0$$

Thus a neutron colliding with a carbon nucleus can lose up to 28% of its original energy. That is, a 1-Mev neutron can lose up to 0.28 Mev and a 1-ev neutron can lose up to 0.28 ev in such a collision. The fact that the maximum fractional loss is constant makes it convenient to use a logarithmic energy scale for calculations. From Figure 2 one can obtain, by applying the law of cosines,

$$v^2 = v_0^2 \left(\frac{M}{M + m}\right)^2 + v_0^2 \left(\frac{m}{M + m}\right)^2 + 2v_0^2 \left(\frac{M}{M + m}\right)\left(\frac{m}{M + m}\right) \cos \phi \tag{3-1}$$

and the ratio of the neutron energy after collision E to its original energy E_0 is

$$\frac{E}{E_0} = \frac{\frac{1}{2}mv^2}{\frac{1}{2}mv_0^2} = \frac{v^2}{v_0^2} = \frac{M^2 + m^2 + 2mM \cos \phi}{(M + m)^2} \tag{3-2}$$

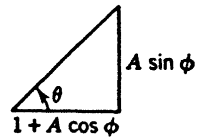
on introducing the mass ratio $A = M/m$ and $r = \left(\frac{A - 1}{A + 1}\right)^2$ this may be transformed into

$$\frac{E}{E_0} = \frac{1 + r}{2} + \frac{1 - r}{2} \cos \phi \tag{3-3}$$

Again the smallest value of E occurs for $\phi = 180°$, when $\cos \phi = -1$ and $E = rE_0$, whereas, for $\phi = 0$, $\cos \phi = 1$ and $E = E_0$.

The angle of deflection in the C system is related by Figure 2 to the corresponding angle θ in the L system according to

$$\cot\theta = \frac{\cos\phi + \dfrac{1}{A}}{\sin\phi}$$



$$\cos\theta = \frac{1 + A\cos\phi}{\sqrt{1 + A^2 + 2A\cos\phi}} \tag{3–4}$$

It may be seen that, if $A \gg 1$, then $\phi = \theta$ approximately, and the C and L systems are almost identical.

In order to get the average properties of neutrons slowing down in a moderator it is necessary to know how the probability of scattering in the C system depends on the angle ϕ. The answer, as given by both theory and experiment, is that for the neutron energies that we shall consider (energies less than 10 Mev) and for A small the scattering is to a good approximation spherically symmetric in the C system.

This means that the differential cross section $d\sigma_s$ for neutrons scattered into a solid angle $d\Omega$ is $\sigma_s/4\pi\, d\Omega$ where σ_s is a constant and is the ordinary scattering cross section as defined previously. Since the element of solid angle between ϕ and $\phi + d\phi$ is

$$2\pi \sin\phi\, d\phi = -2\pi\, d(\cos\phi)$$

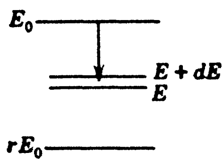


FIG. 3

all values of $\cos\phi$ are equally probable. Owing to the fact that E/E_0 is a simple linear function of $\cos\phi$, it can be seen that all values of E/E_0 from 1 to r are equally probable. The probability $P\,dE$ that a neutron will lose energy in one collision from an initial energy E_0 to a final energy in the range E to $E + dE$ is therefore equal to dE divided by the whole interval $E_0 - rE_0$ into which it can go; that is,

$$P\,dE = \frac{dE}{E_0 - rE_0}$$

We are now in a position to calculate a useful quantity called the "average loss in the logarithm of the energy in one collision," and usually denoted by ξ. The utility of ξ lies in the fact that it is independent of the neutron energy. By definition

$$\xi = \overline{\log E_0 - \log E} = \overline{\log \frac{E_0}{E}} \tag{3-5}$$

$$\xi = \int_{rE_0}^{E_0} \log \frac{E_0}{E} \cdot P\,dE = \int_{rE_0}^{E_0} \log \frac{E_0}{E} \frac{dE}{E_0 - rE_0}$$

Let

$$x = \frac{E}{E_0}$$

Then

$$\xi = \frac{1}{1-r} \int_1^r \log x\,dx$$

or

$$\xi = 1 + \frac{r}{1-r} \log r \tag{3-6}$$

where

$$r = \left(\frac{A-1}{A+1}\right)^2$$

A convenient approximate expression accurate to 1% for $A > 10$ is

$$\xi = \frac{2}{A + \frac{2}{3}} \tag{3-7}$$

For $A = 1$ $(r = 0)$ and for $A = \infty$ $(r = 1)$ the function is indeterminate. We can, however, define it to be equal to the limit of the function for these two values. This procedure gives $\xi = 1$ for $A = 1$, and $\xi = 0$ for $A = \infty$. The first case corresponds to the use of hydrogen as a moderator. The value $\xi = 1$ for this case means that on the average the energy of a neutron colliding with a hydrogen nucleus decreases by a factor e in each collision; that is, its energy after collision is only 37% of its original energy.

On the other hand, if A is very large, $\xi = 0$. The neutron loses practically no energy in an elastic collision with a heavy nucleus, for example U^{238}.

Although the scattering is spherically symmetric in the C system, it is in general not so in the L system. The divergence from spherical symmetry is measured by the average value of $\cos\theta$, where the average is taken over all possible collisions. One obtains, using equation 3–4,

$$\overline{\cos\theta} = \frac{1}{4\pi}\int \cos\theta \, d\Omega$$

$$= \frac{1}{2}\int_0^{\pi} \frac{1 + A\cos\phi}{\sqrt{1 + A^2 + 2A\cos\phi}} \cdot \sin\phi \, d\phi$$

$$= \frac{1}{2}\int_{-1}^{1} \frac{1 + Ax}{\sqrt{1 + A^2 + 2Ax}} dx$$

$$\overline{\cos\theta} = \frac{2}{3A} \tag{3–8}$$

When A is very large, $\overline{\cos\theta}$ is very small, and the distribution in angle of deflection is effectively isotropic. This agrees with the previous result that, when $A \gg 1$, L and C systems are indistinguishable. Neutrons colliding with heavy nuclei, therefore, are as often scattered forward (positive $\cos\theta$) as backward (negative $\cos\theta$). In the case of hydrogen ($A = 1$) $\overline{\cos\theta} = 2/3$, and the scattering is preferentially forward in the L system although it remains spherically symmetric in the C system.

If we know the value ξ (equation 3–6) for a moderator we can calculate very simply the average number of collisions a neutron must make in slowing down from, for instance, an energy of 2 Mev to thermal energy $1/30$ ev. The number of collisions is the total loss in the logarithm of the energy divided by the average loss ξ in one collision. This gives, for the average number of collisions,

$$\frac{\log 2 \times 10^6 - \log \frac{1}{30}}{\xi} = \frac{\log 6 \times 10^7}{\xi} = \frac{18}{\xi} \tag{3–9}$$

Sample values are given in Table 1.

TABLE 1

Moderator	A	ξ	No. of Collisions, 2 Mev to E_{th} = $18/\xi$
Hydrogen	1	1	18
Deuterium	2	0.725	25
Carbon	12	0.158	114
Beryllium	9	0.209	86

4. SLOWING-DOWN DENSITY

Consider a block of moderator so large that we can neglect leakage of neutrons out of it. Imagine that in each cubic centimeter q neutrons, each of energy E_0, are produced each second. We direct our attention to the energy distribution of neutrons and assume a uniform space distribution. Then, provided there is no absorption in the moderator during the slowing-down process, q neutrons per sec per cm^3 will slow down past each energy level E. The quantity q is known as the slowing-down density. If there is absorption in slowing down, fewer neutrons will leave dE than will enter, and q will be a function of the energy E. In general, $q(E)$ is defined as the number of neutrons per second per cubic centimeter that slow down past the energy E.

Under steady-state conditions the number of neutrons that leave the interval dE must equal the number entering. The number that leave is just the number scattered out, namely, $n(E)\,dE\,v\Sigma_s$ where $n(E)\,dE$ is the number of neutrons per cubic centimeter in the energy interval E to $E + dE$, and $v\Sigma_s$ is the probability per second that a neutron is scattered. This equals the number that enter by being scattered into dE from higher energies. Consider those coming from an energy interval dE', which is between energy E and E/r. (Neutrons of energy higher than E/r cannot enter dE.)

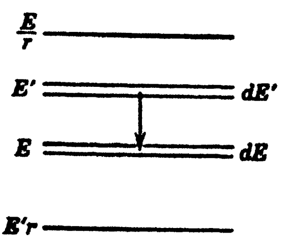


FIG. 4

The number of neutrons scattered into dE from higher energies is then equal to an integral over dE' of

the number of scattering collisions in dE' (primes refer to energy E')

$$= n(E')v'\Sigma_s'\,dE'$$

times

the probability that the energy loss will be such as to transfer the neutron from dE' to dE

$$= \frac{dE}{E' - E'r}$$

The balance equation may then be written:

$$n(E)v\Sigma_s\, dE = \int_E^{E/r} n(E')\, dE'\, v'\Sigma_s' \frac{dE}{E' - E'r} \qquad (4\text{–}1)$$

The equation is satisfied by $n(E)v\Sigma_s = c/E$, where c is a constant, as can be verified by direct substitution:

$$\int_E^{E/r} \frac{c}{E'} \frac{dE'}{E' - E'r} = \frac{c}{E} = n(E)v\Sigma_s \qquad (4\text{–}2)$$

The value of c can be found in terms of the slowing-down density $q(E)$.

We proceed to calculate an expression for $q(E)$. The number of neutrons (per second per cubic centimeter) slowing past E and scattered from the energy interval dE' is equal to

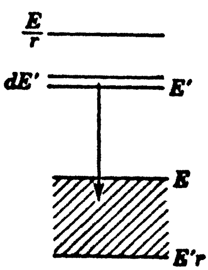


FIG. 5

(the number of scattering collisions in dE')

$$= n(E')v'\Sigma_s'\, dE'$$

times

the fraction of neutrons from dE' that lose energy greater than $E' - E$ and fall in the shaded region of the diagram (past E)

$$= \frac{E - E'r}{E' - E'r}$$

On integrating over E' from E to E/r, one obtains

$$q(E) = \int_E^{E/r} n'v'\Sigma_s' \, dE' \cdot \frac{E - E'r}{E' - E'r} \tag{4-3}$$

and with the substitution of $n(E')v'\Sigma_s' = c/E'$ (equation 4-3) reduces to

$$q = c\left[1 + \frac{r}{1 - r} \log r\right] \tag{4-4}$$

The quantity in the brackets is just ξ, the average loss in $\log E$ in one collision. As expected, q is independent of the energy and the constant $c = q/\xi$. The expression for the "flux" is

$$n(E)v = \frac{q}{\xi\Sigma_s} \cdot \frac{1}{E} \tag{4-5}$$

A picture due to Fermi enables one to obtain this same result in a simpler but less rigorous way. One imagines the neutrons slowing down in a continuous rather than in a discrete manner. We begin by assuming a slowing-down density q. Then how large is $n(E)\, dE$ if q neutrons enter and leave dE each second? The answer depends on how long a time t it takes on the average for a neutron to pass through the interval. If each neutron lingers t sec, then the number of neutrons in the interval is just qt. However, t may be expressed as the product of the average number of collisions the neutron makes in dE $\left(\text{equal to } \frac{1}{\xi}\frac{dE}{E}\right)$ and the time interval between successive collisions (equal to λ_s/v). We then have

$$n(E)\, dE = qt = q \cdot \frac{1}{\xi}\frac{dE}{E}\frac{\lambda_s}{v} \tag{4-6}$$

or since, $\lambda_s = 1/\Sigma_s$,

$$n(E)v = \frac{q}{\xi\Sigma_s} \cdot \frac{1}{E} \tag{4-7}$$

which is identical with equation 4-5. The quantity $\xi\Sigma_s$ is called the "slowing-down power" of a moderator. Since $\Sigma_s = N\sigma_s$ is the probability per centimeter for the neutron to make a collision, and ξ is the average loss in $\log E$ per collision, a simple interpretation of the slowing-down power is that it equals the average loss in log E per centimeter of travel. A good moderator therefore has a relatively large value for $\xi\Sigma_s$.

It should be noted that the neutron "flux" $n(E)v$ in the energy range dE is inversely proportional to the energy E.

5. SLOWING DOWN WITH ABSORPTION

In the previous calculation no absorption of the neutrons slowing down was allowed for, and as a consequence the slowing-down density q was a constant. Now we consider the case in which the moderator has an absorption cross section Σ_a, which in the general case is a function of the energy. The slowing-down density $q(E)$ will decrease as E decreases. The decrease in q, $dq = \frac{dq}{dE} \cdot dE$ is just equal to the number of neutrons per cubic centimeter per second absorbed in dE, namely, $n(E)v\Sigma_a\, dE$. We therefore have

$$n(E)v\Sigma_a = \frac{dq}{dE} \tag{5-1}$$

The previous result (equation 4-1) may be rewritten

$$n(E)v\Sigma_s\, dE = \frac{q}{\xi}\frac{dE}{E}$$

It must now be modified so that the absorption of the moderator is taken into account. This equation represents a balance between the number of neutrons (per second per cubic centimeter) lost to dE by being scattered out and the gain due to those neutrons scattered in from higher energies. In the situation now considered the loss of neutrons in the range dE is by absorption as well as by scattering. We may therefore write

$$n(E)v(\Sigma_s + \Sigma_a)\, dE = \frac{q(E)}{\xi}\frac{dE}{E} \tag{5-2}$$

as the appropriate balance condition. After multiplying equation 5-2 on both sides with $\frac{\Sigma_a}{\Sigma_a + \Sigma_s}$ and combining with equation 5-1 one obtains the simple differential equation for $q(E)$,

$$\frac{dq(E)}{dE} = \frac{\Sigma_a}{\Sigma_s + \Sigma_a}\frac{q(E)}{\xi E} \tag{5-3}$$

which may be integrated between the limits E and E_0 to give

$$\log \frac{q(E_0)}{q(E)} = \frac{1}{\xi} \int_E^{E_0} \frac{\Sigma_a}{\Sigma_s + \Sigma_a} \cdot \frac{dE}{E}$$

On solving for $q(E)$, one obtains

$$\frac{q(E)}{q(E_0)} = e^{-\frac{1}{\xi}\int_E^{E_0} \frac{\Sigma_a}{\Sigma_s + \Sigma_a} \cdot \frac{dE}{E}} \tag{5-4}$$

In the case where Σ_a vanishes, equation 5–4 reduces to equation 4–7, which was derived for the case of no absorption. If Σ_a and Σ_s are known as functions of the energy E, then $q(E)$ can be computed. Since Σ_s is usually a slowly varying function of the energy, one can define a suitable average value $\overline{\Sigma}_s$ and take it out of the integral, thus obtaining

$$\frac{q(E)}{q(E_0)} = e^{-\frac{1}{\xi\overline{\Sigma}_s}\int_E^{E_0} \frac{\Sigma_a}{1 + \Sigma_a/\overline{\Sigma}_s} \frac{dE}{E}} \tag{5-5}$$

The meaning of this formula is readily seen if one also defines an average value of Σ_a. However, since Σ_a is in general a rapidly varying function of the energy (especially in the case where there is resonance absorption), this procedure is not followed in an actual calculation. The integral of dE/E gives $\log E_0/E$, and equation 5–5 reduces to

$$\frac{q(E)}{q(E_0)} = e^{-\frac{\overline{\Sigma}_a}{\xi\overline{\Sigma}_s} \cdot \log \frac{E_0}{E} \cdot \overline{\frac{1}{1 + (\Sigma_a/\Sigma_s)}}} \tag{5-6}$$

As was noted previously, $\xi\overline{\Sigma}_s$ the slowing-down power is the average loss of $\log E$ per centimeter of travel. The factor $X = \frac{1}{\xi\overline{\Sigma}_s} \log \frac{E_0}{E}$ is therefore the average distance traversed by the neutron in slowing down from E_0 to E. Then as before, the fraction of the number of neutrons that slow down past E is

$$\frac{q(E)}{q(E_0)} = e^{-\overline{\Sigma}_a X} \tag{5-7}$$

if $\Sigma_a/\Sigma_s \ll 1$. The factor $\frac{1}{1 + \Sigma_a/\Sigma_s}$ is a measure of the self-protection of the absorber. For a very dilute absorber the number of moderator atoms is so much greater than the number of absorbing atoms that this factor is very close to 1.

So far only homogeneous mixtures of moderator and absorber atoms have been considered. However, an additional degree of freedom enters into the problem if it is possible to dispose the absorber in lumps instead of spreading it out uniformly. The advantage of doing so in a graphite-moderated natural uranium pile is due to the fact that resonance neutrons are very strongly absorbed in U^{238}. By lumping the uranium it is possible to increase the fraction of neutrons that escape this "resonance trap" and that may be subsequently absorbed in U^{235} giving rise to fissions. If a resonance neutron enters a U slug, it is absorbed in the outer thin layer of U, which acts as a filter to exclude all such neutrons from the inner portion of the slug. Higher-energy neutrons are not appreciably slowed down by U. It is therefore apparent that the resonance absorption *per U atom* is much smaller with lumping than in the case of a homogeneous mixture.

Piles are characterized as being slow, fast, or intermediate piles, according to whether the absorption of neutrons takes place when the neutrons are slow, fast, or in the intermediate energy region.

It is important to know in a particular case what fraction of the neutrons are absorbed in the various energy intervals. This depends on how the absorption cross section varies with the energy. From equation 5–7 it can be seen that the important quantity that determines in what energy region the neutrons are absorbed is the ratio of the average macroscopic absorption cross section of the fissionable material to the slowing-down power of the moderator. If this ratio is small (high slowing-down power) the neutron will not have traveled very far before its energy is reduced to a small value, and in this short distance there is not a very good chance to be absorbed. In this case most of the neutrons may reach thermal energies before being absorbed. At the other extreme, if there is no moderator present, neutrons will be absorbed at high energies if they are absorbed at all.

6. INTRODUCTION TO DIFFUSION THEORY

We now turn to the examination of the problem of the space distribution of neutrons, which is important if we wish to compute the leakage from a chain-reacting pile or its critical size.

Suppose that we have to do with a substance characterized by a macroscopic absorption cross section Σ_a, a macroscopic scattering cross section Σ_s, and a multiplication constant k, where k is the number of neutrons produced per neutron absorbed. Then in one cubic centimeter of the material the number of neutrons absorbed per second will be $nv\Sigma_a$ whereas the number of neutrons produced per second will be $nv\Sigma_a k$. The quantity nv is customarily called the "neutron flux." It is obtained by taking the product of n, the number of neutrons per cubic centimeter, and v their velocity. It can be visualized best by noting that nv is simply the total distance traversed by all the neutrons in a cubic centimeter in one second. Since Σ_a is the probability per centimeter for a neutron to be absorbed, the product $nv\Sigma_a$ is just the number of neutrons absorbed per second per cubic centimeter.

If we forget for the moment that these neutrons are born fast, the balance equation 1–1 written for one cubic centimeter is

$$nv\Sigma_a k = nv\Sigma_a + L$$

or

$$-L + (k - 1)\Sigma_a nv = 0 \tag{6-1}$$

The problem is now to obtain an expression for the leakage L per unit volume as a function of nv and its spatial variation.

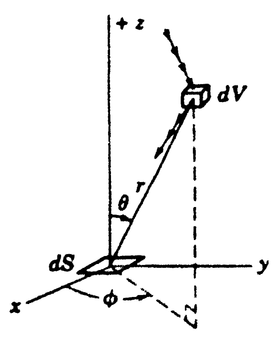


FIG. 6

We can now calculate the number of neutrons that pass per second from above through an element of surface area dS, whose normal is in the z direction as shown in the diagram. The number that arrive from the volume element dV is equal to:

the number of scattering collisions per second in dV,

$$nv\Sigma_s\, dV$$

times

the chance that the scattered particle is headed in the right direction to pass through dS,

$$\frac{|\cos\theta|\, dS}{4\pi r^2}$$

times

the probability that the particle will travel a distance r without being scattered,

$$e^{-r/\lambda_s}$$

To find the total current $J_+\, dS$ from above, we integrate over the total volume lying above dS, obtaining

$$J_+\, dS = dS \int nv\Sigma_s e^{-r/\lambda_s} \frac{|\cos\theta|}{4\pi r^2}\, dV$$

In polar coordinates, $dV = r^2 \sin\theta\, d\theta\, dr\, d\phi$, so that

$$J_+ = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\infty} \int_0^{\pi/2} nv\Sigma_s e^{-r/\lambda_s} \frac{|\cos\theta|}{r^2} r^2 \sin\theta\, d\theta\, dr\, d\phi \quad (6\text{–}2)$$

Now we must make some assumption about the dependence of nv on the coordinates x, y, and z; insert that form into the integral (equation 6–2); and carry through the integration. Owing to the term e^{-r/λ_s} in the integrand, the major contribution to the integral arises from values of nv in the region within a few mean free paths of the origin. It seems reasonable, then, to expand $nv(x, y, z)$ in a MacLaurin's series, giving

$$\begin{aligned} nv(x, y, z) &= nv_0 + x\left(\frac{\partial nv}{\partial x}\right)_0 + y\left(\frac{\partial nv}{\partial y}\right)_0 + z\left(\frac{\partial nv}{\partial z}\right)_0 \\ &+ \frac{1}{2!}\left\{x^2\left(\frac{\partial^2 nv}{\partial x^2}\right)_0 + y^2\left(\frac{\partial^2 nv}{\partial y^2}\right)_0 + z^2\left(\frac{\partial^2 nv}{\partial z^2}\right)_0 + 2xy\left(\frac{\partial^2 nv}{\partial x\, \partial y}\right)_0 \right. \\ &\left. + 2xz\left(\frac{\partial^2 nv}{\partial x\, \partial z}\right)_0 + 2yz\left(\frac{\partial^2 nv}{\partial y\, \partial z}\right)_0\right\} \quad (6\text{–}3) \end{aligned}$$

through terms of the second order. The subscript 0 means that the derivatives are to be evaluated at the origin. We express x, y, and z in terms of the polar coordinates r, θ, and ϕ by

$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r \cos\theta \end{cases}$$

The integration can be performed by elementary methods, giving

$$J_{+} = \frac{nv_0}{4} + \frac{\lambda_s}{6}\left(\frac{\partial nv}{\partial z}\right)_0 + \frac{\lambda_s^2}{16}\left\{\left(\frac{\partial^2 nv}{\partial x^2}\right)_0 + \left(\frac{\partial^2 nv}{\partial y^2}\right)_0 + 2\left(\frac{\partial^2 nv}{\partial z^2}\right)_0\right\} \qquad (6\text{–}4)$$

In a similar way the current density J_{-} due to particles passing through dS from below (z negative) can be evaluated. Here the integrand is the same except that the integration over ϕ is between the limits $\pi/2$ and π.

On carrying out the integration, we obtain

$$J_{-} = \frac{nv_0}{4} - \frac{\lambda_s}{6}\left(\frac{\partial nv}{\partial z}\right)_0 + \frac{\lambda_s^2}{16}\left\{\left(\frac{\partial^2 nv}{\partial x^2}\right)_0 + \left(\frac{\partial^2 nv}{\partial y^2}\right)_0 + 2\left(\frac{\partial^2 nv}{\partial z^2}\right)_0\right\} \qquad (6\text{–}5)$$

Terms in x, y, xy, xz, and yz do not contribute to either integral because the integral over ϕ of these terms vanishes.

The *net* current density J in the $+z$ direction is then

$$J = J_{-} - J_{+} = -\frac{\lambda_s}{3}\left(\frac{\partial nv}{\partial z}\right)_0 \qquad (6\text{–}6)$$

This is correct through terms of the second order.

The condition for the validity of equation 6–6 is that the third-order and higher terms of equation 6–4 which would have given a contribution to J should be small compared to the first-order term.

If the surface element dS is oriented so that its normal makes angles α, β, γ with the x, y, z axes instead of having its normal along the z axis, the expression for the net current through dS is

$$J\,dS = -\frac{\lambda_s}{3}dS\left[\left(\frac{\partial nv}{\partial x}\right)_0 \cos\alpha + \left(\frac{\partial nv}{\partial y}\right)_0 \cos\beta + \left(\frac{\partial nv}{\partial z}\right)_0 \cos\gamma\right] \quad (6\text{–}7)$$

This expression can be put in an abbreviated form by using the differential vector operator $\overrightarrow{\text{grad}}$ (sometimes denoted by $\vec{\nabla}$). The vector grad (nv) has components $\partial nv/\partial x$, $\partial nv/\partial y$, and $\partial nv/\partial z$ in the x, y, and z directions, respectively. If $\overrightarrow{dS}$ is a vector having the magnitude of the surface area and a direction normal to it, then equation 6–7 can be written

$$\vec{J}\cdot\overrightarrow{dS} = -\frac{\lambda_s}{3}\overrightarrow{dS}\cdot\overrightarrow{\text{grad}}\,(nv) \quad (6\text{–}7a)$$

or

$$\vec{J} = -\frac{\lambda_s}{3}\overrightarrow{\text{grad}}\,(nv) \quad (6\text{–}7b)$$

It must be noted in using the expressions for J_+ or J_- that they contain second-order terms. The approximation of using only the first two terms in the expression 6–4 is a good one only if the change in the grad (nv) in moving a distance λ_s in the medium is small compared with grad (nv). This condition is usually not satisfied in a region within a mean free path of the boundary between two dissimilar media or close to a heavy absorber.

In deriving equation 6–6 we have assumed no correlation between the direction of neutron travel before and after collision. In fact we have implicitly assumed that the scattering was spherically symmetric ($\overline{\cos\theta} = 0$) in the laboratory system. As was pointed out previously in Section 3, this is true only for collisions with heavy nuclei. For nuclei of mass A, $\overline{\cos\theta} = 2/3A$. One can correct for this preferential forward scattering by using, instead of the scattering mean free path λ_s, a new quantity called the transport mean free path λ_t defined by

$$\lambda_t = \frac{\lambda_s}{1 - \overline{\cos\theta}} \quad (6\text{–}8)$$

If the scattering is predominantly forward, then $\overline{\cos\theta}$ is positive, and the transport mean free path is longer than the mean free path for scattering. This is understandable since it means that on the average the neutron will travel further in a given number of collisions if this type of correlation exists between the directions of travel before and after collision.

Similarly, one may define a transport cross section σ_t and a macroscopic transport cross section Σ_t by

$$\sigma_t = \sigma_s(1 - \overline{\cos\theta}) \tag{6–8a}$$

and

$$\Sigma_t = \Sigma_s(1 - \overline{\cos\theta}) \tag{6–8b}$$

In the case of graphite, $\overline{\cos\theta} = 2/3A = 2/3 \times 12 = 0.056$, and therefore the transport mean free path in graphite $\lambda_t = 2.70/1 - 0.056 = 2.86$ cm.

We may then write the expression for the net current through an area dS,

$$J\,dS = -\frac{\lambda_t}{3}(nv)'\,dS \tag{6–9}$$

where the derivative $(nv)'$ is taken along a direction perpendicular to dS and λ_t has been substituted for λ_s.

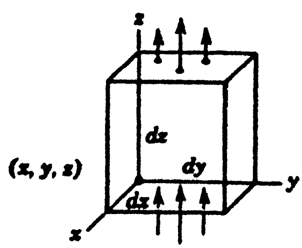


FIG. 7

Suppose now that the neutron flux $nv(x, y, z)$ is known. We wish to compute the number of neutrons that leak out of a volume element $dV = dx\,dy\,dz$ located at (x, y, z). Consider first the leakage from the two faces of area $dx\,dy$ perpendicular to the z direction. We have

$$L_z\,dV = (J_{z+dz} - J_z)\,dx\,dy$$
$$= -\frac{\lambda_t}{3}\left\{\left(\frac{\partial nv}{\partial z}\right)_{z+dz} - \left(\frac{\partial nv}{\partial z}\right)_z\right\} dx\,dy$$
$$= -\frac{\lambda_t}{3}\frac{\partial^2 nv}{\partial z^2}\,dx\,dy\,dz$$

In a similar way one can obtain, for the contribution to the leakage of the faces perpendicular to the x and y directions,

$$L_x\,dV = -\frac{\lambda_t}{3}\frac{\partial^2 nv}{\partial x^2}\,dV$$

$$L_y\,dV = -\frac{\lambda_t}{3}\frac{\partial^2 nv}{\partial y^2}\,dV$$

or, for the total leakage per unit volume,

$$L = L_x + L_y + L_z = -\frac{\lambda_t}{3}\left[\frac{\partial^2 nv}{\partial x^2} + \frac{\partial^2 nv}{\partial y^2} + \frac{\partial^2 nv}{\partial z^2}\right] \quad (6\text{–}10)$$

It is usual to abbreviate this expression thus:

$$L = -\frac{\lambda_t}{3}\Delta nv \quad (6\text{–}11)$$

where the differential operator $\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian in Cartesian coordinates.

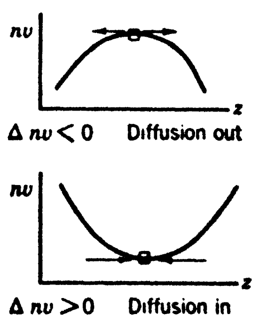


Fig. 8

From the sign of equation 6–11 it can be seen that there is a net leakage *out* of a volume element if Δnv is *negative*, for in this case L is positive. This corresponds to a *convex* neutron distribution as shown in the diagram. The neutrons diffuse from the place where the neutron density is high toward the place where the neutron density is low. Equation 6–11 may be obtained by taking the divergence of the vector equation 6–7*b*, giving, as before,

$$L = \text{div}\, J = -\frac{\lambda_t}{3}\,\text{div grad}\, nv = -\frac{\lambda_t}{3}\,\Delta nv$$

7. SOLUTIONS OF DIFFUSION EQUATIONS—BOUNDARY CONDITIONS

Our balance equation 1–1 becomes now

$$\underset{\text{Leakage}}{-\frac{\lambda_t}{3}\Delta(nv)} + \underset{\text{Absorption}}{nv\Sigma_a} = \underset{\text{Production}}{Q}$$

where Q is the number of neutrons produced per second per cubic centimeter and may also be a function of the coordinates. Then we have

$$\frac{\lambda_t}{3}\Delta nv - nv\Sigma_a + Q = 0 \qquad (7\text{–}1)$$

To avoid complications we shall assume that all neutrons have thermal energies and that all our sources emit thermal neutrons only. In the actual case thermal neutrons are produced only by slowing down of fast neutrons. We shall first consider the special case ($Q = 0$) in which there is no production of neutrons inside a particular region. We may have, for example, a neutron source outside a block of graphite and ask for the spatial dependence of the neutron flux inside. The problem is similar in its mathematical form to the problem of the temperature distribution inside a conductor of heat with specified boundary conditions. In the case of neutron diffusion the boundary conditions will determine which solution or combinations of solutions of equation 7–1 will solve the problem. With $Q = 0$ equation 7–1 may be written

$$\Delta nv - K^2 nv = 0 \qquad (7\text{–}1a)$$

where we have put

$$K^2 = \frac{3\Sigma_a}{\lambda_t}$$

For convenience we list the solutions of the equation,

$$\Delta u - \alpha^2 u = 0 \tag{7–1b}$$

in the coordinate systems appropriate for the various shapes of boundary. The solutions for $\alpha^2 > 0$ are to be chosen for the neutron flux inside a medium which cannot sustain a chain reaction; solutions for $\alpha^2 < 0$ are, as may be seen, appropriate for substances in which a chain reaction may take place.

TABLE 2

Shape of Boundary	Variables	Solutions for $\alpha^2 > 0$	Solutions for $\alpha^2 < 0$ let $\beta = i\alpha$
Plane	 $\Delta = \frac{\partial^2}{\partial x^2}$	$e^{\pm \alpha x}$ or $\left.\begin{matrix}\sinh \\ \cosh\end{matrix}\right\} \alpha x$	$e^{\pm i\beta x}$ or $\left.\begin{matrix}\sin \\ \cos\end{matrix}\right\} \beta x$
Cylindrical	 $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$	$e^{\pm \alpha_1 z} \times \left\{\begin{matrix} I_0(\alpha_2 r) \\ K_0(\alpha_2 r)\end{matrix}\right.$ where $\alpha_1^2 + \alpha_2^2 = \alpha^2$	$e^{\pm i\beta_1 z} \times \left\{\begin{matrix} J_0(\beta_2 r) \\ Y_0(\beta_2 r)\end{matrix}\right.$ where $\beta_1^2 + \beta_2^2 = \beta^2$
Spherical	 $\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$	$\frac{1}{r} e^{\pm \alpha r}$ or $\frac{1}{r} \times \left\{\begin{matrix}\sinh \alpha r \\ \cosh \alpha r\end{matrix}\right.$	$\frac{1}{r} e^{\pm i\beta r}$ or $\frac{1}{r} \times \left\{\begin{matrix}\sin \beta r \\ \cos \beta r\end{matrix}\right.$

The functions I_0, K_0, J_0, and Y_0 are zero-order Bessel functions * which are defined and tabulated in Watson, *Theory of Bessel Functions*. The general solution in each case is obtained by taking a linear combination of the functions given in the table. For example, in the spherical case the general solution is

$$nv = A \frac{e^{-\alpha r}}{r} + B \frac{e^{\alpha r}}{r}$$

where A and B are arbitrary constants.

* See also Karman and Biot, *Mathematical Methods in Engineering*, Chapter II, for excellent brief summary of properties of these functions.

Case 1. For simplicity we take first the case of an infinite slab of material with a source producing Q_0 neutrons per sec per cm^2 spread uniformly over a plane boundary at $x = 0$. Owing to

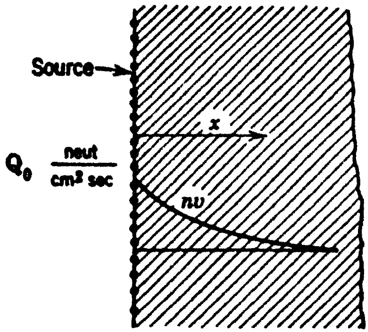


FIG. 9

symmetry nv is independent of y and z, and $\Delta(nv)$ reduces to $\frac{d^2}{dx^2}(nv)$. The equation to be satisfied is

$$\frac{d^2(nv)}{dx^2} - K^2 nv = 0 \quad (x > 0) \tag{7-2}$$

where we have put $Q = 0$ and have again introduced $K^2 = 3\Sigma_a/\lambda_t$. The general solution of this equation is

$$nv = ae^{-Kx} + be^{Kx} \tag{7-3}$$

where the arbitrary constants a and b are to be determined by the boundary conditions. The first condition is that nv is everywhere finite. This gives us $b = 0$, since otherwise $nv \rightarrow \infty$ as $x \rightarrow \infty$. The other boundary condition is that the current density J at $x = 0$ is just $\frac{1}{2}Q_0$, since only half the neutrons produced at the boundary travel to the right. This determines the constant a. We have, using equation 6–6,

$$J(0) = \frac{1}{2}Q_0 = -\frac{\lambda_t}{3}\left[\frac{d}{dx}(nv)\right]_{x=0} = \frac{\lambda_t}{3}Ka$$

from which

$$a = \frac{3}{2\lambda_t K} \cdot Q_0$$

The solution of the problem is then given by

$$nv = \frac{3}{2\lambda_t K} Q_0 e^{-Kx} \tag{7-4}$$

It may seem that the neutron flux inside the substance decreases exponentially with the distance from the plane source. The flux is, so to speak, attenuated so that it is reduced by a factor e in a distance $L = 1/K$. This distance L is called the "diffusion length" and is defined by

$$L^2 = \frac{1}{K^2} = \frac{\lambda_t}{3\Sigma_a} = \frac{1}{3\Sigma_a\Sigma_t} = \frac{\lambda_t\lambda_a}{3} \tag{7-5}$$

Except for the factor $\sqrt{3}$ the diffusion length is a geometric mean between the transport mean free path and the mean free path for absorption. For graphite,

$$L = \sqrt{\frac{2.86 \times 2700}{3}} \sim 50 \text{ cm}; \quad K = \frac{1}{50} = 0.02 \text{ cm}^{-1}$$

Case 2. In a similar way one can solve the problem of finding the neutron distribution in an infinite medium with a single point

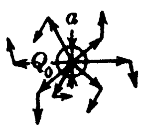


FIG. 10

source of strength Q_0 neutrons per sec. Here nv is a function only of the distance r from the source. In spherical coordinates, the operator,

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$$

plus other terms which depend only on derivatives with respect to the azimuthal and polar angles. If we assume spherical symmetry for the function nv, these terms give no contribution. We must solve the equation,

$$\frac{d^2}{dr^2}(nv) + \frac{2}{r}\frac{d}{dr}(nv) - K^2 nv = 0 \quad (r > 0) \tag{7-6}$$

subject to the boundary conditions: (A) the total neutron current through a small sphere of radius a surrounding the source has a limit Q_0 as the radius of the sphere approaches zero, and (B) the function nv is everywhere finite.

The two solutions of this equation are

$$\frac{C}{r}e^{-Kr} \quad \text{and} \quad \frac{D}{r}e^{Kr}$$

but the second function does not satisfy boundary condition B for infinite r. We might be tempted also to reject the remaining solution on the grounds that it becomes infinite for $r = 0$. This procedure is incorrect because the source-free equation 7–6 does not apply for $r = 0$ since there is a source at this point.

Boundary condition A can be written

$$Q_0 = \lim_{a \to 0} 4\pi a^2 J(a) = \lim_{a \to 0} 4\pi a^2 \left[-\frac{\lambda_t}{3}\frac{d}{dr}(nv) \right]_{r=a}$$

and on substituting

$$nv = C\frac{e^{-Kr}}{r}$$

we obtain

$$Q_0 = \lim_{a \to 0} \left[-\frac{\lambda_t}{3} C \left(-\frac{Ke^{-Ka}}{a} - \frac{e^{-Ka}}{a^2} \right) 4\pi a^2 \right]$$

$$= \lim_{a \to 0} \left[\frac{4\pi C\lambda_t}{3}(aK + 1)e^{-Ka} \right] = \frac{4\pi C\lambda_t}{3}$$

The complete solution is, therefore,

$$nv = \frac{3Q_0}{4\pi\lambda_t} \cdot \frac{e^{-Kr}}{r} \tag{7-7}$$

The meaning of the diffusion length $L = 1/K$ can be seen if we use the solution 7–7 for a point source (of thermal neutrons) in an infinite medium. The mean square distance $\overline{r^2}$ from the point source through which the neutrons diffuse (before being absorbed) can be computed. We have

$$\overline{r^2} = \frac{\int r^2 nv\, dV}{\int nv\, dV} = \frac{\int_0^\infty r^2 \frac{e^{-r/L}}{r} 4\pi r^2\, dr}{\int_0^\infty \frac{e^{-r/L}}{r} 4\pi r^2\, dr} = \frac{6L^4}{L^2} = 6L^2$$

where the integrals are evaluated by the general formula,

$$\int_0^\infty x^n e^{-x/L}\,dx = n!\;\; L^{n+1} \quad (n \text{ is any positive integer})$$

Thus the square of the diffusion length L is just one sixth of the mean square (crow-flight) distance traversed by the neutron from the position of the source (where the neutron becomes thermal) to the point where it is absorbed.

Case 3. Next we consider the neutron distribution in a flat slab of material which is infinite in extent in the y and z directions but has a finite thickness t in the x direction. The planes $x = 0$ and $x = t$ are the boundaries of the slab. The equation to be

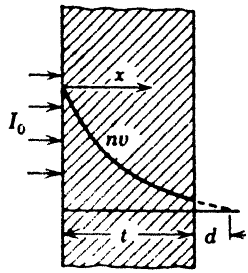


FIG. 11

solved is again equation 7·2, but the boundary conditions are changed. Suppose that the net neutron current density in the positive x direction at $x = 0$ is given and is equal to I_0. The boundary condition imposed at $x = 0$ is then $J(0) = I_0$. At the boundary $x = t$ which separates the substance from empty space an *approximate* boundary condition is that the neutron flux has the value zero at the boundary. A more exact treatment is based on the following consideration: all the neutrons which diffuse past the boundary $x = t$ are lost to the medium since they cannot be scattered back. The empty space plays, therefore, the role of a perfect absorber. The boundary condition C is that the neutron current density J_+ in the $-x$ direction vanishes at the boundary. Then,

$$J_+(t) = \left[\frac{nv}{4} + \frac{\lambda_t}{6}\frac{d}{dx}(nv)\right]_{x=t} = 0 \qquad (7\text{–}8)$$

from which we obtain

$$\left[\frac{nv}{\frac{d(nv)}{dx}}\right]_{x=t} = -\frac{2}{3}\lambda_t$$

If we suppose that nv *inside* the medium may be represented in the vicinity of $x = t$ (for $x < t$) by a linear function which vanishes for $x = t + d$ (at a distance d beyond the boundary) we have

$$nv(x) = nv(t)\left[1 - \frac{x - t}{d}\right] \qquad (7\text{–}9)$$

from which we obtain

$$\left[\frac{nv}{\frac{d(nv)}{dx}}\right]_{x=t} = -d$$

whence

$$d = \tfrac{2}{3}\lambda_t \qquad (7\text{–}10)$$

It follows then that the boundary condition C is equivalent to the following: The neutron flux inside a medium with a plane boundary (with vacuum) varies in such a way that its linear extrapolation from the boundary vanishes at a distance $d = \frac{2}{3}\lambda_t$ beyond the boundary. This distance is called the augmentation or extrapolation distance. A more accurate formulation of the problem according to transport theory leads to

$$d = 0.71\lambda_t \qquad (7\text{–}11)$$

It should be noted that equation 7–9 does not really represent the neutron flux *beyond* the boundary. The pseudo- boundary condition $nv = 0$ can be used for this problem if we also introduce a fictitious boundary of the medium at $x = t' = t + d$. Since the diffusion equation 7–1 itself is not accurate for the neutron flux within a few mean free paths of the boundary, the neutron flux calculated from the equation will not be precise. It happens, however, that the calculation of the critical size of a bare pile by setting the neutron flux equal to zero at a distance $d = 0.71\lambda_t$ beyond the boundary is a very good approximation. It will be convenient in what follows to use the boundary condition $nv = 0$. When this condition is used, it is to be understood that the dimensions are to be later corrected for the augmentation distance by equation 7–11.

The solution in case 3 is now straightforward. It is just the linear combination of exponentials (equation 7–3) with the arbitrary constants a and b chosen to satisfy the boundary conditions. This gives

$$I_0 = J(0) = -\frac{\lambda_t}{3}\left[\frac{d}{dx}(nv)\right]_0 = -\frac{\lambda_t}{3}(-a\mathrm{K} + b\mathrm{K})$$

$$nv(t') = ae^{-\mathrm{K}t'} + be^{\mathrm{K}t'} = 0$$

where t' is now the augmented thickness. We then obtain, by solving the second equation for b,

$$b = -ae^{-2\mathrm{K}t'}$$

and, on substituting b into the first,

$$I_0 = \frac{\lambda_t \mathrm{K} a}{3}(1 + e^{-2\mathrm{K}t'}) \quad \text{or} \quad a = \frac{3I_0}{\lambda_t \mathrm{K}}\frac{1}{1 + e^{-2\mathrm{K}t'}}$$

Then equation 7–3 becomes

$$\begin{aligned} nv &= ae^{-\mathrm{K}x} + be^{\mathrm{K}x} = a(e^{-\mathrm{K}x} - e^{-2\mathrm{K}t'}e^{\mathrm{K}x}) \\ &= ae^{-\mathrm{K}t'}[e^{\mathrm{K}(t'-x)} - e^{-\mathrm{K}(t'-x)}] \\ &= 2ae^{-\mathrm{K}t'} \sinh \mathrm{K}(t' - x) \end{aligned}$$

where the hyperbolic sine function $\sinh z \equiv \dfrac{e^z - e^{-z}}{2}$ has been introduced. The complete solution is therefore

$$nv = \frac{3I_0}{\lambda_t \mathrm{K} \cosh \mathrm{K}t'} \sinh \mathrm{K}(t' - x) \tag{7–12}$$

As $t' \to \infty$ this solution approaches equation 7–4 with $I_0 = Q/2$.

Case 4. The solution of the neutron diffusion problem in a medium consisting of flat slabs of different materials can be carried through by the methods of case 3. The only new feature relates to the boundary conditions at the plane separating two dissimilar media. Here we must require that the neutron flux $nv(x)$ and the current density $J(x)$ be continuous at such a boundary. That these conditions are physically reasonable can be seen immediately. If the neutron current were not continuous, it would mean that neutrons would be absorbed or created at the infinitesimally thin interface; a discontinuity of the neutron flux, on the other hand, if it existed at all, would be quickly ironed out by diffusion from the side of large flux to the side of small flux. We can write, then, the boundary conditions in the form evaluated at each boundary:

(i) $$nv_{\mathrm{I}} = nv_{\mathrm{II}}$$

(ii) $$-\frac{\lambda_{\mathrm{I}}}{3}(nv_{\mathrm{I}})' = -\frac{\lambda_{\mathrm{II}}}{3}(nv_{\mathrm{II}})' \qquad (7\text{–}13)$$

Suppose that the problem to be solved is to find the neutron flux inside a series of n plane slabs of any thickness, given that the *net* current density at $x = 0$ is I_0. The method of treating this prob-

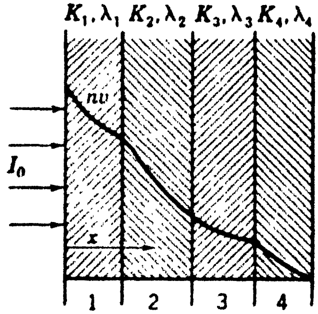


FIG. 12

lem will be presented, but the solution will not be carried through in detail. In each of the n slabs (labeled 1, 2, etc.) we will have a solution of the type 7–3,

$$nv_i = a_i e^{-K_i x} + b_i e^{K_i x}$$

with two arbitrary constants. There will be altogether $2n$ arbitrary constants ($a_1 b_1$, $a_2 b_2$, etc.) to be determined by the boundary conditions. We need, therefore, $2n$ independent equations to determine the constants. Since there are two boundary conditions (of types i and ii) at each of the $n - 1$ inside boundaries and one boundary condition at each of the outside faces, we have the required number of linear equations. The complete solution giving the neutron flux at every point inside the medium is then reduced to the solution of $2n$ linear equations in $2n$ unknowns.

8. THE ALBEDO

Not all the neutrons that are incident on a block of graphite, for example, remain in the graphite. Some are scattered back and leave the graphite at the same face by which they entered. The ratio of the back current J_+ to the forward current J_- is called the albedo A or reflection coefficient of the medium. By definition then,

$$A = \frac{J_+}{J_-} = \frac{\dfrac{nv_0}{4} + \dfrac{\lambda_t}{6}(nv)_0'}{\dfrac{nv_0}{4} - \dfrac{\lambda_t}{6}(nv)_0'} = \frac{1 + \dfrac{2}{3}\lambda_t\left(\dfrac{nv'}{nv}\right)_0}{1 - \dfrac{2}{3}\lambda_t\left(\dfrac{nv'}{nv}\right)_0} \quad (8\text{–}1)$$

where the expressions for J_+ and J_- are obtained from equation 6–5 by using only the first two terms and substituting λ_t for λ_s. The albedo is then a function of the transport mean free path and of the logarithmic derivative of the neutron flux.

In the one-dimensional case with an infinitely thick slab the neutron flux,

$$nv \sim e^{-Kx}$$

and, therefore, the logarithmic derivative is

$$\left(\frac{nv'}{nv}\right)_0 = -K$$

giving

$$A = \frac{1 - \frac{2}{3}\lambda_t K}{1 + \frac{2}{3}\lambda_t K} \quad (8\text{–}2)$$

In the case of graphite the albedo of an infinitely thick slab may be computed using the values $\lambda_t = 2.86$, $K = 0.02$. Then

$$A = \frac{1 - 0.0375}{1 + 0.0375} = 0.93$$

This means that 93% of the neutrons entering a large (infinite) block of graphite are reflected, and only 7% are absorbed in the graphite.

For a slab of finite thickness t, nv is given by equation 7–12.

$$nv = C \sinh K(t - x); \quad (nv)' = -KC \cosh K(t - x)$$

Substitution in equation 8–1 leads to

$$A = \frac{1 - \frac{2}{3}\lambda_t \mathrm{K} \coth \mathrm{K}t}{1 + \frac{2}{3}\lambda_t \mathrm{K} \coth \mathrm{K}t} \tag{8–3}$$

As $\mathrm{K}t \to \infty$, $\coth \mathrm{K}t \to 1$, and, therefore, equation 8–3 reduces again to equation 8–2 for infinite thickness.

From equation 8–3 it can be seen that for the albedo of a medium to be high the quantity $\mathrm{K}\lambda_t$ should be small. Stated differently, this means that the transport mean free path should be small compared with the diffusion length of the medium.

The albedo as defined previously is a property of both the nuclear constants of a material and its shape. This can be most readily seen for the case of a reflector (hole) having spherical symmetry. The neutron flux for $r > R$ as given by equation 7–7 is

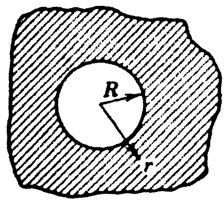


FIG. 13

$$nv = \mathrm{C}\frac{e^{-\mathrm{K}r}}{r}$$

From this we obtain

$$\left(\frac{nv'}{nv}\right)_{r=R} = -\mathrm{K} - \frac{1}{R}$$

and, on substitution in equation 8–1, this gives

$$A = \frac{1 - \frac{2}{3}\lambda_t\left(\mathrm{K} + \frac{1}{R}\right)}{1 + \frac{2}{3}\lambda_t\left(\mathrm{K} + \frac{1}{R}\right)} \tag{8–4}$$

as an expression for the albedo.

The fact that the albedo of a spherical hole is less than the albedo of a flat slab can be explained in the following way: Neutrons that diffuse through the medium from the hole have a smaller probability of being scattered back to the hole the farther they diffuse into the medium, since the probability of getting back (even without absorption) is roughly proportional to the solid angle subtended by the hole at the neutron position.

The albedo of a medium is intimately related to the number of times on the average that a neutron crosses a given plane in the medium. In diffusing past a boundary a neutron must cross the boundary an odd number of times. Let the probability that the neutron crosses the boundary 1, 3, 5 $\cdots$ times be $p(1)$, $p(3)$, $p(5)$ $\cdots$. Then it can be shown, if the albedo is A, that

$$\begin{aligned} p(1) &= 1 - A \\ p(3) &= A(1 - A) \\ p(5) &= A^2(1 - A) \\ &\vdots \end{aligned}$$

or, in general,

$$p(2i + 1) = A^i(1 - A) \quad \text{for any } i$$

Then the sum of the probabilities is just

$$\begin{aligned} \sum_{n \text{ odd}} p(n) &= 1 - A + A(1 - A) + A^2(1 - A) + \cdots \\ &= (1 - A)(1 + A + A^2 + A^3 + \cdots) \\ &= \frac{1 - A}{1 - A} = 1 \end{aligned}$$

where the geometrical series has been summed for $A < 1$.

The average number of crossings $\bar{n}$ is given by

$$\begin{aligned} \bar{n} &= \sum_{\text{odd}} np(n) = p(1) + 3p(3) + 5p(5) + \cdots \\ &= 1 - A + 3A(1 - A) + 5A^2(1 - A) + \cdots \\ &= (1 - A)(1 + 3A + 5A^2 + \cdots) = 1 + 2A + 2A^2 + 2A^3 + \cdots \\ &= 1 + \frac{2A}{1 - A} = \frac{1 + A}{1 - A} \end{aligned}$$

For graphite (infinite thickness)

$$A = 0.93 \quad \text{and} \quad \bar{n} = \frac{1 + 0.93}{1 - 0.93} = 27$$

Thus a neutron diffusing in a thick slab of graphite is expected to cross a given boundary plane on the average 27 times.

9. SPACE DISTRIBUTION OF SLOWING-DOWN DENSITY

In Sections 4 and 5 the slowing down of neutrons was treated insofar as their loss of energy was concerned, but the spatial distribution of the neutrons as they slowed down was ignored in the discussion. Actually the slowing-down density q is a function of the coordinates x, y, z, as well as of the energy E. That this is a very important consideration for chain reactors arises from the circumstance that, if a neutron diffuses too far from its birth as a fast neutron, it may not become thermal (and therefore susceptible of absorption to produce new fissions) before it has left the pile.

The production term $Q(E_0)$, where E_0 is the original energy of a neutron emitted in the act of fission, is proportional to the thermal neutron flux nv_{th}. We assume that, for every neutron absorbed, k neutrons are produced on the average. Since the number of neutrons absorbed per second per cubic centimeter is $nv_{th}\Sigma_a$, the number of neutrons produced per second per cubic centimeter is just $nv_{th}\Sigma_a k$. If we had a uniform spatial distribution of the thermal neutron flux and no absorption in slowing down, all these neutrons would reach thermal velocities. Then we could have $Q(E_{th}) = Q(E_0)$. That this equation is *not* valid for a real pile is due to (1) absorption in slowing down and (2) leakage in slowing down. A discussion of cause 1 has been given in Section 5; in this section we shall discuss cause 2.

In a chain-reacting pile the thermal neutron flux, as may be seen presently, is *not* a constant, but has a higher value at the center than at the edges of the pile. Since there will be a continuous outward streaming of neutrons from the center to the edge where they escape, the multiplicative properties of the chain reactor must be such as to compensate for this loss as well as the loss due to nonproductive absorption in the pile itself. The loss will occur also for fast neutrons, particularly for those produced near the edge of the pile. However, since the thermal neutron flux is low at the edge, the production of fast neutrons is also low there. In any case, both the fast and slow flux will be low at the pile boundary. In general, if we examine the space distribution of neutrons of various energies coming from a point source, we shall find that, the lower the neutron energy, the greater will they be spread out in space. A precise formulation of this appears later in the discussion. It can be seen that each point source of fast neutrons (from fission) gives rise to a diffuse or distributed source of thermal neutrons. In a steady state the number of neutrons in the energy

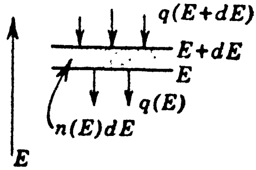


FIG. 14

range E to $E + dE$ that leak out of a cubic centimeter in one second is $-\frac{\lambda_t}{3}\Delta n(E)v\,dE$. If we neglect the absorption in slowing down, this is balanced by $Q(E)\,dE$, the excess of the number entering over the number leaving the energy range of dE. We have, therefore,

$$Q(E)\,dE = q(E + dE) - q(E) = \frac{\partial q}{\partial E}dE$$

where $q(E)$ is again the slowing-down density. Then we can write

$$-\frac{\lambda_t}{3}\Delta n(E)v\,dE = \frac{\partial q}{\partial E}\cdot dE \tag{9-1}$$

and, on substituting $n(E)v = q/\xi\Sigma_s E$ (equation 4-7) into this equation, we obtain

$$\frac{\lambda_t}{3}\frac{1}{\xi\Sigma_s E}\Delta q + \frac{\partial q}{\partial E} = 0 \tag{9-2}$$

This equation can be written in a simplified fashion if we introduce a quantity τ such that

$$d\tau = \frac{\lambda_t}{3\xi\Sigma_s}dE \tag{9-3}$$

Then equation 9–2 becomes

$$\Delta q + \frac{\partial q}{\partial \tau} = 0 \qquad (9\text{–}4)$$

Note that q is a function of the coordinates and of the energy, which enters into the equation through τ. On integrating equation 9–3 we choose the constant of integration so that for thermal energies $\tau = 0$. Then $\tau(E_{th}) = 0$. This gives us as the defining equation for $\tau(E)$:

$$\tau(E) = \int_{E_{th}}^{E} \frac{\lambda_t}{3\xi\Sigma_s} \frac{dE}{E} = \int_{E_{th}}^{E} \frac{1}{3\xi\Sigma_s\Sigma_t} \frac{dE}{E} \qquad (9\text{–}5)$$

The quantity τ, which plays a central role in the slowing-down theory, is called the "Fermi age" or just "age" because, although it has nothing to do with time, it appears in equation 9–4 in a way analogous to the appearance of *time* in the equation for the diffusion of heat.

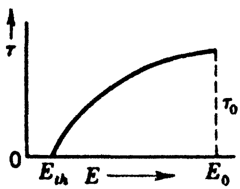


FIG. 15

Equation 9–5 can be integrated if we define suitable average values λ_t and Σ_s. Then

$$\tau(E) = \frac{\overline{\lambda_t}}{3}\left[\frac{\log E/E_{th}}{\xi\Sigma_s}\right]$$

As we saw in deriving equation 5–7, the quantity in brackets is just the average (zigzag) distance X traveled by the neutron in slowing down from E to E_{th}. Then τ can be written $\tau(E) = \dfrac{\overline{\lambda_t}}{3} X$. It can be seen that $\tau(E)$ has the dimensions of length squared and is essentially proportional to the logarithm of the neutron energy.

If an appreciable fraction of neutrons are absorbed in slowing down, equation 9–1 is no longer correct. We must add a term

$-n(E)v\Sigma_a\, dE$ to account for the absorption in the energy interval $(E, E + dE)$. We then have

$$\frac{\lambda_t}{3} \Delta n(E)v - n(E)v\Sigma_a + \frac{\partial q'}{\partial E} = 0 \tag{9-6}$$

where $q'(E)$ is the slowing-down density in the case with absorption. On substituting

$$q'(E) = n(E)v\xi\Sigma_s E \quad \text{and} \quad d\tau = \frac{\lambda_t}{3\xi\Sigma_s} \frac{dE}{E}$$

as before we obtain

$$\Delta q' - \frac{3\Sigma_a}{\lambda_t} q' + \frac{\partial q'}{\partial \tau} = 0$$

If we multiply this equation by $e^{+\int_\tau^{\tau_0} \frac{3\Sigma_a}{\lambda_t} d\tau}$ and then write

$$q(E) = q'(E)e^{+\int_\tau^{\tau_0} \frac{3\Sigma_a}{\lambda_t} d\tau} \tag{9-7}$$

we then obtain $\Delta q + (\partial q/\partial \tau) = 0$ which is the same as equation 9–4. The meaning of the substitution 9–7 can be seen if we solve for $q'(E)$ and express the integral in terms of E rather than τ. The resulting expression,

$$q'(E) = q(E)e^{-\int_E^{E_0} \frac{\Sigma_a}{\xi\Sigma_s} \frac{dE}{E}} \tag{9-8}$$

is identical, except for a factor $\dfrac{1}{1 + \Sigma_a/\Sigma_s}$ inside the integral, with the more rigorous expression 5–4, due to E. P. Wigner.

It is apparent, therefore, that the effect of absorption in slowing down on the slowing-down density q is to multiply it by

$$p(E) \equiv e^{-\int_E^{E_0} \frac{\Sigma_a}{\xi\Sigma_s} \frac{dE}{E}} \tag{9-9}$$

a factor that depends only on the energy and does not alter the spatial dependence of q. The quantity $p(E_{th})$ represents the probability that a neutron will reach thermal energies before being captured and is called the "resonance escape probability."

To solve equation 9–4 for $q(x, y, z, \tau)$ we try a solution of the type:

$$q = e^{\alpha x + \beta y + \gamma z + \epsilon\tau}$$

On substituting it into the equation, we find that it is a solution provided that

$$\alpha^2 + \beta^2 + \gamma^2 + \epsilon = 0$$

The solution may be written

$$q = e^{\alpha x + \beta y + \gamma z} \cdot e^{-(\alpha^2 + \beta^2 + \gamma^2)\tau} \tag{9-10}$$

where α, β, γ are arbitrary. Since equation 9–4 is a linear equation, any linear combination of such solutions is also a solution. It is possible to find a linear combination that will satisfy particular boundary conditions by using the method of the Fourier integral.

In particular, if we have a point source at $(x, y, z) = (0, 0, 0)$ emitting Q_0 neutrons per second all of energy E_0 (corresponding to $\tau = \tau_0$) the boundary condition is

$$q(x, y, z, \tau_0) = Q_0\, \delta(x, y, z)$$

where $\delta(x, y, z)$ is the "delta function" which is zero everywhere except at the origin and $\int \delta(x, y, z)\, dV = 1$ if the region of integration contains the origin.

It can be shown that with this boundary condition the solution of equation 9–4, obtained by the superposition (actually integration over α, β, γ) of solutions of the type 9–10, is

$$q(x, y, z, \tau) = Q_0 \frac{e^{-\frac{x^2+y^2+z^2}{4(\tau_0-\tau)}}}{[4\pi(\tau_0 - \tau)]^{3/2}} = \frac{Q_0}{[4\pi(\tau_0 - \tau)]^{3/2}}\, e^{-\frac{r^2}{4(\tau_0-\tau)}} \tag{9-11}$$

where $r^2 = x^2 + y^2 + z^2$.

For each energy (corresponding to a particular value of τ), $q(r)$ is a Gauss error curve, the width and height of which depend on τ (or on the energy.) The width to the $1/e$ value is $2\sqrt{\tau_0 - \tau}$.

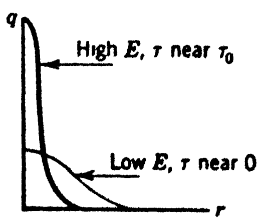


FIG. 16

In Figure 16 this function is sketched for various values of the energy. It can be seen that for high energies $q(r)$ is high and narrow, whereas for low energies $q(r)$ is low and broad. This corresponds to our physical picture in the following way: neutrons that have made very few collisions and have lost very little of their initial energy are found almost entirely in the vicinity of the origin; those that have lost a large amount of energy are distributed over a wide region, since they have made on the average many more collisions and have traveled further.

The meaning of τ_0 is clarified if we calculate the mean square (crow-flight) distance from the source for neutrons entering the thermal energy region for which $\tau = 0$. We obtain

$$\overline{r^2} = \frac{\int r^2 q_{th}(r)\, dV}{\int q_{th}(r)\, dV} = \frac{\int_0^\infty r^2 e^{-\frac{r^2}{4\tau_0}} \cdot 4\pi r^2\, dr}{\int_0^\infty e^{-\frac{r^2}{4\tau_0}} \cdot 4\pi r^2\, dr} = 6\tau_0 \qquad (9\text{–}12)$$

where τ_0 is the age of fission neutrons. The age of fission neutrons can be measured experimentally in various substances by using as a detector a cadmium-covered indium foil, which becomes radioactive (54-min beta decay) when neutrons of the indium resonance energy (1.4 ev) strike the foil. The activity of the foil as a function of position gives the neutron distribution and from this can be obtained by use of equation 9–12.

Equation 9–4 and the results based on it involve an assumption that the slowing down of neutrons is a continuous rather than a discrete process. Although the description is therefore not exact, it is adequate for those moderators, like graphite, for which ξ is small. For neutrons slowing down in water the distribution is far from Gaussian, since in a single collision with a hydrogen atom a neutron may lose a large fraction of its initial energy. In this case the continuous slowing-down picture is not good.

10. THE PILE EQUATIONS—SOLUTIONS

In a chain-reacting pile operating at a constant level of neutron density there must be a balance between production of neutrons on the one hand and leakage plus absorption on the other. This condition must be satisfied for every volume element of the pile and for each energy range dE. We have in effect an infinite number of equations to be solved, which may be written

$$\frac{\lambda_t}{3}\Delta n(E)v - n(E)v\Sigma_a + Q(E) = 0 \quad \text{for all } E \text{ from } E_0 \text{ to } E_{th} \qquad (10\text{–}1)$$

It is important to distinguish between the slowing-down density $q(E)$ and $Q(E)\,dE$, the number of neutrons (per second per cm^3) which compensate for the leakage and absorption in the energy range dE. As was shown previously in deriving equation 10–1, $Q(E)\,dE = dq = \frac{\partial q}{\partial E}dE$. In evaluating $Q(E_{th})$ it must be noted that when the neutrons reach thermal energies they cannot slow down any further. In this case, therefore, the number of neutrons becoming thermal per second per cubic centimeter is just equal to the slowing-down density evaluated at an energy slightly higher than thermal.

$$Q(E_{th}) = q(E_{th}) \qquad (10\text{–}2)$$

With absorption in slowing down we should substitute for this expression, using equation 9–9

$$Q(E_{th}) = q'(E_{th}) = q(E_{th})p(E_{th}) \qquad (10\text{–}2a)$$

when p is again the resonance escape probability.

It is customary to group together all those neutrons in the thermal energy region even though they have slightly different energies. We can then write a separate equation for the thermal neutron flux nv_{th}.

$$\frac{\lambda_{th}}{3}\Delta nv_{th} - nv_{th}\Sigma_{a_{th}} + Q(E_{th}) = 0 \qquad (10\text{–}3)$$

For the fast neutron flux we use equation 9–4,

$$\Delta q + \frac{\partial q}{\partial \tau} = 0 \qquad (9\text{–}4)$$

which is equivalent to equation 10–1 but is written in terms of the slowing-down density $q(E, x, y, z)$.

We do not yet know $Q(E_{th}) = q(E_{th})$ which must be inserted in equation 10–3. This must appear as a solution of equation 9–4. We can, however, obtain $q(E_0)$ in the following way. If the cross section for fission Σ_f and the total absorption cross section Σ_a of the material in the pile and ν, the average number of neutrons given off per fission, are known, it is possible to calculate the multiplication constant k_{th}, defined as number of fission neutrons produced per thermal neutron absorbed. The usual definition of the multiplication constant is slightly different from this and appears in what follows. If we start with one thermal neutron absorbed, the probability that it will produce a fission process is evidently Σ_f/Σ_a. We then have $\nu \cdot \Sigma_f/\Sigma_a$ fast neutrons produced per thermal neutron absorbed. Therefore,

$$k_{th} = (\Sigma_f/\Sigma_a)\nu$$

We may write then as an expression for the slowing-down density at fission energies E_0 (age, τ_0):

$$q(\tau_0) = q(E_0) = nv_{th}\Sigma_{a_{th}}k_{th} \tag{10–4}$$

We now wish to find the solutions of the steady pile equations which we rewrite below:

$$\frac{\lambda_{th}}{3}\,\Delta nv_{th} - nv_{th}\Sigma_{a_{th}} + Q_{th} = 0 \tag{10–3}$$

$$\Delta q + \frac{\partial q}{\partial \tau} = 0 \tag{9–4}$$

These are to be solved subject to the conditions:

$$\begin{cases} nv_{th} = 0 \quad \text{at the boundary} \\ q(E) = 0 \end{cases} \tag{10–5}$$

$$q(E_0) = q(\tau_0) = nv_{th}\Sigma_{a_{th}} \cdot k_{th} \tag{10–4}$$

Suppose that

$$q(E) = nv_{th}\Sigma_{a_{th}} \cdot k_{th} \cdot f(\tau)$$

where $f(\tau)$ is a function of τ on which we impose the condition $f(\tau_0) = 1$ so that equation 10–4 is satisfied for $\tau = \tau_0$. On substituting this expression in equation 9–4 we obtain, after dividing by $nv_{th}\Sigma_{a_{th}}k_{th}f$,

$$\frac{\Delta nv_{th}}{nv_{th}} + \frac{f'(\tau)}{f(\tau)} = 0 \tag{10–6}$$

Since the first term is a function only of x, y, and z and the second term a function only of τ, it is satisfied only if each term is a constant. We write, therefore,

$$\frac{\Delta n v_{th}}{n v_{th}} = \Delta \tag{10-7}$$

and

$$\frac{f'(\tau)}{f(\tau)} = -\Delta \tag{10-8}$$

where Δ is a constant. On integrating equation 10–8 from τ to τ_0 we obtain

$$f(\tau) = f(\tau_0)e^{\Delta(\tau_0 - \tau)} = e^{\Delta(\tau_0 - \tau)} \tag{10-9}$$

since $f(\tau_0) = 1$. Then on putting f back in the expression for $q(E)$ we have

$$q(E) = n v_{th} \Sigma_{a_{th}} k_{th} e^{\Delta(\tau_0 - \tau)} \tag{10-10}$$

Then the thermal neutron production term becomes

$$Q(E_{th}) = q(E_{th}) \cdot p = n v_{th} \Sigma_{a_{th}} k_{th} \cdot p e^{\Delta \tau_0} \tag{10-11}$$

where $\tau = 0$ corresponds to $E = E_{th}$.

The product $k_{th} \cdot p \equiv k$ is what has usually been called the multiplication constant and means the number of thermal neutrons produced per thermal neutron absorbed under the assumption that there is no leakage of fast neutrons while slowing down. The actual number of thermal neutrons produced per thermal neutron absorbed is $k_{th} \cdot p e^{\tau_0 \Delta}$ where the factor $e^{\tau_0 \Delta}$ is the fraction of fast neutrons that escape leakage.

This expression is now inserted in equation 10–3. After equation 10–7 is used, this becomes

$$\frac{\lambda_{th}}{3} \Delta n v_{th} - n v_{th} \Sigma_{a_{th}} + n v_{th} \Sigma_{a_{th}} k e^{\tau_0 \Delta} = 0 \tag{10-12}$$

In this equation the function $n v_{th}$ is a factor of each term. If we divide by $n v_{th} \Sigma_{a_{th}}$ and insert $L^2 = \dfrac{\lambda_{th}}{3\Sigma_{a_{th}}}$ (equation 7–5), we obtain, after rearranging terms,

$$k - (1 - L^2 \Delta)e^{-\tau_0 \Delta} = 0 \tag{10-13}$$

This transcendental equation for Δ is called the *critical equation.* The quantity Δ is a function only of the pile constants k, L^2, τ_0, each of which is, in principle, calculable from fundamental nuclear constants. To remind ourselves that it is a property of the pile material and independent of the shape and size of the pile, we call it Δ_m where the subscript m refers to material of the pile. It is defined simply as the solution of equation 10–13.

If the multiplication constant k is greater than one, equation 10–13 has one real negative solution, as can be seen by elementary reasoning.

There remains the problem of finding solutions of the equation

$$\Delta nv_{th} = \Delta nv_{th} \tag{10–7}$$

subject to the boundary condition that $nv_{th} = 0$ at the (augmented) boundary of the pile. We notice that this is essentially the same equation for which solutions ($\alpha^2 < 0$) were given in the table of Section 7, page 23.

For a particular size and shape of pile the solution of equation 10–7 which goes to zero at the boundary will correspond to a certain value Δ. This quantity depends only on the shape and size of the pile. We label it Δ_g, where the subscript g stands for geometry.

The condition that a pile is just critical is then

$$\Delta_g = \Delta_m$$

In this case only is nv_{th} a solution of the steady-state pile equations. The distinction between Δ_g and Δ_m is made so that we may later treat in a general manner the kinetics of the pile. When the power level of a pile is rising, we say that the pile is supercritical, and then $|\Delta_g| < |\Delta_m|$. If the power level is falling, the pile is subcritical and $|\Delta_g| > |\Delta_m|$. The quantity $-\Delta_g$, sometimes called the *buckling*, gives essentially the curvature of the neutron flux distribution. If the curvature is too large ($-\Delta_g > -\Delta_m$) corresponding to too small a pile, the pile will be subcritical.

In the next section the quantity Δ_g is given explicitly for various simple shapes of pile as a function of the pile dimensions.

11. CRITICAL PILE DIMENSIONS FOR SIMPLE PILE SHAPES

In this section we list the solutions of the equation,

$$\Delta nv = \Delta nv \tag{10–7}$$

subject to

$$nv = 0$$

at the boundary. The solutions are taken directly from the table of Section 7. The critical dimensions for piles of simple shapes are also given.

I. Slab Pile—infinite in y and z dimensions—thickness t in x direction

Boundary: Planes $x = \pm\frac{1}{2}t$

Equation: $$\frac{d^2nv}{dx^2} = \Delta_g nv$$

Solution: $$nv = A\cos\frac{\pi x}{t}; \qquad \Delta_g = -\frac{\pi^2}{t^2}$$

Critical size: $$\Delta_g = \Delta_m \rightarrow t_0 = \frac{\pi}{\sqrt{-\Delta_m}}$$

Critical volume: $$V_0 \sim \infty$$

II. Spherical Pile—radius R

Boundary: Sphere of radius R

Equation: $$\frac{d^2nv}{dr^2} + \frac{2}{r}\frac{dnv}{dr} = \Delta_g nv$$

Solution: $$nv = \frac{A}{r}\sin\frac{\pi r}{R}; \qquad \Delta_g = -\frac{\pi^2}{R^2}$$

Critical size: $$\Delta_g = \Delta_m \rightarrow R_0 = \frac{\pi}{\sqrt{-\Delta_m}}$$

Critical volume: $$V_0 = \frac{4\pi}{3}R_0^3 = \frac{4\pi^4}{3(-\Delta_m)^{3/2}} = \frac{130}{(-\Delta_m)^{3/2}}$$

III. Cylindrical Pile—radius R, height H

Boundary: Circular cylinder $r = R$, planes $z = \pm\frac{1}{2}H$

Equation: $$\frac{\partial^2 nv}{\partial r^2} + \frac{1}{r}\frac{\partial nv}{\partial r} + \frac{\partial^2 nv}{\partial z^2} = \Delta_g nv$$

Solution:

$$nv = A\cos\frac{\pi z}{H}\, J_0\left(\frac{2.405r}{R}\right); \qquad \Delta_g = -\left(\frac{\pi^2}{H^2} + \frac{(2.405)^2}{R^2}\right)$$

Critical dimensions having least volume:

$$\Delta_g = \Delta_m \rightarrow$$

$$H_0 = \frac{\pi\sqrt{2}}{2.405} R_0 = 1.847 R_0 \left\{ \begin{array}{l} R_0 = \dfrac{2.405\sqrt{\frac{3}{2}}}{\sqrt{-\Delta_m}} = \dfrac{2.495}{\sqrt{-\Delta_m}} \\ \\ H_0 = \dfrac{\pi\sqrt{3}}{\sqrt{-\Delta_m}} = \dfrac{5.441}{\sqrt{-\Delta_m}} \end{array} \right.$$

Least critical volume: $V_0 = \pi R_0{}^2 H_0 = \dfrac{148.2}{(-\Delta_m)^{3/2}}$

IV. Rectangular Block Pile—slides a, b, and c in x, y, z directions

Boundary: Planes $x = \pm\frac{1}{2}a, \quad y = \pm\frac{1}{2}b, \quad z = \pm\frac{1}{2}c$

Equation: $$\frac{\partial^2 nv}{\partial x^2} + \frac{\partial^2 nv}{\partial y^2} + \frac{\partial^2 nv}{\partial z^2} = \Delta_g nv$$

Solution:

$$nv = A \cos\frac{\pi x}{a} \cos\frac{\pi y}{b} \cos\frac{\pi z}{c}; \qquad \Delta_g = -\left(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} + \frac{\pi^2}{c^2}\right)$$

Critical dimensions for least volume:

$$\Delta_g = \Delta_m \rightarrow$$

$$a_0 = b = c = \frac{\pi\sqrt{3}}{\sqrt{-\Delta_m}} = \frac{5.44}{\sqrt{-\Delta_m}}$$

Least critical volume: $V_0 = a_0{}^3 = \dfrac{161}{(-\Delta_m)^{3/2}}$

Suppose, for example, that the value of Δ_m obtained by solving the critical equation 10–13 is

$$\Delta_m = -10^{-4}\ \text{cm}^{-2}$$

Then the critical dimensions will be

For a		Volume
Slab pile	$t = 314$ cm	
Spherical pile	$R_0 = 314$ cm	1.30×10^8 cm^3
Cylinder pile of least volume	$R_0 = 294$ cm $H_0 = 544$ cm	1.48×10^8 cm^3 (14% greater than sphere)
Cube pile	$a_0 = 544$ cm	1.61×10^8 cm^3 (24% greater than sphere)

The relative sizes of these various shapes are shown in the scale drawing of Figure 17.

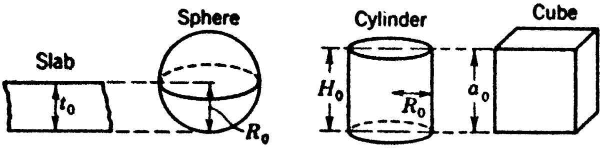


FIG. 17

The three important functions used in the description of piles are sketched in Figure 18. These are:

$\cos (\pi/2)u$ where $u = 2x/t$ for slab or $u = 2z/H$ for cylinder

$(1/\pi u) \sin \pi u$ where $u = r/R$ for sphere

$J_0(2.405u)$ where $u = r/R$ for cylinder

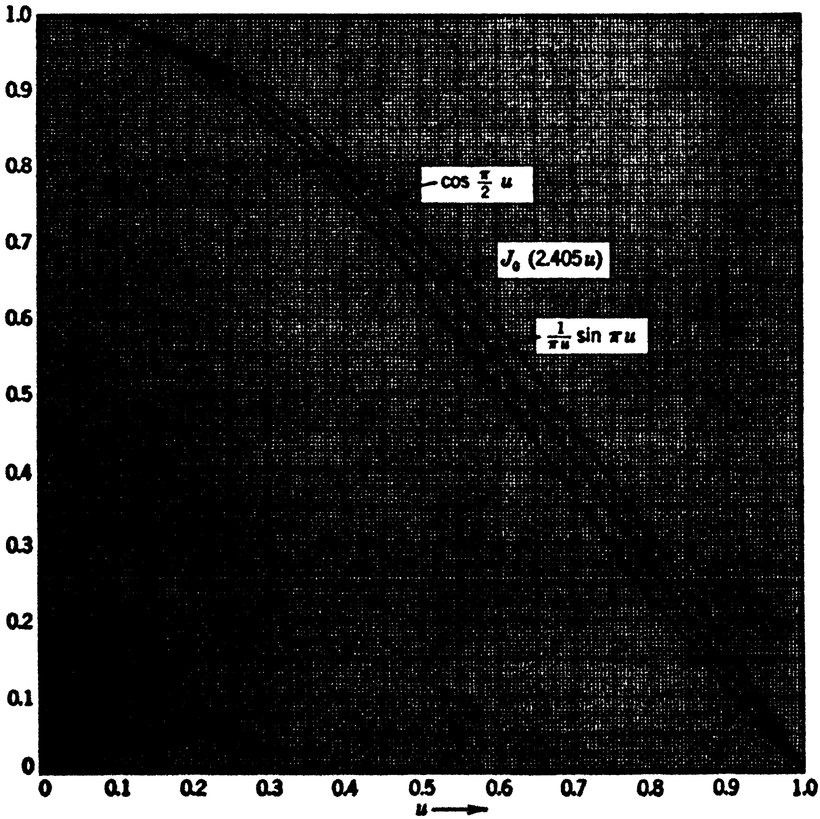


FIG. 18

It is to be noted that the curves representing these three functions differ very little in shape; in certain rough calculations one may use the cosine function instead of the other two without introducing too large an error.

12. THE NEUTRON CYCLE

In a critical pile neutron production is just balanced by absorption and leakage. Suppose that we throw N fission neutrons (energy E_0) into such a pile and follow them through one life cycle. These neutrons will slow down. Some will leak from the pile as fast neutrons, whereas others will be absorbed at energies greater than thermal. The remainder will become thermal neutrons inside the pile.

The number of neutrons reaching the energy $E + dE$ inside the pile is

$$Np(E_0, E + dE)e^{[\tau_0 - \tau(E+dE)]\Delta}$$

whereas the number of neutrons reaching energy E is given by

$$Np(E_0, E)e^{[\tau_0 - \tau(E)]\Delta}$$

The difference, which represents the loss in the energy interval dE due to absorption and leakage is

$$N\left[e^{[\tau_0 - \tau(E)]\Delta}\,\frac{dp\,(E_0, E)}{dE}\,dE\right] + Np(E_0, E)e^{[\tau_0 - \tau(E)]\Delta}\left(-\Delta\,\frac{d\tau(E)}{dE}\,dE\right)$$

where the first term represents the number of the original N neutrons that are absorbed in dE and the second, the number that leak from the pile while in the energy range dE.

The number which reach thermal energies is $Ne^{\tau_0\Delta}p(E_0, E_{th})$. Of these the fraction $\dfrac{1}{1 - L^2\Delta}$ are absorbed as thermal neutrons, whereas the remainder, the fraction $\dfrac{-L^2\Delta}{1 - L^2\Delta}$, leak from the pile as thermal neutrons. This can be seen from the following simple argument.

For any unit volume of the pile (and therefore for the pile as a whole) the ratio of thermal leakage to thermal absorption is

$$\frac{L}{A} = \frac{-\frac{\lambda_t}{3}\Delta nv}{nv\Sigma_a} = \frac{-\lambda_t\Delta}{3\Sigma_a} = -L^2\Delta \qquad (12\text{-}1)$$

Therefore, the fraction of the thermal neutrons absorbed will be

$$\frac{A}{L+A} = \frac{1}{1+L/A} = \frac{1}{1-L^2\Delta}$$

whereas the fraction that leaks out will be

$$\frac{L}{L+A} = \frac{L/A}{1+L/A} = \frac{-L^2\Delta}{1-L^2\Delta}$$

We know therefore how many of the original N neutrons are absorbed in the pile and at which energies. The number

$$N\frac{e^{\tau_0\Delta}p(E_0, E_{th})}{1-L^2\Delta}$$

are absorbed at thermal energies and the number

$$N\int_{E_{th}}^{E_0} e^{[\tau_0-\tau(E)]\Delta}\frac{dp(E_0, E)}{dE}\,dE$$

are absorbed at energies above thermal.

Let us define $k(E)$ as the average number of fission neutrons produced per neutron absorbed at energy E. Then

$$k(E) = \frac{\Sigma_f(E)}{\Sigma_a(E)}\,\nu(E)$$

We call k_{th} the value of $k(E)$ for thermal energies.

The neutron cycle is now complete. The final result is as follows: Starting with N fission neutrons, we obtain, after one cycle,

$$N\left[\frac{e^{\tau_0\Delta}p(E_0, E_{th})k_{th}}{1-L^2\Delta} + \int_{E_{th}}^{E_0} e^{[\tau_0-\tau(E)]\Delta}\frac{dp(E, E_0)}{dE}\,k(E)\,dE\right]$$

neutrons. This can also be written as

$$N\left[\frac{e^{\tau_0\Delta}p_{th}k_{th}}{1 - L^2\Delta} + e^{\overline{[\tau_0-\tau(E)]}\Delta}(1 - p_{th})\overline{k(E)}\right]$$

where $\overline{\tau_0 - \tau(E)}$ is the average "age" from fission energy to the energy at which the neutrons are absorbed (while slowing down) $\overline{k(E)}$ is an average multiplication constant for the neutrons absorbed above thermal energies, and $p_{th} \equiv p(E_0, E_{th})$.

If the pile is to be critical, the same number of fission neutrons must be present at the end of the cycle as at the beginning. Thus the critical equation for the general case is

$$\frac{e^{\tau_0\Delta}p_{th}k_{th}}{1 - L^2\Delta} + e^{\overline{[\tau_0-\tau(E)]}\Delta}(1 - p_{th})\overline{k(E)} = 1 \qquad (12\text{–}2)$$

A thermal pile, that is, a pile in which fissions are caused only by the absorption of thermal neutrons, can be realized in two ways: either $p_{th} = 1$ (neutrons are absorbed only at thermal energies) or else $p_{th} < 1$ but $\overline{k(E)}$ vanishes. The latter corresponds closely to the case of the natural uranium piles where there is considerable resonance absorption (not leading to fission) by U^{238} at energies above thermal.

In such a thermal pile the critical equation is then

$$\frac{ke^{\tau_0\Delta}}{1 - L^2\Delta} = 1 \qquad (12\text{–}3)$$

or

$$k - (1 - L^2\Delta)e^{-\tau_0\Delta} = 0 \qquad (12\text{–}4)$$

where we define

$$k = k_{th}p_{th} = \frac{\Sigma_f(E_{th})}{\Sigma_a(E_{th})}\nu p_{th} \qquad (12\text{–}5)$$

The multiplication constant k can also be written as

$$k = f\eta p_{th} \qquad (12\text{–}6)$$

where

$$f = \frac{\text{absorption cross section of fissionable material at } E_{th}}{\text{absorption cross section of all pile material at } E_{th}}$$

$$= \frac{\Sigma_a^{fiss}(E_{th})}{\Sigma_a(E_{th})}$$

is the fraction of thermal neutrons absorbed that are absorbed in fissionable material and

$$\eta = \frac{\text{fission cross section of fissionable material at } E_{th}}{\text{absorption cross section of fissionable material at } E_{th}} \cdot \nu$$
$$= \frac{\Sigma_f^{fiss}(E_{th})}{\Sigma_a^{fiss}(E_{th})} \nu$$

is the number of fission neutrons produced for each thermal neutron absorbed in fissionable material.

In a nonthermal pile, $p_{th} = 0$. In such a pile no neutrons will reach thermal energies; this condition may be brought about by making the concentration of fissionable material high or the concentration of moderator low, or both. Then the critical equation may be written

$$\overline{k(E)e^{[\tau_0 - \tau(E)]\Delta}} = 1 \tag{12-7}$$

For any of these cases, thermal, nonthermal, or mixed (semithermal) the critical size of the bare pile is obtained by setting $\Delta_g = \Delta_m$ where Δ_m is the real root of the critical equation 12–2, 12–4, or 12–7 and Δ_g is the geometrical Laplacian, determined (Section 11) by the size and shape of the pile.

If the pile is not of the critical size, $\Delta = \Delta_g$ will be greater or smaller than Δ_m, and the number of neutrons at the end of a cycle will be different from the number at the beginning. This means that the neutron density in the pile will change with time. If γ is the fractional increase in the number of neutrons per cycle, we then have, for the mixed noncritical pile,

$$\frac{e^{\tau_0\Delta}p_{th}k_{th}}{1 - L^2\Delta} + \overline{e^{[\tau_0 - \tau(E)]\Delta}(1 - p_{th})k(E)} = 1 + \gamma \tag{12-8}$$

If $\gamma > 0$, the pile is supercritical, and the neutron density will rise; if $\gamma < 0$, the pile is subcritical, and the neutron density will fall.

For a thermal pile γ is given by

$$\gamma = \rho \frac{e^{\tau_0\Delta}}{1 - L^2\Delta} \tag{12-9}$$

where

$$\rho \equiv k - (1 - L^2\Delta)e^{-\tau_0\Delta} \tag{12-10}$$

is called the reactivity of the pile. The reactivity of the pile is an important quantity in the later discussion of pile kinetics.

In the foregoing discussion of the neutron cycle we assumed that all fission neutrons came off with the same energy E_0. The introduction of the observed continuous fission spectrum into the picture requires that the quantities τ_0 and $\tau_0 - \tau(E)$ in equations 12–2, 12–7, and 12–8 represent averages over the fission spectrum.

13. PILE WITH REFLECTOR—GROUP THEORIES

So far only the case of the bare homogeneous pile has been considered. However, a considerable practical advantage results from surrounding a pile with a reflector having a high albedo. Since the leakage of both fast and slow neutrons may be thereby reduced, such a pile will have a smaller critical volume. Indeed, if a very good reflector is used, it is possible that the total size of critical pile plus reflector is smaller than that of the bare pile alone. Thus with a reflector one can achieve a considerable saving in the initial charge of fissionable material for a pile.

Another advantage results from the flattening of the neutron distribution in a pile with reflector. In a bare pile the fissionable material at the edge of the pile is used with a very low efficiency since the neutron flux is very low there. With a reflector the neutron flux is made more uniform over the pile, and therefore the *average* flux can be considerably greater than in a pile without reflector. If the limit to the power level is determined by the temperature at the center of the pile, for example, a higher power level may thus be reached in the pile with reflector. It is possible to achieve this end also by a nonuniform distribution of the fissionable material in the pile.

Unfortunately, it has not been found possible to give a precise solution for the pile equations for the pile plus reflector, in the interesting case where the properties of the reflector differ from those of the reactor. It is necessary to use approximate methods to solve such problems.

A general class of such approximate methods is called the multigroup method, in which the various groups of neutrons in the pile and reflector are sorted according to their average energy; the materials in pile and reflector are assigned average properties for each of these different neutron groups. Such a method is very useful for computing the effects of control rods which absorb thermal neutrons strongly but have little effect on fast neutrons.

An equation of the type 10–1 is written for the neutron flux of each energy group. The equations are coupled together owing to the fact that the fast neutrons slow down; the disappearance of a neutron from a certain group is accompanied by its appearance in a group of lower energy. As before, the equations for the lowest and highest energy groups are coupled together by the fission process.

The simplest theory of this type is the *one-group theory*, in which we imagine that all absorption and production occurs at one neutron energy. This is not too bad an approximation for a fast pile, for example, in which absorption and production take place close to fission energies. It would be an accurate representation of what happens in a thermal pile only if fission neutrons were born as thermals instead of as fast neutrons. To illustrate the method we shall apply it to the case of slab and spherical piles with reflectors; in the next section the *two-group theory* is considered. It is to be noted that the critical size of the *bare* pile is not given correctly by either of these approximate formulations. It is convenient, however, to "adjust" the pile constants in such a way that the correct critical size is given by the one- and two-group approximations.

We begin with equation 6–1 which we write in terms of the neutron flux $\phi = nv$. This quantity satisfies the relations,

$$-\frac{\lambda_t^0}{3}\Delta\phi + \Sigma_a^0\phi - k\Sigma_a^0\phi = 0 \qquad (13\text{–}1)$$

inside the reactor, characterized by the constants λ_t^0, Σ_a^0, and k, and

$$-\frac{\lambda_t^1}{3}\Delta\phi + \Sigma_a^1\phi = 0 \qquad (13\text{–}2)$$

inside the reflector, characterized by the constants λ_t^1, Σ_a^1, $k = 0$ (no production). These can be put in the form,

$$\begin{cases} \Delta\phi + K_0^2\phi = 0 & \text{in reactor} \qquad (13\text{–}3) \\ \Delta\phi - K_1^2\phi = 0 & \text{in reflector} \qquad (13\text{–}4) \end{cases}$$

where

$$\begin{cases} K_0^2 = (k-1)\dfrac{3\,\Sigma_a^0}{\lambda_t^0} = \dfrac{k-1}{L_0^2} & (13\text{–}5) \\[2ex] K_1^2 = \dfrac{3\,\Sigma_a^1}{\lambda_a^1} = \dfrac{1}{L_1^2} & (13\text{–}6) \end{cases}$$

These are to be solved subject to the boundary conditions:

I. $\phi = 0$ at the outer boundary of the reflector

II. ϕ and $\frac{\lambda_t}{3}\phi'$ are continuous at the boundary between reactor and reflector.

As an example consider a slab pile of half-thickness a covered (both sides) by a reflector of thickness t. The solutions of equations 13–3 and 13–4 are:

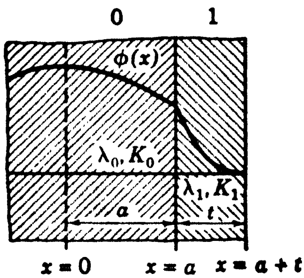


FIG. 19

$$\phi_0 = \cos K_0 x \qquad (|x| \leq a) \tag{13–7}$$

$$\phi_1 = C \sinh [K_1(t + a - x)] \ (a \leq |x| \leq a + t) \tag{13–8}$$

where boundary condition I has been satisfied by choosing the sinh function, which vanishes at the boundary of the reflector. Condition II applied at $x = a$ gives

$$\lambda_t{}^0 K_0 \tan K_0 a = \lambda_t{}^1 K_1 \coth K_1 t \tag{13–9}$$

This is a transcendental equation for the critical half-thickness of the pile as a function of the thickness of reflector. If $t \to 0$, $\coth K_1 t \to \infty$, $K_0 a = \pi/2$; the critical half-thickness is of the same form as was previously obtained for the bare pile in Section 11, namely, $a_0 = \pi/2K_0$. They are not identical since K_0, as defined by equation 13–5, is not identical with $\sqrt{-\Delta_m}$ defined as the solution of equation 10–13.

The reflector savings $S = a_0 - a$ is the amount by which the critical half-thickness is decreased (owing to the reflector) from what it would be for a bare pile. Evidently, S depends on the reflector thickness as well as the constants of pile and reflector. From equation 13–9 we have directly

$$S = a_0 - a - \frac{1}{K_0}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\lambda_t^1 K_1}{\lambda_t^0 K_0} \coth K_1 t\right)\right] \quad (13\text{–}10)$$

In the limiting case for small reflector thickness ($K_1 t \ll 1$) equation 13–10 becomes

$$S = a_0 - a = \frac{\lambda_t^0}{\lambda_t^1} t \quad (13\text{–}11)$$

If the reflector is such an efficient scatterer that its transport mean free path is smaller than that of the pile, the reflector savings will be greater than the reflector thickness. This means that the total size of pile plus reflector will be less than that of the bare pile alone.

A similar calculation can be made for the more practical case of a spherical reactor of radius R covered by a reflector shell of thickness t. The transcendental equation corresponding to equation 13–9 is then

$$\cot K_0 R = \frac{1}{K_0 R}\left(1 - \frac{\lambda_t^1}{\lambda_t^0}\right) - \frac{\lambda_t^1 K_1}{\lambda_t^0 K_0} \coth K_1 t \quad (13\text{–}12)$$

and the critical radius of the bare pile is $R_0 = \pi/K_0$. The reflector savings in this case $S = R_0 - R(t)$ can be computed by solving the equation numerically for given values of the pile constants.

This procedure is carried through for both the spherical and slab pile having the dimensions given in Section 11. We imagine the pile to be moderated by graphite and to be surrounded by a water reflector. Let us assume the following round number for the Laplacian:

$$\Delta = -10^{-4} = -K_0^2$$

$$K_0 = 10^{-2}\ \text{cm}^{-1}$$

In addition the following constants are used:

$$\lambda_t^0 = 2.7\ \text{cm}, \qquad K_1 = \frac{1}{2.8}\ \text{cm}^{-1}, \qquad \lambda_t^1 = 0.4\ \text{cm}$$

The results in Figure 20 show the reflector savings as a function of the reflector thickness in the two cases. The general character of both curves is similar; the reflector savings "saturate" between 18 and 20 cm for reflector thickness greater than 10 cm. According to the curve, then, a spherical graphite-moderated pile that has a critical radius of 314 cm with no reflector could be made critical with a 10-cm water reflector if the radius were 294 cm.

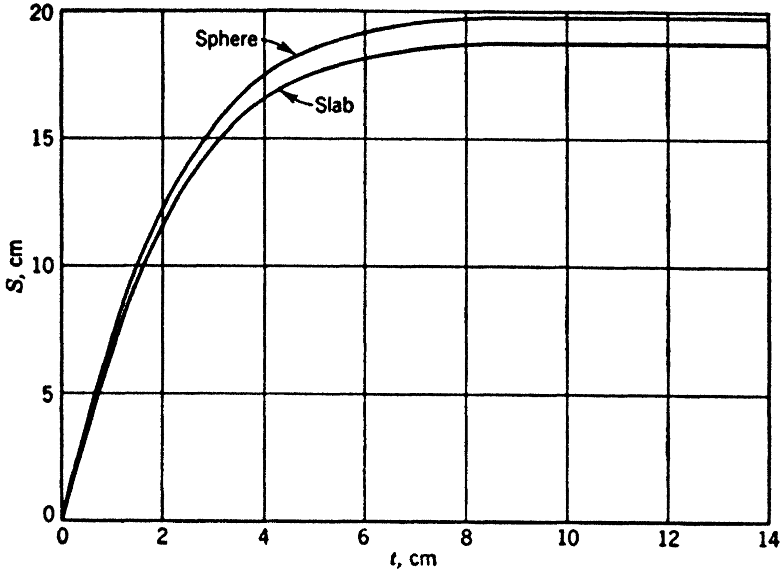


FIG. 20

The reflector savings for an infinite water reflector (thickness greater than 10 cm) are found to be

For a sphere	19.7 cm
For a slab	18.8 cm
For an infinite circular cylinder	19.7 cm

The numerical values are not to be taken too seriously; the general shape of the curves, however, is not very different from what would be given by a more exact theory. It is found that reflector savings calculated on the basis of one-group theory are too low; the more exact two-group calculation gives a larger reflector savings since the albedo of the reflector for fast neutrons exceeds that for thermal neutrons.

In the case that the transport mean free path is the same for reactor and reflector ($\lambda_t^0 = \lambda_t^1$) the equations 13–12 for the sphere and 13–9 for the slab are of the same form; the reflector savings in the half-thickness of the slab and in the radius of the sphere are therefore equal.

14. TWO-GROUP THEORY

Since the pertinent nuclear properties of materials in pile and reflector are in general very different for fast and for slow neutrons, the one-group calculation of the preceding section cannot give very precise results. The next order of approximation is to consider separately the fast and thermal neutron fluxes, ϕ_f and ϕ_{th}. Although the fast neutrons have a large range of energies, we make the approximation of lumping them all in one group and attribute to the substances in which they diffuse appropriate average properties.

Inside a reactor ϕ_{th} and ϕ_f satisfy the equations:

$$\frac{\lambda_f}{3}\,\Delta\phi_f - \Sigma_{a_f}\phi_f + k\Sigma_{a_{th}}\phi_{th} = 0 \qquad (14\text{–}1)$$

$$\frac{\lambda_{th}}{3}\,\Delta\phi_{th} - \Sigma_{a_{th}}\phi_{th} + \Sigma_{a_f}\phi_f = 0 \qquad (14\text{–}2)$$

These are of the same general type as our original equations in that the three terms of each equation stand for leakage, absorption, and production, respectively. We have introduced a *fictitious* fast-absorption cross section Σ_{a_f} which takes into account not the true fast absorption (which we neglect) but the loss of neutrons from the fast group which occurs when fast neutrons in slowing down enter the thermal group. The term $\Sigma_{a_f}\phi_f$ represents the source (number becoming thermal per cubic centimeter per second) of thermal neutrons, therefore, as well as a loss to the fast group. The source of fast (fission) neutrons, on the other hand, is proportional to the thermal flux, and is equal to $K\Sigma_{a_{th}}\phi_{th}$ if $K \equiv K_{th}p_{th}$ thermal neutrons are produced for each thermal neutron absorbed and $p_{th} = 1$.

Inside the reflector ϕ_f and ϕ_{th} satisfy similar equations except that, since we assume there is no fissionable material in the reflector, the multiplication factor k is zero. In general, the other constants in the equation will have in the reflector a different set of values.

It is necessary to choose values of Σ_{a_f} and λ_f which give the same mean square slowing-down distance $6\tau_0$ as in the exact theory. This can be achieved in the following way: we try to solve equation 14–1, assuming a point source of (fission) neutrons at $r = 0$. Let $K_f^2 = 3\Sigma_{a_f}/\lambda_f$. The equation may then be written

$$\Delta\phi_f - K_f^2\phi = 0 \qquad (r > 0) \tag{14-3}$$

and the solution is (see Section 7)

$$\phi_f = A\,\frac{e^{-K_f r}}{r} \tag{14-4}$$

For r^2 we obtain, as in Section 7,

$$\overline{r^2} = \frac{\int r^2\phi_f\,dV}{\int \phi_f\,dV} = \frac{\int_0^\infty r^2\,\frac{e^{-K_f r}}{r}\,4\pi r^2\,dr}{\int_0^\infty \frac{e^{-K_f r}}{r}\,4\pi r^2\,dr} = \frac{6}{K_f^2} \tag{14-5}$$

We therefore choose $K_f^2 = \frac{1}{\tau_0}$ so that equation 14–5 agrees with equation 9–12. The approximation consists of substituting a function $\frac{1}{r}\,e^{-K_f r}$ for the more exact Gauss error function $\frac{e^{-r^2/4\tau_0}}{\sqrt{4\pi\tau_0}}$.

Another way of looking at the problem is to consider the details of the slowing-down process. The average number of collisions made by a neutron is slowing down from E_0 to E_{th} is $N = \frac{1}{\xi}\log\frac{E_0}{E_{th}}$. On the average, then, one "absorption" process will occur for each N collisions. The ratio of the fictitious absorption cross section Σ_{a_f} to the average scattering cross section $\overline{\Sigma}_s$ is therefore

$$\frac{\Sigma_{a_f}}{\overline{\Sigma}_s} = \frac{1}{N} = \frac{\xi}{\log\frac{E_0}{E_{th}}}$$

from which we obtain

$$\Sigma_{a_f} = \frac{\xi\overline{\Sigma}_s}{\log\frac{E_0}{E_{th}}} \tag{14-6}$$

Then

$$K_f^2 = \frac{3\Sigma_{a_f}}{\lambda_f} = \frac{3\xi\Sigma_s}{\lambda_f\log\frac{E_0}{E_{th}}} = \frac{1}{\tau_0} \tag{14-7}$$

if λ_f is defined as a proper average transport mean free path for fast neutrons.

We now seek solutions of equations 14–1 and 14–2 which satisfy the conditions,

$$\begin{cases} \Delta\phi_f = \Delta\phi_f \\ \Delta\phi_{th} = \Delta\phi_{th} \end{cases} \tag{14-8}$$

where Δ is the same constant in both equations. On substitution in equations 14–1 and 14–2, we obtain the two simultaneous linear equations:

$$\left(\frac{\lambda_f}{3}\Delta - \Sigma_{a_f}\right)\phi_f + k\Sigma_{a_{th}}\phi_{th} = 0 \tag{14-9}$$

$$\Sigma_{a_f}\phi_f + \left(\frac{\lambda_{th}}{3}\Delta - \Sigma_{a_{th}}\right)\phi_{th} = 0 \tag{14-10}$$

which have a nontrivial solution only if the determinant of the coefficients vanishes, that is, if

$$\left(\frac{\lambda_f}{3}\Delta - \Sigma_{a_f}\right)\left(\frac{\lambda_{th}}{3}\Delta - \Sigma_{a_{th}}\right) - k\Sigma_{a_{th}}\Sigma_{a_f} = 0 \tag{14-11}$$

After dividing by $\Sigma_{a_{th}}\Sigma_{a_f}$ and introducing $L^2 = \lambda_{th}/3\Sigma_{a_{th}}$ and $\tau_0 = \lambda_f/3\Sigma_{a_f}$, we obtain

$$k - (1 - L^2\Delta)(1 - \tau_0\Delta) = 0 \tag{14-12}$$

This is the critical equation of the two-group theory. It is analogous to equation 10–13 and reduces to this equation if we set $e^{-\tau_0\Delta} = 1 - \tau_0\Delta$, an approximation which is good if $\tau_0\Delta \ll 1$, that is, if the mean square slowing-down distance is small compared with the cross-sectional area of the critical pile. It is different from equation 10–13 in that it is a quadratic equation in Δ instead of a transcendental equation.

For the case that the condition $\tau_0\Delta \ll 1$ is *not* satisfied, the critical size of the bare pile as given by equation 14–12 is too small (Δ too large); one may, however, introduce a modified age τ_0' into the theory which does give the correct critical size with equation 14–12.

In the limit as the number of groups is increased indefinitely, the critical equation for the multigroup theory goes into the form 10–13 of the continuous theory. This is shown as follows. The critical equation for n groups (in addition to the thermal group) may be written in analogy with equation 14–12 as

$$k - (1 - L^2\Delta)(1 - \tau_1\Delta)(1 - \tau_2\Delta) \cdots (1 - \tau_n\Delta) = 0 \quad (14\text{–}12a)$$

where $\tau_1, \tau_2 \cdots$ are to be interpreted as the partial "ages" of the neutrons in the various groups. Clearly, then, as a neutron slows down continuously from fission to thermal energies, it will pass through all these groups, and we will have

$$\tau_0 = \tau_1 + \tau_2 + \tau_2 + \cdots \tau_n = \sum_1^n \tau_i$$

To evaluate the product,

$$f_n = (1 - \tau_1\Delta)(1 - \tau_2\Delta)(1 - \tau_1\Delta) \cdots (1 - \tau_n\Delta)$$

as $n \to \infty$, we take the logarithm of both sides and obtain

$$\log f_n = \sum_1^n \log (1 - \tau_i\Delta)$$

As $n \to \infty$ and each of the τ_i becomes small, we can use the approximation $\log (1 - x) = -x$. Then

$$\log f_\infty = \lim_{n \to \infty} \log f_n = \lim_{n \to \infty} \sum_{i=1}^n (-\tau_i\Delta) = -\Delta \sum_{i=1}^n \tau_i = -\Delta\tau_0$$

$$f_\infty = e^{-\tau_0\Delta}$$

And equation 14–12a assumes the form of the critical equation of the continuous theory, namely,

$$k - (1 - L^2\Delta)e^{-\tau_0\Delta} = 0$$

The critical equation can be written

$$\Delta^2 - \left(\frac{1}{\tau_0} + \frac{1}{L^2}\right)\Delta - \frac{k-1}{L^2\tau_0} = 0 \quad (14\text{–}13)$$

and the two solutions are

$$\Delta_1 = \frac{1}{2}\left(\frac{1}{\tau_0} + \frac{1}{L^2}\right) - \frac{1}{2}\sqrt{\left(\frac{1}{\tau_0} + \frac{1}{L^2}\right)^2 + 4\frac{k-1}{L^2\tau_0}}$$

$$\Delta_2 = \frac{1}{2}\left(\frac{1}{\tau_0} + \frac{1}{L^2}\right) + \frac{1}{2}\sqrt{\left(\frac{1}{\tau_0} + \frac{1}{L^2}\right)^2 + 4\frac{k-1}{L^2\tau_0}} \quad (14\text{–}14)$$

if $k > 1$, the two solutions are of opposite sign. The negative root Δ_1 is given approximately by

$$\Delta_1 \sim -\frac{k-1}{L^2+\tau_0} \tag{14-15}$$

and the positive root by

$$\Delta_2 \sim \frac{1}{\tau_0} + \frac{1}{L^2} \tag{14-15a}$$

if $k - 1$ is small. For each of the two possible values of Δ the ratio ϕ_{th}/ϕ_f is determined by equations 14–9 and 14–10. We call s_1 the ratio corresponding to Δ_1 and s_2 the ratio corresponding to Δ_2. Then, if we define $K_f^2 = 1/\tau_0$, the expressions for s_1 and s_2 are

$$s_1 = -\frac{\lambda_f}{\lambda_{th}} \frac{\Delta_1 - K_f^2}{kK_{th}^2} \qquad s_2 = \frac{\lambda_f}{\lambda_{th}} \frac{\Delta_2 - K_{th}^2}{kK_{th}^2} \tag{14-16}$$

If $\Delta_1 < 0$ and $\Delta_2 > 0$, then $s_1 > 0$ and $s_2 < 0$. The general solution of equations 14–1 and 14–2 for the fast and the thermal neutron flux is a linear combination of terms corresponding to the two values Δ_1, Δ_2 given by equation 14–14.

In the case of a slab pile then we have the solutions inside the reactor,

$$\begin{cases} \phi_f = \cos \mu_1 x + C \cosh \mu_2 x \\ \phi_{th} = s_1 \cos \mu_1 x + C s_2 \cosh \mu_2 x \end{cases} \tag{14-17}$$

where the positive quantities $\mu_1^2 \equiv -\Delta_1$, $\mu_2^2 \equiv \Delta_2$ have been introduced. The equations for the fast and for the thermal flux inside a reflector may be written in the same form as equations 14–1 and 14–2 except that $k = 0$. The critical equation is then the same as equation 14–12 with $k = 0$. We have, therefore,

$$(1 - L'^2\Delta')(1 - \tau_0'\Delta') = 0 \tag{14-18}$$

where the primes indicate that the quantities refer to the reflector. The solutions of equation 14–18 are

$$\begin{cases} \Delta_1' = \dfrac{1}{\tau_0'} = K_f'^2 \\ \Delta_2' = \dfrac{1}{L'^2} = K_{th}'^2 \end{cases} \tag{14-19}$$

In this case the ratio of the thermal to fast flux corresponding to the two values of Δ is

$$\begin{cases} r_1 = -\dfrac{\Sigma_{a_f}'}{\dfrac{\lambda_{th}'}{3}\Delta_1' - \Sigma_{a_{th}}'} = \dfrac{\lambda_f'}{\lambda_{th}'}\dfrac{K_f'^2}{K_f'^2 - K_{th}'^2} \\ r_2 = \infty \end{cases} \tag{14-20}$$

The last condition means simply that in the reflector the fast flux corresponding to the second solution vanishes. The solution in the reflector of thickness t which vanishes at the outer boundary $x = \pm(a + t)$ is then

$$\phi_f' = A \sinh K_f'(a + t - |x|) \tag{14-21}$$

$$\phi_{th}' = r_1 A \sinh K_f'(a + t - |x|) + B \sinh K_{th}'(a + t - |x|) \tag{14-22}$$

Four boundary conditions remain to be satisfied at the interface between reactor and reflector at $x = \pm a$. These express the continuity of the fast and thermal flux and the continuity of the fast and thermal net current. The four equations resulting from these four boundary conditions are in general incompatible since we have only three arbitrary constants A, B, C; only for a particular value of the reactor half-thickness a do these equations have a solution. In this way the critical size of the reactor a is determined as an implicit function of the reflector thickness t, which is also a parameter in the equations. At this point one must resort to rather tedious numerical calculations to obtain a as a function of t. This has been done for certain problems of practical importance and is to be found in the project literature. The results which have been so calculated for particular pile and reflector combinations show that with the two-group theory the calculated reflector savings are usually several per cent larger than with the simpler one-group theory.

15. PILE CONTROL

In a practical case the dimensions of a pile are always greater than the critical dimensions in order to allow some latitude of control. In thermal piles control rods, made of material having a large absorption cross section for thermal neutrons (Cd, Boron steel), are used for this purpose. Essentially, then, instead of changing the pile dimensions one changes the effective pile constants k, L^2 and τ_0 by the introduction or removal of foreign materials in the pile structure.

Another way of looking at this question is to consider that the control rods introduce modifications in the boundary conditions which the neutron flux must satisfy. For example, if the control rods are "black" to thermal neutrons, the neutron flux must approach zero at the (extrapolated) boundary of the control rod.

It is convenient to introduce again the quantity ρ (equation 12–10) called the reactivity defined in the continuous theory by

$$\rho = k - (1 - L^2\Delta)e^{-\tau_0\Delta} \qquad (15\text{–}1)$$

or, in the one-group theory by the analogous expression,

$$\rho = k - (1 - L^2\Delta) \qquad (15\text{–}1a)$$

If $\rho = 0$, the pile is just critical since equation 10–13 is then satisfied. If ρ is positive, the neutron flux will increase with time; if ρ is negative, it will decrease.

The pile and control rods must be designed to allow and control considerable excess reactivity. As the fissionable material in a pile is used up, the pile reactivity will decrease for two reasons: (1) There are fewer fissionable atoms left to absorb neutrons, and (2) the fission fragments produced in the act of fission absorb neutrons. Thus, the productive absorption will decrease and unproductive absorption will increase. To keep the pile operating after a considerable depletion of the fissionable material and accumulation of fission product poisons requires that enough excess reactivity should have been built into the pile that by withdrawal of the control rods critical conditions ($\rho = 0$) can be re-established.

Another effect depends on the circumstance that the pile constants and therefore Δ_m depend on the temperature of the pile structure. There are several temperature effects which depend principally on (1) the variation of absorption and scattering cross sections of the substances in the pile with neutron energy and (2) the variation of the physical constants (dimensions, density) of the pile with temperature. The reactivity may either increase or decrease with temperature according to the pile design. A pile with a negative temperature coefficient of reactivity is desirable since then a small perturbation which increases the temperature will result in a lowering of the reactivity and, therefore, a return to the original temperature. Such a pile is stable, whereas one with a positive temperature coefficient is unstable. In all piles, then, the temperature coefficient is an important factor in determining the stability. In high-temperature piles the importance is even greater since a relatively large reactivity difference exists between the hot and cold pile.

In an experimental pile it is necessary and desirable to have excess reactivity so that the introduction of neutron absorbers for the production of radioisotopes, for example, will not cause the pile to shut down.

In addition to these relatively large effects which change the reactivity of a pile and for which a coarse control mechanism must be provided, there are smaller short-time fluctuations in reactivity which must be controlled if the pile is to operate at a steady power level. For example, a change in the temperature of the coolant may slightly change the pile reactivity.

In general, the excess reactivity built into a pile may be several times the reactivity associated with delayed neutrons (see Section 16).

In general, thermal piles are easiest to control since there exist materials with large absorption cross sections for thermal neutrons. The problem of control of fast piles is more difficult since absorption cross sections for fast neutrons are invariably much smaller.

In order to understand the effect on the neutron flux of a change of pile reactivity we must examine the solutions of the time-dependent pile equations. This is done in the next section.

16. TIME-DEPENDENT PILE EQUATIONS

In deriving the pile equations in Section 10 we made the assumption that there was an exact balance between leakage and absorption on the one hand and production on the other. We must now treat the case in which such a balance is not maintained; if more neutrons are produced in the pile than are lost through leakage and absorption, we should expect that the neutron flux would increase with time. In fact, if these considerations are applied to a cubic centimeter of the pile, we have

Production − (leakage + absorption) = rate of change of thermal neutron density or in symbols

$$Q - \left(-\frac{\lambda_t}{3}\Delta nv + nv\Sigma_a\right) = \frac{\partial n}{\partial t} \qquad (16\text{-}1)$$

or

$$\frac{\lambda_t}{3}\Delta nv - nv\Sigma_a + Q = \frac{\partial n}{\partial t} \qquad (16\text{-}2)$$

For the sake of simplicity let us treat first pile kinetics according to the one-group theory. Then we put

$$Q = nv\Sigma_a k \qquad (16\text{-}3)$$

introduce

$$K^2 = \frac{3\Sigma_a}{\lambda_t}(k - 1) = \frac{k - 1}{L^2} \qquad (16\text{-}4)$$

the neutron lifetime $l = \lambda_a/v = 1/v\Sigma_a$, and divide the equation by v. Equation 16–2 then transforms into

$$\Delta n + K^2 n = \frac{l}{L^2}\frac{\partial n}{\partial t} \qquad (16\text{-}5)$$

To separate the equation let

$$n(x, y, z, t) = n_0(x, y, z)f(t) \qquad (16\text{-}6)$$

After substituting the expression in equation 16–5 and dividing by $n_0 f$, we obtain

$$\frac{\Delta n_0}{n_0} + K^2 = \frac{l}{L^2}\frac{1}{f(t)}\frac{df}{dt} \qquad (16\text{-}7)$$

Since the left side is a function only of xyz and the right side only of t, each side must be a constant which we call $\frac{l}{L^2}\Lambda$.

Then

$$\frac{1}{f}\frac{df}{dt} = \Lambda \qquad (16\text{-}8)$$

on integrating from $t = 0$ to t, we obtain

$$f \sim e^{\Lambda t} \qquad (16\text{-}9)$$

where Λ is determined by the equation,

$$\frac{\Delta n_0}{n_0} + K^2 = \frac{l}{L^2}\Lambda \qquad (16\text{-}10)$$

obtained by introducing Λ into equation 16–7. If we put

$$\frac{\Delta n_0}{n_0} = \Delta \qquad (16\text{–}11)$$

then

$$\Lambda = \frac{L^2}{l}(\Delta + K^2) = \frac{1}{l}[k - (1 - L^2\Delta)] = \frac{\rho}{l} \qquad (16\text{–}12)$$

by equation 15–1a, and $n = n_0 e^{(\rho/l)t}$

The interpretation of equation 16–12 is straightforward. The term $-L^2\Delta$ according to equation 12–1 is the ratio of leakage to absorption, that is, the number of thermal neutrons that leak *from* the pile per thermal neutron absorbed *in* the pile. Since k neutrons are produced by the fission process per neutron absorbed and one neutron has to be used to maintain the chain reaction, the quantity $\rho = k - (1 - L^2\Delta)$ is the net increase (or decrease) in the number of neutrons per neutron absorbed. On the average, then, every l sec (where l is the mean neutron lifetime) there is an increase (or decrease) of ρn neutrons per cm^3. The rate of change of n is therefore $\frac{\partial n}{\partial t} = \frac{\rho}{l} n$, which may be integrated immediately to give $n = n_0 e^{(\rho/l)t}$, an expression identical to that just derived.

The quantity $T = \frac{1}{\Lambda} = \frac{l}{\rho}$ is called the "pile period." It may be very long if ρ is very small compared to l. In a thermal pile $l = 1/v\Sigma_a$ may be of the order of 10^{-3} sec, whereas in a fast pile it is many orders of magnitude smaller.

It appears, then, that, if this discussion presented the whole story, it might be very difficult to control even a thermal pile. For example, to cite an *extreme* case, if the value of k for a critical thermal pile is increased by 0.1 so that $\rho = 0.1$, by pulling out a control rod suddenly, the pile activity would in one second rise by a factor $e^{\rho/l} = e^{0.1/10^{-3}} = e^{100} = 10^{43}$. This catastrophic result is to be understood as a warning that large sudden changes in pile reactivity cannot be made. Actually, all piles are provided with special safety controls which would insure against such a catastrophe. It is clear that piles with high excess reactivity might be very dangerous to operate. A pile with a period of 1/100 sec would be almost certainly impossible to control by ordinary control rods; a pile with $T = 1/10$ sec could be so controlled. This means that control rods should be extracted at such a rate that the excess reactivity is never larger than 1%. As may be seen presently, some of these conclusions must be modified when the part played in pile control by the delayed neutrons is considered.

The fortunate circumstance which aids in the control of piles is that in addition to the prompt neutrons a small fraction (0.76%) of the neutrons emitted in a fission process are delayed. Six groups with half-lives ranging from 0.05 to 55 sec have been found and two periods have been identified as being emitted from certain of the fission fragments following a beta decay.

It is found that a certain number ν_p (ν prompt) neutrons come off immediately after the fission process (within, say, 10^{-12} sec), whereas a smaller number ν_d (ν delayed) are emitted much later. Each group of delayed neutrons is given off with the characteristic radioactive period of the beta decay which it follows. Thus the total number of neutrons emitted per fission ν is equal to the sum of ν_p and ν_d. The energy of the delayed neutron groups is somewhat lower than the average energy of the prompt neutrons. It is evident that only in the nonsteady state is the distinction between prompt and delayed neutrons of importance.

Data on the delayed neutrons from the fission of U^{235} are given in Table 3.

TABLE 3

Half-life $\tau_{1/2}$ (sec)	Decay Constant $\lambda_i = \frac{0.693}{\tau_{1/2}}$ sec^{-1}	Per Cent of Fission Neutrons β_i	Identification of Neutron Emitter
0.05	14	0.029	.....
0.43	1.61	0.084	.....
1.52	0.456	0.24	.....
4.51	0.154	0.21	.....
22.0	0.0315	0.17	$Xe^{(137)}$
55.6	0.0125	0.026	$Kr^{(87)}$

Total $\beta = 0.76\%$

The effect of the delayed neutrons is to increase the effective mean life of neutrons considerably, provided that the change of reactivity is not so large (as it is in the catastrophic example just given) that the pile is supercritical on the prompt neutrons alone. If β is the fraction of neutrons emitted that are delayed and l_d is the average time of delay, then the effective mean life of a neutron is $l_{eff} \sim l + \beta l_d$. Using the approximate values $\beta = 1/100$, $l_d \sim$ 10 sec, we find

$$l_{eff} \sim 10^{-3} + \tfrac{1}{10} \sim \tfrac{1}{10} \text{ sec}$$

17. PILE KINETICS—TRANSIENT BEHAVIOR

In the last section the transient behavior of a pile was worked out on the one-group theory without considering directly the effect of the delayed neutrons. In this section the general treatment will be given of the kinetics of a bare homogeneous pile with delayed neutrons, using the more exact treatment of Section 10.

We denote the decay constants of the various groups of delayed neutrons by $\lambda_1, \lambda_2 \cdots$ or, in general, by λ_i, and the number of delayed neutron emitters per fission by $\nu\beta_i$. Then, if ν is the total number of neutrons emitted per fission, $\nu \sum_i \beta_i = \nu\beta = \nu_d$ is the number of delayed neutrons per fission and $(1 - \beta)\nu$ is the number of prompt neutrons emitted per fission. Similarly, we can split up the multiplication constant,

$$k = k_d + k_p$$

where

$$k_d = \frac{\Sigma_f}{\Sigma_a} \beta\nu \tag{17-1}$$

and

$$k_p = \frac{\Sigma_f}{\Sigma_a} (1 - \beta)\nu$$

The time-dependent pile equation for the thermal flux may be written by analogy with equation 16–1:

$$\frac{\lambda_t}{3} \Delta n v_{th} - n v_{th} \Sigma_a + (n v_{th} \Sigma_{a_{th}} k_p + \lambda c) e^{\tau_0 \Delta} = \frac{\partial n}{\partial t} \tag{17-2}$$

Here c is the number of fission fragments per cubic centimeter that emit delayed neutrons. For simplicity we shall treat the case of only one delayed neutron emitter and indicate later how all the six groups may be treated simultaneously. Let λ be the decay constant (probability per second for emission) for the delayed neutron emitter. Then each second λc delayed neutrons are emitted per cubic centimeter. The source term in equation 17–2 is therefore $(nv_{th}\Sigma_{a_{th}}k_p + \lambda c)e^{\tau_0\Delta}$.

The concentration of these fission fragments c at any instant depends on how many fissions occurred shortly before this instant. For the rate of change of c we write

$$\frac{\partial c}{\partial t} = nv\Sigma_a k_d - \lambda c \tag{17-3}$$

where the first term is the number produced and the second term is the number that decay per second per cubic centimeter. Using $l_p = 1/v\Sigma_a$ and $k_d = \beta k$, we have

$$\frac{\partial c}{\partial t} = \frac{n\beta k}{l_p} - \lambda c \tag{17-4}$$

In the equilibrium case $\partial c/\partial t = 0$, and therefore

$$c = \frac{n\beta k}{l_p\lambda} \tag{17-5}$$

is the concentration of delayed emitters after the pile has been running steadily for a long time and the production and decay rates are exactly balanced.

It is necessary to solve the simultaneous differential equations 17–2 and 17–4. For the time dependence, we assume

$$\begin{cases} n = n_0 e^{\Lambda t} \\ c = c_0 e^{\Lambda t} \end{cases} \tag{17-6}$$

Then $\partial n/\partial t = \Lambda n$, and $\partial c/\partial t = \Lambda c$. On substitution of these relations into equations 17–2 and 17–4, we obtain

$$\lambda c = \frac{n\beta k}{l_p}\frac{\lambda}{\Lambda + \lambda} \tag{17-7}$$

and

$$\frac{\lambda_t}{3}\Delta nv - nv\Sigma_a + \left[nv\Sigma_a(1-\beta)k + \frac{n\beta k}{l_p}\frac{\lambda}{\Lambda+\lambda}\right]e^{\tau_0\Delta} = \Lambda n \tag{17-8}$$

where $\Delta nv/nv$ has been put equal to the constant Δ and the expression 17–5 for λc has been introduced. If we now multiply the equation by $e^{-\tau_0\Delta}$, divide by $nv\Sigma_a = n/l_p$ and use $L^2 = 3\lambda_t/\Sigma_a$, we obtain

$$e^{-\tau_0\Delta}(L^2\Delta - 1) + k - \beta k + \beta k \frac{\lambda}{\Lambda + \lambda} = \Lambda l_p e^{-\tau_0\Delta} \quad (17\text{–}9)$$

We define the reactivity as before by

$$\rho = k - (1 - L^2\Delta)e^{-\tau_0\Delta} \quad (15\text{–}1)$$

Then equation 17–9 becomes

$$\rho + \beta k \frac{\Lambda}{\Lambda + \lambda} = \Lambda l_p e^{-\tau_0\Delta} \quad (17\text{–}10)$$

This is a quadratic equation for Λ which may be written

$$\Lambda = \frac{\rho/k}{\dfrac{l_p}{k} e^{-\tau_0\Delta} + \dfrac{\beta}{\Lambda + \lambda}} \quad (17\text{–}11)$$

In the case where several neutron groups characterized by β_i, λ_i, are considered equation 17–11 is replaced by

$$\Lambda = \frac{\rho/k}{\dfrac{l_p}{k} e^{-\tau_0\Delta} + \sum_i \dfrac{\beta_i}{\Lambda + \lambda_i}} \quad (17\text{–}11a)$$

With all six delayed neutron groups equation 17–11*a* is of order 6 + 1 in Λ and has therefore 7 roots $\Lambda_0\Lambda_1 \cdots \Lambda_6$. The general solution will be expressible as a sum of exponentials of the form $\Sigma a_i e^{\Lambda_i t}$. The equation 17–11 can be written

$$\Lambda^2 l + \Lambda(l\lambda + \beta - \rho/k) - \rho/k\lambda = 0$$

where l is an abbreviation for $\frac{l_p}{k} e^{-\tau_0\Delta}$. Since the last term is negative, the two roots are of opposite signs. Let $\Lambda_1 > 0$ and $\Lambda_2 < 0$. The solutions are

$$\Lambda_1 = -\frac{(\beta + l\lambda - \rho/k)}{2l} + \frac{1}{2l}\sqrt{\left(\beta + l\lambda - \frac{\rho}{k}\right)^2 + \frac{4l\rho\lambda}{k}} \quad (17\text{–}12)$$

$$\Lambda_2 = -\frac{(\beta + l\lambda - \rho/k)}{2} - \frac{1}{2l}\sqrt{\left(\beta + l\lambda - \frac{\rho}{k}\right)^2 + \frac{4l\rho\lambda}{k}} \quad (17\text{–}13)$$

The quantity $\frac{\rho}{k} - \beta = \frac{1}{k}(\rho - k_d)$ is proportional to the reactivity in excess of that which is permissible if the pile is to be controlled by the delayed neutrons and is equal to $1/k$ times the amount of reactivity by which the pile is off prompt-critical. Suppose that the reactivity is small enough so that the pile is not critical on the prompt neutrons alone. Then $\beta > \rho/k$, and, since $l\lambda$ is small, equations 17–12 and 17–13 can be approximated by

$$\Lambda_1 \sim \frac{\rho/k}{l + \dfrac{\beta - \rho/k}{\lambda}} \tag{17–14}$$

$$\Lambda_2 \sim -\frac{\beta - \rho/k}{l} \tag{17–15}$$

The general solution,

$$n = a_1 e^{\Lambda_1 t} + a_2 e^{\Lambda_2 t} \tag{17–16}$$

then consists of two terms; the first represents a slow rise and the second a transient effect which quickly damps out.

As an example we assume the numerical values:

$$\frac{\rho}{k} = 0.005, \qquad \beta = 0.01, \qquad \lambda = \tfrac{1}{10}\ \text{sec}^{-1}, \qquad l = 10^{-3}\ \text{sec}$$

Then

$$\Lambda_1 \sim 0.1\ \text{sec}^{-1}$$

$$\Lambda_2 \sim -5\ \text{sec}^{-1}$$

The arbitrary constants are determined by the initial conditions at $t = 0$ which we take to be

$$n(0) = n_0 = \text{constant}; \qquad \frac{\partial c}{\partial t}(0) = 0 \tag{17–17}$$

and by equations 17–2, 5,

$$\frac{\partial n}{\partial t}(0) = \frac{\rho/k}{l} n_0 \tag{17–18}$$

since the fact that some neutrons are delayed does not affect the initial slope. Then the constants a_1 and a_2 of equation 17–16 are determined by

$$a_1 + a_2 = n_0 \tag{17–19}$$

$$\Lambda_1 a_1 + \Lambda_2 a_2 = \frac{\rho n_0}{kl} \tag{17–20}$$

of which the solutions are

$$a_1 = \frac{\Lambda_2 - \rho/kl}{\Lambda_2 - \Lambda_1} n_0 \approx \frac{\beta}{\beta - \rho/k} n_0 \tag{17-21}$$

$$a_2 = \frac{\rho/kl - \Lambda_1}{\Lambda_2 - \Lambda_1} n_0 \approx - \frac{\rho/k}{\beta - \rho/k} n_0 \tag{17-22}$$

Collecting all these results together, we can write, for the general solution,

$$n(t) = \left[\frac{\Lambda_2 - \rho/kl}{\Lambda_2 - \Lambda_1} e^{\Lambda_1 t} + \frac{\rho/kl - \Lambda_1}{\Lambda_2 - \Lambda_1} e^{\Lambda_2 t} \right] n_0$$

Inserting the numerical values, we obtain

$$a_1 \sim 2n_0$$

$$a_2 \sim -n_0$$

so that for this case the solution is (approximately)

$$n \cong n_0(2e^{0.1t} - e^{-5t})$$

which is sketched in Figure 21.

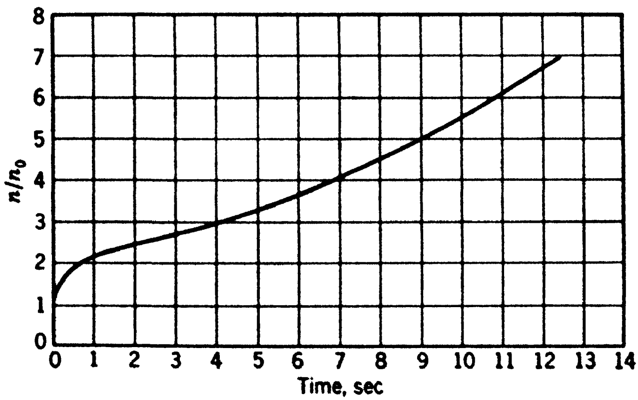


FIG. 21

It may be seen that the neutron density rises by a factor of 2 in the first half-second. In a sense the pile does not know initially that it must wait for the delayed neutrons. The neutron density levels off somewhat, and the slow rise characteristic of the delayed neutrons sets in. In the case that $\rho/k > \beta$, the initial steep rise would be continued, for then the pile would be supercritical on the prompt neutrons alone.

INDEX