# A Study of Certain Graphs Associated to Rings and Semigroups

**THESIS**

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*Under the Supervision of*

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*Dedicated to My Uncle*

**BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE, PILANI**
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## CERTIFICATE

This is to certify that the thesis entitled **A Study of Certain Graphs Associated to Rings and Semigroups** submitted by **Barkha**, ID No. **2018PHXF0026P** for award of Ph.D. of the Institute embodies original work done by her under my supervision.

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# Abstract

The study of graphs associated with algebraic structures is a well-studied research area and has attracted considerable attention of various researchers. Indeed, research in this direction aims to expose the relationship between algebra and graph theory. This thesis aims to understand the connections between the algebraic structures (rings or semigroups) and their associated graphs. The center of attention of this thesis is to explore the cozero-divisor graphs, upper ideal-relation graphs, left ideal-relation graphs of rings and intersection ideal graphs, inclusion ideal graphs of semigroups.

The cozero-divisor graph $\Gamma'(R)$ of the ring $R$ has been studied extensively in the literature. To contribute further on cozero-divisor graphs, this thesis provides a closed-form formula to compute the Wiener index of the cozero-divisor graph of an arbitrary commutative ring with unity. As an application, the Wiener index of the cozero-divisor graph of various classes of rings, namely: reduced ring, ring of integers modulo $n$, the product of the rings of integers modulo $n$, has been determined. Moreover, the Laplacian spectrum, Laplacian spectral radius, vertex connectivity and algebraic connectivity of the $\Gamma'(\mathbb{Z}_n)$ have been investigated. This thesis has introduced and studied the upper ideal-relation $\Gamma_U(R)$ of a ring $R$. Moving forward, the topological graph-theoretic properties of $\Gamma_U(R)$ such as planarity, outerplanarity, toroidality, bitoroidality and projective planar etc., have also been explored. All non-local commutative rings are determined precisely such that $\Gamma_U(R)$ has genus (and crosscap) at most two. Algebraic properties of the rings $R$ have been investigated if the graph $\Gamma_U(R)$ is perfect, bipartite, Eulerian, complete, and vice-versa. Other than the forbidden graph classes of $\Gamma_U(R)$, this thesis also established various graph-theoretic properties, namely: the metric dimension, the strong metric dimension, vertex connectivity, Hamiltonicity, of the upper ideal-relation graph of certain rings.

Other part of the thesis is devoted to the study of the intersection ideal graph $\Gamma(S)$ and the inclusion ideal graph $\mathcal{I}n(S)$ of a semigroup $S$. After ascertaining the connectedness of $\Gamma(S)$ and $\mathcal{I}n(S)$, the semigroups $S$, such that the diameter of $\Gamma(S)$ is two, have been characterized in terms of their ideals. It is ascertained that if $S$ is a completely simple semigroup, then $\Gamma(S)$ is weakly perfect. For an arbitrary semigroup $S$, an upper bound of the chromatic number of $\Gamma(S)$ has been obtained. Further, various graph invariants of $\mathcal{I}n(S)$ including perfectness, planarity, girth etc. are discussed. Moreover, a necessary and sufficient condition on a semigroup $S$ with $n$ minimal left ideals such that the clique number of $\mathcal{I}n(S)$ is $n$, is determined. For a completely simple semigroup $S$ with $n$ minimal left ideals, this thesis reveals graph-theoretic invariants including the independence number, matching number and the Wiener index of $\mathcal{I}n(S)$. The information on ideals of $S$ such that its associated graph $\mathcal{I}n(S)$ is bipartite, triangulated, edge-transitive, retract of a Cayley graph etc., has been exposed.

Additionally, all the automorphisms of these graphs (except cozero-divisor graph) associated to certain class of rings and semigroups have been investigated.

# List of Symbols

| | |
|---|---|
| $\mathbb{N}$ | the set of natural numbers |
| $\|X\|$ | cardinality of a set $X$ |
| $R$ | ring |
| $Z(R)$ | the set of zero divisors of $R$ |
| $Z(R)^*$ | the set of non $-$ zero zero divisors of $R$ |
| $U(R)$ | the set of unit elements of $R$ |
| $\mathbb{Z}_n$ | ring of integer modulo $n$ |
| $\mathbb{F}_q$ | field with $q$ elements |
| $M_n(\mathbb{F}_q)$ | ring of all $n \times n$ matrices over $\mathbb{F}_q$ |
| $S$ | semigroup |
| $E(S)$ | $\{e \in S : e^2 = e\}$ |
| $S_n$ | symmetric group of degree $n$ |
| $\Gamma$ | graph |
| $\overline{\Gamma}$ | complement of a graph $\Gamma$ |
| $a \sim b$ | $a$ and $b$ are adjacent |
| $a \nsim b$ | $a$ and $b$ are not adjacent |
| $V(\Gamma)$ | vertex set of a graph $\Gamma$ |
| $E(\Gamma)$ | edge set of a graph $\Gamma$ |
| $\mathrm{N}(x)$ | $\{y \in V(\Gamma) : y \sim x\}$ |
| $\mathrm{N}[x]$ | $\mathrm{N}(x) \cup \{x\}$ |
| $\alpha(\Gamma)$ | independence number of $\Gamma$ |
| $\beta(\Gamma)$ | metric dimension of $\Gamma$ |
| $\alpha'(\Gamma)$ | matching number of $\Gamma$ |

| | |
|---|---|
| $\beta'(\Gamma)$ | edge covering number of $\Gamma$ |
| $\omega(\Gamma)$ | clique number of a graph $\Gamma$ |
| $\delta(\Gamma)$ | minimum degree of a graph $\Gamma$ |
| $\mathrm{sdim}(\Gamma)$ | strong metric dimension of a graph $\Gamma$ |
| $cr(\Gamma)$ | crosscap of graph $\Gamma$ |
| $\eth(\Gamma)$ | genus of graph $\Gamma$ |
| $\mathbb{S}_{\eth}$ | orientable surface of genus $\eth$ |
| $\mathbb{N}_k$ | non $-$ orientable surface of crosscap $k$ |
| $\chi(\Gamma)$ | chromatic number of $\Gamma$ |
| $\Gamma'(R)$ | cozero $-$ divisor graph of $R$ |
| $\Gamma_U(R)$ | Upper ideal $-$ relation graph of $R$ |
| $\Gamma(S)$ | intersection ideal graph of $S$ |
| $\mathcal{I}n(S)$ | inclusion ideal graph of $S$ |
| $\overrightarrow{\Gamma_L}(R)$ | left ideal $-$ relation graph of $R$ |
| $\kappa(\Gamma)$ | vertex connectivity of $\Gamma$ |
| $\kappa'(\Gamma)$ | edge connectivity of $\Gamma$ |

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# Introduction

Algebraic graph theory is a branch of mathematics which provides connections between algebra and graph theory. Other sub-branch of algebraic graph theory is the spectral graph theory which emphasizes on the study of spectra of matrices (adjacency matrix or Laplacian matrix) associated with graphs. One of the broad research problem in graph theory is the study of graphs associated with algebraic structures. In past few decades, this problem has been investigated by researchers in three aspects: (i) study of graph-theoretic invariants (ii) classification of algebraic structures using its associated graph-theoretic invariants (iii) interconnections between algebraic structures and their associated graphs.

The study of graphs associated with algebraic structures is a large research area and has attracted considerable attention of various researchers. In this direction, widely studied graphs are Cayley graphs of groups and zero-divisor graphs of rings. Initially, the Cayley graphs were introduced as a generic theoretical tool to anaylze the symmetric interconnection networks. Moreover, the symmetry of Cayley graphs of groups provide various applications (see Cooperman et al. [1990]). Commuting graph of a group plays an important role to classify finite simple groups (cf. Aschbacher [2000]). Besides this, Beck [1988] introduced the notion of the zero-divisor graph associated with commutative ring. This graph best illustrates the properties of the set of zero-divisors of a commutative ring. Ganesan [1964] examines the structure of ring and demonstrates when it has a finite number of zero-divisors. Anderson and Livingston [1999] discusses an alternative proof of the theorem given by Ganesan [1964]. It was proved that the ring $R$ is finite or an integral domain if and only if the set of vertices of zero-divisor graph is finite. The zero-divisor graph of a commutative ring is also used as a input model in frequency assignment problem (cf. Radha and Rilwan [2021]). Frequency assignment problem is mathematical optimization techniques used in wireless communication. Moreover, random intersection graphs have several applications, including secure wireless communication, social networks, cryptanalysis, circuit design, recommender systems, and clustering (cf. Zhao et al. [2015]). Motivated by certain applications of graphs associated with the algebraic structures, numerous authors defined and studied various graphs related to algebraic objects. To name a few: cozero-divisor graphs (Afkhami and Khashyarmanesh [2011]), co-maximal graphs (Sharma and Bhatwadekar [1995]), ideal-relation graphs (Ma and Wong [2016]), intersection graphs (Bosák [1964]) and inclusion ideal graph (Akbari, Habibi, Majidinya and Manaviyat [2014]). This directs us to investigate the interconnections between algebraic structures (rings and semigroups) and their associated graphs. This thesis aims to study the cozero-divisor graphs, upper ideal-relation graphs, left ideal-relation graphs associated to rings and the intersection ideal graphs, inclusion ideal graphs of semigroups.

The notion of dual of zero-divisor graph is referred as the cozero-divisor graph and it is introduced by Afkhami and Khashyarmanesh [2011]. They studied the basic graph-theoretic properties including completeness, girth, clique number etc. of the cozero-divisor graphs of rings and also discussed the relation between the zero-divisor graph and the cozero-divisor graph. Afkhami and Khashyarmanesh [2012] characterize all commutative non-local rings whose cozero-divisor graphs are forest, star, double-star, or unicyclic. Moreover, the basic properties of the complement of the cozero-divisor graph have been investigated. Afkhami and

Khashyarmanesh [2013] studied the relation between the cozero-divisor graph and the co-maximal graph of commutative rings. They have shown that if the vertex set of the co-maximal graph is the set of all non-zero non-unit elements of the ring $R$, then the co-maximal graph is a spanning subgraph of the cozero-divisor graph of ring $R$. Akbari and Khojasteh [2013] studied the independence number, domination number, and the maximum degree of the cozero-divisor graph and classified the rings when these parameters are finite. The diameter of the cozero-divisor graph of polynomial rings and rings of power series have been studied by Akbari, Alizadeh and Khojasteh [2014]. They have characterized the commutative ring whose cozero-divisor graph is forest. Further, Akbari and Khojasteh [2014] generalized the result of Afkhami and Khashyarmanesh [2012] and classified all commutative rings whose cozero-divisor graphs are unicyclic and obtained the girth of the cozero-divisor graph of a ring $R$. Further, they characterized all commutative rings whose cozero-divisor graph has maximum degree 3. Afkhami [2014] extended the cozero-divisor graph to the non-commutative ring and studied the cozero-divisor graph of matrix rings. Kavitha and Kala [2017] determined all non-local finite commutative rings whose cozero-divisor graph has genus one. Later, Paknejad and Erfanian [2017] characterized all Artinian rings with claw-free or triangle-free cozero-divisor graphs. They also investigated all Artinian rings whose cozero-divisor graphs are $C_4$-free. Mallika and Kala [2017] classified all finite non-local commutative rings whose cozero-divisor graph has crosscap at most two. They also characterized all finite non-local commutative rings for which the cozero-divisor graph has outerplanarity index two. Bakhtyiari et al. [2020] studied the chromatic number and the clique number of the cozero-divisor graph of commutative von Neumann regular ring. Moreover, they proved that the cozero-divisor graph of von Neumann regular ring with finite clique number is perfect. Barati and Afkhami [2020] provided the full characterization of the cozero-divisor graph concerning their planar and outerplanar indices. Recently, Nikandish et al. [2021] have calculated the metric dimension and the strong metric dimension of the cozero-divisor graphs of non-local commutative rings. Moreover, they discussed the metric dimension of the cozero-divisor graph of an Artinian ring whose maximal ideals are principal.

Ideals play a vital role in the development of algebraic structures (rings and semigroups). Thus, it would be interesting to study graphs associated with the ideals of algebraic structures. The graphs associated to ideals, viz. inclusion ideal graphs (Akbari, Habibi, Majidinya and Manaviyat [2015]), intersection ideal graphs (Chakrabarty et al. [2009]), ideal-relation graphs (Ma and Wong [2016]), co-maximal ideal graphs (Ye and Wu [2012]) etc., of rings have been studied in the literature. Motivated by the study of graphs associated to ideals, in this thesis, we define and study the upper ideal-relation graph of rings.

Frucht [1939] proved that all groups can be viewed as the automorphism group of a connected graph. The symmetries of a graph are described by its automorphism group. In general, automorphism groups are important for studying sizeable graphs since these symmetries allow to simplify and understand the behavior of a graph. However, the determination of the full automorphism group is a challenging problem in algebraic graph theory. The automorphisms of the zero-divisor graphs over the matrix rings attracted a lot of attention by various researchers (see, Ma et al. [2016]; Ou et al. [2020]; Wang [2016]; Wong et al. [2014]; Zhou et al. [2017b]). Also, Xu et al. [2022] determined all the automorphisms of the intersection graph of ideals over matrix rings. Automorphisms of the total graph over matrix rings are also characterized (see, Wang et al. [2020]; Zhou et al. [2017a]). Further, Wang et al. [2017] characterized the automorphisms of the co-maximal ideal graph over matrix ring. Ma and Wong [2016] introduced the ideal-relation graph and discuss all the automorphisms of ideal-relation graph on $n \times n$ upper triangular matrices over a finite field $\mathbb{F}_q$.

The intersection graph of a semigroup was introduced by Bosák [1964]. The *intersection subsemigroup graph* $\Gamma(S)$ of a semigroup $S$ is a simple undirected graph whose vertex set is the collection of proper subsemigroups of $S$ and two distinct vertices $A$ and $B$ are adjacent if and only if $A \cap B \neq \emptyset$. Bosák [1964] proved that if $S$ is a nondenumerable semigroup or a periodic semigroup with more than two elements, then the graph $\Gamma(S)$ is connected. Bosák then raised the following open problem: Does there exists a semigroup with more than two elements whose graph is disconnected? Lin [1969], answered the problem posed by Bosák, in a negative manner and proved that every semigroup with more than two elements has a connected graph. Also, Pondělíček [1967] proved that the diameter of a semigroup with more than two elements does not exceed three. Inspired by the work of Bosák [1964], the intersection graph of groups was studied by Csákány and Pollák [1969] and then they proved that there is an edge between two proper subgroups if they have at least two elements common. Further, Zelinka [1975] continued this work for finite abelian groups. Shen [2010] characterized all finite groups whose intersection graphs are disconnected. The groups whose intersection graphs of normal subgroups are connected, complete, forests or bipartite are classified by Jafari and Rad [2013]. Tamizh Chelvam and Sattanathan [2012] continued the seminal paper of Csákány and Pollák to introduce the subgroup intersection graph of a finite group $G$. Further, Ma [2016] proved that the diameter of the intersection graph of a finite non-abelian simple group has an upper bound 28. Chakrabarty et al. [2009] introduced the notion of intersection ideal graph of rings. The *intersection ideal graph* $\Gamma(R)$ of a ring $R$ is an undirected simple graph whose vertex set is the collection of non-trivial left ideals of $R$ and two distinct vertices $I$ and $J$ are adjacent if and only if $I \cap J \neq \{0\}$. They characterized the rings $R$ for which the graph $\Gamma(R)$ is connected and obtained several necessary and sufficient conditions on a ring $R$ such that $\Gamma(R)$ is complete. Jafari and Rad [2010] studied the planarity of the intersection ideal graphs $\Gamma(R)$ of a commutative ring $R$ with unity. The domination number of $\Gamma(R)$ has been obtained by Jafari and Rad [2011]. Akbari et al. [2013] classified all the rings whose intersection graphs of ideals are not connected and also determined all rings whose clique number is finite. Rad et al. [2014] discussed the intersection graphs of ideals of the direct product of rings. Das [2017] characterized the positive integer $n$ for which the intersection graph of ideals of $\mathbb{Z}_n$ is perfect. Moreover, we refer the reader to Alraqad [2022] and references therein for the graded case. For some more research work on the intersection graph, we refer readers to Ahmadi and Taeri [2016]; Akbari, Heydari and Maghasedi [2015]; Akbari and Nikandish [2014]; Haghi and Ashrafi [2017]; Kayacan [2018]; Kayacan and Yaraneri [2015]; Laison and Qing [2010]; Rad and Jafari [2011]; Shahsavari and Khosravi [2017]; Xu et al. [2022] and references therein.

Akbari, Habibi, Majidinya and Manaviyat [2014] have introduced the notion of inclusion ideal graph associated with ring structure. The *inclusion ideal graph* $\mathcal{I}n(R)$ of a ring $R$ is an undirected simple graph whose vertex set is the collection of non-trivial left ideals of $R$ and two distinct non-trivial left ideals $I$ and $J$ are adjacent if and only if either $I \subset J$ or $J \subset I$. Further, Akbari, Habibi, Majidinya and Manaviyat [2015] have studied various graph invariants including connectedness, perfectness, diameter and the girth of $\mathcal{I}n(R)$. It was shown that $\mathcal{I}n(R)$ is disconnected if and only if $R \cong M_2(D)$ or $D_1 \times D_2$, for some division rings, $D, D_1$ and $D_2$. The subspace inclusion graph $\mathcal{I}n(\mathbb{V})$ associated with vector space $\mathbb{V}$ has been studied by Das [2016]. The *subspace inclusion graph* on a finite-dimensional vector space $\mathbb{V}$ is undirected simple graph whose vertices are all non-trivial proper subspaces of $\mathbb{V}$ and two distinct non-trivial proper subspaces $W_1$ and $W_2$ are adjacent if and only if $W_1 \subset W_2$ or $W_2 \subset W_1$. Das [2016] has studied the diameter, girth, clique number and chromatic number of $\mathcal{I}n(\mathbb{V})$. Moreover, other graph invariants, namely: perfectness and planarity of $\mathcal{I}n(\mathbb{V})$, have been studied by Das [2018]. Also, for a 3-dimensional vector space it was shown that $\mathcal{I}n(\mathbb{V})$ is bipartite, vertex transitive, edge transitive and has a perfect matching. Wong et al. [2018] have proved the following conjectures proposed by Das: If $\mathbb{V}$ is a 3-dimensional vector space over a finite field $F_q$ with $q$ elements, then

(1) the domination number of $\mathcal{I}n(\mathbb{V})$ is $2q$.

(2) $\mathcal{I}n(\mathbb{V})$ is distance regular.

The problem to determine the independence number of vector space $\mathbb{V}$ when the base field is finite is solved by Ma and Wang [2018]. Further, the automorphism group of $\mathcal{I}n(\mathbb{V})$ was obtained by Wang and Wong [2019]. Analogously, the subgroup inclusion graph of a group $G$, denoted by $\mathcal{I}n(G)$, has been studied by Devi and Rajkumar [2016]. They classified the finite groups whose inclusion graph is complete, bipartite, tree, star, path, cycle, disconnected and claw-free. Ou et al. [2019] determined the diameter of $\mathcal{I}n(G)$ when $G$ is nilpotent group and characterized the independent dominating sets as well as automorphism group of $\mathcal{I}n(\mathbb{Z}_n)$.

Motivated with the research work presented earlier, in this thesis, we study the cozero-divisor graphs, upper ideal-relation graphs, left ideal-relation graphs of rings and intersection ideal graphs, inclusion ideal graphs of semigroups. The thesis has been arranged in the following chapters.

Chapter 1: Background

Chapter 2: The Cozero-divisor Graph of a Commutative Ring

Chapter 3: The Upper Ideal-Relation Graphs of Rings

Chapter 4: The Left Ideal-Relation Graph over Full Matrix Ring

Chapter 5: Graphs on Semigroups

Chapter 6: Conclusion and Future Research Work

Chapter 1: This chapter contains all the basic definitions, results and notations which are required to understand the subsequent chapters of the thesis.

Chapter 2: The cozero-divisor graph $\Gamma'(R)$ associated to a commutative ring $R$ has been explored by numerous researchers. In this chapter, we consider the Wiener index of $\Gamma'(R)$, which is one of the important topological graph indices and it has various applications. In this connection, we derive a closed-form formula of the Wiener index of $\Gamma'(R)$ of a finite commutative ring $R$ (see Theorem 2.1.5). Using this, we also obtained the Wiener index of cozero-divisor graph of few classes of commutative rings, namely: product of the ring of integers modulo $n$ (cf. Theorem 2.1.7), reduced ring (cf. Theorem 2.1.10) and ring of integers modulo $n$ (see Theorem 2.1.20). We study the Laplacian spectrum of $\Gamma'(\mathbb{Z}_n)$. We investigate the Laplacian spectral radius, algebraic connectivity of $\Gamma'(\mathbb{Z}_n)$ and characterized the values of $n$ for which the Laplacian spectral radius is equal to the order of $\Gamma'(\mathbb{Z}_n)$ (see Proposition 2.3.3). Moreover, the values of $n$ for which the algebraic connectivity and the vertex connectivity of $\Gamma'(\mathbb{Z}_n)$ coincide are also described (see Theorem 2.3.6).

The content of this chapter (Subsection 2.1.3, Section 2.3 and Section 2.2) is published in journal *"AKCE International Journal of Graphs and Combinatorics"*, Taylor & Francis. The remaining results of this chapter has been submitted for publication.

Chapter 3: In this chapter, we define and study the notion of upper ideal-relation graph $\Gamma_U(R)$ associated to a ring $R$. In order to investigate the algebraic properties of a ring $R$ and the graph-theoretic properties of $\Gamma_U(R)$, first we obtained the girth, minimum degree and the independence number of $\Gamma_U(R)$. We classify all the finite rings $R$ such that the graph $\Gamma_U(R)$ is Eulerian (see Theorem 3.1.15). We provide a necessary and sufficient condition on $R$, in terms of the cardinality of their principal ideals, such that the graph $\Gamma_U(R)$ is bipartite (cf. Theorem 3.1.4), planar (cf. Theorem 3.1.7) and outerplanar, respectively. We also obtained the metric and the strong metric dimension of the graph $\Gamma_U(R)$. We discuss the perfectness of $\Gamma_U(R)$ (cf. Theorem 3.3.6). We also investigated the topological aspects of $\Gamma_U(R)$. In this connection, we classify all the non-local commutative rings $R$ for which $\Gamma_U(R)$ has genus at most 2 (see Theorems 3.4.6 and 3.4.7). Also, we determine precisely all the non-local commutative rings for which $\Gamma_U(R)$ has crosscap at most 2 (cf. Theorems 3.4.9 and 3.4.10). Along with this, we characterize all the non-local commutative rings whose upper ideal-relation graphs are split graphs, threshold graphs and cographs, respectively. Moreover, we have studied the upper ideal-relation graph of the ring $\mathbb{Z}_n$. In this direction, we classify all the values of $n$ for which the graph $\Gamma_U(\mathbb{Z}_n)$ is Hamiltonian (see Theorem 3.6.3). We determine the vertex connectivity (cf. Theorem 3.6.5) and automorphism group (see Theorem 3.6.9) of $\Gamma_U(\mathbb{Z}_n)$. We also investigated the Laplacian and the normalized Laplacian spectrum of the upper ideal-relation graph of the ring $\mathbb{Z}_n$. Two research papers containing the results of this chapter are submitted for publication.

Chapter 4: The automorphisms of graphs associated with rings attracted a lot of attention of researchers. In order to reveal the significant structure of ideal-relation graph on full matrix ring , we study the left ideal-relation graph of full matrix ring. In this chapter, we obtained all the automorphisms of left ideal-relation graph over full matrix ring. The content of this chapter is submitted for publication.

Chapter 5: In this chapter, we discuss the graph associated with ideals of semigroups, namely: the intersection ideal graph and the inclusion ideal graph of semigroups. First, we investigate the connectedness of $\Gamma(S)$ (cf. Theorems 5.1.1 and 5.1.3). We classify the semigroups $S$ in terms of their ideals such that the diameter of $\Gamma(S)$ is two (see Theorem 5.1.7). We obtain the domination number, independence number, girth and the strong metric dimension of $\Gamma(S)$. We have also investigated the completeness, planarity and perfectness of $\Gamma(S)$. We show that if $S$ is a completely simple semigroup, then $\Gamma(S)$ is weakly perfect. Moreover, for an arbitrary semigroup $S$, we give an upper bound of the chromatic number of $\Gamma(S)$ (cf. Theorem 5.1.20). Finally, if $S$ is the union of $n$ minimal left ideals, then we obtain the metric dimension (see Theorem 5.1.25) and the automorphism group of $\Gamma(S)$ (cf. Theorem 5.1.28).

The results on $\Gamma(S)$ of this chapter are published in the journal *"Quasigroups and Related Systems"*.

Besides this, we study the algebraic properties of the semigroup $S$ and graph-theoretic properties of the inclusion ideal graph $\mathcal{I}n(S)$, in particular, when $S$ is a completely simple semigroup. After ascertaining the connectedness of $\mathcal{I}n(S)$, we show that the diameter of $\mathcal{I}n(S)$ is at most 3, if it is connected (see Theorem 5.2.4). We also obtain a necessary and sufficient condition of $S$ such that the clique number of $\mathcal{I}n(S)$ is $n$, where $n$ is the number of minimal left ideals of $S$ (cf. Theorem 5.2.10). Further, various graph invariants of $\mathcal{I}n(S)$, viz. perfectness, planarity, girth etc., are discussed. For a completely simple semigroup $S$, we investigate various properties of $\mathcal{I}n(S)$ including its independence number (see Theorem 5.2.24) and the matching number (see Theorem 5.2.26). Finally, for a completely simple semigroup $S$ with $n(\geq 3)$ minimal left ideals, we prove that the automorphism group $Aut(\mathcal{I}n(S)) \cong S_n \times \mathbb{Z}_2$ (cf. Theorem 5.2.34).

The content of the inclusion ideal graph $\mathcal{I}n(S)$ is accepted for publication in the journal *"Algebra Colloquium"*.

Chapter 6: This chapter summarises the thesis and concludes with future research problems.

# Chapter 1

# Background

In this chapter, we recall necessary definitions and results, which we need for the upcoming chapters. In Section 1.1, we provide fundamental definitions and results related to the theory of semigroups. In Section 1.2, we recall necessary results and definitions of ring theory. Also, we prove the required results related to semigroups and rings used in the thesis. Section 1.3 is devoted to the concepts of graph theory. Also, we recall some basic results on graphs which will be used in the thesis. This chapter also fixes various notations used throughout the thesis.

## 1.1 Semigroups

In this section, we recall necessary definitions and results of semigroup theory from Clifford and Preston [1961] and Howie [1995].

A *semigroup* is a non-empty set $S$ together with an associative binary operation on $S$. We say $S$ to be a *monoid* if it contains an identity element. A *subsemigroup* of a semigroup is a subset that is also a semigroup under the same operation. A semigroup $S$ is said to be *regular* if for each $a \in S$ there exists $x \in S$ such that $axa = a$. If a semigroup $S$ with at least two elements contains an element 0 such that for all $x \in S$, $0x = x0 = 0$, then $0 \in S$ is called the *zero element* of $S$ and in this case, $S$ is known as a *semigroup with zero*. If $S$ does not contain zero, then we say that $S$ is a *semigroup without zero*. If $A$ and $B$ are subsets of a semigroup $S$, then define $AB = \{ab : a \in A, b \in B\}$. A non-empty subset $I$ of $S$ is said to be a *left ideal* if $SI \subseteq I$, a *right ideal* if $IS \subseteq I$, and a (two-sided) *ideal* of $S$ if $SIS \subseteq I$ i.e. it is both left and right ideal. A left [right, two-sided] ideal is called 0-minimal if it is minimal within the non-zero left [right, two-sided] ideals. The union of two left [right] ideals of $S$ is again a left [right] ideal of $S$. A left ideal $I$ of $S$ is *maximal* if it is not contained in any non-trivial left ideal of $S$. If $S$ has a unique maximal left ideal, it contains every non-trivial left ideal of $S$. A left ideal $I$ is *minimal* if it does not properly contain any non-zero left ideal of $S$. If $S$ has a minimal left ideal, then every non-trivial left ideal contains at least one minimal left ideal. If $A$ is any left ideal of $S$ other than minimal left ideal $I$, then either $I \subset A$ or $I \cap A = \emptyset$. Thus, we have the following remark.

**Remark 1.1.1.** Any two distinct minimal left ideals of a semigroup $S$ are disjoint.

The following lemma is useful in the sequel.

**Lemma 1.1.2.** *If the semigroup $S = I_1 \cup I_2$, where $I_1$ and $I_2$ are minimal left ideals of $S$, then $I_1$ and $I_2$ are maximal left ideals.*

*Proof.* Suppose that $S = I_1 \cup I_2$, where $I_1$ and $I_2$ are minimal left ideals of $S$. If there exists a left ideal $I_k$ of $S$ such that $I_1 \subsetneq I_k$, then there exists $y \in I_k$ such that $y \notin I_1$. Since $S = I_1 \cup I_2$, we obtain $y \in I_2$. It follows that $I_2 = Sy \subset I_k$. Thus, $S = I_1 \cup I_2 \subseteq I_k$. Consequently, $I_k = S$ and so $I_1$ is a maximal left ideal. Similarly, for $I_2 \subsetneq I_k$, we obtain $I_k = S$. Therefore, $I_1$ and $I_2$ are maximal left ideals. $\square$

**Example 1.1.3.** (i) Let $X$ be a non-empty set and $\mathcal{T}_X$ be the set of all mappings on $X$. Then $\mathcal{T}_X$ forms a semigroup under the composition of mappings called *full transformation semigroup.*

(ii) The set $I(X)$ of partial injective mappings on $X$ forms a semigroup under the composition of relations, and it is known as *symmetric inverse semigroup.*

(iii) Given a finite group $G$ and a natural number $n$, write $[n] = \{1, 2, \ldots, n\}$ and $B_n(G) = ([n] \times G \times [n]) \cup \{0\}$. Define a binary operation '$\cdot$' on $B_n(G)$ by

$$(i, a, j) \cdot (k, b, l) = \begin{cases} (i, ab, l) & \text{if } j = k; \\ 0 & \text{if } j \neq k \end{cases}$$

and, for all $\alpha \in B_n(G), \alpha \cdot 0 = 0 \cdot \alpha = 0$, is known as *Brandt semigroup*. When $G = \{e\}$ is the trivial group, the Brandt semigroup $B_n(G)$ is denoted by $B_n$. Instead of writing the identity element $e \in G$ in the triplets of elements of $B_n$, we use the following description in the definition of $B_n$. For any integer $n \geq 1$, let $[n] = \{1, 2, \ldots, n\}$. The semigroup $(B_n, \cdot)$, where $B_n = ([n] \times [n]) \cup \{0\}$ and the operation '$\cdot$' is given by

$$(i, j) \cdot (k, l) = \begin{cases} (i, l) & \text{if } j = k; \\ 0 & \text{if } j \neq k; \end{cases}$$

and, $\alpha \cdot 0 = 0 \cdot \alpha = 0$ for all $\alpha \in B_n$.

By $S^1$, we shall mean the monoid obtained from $S$ by adjoining an identity element ( if $S$ does not have such an element). The most fundamental tools in understanding a semigroup are its Green relations. J. A. Green introduced Green's relations in 1951 that characterize the elements of $S$ in terms of their principal ideals. These relations are defined as

(i) $x \,\mathcal{L}\, y$ if and only if $S^1x = S^1y$.

(ii) $x \mathcal{R} y$ if and only if $xS^1 = yS^1$.

(iii) $x \mathcal{J} y$ if and only if $S^1xS^1 = S^1yS^1$.

(iv) $x \mathcal{H} y$ if and only if $x \mathcal{L} y$ and $x \mathcal{R} y$.

(v) $x \mathcal{D} y$ if and only if $x \mathcal{L} z$ and $z \mathcal{R} y$ for some $z \in S$.

Note that the $\mathcal{L}$- class ($\mathcal{R}$-class, $\mathcal{J}$-class, $\mathcal{H}$-class, $\mathcal{D}$-class) containing the element $a$ is denoted by $L_a(R_a, J_a, H_a, D_a)$.

**Remark 1.1.4.** The non-zero elements of a minimal left ideal of $S$ are in the same $\mathcal{L}$-class.

**Lemma 1.1.5.** *A left ideal $K$ of $S$ is maximal if and only if $S \setminus K$ is an $\mathcal{L}$-class.*

*Proof.* First suppose that $S \setminus K$ is an $\mathcal{L}-$class. Let, if possible, $K$ is not the maximal left ideal of $S$. Then there exists a non-trivial left ideal $K'$ of $S$ such that $K \subsetneq K'$. There exists $a \in K'$ but $a \notin K$. Thus, $L_a = S \setminus K$. Consequently, $L_a \subset K'$ gives $S = K'$, a contradiction. Conversely, suppose that $K$ is a maximal left ideal of $S$. For each $a \in S \setminus K$, the maximality of $K$ implies $K \cup S^1a = S$. Consequently, $a \mathcal{L} b$ for every $a, b \in S \setminus K$. Thus $S \setminus K$ is contained in some $\mathcal{L}-$class and this $\mathcal{L}-$class is disjoint from $K$. It follows that $S \setminus K$ is an $\mathcal{L}-$class. □

A semigroup $S$ without zero is said to be *simple* if it has no proper ideals. A semigroup $S$ with zero is called *0-simple* if

(i) $\{0\}$ and $S$ are its only ideals;

(ii) $S^2 \neq \{0\}$.

An element $a$ of a semigroup $S$ is *idempotent* if $a^2 = a$. Let $\mathcal{E}$ be the set of idempotents of a semigroup $S$. If $e, f \in \mathcal{E}$, we define $e \leq f$ to mean $ef = fe = e$. Recall that a semigroup $S$ is called *completely simple* if $S$ is simple and contains a primitive idempotent. By *primitive idempotent*, we mean an idempotent which is minimal within the set of all idempotents under the relation $\leq$. Let $G$ be a group and let $I, \Lambda$ be non-empty sets. Let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in $G^0 (= G \cup \{0\})$, and suppose $P$ is *regular*, in the sense that no row or column of $P$ consists entirely of zeros. Let $S = \mathfrak{M}^0[G, I, \Lambda, P] = (I \times G \times \Lambda) \cup \{0\}$, and define a composition on $S$ by

$$(i, a, \lambda)(j, b, \mu) = \begin{cases} (i, ap_{\lambda j}b, \mu) & \text{if } p_{\lambda j} \neq 0 \\ 0 & \text{if } p_{\lambda j} = 0 \end{cases} \tag{1.1}$$

$$(i, a, \lambda)0 = 0(i, a, \lambda) = 0.$$

**Theorem 1.1.6** ([Howie, 1995, Theorem 3.2.3]). *Let $S$ be a semigroup. Then $S$ is a completely 0-simple semigroup if and only if $S \cong \mathfrak{M}^0[G, I, \Lambda, P]$ for some non-empty index sets $I$, $\Lambda$, regular matrix $P$ and a group $G$.*

**Lemma 1.1.7.** *Let $S$ be the union of $n$ minimal left ideals. Then each non-trivial left ideal of $S$ is the union of some of these minimal left ideals.*

*Proof.* Let $I_1, I_2, \ldots, I_n$ be $n$ minimal left ideals of $S$ such that $S = I_1 \cup I_2 \cup \cdots \cup I_n$. Let $K$ be a non-trivial left ideal of $S$. Then there exists at least one element $x_1 \in K$. It follows that $x_1 \in I_{i_1}$, for some $i_1 \in [n]$. By the minimality of $I_{i_1}$, we obtain $I_{i_1} = Sx_1 \subseteq K$. If $K = I_{i_1}$, then we are done. Otherwise, there exists $x_2 \in K$ such that $x_2 \in I_{i_2}$, for some $i_2 \in [n] \setminus \{i_1\}$. Similar to the above argument, we get $I_{i_2} \subseteq K$. Therefore, $(I_{i_1} \cup I_{i_2}) \subseteq K$. If $I_{i_1} \cup I_{i_2} = K$, then there is nothing to prove. Otherwise, there exists $x_3 \in K$ such that $x_3 \in I_{i_3}$, for some $i_3 \in [n] \setminus \{i_1, i_2\}$. Consequently, $I_{i_3} \subseteq K$. If $I_{i_1} \cup I_{i_2} \cup I_{i_3} = K$, then we are done. On continuing in this way, we get the desired result. $\square$

**Lemma 1.1.8** ([Clifford and Preston, 1961, Corollary 2.49]). *A completely simple semigroup is the union of its minimal left [right] ideals.*

In view of Lemma 1.1.8, we have the following corollary of Lemma 1.1.7.

**Corollary 1.1.9.** *Let $S$ be a completely simple semigroup with $n$ minimal left ideals. Then any non-trivial left ideal of $S$ is the union of some of these minimal left ideals.*

## 1.2 Rings

In this section, we present the necessary definitions and results of ring theory. For more details, we refer the reader to Atiyah and Macdonald [2016].

**Definition 1.2.1.** Let $R$ be a non-empty set with two binary operations $+$ and $\cdot$, respectively. Then $R$ is called a ring with respect to $+$ and $\cdot$, if the following properties hold.

(i) $(R, +)$ is an abelian group;

(ii) $(R, \cdot)$ is a semigroup;

(iii) $\cdot$ distributes over $+$, that is,

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (b + c) \cdot a = b \cdot a + c \cdot a \quad \forall a, b, c \in R.$$

A ring is usually denoted by $(R, +, \cdot)$, and often it is written as $R$ when the operations are understood.

**Example 1.2.2.** (i) The set $\mathbb{Z}_n = \{0, 1, \ldots, n - 1\}$ forms a commutative ring with unity under addition and multiplication integers modulo $n$.

(ii) The set $M_n(\mathbb{F}_q)$ of $n \times n$ matrices with entries in $\mathbb{F}_q$ under the usual matrix addition and multiplication forms a ring.

(iii) Let $R_1$, $R_2$ be rings. Then $R_1 \times R_2$ forms a ring under componentwise $+$ and $\cdot$, respectively.

$R^*$ denotes the set of all non-zero elements of $R$. If there exists an element $1 \in R$ such that $x \cdot 1 = 1 = 1 \cdot x$ for all $x \in R$, then $R$ is said to be a ring with unity. The ring $R$ is said to be *commutative* if $x \cdot y = y \cdot x$, for all $x, y \in R$. An element $x \in R$ is said to be a *unit element* of a ring if there exists an element $y \in R$ such that $xy = 1$. The set $U(R)$ represents the set of all unit elements of $R$. An element $x$ of $R$ is called a zero-divisor if there exists a non-zero element $y \in R$ such that $xy = 0$. The set of all zero-divisors of a ring $R$ is denoted by $Z(R)$ and the set of all non-zero zero-divisors is denoted by $Z(R)^*$. An element $x$ is *nilpotent* if $x^k = 0$ for some $k \in \mathbb{N}$. A commutative ring with unity in which every non-zero element has a multiplicative inverse is said to be a *field*. Let $\mathbb{F}_q$ denotes the field having $q$ elements. A ring $R$ is reduced if it contains no non-zero nilpotent element. A non-empty subset $T$ of a ring $R$ is a subring of $R$ if $a - b \in T$ and $ab \in T$ for all $a, b \in T$. A subring $I$ of $R$ is said to be a *left ideal* if $rx \in I$ for every $x \in I$ and $r \in R$. A subring $I$ is said to be a *right ideal* of $R$ if $xr \in I$ for every $x \in I$ and $r \in R$. For $x \in R$, $Rx = \{rx : r \in R\}$ denotes the principal left ideal and $xR = \{xr : r \in R\}$ denotes the principal right ideal, generated by the element $x$. A subring $I$ of a ring $R$ is said to be *two-sided ideal* or simply an *ideal* of $R$ if it is left ideal as well as right ideal of $R$. An ideal $I$ is said to be a *principal ideal* of $R$ if $I$ is generated by some element $x \in R$, that is, $I = RxR = (x) = \{r_1 x r_2 : r_1, r_2 \in R\}$. For a commutative ring, the left and right ideals coincide. An ideal $I$ of a ring $R$ is called a *proper ideal* if $I \neq \{0\}$ and $I \neq R$. An ideal $\mathcal{M}(\neq R)$ of a ring $R$ is said to be a *maximal ideal* if whenever $K$ is an ideal of $R$ such that $\mathcal{M} \subset K \subset R$ then either $R = K$ or $\mathcal{M} = K$. An ideal of a ring $R$ is said to be a *maximal principal ideal* if it is maximal among all the principal ideals of $R$. The following information of the ring $\mathbb{Z}_n$ is useful for the latter use.

**Remark 1.2.3.** Every element of a finite commutative ring $R$ is either a zero-divisor or a unit. Thus, the number of non-zero zero-divisors in $\mathbb{Z}_n$ is $n - \phi(n) - 1$ and $|U(\mathbb{Z}_n)| = \phi(n)$, where $\phi$ is the Euler's totient function

An integer $d$ such that $1 < d < n$ is called a proper divisor of $n$ if $d|n$. The set $\tau(n)$ denotes the number of positive divisors of $n$. The greatest common divisor of two elements $a$ and $b$ is denoted by $\gcd(a, b)$.

**Remark 1.2.4.** Let $d_1, d_2, \ldots, d_k$ be the distinct proper divisors of $n$. For $1 \leq i \leq k$, define

$$\mathcal{A}_{d_i} = \{x \in \mathbb{Z}_n : \gcd(x, n) = d_i\}.$$

The ideals $(x) = (y) = (d_i)$ if and only if $x, y \in \mathcal{A}_{d_i}$.

**Lemma 1.2.5** (Young [2015]). *$|\mathcal{A}_{d_i}| = \phi(\frac{n}{d_i})$ for $1 \leq i \leq k$.*

*Proof.* Consider the set $X = \{x \in \mathbb{Z}_n : \gcd(x, n) = 1\}$ and $X' = \{x \in \mathbb{Z}_n : \gcd(x, n) \neq 1\}$. Let $x(\neq 0) \in X'$. Then $x \in \mathcal{A}_{d_i}$, for some divisor $d_i$ of $n$. Also, $\mathcal{A}_{d_i} \cap \mathcal{A}_{d_j} = \emptyset$, for any distinct proper divisors $d_i$ and $d_j$ of $n$. Thus, for $n > 2$, note that $X' \setminus \{0\} = \bigcup_{i=1}^{k} \mathcal{A}_{d_i}$. Let $x_i \in \mathcal{A}_{d_i}$ for some divisor $d_i$, where $1 \leq i \leq k$. It follows that $\gcd(x_i, n) = d_i$ which implies that $x_i = cd_i$ for some $c$. Therefore, $\gcd(cd_i, n) = d_i$ so that $\gcd(c, \frac{n}{d_i}) = 1$. Consequently, any element of the set $\mathcal{A}_{d_i}$ of the form $cd_i$, where $1 \leq c \leq \frac{n}{d_i}$, and $\gcd(c, \frac{n}{d_i}) = 1$. Therefore, the number of elements in the set $\mathcal{A}_{d_i}$ is $\phi(\frac{n}{d_i})$, for $1 \leq i \leq k$. $\square$

A ring $R$ is said to satisfy the ascending chain condition of ideals if, for every chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$, there exists a positive integer $m$ such that $I_k = I_m$ for all $k \leq m$ and $R$ satisfies the descending chain condition of ideals if, for every chain of ideals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$, there exists a positive integer $m$ such that $I_k = I_m$ for all $k \geq m$. A ring $R$ is said to be *Noetherian* if it satisfies the ascending chain condition of ideals. A ring $R$ is said to be *Artinian* if it satisfies the descending chain condition of ideals. A ring $R$ is called *local* if it has a unique maximal ideal $\mathcal{M}$ and we abbreviate this by $(R, \mathcal{M})$. Also, recall that a *reduced Artin local ring* is always a field. The following remarks are useful in the sequel.

**Remark 1.2.6.** Let $R$ be a finite local ring and $p$ be a prime number. Then $|R| = p^{\alpha}$ for some positive integer $\alpha$.

The classification of the finite non-isomorphic local rings of order up to 8 is given in the Table 1.1.

| Order | type of rings |
|---|---|
| 2 | $\mathbb{Z}_2$ |
| 3 | $\mathbb{Z}_3$ |
| 4 | $\mathbb{F}_4$, $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{(x^2)}$ |
| 5 | $\mathbb{Z}_5$ |
| 7 | $\mathbb{Z}_7$ |
| 8 | $\mathbb{F}_8$, $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{(x^3)}$, $\frac{\mathbb{Z}_2[x,y]}{(x^2,xy,y^2)}$, $\frac{\mathbb{Z}_4[x]}{(2x,x^2)}$, $\frac{\mathbb{Z}_4[x]}{(2x,x^2-2)}$ |

Table 1.1: Non-isomorphic local rings with unity

The following theorem describes the structure of finite commutative rings with unity.

**Theorem 1.2.7.** *If $R$ is a finite commutative ring, then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each $R_i$ is a local ring.*

**Corollary 1.2.8.** *Let $R$ be a commutative reduced ring. Then*

$$R \cong F_1 \times F_2 \times \cdots \times F_n,$$

*where each $F_i$ is a field.*

## 1.3 Graphs

This section comprises the preliminary definitions and results of graphs which we need for the later discussion. For more details, one can refer to West [1996]. In this thesis, we consider simple graphs, that is, without multiple edges or loops. A *graph* $\Gamma$ is a pair $\Gamma = (V, E)$, where $V = V(\Gamma)$ and $E = E(\Gamma)$ are the set of vertices and edges of the graph $\Gamma$, respectively. We say that two different vertices $u$ and $v$ are *adjacent*, denoted by $u \sim v$ or $(u, v)$, if there is an edge between $u$ and $v$. We write $u \nsim v$, if there is no edge between $u$ and $v$. A *path* in a graph is a sequence of distinct vertices with the property that each vertex in the sequence is adjacent to the next vertex of it. A path graph on $n$ vertices is denoted by $P_n$. A graph $\Gamma$ is said to be *connected* if there is a path between every pair of vertex. A *cycle* is a path that begins and ends on the same vertex. The length of a cycle is the number of edges in it. A cycle of length $n$ is denoted by $C_n$. A *tree* is a connected graph with no cycles. A cycle is said to be a *Hamiltonian cycle* if it covers all the vertices. A graph $\Gamma$ is said to be Hamiltonian if it contains a Hamiltonian cycle. The *girth* of $\Gamma$ is the length of its shortest cycle and is denoted by $g(\Gamma)$. A connected graph $\Gamma$ is *Eulerian* if $\Gamma$ has a closed trail (walk with no repeated edge) containing all the edges of a graph.

**Theorem 1.3.1** ([West, 1996, Theorem 1.2.26]). *A connected graph $\Gamma$ is Eulerian if and only if the degree of every vertex is even.*

The *distance* between two vertices $u$ and $v$ in $\Gamma$ is the number of edges in a shortest path connecting them, and it is denoted by $d(u, v)$. If there is no path between $u$ and $v$, we say that the distance between $u$ and $v$ is *infinity* and we write as $d(u, v) = \infty$. The Wiener index $W(\Gamma)$ of a connected graph $\Gamma$ is defined as the sum of all distances between every pair of vertices in the graph, that is,

$$W(\Gamma) = \frac{1}{2} \sum_{u \in V(\Gamma)} \sum_{v \in V(\Gamma)} d(u, v)$$

The *diameter* of a connected graph $\Gamma$, written as $diam(\Gamma)$, is the maximum of the distances between vertices. If the graph consists of a single vertex, then its diameter is 0. The *degree* of the vertex $v$ in $\Gamma$ is the number of edges incident to $v$ and it is denoted by $\deg(v)$. The smallest degree among the vertices of $\Gamma$ is called the *minimum degree* of $\Gamma$, and it is denoted by $\delta(\Gamma)$. A graph $\Gamma$ is *regular* if degree of every vertex is same. A graph $\Gamma$ is said to be *biregular* whose vertices have two distinct degrees. The set $N_\Gamma(x)$ or $N(x)$ of all the vertices adjacent to $x$ in $\Gamma$ is said to be the *neighbourhood* of $x$. Additionally, we denote $N[x] = N(x) \cup \{x\}$. A *subgraph* $\Gamma'$ of $\Gamma$ is the graph such that $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$. If $X \subseteq V(\Gamma)$, then the subgraph of $\Gamma$ induced by $X$, denoted by $\Gamma(X)$, is the graph with vertex set $X$ and two vertices of $\Gamma(X)$ are adjacent if and only if they are adjacent in $\Gamma$. A subgraph $\Gamma'$ is said to be a *spanning subgraph* of $\Gamma$ if $V(\Gamma') = V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$. A graph $\Gamma$ is said to be *complete* if every two distinct vertices are adjacent. The complete graph on $n$ vertices is denoted by $K_n$. A graph $\Gamma$ is said to be a *complete bipartite graph* if the vertex $V(\Gamma)$ can be partitioned into two disjoint unions of non-empty sets $A$ and $B$, such that two distinct vertices are adjacent if and only if they belong to different sets. Moreover, if $|A| = m$ and $|B| = n$, then we denote it by $K_{m,n}$.

A subset $X$ of $V(\Gamma)$ is said to be *independent* if no two vertices of $X$ are adjacent. The *independence number* of $\Gamma$ is the cardinality of the largest independent set, and it is denoted by $\alpha(\Gamma)$. The *chromatic number* of $\Gamma$, denoted by $\chi(\Gamma)$, is the smallest number of colors needed to color the vertices of $\Gamma$ such that no two adjacent vertices share the same color. A *clique* in $\Gamma$ is a set of pairwise adjacent vertices. The *clique number* of $\Gamma$ is the size of maximum clique in $\Gamma$ and it is denoted by $\omega(\Gamma)$. It is well known that $\omega(\Gamma) \leq \chi(\Gamma)$. A graph $\Gamma$ is *weakly perfect* if $\omega(\Gamma) = \chi(\Gamma)$. A graph $\Gamma$ is *perfect* if $\omega(\Gamma') = \chi(\Gamma')$ for every induced subgraph $\Gamma'$ of $\Gamma$. Recall that the *complement* $\overline{\Gamma}$ of $\Gamma$ is a graph with same vertex set as $\Gamma$ and distinct vertices $u, v$ are adjacent in $\overline{\Gamma}$ if they are not adjacent in $\Gamma$. A subgraph $\Gamma'$ of $\Gamma$ is called *hole* if $\Gamma'$ is a cycle as an induced subgraph, and $\Gamma'$ is called an *antihole* of $\Gamma$ if $\overline{\Gamma'}$ is a hole in $\overline{\Gamma}$.

**Theorem 1.3.2** (Chudnovsky et al. [2006])**.** *A finite graph $\Gamma$ is perfect if and only if it does not contain a hole or antihole of odd length, at least* 5*.*

A subset $D$ of $V(\Gamma)$ is said to be a dominating set if any vertex in $V(\Gamma) \setminus D$ is adjacent to at least one vertex in $D$. If $D$ contains only one vertex, then that vertex is called the dominating vertex. The *domination number* $\gamma(\Gamma)$ of $\Gamma$ is the minimum size of a dominating set in $\Gamma$. A graph $\Gamma$ is said to be *triangulated* if every vertex of $\Gamma$ is a vertex of a triangle. A *vertex (edge) cutset* in a connected graph $\Gamma$ is a set of vertices (edges) whose deletion increases the number of connected components of $\Gamma$. The *vertex connectivity* (*edge connectivity*) of a connected graph $\Gamma$ is the minimum size of a vertex (edge) cutset and it is denoted by $\kappa(\Gamma)$ ($\kappa'(\Gamma)$). It is well known that $\kappa(\Gamma) \leq \kappa'(\Gamma) \leq \delta(\Gamma)$. An edge cover of $\Gamma$ is a subset $L$ of $E(\Gamma)$ such that every vertex of $\Gamma$ is incident to some edge of $L$. The minimum cardinality of an edge cover in $\Gamma$ is called the *edge covering number*. A vertex cover of $\Gamma$ is a subset $Q$ of $V(\Gamma)$ containing at least one endpoint of every edge of $\Gamma$. The minimum cardinality of a vertex cover in $\Gamma$ is called the *vertex covering number*. A *matching* of the graph $\Gamma$ is a set of edges with no share endpoints. The maximum cardinality of a matching in $\Gamma$ is called the *matching number* of $\Gamma$. It is denoted by $\alpha'(\Gamma)$.

In a graph $\Gamma$, a vertex $z$ resolves a pair of distinct vertices $x$ and $y$ if $d(x, z) \neq d(y, z)$. A resolving set of $\Gamma$ is a subset $\mathcal{R} \subseteq V(\Gamma)$ such that every pair of distinct vertices of $\Gamma$ is resolved by some vertex in $\mathcal{R}$. The metric dimension of $\Gamma$, denoted by $\beta(\Gamma)$, is the minimum cardinality of a resolving set of $\Gamma$. Note that the twin vertices in a graph correspond to vertices sharing the same neighbourhood.

**Lemma 1.3.3.** *If $U$ is twin-set in a connected graph $\Gamma$ of order $n$ with $|U| = l \geq 2$, then every resolving set for $\Gamma$ contains at least $l-1$ vertices of $U$.*

**Lemma 1.3.4** ([Chartrand et al., 2000, Theorem 1])**.** *For positive integers $d$ and $m$ with $d < m$, define $f(m,d)$ as the least positive integer $k$ such that $k + d^k \geq m$. Then for a connected graph $\Gamma$ of order $m \geq 2$ and diameter $d$, the metric dimension $\beta(\Gamma) \geq f(m,d)$.*

For vertices $u$ and $v$ in a graph $\Gamma$, we say that $z$ *strongly resolves* $u$ and $v$ if there exists a shortest path from $z$ to $u$ containing $v$, or a shortest path from $z$ to $v$ containing $u$. A subset $U$ of $V(\Gamma)$ is a *strong resolving set* of $\Gamma$ if every pair of vertices of $\Gamma$ is strongly resolved by some vertex of $U$. The least cardinality of a strong resolving set of $\Gamma$ is called the *strong metric dimension* of $\Gamma$ and is denoted by $\mathrm{sdim}(\Gamma)$. For vertices $u$ and $v$ in a graph $\Gamma$, we write $u \equiv v$ if $N[u] = N[v]$. Notice that $\equiv$ is an equivalence relation on $V(\Gamma)$. We denote $\widehat{v}$ by the $\equiv$-class containing a vertex $v$ of $\Gamma$. Consider a graph $\widehat{\Gamma}$ whose vertex set is the set of all $\equiv$-classes, and vertices $\widehat{u}$ and $\widehat{v}$ are adjacent if $u$ and $v$ are adjacent in $\Gamma$. This graph is well-defined because in $\Gamma$, $w \sim v$ for all $w \in \widehat{u}$ if and only if $u \sim v$. We observe that $\widehat{\Gamma}$ is isomorphic to the subgraph $\mathcal{R}_\Gamma$ of $\Gamma$ induced by a set of vertices consisting of exactly one element from each $\equiv$-class.

**Theorem 1.3.5** ([Ma et al., 2018, Theorem 2.2])**.** *For any graph $\Gamma$ with diameter at most 2, we have $\mathrm{sdim}(\Gamma) = |V(\Gamma)| - \omega(\widehat{\Gamma})$.*

A graph $\Gamma$ is a *split* graph if the vertex set is the disjoint union of two sets $A$ and $B$, where $A$ induces a complete subgraph and $B$ is an independent set.

**Lemma 1.3.6** (Foldes and Hammer [1977]). *A graph* $\Gamma$ *is a split graph if and only if it does not have an induced subgraph isomorphic to one of the three forbidden graphs,* $C_4, C_5$ *or* $2K_2$.

A graph $\Gamma$ is said to be a *cograph* if it has no induced subgraph isomorphic to $P_4$. A *threshold* graph is the graph which does not contain an induced subgraph isomorphic to $P_4, C_4$ or $2K_2$. Every threshold graph is a cograph as well as a split graph. A *cactus* graph is a connected graph where any two simple cycles have at most one vertex in common. A connected graph is said to be *unicyclic* if it contains exactly one cycle. A graph $\Gamma$ is *outerplanar* if it can be embedded in the plane such that all vertices lie on the outer face. A graph $\Gamma$ is *planar* if it can be drawn on a plane without edge crossing. It is well known that every outerplanar graph is a planar graph. In a graph $\Gamma$ the *subdivision* of an edge is the deletion of the edge $(u, v)$ from $\Gamma$ and the addition of two edges $(u, w)$ and $(w, v)$ along with a new vertex $w$. A graph obtained from $\Gamma$ by a sequence of edge subdivisions is called a subdivision of $\Gamma$. Two graphs are said to be *homeomorphic* if both can be obtained from the same graph by subdivisions of edges. Now we have the following known results related to outerplanar and planar graphs.

**Theorem 1.3.7** (West [1996]). *A graph* $\Gamma$ *is outerplanar if and only if it does not contain a subdivision of* $K_4$ *or* $K_{2,3}$.

**Theorem 1.3.8** (West [1996]). *A graph* $\Gamma$ *is planar if and only if it does not contain a subdivision of* $K_5$ *or* $K_{3,3}$.

Recall that a chord in a graph $\Gamma$ is an edge joining two non-adjacent vertices in a cycle of $\Gamma$. A cycle is said to be *primitive* if it has no chord. A graph $\Gamma$ satisfies the primitive cycle property (PCP) if any two primitive cycles intersect in at most one edge. The *free rank* of graph $\Gamma$, denoted by $frank(\Gamma)$, is the number of primitive cycles in $\Gamma$. The cycle rank $rank(\Gamma)$ of $\Gamma$ is the number $|E(\Gamma)|-|V(\Gamma)|+\mathcal{C}$, where $\mathcal{C}$ is the number of connected components of $\Gamma$.

A compact connected topological space such that each point has a neighbourhood homeomorphic to an open disc in $\mathbb{R}^2$ is called a surface. An embedding of a graph $\Gamma$ on a surface $\mathbb{S}$ is $2-$cell embedding if each component of $\mathbb{S}-\Gamma$ is homeomorphic to an open disc in $\mathbb{R}^2$. A $2-$cell embedding is said to be *triangular* if all the faces have boundaries consisting of exactly three edges. Let $\mathbb{S}_{\eth}$ denote the orientable surface with $\eth$ handles, where $\eth$ is a non-negative integer. The *genus* of a graph $\Gamma$, denoted by $\eth(\Gamma)$, is the minimum integer $\eth$ such that $\Gamma$ can be embedded in $\mathbb{S}_{\eth}$ that is the graph $\Gamma$ can be drawn into a surface $\mathbb{S}_{\eth}$ without edge crossing. Note that the graphs with genus zero are planar graphs and those with genus one are toroidal graphs. The following results are useful in the sequel.

**Proposition 1.3.9** ([White, 1984, Ringel and Youngs]). *Let $n \geq 3$ be a positive integer. Then $\eth(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$.*

**Lemma 1.3.10** ([White, 1984, Theorem 5.14]). *Let $\Gamma$ be a connected graph, with a 2-cell embedding in $\mathbb{S}_{\eth}$. Then $v - e + f = 2 - 2\eth$, where $v, e$ and $f$ are the number of vertices, edges and faces embedded in $\mathbb{S}_{\eth}$, respectively and $\eth$ is the genus of surface of graph embedded.*

**Lemma 1.3.11** (White [2001]). *The genus of a connected graph $\Gamma$ is the sum of the genera of its blocks.*

Let $\mathbb{N}_k$ denote the non-orientable surface formed by the connected sum of $k$ projective planes, that is, $\mathbb{N}_k$ is a non-orientable surface with $k$ crosscap. The *crosscap* of a graph $\Gamma$, denoted by $cr(\Gamma)$, is the minimum non-negative integer $k$ such that $\Gamma$ can be embedded in $\mathbb{N}_k$. For instance, a graph $\Gamma$ is planar if $cr(\Gamma) = 0$ and the $\Gamma$ is projective if $cr(\Gamma) = 1$. The following results are useful to determine the crosscap of a graph.

**Proposition 1.3.12** ([Mohar and Thomassen, 2001, Ringel and Youngs]). *Let $n$ be a positive integer. Then*

$$cr(K_n) = \begin{cases} \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil, & if \quad n \geq 3 \\ 3, & if \quad n = 7. \end{cases}$$

**Lemma 1.3.13** ([Mohar and Thomassen, 2001, Lemma 3.1.4]). *Let $\phi : \Gamma \to \mathbb{N}_k$ be a 2-cell embedding of a connected graph $\Gamma$ to the non-orientable surface $\mathbb{N}_k$. Then $v - e + f = 2 - k$, where $v, e$ and $f$ are the number of vertices, edges and faces of $\phi(\Gamma)$ respectively, and $k$ is the crosscap of $\mathbb{N}_k$.*

**Definition 1.3.14** (White [2001]). A graph $\Gamma$ is orientably simple if the manifold number $\Theta(\Gamma) \neq 2 - cr(\Gamma)$, where $\Theta(\Gamma) = \max\{2 - 2\eth(\Gamma), 2 - \mathrm{cr}(\Gamma)\}$.

**Lemma 1.3.15** (White [2001]). *Let $\Gamma$ be a graph with blocks $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$. Then*

$$cr(\Gamma) = \begin{cases} 1 - k + \sum\limits_{i=1}^{k} cr(\Gamma_i), & \text{if } \Gamma \text{ is orientably simple} \\ 2k - \sum\limits_{i=1}^{k} \Theta(\Gamma_i), & \text{otherwise.} \end{cases}$$

We shall use the following remark explicitly without referring to it.

**Remark 1.3.16.** For a simple graph $\Gamma$, every face has at least three boundary edges, and every edge is a boundary of two faces, that is, $2e \geq 3f$. Moreover, the equality holds if and only if $\Gamma$ has a triangular embedding.

A *homomorphism* of graph $\Gamma$ to a graph $\Gamma'$ is a mapping $f$ from $V(\Gamma)$ to $V(\Gamma')$ with the property that if $u \sim v$, then $uf \sim vf$ for all $u, v \in V(\Gamma)$. A retraction is a homomorphism $f$ from a graph $\Gamma$ to a subgraph $\Gamma'$ of $\Gamma$ such that $vf = v$ for each vertex $v$ of $\Gamma'$. In this case, the subgraph $\Gamma'$ is called a retract of $\Gamma$. An *isomorphism* of a simple graphs $\Gamma$ and $\Gamma'$ is a bijection $f : V(\Gamma) \mapsto V(\Gamma')$ such that $u \sim v$ in $\Gamma$ if and only if $uf \sim vf$ in $\Gamma'$. Graphs $\Gamma$ and $\Gamma'$ are isomorphic, if there is an isomorphism between them and we write it as $\Gamma \cong \Gamma'$. An *automorphism* of a graph $\Gamma$ is a permutation $f$ on $V(\Gamma)$ with the property that, for any vertices $u$ and $v$, we have $uf \sim vf$ if and only if $u \sim v$. The set $Aut(\Gamma)$ of all the graph automorphisms of a graph $\Gamma$ forms a group with respect to the composition of mappings. A graph $\Gamma$ is *vertex transitive* if for every two vertices $u$ and $v$ there exists a graph automorphism $f$ such that $uf = v$. For a subset $X$, $S_{|X|}$ denotes the symmetric group of degree $|X|$.

Let $\Gamma_1$ and $\Gamma_2$ be two graphs. The *union* $\Gamma_1 \cup \Gamma_2$ is the graph with $V(\Gamma_1 \cup \Gamma_2) = V(\Gamma_1) \cup V(\Gamma_2)$ and $E(\Gamma_1 \cup \Gamma_2) = E(\Gamma_1) \cup E(\Gamma_2)$. The *join* $\Gamma_1 \vee \Gamma_2$ of $\Gamma_1$ and $\Gamma_2$ is the graph obtained from the union of $\Gamma_1$ and $\Gamma_2$ by adding new edges from each vertex of $\Gamma_1$ to every vertex of $\Gamma_2$. Let $\Gamma$ be a graph on $k$ vertices and $V(\Gamma) = \{u_1, u_2, \ldots, u_k\}$. Suppose that $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ are $k$ pairwise disjoint graphs. Then the *generalised join graph* $\Gamma[\Gamma_1, \Gamma_2, \ldots, \Gamma_k]$ of $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ is the graph formed by replacing each vertex $u_i$ of $\Gamma$ by $\Gamma_i$ and then joining each vertex of $\Gamma_i$ to every vertex of $\Gamma_j$ whenever $u_i \sim u_j$ in $\Gamma$ (cf. Schwenk [1974]). For a finite simple undirected graph $\Gamma$ with vertex set $V(\Gamma) = \{u_1, u_2, \ldots, u_k\}$, the *adjacency matrix* $A(\Gamma)$ is defined as the $k \times k$ matrix whose $(i,j)th$ entry is 1 if $u_i \sim u_j$ and 0 otherwise. The adjacency eigenvalues are the eigenvalues of the matrix $A(\Gamma)$. We denote the diagonal matrix by $D(\Gamma) = \operatorname{diag}(d_1, d_2, \ldots, d_k)$, where $d_i$ is the degree of the vertex $u_i$ of $\Gamma$. The *Laplacian matrix* $\mathcal{L}(\Gamma)$ of $\Gamma$ is the matrix $D(\Gamma) - A(\Gamma)$. The matrix $\mathcal{L}(\Gamma)$ is symmetric and positive semidefinite, so its eigenvalues are real and non-negative. Furthermore, the sum of each row (column) of $\mathcal{L}(\Gamma)$ is zero. The eigenvalues of $\mathcal{L}(\Gamma)$ are called the *Laplacian eigenvalues* of $\Gamma$ and are taken as $\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots \geq \lambda_n(\Gamma) = 0$. The second smallest Laplacian eigenvalue of $\mathcal{L}(\Gamma)$, denoted by $\mu(\Gamma)$, is called the *algebraic connectivity* of $\Gamma$. The largest Laplacian eigenvalue $\lambda(\Gamma)$ of $\mathcal{L}(\Gamma)$ is called the *Laplacian spectral radius* of $\Gamma$. Now let $\lambda_1(\Gamma), \lambda_2(\Gamma), \ldots, \lambda_r(\Gamma)$ be the distinct eigenvalues of $\Gamma$ with multiplicities $\mu_1, \mu_2, \ldots, \mu_r$, respectively. The *Laplacian spectrum* of $\Gamma$, that is, the spectrum of

$\mathcal{L}(\Gamma)$, is represented as

$$\Phi(\mathcal{L}(\Gamma)) = \begin{pmatrix} \lambda_1(\Gamma) & \lambda_2(\Gamma) & \cdots & \lambda_r(\Gamma) \\ \mu_1 & \mu_2 & \cdots & \mu_r \end{pmatrix}.$$

Sometime we write $\Phi(\mathcal{L}(\Gamma)$ as $\Phi_{\mathcal{L}}(\Gamma)$ also. The following results are useful in the sequel.

**Theorem 1.3.17** (Cardoso et al. [2013]). *Let $\Gamma$ be a graph on $k$ vertices having $V(\Gamma) = \{u_1, u_2, \ldots, u_k\}$ and let $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ be $k$ pairwise disjoint graphs on $n_1, n_2, \ldots, n_k$ vertices, respectively. Then the Laplacian spectrum of $\Gamma[\Gamma_1, \Gamma_2, \ldots, \Gamma_k]$ is given by*

$$\Phi_L(\Gamma[\Gamma_1, \Gamma_2, \cdots, \Gamma_k]) = \bigcup_{i=1}^{k}(D_i + (\Phi_L(\Gamma_i) \setminus \{0\})) \bigcup \Phi(\mathbb{L}(\Gamma)) \tag{1.2}$$

*where*

$$D_i = \begin{cases} \sum\limits_{u_j \sim u_i} n_j & \textit{if} \;\; N_\Gamma(u_i) \neq \emptyset; \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{L}(\Gamma) = \begin{bmatrix} D_1 & -p_{1,2} & \cdots & -p_{1,k} \\ -p_{2,1} & D_2 & \cdots & -p_{2,k} \\ \cdots & \cdots & \cdots & \cdots \\ -p_{k,1} & -p_{k,2} & \cdots & D_k \end{bmatrix} \tag{1.3}$$

*such that*

$$p_{i,j} = \begin{cases} \sqrt{n_i n_j} & \textit{if} \;\; u_i \sim u_j \;\; \textit{in} \;\; \Gamma \\ 0 & \text{otherwise.} \end{cases}$$

*in* (1.2), $(\Phi_L(\Gamma_i) \setminus \{0\}))$ means that one copy of the eigenvalue 0 is removed from the multiset $\Phi_L(\Gamma_i)$, and $D_i + (\Phi_L(\Gamma_i) \setminus \{0\}))$ means $D_i$ is added to each element of $(\Phi_L(\Gamma_i) \setminus \{0\}))$.

Let $\Gamma$ be a weighted graph by assigning the weight $n_i = |V(\Gamma_i)|$ to the vertex $u_i$ of $\Gamma$ and $i$ varies from 1 to $k$. Consider $L(\Gamma) = (l_{i,j})$ to be a $k \times k$ matrix, where

$$l_{i,j} = \begin{cases} -n_j & \text{if } \ i \neq j \ \text{ and } \ u_i \sim u_j; \\ \sum\limits_{u_i \sim u_r} n_r & \text{if } \ i = j; \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $L(\Gamma)$ is called the vertex weighted Laplacian matrix of $\Gamma$, which is a zero row sum matrix but not a symmetric matrix in general. Though the $k \times k$ matrix $\mathbb{L}(\Gamma)$ defined in Theorem 1.3.17, is a symmetric matrix, it need not be a zero row sum matrix. Since the matrices $\mathbb{L}(\Gamma)$ and $L(\Gamma)$ are similar, we have the following remark.

**Remark 1.3.18.** $\Phi(\mathbb{L}(\Gamma)) = \Phi(L(\Gamma))$.

The normalized Laplacian introduced by Chung [1997] and it is defined as $L(\Gamma) = D(\Gamma)^{\frac{-1}{2}}\mathcal{L}(\Gamma)D(\Gamma)^{\frac{-1}{2}}$. The following results are useful in the sequel.

**Theorem 1.3.19** (Mohar [1991])**.** *Let $\Gamma_1 \vee \Gamma_2$ denotes the join of two graphs $\Gamma_1$ and $\Gamma_2$. Then the characteristic polynomial of the $\mathcal{L}(\Gamma_1 \vee \Gamma_2)$ is*

$$\mu(\Gamma_1 \vee \Gamma_2, x) = \tfrac{x(x-n_1-n_2)}{(x-n_1)(x-n_2)}\mu(\Gamma_1, x-n_2)\mu(\Gamma_2, x-n_1),$$

*where $n_1, n_2$ are the orders of graph $\Gamma_1$ and $\Gamma_2$, respectively.*

**Theorem 1.3.20** (Mohar [1991])**.** *Let $\Gamma$ be the disjoint union of graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$. Then the characteristic polynomial of the $\mathcal{L}(\Gamma)$ is*

$$\mu(\Gamma, x) = \prod_{i=1}^{k} \mu(\Gamma_i, x).$$

The proof of the following lemma is straightforward.

**Lemma 1.3.21.** *The adjacency eigenvalues of the complete graph $K_n$ are $n-1$ and $-1$ with multiplicities $1$ and $n-1$, respectively.*

**Lemma 1.3.22** (Rather et al. [2022]). *Let $\Gamma$ be a graph of order $n$ and let $\Gamma_i$ be the $r_i$ regular graph on $n_i$ vertices, having adjacency eigenvalues $\lambda_{i1} = r_i \geq \lambda_{i2} \geq \ldots \geq \lambda_{in_i}$, where $i = 1, 2, \ldots, n$. Then the normalized Laplacian eigenvalues of the graph $\Gamma[\Gamma_1, \Gamma_2, \ldots \Gamma_n]$ consists of the eigenvalues $1 - \dfrac{1}{r_i + \alpha_i}\lambda_{ik}(\Gamma_i)$, for $i = 1, 2, \ldots, n$ and $k = 2, 3, \ldots, n_i$, where $\alpha_i = \sum_{v_j \in N_\Gamma(v_i)} n_i$ is the sum of the orders of the graphs $\Gamma_j, j \neq i$, which correspond to the neighbours of the vertex $v_i \in \Gamma$. The remaining $n$ eigenvalues are the eigenvalues of the matrix*

$$\begin{bmatrix} \dfrac{\alpha_1}{\alpha_1 + r_1} & \dfrac{-n_2 a_{12}}{\sqrt{(r_1+\alpha_1)(r_2+\alpha_2)}} & \cdots & \dfrac{-n_n a_{1n}}{\sqrt{(r_1+\alpha_1)(r_n+\alpha_n)}} \\ \dfrac{-n_1 a_{21}}{\sqrt{(r_2+\alpha_2)(r_1+\alpha_1)}} & \dfrac{\alpha_2}{\alpha_2 + r_2} & \cdots & \dfrac{-n_n a_{2n}}{\sqrt{(r_2+\alpha_2)(r_n+\alpha_n)}} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{-n_1 a_{n1}}{\sqrt{(r_n+\alpha_n)(r_1+\alpha_1)}} & \dfrac{-n_2 a_{n2}}{\sqrt{(r_n+\alpha_n)(r_2+\alpha_2)}} & \cdots & \dfrac{\alpha_n}{\alpha_n + r_n} \end{bmatrix}$$

*where,*

$$a_{ij} = \begin{cases} 1, & v_i \sim v_j \\ 0, & \textit{otherwise.} \end{cases}$$

# Chapter 2

# The Cozero-divisor Graph of a Commutative Ring

Afkhami and Khashyarmanesh [2011] introduced the cozero-divisor graph of a commutative ring. Let $R$ be a commutative ring with unity. The cozero-divisor graph of a ring $R$, denoted by $\Gamma'(R)$, is a simple undirected graph whose vertices are the set of all non-zero, non-unit elements of $R$, and two distinct vertices $x$ and $y$ are adjacent if and only if $x \notin Ry$ and $y \notin Rx$, where $Rx$ is the ideal generated by the element $x$ of $R$. They explored the interrelation between the algebraic properties of the ring $R$ and graph-theoretic properties of $\Gamma'(R)$. The relation between the cozero-divisor and zero-divisor graphs are also discussed in Afkhami and Khashyarmanesh [2011]. Then Afkhami and Khashyarmanesh [2012] investigated the complement of the cozero-divisor graph and characterised all commutative rings $R$ whose $\Gamma'(R)$ is unicyclic, star, double-star or forest. Akbari and Khojasteh [2013] proved that if the independence number $\alpha(\Gamma'(R))$ is finite, then the ring $R$ is Artinian if and only if $R$ is Noetherian. They also showed that if the maximum degree of $\Gamma'(R)$, for a commutative ring $R$, is finite, then the ring $R$ is finite. Akbari, Alizadeh and Khojasteh [2014] showed that $diam(\Gamma'(R[x])) = 2$, for every commutative ring $R$. Particularly, they proved that if $R$ is a commutative non-local ring, then $diam(\Gamma'(R[[x]])) \leq 3$. Recently, Bakhtyiari et al. [2020] discussed the perfectness of $\Gamma'(R)$, when $R$ is a von Neumann regular ring. They also gave an explicit formula for the clique number of the graph $\Gamma'(R)$. Nikandish et al. [2021] obtained the metric and strong metric dimension $\Gamma'(R)$, for some classes of ring $R$.

This chapter seeks to present some more insight to the study of the cozero-divisor graph $\Gamma'(R)$ of a commutative ring $R$. In this connection, first we obtain a closed-form formula of the Wiener index of the cozero-divisor graph $\Gamma'(R)$ of a finite commutative ring $R$ in Section 2.1. As an application, we determine the Wiener index of $\Gamma'(R)$ when $R$ is either the product of the rings of integers modulo $n$ (see Subsection 2.1.1) or a reduced ring (see Subsection 2.1.2). Subsection 2.1.3 deals with the computation of Wiener index of $\Gamma'(\mathbb{Z}_n)$ independently. We also derive a SageMath code to compute the Wiener index of these class of rings in Subsection 2.1.4. In Section 2.2, we study the Laplacian spectrum of $\Gamma'(\mathbb{Z}_n)$. We show that the graph $\Gamma'(\mathbb{Z}_{pq})$ is Laplacian integral. Further, we obtain the Laplacian spectrum of $\Gamma'(\mathbb{Z}_n)$ for $n = p^{n_1}q^{n_2}$, where $n_1, n_2 \in \mathbb{N}$ and $p, q$ are distinct primes. Section 2.3 deals with the investigation of the Laplacian spectral radius and algebraic connectivity of $\Gamma'(\mathbb{Z}_n)$. We characterized the values of $n$ for which the Laplacian spectral radius is equal to the order of $\Gamma'(\mathbb{Z}_n)$. Moreover, the values of $n$ for which the algebraic connectivity and vertex connectivity of $\Gamma'(\mathbb{Z}_n)$ coincide are also described. The content (Subsection 2.1.3, Section 2.3 and Section 2.2) of this chapter is published in the journal *"AKCE International Journal of Graphs and Combinatorics"*. However, the results of Section 2.1 (except Subsection 2.1.3) is submitted for the publication.

## 2.1 The Wiener Index of $\Gamma'(R)$

The purpose of this section is to provide a closed-form formula of the Wiener index of the cozero-divisor graph of a finite commutative ring (see Theorem 2.1.5). Let $R$ be a finite commutative ring with unity. Define a relation $\equiv$ on $V(\Gamma'(R))$ such that $x \equiv y$ if and only if $(x) = (y)$. Note that the relation $\equiv$ is an equivalence relation. Let $x_1, x_2, \ldots, x_k$ be the representatives of the equivalence classes of $X_1, X_2, \ldots, X_k$, respectively, under the relation $\equiv$. We begin with the following lemma.

**Lemma 2.1.1.** *A vertex of $X_i$ is adjacent to a vertex of $X_j$ if and only if $(x_i) \nsubseteq (x_j)$ and $(x_j) \nsubseteq (x_i)$.*

*Proof.* Suppose $a \in X_i$ and $b \in X_j$. Then $(a) = (x_i)$ and $(b) = (x_j)$ in $R$. If $a \sim b$ in $\Gamma'(R)$, then $(a) \not\subset (b)$ and $(b) \not\subset (a)$. It follows that $(x_i) \not\subset (x_j)$ and $(x_j) \not\subset (x_i)$. The converse holds by the definition of $\Gamma'(R)$. □

**Corollary 2.1.2.** (i) *For $i \in \{1, 2, \ldots, k\}$, the induced subgraph $\Gamma'(X_i)$ of $\Gamma'(R)$ is isomorphic to $\overline{K}_{|X_i|}$.*

(ii) *For distinct $i, j \in \{1, 2, \ldots, k\}$, a vertex of $X_i$ is adjacent to either all or none of the vertices of $X_j$.*

Define a subgraph $\Upsilon'(R)$ (or $\Upsilon'$) induced by the set $\{x_1, x_2, \ldots, x_k\}$ of representatives of the respective equivalence classes $X_1, X_2, \ldots, X_k$ of elements of $V(\Gamma'(R))$ under the relation $\equiv$.

**Lemma 2.1.3.** *The graph $\Upsilon'(R)$, with more than one vertex, is connected if and only if the cozero-divisor graph $\Gamma'(R)$ is connected. Moreover, for a connected graph $\Gamma'(R)$ and $a, b \in V(\Gamma'(R))$, we have*

$$d_{\Gamma'(R)}(a, b) = \begin{cases} 2 & \text{if } a, b \in X_i, \text{for some } i \\ d_{\Upsilon'(R)}(x_i, x_j) & \text{if } a \in X_i, b \in X_j \text{ and } i \neq j. \end{cases}$$

*Proof.* First, suppose that $\Upsilon'(R)$ is connected. Let $a, b$ be two arbitrary vertices of $\Gamma'(R)$. We may now suppose that $a \in X_i$ and $b \in X_j$. If $i = j$, then $a \nsim b$ in $\Gamma'(R)$. Since $\Upsilon'(R)$ is connected, we have $x_t \in X_t$ such that $x_i \sim x_t$ in $\Gamma'(R)$. Consequently, $a \sim x_t \sim b$ in $\Gamma'(R)$ and $d_{\Gamma'(R)}(a, b) = 2$. If $a \sim b$, then there is nothing to prove. Let $a \nsim b$ in $\Gamma'(R)$. Connectedness of $\Upsilon'(R)$ implies that there exists a path $x_i \sim x_{i_1} \sim x_{i_2} \sim \cdots \sim x_{i_t} \sim x_j$, where $i \neq j$. It follows that $a \sim x_{i_1} \sim x_{i_2} \sim \cdots \sim x_{i_t} \sim b$ in $\Gamma'(R)$ and $d_{\Gamma'(R)}(a, b) = d_{\Upsilon'(R)}(x_i, x_j)$. Therefore, $\Gamma'(R)$ is connected. The converse is straightforward. $\square$

In view of Corollary 2.1.2, we have the following proposition.

**Proposition 2.1.4.** *Let $\Gamma'_i$ be the subgraph induced by the set $X_i$ in $\Gamma'(R)$. Then $\Gamma'(R) = \Upsilon'[\Gamma'_1, \Gamma'_2, \ldots, \Gamma'_k]$.*

Let $R$ be a finite commutative ring with unity. As a consequence of Lemma 2.1.3 and Proposition 2.1.4, we have the following theorem.

**Theorem 2.1.5.** *The Wiener index of the cozero-divisor graph $\Gamma'(R)$ of a finite commutative ring with unity is given by*

$$W(\Gamma'(R)) = 2\sum \binom{|X_i|}{2} + \sum_{\substack{i \neq j \\ 1 \leq i < j \leq k}} |X_i||X_j| d_{\Upsilon'(R)}(x_i, x_j),$$

*where, $x_i$ is a representative of the equivalence class $X_i$ under the relation $\equiv$.*

In the subsequent subsections, we use Theorem 2.1.5 to derive the Wiener index of the cozero-divisor graph $\Gamma'(R)$ for various class of rings.

### 2.1.1 The Wiener Index of $\Gamma'(\mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \cdots \times \mathbb{Z}_{p_k^{m_k}})$

First note that $\Gamma'(\mathbb{Z}_4)$ is a graph with one vertex only. Moreover, for a prime $p$, the cozero-divisor graph $\Gamma'(\mathbb{Z}_p)$ is a graph without any vertices and for $\alpha \geq 2$, the cozero-divisor graph of $\mathbb{Z}_{p^\alpha}$, where $p^\alpha \neq 4$, is a graph with $p^{\alpha-1}-1$ vertices without any edges. Consequently, in this subsection, we obtain the Wiener index of the cozero-divisor graph $\Gamma'(R)$, when $R \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$ or $R \cong \mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \cdots \times \mathbb{Z}_{p_k^{m_k}}$, except $\mathbb{Z}_{p^\alpha}$, where $\alpha \geq 1$.

Define the relation $\equiv$ on the elements of $\mathbb{Z}_{p_i^{m_i}}$ such that $x \equiv y$ if and only if $(x) = (y)$. Note that $\equiv$ equivalence relation on $\mathbb{Z}_{p_i^{m_i}}$. Let $X_i^0, X_i^1, \ldots, X_i^{m_i}$ be the corresponding equivalence classes $\equiv$, where $X_i^0 = \{0\}$, $X_i^1 = U(\mathbb{Z}_{p_i^{m_i}})$ and $X_i^j = \mathcal{A}_{p_i{}^{j-1}}$ for $2 \leq j \leq m_i$. Now we have

$$|X_i^j| = \begin{cases} 1 & \text{if } j = 0, \\ p_i^{m_i} - p_i^{m_i-1} & \text{if } j = 1, \\ p_i^{m_i-j+1} - p_i^{m_i-j} & \text{if } 2 \leq j \leq m_i. \end{cases}$$

Let $x = (x_1, x_2, \ldots, x_r, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_r, \ldots, y_k) \in R$. Notice that $(x) = (y)$ if and only if $(x_i) = (y_i)$ for each $i$. It follows that the equivalence classes of the ring $R$ is of the form $X_1^{j_1} \times X_2^{j_2} \times \cdots \times X_k^{j_k}$. Consequently, $|X_1^{j_1} \times X_2^{j_2} \times \cdots \times X_k^{j_k}| = \prod_{i=1}^{k} |X_i^{j_i}|$.

**Lemma 2.1.6.** *Let $R \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$ and let $x = (x_1, x_2, \ldots, x_r, \ldots, x_k)$, $y = (y_1, y_2, \ldots, y_r, \ldots, y_k) \in V(\Gamma'(R))$. Define $S_r = \{\{x, y\} : \ x_r, y_r \in Z(\mathbb{Z}_{n_r})^* \text{ and } (x_r) \subseteq (y_r), x_i = 0, y_i \in U(\mathbb{Z}_{n_i}) \text{ for each } \ i \neq r\}$. Then*

$$d_{\Gamma'(R)}(x, y) = \begin{cases} 1 & \text{if } x \sim y, \\ 2 & \text{if } x \nsim y \text{ and } \ \{x, y\} \notin S_r \text{ for all } r, \\ 3 & \text{if } \{x, y\} \in S_r \text{ for some } r. \end{cases}$$

*Proof.* To prove the result, we discuss the following cases.

**Case-1.** $x_i \in Z(\mathbb{Z}_{n_i})^*$ for each $i \in [k]$. If $x \sim y$ in $\Gamma'(R)$, then $d(x, y) = 1$. Otherwise, either $(x) \subseteq (y)$ or $(y) \subseteq (x)$. Suppose that $y_i \in Z(\mathbb{Z}_{n_i})^*$, for each

$i \in [k]$. Then for $z = (1, 0, \ldots, 0) \in R$, we obtain $x \sim z \sim y$ in $\Gamma'(R)$. It follows that $d(x, y) = 2$. Now assume that $y_i \in Z(\mathbb{Z}_{n_i})$ for each $i \in [k]$ and $y_j = 0$ for some $j \in [k]$. If $x \nsim y$ in $\Gamma'(R)$, then $(y_i) \subseteq (x_i)$ for each $i$. Choose $z = (z_1, z_2, \ldots, z_k) \in R$ such that $z_i = 0$ whenever $y_i \in Z(\mathbb{Z}_{n_i})^*$ and $z_j \in U(\mathbb{Z}_{n_j})$ whenever $y_j = 0$, for some $i, j \in [k]$. Consequently, $x \sim z \sim y$ in $\Gamma'(R)$. It follows that $d(x, y) = 2$. If $y_i \in U(\mathbb{Z}_{n_i})$ and $y_j = 0$, for some $i, j \in [k]$, then note that $d(x, y) = 1$. Now, let $y_i \in U(\mathbb{Z}_{n_i})$ and $y_j \in Z(\mathbb{Z}_{n_i})^*$, for some $i, j \in [k]$. If $x \nsim y$ in $\Gamma'(R)$, then $(x_i) \subsetneq (y_i)$ for each $i \in [k]$. Choose $z = (z_1, z_2, \ldots, z_k) \in R$ such that $z_i = 0$ whenever $y_i \in U(\mathbb{Z}_{n_i})$ and $z_j \in U(\mathbb{Z}_{n_j})$ whenever $y_j \in Z(\mathbb{Z}_{n_j})^*$. It follows that $x \sim z \sim y$ in $\Gamma'(R)$ and so $d(x, y) = 2$. Further, assume that $y_i \in U(\mathbb{Z}_{n_i}), y_j \in Z(\mathbb{Z}_{n_j})^*$ and $y_l = 0$ for some $i, j, l \in [k]$. Then $x \sim y$ in $\Gamma'(R)$ and so $d(x, y) = 1$.

**Case-2.** $x_i \in U(\mathbb{Z}_{n_i})$ and $x_j = 0$ for some $i, j \in [k]$. Suppose $y_i \in U(\mathbb{Z}_{n_i})$ and $y_j = 0$ for some $i, j \in [k]$. If $x \sim y$ in $\Gamma'(R)$, then $d(x, y) = 1$. Otherwise, choose $z = (z_1, z_2, \ldots, z_k) \in R$ such that

$$z_i = \begin{cases} 1 & \text{when both } \; x_i = y_i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $d(x, y) = 2$. Further, suppose that $y_i \in Z(\mathbb{Z}_{n_i})$ for each $i \in [k]$ and $y_j = 0$ for some $j \in [k]$. If $x \nsim y$ in $\Gamma'(R)$, then choose $z = (z_1, z_2, \ldots, z_k)$ such that $z_i \in U(\mathbb{Z}_{n_i})$ whenever $y_i = 0$ and $z_j = 0$ whenever $y_j \in Z(\mathbb{Z}_{n_j})^*$. Consequently, $d(x, y) = 2$. Suppose that $y_i \in Z(\mathbb{Z}_{n_i})^*$ and $y_j \in U(\mathbb{Z}_{n_j})$ for some $i, j \in [k]$. If $x \sim y$ in $\Gamma'(R)$, then $d(x, y) = 1$. Otherwise, consider $z = (z_1, z_2, \ldots, z_k) \in R$ such that

$$z_i = \begin{cases} 0 & \text{if } y_i \in U(\mathbb{Z}_{n_i}), \\ 1 & \text{if } y_i \in Z(\mathbb{Z}_{n_i})^*. \end{cases}$$

Note that $x \sim z \sim y$ in $\Gamma'(R)$. It follows that $d(x,y) = 2$. Assume that $y_i \in U(\mathbb{Z}_{n_i}), y_j \in Z(\mathbb{Z}_{n_j})^*$ and $y_l = 0$ for some $i,j,l \in [k]$. If $x \nsim y$ in $\Gamma'(R)$, then choose $z = (z_1, z_2, \ldots, z_k) \in R$ such that $z_i \in U(\mathbb{Z}_{n_i})$ whenever $x_i = 0$ and $z_j = 0$ whenever $x_j \in U(\mathbb{Z}_{n_j})$, for some $i,j \in [k]$. Consequently, $d(x,y) = 2$.

**Case-3.** $x_i \in Z(\mathbb{Z}_{n_i})$ for each $i \in [k]$ and $x_j = 0$ for some $j \in [k]$. Suppose $y_i \in Z(\mathbb{Z}_{n_i})$ for each $i \in [k]$ and $y_j = 0$ for some $j \in [k]$. If $x \sim y$ in $\Gamma'(R)$, then $d(x,y) = 1$. Let $x \nsim y$ in $\Gamma'(R)$. Then choose $z = (z_1, z_2, \ldots, z_k) \in R$ such that $z_i = 0$, whenever $x_i \in Z(\mathbb{Z}_{n_i})^*$, and $z_j = 1$, whenever $x_j = 0$ for some $i,j \in [k]$. It follows that $x \sim z \sim y$ and so $d(x,y) = 2$. Next, assume that $y_i \in Z(\mathbb{Z}_{n_i})^*, y_j \in U(\mathbb{Z}_{n_j})$ and $y_l = 0$ for some $i,j,l \in [k]$. If $x \nsim y$ in $\Gamma'(R)$, then choose $z = (z_1, z_2, \ldots, z_k)$ such that $z_i = 1$ when $x_i = 0$, and $z_j = 0$ when $x_j \in Z(\mathbb{Z}_{n_j})^*$ for some $i,j \in [k]$. Consequently, we have $x \sim z \sim y$ in $\Gamma'(R)$. It implies that $d(x,y) = 2$. Further, assume that $y_i \in U(\mathbb{Z}_{n_i})$ and $y_j \in Z(\mathbb{Z}_{n_j})^*$ for some $i,j \in [k]$. Let $x \nsim y$ in $\Gamma'(R)$. Suppose that there exists $r \in [k]$ such that $x_r \in Z(\mathbb{Z}_{n_r})^*$ and $x_i = 0$ for each $i \in [k] \setminus \{r\}$. Also, $y_i \in U(\mathbb{Z}_{n_i})$ and $y_r \in Z(\mathbb{Z}_{n_r})^*$ for each $i \in [k] \setminus \{r\}$. Then $(x_r) \subsetneq (y_r)$. If there exists $a = (a_1, a_2, \ldots, a_r, \ldots, a_k)$ such that $a \sim y$, then $(y_r) \subsetneq (a_r)$. It follows that $(x_r) \subsetneq (y_r) \subsetneq (a_r)$. Consequently, $a \nsim x$ in $\Gamma'(R)$. Therefore, $d(x,y) > 2$. Consider $z = (z_1, z_2, \ldots, z_k)$ and $z' = (z'_1, z'_2, \ldots, z'_k) \in R$ such that

$$z_i = \begin{cases} 1 & \text{if } x_i = 0, \\ 0 & \text{if } x_i \in Z(\mathbb{Z}_{n_i})^* \end{cases}$$

and

$$z'_i = \begin{cases} 0 & \text{if } y_i \in U(\mathbb{Z}_{n_i}), \\ 1 & \text{if } y_i \in Z(\mathbb{Z}_{n_i})^*. \end{cases}$$

It follows that $x \sim z \sim z' \sim y$ in $\Gamma'(R)$. Therefore, $d(x,y) = 3$. Next, we claim that if there exist $t$ and $r \in [k]$ such that $x_t \in Z(\mathbb{Z}_{n_t})^*, x_r \in Z(\mathbb{Z}_{n_r})^*$ then $d(x,y) \leq 2$.

If $x \sim y$ in $\Gamma'(R)$, then $d(x,y) = 1$. Next, assume that $x \nsim y$ in $\Gamma'(R)$. Since $x \nsim y$, we have $(x) \subsetneq (y)$. If there exists $i_1 \in [k]$ such that $x_{i_1}, y_{i_1} \in Z(\mathbb{Z}_{n_{i_1}})^*$, then take $r = i_1$. Now consider $z = (z_1, z_2, \ldots, z_k) \in R$ such that $z_t = 0, z_r = 1$ and, for $i \neq \{t, r\}$ whenever $y_i \in U(\mathbb{Z}_{n_i})$ take $z_i = 0$ and, whenever $y_i \in Z(\mathbb{Z}_{n_i})^*$ then choose $z_i = 1$. It follows that $x \sim z \sim y$ in $\Gamma'(R)$. Therefore, $d(x,y) \leq 2$.

**Case-4.** $x_i \in Z(\mathbb{Z}_{n_i})^*$ and $x_j \in U(\mathbb{Z}_{n_j})$ for some $i, j \in [k]$. Let $y_i \in Z(\mathbb{Z}_{n_i})^*$ and $y_j \in U(\mathbb{Z}_{n_j})$ for some $i, j \in [k]$. If $x \sim y$ in $\Gamma'(R)$, then $d(x,y) = 1$. Let $x \nsim y$ in $\Gamma'(R)$. Then choose $z = (z_1, z_2, \ldots, z_k) \in R$ such that $z_i = 0$ whenever $x_i \in U(\mathbb{Z}_{n_i})$, and $z_j = 1$ whenever $x_j \in Z(\mathbb{Z}_{n_j})^*$ for some $i, j \in [k]$. It follows that $d(x,y) = 2$. Next, let $y_i \in Z(\mathbb{Z}_{n_i})^*, y_j \in U(\mathbb{Z}_{n_j})$ and $y_l = 0$ for some $i, j \in [k]$. If $x \nsim y$ in $\Gamma'(R)$, then choose $z = (z_1, z_2, \ldots, z_k)$ such that $z_i = 1$ whenever $x_i \in Z(\mathbb{Z}_{n_i})^*$, and $z_j = 0$ whenever $x_j \in U(\mathbb{Z}_{n_j})$ for some $i, j \in [k]$. Therefore, $d(x,y) = 2$.

**Case-5.** $x_i \in Z(\mathbb{Z}_{n_i})^*, x_j \in U(\mathbb{Z}_{n_j})$ and $x_l = 0$ for some $i, j, l \in [k]$. Assume that $y_i \in Z(\mathbb{Z}_{n_i})^*, y_j \in U(\mathbb{Z}_{n_j})$ and $y_l = 0$ for some $i, j, l \in [k]$. If $x \sim y$ in $\Gamma'(R)$, then $d(x,y) = 1$. Otherwise, choose $z = (z_1, z_2, \ldots, z_k) \in R$ as follows

$$z_i = \begin{cases} 0 & \text{if } x_i \in Z(\mathbb{Z}_{n_i})^* \ \text{ and } x_i \in U(\mathbb{Z}_{n_i}), \\ 1 & \text{if } x_i = 0. \end{cases}$$

Then $x \sim z \sim y$ in $\Gamma'(R)$. It follows that $d(x,y) = 2$. $\square$

In view of Lemma 2.1.6, now we calculate the Wiener index of $\Gamma'(R)$, where $R \cong \mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \cdots \times \mathbb{Z}_{p_k^{m_k}}$. Let $x = (x_1^{j_1}, x_2^{j_2}, \ldots, x_k^{j_k})$ and $y = (y_1^{l_1}, y_2^{l_2}, \ldots, y_k^{l_k})$ be the representatives of two distinct equivalence classes $X_1^{j_1} \times X_2^{j_2} \times \cdots \times X_k^{j_k}$ and $X_1^{l_1} \times X_2^{l_2} \times \cdots \times X_k^{l_k}$, respectively.

**Theorem 2.1.7.** *The Wiener index of the cozero-divisor graph $\Gamma'(R)$, where $R \cong$*

$\mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \cdots \times \mathbb{Z}_{p_k^{m_k}}$, *is given below:*

$$\begin{aligned} W(\Gamma'(R)) = 2 & \sum_{(x_1^{j_1}, x_2^{j_2}, \ldots x_k^{j_k}) \in \Upsilon'} \binom{\prod_{\substack{i=1\\ j_i \geq 1}}^{k} (p_i^{m_i - j_i + 1} - p_i^{m_i - j_i})}{2} \\ & + \sum_{x \sim y} \left( \prod_{\substack{i=1\\ j_i \geq 1}}^{k} (p_i^{m_i - j_i + 1} - p_i^{m_i - j_i}) \right) \left( \prod_{\substack{i=1\\ j_i \geq 1}}^{k} (p_i^{m_i - l_i + 1} - p_i^{m_i - l_i}) \right) \\ & + 2 \sum_{\substack{x \nsim y\\ \{x,y\} \notin S_r}} \left( \prod_{\substack{i=1\\ j_i \geq 1}}^{k} (p_i^{m_i - j_i + 1} - p_i^{m_i - j_i}) \right) \left( \prod_{\substack{i=1\\ j_i \geq 1}}^{k} (p_i^{m_i - l_i + 1} - p_i^{m_i - l_i}) \right) \\ & + 3 \sum_{\{x,y\} \in S_r} \left( \prod_{\substack{i=1\\ j_i \geq 1}}^{k} (p_i^{m_i - j_i + 1} - p_i^{n_i - j_i}) \right) \left( \prod_{\substack{i=1\\ j_i \geq 1}}^{k} (p_i^{m_i - l_i + 1} - p_i^{m_i - l_i}) \right). \end{aligned}$$

In the following example, we compute the Wiener index of the $\Gamma'(R)$, for a specific ring $R$.

**Example 2.1.8.** Let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9$. Then $|X_1^0| = 1$, $|X_2^0| = 1$, $|X_3^0| = 1$, $|X_1^1| = 1$, $|X_2^1| = 2$, $|X_3^1| = 6$, $|X_1^2| = 0$, $|X_2^2| = 1$ and $|X_3^2| = 2$. Let $Y_1 = X_1^0 \times X_2^0 \times X_3^1$, $Y_2 = X_1^0 \times X_2^0 \times X_3^2$, $Y_3 = X_1^0 \times X_2^1 \times X_3^0$, $Y_4 = X_1^0 \times X_2^2 \times X_3^0$, $Y_5 = X_1^0 \times X_2^1 \times X_3^1$, $Y_6 = X_1^0 \times X_2^2 \times X_3^1$, $Y_7 = X_1^0 \times X_2^1 \times X_3^2$, $Y_8 = X_1^0 \times X_2^2 \times X_3^2$, $Y_9 = X_1^1 \times X_2^0 \times X_3^0$, $Y_{10} = X_1^1 \times X_2^0 \times X_3^1$, $Y_{11} = X_1^1 \times X_2^0 \times X_3^2$, $Y_{12} = X_1^1 \times X_2^1 \times X_3^0$, $Y_{13} = X_1^1 \times X_2^1 \times X_3^2$, $Y_{14} = X_1^1 \times X_2^2 \times X_3^0$, $Y_{15} = X_1^1 \times X_2^2 \times X_3^1$ and $Y_{16} = X_1^1 \times X_2^2 \times X_3^2$. Then $S_3 = \{\{Y_2, Y_{13}\}\}$, $S_2 = \{\{Y_4, Y_{15}\}\}$ and the pair of sets whose elements are at distance two $\{\{Y_1, Y_2\}, \{Y_1, Y_5\}, \{Y_1, Y_6\}, \{Y_1, Y_{10}\}, \{Y_1, Y_{15}\}, \{Y_2, Y_5\}, \{Y_2, Y_6\}, \{Y_2, Y_7\}, \{Y_2, Y_8\}, \{Y_2, Y_{10}\}, \{Y_2, Y_{11}\}, \{Y_2, Y_{15}\}, \{Y_2, Y_{16}\}, \{Y_3, Y_4\}, \{Y_3, Y_5\}, \{Y_3, Y_7\}, \{Y_3, Y_{12}\}, \{Y_3, Y_{13}\}, \{Y_4, Y_5\}, \{Y_4, Y_6\}, \{Y_4, Y_7\}, \{Y_4, Y_8\}, \{Y_4, Y_{12}\}, \{Y_4, Y_{13}\}, \{Y_4, Y_{14}\}, \{Y_4, Y_{16}\}, \{Y_5, Y_6\}, \{Y_5, Y_7\}, \{Y_5, Y_8\}, \{Y_6, Y_8\}, \{Y_6, Y_{15}\}, \{Y_7, Y_8\}, \{Y_7, Y_{13}\}, \{Y_8, Y_{13}\}, \{Y_8, Y_{15}\}, \{Y_8, Y_{16}\}, \{Y_9, Y_{10}\}, \{Y_9, Y_{11}\}, \{Y_9, Y_{12}\}, \{Y_9, Y_{13}\}, \{Y_9, Y_{14}\}, \{Y_9, Y_{15}\}, \{Y_9, Y_{16}\}, \{Y_{10}, Y_{11}\}, \{Y_{10}, Y_{15}\}, \{Y_{11}, Y_{13}\}, \{Y_{11}, Y_{15}\}, \{Y_{11}, Y_{16}\}, \{Y_{12}, Y_{13}\}, \{Y_{12}, Y_{14}\}, \{Y_{13}, Y_{14}\}, \{Y_{13}, Y_{16}\}, \{Y_{14}, Y_{15}\}, \{Y_{14}, Y_{16}\}, \{Y_{15}, Y_{16}\}\}$.

Thus, the Wiener index of the cozero-divisor graph of the ring $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9$ is

$$\begin{aligned}
W(\Gamma'(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9)) &= 2 \times \frac{1}{2}[30 + 2 + 2 + 0 + 132 + 30 + 12 + 2 + 0 + 30 + 2 + 2 \\
&+ 12 + 0 + 30 + 2] + [6(2 + 1 + 4 + 2 + 1 + 2 + 2 + 4 + 1 + 2) \\
&+ 2(2 + 1 + 1 + 2 + 1) + 2(6 + 2 + 1 + 6 + 2 + 1 + 6 + 2) \\
&+ (1 + 6 + 2) + 12(1 + 6 + 2 + 2 + 4 + 1 + 6 + 2) + 6(4 + 1 + 6 \\
&+ 2 + 2 + 4 + 1 + 2) + 4(1 + 6 + 2 + 2 + 1 + 6 + 2) + 2(1 + 6+ \\
&2 + 2 + 1) + (0) + 6(2 + 4 + 1 + 2) + 2(2 + 1) + 2(6 + 2) + 4(6) \\
&+ (0)] + 2[6(2 + 12 + 6 + 6 + 6) + 2(12 + 6 + 4 + 2 + 6 + 2 \\
&+ 6 + 2) + 2(1 + 12 + 4 + 2 + 4) + (12 + 6 + 4 + 2 + 2 + 4 \\
&+ 1 + 2) + 12(6 + 4 + 2) + 6(2 + 6) + 4(2 + 4) + 2(4 + 6 + 2) \\
&+ (6 + 2 + 2 + 4 + 1 + 6 + 2) + 6(2 + 6) + 2(4 + 6 + 2) \\
&+ 2(4 + 1) + 4(1 + 2) + (6 + 2) + 6(2)] + 3[(2 \times 4) + (1 \times 6)] \\
&= 2611.
\end{aligned}$$

### 2.1.2 The Wiener Index of Cozero-divisor Graph of Reduced ring

In this subsection, we obtain the Wiener index of the cozero-divisor graph of a finite commutative reduced ring. Let $R$ be a reduced ring i.e. $R \cong \mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \times \cdots \times \mathbb{F}_{q_k}$ with $k \geq 2$. Notice that, for $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_k) \in R$ such that $(x) = (y)$, we have $x_i = 0$ if and only if $y_i = 0$ for each $i$. For $i_1, i_2, \ldots, i_r \in [k]$, define

$$X_{\{i_1,i_2,\ldots,i_r\}} = \{(x_1, x_2, \ldots, x_k) \in R : \text{only } x_{i_1}, x_{i_2}, \ldots, x_{i_r} \text{ are non-zero}\}.$$

Note that the sets $X_A$, where $A$ is a non-empty proper subset of $[k]$, are the equivalence classes under the relation $\equiv$. We write $x_A$ by the representative of equivalence class $X_A$. Now we obtain the possible distances between the vertices of $\Upsilon'(R)$.

**Lemma 2.1.9.** *For the distinct vertices $x_A$ and $x_B$ of $\Upsilon'(R)$, we have*

$$d_{\Upsilon'(R)}(x_A, x_B) = \begin{cases} 1 & \text{if } A \nsubseteq B \text{ and } B \nsubseteq A \\ 2 & \text{otherwise.} \end{cases}$$

*Proof.* First assume that $A \nsubseteq B$ and $B \nsubseteq A$. Then $(x_A) \nsubseteq (x_B)$ and $(x_B) \nsubseteq (x_A)$. It follows that $d_{\Upsilon'}(x_A, x_B) = 1$. Now without loss of generality let $A \subsetneq B$. Then there exists $i \in [k]$ such that $i \notin B$ and so $i \notin A$. Then by Lemma 2.1.1, we have $x_A \sim x_{\{i\}} \sim x_B$. Thus, $d_{\Upsilon'}(x_A, x_B) = 2$. □

For distinct $A, B \subsetneq [k]$, we define

$$D_1 = \{\{A, B\} : \ A \nsubseteq B\} \text{ and } D_2 = \{\{A, B\} : \ A \subsetneq B\}.$$

Using Theorem 2.1.5 and the sets $D_1$ and $D_2$, we obtain the Wiener index of the cozero-divisor $\Gamma'(R)$ of a reduced ring $R$ in the following theorem.

**Theorem 2.1.10.** *The Wiener index of the cozero-divisor graph of a finite commutative reduced ring $R \cong \mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \times \cdots \times \mathbb{F}_{q_k}$, $k \geq 2$, is given by*

$$W(\Gamma'(R)) = 2 \sum_{A \subset [k]} \binom{\prod_{i \in A} (q_i - 1)}{2} + \sum_{\{A,B\} \in D_1} \left( \prod_{i \in A} (q_i - 1) \right) \left( \prod_{j \in B} (q_j - 1) \right)$$
$$+ 2 \sum_{\{A,B\} \in D_2} \left( \prod_{i \in A} (q_i - 1) \right) \left( \prod_{j \in B} (q_j - 1) \right).$$

*Proof.* The proof follows from Lemma 2.1.9. □

Now we provide few examples to compute the Wiener index of the cozero-divisor graph of some finite reduced rings.

**Example 2.1.11.** Let $R = \mathbb{Z}_{pq} \cong \mathbb{Z}_p \times \mathbb{Z}_q$, where $p, q$ are distinct prime numbers. Then we have two distinct equivalence classes, $X_{\{1\}} = \{(a, 0) : a \in \mathbb{Z}_p \setminus \{0\}\}$ and $X_{\{2\}} = \{(0, b) : b \in \mathbb{Z}_q \setminus \{0\}\}$, of the equivalence relation $\equiv$. Moreover, $D_1 = \{\{\{1\}, \{2\}\}\}$ and $D_2 = \emptyset$. Note that $|X_{\{1\}}| = p - 1$ and $|X_{\{2\}}| = q - 1$. Consequently, by Theorem 2.1.10, we get

$W(\Gamma'(\mathbb{Z}_{pq})) = (p-1)(p-2)+(q-1)(q-2)+(p-1)(q-1) = p^2+q^2-4p-4q+pq+5.$

**Example 2.1.12.** Let $R = \mathbb{Z}_{pqr} \cong \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$, where $p, q, r$ are distinct prime numbers. For $a \in \mathbb{Z}_p \setminus \{0\}$, $b \in \mathbb{Z}_q \setminus \{0\}$ and $c \in \mathbb{Z}_q \setminus \{0\}$, we have the equivalence classes : $X_{\{1\}} = \{(a, 0, 0)\}$, $X_{\{2\}} = \{(0, b, 0)\}$, $X_{\{3\}} = \{(0, 0, c)\}$, $X_{\{1,2\}} = \{(a, b, 0)\}$, $X_{\{1,3\}} = \{(a, 0, c)\}$, $X_{\{2,3\}} = \{(0, b, c)\}$. Moreover,

$$D_1 = \Big\{\{\{1\},\{2\}\},\ \{\{1\},\{3\}\},\{\{2\},\{3\}\},\{\{1,2\},\{1,3\}\},\{\{1,2\},\{2,3\}\},$$
$$\{\{1,3\},\{2,3\}\},\ \{\{1\},\{2,3\}\},\ \{\{2\},\{1,3\}\},\{\{3\},\{1,2\}\}\Big\} \text{ and}$$
$$D_2 = \Big\{\{\{1\},\{1,2\}\},\{\{1\},\{1,3\}\},\{\{2\},\{1,2\}\},\ \{\{2\},\{2,3\}\},\{\{3\},\{1,3\}\},$$
$$\{\{3\},\{2,3\}\}\Big\}.$$

Also, $|X_{\{1\}}| = (p-1)$, $|X_{\{2\}}| = (q-1)$, $|X_{\{3\}}| = (r-1)$, $|X_{\{1,2\}}| = (p-1)(q-1)$, $|X_{\{1,3\}}| = (p-1)(r-1)$, $|X_{\{2,3\}}| = (q-1)(r-1)$. Then, by Theorem 2.1.10, the Wiener index of $\Gamma'(R)$ is given by

$$\begin{aligned} W(\Gamma'(\mathbb{Z}_{pqr})) &= 2\binom{p-1}{2} + 2\binom{q-1}{2} + 2\binom{r-1}{2} + 2\binom{(p-1)(q-1)}{2} \\ &+ 2\binom{(p-1)(r-1)}{2} + 2\binom{(q-1)(r-1)}{2} + (p-1)(q-1) \\ &+ (p-1)(r-1) + (q-1)(r-1) + (p-1)(q-1)(p-1)(r-1) \\ &+ (p-1)(q-1)(q-1)(r-1) + (p-1)(r-1)(q-1)(r-1) \\ &+ (p-1)(q-1)(r-1) + (q-1)(p-1)(r-1) + (r-1)(p-1)(q-1) \\ &+ 2(p-1)\left[(p-1)(q-1)\right] + 2(p-1)\left[(p-1)(r-1)\right] \\ &+ 2(q-1)\left[(p-1)(q-1)\right] + 2(q-1)\left[(q-1)(r-1)\right] \\ &+ 2(r-1)\left[(p-1)(r-1)\right] + 2(r-1)\left[(q-1)(r-1)\right]. \end{aligned}$$

simplifying this expression, we get

$$W(\Gamma'(\mathbb{Z}_{pqr})) = pqr(p+q+r-3) + p^2q^2 + p^2r^2 + q^2r^2 - p^2(q+r) - q^2(p+r) - r^2(p+q) - 2(pq+pr+qr) + 4(p+q+r) - 3.$$

### 2.1.3 The Wiener Index of $\Gamma'(\mathbb{Z}_n)$

In this subsection, we obtain the Wiener index of the cozero-divisor graph of the ring $\mathbb{Z}_n$ for arbitrary $n \in \mathbb{N}$. The Wiener index of the cozero-divisor graph of $\mathbb{Z}_n$ can be deduced from the Subsection 2.1.1. But we prefer to give the independent proof for the ring $\mathbb{Z}_n$ because it provides us a different approach as well as some more insight of the structure of $\Gamma'(\mathbb{Z}_n)$ to study various aspects of $\Gamma'(\mathbb{Z}_n)$ in the subsequent sections.

We begin with the structure of the cozero-divisor graph $\Gamma'(\mathbb{Z}_n)$. In view of Remark 1.2.4, we have the following result.

**Remark 2.1.13.** The sets $\mathcal{A}_{d_1}, \mathcal{A}_{d_2}, \ldots, \mathcal{A}_{d_k}$ forms a partition of the vertex set of the graph $\Gamma'(\mathbb{Z}_n)$. Thus, $V(\Gamma'(\mathbb{Z}_n)) = \mathcal{A}_{d_1} \cup \mathcal{A}_{d_2} \cup \cdots \cup \mathcal{A}_{d_k}$.

**Lemma 2.1.14.** *Let $x \in \mathcal{A}_{d_i}$, $y \in \mathcal{A}_{d_j}$, where $i, j \in \{1, 2, \ldots, \tau(n) - 2\}$. Then $x \sim y$ in $\Gamma'(\mathbb{Z}_n)$ if and only if $d_i \nmid d_j$ and $d_j \nmid d_i$.*

*Proof.* First note that in $\mathbb{Z}_n$, $x \in (y)$ if and only if $y \mid x$. Let $x \in \mathcal{A}_{d_i}$ and $y \in \mathcal{A}_{d_j}$ be two distinct vertices of $\Gamma'(\mathbb{Z}_n)$. Suppose that $x \sim y$ in $\Gamma'(\mathbb{Z}_n)$. Then $x \notin (y)$ and $y \notin (x)$. If $d_i \mid d_j$, then $d_j \in (d_i) = (x)$. It follows that $(y) = (d_j) \subseteq (x)$ and so $y \in (x)$, which is not possible. Similarly, if $d_j \mid d_i$, then we get $x \in (y)$, again a contradiction. Thus, neither $d_i \mid d_j$ nor $d_j \mid d_i$. Conversely, if $d_i \nmid d_j$ and $d_j \nmid d_i$ then we obtain $x \notin (y)$ and $y \notin (x)$. It follows that $x \sim y$. The result holds. $\square$

For distinct vertices $x, y$ of $\mathcal{A}_{d_i}$, by Remark 1.2.4, clearly $x \in (y)$ and $y \in (x)$. It follows that $x \nsim y$ in $\Gamma'(\mathbb{Z}_n)$. Using Lemma 1.2.5, we have the following corollary.

**Corollary 2.1.15.** *The following statements hold:*

(i) *For $i \in \{1, 2, \ldots, \tau(n) - 2\}$, the induced subgraph $\Gamma'(\mathcal{A}_{d_i})$ of $\Gamma'(\mathbb{Z}_n)$ is isomorphic to $\overline{K}_{\phi(\frac{n}{d_i})}$.*

(ii) *For* $i, j \in \{1, 2, \ldots, \tau(n) - 2\}$ *and* $i \neq j$*, a vertex of* $\mathcal{A}_{d_i}$ *is adjacent to either all or none of the vertices of* $\mathcal{A}_{d_j}$.

Thus, the partition $\mathcal{A}_{d_1}, \mathcal{A}_{d_2}, \ldots, \mathcal{A}_{d_{\tau(n)-2}}$ of $V(\Gamma'(\mathbb{Z}_n))$ is an equitable partition in such a way that every vertex of the $\mathcal{A}_{d_i}$ has equal number of neighbors in $\mathcal{A}_{d_j}$ for every $i, j \in \{1, 2, \ldots, \tau(n) - 2\}$.

We define $\Upsilon'_n$ by the simple undirected graph whose vertex set is the set of all proper divisors $d_1, d_2, \ldots, d_k$ of $n$ and two distinct vertices $d_i$ and $d_j$ are adjacent if and only if $d_i \nmid d_j$ and $d_j \nmid d_i$.

**Lemma 2.1.16.** *For a prime* $p$*, the graph* $\Upsilon'_n$ *is connected if and only if* $n \neq p^t$*, where* $t \geq 3$.

*Proof.* Suppose that $\Upsilon'_n$ is a connected graph and $V(\Upsilon'_n) = \{d_1, d_2, \ldots, d_k\}$. If $n = p^t$ for $t \geq 3$, then $V(\Upsilon'_{p^t}) = \{p, p^2, \ldots, p^{t-1}\}$. The definition of $\Upsilon'_n$ gives that $\Upsilon'_{p^t}$ is a null graph on $t - 1$ vertices. Thus, $\Upsilon'_n$ is not connected; a contradiction. Conversely, suppose that $n \neq p^t$, where $t \geq 3$. If $n = p^t$ for $t \in \{1, 2\}$, then there is nothing to prove because $\Upsilon'_p$ is an empty graph whereas $\Upsilon'_{p^2}$ is a graph with one vertex only. We may now suppose that $n = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$, where $p_i$'s are distinct primes and $m \geq 2$. Now let $d, d' \in V(\Upsilon'_n)$. If $d \nmid d'$ and $d' \nmid d$, then $d \sim d'$. Without loss of generality, assume that $d \mid d'$ with $d = p_1^{\beta_1} p_2^{\beta_2} \cdots p_m^{\beta_m}$ and $d' = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$. Note that $\alpha_i, \beta_i \in \mathbb{N} \cup \{0\}$ such that $\beta_i \leq \alpha_i$. Since $d'$ is a proper divisor of $n$ there exists $r \in \{1, 2, \ldots, m\}$, where $\alpha_r \leq n_r$, such that $p_r^{n_r} \nmid d'$ and $d' \nmid p_r^{n_r}$. Clearly, $p_r^{n_r} \nmid d$. If $d \nmid p_r^{n_r}$, then $d \sim p_r^{n_r} \sim d'$. If $d \mid p_r^{n_r}$, then there exists $s \in \{1, 2, \ldots, m\} \setminus \{r\}$ such that $d \nmid p_s$ and $p_s \nmid d$. Also, $p_r^{n_r} \nmid p_s$ and $p_s \nmid p_r^{n_r}$. It follows that $d' \sim p_r^{n_r} \sim p_s \sim d$. Hence, the graph $\Upsilon'_n$ is connected. $\square$

**Lemma 2.1.17.** $\Gamma'(\mathbb{Z}_n) = \Upsilon'_n[\Gamma'(\mathcal{A}_{d_1}), \Gamma'(\mathcal{A}_{d_2}), \ldots, \Gamma'(\mathcal{A}_{d_k})]$*, where* $d_1, d_2, \ldots, d_k$ *are all the proper divisors of* $n$.

*Proof.* Replace the vertex $d_i$ of $\Upsilon'_n$ by $\Gamma'(\mathcal{A}_{d_i})$ for $1 \leq i \leq k$. Consequently, the result can be obtained by using Lemma 2.1.14. $\square$

**Lemma 2.1.18.** *For a prime $p$, we have $\Gamma'(\mathbb{Z}_n)$ is connected if and only if either $n = 4$ or $n \neq p^t$, where $t \geq 2$.*

*Proof.* Suppose that $\Gamma'(\mathbb{Z}_n)$ is a connected graph and $n \neq 4$. If possible, let $n = p^t$ for $t \geq 2$, then note that $V(\Gamma'(\mathbb{Z}_n)) = \Gamma'(\mathcal{A}_p) \cup \Gamma'(\mathcal{A}_{p^2}) \cup \cdots \cup \Gamma'(\mathcal{A}_{p^{t-1}})$ and so $x \nsim y$ for any $x, y \in V(\Gamma'(\mathbb{Z}_n))$ (see Lemma 2.1.14 and Corollary 2.1.15). Consequently, $\Gamma'(\mathbb{Z}_n)$ is a null graph; a contradiction. Thus, $n \neq p^t$, where $t \geq 2$. Converse follows by the proof of Lemma 2.1.16 and Lemma 2.1.17. $\square$

**Example 2.1.19.** The cozero-divisor graph $\Gamma'(\mathbb{Z}_{30})$ is shown in Figure 2.1

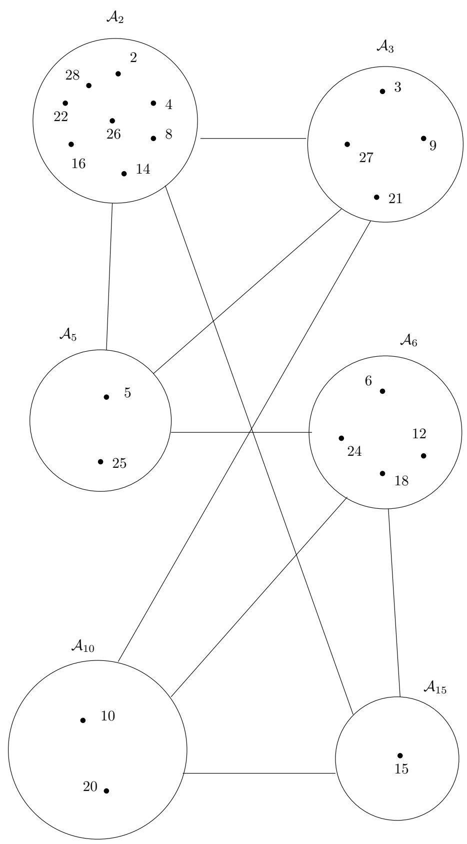



Figure 2.1: The graph $\Gamma'(\mathbb{Z}_{30})$

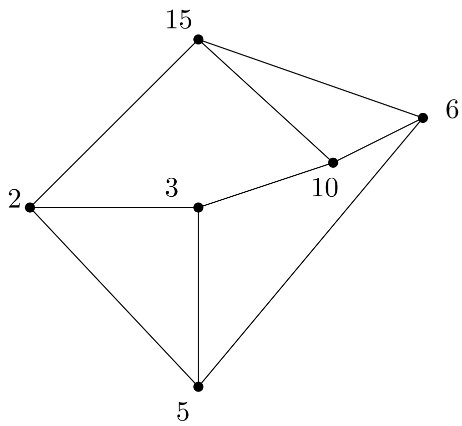



FIGURE 2.2: The graph $\Upsilon'_{30}$

By Lemma 2.1.17, note that $\Gamma'(\mathbb{Z}_{30}) = \Upsilon'_{30}[\Gamma'(\mathcal{A}_2), \Gamma'(\mathcal{A}_3), \Gamma'(\mathcal{A}_5), \Gamma'(\mathcal{A}_6), \Gamma'(\mathcal{A}_{10}), \Gamma'(\mathcal{A}_{15})]$, where $\Upsilon'_{30}$ is shown in Figure 2.2 and $\Gamma'(\mathcal{A}_2) = \overline{K}_8$, $\Gamma'(\mathcal{A}_3) = \overline{K}_4 = \Gamma'(\mathcal{A}_6)$, $\Gamma'(\mathcal{A}_5) = \overline{K}_2 = \Gamma'(\mathcal{A}_{10})$, $\Gamma'(\mathcal{A}_{15}) = \overline{K}_1$.

**Theorem 2.1.20.** *For $1 \le i \le \tau(n) - 2$, let $d_i$'s be the proper divisors of $n$. If $n = p_1p_2 \cdots p_k$, where $p_i$'s are distinct primes and $2 \le k \in \mathbb{N}$, then*

$$W(\Gamma'(\mathbb{Z}_n)) = \sum_{i=1}^{2^k-2} \phi(\frac{n}{d_i}) \left(\phi(\frac{n}{d_i}) - 1\right) \;+\; \frac{1}{2} \sum_{\substack{d_i \nmid d_j \\ d_j \nmid d_i}} \phi(\frac{n}{d_i})\phi(\frac{n}{d_j}) \;+\; 2 \sum_{\substack{d_i \mid d_j \\ i \neq j}} \phi(\frac{n}{d_i})\phi(\frac{n}{d_j}).$$

*Proof.* To determine the Wiener index of $\Gamma'(\mathbb{Z}_n)$, we first obtain the distances between the vertices of each $\mathcal{A}_{d_i}$ and two distinct $\mathcal{A}_{d_i}$'s, respectively. For a proper divisor $d_i$ of $n$, let $x, y \in \mathcal{A}_{d_i}$. Since $\Gamma'(\mathbb{Z}_n)$ is connected, by Corollary 2.1.15, there exists a proper divisor $d_r$ of $n$ such that $x \sim z$ for each $x \in \mathcal{A}_{d_i}$ and $z \in \mathcal{A}_{d_r}$. Consequently, $d(x, y) = 2$ for any two distinct $x, y \in \mathcal{A}_{d_i}$. Now we obtain the distances between the vertices of any two distinct $\mathcal{A}_{d_i}$'s through the following cases.

**Case-1.** Neither $d_i \mid d_j$ nor $d_j \mid d_i$. By Lemma 2.1.14, $d(x, y) = 1$ for every $x \in \mathcal{A}_{d_i}$ and $y \in \mathcal{A}_{d_j}$.

**Case-2.** $d_i \mid d_j$. For $x \in \mathcal{A}_{d_i}$ and $y \in \mathcal{A}_{d_j}$ we have $x \nsim y$. Without loss of generality, assume that $d_j = p_1p_2\cdots p_m d_i$, where $1 \leq m \leq k-2$. Since $d_j$ is a proper divisor of $n$ there exists a prime $p$ such that $p \nmid d_j$. Consequently, $p \nmid d_i$. It follows that for $x \in \mathcal{A}_{d_i}$ and $y \in \mathcal{A}_{d_j}$ there exists a $z \in \mathcal{A}_p$ such that $x \sim z \sim y$. Thus, $d(x,y) = 2$ for each $x \in \mathcal{A}_{d_i}$ and $y \in \mathcal{A}_{d_j}$.

Thus, in view of all the possible distances between the vertices of $\Gamma'(\mathbb{Z}_n)$, we get

$$
\begin{aligned}
W(\Gamma'(\mathbb{Z}_n)) &= \frac{1}{2}\sum_{u\in V(\Gamma'(\mathbb{Z}_n))}\sum_{v\in V(\Gamma'(\mathbb{Z}_n))} d(u,v)\\
&= \frac{1}{2}\left[\sum_{i=1}^{2^k-2} 2|\mathcal{A}_{d_i}|\left(|\mathcal{A}_{d_i}|-1\right) + \sum_{\substack{d_i\nmid d_j\\ d_j\nmid d_i}} |\mathcal{A}_{d_i}||\mathcal{A}_{d_j}|\right] + 2\sum_{\substack{d_i|d_j\\ i\neq j}} |\mathcal{A}_{d_i}||\mathcal{A}_{d_j}|\\
&= \sum_{i=1}^{2^k-2}\phi(\frac{n}{d_i})\left(\phi(\frac{n}{d_i})-1\right) + \frac{1}{2}\sum_{\substack{d_i\nmid d_j\\ d_j\nmid d_i}}\phi(\frac{n}{d_i})\phi(\frac{n}{d_j}) + 2\sum_{\substack{d_i|d_j\\ i\neq j}}\phi(\frac{n}{d_i})\phi(\frac{n}{d_j}).
\end{aligned}
$$

□

**Corollary 2.1.21.** *If $n = pq$, where $p, q$ are distinct primes, then*

$$W(\Gamma'(\mathbb{Z}_n)) = (p-1)(q-1) + (p-1)(p-2) + (q-1)(q-2).$$

**Theorem 2.1.22.** *Let $n = p_1^{n_1}p_2^{n_2}\cdots p_r^{n_r}\cdots p_k^{n_k}$ with $k \geq 2$, where $p_i$'s are distinct primes and let $D = \{d_1, d_2, \ldots, d_{\tau(n)-2}\}$ be the set of all proper divisors of $n$. For $d_i \mid d_j$, define*

$$
\begin{aligned}
A &= \{(d_i,d_j) \in D\times D \mid d_i \neq p_r^s\};\\
B &= \{(d_i,d_j) \in D\times D \mid d_i = p_r^s \text{ and } \frac{n}{d_j} \neq p_r^t\};\\
C &= \{(d_i,d_j) \in D\times D \mid d_i = p_r^s \text{ and } \frac{n}{d_j} = p_r^t\}.
\end{aligned}
$$

*Then*

$$W(\Gamma'(\mathbb{Z}_n)) = \sum_{i=1}^{\tau(n)-2} \phi(\frac{n}{d_i})\left(\phi(\frac{n}{d_i})-1\right) \ + \ \frac{1}{2}\sum_{\substack{d_i\nmid d_j\\ d_j\nmid d_i}} \phi(\frac{n}{d_i})\phi(\frac{n}{d_j})$$
$$+ 2\sum_{(d_i,d_j)\in A} \phi(\frac{n}{d_i})\phi(\frac{n}{d_j}) + \ 2\sum_{(d_i,d_j)\in B} \phi(\frac{n}{d_i})\phi(\frac{n}{d_j}) + 3\sum_{(d_i,d_j)\in C} \phi(\frac{n}{d_i})\phi(\frac{n}{d_j}).$$

*Proof.* In view of Remark 2.1.13, first we obtain all the possible distances between the vertices of $\mathcal{A}_{d_i}$ and $\mathcal{A}_{d_j}$, where $d_i$ and $d_j$ are proper divisors of $n$. If $i = j$, then by the proof of Theorem 2.1.20, we get $d(x, y) = 2$ for any two distinct $x, y \in \mathcal{A}_{d_i}$. Now suppose that $i \neq j$. If $d_i \nmid d_j$ and $d_j \nmid d_i$, then by Lemma 2.1.14, we get $d(x, y) = 1$ for every $x \in \mathcal{A}_{d_i}$ and $y \in \mathcal{A}_{d_j}$. If $d_i \mid d_j$, then we obtain the possible distances through the following cases.

**Case-1.** $(d_i, d_j) \in A$. Since $d_i \mid d_j$, we have $x \nsim y$ for any $x \in \mathcal{A}_{d_i}$ and $y \in \mathcal{A}_{d_j}$. Note that $d_i \neq p_r^s$ implies that $d_i = p_1^{\beta_1}p_2^{\beta_2}\cdots p_m^{\beta_m}$ for some $\beta_i$'s $\in \mathbb{N}\cup\{0\}$ and $m \geq 2$. Consequently, $d_j = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$ for some $\alpha_i$'s $\in \mathbb{N}\cup\{0\}$. Since $d_j$ is a proper divisor of $n$ there exists $l \in \{1, 2, \ldots, k\}$ such that $p_l^{n_l} \nmid d_j$. Also, $p_l^{n_l} \nmid d_i$. Further, $m \geq 2$ follows that $d_i \nmid p_l^{n_l}$ and $d_j \nmid p_l^{n_l}$. Now for any $x \in \mathcal{A}_{d_i}$, $y \in \mathcal{A}_{d_j}$ there exists a $z \in \mathcal{A}_{p_l^{n_l}}$ such that $x \sim z \sim y$. Thus $d(x, y) = 2$ for every $x \in \mathcal{A}_{d_i}$ and $y \in \mathcal{A}_{d_j}$.

**Case-2.** $d_i = p_r^s$ for some $r \in \{1, 2, \ldots, k\}$ and $1 \leq s \leq n_r$. Suppose $x \in \mathcal{A}_{d_i}$ and $y \in \mathcal{A}_{d_j}$. Then we obtain $d(x, y)$ in the following subcases:

**Subcase-2.1.** $(d_i, d_j) \in B$. Suppose $d_j = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}\cdots p_k^{\alpha_k}$. Since $\dfrac{n}{d_j} \neq p_r^t$ there exists a prime $p_m \in \{p_1, p_2, \ldots, p_k\} \setminus \{p_r\}$ and $\alpha_m \leq n_m$ such that $p_m^{n_m} \nmid d_j$. Consequently, $p_m^{n_m} \nmid d_i$. Moreover, $d_i \nmid p_m^{n_m}$ and $d_j \nmid p_m^{n_m}$. Thus, for every $x \in \mathcal{A}_{d_i}$ and $y \in \mathcal{A}_{d_j}$, we get $x \sim z \sim y$ for some $z \in \mathcal{A}_{p_m^{n_m}}$. Hence, $d(x, y) = 2$ for each $x \in \mathcal{A}_{d_i}$ and $y \in \mathcal{A}_{d_j}$.

**Subcase-2.2.** $(d_i, d_j) \in C$. Then $d_j = p_1^{n_1}p_2^{n_2}\cdots p_r^{\alpha_r}\cdots p_k^{n_k}$, where $n_r - \alpha_r = t \geq 1$. Since $d_i \mid d_j$, for each $x \in \mathcal{A}_{d_i}$ and $y \in \mathcal{A}_{d_j}$, we have $d(x, y) \geq 2$ (cf. Lemma

2.1.14). First, we show that $d(x,y) > 2$ for any $x \in \mathcal{A}_{d_i}$ and $y \in \mathcal{A}_{d_j}$. In this connection, it is sufficient to prove that for any proper divisor $d$ of $n$, we have either $d \mid d_j$ or $d_i \mid d$. Suppose that $d \nmid d_j$. Then $d = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_r^{\gamma_r} \cdots p_k^{\gamma_k}$ together with $\gamma_r > \alpha_r$. Since $p_r^s = d_i \mid d_j$, we get $s \leq \alpha_r < \gamma_r$. Consequently, $d_i \mid d$.

Since $n = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r} \cdots p_k^{n_k}$ with $k \geq 2$, there exists a prime $q \neq p_r$ such that $q \mid n$. Clearly, $q \nmid d_i$ and $d_i \nmid q$. Also, $p_r^{n_r} \nmid q$ and $q \nmid p_r^{n_r}$. Since $\alpha_r < n_r$, we obtain $d_j \nmid p_r^{n_r}$ and $p_r^{n_r} \nmid d_j$. Thus, in view of Lemma 2.1.14, for any $x \in \mathcal{A}_{d_i}$ and $y \in \mathcal{A}_{d_j}$, there exist $z \in \mathcal{A}_q$ and $w \in \mathcal{A}_{p_r^{n_r}}$ such that $x \sim z \sim w \sim y$. Hence, $d(x,y) = 3$ for every $x \in \mathcal{A}_{d_i}$ and $y \in \mathcal{A}_{d_j}$. In view of the cases and arguments discussed in this proof, we have

$$W(\Gamma'(\mathbb{Z}_n)) = \frac{1}{2} \sum_{u \in V(\Gamma'(\mathbb{Z}_n))} \sum_{v \in V(\Gamma'(\mathbb{Z}_n)} d(u,v)$$

$$= \frac{1}{2}\left[\sum_{i=1}^{\tau(n)-2} 2|\mathcal{A}_{d_i}|\,(|\mathcal{A}_{d_i}|-1) \;+\; \sum_{\substack{d_i \nmid d_j \\ d_j \nmid d_i}} |\mathcal{A}_{d_i}||\mathcal{A}_{d_j}|\right] \;+\; 2\sum_{(d_i,d_j)\in A} |\mathcal{A}_{d_i}||\mathcal{A}_{d_j}|$$
$$+\,2\sum_{(d_i,d_j)\in B} |\mathcal{A}_{d_i}||\mathcal{A}_{d_j}| \quad +\,3\sum_{(d_i,d_j)\in C} |\mathcal{A}_{d_i}||\mathcal{A}_{d_j}|$$
$$= \sum_{i=1}^{\tau(n)-2} \phi(\frac{n}{d_i})\left(\phi(\frac{n}{d_i})-1\right) \;+\; \frac{1}{2}\sum_{\substack{d_i \nmid d_j \\ d_j \nmid d_i}} \phi(\frac{n}{d_i})\phi(\frac{n}{d_j}) \;+\; 2\sum_{(d_i,d_j)\in A} \phi(\frac{n}{d_i})\phi(\frac{n}{d_j})$$
$$+\;2\sum_{(d_i,d_j)\in B} \phi(\frac{n}{d_i})\phi(\frac{n}{d_j}) \;+\; 3\sum_{(d_i,d_j)\in C} \phi(\frac{n}{d_i})\phi(\frac{n}{d_j}).$$

□

Based on all the possible distances obtained in this subsection, the following proposition is easy to observe.

**Proposition 2.1.23.** *The diameter of* $\Gamma'(\mathbb{Z}_n)$ *is given below:*

$$diam(\Gamma'(\mathbb{Z}_n)) = \begin{cases} 0 & n = 4, \\ 2 & n = p_1p_2\cdots p_k,\ k \geq 2 \\ 3 & \text{otherwise.} \end{cases}$$

Now we conclude this subsection with an illustration of Theorem 2.1.22 for the ring $\mathbb{Z}_{72}$.

**Example 2.1.24.** Consider $n = 2^3 \cdot 3^2 = 72$. Then the number of proper divisor $\tau(n)$ of $n$ is $\prod_{i=1}^{k}(n_i + 1) - 2 = 10$. Therefore, $D = \{2,\ 2^2,\ 2^3,\ 3,\ 3^2,\ 2 \cdot 3,\ 2^2 \cdot 3,\ 2^3 \cdot 3,\ 2 \cdot 3^2,\ 2^2 \cdot 3^2\}$. Let $d_1 = 2,\ d_2 = 2^2,\ d_3 = 2^3,\ d_4 = 3,\ d_5 = 3^2,\ d_6 = 2 \cdot 3,\ d_7 = 2^2 \cdot 3,\ d_8 = 2^3 \cdot 3,\ d_9 = 2 \cdot 3^2,\ d_{10} = 2^2 \cdot 3^2$. By Lemma 1.2.5, we obtain $|\mathcal{A}_{d_1}| = 12,\ |\mathcal{A}_{d_2}| = 6,\ |\mathcal{A}_{d_3}| = 6,\ |\mathcal{A}_{d_4}| = 8,\ |\mathcal{A}_{d_5}| = 4,\ |\mathcal{A}_{d_6}| = 4,\ |\mathcal{A}_{d_7}| = 2,\ |\mathcal{A}_{d_8}| = 2,\ |\mathcal{A}_{d_9}| = 2,\ |\mathcal{A}_{d_{10}}| = 1$. Now

$$\frac{1}{2}\sum_{i=1}^{10} 2|\mathcal{A}_{d_i}|\left(|\mathcal{A}_{d_i}| - 1\right) = [132 + 30 + 30 + 56 + 12 + 12 + 2 + 2 + 2 + 0] = 278$$

and

$$\begin{aligned}\frac{1}{2}\sum\nolimits_{\substack{d_i \nmid d_j \\ d_j \nmid d_i}} |\mathcal{A}_{d_i}||\mathcal{A}_{d_j}| = {} & [96 + 48 + 48 + 24 + 24 + 12 + 48 + 24 + 24 + 12 + 12 \\ & +6 + 16 + 8 + 8 + 4 + 4 + 2] = 420\end{aligned}$$

The sets $A,\ B$ and $C$ defined in Theorem 2.1.22 are

$A = \{(d_6, d_7),\ (d_6, d_8),\ (d_6, d_9),\ (d_6, d_{10}),\ (d_7, d_8),\ (d_7, d_{10}),\ (d_9, d_{10})\}$;

$B = \{(d_1, d_2),\ (d_1, d_3),\ (d_1, d_6),\ (d_1, d_7),\ (d_1, d_8),\ (d_2, d_3),\ (d_2, d_7),\ (d_2, d_8),\ (d_3, d_8),$
$(d_4, d_5),\ (d_4, d_6), (d_4, d_7), (d_4, d_9),\ (d_4, d_{10}),\ (d_5, d_9),\ (d_5, d_{10})\}$;

$C = \{(d_1, d_9),\ (d_1, d_{10}),\ (d_2, d_{10}),\ (d_4, d_8)\}$.

Consequently,

$$2\sum_{(d_i,d_j)\in A}|\mathcal{A}_{d_i}||\mathcal{A}_{d_j}| = 2[8+8+8+4+4+2+2] = 72$$

$$2\sum_{(d_i,d_j)\in B}|\mathcal{A}_{d_i}||\mathcal{A}_{d_j}| = 2[72+72+48+24+24+36+12+12+12+32+32$$
$$+16+16+8+8+4] = 856$$

$$3\sum_{(d_i,d_j)\in C}|\mathcal{A}_{d_i}||\mathcal{A}_{d_j}| = 3[24+12+6+16] = 174$$

Hence, the Wiener index of $\Gamma'(\mathbb{Z}_{72})$ is given by

$$W(\Gamma(\mathbb{Z}_{72})) = \frac{1}{2}\left[\sum_{i=1}^{\tau(n)-2} 2|\mathcal{A}_{d_i}|\big(|\mathcal{A}_{d_j}|-1\big) + \sum_{\substack{d_i\nmid d_j\\ d_j\nmid d_i}}|\mathcal{A}_{d_i}||\mathcal{A}_{d_j}|\right] + 2\sum_{(d_i,d_j)\in A}|\mathcal{A}_{d_i}||\mathcal{A}_{d_j}|$$
$$+ 2\sum_{(d_i,d_j)\in B}|\mathcal{A}_{d_i}||\mathcal{A}_{d_j}| + 3\sum_{(d_i,d_j)\in C}|\mathcal{A}_{d_i}||\mathcal{A}_{d_j}|$$
$$= 278+420+72+856+174 = 1800.$$

### 2.1.4 SageMath Code

In this subsection, we produce a SAGE code to compute the Wiener index of the cozero-divisor graph of ring classes considered in this chapter. On providing the value of integer $n$, the following SAGE code computes the Wiener index of the graph $\Gamma'(\mathbb{Z}_n)$.

```
cozero_divisor_graph=Graph()
E=[]
n=72

for i in range(n):
  for j in range(n):
          p=gcd(i,n)
```

```
            q=gcd(j,n)
            if (p%q!=0 and q%p!=0):
                E.append((i,j))

cozero_divisor_graph.add_edges(E)

if(E==[]):
    V=[]
    for i in range(1,n):
        if (gcd(i,n)!=1):
            V.append(i)
    cozero_divisor_graph.add_vertices(V)


W=cozero_divisor_graph.wiener_index();

if (W==oo):
    print("Wiener Index undefined for Null Graph")
else :

    print("Wiener Index:", W)
```

Using the given code, in the Table 2.1, we obtain the Wiener index of $\Gamma'(\mathbb{Z}_n)$ for some values of $n$.

| $n$ | 100 | 500 | 1000 | 1500 | 2000 | 2500 |
|---|---|---|---|---|---|---|
| $W(\Gamma'(\mathbb{Z}_n))$ | 2954 | 77174 | 306202 | 930248 | 1222530 | 1946274 |

Table 2.1: Wiener index of $\Gamma'(\mathbb{Z}_n)$

Let $R$ be a reduced ring i.e. $R \cong \mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \times \cdots \times \mathbb{F}_{q_n}$, where $\mathbb{F}_{q_i}$ is a field with $q_i$ elements. The following code determines the Wiener index of $\Gamma'(R)$ on providing the values of the field size $q_i$ $(1 \leq i \leq n)$.

```
field_orders=[3,5,7]
P=Subsets(range(len(field_orders)))[1:-1]
```

```
P=[Set(i) for i in P]

D1=[]
D2=[]
for i in P:
    for j in P:
        if (not(i.issubset(j) or j.issubset(i)) and P.index(i) > P.index(j)):
            D1.append([i,j])
        if (i.issubset(j) and i!=j):
            D2.append([i,j])

partial_sum=0
for i in P:
    sum_pp=1
    for j in i:
        sum_pp *= field_orders[j]-1
    partial_sum +=((sum_pp*(sum_pp-1))/2)

D1_sum=0
for i in D1:
    D1_pp=1
    for j in i[0]:
        D1_pp *= field_orders[j]-1
    for k in i[1]:
        D1_pp *= field_orders[k]-1
    D1_sum += D1_pp

D2_sum=0
for i in D2:
    D2_pp=1
    for j in i[0]:
        D2_pp *= field_orders[j]-1
    for k in i[1]:
        D2_pp *= field_orders[k]-1
    D2_sum += D2_pp

W = 2*partial_sum + D1_sum + 2*D2_sum
print("Wiener Index:" , W)
```

Using the given code, in the following tables, we obtain the Wiener index of the cozero-divisor graphs of the reduced rings $\mathbb{F}_{q_1} \times \mathbb{F}_{q_2}$ (see Table 2.2) and

$\mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \times \mathbb{F}_{q_3}$ (see Table 2.3), respectively.

| $(q_1, q_2)$ | $(9, 25)$ | $(49, 81)$ | $(101, 121)$ | $(125, 139)$ | $(163, 169)$ | $(289, 343)$ |
|---|---|---|---|---|---|---|
| $W(\Gamma'(\mathbb{F}_{q_1} \times \mathbb{F}_{q_2}))$ | 800 | 12416 | 36180 | 51270 | 81354 | 297774 |

Table 2.2: Wiener index of $\Gamma'(\mathbb{F}_{q_1} \times \mathbb{F}_{q_2})$

| $(q_1, q_2, q_3)$ | $(7, 8, 13)$ | $(9, 25, 49)$ | $(53, 64, 81)$ | $(83, 101, 121)$ | $(125, 131, 169)$ | $(289, 343, 361)$ |
|---|---|---|---|---|---|---|
| $W(\Gamma'(\mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \times \mathbb{F}_{q_3}))$ | 35196 | 2500400 | 108637254 | 620456582 | 2355211790 | 71251552134 |

Table 2.3: Wiener index of $\Gamma'(\mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \times \mathbb{F}_{q_3})$

Let $R \cong \mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \cdots \times \mathbb{Z}_{p_k^{m_k}}$. Then the following SAGE code gives the value of $W(\Gamma'(R))$ after providing the values of $p_i^{m_i}(1 \leq i \leq k)$, where each $p_i$ is a prime number.

```
orders = [2,4,9]
A = cartesian_product([range(i) for i in orders]).list()
units = [{i for i in range(1,j) if gcd(i,j) == 1} for j in orders]

def contQ(lst1, lst2):
    flag = True
    for i in range(len(orders)):
        p=gcd(lst1[i],orders[i])
        q=gcd(lst2[i],orders[i])
        if(not(lst1[i]==0 or {lst2[i]}.issubset(units[i]) or p%q==0)):
            flag = False
    return flag

E=[]
for i in A:
    for j in A:
        if(not(contQ(i,j) or contQ(j,i))and A.index(i) > A.index(j)):
            E.append([i,j])
```

```
G = Graph()
G.add_edges(E)
W=G.wiener_index()
print("Wiener_Index:", W)
```

Using the given code, we obtain the Wiener index of the cozero-divisor graph of the ring $R \cong \mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \cdots \times \mathbb{Z}_{p_k^{m_k}}$ (see Table 2.4).

| $R$ | $W(\Gamma'(R))$ |
|---|---|
| $\mathbb{Z}_4 \times \mathbb{Z}_9$ | 420 |
| $\mathbb{Z}_9 \times \mathbb{Z}_{25}$ | 8808 |
| $\mathbb{Z}_{16} \times \mathbb{Z}_{25}$ | 48870 |
| $\mathbb{Z}_{27} \times \mathbb{Z}_{49}$ | 268022 |
| $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ | 521 |
| $\mathbb{Z}_5 \times \mathbb{Z}_7 \times \mathbb{Z}_{11}$ | 14948 |
| $\mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_{16}$ | 167769 |
| $\mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_{25}$ | 327394 |
| $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_9$ | 232937 |
| $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \mathbb{Z}_8$ | 333963 |

Table 2.4: Wiener index of $\Gamma'(R)$

## 2.2 Laplacian Spectrum of $\Gamma'(\mathbb{Z}_n)$

In this section, we investigated the Laplacian spectrum of the $\Gamma'(\mathbb{Z}_n)$ for various $n$. Consider $d_1, d_2, \ldots, d_k$ as all the proper divisors of $n$. For $1 \le i \le k$, we give the weight $\phi(\frac{n}{d_i}) = |\mathcal{A}_{d_i}|$ to the vertex $d_i$ of the graph $\Upsilon'_n$. Define the integer

$$D_{d_j} = \sum_{d_i \in N_{\Upsilon'_n}(d_j)} \phi(\tfrac{n}{d_i})$$

The $k \times k$ weighted Laplacian matrix $L(\Upsilon'_n)$ of $\Upsilon'_n$ defined in Theorem 1.3.17 is given by

$$L(\Upsilon'_n) = \begin{bmatrix} D_{d_1} & -l_{1,2} & \cdots & -l_{1,k} \\ -l_{2,1} & D_{d_2} & \cdots & -l_{2,k} \\ \cdots & \cdots & \cdots & \cdots \\ -l_{k,1} & -l_{k,2} & \cdots & D_{d_k} \end{bmatrix} \tag{2.1}$$

where

$$l_{i,j} = \begin{cases} \phi(\frac{n}{d_j}) & \text{if } d_i \sim d_j \text{ in } \Upsilon'_n; \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 2.2.1.** *The Laplacian spectrum of* $\Gamma'(\mathbb{Z}_n)$ *is given by*

$$\Phi_L(\Gamma'(\mathbb{Z}_n)) = \bigcup_{i=1}^{k} (D_{d_i} + (\Phi_L(\Gamma'(\mathcal{A}_{d_i})) \setminus \{0\})) \bigcup \Phi(L(\Upsilon'_n)),$$

*where* $D_{d_i} + (\Phi_L(\Gamma'(\mathcal{A}_{d_i})) \setminus \{0\})$ *represents that* $D_{d_i}$ *is added to each element of the multiset* $(\Phi_L(\Gamma'(\mathcal{A}_{d_i})) \setminus \{0\})$.

*Proof.* By Lemma 2.1.17, $\Gamma'(\mathbb{Z}_n) = \Upsilon'_n[\Gamma'(\mathcal{A}_{d_1}), \Gamma'(\mathcal{A}_{d_2}), \ldots, \Gamma'(\mathcal{A}_{d_k})]$. Consequently, by Theorem 1.3.17 and Remark 1.3.18, the result holds. $\square$

If $n = p^t$, where $t > 1$, then the graph $\Gamma'(\mathbb{Z}_n)$ is a null graph. Let $n \neq p^t$ for any $t \in \mathbb{N}$. Then by Lemma 2.1.16, $\Upsilon'_n$ is connected graph so that $D_{d_i} > 0$. By Theorem 2.2.1, out of $n - \phi(n) - 1$ Laplacian eigenvalues of $\Gamma'(\mathbb{Z}_n)$ note that $n - \phi(n) - 1 - k$ eigenvalues are non-zero integers. The remaining $k$ Laplacian eigenvalues of $\Gamma'(\mathbb{Z}_n)$ are the roots of the characteristic equation of the matrix $L(\Upsilon'_n)$ given in equation (2.1).

**Lemma 2.2.2.** *Let* $n = pq$ *be a product of two distinct primes. Then the Laplacian spectrum of* $\Gamma'(\mathbb{Z}_n)$ *is given by*

$$\begin{pmatrix} 0 & p+q-2 & p-1 & q-1 \\ 1 & 1 & q-2 & p-2 \end{pmatrix}.$$

*Proof.* By Lemma 2.1.17, we have $\Gamma'(\mathbb{Z}_{pq}) = \Upsilon'_{pq}[\Gamma'(\mathcal{A}_p), \Gamma'(\mathcal{A}_q)]$, where $\Upsilon'_{pq} = K_2$, $\Gamma'(\mathcal{A}_p) = \overline{K}_{\phi(q)}$ and $\Gamma'(\mathcal{A}_q) = \overline{K}_{\phi(p)}$ (cf. Lemma 1.2.5 and Corollary 2.1.15). Consequently, by Theorem 2.2.1, the Laplacian spectrum of $\Gamma'(\mathbb{Z}_{pq})$ is

$$\begin{aligned}\Phi_L(\Gamma'(\mathbb{Z}_{pq})) &= (D_p + (\Phi_L(\Gamma'(\mathcal{A}_p)) \setminus \{0\})) \bigcup (D_q + (\Phi_L(\Gamma'(\mathcal{A}_q)) \setminus \{0\})) \bigcup \Phi(L(\Upsilon'_{pq})) \\ &= \begin{pmatrix} p-1 & q-1 \\ q-2 & p-2 \end{pmatrix} \bigcup \Phi(L(\Upsilon'_{pq})).\end{aligned}$$

Then the matrix

$$L(\Upsilon'_{pq}) = \begin{bmatrix} p-1 & -(p-1) \\ -(q-1) & q-1 \end{bmatrix}$$

has eigenvalues $p+q-2$ and 0. Thus, we have the result. □

*Notation* 2.2.3. $(\lambda_i)^{[\mu_i]}$ denotes the eigenvalue $\lambda_i$ of $\mathcal{L}(\Gamma'(\mathbb{Z}_n))$ with multiplicity $\mu_i$.

**Lemma 2.2.4.** *For distinct primes $p$ and $q$, if $n = p^2q$ then the Laplacian eigenvalues of $\Gamma'(\mathbb{Z}_n)$ consists of the set*

$$\left\{(p^2-p)^{[(p-1)(q-1)-1]}, (pq-p)^{[p^2-p-1]}, (p^2-1)^{[q-2]}, (q-1)^{[p-2]}\right\}$$

*and the remaining eigenvalues are the roots of the characteristic polynomial*

$x^4 - \{(p-1)(2p+1) + (p+1)(q-1)\}x^3 + \{p(p-1)^2(p+1) + (p-1)(p+1)^2(q-1) + p(q-1)^2 + (p-1)^2(q-1)\}x^2 - p(p-1)(q-1)\{(p-1)(p+1) + p(q-1)\}x.$

*Proof.* First note that $\Upsilon'_{p^2q}$ is the path graph given by $p \sim q \sim p^2 \sim pq$. By Lemma 2.1.17,

$$\Gamma'(\mathbb{Z}_{p^2q}) = \Upsilon'_{p^2q}[\Gamma'(\mathcal{A}_p), \Gamma'(\mathcal{A}_q), \Gamma'(\mathcal{A}_{p^2}), \Gamma'(\mathcal{A}_{pq})],$$

where $\Gamma'(\mathcal{A}_p) = \overline{K}_{\phi(pq)}$, $\Gamma'(\mathcal{A}_q) = \overline{K}_{\phi(p^2)}$, $\Gamma'(\mathcal{A}_{p^2}) = \overline{K}_{\phi(q)}$ and $\Gamma'(\mathcal{A}_{pq}) = \overline{K}_{\phi(p)}$. It follows that $D_p = \phi(p^2) = p^2 - p$ and $D_q = \phi(pq) + \phi(q) = p(q-1)$, $D_{p^2} =$

$\phi(p^2) + \phi(p) = p^2 - 1$ and $D_{pq} = \phi(q) = q - 1$. Therefore, by Theorem 2.2.1, the Laplacian spectrum of $\Gamma'(\mathbb{Z}_{p^2q})$ is

$$\begin{aligned}
\Phi_L(\Gamma'(\mathbb{Z}_{p^2q})) &= (D_p + (\Phi_L(\Gamma'(\mathcal{A}_p)) \setminus \{0\})) \bigcup (D_q + (\Phi_L(\Gamma'(\mathcal{A}_q)) \setminus \{0\})) \\
&\bigcup (D_{p^2} + (\Phi_L(\Gamma'(\mathcal{A}_{p^2})) \setminus \{0\})) \bigcup (D_{pq} + (\Phi_L(\Gamma'(\mathcal{A}_{pq})) \setminus \{0\})) \\
&\bigcup \Phi(L(\Upsilon'_{p^2q})) \\
&= \begin{pmatrix} p^2 - p & pq - p & p^2 - 1 & q - 1 \\ (p-1)(q-1) - 1 & p^2 - p - 1 & q - 2 & p - 2 \end{pmatrix} \bigcup \Phi(L(\Upsilon'_{p^2q})).
\end{aligned}$$

Thus, the remaining Laplacian eigenvalues can be obtained by the characteristic polynomial (given in the statement) of the matrix

$$L(\Upsilon'_{p^2q}) = \begin{bmatrix} p^2 - p & -p^2 + p & 0 & 0 \\ -(p-1)(q-1) & p(q-1) & -(q-1) & 0 \\ 0 & -p^2 + p & p^2 - 1 & -p + 1 \\ 0 & 0 & -q + 1 & q - 1 \end{bmatrix}.$$

□

**Lemma 2.2.5.** *For distinct primes $p$ and $q$, if $n = p^{n_1}q$ then the Laplacian eigenvalues of $\Gamma'(\mathbb{Z}_n)$ consists of the set*

$$\{(\phi(p^{n_1}))^{[\phi(p^{n_1-1}q)-1]}, (\phi(p^{n_1}) + \phi(p^{n_1-1}))^{[\phi(p^{n_1-2}q)-1]},$$
$$\left(\sum_{i=0}^{2} \phi(p^{n_1-i})\right)^{[\phi(p^{n_1-3}q)-1]}, \ldots, \left(\sum_{i=0}^{n_1-1} \phi(p^{n_1-i})\right)^{[\phi(p^{n_1-n_1}q)-1]},$$
$$\left(\sum_{i=1}^{n_1} \phi(p^{n_1-i}q)\right)^{[\phi(p^{n_1})-1]}, \left(\sum_{i=2}^{n_1} \phi(p^{n_1-i}q)\right)^{[\phi(p^{n_1-1})-1]}, \ldots, (\phi(q))^{[\phi(p)-1]}\}.$$

*and the remaining eigenvalues are the eigenvalues of the matrix given in equation* (2.1).

*Proof.* Note that $\{p, p^2, \ldots, p^{n_1}, q, pq, p^2q, \ldots, p^{n_1-1}q\}$ is the vertex set of the graph $\Upsilon'_{p^{n_1}q}$. By Lemma 2.1.17, note that $\Gamma'(\mathbb{Z}_{p^{n_1}q})$ equals to

$\Upsilon'_{p^{n_1}q}[\Gamma'(\mathcal{A}_p), \Gamma'(\mathcal{A}_{p^2}), \ldots, \Gamma'(\mathcal{A}_{p^{n_1}}), \Gamma'(\mathcal{A}_q), \Gamma'(\mathcal{A}_{pq}), \Gamma'(\mathcal{A}_{p^2q}), \ldots, \Gamma'(\mathcal{A}_{p^{n_1-1}q})]$,

where $\Gamma'(\mathcal{A}_p) = \overline{K}_{\phi(p^{n_1-1}q)}$, $\Gamma'(\mathcal{A}_{p^2}) = \overline{K}_{\phi(p^{n_1-2}q)}$, …, $\Gamma'(\mathcal{A}_{p^{n_1}}) = \overline{K}_{\phi(q)}$, $\Gamma'(\mathcal{A}_q) = \overline{K}_{\phi(p^{n_1})}$, $\Gamma'(\mathcal{A}_{pq}) = \overline{K}_{\phi(p^{n_1-1})}$, …, $\Gamma'(\mathcal{A}_{p^{n_1-1}q}) = \overline{K}_{\phi(p)}$. It follows that $D_p = \phi(p^{n_1})$, $D_{p^2} = \phi(p^{n_1}) + \phi(p^{n_1-1})$, …, $D_{p^{n_1}} = \sum\limits_{i=0}^{n_1-1} \phi(p^{n_1-i})$, $D_q = \sum\limits_{i=1}^{n_1} \phi(p^{n_1-i}q)$, $D_{pq} = \sum\limits_{i=2}^{n_1} \phi(p^{n_1-i}q)$, …, $D_{p^{n_1-1}q} = \phi(q)$. Consequently, by Theorem 2.2.1, the Laplacian spectrum of $\Gamma'(\mathbb{Z}_{p^{n_1}q})$ is

$$\begin{aligned}\Phi_L(\Gamma'(\mathbb{Z}_{p^{n_1}q})) = (D_p + (\Phi_L(\Gamma'(\mathcal{A}_p)) \setminus \{0\})) \bigcup (D_{p^2} + (\Phi_L(\Gamma'(\mathcal{A}_{p^2})) \setminus \{0\})) \bigcup \cdots \\ \bigcup (D_{p^{n_1}} + (\Phi_L(\Gamma'(\mathcal{A}_{p^{n_1}})) \setminus \{0\})) \bigcup (D_q + (\Phi_L(\Gamma'(\mathcal{A}_q)) \setminus \{0\})) \\ \bigcup (D_{pq} + (\Phi_L(\Gamma'(\mathcal{A}_{pq})) \setminus \{0\})) \bigcup \cdots \bigcup (D_{p^{n_1-1}q} \\ + (\Phi_L(\Gamma'(\mathcal{A}_{p^{n_1-1}q})) \setminus \{0\})) \bigcup \Phi(L(\Upsilon'_{p^{n_1}q})).\end{aligned}$$

$$= \begin{pmatrix} \phi(p^{n_1}) & \cdots & \sum\limits_{i=0}^{n_1-1} \phi(p^{n_1-i}) & \sum\limits_{i=1}^{n_1} \phi(p^{n_1-i}q) & \sum\limits_{i=2}^{n_1} \phi(p^{n_1-i}q) & \cdots & \phi(q) \\ \phi(p^{n_1-1}q) - 1 & \cdots & \phi(p^{n_1-n_1}q) - 1 & \phi(p^{n_1}) - 1 & \phi(p^{n_1}) - 1 & \cdots & \phi(p) - 1 \end{pmatrix}$$
$\bigcup \Phi(L(\Upsilon'_{p^{n_1}q}))$.

Thus, the remaining $2n_1$ Laplacian eigenvalues are the eigenvalues of the matrix $L(\Upsilon'_{p^{n_1}q}) =$

$$\begin{bmatrix} \phi(p^{n_1}) & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 & -\phi(p^{n_1}) \\ 0 & \sum\limits_{i=0}^{1} \phi(p^{n_1-i}) & 0 & \cdots & \cdots & 0 & \cdots & -\phi(p^{n_1-1}) & -\phi(p^{n_1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \sum\limits_{i=0}^{n_1-1} \phi(p^{n_1-i}) & -\phi(p) & -\phi(p^2) & \cdots & -\phi(p^{n_1}) \\ 0 & 0 & 0 & \cdots & -\phi(q) & \phi(q) & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & -\phi(pq) & -\phi(q) & 0 & \phi(pq) + \phi(q) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\phi(p^{n_1-1}q) & -\phi(p^{n_1-2}q) & \cdots & \cdots & -\phi(q) & 0 & 0 & \cdots & \sum\limits_{i=1}^{n_1} \phi(p^{n_1-i}q) \end{bmatrix}$$

where matrix $L(\Upsilon'_{p^{n_1}q})$ is obtained by indexing the rows and columns as $p, p^2, \ldots, p^{n_1}$, $p^{n_1-1}q, \ldots, pq, q$. □

**Theorem 2.2.6.** *If $n = p^{n_1}q^{n_2}$, where $p$ and $q$ are distinct primes, then the set of*

*Laplacian eigenvalues of* $\Gamma'(\mathbb{Z}_n)$ *consists of*

$$\left(\sum_{i=1}^{n_2}\phi(p^{n_1}q^{n_2-i})\right)^{[\phi(p^{n_1-1}q^{n_2})-1]}, \quad \left(\sum_{i=1}^{n_2}\phi(p^{n_1}q^{n_2-i})+\sum_{i=1}^{n_2}\phi(p^{n_1-1}q^{n_2-i})\right)^{[\phi(p^{n_1-2}q^{n_2})-1]},$$

$$\left(\sum_{i=1}^{n_2}\phi(p^{n_1}q^{n_2-i})+\sum_{i=1}^{n_2}\phi(p^{n_1-1}q^{n_2-i})+\sum_{i=1}^{n_2}\phi(p^{n_1-2}q^{n_2-i})\right)^{[\phi(p^{n_1-3}q^{n_2})-1]},$$

$$\vdots$$

$$\left(\sum_{i=1}^{n_2}\phi(p^{n_1}q^{n_2-i})+\sum_{i=1}^{n_2}\phi(p^{n_1-1}q^{n_2-i})+\cdots+\sum_{i=1}^{n_2}\phi(pq^{n_2-i})\right)^{[\phi(q^{n_2})-1]},$$

$$\left(\sum_{i=1}^{n_1}\phi(p^{n_1-i}q^{n_2})\right)^{[\phi(p^{n_1}q^{n_2-1})-1]}, \left(\sum_{i=1}^{n_1}\phi(p^{n_1-i}q^{n_2})+\sum_{i=1}^{n_1}\phi(p^{n_1-i}q^{n_2-1})\right)^{[\phi(p^{n_1}q^{n_2-2})-1]},$$

$$\left(\sum_{i=1}^{n_1}\phi(p^{n_1-i}q^{n_2})+\sum_{i=1}^{n_1}\phi(p^{n_1-i}q^{n_2-1})+\sum_{i=1}^{n_1}\phi(p^{n_1-i}q^{n_2-2})\right)^{[\phi(p^{n_1}q^{n_2-3})-1]},$$

$$\vdots$$

$$\left(\sum_{i=1}^{n_1}\phi(p^{n_1-i}q^{n_2})+\sum_{i=1}^{n_1}\phi(p^{n_1-i}q^{n_2-1})+\cdots+\sum_{i=1}^{n_1}\phi(p^{n_1-i}q)\right)^{[\phi(p^{n_1})-1]},$$

$$\left(\sum_{i=2}^{n_1}\phi(p^{n_1-i}q^{n_2})+\sum_{i=2}^{n_2}\phi(p^{n_1}q^{n_2-i})\right)^{[\phi(p^{n_1-1}q^{n_2-1})-1]},$$

$$\left(\sum_{i=3}^{n_1}\phi(p^{n_1-i}q^{n_2})+\sum_{i=2}^{n_2}\phi(p^{n_1}q^{n_2-i})+\sum_{i=1}^{n_2}\phi(p^{n_1-1}q^{n_2-i})\right)^{[\phi(p^{n_1-2}q^{n_2-1})-1]},$$

$$\vdots$$

$$\left(\sum_{i=2}^{n_2}\phi(p^{n_1}q^{n_2-i})+\sum_{i=1}^{n_1}\phi(p^{n_1-i}q^{n_2-2})+\sum_{i=1}^{n_1}\phi(p^{n_1-i}q^{n_2-3})+\cdots+\sum_{i=1}^{n_1-1}\phi(p^{n_1-i})\right)^{[\phi(q^{n_2-1})-1]},$$

$$\left(\sum_{i=2}^{n_1}\phi(p^{n_1-i}q^{n_2})+\sum_{i=3}^{n_2}\phi(p^{n_1}q^{n_2-i})+\sum_{i=2}^{n_1}\phi(p^{n_1-i}q^{n_2-1})\right)^{[\phi(p^{n_1-1}q^{n_2-2})-1]},$$

$$\vdots$$

$$\left(\sum_{i=2}^{n_1}\phi(p^{n_1-i}q^{n_2})+\sum_{i=1}^{n_2}\phi(p^{n_1-2}q^{n_2-i})+\sum_{i=1}^{n_2}\phi(p^{n_1-3}q^{n_2-i})+\cdots+\sum_{i=1}^{n_2-1}\phi(q^{n_2-i})\right)^{[\phi(p^{n_1-1})-1]},$$

$$\vdots$$

$$\left(\sum_{i=1}^{n_2-1}\phi(q^{n_2-i})+\phi(q^{n_2})\right)^{[\phi(p)-1]}$$

*and the remaining* $(n_1+1)(n_2+1)-2$ *eigenvalues are given by the zeros of the characteristic polynomial of the matrix given in equation* (2.1).

*Proof.* The set of proper divisors of $n = p^{n_1}q^{n_2}$ is

$$\{p, p^2, \ldots, p^{n_1}, q, q^2, \ldots, q^{n_2}, pq, p^2q, \ldots, \; p^{n_1}q, pq^2, p^2q^2, \ldots, p^{n_1}q^2, \\ \ldots, pq^{n_2}, p^2q^{n_2}, \ldots, p^{n_1-1}q^{n_2}\}.$$

By the definition of $\Upsilon'_n$, note that

- $p^i \sim q^j$ for all $i, j$.
- $p^i \sim p^{i_1}q^{j_1}$ for $i > i_1$ and $j_1 > 0$.
- $q^j \sim p^i q^{j_1}$ for $j > j_1$ and $i > 0$.
- If either $i_1 > i_2$, $j_1 < j_2$ or $j_1 > j_2$, $i_1 < i_2$, then $p^{i_1}q^{j_1} \sim p^{i_2}q^{j_2}$ .

In view of Lemma 2.1.17,

$$\Gamma'(\mathbb{Z}_{p^{n_1}q^{n_2}}) = \Upsilon'_{p^{n_1}q^{n_2}}[\Gamma'(\mathcal{A}_p), \Gamma'(\mathcal{A}_{p^2}), \ldots, \Gamma'(\mathcal{A}_{p^{n_1}}), \Gamma'(\mathcal{A}_q), \\ \Gamma'(\mathcal{A}_{q^2}), \ldots, \Gamma'(\mathcal{A}_{q^{n_2}}), \Gamma'(\mathcal{A}_{pq}), \Gamma'(\mathcal{A}_{p^2q}), \ldots, \Gamma'(\mathcal{A}_{p^{n_1}q}), \ldots, \Gamma'(\mathcal{A}_{pq^{n_2}}), \ldots, \Gamma'(\mathcal{A}_{p^{n_1-1}q^{n_2}})].$$

Therefore, by Lemma 1.2.5 and Corollary 2.1.15, we get

$$\Gamma'(\mathcal{A}_{p^i}) = \overline{K}_{\phi(p^{n_1-i}q^{n_2})}, \text{ where } 1 \leq i \leq n_1,$$

$$\Gamma'(\mathcal{A}_{q^j}) = \overline{K}_{\phi(p^{n_1}q^{n_2-j})} \text{ where } 1 \leq j \leq n_2,$$

$$\Gamma'(\mathcal{A}_{p^iq^j}) = \overline{K}_{\phi(p^{n_1-i}q^{n_2-j})}.$$

Consequently, we have

$$D_p = \sum_{i=1}^{n_2} \phi(p^{n_1}q^{n_2-i}), \; D_{p^2} = \sum_{i=1}^{n_2} \phi(p^{n_1}q^{n_2-i}) + \sum_{i=1}^{n_2} \phi(p^{n_1-1}q^{n_2-i}),$$
$$\vdots$$
$$D_{p^{n_1}} = \sum_{i=1}^{n_2} \phi(p^{n_1}q^{n_2-i}) + \sum_{i=1}^{n_2} \phi(p^{n_1-1}q^{n_2-i}) + \cdots + \sum_{i=1}^{n_2} \phi(pq^{n_2-i}),$$
$$D_q = \sum_{i=1}^{n_1} \phi(p^{n_1-i}q^{n_2}),$$
$$\vdots$$
$$D_{q^{n_2}} = \sum_{i=1}^{n_1} \phi(p^{n_1-i}q^{n_2}) + \sum_{i=1}^{n_1} \phi(p^{n_1-i}q^{n_2-1}) + \cdots + \sum_{i=1}^{n_1} \phi(p^{n_1-i}q),$$
$$D_{pq} = \sum_{i=2}^{n_1} \phi(p^{n_1-i}q^{n_2}) + \sum_{i=2}^{n_2} \phi(p^{n_1}q^{n_2-i}),$$

$$\vdots$$

$$D_{p^{n_1}q} = \sum_{i=2}^{n_2} \phi(p^{n_1}q^{n_2-i}) + \sum_{i=1}^{n_1} \phi(p^{n_1-i}q^{n_2-2}) + \sum_{i=1}^{n_1} \phi(p^{n_1-i}q^{n_2-3}) + \cdots + \sum_{i=1}^{n_1-1} \phi(p^{n_1-i}),$$

$$D_{pq^2} = \sum_{i=2}^{n_1} \phi(p^{n_1-i}q^{n_2}) + \sum_{i=3}^{n_2} \phi(p^{n_1}q^{n_2-i}) + \sum_{i=2}^{n_1} \phi(p^{n_1-i}q^{n_2-1}),$$

$$\vdots$$

$$D_{pq^{n_2}} = \sum_{i=2}^{n_1} \phi(p^{n_1-i}q^{n_2}) + \sum_{i=1}^{n_2} \phi(p^{n_1-2}q^{n_2-i}) + \sum_{i=1}^{n_2} \phi(p^{n_1-3}q^{n_2-i}) + \cdots + \sum_{i=1}^{n_2-1} \phi(q^{n_2-i}),$$

$$\vdots$$

$$D_{p^{n_1-1}q^{n_2}} = \sum_{i=1}^{n_2-1} \phi(q^{n_2-i}) + \phi(q^{n_2}).$$

Therefore, by Theorem 2.2.1, the Laplacian spectrum of $\Gamma'(\mathbb{Z}_{p^{n_1}q^{n_2}})$ is

$$\begin{aligned}
\Phi_L(\Gamma'(\mathbb{Z}_{p^{n_1}q^{n_2}})) = {} & (D_p + (\Phi_L(\Gamma'(\mathcal{A}_p)) \setminus \{0\})) \bigcup (D_{p^2} + (\Phi_L(\Gamma'(\mathcal{A}_{p^2})) \setminus \{0\})) \bigcup \\
& \cdots \bigcup (D_{p^{n_1}} + (\Phi_L(\Gamma'(\mathcal{A}_{p^{n_1}})) \setminus \{0\})) \bigcup (D_q + (\Phi_L(\Gamma'(\mathcal{A}_q)) \setminus \{0\})) \\
& \bigcup (D_{q^2} + (\Phi_L(\Gamma'(\mathcal{A}_{q^2})) \setminus \{0\})) \bigcup \cdots \bigcup (D_{q^{n_2}} + (\Phi_L(\Gamma'(\mathcal{A}_{q^{n_2}})) \setminus \{0\})) \\
& \bigcup (D_{pq} + (\Phi_L(\Gamma'(\mathcal{A}_{pq})) \setminus \{0\})) \bigcup \cdots \bigcup (D_{p^{n_1}q} + (\Phi_L(\Gamma'(\mathcal{A}_{p^{n_1}q})) \setminus \{0\})) \\
& \bigcup \cdots \bigcup (D_{pq^{n_2}} + (\Phi_L(\Gamma'(\mathcal{A}_{pq^{n_2}})) \setminus \{0\})) \bigcup \cdots \bigcup (D_{p^{n_1-1}q^{n_2}} \\
& + (\Phi_L(\Gamma'(\mathcal{A}_{p^{n_1-1}q^{n_2}})) \setminus \{0\})) \bigcup \Phi(L(\Upsilon'_{p^{n_1}q^{n_2}})).
\end{aligned}$$

The remaining $(n_1 + 1)(n_2 + 1) - 2$ eigenvalues are the zeros of the characteristic polynomial of the matrix $L(\Upsilon'_{p^{n_1}q^{n_2}})$ given in equation (2.1). □

## 2.3 Laplacian Spectral Radius and Algebraic Connectivity

In this section, we study the algebraic connectivity and the Laplacian spectral radius of $\Gamma'(\mathbb{Z}_n)$. We classify all those values of $n$ for which the Laplacian spectral radius of $\Gamma'(\mathbb{Z}_n)$ is equal to the order of $\Gamma'(\mathbb{Z}_n)$. Moreover, the values of $n$ for which the algebraic connectivity and the vertex connectivity coincide are also described.

The following theorem follows from the relation $\lambda(\Gamma) = |V(\Gamma)| - \mu(\overline{\Gamma})$ and the fact $\overline{\Gamma}$ is disconnected if and only if $\Gamma$ is the join of two graphs.

**Theorem 2.3.1** (Fiedler [1973])**.** *If $\Gamma$ is a graph on $m$ vertices, then $\lambda(\Gamma) \leq m$. Further, equality holds if and only if $\overline{\Gamma}$ is disconnected if and only if $\Gamma$ is the join of two graphs.*

In view of Theorem 2.3.1, first we characterize the values of $n$ for which the complement of $\Gamma'(\mathbb{Z}_n)$ is disconnected.

**Proposition 2.3.2.** *The graph $\overline{\Gamma'(\mathbb{Z}_n)}$ is disconnected if and only if $n$ is a product of two distinct primes.*

*Proof.* Let $p$ and $q$ be two distinct primes. If $n = pq$, then by Remark 2.1.13 we get $V(\Gamma'(\mathbb{Z}_n)) = \mathcal{A}_p \cup \mathcal{A}_q$ such that $\mathcal{A}_p \cap \mathcal{A}_q = \emptyset$. In fact, $\Gamma'(\mathbb{Z}_n) = K_{\phi(q),\phi(p)}$ is a complete bipartite graph. Consequently, $\overline{\Gamma'(\mathbb{Z}_n)}$ is a disconnected graph.

Conversely, suppose $\overline{\Gamma'(\mathbb{Z}_n)}$ is disconnected. Clearly, for $n = p$ there is nothing to prove. If $n = p^\alpha$ for some $1 < \alpha \in \mathbb{N}$, then $\Gamma'(\mathbb{Z}_{p^\alpha})$ is a null graph. Consequently, $\overline{\Gamma'(\mathbb{Z}_{p^\alpha})}$ is a complete graph which is not possible. If possible, let $n \neq pq$. Let $d_1$ and $d_2$ be the proper divisors of $n$ and let $x \in \mathcal{A}_{d_1}$, $y \in \mathcal{A}_{d_2}$. If $d_1 = d_2$, then clearly $x \sim y$ in $\overline{\Gamma'(\mathbb{Z}_n)}$. If $d_1 \neq d_2$ such that either $d_1 \mid d_2$ or $d_2 \mid d_1$, then $x \sim y$ in $\overline{\Gamma'(\mathbb{Z}_n)}$ (cf. Lemma 2.1.14 ). If $d_1 \neq d_2$ and neither $d_1 \mid d_2$ nor $d_2 \mid d_1$, then there exist two primes $p_1$ and $p_2$ such that $p_1 \mid d_1$ and $p_2 \mid d_2$. Consequently, $x \sim z_1 \sim z_2 \sim z_3 \sim y$ in $\overline{\Gamma'(\mathbb{Z}_n)}$ for some $z_1 \in \mathcal{A}_{p_1}$, $z_2 \in \mathcal{A}_{p_1p_2}$ and $z_3 \in \mathcal{A}_{p_2}$. Thus, $\overline{\Gamma'(\mathbb{Z}_n)}$ is connected; a contradiction. Hence, $n$ must be a product of two distinct primes. $\square$

Since $|V(\Gamma'(\mathbb{Z}_n))| = n - \phi(n) - 1$, by using the Proposition 2.3.2 in Theorem 2.3.1, we have the following proposition.

**Proposition 2.3.3.** $\lambda(\Gamma'(\mathbb{Z}_n)) = |V(\Gamma'(\mathbb{Z}_n))|$ *if and only if $n$ is a product of two distinct primes. Moreover, if $n = pq$ then $\lambda(\Gamma'(\mathbb{Z}_n)) = p + q - 2$.*

Now we classify all those values of $n$ for which the algebraic connectivity and the vertex connectivity of $\Gamma'(\mathbb{Z}_n)$ are equal. The following theorem is useful in this study.

**Theorem 2.3.4** (Kirkland et al. [2002])**.** *Let $\Gamma$ be a non-complete connected graph on $m$ vertices. Then $\kappa(\Gamma) = \mu(\Gamma)$ if and only if $\Gamma$ can be written as $\Gamma_1 \vee \Gamma_2$, where $\Gamma_1$ is a disconnected graph on $m - \kappa(\Gamma)$ vertices and $\Gamma_2$ is a graph on $\kappa(\Gamma)$ vertices with $\mu(\Gamma_2) \geq 2\kappa(\Gamma) - m$.*

**Lemma 2.3.5.** *For distinct primes $p$ and $q$, if $n = pq$ where $p < q$ then $\kappa(\Gamma'(\mathbb{Z}_n)) = \delta(\Gamma'(\mathbb{Z}_n)) = p - 1$.*

*Proof.* For $n = pq$, $\Gamma'(\mathbb{Z}_n)$ is a complete bipartite graph with partition sets $\mathcal{A}_p$ and $\mathcal{A}_q$. Hence, $\kappa(\Gamma'(\mathbb{Z}_n)) = \delta(\Gamma'(\mathbb{Z}_n)) = \min\{|\mathcal{A}_p|, |\mathcal{A}_q|\} = p - 1$ □

**Theorem 2.3.6.** *For the graph $\Gamma'(\mathbb{Z}_n)$, we have $\mu(\Gamma'(\mathbb{Z}_n)) \leq \kappa(\Gamma'(\mathbb{Z}_n))$. The equality holds if and only if $n$ is a product of two distinct primes.*

*Proof.* By Kirkland et al. [2002], for any graph $\Gamma$ which is not complete, we have $\mu(\Gamma) \leq \kappa(\Gamma)$. If $n = 4$ then there is nothing to prove because $\Gamma'(\mathbb{Z}_4)$ is the graph of one vertex only. If $n \neq 4$, then $\Gamma'(\mathbb{Z}_n)$ is not a complete graph. Consequently, $\mu(\Gamma'(\mathbb{Z}_n)) \leq \kappa(\Gamma'(\mathbb{Z}_n))$.

If $n$ is not a product of two distinct primes, then by Proposition 2.3.2 and by Theorem 2.3.1, $\Gamma'(\mathbb{Z}_n)$ can not be written as the join of two graphs. Thus, by Theorem 2.3.4, we obtain $\mu(\Gamma'(\mathbb{Z}_n)) < \kappa(\Gamma'(\mathbb{Z}_n))$. If $n = pq$, where $p$ and $q$ are distinct primes such that $p < q$, then by Theorem 2.3.1, Proposition 2.3.2, Theorem 2.3.4 and Lemma 2.3.5, we obtain $\mu(\Gamma'(\mathbb{Z}_n)) = \kappa(\Gamma'(\mathbb{Z}_n)) = p - 1$. □

# Chapter 3

# The Upper Ideal-Relation Graphs of Rings

Ma and Wong [2016] introduced the ideal-relation graph of the ring $R$ is a directed graph whose vertex set is $R$ and there is an edge from a vertex $x$ to a distinct vertex $y$ if and only if $(x) \subset (y)$. Analogously, the undirected ideal-relation graph of the ring $R$ is the simple graph whose vertex set is $R$ and two distinct vertices $x$ and $y$ are adjacent if and only if either $(x) \subset (y)$ or $(y) \subset (x)$, that is the principal ideals $(x)$ and $(y)$ are comparable in the poset of principal ideals of $R$. So it is natural to define a graph on a ring $R$ such that its vertices $x$ and $y$ are adjacent if and only if $(x)$ and $(y)$ have an upper bound in the poset of the principal ideals of $R$. In view of this, we define *upper ideal-relation* graph associated with the ring $R$. The upper ideal-relation graph $\Gamma_U(R)$ of the ring $R$ is the simple undirected graph whose vertex set is the set of all non-unit elements of $R$ and two distinct vertices $x$ and $y$ are adjacent if and only if there exists a non-unit element $z \in R$ such that the ideals $(x)$ and $(y)$ contained in the ideal $(z)$. In this chapter, we investigate the algebraic properties of ring $R$ and the graph-theoretic properties of $\Gamma_U(R)$. In Section 3.1, we obtain the girth, minimum degree and the independence number of $\Gamma_U(R)$. We give a necessary and sufficient condition on $R$, in terms of the cardinality of their principal ideals, such that the graph $\Gamma_U(R)$ is bipartite, planar and outerplanar, respectively. We also discuss all the finite rings $R$ such that the graph $\Gamma_U(R)$ is Eulerian. For reduced rings, we obtain the metric and the strong metric dimension of the graph $\Gamma_U(R)$ (cf. Section 3.2). For a non-local commutative ring $R \cong R_1 \times R_2 \times \cdots \times R_n$ ($n \geq 3$), where each $R_i$ is a local ring with maximal ideal $\mathcal{M}_i$, in Section 3.3, we prove that the graph $\Gamma_U(R)$ is perfect if and only if $n \in \{3, 4\}$ and each $\mathcal{M}_i$ is a principal ideal. Section 3.4 classifies all the non-local commutative rings $R$ for which $\Gamma_U(R)$ has genus at most 2. Also, we determine precisely all the non-local commutative rings for which $\Gamma_U(R)$ has crosscap at most 2. In Section 3.5, we classify all the non-local commutative rings whose upper ideal-relation graphs are split graphs, threshold graphs and cographs, respectively. In Section 3.6, we determine the vertex connectivity, automorphism group, Laplacian and the normalized Laplacian spectrum of the upper ideal-relation graph of the ring $\mathbb{Z}_n$. We classify all the values of $n$ for which the graph $\Gamma_U(\mathbb{Z}_n)$ is Hamiltonian.

The content of this chapter excluding Section 3.4 and Section 3.5 is submitted for publication. Whereas the results of Section 3.4 and Section 3.5 are submitted for publication.

## 3.1 Invariants of $\Gamma_U(R)$

In this section, we study the algebraic properties of $R$ as well as graph-theoretic properties of the upper ideal-relation graph $\Gamma_U(R)$. We obtain the girth, minimum degree, independence number of $\Gamma_U(R)$. We obtain a necessary and sufficient condition on $R$, in terms of the cardinality of their principal ideals, such that the graph $\Gamma_U(R)$ is bipartite, planar and outerplanar, respectively. In order to study the basic properties of $\Gamma_U(R)$, the following remark is useful in the sequel.

**Remark 3.1.1.** Let $F_1$ and $F_2$ be fields such that $R \cong F_1 \times F_2$. Then we have

$$\Gamma_U(F_1 \times F_2) \cong K_1 \vee (K_{|F_1|-1} \bigcup K_{|F_2|-1}).$$

For $x, y \in V(\Gamma_U(R))$, note that $x \sim 0 \sim y$. It follows that the graph is connected and hence $diam(\Gamma_U(R)) \leq 2$.

**Theorem 3.1.2.** *The upper ideal-relation graph $\Gamma_U(R)$ contains a cycle if and only if $|(x)| \geq 3$ for some $x \in V(\Gamma_U(R))$. Moreover, $g(\Gamma_U(R)) \in \{3, \infty\}$.*

*Proof.* Assume that $\Gamma_U(R)$ contains a cycle. Let if possible, $|(x)| \leq 2$ for each non-unit element $x$ of $R$. Then being a star graph, $\Gamma_U(R)$ is an acyclic graph, a contradiction. Conversely, assume that there exists a non-unit element $x$ of $R$ such that $|(x)| \geq 3$. Consequently, the elements $x, 0, x'$ of the ideal $(x)$ form a cycle $x \sim x' \sim 0 \sim x$ of length three in $\Gamma_U(R)$. Thus, the result holds. $\square$

**Corollary 3.1.3.** *The girth of $\Gamma_U(\mathbb{Z}_n)$ is given below:*

$$g(\Gamma_U(\mathbb{Z}_n)) = \begin{cases} \infty & \text{if } \ n = 1, 4, \textit{ or } p \\ 3 & \textit{otherwise} \end{cases}$$

**Theorem 3.1.4.** *For the graph $\Gamma_U(R)$, the following conditions are equivalent:*

(i) $\Gamma_U(R)$ *is a bipartite graph.*

(ii) $\Gamma_U(R)$ *is a tree.*

(iii) $|(x)| \leq 2$ *for all* $x \in V(\Gamma_U(R))$.

(iv) $\Gamma_U(R)$ *is a star graph.*

*Proof.* (i) $\Rightarrow$ (ii). Suppose that $\Gamma_U(R)$ is a bipartite graph. It follows that it does not contain a cycle of odd length. By Theorem 3.1.2, $\Gamma_U(R)$ does not contain a cycle of length three. Consequently, $\Gamma_U(R)$ is an acyclic graph and so is a tree.

(ii) $\Rightarrow$ (iii). $\Gamma_U(R)$ is a tree. If there exists a non-unit element $x$ of $R$ such that $|(x)| \geq 3$, then by Theorem 3.1.2, $\Gamma_U(R)$ contains a cycle, a contradiction. Therefore, $|(x)| \leq 2$ for every $x \in V(\Gamma_U(R))$.

(iii) $\Rightarrow$ (iv). Let $|(x)| \leq 2$, for all $x \in V(\Gamma_U(R))$. Then for the non-zero non-unit elements $x_1, x_2$ of $R$ note that $x_1 \nsim x_2$. Also, $0 \sim x$ for every $x \in V(\Gamma_U(R))$. Therefore, $\Gamma_U(R)$ is a star graph.

(iv) $\Rightarrow$ (i). Being a star graph, $\Gamma_U(R)$ is a complete bipartite which is isomorphic to $K_{1,|R \setminus U(R)|-1}$. $\square$

**Theorem 3.1.5.** *The upper ideal-relation graph $\Gamma_U(R)$ is complete if and only if $R$ has a unique maximal principal ideal.*

*Proof.* Suppose that $\Gamma_U(R)$ is a complete graph. On contrary, suppose that $R$ has at least two maximal principal ideals, namely $(x_1)$ and $(x_2)$. Then $x_1 \nsim x_2$ in $\Gamma_U(R)$, a contradiction. Conversely, suppose that $R$ has a unique maximal principal ideal, $(x_1)$. Then for $x, y \in V(\Gamma_U(R))$, note that both $(x)$ and $(y)$ is contained in $(x_1)$. It follows that $x \sim y$ in $\Gamma_U(R)$ and so $\Gamma_U(R)$ is a complete graph. $\square$

**Corollary 3.1.6.** *The graph $\Gamma_U(\mathbb{Z}_n)$ is complete if and only if $n$ is a prime power.*

**Theorem 3.1.7.** *The upper ideal-relation graph $\Gamma_U(R)$ is planar if and only if $|(x)| \leq 4$ for all non-unit element $x$ of $R$.*

*Proof.* Suppose that $\Gamma_U(R)$ is a planar graph. If there exists a non-unit element $x$ of $R$ such that $|(x)| \geq 5$, then the elements of $(x)$ induces a complete subgraph which is isomorphic to $K_5$; a contradiction. Conversely, suppose that $|(x)| \leq 4$ for all $x \in R \setminus U(R)$. First let $|(x_1)| = 2$ and $|(x_2)| = 3$ for some non-unit elements $x_1, x_2$ of $R$. Then note that $x_1 \nsim x_2$ in $\Gamma_U(R)$. Otherwise, there exists a non-unit element $z \in R$ such that $|(z)| \geq 5$, a contradiction. Similarly, if $|(x_1)| \in \{3, 4\}$ and $|(x_2)| \in \{3, 4\}$, then $x_1 \nsim x_2$ in $\Gamma_U(R)$. Next, let $(x_1) \neq (x_2)$ and $|(x_1)| = |(x_2)| = 2$ for some $x_1, x_2 \in R \setminus U(R)$. If $x_1 \sim x_2$, then $x_1, x_2 \in (x)$ for some $x \in R \setminus U(R)$. By hypothesis, we obtain $|(x)| \leq 4$, which is not possible. Therefore, $x_1 \nsim x_2$. Thus, $\Gamma_U(R)$ is a planar graph. □

**Corollary 3.1.8.** *The graph $\Gamma_U(\mathbb{Z}_n)$ is planar if and only if $n = 4, 6, 8, 9$ or $p$.*

In the similar lines of the proof of Theorem 3.1.7, we have the following theorem.

**Theorem 3.1.9.** *The upper ideal-relation graph $\Gamma_U(R)$ is outerplanar if and only if $|(x)| \leq 3$ for all non-unit element $x$ of $R$.*

**Corollary 3.1.10.** *The graph $\Gamma_U(\mathbb{Z}_n)$ is outerplanar if and only if $n = 4, 6, 9$ or $p$.*

**Theorem 3.1.11.** *The minimum degree of the graph $\Gamma_U(R)$ is $m - 1$, where $m$ is the cardinality of smallest maximal principal ideal of the ring $R$.*

*Proof.* Let $x \in V(\Gamma_U(R))$. Then $x$ is contained in some maximal principal ideal $(z)$. Since $(z)$ induces a clique of size $|z| - 1$, we get $\deg(x) \geq |z| - 1$. Let $(y)$ be a maximal principal ideal of the smallest size $m$. Then $\deg(y) = m - 1$. Consequently, $\deg(x) \geq |(z)| - 1 \geq m - 1$. Thus, the minimum degree of $\Gamma_U(R)$ is $m - 1$. □

**Corollary 3.1.12.** *Let $n = p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots p_m^{n_m}$ such that $p_1 < p_2 < \cdots < p_m$. Then the minimum degree of $\Gamma_U(\mathbb{Z}_n)$ is $\frac{n}{p_m} - 1$.*

**Theorem 3.1.13.** *The independence number of the graph $\Gamma_U(R)$ is $|Max(R)|$, where $Max(R)$ is the set of all maximal principal ideals of the ring $R$.*

*Proof.* Note that $(x_1), (x_2) \in Max(R)$, we get $x_1 \nsim x_2$ in $\Gamma_U(R)$. It follows that $\alpha(\Gamma_U(R)) \geq |Max(R)|$. Let $I$ be an arbitrary independent set of $\Gamma_U(R)$ and let $x \in I$. Then $x \in (z)$ for some $(z) \in Max(R)$. Also the subgraph induced by $(z)$ forms a clique. Consequently, $I$ can not contain any element of $(z)$ other than $x$. It implies that $|I| \leq |Max(R)|$. Thus, the result holds. $\square$

**Corollary 3.1.14.** *The independence number of $\Gamma_U(\mathbb{Z}_n)$ is the number of distinct prime factors of $n$.*

Define a relation $x \simeq y$ if and only if $(x) = (y)$. It is easy to observe that $\simeq$ is an equivalence relation and $[x]$ is an equivalence class containing $x$. Note that $V(\Gamma_U(R)) = \bigcup\limits_{x \in R \setminus U(R)} [x]$.

**Theorem 3.1.15.** *The graph $\Gamma_U(R)$ is Eulerian if and only if $|R|$ and $|R \setminus U(R)|$ is odd.*

*Proof.* First suppose that $\Gamma_U(R)$ is Eulerian. Since 0 is adjacent with every element of $\Gamma_U(R)$ we obtain $\deg(0) = |R \setminus U(R)| - 1$. By Theorem 1.3.1, $|R \setminus U(R)|$ is odd. Let $(x)$ be maximal principal ideal of $R$. Then $\deg(x) = |(x)| - 1$. Since $\Gamma_U(R)$ is Eulerian, we get $|(x)|$ is odd. Consequently, for any $y \in V(\Gamma_U(R))$, we get $o(y)$ is always odd in the group $(R, +)$. It follows that $|R|$ is odd.

Conversely, suppose that $|R|$ and $|R \setminus U(R)|$ is odd. Clearly, $\deg(0) = |R \setminus U(R)| - 1$ which is an even number. Let $x \neq 0 \in V(\Gamma_U(R))$. Then $x \in [y]$ for some $y \in R \setminus U(R)$. Note that each equivalence class under the relation $\simeq$, defined above, forms a clique. Moreover, if $x_1 \sim y_1$, where $x_1 \in [z_1]$ and $y_1 \in [z_2]$, then $x' \sim y'$ for each $x' \in [z_1]$ and $y' \in [z_2]$. Consequently, $\deg(x) = (|[x]| - 1) + |[x_1]| + \cdots + |[x_m]| + 1$. Since $|R|$ is odd, each equivalence class of the relation $\simeq$ is of even size. Hence, $\deg(x)$ is even and so $\Gamma_U(R)$ is Eulerian. $\square$

**Theorem 3.1.16.** *Let $R$ be a principal ideal ring having $n$ maximal ideals $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$ of $R$ such that $|\mathcal{M}_1| \geq |\mathcal{M}_2| \geq \cdots \geq |\mathcal{M}_n|$. Then $\omega(\Gamma_U(R)) = |\mathcal{M}_1|$.*

*Proof.* We prove the result by applying induction on $n$. Let $n = 2$. Suppose that $\mathcal{M}_1, \mathcal{M}_2$ are maximal ideals of $R$ with $|\mathcal{M}_1| \geq |\mathcal{M}_2|$. First we cover all the elements of $\mathcal{M}_1$ by using $|\mathcal{M}_1|$ colors. Next, if $x \in \mathcal{M}_2 \setminus \mathcal{M}_1$, then these can be colored using the colors used in the coloring of the set $\mathcal{M}_1 \setminus \mathcal{M}_2$ as $|\mathcal{M}_2 \setminus \mathcal{M}_1| \leq |\mathcal{M}_1 \setminus \mathcal{M}_2|$. It follows that $\chi(\Gamma_U(R)) \leq |\mathcal{M}_1|$. Also, note that all the elements of $\mathcal{M}_1$ forms a clique of size $|\mathcal{M}_1|$. Therefore, $\omega(\Gamma_U(R)) = |\mathcal{M}_1|$. Now, assume that $n \geq 3$. Clearly, $\omega(\Gamma_U(R)) \geq |\mathcal{M}_1|$. Let if possible, $\omega(\Gamma_U(R)) > |\mathcal{M}_1|$. Then there exists a set $T$ of $V(\Gamma_U(R))$ such that $|\mathcal{M}_1| < |T|$ and all the elements of $T$ forms a clique. If $T$ contains any element of $\mathcal{M}_n \setminus \cup_{i=1}^{n-1}\mathcal{M}_i$, then $T \subseteq \mathcal{M}_n$; a contradiction. Therefore, $T \subseteq \cup_{i=1}^{n-1}\mathcal{M}_i$. Suppose that $J = \cap_{i=1}^{n-1}\mathcal{M}_i$ so that $J \neq 0$. Further, assume that $\bar{R} = R/J$. Notice that all the elements of $\bar{T}$ forms a clique in $\bar{R}$. Since $\bar{R}$ is a ring with $n-1$ maximal ideals, by induction hypothesis, we have $\omega(\Gamma_U(\bar{R})) = |\bar{\mathcal{M}_1}| \geq |\bar{T}|$. It follows that $|T| \leq |\mathcal{M}_1|$; a contradiction. Thus, $\omega(\Gamma_U(R)) = |\mathcal{M}_1|$. □

**Corollary 3.1.17.** *Let $n = p_1^{n_1}p_2^{n_2}p_3^{n_3}\cdots p_m^{n_m}$ such that $p_1 < p_2 < \cdots < p_m$. Then $\omega(\Gamma_U(\mathbb{Z}_n)) = \dfrac{n}{p_1}$.*

## 3.2 Metric and Strong Metric Dimension of $\Gamma_U(R)$

In this section, we obtain the metric and the strong metric dimension of $\Gamma_U(R)$ of the reduced ring $R \cong F_1 \times F_2 \times \cdots \times F_n$, where $n \geq 2$. For $i_1, i_2, \ldots, i_k \in [n]$, we define

$$\widehat{A_{i_1 i_2 \cdots i_k}} = \{(a_1, a_2, \ldots, a_n) : \text{ only } a_{i_1}, a_{i_2}, \ldots, a_{i_k} \text{are non-zero}\}.$$

For instance, if $R \cong F_1 \times F_2 \times \cdots \times F_5$, then

$$\widehat{A_{234}} = \{(0, a_2, a_3, a_4, 0) : a_2 \in F_2^*, a_3 \in F_3^*, a_4 \in F_4^*\}.$$

We begin with the following remark.

**Remark 3.2.1.** Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $V(\Gamma_U(R))$ with $(x) = (y)$. Then $x_i = 0$ if and only if $y_i = 0$ for each $i$.

**Lemma 3.2.2.** *Let $R \cong F_1 \times F_2 \times \cdots \times F_n$ such that $n \geq 3$. Then for $x_1, x_2 \in V(\Gamma_U(R))$, we have $(x_1) = (x_2)$ if and only if $N[x_1] = N[x_2]$.*

*Proof.* First suppose that $(x_1) = (x_2)$. If $x \in N[x_1]$, then there exists a non-unit element $z \in R$ such that $(x) \subseteq (z)$ and $(x_1) \subseteq (z)$. Since $(x_1) = (x_2)$, we get $x \sim x_2$ in $\Gamma_U(R)$ and so $x \in N[x_2]$. Thus, $N[x_1] \subseteq N[x_2]$. Similarly, $N[x_2] \subseteq N[x_1]$. To prove the converse, let if possible $(x_1) \neq (x_2)$, where $x_1 = (a_1, a_2, \ldots, a_n), x_2 = (b_1, b_2, \ldots, b_n)$. Since $(x_1) \neq (x_2)$, by Remark 3.2.1, there exists $j \in [n]$ such that $a_j = 0$ but $b_j \neq 0$. Now choose $z = (z_1, \ldots, z_{j-1}, 0, z_{j+1}, \ldots, z_n) \in V(\Gamma_U(R))$ such that $z_i \neq 0$ whenever $b_i = 0$. Note that $x_1 \sim z$ but $x_2 \nsim z$ in $\Gamma_U(R)$. Therefore, $N[x_1] \neq N[x_2]$. Thus, the result holds. $\square$

Define a relation $\equiv$ on $V(\Gamma_U(R))$ such that $x \equiv y$ if and only if $N[x] = N[y]$. Note that $\equiv$ is an equivalence relation. For $R \cong F_1 \times F_2 \times \cdots \times F_n$, where $n \geq 3$, note that $V(\Gamma_U(R))$ has $2^n - 1$ equivalence classes, viz. $\widehat{A_0}, \widehat{A_{i_1}}, \ldots, \widehat{A_{i_1 i_2 \cdots i_{n-1}}}$. Notice that $|\widehat{A_0}| = 1$ and $|\widehat{A_{i_1 i_2 \cdots i_k}}| \geq 2$, whenever $F_i \neq \mathbb{Z}_2$ for each $i$.

**Theorem 3.2.3.** *Let $R \cong F_1 \times F_2 \times \cdots \times F_n$, where $n \geq 2$. Then*

$$\mathrm{sdim}(\Gamma_U(R)) = \begin{cases} |F_1| + |F_2| - 3; & \textit{if } n = 2 \\ |R \setminus U(R)| - 2^{n-1}; & \textit{if } n \geq 3. \end{cases}$$

*Proof.* By Remark 3.1.1, we have $\Gamma_U(F_1 \times F_2) = K_1 \vee (K_{|F_1|-1} \cup K_{|F_2|-1})$. Notice that the reduced graph $\mathcal{R}_{\Gamma_U(R)}$ is isomorphic to a path graph on three vertices. Therefore, $\omega(\mathcal{R}_{\Gamma_U(R)}) = 2$. By Theorem 1.3.5, we get $\mathrm{sdim}(\Gamma_U(F_1 \times F_2)) = |F_1| + |F_2| - 3$. Next, we assume that $R \cong F_1 \times F_2 \times \cdots \times F_n$ such that $n \geq 3$. First suppose that $n$ is odd. Note that the set

$$\mathcal{C} = \widehat{A_0} \cup (\textstyle\bigcup_{i_1 \in [n]} \widehat{A_{i_1}}) \cup (\bigcup_{i_1, i_2 \in [n]} \widehat{A_{i_1 i_2}}) \cup \cdots \cup (\bigcup_{\substack{i_r \in [n] \\ 1 \leq r \leq \lfloor \frac{n}{2} \rfloor}} \widehat{A_{i_1 i_2 \cdots i_{\lfloor \frac{n}{2} \rfloor}}})$$

forms a clique in $\Gamma_U(R)$. To determine the strong metric dimension of $\Gamma_U(R)$, we need to find $\omega(\mathcal{R}_{\Gamma_U(R)})$. Further, by considering exactly one representative of each equivalence class from $\mathcal{C}$, we obtain $\omega(\mathcal{R}_{\Gamma_U(R)}) \geq {}^nC_0 + {}^nC_1 + \cdots + {}^nC_{\lfloor \frac{n}{2} \rfloor} = 2^{n-1}$. Now suppose that $n$ is even. Consider the set

$$\mathcal{C}_1 = \widehat{A_0} \cup (\bigcup_{i_1 \in [n]} \widehat{A_{i_1}}) \cup (\bigcup_{i_1, i_2 \in [n]} \widehat{A_{i_1 i_2}}) \cup \cdots \cup (\bigcup_{\substack{i_r \in [n] \\ 1 \leq r \leq \frac{n}{2}-1}} \widehat{A_{i_1 i_2 \cdots i_{\frac{n}{2}-1}}}).$$

Note that $\mathcal{C}_1$ forms a clique of $\Gamma_U(R)$, whereas the set

$$\mathcal{C}_2 = \{(a_1, a_2, \ldots, a_n) : \text{ only } a_{i_1}, a_{i_2}, \ldots, a_{i_{\frac{n}{2}}} \text{ are non-zero}\}$$

does not form a clique of $\Gamma_U(R)$. Now choose the set $\mathcal{C}_3 = \{(b_1, b_2, \ldots, b_n) \in \mathcal{C}_2 :$ only $b_{j_1}, b_{j_2}, \ldots, b_{j_{\frac{n}{2}}}$ are non-zero , where $j_1, j_2, \ldots, j_{\frac{n}{2}} \in [n] \setminus \{i_1, i_2, \ldots, i_{\frac{n}{2}}\}\}$.

Notice that the set $\mathcal{C}_3$ forms a clique in $\Gamma_U(R)$. Also, observe that the set $\mathcal{C}_1 \cup \mathcal{C}_3$ forms a clique of the graph $\Gamma_U(R)$. Consequently, $\omega(\mathcal{R}_{\Gamma_U(R)}) \geq 2^{n-1}$. To complete the proof, we show that $\chi(\mathcal{R}_{\Gamma_U(R)}) \leq 2^{n-1}$. Let $x \in \widehat{A_{i_1 i_2 \cdots i_k}}$ and $y \in \widehat{A_{j_1 j_2 \cdots j_{n-k}}}$, where $i_1, i_2, \ldots, i_k \in [n] \setminus \{j_1, j_2, \ldots, j_{n-k}\}$. Then note that $x \nsim y$ in $\Gamma_U(R)$. Consequently, we can color such vertices with the same color. Therefore, we can color all the vertices of $\mathcal{R}_{\Gamma_U(R)}$ with $2^{n-1}$ colors. Thus, $\chi(\mathcal{R}_{\Gamma_U(R)}) \leq 2^{n-1}$ and so $\omega(\mathcal{R}_{\Gamma_U(R)}) = 2^{n-1}$. Theorem 1.3.5 yields $\mathrm{sdim}(\Gamma_U(R)) = |R \setminus U(R)| - 2^{n-1}$. $\square$

**Corollary 3.2.4.** *Let $n \geq 2$ be a positive integer and $R \cong \prod_{i=1}^{n} \mathbb{Z}_2$. Then*

$$\mathrm{sdim}(\Gamma_U(R)) = 2^{n-1} - 1.$$

**Corollary 3.2.5.** *Let $n = p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots p_m^{n_m}$, where $m \geq 2$. Then*

$$\mathrm{sdim}(\Gamma_U(\mathbb{Z}_n)) = n - \phi(n) - 2^{m-1}.$$

**Theorem 3.2.6.** *Let $n \geq 2$ be a positive integer and $R \cong \prod_{i=1}^{n} \mathbb{Z}_2$. Then the metric dimension of $\Gamma_U(R)$ is given below:*

$$\beta(\Gamma_U(R)) = \begin{cases} 1; & n = 2 \\ n; & \textit{Otherwise.} \end{cases}$$

*Proof.* For $n = 2$, we have $\Gamma_U(R) \cong K_1 \vee (K_1 \cup K_1)$ (see Remark 3.1.1). Now suppose that $n \geq 3$. Clearly, $|V(\Gamma_U(R))| = 2^n - 1$. By Lemma 1.3.4, we get $n = f(2^n - 1, 2) \leq \beta(\Gamma_U(R))$. To prove the result, we show that there exists a resolving set of size $n$. Consider the set

$$\mathcal{R} = \{(0, 1, 1, \ldots, 1), (1, 0, 1, 1, \ldots, 1), (1, 1, 0, 1, \ldots, 1), \ldots, (1, 1, \ldots, 1, 0)\}.$$

Let $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in V(\Gamma_U(R))$. If one of $x$ and $y$ belongs to $\mathcal{R}$, then there is nothing to prove. We may now suppose that $x, y \notin \mathcal{R}$. Since $x, y \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, we have $(x) \neq (y)$. If $x \nsim y$ in $\Gamma_U(R)$, then there exists $i \in [n]$ such that $x_i = 0$ but $y_i \neq 0$. Now choose $z = (z_1, z_2, \ldots, z_n) \in \mathcal{R}$ such that only $z_i = 0$. Note that $z \sim x$ but $z \nsim y$ in $\Gamma_U(R)$. It follows that $d(x, z) \neq d(y, z)$. We may now suppose that $x \sim y$ in $\Gamma_U(R)$. Without loss of generality, assume that $(x) \subset (y)$. Then there exists $i \in [n]$ such that $x_i = 0$ but $y_i \neq 0$. Choose $z = (z_1, z_2, \ldots, z_n) \in \mathcal{R}$ such that only $z_i = 0$. It follows that $x \sim z$ but $y \nsim z$ in $\Gamma_U(R)$. It follows that $d(x, z) \neq d(y, z)$. If $(x) \not\subset (y)$ and $(y) \not\subset (x)$, then one can find $z \in \mathcal{R}$ such that $x \in (z)$ but $y \notin (z)$. Consequently, $d(x, z) \neq d(y, z)$. Thus, $R$ is a resolving set. Hence, $\beta(\Gamma_U(R)) = n$. $\square$

**Theorem 3.2.7.** *Let $R \cong F_1 \times F_2 \times \cdots \times F_n$ $(n \geq 2)$, where each $F_i \neq \mathbb{Z}_2$. Then*

$$\beta(\Gamma_U(R)) = |R \setminus U(R)| - 2^n + 2.$$

*Proof.* Let $R \cong F_1 \times F_2 \times \cdots \times F_n$, where $n \geq 2$. For each $i_1, i_2, \ldots, i_{n-1} \in [n]$, note that $\Gamma_U(R)$ has $2^n - 1$ equivalence classes, namely $\widehat{A_0}, \widehat{A_{i_1}}, \ldots, \widehat{A_{i_1 i_2 \cdots i_{n-1}}}$ under the relation $\equiv$. Let $T$ be an arbitrary resolving set. Then by Lemma 1.3.3, $T$ contains at least $|\widehat{A_{i_1 i_2 \cdots i_k}}| - 1$ elements from each equivalence class $\widehat{A_{i_1 i_2 \cdots i_k}}$, where $i_1, i_2, \ldots, i_k \in [n]$ and $1 \leq k \leq n - 1$. It follows that $|T| \geq |R \setminus U(R)| - 2^n + 2$. Let $\mathcal{R}$ be a set containing exactly $|\widehat{A_{i_1 i_2 \cdots i_k}}| - 1$ elements from $\widehat{A_{i_1 i_2 \cdots i_k}}$, where $i_1, i_2, \ldots, i_k \in [n]$ and $1 \leq k \leq n - 1$. Note that $|\mathcal{R}| = |R \setminus U(R)| - 2^n + 2$. Now we show that $\mathcal{R}$ is a resolving set. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$

be arbitrary vertices of $\Gamma_U(R)$. If one of $x$ and $y$ belongs to $\mathcal{R}$, then there is nothing to prove. We may now suppose that $x, y \notin \mathcal{R}$. It follows that $(x) \neq (y)$. Then either $x \nsim y$ or $x \sim y$ in $\Gamma_U(R)$. If $x \nsim y$ in $\Gamma_U(R)$, then there exists $z \in \mathcal{R}$ such that $(z) = (x)$. It follows that $d(x, z) \neq d(y, z)$. Now let $x \sim y$ in $\Gamma_U(R)$. Then by the similar argument used in the proof of Theorem 3.2.6, there exists a $z \in \mathcal{R}$ such that $d(x, z) \neq d(y, z)$. Hence, $\mathcal{R}$ is a resolving set and so $\beta(\Gamma_U(R)) = |R \setminus U(R)| - 2^n + 2$. □

**Corollary 3.2.8.** *Let $n = p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots p_m^{n_m}$, $m \geq 2$. Then the metric dimension, $\beta(\Gamma_U(\mathbb{Z}_n)) = n - \phi(n) - 2^m + 1$.*

## 3.3 Perfectness of $\Gamma_U(R)$

Let $R \cong R_1 \times R_2 \times \cdots \times R_n$ be a finite commutative ring with unity, where each $R_i$ is a local ring with maximal ideal $\mathcal{M}_i$. In this section, we have investigated the perfectness of $\Gamma_U(R)$. We write $x_i = (a_{i1}, a_{i2}, \ldots, a_{in}) \in R$, where $a_{ij} \in R_j$ $(1 \leq j \leq n)$. We begin with the following lemma.

**Lemma 3.3.1.** *Let $R$ be a non-local commutative ring such that $R \cong R_1 \times R_2 \times \cdots \times R_n$, where $n \geq 5$. Then $\Gamma_U(R)$ is not a perfect graph.*

*Proof.* Let $n \geq 5$. Consider the set $X = \{(1, 0, 1, 1, 0, 1, 1, \ldots, 1), (1, 0, 0, 1, 1, \ldots, 1), (1, 1, 0, 0, 1, 1, \ldots, 1), (0, 1, 1, 0, 1, 1, \ldots, 1), (0, 1, 1, 1, 0, 1, 1, \ldots, 1)\}$. Note that $\Gamma_U(X) \cong C_5$. Hence, by Theorem 1.3.2, $\Gamma_U(R)$ is not a perfect graph. □

**Lemma 3.3.2.** *Let $(R_i, \mathcal{M}_i)$ be a local commutative ring and let $R \cong R_1 \times R_2 \times \cdots \times R_n$, where $n \geq 3$. If $\Gamma_U(R)$ is a perfect graph, then $n \in \{3, 4\}$ and each $\mathcal{M}_i$ is a principal ideal of $R_i$.*

*Proof.* Suppose that the graph $\Gamma_U(R)$ is perfect. Then by Lemma 3.3.1, we have $n \in \{3, 4\}$. Let $R \cong R_1 \times R_2 \times R_3$. Suppose that one of $\mathcal{M}_i$ is not a principal ideal of $R_i$. Without loss of generality, assume that the maximal ideal $\mathcal{M}_1$ of $R_1$ is not principal. Then $R_1$ has at least two principal maximal ideals $(a_1)$ and $(a_2)$. Then notice that $\overline{\Gamma_U(R)}$ contains an induced cycle $C : (a_1, 0, 1) \sim (a_2, 1, 0) \sim (1, 0, 1) \sim (0, 1, 1) \sim (1, 1, 0) \sim (a_1, 0, 1)$ of length five, which is a contradiction (see Theorem 1.3.2). Further, let $n = 4$, that is, $R \cong R_1 \times R_2 \times R_3 \times R_4$. Let if possible, $\mathcal{M}_i$ is not a principal maximal ideal for some $i$. Without loss of generality, assume that $\mathcal{M}_1$ is not principal. Then there exists at least two principal maximal ideals, viz. $(a_1)$ and $(a_2)$, of $R_1$. The subgraph induced by the set $X = \{(a_1, 0, 1, 1), (a_2, 0, 1, 1), (a_2, 1, 0, 1), (1, 1, 0, 0), (a_1, 1, 1, 0)\}$ is isomorphic to $C_5$ in $\Gamma_U(R)$; again a contradiction. Therefore, each $\mathcal{M}_i$ is principal. Thus the result holds. □

**Lemma 3.3.3.** *Let $(R_i, \mathcal{M}_i)$ be a local ring and let $R \cong R_1 \times R_2 \times \cdots \times R_n$ such that each $\mathcal{M}_i$ is principal. Let $x_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ and $y_l = (b_{l1}, b_{l2}, \ldots, b_{ln})$. Then $x_i \nsim y_l$ in $\Gamma_U(R)$ if and only if both $a_{ij}, b_{lj} \notin Z(R_j)$, for each $j$, $1 \leq j \leq n$.*

*Proof.* If both $a_{ij}, b_{lj} \in Z(R_j)$, then the ideals $(a_{ij})$ and $(b_{lj})$ is contained in $\mathcal{M}_j = (m_j)$. Note that the ideals $(x_i)$ and $(y_l)$ is contained in the principal ideal generated by $(1, 1, \ldots, 1, m_j, 1, , 1, \ldots, 1)$. Thus, $x_i \sim y_l$; a contradiction.

Conversely, assume that both $a_{ij}, b_{lj} \notin Z(R_j)$ for each $j$. If $a_{ij} \in Z(R_j)$, then $b_{lj} \in R_j \setminus Z(R_j) = U(R_j)$. It follows that there does not exists $z_j \in Z(R_j)$ such that $(a_{ij}), (b_{lj}) \subseteq (z_j)$ for each $j$. Therefore, $x_i \nsim y_l$ in $\Gamma_U(R)$. □

**Proposition 3.3.4.** *Let $(R_i, \mathcal{M}_i)$ be a local ring and $R \cong R_1 \times R_2 \times \cdots \times R_n$, where $n \leq 4$ and each $\mathcal{M}_i$ is principal. Then $\Gamma_U(R)$ does not contain any induced cycle of odd length greater than three.*

*Proof.* The result is straightforward for $n = 1$ (cf. Theorem 3.1.5). We first prove the result for $n = 4$ that is $R \cong R_1 \times R_2 \times R_3 \times R_4$. Let if possible, $\Gamma_U(R)$ contains an induced cycle $C : x_1 \sim x_2 \sim x_3 \sim \cdots \sim x_k \sim x_1$, where $k \geq 5$ is an odd integer. For $2 \leq i \leq k-1$, note that $x_{i-1} \sim x_i \sim x_{i+1}$ but $x_i \nsim x_t$ where $t \notin \{i-1, i+1\}$. Consider $x_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4}) \in R$. Since $x_1 \nsim x_3$, by Lemma 3.3.3, both $a_{11}, a_{31} \notin Z(R_1)$. Without loss of generality, assume that $a_{11} \in U(R_1)$. Since $x_1$ is a non-unit element of $R$, we have $a_{1j} \in Z(R_j)$ for some $j \in \{2,3,4\}$. Without loss of generality, assume that $a_{12} \in Z(R_2)$. By Lemma 3.3.3, we get $a_{32} \in U(R_2)$. Since $x_1 \nsim x_3$, we get both $a_{13}, a_{33} \notin Z(R_3)$. Now we have the following cases:

**Case-1.** $a_{13} \in U(R_3)$. First suppose that $a_{14} \in U(R_4)$. Since $x_k \sim x_1 \sim x_2$, we get $a_{k2}, a_{22} \in Z(R_2)$. It follows that $x_2 \sim x_k$ in $\Gamma_U(R)$, which is not possible. We may now suppose that $a_{14} \in Z(R_4)$. By Lemma 3.3.3, we obtain $a_{34}, a_{44} \in U(R_4)$. Since $x_3$ is a non-unit element of $R$ we have either $a_{31} \in Z(R_1)$ or $a_{33} \in Z(R_3)$. Let $a_{31} \in Z(R_1)$. If $a_{33} \in U(R_3)$, then $a_{21}, a_{41} \in Z(R_1)$. It follows that $x_2 \sim x_4$, which is not possible. Therefore, $a_{33} \in Z(R_3)$. Since $x_3 \nsim x_5$, we obtain that $a_{51} \in U(R_1)$ and $a_{53} \in U(R_3)$. Consequently, $x_4 \nsim x_5$; a contradiction. Thus, $a_{31} \in U(R_1)$ and so $a_{33} \in Z(R_3)$. Since $x_2 \sim x_3 \sim x_4$, we must have $a_{23}, a_{43} \in Z(R_3)$. It follows that $x_2 \sim x_4$. Thus, the case $a_{13} \in U(R_3)$ is not possible.

**Case-2.** $a_{33} \in U(R_3)$. Since $x_1 \nsim x_4$, we have $a_{43} \in U(R_3)$. Since $x_3$ is a non-unit element of $R$ we have either $a_{31} \in Z(R_1)$ or $a_{34} \in Z(R_4)$. Let $a_{31} \in Z(R_1)$. If $a_{34} \in U(R_4)$, then both $a_{21}, a_{41} \in Z(R_1)$ so that $x_2 \sim x_4$; a contradiction. Therefore, $a_{34} \in Z(R_4)$. Since $x_3 \nsim x_5$, we must have $a_{51} \in U(R_1)$ and $a_{54} \in U(R_4)$. It follows that $x_4 \nsim x_5$ in $\Gamma_U(R)$; again a contradiction. Therefore, $a_{31} \in U(R_1)$ and $a_{34} \in Z(R_4)$. Consequently, $x_2 \sim x_4$; a contradiction. Therefore, the case $a_{33} \in U(R_3)$ is not possible.

Thus, there does not exists an induced cycle of odd length greater than three. The proof is similar when $R \cong R_1 \times R_2 \times R_3$ or $R \cong R_1 \times R_2$. □

**Proposition 3.3.5.** *Let $(R_i, \mathcal{M}_i)$ be a local ring and $R \cong R_1 \times R_2 \times \cdots \times R_n$, where $n \leq 4$ and each $\mathcal{M}_i$ is principal. Then $\overline{\Gamma_U(R)}$ does not contain any induced* *cycle of odd length greater than three.*

*Proof.* The result is straightforward for $n = 1$ (cf. Theorem 3.1.5). Now, let $R \cong R_1 \times R_2 \times R_3 \times R_4$. On contrary, suppose that $\overline{\Gamma_U(R)}$ contains an induced cycle of odd length greater than three, namely $C : y_1 \sim y_2 \sim y_3 \sim \cdots \sim y_k \sim y_1$ and $k \geq 5$. Let $y_i = (b_{i1}, b_{i2}, b_{i3}, b_{i4}) \in R$. Since $y_1 \sim y_2$ in $\overline{\Gamma_U(R)}$, by Lemma 3.3.3, both $b_{11}, b_{21} \notin Z(R_1)$. Without loss of generality, assume that $b_{11} \in U(R_1)$. Since $y_1$ is a non-unit element of $R$, we have $b_{1j} \in Z(R_j)$, for some $j \in \{2, 3, 4\}$. Without loss of generality, assume that $b_{12} \in Z(R_2)$. By Lemma 3.3.3, we get $b_{22} \in U(R_2)$. Since $y_1 \sim y_2$ in $\overline{\Gamma_U(R)}$, we get both $b_{13}, b_{23} \notin Z(R_3)$. Now we have the following cases:

**Case-1.** $b_{13} \in U(R_3)$. Let $b_{14} \in U(R_4)$. Since $y_1 \nsim y_3$ and $y_1 \nsim y_4$ in $\overline{\Gamma_U(R)}$, we get $b_{32}, b_{42} \in Z(R_2)$. It follows that $y_3 \nsim y_4$, which is not possible. We may now suppose that $b_{14} \in Z(R_4)$. It follows that $b_{24}, b_{k4} \in U(R_4)$. Since $y_2$ is a non-unit element of $R$, we have either $b_{21} \in Z(R_1)$ or $b_{23} \in Z(R_3)$. Let $b_{21} \in Z(R_1)$. If $b_{23} \in U(R_3)$, then $b_{k1}, b_{(k-1)1} \in Z(R_1)$. It follows that $y_k \nsim y_{k-1}$ in $\overline{\Gamma_U(R)}$, which is not possible. Therefore, $b_{23} \in Z(R_3)$. Since $y_2 \sim y_3$, we obtain that $b_{31} \in U(R_1)$ and $b_{33} \in U(R_3)$. The adjacency of $y_1$ with $y_k$ follows that $b_{k2} \in U(R_2)$ and $b_{k4} \in U(R_4)$. It follows that $y_3 \sim y_k$ in $\overline{\Gamma_U(R)}$; a contradiction. Therefore, $b_{21} \notin Z(R_1)$. We may now suppose that $b_{23} \in Z(R_3)$. Since $y_2 \nsim y_k$ and $y_2 \nsim y_{k-1}$ in $\overline{\Gamma_U(R)}$, we have $b_{k3}, b_{(k-1)3} \in Z(R_3)$. It follows that $y_k \nsim y_{k-1}$. Thus, the case $b_{13} \in U(R_3)$ is not possible.

**Case-2.** $b_{23} \in U(R_3)$. First suppose that $b_{24} \in U(R_4)$. Since $y_2$ is a non-unit element of $R$, we have $b_{21} \in Z(R_1)$. Note that $y_2 \nsim y_k$ and $y_2 \nsim y_{k-1}$ in $\overline{\Gamma_U(R)}$ so that $b_{k1}, b_{(k-1)1} \in Z(R_1)$. It follows that $y_k \nsim y_{k-1}$; a contradiction. Therefore, $b_{24} \in Z(R_4)$. Since $y_1 \sim y_2 \sim y_3$ in $\overline{\Gamma_U(R)}$, we have $b_{14}, b_{34} \in U(R_4)$. First suppose that $b_{21} \in U(R_1)$. Since $y_2 \nsim y_k$ and $y_2 \nsim y_{k-1}$ in $\overline{\Gamma_U(R)}$, we have $b_{k4}, b_{(k-1)4} \in Z(R_4)$. It follows that $y_k \nsim y_{k-1}$; a contradiction. We may now suppose that $b_{21} \in Z(R_1)$. It follows that $b_{31} \in U(R_1)$. Since $y_3$ is a non-unit element of $R$, we have either $b_{32} \in Z(R_2)$ or $b_{33} \in Z(R_3)$.

Let $b_{32} \in Z(R_2)$. The adjacency of $y_3$ with $y_4$ implies that $b_{42} \in U(R_2)$. If $b_{33} \in U(R_3)$, then $b_{k2}, b_{12} \in Z(R_2)$. It follows that $y_1 \nsim y_k$ in $\overline{\Gamma_U(R)}$, which is not possible. Therefore, $b_{33} \in Z(R_3)$. Since $y_3 \sim y_4$, we have $b_{42} \in U(R_2)$ and $b_{43} \in U(R_3)$. Since $y_4$ is a non-unit element of $R$, we have either $b_{41} \in Z(R_1)$ or $b_{44} \in Z(R_4)$. If $b_{41} \in Z(R_1)$ and $b_{44} \in U(R_4)$, then $y_1 \sim y_4$; a contradiction. Therefore, $b_{44} \in Z(R_4)$. The adjacency of $y_4$ with $y_5$ follows that $b_{51} \in U(R_1)$ and $b_{54} \in U(R_4)$. It follows that $y_2 \sim y_5$ in $\overline{\Gamma_U(R)}$; a contradiction. Thus, $b_{41} \in U(R_1)$. Consequently, $b_{44} \in Z(R_4)$. Since $y_2 \nsim y_4$ and $y_1 \nsim y_4$, we have $b_{14}, b_{24} \in Z(R_4)$. It follows that $y_1 \nsim y_2$ in $\overline{\Gamma_U(R)}$; a contradiction. Therefore, this case is not possible.

Thus, $\overline{\Gamma_U(R)}$ does not contain an induced cycle of odd length greater than three. The proof is similar when $R \cong R_1 \times R_2 \times R_3$ or $R \cong R_1 \times R_2$. $\square$

By combining Lemma 3.3.2, and Propositions 3.3.4, 3.3.5, we get the following theorem.

**Theorem 3.3.6.** *Let $(R_i, \mathcal{M}_i)$ be a local ring and $R \cong R_1 \times R_2 \times \cdots \times R_n$, where $n \geq 3$. Then the graph $\Gamma_U(R)$ is perfect if and only if $n \in \{3, 4\}$ and each ideal $\mathcal{M}_i$ of $R_i$ is principal.*

In view of Proposition 3.3.4 and Proposition 3.3.5, we have the following lemma.

**Lemma 3.3.7.** *Let $(R_i, \mathcal{M}_i)$ be a local ring and $R \cong R_1 \times R_2 \times \cdots \times R_n$, where $n \in \{1, 2\}$ and each $\mathcal{M}_i$ is principal. Then $\Gamma_U(R)$ is a perfect graph.*

**Corollary 3.3.8.** *The graph $\Gamma_U(\mathbb{Z}_n)$ is perfect if and only if $n = p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4}$, where $p_i$'s are distinct prime numbers and $n_i \in \mathbb{N} \cup \{0\}$.*

**Proposition 3.3.9.** *Let $R \cong R_1 \times R_2$ such that $R_1, R_2$ are local rings with maximal ideals $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. If both $\mathcal{M}_1, \mathcal{M}_2$ are not principal, then $\Gamma_U(R)$ is not a perfect graph.*

*Proof.* Suppose both $\mathcal{M}_1$ and $\mathcal{M}_2$ are not principal ideals of $R_1$ and $R_2$, respectively. Then $R_1$ has at least two principal maximal ideals $(a_1)$ and $(a_2)$. Similarly, $R_2$ contains at least two principal maximal ideals $(b_1)$ and $(b_2)$. Observe that the set

$$X = \{(a_1, b_1), (a_2, b_2), (1, b_1), (a_1, b_2), (a_2, b_1)\}$$

induces $C_5$ in $\overline{\Gamma_U(R)}$. Therefore, $\Gamma_U(R)$ is not a perfect graph. $\square$

Based on our computation for various local rings of small order we propose the following conjecture.

**Conjecture:** Let $R \cong R_1 \times R_2$ such that $R_1, R_2$ are local rings with maximal ideals $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Then $\Gamma_U(R)$ is a perfect graph if and only if either $\mathcal{M}_1$ or $\mathcal{M}_2$ is principal.

## 3.4 Embedding of $\Gamma_U(R)$ on Surfaces

In this section, we study the topological properties of $\Gamma_U(R)$ including planar, projective planar, toroidal, bitoroidal, etc. We begin with the investigation of an embedding of $\Gamma_U(R)$ on a plane.

### 3.4.1 Planarity of $\Gamma_U(R)$

In this subsection, we classify all the non-local commutative rings for which the graph $\Gamma_U(R)$ is outerplanar and planar, respectively. We begin with the following lemma.

**Lemma 3.4.1.** *Let $R$ be a non-local commutative ring such that $R \cong R_1 \times R_2 \times \cdots \times R_n$ for $n \geq 4$. Then the graph $\Gamma_U(R)$ is not planar.*

*Proof.* Consider the set $X = \{(0,0,\cdots,0), (1,0,\cdots,0), (0,1,0,\cdots,0), (0,0,1,0,\cdots,0), (0,0,0,1,0,\cdots,0)\}$. Note that $\Gamma_U(X) \cong K_5$. Therefore, by Theorem 1.3.8, $\Gamma_U(R)$ is not a planar graph. $\square$

**Theorem 3.4.2.** *Let $R$ be a non-local commutative ring. Then $\Gamma_U(R)$ is outerplanar if and only if $R$ is isomorphic to one of the following 3 rings:*

$$\mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_2 \times \mathbb{Z}_3,\ \mathbb{Z}_3 \times \mathbb{Z}_3.$$

*Proof.* Let $R$ be a non-local ring. Then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each $R_i$ is a local ring and $n \geq 2$. Let $\Gamma_U(R)$ be an outerplanar graph. By Lemma 3.4.1, we must have $n \leq 3$. Suppose that $R \cong R_1 \times R_2 \times R_3$. If $|R_i| = 2$ for every $i \in \{1,2,3\}$, then for the set $X = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\}$ note that $\Gamma_U(X) \cong K_4$, which is not possible. We may now suppose that $R \cong R_1 \times R_2$. Let $|R_i| \geq 4$ for some $i \in \{1,2\}$. Without loss of generality, assume that $|R_1| = 4$ such that $R_1 = \{0, a_1, a_2, a_3\}$. Then for $X' = \{(0,0), (a_1,0), (a_2,0), (a_3,0)\}$, we have $\Gamma_U(X') \cong K_4$; again a contradiction. Consequently, $R \cong R_1 \times R_2$ with $|R_i| \leq 3$ for $i \in \{1,2\}$. Converse holds by Theorem 1.3.7 and Remark 3.1.1. $\square$

**Theorem 3.4.3.** *Let $R$ be a non-local commutative ring. Then $\Gamma_U(R)$ is a planar graph if and only if $R$ is isomorphic to one of the following 9 rings:*

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_2 \times \mathbb{Z}_4,\ \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)},\ \mathbb{Z}_2 \times \mathbb{F}_4,\ \mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_2 \times \mathbb{Z}_3,\ \mathbb{Z}_3 \times \mathbb{Z}_3,\ \mathbb{Z}_3 \times \mathbb{F}_4,\ \mathbb{F}_4 \times \mathbb{F}_4.$$

*Proof.* Suppose that $\Gamma_U(R)$ is a planar graph. In the similar lines of the proof of Theorem 3.4.2, we have either $R \cong R_1 \times R_2 \times R_3$ or $R \cong R_1 \times R_2$. Let $R \cong R_1 \times R_2 \times R_3$ such that $|R_i| \geq 3$ for some $i \in \{1,2,3\}$. Without loss of generality, assume that $|R_1| \geq 3$ with $a_1, a_2 \in R_1^*$. For the set

$$X = \{(0,0,0), (a_1,0,0), (a_2,0,0), (0,1,0), (0,0,1)\},$$

we have $\Gamma_U(X) \cong K_5$; a contradiction. Thus, $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ in this case. We may now suppose that $R \cong R_1 \times R_2$. If $|R_i| \geq 5$ for some $i \in \{1, 2\}$ and $a_1, a_2, a_3, a_4 \in R_i^*$, then for $X' = \{(0,0), (a_1, 0), (a_2, 0), (a_3, 0), (a_4, 0)\}$ note that $\Gamma_U(X') \cong K_5$ which is not possible. Thus, for $R \cong R_1 \times R_2$ we must have $|R_i| \leq 4$. Further, if $R_2$ is not a field of cardinality four, then either $R_2 \cong \mathbb{Z}_4$ or $R_2 \cong \frac{\mathbb{Z}_2[x]}{(x^2)}$ (cf. Table 1.1). Then there exists $z \in Z(R_2)^*$ and so $|R_1| \neq 3, 4$. Otherwise, the set $Y = \{(0,0), (a_1, 0), (a_2, 0), (0, z), (a_1, z)\}$, where $a_1, a_2 \in R_1^*$, induces a subgraph $\Gamma_U(Y)$ which is isomorphic to $K_5$. Consequently, $R$ is isomorphic to one of the rings: $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, $\mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_3 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3$. Conversely, if $R$ is isomorphic to one of the given rings then by Figures 3.1, 3.2 Theorem 1.3.8 and Remark 3.1.1, $\Gamma_U(R)$ is planar. $\square$

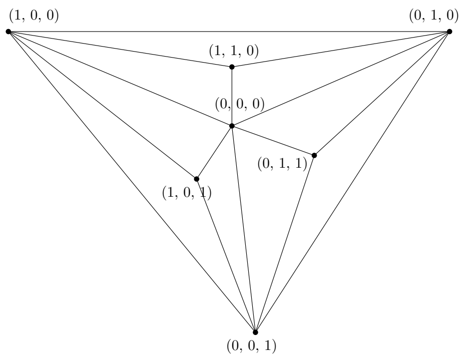



FIGURE 3.1: Planar drawing of $\Gamma_U(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

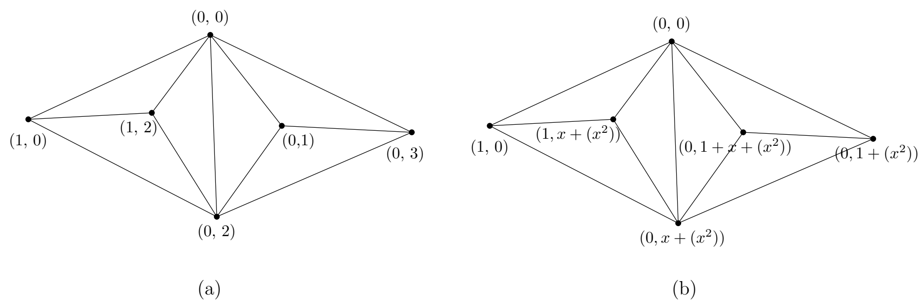



FIGURE 3.2: Planar drawing of (a) $\Gamma_U(\mathbb{Z}_2 \times \mathbb{Z}_4)$ and (b) $\Gamma_U(\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)})$

A graph $\Gamma$ which satisfies the PCP property is said to be a *ring graph* if $rank(\Gamma) = frank(\Gamma)$ and $\Gamma$ does not contain a subdivision of $K_4$ as a subgraph. In the similar lines of the proof of Theorem 3.4.2 and using Remark 3.1.1, we have the following proposition.

**Proposition 3.4.4.** *Let $R$ be a non-local commutative ring. Then $\Gamma_U(R)$ is a ring graph if and only if $R$ is isomorphic to one of the following 3 rings:*

$$\mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_2 \times \mathbb{Z}_3,\ \mathbb{Z}_3 \times \mathbb{Z}_3.$$

### 3.4.2 Genus of $\Gamma_U(R)$

In this subsection, we classify all the non-local commutative rings such that $\Gamma_U(R)$ has genus at most 2.

**Lemma 3.4.5.** *Let $R$ be a non-local commutative ring such that $R \cong R_1 \times R_2 \times \cdots \times R_n$ for $n \geq 4$. Then $\eth(\Gamma_U(R)) > 1$.*

*Proof.* Let $n \geq 4$. Then note that the vertices $x_1 = (0, 0, \ldots, 0)$, $x_2 = (1, 0, \ldots, 0)$, $x_3 = (0, 1, 0, \ldots, 0)$, $x_4 = (0, 0, 1, 0, \ldots, 0)$, $x_5 = (0, 0, 0, 1, 0, \ldots, 0)$, $x_6 = (1, 1, 0, 0, \ldots, 0)$, $x_7 = (1, 0, 1, 0, \ldots, 0)$, $x_8 = (1, 0, 0, 1, 0, \ldots, 0)$ induces a subgraph of $\Gamma_U(R)$ which is isomorphic to $K_8$. By Proposition 1.3.9, we have $\eth(\Gamma_U(R)) > 1$. $\square$

**Theorem 3.4.6.** *Let $R$ be a non-local commutative ring. Then the genus of $\Gamma_U(R)$ is 1 if and only if $R$ is isomorphic to one of the following 8 rings:*

$$\mathbb{Z}_2 \times \mathbb{Z}_7,\ \mathbb{Z}_3 \times \mathbb{Z}_7,\ \mathbb{F}_4 \times \mathbb{Z}_7,\ \mathbb{Z}_2 \times \mathbb{Z}_5,\ \mathbb{Z}_3 \times \mathbb{Z}_5,\ \mathbb{F}_4 \times \mathbb{Z}_5,\ \mathbb{Z}_3 \times \mathbb{Z}_4,\ \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}.$$

*Proof.* Let $R$ be a non-local ring. Then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each $R_i$ is a local ring and $n \geq 2$. First suppose that $\eth(\Gamma_U(R)) = 1$. By Lemma 3.4.5, we get $n \leq 3$. We claim that $n \neq 3$. Let, if possible $n = 3$, that is $R \cong R_1 \times R_2 \times R_3$. If $|R_i| \geq 3$

for each $i$, with $a_1, a_2 \in R_1^*$, $b_1, b_2 \in R_2^*$ and $c_1, c_2 \in R_3^*$, then the subgraph induced by $X = \{(0,0,0), (a_1,0,0), (0,b_1,0), (a_2,0,0), (0,b_2,0), (a_1,b_1,0), (a_1,b_2,0), (a_2,b_1,0)\}$ is isomorphic to $K_8$; a contradiction (cf. Proposition 1.3.9). Consequently, $R \cong R_1 \times R_2 \times R_3$ such that $|R_i| \leq 2$ for some $i$. Without loss of generality, assume that $|R_1| = 2$ and $|R_2| = |R_3| = 3$. Note that for the set

$$Y = \{(0,0,0), (0,b_1,0), (0,b_2,0), (0,0,c_1), (0,b_1,c_1), (0,b_2,c_1), (0,0,c_2), (0,b_1,c_2)\},$$

we obtain $\Gamma_U(Y) \cong K_8$; again a contradiction. Therefore, we may now suppose that $|R_1| = 2 = |R_2|$ and $|R_3| \leq 3$. For $|R_3| = 3$, we have $v = 10$, $e = 31$. By Lemma 1.3.10, we get $f = 21$. It follows that $2e < 3f$; a contradiction. Thus, $R \cong R_1 \times R_2 \times R_3$ such that $|R_i| = 2$ for every $i$. By Figure 3.1, $\Gamma_U(R)$ is planar. This completes our claim and so $R \cong R_1 \times R_2$. Now first note that either $|R_1| \geq 8$ or $|R_2| \geq 8$ then there exists an induced subgraph which is isomorphic to $K_8$; a contradiction. It follows that $R \cong R_1 \times R_2$ with $|R_i| \leq 7$ for $i = 1, 2$. Now we classify the ring $R$ such that $\Gamma_U(R)$ has genus 1 through the following cases.

**Case-1.** $|R_2| = 7$. If $|R_1| = 7$, then note that in $\Gamma_U(R)$, $v = 13$, $e = 42$ and $f = 29$. It follows that $2e < 3f$; a contradiction. We may now suppose that $|R_1| = 5$. By Proposition 1.3.9, Lemma 1.3.11 and Remark 3.1.1, we get $\eth(\Gamma_U(R)) > 1$ which is not possible. Thus, $|R_1| \leq 4$. If $R_1$ is not a field of cardinality four, then either $R_1 \cong \mathbb{Z}_4$ or $R_1 \cong \frac{\mathbb{Z}_2[x]}{(x^2)}$ (see Table 1.1). Consequently, there exists exactly one zero-divisor $z \in Z(R_1)^*$. Then for the set $X' = \{(0,0), (0,1), \ldots, (0,6), (z,0), (z,1)\}$, we get $\Gamma_U(X') \cong K_9$; a contradiction. Thus, in this case $R$ is isomorphic to one of the three rings: $\mathbb{Z}_2 \times \mathbb{Z}_7$, $\mathbb{Z}_3 \times \mathbb{Z}_7$, $\mathbb{F}_4 \times \mathbb{Z}_7$.

**Case-2.** $|R_2| = 5$. If $|R_1| = 5$, then by Proposition 1.3.9, Lemma 1.3.11 and Remark 3.1.1, we get $\eth(\Gamma_U(R)) \neq 1$; a contradiction. We may now suppose that $R_1$ is not a field of cardinality four. Then note that the set

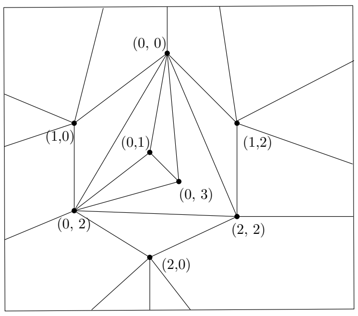
$$\{(0,0), (0,1), (0,2), (0,3), (0,4), (z,0), (z,1), (z,2), (z,3)\}, \text{ where } z \in Z(R_1)^*,$$

induces a subgraph which is isomorphic to $K_9$; again a contradiction. Consequently, $R$ is isomorphic to one of the following 3 rings: $\mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{F}_4 \times \mathbb{Z}_5$.

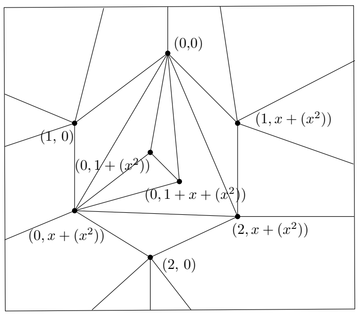
**Case-3.** $|R_2| = 4$. Suppose that $|R_1| = 4$. If both $R_1, R_2$ are fields, then by Theorem 3.4.3, $\Gamma_U(R_1 \times R_2)$ is planar and so $\eth(\Gamma_U(R)) = 0$ which is not possible. Thus, either $R_1$ or $R_2$ is not a field. Without loss of generality, assume that $R_1$ is not a field. By the argument used in **Case-1**, and by choosing $X'' = \{(0,0),(0,1),(0,b_1),(0,b_2),(z,0),(z,1),(z,b_1),(z,b_2)\}$, where $z \in Z(R_1)^*$, $b_1, b_2 \in R_2^*$, note that $\Gamma_U(X'') \cong K_8$. Consequently, $|R_1| \leq 3$. Let $|R_1| \leq 3$ and $R_2$ be a field. Then by Theorem 3.4.3, $\Gamma_U(R)$ is a planar graph; a contradiction. If $R_2$ is not a field and $|R_1| = 2$, then again by Theorem 3.4.3, $\eth(\Gamma_U(R)) = 0$; a contradiction. Thus, $R$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_4$ or $\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$.

**Case-4.** $|R_2| \leq 3$. If $|R_1| \in \{2,3\}$, then by Theorem 3.4.3, $\eth(\Gamma_U(R)) = 0$; a contradiction.

Conversely, if $R$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_4$ or $\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, then by Figure 3.3, we have $\eth(\Gamma_U(R)) = 1$. If $R$ is isomorphic to one of the remaining 6 given rings, then by Proposition 1.3.9, Lemma 1.3.11 and Remark 3.1.1, we get $\eth(\Gamma_U(R)) = 1$. $\square$




(a)




(b)

FIGURE 3.3: Embedding of (a) $\Gamma_U(\mathbb{Z}_3 \times \mathbb{Z}_4)$ and (b) $\Gamma_U(\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)})$ in $\mathbb{S}_1$

**Theorem 3.4.7.** *Let $R$ be a non-local commutative ring. Then $\eth(\Gamma_U(R)) = 2$ if and only if $R$ is isomorphic to one of the following 9 rings:*

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3,\ \mathbb{Z}_2 \times \mathbb{F}_8,\ \mathbb{Z}_3 \times \mathbb{F}_8,\ \mathbb{F}_4 \times \mathbb{F}_8,\ \mathbb{Z}_7 \times \mathbb{Z}_7,\ \mathbb{Z}_5 \times \mathbb{Z}_7,\ \mathbb{Z}_5 \times \mathbb{Z}_5,\ \mathbb{F}_4 \times \mathbb{Z}_4,$$
$$\mathbb{F}_4 \times \frac{\mathbb{Z}_2[x]}{(x^2)}.$$

*Proof.* Let $R$ be a non-local ring. Then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each $R_i$ is a local ring and $n \geq 2$. First assume that $\eth(\Gamma_U(R)) = 2$. If $n \geq 5$, then note that the vertices $x_1 = (0,0,\ldots,0)$, $x_2 = (1,0,\ldots,0)$, $x_3 = (0,1,0,\ldots,0)$, $x_4 = (0,0,1,0,\ldots,0)$, $x_5 = (0,0,0,1,0,\ldots,0)$, $x_6 = (0,0,0,0,1,0,\ldots,0)$, $x_7 = (1,1,0,0,\ldots,0)$, $x_8 = (1,0,1,0,\ldots,0)$, $x_9 = (1,0,0,1,0,\ldots,0)$ induces a subgraph isomorphic to $K_9$; a contradiction. We may now suppose that $n = 4$. If $|R_i| = 2$ for every $i \in \{1,2,3,4\}$, then note that $v = 15$, $e = 80$. Further, by Lemma 1.3.10, we get $f = 63$. It follows that $2e < 3f$ in $\Gamma_U(R)$, which is not possible. Therefore, $n \leq 3$. Let $n = 3$ that is $R \cong R_1 \times R_2 \times R_3$. If $|R_i| \geq 4$, for each $i$, with $a_1, a_2, a_3 \in R_1^*$, $b_1, b_2, b_3 \in R_2^*$ and $c_1, c_2, c_3 \in R_3^*$, then the subgraph of $\Gamma_U(R)$ induced by the set $X = \{(0,0,0), (a_1,0,0), (0,b_1,0), (0,0,c_1), (a_2,0,0), (0,b_2,0), (0,0,c_2), (a_3,0,0), (0,b_3,0)\}$ is isomorphic to $K_9$; a contradiction. Without loss of generality, suppose that $|R_1| \leq 3$ and $|R_2| = 4 = |R_3|$. For $X' = \{(0,0,0), (a_1,0,0), (0,b_1,0), (0,0,c_1), (a_2,0,0), (0,b_2,0), (0,0,c_2), (0,b_3,0), (0,0,c_3)\}$, we get $\Gamma_U(X') \cong K_9$; again a contradiction. Therefore, $|R_i| \leq 3$ for $i = 1,2$ and $|R_3| = 4$. We may now suppose that $|R_1| = 2 = |R_2|$ and $R_3$ is a field of size 4. Then $v = 13$, $e = 54$ and Lemma 1.3.10 follows that $f = 39$ which is not possible. Consequently, $|R_i| \leq 3$ for each $i \in \{1,2,3\}$. Next, let $|R_1| = |R_2| = |R_3| = 3$. Then the subgraph induced by the vertices $(0,0,0), (a_1,0,0), (0,b_1,0), (a_2,0,0), (0,b_2,0), (a_1,b_1,0), (a_1,b_2,0), (a_2,b_1,0)$ and $(a_2,b_2,0)$ is isomorphic to $K_9$; a contradiction. Therefore, we may now suppose that $|R_1| = 2$ and $|R_2| = 3 = |R_3|$. Consider the set $Y = \{(0,0,0), (0,0,c_1), (0,b_1,0), (0,0,c_2), (0,b_2,0), (0,b_1,c_1), (0,b_2,c_1), (0,b_1,c_2), (0,b_2,c_2)\}$. Note that $\Gamma_U(Y) \cong K_9$; a contradiction. If $|R_i| = 2$, for each $i \in \{1,2,3\}$, then $\Gamma_U(R)$ is a planar graph (cf. Theorem 3.4.3). Consequently, $R$ is isomorphic to the ring $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.

Now let $R \cong R_1 \times R_2$. If either $|R_1| \geq 9$ or $|R_2| \geq 9$, then there exists an induced subgraph of $\Gamma_U(R)$ which is isomorphic to $K_9$; a contradiction. It follows that the cardinality of each $R_i$ is at most 8. Now we characterize the rings $R \cong R_1 \times R_2$ such that $\Gamma_U(R)$ has genus 2 through the following cases.

**Case-1.** $|R_2| = 8$. Suppose that $|R_1| = 8$. If both $R_1, R_2$ are fields, then $v = 15$, $e = 56$ and Lemma 1.3.10 gives $f = 39$; a contradiction. If $R_1$ is not a field, then the existence of $K_9$ as an induced subgraph of $\Gamma_U(R)$ gives $\eth(\Gamma_U(R)) > 2$; a contradiction. Consequently, $|R_1| \neq 8$.

**Subcase-1.1.** $R_2$ is not a field. Then $R_2$ is isomorphic to one of the following rings: $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{(x^3)}$, $\frac{\mathbb{Z}_2[x,y]}{(x^2,xy,y^2)}$, $\frac{\mathbb{Z}_4[x]}{(2x,x^2)}$, $\frac{\mathbb{Z}_4[x]}{(2x,x^2-2)}$ (cf. Table 1.1). First suppose that $R_2$ is isomorphic to $\mathbb{Z}_8$ or $\frac{\mathbb{Z}_2[x]}{(x^3)}$. If $|R_1| \in \{3, 4, 5, 7\}$, then we can easily get $K_9$ as an induced subgraph of $\Gamma_U(R)$; a contradiction. If $|R_1| = 2$, then note that $\Gamma_U(R)$ has 12 vertices, 50 edges and 36 faces (cf. Lemma 1.3.10); a contradiction to Remark 1.3.16.

Now suppose that $R_2$ is isomorphic to one of the following 3 rings: $\frac{\mathbb{Z}_2[x,y]}{(x^2,xy,y^2)}$, $\frac{\mathbb{Z}_4[x]}{(2x,x^2)}$, $\frac{\mathbb{Z}_4[x]}{(2x,x^2-2)}$. If $|R_1| \in \{5, 7\}$, then $\Gamma_U(R)$ has an induced subgraph isomorphic to $K_9$; a contradiction. Next, let $R_1$ be not a field of size 4 and $z \in Z(R_1)^*$. Then note that the set $\{(0,0), (0,b_1), \dots, (0,b_7), (z,0)\}$, where $b_1, b_2, \dots, b_7 \in R_2^*$, induces a subgraph which is isomorphic to $K_9$; a contradiction. Let $R_1$ be a field of size 4. Then $\Gamma_U(R)$ has 20 vertices, 97 edges and so Lemma 1.3.10 gives 75 faces; a contradiction (cf. Remark 1.3.16). If $|R_1| = 3$, then $v = 16$, $e = 64$ and by Lemma 1.3.10 we get $f = 46$; a contradiction. If $|R_1| = 2$, then note that $\Gamma_U(R)$ has 12 vertices and 41 edges. Let $u_1, u_2, u_3, u_4 \in U(R_2)$ and $z_1, z_2, z_3 \in Z(R_2)^*$. Note that $V(\Gamma_U(R)) = \{(0,0), (0,u_1), (0,u_2), (0,u_3), (0,u_4), (0,z_1), (0,z_2), (0,z_3), (1,0), (1,z_1), (1,z_2), (1,z_3)\}$. Let $f_i$ be the number of faces of size $i$ in an embedding of $\Gamma_U(R)$. Since $\eth(\Gamma_U(R)) = 2$, we have $f = 27$ (cf. Lemma 1.3.10). Note that $f_4 + 2f_5 = 2e - 3f = 1$. It follows that any embedding of $\Gamma_U(R)$ in $\mathbb{S}_2$ has 26 triangular faces and one quadrilateral face. Now consider the set

$$X = \{(0,0),(0,u_1),(0,u_2),(0,u_3),(0,u_4),(0,z_1),(0,z_2),(0,z_3)\},$$

we get $\Gamma_U(X) \cong K_8$. Consequently, any embedding of $\Gamma_U(X)$ has either one pentagonal, 17 triangular faces or two quadrilateral, 16 triangular faces. Suppose $\Gamma_U(X)$ has one pentagonal, 17 triangular faces in an embedding on $\mathbb{S}_2$. If we insert the set $Y = \{(1,0),(1,z_1),(1,z_2),(1,z_3)\}$ of vertices and their respective edges to the embedding of $\Gamma_U(X)$, then $\Gamma_U(Y)$ must be embedded in the pentagonal face. Consequently, any embedding of $\Gamma_U(R)$ on $\mathbb{S}_2$ leads to an edge crossing. Similarly, the insertion of $\Gamma_U(Y)$ in an embedding of $\Gamma_U(X)$, when it has two quadrilateral and 16 triangular faces, yields to an edge crossing. Therefore, $\eth(\Gamma_U(R)) > 2$; a contradiction and so this subcase is not possible.

**Subcase-1.2.** $R_2$ be a field of size 8. If $|R_1| \in \{5,7\}$, then by Lemma 1.3.11 and Remark 3.1.1, we obtain $\eth(\Gamma_U(R)) \neq 2$; a contradiction. If $|R_1| = 4$ such that $R_1$ is not a field, then there exists a $z \in Z(R_1)^*$. Consequently, for the set $Y' = \{(0,0),(0,1),\cdots,(0,7),(z,0)\}$, we have $\Gamma_U(Y') \cong K_9$; a contradiction. Thus, $R$ is isomorphic to one of the following 3 rings: $\mathbb{Z}_2 \times \mathbb{F}_8$, $\mathbb{Z}_3 \times \mathbb{F}_8$, $\mathbb{F}_4 \times \mathbb{F}_8$.

**Case-2.** $|R_2| \in \{5,7\}$. If $R_1$ is a field and $|R_1| \leq 4$, then by Theorem 3.4.6, we have $\eth(\Gamma_U(R)) = 1$. If $R_1$ is not a field of size 4, then for $z \in Z(R_1)^*$ note that $\Gamma_U(A) \cong K_9$, where $A = \{(0,0),(0,1),(0,2),(0,3),(0,4),(z,0),(z,1),(z,2),(z,3)\}$. Consequently, $R$ is isomorphic to one of the following 3 rings: $\mathbb{Z}_7 \times \mathbb{Z}_7, \mathbb{Z}_5 \times \mathbb{Z}_7, \mathbb{Z}_5 \times \mathbb{Z}_5$.

**Case-3.** $|R_2| = 4$. Assume that $|R_1| = 4$. If both $R_1, R_2$ are fields, then $\Gamma_U(R_1 \times R_2)$ is planar ( cf. Theorem 3.4.3); a contradiction. We may now suppose that both $R_1$ and $R_2$ are not fields. Then note that $\Gamma_U(R)$ has 12 vertices, 50 edges and so 36 faces (cf. Remark 1.3.16); a contradiction. Note that $|R_1| \notin \{2,3\}$ (cf. Theorem 3.4.3 and Theorem 3.4.6). Thus, $|R_1| = 4$ and both $R_1$ and $R_2$ can not be fields. Consequently, either $R \cong \mathbb{F}_4 \times \mathbb{Z}_4$ or $R \cong \mathbb{F}_4 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$.

**Case-4.** $|R_2| \leq 3$. If $|R_1| \leq 3$, then $\Gamma_U(R)$ is a planar graph (cf. Theorem 3.4.3).

Conversely, suppose that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. By Figure 3.4, $\eth(\Gamma_U(R)) =$

2. If either $R \cong \mathbb{F}_4 \times \mathbb{Z}_4$, or $R \cong \mathbb{F}_4 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, then $\Gamma_U(R) \cong K_2 \vee (K_2 \bigcup K_6)$. Consequently, $\eth(\Gamma_U(R)) = 2$ (cf. Proposition 1.3.9 and Lemma 1.3.11). Further, if $R$ is isomorphic to one of the remaining 6 rings, then by Lemma 1.3.11 and Remark 3.1.1, we obtain $\eth(\Gamma_U(R)) = 2$. $\square$

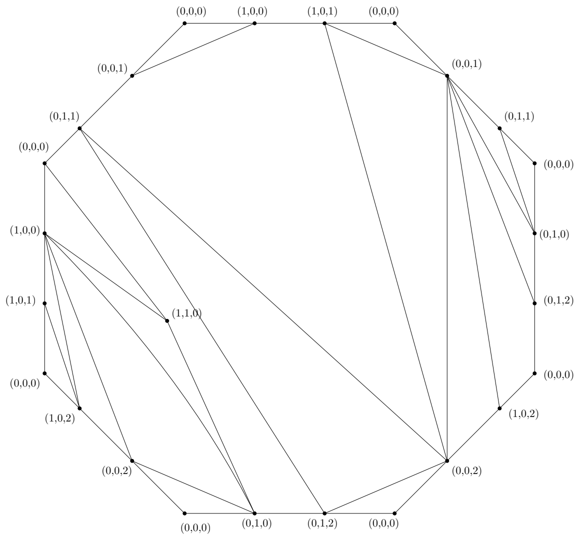



FIGURE 3.4: Embedding of $\Gamma_U(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ in $\mathbb{S}_2$

### 3.4.3 Crosscap of $\Gamma_U(R)$

In this subsection, we characterize all the non-local commutative rings $R$ such that $\Gamma_U(R)$ has crosscap at most 2. We begin with the following lemma.

**Lemma 3.4.8.** *Let $R$ be a non-local commutative ring such that $R \cong R_1 \times R_2 \times \cdots \times R_n$, for $n \geq 4$. Then $cr(\Gamma_U(R) > 2$.*

*Proof.* Let $n \geq 4$. Then note that the vertices $x_1 = (0, 0, \ldots, 0)$, $x_2 = (1, 0, \ldots, 0)$, $x_3 = (0, 1, 0, \ldots, 0)$, $x_4 = (0, 0, 1, 0, \ldots, 0)$, $x_5 = (1, 1, 0, 0, \ldots, 0)$, $x_6 = (1, 0, 1, 0, \ldots, 0)$, $x_7 = (0, 1, 1, 0, \ldots, 0)$, $x_8 = (1, 1, 1, 0, \ldots, 0)$ induces a subgraph of $\Gamma_U(R)$ which is isomorphic to $K_8$. Therefore, by Proposition 1.3.12, we get $cr(\Gamma_U(R)) > 2$. $\square$

**Theorem 3.4.9.** *Let $R$ be a non-local commutative ring. Then the crosscap of $\Gamma_U(R)$ is 1 if and only if $R$ is isomorphic to one of the following 5 rings:*

$$\mathbb{Z}_2 \times \mathbb{Z}_5,\ \mathbb{Z}_3 \times \mathbb{Z}_5,\ \mathbb{F}_4 \times \mathbb{Z}_5,\ \mathbb{Z}_3 \times \mathbb{Z}_4,\ \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}.$$

*Proof.* Let $R$ be a non-local ring. Then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each $R_i$ is a local ring and $n \geq 2$. Suppose that $cr(\Gamma_U(R)) = 1$. In view of Lemma 3.4.8, either $R \cong R_1 \times R_2 \times R_3$ or $R \cong R_1 \times R_2$. First assume that $R \cong R_1 \times R_2 \times R_3$. Suppose that for each $i \in \{1, 2, 3\}$, $|R_i| = 3$ with $a_1, a_2 \in R_1^*$, $b_1, b_2 \in R_2^*$ and $c_1, c_2 \in R_3^*$. Then for the set $X = \{(0, 0, 0), (a_1, 0, 0), (0, b_1, 0), (a_2, 0, 0), (a_1, b_1, 0), (0, b_2, 0), (a_1, b_2, 0)\}$, we get $\Gamma_U(X) \cong K_7$; a contradiction. Without loss of generality, assume that $|R_1| = 2$ and $|R_2| = 3 = |R_3|$. Then the set

$$X' = \{(0, 0, 0), (0, 0, c_1), (0, b_1, 0), (0, 0, c_2), (0, b_1, c_1), (0, b_2, 0), (0, b_2, c_1)\}$$

induces $K_7$ as a subgraph of $\Gamma_U(R)$, which is not possible. Thus, we have $|R_1| = 2 = |R_2|$ and $|R_3| = 3$. It implies that $\Gamma_U(R)$ has 10 vertices, 31 edges and then Lemma 1.3.13 follows that $f = 22$; a contradiction to Remark 1.3.16. Therefore, $|R_i| = 2$ for each $i$ and so $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, which is not possible (see Figure 3.1). Consequently, $R \cong R_1 \times R_2$. If either $|R_1| \geq 7$ or $|R_2| \geq 7$, then we can easily get $K_7$ as an induced subgraph of $\Gamma_U(R)$. By Proposition 1.3.12, $cr(\Gamma_U(R)) \neq 1$; a contradiction. In view of Remark 1.2.6, we classify $R$ through the following cases.

**Case-1.** $|R_2| = 5$. Note that $|R_1| \neq 5$ (cf. Remark 3.1.1 and Lemma 1.3.15). Therefore, $|R_1| \leq 4$. Further, if $R_1$ is not a field of size 4, then there exists a $z \in Z(R_1)^*$. The set $X'' = \{(0, 0), (0, 1), \ldots, (0, 4), (z, 0), (z, 1)\}$ induces $K_7$ as a subgraph of $\Gamma_U(R)$; again a contradiction. Thus, $R$ is isomorphic to one of the following three rings: $\mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{F}_4 \times \mathbb{Z}_5$.

**Case-2.** $|R_2| = 4$. Let $|R_1| = 4$. If both $R_1, R_2$ are fields, then $\Gamma_U(R)$ is a planar graph ( cf. Theorem 3.4.3). Without loss of generality, assume that $R_1$ is not a field. Then for $z \in Z(R_1)^*$, note that the set

$$Y = \{(0,0),(0,b_1),(0,b_2),(0,b_3),(z,0),(z,b_1),(z,b_2)\}$$

induces $K_7$ as a subgraph of $\Gamma_U(R)$; a contradiction. Therefore, $|R_1| \neq 4$. Further, suppose that $|R_1| \in \{2,3\}$ and $R_2$ is a field. Then by Theorem 3.4.3, $\Gamma_U(R)$ is planar. Next, if $|R_1| = 2$ and $R_2$ is not a field, then by Theorem 3.4.3, $cr(\Gamma_U(R)) = 0$; again a contradiction. Thus, either $R \cong \mathbb{Z}_3 \times \mathbb{Z}_4$ or $R \cong \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$.

By Theorem 3.4.3, note that $|R_i| \leq 3$ is not possible. Conversely, if $R \cong \mathbb{F}_4 \times \mathbb{Z}_5$, then from Figure 3.5, $cr(\Gamma_U(R)) = 1$. For $R \cong \mathbb{Z}_2 \times \mathbb{Z}_5$, we get $\Gamma_U(\mathbb{Z}_2 \times \mathbb{Z}_5) \cong \Gamma_U(X)$, where $X = V(\Gamma_U(\mathbb{F}_4 \times \mathbb{Z}_5)) - \{(2,0),(3,0)\}$. It follows that $cr(\Gamma_U(\mathbb{Z}_2 \times \mathbb{Z}_5)) = 1$. Similarly, $cr(\Gamma_U(\mathbb{Z}_3 \times \mathbb{Z}_5)) = 1$ because $\Gamma_U(\mathbb{Z}_3 \times \mathbb{Z}_5) \cong \Gamma_U(Y)$ for $Y = V(\Gamma_U(\mathbb{F}_4 \times \mathbb{Z}_5)) - \{(3,0)\}$. Finally, if either $R \cong \mathbb{Z}_3 \times \mathbb{Z}_4$ or $R \cong \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, then by Figure 3.6, $cr(\Gamma_U(R)) = 1$. □

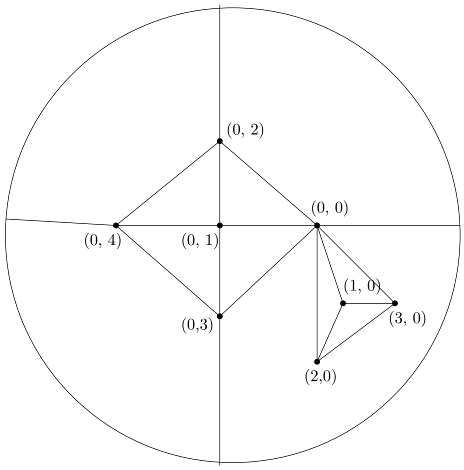



Figure 3.5: Embedding of $\Gamma_U(\mathbb{F}_4 \times \mathbb{Z}_5)$ in $\mathbb{N}_1$

**Theorem 3.4.10.** *Let $R$ be a non-local commutative ring. Then $cr(\Gamma_U(R)) = 2$ if and only if $R$ is isomorphic to the ring $\mathbb{Z}_5 \times \mathbb{Z}_5$.*

*Proof.* Let $cr(\Gamma_U(R)) = 2$ and $R$ be a non-local commutative ring. Then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where $n \geq 2$ and each $R_i$ is a local ring. By Lemma 3.4.8, we get $n \leq 3$. First suppose that $n = 3$, that is $R \cong R_1 \times R_2 \times R_3$. If $|R_1| = |R_2| =$

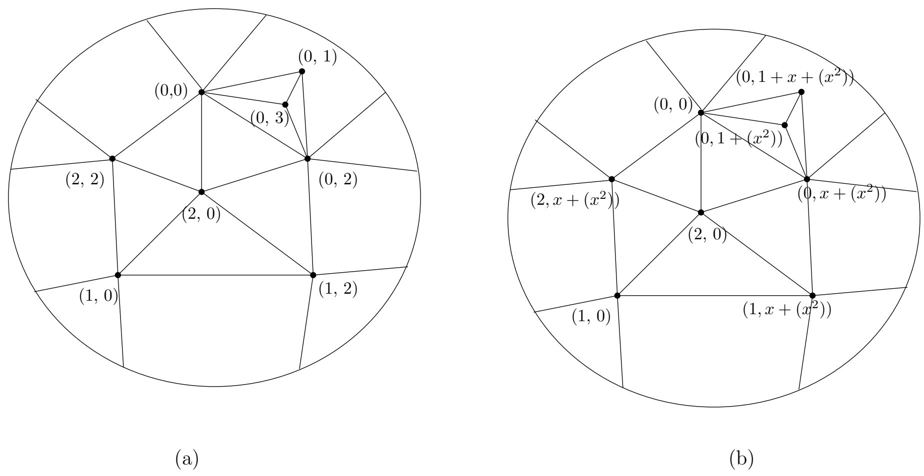



FIGURE 3.6: Embedding of (a) $\Gamma_U(\mathbb{Z}_3 \times \mathbb{Z}_4)$ and (b) $\Gamma_U(\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)})$ in $\mathbb{N}_1$

$|R_3| = 3$, then the set $\{(0,0,0),(1,0,0),(2,0,0),(0,1,0),(0,2,0),(0,0,1),(0,0,2)\}$ induces $K_7$ as a subgraph of $\Gamma_U(R)$; a contradiction. Without loss of generality, assume that $|R_1| = 2$ and $|R_2| = 3 = |R_3|$. For the set

$$X = \{(0,0,0),(0,1,0),(0,2,0),(0,0,1),(0,0,2),(0,1,1),(0,1,2)\},$$

we get $\Gamma_U(X) \cong K_7$; a contradiction. We may now suppose that $|R_1| = 2 = |R_2|$ and $|R_3| = 3$. Note that $v = 10$, $e = 31$ and so Lemma 1.3.13 gives $f = 21$; a contradiction to the Remark 1.3.16. Consequently, $|R_i| = 2$, for each $i \in \{1, 2, 3\}$. This is also not possible (cf. Figure 3.1). Thus, $R \cong R_1 \times R_2$. If either $|R_1| \geq 7$ or $|R_2| \geq 7$, then we can easily get $K_7$ as an induced subgraph of $\Gamma_U(R)$. By Proposition 1.3.12, we get $cr(\Gamma_U(R)) \neq 2$; a contradiction. Therefore, by Remark 1.2.6, $|R_i| \leq 5$ for each $i = 1, 2$. Now we have the following cases:

**Case-1.** $|R_2| = 5$. Let $R_1$ be a field such that $|R_1| \in \{2, 3, 4\}$. Then by Theorem 3.4.9, we get $cr(\Gamma_U(R)) = 1$. If $R_1$ is not a field of size 4, then by the **Case-1** of Theorem 3.4.9, we get a contradiction. Therefore, $R \cong \mathbb{Z}_5 \times \mathbb{Z}_5$.

**Case-2.** $|R_2| = 4$. Suppose that $|R_1| \in \{2, 3, 4\}$. If both $R_1$ and $R_2$ are fields, then $\Gamma_U(R)$ is a planar graph (cf. Theorem 3.4.3). Now suppose that $R_2$ is not

a field with $z \in Z(R_2)^*$. If $|R_1| \in \{2,3\}$, then by Theorem 3.4.3 and Theorem 3.4.9, we have $cr(\Gamma_U(R)) \neq 2$; a contradiction. If $|R_1| = 4$, then the set $Y = \{(0,0),(b_1,0),(b_2,0),$
$(b_3,0),(0,z),\dots,(b_2,z)\}$, where $b_1,b_2,b_3 \in R_1^*$, induces $K_7$ as a subgraph of $\Gamma_U(R)$. Thus, this case is not possible.

In view of above cases and by Theorem 3.4.3, note that the case $|R_2| \leq 3$ is not possible. Conversely if $R \cong \mathbb{Z}_5 \times \mathbb{Z}_5$, then by Remark 3.1.1 and Lemma 1.3.15, we obtain $cr(\Gamma_U(R)) = 2$. □

## 3.5 Forbidden Subgraphs of $\Gamma_U(R)$

In this section, we classify all the non-local commutative rings whose upper ideal relation graphs are split graphs, threshold graphs, cographs, cactus graphs and unicyclic graphs.

**Theorem 3.5.1.** *Let $R$ be a non-local commutative ring. Then $\Gamma_U(R)$ is a split graph if and only if $R$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{F}_q$.*

*Proof.* First suppose that $\Gamma_U(R)$ is a split graph. Since $R$ is a non-local commutative ring, we have $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each $R_i$ is a local ring and $n \geq 2$. If $n \geq 4$, then $\Gamma_U(R)$ has a subgraph induced by $u_1 = (1,1,0,1,\dots,1)$, $u_2 = (0,1,0,1,\dots,1)$ and $v_1 = (1,0,1,1,\dots,1)$, $v_2 = (1,0,1,0,1,\dots,1)$ which is isomorphic to $2K_2$; a contradiction. Thus, either $R \cong R_1 \times R_2 \times R_3$ or $R \cong R_1 \times R_2$. Let $a_1,a_2 \in U(R_1)$, $b_1,b_2 \in U(R_2)$ and $c_1,c_2 \in U(R_3)$. First suppose that $R \cong R_1 \times R_2 \times R_3$. If $|R_i| \geq 3$ for every $i \in \{1,2,3\}$, then notice that $u_1 = (a_1,0,c_1)$, $u_2 = (a_2,0,c_2)$ and $v_1 = (0,b_1,c_1)$, $v_2 = (0,b_2,c_2)$ induces a subgraph of $\Gamma_U(R)$ which is isomorphic to $2K_2$. Without loss of generality, assume that $|R_1| = 2$ and $|R_2| = 3 = |R_3|$. Then for the set $X = \{(0,b_1,c_1),(0,b_2,c_2)(1,b_1,0),(1,b_2,0)\}$ we have $\Gamma_U(X) \cong 2K_2$; a contradiction. Therefore, both $R_2$ and $R_3$ can not have cardinality three. We may now suppose that $|R_1| = 2 = |R_2|$ and $|R_3| = 3$. The set $X' = \{(0,1,c_1),(0,1,c_2),(1,0,c_1),(1,0,c_2)\}$ induces $2K_2$ as a subgraph of $\Gamma_U(R)$ which is not possible. Consequently, $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Now, suppose that $R \cong R_1 \times R_2$. If $|R_1| \geq 3$ and $|R_2| \geq 3$, then the set of vertices $\{(a_1,0),(a_2,0),(0,b_1),(0,b_2)\}$ induces a subgraph isomorphic to $2K_2$; a contradiction. Without loss of generality, assume that $|R_1| = 2$ and $|R_2| \geq 3$. If $R_2$ is not a field, then there exists $z \in Z(R_2)^*$. Consequently, the subgraph induced by $X'' = \{(0,b_1),(0,b_2),(1,0),(1,z)\}$ is isomorphic to $2K_2$, which is not possible. Thus, $R \cong \mathbb{Z}_2 \times \mathbb{F}_q$.

Conversely, suppose that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Note that $V(\Gamma_U(R)) = \mathcal{V}_1 \cup \mathcal{V}_2$, where $\mathcal{V}_1 = \{(1,1,0),(1,0,1),(0,1,1)\}$ is an independent set and $\mathcal{V}_2 = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\}$ forms a complete subgraph of $\Gamma_U(R)$. Moreover, $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$. Thus, by definition, $\Gamma_U(R)$ is a split graph. If $R \cong \mathbb{Z}_2 \times \mathbb{F}_q$, then note that $V(\Gamma_U(R)) = \mathcal{V}_1 \cup \mathcal{V}_2$, where $\mathcal{V}_1 = \{(1,0)\}$ and $\mathcal{V}_2 = \{(0,b) : b \in \mathbb{F}_q\}$. Consequently, the result holds. □

**Theorem 3.5.2.** *Let $R$ be a non-local commutative ring. Then $\Gamma_U(R)$ is a threshold graph if and only if $R$ is isomorphic to the ring $\mathbb{Z}_2 \times \mathbb{F}_q$.*

*Proof.* Since $R$ is a non-local commutative ring, we have $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each $R_i$ is a local ring and $n \geq 2$. Suppose that $\Gamma_U(R)$ is a threshold graph. Then $\Gamma_U(R)$ is a split graph also. By Theorem 3.5.1, either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $R \cong \mathbb{Z}_2 \times \mathbb{F}_q$. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then there exists an induced subgraph $\Gamma_U(X)$, where $X = \{(1,0,1),(1,0,0),(0,1,0),(0,1,1)\}$, which is isomorphic to $P_4$ which is not possible. Consequently, $R \cong \mathbb{Z}_2 \times \mathbb{F}_q$.

Conversely, if $R \cong \mathbb{Z}_2 \times \mathbb{F}_q$, then $\Gamma_U(R) \cong K_1 \vee (K_1 \cup K_{|\mathbb{F}_q|-1})$ (cf. Remark 3.1.1). It follows that $\Gamma_U(R)$ is a threshold graph. □

**Theorem 3.5.3.** *Let $R$ be a non-local commutative ring such that $R \cong R_1 \times R_2 \times \cdots \times R_n$, $(n \geq 2)$ and each $(R_i, \mathcal{M}_i)$ is a local ring. Then $\Gamma_U(R)$ is a cograph if and only if $R \cong R_1 \times R_2$ and $\mathcal{M}_1, \mathcal{M}_2$ are the maximal principal ideals.*

*Proof.* First suppose that $\Gamma_U(R)$ is a cograph. If $n \geq 3$, consider the set

$$X = \{(1, 0, 1, \ldots, 1), (1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), (0, 1, \ldots, 1)\}.$$

Notice that $\Gamma_U(X) \cong P_4$; a contradiction. Thus, $R \cong R_1 \times R_2$, where $(R_1, \mathcal{M}_1)$ and $(R_2, \mathcal{M}_2)$ are local rings. Now we show that both the ideals $\mathcal{M}_1$ and $\mathcal{M}_2$ are maximal principal. Without loss of generality, assume that $\mathcal{M}_1$ is not a maximal principal ideal. Then $R_1$ has at least two maximal principal ideals, namely: $(x_1)$ and $(x_2)$. For instance, for some $y \in R_1$, if $(y)$ is the only maximal principal ideal of $R_1$ then $R_1 \setminus U(R_1) = \mathcal{M}_1 \subseteq (y)$. Consequently, $\mathcal{M}_1 = (y)$; a contradiction. Further, note that $x_1 \nsim x_2$ in $\Gamma_U(R_1)$. Moreover, $(x_1, 1) \sim (x_1, 0) \sim (x_2, 0) \sim (x_2, 1)$ is an induced subgraph which is isomorphic to $P_4$; a contradiction. Thus, both $\mathcal{M}_1$ and $\mathcal{M}_2$ must be the maximal principal ideals of $R_1$ and $R_2$, respectively.

Conversely, suppose that $R \cong R_1 \times R_2$ and $\mathcal{M}_i$ is the principal ideals of $R_i$. To prove $\Gamma_U(R)$ is a cograph, consider the sets

$$V_1 = \{(z_1, z_2) : z_1 \in \mathcal{M}_1, z_2 \in \mathcal{M}_2\};$$
$$V_2 = \{(z_1, u_2) : z_1 \in \mathcal{M}_1, u_2 \in R_2 \setminus \mathcal{M}_2\};$$
$$V_3 = \{(u_1, z_2) : u_1 \in R_1 \setminus \mathcal{M}_1, z_2 \in \mathcal{M}_2\}.$$

Observe that $V_1, V_2$ and $V_3$ forms a partition of $V(\Gamma_U(R))$. Since $\mathcal{M}_1$ is the principal ideal and $\mathcal{M}_1 = V(\Gamma_U(R_1))$, we obtain $\Gamma_U(R_1) \cong K_{|\mathcal{M}_1|}$. Similarly, $\Gamma_U(R_2) \cong K_{|\mathcal{M}_2|}$. Consequently, $\Gamma_U(V_i) \cong K_{|V_i|}$ for each $i \in \{1, 2, 3\}$. Let $x = (z_1, z_2) \in V_1$. If $y = (t_1, t_2) \in V_2$, then note that $(x), (y) \subseteq ((z, t_2))$, where $\mathcal{M}_1 = (z)$. It follows that $x \sim y$ in $\Gamma_U(R_1 \times R_2)$. Similarly, $x \sim y$ for every $y \in V_3$. Consequently, $x \sim y$ for every $y \in V(\Gamma_U(R_1 \times R_2))$. Further, note that for each $x \in V_2$ and $y \in V_3$, we have $x \nsim y$ in $\Gamma_U(R_1 \times R_2)$. Thus, $\Gamma_U(R) \cong K_{|V_1|} \vee (K_{|V_2|} \bigcup K_{|V_3|})$. Hence, $\Gamma_U(R)$ is a cograph. □

**Theorem 3.5.4.** *Let $R$ be a non-local commutative ring. Then $\Gamma_U(R)$ is a cactus graph if and only if $R$ is isomorphic to one of the following 3 rings: $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3$.*

*Proof.* Let $R$ be a non-local ring. Then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each $R_i$ is a local ring and $n \geq 2$. First suppose that $\Gamma_U(R)$ is a cactus graph. For $n \geq 3$, the graph $\Gamma_U(R)$ has two cycles (infact, triangles)

$$\mathcal{C}_1 : (1, 0, \ldots, 0) \sim (1, 1, 0, \ldots, 0) \sim (0, 1, 0, \ldots, 0) \sim (1, 0, \ldots, 0); \text{ and}$$
$$\mathcal{C}_2 : (0, 1, 0, \ldots, 0) \sim (0, 0, 1, \ldots, 0) \sim (1, 0, \ldots, 0) \sim (0, 1, 0, \ldots, 0),$$

which has a common edge $(1, 0, \ldots, 0) \sim (0, 1, 0, \ldots, 0)$; a contradiction. Thus, $R \cong R_1 \times R_2$. Without loss of generality, assume that $|R_1| \geq 4$ with $a_1, a_2, a_3 \in R_1^*$. Then the cycles $\mathcal{C}_1 : (a_1, 0) \sim (0, 0) \sim (a_2, 0) \sim (a_1, 0)$ and $\mathcal{C}_2 : (0, 0) \sim (a_3, 0) \sim (a_1, 0) \sim (0, 0)$ has a common edge. Therefore, $|R_i| \leq 3$ for every $i \in \{1, 2\}$. Thus, the result holds. Converse follows from Remark 3.1.1. □

**Theorem 3.5.5.** *Let $R$ be a non-local commutative ring. Then $\Gamma_U(R)$ is unicyclic if and only if $R$ is isomorphic to the ring $\mathbb{Z}_2 \times \mathbb{Z}_3$.*

*Proof.* Let $R$ be a non-local ring. Then $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each $R_i$ is a local ring and $n \geq 2$. First suppose that $\Gamma_U(R)$ is a unicyclic graph. If $n \geq 3$, then $\Gamma_U(R)$ has two cycles $\mathcal{C}_1$ and $\mathcal{C}_2$, where

$$\mathcal{C}_1 : (1, 0, \ldots, 0) \sim (0, 1, 0, \ldots, 0) \sim (0, 0, 1, 0, \ldots, 0) \sim (1, 0, \ldots, 0); \text{ and}$$
$$\mathcal{C}_2 : (1, 0, \ldots, 0) \sim (1, 0, 1, \ldots, 0) \sim (0, 0, 1, 0, \ldots, 0) \sim (1, 0, \ldots, 0).$$

Therefore, $R \cong R_1 \times R_2$. Now suppose that $|R_i| \geq 3$ for each $i = 1, 2$ with $a_1, a_2 \in R_1^*$ and $b_1, b_2 \in R_2^*$. Note that for the sets $X_1 = \{(a_1, 0), (0, 0), (a_2, 0)\}$ and $X_2 = \{(0, 0), (0, b_1), (0, b_2)\}$, we get $\Gamma_U(X_i) \cong C_3$, where $i \in \{1, 2\}$; a contradiction to the fact that $\Gamma_U(R)$ has a unique cycle. We may now suppose that $R \cong R_1 \times R_2$ with $|R_i| \leq 2$ for some $i$. Without loss of generality, assume that $|R_1| = 2$. If $|R_2| \geq 4$ and $b_1, b_2, b_3 \in R_2^*$, then for the set $Y_1 = \{(0,0),(0,b_1),(0,b_2)\}$ and $Y_2 = \{(0,0),(0,b_3),(0,b_2)\}$, we get $\Gamma_U(Y_i) \cong C_3$ for $i \in \{1,2\}$; a contradiction. Consequently, $|R_1| = 2$ and $|R_2| \leq 3$. If $|R_2| = 2$, then clearly $\Gamma_U(R)$ does not have cycle (cf. Remark 3.1.1). Thus, $|R_2| = 3$ and so $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. Converse is straightforward from Remark 3.1.1. □

## 3.6 The Upper Ideal-Relation Graph of the Ring $\mathbb{Z}_n$

In this section, we study the upper ideal-relation graph of the ring $\mathbb{Z}_n$. We obtain the vertex connectivity and the automorphism group of $\Gamma_U(\mathbb{Z}_n)$. Moreover, we classify all the values of $n$ such that $\Gamma_U(\mathbb{Z}_n)$ is Hamiltonian. Finally, we obtain the spectrum of $\Gamma_U(\mathbb{Z}_n)$ for $n = p^\alpha q^\beta$, where $\alpha, \beta$ are non-negative integers and $p, q$ are prime numbers. First, we discuss about the structure of the graph $\Gamma_U(\mathbb{Z}_n)$.

Note that the sets $\mathcal{A}_{d_1}, \mathcal{A}_{d_2}, \ldots, \mathcal{A}_{d_k}$ and $\mathcal{A}_n (= \{0\})$ form a partition of $V(\Gamma_U(\mathbb{Z}_n))$ (see Remark 1.2.4).

**Lemma 3.6.1.** *Let $x \in \mathcal{A}_{d_i}$ and $y \in \mathcal{A}_{d_j}$, where $1 \leq i, j \leq k$. Then $x \sim y$ in $\Gamma_U(\mathbb{Z}_n)$ if and only if there exists a proper divisor $d$ of $n$ such that $d|x$ and $d|y$.*

*Proof.* First suppose that $x \sim y$. Then $x, y \in (z)$ for some $z \in V(\Gamma_U(\mathbb{Z}_n))$. It follows that $z \in \mathcal{A}_d$ for some proper divisor $d$ of $n$. Therefore, $x, y \in (d)$. Consequently, $d|x$ and $d|y$. Conversely, assume that $d|x$ and $d|y$. It implies that $x, y \in (d)$. Consequently, $x \sim y$. □

**Corollary 3.6.2.** *For every $x, y \in \mathcal{A}_{d_i}$, we have $x \sim y$ in $\Gamma_U(\mathbb{Z}_n)$.*

**Theorem 3.6.3.** *For distinct primes $p$ and $q$, the graph $\Gamma_U(\mathbb{Z}_n)$ is Hamiltonian if and only if $n \neq 4$ and $n \neq pq$.*

*Proof.* First suppose that the graph $\Gamma_U(\mathbb{Z}_n)$ is Hamiltonian. Let $n = pq$. Note that $\mathbb{Z}_{pq} \cong K_1 \vee (K_{q-1} \cup K_{p-1})$, which is not a Hamiltonian graph. For $n = 4$, we have $\Gamma_U(\mathbb{Z}_4) \cong K_2$ and therefore $\Gamma_U(\mathbb{Z}_4)$ is not a Hamiltonian graph. Conversely, assume that $n \neq 4$ and $n \neq pq$. If $n = p$, then $V(\Gamma_U(\mathbb{Z}_n)) = \{0\}$, which is trivially a Hamiltonian graph. If $n = p^r$, where $r \geq 2$, then $\Gamma_U(\mathbb{Z}_{p^r}) \cong K_{p^{r-1}}$. Since $n \neq 4$, we have $\Gamma_U(\mathbb{Z}_{p^r})$ is a complete graph of at least three vertices. Thus $\Gamma_U(\mathbb{Z}_{p^r})$ is Hamiltonian. We may now suppose that $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_m^{\alpha_m}$, where $m \geq 2$. Note that $(p_1), (p_2), \ldots, (p_m)$ are $m$ distinct maximal ideals of $\mathbb{Z}_n$ and the subgraph of $\Gamma_U(\mathbb{Z}_n)$ induced by each of these ideals forms a clique in $\Gamma_U(\mathbb{Z}_n)$. Further, note that for any two distinct maximal ideals $(p_i)$ and $(p_j)$, we have $(p_i) \sim p_i p_j \sim (p_j)$. One can construct a Hamiltonian cycle using the adjacency $0 \sim (p_1) \sim p_1 p_2 \sim (p_2) \sim \cdots \sim p_{m-1} p_m \sim (p_m)$. Hence, the graph $\Gamma_U(\mathbb{Z}_n)$ is Hamiltonian. $\square$

Now we obtain the vertex connectivity of $\Gamma_U(\mathbb{Z}_n)$. By Corollary 3.1.6, we have $\kappa(\Gamma_U(\mathbb{Z}_{p^\alpha})) = p^{\alpha-1} - 1$. If $n = p^\alpha q^\beta$, then note that

$$\Gamma_U(\mathbb{Z}_n) \cong K_{|(pq)|} \vee (K_{|(p)\setminus(pq)|} \cup K_{|(q)\setminus(pq)|}).$$

It follows that $\kappa(\Gamma_U(\mathbb{Z}_n)) = |(pq)| = p^{\alpha-1} q^{\beta-1}$. For distinct primes divisors $p_1, p_2, \ldots, p_r$ of $n$, we define the set

$$(p_1 p_2 \cdots p_r)^\flat = \{u p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} : u \in U(\mathbb{Z}_n) \text{ and } \alpha_i > 0 \text{ for each } i \in \{1, 2, \ldots, r\}\}.$$

**Lemma 3.6.4.** *Let $n = p_1^{n_1} p_2^{n_2} \cdots p_{r-1}^{n_{r-1}} p_r^{n_r} p_{r+1}^{n_{r+1}} \cdots p_m^{n_m}$, where $p_i$'s are distinct primes and $r \leq m-2$. If $p_k < p_t$ with $k, t \notin \{1, 2, \ldots, r\}$, then $|(p_1 p_2 \cdots p_r p_t)^\flat| \leq |(p_1 p_2 \cdots p_r p_k)^\flat|$.*

*Proof.* Let $x = up_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}p_t^{\alpha_t}$ and $y = up_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}p_k^{\alpha_t}$ such that $x < n$. Then note that the map $\psi : (p_1p_2\cdots p_rp_t)^\flat \to (p_1p_2\cdots p_rp_k)^\flat$ defined by $\psi(x) = y$ is a one-one map. Thus, the result holds. $\square$

**Theorem 3.6.5.** *For $m \geq 3$, let $n = p_1^{n_1}p_2^{n_2}\cdots p_m^{n_m}$, where $p_i$'s are distinct primes and $p_1 < p_2 < \cdots < p_m$. Then the set*

$$T = (p_1p_m) \cup (p_2p_m) \cup \cdots \cup (p_{m-1}p_m)$$

*is a cut set of $\Gamma_U(\mathbb{Z}_n)$. Moreover, the vertex connectivity of $\Gamma_U(\mathbb{Z}_n)$ is $|T|$.*

*Proof.* To prove, $T$ is a cut set of $\Gamma_U(\mathbb{Z}_n)$, we show that the subgraph induced by $V(\Gamma_U(\mathbb{Z}_n)) \setminus T$ is disconnected. Let $x \in (p_m) \setminus T$ and $y \in (p_r) \setminus T$, where $r \neq m$. Assume that $x \sim y$. Then there exists a proper divisor $d$ of $n$ such that $d|x$ and $d|y$. Consequently, $d \in (p_rp_m) \subseteq T$. It follows that $x, y \in T$; a contradiction. Thus, $x \in (p_m) \setminus T$ is not adjacent to any $y \in (p_r) \setminus T$, where $r \neq m$. Therefore, there is no path between $x$ and $y$ in $\Gamma_U(\mathbb{Z}_n)$. Hence, $T$ is a cut set. To obtain the vertex connectivity of $\Gamma_U(\mathbb{Z}_n)$, we show that $T$ is a minimum cut set. In this connection, we show that there are at least $|T|$ vertex disjoint paths between $x$ and $y$ in $\Gamma_U(\mathbb{Z}_n)$ (see [West, 1996, Theorem 4.2.21]). We discuss the following cases.

**Case-1.** $x, y \in (p_i)$. Since the subgraph induced by $(p_i)$ forms a complete subgraph; we get $|(p_i)| - 1$ vertex disjoint paths between $x$ and $y$. Note that $T \subset (p_m)$ and for each $p_i$, we have $|(p_i)| \geq |(p_m)|$. Consequently, there exist at least $|T|$ vertex disjoint paths between $x$ and $y$.

**Case-2.** $x \in (p_i)$, $y \in (p_j)$ such that $p_i > p_j$ and $i \neq m$. Then for any prime divisor $p_r(r \neq i, j, m)$ of $n$, we get the paths $x \sim u \sim v \sim w \sim y$, where $u \in (p_ip_r)^\flat$, $v \in (p_rp_m)^\flat$, $w \in (p_rp_j)^\flat$. By Lemma 3.6.4, $|(p_ip_r)^\flat|$, $|(p_rp_j)^\flat| > |(p_rp_m)^\flat|$, we obtain at least $|(p_rp_m)^\flat|$ vertex disjoint paths between $x$ and $y$. Analogously, for $z \in (p_ip_m)^\flat$, $z' \in (p_jp_m)^\flat$, we get the paths $x \sim z \sim z' \sim y$ between $x$ and $y$ in $\Gamma_U(\mathbb{Z}_n)$. Note that we have $|(p_ip_m)^\flat|$ vertex disjoint paths between $x$ and $y$. In addition to these paths, we also have $x \sim z \sim y$, where $z \in (p_ip_j)^\flat$. It follows that we have at least $|(p_jp_m)^\flat|$ vertex disjoint paths because $|(p_ip_j)^\flat| > |(p_jp_m)^\flat|$. Consequently, we get at least $\sum_{i_1=1}^{m-1} |(p_{i_1}p_m)^\flat|$ vertex disjoint paths between $x$ and $y$ in $\Gamma_U(\mathbb{Z}_n)$. Similarly, we have following additional vertex disjoint paths between $x$ and $y$

$$x \sim u \sim v \sim w \sim y, \text{ where } u \in (p_ip_rp_{r'})^\flat, v \in (p_rp_{r'}p_m)^\flat, u \in (p_rp_{r'}p_j)^\flat$$
$$x \sim z \sim z' \sim y, \text{ where } z \in (p_ip_rp_m)^\flat, z' \in (p_jp_rp_m)^\flat$$
$$x \sim z'' \sim y, \text{ where } z'' \in (p_ip_rp_j)^\flat$$
$$x \sim z''' \sim y, \text{ where } z''' \in (p_ip_jp_m)^\flat.$$

Consequently, we get at least $\sum_{i_1<i_2} |(p_{i_1}p_{i_2}p_m)^\flat|$ vertex disjoint paths between $x$ and $y$. On continuing in this way, we shall get at least

$$l = \sum_{i_1=1}^{m-1} |(p_{i_1}p_m)^\flat| + \sum_{i_1<i_2} |(p_{i_1}p_{i_2}p_m)^\flat| + \cdots + \sum_{i_1<i_2<\cdots<i_{m-2}} |(p_{i_1}p_{i_2}\cdots p_{i_{m-2}}p_m)^\flat| + |(p_{i_1}p_{i_2}\cdots p_{m-1}p_m)^\flat|$$

vertex disjoint paths between $x$ and $y$. Also note that $|T| = l$. Thus, we have at least $|T|$ vertex disjoint paths between $x$ and $y$ in $\Gamma_U(\mathbb{Z}_n)$.

**Case-3**: $x \in (p_i), y \in (p_m)$ and $i \neq m$. Then, for any prime divisor $p_r (r \neq i, m)$ of $n$, we get the following vertex disjoint paths

$$x \sim u \sim v \sim y, \text{ where } u \in (p_ip_r)^\flat, \ v \in (p_rp_m)^\flat$$
$$x \sim z \sim y, \text{ where } z \in (p_ip_m)^\flat.$$

Consequently, we get at least $\sum_{i_1=1}^{m-1} |(p_{i_1}p_m)^\flat|$ vertex disjoint path between $x$ and $y$. Now in the similar lines of the **Case-2**, we shall get at least $l = |T|$ vertex disjoint paths between $x$ and $y$. Thus, the result follows. $\square$

### 3.6.1 Automorphism Group of $\Gamma_U(\mathbb{Z}_n)$

In this subsection, we obtain the automorphism group $Aut(\Gamma_U(\mathbb{Z}_n))$ of the ring $\mathbb{Z}_n$. For $n = p_1^{n_1}p_2^{n_2}p_3^{n_3}\cdots p_m^{n_m}$, consider the following sets

$$X_{p_1} = \bigcup_{i=1}^{n_1} \mathcal{A}_{p_1^i}, \quad X_{p_2} = \bigcup_{i=1}^{n_2} \mathcal{A}_{p_2^i}, \ldots, X_{p_m} = \bigcup_{i=1}^{n_m} \mathcal{A}_{p_m^i},$$

$$X_{p_1p_2} = \bigcup_{i=1}^{n_1}\left(\bigcup_{j=1}^{n_2} \mathcal{A}_{p_1^ip_2^j}\right), \ldots, X_{p_\alpha p_\beta} = \bigcup_{i=1}^{n_\alpha}\left(\bigcup_{j=1}^{n_\beta} \mathcal{A}_{p_\alpha^ip_\beta^j}\right),$$

$$\vdots$$

$$X_{p_{i_1}p_{i_2}\cdots p_{i_k}} = \bigcup_{j_1=1}^{n_{i_1}}\bigcup_{j_2=1}^{n_{i_2}}\cdots\bigcup_{j_k=1}^{n_{i_k}} \mathcal{A}_{p_{i_1}^{j_1}p_{i_2}^{j_2}\cdots p_{i_k}^{j_k}}, \qquad k < m$$

$$\vdots$$

$$X_{p_1p_2\cdots p_m} = \bigcup_{i_1=1}^{n_1}\bigcup_{i_2=1}^{n_2}\cdots\bigcup_{i_m=1}^{n_m} \mathcal{A}_{p_1^{i_1}p_2^{i_2}\cdots p_m^{i_m}}.$$

**Lemma 3.6.6.** *Let $x \in X_r$. Then $x \in (r)$.*

*Proof.* Without loss of generality, let $r = p_1p_2\cdots p_k$. Then $x \in \mathcal{A}_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}}$, where $1 \leq \alpha_i \leq n_i$. It follows that $(x) = (p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k})$. Thus, the result holds. □

**Lemma 3.6.7.** *Let $x, y \in V(\Gamma_U(\mathbb{Z}_n))$. Then $x, y \in X_r$ if and only if $N[x] = N[y]$.*

*Proof.* Let $x, y \in X_r$. Then by Lemma 3.6.1 and Corollary 3.6.2, $x \sim y$. Now suppose $z \sim x$ for some $z \in V(\Gamma_U(\mathbb{Z}_n))$. By Lemma 3.6.1, there exists a prime divisor $p$ of $n$ such that $p|x$ and $p|z$. It follows that $x, z \in (p)$. Since $x \in X_r$, by Lemma 3.6.6, we have $p|r$. Consequently, we get $y \in (p)$. It implies that $y \sim z$. Conversely, let $N[x] = N[y]$. For any prime divisor $p$ of $n$, note that if $p|x$ then $p|y$ and vice-versa. Now suppose $x \in X_r$ and $y \in X_{r'}$ with $r \neq r'$. Then, without loss of generality, there exists a prime divisor $q$ of $n$ such that $q|r$ but $q \nmid r'$; a contradiction. □

**Corollary 3.6.8.** *Let* $x, y \in X_r$. *Then* $\deg(x) = \deg(y)$.

**Theorem 3.6.9.** *Let* $n = p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots p_m^{n_m}$. *Then*

$$Aut(\Gamma_U(\mathbb{Z}_n)) \cong S_{|X_{p_1}|} \times S_{|X_{p_2}|} \times \cdots \times S_{|X_{p_1 p_2 \cdots p_m}|}.$$

*Proof.* By Lemma 3.6.7, the vertices of each partition set $X_r$ have same adjacency with the other vertices of $\Gamma_U(\mathbb{Z}_n)$. Thus, for each $r$, the vertices of $X_r$ can be permuted among themselves.

Now we show that a vertex of $X_r$ cannot be mapped to a vertex of $X_{r'}$ $(r \neq r')$ under an automorphism $f$ of $\Gamma_U(\mathbb{Z}_n)$. Let $x \in X_r$ and $y \in X_{r'}$ with $r \neq r'$. Suppose $f(x) = y$. Let $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$ be the prime divisors of $n$ which divides either $x$ or $y$ but not both. Let $p_i = \max\{p_{i_1}, p_{i_2}, \ldots, p_{i_k}\}$. Without loss of generality assume that $p_i$ divides $x$ but not $y$. Thus, we have $x \sim p_i$. We know that $\deg(p_i) = \frac{n}{p_i} - 1$. Let $f(p_i) = t$. Since $f$ is an automorphism, we get $t \sim y$ and $\deg(t) = \frac{n}{p_i} - 1$. It implies that $t \in (p_i)$. Consequently, we get $y \in (p_i)$ and $p_i | y$; a contradiction. Hence, the result holds. $\square$

### 3.6.2 The Laplacian Spectrum of $\Gamma_U(\mathbb{Z}_n)$

In this subsection, we find the Laplacian spectra of $\Gamma_U(\mathbb{Z}_n)$ for $n = p^\alpha q^\beta$, where $p, q$ are distinct primes. By Corollary 3.1.6, the Laplacian eigenvalues of $\Gamma_U(\mathbb{Z}_{p^\alpha})$ are 0 and $p^{\alpha-1}$ with multiplicity 1 and $p^{\alpha-1} - 1$, respectively.

**Theorem 3.6.10.** *Let* $n = p^\alpha q^\beta$. *Then the Laplacian spectrum of* $\Gamma_U(\mathbb{Z}_n)$ *is given by*

$$\begin{pmatrix} 0 & t & t(p+q-1) & tq & tp \\ 1 & 1 & t & t(q-1)-1 & t(p-1)-1 \end{pmatrix}$$

*where* $t = p^{\alpha-1} q^{\beta-1}$.

*Proof.* Let $n = p^\alpha q^\beta$. Then the proper divisors of $n$ are of the form $p^i q^j$, where $0 \leq i \leq \alpha$, $0 \leq j \leq \beta$. Note that

$$V(\Gamma_U(\mathbb{Z}_n)) = (\mathcal{A}_p \bigcup \mathcal{A}_{p^2} \bigcup \cdots \bigcup \mathcal{A}_{p^\alpha}) \bigcup (\mathcal{A}_q \bigcup \mathcal{A}_{q^2} \bigcup \cdots \bigcup \mathcal{A}_{q^\beta}) \bigcup (\bigcup_{j=1}^{\beta} \mathcal{A}_{pq^j}) \bigcup (\bigcup_{j=1}^{\beta} \mathcal{A}_{p^2q^{j_1}})$$
$$\bigcup \cdots \bigcup (\bigcup_{j=1}^{\beta-1} \mathcal{A}_{p^\alpha q^j}).$$

Let $x_1 \in \mathcal{A}_{p^{i_1}}, x_2 \in \mathcal{A}_{p^{i_2}}$ and $y_1 \in \mathcal{A}_{q^{j_1}}, y_2 \in \mathcal{A}_{q^{j_2}}$. Then by Lemma 3.6.1, we get $x_1 \sim x_2$ and $y_1 \sim y_2$ in $\Gamma_U(\mathbb{Z}_n)$. Also, note that for $x \in \mathcal{A}_{p^i}$ and $y \in \mathcal{A}_{q^j}$, we have $x \nsim y$. Further, for any $x \in \mathcal{A}_{p^iq^j}$ and $y \in \mathcal{A}_{p^i} \cup \mathcal{A}_{q^j}$ such that $1 \leq i \leq \alpha$, $1 \leq j \leq \beta$ with $i + j \neq \alpha + \beta$, we have $x \sim y$ in $\Gamma_U(\mathbb{Z}_n)$ (cf. Lemma 3.6.1). Also, for any proper divisor $p^iq^j$ $(i, j \neq 0)$ of $n$, we obtain $x \sim y$ in $\Gamma_U(\mathbb{Z}_n)$. Next, we have

$$\begin{aligned}
\sum_{i=1}^{\alpha} |\mathcal{A}_{p^i}| &= |\mathcal{A}_p| + |\mathcal{A}_{p^2}| + \cdots + |\mathcal{A}_{p^\alpha}| \\
&= \phi(p^{\alpha-1}q^\beta) + \phi(p^{\alpha-2}q^\beta) + \cdots + \phi(q^\beta) \\
&= p^{\alpha-1}q^{\beta-1}(q-1) = t(q-1)
\end{aligned}$$

Similarly, we obtain

$$\sum_{j=1}^{\beta} |\mathcal{A}_{q^j}| = p^{\alpha-1}q^{\beta-1}(p-1) = t(p-1)$$

Also, we have $0 \sim x$, for each $x \in \Gamma_U(\mathbb{Z}_n)$. Note that the graph

$$\Gamma_U(\mathbb{Z}_n) \cong K_t \vee (K_{t(q-1)} \cup K_{t(p-1)}).$$

By Theorem 1.3.20, the characteristic polynomial of the Laplacian matrix of $K_{t(q-1)} \cup K_{t(p-1)}$ is

$$\mu(K_{t(q-1)} \cup K_{t(p-1)}, x) = x(x - t(q-1))^{t(q-1)-1} \times x(x - t(p-1))^{t(p-1)-1}.$$

Consequently, by Theorem 1.3.19, the characteristic polynomial of the Laplacian matrix of $K_t \vee (K_{t(q-1)} \cup K_{t(p-1)})$ is

$$x(x - (t(p+q-1))^t \times (x - tq)^{t(q-1)-1} \times (x-t)(x-tp)^{t(p-1)-1}.$$

Therefore, the Laplacian eigenvalues of $\Gamma_U(\mathbb{Z}_n)$ are 0, $t$, $t(p+q-1)$, $tq$, and $tp$ with multiplicities 1, 1, $t$, $t(q-1)-1$ and $t(p-1)-1$, respectively. Thus, the result holds. $\square$

### 3.6.3 The Normalized Laplacian Spectrum of $\Gamma_U(\mathbb{Z}_n)$

In this subsection, we investigate the normalized Laplacian eigenvalues of the graph $\Gamma_U(\mathbb{Z}_n)$ for $n = p^\alpha q^\beta$. Note that $\Gamma_U(\mathbb{Z}_{p^\alpha q^\beta}) = P_3[K_{|M_1|}, K_{|M_2|}, K_{|M_3|}]$, where $M_1 = (p) \setminus (pq)$, $M_2 = (pq)$ and $M_3 = (q) \setminus (pq)$. Also, $|M_1| = \sum\limits_{i=1}^{\alpha} |\mathcal{A}_{p^i}| = p^{\alpha-1}q^\beta - p^{\alpha-1}q^{\beta-1}$, $|M_2| = p^{\alpha-1}q^{\beta-1}$ and $|M_3| = \sum\limits_{j=1}^{\beta} |\mathcal{A}_{q^j}| = p^\alpha q^{\beta-1} - p^{\alpha-1}q^{\beta-1}$.

**Example 3.6.11.** Let $p$ be a prime and $\alpha$ be a positive integer. Then $\Gamma_U(\mathbb{Z}_{p^\alpha}) \cong K_{p^{\alpha-1}}$. Therefore, the normalized Laplacian spectrum of $\Gamma_U(\mathbb{Z}_{p^\alpha})$ is $\begin{pmatrix} 0 & \frac{p^{\alpha-1}}{p^{\alpha-1}-1} \\ 1 & p^{\alpha-1}-1 \end{pmatrix}$.

**Example 3.6.12.** If $n = p^2q$, then $\Gamma_U(\mathbb{Z}_n) \cong P_3[K_{pq-p}, K_p, K_{p^2-p}]$. By Lemma 1.3.21 and Lemma 1.3.22, the eigenvalues of $\Gamma_U(\mathbb{Z}_n)$ are $\frac{pq}{pq-1}$, $\frac{p(p+q-1)}{p(p+q-1)-1}$ and $\frac{p^2}{p^2-1}$ with multiplicities $pq-p-1$, $p-1$ and $p^2-p-1$, respectively. The remaining 3 eigenvalues are the eigenvalues of the matrix

$$\begin{bmatrix} \frac{p}{pq-1} & \frac{-p}{\sqrt{(pq-1)(p(p+q-1)-1)}} & 0 \\ \frac{-p(q-1)}{\sqrt{(pq-1)(p(p+q-1)-1)}} & \frac{p(p+q-2)}{p(p+q-1)-1} & \frac{-p(p-1)}{\sqrt{(p^2-1)(p(p+q-1)-1)}} \\ 0 & \frac{-p}{\sqrt{(p^2-1)(p(p+q-1)-1)}} & \frac{p}{p^2-1} \end{bmatrix}.$$

**Theorem 3.6.13.** *The normalized Laplacian eigenvalues of $\Gamma_U(\mathbb{Z}_{p^\alpha q^\beta})$ consists of the eigenvalues $\dfrac{|M_1|+|M_2|}{|M_1|+|M_2|-1}$ with multiplicity $|M_1|-1$, $\dfrac{|M_1|+|M_2|+|M_3|}{|M_1|+|M_2|+|M_3|-1}$ with multiplicity $|M_2|-1$ and $\dfrac{|M_2|+|M_3|}{|M_2|+|M_3|-1}$ with multiplicity $|M_3|-1$ and the remaining 3 eigenvalues are the eigenvalues of the matrix*

$$\begin{bmatrix} \frac{|M_2|}{|M_1|+|M_2|-1} & \frac{-|M_2|}{\sqrt{(|M_1|+|M_2|-1)(|M_1|+|M_2|+|M_3|-1)}} & 0 \\ \frac{-|M_1|}{\sqrt{(|M_1|+|M_2|+|M_3|-1)(|M_1|+|M_2|-1)}} & \frac{|M_1|+|M_3|}{|M_1|+|M_2|+|M_3|-1} & \frac{-|M_3|}{\sqrt{(|M_1|+|M_2|+|M_3|-1)(|M_2|+|M_3|-1)}} \\ 0 & \frac{-|M_2|}{\sqrt{(|M_2|+|M_3|-1)(|M_1|+|M_2|+|M_3|-1)}} & \frac{|M_2|}{|M_2|+|M_3|-1} \end{bmatrix}$$

*where,* $|M_1| = p^{\alpha-1}q^{\beta} - p^{\alpha-1}q^{\beta-1}, |M_2| = p^{\alpha-1}q^{\beta-1}$ *and* $|M_3| = p^{\alpha}q^{\beta-1} - p^{\alpha-1}q^{\beta-1}$.

*Proof.* Note that $\Gamma_U(\mathbb{Z}_{p^\alpha q^\beta}) = \Gamma[\Gamma_1, \Gamma_2, \Gamma_3]$, where $\Gamma = P_3$, $\Gamma_1 = K_{|M_1|}, \Gamma_2 = K_{|M_2|}$ and $\Gamma_3 = K_{|M_3|}$. In view of Lemma 1.3.22, we have $r_i = |M_i|-1$ for each $i = 1, 2, 3$. By Lemma 1.3.21, for $1 \le i \le 3$, the adjacency spectrum of $K_{|M_i|}$ is given by

$$\begin{pmatrix} |M_i|-1 & -1 \\ 1 & |M_i|-1 \end{pmatrix}.$$

Let $\{v_1, v_2, v_3\}$ is the vertex set of $P_3$ corresponding to the graph $\Gamma_1, \Gamma_2, \Gamma_3$. Since $\Gamma_U(\mathbb{Z}_n) \cong K_{|M_2|} \vee (K_{|M_1|} \cup K_{|M_3|})$, we have

$$\alpha_1 = \sum_{v_j \in N_\Gamma(v_1)} n_i = |\Gamma_2| = |M_2|,$$

$$\alpha_2 = \sum_{v_j \in N_\Gamma(v_2)} n_i = |\Gamma_1| + |\Gamma_3| = |M_1| + |M_3|,$$

$$\alpha_3 = \sum_{v_j \in N_\Gamma(v_3)} n_i = |\Gamma_2| = |M_2|.$$

Now by Lemma 1.3.22, the normalized Laplacian eigenvalues of $\Gamma_U(\mathbb{Z}_{p^\alpha q^\beta})$ are given by $1 - \dfrac{1}{r_i + \alpha_i}\lambda_{ik}(\Gamma_i)$, for $i = 1, 2, 3$ and $k = 2, 3, \cdots, n_i$. Since $\lambda_{ik} = -1$, where $i = 1, 2, 3$ and $2 \le k \le n_i$, by Lemma 1.3.22, the normalized Laplacian eigenvalues of $\Gamma_U(\mathbb{Z}_n)$ are

$$\frac{|M_1|+|M_2|}{|M_1|+|M_2|-1},\ 1 + \frac{1}{|M_1|+|M_2|+|M_3|-1} \text{ and } \frac{|M_2|+|M_3|}{|M_2|+|M_3|-1}$$

with multiplicities $|M_1|-1$, $|M_2|-1$ and $|M_3|-1$, respectively. Since $a_{12} = a_{21} = a_{23} = a_{32} = 1$, $a_{13} = a_{31} = 0$, $\alpha_1 = |M_2|$, $\alpha_2 = |M_1| + |M_3|$, $\alpha_3 = |M_2|$ and

$r_1 = |M_1| - 1,\ r_2 = |M_2| - 1,\ r_3 = |M_3| - 1$, by Lemma 1.3.22 the remaining 3 eigenvalues are the eigenvalues of the following matrix

$$\begin{bmatrix} \frac{|M_2|}{|M_1|+|M_2|-1} & \frac{-|M_2|}{\sqrt{(|M_1|+|M_2|-1)(|M_1|+|M_2|+|M_3|-1)}} & 0 \\ \frac{-|M_1|}{\sqrt{(|M_1|+|M_2|+|M_3|-1)(|M_1|+|M_2|-1)}} & \frac{|M_1|+|M_3|}{|M_1|+|M_2|+|M_3|-1} & \frac{-|M_3|}{\sqrt{(|M_1|+|M_2|+|M_3|-1)(|M_2|+|M_3|-1)}} \\ 0 & \frac{-|M_2|}{\sqrt{(|M_2|+|M_3|-1)(|M_1|+|M_2|+|M_3|-1)}} & \frac{|M_2|}{|M_2|+|M_3|-1} \end{bmatrix}.$$

$\square$

# Chapter 4

# The Left Ideal-Relation Graph over Full Matrix Ring

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Graph automorphism describes the symmetry of a graph. In general, graph automorphism groups are essential for studying sizeable graphs since these symmetries allow one to simplify and understand the behaviour of a graph. However, determining the full automorphism group is a challenging problem in algebraic graph theory. Ma and Wong [2016] introduced the ideal-relation graph of the ring $R$, denoted by $\overrightarrow{\Gamma}_i(R)$, which is a directed graph whose vertex set is $R$ and there is an edge from a vertex $x$ to a distinct vertex $y$ if and only if the ideal generated by $x$ is properly contained in the ideal generated by $y$. All the automorphisms of $\overrightarrow{\Gamma}_i(R)$, where $R$ is the ring of all $n \times n$ upper triangular matrices over a finite field $\mathbb{F}_q$, were obtained by Ma and Wong [2016]. Motivated by this work and to reveal the significant structure of ideal-relation graph on full matrix ring, in this chapter, we define and study the left ideal-relation graph of the full matrix ring. Analogously, it can be defined for arbitrary rings. The left ideal-relation graph $\overrightarrow{\Gamma}_L(M_n(\mathbb{F}_q))$, where $M_n(\mathbb{F}_q)$ is the ring of all $n \times n$ matrices over a finite field $\mathbb{F}_q$, is a directed simple graph whose vertex set consists of all the elements of $M_n(\mathbb{F}_q)$ and there is a directed edge from a vertex $X$ to $Y$ if and only if the left ideal generated by $X$ is properly contained in the left ideal generated by $Y$. In this chapter, all the automorphisms of $\overrightarrow{\Gamma_L}(R)$ are characterized, where $R$ is the ring of all $n \times n$ matrices over a finite field $\mathbb{F}_q$. The content of this chapter is submitted for publication.

We denote $M_n^*(\mathbb{F}_q)$ by the set of all invertible matrices over finite field $\mathbb{F}_q$. Let $E$ be the $n \times n$ identity matrix. We write $E_{s,t}$ by the $n \times n$ matrix which has 1 at $(s,t)$- position and 0 elsewhere. The matrix $E(i,j)$ denotes the matrix obtained by interchanging $i^{\text{th}}$ and $j^{\text{th}}$ columns of the identity matrix $E$ and $E(i(a))$ denotes the matrix obtained by multiplying $i^{\text{th}}$ column of the identity matrix $E$ by $a \in \mathbb{F}_q$. Let $E_r$ be the matrix $\sum_{i=0}^{r} E_{i,i}$ in $M_n(\mathbb{F}_q)$. For $X \in M_n(\mathbb{F}_q)$, we write by $I_X$ (or $[X]$) the left ideal generated by $X$. Also, by $S_X$, we mean the subspace of $\mathbb{F}_q^n$ spanned by row vectors of $X \in M_n(\mathbb{F}_q)$. Let $\Gamma$ be a directed graph with vertex set $V(\Gamma)$ and $u, v \in V(\Gamma)$. Moreover, $N_i(v) = \{u \mid u \to v\}$ and $N_o(v)) = \{u \mid v \to u\}$. The in-degree $d_i(v)$ of $v$ is the number of vertices in $N_i(v)$. Analogusly, the out-degree $d_o(v)$ of $v$ can be defined as the number of vertices in $N_o(v)$.

## 4.1 Automorphisms of the left-ideal relation graph of $M_n(F_q)$

In this section, we obtain the automorphisms of the left ideal-relation graph of ideals over $M_n(\mathbb{F}_q)$ (see Theorem 4.1.11 and Theorem 4.1.31). Before characterizing all the automorphisms of $\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q))$, we introduce two kinds of standard automorphisms for $\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q))$. Using them, the automorphisms of $\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q))$ can be characterized.

**Lemma 4.1.1.** *For $P \in M_n^*(\mathbb{F}_q)$, the map $\varphi_P : \ M_n(\mathbb{F}_q) \to M_n(\mathbb{F}_q)$ defined by $\varphi_P(X) = XP$ for each $X \in M_n(\mathbb{F}_q)$, is an automorphism of $\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q))$.*

*Proof.* Invertibility of $P$ proves that $\varphi_P$ is one-one. Since $M_n(\mathbb{F}_q)$ is finite, we get $\varphi_P$ is onto also. For $X \to Y$, we have $[X] \subset [Y]$. Let $Z \in [XP]$. Then $Z = WXP$ for some $W \in M_n(\mathbb{F}_q)$. Consequently, $ZP^{-1} \in [Y]$ and $ZP^{-1} = VY$ for some $V \in M_n(\mathbb{F}_q)$. This implies $Z = VYP$ so that $Z \in [YP]$. Therefore, $[XP] \subseteq [YP]$. Since $[X] \subset [Y]$ there exists $L \in [Y]$ but $L \notin [X]$. Then $L = UY$ for some $U \in M_n(\mathbb{F}_q)$. Consequently, $LP = UYP$ follows that $LP \in [YP]$. If $LP \in [XP]$, then there exists $N \in M_n(\mathbb{F}_q)$ such that $LP = NXP$. It follows that $L = NX$ and so $L \in [X]$, a contradiction. Thus, $[XP] \subset [YP]$ and $\varphi_P(X) \to \varphi_P(Y)$. Hence, $\varphi_P$ is an automorphism of $\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q))$. □

For $P \in M_n^*(\mathbb{F}_q)$, we define $[X]P = I_X P = \{AP |\ A \in [X]\}$. Notice that $[\varphi_P(X)] = I_{XP}$ and $I_{XP} = I_X P$. We recall some necessary lemmas for latter use.

**Lemma 4.1.2** ([Wang et al., 2017, Lemma 2.2])**.** *Let $I$ be a left ideal of $M_n(\mathbb{F}_q)$. Then there exists $X \in M_n(\mathbb{F}_q)$ such that $I = [X]$.*

**Lemma 4.1.3** ([Wang et al., 2017, Lemma 2.2])**.** *Let $I$ be a left ideal of $M_n(\mathbb{F}_q)$. Then there exists $P \in M_n^*(\mathbb{F}_q)$ such that $I = I_{E_r P}$, where $0 \le r \le n$ and $E_0 = 0$.*

**Lemma 4.1.4** ([Xu et al., 2022, Lemma 3.3])**.** *Let $X, Y \in M_n(\mathbb{F}_q)$. Then $S_X = S_Y$ if and only if $I_X = I_Y$.*

**Lemma 4.1.5.** *Let $X, Y \in M_n(\mathbb{F}_q)$. Then $S_X \subset S_Y$ if and only if $I_X \subset I_Y$.*

*Proof.* First assume that $S_X \subset S_Y$. Let $A \in I_X$. Then there exists $B \in M_n(\mathbb{F}_q)$ such that $A = BX$. Note that $\begin{pmatrix} X_1' \\ X_2' \\ \vdots \\ X_n' \end{pmatrix} = B \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$ where $X_i$ and $X_i'$ are row vectors of the matrix $X$ and $A$, respectively. It follows that $X_i^{'} \in S_X$. Since $S_X \subset S_Y$, we get $X_i^{'} \in S_Y$. Consequently, there exists $B^{'} \in M_n(\mathbb{F}_q)$ such that $B^{'}Y = \begin{pmatrix} X_1^{'} \\ X_2^{'} \\ \vdots \\ X_n^{'} \end{pmatrix}$. It implies that $B^{'}Y = BX$ so that $BX \in I_Y$. Therefore, $I_X \subseteq I_Y$. If $I_X = I_Y$, then by Lemma 4.1.4, we have $S_X = S_Y$, a contradiction. Thus, $I_X \subset I_Y$. Conversely, let $I_X \subset I_Y$. Consider $a \in S_X$ such that $a = \sum_{i=1}^{n} c_i X_i$, where $X_i$ is the $i^{th}$ row of matrix $X$ and $c_i \in \mathbb{F}_q$. It follows that

$$[c_1\ c_2\ \cdots\ c_n]X = a$$

Since $I_X \subset I_Y$ so that $X \in I_Y$. It follows that $\begin{pmatrix} c \\ \mathbf{0} \end{pmatrix} X = \begin{pmatrix} a \\ \mathbf{0} \end{pmatrix} \in I_Y$. Therefore, there exists $d \in \mathbb{F}_q^n$ we have $\begin{pmatrix} d \\ \mathbf{0} \end{pmatrix} Y = \begin{pmatrix} a \\ \mathbf{0} \end{pmatrix}$. Therefore, $a \in S_Y$ so that $S_X \subseteq S_Y$. If $S_X = S_Y$, then by Lemma 4.1.4 $I_X = I_Y$, again a contradiction. Hence, $S_X \subset S_Y$. $\square$

Let $\Omega$ be the set of $r$ linearly independent vectors belongs to $\mathbb{F}_q^n$ and let $E_\Omega$ be the matrix such that it's first $|\Omega|$ rows are from the set $\Omega$ and rest all rows are zero vectors. Let us denote $I_\Omega$ as the left ideal generated by $E_\Omega$ and let $I$ be a left ideal of $M_n(\mathbb{F}_q)$. By Lemma 4.1.2, we have, $I = I_X$ for some $X \in M_n(\mathbb{F}_q)$. Let $\Omega$ be the set of maximal linearly independent row vectors of $X$. Then $S_X = S_{E_\Omega}$ so that $I_X = I_{E_\Omega}$ and $I = I_\Omega$. From the above discussion, it is easy to observe the following lemma.

**Lemma 4.1.6.** *Let $I$ be a left ideal of $M_n(\mathbb{F}_q)$. Then there exists a set $\Omega \subseteq \mathbb{F}_q^n$ of linearly independent row vectors such that $I = I_\Omega$.*

For $1 \leq i \leq n$, let $E_{e_i}$ be the matrix such that its first row is $e_i \in \mathbb{F}_q^n$ and rest all rows are zero vectors. Let $X \in M_n(\mathbb{F}_q)$. Then by Lemma 4.1.6, there exists

$\Omega \subseteq \mathbb{F}_q^n$ such that $I_X = I_\Omega$. From this we get that $rank(X) = rank(E_\Omega) = |\Omega|$. Define $E_a$ to be the matrix with its first row as $a \in \mathbb{F}_q^n$ and rest all rows are zero vectors. Clearly $rank(E_a) = 1$. Let $I$ be a left ideal of $M_n(\mathbb{F}_q)$. Then $X \in I$ and $a \in \mathbb{F}_q$, define $aX = (aE) \cdot X \in I$. Then $I$ becomes a vector space so the dimension of $I$ is well defined. We claim that $\dim(I) = n \cdot rank(I)$, where $rank(I) = rank(X)$ such that $I_X = I$. It follows that $I = [E_r P]$, where $P \in M_n^*(\mathbb{F}_q)$. Therefore, by Lemma 4.1.2, we have $rank(I) = rank(E_r P) = r$. Notice that $I_{E_r}$ is a vector space with bases as $\{E_{i,j} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq r\}$. Thus, $\dim(I_{E_r}) = nr = n \cdot rank(I)$ and vector space $I \cong I_{E_r}$ as $I = \varphi_P(E_r)$. By Lemma 4.1.1, we get that $\varphi_P$ is bijective and one can verify that $\varphi_P$ is a linear transformation. Thus, $\dim(I) = \dim(I_{E_r}) = n \cdot rank(I)$.

**Lemma 4.1.7.** *Let $X, Y \in M_n(\mathbb{F}_q)$ such that $rank(X) < rank(Y)$. Then $d_i(X) < d_i(Y)$ and $d_o(X) > d_o(Y)$. Moreover,*

(i) *$d_i(X) = d_i(Y)$ if and only if $rank(X) = rank(Y)$*

(ii) *$d_o(X) = d_o(Y)$ if and only if $rank(X) = rank(Y)$.*

*Proof.* (i) If $rank(X) = s$ and $rank(Y) = t$, then $s < t$. Assume that $[X] = [E_s A]$ and $[Y] = [E_t B]$, where $A, B \in M_n^*(\mathbb{F}_q)$. It follows that $d_i(X) = d_i(E_s A)$ so that $d_i(X) = d_i(E_s)$ because $\varphi_A(E_s) = E_s A$. Similarly, we have $d_i(Y) = d_i(E_t)$. Next, we need to prove that $d_i(E_s) < d_i(E_t)$. Since $E_s \in N_i(E_t)$ and $E_s \notin N_i(E_s)$, we get $N_i(E_s) \subset N_i(E_t)$. Then $d_i(E_s) < d_i(E_t)$ implies that $d_i(X) < d_i(Y)$. Further, assume that $d_i(X) = d_i(Y)$. From the above discussion, if we assume that $rank(X) \neq rank(Y)$, then $d_i(X) \neq d_i(Y)$, a contradiction. Thus, $rank(X) = rank(Y)$. Now let $rank(X) = rank(Y) = r$. This implies that $d_i(X) = d_i(E_r) = d_i(Y)$.

(ii) The proof is similar to Part (i), hence we omit the details. □

**Corollary 4.1.8.** *Let $\Psi \in Aut(\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q)))$ and $X \in M_n(\mathbb{F}_q)$. Then $rank(\Psi(X)) = rank(X)$.*

First we obtain all the automorphisms of the graph $\overrightarrow{\Gamma_L}(M_2(\mathbb{F}_q))$. For $X \in M_2(\mathbb{F}_q)$, we have $rank(X) \in \{0, 1, 2\}$. Further, note that $rank(X) = 0$ if and only if $[X] = [\mathbf{0}]$. Also, $rank(X) = 2$ if and only if $[X] = M_2(\mathbb{F}_q)$.

Let $R_i = \{X \in M_2(\mathbb{F}_q) : rank(X) = i\}$. The proof of the following lemma is straightforward.

**Lemma 4.1.9.** *For $X, Y \in M_2(\mathbb{F}_q)$, $X \to Y$ if and only if $rank(X) < rank(Y)$.*

Define the set of mappings

$$\Phi = \{\rho : M_2(\mathbb{F}_q) \to M_2(\mathbb{F}_q) \ \mid \ \rho \text{ permutes vertices of } R_i \text{ for each } i = 0, 1, 2\}.$$

For each $X, Y \in R_i$, observe that $N_i(X) = N_i(Y)$ and $N_o(X) = N_o(Y)$. Thus, we get the following lemma.

**Lemma 4.1.10.** *If $\alpha \in \Phi$, then $\alpha \in Aut(\overrightarrow{\Gamma_L}(M_2(\mathbb{F}_q)))$.*

*Proof.* Clearly, $\alpha$ is a bijective map. Let $X \to Y$, where $X, Y \in M_2(\mathbb{F}_q)$. Then by Lemma 4.1.9, we have $rank(X) < rank(Y)$. By Corollary 4.1.8, $rank(\alpha(X)) = rank(X)$ and $rank(\alpha(Y)) = rank(Y)$. Lemma 4.1.9 gives $rank(\alpha(X)) < rank(\alpha(Y))$ so that $\alpha(X) \to \alpha(Y)$. Thus, $\alpha \in Aut(\overrightarrow{\Gamma_L}(M_2(\mathbb{F}_q)))$. □

The following theorem determines all the automorphisms of $\overrightarrow{\Gamma_L}(M_2(\mathbb{F}_q))$.

**Theorem 4.1.11.** *If $\Psi \in Aut(\overrightarrow{\Gamma_L}(M_2(\mathbb{F}_q)))$, then $\Psi \in \Phi$.*

*Proof.* Since $rank(\Psi(X)) = rank(X)$ (cf. Corollary 4.1.8), there exists a map $\alpha_1 \in \Phi$ such that $\alpha_1(X) = \Psi(X)$. It follows that $\Psi \in \Phi$. Thus, the result holds. □

In the remaining chapter, we assume that $n \geq 3$.

**Lemma 4.1.12.** *Let $\Psi \in Aut(\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q)))$. Then there exist $P \in M_n^*(\mathbb{F}_q)$ such that $[(\varphi_P \cdot \Psi)(E_{e_1})] = [E_{e_1}]$.*

*Proof.* Since $rank(E_{e_1}) = 1$, we get $rank(\Psi(E_{e_1})) = 1$. Asume that $[\Psi(E_{e_1})] = [E_a]$, where $0 \neq a \in \mathbb{F}_q^n$. Let $a_k$ be the first non-zero element in row vector $a$. Define

$$P = (E - \textstyle\sum_{i \neq k} a_k^{-1} a_i E_{k,i}) E(k(a_k^{-1})) E(1,k).$$

Note that $P \in M_n^*(\mathbb{F}_q)$. Moreover, $[\Psi(E_{e_1})]P = [E_a]P$ implies that $[\Psi(E_{e_1})P] = [E_a P]$. Therefore, $[(\varphi_P \cdot \Psi)(E_{e_1})] = \left[ \begin{pmatrix} e_1 \\ \mathbf{0} \end{pmatrix} \right]$ and hence, $[(\varphi_P \cdot \Psi)(E_{e_1})] = [E_{e_1}]$. □

**Lemma 4.1.13.** *Let $\Psi \in Aut(\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q)))$ such that $[E_{e_k}] = [\Psi(E_{e_k})]$, for each $k$, where $1 \leq k \leq i$. Then $[E_i] = [\Psi(E_i)]$.*

*Proof.* The result holds for $i = 1$ because $E_{e_1} = E_1$. If $i \geq 2$, then $[E_{e_k}] \subset [E_i]$. This implies that $E_{e_k} \to E_i$ and so $\Psi(E_{e_k}) \to \Psi(E_i)$. Thus, $[\Psi(E_{e_k})] \subset [\Psi(E_i)]$ gives $[E_{e_k}] \subset [\Psi(E_i)]$. It follows that $\sum_{k=1}^{i} [E_{e_k}] \subseteq [\Psi(E_i)]$. Therefore, $[E_i] \subseteq [\Psi(E_i)]$. Since $\dim([E_i]) = \dim([\Psi(E_i)]) = n \cdot i$, we get $[E_i] = [\Psi(E_i)]$. □

**Lemma 4.1.14.** *Let $\Psi \in Aut(\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q)))$ such that $[E_{e_k}] = [\Psi(E_{e_k})]$ and $1 \leq k \leq i-1$. Then there exists $P \in M_n^*(\mathbb{F}_q)$ such that $[(\varphi_P \cdot \Psi)(E_{e_s})] = [E_{e_s}]$, for each $s$, where $1 \leq s \leq i$.*

*Proof.* Let $[\Psi(E_{e_i})] = [E_a]$, where $a(\neq 0) \in \mathbb{F}_q^n$. Note that $[E_{e_i}]$ is not a proper subset of $[E_{i-1}]$. Consequently, $[\Psi(E_{e_i})]$ is not a proper subset of $[\Psi(E_{i-1}]$. By Lemma 4.1.13, we have $[E_{i-1}] = [\Psi(E_{i-1})]$. It follows that $[\Psi(E_{e_i})]$ is not a proper subset of $[E_{i-1}]$. There exists $a_l(\neq 0) \in \mathbb{F}_q$ such that $i \leq l \leq n$, we define

$$P = (E - \textstyle\sum_{j \neq l} a_l^{-1} a_j E_{l,j}) E(l(a_l^{-1})) E(i,l).$$

Note that $P \in M_n^*(\mathbb{F}_q)$. Moreover, $[\Psi(E_{e_i})]P = [E_a]P$ and $[\Psi(E_{e_i})P] = [E_aP]$. Therefore, $[(\varphi_P \cdot \Psi)(E_{e_i})] = \left[\begin{pmatrix} e_i \\ \mathbf{0} \end{pmatrix}\right]$ so that $[(\varphi_P \cdot \Psi)(E_{e_i})] = [E_{e_i}]$. For $1 \leq k < i$, $[\Psi(E_{e_k})]P = [E_{e_k}]P$, we have $[\Psi(E_{e_k})P] = [E_{e_k}P]$. Therefore, $[(\varphi_P \cdot \Psi)(E_{e_k})] = [\varphi_P(E_{e_k})]$ gives $[\varphi_P(E_{e_k})] = [E_{e_k}]$. Thus, $[(\varphi_P \cdot \Psi)(E_{e_k})] = [E_{e_k}]$, where $1 \leq k \leq i-1$. Hence, $[(\varphi_P \cdot \Psi)(E_{e_s})] = [E_{e_s}]$, where $1 \leq s \leq i$. □

In view of the above lemmas, we have the following corollary.

**Corollary 4.1.15.** *Let* $\Psi \in Aut(\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q)))$. *Then there exists* $P \in M_n^*(\mathbb{F}_q)$ *such that* $[(\varphi_P \cdot \Psi)(E_{e_k})] = [E_{e_k}]$, *for all* $k = 1, 2, \ldots, n$.

*Proof.* In view of Lemma 4.1.12, there exists $P_1 \in M_n^*(\mathbb{F}_q)$ such that $[(\varphi_{P_1} \cdot \Psi)(E_{e_1})] = [E_{e_1}]$. Then by Lemma 4.1.14, we get $P_2 \in M_n^*(\mathbb{F}_q)$ such that $[(\varphi_{P_2} \cdot \varphi_{P_1} \cdot \Psi)(E_{e_k})] = [E_{e_k}]$, where $k = 1, 2$. On continuing in this way there exists $P_1, P_2, \cdots, P_n$ such that $[(\varphi_{P_n} \cdots \varphi_{P_2} \cdot \varphi_{P_1} \cdot \Psi)(E_{e_k})] = [E_{e_k}]$, where $k = 1, 2, \ldots, n$. On taking $P = P_1P_2 \cdots P_n \in M_n^*(\mathbb{F}_q)$, we get the result. □

Let $\Delta = \{e_1, e_2, \ldots, e_n\}$ be the set of all unit vectors of $\mathbb{F}_q^n$.

**Lemma 4.1.16.** *Let* $\delta \subseteq \Delta$. *If* $\Psi \in Aut(\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q)))$ *such that* $[\Psi(E_{e_k})] = [E_{e_k}]$ *for all* $k = 1, 2, \ldots, n$. *Then* $[\Psi(E_\delta)] = [E_\delta]$.

*Proof.* If $|\delta| = 1$, then $E_\delta = E_{e_k}$. We may now suppose that $|\delta| \geq 2$. For each $e_k \in \delta$ note that $[E_{e_k}] \subset [E_\delta]$ and so $[\Psi(E_{e_k})] \subset [\Psi(E_\delta)]$. Consequently, we obtain $[E_{e_k}] \subset [\Psi(E_\delta)]$. Let $I(\delta) = \{i \mid e_i \in \delta\}$. Then we have $\sum_{k \in I(\delta)} [E_{e_k}] \subseteq [\Psi(E_\delta)]$. This implies that $[E_\delta] \subseteq [\Psi(E_\delta)]$. Since $\dim([E_\delta]) = \dim([\Psi(E_\delta)]) = n \cdot |\delta|$, we get $[E_\delta] = [\Psi(E_\delta)]$. □

For any $X \in M_n(\mathbb{F}_q)$ such that $rank(X) = 1$, we have $[X] = [E_a]$, where $a \in \mathbb{F}_q^n$. Note that such representation of $[X]$ is not unique and there can be some other $b \in \mathbb{F}_q^n$ such that $[X] = [E_b]$. But if we impose a condition on $a$ that its first non-zero element is 1, then we get a unique $a \in \mathbb{F}_q^n$ such that $[X] = [E_a]$. In next few lemmas, we consider rank 1 matrices of $M_n(\mathbb{F}_q)$.

**Lemma 4.1.17.** *Let $\Psi \in Aut(\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q)))$ such that $[\Psi(E_{e_k})] = [E_{e_k}]$, for each $E_{e_k}$, where $k = 1, 2, \ldots, n$ and let $X \in M_n(\mathbb{F}_q)$ such that $rank(X) = 1$. Suppose that $[X] = [E_a]$ and $[\Psi(X)] = [E_b]$, where $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n) \in \mathbb{F}_q^n$. Then $a_l = 0$ if and only if $b_l = 0$ for all $l$, where $1 \leq l \leq n$.*

*Proof.* First suppose that $a_l = 0$. For each $Y \in [E_a]$, note that the $l^{th}$ column of $Y$ is zero vector. Let us denote $\Delta \backslash \{e_l\}$ by $\Delta_l$. Then $[E_a] \subset [E_{\Delta_l}]$ implies that $[\Psi(X)] \subset [\Psi(E_{\Delta_l})]$. Consequently, $[E_b] \subset [\Psi(E_{\Delta_l})]$. By Lemma 4.1.16, we obtain $[E_{\Delta_l}] = [\Psi(E_{\Delta_l})]$. Therefore, $[E_b] \subset [E_{\Delta_l}]$ which yields $b_l = 0$. Conversely, assume that $b_l = 0$. This implies that $[E_b] \subset [E_{\Delta_l}]$. Next, on contrary suppose that $a_l \neq 0$. It follows that $[E_a] \not\subset [E_{\Delta_l}]$. Consequently, $[\Psi(X)] \not\subset [\Psi(E_{\Delta_l})]$ and so $[E_b] \not\subset [\Psi(E_{\Delta_l})]$. By Lemma 4.1.16, we get $[E_b] \not\subset [E_{\Delta_l}]$, a contradiction. $\square$

**Lemma 4.1.18.** *Let $\Psi \in Aut(\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q)))$ such that $[\Psi(E_{e_k})] = [E_{e_k}]$, for each $E_{e_k}$, where $k = 1, 2, \ldots, n$. Suppose that $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ and $b_i = (b_{i1}, b_{i2}, \ldots, b_{in})$, where $i = 1, 2$ with $[X_i] = [E_{a_i}]$ and $[\Psi(X_i)] = [E_{b_i}]$. Then $b_{1s}b_{2k} = b_{1k}b_{2s}$ if and only if $a_{1s}a_{2k} = a_{1k}a_{2s}$, where $1 \leq s < k \leq n$.*

*Proof.* If $a_{1s}a_{2k}a_{1k}a_{2s} = 0$, then $b_{1s}b_{2k}b_{1k}b_{2s} = 0$ (cf. Lemma 4.1.17). Thus, in this case $b_{1s}b_{2k} = b_{1k}b_{2s}$ if and only if $a_{1s}a_{2k} = a_{1k}a_{2s}$ for $1 \leq s < k \leq n$. We may now suppose that $a_{1s}a_{2k}a_{1k}a_{2s} \neq 0$. By Lemma 4.1.17, we get $b_{1s}b_{2k}b_{1k}b_{2s} \neq 0$. Now, consider the sets

$$\alpha = \{e_1, e_2, \ldots, e_{s-1}, a_{1s}e_s + a_{1k}e_k, e_{s+1}, \ldots, e_{k-1}, e_{k+1}, \ldots, e_n\}, \text{and}$$

$$\beta = \{e_1, e_2, \ldots, e_{s-1}, b_{1s}e_s + b_{1k}e_k, e_{s+1}, \ldots, e_{k-1}, e_{k+1}, \ldots, e_n\}$$

Note that $[E_{a_1}] \subset [E_\alpha]$. It follows that $[\Psi(X_1)] \subset [\Psi(E_\alpha)]$ and so $[E_{b_1}] \subset [E_\alpha]$. Note that $[E_{\Delta \backslash \{e_s, e_k\}}] \subset [E_\alpha]$ so that $[E_{\Delta \backslash \{e_s, e_k\}}] \subset [\Psi(E_\alpha)]$. Consequently,

$[E_{\Delta\setminus\{e_s,e_k\}}] + [E_{b_1}] \subseteq [\Psi(E_\alpha)]$. Since $\dim([E_{\Delta\setminus\{e_s,e_k\}}] + [E_{b_1}]) = \dim([\Psi(E_\alpha)]) = n(n-1)$, we have $[E_{\Delta\setminus\{e_s,e_k\}}] + [E_{b_1}] = [\Psi(E_\alpha)]$. Notice that $[E_{\Delta\setminus\{e_s,e_k\}}] + [E_{b_1}] = [E_\beta]$. Thus, $[E_\beta] = [\Psi(E_\alpha)]$.

If $a_{1s}a_{2k} = a_{1k}a_{2s}$, then $[E_{a_2}] \subset [E_\alpha]$. Consequently, $[E_{b_2}] \subset [E_\beta]$. Therefore, $b_{1s}b_{2k} = b_{1k}b_{2s}$. Conversely, suppose that $b_{1s}b_{2k} = b_{1k}b_{2s}$. It follows that $[E_{b_2}] \subset [E_\beta]$. On contrary suppose $a_{1s}a_{2k} \neq a_{1k}a_{2s}$. This implies that $[E_{a_2}] \not\subset [E_\alpha]$. Consequently, $[X_2] \not\subset [E_\alpha]$ gives $[E_{b_2}] \not\subset [E_\beta]$, a contradiction. □

**Lemma 4.1.19.** *Let $\Psi \in Aut(\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q)))$. Then there exists $P \in M_n^*(\mathbb{F}_q)$ such that $[(\varphi_P \cdot \Psi)(E_{\mathbf{1}})] = [E_{\mathbf{1}}]$, where $E_{\mathbf{1}} = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}$ with $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{F}_q^n$. Moreover, $[(\varphi_P \cdot \Psi)(E_{e_k})] = [E_{e_k}]$, for all $k = 1, 2, \ldots, n$.*

*Proof.* In view of Corollary 4.1.15, there exists a matrix $P_1 \in M_n^*(\mathbb{F}_q)$ such that $[(\varphi_{P_1} \cdot \Psi)(E_{e_k})] = [E_{e_k}]$. Let $[(\varphi_{P_1} \cdot \Psi)(E_{\mathbf{1}})] = [E_a]$, where $a \in \mathbb{F}_q^n$. Then by Lemma 4.1.17, $a_l \neq 0$ for each $1 \leq l \leq n$. Now, consider $P_2 \in M_n^*(\mathbb{F}_q)$ such that $P_2 = diag(a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1})$. Since $[(\varphi_{P_1} \cdot \Psi)(E_{\mathbf{1}})]P_2 = [E_a]P_2$, we get $[(\varphi_{P_1} \cdot \Psi)(E_{\mathbf{1}})P_2] = [E_a P_2]$. It implies that $[(\varphi_{P_2} \cdot \varphi_{P_1} \cdot \Psi)(E_{\mathbf{1}})] = [E_{\mathbf{1}}]$ so that $[(\varphi_{P_1P_2} \cdot \Psi)(E_{\mathbf{1}})] = [E_{\mathbf{1}}]$. Let $P = P_1P_2 \in M_n^*(\mathbb{F}_q)$. Then $[(\varphi_P \cdot \Psi)(E_{\mathbf{1}})] = [E_{\mathbf{1}}]$. Also, $[(\varphi_{P_1P_2} \cdot \Psi)(E_{e_k})] = [\varphi_{P_2}(E_{e_k})] = [E_{e_k}]$. Thus, $[(\varphi_P \cdot \Psi)(E_{e_k})] = [E_{e_k}]$, where $1 \leq k \leq n$. □

**Lemma 4.1.20.** *Let $\Psi \in Aut(\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q)))$ such that $[\Psi(E_{\mathbf{1}})] = [E_{\mathbf{1}}]$ and $[\Psi(E_{e_k})] = [E_{e_k}]$, for each $E_{e_k}$, where $1 \leq k \leq n$. Suppose $X \in M_n(\mathbb{F}_q)$ such that $[X] = [E_a]$ and $[\Psi(X)] = [E_b]$, where $a, b \in \mathbb{F}_q^n$. Then $a_s = a_k$ if and only if $b_s = b_k$ for $1 \leq s < k \leq n$.*

*Proof.* On applying Lemma 4.1.18, by taking $X_1 = X$ and $X_2 = E_1$, the result holds. □

**Remark 4.1.21.** Let $\Psi \in Aut(\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q)))$ such that $[\Psi(E_{\mathbf{1}})] = [E_{\mathbf{1}}]$ and $[\Psi(E_{e_k})] = [E_{e_k}]$, where $1 \leq k \leq n$. Let $X \in M_n(\mathbb{F}_q)$ such that $[X] = [E_\alpha]$ for $\alpha = (e_1 + ae_2)$, where $a \in \mathbb{F}_q$. Further, assume that $[\Psi(X)] = [E_\beta]$, where $\beta = e_1 + a'e_2$ and $a' \in \mathbb{F}_q$. Note that $a'$ depends upon $a$. Hence, we can define a mapping $\Upsilon$ on $\mathbb{F}_q$ such that $\Upsilon(a) = a'$. Notice that $\Upsilon$ is a one -one map on a finite field $\mathbb{F}_q$ and so $\Upsilon$ is a bijective map on $\mathbb{F}_q$. Moreover, by Lemma 4.1.17 and Lemma 4.1.18, we get $\Upsilon(0) = 0$ and $\Upsilon(1) = 1$.

**Lemma 4.1.22.** *Let $\Psi \in Aut(\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q)))$ such that $[\Psi(E_{\mathbf{1}})] = [E_{\mathbf{1}}]$ and $[\Psi(E_{e_k})] = [E_{e_k}]$, for each $E_{e_k}$, where $1 \leq k \leq n$. Suppose that $\Upsilon$ is a map as defined in Remark 4.1.21. Then the following holds:*

(i) *If $[X] = [E_{\{e_1+ae_k\}}]$, then $[\Psi(X)] = [E_{\{e_1+\Upsilon(a)e_k\}}]$ for $2 \leq k \leq n$ and $a \in \mathbb{F}_q$.*

(ii) *If $[X] = [E_{\{e_1+a_2e_2+\cdots+a_ne_n\}}]$, then $[\Psi(X)] = [E_{\{e_1+\Upsilon(a_2)e_2+\cdots+\Upsilon(a_n)e_n\}}]$, for $a_i \in \mathbb{F}_q$.*

(iii) *If $[X] = [E_{\{e_i+a_{i+1}e_{i+1}+\cdots+a_ne_n\}}]$, where $X \in M_n(\mathbb{F}_q)$, then*

$$[\Psi(X)] = [E_{\{e_i+\Upsilon(a_{i+1})e_{i+1}+\cdots+\Upsilon(a_n)e_n\}}].$$

*Proof.* (i) For $k = 2$, the result holds (cf. Remark 4.1.21). We now suppose that $k \geq 3$. Let $Y \in M_n(\mathbb{F}_q)$ such that $[Y] = [E_{\{e_1+ae_2+ae_k\}}]$ and $[\Psi(Y)] = [E_{\{e_1+a'e_2+a''e_k\}}]$. By Lemma 4.1.20, we get $a' = a''$ so $[\Psi(Y)] = [E_{\{e_1+a'e_2+a'e_k\}}]$. Let $[Z] = [E_{\{e_1+ae_2\}}]$. Then $[\Psi(Z)] = [E_{\{e_1+\Upsilon(a)e_2\}}]$. By Lemma 4.1.18, and using $[Y]$, $[Z]$, we have $a' = \Upsilon(a)$. Thus, $[\Psi(Y)] = [E_{\{e_1+\Upsilon(a)e_2+\Upsilon(a)e_k\}}]$. If $[\Psi(X)] = [E_{\{e_1+de_k\}}]$, then using $[X]$, $[Y]$ and Lemma 4.1.18, we get $d = \Upsilon(a)$.

(ii) Suppose that $[\Psi(X)] = [E_{\{e_1+c_2e_2+\cdots+c_ne_n\}}]$. Let $Y_k \in M_n(\mathbb{F}_q)$ such that $[Y_k] = [E_{\{e_1+a_ke_k\}}]$. By Part (i), we get $[\Psi(Y_k)] = [E_{\{e_1+\Upsilon(a_k)e_k\}}]$. Now by using $[Y_k]$, $[X]$ and Lemma 4.1.18, we have $c_k = \Upsilon(a_k)$ for each $2 \leq k \leq n$.

(iii) Let $Y, Z \in M_n(\mathbb{F}_q)$ such that $[Y] = [E_{\{e_1+a_2e_2+\cdots+a_ne_n\}}]$ and $[Z] = [E_{\{e_1+e_i+a_{i+1}e_{i+1}+\cdots+a_ne_n\}}]$. Then by Part (ii), we get $[\Psi(Y)] = [E_{\{e_1+\Upsilon(a_2)e_2+\cdots+\Upsilon(a_n)e_n\}}]$ and $[\Psi(Z)] = [E_{\{e_1+e_i+\Upsilon(a_{i+1})e_{i+1}+\cdots+\Upsilon(a_n)e_n\}}]$. If $[\Psi(X)] = [E_{\{e_i+d_{i+1}e_{i+1}+\cdots+d_ne_n\}}]$, then using $[X]$ and $[Z]$ in Lemma 4.1.18, we get $d_j = \Upsilon(a_j)$, where $i < j \leq n$. The result holds.

□

**Lemma 4.1.23.** *Let $\Psi \in Aut(\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q)))$ such that $[\Psi(E_{\mathbf{1}})] = [E_{\mathbf{1}}]$ and $[\Psi(E_{e_k})] = [E_{e_k}]$, for each $E_{e_k}$, where $1 \leq k \leq n$. Let $\Upsilon$ be a map defined in Remark 4.1.21. Then $\Upsilon$ is field automorphism of $\mathbb{F}_q$.*

*Proof.* Since $\Upsilon$ is a bijective map on $\mathbb{F}_q$ (see Remark 4.1.21). Next, we prove that $\Upsilon(a+b) = \Upsilon(a)+\Upsilon(b)$ and, $\Upsilon(ab) = \Upsilon(a)\Upsilon(b)$. Let $a, b \in \mathbb{F}_q$. If $ab = 0$, then clearly $\Upsilon(a)\Upsilon(b) = 0 = \Upsilon(ab)$. We may now assume that $ab \neq 0$. It follows that $\Upsilon(a)\Upsilon(b) \neq 0$. First we claim that $\Upsilon(a^{-1}) = \Upsilon(a)^{-1}$. Let $X_1, X_2 \in M_n(\mathbb{F}_q)$ such that $[X_1] = [E_{\{e_1+ae_2+e_3\}}]$ and $[X_2] = [E_{\{e_2+a^{-1}e_3\}}]$. Then $[\Psi(X_1)] = [E_{\{e_1+\Upsilon(a)e_2+e_3\}}]$ and $[\Psi(X_2)] = [E_{\{e_2+\Upsilon(a^{-1})e_3\}}]$. Therefore, by Lemma 4.1.18 and by using $[X_1]$, $[X_2]$, we obtain $\Upsilon(a)\Upsilon(a^{-1}) = 1$. This proves our claim. Now suppose that $X_3, X_4 \in M_n(\mathbb{F}_q)$ such that $[X_3] = [E_{\{e_1+abe_2+e_3\}}]$ and $[X_4] = [E_{\{e_1+be_2+a^{-1}e_3\}}]$. It follows that $[\Psi(X_3)] = [E_{\{e_1+\Upsilon(ab)e_2+e_3\}}]$ and $[\Psi(X_4)] = [E_{\{e_1+\Upsilon(b)e_2+\Upsilon(a^{-1})e_3\}}]$. On applying Lemma 4.1.18 together with $[X_3]$ and $[X_4]$, we obtain $\Upsilon(ab)\Upsilon(a^{-1}) = \Upsilon(b)$. Consequently, $\Upsilon(ab) = \Upsilon(a)\Upsilon(b)$.

Let $Y_1, Y_2 \in M_n(\mathbb{F}_q)$ such that $[Y_1] = [E_{\{e_1+ae_3\}}]$ and $[Y_2] = [E_{\{e_2+be_3\}}]$. Then $[\Psi(Y_1)] = [E_{\{e_1+\Upsilon(a)e_3\}}]$ and $[\Psi(Y_2)] = [E_{\{e_2+\Upsilon(b)e_3\}}]$. Since $[Y_1] \subset [E_{\{e_1+ae_3, e_2+be_3\}}]$ and $[Y_2] \subset [E_{\{e_1+ae_3, e_2+be_3\}}]$, we have $[E_{\{e_1+\Upsilon(a)e_3\}}] \subset [\Psi(E_{\{e_1+ae_3, e_2+be_3\}})]$ and $[E_{\{e_2+\Upsilon(b)e_3\}}] \subset [\Psi(E_{\{e_1+ae_3, e_2+be_3\}})]$. It follows that $[E_{\{e_1+\Upsilon(a)e_3\}}] + [E_{\{e_2+\Upsilon(b)e_3\}}] \subseteq [\Psi(E_{\{e_1+ae_3, e_2+be_3\}})]$. Since

$$\dim([E_{\{e_1+\Upsilon(a)e_3\}}] + [E_{\{e_2+\Upsilon(b)e_3\}}]) = \dim([\Psi(E_{\{e_1+ae_3, e_2+be_3\}})]) = 2n$$

so $[E_{\{e_1+\Upsilon(a)e_3\}}] + [E_{\{e_2+\Upsilon(b)e_3\}}] = [\Psi(E_{\{e_1+ae_3,e_2+be_3\}})]$. Assume that $Y_3 \in M_n(\mathbb{F}_q)$ with $[Y_3] = [E_{\{e_1+e_2+(a+b)e_3\}}]$. Therefore, $[Y_3] \subset [E_{\{e_1+ae_3,e_2+be_3\}}]$ implies that $[\Psi(Y_3)] \subset [\Psi(E_{\{e_1+ae_3,e_2+be_3\}})]$. It follows that $[E_{\{e_1+e_2+\Upsilon(a+b)e_3\}}] \subset [\Psi(E_{\{e_1+ae_3,e_2+be_3\}})]$ and so $[E_{\{e_1+e_2+\Upsilon(a+b)e_3\}}] \subset [E_{\{e_1+\Upsilon(a)e_3\}}]+[E_{\{e_2+\Upsilon(b)e_3\}}]$. Consequently, $\Upsilon(a+b) = \Upsilon(a) + \Upsilon(b)$. Thus, $\Upsilon$ is field automorphism of $\mathbb{F}_q$. □

Next, we define a relation $\equiv_E$ on $M_n(\mathbb{F}_q)$ such that $X \equiv_E Y$ if and only if $[X] = [Y]$. Note that $\equiv_E$ is an equivalence relation on $M_n(\mathbb{F}_q)$. Let $\mathcal{R}$ be a complete set of distinct representative element of $\equiv_E$ and for $X = (x_{ij})_{n\times n} \in \mathcal{R}$, we write $cl(X)$ to denote the class of $X$.

**Lemma 4.1.24.** *The map* $\Upsilon_E : \mathcal{R} \to \mathcal{R}$ *with* $\Upsilon_E(X) = Y$*, where* $\Upsilon_E(X) = (\Upsilon x_{ij})_{n\times n} \in cl(Y)$*, is an automorphism of* $\overrightarrow{\Gamma_L}(\mathcal{R})$*.*

*Proof.* Let $X = (x_{ij})_{n\times n}$ and $Y = (y_{ij})_{n\times n}$ belongs to $\mathcal{R}$ such that $\Upsilon_E(X) = \Upsilon_E(Y) = Z$, where $\Upsilon_E(X) = (\Upsilon x_{ij})_{n\times n}$ and $\Upsilon_E(Y) = (\Upsilon y_{ij})_{n\times n} \in cl(Z)$. Since $\Upsilon$ is a bijective map over $\mathbb{F}_q$, we have $[X] = [Y]$. It follows that $cl(X) = cl(Y)$ and so that $X = Y$ in $\mathcal{R}$. Being a one-one map on the finite set $\mathcal{R}$, we get $\Upsilon_E$ is a bijection.

Next, let $X \to Y$ in $\overrightarrow{\Gamma_L}(\mathcal{R})$, where $X, Y \in \mathcal{R}$. It follows that $[X] \subset [Y]$. Assume that $\Upsilon_E(X) = (\Upsilon x_{ij})_{n\times n} = Z_1$ and $\Upsilon_E(Y) = Z_2$. Let $A \in [\Upsilon_E(X)]$. Then there exist $T' \in \mathcal{R}$ such that $A \in cl(T')$. Since $\Upsilon_E$ is a bijective map so that there exists $T \in \mathcal{R}$ with $\Upsilon_E(T) = (\Upsilon t_{ij})_{n\times n} = T'$. Since $[(\Upsilon t_{ij})_{n\times n}] = [T'] \subseteq [\Upsilon_E(X)]$, we get $\Upsilon(T) = \Upsilon(C_1)\Upsilon(X)$. Consequently, $T \in [X] \subset [Y]$. There exists $C_2 \in M_n(\mathbb{F}_q)$ such that $T = C_2Y$. It follows that $\Upsilon(T) = \Upsilon(C_2)\Upsilon(Y)$. Now $[\Upsilon(T)] = [A]$ implies that $A \in [\Upsilon_E(Y)]$. Therefore, $[\Upsilon_E(X)] \subseteq [\Upsilon_E(Y)]$. Further, assume that there exists $B \in [Y]$ but $B \notin [X]$. Then $B = C'Y$. Consequently, $\Upsilon_E(Z'') \in [\Upsilon_E(Y)]$, for some $Z'' \in \mathcal{R}$ such that $B \in cl(Z'')$. If $\Upsilon_E(Z'') \in [\Upsilon_E(X)]$, then $Z'' \in [X]$, so that $B \in [X]$; a contradiction. Consequently, $\Upsilon_E(X) \to \Upsilon_E(Y)$. Thus, $\Upsilon_E$ is an automorphism of $\overrightarrow{\Gamma_L}(\mathcal{R})$. $\square$

Now extend $\Upsilon_E$ to $\Upsilon'$ on $M_n(\mathbb{F}_q)$ in the following remark.

**Remark 4.1.25.** Define a permutation $\Upsilon' : M_n(\mathbb{F}_q) \to M_n(\mathbb{F}_q)$ such that

$$\Upsilon'(A) = B \Longleftrightarrow \Upsilon_E(X_1) = X_2$$

where $A \in cl(X_1)$, $B \in cl(X_2)$. Observe that $\Upsilon'$ is an automorphism of $\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q))$.

**Lemma 4.1.26.** *Let $X, Y \in M_n(\mathbb{F}_q)$. If $[X] = [Y]$, then $[\Upsilon'(X)] = [\Upsilon'(Y)]$.*

*Proof.* Let $[X] = [Y]$. Then $X, Y \in cl(X_1)$ for $X_1 \in \mathcal{R}$. If $\Upsilon_E(X_1) = Z$, then by Remark 4.1.25, $\Upsilon'(X) = Z_1$ and $\Upsilon'(Y) = Z_2$, where $X, Y \in cl(X_1)$ and $Z_1, Z_2 \in cl(Z)$. Thus, $[\Upsilon'(X)] = [\Upsilon'(Y)]$. $\square$

**Lemma 4.1.27.** *Let $\Psi_E \in Aut(\overrightarrow{\Gamma_L}(\mathcal{R}))$ such that $\Psi_E(E_{\mathbf{1}}) = E_{\mathbf{1}}$ and $\Psi_E(E_{e_k}) = E_{e_k}$, for each $E_{e_k}$, where $k = 1, 2, \ldots, n$. Then $(\Upsilon_E^{-1}\Psi_E)(X) = X$ for each $X \in \mathcal{R}$, where $\Upsilon_E$ is a map defined in Lemma 4.1.24.*

*Proof.* In view of Corollary 4.1.8, the result holds for $X = \mathbf{0}$. Next, let $X \in \mathcal{R}$ such that $rank(X) = 1$. Then $[X] = [E_a]$ for some $0 \neq a \in \mathbb{F}_q^n$, where $a = (e_i + a_{i+1}e_{i+1} + \cdots + a_n e_n)$ and $1 \leq i \leq n$. By Lemma 4.1.22, we obtain $\Psi_E(X) = E_{\{e_i+\Upsilon(a_{i+1})e_{i+1}+\cdots+\Upsilon(a_n)e_n\}}$. By Lemma 4.1.26, we have $({\Upsilon_E}^{-1} \cdot \Psi_E)(X) = \Upsilon^{-1}(E_{\{e_i+\Upsilon(a_{i+1})e_{i+1}+\cdots+\Upsilon(a_n)e_n\}})$. It follows that $({\Upsilon_E}^{-1} \cdot \Psi_E)(X) = E_{\{e_i+a_{i+1}e_{i+1}+\cdots+a_n e_n\}}$. Therefore, $({\Upsilon_E}^{-1} \cdot \Psi_E)(X) = X$. Further, let $rank(X) \geq 2$. By Lemma 4.1.3, there exists $P \in M_n^*(\mathbb{F}_q)$ such that $[X] = [E_r P]$, where $rank(X) = r$. Since $[E_{e_i}P] \subset [E_r P]$, where $1 \leq i \leq r$, we get $[({\Upsilon_E}^{-1} \cdot \Psi_E)(E_{e_i}P)] \subset [(\Upsilon_E^{-1} \cdot \Psi_E)(X)]$. Note that $rank(E_{e_i}P) = 1$ gives $[E_{e_i}P] \subset [({\Upsilon_E}^{-1} \cdot \Psi_E)(X)]$. It follows that $\sum_{i=1}^{r}[E_{e_i}P] \subseteq [({\Upsilon_E}^{-1} \cdot \Psi_E)(X)]$ and so $[E_r P] \subseteq [({\Upsilon_E}^{-1} \cdot \Psi_E)(X)]$. Since $\dim([E_r P]) = \dim([({\Upsilon_E}^{-1} \cdot \Psi_E)(X)]) = nr$, we have $[E_r P] = [({\Upsilon_E}^{-1} \cdot \Psi_E)(X)]$. Therefore, $[X] = [(\Upsilon_E^{-1} \cdot \Psi_E)(X)]$ for $X \in M_n(\mathbb{F}_q)$. Hence, $(\Upsilon_E^{-1} \cdot \Psi_E)(X) = X$, for each $X \in \mathcal{R}$. $\square$

**Remark 4.1.28.** Let $\Psi \in Aut(\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q)))$ such that $[\Psi(E_{\mathbf{1}})] = [E_{\mathbf{1}}]$ and $[\Psi(E_{e_k})] = [E_{e_k}]$, for each $E_{e_k}$, where $1 \leq k \leq n$. Further, let $\Upsilon'$ be a map defined in Remark 4.1.25. Then by Lemma 4.1.27, for each $X \in \mathcal{R}$ we get $(\Upsilon_E^{-1}\Psi_E)(X) = X$. Consequently, we obtain $(\Upsilon'^{-1}\Psi)(A) = B$, where $A \in cl(\Psi_E(X))$ and $B \in cl(X)$.

Let $\sigma_E$ be an identity map on $\mathcal{R}$ i.e. for each $X \in \mathcal{R}, \sigma_E(X) = X$. Now we extend $\sigma_E$ to a permutation $\sigma$ on $M_n(\mathbb{F}_q)$ such that $\sigma(A) = B$, where $A, B \in cl(X)$.

**Lemma 4.1.29.** *The map $\sigma$ is an automorphism of $\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q))$.*

*Proof.* By the definition, $\sigma$ is a bijection on $M_n(\mathbb{F}_q)$. Let $X, Y \in M_n(\mathbb{F}_q)$ such that $X \to Y$. Then $[X] \subset [Y]$. Since $[\sigma(X)] = [X]$ and $[\sigma(Y)] = [Y]$ implies that $[\sigma(X)] \subset [\sigma(Y)]$. It follows that $\sigma(X) \to \sigma(Y)$. Thus, $\sigma$ is an automorphism of $\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q))$. $\square$

**Theorem 4.1.30.** *Let $n \geq 3$ and $\Psi_E \in Aut(\overrightarrow{\Gamma_L}(\mathcal{R}))$. Then there exist $P \in M_n^*(\mathbb{F}_q)$, a field automorphism $\Upsilon_E$ (defined in Lemma 4.1.24) and $\sigma_E$ (defined above) such that $\Psi_E = \varphi_P\big|_{\mathcal{R}} \cdot \Upsilon_E \cdot \sigma_E$.*

*Proof.* By Lemma 4.1.14, there exist $P \in M_n^*(\mathbb{F}_q)$ such that $(\varphi_{P^{-1}}\big|_{\mathcal{R}} \cdot \Psi_E)(E_{\mathbf{1}}) = E_{\mathbf{1}}$ and $(\varphi_{P^{-1}}\big|_{\mathcal{R}} \cdot \Psi_E)(E_{e_k}) = E_{e_k}$, where $k = 1, 2, \ldots, n$. In view of Lemma 4.1.27, there exist $\Upsilon_E$ such that $X = ({\Upsilon_E}^{-1} \cdot \varphi_{P^{-1}}\big|_{\mathcal{R}} \cdot \Psi_E)(X)$ for all $X \in \mathcal{R}$. Thus, ${\Upsilon_E}^{-1} \cdot \varphi_{P^{-1}}\big|_{\mathcal{R}} \cdot \Psi_E = \sigma_E$. Hence, $\Psi_E = \varphi_P\big|_{\mathcal{R}} \cdot \Upsilon_E \cdot \sigma_E$. $\square$

**Theorem 4.1.31.** *Let $\Psi \in Aut(\overrightarrow{\Gamma_L}(M_n(\mathbb{F}_q)))$, where $n \geq 3$. Then there exist $P \in M_n^*(\mathbb{F}_q)$, a field automorphism $\Upsilon'$ (defined in Remark 4.1.25) and a permutation $\sigma$ (defined above) such that $\Psi = \varphi_P \cdot \Upsilon' \cdot \sigma$.*

*Proof.* The proof follows from Lemma 4.1.14, Lemma 4.1.27 and Theorem 4.1.30.

$\square$

# Chapter 5

# Graphs on Semigroups

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The intersection graph of a semigroup was introduced by Bosák [1964]. The *intersection subsemigroup graph* $\Gamma(S)$ of semigroup $S$ is a simple undirected graph whose vertex set is the collection of proper subsemigroups of $S$ and two distinct vertices $A$ and $B$ are adjacent if and only if $A \cap B \neq \emptyset$. Inspired by the work of Bosák [1964], Csákány and Pollák [1969] studied the intersection graphs of groups and proved that there is an edge between two proper subgroups if they have at least two elements in common. The intersection graph on cyclic subgroups of a group has been studied by Haghi and Ashrafi [2017]. Chakrabarty et al. [2009] introduced the notion of the intersection ideal graph of rings. The *intersection ideal graph* $\Gamma(R)$ of a ring $R$ is an undirected simple graph whose vertex set is the collection of non-trivial left ideals of $R$ and two distinct vertices $I$ and $J$ are adjacent if and only if $I \cap J \neq \{0\}$. They characterized the rings $R$ for which the graph $\Gamma(R)$ is connected and obtained several necessary and sufficient conditions on a ring $R$ such that $\Gamma(R)$ is complete. Jafari and Rad [2010] studied the planarity of the intersection ideal graphs $\Gamma(R)$ of a commutative ring $R$ with unity. Akbari et al. [2013] classified all rings whose intersection graphs of ideals are not connected and also determined all rings whose clique number is finite. The intersection ideal graphs of semigroups can be defined analogously. The *intersection ideal graph* $\Gamma(S)$ of a semigroup $S$ is an undirected simple graph whose vertex set consists of non-trivial left ideals of $S$ and two distinct non-trivial left ideals $I$ and $J$ are adjacent if and only if their intersection is nontrivial.

Akbari, Habibi, Majidinya and Manaviyat [2014] have introduced the notion of inclusion ideal graph associated with the ring structure. The *inclusion ideal graph* $\mathcal{I}n(R)$ of a ring $R$ is an undirected simple graph whose vertex set is the collection of non-trivial left ideals of $R$ and two distinct non-trivial left ideals $I$ and $J$ are adjacent if and only if either $I \subset J$ or $J \subset I$. Further, Akbari, Habibi, Majidinya and Manaviyat [2015] have studied various graph invariants including connectedness, perfectness, diameter, girth etc. of $\mathcal{I}n(R)$. We consider the inclusion ideal graph associated with semigroups. The *inclusion ideal graph* $\mathcal{I}n(S)$ of a semigroup $S$ is an undirected simple graph whose vertex set is non-trivial left ideals of $S$ and two distinct non-trivial left ideals $I$ and $J$ of $S$ are adjacent if and only if $I \subset J$ or $J \subset I$.

This chapter aims to study the intersection ideal graph $\Gamma(S)$ of a semigroup $S$ and its spanning subgraph $\mathcal{I}n(S)$. In Section 5.1, we investigate the connectedness of $\Gamma(S)$. We classify the semigroups $S$ in terms of their ideals such that the diameter of $\Gamma(S)$ is two. We obtain the domination number, independence number, girth and the strong metric dimension of $\Gamma(S)$. We have also investigated the completeness, planarity and perfectness of $\Gamma(S)$. We show that if $S$ is a completely simple semigroup, then $\Gamma(S)$ is weakly perfect. Moreover, we give an upper bound of the chromatic number of $\Gamma(S)$. Finally, if $S$ is the union of $n$ minimal left ideals, then we obtain the metric dimension and the automorphism group of $\Gamma(S)$. In Section 5.2, we study the inclusion ideal graph $\mathcal{I}n(S)$ of a semigroup which is a spanning subgraph of $\Gamma(S)$. In this connection, we show that $\Gamma(S)$

is disconnected if and only if $\mathcal{I}n(S)$ is disconnected. For an arbitrary semigroup $S$, we investigate the perfectness, girth, planarity of $\mathcal{I}n(S)$. Moreover, various graph invariants including the dominance number, clique number, independence number and autmorphism group of the inclusion ideal graph of a completely simple semigroup $S$ is obatined. We also prove that the graph $\mathcal{I}n(S)$ has a perfect matching.

The content of Section 5.1 is published in the journal *"Quasigroups and Related Systems"* and the content of Section 5.2 is accepted for publication in the journal *"Algebra Colloquium"*.

## 5.1 The Intersection Ideal Graph of a Semigroup

In this section, we study the connectedness of intersection ideal graph $\Gamma(S)$. We show that if $\Gamma(S)$ is connected, then the diameter of $\Gamma(S)$ is at most two (see Theorem 5.1.4). Further, we classify the semigroups $S$ in terms of their ideals such that the diameter of $\Gamma(S)$ is two (see Theorem 5.1.7). We obtain the domination number, independence number, girth and the strong metric dimension of $\Gamma(S)$. We have also investigated the completeness, planarity and perfectness of $\Gamma(S)$. We show that if $S$ is a completely simple semigroup, then $\Gamma(S)$ is weakly perfect. Moreover, we give an upper bound of the chromatic number of $\Gamma(S)$. Finally, if $S$ is the union of $n$ minimal left ideals, then we obtain the metric dimension and the automorphism group of $\Gamma(S)$.

### 5.1.1 Connectivity of the Intersection Ideal Graph

In this subsection, we investigate the connectedness of $\Gamma(S)$. We show that $diam(\Gamma(S)) \leq 2$ if it is connected. Also, we classify the semigroups, in terms of their left ideals, such that the diameter of $\Gamma(S)$ is two.

**Theorem 5.1.1.** *The intersection ideal graph* $\Gamma(S)$ *is disconnected if and only if* $S$ *contains at least two minimal left ideals and every non-trivial left ideal of* $S$ *is minimal as well as maximal.*

*Proof.* First suppose that $\Gamma(S)$ is not connected. Then $S$ has at least two non-trivial left ideals $I_1$ and $I_2$. Without loss of generality, assume that $I_1 \in C_1$ and $I_2 \in C_2$, where $C_1$ and $C_2$ are distinct components of $\Gamma(S)$. If $I_1$ is not minimal, then there exists at least one non-trivial left ideal $I_k$ of $S$ such that $I_k \subset I_1$ so that their intersection is nontrivial. Therefore, $I_1 \sim I_k$. Now if the intersection of $I_2$ and $I_k$ is non-trivial then $I_1 \sim I_k \sim I_2$, a contradiction. Therefore the intersection of $I_2$ and $I_k$ is trivial. If $I_2 \cup I_k \neq S$, then $I_1 \sim (I_2 \cup I_k) \sim I_2$, a contradiction. Thus, $I_k \cup I_2 = S$. It follows that $I_1 \sim I_2$, again a contradiction. Thus, $I_1$ is minimal. Similarly, we get $I_2$ is minimal.

Further assume that $I_1$ is not maximal. Then there exists a non-trivial left ideal $I_k$ of $S$ such that $I_1 \subset I_k$ so that $I_1 \sim I_k$. If $I_1 \cup I_2 \neq S$, then $I_1 \sim I_1 \cup I_2 \sim I_2$, a contradiction to the fact that $\Gamma(S)$ is disconnected. It follows that $I_1 \cup I_2 = S$ so that the intersection of $I_k$ and $I_2$ is nontrivial. Thus we have $I_1 \sim I_k \sim I_2$, a contradiction. Hence, $I_1$ is maximal. Similarly, we observe that $I_2$ is maximal. The converse follows from the Remark 1.1.1. □

**Corollary 5.1.2.** *If the graph* $\Gamma(S)$ *is disconnected, then it is a null graph (i.e. it has no edge).*

**Theorem 5.1.3.** *The intersection ideal graph* $\Gamma(S)$ *is disconnected if and only if* $S$ *is the union of exactly two minimal left ideals.*

*Proof.* Suppose first that $\Gamma(S)$ is disconnected. Then by Theorem 5.1.1, each non-trivial left ideal of $S$ is minimal. Suppose $S$ has at least three minimal left ideals, namely $I_1, I_2$ and $I_3$. Then $I_1 \cup I_2$ is a non-trivial left ideal of $S$ which is not minimal. Consequently, by Theorem 5.1.1, we get a contradiction of the fact that

$\Gamma(S)$ is disconnected. Thus, $S$ has exactly two minimal left ideals. If $S \neq I_1 \cup I_2$, then $I_1 \cup I_2$ is a non-trivial left ideal which is not minimal, a contradiction ( cf. Theorem 5.1.1). Thus, $S = I_1 \cup I_2$.

Converse part follows from Theorem 5.1.1 and Lemma 1.1.2. □

**Theorem 5.1.4.** *If the intersection ideal graph* $\Gamma(S)$ *is connected, then* $diam(\Gamma(S)) \leq 2$.

*Proof.* Let $I_1, I_2$ be two non-trivial left ideals of $S$. If $I_1 \sim I_2$, then $d(I_1, I_2) = 1$. If $I_1 \nsim I_2$ i.e. $I_1 \cap I_2$ is trivial, then in the following cases we show that $d(I_1, I_2) \leq 2$.

**Case-1.** $I_1 \cup I_2 \neq S$. Then $I_1 \sim (I_1 \cup I_2) \sim I_2$ so that $d(I_1, I_2) = 2$.

**Case-2.** $I_1 \cup I_2 = S$. Since $\Gamma(S)$ is a connected graph, there exists a non-trivial left ideal $I_k$ of $S$ such that either $I_1 \cap I_k$ is non-trivial or $I_2 \cap I_k$ is nontrivial. Now we have the following subcases.

**Subcase 1.** $I_1 \not\subset I_k$ and $I_k \not\subset I_1$. Since $I_1 \not\subset I_k$ it follows that there exists $x \in I_k$ but $x \notin I_1$ so that $x \in I_2$. Consequently, $I_2 \cap I_k$ is nontrivial. Therefore, we get a path $I_1 \sim I_k \sim I_2$ of length two. Thus, $d(I_1, I_2) = 2$.

**Subcase 2.** $I_k \subset I_1$. There exists $x \in I_1$ but $x \notin I_k$. If $I_2 \cup I_k = S$, then $x \in I_2$. Thus, we get $I_1 \cap I_2$ is nontrivial, a contradiction. Consequently, $I_2 \cup I_k \neq S$. Further, we get a path $I_1 \sim (I_2 \cup I_k) \sim I_2$ of length two. Thus, $d(I_1, I_2) = 2$.

**Subcase 3.** $I_1 \subset I_k$. Since $I_1 \cup I_2 = S$ we get $I_k \cup I_2 = S$. Further, the intersection of $I_k$ and $I_2$ is nontrivial. Consequently, $I_1 \sim I_k \sim I_2$ gives a path of length two between $I_1$ and $I_2$. Thus, $d(I_1, I_2) = 2$. Hence, $diam(\Gamma(S)) \leq 2$. □

**Lemma 5.1.5.** *Let $S$ be a semigroup having minimal left ideals. Then* $\Gamma(S)$ *is complete if and only if $S$ has a unique minimal left ideal.*

*Proof.* Suppose that $S$ contains a unique minimal left ideal $I_1$. Note that every non-trivial left ideal of $S$ contains at least one minimal left ideal. Since $I_1$ is unique then it must contained in every non-trivial left ideals of $S$. Thus, the graph $\Gamma(S)$ is complete.

Conversely, suppose that $\Gamma(S)$ is a complete graph. On the contrary, if $S$ has at least two minimal left ideals $I_1$ and $I_2$, then $I_1 \nsim I_2$ by Remark 1.1.1, a contradiction to the fact that $\Gamma(S)$ is complete. Thus $S$ has a unique minimal left ideal. $\square$

**Lemma 5.1.6.** *The graph $\Gamma(S)$ is regular if and only if either $\Gamma(S)$ is null or a complete graph.*

*Proof.* First suppose that $\Gamma(S)$ is not a null graph. Suppose $S$ has at least two minimal left ideals $I_1$ and $I_2$. Since $\Gamma(S)$ is not a null graph, we get $I_1$ and $I_1 \cup I_2$ form non-trivial left ideals of $S$ and $I_1 \sim (I_1 \cup I_2)$. If $J$ is any non-trivial left ideal of $S$ such that $J \sim I_1$, then $J \sim (I_1 \cup I_2)$. It follows that every non-trivial left ideal of $S$ which is adjacent with $I_1$ is also adjacent with $I_1 \cup I_2$ and $I_2 \sim (I_1 \cup I_2)$ but $I_2 \nsim I_1$ implies that $\deg(I_1) < \deg(I_1 \cup I_2)$, a contradiction. Therefore, $\Gamma(S)$ is a complete graph. $\square$

Next we classify the semigroups such that the diameter of the intersection ideal graph $\Gamma(S)$ is two.

**Theorem 5.1.7.** *Let $S$ be a semigroup having minimal left ideals. Then for a connected graph $\Gamma(S)$, we have $diam(\Gamma(S)) = 2$ if and only if $S$ has at least two minimal left ideals.*

*Proof.* Suppose that $diam(\Gamma(S)) = 2$. Assume that $I_1$ is the only minimal left ideal of $S$. Since $I_1$ is a unique minimal left ideal, we have $I_1 \subset K$, for any non-trivial left ideal $K$ of $S$. Therefore, for any non-trivial left ideals $J$ and $K$, we have $I_1 \subset (J \cap K)$. Consequently, $d(J, K) = 1$ for any $J, K \in V(\Gamma(S))$. Therefore, $S$ has at least two minimal left ideals. Conversely, suppose that $S$ has at least two minimal left ideals $I_1$ and $I_2$. Then by Remark 1.1.1, we have $I_1 \nsim I_2$. Consequently, by Theorem 5.1.4, $d(I_1, I_2) = 2$. Thus, $diam(\Gamma(S)) = 2$. □

### 5.1.2 Invariants of $\Gamma(S)$

In this subsection, first we obtain the girth of $\Gamma(S)$. Then we discuss planarity and perfectness of $\Gamma(S)$. Also, we classify the semigroup $S$ such that $\Gamma(S)$ is bipartite, star graph and tree, respectively. Further, we investigate the other graph invariants viz. clique number, independence number and strong metric dimension of $\Gamma(S)$.

**Theorem 5.1.8.** *Let $S$ be a semigroup such that $\Gamma(S)$ contains a cycle. Then $g(\Gamma(S)) = 3$.*

*Proof.* If $\Gamma(S)$ is disconnected or a tree, then clearly $g(\Gamma(S)) = \infty$. Suppose that the semigroup $S$ has $n$ minimal left ideals. Now we prove the result by observing the following cases.

**Case-1.** $n = 0$. If $S$ has no non-trivial left ideals, then there is nothing to prove. Otherwise, there exists a chain of non-trivial left ideals of $S$ such that $I_1 \supset I_2 \supset \cdots \supset I_k \supset \cdots$. Thus, $g(\Gamma(S)) = 3$.

**Case-2.** $n = 1$. Suppose that $I_1$ is the only minimal left ideal of $S$. Since $I_1$ is a unique minimal left ideal, we obtain $I_1 \subset K$, for any non-trivial left ideal $K$ of $S$. Therefore, for any non-trivial left ideals $I$ and $J$, we get $I_1 \subset (I \cap J) \neq \emptyset$. If $S$ has at least three non-trivial left ideals, then $g(\Gamma(S)) = 3$. Otherwise, $g(\Gamma(S)) = \infty$.

**Case-3.** $n = 2$. Let $I_1, I_2$ be two minimal left ideals of $S$. If $I_1 \cup I_2 = S$, then by Theorem 5.1.3 and Corollary 5.1.2, $g(\Gamma(S)) = \infty$. If $I_1 \cup I_2 \neq S$, then $J = I_1 \cup I_2$ is a non-trivial left ideal of $S$. Suppose $I_1, I_2$ and $J$ are the only non-trivial left ideals of $S$. Then $I_1 \sim J \sim I_2$ and so $g(\Gamma(S)) = \infty$. Further, assume that $S$ has a non-trivial left ideal $K$ other than $I_1, I_2$ and $J$. Since $I_1, I_2$ are minimal left ideals of $S$, we have either $I_1 \subset K$ or $I_2 \subset K$. Without loss of generality, assume that

$I_1 \subset K$. Then $I_1 \sim K \sim J \sim I_1$. It follows that $g(\Gamma(S)) = 3$.

**Case-4.** $n \geq 3$. Let $I_1, I_2, I_3$ be the minimal left ideals of $S$. Then we have a cycle $(I_1 \cup I_2) \sim (I_2 \cup I_3) \sim (I_1 \cup I_3) \sim (I_1 \cup I_2)$ of length 3. Thus, $g(\Gamma(S)) = 3$. □

Let Min$(S)$ (Max$(S)$) be the set of all minimal (maximal) left ideals of $S$. By a non-trivial left ideal $I_{i_1 i_2 \cdots i_k}$, we mean $I_{i_1} \cup I_{i_2} \cup \cdots \cup I_{i_k}$, where $I_{i_1}, I_{i_2}, \ldots, I_{i_k} \in$ Min$(S)$.

**Theorem 5.1.9.** *Let $\Gamma(S)$ be the intersection ideal graph of $S$. Then the following statements hold:*

(i) *If $\Gamma(S)$ is planar, then $|\mathrm{Min}(S)| \leq 3$.*

(ii) *Let $S$ be a semigroup such that it is a union of $n$ minimal left ideals. Then $\Gamma(S)$ is planar if and only if $n \leq 3$.*

*Proof.* (i) Suppose that $|\mathrm{Min}(S)| = 4$ with $\mathrm{Min}(S) = \{I_1, I_2, I_3, I_4\}$. Then note that the subgraph induced by the vertices $I_1, I_{12}, I_{123}, I_{14}$ and $I_{124}$ is isomorphic to $K_5$. Thus, $\Gamma(S)$ is nonplanar.

(ii) The proof for $\Gamma(S)$ is nonplanar for $n \geq 4$ follows from part (i). If $n = 2$, then by Corollary 5.1.2 and Theorem 5.1.3, $\Gamma(S)$ is planar. For $n = 3$, $\Gamma(S)$ is planar as shown in Figure 5.1. □

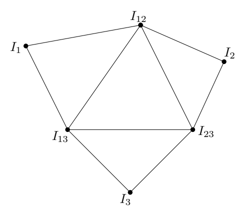


FIGURE 5.1: Planar drawing of $\Gamma(S)$ for $S = I_{123}$.

**Theorem 5.1.10.** *Let $\Gamma(S)$ be the intersection ideal graph of $S$. Then the following statements hold:*

(i) *If* $\Gamma(S)$ *is a perfect graph, then* $|\mathrm{Min}(S)| \leq 4$.

(ii) *Let* $S$ *be the union of* $n$ *minimal left ideals. Then* $\Gamma(S)$ *is perfect if and only if* $n \leq 4$.

*Proof.* (i) Suppose that $|\mathrm{Min}(S)| = 5$ with $\mathrm{Min}(S) = \{I_1, I_2, I_3, I_4, I_5\}$. Note that $I_{12} \sim I_{23} \sim I_{34} \sim I_{45} \sim I_{15} \sim I_{12}$ induces a cycle of length 5. Then by Theorem 1.3.2, $\Gamma(S)$ is not perfect.

(ii) The proof for $\Gamma(S)$ is not a perfect graph for $n \geq 5$ follows from part (i). If $n = 2$, then by Corollary 5.1.2 and Theorem 5.1.1, $\Gamma(S)$ is disconnected. Thus, being a null graph, $\Gamma(S)$ is perfect. For $n \in \{3, 4\}$, we show that $\Gamma(S)$ does not contain a hole or an antihole of odd length at least five (cf. Theorem 1.3.2). If $n = 3$, then $\Gamma(S)$ is perfect as shown in Figure 5.1. If $n = 4$, then from Figure 5.2, note that $\Gamma(S)$ does not contain a hole or an antihole of odd length at least five. $\square$

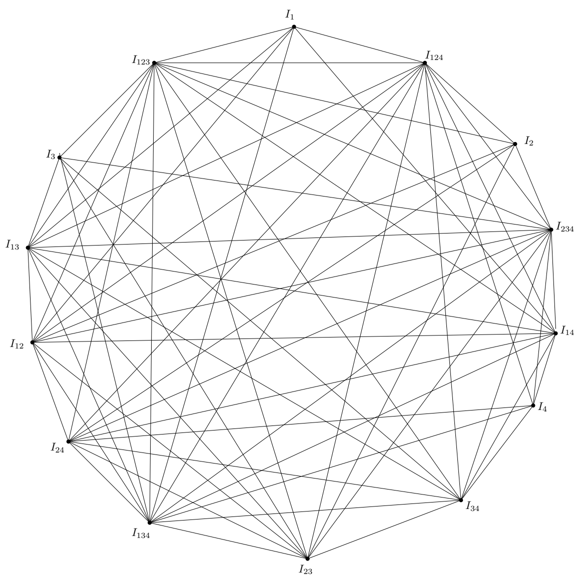


Figure 5.2: The intersection graph $\Gamma(S)$ for $S = I_{1234}$.

**Theorem 5.1.11.** *Let $S$ be a semigroup having minimal left ideals such that $V(\Gamma(S)) > 1$. Then the following conditions are equivalent:*

(i) $\Gamma(S)$ *is a star graph.*

(ii) $\Gamma(S)$ *is a tree.*

(iii) $\Gamma(S)$ *is bipartite.*

(iv) *Either $S$ has exactly three non-trivial left ideals $I_1$, $I_2$ and $I_{12}$ such that $I_1$ and $I_2$ are minimal or $S$ has two non-trivial left ideals $I_1, I_2$ such that $I_1 \subset I_2$.*

*Proof.* We prove (ii), (iii) $\Rightarrow$ (iv). The proof of remaining parts is straightforward. Suppose $\Gamma(S)$ is a tree. Then clearly $|\mathrm{Min}(S)| \leq 2$. Otherwise, for minimal left ideals $I_1, I_2, I_3$ we have $I_{12} \sim I_{13} \sim I_{23} \sim I_{12}$ a cycle, a contradiction. Suppose that $|\mathrm{Min}(S)| = 1$. Let $I_1$ be the unique minimal left ideal of $S$. Consequently, $I_1$ is contained in all the other non-trivial left ideals of $S$. If $S$ has at least three non-trivial left ideals, then we get a cycle, a contradiction. Thus $|V(\Gamma(S))| = 2$. Now we assume that $|\mathrm{Min}(S)| = 2$. Let $I_1, I_2$ be two minimal left ideals of $S$. Let $S = I_{12}$. Then by Corollary 5.1.2 and Theorem 5.1.3, $\Gamma(S)$ is disconnected so is not a tree. Thus $S \neq I_{12}$. Then $J = I_{12}$ is a non-trivial left ideal of $S$. Suppose $S$ has a non-trivial left ideal $K$ other than $I_1, I_2$ and $J$. Without loss of generality, assume that $I_1 \subset K$ then we get a cycle $I_1 \sim I_{12} \sim K \sim I_1$, a contradiction. Thus, for $S \neq I_{12}$, we have $V(\Gamma(S)) = \{I_1, I_2, I_{12}\}$. (iii) $\Rightarrow$ (iv). If $\Gamma(S)$ is bipartite, then we have $|\mathrm{Min}(S)| \leq 2$. In the similar lines of the work discussed above, (iv) holds. □

**Theorem 5.1.12.** *If $S$ is the union of $n$ minimal left ideals, then $\gamma(\Gamma(S)) = 2$. Otherwise, $\gamma(\Gamma(S)) = 1$.*

*Proof.* Suppose that $S$ is the union of $n$ minimal left ideals, that is, $S = I_{12\cdots n}$. Note that there is no dominating vertex in $\Gamma(S)$ so that $\gamma(\Gamma(S)) \geq 2$. Now we show that $D = \{I_1, I_{23\cdots n}\}$ is a dominating set. Since $S$ is the union of $n$ minimal left ideals so any non-trivial left ideal of $S$ is the union of some of these minimal left ideals (cf. Lemma 1.1.7). Let $J \in V(\Gamma(S)) \setminus D$ be any non-trivial left ideal of $S$. Then $J$ is a union of $k$ minimal left ideals of $S$, where $1 \le k \le n-1$. If $I_1 \subset J$, then we are done. Otherwise, $J$ must be the union of $I_2, I_3, \ldots, I_n$. It follows that the intersection of $J$ and $I_{23\cdots n}$ is nontrivial. Consequently, $J \sim I_{23\cdots n}$. Thus, $D$ is a dominating set. Further, suppose that $S \ne I_{12\cdots n}$. It follows that $J = I_{12\cdots n}$ is a non-trivial left ideal of $S$. It is well known that every non-trivial left ideal of $S$ contains at least one minimal left ideal. Consequently, for any non-trivial left ideal $K$ of $S$, we have $J \cap K$ is nontrivial. Thus, $J$ is a dominating vertex. Hence, $\gamma(\Gamma(S)) = 1$. This completes the proof. □

**Theorem 5.1.13.** *Let $S$ be a semigroup with $n$ minimal left ideals. Then the independence number of $\Gamma(S)$ is $n$.*

*Proof.* Let $\mathrm{Min}(S) = \{I_{i_1} : i_1 \in [n]\}$ be the set of all minimal left ideals of $S$. Then, by Remark 1.1.1, $\mathrm{Min}(S)$ is an independent set of $\Gamma(S)$. It follows that $\alpha(\Gamma(S)) \ge n$. Now we prove that for any arbitrary independent set $U$, we have $|U| \le n$. Assume that $I \in V(\Gamma(S))$ such that $I \in U$. Since every non-trivial left ideal contains at least one minimal left ideal. Without loss of generality, assume that $I_{i_1 i_2 \cdots i_k} \subseteq I$ for some $i_1, i_2, \ldots, i_k \in [n]$. Then note that $|U| \le n - k + 1$. Otherwise, there exist at least two non-trivial left ideals which are adjacent, a contradiction. Consequently, we have $|U| \le n$. Thus, $\alpha(\Gamma(S)) = n$. □

**Lemma 5.1.14.** *Let $S$ be a semigroup with $n$ $(\ge 3)$ minimal left ideals. Then there exists a clique in $\Gamma(S)$ of size $n$.*

*Proof.* Let $I_1, I_2, \ldots, I_n$ be $n$ minimal left ideals. Consider the set

$$\mathcal{C} = \{I_{i_1 i_2 \cdots i_{n-1}} : i_1, i_2, \ldots, i_{n-1} \in [n]\}.$$

Clearly, $|\mathcal{C}| = n$. Notice that for any $J, K \in \mathcal{C}$, we have $J \cap K$ is non-trivial so that $J \sim K$. Thus, $\mathcal{C}$ becomes a clique of size $n$. □

**Theorem 5.1.15.** *Let $S$ be a semigroup with $n(> 1)$ minimal left ideals. Then $\omega(\Gamma(S)) = n$ if and only if one of the following holds:*

(i) *$S$ is the union of exactly three minimal left ideals.*

(ii) *$S$ has only two minimal left ideals $I_1, I_2$ and a unique maximal left ideal $I_{12}$.*

*Proof.* First suppose that $\omega(\Gamma(S)) = n$. Assume that $S$ has $n(\geq 4)$ minimal left ideals, namely $I_1, I_2, \ldots, I_n$. Then $\mathcal{C} = \{I_{i_1 i_2 \cdots i_{n-1}}, I_{i_1 i_2} : i_1, i_2, \ldots, i_n \in [n]\}$ forms a clique of size greater than $n$ of $\Gamma(S)$. It follows that $\omega(\Gamma(S)) > n$. If $n = 3$ and assume that $S \neq I_{123}$, then $\mathcal{C} = \{I_{12}, I_{13}, I_{23}, I_{123}\}$ forms a clique of size four of $\Gamma(S)$. It follows that $S = I_{123}$. For $n = 2$, we have either $S = I_{12}$ or $S \neq I_{12}$. For $S = I_{12}$, by Corollary 5.1.2 and by Theorem 5.1.3, $\Gamma(S)$ is disconnected. Thus, $\omega(\Gamma(S)) < n$. Thus $S \neq I_{12}$. If $S$ has a non-trivial left ideal $K \notin \{I_1, I_2, I_{12}\}$, then we get a clique of size three. Therefore, $I_{12}$ is a unique maximal left ideal. Converse follows trivially. □

**Lemma 5.1.16.** *If $\Gamma(S)$ is connected, then $\mathrm{Max}(S)$ forms a clique of $\Gamma(S)$.*

*Proof.* We prove the result by showing that if $J, K \in \mathrm{Max}(S)$ then $J \sim K$. Let $J \nsim K$. The maximality of $J$ and $K$ follows that $J \cup K = S$. By Lemma 1.1.5, $S \setminus J$ and $S \setminus K$ are $\mathcal{L}-$classes of $S$. It follows that $J$ and $K$ are only non-trivial left ideals of $S$. Thus, being a null graph $\Gamma(S)$ is disconnected, a contradiction. □

**Theorem 5.1.17.** *If $K$ is a maximal left ideal of $S$ such that $\deg(K)$ is finite, then $\chi(\Gamma(S)) < \infty$.*

*Proof.* Let $J$ be an arbitrary non-trivial left ideal of $S$ such that $J \nsim K$. Note that $J$ is the minimal left ideal of $S$. On the contrary, suppose that $J$ is not a minimal left ideal of $S$. Then there exists a non-trivial left ideal $J'$ of $S$ such that $J' \subset J$. Since $K$ is the maximal left ideal of $S$, we get $J' \cup K = S$. It follows that the intersection of $J$ and $K$ is non-trivial, a contradiction. By Remark 1.1.1, we can color all the vertices which are not adjacent with $K$ with one color. Since $\deg(K)$ is finite, we have $\chi(\Gamma(S)) < \infty$. $\square$

**Proposition 5.1.18.** *If $S$ is the union of $n$ minimal left ideals, then $\omega(\Gamma(S)) = \chi(\Gamma(S)) = 2^{n-1} - 1$. Moreover, $\Gamma(S)$ is weakly perfect.*

*Proof.* First note that $S$ has $2^n - 2$ non-trivial left ideals and every non-trivial left ideal of $S$ is of the form $I_{i_1 i_2 \cdots i_k}$ and $1 \leq k \leq n-1$ (cf. Lemma 1.1.7). If $n$ is odd, then consider $\mathcal{C} = \{I_{j_1 j_2 \cdots j_t} : \lceil \frac{n}{2} \rceil \leq t \leq n-1\}$. Note that $\mathcal{C}$ forms a clique of size $2^{n-1} - 1$. We may now suppose that $n$ is even. Consider $\mathcal{C}_1 = \{I_{j_1 j_2 \cdots j_t} : \frac{n}{2} + 1 \leq t \leq n-1\}$. Notice that $\mathcal{C}_1$ forms a clique. Further, observe that $\mathcal{C}' = \{I_{i_1 i_2 \cdots i_{\frac{n}{2}}} : i_1, i_2, \ldots, i_{\frac{n}{2}} \in [n]\}$ do not form a clique because for $j_1, j_2, \ldots, j_{\frac{n}{2}} \in [n] \setminus \{i_1, i_2, \ldots, i_{\frac{n}{2}}\}$, we get $I_{i_1 i_2 \cdots i_{\frac{n}{2}}} \nsim I_{j_1 j_2 \cdots j_{\frac{n}{2}}}$. However, $\mathcal{C}'' = \{I_{i_1 i_2 \cdots i_{\frac{n}{2}}} \in \mathcal{C}' \setminus \{I_{j_1 j_2 \cdots j_{\frac{n}{2}}}\} : j_1, j_2, \ldots, j_{\frac{n}{2}} \notin \{i_1, i_2, \ldots, i_{\frac{n}{2}}\}\}$ forms a clique of size $\frac{|\mathcal{C}'|}{2}$. Further note that the set $\mathcal{C}_1 \cup \mathcal{C}''$ also forms a clique of size $2^{n-1} - 1$. Thus, $\omega(\Gamma(S)) \geq 2^{n-1} - 1$. To complete the proof, we show that $\chi(\Gamma(S)) \leq 2^{n-1} - 1$. For $I = I_{i_1 i_2 \cdots i_k}$ and $J = I_{j_1 j_2 \cdots j_{n-k}}$, where $i_1, i_2, \ldots, i_k \in [n] \setminus \{j_1, j_2, \ldots, j_{n-k}\}$ we have $I \cap J$ is trivial. Consequently, we can color these vertices with same color so that we can cover all the vertices with $2^{n-1} - 1$ colors. Thus $\chi(\Gamma(S)) \leq 2^{n-1} - 1$. Hence $\omega(\Gamma(S)) = \chi(\Gamma(S)) = 2^{n-1} - 1$. $\square$

**Corollary 5.1.19.** *Let $S$ be a completely simple semigroup. Then the graph $\Gamma(S)$ is weakly perfect.*

In order to find an upper bound of the chromatic number of $\Gamma(S)$, where $S$ is an arbitrary semigroup, first we define

$$X_1 = \{I \in V(\Gamma(S)) : I_{i_1 i_2 \cdots i_n} \subseteq I\},$$
$$X_2 = \{I \in V(\Gamma(S)) : I \subset I_{i_1 i_2 \cdots i_n} \text{ and } I \neq I_{i_1 i_2 \cdots i_n}\},$$
$$X_3 = V(\Gamma(S)) \setminus (X_1 \cup X_2).$$

Let $\widetilde{\mathrm{Min}(I)}$ be the set of all minimal left ideals contained in $I$ . Further define a relation $\rho$ on $X_3$ such that

$$J \;\; \rho \;\; K \Longleftrightarrow \widetilde{\mathrm{Min}(J)} = \widetilde{\mathrm{Min}(K)}.$$

Note that $\rho$ is an equivalence relation.

**Theorem 5.1.20.** *Let $S$ be a semigroup with $n$ minimal left ideals such that $\chi(\Gamma(S))$ is finite. Then*

$$\chi(\Gamma(S)) \leq |X_1| + (2^{n-1} - 1) + (2^{n-1} - 1)m,$$

*where $m = \max\{|C(I)| : C(I)$ is an equivalence class of $\rho\}$.*

*Proof.* Note that for any $I, J \in X_1$, we have $I \sim J$. Since every non-trivial left ideal contains at least one minimal left ideal, consequently each element of $X_1$ is a dominating vertex of $\Gamma(S)$. Therefore, we need at least $|X_1|$ colors in any coloring of $\Gamma(S)$. By proof of Proposition 5.1.18, we can color all the vertices of $X_2$ with at least $2^{n-1} - 1$ colors so that we need at least $2^{n-1} - 1 + |X_1|$ colors to color $X_1 \cup X_2$.

To prove our result, we need to show that the vertices of $X_3$ can be colored by using $(2^{n-1} - 1)m$ colors. Now let $J, K \in X_3$ such that $I_{i_1 i_2 \cdots i_k} \subset J$ and $I_{j_1 j_2 \cdots j_t} \subset K$. Note that $J \cap K$ is non-trivial if and only if $I_{i_1 i_2 \cdots i_k} \cap I_{j_1 j_2 \cdots j_t}$ is nontrivial. It follows that $J \sim K$ in $\Gamma(S)$ if and only if either $I_{i_1 i_2 \cdots i_k} = I_{j_1 j_2 \cdots j_t}$ or $I_{i_1 i_2 \cdots i_k} \sim I_{j_1 j_2 \cdots j_t}$.

Note that the equivalence class of $I \in X_3$ under $\rho$ is

$$C(I) = \{J \in X_3 : \widetilde{\mathrm{Min}(I)} = \widetilde{\mathrm{Min}(J)}\}.$$

Since $\chi(\Gamma(S)) < \infty$ we get $|C(I)| < \infty$. Consequently, $|C(I)| \leq m$. Observe that $C(I)$ forms a clique, we require maximum $m$ colors to color each class under $\rho$. Note that $J \in C(J)$ and $K \in C(K)$ such that $J \sim K$ if and only if $I_{i_1 i_2 \cdots i_k} \sim I_{j_1 j_2 \cdots j_t}$ in $\Gamma(S)$. Consequently, we can color the vertices in $X_3$ by using $(2^{n-1} - 1)m$ colors. □

**Theorem 5.1.21.** *Let $S$ be a semigroup with $n$ minimal left ideals. Then*

$$\mathrm{sdim}(\Gamma(S)) = \begin{cases} 2^{n-1} - 1; & \text{if } S \text{ is a union of } n \text{ minimal left ideals;} \\ |X_1| + |X_3| + 2^{n-1} - 2; & \text{Otherwise.} \end{cases}$$

*Proof.* Let $I, J \in V(\Gamma(S))$ such that $I_{i_1 i_2 \cdots i_k} \subseteq I$ and $I_{j_1 j_2 \cdots j_t} \subseteq J$. Then $I \sim J$ if and only if either $I_{i_1 i_2 \cdots i_k} = I_{j_1 j_2 \cdots j_t}$ or $I_{i_1 i_2 \cdots i_k} \sim I_{j_1 j_2 \cdots j_t}$. Define a relation $\rho_1$ on $V(\Gamma(S))$ such that $I \ \rho_1 \ J$ if and only if $\widetilde{\mathrm{Min}(I)} = \widetilde{\mathrm{Min}(J)}$. Clearly, $\rho_1$ is an equivalence relation on $V(\Gamma(S))$. Let $N[I_{i_1 i_2 \cdots i_k}] = \{I \in V(\Gamma(S)) : \widetilde{\mathrm{Min}(I)} = I_{i_1 i_2 \cdots i_k}\}$ be equivalence class containing $I_{i_1 i_2 \cdots i_k}$. If $S \neq I_{i_1 i_2 \cdots i_n}$, then by Theorem 1.3.5, we have $\mathcal{R}_{\Gamma(S)}$ whose vertex set $V(\mathcal{R}_{\Gamma(S)}) = \{I_{i_1 i_2 \cdots i_k} : i_1, i_2, \ldots, i_k \in [n] \text{ and } 1 \leq k \leq n\}$. By using Proposition 5.1.18, note that $\omega(\mathcal{R}_{\Gamma(S)}) = 2^{n-1}$. Then $\mathrm{sdim}(\Gamma(S)) = |X_1| + |X_3| + 2^{n-1} - 2$. Next, if $S = I_{i_1 i_2 \cdots i_n}$, then $V(\mathcal{R}_{\Gamma(S)}) = \{I_{i_1 i_2 \cdots i_k} : i_1, i_2, \ldots, i_k \in [n] \text{ and } 1 \leq k \leq n - 1\}$. By using Proposition 5.1.18, note that $\omega(\mathcal{R}_{\Gamma(S)}) = 2^{n-1} - 1$. Then $\mathrm{sdim}(\Gamma(S)) = 2^{n-1} - 1$. □

In the rest of the subsection, we consider a class of those semigroups which are the union of $n$ minimal left ideals. In particular, completely simple semigroups belongs to this class. In what follows, the semigroup $S$ is assumed to be the union of $n$ minimal left ideals $I_{i_1}, I_{i_2}, \ldots, I_{i_n}$ i.e. $S = I_{i_1 i_2 \cdots i_n}$.

**Theorem 5.1.22.** *Let $S$ be the union of $n$ minimal left ideals. Then $\Gamma(S)$ is a graph of order $2^n - 2$.*

*Proof.* In view of Corollary 1.1.9, the vertices of $\Gamma(S)$ are either minimal left ideals or union of minimal left ideals. In addition to $n$ minimal left ideals, we have $\binom{n}{2}$ + $\binom{n}{3}+\cdots+\binom{n}{n-1}$ non-trivial left ideals as a union of minimal left ideals. Thus, we obtain $\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n-1} = 2^n-2$ non-trivial left ideals of $S$. Hence, $|V(\Gamma(S)))| = 2^n - 2$. □

**Lemma 5.1.23.** *Let $S$ be the union of $n$ minimal left ideals and let $K = I_{i_1 i_2 \cdots i_k}$ be a non-trivial left ideal of $S$. Then* $\deg(K) = (2^k-2)+(2^{n-k}-2)+(2^{n-k}-1)(2^k-2)$.

*Proof.* Let $J$ be a non-trivial left ideal of $S$ such that $J \sim K$. Clearly, $J \cap K$ is a non-trivial left ideal. We have the following cases:

**Case-1.** $J \not\subset K$ and $K \not\subset J$. Since $J \sim K$ and $K = I_{i_1 i_2 \cdots i_k}$, we obtain the number of non-trivial left ideals such that $J \not\subset K$ and $K \not\subset J$ is

$$= \left(\sum_{i=1}^{n-k} \binom{n-k}{i}\right)\left(\sum_{i=1}^{k-1} \binom{k}{i}\right) = (2^{n-k}-1)(2^k-2).$$

**Case-2.** $J \subset K$. By the proof of Theorem 5.1.22, we have $\binom{k}{1}+\binom{k}{2}+\binom{k}{3}+\cdots+\binom{k}{k-1} = 2^k - 2$ non-trivial left ideals which are properly contained in $K$.

**Case-3.** $K \subset J$. Let $J = I_{i_1 i_2 \cdots i_k i_{k+1} i_{k+2} \cdots i_s}$ such that $i_{k+1}, i_{k+2}, \ldots, i_s \in [n] \setminus \{i_1, i_2, \ldots, i_k\}$ and $1 \leq s \leq n-k-1$. Consequently, we have $\sum_{i=1}^{n-k-1} \binom{n-k}{i} = 2^{n-k} - 2$ non-trivial left ideals which properly contains $K$. Thus, from the above cases we have the result. □

**Corollary 5.1.24.** *If $S$ is the union of $n$ minimal left ideals, then the graph $\Gamma(S)$ is Eulerian for $n \geq 3$.*

**Theorem 5.1.25.** *If $S$ is the union of $n$ minimal left ideals, then the metric dimension of $\Gamma(S)$ is given below:*

$$\beta(\Gamma(S)) = \begin{cases} 2 & \text{if } n = 3; \\ n & \text{if } n \geq 4. \end{cases}$$

*Proof.* For $n = 3$, it is easy to observe that $\{I_{i_1}, I_{i_2}\}$ forms a minimum resolving set. If $n \geq 4$, then by Lemma 1.1.7, we have $|V(\Gamma(S))| = 2^n - 2$. In view of Lemma 1.3.4, we get

$$n = f(2^n - 2, 2) \leq \beta(\Gamma(S)).$$

It is easy to observe that for $k = n - 1$, we have $2^k + k \not\geq 2^n - 2$. Therefore, the least positive integer $k$ is $n$ for which $k + 2^k \geq 2^n - 2$. Thus $n \leq \beta(\Gamma(S))$. To obtain upper bound of $\beta(\Gamma(S))$, let $J = I_{i_1 i_2 \cdots i_k}$ and $K = I_{j_1 j_2 \cdots j_t}$ be distinct arbitrary vertices $\Gamma(S)$. Since $J \neq K$, there exists at least $I_{i_s} \in \text{Min}(S)$ such that $I_{i_s} \sim J$ and $I_{i_s} \nsim K$. It follows that $d(J, I_{i_s}) \neq d(K, I_{i_s})$. Thus $\text{Min}(S) = \{I_{i_1} : i_1 \in [n]\}$ forms a resolving set for $\Gamma(S)$ of size $n$. It follows that $\beta(\Gamma(S)) \leq n$. This completes our proof. $\square$

Now we obtain the automorphism group of $\Gamma(S)$, when $S$ is the union of $n$ minimal left ideal.

**Lemma 5.1.26.** *For $\sigma \in S_n$, let $\phi_\sigma : V(\Gamma(S)) \to V(\Gamma(S))$ defined by $\phi_\sigma(I_{i_1 i_2 \cdots i_k}) = I_{\sigma(i_1)\sigma(i_2)\cdots\sigma(i_k)}$. Then $\phi_\sigma \in Aut(\Gamma(S))$.*

*Proof.* It is easy to verify that $\phi_\sigma$ is a permutation on $V(\Gamma(S))$. Now we show that the map $\phi_\sigma$ preserves adjacency. Let $I_{i_1 i_2 \cdots i_t}$ and $I_{j_1 j_2 \cdots j_k}$ be arbitrary vertices of $\Gamma(S)$ such that $I_{i_1 i_2 \cdots i_t} \sim I_{j_1 j_2 \cdots j_k}$. Then $I_{i_1 i_2 \cdots i_t} \cap I_{j_1 j_2 \cdots j_k} \neq \emptyset$. Now

$$\begin{aligned} I_{i_1 i_2 \cdots i_t} \sim I_{j_1 j_2 \cdots j_k} &\Longleftrightarrow I_{\sigma(i_1)\sigma(i_2)\cdots\sigma(i_t)} \sim I_{\sigma(j_1)\sigma(j_2)\cdots\sigma(j_k)} \\ &\Longleftrightarrow \phi_\sigma(I_{i_1 i_2 \cdots i_t}) \sim \phi_\sigma(I_{j_1 j_2 \cdots j_k}). \end{aligned}$$

Thus, $\phi_\sigma \in Aut(\Gamma(S))$. $\square$

**Proposition 5.1.27.** *For each $f \in Aut(\Gamma(S))$, we have $f = \phi_\sigma$ for some $\sigma \in S_n$.*

*Proof.* In view of Lemma 5.1.23 and Lemma 5.1.26, suppose that $f(I_{i_1}) = I_{j_1}$, $f(I_{i_2}) = I_{j_2}$, $\ldots$, $f(I_{i_n}) = I_{j_n}$. Consider $\sigma \in S_n$ such that $\sigma(i_1) = j_1, \sigma(i_2) =$

$j_2, \ldots, \sigma(i_n) = j_n$. Then $\phi_\sigma(I_{i_1 i_2 \cdots i_k}) = I_{\sigma(i_1)\sigma(i_2)\cdots\sigma(i_k)} = I_{j_1 j_2 \cdots j_k}$ (cf. Lemma 5.1.26). Clearly, $I_{i_1} \sim I_{i_1 i_2 \cdots i_k}$, $I_{i_2} \sim I_{i_1 i_2 \cdots i_k}$, $\ldots$, $I_{i_k} \sim I_{i_1 i_2 \cdots i_k}$. Also note that $I_{i_t} \cap I_{i_1 i_2 \cdots i_k}$ is trivial for $i_t \in \{i_{k+1}, i_{k+2}, \ldots, i_n\}$ where $i_{k+1}, i_{k+2}, \ldots, i_n \in [n] \backslash \{i_1, i_2, \ldots, i_k\}$. It follows that $I_{i_{k+1}} \nsim I_{i_1 i_2 \cdots i_k}$, $I_{i_{k+2}} \nsim I_{i_1 i_2 \cdots i_k}$, $\ldots$, $I_{i_n} \nsim I_{i_1 i_2 \cdots i_k}$. Thus, $f(I_{i_1}) \sim f(I_{i_1 i_2 \cdots i_k})$, $f(I_{i_2}) \sim f(I_{i_1 i_2 \cdots i_k})$, $\ldots$, $f(I_{i_k}) \sim f(I_{i_1 i_2 \cdots i_k})$ and $f(I_{i_{k+1}}) \nsim f(I_{i_1 i_2 \cdots i_k})$, $f(I_{i_{k+2}}) \nsim f(I_{i_1 i_2 \cdots i_k})$, $\ldots$, $f(I_{i_n}) \nsim f(I_{i_1 i_2 \cdots i_k})$. Consequently, $I_{j_1} \subset f(I_{i_1 i_2 \cdots i_k})$, $I_{j_2} \subset f(I_{i_1 i_2 \cdots i_k})$, $\ldots$, $I_{j_k} \subset f(I_{i_1 i_2 \cdots i_k})$ and $I_{j_{k+1}} \not\subset f(I_{i_1 i_2 \cdots i_k})$, $I_{j_{k+2}} \not\subset f(I_{i_1 i_2 \cdots i_k})$, $\ldots$, $I_{j_n} \not\subset f(I_{i_1 i_2 \cdots i_k})$. It follows that $f(I_{i_1 i_2 \cdots i_k}) = I_{j_1 j_2 \cdots j_k} = \phi_\sigma(I_{i_1 i_2 \cdots i_k})$. Thus, $f = \phi_\sigma$. $\square$

**Theorem 5.1.28.** *Let $S$ be the union of $n$ minimal left ideals. Then for $n \geq 2$, we have $Aut(\Gamma(S)) \cong S_n$. Moreover, $|Aut(\Gamma(S))| = n!$.*

*Proof.* In view of Lemma 5.1.26 and by Proposition 5.1.27, note that the underlying set of the automorphism group of $\Gamma(S)$ is $Aut(\Gamma(S)) = \{\phi_\sigma : \sigma \in S_n\}$, where $S_n$ is a symmetric group of degree $n$. The groups $Aut(\Gamma(S))$ and $S_n$ are isomorphic under the assignment $\phi_\sigma \mapsto \sigma$. Since all the elements in $Aut(\Gamma(S))$ are distinct, we have $|Aut(\Gamma(S))| = n!$. $\square$

## 5.2 The Inclusion Ideal Graph of a Semigroup

In this section, we discuss the inclusion ideal graph of $\mathcal{I}n(S)$. The *inclusion ideal graph* $\mathcal{I}n(S)$ of a semigroup $S$ is an undirected simple graph whose vertex set is non-trivial left ideals of $S$ and two distinct non-trivial left ideals $I$ and $J$ are adjacent if and only if $I \subset J$ or $J \subset I$. Note that $\mathcal{I}n(S)$ is a spanning subgraph of $\Gamma(S)$. We study an interplay between algebraic properties of the semigroup $S$ and graph-theoretic properties of $\mathcal{I}n(S)$. We also investigate the connectedness of $\mathcal{I}n(S)$. We show that the diameter of $\mathcal{I}n(S)$ is at most 3, if it is connected. We also obtain a necessary and sufficient condition of $S$ such that the clique number of $\mathcal{I}n(S)$ is $n$, where $n$ is the number of minimal left ideals of $S$. Further, various graph invariants of $\mathcal{I}n(S)$, viz. perfectness, planarity, girth etc., are discussed. For a completely simple semigroup $S$, we investigate various properties of $\mathcal{I}n(S)$, including its independence number and matching number. Finally, we obtain the automorphism group of $\mathcal{I}n(S)$.

### 5.2.1 Graph-theoretic Properties of $\mathcal{I}n(S)$

In this subsection, we study the algebraic properties of $S$ as well as graph-theoretic properties of the inclusion ideal graph $\mathcal{I}n(S)$. First we investigate the connectedness of $\mathcal{I}n(S)$. We show that the intersection ideal graph $\Gamma(S)$ is disconnected if and only if $\mathcal{I}n(S)$ is disconnected. We also prove that $diam(\mathcal{I}n(S)) \leq 3$ if it is connected. Moreover, the clique number, planarity, perfectness and the girth of $\mathcal{I}n(S)$ are investigated.

**Theorem 5.2.1.** *The inclusion ideal graph $\mathcal{I}n(S)$ is disconnected if and only if $S$ contains at least two minimal left ideals and every non-trivial left ideals of $S$ is minimal as well as maximal.*

*Proof.* Suppose that the graph $\mathcal{I}n(S)$ is disconnected. Without loss of generality, we may assume that there exist at least two non-trivial left ideals $I_1, I_2$ of $S$ such that $I_1 \in C_1$ and $I_2 \in C_2$, where $C_1$ and $C_2$ are two distinct components of $\mathcal{I}n(S)$. Let if possible, $I_1$ is not minimal. Then there exists a non-trivial left ideal $I_k$ of $S$ such that $I_k \subset I_1$. Now, we have the following cases.

**Case-1.** $I_k \cup I_2 \neq S$. Then we have $I_1 \sim I_k \sim (I_k \cup I_2) \sim I_2$, a contradiction.

**Case-2.** $I_k \cup I_2 = S$. Then clearly $I_1 \cup I_2 = S$. Let $x \in I_1$ but $x \notin I_k$. Thus, $x \in I_2$ so that $x \in I_1 \cap I_2$. We get $I_1 \sim (I_1 \cap I_2) \sim I_2$, again a contradiction. Thus, $I_1$ is minimal. Similarly, we obtain $I_2$ is minimal.

Further, on contrary suppose that $I_1$ is not maximal. Then there exists a non-trivial left ideal $I_k$ of $S$ such that $I_1 \subset I_k$. Now we get a contradiction in the following possible cases.

**Case-1.** $I_1 \cup I_2 \neq S$. Then $I_1 \sim (I_1 \cup I_2) \sim I_2$ gives a contradiction.

**Case-2.** $I_1 \cup I_2 = S$. Then clearly $I_k \cup I_2 = S$ so that there exists $x \in I_k \cap I_2$. Thus, we have $I_1 \sim I_k \sim (I_k \cap I_2) \sim I_2$, again a contradiction. Hence, $I_1$ is maximal. Similarly, one can observe that the left ideal $I_2$ is maximal.

The converse follows from Remark 1.1.1. □

**Corollary 5.2.2.** *If the graph $\mathcal{I}n(S)$ is disconnected, then it is a null graph (i.e. it has no edge).*

**Theorem 5.2.3.** *The graph $\mathcal{I}n(S)$ is disconnected if and only if $S$ is the union of exactly two minimal left ideals.*

*Proof.* The proof follows from the proof of Theorems 5.1.3, 5.2.1 and Lemma 1.1.2. □

**Theorem 5.2.4.** *If $\mathcal{I}n(S)$ is a connected graph, then $diam(\mathcal{I}n(S)) \leq 3$.*

*Proof.* Let $I_1, I_2$ be two non-trivial left ideals of $S$. Let $I_1 \sim I_2$. Then $d(I_1, I_2) = 1$. If $I_1 \nsim I_2$, then in the following cases we show that $d(I_1, I_2) \leq 3$.

**Case-1.** $I_1 \cup I_2 \neq S$. Then $I_1 \sim (I_1 \cup I_2) \sim I_2$ so that $d(I_1, I_2) = 2$.

**Case-2.** $I_1 \cup I_2 = S$. If $I_1 \cap I_2 \neq \emptyset$, then $I_1 \sim (I_1 \cap I_2) \sim I_2$ gives $d(I_1, I_2) = 2$. We may now suppose that $I_1 \cap I_2 = \emptyset$. Since $\mathcal{I}n(S)$ is a connected graph, there exists a non-trivial left ideal $I_k$ of $S$ such that either $I_k \subset I_1$ or $I_1 \subset I_k$. If $I_k \subset I_1$, then there exists $x \in I_1$ but $x \notin I_k$. If $I_k \cup I_2 = S$, then $x \in I_2$. Thus, we get $I_1 \cap I_2 \neq \emptyset$, a contradiction. Consequently, $I_2 \cup I_k \neq S$. Further, we get a path $I_1 \sim I_k \sim (I_2 \cup I_k) \sim I_2$ of length 3. Thus, $d(I_1, I_2) = 3$. Now if $I_1 \subset I_k$ then note that $I_k \cap I_2 \neq \emptyset$. Consequently, we get $I_1 \sim I_k \sim (I_k \cap I_2) \sim I_2$ between $I_1$ and $I_2$. Hence, $diam(\mathcal{I}n(S)) \leq 3$. □

**Lemma 5.2.5.** *Let $\mathcal{I}$ and $\mathcal{I}'$ be two distinct left ideals of $S$ such that both $\mathcal{I}$ and $\mathcal{I}'$ are the union of $k$ minimal left ideals of $S$. Then $\mathcal{I} \nsim \mathcal{I}'$ in $\mathcal{I}n(S)$.*

*Proof.* On contrary suppose that $\mathcal{I} \sim \mathcal{I}'$. Without loss of generality assume that $\mathcal{I} \subset \mathcal{I}'$. Since $\mathcal{I}$ and $\mathcal{I}'$ are the union of $k$ minimal left ideals of $S$ and $\mathcal{I} \subset \mathcal{I}'$, by Remark 1.1.1 we get $\mathcal{I} = \mathcal{I}'$, a contradiction. □

**Lemma 5.2.6.** *If $\mathcal{I}n(S)$ has a cycle of length 4 or 5, then $\mathcal{I}n(S)$ has a triangle.*

*Proof.* Suppose first that $\mathcal{I}n(S)$ has a cycle of length 5 such that $C : I_1 \sim I_2 \sim I_3 \sim I_4 \sim I_5 \sim I_1$. From the adjacency of ideals in $C$, note that there exists a chain $I_i \subset I_j \subset I_k$ in $S$. Thus $\mathcal{I}n(S)$ has a triangle.

Now we suppose that there exists a cycle $C : I_1 \sim I_2 \sim I_3 \sim I_4 \sim I_1$ of length 4 in $\mathcal{I}n(S)$. Assume that $I_1 \nsim I_3$ and $I_2 \nsim I_4$. Since $I_1 \sim I_2$ we have either $I_1 \subset I_2$ or $I_2 \subset I_1$. If $I_1 \subset I_2$, then $I_3 \subset I_2$ and $I_3 \subset I_4$. It follows that $I_1 \cup I_3 \subseteq I_2$ and $I_3 \subseteq I_2 \cap I_4$. Consequently, we obtain either $I_2 \sim (I_1 \cup I_3) \sim I_3 \sim I_2$ or $I_2 \sim (I_2 \cap I_4) \sim I_3 \sim I_2$. Thus, $\mathcal{I}n(S)$ has a triangle. If $I_2 \subset I_1$, then $I_2 \subset I_3$ and $I_4 \subset I_3$. It follows that $I_2 \subseteq I_1 \cap I_3$ and $I_2 \cup I_4 \subseteq I_3$. Further, we get a cycle either $I_2 \sim I_1 \sim (I_1 \cap I_3) \sim I_2$ or $I_2 \sim (I_2 \cup I_4) \sim I_3 \sim I_2$. Hence, we have the result. □

In the following theorem, we determine the girth of $\mathcal{I}n(S)$.

**Theorem 5.2.7.** *For a semigroup $S$, we have $g(\mathcal{I}n(S)) \in \{3, 6, \infty\}$.*

*Proof.* If $\mathcal{I}n(S)$ is disconnected or a tree, then clearly $g(\mathcal{I}n(S)) = \infty$. Suppose that $S$ has $n$ minimal left ideals. Now we prove the result through following cases.

**Case-1.** $n = 0$. If $S$ has no non-trivial left ideal, then there is nothing to prove. Otherwise there exists a chain of non-trivial left ideals of $S$ such that $I_1 \supset I_2 \supset \cdots \supset I_k \supset \cdots$. Thus, $g(\mathcal{I}n(S)) = 3$.

**Case-2.** $n = 1$. Suppose that $I_1$ is the only minimal left ideal of $S$. Since $I_1$ is unique minimal left ideal, it is contained in all other non-trivial left ideals of $S$.

If any two non minimal left ideals are adjacent, then $g(\mathcal{I}n(S)) = 3$. Otherwise, being a star graph, $g(\mathcal{I}n(S)) = \infty$.

**Case-3.** $n = 2$. Let $I_1, I_2$ be two minimal left ideals of $S$. If $I_1 \cup I_2 = S$, then by Theorem 5.2.1 and Corollary 5.2.2, $g(\mathcal{I}n(S)) = \infty$. If $I_1 \cup I_2 \neq S$, then $J = I_1 \cup I_2$ is a non-trivial left ideal of $S$. If $S$ has only these three, namely $I_1, I_2$ and $J$, left ideals, then we obtain $I_1 \sim J \sim I_2$. Therefore, $g(\mathcal{I}n(S)) = \infty$. Now suppose that $S$ has a non-trivial left ideal $K$ other than $I_1, I_2$ and $J$. Now we have the following subcases.

**Subcase-1.** Both $I_1, I_2$ contained in $K$. Then $I_1 \cup I_2 = J \subset K$. Consequently, $I_1 \sim J \sim K \sim I_1$ so that $g(\mathcal{I}n(S)) = 3$.

**Subcase-2.** Either $I_1 \subset K$ or $I_2 \subset K$. Without loss of generality, assume that $I_1 \subset K$. If there exists a non-trivial left ideal $K'$ of $S$ such that $K \subset K'$, then $I_1 \sim K \sim K' \sim I_1$ follows that $g(\mathcal{I}n(S)) = 3$. Otherwise, $K$ is a maximal left ideal of $S$. Consequently, $I_2 \cup K = S$. If $J$ is not maximal, then by **Subcase-1** we get $g(\mathcal{I}n(S)) = 3$. We may now suppose that $J$ is also a maximal left ideal of $S$. Now we claim that $V(\mathcal{I}n(S)) = \{I_1, I_2, J, K\}$. Let, if possible $J' \in V(\mathcal{I}n(S))$ but $J' \notin \{I_1, I_2, J, K\}$. Since $J$ is maximal, for any $a \in K \setminus I_1$, we have $Sa = K$. Since $J'$ is a non-trivial left ideal of $S$, there exists an element $b \in J'$ so that $b$ is either in $I_2$ or in $K$. If $b \notin I_2$, then $b \in K$. Consequently, $K \subset J'$, a contradiction to the maximality of $K$. Now suppose that $b \in I_2$. By the minimality of $I_2$, we obtain $I_2 \subset J'$. Since $I_2 \neq J'$, there exists an element $c \in J'$ but $c \notin I_2$. Consequently, $I_2 \cup K = S$. Thus, $c \in K$ so that $K \subset J'$. It follows that $I_2 \cup K = S \subset J'$ and so $J' = S$. Thus, we get $V(\mathcal{I}n(S)) = \{I_1, I_2, J, K\}$ such that $K \sim I_1 \sim J \sim I_2$. Therefore, $g(\mathcal{I}n(S)) = \infty$.

**Case-4.** $n = 3$. Let $I_1, I_2, I_3$ be the minimal left ideals of $S$. If $I_1 \cup I_2 \cup I_3 \neq S$, then we have $I_1 \subset (I_1 \cup I_2) \subset (I_1 \cup I_2 \cup I_3)$. It follows that, $g(\mathcal{I}n(S)) = 3$. Further, suppose that $I_1 \cup I_2 \cup I_3 = S$. Then all the non-trivial left ideals of $S$ are

$I_1, I_2, I_3, I_1 \cup I_2, I_1 \cup I_3$ and $I_2 \cup I_3$. Infact $\mathcal{I}n(S) \cong C_6$. Thus, $g(\mathcal{I}n(S)) = 6$.

**Case-5.** $n \geq 4$. Then for minimal left ideals $I_1, I_2$ and $I_3$, we get a triangle $I_1 \sim (I_1 \cup I_2) \sim (I_1 \cup I_2 \cup I_3) \sim I_1$ so that $g(\mathcal{I}n(S)) = 3$.

Hence, from above cases, we have $g(\mathcal{I}n(S)) \in \{3, 6, \infty\}$. □

**Theorem 5.2.8.** *Let $S$ be a semigroup with finite number of left ideals. Then the graph $\mathcal{I}n(S)$ is perfect.*

*Proof.* In view of Theorem 1.3.2, we show that $\mathcal{I}n(S)$ does not contain a hole or an antihole of odd length at least five. On contrary, assume that $\mathcal{I}n(S)$ contains a hole $C : I_1 \sim I_2 \sim I_3 \sim \cdots \sim I_{2n+1} \sim I_1$, where $n \geq 2$. Since $I_1 \sim I_2$, we have either $I_1 \subset I_2$ or $I_2 \subset I_1$. Without loss of generality, suppose that $I_1 \subset I_2$. Then clearly $I_3 \subset I_2$. Otherwise $I_1 \sim I_3$, a contradiction. Further for $2 \leq i \leq n$ note that $I_{2i-1} \subset I_{2i}$. Since $I_{2n} \sim I_{2n+1}$, we have either $I_{2n} \subset I_{2n+1}$ or $I_{2n+1} \subset I_{2n}$. But $I_{2n} \subset I_{2n+1}$ is not possible because $I_{2n-1} \subset I_{2n}$ follows that $I_{2n-1} \sim I_{2n+1}$, a contradiction. Also $I_{2n+1} \sim I_1$ will give $I_{2n+1} \subset I_1$. Consequently, from $I_1 \subset I_2$ we obtain $I_{2n+1} \subset I_2$. Thus, $I_{2n+1} \sim I_2$, a contradiction.

Next, suppose that $\mathcal{I}n(S)$ contains an antihole $C$ of length at least five. Then $\overline{C}$: $I_1 \sim I_2 \sim I_3 \sim \cdots \sim I_{2n+1} \sim I_1$, where $n \geq 2$, is a hole in $\overline{\mathcal{I}n(S)}$. Since $I_1 \sim I_3$ in $\mathcal{I}n(S)$, we have either $I_1 \subset I_3$ or $I_3 \subset I_1$. Without loss of generality, assume that $I_1 \subset I_3$. Then, for $4 \leq j \leq 2n$, note that $I_1 \subset I_j$. Moreover, $I_2 \subset I_j$ for $4 \leq j \leq 2n + 1$. Since $I_3 \sim I_5$ in $\mathcal{I}n(S)$ we have either $I_3 \subset I_5$ or $I_5 \subset I_3$. If $I_5 \subset I_3$, then $I_2 \subset I_3$ as $I_2 \subset I_5$. Thus, $I_2 \sim I_3$ in $\mathcal{I}n(S)$ which is not possible. Consequently, $I_3 \subset I_5$. Also, it is easy to observe that $I_3 \subset I_{2n+1}$. Since $I_1 \subset I_3$ we have $I_1 \subset I_{2n+1}$ so that $I_1 \sim I_{2n+1}$ in $\mathcal{I}n(S)$, a contradiction. Thus, $\mathcal{I}n(S)$ does not contain an antihole of length at least five. □

Let $S$ be a semigroup with $n$ minimal left ideals. Now we classify the semigroups $S$ for which $\omega(\mathcal{I}n(S))$ is $n$.

**Lemma 5.2.9.** *Let $S$ be a semigroup such that $S = I_{i_1 i_2 \cdots i_n}$. Then $\omega(\mathcal{I}n(S)) = n-1$.*

*Proof.* Let $S$ has $n$ minimal left ideals, namely $I_{i_1}, I_{i_2}, \ldots, I_{i_n}$. Note that $\mathcal{C} = \{I_{i_1}, I_{i_1 i_2}, \ldots, I_{i_1 i_2 \cdots i_{n-1}}\}$ be a clique of size $n-1$ in $\mathcal{I}n(S)$. Let $\mathcal{C} \cup \{J\}$ be a clique in $\mathcal{I}n(S)$, where $J = I_{i_1 i_2 \cdots i_k}$ for some $k$, $1 \leq k \leq n-1$. Then $J$ is adjacent with every element of $\mathcal{C}$, a contradiction (see Lemma 5.2.5). Consequently, $\mathcal{C}$ is a maximal clique in $\mathcal{I}n(S)$. Now if $\mathcal{C}'$ be a clique of size $n$, then there exist two non-trivial left ideals in $\mathcal{C}'$ which are union of $k$ minimal left ideals, for some $k$, where $1 \leq k \leq n-1$. By Lemma 5.2.5, a contradiction for $\mathcal{C}'$ to be a clique in $\mathcal{I}n(S)$. Hence, $\omega(\mathcal{I}n(S)) = n-1$. □

**Theorem 5.2.10.** *Let $S$ be a semigroup with $n$ minimal left ideals. Then $\omega(\mathcal{I}n(S)) = n$ if and only if $I_{i_1 i_2 \cdots i_n}$ is a maximal left ideal.*

*Proof.* Suppose that $\omega(\mathcal{I}n(S)) = n$. Clearly, by Lemma 5.2.9, we have $S \neq I_{i_1 i_2 \cdots i_n}$.Let if possible, $I_{i_1 i_2 \cdots i_n}$ is not a maximal left ideal of $S$. Then there exists a non-trivial left ideal $K$ of $S$ such that $I_{i_1 i_2 \cdots i_n} \subset K$. Note that $\mathcal{C} = \{I_{i_1}, I_{i_1 i_2}, \ldots, I_{i_1 i_2 \cdots i_{n-1}}, I_{i_1 i_2 \cdots i_n}, K\}$ forms a clique of size $n+1$. Consequently, $\omega(\mathcal{I}n(S)) \neq n$, a contradiction. Thus, $I_{i_1 i_2 \cdots i_n}$ is a maximal left ideal of $S$.

Conversely, suppose that $I_{i_1 i_2 \cdots i_n}$ is a maximal left ideal of $S$. Then by Lemma 1.1.5, $S \setminus I_{i_1 i_2 \cdots i_n}$ is an $\mathcal{L}$-class. Thus, for each $a \in S \setminus I_{i_1 i_2 \cdots i_n}$, we get either $S^1 a = S$ or $S^1 a$ is a non-trivial left ideal of $S$. First suppose that $S^1 a = S$. Therefore, for any non-trivial left ideal $I$ of $S$, note that if for some $a \in S \setminus I_{i_1 i_2 \cdots i_n}$ such that $a \in I$, then $I = S$, a contradiction. Thus, every non-trivial left ideal $I$ of $S$ is either a minimal left ideal or a union of minimal left ideals. Consequently, in the similar lines of the proof of Lemma 5.2.9, we get a clique $\mathcal{C} = \{I_{i_1}, I_{i_1 i_2}, \ldots, I_{i_1 i_2 \cdots i_{n-1}}, I_{i_1 i_2 \cdots i_n}\}$ of maximum size $n$ so that $\omega(\mathcal{I}n(S)) = n$.

We may now suppose that for each $a \in S \setminus I_{i_1 i_2 \cdots i_n}$, we have $S^1 a = I$ is a non-trivial left ideal of $S$. Assume that $J$ is any non-trivial left ideal of $S$ such that $a \in J$ for some $a \in S \setminus I_{i_1 i_2 \cdots i_n}$. Then $S \setminus I_{i_1 i_2 \cdots i_n} \subset I \subseteq J$. Consequently, all the vertices of $\mathcal{I}n(S)$ are minimal left ideals, union of minimal left ideals and of the form $(S \setminus I_{i_1 i_2 \cdots i_n}) \cup I_{j_1 j_2 \cdots j_k}$. Note that $I_{i_1} \sim (S \setminus I_{i_1 i_2 \cdots i_n}) \cup I_{j_1 j_2 \cdots j_k}$ implies $i_1 \in \{j_1, j_2, \ldots, j_k\}$ and $(S \setminus I_{i_1 i_2 \cdots i_n}) \cup I_{s_1 s_2 \cdots s_k} \sim (S \setminus I_{i_1 i_2 \cdots i_n}) \cup I_{t_1 t_2 \cdots t_p}$ implies $I_{s_1 s_2 \cdots s_k} \sim I_{t_1 t_2 \cdots t_p}$. Suppose that $K$ is of the form $(S \setminus I_{i_1 i_2 \cdots i_n}) \cup I_{j_1 j i_2 \cdots j_k}$. Let $\mathcal{C}$ be an arbitrary clique such that $K \in \mathcal{C}$. Note that $\mathcal{C} = \{I_{j_1}, I_{j_1 j_2}, \ldots, I_{j_1 j_2 \cdots j_k}, K, K \cup I_{j_{k+1}}, \ldots, K \cup I_{j_{k+1} j_{k+2} \cdots j_{n-k-1}}\}$ is a clique of size $n$. If $\mathcal{C} \cup \{K'\}$ is a clique of size $n+1$, then either $K' = I_{j_1 j_2 \cdots j_t}$ or $K' = (S \setminus I_{i_1 i_2 \cdots i_n}) \cup I_{j_1 j_2 \cdots j_s}$. Then $K'$ is not adjacent with at least one vertex of $\mathcal{C}$, a contradiction of the fact that $\mathcal{C} \cup \{K'\}$ is a clique. Consequently, $\mathcal{C}$ is a maximal clique in $\mathcal{I}n(S)$. Further, suppose that $\mathcal{C}$ do not contain any non-trivial left ideal of the form $(S \setminus I_{i_1 i_2 \cdots i_n}) \cup I_{j_1 j_2 \cdots j_k}$. Note that $\mathcal{C} = \{I_{i_1}, I_{i_1 i_2}, \ldots, I_{i_1 i_2 \cdots i_{n-1}}, I_{i_1 i_2 \cdots i_n}\}$ is a maximal clique of size $n$. Now suppose that $\mathcal{C}'$ is an arbitrary clique of size at least $n+1$. Then by the adjacency of vertices in $\mathcal{I}n(S)$ mentioned above and in Lemma 5.2.5, there exist at least two vertices $U$ and $U'$ such that $U \nsim U'$. Thus, $\omega(\mathcal{I}n(S)) = n$ and the proof is complete. □

**Theorem 5.2.11.** *For the graph $\mathcal{I}n(S)$, we have the following results:*

(i) *If $\mathcal{I}n(S)$ is a planar graph, then $|\mathrm{Min}(S)| \leq 4$.*

(ii) *Let $S$ be the union of $n$ minimal left ideals. Then $\mathcal{I}n(S)$ is a planar graph if and only if $n \leq 4$.*

*Proof.* (i) Suppose that $|\mathrm{Min}(S)| = 5$ with $\mathrm{Min}(S) = \{I_1, I_2, I_3, I_4, I_5\}$. Then, from the graph given in Figure 5.3, note that $\mathcal{I}n(S)$ contains a subdivision of complete bipartite $K_{3,3}$ as a subgraph.

For $|\mathrm{Min}(S)| \geq 6$, note that $I_1 \subset I_{12} \subset I_{123} \subset I_{1234} \subset I_{12345}$ be a chain of non-trivial left ideals of $S$. Consequently, $\mathcal{I}n(S)$ contains a subgraph isomorphic to $K_5$. Thus, by Kurwatowski theorem (cf. Theorem 1.3.8), $\mathcal{I}n(S)$ is nonplanar.

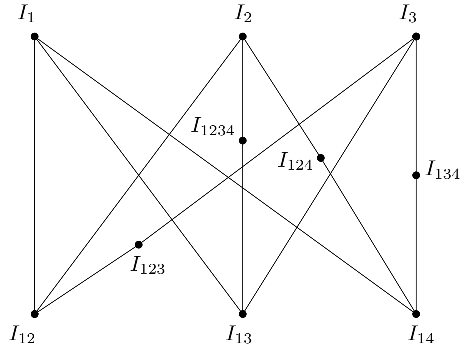


FIGURE 5.3: Subgraph of $\mathcal{I}n(S)$ homeomorphic to $K_{3,3}$

(ii) The proof for $\mathcal{I}n(S)$ is nonplanar for $n \geq 5$ follows from part (i). By Corollary 5.2.2 and Theorem 5.2.3, $\mathcal{I}n(S)$ is planar for $n = 2$. For $n = 3$, note that $\mathcal{I}n(S) \cong C_6$ so that $\mathcal{I}n(S)$ is planar. For $n = 4$, the planarity of $\mathcal{I}n(S)$ can be seen from Figure 5.4

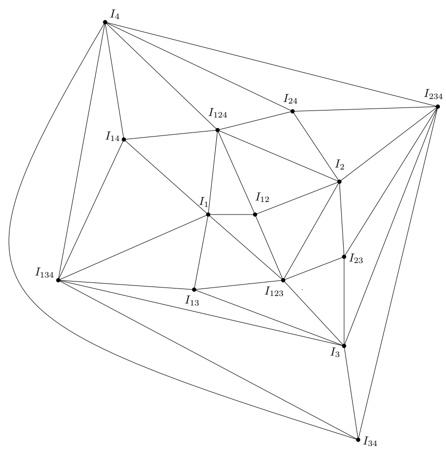


FIGURE 5.4: Planar drawing of $\mathcal{I}n(S)$

□

### 5.2.2 The Inclusion Ideal Graph of Completely Simple Semigroup

In this subsection, we study various graph invariants including the dominance number, clique number, independence number of the inclusion ideal graph of a completely simple semigroup $S$. We also prove that the graph $\mathcal{I}n(S)$ has a perfect matching (cf. Theorem 5.2.26). In what follows, for $n \in \mathbb{N}$, we denote $[n] = \{1, 2, \ldots, n\}$. Recall that, for a completely simple semigroup having $n$ minimal left ideals, we write a non-trivial left ideal $I_{i_1 i_2 \cdots i_k} = I_{i_1} \cup I_{i_2} \cup I_{i_3} \cup \cdots \cup I_{i_k}$ such that $i_1, i_2, \ldots, i_k \in [n]$ and $1 \leq k \leq n-1$, where $I_{i_1}, I_{i_2}, \ldots, I_{i_k}$ are minimal left ideals of $S$.

**Lemma 5.2.12.** *Let $S$ be a completely simple semigroup with $n$ minimal left ideals. Then $\mathcal{I}n(S)$ is disconnected for $n = 2$, and connected for $n \geq 3$. Moreover, if $\mathcal{I}n(S)$ is connected, then $diam(\mathcal{I}n(S)) = 3$.*

*Proof.* By Corollary 1.1.9 and Theorem 5.2.3, $\mathcal{I}n(S)$ is disconnected for $n = 2$. For $n \geq 3$, as a consequence of Theorem 5.2.3, $\mathcal{I}n(S)$ is connected. Let $I_1, I_{234\cdots n}$ be two non-trivial left ideals. Then there exists a shortest path $I_1 \sim I_{12} \sim I_2 \sim I_{234\cdots n}$ such that $d(I_1, I_{234\cdots n}) = 3$. By Theorem 5.2.4, $diam(\mathcal{I}n(S)) = 3$. $\square$

The following theorem can be obtained from Theorem 5.1.22

**Theorem 5.2.13.** *Let $S$ be a completely simple semigroup with $n$ minimal left ideals. Then $\mathcal{I}n(S)$ is a graph of order $2^n - 2$.*

The following lemma gives the degree of each vertex of $\mathcal{I}n(S)$.

**Lemma 5.2.14.** *Let $S$ be a completely simple semigroup with $n$ minimal left ideals and let $K = I_{i_1 i_2 \cdots i_k}$ be a non-trivial left ideal of $S$. Then* $\deg(K) = (2^k - 2) + (2^{n-k} - 2)$.

*Proof.* The proof follows from **Case-2** and **Case-3** of Lemma 5.1.23. □

**Corollary 5.2.15.** *Let $S$ be a completely simple semigroup with $n$ minimal left ideals. Then the graph $\mathcal{I}n(S)$ is Eulerian for $n \geq 3$.*

**Theorem 5.2.16.** *Let $S$ be a completely simple semigroup with $n(\geq 3)$ minimal left ideals. Then the Wiener index $W(\mathcal{I}n(S)) = 2(4^n - 3^n) + 2^n - 4$.*

*Proof.* Let $I = I_{i_1 i_2 \cdots i_k}$ be an arbitrary non-trivial left ideal of $S$, where $1 \leq k \leq n-1$. In view of Lemma 5.2.14, we get $\deg(I) = (2^k - 2) + (2^{n-k} - 2)$. Now suppose that $J$ is a non-trivial left ideal of $S$ such that $J \nsim I$. Then we have the following cases.

**Case-1.** $J = I_{j_1 j_2 \cdots j_{n-k}}$ such that $j_1, j_2, \ldots, j_{n-k} \in [n] \setminus \{i_1, i_2, \ldots, i_k\}$. Then by the proof of Lemma 5.2.12, we have $d(I, J) = 3$.

**Case-2.** $J \neq I_{j_1 j_2 \cdots j_{n-k}}$ then $I \cap J$ is a non-trivial left ideal of $S$. It follows that $d(I, J) = 2$. Hence,

$$\begin{aligned} W(I) &= 2^{n-k} + 2^k - 4 + 3 + 2(2^n - 2 - (2^{n-k} + 2^k - 4) - 1) \\ &= 2^{n+1} - 2^{n-k} - 2^k + 3. \end{aligned}$$

Therefore,

$$\begin{aligned} W(\mathcal{I}n(S)) &= (2^n - 2)(2^{n+1} - 2^{n-k} - 2^k + 3) \\ &= 2(4^n - 3^n) + 2^n - 4. \end{aligned}$$

□

**Theorem 5.2.17.** *Let $S$ be a completely simple semigroup with $n$ minimal left ideals. Then*

$$g(\mathcal{I}n(S)) = \begin{cases} \infty & \text{if } n = 2 \\ 6 & \text{if } n = 3 \\ 3 & \textit{otherwise.} \end{cases}$$

*Proof.* If $n = 2$, then by Theorem 5.2.3 and Corollary 5.2.2, the graph $\mathcal{I}n(S)$ is disconnected. It follows that $g(\mathcal{I}n(S)) = \infty$. If $n = 3$, then by Corollary 1.1.9, $\mathcal{I}n(S) \cong C_6$. Consequently, $g(\mathcal{I}n(S)) = 6$. For $n \geq 4$, we have at least $I_1, I_2, I_3, I_4$ minimal left ideals of $S$ so that we obtain a cycle $I_1 \sim (I_1 \cup I_2) \sim (I_1 \cup I_2 \cup I_3) \sim I_1$ of length 3. Thus, $g(\mathcal{I}n(S)) = 3$. $\square$

**Theorem 5.2.18.** *Let $S$ be a completely simple semigroup with $n$ minimal left ideals. Then*

(i) *$\mathcal{I}n(S)$ is a bipartite graph if and only if $n = 3$.*

(ii) *the dominance number of $\mathcal{I}n(S)$ is 2.*

(iii) *for $n \geq 4$, $\mathcal{I}n(S)$ is triangulated.*

(iv) *the clique number of $\mathcal{I}n(S)$ is $n - 1$.*

*Proof.* (i) If $n = 3$, then by Corollary 1.1.9, $\mathcal{I}n(S) \cong C_6$ which is a bipartite graph. Conversely, suppose that $\mathcal{I}n(S)$ is a bipartite graph. Let if possible, $n = 2$. Then by Corollary 1.1.9 and Theorem 5.2.3, $\mathcal{I}n(S)$ is disconnected, a contradiction for $\mathcal{I}n(S)$ to be bipartite. Suppose $n \geq 4$ and $I_1, I_2, I_3$ are the minimal left ideals of $S$. Since $I_1 \subset I_{12} \subset I_{123}$, we get a cycle $I_1 \sim I_{12} \sim I_{123} \sim I_1$ of odd length. Thus, $\mathcal{I}n(S)$ is not a bipartite graph, a contradiction.

(ii) Since there is no dominating vertex in $\mathcal{I}n(S)$, we have $\gamma(\mathcal{I}n(S)) \geq 2$. To prove the result we show that there exists a dominating set of size two in $\mathcal{I}n(S)$. We claim that the set $D = \{I_1, I_{234\cdots n}\}$ is a dominating set. Let $J = I_{i_1 i_2 \cdots i_k}$, where $i_1, i_2, \ldots, i_k \in [n]$ and $1 \leq k \leq n - 1$ be a non-trivial left ideal of $S$ such that $J \in V(\mathcal{I}n(S)) \setminus D$. If some $i_s = 1$, then $I_1 \sim J$. Otherwise, for $1 \leq k \leq n-2$, $J \sim I_{234\cdots n}$. Thus, $D$ is a dominating set of size two.

(iii) We show that any vertex of $\mathcal{I}n(S)$ is a vertex of a triangle. Let $J = I_{i_1 i_2 \cdots i_k}$, where $i_1, i_2, \ldots, i_k \in [n]$ and $1 \leq k < n$. If $k = 1$, then $J = I_{i_1} \sim I_{i_1 i_2} \sim I_{i_1 i_2 i_3} \sim J$

gives a triangle. If $k = 2$, then we have $J = I_{i_1i_2} \sim I_{i_1i_2i_3} \sim I_{i_1} \sim J$. Consequently, we get a triangle. If $k \geq 3$, then note that $J = I_{i_1i_2i_3\cdots i_k} \sim I_{i_1} \sim I_{i_1i_2} \sim J$ is a triangle. Hence, $\mathcal{I}n(S)$ is triangulated.

(iv) The result follows from Lemma 5.2.9. □

In view of Theorem 5.2.8, we have the following corollary of Theorem 5.2.18.

**Corollary 5.2.19.** *Let $S$ be a completely simple semigroup with $n$ minimal left ideals. Then $\chi(\mathcal{I}n(S)) = n - 1$.*

**Theorem 5.2.20.** *Let $S$ be a completely simple semigroup with $n$ minimal left ideals. Then the graph $\mathcal{I}n(S)$ is edge transitive if and only if $n \in \{2, 3\}$.*

*Proof.* It is well known that edge transitive graphs are either vertex transitive or bipartite. For $n \geq 4$, by Lemma 5.2.14, $\mathcal{I}n(S)$ is not a regular graph so is not vertex transitive. Also for $n \geq 4$, $g(\mathcal{I}n(S)) = 3$, hence $\mathcal{I}n(S)$ is not a bipartite graph. Thus, $\mathcal{I}n(S)$ is not an edge transitive graph. Conversely, suppose that $n \in \{2, 3\}$. If $n = 2$, then by Corollary 5.2.2 and Theorem 5.2.3, $\mathcal{I}n(S)$ is edge transitive. By Corollary 1.1.9, $\mathcal{I}n(S) \cong C_6$, for $n = 3$, which is an edge transitive graph. □

Now we determine the independence number of the graph $\mathcal{I}n(S)$.

**Remark 5.2.21.** It is well known that

(i) $\binom{n}{1} \leq \binom{n}{2} \leq \binom{n}{3} \cdots \leq \binom{n}{p} \geq \binom{n}{p+1} \geq \binom{n}{p+2} \cdots \geq \binom{n}{n-1}$, if $n = 2p$

(ii) $\binom{n}{1} \leq \binom{n}{2} \leq \binom{n}{3} \cdots \leq \binom{n}{p} = \binom{n}{p+1} \geq \binom{n}{p+2} \cdots \geq \binom{n}{n-1}$, if $n = 2p + 1$.

**Lemma 5.2.22** ([West, 1996, Theorem 3.1.11])**.** *(Hall's theorem) Let $\Gamma$ be a finite bipartite graph with bipartite sets $X$ and $Y$. For a set $X'$ of vertices in $X$, let $N_\Gamma(X')$ denote the neighbourhood of $X'$ in $\Gamma$, i.e. the set of all vertices in $Y$*

*adjacent to some elements of $X'$. There is a matching that entirely covers $X$ if and only if every subset $X'$ of $X$ : $|X'| \leq |N_\Gamma(X')|$.*

Define $T_k = \{I_{i_1 i_2 \cdots i_k} : i_1, i_2, \ldots, i_k \in [n]\}$ and $M_k$ (where $k = 1, 2, \ldots, n-2$) be the induced bipartite subgraph of $\mathcal{I}n(S)$ with vertex set $T_k$ and $T_{k+1}$.

**Lemma 5.2.23.** *Let $n = 2p$ or $n = 2p+1$. If $1 \leq k \leq p-1$, then $M_k$ has a matching that covers all the vertices of $T_k$. If $p \leq k \leq n-2$, then $M_k$ has a matching that covers all the vertices of $T_{k+1}$.*

*Proof.* First suppose $1 \leq k \leq p-1$. Then by Lemma 5.2.5, $M_k$ is a bipartite graph with vertex set $T_k$ and $T_{k+1}$. By Remark 5.2.21, we have

$$|T_k| = \tbinom{n}{k} \leq \tbinom{n}{k+1} = |T_{k+1}|.$$

By Lemma 5.2.14, observe that $M_k$ is a biregular graph in which all vertices in $T_k$ (respectively, in $T_{k+1}$) have the same vertex degree. Therefore, for any $J \in T_k$ and $J' \in T_{k+1}$, we have

$$\tbinom{n-k}{1} \tbinom{n}{k} = \deg_{M_k}(J) \cdot |T_k| = \deg_{M_k}(J') \cdot |T_{k+1}| = \tbinom{k+1}{k} \tbinom{n}{k+1}.$$

where $\deg_{M_k}(J)$ and $\deg_{M_k}(J')$ is the degree of $J$ and $J'$ in the induced subgraph $M_k$ of $\mathcal{I}n(S)$. Thus, $\deg_{M_k}(J) \geq \deg_{M_k}(J')$. Let $T$ be any arbitrary subset of $T_k$ and consider the induced subgraph of $M_k$ with vertex set $T$ and $N_{M_k}(T)$. The number of edges of this graph is $|T| \cdot \deg_{M_k}(L) \leq |N_{M_k}(T)| \cdot \deg_{M_k}(L')$, where $L$ and $L'$ are vertices of $T$ and $N_{M_k}(T)$, respectively. Thus, we have $|T| \leq |N_{M_k}(T)|$. Then by Lemma 5.2.22, $M_k$ has a matching that covers all the vertices of $T_k$.

The proof for $p \leq k \leq n-2$, is similar. Hence, omitted. $\square$

**Theorem 5.2.24.** *Let $S$ be a completely simple semigroup with $n$ minimal left ideals, where $n = 2p$ or $n = 2p+1$. Then $\alpha(\mathcal{I}n(S)) = \tbinom{n}{p}$.*

*Proof.* Let $n = 2p$ or $n = 2p + 1$. By Lemma 5.2.5, note that $T_p$ forms an independent set of $\mathcal{I}n(S)$. Consequently, $\alpha(\mathcal{I}n(S)) \geq |T_p| = \binom{n}{p}$. Let $\mathcal{U}$ be an arbitrary independent set of $\mathcal{I}n(S)$. We need to show that $|\mathcal{U}| \leq |T_p|$.

If $1 \leq k \leq p-1$, by Lemma 5.2.23, consider $Q_k$ to be a fixed matching of $M_k$ that covers all the vertices of $T_k$. Assume that $\phi_k$ is a mapping from $T_k$ to $T_{k+1}$ which sends a vertex $J \in T_k$ to a vertex $J'$ of $T_{k+1}$ such that $(J, J')$ is an edge in $Q_k$. Since $Q_k$ is a matching of $M_k$, we get $\phi_k$ is a one-one map for any $k$. Now, consider $\mathcal{U}_1$ to be $\mathcal{U}$ and recursively define $\mathcal{U}_2, \mathcal{U}_3, \ldots, \mathcal{U}_k, \ldots, \mathcal{U}_p$ for $2 \leq k \leq p$ as follows:

$$\begin{aligned}
\mathcal{U}_2 &= (\mathcal{U}_1 \setminus (\mathcal{U}_1 \cap T_1)) \cup \phi_1(\mathcal{U}_1 \cap T_1)\\
\mathcal{U}_3 &= (\mathcal{U}_2 \setminus (\mathcal{U}_2 \cap T_2)) \cup \phi_2(\mathcal{U}_2 \cap T_2)\\
&\vdots\\
\mathcal{U}_k &= (\mathcal{U}_{k-1} \setminus (\mathcal{U}_{k-1} \cap T_{k-1})) \cup \phi_{k-1}(\mathcal{U}_{k-1} \cap T_{k-1})\\
&\vdots\\
\mathcal{U}_p &= (\mathcal{U}_{p-1} \setminus (\mathcal{U}_{p-1} \cap T_{p-1})) \cup \phi_{p-1}(\mathcal{U}_{p-1} \cap T_{p-1}).
\end{aligned}$$

Observe that $\mathcal{U}_k \cap (T_1 \cup T_2 \cup \cdots \cup T_{k-1}) = \emptyset$, for any $2 \leq k \leq p$. First we show that $\mathcal{U}_k$ is an independent set and $|\mathcal{U}_k| = |\mathcal{U}|$. To do this we proceed by induction on $k$, where $k = 1, 2, \ldots, p$. Clearly, for $k = 1$, $\mathcal{U}_1$ is an independent set and $|\mathcal{U}_1| = |\mathcal{U}|$ and assume for $\mathcal{U}_{k-1}$ i.e., $\mathcal{U}_{k-1}$ is an independent set and $|\mathcal{U}_{k-1}| = |\mathcal{U}|$. Now, we prove it for $\mathcal{U}_k$. First we show that $|\mathcal{U}_k| = |\mathcal{U}|$. For this purpose we prove that $\phi_{k-1}(\mathcal{U}_{k-1} \cap T_{k-1})$ and $(\mathcal{U}_{k-1} \setminus (\mathcal{U}_{k-1} \cap T_{k-1}))$ have no common vertices. Assume that there exists $J \in (\mathcal{U}_{k-1} \setminus (\mathcal{U}_{k-1} \cap T_{k-1})) \cap \phi_{k-1}(\mathcal{U}_{k-1} \cap T_{k-1})$. Then there is $J' \in (\mathcal{U}_{k-1} \cap T_{k-1})$ such that $(J', J) \in Q_{k-1}$ and $J' \subset J$. Since $J \sim J'$ in $\mathcal{U}_{k-1}$, we get a contradiction. Hence,

$$(\mathcal{U}_{k-1} \setminus (\mathcal{U}_{k-1} \cap T_{k-1})) \cap \phi_{k-1}(\mathcal{U}_{k-1} \cap T_{k-1}) = \emptyset.$$

Thus, $|\mathcal{U}_k| = |\mathcal{U}_{k-1}| = |\mathcal{U}|$. Now if $\mathcal{U}_k$ is not an independent set then for any vertex $J \in \phi_{k-1}(\mathcal{U}_{k-1} \cap T_{k-1})$ and $J' \in (\mathcal{U}_{k-1} \setminus (\mathcal{U}_{k-1} \cap T_{k-1}))$, $J \sim J'$ and $J \subset J'$. Since $Q_k$ is matching, then there exists $J'' \in \mathcal{U}_{k-1} \cap T_{k-1}$ such that $\phi_{k-1}(J'') = J$ and $J'' \subset J$. Consequently, $J'' \sim J'$ in $\mathcal{U}_{k-1}$, a contradiction. Thus, $\mathcal{U}_k$ is an independent set.

For $p \leq k \leq n-2$. By Lemma 5.2.23, consider $Q'_k$ to be a fixed matching of $M_k$ that covers all the vertices of $T_{k+1}$. Assume that $\phi'_k$ is a mapping from $T_{k+1}$ to $T_k$ which sends a vertex $J \in T_{k+1}$ to a vertex $J'$ of $T_k$ such that $(J, J')$ is an edge in $Q'_k$. As $Q'_k$ is a matching of $M_k$, so $\phi'_k$ is one-one map for any $k$. Now, consider $\mathcal{V}_{n-1}$ to be $\mathcal{U}_p$ and analogously define $\mathcal{V}_{n-2}$, $\mathcal{V}_{n-3}$, …, $\mathcal{V}_p$ for $p \leq k \leq n-2$ as follows:

$$\begin{gathered}
\mathcal{V}_{n-2} = (\mathcal{V}_{n-1} \setminus (\mathcal{V}_{n-1} \cap T_{n-1})) \cup \phi'_{n-1}(\mathcal{V}_{n-1} \cap T_{n-1}) \\
\vdots \\
\mathcal{V}_k = (\mathcal{V}_{k-1} \setminus (\mathcal{V}_{k-1} \cap T_{k-1})) \cup \phi'_{k-1}(\mathcal{V}_{k-1} \cap T_{k-1}) \\
\vdots \\
\mathcal{V}_p = (\mathcal{V}_{p-1} \setminus (\mathcal{V}_{p-1} \cap T_{p-1})) \cup \phi'_{p-1}(\mathcal{V}_{p-1} \cap T_{p-1}).
\end{gathered}$$

Note that, $\mathcal{V}_k \cap (T_{n-1} \cup T_{n-2} \cup \cdots \cup T_{k+1}) = \emptyset$, for any $p \leq k \leq n-2$. Similarly, as shown above we can prove that $\mathcal{V}_k$ is an independent set and $|\mathcal{V}_k| = |\mathcal{V}_{n-1}| = |\mathcal{U}_p| = |\mathcal{U}|$ for $k = n-2, n-3, \ldots, p$. Since $\mathcal{V}_p \subseteq T_p$, we have $|\mathcal{V}_p| \leq |T_p|$. Consequently, we have $|\mathcal{U}| \leq |T_p| = \binom{n}{p}$. Hence, $\alpha(\mathcal{I}n(S)) = \binom{n}{p}$. □

**Corollary 5.2.25.** *Let $S$ be a completely simple semigroup with $n$ minimal left ideals, where $n = 2p$ or $n = 2p+1$. Then the vertex covering number is $(2^n - 2) - \binom{n}{p}$.*

**Theorem 5.2.26.** *Let $S$ be a completely simple semigroup with $n$ minimal left ideals. Then $\mathcal{I}n(S)$ has a perfect matching.*

*Proof.* In view of Theorem 5.2.13, it is sufficient to provide a matching of size $2^{n-1} - 1$. We have the following cases.

**Case-1.** $n = 2p + 1$. By Lemma 5.2.5, note that the set of edges

$$M = \{(I_{i_1 i_2 \cdots i_k}, I_{j_1 j_2 \cdots j_{n-k}}) : i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_{n-k} \in [n]\}$$

forms a matching and $2|M| = |V(\mathcal{I}n(S))|$. Thus, $|M| = \frac{|V(\mathcal{I}n(S))|}{2}$. Consequently, $\mathcal{I}n(S)$ has a perfect matching.

**Case-2.** $n = 2p$. Consider

$$\begin{gathered}
T_1 = \{I_{i_1} : i_1 \in [n]\} \\
T_2 = \{I_{i_1 i_2} : i_1, i_2 \in [n]\} \\
\vdots \\
T_k = \{I_{i_1 i_2 \cdots i_k} : i_1, i_2, \ldots, i_k \in [n]\} \\
\vdots \\
T_{n-1} = \{I_{i_1 i_2 \cdots i_{n-1}} : i_1, i_2, \ldots, i_{n-1} \in [n]\}
\end{gathered}$$

Note that $T_1, T_2, \ldots, T_{n-1}$ forms a partition of $V(\mathcal{I}n(S))$. Consider the following injective maps

$$\begin{gathered}
\phi_1 : T_1 \setminus \{I_1\} \to T_2 \\
\phi_2 : T_2 \setminus im(\phi_1) \to T_3 \\
\vdots \\
\phi_{n-2} : T_{n-2} \setminus im(\phi_{n-3}) \to T_{n-1} \setminus \{I_{12\cdots(n-1)}\}.
\end{gathered}$$

under the assignment $J \mapsto J'$ such that $(J, J')$ is an edge in $\mathcal{I}n(S)$. The set $M = \{(I_1, I_{12\cdots(n-1)})\} \cup \{(\alpha_1, \phi_1(\alpha_1)) : \alpha_1 \in dom(\phi_1)\} \cup \{(\alpha_2, \phi_2(\alpha_2)) : \alpha_2 \in dom(\phi_2)\} \cup \{(\alpha_3, \phi_3(\alpha_3)) : \alpha_3 \in dom(\phi_3)\} \cup \cdots \cup \{(\alpha_{n-2}, \phi_{n-2}(\alpha_{n-2})) : \alpha_{n-2} \in dom(\phi_{n-2})\}$ forms a matching and no edge in $M$ share same end vertices. In the above, by $im(\phi_i)$ and $dom(\phi_i)$, we mean the image set and domain of $\phi_i$ respectively. Further, note that $2|M| = |V(\mathcal{I}n(S))|$. Thus, $|M| = \frac{|V(\mathcal{I}n(S))|}{2}$. Consequently, $\mathcal{I}n(S)$ has a perfect matching. $\square$

**Corollary 5.2.27.** *Let $S$ be a completely simple semigroup with $n$ minimal left ideals. Then the edge covering number is $2^{n-1} - 1$.*

### 5.2.3 The Automorphism Group of $\mathcal{I}n(S)$

In order to study algebraic properties of $\mathcal{I}n(S)$, we obtain the automorphism group of $\mathcal{I}n(S)$, where $S$ is completely simple semigroup. For a completely simple semigroup $S$ having two minimal left ideals, $\mathcal{I}n(S)$ is disconnected (cf. Theorem 5.2.3). It follows that $Aut(\mathcal{I}n(S)) \cong \mathbb{Z}_2$. Now in the remaining subsection, we find the automorphism group of the inclusion ideal graph of completely simple semigroup having at least three minimal left ideals. In view of Lemma 5.2.14, we have the following remark.

**Remark 5.2.28.** In $\mathcal{I}n(S)$, we have $\deg(I_{i_1 i_2 \cdots i_k}) = \deg(I_{j_1 j_2 \cdots j_{n-k}}) = \deg(I_{j_1 j_2 \cdots j_k})$.

**Lemma 5.2.29.** *For $\sigma \in S_n$, let $\phi_\sigma : V(\mathcal{I}n(S)) \to V(\mathcal{I}n(S))$ defined by $\phi_\sigma(I_{i_1 i_2 \cdots i_k}) = I_{\sigma(i_1)\sigma(i_2)\cdots\sigma(i_k)}$. Then $\phi_\sigma \in Aut(\mathcal{I}n(S))$.*

*Proof.* Let $I_{i_1 i_2 \cdots i_t}$ and $I_{j_1 j_2 \cdots j_k}$ be arbitrary vertices of $\mathcal{I}n(S)$ such that $I_{i_1 i_2 \cdots i_t} \sim I_{j_1 j_2 \cdots j_k}$. Without loss of generality, assume that $I_{i_1 i_2 \cdots i_t} \subset I_{j_1 j_2 \cdots j_k}$. This implies that $I_{i_1}, I_{i_2}, \ldots, I_{i_t} \subset I_{j_1 j_2 \cdots j_k}$. Now

$$\begin{aligned} I_{i_1 i_2 \cdots i_t} \sim I_{j_1 j_2 \cdots j_k} &\Longleftrightarrow I_{\sigma(i_1)\sigma(i_2)\cdots\sigma(i_t)} \sim I_{\sigma(j_1)\sigma(j_2)\cdots\sigma(j_k)} \\ &\Longleftrightarrow \phi_\sigma(I_{i_1 i_2 \cdots i_t}) \sim \phi_\sigma(I_{j_1 j_2 \cdots j_k}). \end{aligned}$$

Thus, $\phi_\sigma \in Aut(\mathcal{I}n(S))$. □

**Lemma 5.2.30.** *Let $f \in Aut(\mathcal{I}n(S))$ such that $f(I_{i_s}) = I_{j_1 j_2 \cdots j_{n-1}}$ for some $i_s \in [n]$. Then $f(I_{i_1 i_2 \cdots i_k}) = I_{i'_1 i'_2 \cdots i'_{n-k}}$ for all $I_{i_1 i_2 \cdots i_k} \in V(\mathcal{I}n(S))$.*

*Proof.* Suppose $f(I_{i_s}) = I_{j_1 j_2 \cdots j_{n-1}}$. Since $I_{j_1} \sim I_{j_1 j_2 \cdots j_{n-1}}$, $I_{j_2} \sim I_{j_1 j_2 \cdots j_{n-1}}$, $\ldots$, $I_{j_{n-1}} \sim I_{j_1 j_2 \cdots j_{n-1}}$. It follows that $f(I_{j_1}) \sim f(I_{j_1 j_2 \cdots j_{n-1}})$, $f(I_{j_2}) \sim f(I_{j_1 j_2 \cdots j_{n-1}})$,

$\ldots$, $f(I_{j_{n-1}}) \sim f(I_{j_1 j_2 \cdots j_{n-1}})$. By Lemma 5.2.14 and Remark 5.2.28, we get either $f(I_{j_1}) = I_{j_\ell}$ or $f(I_{j_1}) = I_{j_1' j_2' \cdots j_{n-1}'}$. First suppose that $f(I_{j_1}) = I_{j_\ell}$. Then by Remark 5.2.28, either $f(I_{j_1 j_2 \cdots j_{n-1}}) = I_{i_p}$ or $f(I_{j_1 j_2 \cdots j_{n-1}}) = I_{\ell_1 \ell_2 \cdots \ell_{n-1}}$. For $f(I_{j_1 j_2 \cdots j_{n-1}}) = I_{i_p}$ since $I_{j_1} \sim I_{j_1 j_2 \cdots j_{n-1}}$, we get $I_{j_\ell} = f(I_{j_1}) \sim f(I_{j_1 j_2 \cdots j_{n-1}}) = I_{i_p}$, a contradiction (see Lemma 5.2.5). Consequently, $f(I_{j_1 j_2 \cdots j_{n-1}}) = I_{\ell_1 \ell_2 \cdots \ell_{n-1}}$. Now if $I_{i_s} \sim I_{j_1 j_2 \cdots j_{n-1}}$, then $I_{j_1 j_2 \cdots j_{n-1}} = f(I_{i_s}) \sim f(I_{j_1 j_2 \cdots j_{n-1}}) = I_{\ell_1 \ell_2 \cdots \ell_{n-1}}$, which is not possible. If $I_{i_s} \nsim I_{j_1 j_2 \cdots j_{n-1}}$ and $I_{i_s} \neq I_{j_\ell}$, then $I_{j_1} \nsim I_{i_s}$ implies that $f(I_{j_1}) \nsim f(I_{i_s})$. Consequently, $I_{j_\ell} \nsim I_{j_1 j_2 \cdots j_{n-1}}$, a contradiction because $j_\ell \in \{j_1, j_2, \ldots, j_{n-1}\}$. We may now suppose that $I_{j_\ell} = I_{i_s}$. Since $I_{j_2} \sim I_{j_1 j_2 \cdots j_{n-1}}$, it follows that $f(I_{j_2}) \sim f(I_{j_1 j_2 \cdots j_{n-1}})$. If $f(I_{j_2}) = I_{i_t}$, then $I_{j_2} \nsim I_{i_s}$ yields $f(I_{j_2}) \nsim f(I_{i_s})$, which is not possible as $i_t \in \{j_1, j_2, \ldots, j_{n-1}\}$. Now let $f(I_{j_2}) = I_{i_1' i_2' \cdots i_{n-1}'}$. Since $I_{j_1} \nsim I_{j_2}$, we have $f(I_{j_1}) \nsim f(I_{j_2})$. It follows that $I_{i_s} \nsim I_{i_1' i_2' \cdots i_{n-1}'}$ as $j_1, j_2, \ldots, j_{n-1} \in [n] \setminus \{i_s\}$. This implies that $j_1, j_2, \ldots, j_{n-1} \in \{i_1', i_2', \ldots, i_{n-1}'\}$. Thus $f(I_{i_s}) = f(I_{j_2})$, a contradiction. Therefore, for any $j_1 \in [n]$, $f(I_{j_1}) = I_{j_1' j_2' \cdots j_{n-1}'}$, where $j_1', j_2', \ldots, j_{n-1}' \in [n]$.

Next we show that $f(I_{i_1 i_2 \cdots i_k}) = I_{i_1' i_2' \cdots i_{n-k}'}$ for $1 < k \leq n-1$. Since $I_{i_1} \sim I_{i_1 i_2 \cdots i_k}$, $I_{i_2} \sim I_{i_1 i_2 \cdots i_k}$, $\ldots$, $I_{i_k} \sim I_{i_1 i_2 \cdots i_k}$ we get $f(I_{i_1}) \sim f(I_{i_1 i_2 \cdots i_k})$, $f(I_{i_2}) \sim f(I_{i_1 i_2 \cdots i_k})$, $\ldots$, $f(I_{i_k}) \sim f(I_{i_1 i_2 \cdots i_k})$. Then either $f(I_{i_1}) \subset f(I_{i_1 i_2 \cdots i_k})$ or $f(I_{i_1 i_2 \cdots i_k}) \subset f(I_{i_1})$. Since $f(I_{i_1}) = I_{j_1 j_2 \cdots j_{n-1}}$, for some $j_1, j_2, \ldots, j_{n-1} \in [n]$, it follows that $f(I_{i_1}) \not\subset f(I_{i_1 i_2 \cdots i_k})$. Consequently, we get $f(I_{i_1 i_2 \cdots i_k}) \subset f(I_{i_1})$. Thus, there exists $I_{\ell_1} \not\subset f(I_{i_1})$ such that $I_{\ell_1} \not\subset f(I_{i_1 i_2 \cdots i_k})$. Similarly, one can get $f(I_{i_1 i_2 \cdots i_k}) \subset f(I_{i_2})$, $f(I_{i_1 i_2 \cdots i_k}) \subset f(I_{i_3})$, $\ldots$, $f(I_{i_1 i_2 \cdots i_k}) \subset f(I_{i_k})$. Consequently, there exist $I_{\ell_2} \not\subset f(I_{i_2})$, $I_{\ell_3} \not\subset f(I_{i_3})$, $\ldots$, $I_{\ell_k} \not\subset f(I_{i_k})$ such that $I_{\ell_2}, I_{\ell_3}, \ldots, I_{\ell_k} \not\subset f(I_{i_1 i_2 \cdots i_k})$. Therefore, $I_{\ell_1 \ell_2 \cdots \ell_k} \not\subset f(I_{i_1 i_2 \cdots i_k})$. Clearly, there exist $i_{k+1}, i_{k+2}, \ldots, i_{n-k} \notin \{i_1, i_2, \ldots, i_k\}$ such that $I_{i_{k+1}}, I_{i_{k+2}}, \ldots, I_{i_{n-k}} \nsim I_{i_1 i_2 \cdots i_k}$. It follows that $f(I_{i_{k+1}}) \nsim f(I_{i_1 i_2 \cdots i_k})$, $f(I_{i_{k+2}}) \nsim f(I_{i_1 i_2 \cdots i_k})$, $\ldots$, $f(I_{i_{n-k}}) \nsim f(I_{i_1 i_2 \cdots i_k})$ so that $f(I_{i_1 i_2 \cdots i_k}) \not\subset f(I_{i_{k+1}})$, $f(I_{i_1 i_2 \cdots i_k}) \not\subset f(I_{i_{k+2}})$, $\ldots$, $f(I_{i_1 i_2 \cdots i_k}) \not\subset f(I_{i_{n-k}})$. Thus, there exists $I_{i_1'} \not\subset f(I_{i_{k+1}})$ such that $I_{i_1'} \subset f(I_{i_1 i_2 \cdots i_k})$. Similarly, there exist $I_{i_2'} \subset f(I_{i_1 i_2 \cdots i_k})$, $I_{i_3'} \subset f(I_{i_1 i_2 \cdots i_k})$, $\ldots$, $I_{i_{n-k}'} \subset$

$f(I_{i_1i_2\cdots i_k})$ such that $I_{i'_2}, I_{i'_3}, \ldots, I_{i'_{n-k}} \subset f(I_{i_1i_2\cdots i_k})$. It follows that $I_{i'_1i'_2\cdots i'_{n-k}} \subset f(I_{i_1i_2\cdots i_k})$ and $I_{\ell_1\ell_2\cdots\ell_k} \not\subset f(I_{i_1i_2\cdots i_k})$. Thus, $f(I_{i_1i_2\cdots i_k}) = I_{i'_1i'_2\cdots i'_{n-k}}$. □

**Lemma 5.2.31.** *Let $\alpha : V(\mathcal{I}n(S)) \to V(\mathcal{I}n(S))$ be a mapping defined by $\alpha(I_{i_1i_2\cdots i_k}) = I_{i'_1i'_2\cdots i'_{n-k}}$ such that $i'_1, i'_2, \ldots, i'_{n-k} \in [n] \setminus \{i_1, i_2, \ldots, i_k\}$. Then $\alpha \in Aut(\mathcal{I}n(S))$.*

*Proof.* It is straightforward to verify that $\alpha$ is one-one and onto map on $V(\mathcal{I}n(S))$. For any $I_{j_1j_2\cdots j_t}, I_{j'_1j'_2\cdots j'_s} \in V(\mathcal{I}n(S))$, suppose that $I_{j_1j_2\cdots j_t} \sim I_{j'_1j'_2\cdots j'_s}$. Without loss of generality, assume that $I_{j_1j_2\cdots j_t} \subset I_{j'_1j'_2\cdots j'_s}$. Thus, $j_1, j_2, \ldots, j_t \in \{j'_1, j'_2, \ldots, j'_s\}$. Let $\alpha(I_{j_1j_2\cdots j_t}) = I_{l_1l_2\cdots l_{n-t}}$, where $j_1, j_2, \ldots, j_t \in [n]\setminus\{l_1, l_2, \ldots, l_{n-t}\}$ and $\alpha(I_{j'_1j'_2\cdots j'_s}) = I_{l'_1l'_2\cdots l'_{n-s}}$, where $j'_1, j_2', \ldots, j'_s \in [n] \setminus \{l'_1, l'_2, \ldots, l'_{n-s}\}$. Since $l_1, l_2, \ldots, l_{n-t} \in [n] \setminus \{j_1, j_2, \ldots, j_t\}$ and $l'_1, l'_2, \ldots, l'_{n-s} \in [n]\setminus\{j'_1, j_2', \ldots, j'_s\}$, implies that $\{l'_1, l'_2, \ldots, l'_{n-s}\} \subset \{l_1, l_2, \ldots, l_{n-t}\}$. It follows that $I_{l'_1l'_2\cdots l'_{n-s}} \subset I_{l_1l_2\cdots l_{n-t}}$. Consequently, $\alpha(I_{j'_1j'_2\cdots j'_s}) \subset \alpha(I_{j_1j_2\cdots j_t})$. Thus, $I_{j_1j_2\cdots j_t} \sim I_{j'_1j'_2\cdots j'_s}$ implies that $\alpha(I_{j_1j_2\cdots j_t}) \sim \alpha(I_{j'_1j'_2\cdots j'_s})$. Now, suppose that $\alpha(I_{j_1j_2\cdots j_t}) \sim \alpha(I_{j'_1j'_2\cdots j'_s})$. Without loss of generality, assume that $\alpha(I_{j_1j_2\cdots j_t}) \subset \alpha(I_{j'_1j'_2\cdots j'_s})$. Similar to the argument discussed above, we obtain that if $\alpha(I_{j_1j_2\cdots j_t}) \subset \alpha(I_{j'_1j'_2\cdots j'_s})$, then $I_{j'_1j'_2\cdots j'_s} \subset I_{j_1j_2\cdots j_t}$. Thus, $\alpha$ is an automorphism. □

**Remark 5.2.32.** For $\phi_\sigma$ and $\alpha$, defined in Lemma 5.2.29 and 5.2.31, we have $\phi_\sigma \circ \alpha = \alpha \circ \phi_\sigma$.

**Proposition 5.2.33.** *For each $f \in Aut(\mathcal{I}n(S))$, we have either $f = \phi_\sigma$ or $f = \phi_\sigma \circ \alpha$ for some $\sigma \in S_n$.*

*Proof.* In view of Remark 5.2.28 and Lemma 5.2.30, we prove the result through the following cases.

**Case-1.** $f(I_{i_1}) = I_{j_1}$, $f(I_{i_2}) = I_{j_2}$, $\ldots$, $f(I_{i_n}) = I_{j_n}$. Consider $\sigma \in S_n$ such that $\sigma(i_1) = j_1, \sigma(i_2) = j_2, \ldots, \sigma(i_n) = j_n$. Then $\phi_\sigma(I_{i_1i_2\cdots i_k}) = I_{\sigma(i_1)\sigma(i_2)\cdots\sigma(i_k)} = I_{j_1j_2\cdots j_k}$

(cf. Lemma 5.2.29). Clearly, $I_{i_1} \sim I_{i_1 i_2 \cdots i_k}$, $I_{i_2} \sim I_{i_1 i_2 \cdots i_k}$, …, $I_{i_k} \sim I_{i_1 i_2 \cdots i_k}$. Also note that for $i_{k+1}, i_{k+2}, \ldots, i_n \in [n] \setminus \{i_1, i_2, \ldots, i_k\}$, we have $I_{i_{k+1}} \nsim I_{i_1 i_2 \cdots i_k}$, $I_{i_{k+2}} \nsim I_{i_1 i_2 \cdots i_k}$, …, $I_{i_n} \nsim I_{i_1 i_2 \cdots i_k}$. Therefore, $f(I_{i_1}) \sim f(I_{i_1 i_2 \cdots i_k})$, $f(I_{i_2}) \sim f(I_{i_1 i_2 \cdots i_k})$, …, $f(I_{i_k}) \sim f(I_{i_1 i_2 \cdots i_k})$ and $f(I_{i_{k+1}}) \nsim f(I_{i_1 i_2 \cdots i_k})$, $f(I_{i_{k+2}}) \nsim f(I_{i_1 i_2 \cdots i_k})$, …, $f(I_{i_n}) \nsim f(I_{i_1 i_2 \cdots i_k})$. Consequently, $I_{j_1} \subset f(I_{i_1 i_2 \cdots i_k})$, $I_{j_2} \subset f(I_{i_1 i_2 \cdots i_k})$, …, $I_{j_k} \subset f(I_{i_1 i_2 \cdots i_k})$ and $I_{j_{k+1}} \not\subset f(I_{i_1 i_2 \cdots i_k})$, $I_{j_{k+2}} \not\subset f(I_{i_1 i_2 \cdots i_k})$, …, $I_{j_n} \not\subset f(I_{i_1 i_2 \cdots i_k})$. It follows that $f(I_{i_1 i_2 \cdots i_k}) = I_{j_1 j_2 \cdots j_k} = \phi_\sigma(I_{i_1 i_2 \cdots i_k})$. Thus, $f = \phi_\sigma$.

**Case-2.** $f(I_{i_1}) = I_{j_1 j_2 \cdots j_{n-1}}$, $f(I_{i_2}) = I_{j'_1 j'_2 \cdots j'_{n-1}}$, …, $f(I_{i_n}) = I_{\ell_1 \ell_2 \cdots \ell_{n-1}}$. Assume that $I_{i'_1} \not\subset f(I_{i_1})$, $I_{i'_2} \not\subset f(I_{i_2})$, …, $I_{i'_n} \not\subset f(I_{i_n})$. Since $I_{i_1} \sim I_{i_1 i_2 \cdots i_k}$, $I_{i_2} \sim I_{i_1 i_2 \cdots i_k}$, …, $I_{i_k} \sim I_{i_1 i_2 \cdots i_k}$, we obtain $f(I_{i_1}) \sim f(I_{i_1 i_2 \cdots i_k})$, $f(I_{i_2}) \sim f(I_{i_1 i_2 \cdots i_k})$, …, $f(I_{i_k}) \sim f(I_{i_1 i_2 \cdots i_k})$. Consequently, $I_{i'_1} \not\subset f(I_{i_1 i_2 \cdots i_k})$, $I_{i'_2} \not\subset f(I_{i_1 i_2 \cdots i_k})$, …, $I_{i'_k} \not\subset f(I_{i_1 i_2 \cdots i_k})$. It follows that $I_{i'_1 i'_2 \cdots i'_k} \not\subset f(I_{i_1 i_2 \cdots i_k})$. For $i_{k+1}, i_{k+2}, \ldots, i_n \in [n] \setminus \{i_1, i_2, \ldots, i_k\}$, we have $I_{i_{k+1}} \nsim I_{i_1 i_2 \cdots i_k}$, $I_{i_{k+2}} \nsim I_{i_1 i_2 \cdots i_k}$, …, $I_{i_n} \nsim I_{i_1 i_2 \cdots i_k}$ so that $f(I_{i_{k+1}}) \nsim f(I_{i_1 i_2 \cdots i_k})$, $f(I_{i_{k+2}}) \nsim f(I_{i_1 i_2 \cdots i_k})$, …, $f(I_{i_n}) \nsim f(I_{i_1 i_2 \cdots i_k})$. This implies that $I_{i'_{k+1}} \subset f(I_{i_1 i_2 \cdots i_k})$, $I_{i'_{k+2}} \subset f(I_{i_1 i_2 \cdots i_k})$, …, $I_{i'_n} \subset f(I_{i_1 i_2 \cdots i_k})$. As a result, $I_{i'_{k+1} i'_{k+2} \cdots i'_n} \subset f(I_{i_1 i_2 \cdots i_k})$ and $I_{i'_1 i'_2 \cdots i'_k} \not\subset f(I_{i_1 i_2 \cdots i_k})$. Thus, $f(I_{i_1 i_2 \cdots i_k}) = I_{i'_{k+1} i'_{k+2} \cdots i'_n}$. Define $\sigma(i_1) = i'_1$, $\sigma(i_2) = i'_2$, …, $\sigma(i_n) = i'_n$. Now, $(\phi_\sigma \circ \alpha)(I_{i_1 i_2 \cdots i_k}) = \phi_\sigma(I_{i_{k+1} i_{k+2} \cdots i_n}) = I_{\sigma(i_{k+1}) \sigma(i_{k+2}) \cdots \sigma(i_n)} = I_{i'_{k+1} i'_{k+2} \cdots i'_n} = f(I_{i_1 i_2 \cdots i_k})$. Hence $f = \phi_\sigma \circ \alpha$. □

**Theorem 5.2.34.** *Let $S$ be a completely simple semigroup with $n$ minimal left ideals. Then for $n \geq 3$, we have $Aut(\mathcal{I}n(S)) \cong S_n \times \mathbb{Z}_2$. Moreover, $|Aut(\mathcal{I}n(S))| = 2(n!)$.*

*Proof.* In view of Lemmas 5.2.5, 5.2.30 and by Proposition 5.2.33, note that the underlying set of the automorphism group of $\mathcal{I}n(S)$ is $Aut(\mathcal{I}n(S)) = \{\phi_\sigma : \sigma \in S_n\} \cup \{\phi_\sigma \circ \alpha : \sigma \in S_n\}$, where $S_n$ is a symmetric group of degree $n$. The groups $Aut(\mathcal{I}n(S))$ and $S_n \times \mathbb{Z}_2$ are isomorphic under the assignment $\phi_\sigma \mapsto (\sigma, \bar{0})$ and $\phi_\sigma \circ \alpha \mapsto (\sigma, \bar{1})$. Since all the elements in $Aut(\mathcal{I}n(S))$ are distinct, we have $|Aut(\mathcal{I}n(S))| = 2(n!)$. □

**Theorem 5.2.35.** *Let $S$ be a completely simple semigroup with $n$ minimal left ideals. Then the graph $\mathcal{I}n(S)$ is vertex transitive if and only if $n \in \{2, 3\}$.*

*Proof.* Suppose that $n \geq 4$. By Remark 5.2.28, there exist at least two vertices whose degree is not equal. Thus, $\mathcal{I}n(S)$ is not a regular graph and so is not a vertex transitive graph. Conversely, suppose that $n \in \{2, 3\}$. If $n = 2$, then we have $V(\mathcal{I}n(S)) = \{I_1, I_2\}$. Then by Lemma 5.2.29, $\mathcal{I}n(S)$ is vertex transitive. If $n = 3$, then we have $V(\mathcal{I}n(S)) = \{I_1, I_2, I_3, I_{12}, I_{13}, I_{23}\}$. Let $J$ and $J'$ be two non-trivial left ideals of $S$. If both $J$ and $J'$ are minimal (or non minimal), then by Theorem 5.2.34, there exist a graph automorphism $\phi_\sigma$ such that $\phi_\sigma(J) = J'$. Now suppose that one of them is minimal. Without loss of generality, assume that $J$ is minimal and $J'$ is not a minimal left ideal of $S$. Then again by Theorem 5.2.34, there exist a graph automorphism $\phi_\sigma \circ \alpha$ for some $\sigma \in S_n$ such that $(\phi_\sigma \circ \alpha)(J) = J'$. Thus, $\mathcal{I}n(S)$ is vertex transitive. $\square$

Since every connected vertex transitive graph is a retract of Cayley graph (cf. Godsil and Royle [2001]), by Theorem 5.2.3 and 5.2.35, we have the following corollary.

**Corollary 5.2.36.** *Let $S$ be a completely simple semigroup with 3 minimal left ideals. Then the graph $\mathcal{I}n(S)$ is a retract of Cayley graph.*

# Chapter 6

# Conclusion and Future Research Work

## 6.1 Contribution of the Thesis

In this thesis, we consider certain algebraic graphs, namely the cozero-divisor graphs, upper ideal-relation graphs, left ideal-relation graphs of rings and intersection ideal graphs, inclusion ideal graphs of semigroups. In this chapter, we summarize the main findings of the research work presented in the earlier chapters along with the possible extensions and future scope. The contribution of the thesis are highlighted below.

**In Chapter 2**, we derived a closed-form formula of the Wiener index of the cozero-divisor graph of a finite commutative ring $R$. As applications, we calculated the Wiener index of $\Gamma'(R)$, when either $R$ is the product of ring of integers modulo $n$ or a reduced ring. We also provided a SageMath code to compute the Wiener index of the cozero-divisor graph of these class of rings including the ring $\mathbb{Z}_n$ of integers modulo $n$. Moreover, we investigated the Laplacian spectrum of $\Gamma'(\mathbb{Z}_n)$. We proved that the graph $\Gamma'(\mathbb{Z}_{pq})$ is Laplacian integral. Further, we obtained the Laplacian spectrum of $\Gamma'(\mathbb{Z}_n)$ for $n = p^{n_1}q^{n_2}$, where $n_1, n_2 \in \mathbb{N}$ and $p, q$ are distinct primes. In order to study the Laplacian spectral radius and algebraic connectivity of $\Gamma'(\mathbb{Z}_n)$, we characterized the values of $n$ for which the Laplacian spectral radius is equal to the order of $\Gamma'(\mathbb{Z}_n)$. In addition to this, the values of $n$ for which the algebraic connectivity and vertex connectivity of $\Gamma'(\mathbb{Z}_n)$ coincide are also described.

**In Chapter 3**, we define the upper ideal-relation $\Gamma_U(R)$ of a ring $R$. We obtained the girth, minimum degree and the independence number of $\Gamma_U(R)$. We provided a necessary and sufficient condition on $R$, in terms of the cardinality of their principal ideals, such that the graph $\Gamma_U(R)$ is bipartite, planar and outerplanar, respectively. For a non-local commutative ring $R \cong R_1 \times R_2 \times \cdots \times R_n$, where $R_i$ is a local ring with maximal ideal $\mathcal{M}_i$ and $n \geq 3$, we proved that the graph $\Gamma_U(R)$ is perfect if and only if $n \in \{3, 4\}$ and each $\mathcal{M}_i$ is a principal ideal. We also discussed all the finite rings $R$ such that the graph $\Gamma_U(R)$ is Eulerian. Moreover, we obtained the metric dimension and strong metric dimension of $\Gamma_U(R)$, when $R$ is a reduced ring. Finally, the vertex connectivity, automorphism group, Laplacian and the normalized Laplacian spectrum of $\Gamma_U(\mathbb{Z}_n)$ are determined. We characterized all the values of $n$ for which the graph $\Gamma_U(\mathbb{Z}_n)$ is Hamiltonian. Besides this, we explored the topological aspects of $\Gamma_U(R)$. In order to study topological properties of $\Gamma_U(R)$, all the non-local commutative rings $R$ for which $\Gamma_U(R)$ has genus at most 2 are classified. We precisely characterized all the non-local commutative rings for which the crosscap of $\Gamma_U(R)$ is at most 2. We obtained all the non-local commutative rings whose upper ideal-relation graphs are split graphs, threshold graphs and cographs, respectively.

**In Chapter 4**, we define and study the left ideal-relation graph $\overrightarrow{\Gamma_L}(R)$ of the full matrix ring. We obtained all the automorphisms of $\overrightarrow{\Gamma_L}(R)$, where $R$ is the ring of all $n \times n$ matrices over a finite field $\mathbb{F}_q$.

**In Chapter 5**, we have investigated the intersection ideal graph $\Gamma(S)$ and spanning subgraph of $\Gamma(S)$. Indeed, the inclusion ideal graph $\mathcal{I}n(S)$ of a semigroup $S$ is a spanning subgraph of $\Gamma(S)$. First, we established the connectedness of $\Gamma(S)$. We proved that if $\Gamma(S)$ is connected, then the diameter of $\Gamma(S)$ is at most two. Further, the semigroups $S$ in terms of their ideals are classified such that the diameter of $\Gamma(S)$ is two. We obtained the domination number, independence number, girth and the strong metric dimension of $\Gamma(S)$. We have also investigated the completeness, planarity and perfectness of $\Gamma(S)$. We show that if $S$ is a completely simple semigroup, then $\Gamma(S)$ is weakly perfect. Moreover, we obtained an upper bound of the chromatic number of $\Gamma(S)$. If $S$ is the union of $n$ minimal left ideals, then the metric dimension and the automorphism group of $\Gamma(S)$ are also determined.

We study algebraic properties of the semigroup $S$ and graph-theoretic properties of the inclusion ideal graph $\mathcal{I}n(S)$. We also investigated the connectedness of $\mathcal{I}n(S)$. We showed that the diameter of $\mathcal{I}n(S)$ is at most 3 if it is connected. We also obtained a necessary and sufficient condition of $S$ such that the clique number of $\mathcal{I}n(S)$ is $n$, where $n$ is the number of minimal left ideals of $S$. Further, various graph invariants of $\mathcal{I}n(S)$, viz. perfectness, planarity, girth etc., are discussed. For a completely simple semigroup $S$, we studied various properties of $\mathcal{I}n(S)$, including its independence number and matching number. Finally, we obtained the automorphism group of $\mathcal{I}n(S)$.

The work embedded in the thesis has its own limitations. During the investigation of graphs associated with semigroups, we noticed that the research problem "Classification of semigroups such that its associated graph satisfy certain property viz. metric dimension, chromatic number, planar etc." is not easy to handle.

Moreover, the investigation of graphs associated with semigroups becomes limited because complete classification of finite semigroups of the given cardinality is not known. We also observed that the study of upper ideal-relation graph associated with local rings is not easy to handle.

## 6.2 Scope for Future Research

We conclude this thesis with some research problems which can be addressed in the future.

- Investigation of the automorphisms of the cozero-divisor graphs of some classes of rings.

- Classification of all the commutative rings whose cozero-divisor graph has genus two.

- Classification of commuative local rings whose upper ideal-relation graph is perfect. Also, prove the necessary and sufficient condition on $R \cong R_1 \times R_2$, where $R_1, R_2$ are local rings such that the graph $\Gamma_U(R)$ is perfect.

- Characterization of local rings $R$ such that the graph $\Gamma_U(R)$ is of genus (or crosscap) at most two.

- Determine the independence number, chromatic number, automorphism group of the inclusion ideal graph $\mathcal{I}n(S)$, when $S$ is not the union of $n$ minimal left ideals.

- Classify the semigroup $S$ when the diameter of the inclusion ideal graph $\mathcal{I}n(S)$ is three.

- Investigation of the metric dimension, fixing number, determining sets and determining number of the inclusion ideal graph of a completely simple semigroup.

- Investigation of the topological aspects, viz., embedding on a orientable or non-orientable surfaces, of the intersection ideal graph of semigroup.

- Classify all the semigroups $S$ such that the $\mathcal{I}n(S)$ and $\Gamma(S)$ are equal.

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# List of research publications

The following research papers are published/accepted/communicated in international journals:

1. **Barkha Baloda** and Jitender Kumar, On the inclusion ideal graph of semigroups, *Algebra Colloquium*, **Accepted**, arXiv.2110.14194

2. Praveen Mathil, **Barkha Baloda** and Jitender Kumar, On the cozero-divisor graphs associated to rings, *AKCE International Journal of Graphs and Combinatorics*, $19(3), 238-248$, 2022.

3. **Barkha Baloda** and Jitender Kumar, On the intersection ideal graph of semigroups, *Quasigroups and Related systems* $31(1), 1-20$, 2023.

4. **Barkha Baloda**, Praveen Mathil, Jitender Kumar and Aryan Barapatre, Wiener index of the cozero-divisor graph of a finite commutative ring, **Submitted**, arXiv.2210.01570

5. Jitender Kumar, **Barkha Baloda** and Sanjeet Malhotra, Automorphisms of left ideal-relation graph over full matrix ring, **Submitted**, arXiv.2201.02345

6. **Barkha Baloda** and Jitender Kumar, Upper ideal-relation graphs associated to rings, **Submitted**.

7. **Barkha Baloda**, Praveen Mathil and Jitender Kumar, A study of upper ideal-relation graphs of rings, **Submitted**.

# Paper Presentations in Conferences:

1. *A study of upper ideal-relation graph of rings* at the International Conference on Algebra and Discrete Mathematics (ICADM-2022), organized by

Department of Mathematics, Savitribai Phule Pune University, May 26-28, 2022.

2. *Laplacian Spectrum of the Cozero-divisor Graph of Ring* at the International Conference on Graphs, Combinatorics, and Optimization (ICGCO-2022), Organized by Department of Mathematics, Birla Institute of Technology & Science Pilani, Dubai Campus, Dubai, February 06-08, 2022.

3. *On the Graphs associated with ideals of Semigroups* at the 52nd Southeastern International Conference on Combinatorics, Graph Theory and Computing (52nd SEICCGTC), organized by Department of Mathematics, Florida Atlantic University, USA, March 08-12, 2021.

4. *On the Inclusion Ideal Graph of Semigroups* at the International Conference on Algebraic Graph Theory and Application (ICAGTA 2020), organized by Department of Mathematics, Presidency University, Kolkata, India, November 21-22, 2020.

5. *On the Inclusion ideal graph of Completely Simple Semigroup* at the Second International Conference on Algebra and Discrete Mathematics (ICADM 2020), organized by Department of Mathematics, Madurai Kamaraj University, Madurai, India, June 24-26, 2020.

# Brief biography of the candidate

Barkha earned a Bachelor of Science with Honors in Mathematics from Miranda House and a Master of Science in Mathematics from Ramjas College, University of Delhi. In 2018, she began her PhD studies under Prof. Jitender Kumar's guidance in the algebraic graph theory at the Department of Mathematics, Birla Institute of Technology and Science, Pilani, Pilani Campus. She passed the National Eligibility Test (NET) and the Junior Research Fellowship (JRF) in December 2016 and June 2017, respectively. She also qualified Graduate Aptitude Test for Engineering (GATE) for Mathematics in 2018.

# Brief biography of the supervisor

Prof. Jitender Kumar is presently working as an Associate Professor in the Department of Mathematics, Birla Institute of Technology and Science, Pilani, Pilani Campus. He received his Ph.D. degree from IIT Guhawati. He has one year post doctoral research experience from Harish-Chandra Research Institute, Prayagraj (Allahabad). He has more than 7 years of teaching experience in Birla Institute of Technology and Science, Pilani, Pilani Campus. He has completed a three year research project funded by SERB-DST (India) under MATRICS scheme. He has published more than fifteen research papers in international journals of repute.