A Study on Existence of Solution for Some Non-Local Elliptic Problems via Variational Techniques

THESIS

Submitted in partial fulfilment
of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

SHILPA

Under the Supervision of

DR. GAURAV DWIVEDI

BIRLA INSTITUTE OF TECHNOLOGY & SCIENCE,
PILANI - 333031 (RAJASTHAN) INDIA

2023

BIRLA INSTITUTE OF TECHNOLOGY & SCIENCE, PILANI
PILANI CAMPUS, RAJASTHAN, INDIA

CERTIFICATE

This is to certify that the thesis entitled, **"A Study on Existence of Solution for Some Non-Local Elliptic Problems via Variational Techniques"** submitted by **Ms. Shilpa**, ID No. **2018PHXF0442P** for the award of Ph.D. degree of the institute embodies original work done by her under my supervision.

____

Signature of the Supervisor
Name : **Dr. Gaurav Dwivedi**
Designation : **Assistant Professor,**
Department of Mathematics, BITS Pilani, Pilani Campus

Date:

Acknowledgments

I am deeply indebted to a multitude of individuals who have supported me throughout this challenging yet rewarding journey of completing this thesis.

First and foremost, I would like to express my deep appreciation and sincere gratitude to my thesis supervisor Dr. Gaurav Dwivedi, for his valuable guidance and continuous encouragement throughout my Ph.D. duration. He not only gave me the freedom to explore on my own but, at the same time, helped me plan the research work with timely reviews, comments and suggestions wherever required.

I feel fortunate to have my Doctoral Advisory Committee (DAC) members as Dr. Sangita Yadav and Dr. Rajesh Kumar, whose valuable feedback and support made my work focused. I am grateful to them for always giving me time for all the official presentations. I am also thankful to all the professors of the Department of Mathematics of BITS Pilani, Pilani Campus, for their guidance and valuable suggestions in my research work. I owe a special thanks to Dr. Sweta Tiwari (IIT Guwahati) for giving me a chance to visit her department and giving valuable suggestions from time to time.

Not to forget, I could not have completed this thesis without my friends, who provided stimulating discussions and happy distractions to rest my mind outside the research. A special mention goes to Neha, Sakshi, Ankit sir, Kapil sir, Chandan, Sarita and Sajan for making my BITS life more enjoyable and memorable. I also thank my juniors Anupma, Vikram and Ravindra for their valuable interactions related to research work. Also, I would like to acknowledge BITS Pilani, Pilani campus for providing me financial assistance throughout my research. I am thankful to Science and Engineering Research Board (SERB), India for providing me travel support to attend the international conference.

I would also like to extend my sincere gratitude to my parents (Mr. Gauri Shankar and Mrs. Shankuntla) for believing in me more than myself. Their constant support and tremendous understanding kept me motivated throughout my Ph.D. and my life in general. I would also like to thank my brother Krishan and my sister Deepika for their understanding, and willingness to be a sympathetic ear during both the highs and lows of this academic pursuit have been a source of immeasurable strength.

Although it is not possible to name everyone who has played a role in shaping this thesis, I am sincerely grateful to each and every person who has contributed in any way. Your support, whether big or small, has left an indelible mark on this work.

Thank you almighty God!

Place: BITS Pilani, Pilani Campus

Date: **(Shilpa)**

Abstract

This thesis aims to study existence of solutions to non-local equations with non-standard growth conditions via variational techniques. Specifically, the thesis focuses on some well known non-local equations including the Kirchhoff type equations, Choquard type equations, and problems involving generalized fractional Laplace operators. To establish the main results, variational methods such as the Nehari manifold method, mountain pass theorem and its variants are employed.

The thesis begins by examining the existence of a positive weak solution to the Kirchhoff problem:

$$\begin{cases} -A\left(x, \int_\Omega |\nabla u|^2 dx\right) \Delta u = f(x,u,\nabla u) + \lambda h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded and smooth domain in $\mathbb{R}^N (N \geq 2)$, f, h and A are continuous functions. The salient features of (1) are that the growth of the nonlinearity f depends on u and ∇u and the perturbation term h does not assume any growth condition. In the borderline case, i.e., $N = 2$, we assume the exponential growth in the second variable of f.

Next, we study the existence and multiplicity of weak solutions for the non-local problems:

$$\begin{cases} -a\left(\int_\Omega |\nabla u|^N dx\right) \Delta_N u = \dfrac{f(x,u)}{|x|^b} + \lambda h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

and

$$-a\left(\int_{\mathbb{R}^N} |\nabla u|^N + V(x)|u|^N dx\right)(\Delta_N u + V(x)|u|^{N-2}u) = \frac{g(x,u)}{|x|^b} + \lambda h(x) \text{ in } \mathbb{R}^N \tag{3}$$

where $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain containing origin, $0 \leq b < N$, $N \geq 2$, $\Delta_N = \operatorname{div}(|\nabla u|^{N-2}\nabla u)$ is the N-Laplace operator, λ is a suitably small real parameter, and the perturbation term $h > 0$ belongs to the dual of some suitable Sobolev space. The main highlights of the equations (2) and (3) are that their nonlinear functions f and g assume the critical exponential growth at infinity, which creates the possible loss of compactness. On the other hand, (3) is considered in the entire $\mathbb{R}^N$; hence, it faces the double loss of compactness.

Next, we discuss the existence of a weak solution to the non-local problem:

$$\begin{cases} -a\left(\int_\Omega \mathscr{H}(x,|\nabla u|)dx\right) \Delta_{\mathscr{H}} u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded and smooth domain, $N \geq 2$, $\Delta_{\mathscr{H}} u = \operatorname{div}(h(x,|\nabla u|)\nabla u)$, $\mathscr{H}(x,t) =$

$\int_0^{|t|} h(x,s)s\, ds$ and $h : \Omega \times [0,\infty) \to [0,\infty)$ is a generalized N-function. The most challenging aspect of investigating (4) is that the operator $\Delta_{\mathscr{H}}$ and growth of the nonlinearity f change its behavior from one domain to another. The nonlinear function f assumes variable exponent critical growth on some part of the domain and exponential growth on the other part of the domain. We utilize concentration compactness principle for variable exponent Sobolev spaces and a version of the Moser-Trudinger inequality to overcome these issues.

In the later part of the thesis, the focus is on the study of non-local equations involving generalized fractional Laplace operator. In this order, we introduce the homogeneous fractional Musielak-Sobolev spaces and discuss the existence of solutions to the following fractional problems:

$$(-\Delta)^s_{\mathscr{H}} u(x) + V(x)h(x,x,|u|)u(x) = \left(\int_{\mathbb{R}^N} \frac{K(y)F(u(y))}{|x-y|^{\mu}} dy \right) K(x) f(u(x)) \;\text{ in }\; \mathbb{R}^N, \tag{5}$$

and

$$(-\Delta)^s_{\mathscr{H}} u(x) = \lambda g(x,|u|)u + f(x,u) \;\text{ in }\; \mathbb{R}^N, \tag{6}$$

where $N \geq 1$, $s \in (0,1), \mu \in (0,N)$, $\mathscr{H}(x,y,t) = \int_0^{|t|} h(x,y,r)r\, dr$, and $h : \mathbb{R}^N \times \mathbb{R}^N \times [0,\infty) \to [0,\infty)$ is a generalized N-function. The operator $(-\Delta)^s_{\mathscr{H}}$ is called the generalized fractional Laplace operator and is defined as:

$$(-\Delta)^s_{\mathscr{H}} u(x) = \lim_{\varepsilon \to 0} \int_{R^N \setminus B_\varepsilon(x)} h\left(x, y, \frac{|u(x)-u(y)|}{|x-y|^s} \right) \frac{u(x)-u(y)}{|x-y|^s} \frac{dy}{|x-y|^{N+s}},$$

the functions V, Kf, g are continuous and $F(t) = \int_0^t f(r)dr$.

The special feature of Schrödinger equation (5) is that its potential V vanishes at infinity, and the nonlinearity f is of Choquard type. We prove a suitable version of the Hardy-Littlewood-Sobolev inequality for Lebesgue-Musielak spaces and use it establish an existence result. The existence of a ground state solution is also discussed by the method of Nehari manifold. The main difficulty that arises while studying (6) is that the nonlinearity f assumes the critical growth at infinity, which causes the lack of compactness issue. To overcome this issue, a suitable version of the concentration compactness principle and its variant at infinity are proved for fractional Musielak-Sobolev spaces.

List of symbols

Symbol	Meaning
$\rightarrow$	Strong convergence
$\rightharpoonup$	Weak convergence
$\hookrightarrow$	Continuous embedding
$\hookrightarrow\hookrightarrow$	Compact embedding
a.e.	Almost everywhere
Ω	Subset of $\mathbb{R}^N$
$c, c_i (i = 1, 2, 3, \cdots)$	Generic positive constants which may vary from line to line
ω_{N-1}	Volume of the unit sphere S^{N-1}
$\|u\|_X$	Norm of u in the normed space X
$C(\Omega)$	The space of continuous functions on Ω
$C^k(\Omega)$	The space of k times continuously differentiable functions on Ω
$C^\infty(\Omega)$	The spaces of infinitely differentiable functions on Ω
$C_c^\infty(\Omega)$	The spaces of infinitely differentiable functions with compact support in Ω
$L^p(\Omega)$	Lebesgue space
$L^{p(x)}(\Omega)$	Variable exponent Lebesgue space
$L^{\mathscr{H}}(\Omega)$	Lebesgue Musielak space
$W^{m,p}(\Omega)$	Sobolev space with order m and exponent p

$W^{1,\mathscr{H}}(\Omega)$	Musielak Sobolev space with generalized N- function $\mathscr{H}$
$W^{s,p}(\Omega)$	Fractional Sobolev space with order s and exponent p
$W^{s,\mathscr{H}}(\Omega)$	Fractional Musielak Sobolev space with order s and generalized N-function $\mathscr{H}$
∇u	Gradient of u
Δ	Laplacian
Δ_p	p-Laplacian
$\Delta_{p(x)}$	$p(x)$-Laplacian
$\Delta_{\mathscr{H}}$	$\mathscr{H}$-Laplacian
$(-\Delta)^s$	Fractional Laplacian
$(-\Delta)^s_p$	Fractional p-Laplacian
$(-\Delta)^s_{\mathscr{H}}$	Fractional $\mathscr{H}$-Laplacian
CCP	Concentration Compactness Principle
HLS	Hardy-Littlewood-Sobolev
MPT	Mountain Pass Theorem
(AR)	Ambrosetti-Rabinowitz

Contents

Dedicated to
My Beloved Family

Chapter 1

Introduction

Nonlinear partial differential equations (PDEs) are a fundamental topic and the active area of research due to their wide range of applications in science and engineering. PDEs arise in various real-world applications, such as chemotaxis, population genetics, porous medium, fluid dynamics, chemical reactor theory and the study of standing wave solutions of certain nonlinear Schrödinger equations see, for instance, [18, 157, 135, 48] and references therein. They are hard to contemplate as there are no broad procedures that work for every single such problem, and typically every individual problem must be examined as a different issue. Nevertheless, the study of these equations has led to significant advances in our understanding of nonlinear phenomena.

A classical solution of a partial differential equation should have partial derivatives at least as many times as the order of that equation and should satisfy the equation point-wise throughout in the domain together with boundary conditions. Due to this restriction, only a small class of nonlinear equations possess a classical solution, and many problems in applications do not possess the classical solution. This motivates to develop the concept of a weak solution in such a way that we can get back the classical solution from the weaker one if this is smooth enough. All these ideas led to establishing a new setup to study the partial differential equation, called Sobolev spaces introduced by S. L. Sobolev in the mind of the 1930s.

The basic form of a second-order partial differential equation is

$$-\operatorname{div}(a(x,u,\nabla u)) + b(x,u,\nabla u) = f(u)$$

where u is an unknown function, a, b, and f are given functions, and ∇ denotes the gradient operator.

In spite of these obstacles, substantial progress has been made over the past few decades in comprehending the behavior of nonlinear PDEs. To investigate the existence, uniqueness, and regularity of solutions to these equations, researchers have developed powerful analytical tools.

1.1 Non-local problems

The non-local problem refers to a type of boundary or initial value problem where the given problem is not a point-wise identity of the dependent variable. In contrast to classical PDEs, non-local equations entail interactions over a range of spatial or temporal scales and thus can capture long-range correlations and non-local effects.

In the 19th century, mathematicians such as Joseph Fourier, Augustin-Louis Cauchy and Adrien-Marie Legendre began studying non-local problems. In early 20th century, Italian mathematician Vito Volterra introduced the concept of integral equations, which led to the first systematic study of non-local PDEs. Integral equations involve non-local terms, where the value of the solution at a point is given by an integral over the entire domain. In mid-20th century, the study of non-local PDEs gained renewed interest with the introduction of fractional calculus, which is a generalization of classical calculus that deals with derivatives and integrals of non-integer order.

Today, non-local PDEs continue to be an active area of research in mathematics, physics, and engineering. Non-local problems arise in a wide range of applications, including finance, biology, fluid dynamics, image processing and among others see, for instance, [18, 36, 37] and references therein. Next, we discuss some particular non-local problems studied in this thesis.

1.1.1 Kirchhoff type problems

An example of a non-local problem is the Kirchhoff equation. The study of Kirchhoff-type problems started with the work of Kirchhoff [87], where the author studied the equation:

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_\Omega \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

which extends the classical D'Alembert's wave equation for free vibrations of elastic strings, where ρ, ρ_0, h, E and L are constants. The Kirchhoff's model takes into account the length changes of the string produced by transverse vibrations. Kirchhoff-type equations received a lot of attention after the work of Lions [98], where the author proposed an abstract framework for the study of such types of problems. We refer to [7, 8, 9, 29, 38, 55, 60, 69, 83, 91, 95, 100, 140, 141, 145, 146, 148, 152, 154] and references therein for some existence results related to the problem

$$-A\left(x, \int_\Omega |\nabla u|^p dx\right) \Delta_p u = f(x,u) \text{ in } \Omega;\ u = 0 \text{ on } \partial\Omega, \tag{1.2}$$

with various conditions on the Kirchhoff term A and the nonlinearity f. Problem (1.2) contains the term $A\left(x, \int_\Omega |\nabla u|^p dx\right)$, consequently it is not a point-wise identity. Due to this, the Problem

(1.2) is called a non-local problem. For some existence results related to the problems of type (1.2) in $\mathbb{R}^N$, we refer to [5, 61, 74, 78, 142] and references therein.

1.1.1.1 Kirchhoff equations with gradient nonlinearity

In Chapter 3, we establish the existence of a weak solution to the following non-local problem:

$$\begin{cases} -A\left(x, \int_\Omega |\nabla u|^2 dx\right) \Delta u = f(x,u,\nabla u) + \lambda h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded and smooth domain in $\mathbb{R}^N (N \geq 2)$. The functions f, h and A are continuous functions and the growth of the nonlinearity $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is dependent on u and ∇u.

Due to the presence of ∇u in the nonlinearity f, the Problem (1.3) is not variational. Several authors have studied such types of problems through various non-variational techniques such as the method of sub-solution and super-solution [53, 107, 108] and degree theory [146] etc. de Figueiredo *et al*. [102] established an iterative technique via mountain pass theorem to develop the existence result for the problem:

$$\begin{cases} -\Delta u = f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where Ω is a bounded and smooth domain in $\mathbb{R}^N$. They assumed the following growth condition on f:

(F_1) there exist constants $c > 0$ and $p \in \left(1, \frac{N+2}{N-2}\right)$ such that

$$|f(x,t,\xi)| \leq c(1+|t|^p) \ \ \forall x \in \bar{\Omega}, \ \ t \in \mathbb{R}, \ \ \xi \in \mathbb{R}^N.$$

Several authors further utilized this technique to establish existence results for nonlinear Laplace and p-Laplace equations with dependence on the gradient; see [54, 63, 66, 103, 104, 133, 159] for further details. Recently, Wei and Tian [149] obtained the existence of solution to (1.4) when f satisfy superlinear or asymptotically linear growth condition.

Girardi and Matzeu [63] discussed the existence of a solution to (1.4) under the following growth condition on the nonlinearity f :

(F_2) there exist constants $c > 0$, $p \in (1, \frac{N+2}{N-2})$ and $r \in (0,1)$ such that

$$|f(x,t,\xi)| \leq c(1+|t|^p)(1+|\xi|^r), \ \forall \ (x,t,\xi) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N.$$

They used an iteration technique in the spirit of de Figueiredo *et al*. [102] to obtain their existence result. The problems in which the growth condition of the nonlinearity is dependent on the gradient have been studied extensively in literature by many authors. Matzeu and Servadei [107] studied a general quasilinear elliptic equations in bounded domains of $\mathbb{R}^3$. Thereafter, Servadei [133] has generalized the work of [107] for a bounded domain in $\mathbb{R}^N$. Later on, Wang et al. [147] and Ru et al. [131] derived the existence results for the fourth-order quasilinear elliptic equations. Pimenta and Servadei [124] examined the existence of a solution for the non-local fractional variational inequality.

Recently, Liu and Wang [104] used an iterative technique of de Figueiredo *et al*. [102] to obtain the existence of a solution to the following Kirchhoff problem

$$\begin{cases} -A\left(x, \int_\Omega |\nabla u|^p dx\right) \Delta_p u = f(x,u,|\nabla u|^{p-2}\nabla u) + \lambda h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded and smooth domain in $\mathbb{R}^N$, $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator and λ is a sufficiently small real parameter. They used the following growth condition on the nonlinearity f :

$$|f(x,t,|\xi|^{p-2}\xi)| \leq c(1+|t|^{q-1}), \text{ where } q \in \left(p, \frac{Np}{N-p}\right),$$

but did not assume any growth condition on h.

1.1.1.2 N-Kirchhoff equations with critical exponential growth

In Chapter 4, we establish the existence of a weak solution to the following non-local problems:

$$\begin{cases} -a\left(\int_\Omega |\nabla u|^N dx\right) \Delta_N u &= \dfrac{f(x,u)}{|x|^b} + \lambda h(x) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{cases} \tag{1.5}$$

and

$$-a\left(\int_{\mathbb{R}^N} |\nabla u|^N + V(x)|u|^N dx\right)(\Delta_N u + V(x)|u|^{N-2}u) = \frac{g(x,u)}{|x|^b} + \lambda h(x) \text{ in } \mathbb{R}^N$$

where $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain which contains the origin, $0 \leq b < N$, $N \geq 2$, $\Delta_N = \operatorname{div}(|\nabla u|^{N-2}\nabla u)$ is the N-Laplace operator, λ is a suitably small real parameter and the perturbation term $h > 0$ belongs to the dual of some suitable Sobolev space.

In [82] Corrêa and Figueiredo, considered the problem (1.2) with the following growth condition on f:

$$|f(x,t)| \leq c(1+|t|^p) \;\; \forall x \in \bar{\Omega}, \;\; t \in \mathbb{R}, \;\; \xi \in \mathbb{R}^N,$$

where $c > 0$ is a constant and $p \in \left(1, \frac{N+2}{N-2}\right)$. For the critical case $N = 2$, Figueiredo and Severo [57] obtained the existence of solution to

$$-a\left(\int_\Omega |\nabla u|^2 dx\right) \Delta u = f(x,u) \text{ in } \Omega \subseteq \mathbb{R}^2; \, u = 0 \text{ on } \partial\Omega, \tag{1.6}$$

under the assumption that f has critical exponential growth at infinity. Goyal et. al. [65] generalized the result of Figueiredo and Severo [57] for any arbitrary dimension $N \geq 2$. For some research works on problems of the type (1.2) with exponential nonlinearity in the whole of $\mathbb{R}^N$, we refer to [17, 62, 110] and references therein. Recently, Chen and Yu [35] obtained the existence and multiplicity results for the following problem with a perturbation term:

$$-a\left(\int_\Omega |\nabla u|^2 dx\right) \Delta u = f(x,u) + \varepsilon h(x) \text{ in } \Omega \subseteq \mathbb{R}^2; \, u = 0 \text{ on } \partial\Omega, \tag{1.7}$$

where the nonlinearity f has critical exponential growth and the $h \in (W_0^{1,2}(\Omega))^*$, the dual space of $W_0^{1,2}(\Omega)$.

Existence results for the Kirchhoff type problems, specifically for (1.2) with $p = N = 2$ were established in [57, 75] with the following condition on nonlinear function f instead of (AR) condition:

$H(x,t_0) \leq H(x,t_1)$, $\forall\, 0 < t_0 < t_1$, $x \in \Omega$ where $H(x,t) = tf(x,t) - 2\theta F(x,t)$ and $F(x,t) = \int_0^t f(x,s)ds$, for some $\theta > 0$.

1.1.1.3 Kirchhoff equations with double criticality

In Chapter 5, we establish the existence of a weak solution to the following non-local problem:

$$\begin{cases} -a\left(\int_\Omega \mathscr{H}(x,|\nabla u|)dx\right)\Delta_{\mathscr{H}} u &= f(x,u) \;\text{ in }\; \Omega, \\ \qquad\qquad u &= 0 \;\text{ on }\; \partial\Omega, \end{cases} \tag{1.8}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded and smooth domain, $N \geq 2$, $\Delta_{\mathscr{H}} u = \operatorname{div}(h(x,|\nabla u|)\nabla u)$, $\mathscr{H}(x,t) = \int_0^{|t|} h(x,s)s\, ds$ and $h : \Omega \times [0,\infty) \to [0,\infty)$ is a generalized N-function. The nonlinear function f has double criticality, i.e., growth of the nonlinearity f change its behaviour from one domain to another. The nonlinear function f assumes variable exponent critical growth on some part of the domain and exponential growth on the other part of the domain.

When $\mathscr{H}(x,t) = t^p$, (1.8) reduces to a Kirchhoff type problem for p-Laplace operator, several authors obtained existence results for p-Kirchhoff type equations, see [7, 8, 9, 38, 67, 60, 69, 140, 151, 154] and references therein. If $\mathscr{H}(x,t) = t^{p(x)}$, (1.8) transforms into a Kirchhoff-type problem with variable exponent and existence results are such problems are studied in the variable exponent Sobolev spaces. For some such results, one can refer to, [40, 41, 44, 45, 68, 92] and references therein. Shi and Wu [134] studied the existence result for Kirchhoff-type problems in Musielak-Sobolev spaces.

1.1.2 Problems involving generalized fractional Laplace operator

The operator $(-\Delta)^s_{\mathscr{H}}$ is called the generalized fractional Laplace operator and is defined as:

$$(-\Delta)^s_{\mathscr{H}} u(x) = \lim_{\varepsilon \to 0} \int_{R^N \setminus B_\varepsilon(x)} h\left(x, y, \frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{u(x)-u(y)}{|x-y|^s} \frac{dy}{|x-y|^{N+s}},$$

where $\mathscr{H}$ is a generalized N-function.

The operator $(-\Delta)^s_{\mathscr{H}}$ generalizes fractional p-Laplace operator, fractional (p,q)-Laplace operator, weighted fractional Laplace operator and fractional double-phase operator. More specifically, if we replace $\mathscr{H}$ by $t^p, t^p + t^q$, $a(x)t^p$ and $t^p + a(x)t^q$, then $(-\Delta)^s_{\mathscr{H}}$ reduces to the fractional p-Laplacian, fractional (p,q)-Laplacian, weighted fractional Laplace operator and fractional double-phase operator, respectively.

The existence results for problems involving $(-\Delta)^s_{\mathscr{H}}$ are examined in fractional Orlicz-Sobolev spaces when $\mathscr{H}(x,y,t)$ is independent of x,y. In this context, we quote the work of Bahrouni-Ounaies [20], Bahrouni-Ounaies-Tavares [21], Missaoui-Ounaies [109], and Silva-Carvalho-Albuquerque-Bahrouni [136]. In the case when $\mathscr{H}(x,y,t)$ depends on all x,y and t, the existence of a solution for the problems involving generalized fractional Laplace operator is studied in fractional Musielak-Sobolev spaces.

In Chapter 6, we study existence of a weak solution to the following problem:

$$(-\Delta)^s_{\mathscr{H}} u(x) + V(x) h(x,x,|u|) u(x) = \left(\int_{\mathbb{R}^N} \frac{K(y) F(u(y))}{|x-y|^\lambda} dy \right) K(x) f(u(x)) \text{ in } \mathbb{R}^N, \tag{1.9}$$

where $N \geq 1$, $s \in (0,1), \lambda \in (0,N)$ and $\mathscr{H}$ is a generalized N-function. The functions $V, K : \mathbb{R}^N \to (0,\infty)$, nonlinear function $f : \mathbb{R} \to \mathbb{R}$ are continuous and $F(t) = \int_0^t f(r)dr$.

1.1.3 Choquard type problems

Another example of the non-local problem is the Choquard equation; the prototype of this type of equation is

$$-\Delta u = (|x|^{\lambda} * F(u))f(u) \text{ in } \mathbb{R}^N.$$

One of the main tools to deal with such types of equations is Hardy-Littlewood-Sobolev [97] inequality. Choquard type of equations have been studied extensively in the literature; we refer to [113] for the physical interpretation and survey of such types of equations. For some existence results involving Choquard-type equations, we refer to the works of Moroz-Schaftingen [111, 112] (Laplace operator), Avenia-Siciliano-Squassina [46], Mukherjee-Sreenadh [116](fractional Laplace operator), Xie-Wang-Zhang [153] ((p,q)-Laplacian) and Pucci-Xiang-Zhang [125], Belchior-Bueno-Miya-gaki-Pereira [22] (fractional p-Laplacian).

Existence results for Choquard-type equations with vanishing potential have been obtained by Chen-Yuan [34], Alves-Figueiredo-Yang [11] (for Laplace operator), Albu-querque-Silva-Sousa [6](fractional coupled Choquard-type systems). Alves-Rădulescu-Tavares [13] discussed the generalized choquard problem in Orlicz-Sobolev spaces.

In Chapter 6, we study the class of generalized Choquard Schrödinger equations with vanishing potential.

1.2 Objectives of the thesis

The main objective of the thesis work is to study some non-local equations with non-standard growth conditions via variational techniques. Specifically, the objectives of the thesis are framed as follows:

- To study the Kirchhoff equation involving non-standard growth conditions such as critical growth, exponential critical growth, gradient-dependent growth and double critical growth.

- To introduce homogeneous fractional Musielak-Sobolev spaces for studying the class of generalized non-local problems with Choquard-type nonlinearity.

- To establish the concentration compactness principle and its variant at infinity for fractional Musielak-Sobolev spaces in order to investigate generalized critical growth problems.

1.3 Structure of the thesis

The thesis consists of seven chapters. Having discussed the main objectives of the thesis, this section provides a brief thematic overview of the chapter-wise road map.

Chapter 1 provides a brief overview of those non-local problems which are discussed in this thesis. A short survey of the Moser-Trudinger inequality and concentration compactness principle is also presented in this chapter. In addition, it includes the thesis objective and a brief summary of each chapter.

Chapter 2 supplies basic definitions and results utilized in subsequent chapters.

Chapter 3 presents the proof of existence of a positive weak solution to the Kirchhoff-type problem with gradient nonlinearity and a perturbation term. The proof of the main existence result uses an iterative technique based on the mountain pass theorem.

Chapter 4 studies the existence and multiplicity of weak solutions to the N-Kirchhoff equations with critical exponential growth in the bounded domain and entire $\mathbb{R}^N$. The proof of main existence results uses the mountain pass theorem, Ekeland variational principle and Moser-Trudinger inequality.

Chapter 5 proves the existence of a weak solution for the Kirchhoff-type elliptic equations with double criticality. The existence result is discussed in the Musielak Sobolev spaces.

Chapter 6 introduces the homogeneous fractional Musielak-Sobolev spaces, in order to prove the existence of a weak solution for the generalized Choquard Schrödinger equation with vanishing potential. The proof of Hardy-Littlewood-Sobolev inequality for Lebesgue Musielak spaces is also presented. In addition, this chapter proves the existence of a ground state solution by the method of Nehari manifold.

Chapter 7 focuses on the class of generalized fractional problems with critical growth in $\mathbb{R}^N$. In this order, the proof of concentration compactness principle for fractional Musielak Sobolev spaces is also presented in this chapter.

Chapter 2

Mathematical preliminaries

In this chapter, we recall some basic definitions and results which are used in the subsequent chapters.

2.1 Sobolev spaces

Definition 2.1.1. *A multi-index is an N-tuple* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$*,* $\alpha_i \geq 0$*, where* α_i*,* $1 \leq i \leq N$ *are integers. For a multi-index* α*, we use the following notations:*

$$\begin{aligned} |\alpha| &= \alpha_1 + \alpha_2 + \ldots + \alpha_N. \\ \alpha! &= \alpha_1! \ldots \alpha_N!. \\ x^\alpha &= x_1^{\alpha_1} \ldots x_N^{\alpha_N}, x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N. \end{aligned}$$

Definition 2.1.2. *Let* f *and* g *be two locally integrable functions in some open set* $\Omega \subseteq \mathbb{R}^N$*, and if* α *is a multi-index then we say* g *is the* α^{th}*-weak derivative of* f *if*

$$\int_\Omega f D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega g \varphi,$$

for all $\varphi \in C_c^\infty(\Omega)$*, that is, for all infinitely differentiable functions* φ *with compact support in* Ω*. Here* $D^\alpha \varphi$ *is defined as*

$$D^\alpha \varphi = \frac{\partial^{|\alpha|} \varphi}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}.$$

Definition 2.1.3. *[86] Let* $m > 0$ *be an integer and let* $1 \leq p \leq \infty$*. The Sobolev space is denoted by* $W^{m,p}(\Omega)$ *and defined as*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega), \text{ for all } |\alpha| \leq m\}.$$

The norm on $W^{m,p}(\Omega)$ is defined as:

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha|\leq m} \|D^{\alpha}u\|^{p}_{L^p(\Omega)}\right)^{\frac{1}{p}}.$$

We define $W^{m,p}_0(\Omega)$ space as the closure of $C^{\infty}_c(\Omega)$ in the $W^{m,p}$ norm. $W^{m,p}_0(\Omega)$ is normed space with the norm

$$||u||_{W^{m,p}_0(\Omega)} = \left(\sum_{|\alpha|=m} ||D^{\alpha}u||^{p}_{L^p(\Omega)}\right)^{\frac{1}{p}} \quad 1 \leq p < \infty.$$

Theorem 2.1.4. *[86] Let $m > 0$ be an integer and $1 \leq p \leq \infty$. Then $W^{m,p}(\Omega)$ is a Banach space. Moreover, $W^{m,p}(\Omega)$ is separable if $1 \leq p < \infty$, and is reflexive if $1 < p < \infty$.*

Theorem 2.1.5 (Sobolev Embedding Theorem [2, 86])**.** *Let $m \in \mathbb{N}$ and $p \in [1,\infty)$.*

1. *If $m < \frac{N}{p}$, then $W^{m,p}_0(\Omega) \hookrightarrow$ into $L^{p^*}(\Omega)$, for $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{m}{N}$.*
2. *If $m = \frac{n}{p}$, then $W^{m,p}_0(\Omega) \hookrightarrow$ into $L^r(\Omega)$, for $r \in [1,\infty)$.*
3. *If $m > \frac{N}{p}$, then $W^{m,p}_0(\Omega) \hookrightarrow$ into $C^{0,\gamma}(\bar{\Omega})$, for all $0 \leq \gamma < m - \frac{N}{p}$.*

Theorem 2.1.6 (Rellich-Kondrachov [2, 86])**.** *Let Ω be a bounded subset of $\mathbb{R}^N$. Let $m \in \mathbb{N}$ and $p \in [1,\infty)$.*

1. *If $m < \frac{N}{p}$, then $W^{m,p}_0(\Omega) \hookrightarrow\hookrightarrow$ into $L^q(\Omega)$, for $q \in [1, p^*)$, with $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{m}{N}$.*
2. *If $m = \frac{N}{p}$, then $W^{m,p}_0(\Omega) \hookrightarrow\hookrightarrow$ into $L^r(\Omega)$, for $r \in [1,\infty)$.*
3. *If $m > \frac{N}{p}$, then $W^{m,p}_0(\Omega) \hookrightarrow\hookrightarrow$ into $C^{0,\gamma}(\bar{\Omega})$, for all $0 \leq \gamma < m - \frac{N}{p}$.*

Theorem 2.1.7. *Let $x, y \in \mathbb{R}^N$ be any arbitrary elements then*

$$\langle |x|^{N-2}x - |y|^{N-2}y, x - y\rangle \geq C_N|x-y|^N, \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ is the usual inner product in $\mathbb{R}^N$ and $C_N > 0$.

2.2 Musielak spaces

The study of Musielak spaces started in the mid-1970s with the work of Musielak [117] and Hudzik [76, 77], where the authors provide the general framework for Musielak spaces in terms of modular function.

2.2.1 Generalized N-function

Let $\Omega \subseteq \mathbb{R}^N$ be any open set. Define,

$$\mathscr{H}(x,y,t) = \int_0^{|t|} h(x,y,s)s\, ds,$$

where $h : \Omega \times \Omega \times [0,\infty) \to [0,\infty)$.

Definition 2.2.1. $\mathscr{H} : \Omega \times \Omega \times \mathbb{R} \to [0,\infty)$ *is called a generalized N-function if it satisfies the following conditions:*

1. *$\mathscr{H}$ is continuous, even and convex function of t.*

2. *$\mathscr{H}(x,y,t) = 0$ if and only if $t = 0$, $\forall x,y \in \Omega$.*

3. *$\lim\limits_{t\to 0} \frac{\mathscr{H}(x,y,t)}{t} = 0$ and $\lim\limits_{t\to\infty} \frac{\mathscr{H}(x,y,t)}{t} = \infty$, $\forall x,y \in \Omega$.*

For any generalized N-function $\mathscr{H} : \Omega \times \Omega \times \mathbb{R} \to [0,\infty)$, we define the function $h_x : \Omega \times [0,\infty) \to [0,\infty)$ such that

$$h_x(x,t) = h(x,x,t)\ \forall (x,t) \in \Omega \times [0,\infty)$$

and

$$\mathscr{H}_x(x,t) = \int_0^{|t|} h_x(x,s)s\, ds.$$

It can be observed that $\mathscr{H}_x : \Omega \times \mathbb{R} \to [0,\infty)$ is also a generalized N-function.

We say that a generalized N-function $\mathscr{H}$, satisfy the weak Δ_2 condition if there exist $\delta_0 > 0$ and a non-negative function $k \in L^1(\Omega)$ such that

$$\mathscr{H}(x,y,2t) \le \delta_0 \mathscr{H}(x,y,t) + k(x)\ \ \forall (x,y,t) \in \Omega \times \Omega \times [0,\infty).$$

If $k = 0$, then $\mathscr{H}$ is said to satisfy Δ_2 condition. From the convexity of $\mathscr{H}$ and Δ_2 condition, we have

$$\mathscr{H}(x,y,t_1+t_2) \le \delta_1(\mathscr{H}(t_1) + \mathscr{H}(t_2)), \forall t_1,t_2 > 0 \tag{2.2}$$

where, $\delta_1 = \frac{\delta_0}{2}$.

Throughout this thesis, we assume that $\mathscr{H}$ is a locally integrable generalized N-function which satisfy Δ_2 condition.

Assume the following conditions on the functions $\mathscr{H}$ and h :

$(\mathscr{H}_1)$ $h_1 \le \frac{h(x,y,|t|)|t|^2}{\mathscr{H}(x,y,|t|)} \le h_2 < N$ for all $x,y \in \mathbb{R}^N$ and $t \ne 0$ for some $1 < h_1 < h_2 < h_1^*$, where $h_1^* = \frac{Nh_1}{N-h_1} \le h_2^* = \frac{Nh_2}{N-h_2}$.

$(\mathscr{H}_2)$ $\inf_{x,y\in\mathbb{R}^N} \mathscr{H}(x,y,1) = b_1$ and $\sup_{x,y\in\mathbb{R}^N} \mathscr{H}(x,y,1) = b_2$ for some $b_1, b_2 > 0$.

Definition 2.2.2. *We define the complementary function $\widehat{\mathscr{H}}$ corresponding to generalized N-function $\mathscr{H}$ as*

$$\widehat{\mathscr{H}}(x,y,t) = \int_0^{|t|} \widehat{h}(x,y,s)s\, ds,$$

where $\widehat{h}$ is defined as $\widehat{h}(x,y,t) = \sup\{s : h(x,y,s)s \leq t\}$ $\forall (x,y,t) \in \Omega\times\Omega\times[0,\infty)$.

It can be observed that $\widehat{\mathscr{H}}$ is also a generalized N-function and the condition $(\mathscr{H}_2)$ implies that the generalized N-function $\mathscr{H}$ and its complementary function $\widehat{\mathscr{H}}$ satisfy the Δ_2-condition.

Moreover, the function $\mathscr{H}$ and its complementary function $\widehat{\mathscr{H}}$ satisfy the following Young's inequality [101, Proposition 2.1]:

$$t_1t_2 \leq \mathscr{H}(x,y,t_1) + \widehat{\mathscr{H}}(x,y,t_2)\ \forall x,y \in \Omega, t_1,t_2 > 0.$$

Further, proceeding as [59, Lemma A2], we have

$$\widetilde{\mathscr{H}}(x,y,h(x,y,t)t) \leq \mathscr{H}(x,y,2t),\ \ \forall\,(x,y,t) \in \Omega\times\Omega\times[0,\infty). \tag{2.3}$$

Moreover, the function $\mathscr{H}$ and its complementary function $\widehat{\mathscr{H}}$ satisfy the following inequalities [59, Lemma A.2]:

$$\widehat{\mathscr{H}}(h(x,y,t)t) \leq \mathscr{H}(x,y,2t)\ \forall x,y \in \Omega, t > 0. \tag{2.4}$$

$$\widehat{\mathscr{H}}\left(\frac{\mathscr{H}(x,y,t)}{t}\right) \leq \mathscr{H}(x,y,t)\ \forall x,y \in \Omega, t > 0. \tag{2.5}$$

Next, we provide some examples of the generalized N-function:

- $\mathscr{H}(x,y,t) = t^p$, where $1 < p < \infty$ is a generalized N-function. It also satisfy the Δ_2-condition and its complementary function $\widehat{\mathscr{H}}(x,y,t) = t^q$ where, $1 < q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.
- $\mathscr{H}(x,y,t) = e^t - t - 1$ is a generalized N-function but does not satisfy the Δ_2-condition. Its complementary function $\widehat{\mathscr{H}}(x,y,t) = (1+t)\log(1+t) - t$.

2.2.2 Lebesgue Musielak Spaces

Let $\mathscr{H}_x : \Omega\times\mathbb{R} \to [0,\infty)$ be any generalized N-function.

The Lebesgue-Musielak space $L^{\mathscr{H}_x}(\Omega)$ is defined as:

$$L^{\mathscr{H}_x}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is measurable} \,\middle|\, \int_\Omega \mathscr{H}_x(x, \tau|u|)\, dx < \infty, \text{ for some } \tau > 0 \right\}.$$

$L^{\mathscr{H}_x}(\Omega)$ is a normed space [117] with the Luxemburg norm

$$\|u\|_{L^{\mathscr{H}_x}(\Omega)} = \inf\left\{\tau > 0 \,\middle|\, \int_\Omega \mathscr{H}_x\left(x, \frac{|u|}{\tau}\right) dx \leq 1\right\}.$$

Theorem 2.2.3. *[117] The space $L^{\mathscr{H}_x}(\Omega)$ is separable and reflexive Banach space.*

Theorem 2.2.4. *[156, Theorem 2.2] Let $\mathscr{H}$ be any generalized N-function. If $\mathscr{H}$ is locally integrable then $C_c^\infty(\Omega)$ is dense in $L^{\mathscr{H}_x}(\Omega)$.*

Proposition 2.2.5. *[2] Let $\mathscr{H}$ and $\widehat{\mathscr{H}}$ be complimentary N-functions. Then, for any $u \in L^{\mathscr{H}_x}(\Omega)$ and $v \in L^{\widehat{\mathscr{H}_x}}(\Omega)$, we have*

$$\left|\int_\Omega uv\, dx\right| \leq 2\|u\|_{L^{\mathscr{H}_x}(\Omega)}\|v\|_{L^{\widehat{\mathscr{H}_x}}(\Omega)}.$$

Lemma 2.2.6. *[156, Lemma B.5] Let $v \in L^{\widehat{\mathscr{H}_x}}(\Omega)$. Then*

$$G_v(u) = \int_\Omega u(x)v(x)dx \tag{2.6}$$

is a bounded linear functional on $L^{\mathscr{H}_x}(\Omega)$, i.e., $G_v \in (L^{\mathscr{H}_x}(\Omega))^*$. Also, every bounded linear functional in $L^{\mathscr{H}_x}(\Omega)$ is of the form (2.6) for some $v \in L^{\widehat{\mathscr{H}_x}}(\Omega)$. Moreover, $(L^{\mathscr{H}_x}(\Omega))^*$ is isomorphic to $L^{\widehat{\mathscr{H}_x}}(\Omega)$.

Remark 2.2.7. *By Lemma 2.2.6, the norm $\|\cdot\|_{L^{\widehat{\mathscr{H}_x}}(\Omega)}$ is equivalent to the norm $\|\cdot\|_{(L^{\mathscr{H}_x}(\Omega))^*}$, i.e.,*

$$\|v\|_{L^{\widehat{\mathscr{H}_x}}(\Omega)} \leq \|G_v\|_{(L^{\mathscr{H}_x}(\Omega))^*} = \sup_{\|u\|_{L^{\mathscr{H}_x}(\Omega)}\leq 1}\left\{\left|\int_\Omega u(x)v(x)dx\right|\right\} \leq 2\|v\|_{L^{\widehat{\mathscr{H}_x}}(\Omega)}.$$

Assume the following conditions on the functions $\mathscr{H}$ and h :

($\mathscr{H}_1$) $h_1 \leq \frac{h(x,y,|t|)|t|^2}{\mathscr{H}(x,y,|t|)} \leq h_2 < N$ for all $x, y \in \mathbb{R}^N$ and $t \neq 0$ for some $1 < h_1 < h_2 < h_1^*$, where $h_1^* = \frac{Nh_1}{N-h_1} \leq h_2^* = \frac{Nh_2}{N-h_2}$.

($\mathscr{H}_2$) $\inf\limits_{x,y\in\mathbb{R}^N} \mathscr{H}(x,y,1) = b_1$ and $\sup\limits_{x,y\in\mathbb{R}^N} \mathscr{H}(x,y,1) = b_2$ for some $b_1, b_2 > 0$.

Proposition 2.2.8. *Let $\mathscr{H}$ be any generalized N-function satisfying $(\mathscr{H}_1)-(\mathscr{H}_2)$. Assume that $u \in L^{\mathscr{H}_x}(\mathbb{R}^N)$. Then, we have*

1. $\min\left\{\rho^{h_1}, \rho^{h_2}\right\}\mathscr{H}_x(x,t) \leq \mathscr{H}_x(x,\rho t) \leq \max\left\{\rho^{h_1}, \rho^{h_2}\right\}\mathscr{H}_x(x,t)$, $\forall \rho, t > 0$,
2. $\min\left\{\|u\|^{h_1}_{L^{\mathscr{H}_x}(\mathbb{R}^N)}, \|u\|^{h_2}_{L^{\mathscr{H}_x}(\mathbb{R}^N)}\right\} \leq \int_{\mathbb{R}^N} \mathscr{H}_x(x,|u|)dx \leq \max\left\{\|u\|^{h_1}_{L^{\mathscr{H}_x}(\mathbb{R}^N)}, \|u\|^{h_2}_{L^{\mathscr{H}_x}(\mathbb{R}^N)}\right\}.$

Proof. Proof of 1 is similar to the proof of [59, Lemma 2.1]. By $(\mathscr{H}_2)$ and Proposition 2.2.8, we have

$$b_1 \min\left\{\rho^{h_1}, \rho^{h_2}\right\} \leq \mathscr{H}_x(x,\rho) \leq b_2 \max\left\{\rho^{h_1}, \rho^{h_2}\right\} \ \forall \rho > 0. \tag{2.7}$$

Hence, (2.7) and the definition of norm implies 2. □

By the above lemma, we have the following result:

Corollary 2.2.9. *Let $\mathscr{H}$ be any generalized N-function satisfying $(\mathscr{H}_1) - (\mathscr{H}_2)$. Then*

$$\min\left\{\rho^{h_1}, \rho^{h_2}\right\} \mathscr{H}(x,y,t) \leq \mathscr{H}(x,y,\rho t) \leq \max\left\{\rho^{h_1}, \rho^{h_2}\right\} \mathscr{H}(x,y,t),$$

$\forall \rho, t > 0$ *and* $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$. *Also,*

$$b_1 \min\left\{\rho^{h_1}, \rho^{h_2}\right\} \leq \mathscr{H}(x,y,\rho) \leq b_2 \max\left\{\rho^{h_1}, \rho^{h_2}\right\},$$

$\forall \rho > 0$ *and* $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.

2.2.2.1 Variable exponent Lebesgue spaces

Let $\mathscr{H} : \Omega \times \mathbb{R} \to [0,\infty)$ be any generalized N-function. If we take $\mathscr{H}(x,t) = t^{p(x)}$ then we denote $L^{\mathscr{H}}(\Omega)$ as $L^{p(x)}(\Omega)$, such spaces are called variable exponent Lebesgue spaces. To know more about these spaces, one can check [43, 52, 129].

For any $r \in C(\overline{\Omega}, (1,\infty))$, we denote $r^- = \min_{x\in\Omega} r(x)$ and $r^+ = \max_{x\in\Omega} r(x)$.

Proposition 2.2.10. *[88, Theorem 2.1] Let $r \in C(\overline{\Omega}, (1,\infty))$ and $s \in C(\overline{\Omega}, (1,\infty))$ be the conjugate exponents, i.e., $1/r(x) + 1/s(x) = 1 \ \forall x \in \Omega$. Then, for any $u \in L^{r(x)}(\Omega)$ and $v \in L^{s(x)}(\Omega)$, we have*

$$\left| \int_\Omega uv \, dx \right| \leq \left(\frac{1}{r^-} + \frac{1}{s^-} \right) \|u\|_{L^{r(x)}(\Omega)} \|v\|_{L^{s(x)}(\Omega)}.$$

Proposition 2.2.11. *[52] For any $u \in L^{p(x)}(\Omega)$, the followings are true:*

1. $\|u\|^{p^-}_{L^{p(x)}(\Omega)} \leq \rho(u) \leq \|u\|^{p^+}_{L^{p(x)}(\Omega)}$ *whenever* $\|u\|_{L^{p(x)}(\Omega)} > 1$,
2. $\|u\|^{p^+}_{L^{p(x)}(\Omega)} \leq \rho(u) \leq \|u\|^{p^-}_{L^{p(x)}(\Omega)}$ *whenever* $\|u\|_{L^{p(x)}(\Omega)} < 1$,
3. $\|u\|_{L^{p(x)}(\Omega)} < 1 (= 1; > 1)$ *iff* $\rho(u) < 1 (= 1; > 1)$,

where $\rho(u) = \int_\Omega |u|^{p(x)} dx$.

Proposition 2.2.12. *[88] Let $r, s \in C(\overline{\Omega}, (1,\infty))$ such that $1 < r(x)s(x) < \infty$. Then, for any $u \in L^{s(x)}(\Omega)$, the followings are true:*

1. $\|u\|^{r^-}_{L^{r(x)s(x)}(\Omega)} \leq \||u|^{r(x)}\|_{L^{s(x)}(\Omega)} \leq \|u\|^{r^+}_{L^{r(x)s(x)}(\Omega)}$ *whenever* $\|u\|_{L^{r(x)s(x)}(\Omega)} \geq 1$,

2. $\|u\|^{r^+}_{L^{r(x)s(x)}(\Omega)} \leq \||u|^{r(x)}\|_{L^{s(x)}(\Omega)} \leq \|u\|^{r^-}_{L^{r(x)s(x)}(\Omega)}$ *whenever* $\|u\|_{L^{r(x)s(x)}(\Omega)} \leq 1$.

2.2.3 Musielak-Sobolev spaces

To define the Musielak-Sobolev space, we consider the generalized N-function $\mathscr{H} : \Omega \times \mathbb{R} \to [0,\infty)$.

Definition 2.2.13. *The Musielak-Sobolev space $W^{1,\mathscr{H}}(\Omega)$ is defined as*

$$W^{1,\mathscr{H}}(\Omega) = \{u \in L^{\mathscr{H}}(\Omega) \,\Big|\, |\nabla u| \in L^{\mathscr{H}}(\Omega)\}.$$

$W^{1,\mathscr{H}}(\Omega)$ is a Banach space with the norm [117, Theorem 10.2]

$$\|u\|_{1,\mathscr{H}} = \|u\|_{L^{\mathscr{H}}(\Omega)} + \|\nabla u\|_{L^{\mathscr{H}}(\Omega)}.$$

The space $W_0^{1,\mathscr{H}}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in $W^{1,\mathscr{H}}(\Omega)$. Moreover, the space $W_0^{1,\mathscr{H}}(\Omega)$ is equipped with the norm $\|\nabla u\|_{L^{\mathscr{H}}(\Omega)}$, which is equivalent to the norm $\|\cdot\|_{1,\mathscr{H}}$ [64, Lemma 5.7].

Theorem 2.2.14. *[117] The spaces $L^{\mathscr{H}}(\Omega)$ and $W_0^{1,\mathscr{H}}(\Omega)$ are reflexive and separable Banach spaces.*

In particular, if we take $\mathscr{H}(x,t) = t^{p(x)}$ then we denote $W^{1,\mathscr{H}}(\Omega)$ as $W^{1,p(x)}(\Omega)$ which is known as variable exponent Sobolev space. To know more about these spaces, one can check [43, 52, 129]. If we take $\mathscr{H}(x,t) = t^p,\ 1 < p < \infty$ then $W^{1,\mathscr{H}}(\Omega)$ becomes the well known Sobolev space $W^{1,p}(\Omega)$. Hence, the Musielak-Sobolev spaces are the generalization of Sobolev spaces.

Define the function $m : W_0^{1,\mathscr{H}}(\Omega) \to \mathbb{R}$ as

$$m(u) = \int_\Omega \mathscr{H}(x, |\nabla u|)dx.$$

Proposition 2.2.15. *[12] Let $\mathscr{H}$ be any generalized N-function which satisfy $(\mathscr{H}_1)$. For any $u \in W_0^{1,\mathscr{H}}(\Omega)$, the followings are true:*

1. $\|u\|^{h_1}_{1,\mathscr{H}} \leq m(u) \leq \|u\|^{h_2}_{1,\mathscr{H}}$ *whenever* $\|u\|_{1,\mathscr{H}} \geq 1$.

2. $\|u\|^{h_2}_{1,\mathscr{H}} \leq m(u) \leq \|u\|^{h_1}_{1,\mathscr{H}}$ *whenever* $\|u\|_{1,\mathscr{H}} \leq 1$.

In particular, $m(u) = 1$ iff $\|u\|_{1,\mathscr{H}} = 1$. Moreover, if $\{u_n\} \subset W_0^{1,\mathscr{H}}(\Omega)$, then $\|u\|_{1,\mathscr{H}} \to 0$ iff $m(u_n) \to 0$.

2.2.4 Fractional Musielak-Sobolev spaces

For a given generalized N-function $\mathscr{H}$ and $s \in (0,1)$, fractional Musielak-Sobolev space is denoted by $W^{s,\mathscr{H}}(\Omega)$ and is defined as

$$W^{s,\mathscr{H}}(\Omega) = \left\{ u \in L^{\mathscr{H}_x}(\Omega) : \int_\Omega \int_\Omega \mathscr{H}\left(x, y, \frac{\tau |u(x) - u(y)|}{|x-y|^s}\right) \frac{dx\, dy}{|x-y|^N} < \infty, \text{ for some } \tau > 0 \right\}.$$

$W^{s,\mathscr{H}}(\Omega)$ is a normed space with the norm

$$\|u\|_{s,\mathscr{H}} = \|u\|_{L^{\mathscr{H}_x}(\Omega)} + [u]_{s,\mathscr{H}},$$

where

$$[u]_{s,\mathscr{H}} = \inf\left\{ \tau > 0 \,\middle|\, \int_\Omega \int_\Omega \mathscr{H}\left(x, y, \frac{|u(x) - u(y)|}{\tau |x-y|^s}\right) \frac{dx\, dy}{|x-y|^N} \leq 1 \right\}.$$

We define the Lebesgue-Musielak space $L^{\mathscr{H}}(\Omega \times \Omega; d\mu)$ as:

$$L^{\mathscr{H}}(\Omega \times \Omega; d\mu) = \left\{ u : \Omega \times \Omega \to \mathbb{R} \text{ is measurable} \,\middle|\, \int_\Omega \int_\Omega \mathscr{H}(x, y, \tau |u(x,y)|)\, d\mu < \infty, \text{ for some } \tau > 0 \right\},$$

where $d\mu = \dfrac{dxdy}{|x-y|^N}$ is a measure on the set $\Omega \times \Omega$.

Remark 2.2.16. *$[u]_{s,\mathscr{H}}$ is finite if and only if $\dfrac{(u(x) - u(y))}{|x-y|^s} \in L^{\mathscr{H}}(\Omega \times \Omega; d\mu)$ and $[u]_{s,\mathscr{H}} = \left\| \dfrac{u(x) - u(y)}{|x-y|^s} \right\|_{L^{\mathscr{H}}(\Omega \times \Omega; d\mu)}$.*

Theorem 2.2.17. *[19] $W^{s,\mathscr{H}}(\Omega)$ is a separable and reflexive Banach space.*

The space $W_0^{s,\mathscr{H}}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in $W^{s,\mathscr{H}}(\Omega)$. Moreover, the space $W_0^{s,\mathscr{H}}(\Omega)$ is equipped with the norm $[u]_{s,\mathscr{H}}$.

Next, we state the generalized Poincaré's inequality:

Theorem 2.2.18. *[19] Let Ω be a bounded open subset of $\mathbb{R}^N$ and $0 < s < 1$. Then there exists a positive constant $c > 0$ such that*

$$\|u\|_{L^{\mathscr{H}_x}(\Omega)} \leq c[u]_{s,\mathscr{H}}, \ \ \forall\, u \in W_0^{s,\mathscr{H}}(\Omega).$$

This implies that, $[\cdot]_{s,\mathscr{H}}$ is the norm on $W_0^{s,\mathscr{H}}(\Omega)$, which is equivalent to the norm $\|\cdot\|_{s,\mathscr{H}}$.

Next, we define the Sobolev conjugate function corresponding to a generalized N-function $\mathscr{H}$, for this, we need the following condition:

$(\mathscr{H}_3)$ $\displaystyle\int_a^\infty \left(\frac{t}{\mathscr{H}_x(x,t)}\right)^{\frac{s}{N-s}} dt = \infty$ and $\displaystyle\int_0^b \left(\frac{t}{\mathscr{H}_x(x,t)}\right)^{\frac{s}{N-s}} dt < \infty$, for some $a,b > 0$.

For a given generalized N-function $\mathscr{H} : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to [0,\infty)$, we define the Sobolev conjugate function $\mathscr{H}^* : \mathbb{R}^N \times \mathbb{R} \to [0,\infty)$ as:

$$\mathscr{H}^*(x,t) = \mathscr{H}_x(x, L^{-1}(t)), \forall\, t \geq 0, \tag{2.8}$$

where

$$L(x,t) = \left(\int_0^t \left(\frac{r}{\mathscr{H}_x(x,r)}\right)^{\frac{s}{N-S}} dr\right)^{\frac{N-s}{N}}, \forall\, t \geq 0.$$

One can verify that $\mathscr{H}^*$ is a generalized N-function.

There exists a function $h^* : \mathbb{R}^N \times [0,\infty) \to [0,\infty)$ [109, Lemma 4.5] such that

$$h_1^* \leq \frac{h^*(x,|t|)|t|^2}{\mathscr{H}^*(x,|t|)} \leq h_2^*,\ \forall x,y \in \mathbb{R}^N \text{ and } t \neq 0 \text{ where } \mathscr{H}^*(x,t) = \int_0^{|t|} h^*(x,r)r\, dr. \tag{2.9}$$

Theorem 2.2.19. *[4, Theorem 6.1] Let $s \in (0,1)$ and $\mathscr{H}$ be any generalized N-function satisfying $(\mathscr{H}_3)$. Then the embedding $W^{s,\mathscr{H}}(\mathbb{R}^N) \hookrightarrow L^{\mathscr{H}^*}(\mathbb{R}^N)$ is continuous. Moreover, in this embedding the space $L^{\mathscr{H}^*}(\mathbb{R}^N)$ is optimal among all the Musielak spaces.*

Proposition 2.2.20. *[109, Lemma 4.3] Let $\mathscr{H}$ be any generalized N-function satisfying $(\mathscr{H}_1)$. Assume that $u \in \mathscr{H}^*(\mathbb{R}^N)$ and $\rho, t \geq 0$. Then, we have*

1. $\min\left\{\rho^{h_1^*}, \rho^{h_2^*}\right\} \mathscr{H}^*(x,t) \leq \mathscr{H}^*(x,\rho t) \leq \max\left\{\rho^{h_1^*}, \rho^{h_2^*}\right\} \mathscr{H}^*(x,t)$,
2. $\min\left\{\|u\|^{h_1^*}_{L^{\mathscr{H}^*}(\mathbb{R}^N)}, \|u\|^{h_2^*}_{L^{\mathscr{H}^*}(\mathbb{R}^N)}\right\} \leq \int_{\mathbb{R}^N} \mathscr{H}^*(x,|u|)dx \leq \max\left\{\|u\|^{h_1^*}_{L^{\mathscr{H}^*}(\mathbb{R}^N)}, \|u\|^{h_2^*}_{L^{\mathscr{H}^*}(\mathbb{R}^N)}\right\}$,

where, $h_1^* = \frac{Nh_1}{N-sh_1}$ and $h_2^* = \frac{Nh_2}{N-sh_2}$.

Proposition 2.2.21. *[59, Lemma 2.6] Let $\mathscr{H}$ be any generalized N-function which satisfy $(\mathscr{H}_2)$. Suppose $\widehat{\mathscr{H}}$ be complimentary function of $\mathscr{H}$. Assume that $u \in \widehat{\mathscr{H}^*}(\mathbb{R}^N)$ and $\rho, t \geq 0$. Then, we have*

1. $\min\left\{\rho^{h_3^*}, \rho^{h_4^*}\right\} \widehat{\mathscr{H}^*}(x,t) \leq \widehat{\mathscr{H}^*}(x,\rho t) \leq \max\left\{\rho^{h_3^*}, \rho^{h_4^*}\right\} \widehat{\mathscr{H}^*}(x,t)$,
2. $\min\left\{\|u\|^{h_3^*}_{L^{\widehat{\mathscr{H}^*}}(\mathbb{R}^N)}, \|u\|^{h_4^*}_{L^{\widehat{\mathscr{H}^*}}(\mathbb{R}^N)}\right\} \leq \int_{\mathbb{R}^N} \widehat{\mathscr{H}^*}(x,|u|)dx \leq \max\left\{\|u\|^{h_3^*}_{L^{\widehat{\mathscr{H}^*}}(\mathbb{R}^N)}, \|u\|^{h_4^*}_{L^{\widehat{\mathscr{H}^*}}(\mathbb{R}^N)}\right\}$,

where, $h_3^* = \frac{h_1^*}{h_1^*-1}$ and $h_4^* = \frac{h_2^*}{h_2^*-1}$.

2.3 Variational methods

The modern version of the calculus of variation is known as critical point theory or variational methods. Critical point theory is a mixture of nonlinear analysis and calculus of variations which is a powerful tool for studying nonlinear elliptic equations. As an application of variational methods in nonlinear analysis, the main motive is to convert the problem of proving the existence of solution for nonlinear equations into proving the existence of critical point for some differentiable functional. In this way, the fundamental question of the variational problem is to show the existence of a critical point.

In 1971, Hemple [70] considered the variational boundary value problem of the general form

$$\begin{aligned} -\operatorname{div}(a(x)\nabla u) + c(x)u - f(x,u) = 0 \ \text{ in } \ \Omega \\ u = 0 \ \text{ on } \ \partial\Omega, \end{aligned} \tag{2.10}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and proved the existence of multiple solutions for the same under some assumptions on $f(x,u)$. In 1973, Ambrosetti and Rabinowitz [15] obtained some general existence theorems for nonlinear equations by critical point theory for bounded domains. They proved the existence and estimated the number of critical points possessed by a real-valued continuously differentiable function on a real Banach space. After that, many authors [127, 128], dealt with nonlinear problems in bounded domains. In 1992, Rabinowitz [126] established the existence result for unbounded domains by using variational methods. He examined the equation

$$-\Delta u + b(x)u = f(x,u) \ \text{ in } \ \Omega \tag{2.11}$$

where Ω is any subset of $\mathbb{R}^N$.

2.3.1 Mountain pass theorem

In 1973, Ambrosetti and Rabinowitz [16] provided a significant result called the mountain pass theorem to show the existence of critical point. Mountain pass theorem is considered as a crucial tool for studying nonlinear equations from the last 50 years. To know more details about the MPT, please refer [150].

Definition 2.3.1 (Palais-Smale condition)**.** *Let X be a Banach space and let $J : X \to \mathbb{R}$ be a C^1 functional. We say that J is a Palais-Smale sequence, in notation, $(PS)_c$ sequence (where $c \in \mathbb{R}$ is a constant) if $J(u_n) \to c$ and $J'(u_n) \to 0$ in the dual space X^*. We say J satisfies the $(PS)_c$ condition if every $(PS)_c$ sequence has a convergent subsequence.*

Theorem 2.3.2 (Mountain pass theorem)**.** *Let X be a Banach space and let $J : X \to \mathbb{R}$ be a C^1 functional. Assume that there exist positive numbers ρ and α such that*

1. *$J(u) \geq \alpha$ if $\|u\| = \rho$*
2. *There exists $v \in X$ such that $J(v) \leq 0$ when $\|v\| > \rho$*

Then, there exists a $(PS)_c$ sequence for J where $c > \alpha$. Moreover, if J satisfies the $(PS)_c$ condition then J has a critical point at the level c defined by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)),$$

where Γ is the collection of all continuous paths $\gamma : [0,1] \to X$ such that $\gamma(0) = 0$ and $\gamma(1) = v$.

Definition 2.3.3 (Cerami sequence)**.** *Let $c \in \mathbb{R}$, X be a Banach space and $J : X \to \mathbb{R}$ be a C^1 functional. We say $\{u_n\}$ is a Cerami sequence, in notation, $(C)_c$ sequence if $J(u_n) \to c$ and $(1 + \|u_n\|)|J'(u_n)| \to 0$ in the dual space X^*. We say J satisfies the $(C)_c$ condition if every $(C)_c$ sequence has a convergent subsequence.*

We also, use the following version of MPT given by Cerami [30, 31] to prove our results.

Theorem 2.3.4. *[42] Let X be a Banach space and $J : X \to \mathbb{R}$ be a C^1 functional. Suppose that there exist $\rho, \alpha > 0$ such that*

1. *$J(u) \geq \alpha$ if $\|u\| = \rho$.*
2. *There exists $v \in X$ such that $J(v) \leq 0$ when $\|v\| > \rho$.*

Then, J has a $(C)_c (c \geq \alpha)$ sequence. Moreover, if J satisfies the $(C)_c$ condition then J has a critical point at the level c defined by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)),$$

here Γ is the set of all continuous functions $\gamma : [0,1] \to X$ such that $\gamma(0) = 0$ and $\gamma(1) = v$.

2.3.2 Nehari manifold method

Definition 2.3.5 (Ground state solution)**.** *Let $J : X \to \mathbb{R}$ be a functional. We say, the solution $u_0 \in X$ is a ground state solution associated to J, if it has the least energy, i.e., we say, u_0 is ground state solution of J if*

$$J(u_0) = r = \inf_{u \in S} J(u) \text{ and } J'(u_0) = 0, \tag{2.12}$$

where S is the set of all critical points of the functional J.

To prove the existence of a ground state solution, we use the minimization method, in particular, Nehari manifold method. We define

$$\aleph = \{u \in X\backslash\{0\} | J'(u)u = 0\} \text{ and } b = \inf_{u\in\aleph} J(u).$$

The set $\aleph$ is called the Nehari manifold. It can be observed that $S \subseteq \aleph$. The key idea of this method is to search a non-trivial critical point of J in $\aleph$ instead of the whole space W. To know more about this method, one can refer to [143]. The existence of a ground state solution is proved by many researchers; we refer to, [22, 34, 105, 109, 111, 112] and reference therein.

2.4 Moser-Trudinger inequality: A short review

The shortcoming of the Sobolev embedding theorem is that there is no smallest space into which $W^{m,p}(\Omega)$ can be embedded when $mp = N$ and $p > 1$. In fact,

$$W^{m,p}(\Omega) \hookrightarrow \cdots \hookrightarrow L^3(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L^1(\Omega)$$

but $W^{m,p}(\Omega) \nsubseteq L^\infty(\Omega)$.

If the class of target spaces is expanded to include Orlicz spaces, the minimum largest space can be achieved. In fact, Trudinger [144] proved that $W_0^{1,N}(\Omega)$ is continuously embedded in the Orlicz space $L_\Phi(\Omega)$, where $\Phi = \exp(t^{\frac{N}{N-1}}) - 1$. Moreover, the result of Trudinger [144] is optimal [71] in the sense that the space $L_\Phi(\Omega)$, where $\Phi = \exp(t^{\frac{N}{N-1}}) - 1$ is smallest Orlicz space into which $W^{m,p}(\Omega)$ is embedded. The inequality of Trudinger was later sharpened by J. Moser [114] (known as Moser-Trudinger inequality), is as follows:

Theorem 2.4.1. *[114] If $d \leq \alpha_N$, $u \in W_0^{1,N}(\Omega)$ and $\|\nabla u\|_{L^N(\Omega)} \leq 1$, where $\alpha_N = N\omega_{N-1}^{1/(N-1)}$, ω_{N-1} is the volume of the unit sphere S^{N-1} and Ω is bounded domain in $\mathbb{R}^N$ then there exists c (depending on d and N) such that*

$$\int_\Omega e^{d|u|^{N/(N-1)}} dx \leq c(d,N)|\Omega|.$$

The above integral can be made arbitrary large for $d > \alpha_N$ by the appropriate choice of $u \in W_0^{1,N}(\Omega)$.

After that, Adimurthi and Sandeep [132] proved the following singular Moser-Trudinger inequality for bounded domains:

Theorem 2.4.2. *[132] Let $N \geq 2$ and Ω be an open and bounded set in $\mathbb{R}^N$. If $u \in W_0^{1,N}(\Omega)$, $b \in [0,N)$ and $\alpha > 0$ then*

$$\int_\Omega \frac{\exp\left(\alpha |u|^{\frac{N}{N-1}}\right)}{|x|^b} dx < \infty.$$

Moreover, there exists a constant $c_0 = c_0(N, |\Omega|) > 0$ such that

$$\sup_{\|u\|_{W_0^{1,N}(\Omega)} \leq 1} \int_\Omega \frac{\exp\left(\alpha |u|^{\frac{N}{N-1}}\right)}{|x|^b} dx \leq c_0$$

for any $b \in [0,N)$, $0 \leq \alpha \leq \left(1 - \frac{b}{N}\right) \alpha_N$. Moreover, this constant $\left(1 - \frac{b}{N}\right) \alpha_N$ is sharp in the sense that if $\alpha > \left(1 - \frac{b}{N}\right) \alpha_N$, then the above inequality can no longer hold with some c_0 independent of u.

In case of whole space $\mathbb{R}^N$, Ruf ($N = 2$) [130] and Li and Ruf ($N \geq 2$) [96] obtained Moser-Trudinger inequality. Their result is as follows:

Further, in this section, Φ represents as

$$\Phi(t) = \exp(t) - \sum_{n=0}^{N-2} \frac{t^n}{n!}. \tag{2.13}$$

Theorem 2.4.3. *Let $N \geq 2$. Then there exists a constant $c > 0$ such that*

$$\sup_{u \in W^{1,N}(\mathbb{R}^N);\, \|u\|_{L^N}^N + \|\nabla u\|_{L^N}^N \leq 1} \int_{\mathbb{R}^N} \Phi\left(\alpha |u|^{\frac{N}{N-1}}\right) dx \leq c$$

whenever $\alpha \leq \alpha_N$. Moreover, the constant α_N is sharp in the sense that for any $\alpha > \alpha_N$, the supremum becomes infinite.

For a proof of Theorem 2.4.3 in case of $\alpha < \alpha_N$, we also refer to [1, 28, 119, 120, 122]. Later on, Adimurthi and Yang [155] derived the singular Moser-Trudinger inequality for unbounded domains. Their result is as follows:

Theorem 2.4.4. *Let $N \geq 2$. For all $\alpha > 0$, $0 \leq \beta < N$ and $u \in W^{1,N}(\mathbb{R}^N)$, we have*

$$\int_{\mathbb{R}^N} \frac{1}{|x|^\beta} \Phi(\alpha |u|^{\frac{N}{N-1}}) dx < \infty.$$

Furthermore, if $\alpha \leq \left(1 - \frac{\beta}{N}\right) \alpha_N$ and $\tau > 0$,

$$\sup_{\|u\|_{1,\tau} \leq 1} \int_{\mathbb{R}^N} \frac{1}{|x|^\beta} \Phi(\alpha |u|^{\frac{N}{N-1}}) dx < \infty,$$

where

$$\|u\|_{1,\tau} = \left(\int_{\mathbb{R}^N} (|\nabla u|^N + \tau |u|^N) dx \right)^{1/N}.$$

Moreover, the constant $\left(1 - \frac{\beta}{N}\right) \alpha_N$ *is sharp in the sense that the supremum becomes infinite for any* $\alpha > \left(1 - \frac{\beta}{N}\right) \alpha_N$.

Inspired by the work of Adimurthi and Yang [155], Lam and Lu [89] obtained the following lemmas in the weighted Sobolev spaces.

Lemma 2.4.5. *[89] If* $N \geq 2, \alpha > 0$, $s > N, u \in W'$ *and* $\|u\|_{W'} \leq K$ *with* K *is sufficiently small then*

$$\int_{\mathbb{R}^N} \frac{|u|^s \Phi(\alpha |u|^{N/N-1})}{|x|^b} dx \leq c \|u\|_{W'}^s,$$

for some $c > 0$ *and*

$$W' = \{u \in W^{1,N}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x) |u|^N dx < \infty\}.$$

Lions [99] established a concentration-compactness principle associated with Moser-Trudinger inequality, which reads as follows:

Theorem 2.4.6. *[99, Theorem I.6 and Remark I.18] Let* $\{u_n\}$ *be a sequence of functions in* $W_0^{1,N}(\Omega)$ *with* $\|\nabla u_n\|_{L^N(\Omega)} = 1$, *where* Ω *is bounded domain in* $\mathbb{R}^N$. *If* $u_n \rightharpoonup u \neq 0$ *weakly, then*

$$\sup_n \int_\Omega e^{p\alpha_N |u_n|^{N/(N-1)}} dx < \infty,$$

for any $0 < p < \frac{1}{(1 - \|\nabla u\|_{L^N(\Omega)}^N)^{1/(N-1)}}$, *where* $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ *and* ω_{N-1} *is the volume of the unit sphere* S^{N-1}.

Lam and Lu [89] proved the following version of the concentration compactness principle, which is a generalization of Theorem 2.4.6:

Theorem 2.4.7. *[89] Let* $\{u_n\}$ *be a sequence of functions in* $W^{1,N}(\Omega)$ *with* $\|\nabla u_n\|_{L^N(\Omega)} = 1$, *where* Ω *is bounded domain in* $\mathbb{R}^N$. *If* $0 \leq a < N$, $u_n \rightharpoonup u \neq 0$ *weakly and* $\nabla u_n \to \nabla u$ *a.e. then*

$$\sup_n \int_\Omega \frac{e^{\alpha |u_n|^{N/(N-1)}}}{|x|^a} dx < \infty,$$

for all $0 < \alpha < \frac{\left(\frac{N-a}{N}\right)\alpha_N}{(1 - \|\nabla u\|_{L^N(\Omega)}^N)^{1/(N-1)}}$, *where* $\alpha_N = N\omega_{N-1}^{1/(N-1)}$, ω_{N-1} *is the volume of the unit sphere* S^{N-1}.

The Lions concentration-compactness principle in finite domains was further extended to the whole space $\mathbb{R}^N$ by do Ó et al. [138]. They proved the following theorem:

Theorem 2.4.8. *Let $\{u_n\} \subseteq W^{1,N}(\mathbb{R}^N)$ with $\|u_n\|_{W^{1,N}(\mathbb{R}^N)} = 1$. If $u_n \rightharpoonup u \neq 0$ weakly, then*

$$\sup_n \int_{\mathbb{R}^N} \Phi(p\alpha_N |u_n|^{N/(N-1)})dx < \infty,$$

for all $0 < p < p_N(u) := \dfrac{1}{\left(1 - \|u\|^N_{W^{1,N}(\mathbb{R}^N)}\right)^{1/(N-1)}}$, where $\alpha_N = N\omega_{N-1}^{1/(N-1)}$. Moreover, the constant p is sharp in the sense that if $p > p_N(u)$, then the supremum becomes infinite.

Zhang and Chen [158] established a singular version of the Lions concentration-compact principle in the whole $\mathbb{R}^N$. They obtained the following result:

Theorem 2.4.9. *Let $\{u_n\} \subseteq W^{1,N}(\mathbb{R}^N)$ with $\|u_n\|_{W^{1,N}(\mathbb{R}^N)} = 1$. If $0 \leq a < N$, $u_n \rightharpoonup u \neq 0$ weakly then*

$$\sup_n \int_{\mathbb{R}^N} \frac{\Phi(\alpha_{N,a} p |u_n|^{N/(N-1)})}{|x|^a} dx < \infty,$$

for all $0 < p < p_N(u) := \dfrac{1}{\left(1 - \|u\|^N_{W^{1,N}(\mathbb{R}^N)}\right)^{1/(N-1)}}$, where $\alpha_{N,a} = \alpha_N \left(1 - \frac{a}{N}\right)$. Moreover, the constant p is sharp in the sense that if $p > p_N(u)$, then the supremum becomes infinite.

Zhang and Chen [158], also obtained a concentration-compactness result in weighted Sobolev spaces.

Lemma 2.4.10. *Let $\{u_n\} \subseteq W'$ with $\|u_n\|_{W'} = 1$. If $0 \leq a < N$, $u_n \rightharpoonup u \neq 0$ weakly then*

$$\sup_n \int_{\mathbb{R}^N} \frac{\Phi(\alpha_{N,a} p |u_n|^{N/(N-1)})}{|x|^a} dx < \infty,$$

for all $0 < p < p_N(u) := \dfrac{1}{(1 - \|u\|^N_{W'})^{1/(N-1)}}$, where $\alpha_{N,a} = \alpha_N \left(1 - \frac{a}{N}\right)$,

$$W' = \{u \in W^{1,N}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^N dx < \infty\}$$

and $V : \mathbb{R}^N \to (0,\infty)$ such that $V(x) \geq V_0$ for some $V_0 > 0$. Moreover, the constant p is sharp in the sense that if $p > p_N(u)$, then the supremum becomes infinite.

Alves et at. [12] proved and used the following version of Moser-Trudinger inequality:

Lemma 2.4.11. *Let $\alpha > 0$ and $s > 1$ then $\exists\, 0 < r < 1$ and $c > 0$ such that*

$$\sup \int_\Omega e^{s\alpha|u|^{\frac{N}{N-1}}} dx \leq c,$$

for any $u \in W_0^{1,N}(\Omega)$ such that $\|u\|_{W_0^{1,N}(\Omega)} \leq r$.

Motivated by these Moser-Trudinger type inequalities and Lions type concentration-compactness results, several authors obtained the existence results for the problems involving exponential-type nonlinearities in bounded as well as unbounded domains. Interested readers can refer to [3, 132, 58, 120, 121, 56, 65, 89, 90, 94, 158] and references cited therein.

2.5 Concentration compactness principle

The Rellich-Kondrachov theorem tells that

$$W^{m,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega), q \in [1, p^*)$$

where $\Omega \subseteq \mathbb{R}^N$ is bounded, $mp < N$ and $p^* = \frac{Np}{N-mp}$ is called the critical exponent. But $W^{m,p}(\Omega)$ does not compactly embedded in $L^{p^*}(\Omega)$.

The nonlinear equation having critical growth nonlinearity, faces a lack of compactness issue. Lions provided a systematic way to address such issues. He established concentration compactness principle (CCP) [99, Lemma 1.1] which is stated as:

Lemma 2.5.1. *Let $D^{m,p}(\mathbb{R}^N)$ denotes the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\| = (\int_{\mathbb{R}^N} |D^m u|^p dx)^{\frac{1}{p}}$. Suppose $\{u_n\}$ be a bounded sequence in $D^{m,p}(\mathbb{R}^N)$ which converges weakly to limit u such that $|D^m u_n|^p$ converges weakly to a measure μ, $|u_n|^{p^*}$ converges tightly to a measure ν, where μ and ν are bounded measures on $\mathbb{R}^N$. Then there exist at most countable index set J and two families $(x_j)_{j\in J} \in \mathbb{R}^N$ and $(\nu_j)_{j\in J} > 0$ such that*

(1) $\mu \geq |D^m u|^p + \sum_{j\in J} u_j \delta_{x_j}$

(2) $\nu = |u|^{p^*} + \sum_{j\in J} \nu_j \delta_{x_j}$ *with* $\nu_j^{p/p^*} \leq \mu_j/S$, *where S is the best constant that appears in the Sobolev embedding theorem.*

The above result does not provide any information about a possible loss of mass at infinity. This issue was taken care by Chabrowski [32], who proved the following result:

Theorem 2.5.2. *Let the assumptions of the Theorem 2.5.1 are true. If*

$$\mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |D^m u_n|^p dx$$

and

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |u_n|^{p^*} dx$$

then

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |D^m u|^p dx = \mu(\mathbb{R}^N) + \mu_\infty \tag{2.14}$$

and

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p^*} dx = \nu(\mathbb{R}^N) + \nu_\infty \tag{2.15}$$

with $\nu_\infty^{p/p^*} \leq \mu_\infty / S$.

Many authors proved and used the different versions of CCP according to their needs. Bonder-Saintier-Silva [24] proved the fractional version of CCP in fractional order Sobolev spaces. Ho-Kim [72] obtained the fractional version of CCP with variable exponent in fractional order Sobolev spaces with variable exponents. Ho-Sim [73] established the weighted CCP in weighted variable exponent Sobolev spaces.

In this sequence, Bonder-Silva [25] proved the variable exponent version of CCP in Variable exponent Sobolev spaces. Their result is as follows:

Lemma 2.5.3. *Let* $\{u_n\}$ *in* $W_0^{1,p(x)}(\Omega)$ *which converges weakly to limit u such that*

- $|\nabla u_n|^{p(x)}$ *converges weakly to a measure* μ,
- $|u_n|^{p^*(x)}$ *converges weakly to a measure* ν, *where* μ *and* ν *are bounded non-negative measures on* Ω.

Then there exist at most countable index set I and $(x_i)_{i\in I} \in \Omega$ *such that*

(1) $\nu = |u|^{p^*(x)} + \sum_{i\in I} \nu_i \delta_{x_i},\ \nu_i > 0$

(2) $\mu \geq |\nabla u|^{p(x)} + \sum_{i\in I} \mu_i \delta_{x_i},\ \mu_i > 0$

with $S\nu_i^{1/p^*(x_i)} \leq \mu_i^{1/p(x_i)}, \forall i \in I$,

where

$$S = \inf_{u\in C_c^\infty(\Omega)} \left\{ \frac{\||\nabla u|\|_{L^{p(x)}(\Omega)}}{\|u\|_{L^{p^*(x)}(\Omega)}} \right\} > 0.$$

Bonder-Silva [26] proved the CCP for Orlicz Sobolev spaces.

In Chapter 7, in order to prove the existence of a weak solution for the generalized non-local problem, we derive the CCP for fractional Musielak Sobolev spaces.

Chapter 3

Kirchhoff type problem with gradient nonlinearity

In this chapter[1], we prove the existence of a weak solution to a Kirchhoff problem:

$$\begin{cases} -A\left(x, \int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u,\nabla u) + \lambda h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded and smooth domain in $\mathbb{R}^N (N \geq 2)$. We assume that f, h and A are continuous functions and the growth of the nonlinearity $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is dependent on u and ∇u. We do not assume any growth condition on the perturbation term h. In the case of $N = 2$, we consider the exponential growth in the second variable of f. The proof of our main existence result uses an iterative technique based on the mountain pass theorem.

In Section 3.1, we provide the main assumptions on the Kirchhoff term, the nonlinearity and the perturbation term. We also state the main result of this chapter in Section 3.1. In Section 3.2, we establish existence of a weak solution to a truncated variational problem corresponding to a given equation and later on with the help of the existence result for this truncated problem, the existence of solution to (3.1) is proved in Section 3.3. Finally, we construct some examples illustrating the main theorem in Section 3.4.

3.1 Hypotheses and main result

The nonlinearity f satisfies the following growth conditions:

(f_0) There exist $a_1 > 0$, $q \in (1, 2^* - 1)$ and $s \in (0, \frac{1}{q+1})$ such that

$$|f(x,t,\xi)| \leq a_1(1+|t|^q)(1+|\xi|^s) \;\; \forall\, (x,t,\xi) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N$$

where, $2^* = \frac{2N}{N-2}$ and $N > 2$.

[1]Gaurav Dwivedi, Shilpa Gupta, Existence of solution to Kirchhoff type problem with gradient nonlinearity and a perturbation term, *Journal of Elliptic and Parabolic Equations*, 8 (2022), 533–553.

For the case $N = 2$, we assume the following growth condition on f:

(f_0') There exist $a_2 > 0$ and $r \in (0,1)$ such that

$$|f(x,t,\xi)| \leq a_2 \exp(\tau t^2)(1+|\xi|^r) \ \ \forall\, (x,t,\xi) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^2,$$

for all $\tau > 0$.

The growth condition (f_0') is motivated by the celebrated result of Trudinger [144].

Next, we state our hypotheses on the non-local term A, the perturbation term h and the remaining hypotheses on the nonlinearity f. We assume that $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is continuous and satisfies the following conditions:

(f_1) $f(x,t,\xi) = 0$, $\forall\, t < 0$ and $(x,\xi) \in \bar{\Omega} \times \mathbb{R}^N$.

(f_2) $\displaystyle\lim_{t \to 0} \frac{f(x,t,\xi)}{t} = 0, \ \ \forall\, (x,\xi) \in \bar{\Omega} \times \mathbb{R}^N$.

(f_3) There exists $\theta > 2$ such that

$$0 < \theta F(x,t,\xi) = \theta \int_0^t f(x,s,\xi)ds \leq t f(x,t,\xi),$$

for all $t > 0,\ (x,\xi) \in \bar{\Omega} \times \mathbb{R}^N$.

(f_4) There exist positive constants L_1 and L_2 (depending on ρ_1 and ρ_2) such that

$$|f(x,t_1,\xi) - f(x,t_2,\xi)| \leq L_1 |t_1 - t_2|$$

for all $x \in \overline{\Omega},\ t_1,t_2 \in [0,\rho_1],\ |\xi| \leq \rho_2$,

$$|f(x,t,\xi_1) - f(x,t,\xi_2)| \leq L_2 |\xi_1 - \xi_2|$$

for all $x \in \overline{\Omega},\ t \in [0,\rho_1]$ and $|\xi_1|, |\xi_2| \leq \rho_2$, for some $\rho_1, \rho_2 > 0$.

The function $A : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following conditions:

(A_1) There exist positive real numbers a_0 and a_∞ such that

$$a_0 \leq A(x,t) \leq a_\infty, \ \ \forall\, (x,t) \in \overline{\Omega} \times \mathbb{R}.$$

(A_2) For a given $R_1 > 0$, there exists a positive real number L_3 such that

$$|A(x,t_1^2) - A(x,t_2^2)| \leq L_3 |t_1 - t_2|, \ \ \forall\, x \in \overline{\Omega} \text{ and } |t_1|, |t_2| \leq R_1.$$

The function $h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and fulfils the following condition:

(H) For a given $\rho_3 > 0$, there exists positive constant L_4 such that

$$|h(x,t_1) - h(x,t_2)| \leq L_4|t_1 - t_2|$$

for all $x \in \overline{\Omega}$, $t_1, t_2 \in [0, \rho_3]$.

We do not assume any growth or sign condition on the perturbation term h. To handle this term, the truncation technique is used here; interested readers may look at [82, 84].

The growth conditions (f_0), (f_0'), (f_2) and continuity of f and h imply that these functions are bounded in bounded domains, i.e., given $\delta_1, \delta_2 > 0$, there exist constants c_f and c_h such that

$$f(x,t,\xi) \leq c_f, \quad h(x,t) \leq c_h, \quad \text{for all } |t| < \delta_1, x \in \Omega, |\xi| < \delta_2. \tag{3.2}$$

We discuss the existence of weak solution for the Problem (3.1) in the Sobolev space $W_0^{1,2}(\Omega)$ which is equipped with the norm:

$$\|u\| = \left(\int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}}.$$

Notations: We denote by S_2 the best Sobolev constant in the embedding of $W_0^{1,2}(\Omega)$ into the space $L^p(\Omega)$, where $p \in [2, \infty)$ when $N > 2$ and into the space $L^p(\Omega)$, where $p \in [1, \infty)$ when $N = 2$.

The statement of the main result of this chapter is as follows:

Theorem 3.1.1. *Suppose that the conditions* (f_1)-(f_4)*,* (A_1)-(A_2) *and* (H) *are satisfied. Further assume that* (f_0) *holds if* $N > 2$ *and* (f_0') *holds if* $N = 2$*. Then, there exists a positive real number* λ_0 *such that, for all* $|\lambda| < \lambda_0$ *the Problem* (3.1) *has a positive solution, provided*

$$0 < \frac{L_2 a_0 S_2 + L_3 c_f |\Omega|^{1/2} S_2 + \lambda L_3 c_h |\Omega|^{1/2} S_2}{C_N S_2^2 a_0^2 - (L_1 a_0 + L_4 \lambda a_0)} < 1.$$

3.2 Truncated problem without gradient nonlinearity

Since the Problem (3.1) is non-variational; one can not apply variational methods directly. The technique used to prove our main Theorem 3.1.1 is inspired by [102]. For a fixed $w \in W_0^{1,2}(\Omega)$, we consider the following freezed problem:

$$\begin{cases} -\Delta u = \dfrac{f(x,u,\nabla w)+\lambda h(x,u)}{A(x,\|w\|^2)} & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

We say that $u \in W_0^{1,2}(\Omega)$ is a weak solution of (3.3) if the following holds:

$$\int_\Omega \nabla u \, \nabla \varphi \, dx = \int_\Omega \frac{f(x,u,\nabla w)\ \varphi + \lambda h(x,u)\varphi}{A(x,\|w\|^2)} dx \quad \forall \varphi \in W_0^{1,2}(\Omega).$$

The Problem (3.3) is variational, i.e., the critical points of the functional

$$I_{w_\lambda}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega \frac{F(x,u,\nabla w)+\lambda H(x,u)}{A(x,\|w\|^2)} dx$$

are the weak solutions to (3.3), where $F(x,t,\xi) = \int_0^t f(x,s,\xi)ds$ and $H(x,t) = \int_0^t h(x,s)ds$.

Due to the presence of gradient term in the growth condition of the nonlinearity f, we consider the following truncated problem for any fixed $T > 0$ as:

$$\begin{cases} -\Delta u = \dfrac{f_T(x,u,\nabla w)+\lambda h(x,u)}{A(x,\|w\|^2)} & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

where

$$f_T(x,t,\xi) = f(x,t,\xi\varphi_T(\xi)) \ \ \forall\, (x,t,\xi) \in \bar{\Omega}\times\mathbb{R}\times\mathbb{R}^N, \varphi_T \in C^1(\mathbb{R}^N),\ |\varphi_T(\xi)| \le 1$$

and

$$\varphi_T(\xi) = \begin{cases} 1, & |\xi| \le T \\ 0, & |\xi| \ge T+1. \end{cases}$$

For $N > 2$, we have

$$|f_T(x,t,\xi)| = |f(x,t,\xi\varphi_T(\xi))| \le a_1(1+|t|^q)(1+|\xi\varphi_T(\xi)|^s).$$

By using the definition of φ_T, we get

$$|f_T(x,t,\xi)| \le a_1(T+2)^s(1+|t|^q). \tag{3.5}$$

For $N = 2$, we have

$$|f_T(x,t,\xi)| = |f(x,t,\xi\varphi_T(\xi))| \le a_2 e^{(\tau t^2)}(1 + |\xi\varphi_T(\xi)|^r).$$

By using the definition of φ_T, we get

$$|f_T(x,t,\xi)| \le a_2(T+2)^r e^{\tau t^2}. \tag{3.6}$$

The Problem (3.4) is variational, i.e., the critical points of the functional

$$I^T_{w_\lambda}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega \frac{F_T(x,u,\nabla w) + \lambda H(x,u)}{A(x,\|w\|^2)} dx$$

are the weak solutions to the Problem (3.4), where $F_T(x,t,\xi) = \int_0^t f_T(x,s,\xi)ds$.
We consider a C^1 functional $I^T_{w_0}$ defined as

$$I^T_{w_0}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega \frac{F_T(x,u,\nabla w)}{A(x,\|w\|^2)} dx.$$

It is easy to see that the critical points of $I^T_{w_0}$ are the solutions to the Problem (3.4) with $\lambda = 0$.

Next, we provide the statement of auxiliary results.

Theorem 3.2.1. *Let $N > 2$ and the conditions (f_0)-(f_3) and (A_1) are satisfied. Then, for fixed $w \in W_0^{1,2}(\Omega) \cap C^{1,\beta}(\Omega)$, there exists $\lambda_0 > 0$ such that the Problem (3.3) has positive solution, say u_w, for all $|\lambda| < \lambda_0$. Moreover, there exist $k_1 > 0$ and $0 < \beta < 1$, independent of w, such that $\|u_w\|_{C^{0,\beta}(\Omega)} \le k_1$.*

Theorem 3.2.2. *Let $N = 2$ and the conditions (f_0'), (f_1)-(f_3) and (A_1) are satisfied. Then, for fixed $w \in W_0^{1,2}(\Omega) \cap C^{1,\beta}(\Omega)$, there exists $\lambda_0 > 0$ such that the Problem (3.3) has positive solution, say u_w, for all $|\lambda| < \lambda_0$. Moreover, there exist $k_1 > 0$ and $0 < \beta < 1$, independent of w, such that $\|u_w\|_{C^{0,\beta}(\Omega)} \le k_1$.*

3.2.1 The general case: $N > 2$

In this section, first, we deal with the existence of a positive solution to (3.3) with $\lambda = 0$ and then we use it to establish the existence result for (3.3). Due to the presence of ∇u in the growth condition of nonlinearity f, first we consider (3.4). Since no growth condition is assumed on h, it is difficult to discuss the existence of the solution to (3.4) directly. Therefore, we prove the existence of a solution to a truncated problem in which the perturbation term is bounded. Then, with the help of the existence result for this truncated problem, the existence of solution to (3.3) is established, infact the Theorem 3.2.1 is proved.

Next, we prove a series of lemmas.

Lemma 3.2.3. *Let u be a non-trivial solution to the Problem* (3.4) *with $\lambda = 0$. Then u is strictly positive in Ω, and its outward normal derivative is strictly negative on $\partial\Omega$.*

Proof. Let $D = \{x \in \Omega \mid u(x) < 0\}$. Suppose that $D \neq \phi$. Then,

$$-\Delta u = \frac{f_T(x,u,\nabla w)}{A(x,\|w\|^2)} = 0 \text{ in } D,$$
$$u(x) = 0 \text{ on } \partial D.$$

This shows that $u = 0$ in D, which is a contradiction. Hence, $u \geq 0$.

By (f_2), there exists a constant $c_1 > 0$ such that

$$f_T(x,t,\nabla w) \geq -c_1 t$$

for all $0 \leq t \leq \|u\|_{L^\infty(\Omega)}$, $x \in \Omega$. This implies that

$$\frac{c_1}{a_0}u - \Delta u = \frac{c_1}{a_0}u + \frac{f_T(x,u,\nabla w)}{A(x,\|w\|^2)} \geq 0,$$

for all $x \in \Omega$ such that $f_T(x,u,\nabla w) \leq 0$ and

$$c_1 u - \Delta u = c_1 u + \frac{f_T(x,u,\nabla w)}{A(x,\|w\|^2)} \geq 0,$$

for all $x \in \Omega$ such that $f_T(x,u,\nabla w) \geq 0$. Therefore,

$$\Delta u \leq c_2 u \text{ in } \Omega,$$

for some $c_2 > 0$. Using [115, Theorem 8.27], we get $u > 0$ in Ω and $\frac{\partial u}{\partial \eta} < 0$ on $\partial\Omega$. □

Lemma 3.2.4. *The Palais-Smale condition is satisfied for $I^T_{w_0}$.*

Proof. Let $\{u_n\} \subseteq W_0^{1,2}(\Omega)$ be any Palais-Smale sequence, i.e., $I^T_{w_0}(u_n) \to \ell < \infty$ and $(I^T_{w_0})'(u_n) \to 0$ in dual space of $W_0^{1,2}(\Omega)$. This implies

$$\frac{1}{2}\|u_n\|^2 - \int_\Omega \frac{F_T(x,u_n,\nabla w)}{A(x,\|w\|^2)}dx = \ell + \delta_n, \tag{3.7}$$

where $\delta_n \to 0$ as $n \to \infty$ and

$$\left|\int_\Omega \nabla u_n \nabla \varphi dx - \int_\Omega \frac{f_T(x,u_n,\nabla w)\varphi}{A(x,\|w\|^2)}dx\right| \leq \varepsilon_n \|\varphi\|, \ \forall \varphi \in W_0^{1,2}(\Omega), \tag{3.8}$$

where $\varepsilon_n \to 0$ as $n \to \infty$. On taking $\varphi = u_n$, from (3.7) and (3.8), we obtain

$$\left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|^2 - \int_\Omega \frac{F_T(x,u_n,\nabla w) - \frac{1}{\theta} f_T(x,u_n,\nabla w)u_n}{A(x,\|w\|^2)}dx \le c_3(1+\|u_n\|),$$

for some $c_3 > 0$. It follows from (f_3) that

$$\left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|^2 \le c_3(1+\|u_n\|).$$

This implies that $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$. As $W_0^{1,2}(\Omega)$ is a reflexive space, there exists $u \in W_0^{1,2}(\Omega)$ such that, up to a subsequence

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,2}(\Omega),$$
$$u_n(x) \to u(x) \text{ a.e. } x \in \Omega,$$
$$u_n \to u \text{ in } L^p(\Omega) \text{ for } p \in [1,2^*).$$

As a consequence, $(I_{w_0}^T)'(u_n)(u_n-u) \to 0$ as $n \to \infty$, i.e.,

$$\int_\Omega \nabla u_n(\nabla u_n - \nabla u)dx - \int_\Omega \frac{f_T(x,u_n,\nabla w)(u_n-u)}{A(x,\|w\|^2)}dx \to 0 \text{ as } n \to \infty. \tag{3.9}$$

Further, (3.5) and (A_1) imply

$$\int_\Omega \frac{f_T(x,u_n,\nabla w)(u_n-u)}{A(x,\|w\|^2)}dx \le \frac{(T+2)^s a_1}{a_0}\int_\Omega (1+|u_n|^q)(u_n-u)dx.$$

Now, by Hölder's inequality (we choose p such that, $p \in [1,2^*)$), one gets

$$\begin{aligned}\int_\Omega \frac{f_T(x,u_n,\nabla w)(u_n-u)}{A(x,\|w\|^2)}dx \le \frac{(T+2)^s a_1}{a_0}\|u_n-u\|_{L^1(\Omega)} + \\ \frac{(T+2)^s a_1}{a_0}\left(\int_\Omega |u_n|^{qp'}dx\right)^{\frac{1}{p'}}\|u_n-u\|_{L^p(\Omega)} \\ \to 0 \text{ as } n \to \infty \text{ (Since } u_n \to u \text{ in } L^p(\Omega)).\end{aligned} \tag{3.10}$$

It follows from (3.9) and (3.10) that,

$$\lim_{n\to\infty}\int_\Omega \nabla u_n(\nabla u_n - \nabla u)dx \to 0.$$

Hence, using [115, Proposition 2.72]

$$u_n \to u \text{ in } W_0^{1,2}(\Omega) \text{ (up to a subsequence).}$$

This completes the proof. □

Lemma 3.2.5. *There exist positive real numbers* α *and* ρ *independent of* w *such that*

$$I^T_{w_0}(u) \geq \alpha > 0, \ \ \forall u \in W^{1,2}_0(\Omega) \text{ such that } \|u\| = \rho.$$

Proof. It follows from (f_0) and (f_2) that,

$$|F_T(x,t,\xi)| \leq \frac{1}{2}\varepsilon|t|^2 + c_4|t|^{q+1} \tag{3.11}$$

for some $\varepsilon > 0,\ c_4 > 0$ and for all $(x,t,\xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$.
By using (A_1) and (3.11), we obtain

$$I^T_{w_0}(u) \geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2a_0}\|u\|^2_{L^2(\Omega)} - \frac{c_4}{a_0}\int_\Omega |u|^{q+1}dx.$$

By Sobolev embedding theorem, we get

$$I^T_{w_0}(u) \geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon c_5}{2a_0}\|u\|^2 - \frac{c_6}{a_0}\|u\|^{q+1}.$$

Now choose ε such that $\frac{1}{2} - \frac{c_5\varepsilon}{2a_0} > 0$ and choose $\rho > 0$ sufficiently small such that

$$\alpha = \left(\frac{1}{2} - \frac{c_5\varepsilon}{2a_0}\right)\rho^2 - \frac{c_6}{a_0}\rho^{q+1} > 0.$$

This completes the proof. □

Lemma 3.2.6. *There exist* $u_1 \in W^{1,2}_0(\Omega)$ *and a positive real number* β *independent of* w *such that*

$$I^T_{w_0}(u_1) < 0 \ \ \text{and } \|u_1\| > \beta.$$

Proof. Let $u \in W^{1,2}_0(\Omega)\setminus\{0\}$ and $u \geq 0$. By (f_3), there exist $b,d > 0$ such that

$$F_T(x,t,\xi) \geq bt^\theta - d, \ \ \forall\ (x,t,\xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N. \tag{3.12}$$

By the help of (A_1) and (3.12), we have

$$\begin{aligned} I^T_{w_0}(tu) &= \frac{1}{2}\int_\Omega |\nabla(tu)|^2 - \int_\Omega \frac{F_T(x,tu,\nabla w)}{A(x,\|w\|^2)}dx \\ &\leq \frac{t^2}{2}\|u\|^2 - \frac{bt^\theta}{a_\infty}\int_\Omega u^\theta \ dx + d|\Omega|, \end{aligned}$$

this implies that $I^T_{w_0}(tu) \to -\infty$ as $n \to \infty$, since $\theta > 2$. Now, by setting $u_1 = tu$ for sufficiently large t, we get the desired result. □

Theorem 3.2.7. *Suppose that the conditions* $(f_0)-(f_3)$ *and* (A_1) *are satisfied. Then the Problem* (3.4) *has positive weak solution with* $\lambda = 0$.

Proof. By Lemmas 3.2.4, 3.2.5 and 3.2.6 all the conditions of mountain pass theorem are satisfied for the functional $I^T_{w_0}$. Hence, $I^T_{w_0}$ has a critical point $u \in W^{1,2}_0(\Omega)$ such that $I^T_{w_0}(u) = c_0$ $(c_0 \geq \alpha)$ and $(I^T_{w_0})'(u) = 0$ in $W^{1,2}_0(\Omega)^*$, where $W^{1,2}_0(\Omega)^*$ is the dual space of $W^{1,2}_0(\Omega)$,

$$c_0 = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I^T_{w_0}(\gamma(t)) > 0,$$

and

$$\Gamma = \{\gamma \in C([0,1], W^{1,2}_0(\Omega)) : \gamma(0) = 0,\ \gamma(1) = u_1\},$$

here u_1 is as defined in Lemma 3.2.6. Therefore, (3.4) has the non-trivial solution with $\lambda = 0$. Moreover, by Lemma 3.2.3 $u > 0$ in Ω. □

Lemma 3.2.8. *Let* u^T_w *be the any mountain pass solution of the Problem* (3.4) *with* $\lambda = 0$*, obtained in the Theorem 3.2.7. Then there exist positive constants* k *and* k_0*, independent of* w*, such that*

$$\|u^T_w\|_{L^\infty(\Omega)} \leq k \text{ and } \|u^T_w\|_{C^{1,\beta}(\Omega)} \leq k_0,$$

where $0 < \beta < 1$. *Moreover,* $\|\nabla u^T_w\|_{C^{0,\beta}(\Omega)} \leq T$ *for sufficiently large* T.

Proof. It is given that $(I^T_{w_0})'(u^T_w) = 0$ and $I^T_{w_0}(u^T_w) = c_0$. Now, proceeding as Lemma 3.2.4, there exist $c_7 > 0$ independent of w and T such that $\|u^T_w\| \leq c_7$. On using the regularity result [115, Theorem 8.10], we get $u^T_w \in C^{1,\beta}(\overline{\Omega})$, for some $0 < \beta < 1$.

By taking $p = 2$ and $\varepsilon = 1 - \frac{q(N-2)}{4}$ in [125, Theorem 2.4], we have

$$|u^T_w(x)| \leq c_8^{N/2\varepsilon}(\|u^T_w\|_{L^2(\Omega)} + |\Omega| \|(a_1(T+2)^s)\|_{L^{2^*/q}(\Omega)}) \text{ in } \Omega,$$

for some $c_8 > 0$, dependent only on ε. Finally, by Sobolev embedding theorem, we have

$$\|u^T_w\|_{L^\infty(\Omega)} \leq c_9(T+2)^s = k,$$

where c_9 is independent of T and w.
For any $p \in [1,\infty)$ and from (3.5), we have

$$\begin{aligned}\|f_T\|_{L^p(\Omega)} &\leq a_1(T+2)^s\|(1+|u_w^T|^q)\|_{L^p(\Omega)}\\ &\leq a_1(T+2)^s(|\Omega|^{1/p}+\|u_w^T\|^q_{L^\infty(\Omega)}|\Omega|^{1/p})\\ &\leq a_1(T+2)^s(|\Omega|^{1/p}+c_9^q(T+2)^{qs}|\Omega|^{1/p})\\ &\leq c_{10}(T+2)^{s(1+q)}.\end{aligned}$$

Now, by taking $p > N/2$ and using Morrey's Theorem [50], one gets $\|u_w^T\|_{C^{1,\beta}(\Omega)} \leq c_{11}\|u_w^T\|_{2,p}$ and by [81, Theorem 9.2.1], we have $\|u_w^T\|_{2,p} \leq c_{12}\|f_T\|_{L^p(\Omega)}$. Therefore, $\|u_w^T\|_{C^{1,\beta}(\Omega)} \leq c_{10}c_{11}c_{12}(T+2)^{s(1+q)} = k_0$. Since, $s < \frac{1}{1+q}$, by choosing sufficiently large T, we have $\|u_w^T\|_{C^{1,\beta}(\Omega)} \leq T$. Consequently, $\|u_w^T\|_{C^{0,\beta}(\Omega)} \leq T$ and $\|\nabla u_w^T\|_{C^{0,\beta}(\Omega)} \leq T$ for sufficiently large T. □

With help of the function h, we construct a new function h_k such that:

$$h_k(x,t) = \begin{cases} h(x,0) & if\ t \leq 0\\ h(x,t) & if\ t \in [0,2k]\\ h(x,2k) & if\ t \geq 2k.\end{cases}$$

One can observe that h_k is continuous and bounded on $\overline{\Omega}\times\mathbb{R}$. Now, select a cut-off function $g:\mathbb{R}\to[0,1]$ such that $0 \leq g \leq 1$ in $\mathbb{R}$, $g(t) = 1$ for all $t \in \overline{B(0,2k)}$ and $g(t) = 0$ for all $t \in \mathbb{R}-B(0,4k)$.

Now, consider the truncated problem

$$\begin{cases} -\Delta u = \dfrac{f_T(x,u,\nabla w)+\lambda g(u)h_k(x,u)+\lambda g'(u)H_k(x,u)}{A(x,\|w\|^2)} & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega.\end{cases} \tag{3.13}$$

where, $H_k(x,u) = \int_0^u h_k(x,t)dt$.
Corresponding to the Problem (3.13), we consider the following C^1 functional:

$$I_\lambda^T(u) = \int_\Omega \frac{1}{2}|\nabla u|^2dx - \int_\Omega \frac{F_T(x,u,\nabla w)+\lambda g(u)H_k(x,u)}{A(x,\|w\|^2)}dx.$$

We see that the critical points of the functional I_λ^T are the solutions to (3.13).

In the next theorem, we establish the existence of a bounded mountain pass solution to the Problem (3.13).

Theorem 3.2.9. *Suppose that the conditions* $(f_0)-(f_3)$ *and* (A_1) *are satisfied. Then there exists* $\lambda_1 > 0$ *such that the Problem* (3.13) *has a weak solution for all* $|\lambda| < \lambda_1$.

Proof. We will show the existence of a weak solution with the help of MPT. Let us verify the conditions for MPT. By proceeding as in Lemma 3.2.4 and using the argument that $g(t)H_k(x,t)$ and its partial derivatives are bounded, we obtain that I_λ satisfies the Palais-Smale condition.

Since, $g(t)H_k(x,t)$ is bounded, we get

$$I_{w_0}^T(u) - |\lambda|\kappa \le I_\lambda^T(u) \le I_{w_0}^T(u) + |\lambda|\kappa, \ \forall u \in W_0^{1,2}(\Omega), \tag{3.14}$$

where $\kappa = \max_{(x,t)\in\overline{\Omega}\times\mathbb{R}} \left| \dfrac{g(t)H_k(x,t)}{A(x,\|w\|^2)} |\Omega| \right|$.

Choose $|\lambda|$ sufficiently small, such that

$$I_\lambda^T(u) \ge \alpha - |\lambda|\kappa \ge \alpha/2, \ \forall \ \|u\| = \rho$$

$$I_\lambda^T(u_1) \le I_{w_0}^T(u_1) + |\lambda|\kappa < 0,$$

where α and ρ are as defined in Lemma 3.2.5 and u_1 is as defined in Lemma 3.2.6. This implies, there exists $\lambda_1 > 0$ such that geometric conditions of MPT are fulfilled for all $|\lambda| < \lambda_1$.

Hence, I_λ^T has a critical point $u_\lambda \in W_0^{1,2}(\Omega)$, i.e., $(I_\lambda^T)'(u_\lambda) = 0$ in the dual space of $W_0^{1,2}(\Omega)$ and $I_\lambda^T(u_\lambda) = c_\lambda$, where

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda^T(\gamma(t)) > 0,$$

with

$$\Gamma = \{\gamma \in C([0,1], W_0^{1,2}(\Omega)) : \gamma(0) = 0, \ \gamma(1) = u_1\}.$$

Hence, the Problem (3.13) has a solution for all $|\lambda| < \lambda_1$. □

Lemma 3.2.10. *Let u_{λ_n} be any solution of the Problem (3.13) by replacing λ to λ_n obtained in the Theorem 3.2.9, where $\{\lambda_n\}$ is a sequence of real numbers which converge to zero. Then for every n, $\|\nabla u_{\lambda_n}\|_{C^{0,\beta}(\Omega)} \le T$ for sufficiently large T. Moreover, up to a subsequence $\{u_{\lambda_n}\}$ converges to u in $C^{1,\beta}(\Omega)$ where $0 < \beta < 1$ and u is the critical point of $I_{w_\lambda}^T$.*

Proof. It is given that $(I_{\lambda_n}^T)'(u_{\lambda_n}) = 0$ and $I_{\lambda_n}^T(u_{\lambda_n}) = c_{\lambda_n}$. Again using the fact that $g(t)H_k(x,t)$ is bounded and proceeding as in Lemma 3.2.8, one can prove that $\|u_{\lambda_n}\|_{C^{1,\beta}(\Omega)} \le T$ for sufficiently large T. Consequently, $\|u_{\lambda_n}\|_{C^{0,\beta}(\Omega)} \le T$ and $\|\nabla u_{\lambda_n}\|_{C^{0,\beta}(\Omega)} \le T$ for sufficiently large T. Therefore, up to a subsequence $\{u_{\lambda_n}\}$ converges to u in $C^{1,\beta}(\Omega)$ where $0 < \beta < 1$. By (3.14), we get that $c_{\lambda_n} \to c_0$ as $\lambda_n \to 0$. Then $I_{w_\lambda}^T(u) = c_0$ and $(I_{w_\lambda}^T)'(u) = 0$, hence u is the critical point of $I_{w_\lambda}^T$. □

Now, we are ready to prove the Theorem 3.2.1.

Proof of the Theorem 3.2.1. On proceeding as in [84, Lemma 2.9] once can show that, there exists $\lambda_2 > 0$ such that any mountain pass solution u of the Problem (3.13) with $|\lambda| < \lambda_2$ satisfies

$$0 < u(x) \leq 2k \text{ in } \Omega,$$

where k is as defined in Lemma 3.2.8. Put $\lambda_0 = \min\{\lambda_1, \lambda_2\}$, where λ_1 as in Theorem 3.2.9. Let u_λ be the solution of the Problem (3.13) with $|\lambda| < \lambda_0$, obtained in Theorem 3.2.9. Then $0 < u_\lambda < 2k$. Therefore, $g(u_\lambda) = 1, g'(u_\lambda) = 0$ and $h_k(x, u_\lambda) = h(x, u_\lambda)$ and consequently, u_λ becomes the solution of the Problem (3.4). By using the definition of f_T and Lemma 3.2.10, we have $f_T(x, u, \nabla w) = f(x, u, \nabla w)$. Thus, u_λ is a solution to the Problem (3.3). Let λ_n be the sequence of real numbers which converges to zero and u_{λ_n} be any critical point of the functional $I^T_{\lambda_n}$ then by Lemma 3.2.10, up to a subsequence $\{u_{\lambda_n}\}$ converges to a critical point u of I_{w_λ}. This completes the proof. □

3.2.2 The border line case: $N = 2$

In this section, we discuss the existence of positive solution of the Problem (3.3) when $N = 2$, infact we prove the Theorem 3.2.2.

Lemma 3.2.11. *The Palais-Smale condition is satisfied for $I^T_{w_0}$.*

Proof. Let $\{u_n\} \subseteq W_0^{1,2}(\Omega)$ be any Palais-Smale sequence, i.e., $I^T_{w_0}(u_n) \to \ell < \infty$ and $(I^T_{w_0})'(u_n) \to 0$ in dual space of $W_0^{1,2}(\Omega)$. On proceeding as in the proof of Lemma 3.2.4, we conclude that $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$. Since $W_0^{1,2}(\Omega)$ is a reflexive space, up to a subsequence there exists $u \in W_0^{1,2}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,2}(\Omega)$, $u_n(x) \to u(x)$ a.e. $x \in \Omega$, $u_n \to u$ in $L^p(\Omega)$ for $p \in [1, \infty)$.

As a consequence, $(I^T_{w_0})'(u_n)(u_n - u) \to 0$ as $n \to \infty$, i.e.,

$$\int_\Omega \nabla u_n(\nabla u_n - \nabla u)dx - \int_\Omega \frac{f_T(x, u_n, \nabla w)(u_n - u)}{A(x, \|w\|^2)}dx \to 0 \text{ as } n \to 0. \tag{3.15}$$

Further, (3.6) and (A_1) imply

$$\int_\Omega \frac{f_T(x, u_n, \nabla w)(u_n - u)}{A(x, \|w\|^2)}dx \leq \frac{a_2(T+2)^r}{a_0} \int_\Omega e^{\tau u_n^2}(u_n - u)dx.$$

Now, by Hölder's inequality and Theorem 2.4.1 (we choose τ such that $\tau p' \|u_n\|^2 \leq \alpha_2$), one gets

$$\int_\Omega \frac{f_T(x, u_n, \nabla w)(u_n - u)}{A(x, \|w\|^2)}dx \leq \frac{a_2(T+2)^r}{a_0} \left(\int_\Omega e^{p' \tau \|u_n\|^2 \frac{|u_n|^2}{\|u_n\|^2}} dx \right)^{\frac{1}{p'}} \left(\int_\Omega |u_n - u|^p dx \right)^{\frac{1}{p}}$$

$$= \frac{c_1'}{a_0} \|u_n - u\|_{L^p(\Omega)}$$
$$\to 0 \text{ as } n \to \infty \text{ (Since } u_n \to u \text{ in } L^p(\Omega)).$$

Hence, by using [115, Proposition 2.72]

$$u_n \to u \text{ in } W_0^{1,2}(\Omega) \text{ (up to a subsequence).}$$

This completes the proof. □

Lemma 3.2.12. *There exist positive real numbers α and ρ, independent of w, such that*

$$I^T_{w_0}(u) \geq \alpha > 0, \ \forall u \in W_0^{1,2}(\Omega) : \|u\| = \rho.$$

Proof. It follows from (f_0') and (f_2) that,

$$|F_T(x,t,\xi)| \leq \frac{1}{2}\varepsilon|t|^2 + a_3 e^{a_4 t^2}|t|^m \tag{3.16}$$

for some $\varepsilon > 0$, $a_3 > 0, a_4 > 0$, $m > 2$ and for all $(x,t,\xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2$.
By using (A_1) and (3.16), we obtain

$$I^T_{w_0}(u) \geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{a_0 2}\|u\|^2_{L^2(\Omega)} - \frac{a_3}{a_0}\int_\Omega |u|^m e^{a_4 u^2} dx.$$

On using, Sobolev embedding theorem and Hölder's inequality (we will choose $\|u\| \leq \sigma$ such that $pa_4\sigma^2 \leq \alpha_2$), we get

$$I^T_{w_0}(u) \geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon a_5}{a_0 2}\|u\|^2 - \frac{a_3}{a_0}\left(\int_\Omega e^{pa_4\|u\|^2 \frac{|u|^2}{\|u\|^2}} dx\right)^{1/p} \|u\|^m_{L^{p'm}(\Omega)}.$$

By Theorem 2.4.1 and Sobolev embedding theorem, we get

$$I^T_{w_0}(u) \geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon a_5}{a_0 2}\|u\|^2 - \frac{a_6}{a_0}\|u\|^m.$$

Now choose ε such as $\frac{1}{2} - \frac{a_5\varepsilon}{a_0 2} > 0$ and choose $\rho > 0$ sufficiently small such that

$$\alpha = \left(\frac{1}{2} - \frac{a_5\varepsilon}{a_0 2}\right)\rho^2 - \frac{a_6}{a_0}\rho^m > 0.$$

This completes the proof. □

Lemma 3.2.13. *There exist $u_1 \in W_0^{1,2}(\Omega)$ and positive real number β independent of w such that*

$$I_{w_0}^T(u_1) < 0 \text{ and } \|u_1\| > \beta.$$

Proof. The proof is similar to the proof of the Lemma 3.2.6. We omit the details. □

Theorem 3.2.14. *Suppose that the conditions (f_0'), $(f_1)-(f_3)$ and (A_1) are satisfied. Then the Problem* (3.4) *has positive weak solution with $\lambda = 0$.*

Proof. By Lemmas 3.2.11, 3.2.12 and 3.2.13 all the conditions of mountain pass theorem are satisfied for the functional $I_{w_0}^T$. Hence, there exists $u \in W_0^{1,2}(\Omega)$ such that $I_{w_0}^T(u) = c_0 (c_0 \geq \alpha)$ and $(I_{w_0}^T)'(u) = 0$ in $W_0^{1,2}(\Omega)^*$, where $W_0^{1,2}(\Omega)^*$ is the dual space of $W_0^{1,2}(\Omega)$. On using the arguments similar to the proof of Lemma 3.2.3, we conclude that $u > 0$ in Ω. □

Lemma 3.2.15. *Let u_w^T be the any mountain pass solution of the Problem* (3.4) *with $\lambda = 0$, obtained in the Theorem 3.2.14. Then there exist $k, k_0 > 0$ independent of w such that*

$$\|u_w^T\|_{L^\infty(\Omega)} \leq k \text{ and } \|u_w^T\|_{C^{1,\beta}(\Omega)} \leq k_0,$$

where $0 < \beta < 1$. Moreover, $\|\nabla u_w^T\|_{C^{0,\beta}(\Omega)} \leq T$ for sufficiently large T.

Proof. It is given that $(I_{w_0}^T)'(u_w^T) = 0$ and $I_{w_0}^T(u_w^T) = c_0$. Now, proceeding as Lemma 3.2.4, there exist $c_2' > 0$ independent of w and T such that $\|u_w^T\| \leq c_2'$. For any $p \in [1, \infty)$, we have

$$\begin{aligned}
\|f_T\|_{L^p(\Omega)} &\leq a_2 (T+2)^r \|e^{\tau |u_w^T|^2}\|_{L^p(\Omega)} \\
&\leq a_2 (T+2)^r e^{\tau |u_w^T|^2} |\Omega| \\
&= a_2 (T+2)^r e^{\left(\tau \|u_w^T\|^2 \frac{|u_w^T|^2}{\|u_w^T\|^2}\right)} |\Omega|.
\end{aligned}$$

Now, by Theorem 2.4.1 (we will choose τ such that $\tau p \|u_w^T\|^2 \leq \alpha_2$), one gets $\|f_T\|_{L^p(\Omega)} \leq c_3'(T+2)^r$. Now, by taking $p > 1$ and using Morrey's Theorem [50], one gets $\|u_w^T\|_{C^{1,\beta}(\Omega)} \leq c_4' \|u_w^T\|_{2,p}$ and by [81, Theorem 9.1.1], we have $\|u_w^T\|_{2,p} \leq c_5' \|f_T\|_{L^p(\Omega)}$. Therefore, $\|u_w^T\|_{C^{1,\beta}(\Omega)} \leq c_3' c_4' c_5' (T+2)^r = k_0$, for some $k_0 > 0$. Since, $r \in (0,1)$, by choosing sufficiently large T, we have $\|u_w^T\|_{C^{1,\beta}(\Omega)} \leq T$. Consequently, $\|u_w^T\|_{L^\infty(\Omega)} \leq k$, $\|u_w^T\|_{C^{0,\beta}(\Omega)} \leq T$ and $\|\nabla u_w^T\|_{C^{0,\beta}(\Omega)} \leq T$ for some $k > 0$ and for sufficiently large T. □

With help of the function h, we construct a new function h_k such that:

$$h_k(x,t) = \begin{cases} h(x,0) & if\ t \leq 0 \\ h(x,t) & if\ t \in [0, 2k] \\ h(x, 2k) & if\ t \geq 2k. \end{cases}$$

One can observe that h_k is continuous and bounded on $\overline{\Omega}\times\mathbb{R}$. Now, select a cut-off function $g:\mathbb{R}\to[0,1]$ such that $0\leq g\leq 1$ in $\mathbb{R}$, $g(t)=1$ for all $t\in\overline{B(0,2k)}$ and $g(t)=0$ for all $t\in\mathbb{R}-B(0,4k)$.

Now, we consider the truncated problem

$$\begin{cases} -\Delta u = \dfrac{f_T(x,u,\nabla w)+\lambda g(u)h_k(x,u)+\lambda g'(u)H_k(x,u)}{A(x,\|w\|^2)} & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{3.17}$$

where, $H_k(x,u)=\int_0^u h_k(x,t)dt$.
Corresponding to the Problem (3.17), consider the following C^1 functional:

$$I_\lambda^T(u)=\int_\Omega \frac{1}{2}|\nabla u|^2dx-\int_\Omega \frac{F_T(x,u,\nabla w)+\lambda g(u)H_k(x,u)}{A(x,\|w\|^2)}dx.$$

It is observed, the critical points of the functional I_λ^T are the solutions of the equation (3.17).

Theorem 3.2.16. *Suppose that the conditions (f_0'), $(f_1)-(f_3)$ and (A_1) are satisfied. Then there exists $\lambda_1>0$ such that the Problem (3.17) has a weak solution for all $|\lambda|<\lambda_1$.*

Proof. The proof is similar to the proof of Theorem 3.2.9. We omit the details. □

Lemma 3.2.17. *Let u_{λ_n} be any solution of the Problem (3.17) by replacing λ to λ_n obtained in the Theorem 3.2.16, where $\{\lambda_n\}$ is a sequence of real numbers which converge to zero. Then for every n, $\|\nabla u_{\lambda_n}\|_{C^{0,\beta}(\Omega)}\leq T$ for sufficiently large T. Moreover, up to a subsequence $\{u_{\lambda_n}\}$ converges to u in $C^{1,\beta}(\Omega)$ where $0<\beta<1$ and u is the critical point of $I_{w_\lambda}^T$.*

Proof. The proof is similar to Lemma 3.2.10. For the sake of brevity, we omit the details. □

Proof of the Theorem 3.2.2. The proof is similar to the proof of Theorem 3.2.1. For the sake of brevity, we omit the details. □

3.3 Proof of the Theorem 3.1.1

Proof. Let $u_0\in W_0^{1,2}(\Omega)\cap C^{1,\beta}(\Omega)$ be arbitrary. We construct a sequence $\{u_n\}\subseteq W_0^{1,2}(\Omega)$ satisfying

$$\begin{aligned} -\Delta u_n &= \frac{f(x,u_n,\nabla u_{n-1})+\lambda h(x,u_n)}{A(x,\|u_{n-1}\|^2)}, \quad \text{in } \Omega\\ u_n(x) &= 0, \qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega. \end{aligned} \tag{P_n}$$

Moreover, $\|\nabla u_n\|_{C^{0,\beta}(\Omega)} \leq k_1$ and $\|u_n\|_{C^{0,\beta}(\Omega)} \leq k_1$. Since, u_{n+1} is the weak solution of (P_{n+1}), we have

$$\int_\Omega \nabla u_{n+1} \nabla \varphi \, dx = \int_\Omega \frac{f(x,u_{n+1},\nabla u_n)\ \varphi \, dx}{A(x,\|u_n\|^2)} + \lambda \int_\Omega \frac{h(x,u_{n+1})\ \varphi \, dx}{A(x,\|u_n\|^2)} \quad \forall \varphi \in W_0^{1,2}(\Omega). \tag{3.18}$$

Similarly, u_n is the weak solution of (P_n), we have

$$\int_\Omega \nabla u_n \nabla \varphi \, dx = \int_\Omega \frac{f(x,u_n,\nabla u_{n-1})\ \varphi \, dx}{A(x,\|u_{n-1}\|^2)} + \lambda \int_\Omega \frac{h(x,u_n)\ \varphi \, dx}{A(x,\|u_{n-1}\|^2)} \quad \forall \varphi \in W_0^{1,2}(\Omega). \tag{3.19}$$

By subtracting (3.19) from (3.18), using (2.1) and taking $\varphi = u_{n+1} - u_n$, we get

$$\begin{aligned} C_N\|u_{n+1}-u_n\|^2 \leq & \int_\Omega \frac{f(x,u_{n+1},\nabla u_n)\ (u_{n+1}-u_n)\, dx}{A(x,\|u_n\|^2)} \\ & + \lambda \int_\Omega \frac{h(x,u_{n+1})\ (u_{n+1}-u_n)\, dx}{A(x,\|u_n\|^2)} \\ & - \int_\Omega \frac{f(x,u_n,\nabla u_{n-1})\ (u_{n+1}-u_n)\, dx}{A(x,\|u_{n-1}\|^2)} \\ & - \lambda \int_\Omega \frac{h(x,u_n)\ (u_{n+1}-u_n)\, dx}{A(x,\|u_{n-1}\|^2)}. \end{aligned}$$

By some simple manipulations, we have

$$\begin{aligned} C_N\|u_{n+1}-u_n\|^2 \leq & \int_\Omega [f(x,u_{n+1},\nabla u_n) \\ & - f(x,u_n,\nabla u_{n-1})] \left(\frac{1}{A(x,\|u_n\|^2)}\right)(u_{n+1}-u_n)dx \\ & + \int_\Omega f(x,u_n,\nabla u_{n-1}) \\ & \quad \left(\frac{|A(x,\|u_{n-1}\|^2)-A(x,\|u_n\|^2)|}{A(x,\|u_n\|^2)A(x,\|u_{n-1}\|^2)}\right)(u_{n+1}-u_n)dx \\ & + \lambda \int_\Omega [h(x,u_{n+1})-h(x,u_n)] \left(\frac{1}{A(x,\|u_n\|^2)}\right)(u_{n+1}-u_n)dx \\ & + \lambda \int_\Omega h(x,u_n) \left(\frac{|A(x,\|u_{n-1}\|^2)-A(x,\|u_n\|^2)|}{A(x,\|u_n\|^2)A(x,\|u_{n-1}\|^2)}\right)(u_{n+1}-u_n)dx. \end{aligned}$$

By using $(A_1)-(A_2)$ and (H), we obtain

$$\begin{aligned}
C_N\|u_{n+1}-u_n\|^2 \leq \frac{1}{a_0}\int_\Omega & [f(x,u_{n+1},\nabla u_n) \\
& - f(x,u_n,\nabla u_n)](u_{n+1}-u_n)dx \\
& + \frac{1}{a_0}\int_\Omega [f(x,u_n,\nabla u_n) \\
& - f(x,u_n,\nabla u_{n-1})](u_{n+1}-u_n)dx \\
& + \frac{L_3}{a_0^2}\|u_n-u_{n-1}\|\int_\Omega f(x,u_n,\nabla u_{n-1})(u_{n+1}-u_n)dx \\
& + \frac{L_4\lambda}{a_0}\int_\Omega |u_{n+1}-u_n|(u_{n+1}-u_n)dx \\
& + \frac{L_3\lambda}{a_0^2}\|u_n-u_{n-1}\|\int_\Omega h(x,u_n)(u_{n+1}-u_n)dx.
\end{aligned}$$

Now, Hölder's inequality and (f_4) imply,

$$\begin{aligned}
C_N\|u_{n+1}-u_n\|^2 \leq & \frac{L_1}{a_0}\int_\Omega |u_{n+1}-u_n|^2dx + \frac{L_2}{a_0}\int_\Omega |\nabla u_n-\nabla u_{n-1}|(u_{n+1}-u_n)dx \\
& + \frac{L_3}{a_0^2}\|u_n-u_{n-1}\|\left(\int_\Omega (f(x,u_n,\nabla u_{n-1}))^2dx\right)^{1/2} \\
& \left(\int_\Omega |u_{n+1}-u_n|^2dx\right)^{1/2} + \frac{L_4\lambda}{a_0}\int_\Omega |u_{n+1}-u_n|^2dx \\
& + \frac{L_3\lambda}{a_0^2}\|u_n-u_{n-1}\|\left(\int_\Omega (h(x,u_n))^2dx\right)^{1/2} \\
& \left(\int_\Omega |u_{n+1}-u_n|^2dx\right)^{1/2}.
\end{aligned}$$

Again, using Hölder's inequality, Sobolev embedding theorem and (3.2), we have

$$\begin{aligned}
C_N\|u_{n+1}-u_n\|^2 \leq & \frac{L_1}{S_2^2a_0}\|u_{n+1}-u_n\|^2 + \frac{L_2}{S_2a_0}\|u_n-u_{n-1}\|\|u_{n+1}-u_n\| \\
& + \frac{L_3c_f|\Omega|^{1/2}}{S_2a_0^2}\|u_n-u_{n-1}\|\|u_{n+1}-u_n\| + \frac{L_4\lambda}{S_2^2a_0}\|u_{n+1}-u_n\|^2 \\
& + \frac{L_3\lambda c_h|\Omega|^{1/2}}{a_0^2S_2}\|u_n-u_{n-1}\|\|u_{n+1}-u_n\|dx.
\end{aligned}$$

On simplification, we have

$$(C_NS_2^2a_0^2-(L_1a_0+L_4\lambda a_0))\|u_{n+1}-u_n\|$$

$$\leq (L_2a_0S_2 + L_3c_f|\Omega|^{1/2}S_2 + \lambda L_3c_h|\Omega|^{1/2}S_2)\|u_n - u_{n-1}\|.$$

This follows,

$$\|u_{n+1} - u_n\| \leq e\|u_n - u_{n-1}\|,$$

where $e = \left(\frac{L_2a_0S_2 + L_3c_f|\Omega|^{1/2}S_2 + \lambda L_3c_h|\Omega|^{1/2}S_2}{C_NS_2^2a_0^2 - (L_1a_0 + L_4\lambda a_0)}\right)$. Since $e < 1$, one can have $\{u_n\}$ is a Cauchy sequence in $W_0^{1,2}(\Omega)$, hence $\{u_n\}$ converges to some function $u \in W_0^{1,2}(\Omega)$, which is a solution to the Problem (3.1). By [84, Lemma 2.9], we have $u > 0$ in Ω. □

3.4 Examples

In order to give a clear view of our hypotheses on the nonlinearity f, the non-local term A and the perturbation term h, we provide some examples. We can take f as following function

$$f(x,t,\xi) = \begin{cases} 0 & if\ t < 0, x \in \bar{\Omega}, \xi \in \mathbb{R}^N \\ c_1t^q\ (1+|\xi|^s) & if\ t \geq 0, x \in \bar{\Omega}, \xi \in \mathbb{R}^N \end{cases}$$

where, $c_1 = \frac{1}{q(1+k_2^s)k_2^q}$, $N \geq 2$, $q \in (1, 2^* - 1)$, $s \in (0, \frac{1}{q+1})$ and $k_2 = \max\{1, k_1\}$, here k_1 as obtained in Theorem 3.2.1.

It can be observed that f satisfies all the conditions (f_1') and $(f_1) - (f_4)$. Moreover,

$$|f(x,t_1,\xi) - f(x,t_2,\xi)| \leq |t_1 - t_2|$$

whenever $|t_1|, |t_2| < k_2$, $|\xi| \leq k_2$ and

$$|f(x,t,\xi_1) - f(x,t,\xi_2)| \leq |\xi_1 - \xi_2|$$

whenever $|t| < k_2$, $|\xi_1|, |\xi_2| \leq k_2$. Hence f is locally Lipschitz in second and third variables with Lipschitz constants $L_1 = 1$ and $L_2 = 1$, respectively. Additionally, $|f(x,t,\xi)| \leq c_1(1+k_2^s)k_2^q \leq 1 = c_f$ whenever $|t| < k_2, |\xi| \leq k_2$.

The model for A and h is given below:

$$A(x,t) = c_2(\sin(\sqrt{t}) + 1 + ((k_2+2)\lambda + 1)|\Omega|^{1/2}S_2) \text{ and } h(x,t) = \sqrt{t^2+4},$$

where $c_2 = \frac{a_0}{((k_2+2)\lambda+1)|\Omega|^{1/2}S_2}$ and $a_0 > \frac{S_2+\lambda+2}{C_NS_2^2}$.

Furthermore,

$$|A(x,t_1^2) - A(x,t_2^2)| \leq c_2|\sin(t_1) - \sin(t_2)| \leq c_2|t_1 - t_2|$$

and

$$|h(x,t_1) - h(x,t_2)| \leq |t_1 - t_2|$$

whenever $|t_1|, |t_2| < k_2$. Therefore, A is locally Lipschitz in second variable with Lipschitz constant $L_3 = c_2$ and h is locally Lipschitz in second variable with Lipschitz constant $L_4 = 1$. Moreover,

$$a_0 \leq A(x,t) \leq a_1 = \frac{2a_0}{((k_2+2)\lambda + 1)|\Omega|^{1/2}S_2} + a_0$$

and

$$|h(x,t)| \leq \sqrt{(k_2^2+4)} < k_2 + 2 = c_h, \text{ when } |t| < k_2.$$

Consequently, we get

$$0 < \left(\frac{L_2 a_0 S_2 + L_3 c_f |\Omega|^{1/2} S_2 + \lambda L_3 c_h |\Omega|^{1/2} S_2}{C_N S_2^2 a_0^2 - (L_1 a_0 + L_4 \lambda a_0)} \right) = \frac{S_2 + 1}{C_N S_2^2 a_0 - 1 - \lambda} < 1,$$

since $a_0 > \frac{S_2+\lambda+2}{C_N S_2^2}\cdot$ Hence, all the conditions of Theorem 3.1.1 are satisfied.

Chapter 4

N-Kirchhoff equations with critical exponential growth

This chapter[1] consists of two main sections. Section 4.1, deals with the existence and multiplicity result in the bounded domain. In addition, the existence result is demonstrated without using the Ambrosetti-Rabinowitz condition, and the existence of the ground state solution is addressed in Section 4.1. Section 4.2 deals with the existence and multiplicity result in the whole $\mathbb{R}^N$.

4.1 A problem in bounded domain

In this section, we establish the existence and multiplicity results of weak solutions for the non-local problem:

$$\begin{cases} -a\left(\int_\Omega |\nabla u|^N dx\right)\Delta_N u &= \dfrac{f(x,u)}{|x|^b} + \lambda h(x) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain which contains the origin, $0 \leq b < N$, $N \geq 2$, $\Delta_N = \operatorname{div}(|\nabla u|^{N-2}\nabla u)$ is the N-Laplace operator, λ is a suitably small real parameter and the perturbation term $h > 0$ belongs to the dual of some suitable Sobolev space.

4.1.1 Hypotheses

We assume the following growth condition on the nonlinearity f:

(L_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and has a critical exponential growth at infinity, i.e., there exists $\alpha_0 > 0$ such that

[1]Shilpa Gupta, Gaurav Dwivedi, Existence and multiplicity of solutions to N-Kirchhoff equations with critical exponential growth and perturbation term, *Complex Variables and Elliptic Equations*, 2022.
Shilpa Gupta, Gaurav Dwivedi, Ground state solution for N-Kirchhoff equation with critical exponential growth without Ambrosetti-Rabinowitz condition, *Rendiconti del Circolo Matematico di Palermo Series 2*, 2023.

$$\lim_{t\to\infty}\frac{f(x,t)}{e^{\alpha|t|^{N/(N-1)}}}=0\ \forall\ \alpha>\alpha_0 \text{ and } \lim_{t\to\infty}\frac{f(x,t)}{e^{\alpha|t|^{N/(N-1)}}}=\infty\ \forall\ \alpha<\alpha_0.$$

(L_2) $\lim_{t\to 0}\dfrac{f(x,t)}{t^{N-1}}=0,\ \ \forall\ x\in\bar{\Omega}.$

(L_3) $f(x,t)=0$ for all $x\in\Omega, t\leq 0$

(L_4) $\lim_{t\to\infty}\dfrac{tf(x,t)}{e^{\alpha_0 t^{N/(N-1)}}}\geq\beta_0>0\ \forall\ x\in\bar{\Omega}$,
for some $\beta_0>\dfrac{N-b}{Nd^{N-b}}\left(\dfrac{N-b}{\alpha_0}\right)^{N-1} a\left(\left(\dfrac{N-b}{N}\dfrac{\alpha_N}{\alpha_0}\right)^{N-1}\right)$, where d is the radius of the ball $B(0,d)$ such that it is the largest ball contained in Ω.

(L_5) Ambrosetti-Rabinowitz condition is true, i.e., there exists $\sigma>N\theta$ such that

$$0<\sigma F(x,t)=\sigma\int_0^t f(x,s)ds\leq tf(x,t),$$

for all $t>0,\ x\in\bar{\Omega}$.

It is noted from (L_1) that the nonlinear function f has critical exponential growth, which is motivated by the celebrated result of Trudinger [144].

Next, we state our hypotheses on the non-local term a. The function $a:\mathbb{R}^+\to\mathbb{R}^+$ is continuous and satisfies the following conditions:

(a_1) There exists a positive real number a_0 such that $a(s)\geq a_0$ and a is non-decreasing $\forall\ s>0$.

(a_2) There exists $\theta>1$ such that $a(s)/s^{\theta-1}$ is non-increasing for $s>0$.

Remark 4.1.1. *By condition (a_2), we have*

(a_2') $\theta A(s)-a(s)s$ *is non-decreasing,* $\forall\ s>0$, *where* $A(s)=\int_0^s a(t)dt$.

In particular,

$$\theta A(s)-a(s)s\geq 0\ \ \forall\ s>0. \tag{4.2}$$

Again by (a_2) and (4.2), one get

(a_2'') $A(s)\leq s^\theta A(1)\ \forall\ s\geq 1.$

Corresponding to the problem (4.1), the perturbation term $h\in(W_0^{1,N}(\Omega))^*$. The space $(W_0^{1,N}(\Omega))^*$ is norm space with the norm $\|.\|_*$.

We discuss the existence of weak solution for the Problem (4.1) in the Sobolev space $W_0^{1,N}(\Omega)$, which is equipped with the norm:

$$\|u\| = \left(\int_\Omega |\nabla u|^N dx\right)^{\frac{1}{N}}.$$

Notation:

We denote,

$$\lambda_1(N) = \inf_{u\in W_0^{1,N}(\Omega)\setminus\{0\}} \left\{ \frac{\|u\|^N}{\int_\Omega \frac{|u|^N}{|x|^b} dx} \right\} > 0 \text{ for any } 0 \le b < N. \tag{4.3}$$

4.1.2 Variational framework and main results

Definition 4.1.2. We say that $u \in W_0^{1,N}(\Omega)$ is a weak solution of (4.1) if the following holds:

$$a(\|u\|^N)\int_\Omega |\nabla u|^{N-2}\nabla u\,\nabla\varphi\,dx = \int_\Omega \frac{f(x,u)\ \varphi}{|x|^b}dx + \lambda\int_\Omega h(x)\varphi\,dx, \tag{4.4}$$

for all $\varphi \in W_0^{1,N}(\Omega)$.

Thus the energy functional $I_\lambda : W_0^{1,N}(\Omega) \to \mathbb{R}$ corresponding to (4.4) is given by

$$I_\lambda(u) = \frac{1}{N}A(\|u\|^N) - \int_\Omega \frac{F(x,u)}{|x|^b}dx - \lambda\int_\Omega h(x)u\,dx,$$

where $F(x,t) = \int_0^t f(x,s)ds$ and $A(t) = \int_0^t a(s)ds$. It can be seen that I_λ is C^1 and the derivative of I_λ at any point $u \in W_0^{1,N}(\Omega)$ is given by

$$I'_\lambda(u)(v) = a(\|u\|^N)\int_\Omega |\nabla u|^{N-2}\nabla u\,\nabla v\,dx - \int_\Omega \frac{f(x,u)\ v}{|x|^b}dx - \lambda\int_\Omega h(x)v\,dx.$$

Moreover, the critical points of I_λ are the weak solutions to (4.1).

More precisely, we prove the following:

Theorem 4.1.3. *Suppose that the conditions $(L_1)-(L_5)$ and $(a_1)-(a_2)$ are satisfied. Then there exists $\lambda_2 > 0$, such that for each $0 < \lambda < \lambda_2$, the Problem (4.1) has a non-trivial weak solution.*

Theorem 4.1.4. *Suppose $(L_1)-(L_5)$ and $(a_1)-(a_2)$ hold; then there exists $\lambda_3 > 0$, such that for each $0 < \lambda < \lambda_3$, the Problem (4.1) has a non-trivial minimum type solution with negative* *energy. Moreover, the solutions of the Problem* (4.1) *obtained in this theorem and Theorem 4.1.3 are different.*

4.1.3 Proof of the Theorems 4.1.3 and 4.1.4

Lemma 4.1.5. *Suppose* $(L_1)-(L_2)$ *and* (a_1) *hold. Then there exists* $\lambda_0 > 0$ *such that for each* $0 < \lambda < \lambda_0$*, there are positive real numbers* η *and* ρ_λ *such that*

$$I_\lambda(u) \geq \eta > 0, \ \ \forall u \in W_0^{1,N}(\Omega) \text{ such that } \|u\| = \rho_\lambda.$$

Moreover, one can choose ρ_λ *such that* $\rho_\lambda \to 0$ *as* $\lambda \to 0$.

Proof. It follows from (L_1) and (L_2) that, for a given $\varepsilon > 0$ and $s > N$, $\exists\, c_1 > 0$ (depending on ε and s) such that,

$$|F(x,t)| \leq \frac{1}{N}\varepsilon|t|^N + c_1|t|^s e^{\alpha|t|^{N/(N-1)}} \tag{4.5}$$

for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$. By using (4.5), we obtain

$$I_\lambda(u) \geq \frac{1}{N}A(\|u\|^N) - \frac{\varepsilon}{N}\int_\Omega \frac{|u|^N}{|x|^b}dx - c_1\int_\Omega \frac{|u|^s e^{\alpha|u|^{N/(N-1)}}}{|x|^b}dx - \lambda\int_\Omega h(x)u\, dx.$$

On using (a_1), (4.3) and Hölder's inequality (we will choose $r > 1$ sufficiently close to 1 and $\|u\| \leq K$ such that $r(\alpha K^{N/(N-1)}/\alpha_N + b/N) < 1$), one get

$$\begin{aligned} I_\lambda(u) \geq \frac{a_0\|u\|^N}{N} - \frac{\varepsilon\|u\|^N}{N\lambda_1(N)} - c_1\left(\int_\Omega \frac{e^{r\alpha\|u\|^{N/(N-1)}\left(\frac{|u|}{\|u\|}\right)^{\frac{N}{N-1}}}}{|x|^{br}}dx\right)^{\frac{1}{r}}\left(\int_\Omega |u|^{sr'}dx\right)^{\frac{1}{r'}} \\ - \lambda\|h\|_*\|u\|. \end{aligned} \tag{4.6}$$

Then, by Lemma 2.4.2 and Sobolev embedding theorem, we have

$$I_\lambda(u) \geq \left(\frac{a_0}{N}\|u\|^{N-1} - \frac{\varepsilon}{N\lambda_1(N)}\|u\|^{N-1} - c_2\|u\|^{s-1} - \lambda\|h\|_*\right)\|u\|. \tag{4.7}$$

Now choose ε such as $\frac{a_0}{N} - \frac{\varepsilon}{N\lambda_1(N)} > 0$ and $\rho_\lambda > 0$ sufficiently small such that $\rho_\lambda < K$ and

$$\eta = \left(\frac{a_0}{N} - \frac{\varepsilon}{N\lambda_1(N)}\right)\rho_\lambda^{N-1} - c_2\rho_\lambda^{s-1} > 0.$$

This implies, there exists $\lambda_0 > 0$ such that for each $0 < \lambda < \lambda_0$ we have

$$I_\lambda(u) \geq \eta > 0, \;\; \forall u \in W_0^{1,N}(\Omega) \text{ such that } \|u\| = \rho_\lambda.$$

Moreover, we can chose ρ_λ such that $\rho_\lambda \to 0$ as $\lambda \to 0$. This completes the proof. □

Lemma 4.1.6. *Assume that (L_5) and (a_2) hold. There exist $v_0 \in W_0^{1,N}(\Omega)$ and positive real number β such that*

$$I_\lambda(v_0) < 0 \;\; \textit{and} \; \|v_0\| > \beta.$$

Proof. Let $u \in W_0^{1,N}(\Omega)\backslash\{0\}$ and $u \geq 0$ with compact support $K_1 \subseteq \Omega$. By (L_5), there exist $b_1, b_2 > 0$ such that

$$F(x,s) \geq b_1 s^\sigma - b_2, \;\; \forall\, (x,s) \in K_1 \times [0,\infty). \tag{4.8}$$

By (a_2'') and (4.8), we have

$$\begin{aligned} I_\lambda(tu) &= \frac{1}{N}A(\|tu\|^N) - \int_\Omega \frac{F(x,tu)}{|x|^b}dx - \lambda \int_\Omega h(x)tu\,dx \\ &\leq \frac{A(1)}{N}\|tu\|^{N\theta} - b_1 t^\sigma \int_{K_1} \frac{u^\sigma}{|x|^b}\,dx + b_2 \int_{K_1} \frac{dx}{|x|^b} - \lambda t \int_\Omega h(x)u\,dx \;\forall\, t > 1, \end{aligned}$$

this implies that $I_\lambda(tu) \to -\infty$ as $t \to \infty$, since $\sigma > N\theta$ and $h > 0$. Now, by setting $v_0 = tu$ for sufficiently large t, we get the desired result. □

Lemma 4.1.7. *There exist $\kappa > 0$ and $\vartheta_0 \in W_0^{1,N}(\Omega)$ with $\|\vartheta_0\| = 1$ such that*

$$I_\lambda(t\vartheta_0) < 0 \;\; \textit{for all } t \in [0,\kappa).$$

Moreover,

$$\inf_{\|u\| \leq \kappa} I_\lambda(u) < 0.$$

Proof. Choose $\vartheta_0 \in W_0^{1,N}(\Omega)\backslash\{0\}$ such that $\vartheta_0 \geq 0$ with $\|\vartheta_0\| = 1$. Then, for all $t \geq 0$, we have

$$\frac{d}{dt}(I_\lambda(t\vartheta_0)) = t^{N-1}a(t^N\|\vartheta_0\|^N) - \int_\Omega \frac{\vartheta_0 f(x,t\vartheta_0)}{|x|^b}dx - \lambda \int_\Omega h(x)\vartheta_0\,dx.$$

This implies $(d/dt)(I_\lambda(t\vartheta_0)) < 0$ at $t = 0$, since $h > 0$. Next, by using the continuity property of I_λ', there exists $\kappa > 0$ such that $(d/dt)(I_\lambda(t\vartheta_0)) < 0$ for all $t \in [0,\kappa)$. Consequently, $I_\lambda(t\vartheta_0)$ is strictly decreasing for all $t \in [0,\kappa)$. Since $I_\lambda(0) = 0$, we have $I_\lambda(t\vartheta_0) < 0$ for all $t \in [0,\kappa)$. Moreover, we have

$$\inf_{\|u\| \leq \kappa} I_\lambda(u) \leq \inf_{t \in [0,\kappa)} I_\lambda(t\vartheta_0) < 0,$$

which completes the proof of the lemma. □

Remark 4.1.8. *By* (4.7) *and Lemma 4.1.7, we have*

$$-\infty < c_\rho = \inf_{\|u\|\le\rho} I_\lambda(u) < 0.$$

By Lemmas 4.1.5 and 4.1.6, the geometric conditions of the mountain pass theorem are satisfied for the functional I_λ. Hence, by the version of the mountain pass theorem without (PS) condition, there exists a sequence $\{u_n\} \subseteq W_0^{1,N}(\Omega)$ such that $I_\lambda(u_n) \to c_M$ and $I'_\lambda(u_n) \to 0$ as $n \to \infty$, where

$$c_M = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)) > 0,$$

and

$$\Gamma = \{\gamma \in C([0,1], W_0^{1,N}(\Omega)) : \gamma(0) = 0,\ \gamma(1) < 0\}.$$

Due to the lack of compactness, we are not able to prove directly that (PS) condition holds for I_λ. We need some extra information for the mountain pass level c_M. Hence, we consider Green's function, $m_n : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ given by Moser [114] as:

$$m_n(x) = \frac{1}{\omega_{N-1}^{1/N}} \begin{cases} (\log n)^{(N-1)/N} & if\ |x| \le \frac{d}{n}, \\ \frac{\log \frac{d}{|x|}}{(\log n)^{1/N}} & if\ \frac{d}{n} \le |x| \le d, \\ 0 & if\ |x| \ge d, \end{cases}$$

where d is the radius of the ball $B(0,d)$ such that it is the largest ball contained in Ω.

Remark 4.1.9. *One can observe that, $m_n(x) \in W_0^{1,N}(\Omega)$ and $\|m_n(x)\| = 1$. Also, support of $m_n(x)$ contained in $\overline{B(0,d)}$.*

Lemma 4.1.10. *Suppose (L_4) and $(a_1)-(a_2)$ hold. There exists $n \in \mathbb{N}$ such that*

$$\max_{t\ge0}\left\{\frac{1}{N}A(t^N) - \int_\Omega \frac{F(x,tm_n)}{|x|^b}dx\right\} < \frac{1}{N}A\left(\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}\right).$$

Proof. Let us contrary that, for all $n \in \mathbb{N}$, we have

$$\max_{t\ge0}\left\{\frac{1}{N}A(t^N) - \int_\Omega \frac{F(x,tm_n)}{|x|^b}dx\right\} \ge \frac{1}{N}A\left(\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}\right).$$

By (4.8) and (a''_2), for each n there exists $t_n > 0$ such that

$$\frac{1}{N}A(t_n^N) - \int_\Omega \frac{G(x,t_n g_n)}{|x|^b}dx = \max_{t\ge0}\left\{\frac{1}{N}A(t^N) - \int_\Omega \frac{G(x,tg_n)}{|x|^b}dx\right\}.$$

Therefore,

$$\frac{1}{N}A(t_n^N) - \int_\Omega \frac{F(x,t_n m_n)}{|x|^b}dx \geq \frac{1}{N}A\left(\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}\right).$$

By using (a_1) and the fact that $F(x,t) \geq 0$, we have

$$t_n^N \geq \left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \tag{4.9}$$

Next, we claim that

$$t_n^N \to \left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Since at $t = t_n$, we have

$$\frac{d}{dt}\left(\frac{1}{N}A(t_n^N) - \int_\Omega \frac{F(x,t_n m_n)}{|x|^b}dx\right) = 0,$$

it follows that

$$t_n^N a(t_n^N) = \int_\Omega \frac{t_n m_n f(x,t_n m_n)}{|x|^b}dx = \int_{|x|\leq d} \frac{t_n m_n f(x,t_n m_n)}{|x|^b}dx. \tag{4.10}$$

By (L_4), for a given $\tau > 0$, $\exists\, \rho_\tau > 0$ such that

$$tf(x,t) \geq (\beta_0 - \tau)e^{\alpha_0 |t|^{N/(N-1)}},\ \forall (x,t) \in \overline{B(0,d)} \times [\rho_\tau, \infty). \tag{4.11}$$

It follows from (4.11), for large n, that

$$t_n^N a(t_n^N) \geq (\beta_0 - \tau)\int_{|x|\leq d/n} \frac{\exp(\alpha_0 |t_n^{N/(N-1)} \omega_{N-1}^{-1/(N-1)} \log n|)}{|x|^b}dx.$$

From (4.2) and (a_2''), we have

$$t_n^{N\theta}A(1)\theta \geq (\beta_0 - \tau)e^{\alpha_0 |t_n^{N/(N-1)} \omega_{N-1}^{-1/(N-1)} \log n|}\int_{|x|\leq d/n} \frac{1}{|x|^b}dx.$$

Therefore, we have

$$A(1)\theta \geq (\beta_0 - \tau)\frac{\omega_{N-1}}{N-b}d^{N-b}\exp(G_n), \tag{4.12}$$

where $G_n = \frac{N\alpha_0 \log n}{\alpha_N} t_n^{N/(N-1)} - (N\theta)\log t_n - (N-b)\log n$. This implies $\{t_n\}$ is bounded. If not, upto a subsequence $G_n \to \infty$, which is a contradiction to (4.12). Also, we have

$$A(1)\theta t_n^{N\theta} \geq (\beta_0 - \tau)\frac{\omega_{N-1}}{N-b} d^{N-b} \exp\left[\left(\frac{N\alpha_0 t_n^{\frac{N}{N-1}}}{\alpha_N} - (N-b)\right)\log n\right]. \tag{4.13}$$

By the help of (4.9), (4.13) and using the fact that $\{t_n\}$ is bounded, our claim

$$t_n^N \to \left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1} \tag{4.14}$$

is proved. Next, we construct two sets

$$X_n = \{x \in \overline{B(0,d)} : t_n m_n \geq \rho_\tau\} \text{ and } Y_n = \overline{B(0,d)} - X_n.$$

By (4.10) and (4.11), we have

$$\begin{aligned} t_n^N a(t_n^N) \geq &(\beta_0 - \tau)\int_{|x|\leq d} \frac{e^{\alpha_0 |t_n m_n|^{N/(N-1)}}}{|x|^b} dx + \int_{Y_n} \frac{t_n m_n f(x, t_n m_n)}{|x|^b} dx \\ &- (\beta_0 - \tau)\int_{Y_n} \frac{e^{\alpha_0 |t_n m_n|^{N/(N-1)}}}{|x|^b} dx. \end{aligned} \tag{4.15}$$

By using the fact that $m_n \to 0$ and the characteristic function $\chi_{Y_n} \to 1$ a.e. in $\overline{B(0,d)}$, we have

$$\int_{Y_n} \frac{t_n m_n f(x, t_n m_n)}{|x|^b} dx \to 0$$

and

$$\int_{Y_n} \frac{e^{\alpha_0 |t_n m_n|^{N/(N-1)}}}{|x|^b} dx \to \frac{\omega_{N-1}}{N-b} d^{N-b}.$$

Next, we have,

$$\int_{|x|\leq d} \frac{e^{\alpha_0 |t_n m_n|^{N/(N-1)}}}{|x|^b} dx = \int_{|x|\leq d/n} \frac{e^{\alpha_0 |t_n m_n|^{N/(N-1)}}}{|x|^b} dx + \int_{d/n\leq |x|\leq d} \frac{e^{\alpha_0 |t_n m_n|^{N/(N-1)}}}{|x|^b} dx.$$

By the definition of m_n, we have

$$\int_{|x|\leq d/n} \frac{e^{\alpha_0 |t_n m_n|^{N/(N-1)}}}{|x|^b} dx \to \frac{\omega_{N-1}}{N-b} d^{N-b}.$$

Now, proceeding as [137, Lemma 4.2], we have

$$\int_{d/n\le|x|\le d}\frac{e^{\alpha_0|t_n m_n|^{N/(N-1)}}}{|x|^b}dx \to N\frac{\omega_{N-1}}{N-b}d^{N-b}.$$

Hence, taking $n\to\infty$ in (4.15) and using (4.14), we have

$$\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1} a\left(\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}\right) \ge (\beta_0-\tau)\frac{\omega_{N-1}}{N-b}d^{N-b}N,$$

consequently,

$$\beta_0 \le \frac{N-b}{Nd^{N-b}}\left(\frac{N-b}{\alpha_0}\right)^{N-1} a\left(\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}\right),$$

which is contradiction to (L_4). This completes the proof. □

Lemma 4.1.11. *[137] Let $\{u_n\}\subseteq W_0^{1,N}(\Omega)$ be a sequence such that $u_n \rightharpoonup u_0$ in $W_0^{1,N}(\Omega)$ and $\frac{f(x,u_n)}{|x|^b}\to\frac{f(x,u_0)}{|x|^b}$ in $L^1(B_R)$ for any $R>0$, then*

$$\frac{F(x,u_n)}{|x|^b}\to\frac{F(x,u_0)}{|x|^b} \text{ in } L^1(\mathbb{R}^N).$$

Lemma 4.1.12. *Suppose (L_5) and $(a_1)-(a_2)$ hold. If $\{u_n\}\subset W_0^{1,N}(\Omega)$ is a $(PS)_c$ sequence of I_λ with weak limit u_0 and*

$$c<\frac{1}{N}A\left(\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}\right) \tag{4.16}$$

then there exists $\lambda_1>0$, such that for each $0<\lambda<\lambda_1$,

$$\lim_{n\to\infty}\|u_n\|^N-\|u_0\|^N<\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Proof. Let us assume on the contrary that, $\lim_{n\to\infty}\|u_n\|^N-\|u_0\|^N\ge\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}$. First, we claim that $\{u_n\}$ is a bounded sequence. Since $\{u_n\}$ is a $(PS)_c$ sequence of I_λ, we have $I_\lambda(u_n)\to c$ and $I'_\lambda(u_n)\to 0$ as $n\to\infty$, i.e.,

$$\frac{1}{N}A(\|u_n\|^N)-\int_\Omega\frac{F(x,u_n)}{|x|^b}dx-\int_\Omega\lambda h(x)u_n dx = c+\delta_n, \tag{4.17}$$

where $\delta_n\to 0$ as $n\to\infty$ and

$$\left|a(\|u_n\|^N)\int_\Omega|\nabla u_n|^{N-2}\,\nabla u_n\,\nabla\varphi\,dx-\int_\Omega\frac{f(x,u_n)\;\varphi}{|x|^b}dx-\int_\Omega\lambda h(x)\varphi\,dx\right|\le\varepsilon_n\|\varphi\|, \tag{4.18}$$

$\forall \varphi \in W_0^{1,N}(\Omega)$, where $\varepsilon_n \to 0$ as $n \to \infty$. On taking $\phi = u_n$, by (4.17), (4.18) and using (L_5), we obtain

$$\left(\frac{A(\|u_n\|^N)}{N} - \frac{a(\|u_n\|^N)\|u_n\|^N}{\sigma}\right) + \left(\frac{1-\sigma}{\sigma}\right)\lambda\|h\|_*\|u_n\| \leq c_3(1+\|u_n\|),$$

for some $c_3 > 0$. It follows from (a_1) and (4.2) that

$$a_0\left(\frac{1}{N\theta} - \frac{1}{\sigma}\right)\|u_n\|^N + \left(\frac{1-\sigma}{\sigma}\right)\lambda\|h\|_*\|u_n\| \leq c_3(1+\|u_n\|).$$

Consequently, there exists $\lambda_1^* > 0$, such that for each $0 < \lambda < \lambda_1^*$, we have

$$\|u_n\| \leq C,\ \int_\Omega \frac{f(x,u_n)u_n}{|x|^b}dx \leq C,\ \int_\Omega \frac{F(x,u_n)}{|x|^b}dx \leq C$$

for some $C > 0$. This implies, $\{u_n\}$ is bounded in $W_0^{1,N}(\Omega)$. As $W_0^{1,N}(\Omega)$ is a reflexive space, $\exists\, u_0 \in W_0^{1,N}(\Omega)$ such that up to a subsequence, we have

$$\begin{aligned} u_n &\rightharpoonup u_0 \text{ weakly in } W_0^{1,N}(\Omega),\\ u_n(x) &\to u_0(x) \ a.e.\ x \in \Omega,\\ u_n &\to u_0 \text{ in } L^q(\Omega) \text{ for } q \in [1,\infty). \end{aligned}$$

Further, proceeding in the similar ways to [155, Lemma 4.4] and [65, Lemma 2.7], we have

$$\frac{f(x,u_n)}{|x|^b} \to \frac{f(x,u_0)}{|x|^b} \text{ in } L^1(\Omega)$$

$$|\nabla u_n|^{N-2}\nabla u_n \rightharpoonup |\nabla u_0|^{N-2}\nabla u_0 \text{ weakly in } (L_{loc}^{N/(N-1)}(\Omega))^N.$$

By Lemma 4.1.11, we have

$$\frac{F(x,u_n)}{|x|^b} \to \frac{F(x,u_0)}{|x|^b} \text{ in } L^1(\Omega). \tag{4.19}$$

By passing the limits in (4.18), we get u_0 is weak solution of the problem

$$\begin{aligned} -a(E)\Delta_N u &= \frac{f(x,u)}{|x|^b} + \lambda h(x) \ \text{ in } \ \Omega,\\ u &= 0 \ \text{ on } \ \partial\Omega, \end{aligned} \tag{4.20}$$

where $E = \lim_{n\to\infty} \|u_n\|^N$. This implies,

$$a(E)E - \lim_{n\to\infty} \int_\Omega \frac{f(x,u_n)u_n}{|x|^b} dx - \lambda \int_\Omega h(x)u_0\, dx = 0 \tag{4.21}$$

and

$$a(E)\|u_0\|^N - \int_\Omega \frac{f(x,u_0)u_0}{|x|^b} dx - \lambda \int_\Omega h(x)u_0\, dx = 0. \tag{4.22}$$

By subtracting (4.22) from (4.21), one gets

$$a(E)(E - \|u_0\|^N) - \lim_{n\to\infty} \int_\Omega \left(\frac{f(x,u_n)u_n}{|x|^b} - \frac{f(x,u_0)u_0}{|x|^b} \right) dx = 0.$$

By using (a_1), we get

$$a(E - \|u_0\|^N)(E - \|u_0\|^N) - \lim_{n\to\infty} \int_\Omega \left(\frac{f(x,u_n)u_n}{|x|^b} - \frac{f(x,u_0)u_0}{|x|^b} \right) dx \leq 0.$$

Again using (a_1) and from supposition $E - \|u_0\|^N \geq \left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}$, we have

$$a\left(\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}\right)\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1} - \lim_{n\to\infty} \int_\Omega \left(\frac{f(x,u_n)u_n}{|x|^b} - \frac{f(x,u_0)u_0}{|x|^b} \right) dx \leq 0. \tag{4.23}$$

Next, by using (L_5), there exists $\lambda_1 > 0$ such that for each $0 < \lambda < \lambda_1 < \lambda_1^*$, we have

$$\int_\Omega \frac{1}{N\theta} \left(\frac{f(x,u_0)u_0}{|x|^b} - N\theta \frac{F(x,u_0)}{|x|^b} \right) dx - \left(1 - \frac{1}{N\theta}\right) \lambda \int_\Omega h(x)u_0\, dx \geq 0. \tag{4.24}$$

Again, by using the fact that $\{u_n\}$ is a $(PS)_c$ sequence of I_λ and by (4.19), we have

$$\begin{aligned}
c &= \lim_{n\to\infty} \left[I_\lambda(u_n) - \frac{1}{N\theta} I'_\lambda(u_n)(u_n) \right] \\
&= \frac{1}{N} A(E) - \frac{1}{N\theta} a(E)E + \lim_{n\to\infty} \int_\Omega \left(\frac{1}{N\theta} \frac{f(x,u_n)u_n}{|x|^b} - \frac{F(x,u_n)}{|x|^b} \right) dx \\
&\quad - \lim_{n\to\infty} \left(1 - \frac{1}{N\theta}\right) \int_\Omega h(x)u_n\, dx \\
&= \frac{1}{N} A(E) - \frac{1}{N\theta} a(E)E + \lim_{n\to\infty} \frac{1}{N\theta} \int_\Omega \left(\frac{f(x,u_n)u_n}{|x|^b} - \frac{f(x,u_0)u_0}{|x|^b} \right) dx \\
&\quad + \int_\Omega \frac{1}{N\theta} \left(\frac{f(x,u_0)u_0}{|x|^b} - N\theta \frac{F(x,u_0)}{|x|^b} \right) dx - \left(1 - \frac{1}{N\theta}\right) \lambda \int_\Omega h(x)u_0\, dx.
\end{aligned}$$

By using (4.24) and (a_2'), we obtain

$$c \geq \frac{1}{N\theta}\left\{\theta A\left(\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}\right) - a\left(\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}\right)\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}\right\} + \lim_{n\to\infty}\frac{1}{N\theta}\int_\Omega\left(\frac{f(x,u_n)u_n}{|x|^b} - \frac{f(x,u_0)u_0}{|x|^b}\right)dx$$

Next, by using (4.23), one obtains

$$c \geq \frac{1}{N}A\left(\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}\right),$$

which contradicts to (4.16). This completes the proof. □

Lemma 4.1.13. *Suppose $(L_1)-(L_2)$ and (a_1) holds. If $\{u_n\} \subset W_0^{1,N}(\Omega)$ is a $(PS)_c$ sequence of I_λ with weak limit u_0 and*

$$\lim_{n\to\infty}\|u_n\|^N - \|u_0\|^N < \left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1} \tag{4.25}$$

then $u_n \to u_0$ in $W_0^{1,N}(\Omega)$ up to subsequence.

Proof. First, we will prove that $\frac{|f(x,u_n)|^r}{|x|^b} \in L^1(\Omega)$, for some $r > 1$. Define

$$w_n = \frac{u_n}{\|u_n\|} \text{ and } w_0 = \frac{u_0}{E^{1/N}},$$

where $E = \lim_{n\to\infty}\|u_n\|^N$. One can observe that $\|w_n\| = 1$, $w_n \rightharpoonup w_0$ weakly in $W_0^{1,N}(\Omega)$. Without loss of generality, we have $0 < \|w_0\| < 1$. Therefore, $E = \frac{E-\|u_0\|^N}{1-\|w_0\|^N}\cdot$ By (4.25), we have

$$E < \frac{\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}}{1-\|w_0\|^N}.$$

This implies, $\|u_n\|^N < \left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}(1-\|w_0\|^N)^{-1}$ for sufficiently large n. Taking $r > 1$ (sufficiently close to 1) and $\alpha > \alpha_0$ (sufficiently close to α_0), we have

$$r\alpha\|u_n\|^{N/(N-1)} < \alpha_N\left(1-\frac{b}{N}\right)(1-\|w_0\|^N)^{-1/(N-1)}. \tag{4.26}$$

By using (L_1) and (L_2), there exists $c_4 > 0$ such that

$$|f(x,u_n)|^r \le c_4 \exp(r\alpha u_n^{N/(N-1)})$$

for some $\alpha > \alpha_0$ and $r > 1$. Therefore, for large n, we have

$$|f(x,u_n)|^r \le c_4 \exp(r\alpha \|u_n\|^{N/(N-1)} w_n^{N/(N-1)}).$$

Thus, using Lemma 2.4.7 and (4.26), one gets

$$\int_\Omega \frac{|f(x,u_n)|^r}{|x|^b} dx \le c_5, \tag{4.27}$$

for some constant $c_5 > 0$. Next, using Hölder's inequality (with exponents r and r' such that $1/r + 1/r' = 1$), we have

$$\int_\Omega \frac{f(x,u_n)(u_n-u_0)}{|x|^b} dx \le \left(\int_\Omega \frac{|f(x,u_n)|^r}{|x|^b} dx\right)^{1/r} \left(\int_\Omega \frac{|u_n-u_0|^{r'}}{|x|^b} dx\right)^{1/r'}. \tag{4.28}$$

Define two sets:

$$\Omega_1 = \{x \in \Omega : |x| < 1\} \text{ and } \Omega_2 = \{x \in \Omega : |x| \ge 1\}.$$

Using (4.27) in (4.28), we have

$$\int_\Omega \frac{f(x,u_n)(u_n-u_0)}{|x|^b} dx \le c_5^{1/r} \left(\int_{\Omega_1} \frac{|u_n-u_0|^{r'}}{|x|^b} dx + \int_{\Omega_2} \frac{|u_n-u_0|^{r'}}{|x|^b} dx\right)^{1/r'}.$$

Again using Hölder's inequality(with exponents s and s' such that $1/s + 1/s' = 1$ and $N > bs$), we get

$$\begin{aligned}
&\int_\Omega \frac{f(x,u_n)(u_n-u_0)}{|x|^b} dx \\
&\le c_5^{1/r} \left(\left(\int_{\Omega_1} |u_n-u_0|^{sr'} dx\right)^{1/s} \left(\int_{\Omega_1} \frac{1}{|x|^{bs'}} dx\right)^{1/s'} + \int_{\Omega_2} |u_n-u_0|^{r'} dx \right)^{1/r'} \\
&\le c_5^{1/r} \left(c_6 \|u_n-u_0\|^{r'}_{L^{sr'}(\Omega)} + \|u_n-u_0\|^{r'}_{L^{r'}(\Omega)} \right)^{1/r'}.
\end{aligned} \tag{4.29}$$

Since $\{u_n\}$ is a $(PS)_c$ sequence of I_λ, we have $I'_\lambda(u_n)(u_n-u_0) \to 0$ as $n \to \infty$, i.e.,

$$a(\|u_n\|^N) \int_\Omega |\nabla u_n|^{N-2} \nabla u_n \nabla(u_n-u_0)\, dx = \int_\Omega \frac{f(x,u_n)\,(u_n-u_0)}{|x|^b} dx$$

$$+\int_\Omega \lambda h(x)(u_n-u_0)\,dx+\varepsilon_n,$$

where $\varepsilon_n \to 0$ as $n \to \infty$. By the help of (4.29), we have

$$\begin{aligned} a(\|u_n\|^N)\int_\Omega |\nabla u_n|^{N-2}\,\nabla u_n\,\nabla(u_n-u_0)\,dx \\ \le c_5^{1/r}\left(c_6\|u_n-u_0\|^{r'}_{L^{sr'}(\Omega)}+\|u_n-u_0\|^{r'}_{L^{r'}(\Omega)}\right)^{1/r'} \\ +\lambda\|h\|_*\|u_n-u_0\|+\varepsilon_n. \end{aligned} \tag{4.30}$$

On the other side, since $u_n \rightharpoonup u_0$

$$\int_\Omega |\nabla u_0|^{N-2}\,\nabla u_0\,\nabla(u_n-u_0)\,dx \to 0. \tag{4.31}$$

Taking $n \to \infty$ in (4.30) and using Sobolev embedding theorem, (a_1), (2.1) and (4.31), one has

$$0 \le \lim_{n\to\infty} a_0\|u_n-u_0\|^N \le 0,$$

hence, $u_n \to u_0$ in $W_0^{1,N}(\Omega)$. □

Corollary 4.1.14. *Suppose $(L_1)-(L_5)$ and $(a_1)-(a_2)$ hold. If $\{u_n\} \subset W_0^{1,N}(\Omega)$ is a $(PS)_c$ of I_λ with weak limit u_0 and*

$$c < \frac{1}{N}A\left(\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}\right)$$

then there exists $\lambda_1 > 0$, such that for each $0 < \lambda < \lambda_1$, $u_n \to u_0$ in $W_0^{1,N}(\Omega)$ up to subsequence.

Proof. Proof follows from Lemmas 4.1.12 and 4.1.13. □

Now, we are ready to prove Theorems 4.1.3 and 4.1.4.

Proof of the Theorem 4.1.3. By Lemmas 4.1.5 and 4.1.6, the geometric conditions of the mountain pass theorem are satisfied for the functional I_λ. Hence, by the version of MPT without (PS) condition, there exists a $(\mathrm{PS})_{c_M}$ sequence, say, $\{u_n\}$ where

$$c_M = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)) > 0,$$

and

$$\Gamma = \{\gamma \in C([0,1],W_0^{1,N}(\Omega)) : \gamma(0)=0,\ \gamma(1)<0\}.$$

By Lemma 4.1.10, we have

$$c_M < \frac{1}{N} A\left(\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}\right).$$

Set $\lambda_2 = \min\{\lambda_0, \lambda_1\}$ (where λ_0 and λ_1 as in the Lemma 4.1.5 and Lemma 4.1.12 respectively); then proceeding as Lemma 4.1.12, $\{u_n\}$ converges weakly in $W_0^{1,N}(\Omega)$, say to, u_M. By Corollary 4.1.14, I_λ satisfy the $(\mathrm{PS})_{c_M}$ condition. Hence, u_M is the critical point of I_λ of level c_M, i.e., $I'_\lambda(u_M) = 0$ and $I_\lambda(u_M) = c_M$. Thus u_M is the weak solution of the problem (4.1). Furthermore, u_M is non-trivial solution since $h \neq 0$.

Proof of the Theorem 4.1.4. Let ρ_λ be as in Lemma 4.1.5, therefore $\rho_\lambda \to 0$ as $\lambda \to 0$. Hence, we will choose $0 < \lambda_3 \leq \lambda_2$ (where λ_2 as in the Theorem 4.1.3) such that for any $\lambda \in (0, \lambda_3)$,

$$\rho_\lambda < \left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1/N}.$$

As $\overline{B}_{\rho_\lambda}$ is convex and complete subspace of $W_0^{1,N}(\Omega)$, the functional I_λ is C^1 and bounded below on $\overline{B}_{\rho_\lambda}$, by using Ekeland's variational principle, there exists a sequence $\{u_n\}$ in $\overline{B}_{\rho_\lambda}$ such that

$$I_\lambda(u_n) \to c_\rho = \inf_{\|u\| \leq \rho_\lambda} I_\lambda(u) \text{ and } \|I'_\lambda(u_n)\| \to 0.$$

For sufficiently large n, we get

$$\|u_n\|^N \leq \rho_\lambda^N < \left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1} \leq \left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1} + \|u_0\|^N.$$

By Lemma 4.1.13, $\{u_n\}$ converges to a member of $W_0^{1,N}(\Omega)$, say u_0. Consequently, By Remark 4.1.8 we have, $I_\lambda(u_0) = c_\rho < 0$. On the other hand, the solution u_M obtained in the Theorem 4.1.3 is of mountain pass level $c_M > 0$; therefore $I_\lambda(u_M) = c_M$. Hence, both solutions are different.

4.1.4 Without Ambrosetti-Rabinowitz condition

In this section, we prove the existence of a weak solution for the Problem (4.1) when the Ambrosetti-Rabinowitz condition is not satisfied. As a result of Ambrosetti and Rabinowitz's introduction of the MPT in their renowned paper [16], variational methods have become one of the primary methods to demonstrate the existence of a solution to nonlinear elliptic partial differential equations. They [16] proposed the (L_5) condition on the nonlinearity f which is known as (AR) condition and it ensures that the Palais-Smale sequence is bounded, which plays a crucial role in proving the Palais-Smale condition- a condition needed for the MPT.

Without (AR) condition, one can not use the classical MPT [16]. We refer to [57, 75, 90, 89] and references therein for some existence results when the nonlinear term f does not satisfy the (AR) condition. We use the following condition on nonlinear function f instead of (AR) condition:

(L_5') $H(x,t_0) \leq H(x,t_1)$, $\forall\, 0 < t_0 < t_1$, $x \in \Omega$ where $H(x,t) = tf(x,t) - N\theta F(x,t)$ and $F(x,t) = \int_0^t f(x,s)ds$, where θ is defined in the condition (a_2).

The condition (L_5') is weaker than the (AR) condition.

In addition, we assume the following condition on the nonlinearity f :

(L_0) $\lim\limits_{t\to\infty} \dfrac{F(x,t)}{|x|^b t^{N\theta}} = \infty, \ \forall\, x \in \bar{\Omega}.$

The main existence results of this section are as follows:

Theorem 4.1.15. *Assume that the conditions $(L_0)-(L_4)$, (L_5') and $(a_1)-(a_2)$ are satisfied. Then there exists $\lambda_2 > 0$ such that for each $0 < \lambda < \lambda_2$, the Problem (4.1) has a non-trivial weak solution.*

To prove the existence of a ground state solution, we need the following additional assumption on f:

(GS) For each $x \in \Omega$, the map $t \mapsto \frac{f(x,t)+\lambda h(x)|x|^b}{t^{N\theta-1}|x|^b}$ is strictly increasing for $t > 0$.

We define

$$\aleph = \{u \in W_0^{1,N}(\Omega)\backslash\{0\} | I_\lambda'(u)u = 0\} \text{ and } k = \inf_{u\in\aleph} I_\lambda(u).$$

The set $\aleph$ is called the Nehari manifold.

Theorem 4.1.16. *If $(L_0)-(L_4)$, (L_5') , (GS) and $(a_1)-(a_2)$ are satisfied, then*

$$c_M = k = r,$$

which implies that the solution obtained in Theorem 4.1.15 is a ground state solution, where $r = \inf\limits_{u\in S} I_\lambda(u)$ and S is the set of all critical points of the functional I_λ.

Lemma 4.1.17. *Suppose that $(L_0)-(L_4)$, (L_5') and $(a_1)-(a_2)$ hold. Then the following are true:*

1. *There exists $\lambda_0 > 0$ such that for each $0 < \lambda < \lambda_0$, there are $\eta, \rho > 0$ such that*

$$I_\lambda(u) \geq \eta > 0, \ \ \forall u \in W_0^{1,N}(\Omega) \text{ such that } \|u\| = \rho.$$

2. *There exist* $v_0 \in W_0^{1,N}(\Omega)$ *and* $\beta > 0$ *such that*

$$I_\lambda(v_0) < 0 \ \textit{and} \ \|v_0\| > \beta.$$

Proof. Proof of this lemma resembles to the proof of Lemmas 4.1.5 and 4.1.6. We omit the details. □

By Theorem 2.3.4, there exists a sequence $\{u_n\} \subseteq W_0^{1,N}(\Omega)$ such that $I_\lambda(u_n) \to c_M$ and $(1+\|u_n\|)|I'_\lambda(u_n)| \to 0$ as $n \to \infty$, where

$$c_M = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > 0,$$

and

$$\Gamma = \{\gamma \in C([0,1], W_0^{1,N}(\Omega)) : \gamma(0) = 0,\ I_\lambda(\gamma(1)) < 0\}.$$

To verify that $(C)_{c_M}$ condition hold for I_λ, we require the following result for the mountain pass level c_M.

4.1.4.1 Cerami compactness condition

Lemma 4.1.18. *Any* $(C)_{c_M}$ *sequence is bounded in* $W_0^{1,N}(\Omega)$*, if* f *satisfies* (L'_5) *condition.*

Proof. Let $\{u_n\} \subseteq W_0^{1,N}(\Omega)$ be any $(C)_{c_M}$ sequence, i.e., $I_\lambda(u_n) \to c$ and $(1+\|u_n\|)I'_\lambda(u_n) \to 0$ in $W_0^{1,N}(\Omega)^*$. Thus, we have

$$\frac{1}{N}A(\|u_n\|^N) - \int_\Omega \frac{F(x,u_n)}{|x|^b}dx - \lambda \int_\Omega h(x)u_n\, dx = c_M + O(1), \tag{4.32}$$

and

$$\left| a(\|u_n\|^N) \int_\Omega |\nabla u_n|^{N-2}\nabla u_n \nabla \varphi\, dx - \int_\Omega \frac{f(x,u_n)v}{|x|^b}dx - \lambda \int_\Omega h(x)v dx \right| \le \frac{O(1)\|v\|}{(1+\|u_n\|)}, \tag{4.33}$$

$\forall v \in W_0^{1,N}(\Omega)$.

Let on contrary that $\|u_n\| \to \infty$. Define, $y_n = \frac{u_n}{\|u_n\|}$ then $\|y_n\| = 1$, which implies the boundedness of $\{y_n\}$. By reflexivity of $W_0^{1,N}(\Omega)$, $\exists\, y \in W_0^{1,N}(\Omega)$ such that $y_n \rightharpoonup y$ (up to a subsequence) in $W_0^{1,N}(\Omega)$, this implies that

$$y_n^+ \rightharpoonup y^+ \text{ in } W_0^{1,N}(\Omega),$$
$$y_n^+(x) \to y^+(x) \text{ a.e. } x \in \Omega,$$
$$y_n^+ \to y^+ \text{ in } L^q(\Omega) \text{ for } q \in [1,\infty),$$

where, $y^+ = \max\{y, 0\}$. We will prove that $\Omega^+ = \{x \in \Omega : y^+(x) > 0\}$ has measure zero, it will imply that $y^+ \equiv 0$ a.e. in Ω. Let us assume that Ω^+ has a positive measure, then

$$\lim_{n\to\infty} |u_n(x)| = \lim_{n\to\infty} |y_n(x)| \|u_n\| \to \infty \text{ in } \Omega^+.$$

By using (a_2'') and (L_0), one gets

$$\lim_{n\to\infty} \frac{F(x,u_n(x))}{|x|^b A(\|u_n\|^N)} \geq \lim_{n\to\infty} \frac{F(x,u_n^+(x))|y_n^+(x)|^{N\theta}}{A(1)|x|^b |u_n^+|^{N\theta}} = \infty \text{ in } \Omega^+.$$

Thanks to the Fatou's lemma, we obtain

$$\liminf_{n\to\infty} \int_\Omega \frac{F(x,u_n(x))}{|x|^b A(\|u_n\|^N)} = \infty. \tag{4.34}$$

(4.32) implies that

$$\begin{aligned} N\left(\int_\Omega \frac{F(x,u_n)}{|x|^b} dx + \lambda \int_\Omega h(x) u_n dx + c_M + O(1)\right) &= A(\|u_n\|^N) \\ \int_\Omega \frac{F(x,u_n)}{|x|^b} dx &\geq \frac{a_0}{N\theta} \|u_n\|^N - \lambda \|u_n\| \int_\Omega h(x) y_n dx - c_M - O(1) \\ &\geq \|u_n\| \left(\frac{a_0}{N\theta} \|u_n\|^{N-1} - c\lambda g_1\right) - c_M - O(1), \end{aligned}$$

where $g_1 = \max_{x\in\Omega} h(x)$. On using the assumption that $\|u_n\| \to \infty$, we have

$$\int_\Omega \frac{F(x,u_n)}{|x|^b} dx \to \infty. \tag{4.35}$$

Next, (4.32), (4.34) and (4.35) imply

$$\begin{aligned} \infty = \liminf_{n\to\infty} \int_\Omega \frac{F(x,u_n)}{|x|^b A(\|u_n\|^N)} &= \liminf_{n\to\infty} \frac{\int_\Omega \frac{F(x,u_n)}{|x|^b}}{N\left(\int_\Omega \frac{F(x,u_n)}{|x|^b} dx + \lambda \int_\Omega h(x) u_n \, dx + c_M + O(1)\right)} \\ &\leq \liminf_{n\to\infty} \frac{1}{N + N \frac{\lambda \int_\Omega h(x) u_n \, dx}{\int_\Omega \frac{F(x,u_n)}{|x|^b}}} \\ &\leq \frac{1}{N}. \end{aligned}$$

which is a contradiction. Hence, $y \leq 0$ a.e. in Ω. From (4.33), we obtain

$$a(\|u_n\|^N) \int_\Omega |\nabla u_n|^{N-2} \nabla u_n \nabla y \, dx - \lambda \|h\|_* \|y\| \leq a(\|u_n\|^N) \int_\Omega |\nabla u_n|^{N-2} \nabla u_n \nabla y \, dx$$

$$
\begin{aligned}
&-\int_\Omega \frac{f(x,u_n)y}{|x|^b}dx - \lambda \int_\Omega h(x)ydx \\
&\leq \frac{\varepsilon_n\|y\|}{(1+\|u_n\|)} \to 0.
\end{aligned}
$$

Thus

$$
\begin{aligned}
a(\|u_n\|^N)\int_\Omega |\nabla y_n|^{N-2}\nabla y_n \nabla y\, dx - \frac{\lambda\|h\|_*\|y\|}{\|u_n\|^{N-1}} \\
= \frac{a(\|u_n\|^N)\int_\Omega |\nabla u_n|^{N-2}\nabla u_n \nabla y\, dx}{\|u_n\|^{N-1}} - \frac{\lambda\|h\|_*\|y\|}{\|u_n\|^{N-1}} \\
\leq \frac{\varepsilon_n\|y\|}{(1+\|u_n\|)\|u_n\|^{N-1}} \to 0.
\end{aligned}
$$

Hence, we have $y \equiv 0$, since $y_n \rightharpoonup y$ in $W_0^{1,N}(\Omega)$. Let $0 \leq t_n \leq 1$ be such that

$$
I_\lambda(t_n u_n) = \max_{0\leq t\leq 1} I_\lambda(tu_n).
$$

By choosing $R \in \left(0, \left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{\frac{N-1}{N}}\right)$ we have $\frac{R}{\|u_n\|} \to 0$, hence

$$
I_\lambda(t_n u_n) \geq I_\lambda\left(\frac{R}{\|u_n\|}u_n\right) = I_\lambda(Ry_n),
$$

for sufficiently large n. By using $(L_1)-(L_2)$, we get

$$
\begin{aligned}
I_\lambda(Ry_n) \geq \frac{1}{N}A(\|Ry_n\|^N) - \frac{\varepsilon}{N}\int_\Omega \frac{|Ry_n|^N}{|x|^b}dx - c_1\int_\Omega \frac{|Ry_n|^s e^{\alpha|Ry_n|^{N/(N-1)}}}{|x|^b}dx \\
- \lambda\int_\Omega h(x)Ry_n\, dx.
\end{aligned}
$$

Using Hölder's inequality (Choosing $r > 1$ enough close to 1),

$$
\begin{aligned}
I_\lambda(Ry_n) \geq \frac{1}{N}A(R^N) - \frac{\varepsilon R^N}{N}\left(\int_\Omega |y_n|^{sN}dx\right)^{1/s}\left(\int_\Omega \frac{1}{|x|^{bs'}}dx\right)^{1/s'} \\
-c_1\left(\int_\Omega \frac{e^{r\alpha R^{\frac{N}{N-1}} y_n^{\frac{N}{N-1}}}}{|x|^{br}}dx\right)^{\frac{1}{r}}\left(\int_\Omega |Ry_n|^{sr'}dx\right)^{\frac{1}{r'}} - \lambda\int_\Omega h(x)Ry_n\, dx.
\end{aligned}
$$

Then, by Lemma 2.4.2, we have

$$I_\lambda(Ry_n) \geq \frac{1}{N}A(R^N) - \frac{\varepsilon C}{N}\left(\int_\Omega |y_n|^{sN} dx\right)^{1/s} - c_2\left(\int_\Omega |y_n|^{sr'} dx\right)^{\frac{1}{r'}} - \lambda R g_1 \int_\Omega y_n\, dx. \quad (4.36)$$

Since, $\int_\Omega |y_n|^p dx \to 0$ for all $p \geq 1$, therefore by letting $n \to \infty$ and $R \to \left(\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{\frac{N-1}{N}}\right)^-$ in (4.36), we have

$$\liminf_{n\to\infty} I_\lambda(t_n u_n) \geq \liminf_{n\to\infty} I_\lambda(Ry_n) \geq \frac{1}{N}A\left(\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{\frac{N-1}{N}}\right) > c_M. \quad (4.37)$$

As $\{u_n\}$ is a $(C)_{c_M}$ sequence, we have $I'_\lambda(t_n u_n)(t_n u_n) = 0$, i.e.,

$$a(\|t_n u_n\|^N)\|t_n u_n\|^N\, dx = \int_\Omega \frac{f(x,t_n u_n)t_n u_n}{|x|^b}dx + \lambda \int_\Omega h(x)t_n u_n dx. \quad (4.38)$$

Consider,

$$I_\lambda(t_n u_n) = \frac{1}{N}A(\|t_n u_n\|^N) - \int_\Omega \frac{F(x,t_n u_n)}{|x|^b}dx - \lambda \int_\Omega h(x)t_n u_n\, dx. \quad (4.39)$$

By (a'_2), (L'_5), (4.38) and (4.39), we have

$$\begin{aligned}
I_\lambda(t_n u_n) &\leq \frac{1}{N\theta}(\theta A(\|t_n u_n\|^N) - a(\|t_n u_n\|^N)\|t_n u_n\|^N dx) \\
&\quad + \frac{1}{N\theta}\int_\Omega \frac{f(x,t_n u_n)t_n u_n - N\theta F(x,t_n u_n)}{|x|^b}dx, \\
&\leq \frac{1}{N\theta}(\theta A(\|u_n\|^N) - a(\|u_n\|^N)\|u_n\|^N dx) \\
&\quad + \frac{1}{N\theta}\int_\Omega \frac{f(x,u_n)u_n - N\theta F(x,u_n)}{|x|^b}dx \\
&\leq I_\lambda(u_n) - \frac{1}{N\theta}I'_\lambda(u_n)(u_n) \leq c_M,
\end{aligned}$$

which contradicts (4.37). Thus $\{u_n\}$ must be a bounded sequence in $W_0^{1,N}(\Omega)$. □

Lemma 4.1.19. *Suppose $(L_0)-(L_2)$, (L'_5) and $(a_1)-(a_2)$ hold. If $\{u_n\} \subseteq W_0^{1,N}(\Omega)$ is a $(C)_{c_M}$ sequence for I_λ with weak limit u_0 and*

$$c_M < \frac{1}{N}A\left(\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}\right), \quad (4.40)$$

then $\exists\ \lambda_1 > 0$ such that for each $0 < \lambda < \lambda_1$, $u_n \to u_0$(up to subsequence) in $W_0^{1,N}(\Omega)$.

Proof. We claim that

$$\lim_{n\to\infty}\|u_n\|^N-\|u_0\|^N<\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \tag{4.41}$$

Suppose on the contrary that, $\lim_{n\to\infty}\|u_n\|^N-\|u_0\|^N\geq\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}$.

Lemma 4.1.18 implies the boundedness of $\{u_n\}$. Thus, by reflexivity of $W_0^{1,N}(\Omega)$, $\exists\, u_0\in W_0^{1,N}(\Omega)$ such that $u_n\rightharpoonup u_0$ in $W_0^{1,N}(\Omega)$. Also, by using (L_5'), there exists $\lambda_1>0$ such that for each $0<\lambda<\lambda_1<\lambda_1^*$, we have

$$\int_\Omega\frac{1}{N\theta}\left(\frac{f(x,u_0)u_0}{|x|^b}-N\theta\frac{F(x,u_0)}{|x|^b}\right)dx-\left(1-\frac{1}{N\theta}\right)\lambda\int_\Omega h(x)u_0\,dx\geq 0. \tag{4.42}$$

Further, proceeding as [67, Lemma 3.6 and Lemma 3.7], we get $u_n\to u_0$ in $W_0^{1,N}(\Omega)$ as $n\to\infty$. □

Proof of the Theorem 4.1.15. By Lemma 4.1.17, the functional I_λ satisfied all the geometric conditions of the Theorem 2.3.4. Thus, by the Theorem 2.3.4, $\exists$ a $(C)_{c_M}$ sequence, say, $\{u_n\}$ where

$$c_M=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I_\lambda(\gamma(t))>0,$$

and

$$\Gamma=\{\gamma\in C([0,1],W_0^{1,N}(\Omega)):\gamma(0)=0,\ I_\lambda(\gamma(1))<0\}.$$

By Lemma 4.1.10, we have

$$c_M<\frac{1}{N}A\left(\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}\right).$$

By Lemma 4.1.19, $u_n\to u_0$. Hence, by MPT, u_0 is the critical point of I_λ of level c_M, i.e., $I_\lambda'(u_0)=0$ and $I_\lambda(u_0)=c_M>0$. Therefore, u_0 is the non-trivial weak solution of (4.1). □

4.1.4.2 Ground state solution

In this section, we show that the solution obtained in Theorem 4.1.15 is a ground-state solution.

Lemma 4.1.20. *Let $(L_1)-(L_4)$ and $(a_1)-(a_2)$ hold. If $w\in W_0^{1,N}(\Omega)\backslash\{0\}$, then there exists unique $t_w>0$ such that $t_w w\in\aleph$. Moreover, $\max_{t\in[0,\infty]}h_w(t)=h_w(t_w)=I_\lambda(wt_w)$.*

Proof. Define the function, $h_w:\mathbb{R}^+\to\mathbb{R}$ such that $h_w(t)=I_\lambda(tw)$. We observe that $h_w'(t)=0$ if and only if $tw\in\aleph$. Lemma 4.1.17 implies that $h_w(t)>0$ for sufficiently small t and $h_w(t)<0$ for sufficiently large t. Thus, $\exists\, t_w\in(0,\infty)$ such that $\max_{t\in[0,\infty]}h_w(t)=h_w(t_w)=I_\lambda(wt_w)$.

Consequently, $h'_w(t_w)=0$ and $t_w w \in \aleph$. Next, we will prove the uniqueness of t_w. If t is the critical point of h_w, then we have

$$h'_w(t)=\frac{a(\|tw\|^N)\|tw\|^N}{t}-\int_\Omega \frac{f(x,tw)\,w}{|x|^b}dx-\lambda\int_\Omega h(x)w\,dx=0,$$

which implies that

$$\begin{aligned}\frac{a(\|tw\|^N)\|w\|^{N\theta}}{\|tw\|^{N(\theta-1)}}&=\int_\Omega \frac{f(x,tw)w}{t^{N\theta-1}|x|^b}dx+\lambda\int_\Omega \frac{h(x)w}{t^{N\theta-1}}dx\\&=\int_\Omega\left[\frac{f(x,tw)}{(tw)^{N\theta-1}|x|^b}+\frac{\lambda h(x)}{(tw)^{N\theta-1}}\right]w^{N\theta}dx.\end{aligned}\tag{4.43}$$

By (a_2), the left-hand side of (4.43) is decreasing for $t>0$ while the right-hand side is increasing strictly by (GS). Therefore, t_w is a unique critical point of h_w. □

Proof of the Theorem 4.1.16. By using the fact that $S\subseteq\aleph$, we have $k\le r$. Also, it can be seen $r\le c_M$. It will be sufficient to prove that $k\ge c_M$.

If $v\in\aleph$, then $h'_v(1)=0$. By Lemma 4.1.20, we have $\max\limits_{t\in[0,\infty]} h_v(t)=h_v(1)=I_\lambda(v)$.

Choose a function $\gamma\colon[0,1]\to W_0^{1,N}(\Omega)$ such that $\gamma(t)=tt_0v$, where $t_0>0$ such that $I_\lambda(t_0v)<0$, which implies that $\gamma\in\Gamma$. Therefore, we have $c_M\le\max\limits_{t\in[0,1]}I_\lambda(\gamma(t))=\max\limits_{t\in[0,1]}I_\lambda(tt_0v)\le\max\limits_{t\ge0}I_\lambda(tv)=I_\lambda(v)$, which is true for every element $v\in\aleph$. Hence, $k\ge c_M$ and that concludes the proof. □

4.1.4.3 Examples

In order to give a clear view of our hypotheses on the nonlinearity f, here we provide some examples.

Example 4.1.21. *Consider the function,* $f(x,t)=\left(N\theta t^{N\theta-1}+\frac{t^{N\theta-1}\sqrt{t}}{2}\right)e^{\sqrt{t}}$ *then* $F(x,t)=t^{N\theta}e^{\sqrt{t}}$ *and* $H(x,t)=tf(x,t)-N\theta F(x,t)=\left(\frac{t^{N\theta}\sqrt{t}}{2}\right)e^{\sqrt{t}}$*. Now, the function*

$$H'(x,t)=\left(\left(N\theta+\frac{1}{2}\right)t^{N\theta-1}\sqrt{t}+\frac{t^{N\theta}}{4}\right)e^{\sqrt{t}}\ge0,\forall t>0,$$

consequently f *satisfy* (L'_5) *condition. But for any* $\omega>N\theta$ *there exists* $t_0>0$ *(small enough) such that*

$$\omega t^{N\theta}e^{\sqrt{t}}>\left(N\theta t^{N\theta}+\frac{t^{N\theta}\sqrt{t}}{2}\right)e^{\sqrt{t}},\forall\,0<t<t_0,$$

hence, f *does not satisfy the (AR) condition. For more information, one can refer to [90, Appendix A].*

Example 4.1.22. *A typical example of a function f satisfying the conditions $(L_0)-(L_4)$, (L_5') and (GS) is given by $f(x,t)=0\ \forall\ (x,t)\in\Omega\times(-\infty,0]$ and*

$$f(x,t)=N\theta|x|^b t^{N\theta-1}\left[\exp\left(t^{\frac{N}{N-1}}\right)-1\right]+\frac{N}{N-1}|x|^b\exp\left(t^{\frac{N}{N-1}}\right)t^{N\theta+\frac{1}{N-1}},$$

$\forall\ (x,t)\in\Omega\times(0,\infty)$ and h is any negative, continuous function which belongs to the dual of $W_0^{1,N}(\Omega)$. Indeed, deriving we get

$$F(x,t)=|x|^b t^{N\theta}\left[\exp\left(t^{\frac{N}{N-1}}\right)-1\right]$$

and

$$H(x,t)=tf(x,t)-N\theta F(x,t)=\frac{N}{N-1}|x|^b\exp\left(t^{\frac{N}{N-1}}\right)t^{N\theta+\frac{N}{N-1}}.$$

Clearly, H is increasing function, consequently f satisfy (L_5') condition. We take $\alpha_0=1$ then the condition (L_1) is true. It can be easily verify that the conditions (L_0), $(L_2)-(L_4)$ and (GS) are also true.

4.2 A problem in $\mathbb{R}^N$

In this section, we consider the following problem:

$$-a\left(\int_{\mathbb{R}^N}|\nabla u|^N+V(x)|u|^N dx\right)(\Delta_N u+V(x)|u|^{N-2}u)=\frac{g(x,u)}{|x|^b}+\lambda h(x) \text{ in } \mathbb{R}^N \tag{4.44}$$

where $0\le b<N$, $N\ge 2$, $\Delta_N=\mathrm{div}(|\nabla u|^{N-2}\nabla u)$ is the N-Laplace operator, λ is a suitably small real parameter and the perturbation term $h>0$ belongs to the dual of some suitable Sobolev space.

4.2.1 Hypotheses

We assume the following growth condition on the nonlinearity g:

(L_1') There exist $C_1, C_2, \alpha_0>0$ such that

$$|g(x,t)|\le C_1|t|^{N-1}+C_2\Phi(\alpha_0|t|^{N/N-1}),\ \ \forall\ (x,t)\in\mathbb{R}^N\times\mathbb{R},$$

where

$$\Phi(t)=\exp(t)-\sum_{n=0}^{N-2}\frac{t^n}{n!}\cdot \tag{4.45}$$

(L_2') $\lim_{t\to 0} \frac{g(x,t)}{t^{N-1}} = 0, \ \forall\, x \in \mathbb{R}^N.$

(L_3') There exists $\sigma > N\theta$ such that

$$0 < \sigma G(x,t) = \sigma \int_0^t g(x,s)ds \le tg(x,t),$$

for all $t > 0,\ x \in \mathbb{R}^N$.

(L_4') $\lim_{t\to\infty} \frac{tg(x,t)}{e^{\alpha t^{N/(N-1)}}} \ge \beta_1 > 0$ uniformly on compact subset of $\mathbb{R}^N$, for some $\beta_1 > 0$.

(L_5') $g(x,t) = 0$ for all $x \in \mathbb{R}^N, t \le 0$.

Next, we state our hypotheses on the non-local term a. The function $a : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and satisfies the following conditions:

(a_1) There exists positive real number a_0 such that $a(s) \ge a_0$ and a is non-decreasing $\forall\, s > 0$.

(a_2) There exists $\theta > 1$ such that $a(s)/s^{\theta-1}$ is non-increasing for $s > 0$.

Remark 4.2.1. *By condition (a_2), we have*

(a_2') *$\theta A(s) - a(s)s$ is non-decreasing, $\forall\, s > 0$, where $A(s) = \int_0^s a(t)dt$.*

In particular,

$$\theta A(s) - a(s)s \ge 0 \ \ \forall\, s > 0. \tag{4.46}$$

Again by (a_2) and (4.46), one get

(a_2'') $A(s) \le s^\theta A(1)\ \forall\, s \ge 1.$

Corresponding to the problem (4.1), the perturbation term $h \in (W_0^{1,N}(\Omega))^*$. The space $(W_0^{1,N}(\Omega))^*$ is norm space with the norm $\|.\|_*$.

Next, we state the condition on the potential V :

(V_1) $V(x) \ge V_0 > 0$ for all $x \in \mathbb{R}^N$.

(V_2) $V(x) \to \infty$ as $|x| \to \infty$.

Due to the presence of potential V, the natural space in which we look for a weak solution to the problem (4.44) is

$$W = \{u \in W^{1,N}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^N dx < \infty\}$$

which is a reflexive Banach space, under the condition (V_1), with the norm

$$\|u\|_W = \left(\int_{\mathbb{R}^N} (|\nabla u|^N + V(x)|u|^N) dx \right)^{1/N}.$$

Moreover, we have the compact embedding $W \hookrightarrow W^{1,N}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ for all $s \in [p, p^*)$. For the details; we refer to [139, Lemma 2.1].

Notation: We denote,

$$\lambda_1'(N) = \inf_{u \in W \setminus \{0\}} \left\{ \frac{\|u\|_W^N}{\int_{\mathbb{R}^N} \frac{|u|^N}{|x|^b} dx} \right\} > 0 \text{ for any } 0 \leq b < N. \tag{4.47}$$

4.2.2 Variational framework and main results

Definition 4.2.2. We say that $u \in W$ is a weak solution of (4.44) if the following holds:

$$a(\|u\|_W^N) \int_{\mathbb{R}^N} (|\nabla u|^{N-2} \nabla u \nabla \varphi + V(x)|u|^{N-2} u \varphi)\, dx = \int_{\mathbb{R}^N} \frac{g(x,u)\varphi dx}{|x|^b} + \lambda \int_{\mathbb{R}^N} h(x)\varphi\, dx \tag{4.48}$$

$\forall \varphi \in W$.

Thus the energy functional $J_\lambda : W \to \mathbb{R}$ corresponding to (4.48) is given by

$$J_\lambda(u) = \frac{1}{N} A(\|u\|_W^N) - \int_{\mathbb{R}^N} \frac{G(x,u)}{|x|^b} dx - \lambda \int_{\mathbb{R}^N} h(x) u\, dx,$$

where $G(x,t) = \int_0^t g(x,s) ds$ and $A(t) = \int_0^t a(s) ds$. It can be seen that the functional J_λ is C^1 and the derivative of J_λ at any point $u \in W$ is

$$J_\lambda'(u)(v) = a(\|u\|_W^N) \int_{\mathbb{R}^N} (|\nabla u|^{N-2} \nabla u \nabla v + V(x)|u|^{N-2} uv)\, dx = \int_{\mathbb{R}^N} \frac{g(x,u) v dx}{|x|^b} + \lambda \int_{\mathbb{R}^N} h(x) v\, dx.$$

Further, the critical points of J_λ are the weak solutions to (4.44).

The statement of our main results is as follows:

Theorem 4.2.3. *Suppose that the conditions $(L_1') - (L_5')$ and $(a_1) - (a_2)$ are satisfied. Then there exists $\lambda_2' > 0$, such that for each $0 < \lambda < \lambda_2'$, the Problem (4.44) has a non-trivial weak solution.*

Theorem 4.2.4. *Suppose $(L_1') - (L_5')$ and $(a_1) - (a_2)$ hold; then there exists $\lambda_3' > 0$, such that for each $0 < \lambda < \lambda_3'$, the Problem (4.44) has a non-trivial minimum type solution with negative* *energy. Moreover, the solutions of the Problem* (4.44) *obtained in this theorem and Theorem 4.2.3 are different.*

4.2.3 Proof of the Theorems 4.2.3 and 4.2.4

Lemma 4.2.5. *Suppose* $(L_1') - (L_2')$ *and* (a_1) *hold. Then there exists* $\lambda_0' > 0$ *such that for each* $0 < \lambda < \lambda_0'$, *there are positive real numbers* η *and* ρ_λ *such that*

$$J_\lambda(u) \geq \eta > 0, \ \ \forall u \in W \text{ such that } \|u\|_W = \rho_\lambda.$$

Moreover, one can choose ρ_λ *such that* $\rho_\lambda \to 0$ *as* $\lambda \to 0$.

Proof. It follows from (L_1') and (L_2') that, for a given $\varepsilon > 0$ and $s > N$, $\exists\ \alpha \geq \alpha_0, C_3 > 0$(depending on ε and s) such that,

$$|G(x,t)| \leq \frac{1}{N}\varepsilon|t|^N + C_3|t|^s\Phi(\alpha|t|^{N/N-1}) \tag{4.49}$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. By using (4.49), we obtain

$$J_\lambda(u) \geq \frac{1}{N}A(\|u\|_W^N) - \frac{\varepsilon}{N}\int_{\mathbb{R}^N}\frac{|u|^N}{|x|^b}dx - C_3\int_{\mathbb{R}^N}\frac{|u|^s\Phi(\alpha|u|^{N/N-1})}{|x|^b}dx - \lambda\int_{\mathbb{R}^N}h(x)u\,dx.$$

On using (a_1), (4.47) and Lemma 2.4.5 (we will choose $\|u\|_W \leq K$ with K is sufficiently small), one get

$$J_\lambda(u) \geq \frac{a_0\|u\|_W^N}{N} - \frac{\varepsilon\|u\|_W^N}{N\lambda_1'(N)} - C_4\|u\|_W^s - \lambda\|h\|_*\|u\|_W.$$

Therefore, we have

$$J_\lambda(u) \geq \left(\frac{a_0}{N}\|u\|_W^{N-1} - \frac{\varepsilon}{N\lambda_1'(N)}\|u\|_W^{N-1} - C_4\|u\|_W^{s-1} - \lambda\|h\|_*\right)\|u\|_W. \tag{4.50}$$

Now choose ε such as $\frac{a_0}{N} - \frac{\varepsilon}{N\lambda_1'(N)} > 0$ and $\rho_\lambda > 0$ sufficiently small such that $\rho_\lambda < K$ and

$$\eta = \left(\frac{a_0}{N} - \frac{\varepsilon}{N\lambda_1'(N)}\right)\rho_\lambda^{N-1} - C_4\rho_\lambda^{s-1} > 0.$$

This implies, there exists $\lambda_0' > 0$ such that for each $0 < \lambda < \lambda_0'$ we have

$$J_\lambda(u) \geq \eta > 0, \ \ \forall u \in W \text{ such that } \|u\|_W = \rho_\lambda.$$

Moreover, we can chose ρ_λ such that $\rho_\lambda \to 0$ as $\lambda \to 0$. This completes the proof. □

Lemma 4.2.6. *Assume that* (L_3') *and* (a_2) *hold. There exist* $v_0 \in W$ *and positive real number* β *such that*

$$J_\lambda(v_0) < 0 \ \text{ and } \|v_0\|_W > \beta.$$

Proof. The proof is similar to the proof of the Lemma 4.1.6. We omit the details. □

Lemma 4.2.7. *There exist* $\kappa > 0$ *and* $\vartheta_0 \in W$ *with* $\|\vartheta_0\|_W = 1$ *such that*

$$J_\lambda(t\vartheta_0) < 0 \ \text{ for all } t \in [0,\kappa).$$

Moreover,

$$\inf_{\|u\|_W \leq \kappa} J_\lambda(u) < 0.$$

Proof. The proof is similar to the proof of the Lemma 4.1.7. We omit the details. □

Remark 4.2.8. *By* (4.50) *and Lemma 4.2.7, we have*

$$-\infty < c_\rho' = \inf_{\|u\|_W \leq \rho} J_\lambda(u) < 0.$$

By Lemmas 4.2.5 and 4.2.6, the geometric conditions of the mountain pass theorem are satisfied for the functional J_λ. Hence, there exists a sequence $\{u_n\} \subseteq W$ such that $J_\lambda(u_n) \to c_M'$ and $J_\lambda'(u_n) \to 0$ as $n \to \infty$, where

$$c_M' = \inf_{\gamma\in\Gamma'} \max_{t\in[0,1]} J_\lambda(\gamma(t)) > 0,$$

and

$$\Gamma' = \{\gamma \in C([0,1],W) : \gamma(0) = 0,\ \gamma(1) < 0\}.$$

Due to the lack of compactness, we are not able to prove directly that (PS) condition holds for J_λ. We need some extra information for the mountain pass level c_M'. Hence, we consider the Green's function, $\widetilde{g}_n : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ given by Moser [114] as:

$$\widetilde{g}_n(x,r) = \frac{1}{\omega_{N-1}^{1/N}} \begin{cases} (\log n)^{(N-1)/N} & if\ |x| \leq \frac{r}{n}, \\ \frac{\log \frac{r}{|x|}}{(\log n)^{(1)/N}} & if\ \frac{r}{n} \leq |x| \leq r, \\ 0 & if\ |x| \geq r. \end{cases}$$

Set, $g_n(x,r) = \dfrac{\widetilde{g}_n(x,r)}{\|\widetilde{g}_n(x,r)\|_W}\cdot$ We can write,

$$g_n^{N/(N-1)}(x) = \omega_{N-1}^{-1/(N-1)} \log n + p_n^* \text{ if } |x| \leq r/n$$

where, $p_n^* = \omega_{N-1}^{-1/(N-1)} \log n \, (\|\widetilde{g}_n\|_W^{-N/(N-1)} - 1)$.

Remark 4.2.9. *One can observe that, $\int_{\mathbb{R}^N} |\nabla \widetilde{g}_n(x,r)|^N dx = 1$ and $\int_{\mathbb{R}^N} |\widetilde{g}_n(x,r)|^N dx = O(1/\log n)$, consequently $\|\widetilde{g}_n\|_W \to 1$. Support of $\widetilde{g}_n(x,r)$ contained in $\overline{B(0,r)}$. Also, $g_n(\cdot,r) \in W$ and $\|g_n(x,r)\|_W = 1$.*

Lemma 4.2.10. *Suppose (L_4') and $(a_1)-(a_2)$ hold. There exists $n \in \mathbb{N}$ such that*

$$\max_{t\geq 0} \left\{ \frac{1}{N} A(t^N) - \int_{\mathbb{R}^N} \frac{G(x,tg_n)}{|x|^b} dx \right\} < \frac{1}{N} A\left(\left(\frac{N-b}{N} \frac{\alpha_N}{\alpha_0} \right)^{N-1} \right).$$

Proof. Choose, $r > 0$ such that

$$\beta_1 > \frac{N-b}{N r^{N-b}} \left(\frac{N-b}{\alpha_0} \right)^{N-1} a\left(\left(\frac{N-b}{N} \frac{\alpha_N}{\alpha_0} \right)^{N-1} \right). \tag{4.51}$$

Let us contrary that, for all $n \in \mathbb{N}$, we have

$$\max_{t\geq 0} \left\{ \frac{1}{N} A(t^N) - \int_{\mathbb{R}^N} \frac{G(x,tg_n)}{|x|^b} dx \right\} \geq \frac{1}{N} A\left(\left(\frac{N-b}{N} \frac{\alpha_N}{\alpha_0} \right)^{N-1} \right).$$

By (4.8) and (a_2''), for each n there exists $t_n > 0$ such that

$$\frac{1}{N} A(t_n^N) - \int_{\mathbb{R}^N} \frac{G(x,t_n g_n)}{|x|^b} dx = \max_{t\geq 0} \left\{ \frac{1}{N} A(t^N) - \int_{\mathbb{R}^N} \frac{G(x,tg_n)}{|x|^b} dx \right\}.$$

Therefore,

$$\frac{1}{N} A(t_n^N) - \int_{\mathbb{R}^N} \frac{G(x,t_n g_n)}{|x|^b} dx \geq \frac{1}{N} A\left(\left(\frac{N-b}{N} \frac{\alpha_N}{\alpha_0} \right)^{N-1} \right).$$

By using (a_1) and the fact that $G(x,t) \geq 0$, we have

$$t_n^N \geq \left(\frac{N-b}{N} \frac{\alpha_N}{\alpha_0} \right)^{N-1}. \tag{4.52}$$

Next, we claim that

$$t_n^N \to \left(\frac{N-b}{N} \frac{\alpha_N}{\alpha_0} \right)^{N-1}.$$

Since at $t = t_n$ we have

$$\frac{d}{dt} \left(\frac{1}{N} A(t_n^N) - \int_{\mathbb{R}^N} \frac{G(x,t_n g_n)}{|x|^b} dx \right) = 0,$$

it follows that

$$t_n^N a(t_n^N) = \int_{\mathbb{R}^N} \frac{t_n g_n g(x, t_n g_n)}{|x|^b} dx = \int_{|x| \le r} \frac{t_n g_n g(x, t_n g_n)}{|x|^b} dx. \tag{4.53}$$

By (L_4'), for a given $\tau > 0$, $\exists\ \rho_\tau > 0$ such that

$$tg(x,t) \ge (\beta_1 - \tau) e^{\alpha_0 |t|^{N/(N-1)}},\ \forall (x,t) \in \overline{B(0,r)} \times [\rho_\tau, \infty). \tag{4.54}$$

It follows from (4.54), for large n, that

$$t_n^N a(t_n^N) \ge (\beta_1 - \tau) \int_{|x| \le r/n} \frac{\exp(\alpha_0 |t_n^{N/(N-1)} (\omega_{N-1}^{-1/(N-1)} \log n + p_n^*)|)}{|x|^b} dx.$$

From (4.46) and (a_2''), we have

$$t_n^{N\theta} A(1)\theta \ge (\beta_1 - \tau) e^{\alpha_0 |t_n^{N/(N-1)} (\omega_{N-1}^{-1/(N-1)} \log n + p_n^*)|} \int_{|x| \le r/n} \frac{1}{|x|^b} dx.$$

Therefore, we have

$$A(1)\theta \ge (\beta_1 - \tau) \frac{\omega_{N-1}}{(N-b)} r^{N-b} \exp(G_n), \tag{4.55}$$

where $G_n = \frac{N\alpha_0 \log n}{\alpha_N} t_n^{N/(N-1)} + \alpha_0 t_n^{N/(N-1)} p_n^* - (N\theta) \log t_n - (N-b) \log n$. This implies $\{t_n\}$ is bounded. If not, upto a subsequence $G_n \to \infty$, which is a contradiction to (4.55). Also, we have

$$A(1)\theta t_n^{N\theta} \ge (\beta_1 - \tau) \frac{\omega_{N-1} r^{N-b}}{(N-b)} \exp\left[\left(\frac{N\alpha_0 t_n^{\frac{N}{N-1}}}{\alpha_N} - (N-b) \right) \log n + \alpha_0 t_n^{N/(N-1)} p_n^* \right]. \tag{4.56}$$

By the help of (4.52), (4.56) and using the fact that $\{t_n\}$ is bounded, our claim

$$t_n^N \to \left(\frac{N-b}{N} \frac{\alpha_N}{\alpha_0} \right)^{N-1} \tag{4.57}$$

is proved.

Next, we construct two sets

$$X_n = \{x \in \overline{B(0,r)} : t_n g_n \ge \rho_\tau\} \text{ and } Y_n = \overline{B(0,r)} - X_n.$$

By using (4.53) and (4.54), we have

$$t_n^N a(t_n^N) \ge (\beta_1 - \tau) \int_{|x| \le r} \frac{e^{\alpha_0 |t_n g_n|^{N/(N-1)}}}{|x|^b} dx + \int_{Y_n} \frac{t_n g_n g(x, t_n g_n)}{|x|^b} dx$$

$$-(\beta_1-\tau)\int_{Y_n}\frac{e^{\alpha_0|t_ng_n|^{N/(N-1)}}}{|x|^b}dx. \tag{4.58}$$

By using the fact that $g_n \to 0$ and the characteristic function $\chi_{Y_n} \to 1$ a.e. in $\overline{B(0,r)}$, we have

$$\int_{Y_n}\frac{t_ng_ng(x,t_ng_n)}{|x|^b}dx \to 0$$

and

$$\int_{Y_n}\frac{e^{\alpha_0|t_ng_n|^{N/(N-1)}}}{|x|^b}dx \to \frac{\omega_{N-1}}{N-b}r^{N-b}.$$

Now, proceeding as [137, Lemma 4.2], we have

$$\int_{|x|\le r}\frac{e^{\alpha_0|t_ng_n|^{N/(N-1)}}}{|x|^b}dx \to (N+1)\frac{\omega_{N-1}}{N-b}r^{N-b}.$$

Hence, taking $n\to\infty$ in (4.58) and using (4.57), we have

$$\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1} a\left(\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}\right) \ge (\beta_1-\tau)\frac{\omega_{N-1}}{N-b}r^{N-b}N,$$

consequently,

$$\beta_1 \le \frac{N-b}{Nr^{N-b}}\left(\frac{N-b}{\alpha_0}\right)^{N-1} a\left(\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}\right),$$

which is contradiction to (4.51). This completes the proof. □

Lemma 4.2.11. *Suppose* (L_3') *and* $(a_1)-(a_2)$ *hold. If* $\{u_n\}\subset W$ *is a* $(PS)_c$ *sequence of* J_λ *with weak limit* u_0 *and*

$$c<\frac{1}{N}A\left(\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}\right)$$

then there exists $\lambda_1'>0$*, such that for each* $0<\lambda<\lambda_1'$*,*

$$\lim_{n\to\infty}\|u_n\|_0^N-\|u_0\|_0^N<\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Proof. Proceeding as Lemma 4.1.12, we get $\{u_n\}$ is bounded in W. As W is a reflexive space, $\exists\, u_0\in W$ such that up to a subsequence we have

$$u_n \rightharpoonup u_0 \text{ weakly in } W,$$
$$u_n(x)\to u_0(x) \text{ a.e. } x\in\mathbb{R}^N,$$
$$u_n\to u_0 \text{ in } L^q(\mathbb{R}^N) \text{ for } q\in[N,\infty).$$

Further, proceeding in the similar ways to [155, Lemma 4.4] and [65, Lemma 2.7], we have

$$\frac{g(x,u_n)}{|x|^b} \to \frac{g(x,u_0)}{|x|^b} \text{ in } L^1(B_R)$$

$$|\nabla u_n|^{N-2}\nabla u_n \rightharpoonup |\nabla u_0|^{N-2}\nabla u_0 \text{ weakly in } (L^{N/(N-1)}_{loc}(\mathbb{R}^N))^N$$

for any $R > 0$. By the definition of weak convergence, we have

$$\int_{\mathbb{R}^N} |\nabla u_n|^{N-2}\nabla u_n \nabla\varphi\, dx \to \int_{\mathbb{R}^N} |\nabla u_0|^{N-2}\nabla u_0 \nabla\varphi\, dx \quad \text{for all } \varphi \in W.$$

In view of Brezis-Lieb lemma [27], we have

$$\int_{\mathbb{R}^N} V(x)|u_n|^{N-2}u_n\varphi\, dx \to \int_{\mathbb{R}^N} V(x)|u_0|^{N-2}u_0\varphi\, dx \quad \text{for all } \varphi \in W.$$

Since $\{u_n\}$ is a $(PS)_c$ sequence of J_λ, we get u_0 is weak solution of the problem

$$-a(E_0)(\Delta_N u + V(x)|u|^{N-2}u) = \frac{g(x,u)}{|x|^b} + \lambda h(x) \text{ in } \mathbb{R}^N,$$

where $E_0 = \lim_{n\to\infty} \|u_n\|_W^N$. Now, on proceeding as in Lemma 4.1.12, we get the desired result. □

Lemma 4.2.12. *Suppose (L_1') and (a_1) holds. If $\{u_n\} \subset W$ is a $(PS)_c$ sequence of J_λ with weak limit u_0 and*

$$\lim_{n\to\infty} \|u_n\|_W^N - \|u_0\|_W^N < \left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}$$

then $u_n \to u_0$ in W up to subsequence.

Proof. First, we will prove that $\frac{|g(x,u_n)|^r}{|x|^b} \in L^1(\mathbb{R}^N)$, for some $r > 1$. Define

$$w_n = \frac{u_n}{\|u_n\|_W} \text{ and } w_0 = \frac{u_0}{E_0^{1/N}},$$

where $E_0 = \lim_{n\to\infty} \|u_n\|_W^N$. Note that $\|w_n\|_W = 1$, $w_n \rightharpoonup w_0$ weakly in W. Without loss of generality, we have $0 < \|w_0\|_W < 1$. By taking $r > 1$ (sufficiently close to 1) and proceeding as Lemma 4.1.13, there exists $\beta_2 > 0$ such that

$$r\alpha\|u_n\|_W^{N/(N-1)} < \beta_2 < \alpha_N\left(1 - \frac{b}{N}\right)(1 - \|w_0\|_W^N)^{-1/(N-1)}. \tag{4.59}$$

Next, using Hölder's inequality (with exponents r and r' such that $1/r+1/r'=1$), Lemma 2.4.10 and (4.59), we have

$$\begin{aligned}
\int_{\mathbb{R}^N}\frac{\Phi(\alpha_0|u_n|^{\frac{N}{N-1}})(u_n-u_0)dx}{|x|^b} &\leq \left(\int_{\mathbb{R}^N}\frac{(\Phi(\alpha_0|u_n|^{\frac{N}{N-1}}))^r dx}{|x|^b}\right)^{\frac{1}{r}}\left(\int_{\mathbb{R}^N}\frac{|u_n-u_0|^{r'}dx}{|x|^b}\right)^{\frac{1}{r'}}\\
&\leq \left(\int_{\mathbb{R}^N}\frac{\Phi(r\alpha_0|w_n|^{\frac{N}{N-1}}\|u_n\|_W^{\frac{N}{N-1}}))}{|x|^b}dx\right)^{1/r}\left(\int_{\mathbb{R}^N}\frac{|u_n-u_0|^{r'}}{|x|^b}dx\right)^{1/r'}\\
&\leq \left(\int_{\mathbb{R}^N}\frac{\Phi(\beta_2|w_n|^{\frac{N}{N-1}}))}{|x|^b}dx\right)^{1/r}\left(\int_{\mathbb{R}^N}\frac{|u_n-u_0|^{r'}}{|x|^b}dx\right)^{1/r'}\\
&\leq C_5\left(\int_{\mathbb{R}^N}\frac{|u_n-u_0|^{r'}}{|x|^b}dx\right)^{1/r'}, \qquad (4.60)
\end{aligned}$$

for some constant $C_5>0$. Again using Hölder's inequality (with exponents N and $N/(N-1)$), (L_1') and (4.60), we have

$$\begin{aligned}
\int_{\mathbb{R}^N}\frac{g(x,u_n)(u_n-u_0)}{|x|^b}dx &\leq \int_{\mathbb{R}^N}\frac{(C_1|u_n|^{N-1}+C_2\Phi(\alpha_0|u_n|^{N/N-1}))(u_n-u_0)}{|x|^b}dx\\
&\leq C_1\left(\int_{\mathbb{R}^N}\frac{|u_n|^N}{|x|^b}dx\right)^{\frac{N-1}{N}}\left(\int_{\mathbb{R}^N}\frac{|u_n-u_0|^N}{|x|^b}dx\right)^{\frac{1}{N}}\\
&\quad +C_2C_5\left(\int_{\mathbb{R}^N}\frac{|u_n-u_0|^{r'}}{|x|^b}dx\right)^{1/r'}\\
&\to 0 \text{ as } n\to\infty. \qquad (4.61)
\end{aligned}$$

On the other side, since $u_n \rightharpoonup u_0$, we have

$$\int_{\mathbb{R}^N}|\nabla u_0|^{N-2}\,\nabla u_0\,\nabla(u_n-u_0)\,dx\to 0,\int_{\mathbb{R}^N}V(x)|u_0|^{N-2}\,u_0\,(u_n-u_0)\,dx\to 0. \qquad (4.62)$$

Since $\{u_n\}$ is a $(PS)_c$ sequence of J_λ, we have $J_\lambda'(u_n)(u_n-u_0)\to 0$ as $n\to\infty$, i.e.,

$$\begin{aligned}
a(\|u_n\|_W^N)&\left(\int_{\mathbb{R}^N}|\nabla u_n|^{N-2}\,\nabla u_n\,\nabla(u_n-u_0)\,dx+\int_{\mathbb{R}^N}V(x)|u_n|^{N-2}u_n(u_n-u_0)\,dx\right)\\
&=\int_{\mathbb{R}^N}\frac{g(x,u_n)\,(u_n-u_0)}{|x|^b}dx+\int_{\mathbb{R}^N}\lambda h(x)(u_n-u_0)\,dx+\varepsilon_n, \qquad (4.63)
\end{aligned}$$

where $\varepsilon_n\to 0$ as $n\to\infty$.

Using (4.62) and (4.61) and taking $n \to \infty$ in (4.63), we get

$$\begin{aligned}\lim_{n\to\infty} a(\|u_n\|_W^N) \int_{\mathbb{R}^N} (|\nabla u_n|^{N-2} \nabla u_n - |\nabla u_0|^{N-2} \nabla u_0) \nabla (u_n - u_0)\, dx \\ + \lim_{n\to\infty} a(\|u_n\|_W^N) \int_{\mathbb{R}^N} V(x)(|u_n|^{N-2} u_n - |u_0|^{N-2} u_0)(u_n - u_0)\, dx \\ \le \lim_{n\to\infty} \lambda \|h\|_* \|u_n - u_0\|_W + \lim_{n\to\infty} \varepsilon_n.\end{aligned}$$

On using (a_1) and (2.1), one has

$$0 \le \lim_{n\to\infty} a_0 \|u_n - u_0\|_W^N \le 0,$$

hence, $u_n \to u_0$ in W as $n \to \infty$. □

Corollary 4.2.13. *Suppose* $(L_1') - (L_3')$ *and* $(a_1) - (a_2)$ *holds. If* $\{u_n\} \subset W$ *is a* $(PS)_c$ *of* J_λ *with weak limit* u_0 *and*

$$c < \frac{1}{N} A\left(\left(\frac{N-b}{N}\frac{\alpha_N}{\alpha_0}\right)^{N-1}\right)$$

then there exists $\lambda_1' > 0$*, such that for each* $0 < \lambda < \lambda_1'$*,* $u_n \to u_0$ *in* W *up to subsequence.*

Proof. Proof follows from Lemmas 4.2.11 and 4.2.12. □

Now, we are ready to prove the Theorems 4.2.3 and 4.2.4.

Proof of the Theorem 4.2.3. Proceeding as Theorem 4.1.3, there exists non-trivial weak solution of the Problem (4.44), say, u_M, such that $J_\lambda'(u_M) = 0$ and $J_\lambda(u_M) = c_M'$ where

$$c_M' = \inf_{\gamma \in \Gamma'} \max_{t\in[0,1]} J_\lambda(\gamma(t)) > 0,$$

and

$$\Gamma' = \{\gamma \in C([0,1], W) : \gamma(0) = 0,\ \gamma(1) < 0\}.$$

Proof of the Theorem 4.2.4. Proceeding in the same way as Theorem 4.1.4, we get a non-trivial minimum solution of the Problem (4.44), say, u_0, such that $J_\lambda'(u_0) = 0$ and $J_\lambda(u_0) = c_\rho' < 0$. On the other hand, the solution u_M obtained in the Theorem 4.2.3 is of mountain pass level $c_M' > 0$, therefore $J_\lambda(u_M) = c_M'$. Hence, both solutions are different.

Chapter 5

Kirchhoff type elliptic equations with double criticality

In this chapter[1], we prove the existence of a weak solution for the Kirchhoff type elliptic equations with double criticality. To establish the existence results in a Musielak-Sobolev space, we use a variational technique based on the mountain pass theorem.

In Section 5.1, we discuss our model problem. In Section 5.2, we provide the main assumptions and variational framework. We also, state the main result of this chapter in Section 5.2. Section 5.3 deals with the proof of the main result. Finally, we construct some examples illustrating the main theorems in Section 5.4.

5.1 Introduction

We establish the existence of a weak solution for the non-local problem:

$$\begin{cases} -a\left(\int_\Omega \mathscr{H}(x,|\nabla u|)dx\right)\Delta_{\mathscr{H}} u &= f(x,u) \ \text{ in } \ \Omega, \\ \qquad\qquad u &= 0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{5.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded and smooth domain, $N \geq 2$, $\Delta_{\mathscr{H}} u = \operatorname{div}(h(x,|\nabla u|)\nabla u)$, $\mathscr{H}(x,t) = \int_0^{|t|} h(x,s)s\, ds$ and $h : \Omega \times [0,\infty) \to [0,\infty)$ is a generalized N-function. There are two open and connected subsets Ω_p and Ω_N such that $\overline{\Omega}_p \cap \overline{\Omega}_N = \emptyset$ and $\Delta_{\mathscr{H}} u = \operatorname{div}(h(x,|\nabla u|)\nabla u)$ is the $\mathscr{H}$-Laplace operator. We assume that $\Delta_{\mathscr{H}}$ reduces to $\Delta_{p(x)}$ in Ω_p and to Δ_N in Ω_N, the nonlinear function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ acts as $|t|^{p^*(x)-2}t$ on Ω_p and as $e^{\alpha|t|^{N/(N-1)}}$ on Ω_N for sufficiently large $|t|$. The growth of the nonlinearity f changes its behavior from domain to domain; due to this dual nature of f, we called the Problem (5.1) has double criticality.

[1]Shilpa Gupta, Gaurav Dwivedi, Kirchhoff type elliptic equations with double criticality in Musielak-Sobolev spaces, Mathematical Methods in the Applied Sciences, 46 (2023), 8463-8477.

When $\mathscr{H}(x,t)$ is independent of x, the existence results for problems of type (5.1) are discussed in Orlicz-Sobolev spaces, and we refer to the work of Chaharlang and Razani [33], and Chung [118] in this direction. In the case, when $\mathscr{H}(x,t)$ depends on both x and t, the existence of a solution for the problems of the type (5.1) is studied in Musielak-Sobolev spaces. Many authors [51, 64, 106, 134] used such spaces to prove the existence of a solution for problems of the type (5.1) without the Kirchhoff term a. Shi and Wu [134] studied the existence result for Kirchhoff-type problems in Musielak-Sobolev spaces. Chlebicka [39] provides an extensive survey of elliptic partial differential equations in Musielak spaces. Recently, Alves et al. [12] developed the concept of double criticality and studied the quasilinear problem in Musielak-Sobolev spaces, and our existence results are motivated by their work.

We assume that the nonlinear function f has critical growth on Ω_p, which causes a lack of compactness, and hence one can not prove the Palais-Smale condition directly. Lions established concentration compactness principle [99, Lemma 1.1] to address such issues. We use Lemma 2.5.3, the variable exponent version of the concentration compactness principle that was obtained by Bonder and Silva [25]. On the other hand, the nonlinear function f has exponential growth on Ω_N. To deal with such types of nonlinearities, we use Moser-Trudinger inequality stated in the Lemma 2.4.11.

Notations: Throughout this chapter, for any $r \in C(\overline{\Omega},(1,\infty))$, we denote $r^- = \min_{x\in\Omega} r(x)$ and $r^+ = \max_{x\in\Omega} r(x)$. Further, the functions $p,q,p^*,q_1 \in C(\overline{\Omega},(1,\infty))$.

5.2 Hypotheses and variational framework

We consider the following assumptions on the functions $\mathscr{H}$ and h :

$(\mathscr{H}_1)$ $h(x,\cdot) \in C^1$ in $(0,\infty)$, $\forall x \in \Omega$.

$(\mathscr{H}_2)$ $h(x,t), \partial_t(h(x,t)t) > 0$, $\forall x \in \Omega$ and $t > 0$.

$(\mathscr{H}_3)$ $p^- \leq \frac{h(x,|t|)|t|^2}{\mathscr{H}(x,|t|)} \leq q^+$ for $x \in \Omega$ and $t \neq 0$ for some $1 < p^- \leq p(x) < N < q(x) \leq q^+ < (p^*)^-$.

$(\mathscr{H}_4)$ $\inf_{x\in\Omega} \mathscr{H}(x,1) = b_1$ for some $b_1 > 0$.

$(\mathscr{H}_5)$ For each $t_0 \neq 0$, there exists $d_0 > 0$ such that $\frac{\mathscr{H}(x,t)}{t} \geq d_0$ and $\frac{\widetilde{\mathscr{H}}(x,t)}{t} \geq d_0$ for $t \geq t_0$ and $x \in \Omega$, where $\widetilde{\mathscr{H}}(x,t) = \int_0^{|t|} \widetilde{h}(x,s)s\, ds$, $\widetilde{h}$ is the complimentary function of h which is defined as $\widetilde{h}(x,t) = \sup\{s : h(x,s)s \leq t\}$ $\forall (x,t) \in \overline{\Omega} \times [0,\infty)$.

Let $S \subset \Omega$ and $\delta > 0$. The δ neighborhood of S is denoted by S_δ and defined as

$$S_\delta = \{x \in \Omega : \operatorname{dist}(x,S) < \delta\}.$$

Assume that, we have three smooth domains Ω_p, Ω_N and Ω_q with non-empty interiors such that $\Omega = \Omega_p \cup \Omega_N \cup \Omega_q$ and $(\overline{\Omega}_p)_\delta \cap (\overline{\Omega}_N)_\delta = \emptyset$.

Next, we define continuous functions $\psi_p,\ \psi_N,\ \psi_q : \overline{\Omega} \to [0,1]$ such that

$$\psi_p(x) = 1\ \forall x \in \overline{\Omega_p},\ \ \psi_p(x) = 0\ \forall x \in (\overline{\Omega_p})^c_\delta,$$

$$\psi_N(x) = 1\ \forall x \in \overline{\Omega_N},\ \ \psi_N(x) = 0\ \forall x \in (\overline{\Omega_N})^c_\delta,$$

$$\psi_q(x) = 1\ \forall x \in \overline{\Omega_q},\ \ \psi_q(x) = 0\ \forall x \in (\overline{\Omega_q})^c_\delta.$$

We consider that the nonlinear function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and of the following type:

$$f(x,t) = \lambda \psi_N(x)|t|^{\beta-2} t e^{\alpha|t|^{\frac{N}{N-1}}} + \widetilde{\psi}_q(x)\varphi(x,t) + \psi_p(x)|t|^{p^*-2}t\ \ \forall (x,t) \in \Omega \times \mathbb{R}, \tag{f_1}$$

where $(p^*)^+ \geq p^*(x) \geq (p^*)^- > q^+ \geq q(x) \geq q^- > N > p^+ \geq p(x) \geq p^- > N/2$, $\beta > q^-$, $\lambda > 0$ and $\alpha > 0$. Moreover, $\widetilde{\psi}_q : \overline{\Omega} \to [0,1]$, $\varphi : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are continuous functions such that

$$\widetilde{\psi}_q(x) = 1,\ \forall x \in \Omega_q \text{ and } \widetilde{\psi}_q(x) = 0,\ \forall x \in (\overline{\Omega}_q)^c_{\delta/2},$$

and $\varphi(x,t) = o(|t|^{q_1(x)-1})$ as $t \to 0$ uniformly on $(\overline{\Omega}_q)_{\delta/2}$ for some $q_1^+ \geq q_1(x) \geq q_1^- > q^-$, and there exists $\chi > q^-$ such that

$$0 < \chi\Phi(x,t) \leq \varphi(x,t)t,\ \forall x \in (\overline{\Omega}_q)_{\delta/2},$$

where $\Phi(x,t) = \int_0^{|t|} \varphi(x,s)ds$.

Along with the above notations, h also satisfies the following conditions for each $t > 0$:

($\mathscr{H}_6$) $h(x,t) \geq t^{N-2}\ \ \forall x \in \Omega_N$ and $C_1 t^{N-2} \geq h(x,t)\ \forall x \in \Omega_N \backslash (\overline{\Omega}_q)_\delta$ for some $C_1 > 0$.

($\mathscr{H}_7$) There exists a continuous function $\eta_1 : \overline{\Omega} \to \mathbb{R}$ such that $h(x,t) \geq \eta_1(x)t^{q(x)-2}\ \forall x \in (\Omega_q)_\delta$ and $\eta_1(x) > 0,\ \forall x \in (\Omega_q)_\delta$ and $\eta_1(x) = 0,\ \forall x \in ((\Omega_q)_\delta)^c$.

($\mathscr{H}_8$) There exists a non-negative continuous function $\eta_2 : \overline{\Omega_p} \to \mathbb{R}$ such that $\eta_2(x)t^{q(x)-2} + C_2 t^{p(x)-2} \geq h(x,t) \geq t^{p(x)-2}\ \ \forall x \in \Omega_p$ and $\eta_2(x) > 0,\ \forall x \in (\Omega_q)_\delta$ and $\eta_2(x) = 0,\ \forall x \in \overline{\Omega_p} \backslash \overline{(\Omega_q)_\delta}$, for some $C_2 > 0$.

Next, we state our hypotheses on the non-local term a. The continuous function $a : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies following conditions:

(a_1) There exists positive real number a_0 such that $a(s) \geq a_0$ and a is non-decreasing $\forall\ s > 0$.

(a_2) There exists $\theta > 1$ such that $\beta > N\theta$ and $a(s)/s^{\theta-1}$ is non-increasing for $s > 0$.

Remark 5.2.1. *By* (a_2), *we have*

(a_2') $\theta A(s) - a(s)s$ *is non-decreasing,* $\forall\, s > 0$, *where* $A(s) = \int_0^s a(t)dt$.

In particular,

$$\theta A(s) - a(s)s \geq 0 \ \ \forall\, s > 0. \tag{5.2}$$

Again by (a_2) *and* (5.2), *one gets*

(a_2'') $A(s) \leq s^{\theta} A(1)\ \forall\, s \geq 1.$

Due to the double criticality, we discuss our problem in the Musielak Sobolev space $W_0^{1,\mathscr{H}}(\Omega)$. To know more about these spaces, kindly refer Subsection 2.2.3 of Chapter 2.

Definition 5.2.2. *We say that* $u \in W_0^{1,\mathscr{H}}(\Omega)$ *is a weak solution of* (5.1) *if the following holds:*

$$a\,(m(u)) \int_{\Omega} h(x,|\nabla u|)\ \nabla u\ \nabla v = \int_{\Omega} f(x,u)v \tag{5.3}$$

for all $v \in W_0^{1,\mathscr{H}}(\Omega)$.

Thus, the energy functional $J : W_0^{1,\mathscr{H}}(\Omega) \to \mathbb{R}$ corresponding to (5.3) is given by

$$J(u) = A\,(m(u)) - \int_{\Omega} F(x,u)\ dx,$$

where $F(x,t) = \int_0^t f(x,s)ds$ and $A(t) = \int_0^t a(s)ds$. It can be seen that J is C^1 [12, Lemma 3.8] and the derivative of J at any point $u \in W_0^{1,\mathscr{H}}(\Omega)$ is given by

$$J'(u)(v) = a\,(m(u)) \int_{\Omega} h(x,|\nabla u|)\ \nabla u\ \nabla v - \int_{\Omega} f(x,u)v$$

for all $v \in W_0^{1,\mathscr{H}}(\Omega)$. Moreover, the critical points of J are the weak solutions to (5.1). The statement of the main result is as follows:

Theorem 5.2.3. *Suppose that the conditions* (f_1), $(\mathscr{H}_1) - (\mathscr{H}_8)$ *and* $(a_1) - (a_2)$ *are satisfied. Then there exists* $\lambda_1 > 0$ *such that for any* $\lambda \geq \lambda_1$, *Problem (5.1) has non-trivial weak solution via mountain pass theorem.*

5.3 Proof of the Theorem 5.2.3

To demonstrate the existence result, we use a variational technique based on the mountain pass theorem.

First, we demonstrate a few supporting results that help in establishing our main result.

Proposition 5.3.1. *Let Ω be a bounded smooth domain. Then the following embeddings are continuous:*

(a) $W_0^{1,\mathscr{H}}(\Omega) \hookrightarrow L^{\gamma}(\Omega_N)$, $1 \leq \gamma < \infty$,

(b) $W_0^{1,\mathscr{H}}(\Omega) \hookrightarrow L^{s(x)}((\Omega_p)_\delta)$, *where* $s(x) \leq \frac{Np(x)}{N-p(x)}$.

(c) $W_0^{1,\mathscr{H}}(\Omega) \hookrightarrow W_0^{1,q^-}((\Omega_q)_\delta)$.

Moreover, the embedding

$$W_0^{1,\mathscr{H}}(\Omega) \hookrightarrow C(\overline{(\Omega_q)_\delta}) \text{ is compact }. \tag{5.4}$$

Proof. By using the conditions $(\mathscr{H}_6), (\mathscr{H}_7), (\mathscr{H}_8)$ and the definition of $W_0^{1,\mathscr{H}}(\Omega)$, we have continuous embeddings

$$W_0^{1,\mathscr{H}}(\Omega) \hookrightarrow W_0^{1,N}(\Omega_N)$$
$$W_0^{1,\mathscr{H}}(\Omega) \hookrightarrow W_0^{1,q(x)}((\Omega_q)_\delta),$$
$$W_0^{1,\mathscr{H}}(\Omega) \hookrightarrow W_0^{1,p(x)}((\Omega_p)_\delta).$$

Further, $W_0^{1,N}(\Omega_N) \hookrightarrow L^{\gamma}(\Omega_N)$ is continuous for any $1 \leq \gamma < \infty$ [86, Theorem 2.4.4], which proves (a).

We know that $W_0^{1,p(x)}((\Omega_p)_\delta) \hookrightarrow L^{s(x)}((\Omega_p)_\delta)$ is continuous for $s(x) \leq \frac{Np(x)}{N-p(x)}$ [52, Theorem 2.3]. This proves (b).

For (c), as $q^- \leq q(x)$, $W_0^{1,q(x)}((\Omega_q)_\delta) \hookrightarrow W_0^{1,q^-}((\Omega_q)_\delta)$ is continuous. Moreover, since $q^- > N$, $W_0^{1,q^-}((\Omega_q)_\delta) \hookrightarrow C(\overline{(\Omega_q)_\delta})$ is compact [86, Theorem 2.5.3] and this implies that $W_0^{1,\mathscr{H}}(\Omega) \hookrightarrow C(\overline{(\Omega_q)_\delta})$ is compact. □

Lemma 5.3.2. *[51, Theorem 2.2] Suppose that $(\mathscr{H}_1) - (\mathscr{H}_8)$ hold. If $u_n \rightharpoonup u$ in $W_0^{1,\mathscr{H}}(\Omega)$ and*

$$\lim_{n\to\infty} \int_\Omega h(x, |\nabla u_n|)\, \nabla u_n\, \nabla(u_n - u) \leq 0,$$

then $u_n \to u$ in $W_0^{1,\mathscr{H}}(\Omega)$.

Lemma 5.3.3. *There exist positive real numbers α and ρ such that for each $\lambda \geq 1$ we have*

$$J(u) \geq \alpha > 0, \ \forall u \in W_0^{1,\mathscr{H}}(\Omega) : \|u\| = \rho.$$

Proof. It follows, from the definition of f that

$$\int_\Omega F(x,t)dx = \int_{(\Omega_q)_{\delta/2}} F(x,u)dx + \lambda \int_{\Omega_N \setminus (\Omega_q)_{\delta/2}} F_1(x,u)dx + \int_{\Omega_p \setminus (\Omega_q)_{\delta/2}} \frac{|u|^{p^*(x)}}{p^*(x)} dx \tag{5.5}$$

where, $F_1(x,t) = \int_0^t |s|^{\beta-2} s e^{\alpha|s|^{N/(N-1)}} ds$. Again, from the definition of f, we get

$$\int_{(\Omega_q)_{\delta/2}} F(x,u)dx \le c_1 \int_{(\Omega_q)_{\delta/2}} (|u|^{q_1(x)} + |u|^{\beta} + |u|^{p^*(x)}),$$

for $\|u\| = r$, where $r < 1$ is small enough and for some $c_1 > 0$. Using (5.4) and the fact that $\|u\| = r$, where $r < 1$ is small enough, one gets

$$\begin{aligned}\int_{(\Omega_q)_{\delta/2}} F(x,u)dx &\le c_2(\|u\|^{q_1^-}_{L^{q_1^-}(\Omega_q)_{\delta/2}} + \|u\|^{\beta} + \|u\|^{(p^*)^-}_{L^{(p^*)^-}(\Omega_q)_{\delta/2}}) \\ &\le c_3(\|u\|^{q_1^-} + \|u\|^{\beta} + \|u\|^{(p^*)^-}) \end{aligned} \tag{5.6}$$

for some $c_2, c_3 > 0$.

Next, by using the Hölder's inequality, one gets

$$\lambda \int_{\Omega_N \setminus (\Omega_q)_{\delta/2}} F_1(x,u)dx \le \lambda \left(\int_{\Omega_N} |u|^{2\beta} \right)^{\frac{1}{2}} \left(\int_{\Omega_N} e^{2\alpha|u|^{\frac{N}{N-1}}} \right)^{\frac{1}{2}}.$$

Letting, $\|u\| = r < 1$, by Proposition 5.3.1 (a) and Lemma 2.4.11, we obtain

$$\lambda \int_{\Omega_N \setminus (\Omega_q)_{\delta/2}} F_1(x,u)dx \le c_4 \|u\|^{\beta}, \tag{5.7}$$

for some $c_4 > 0$.

Again, using Proposition 5.3.1 (b) and Proposition 2.2.11, we get

$$\int_{\Omega_p \setminus (\Omega_q)_{\delta/2}} \frac{|u|^{p^*(x)}}{p^*(x)} dx \le c_5 \|u\|^{(p^*)^-}, \tag{5.8}$$

for $\|u\| = r$, where $r < 1$ and $c_5 > 0$. By the help of (5.6), (5.7), (5.8), (a_1) and the Proposition 2.2.15, we have

$$J(u) \ge a_0 \|u\|^{q^+} - c_6 \|u\|^{q_1^-} - c_7 \|u\|^{\beta} - c_8 \|u\|^{(p^*)^-},$$

for some $c_6, c_7, c_8 > 0$. We can conclude the result by the fact that $q_1^-, (p^*)^-, \beta > q^+$. □

Lemma 5.3.4. *There exist $v_0 \in W_0^{1,\mathscr{H}}(\Omega)$ and $\beta > 0$ such that for each $\lambda \ge 1$, we have*

$$J(v_0) < 0 \text{ and } \|v_0\| > \beta.$$

Proof. By the definition of f and as $\lambda \ge 1$, we get

$$f(x,s) \ge |s|^{\beta-2}s, \ \forall \ (x,s) \in (\Omega_N \setminus \overline{(\Omega_q)_{\delta}}) \times [0,\infty). \tag{5.9}$$

Let $u \in C_c^\infty(\Omega_N \backslash \overline{(\Omega_q)_\delta}) \backslash \{0\}$ with $\|u\| = 1$, using $(\mathscr{H}_6)$, (a_2'') and (5.9), we have

$$\begin{aligned} J(tu) &= A(m(tu)) - \int_\Omega F(x,tu)dx \\ &\leq A(1)\left(\int_{\Omega_N} \mathscr{H}(x,|\nabla tu|)dx\right)^\theta - \frac{|t|^\beta}{\beta}\int_{\Omega_N}|u|^\beta dx,\ \forall\, t>1, \\ &\leq A(1)c_1 t^{N\theta}\left(\int_{\Omega_N}|\nabla u|^N\right)^\theta - \frac{|t|^\beta}{\beta}\int_{\Omega_N}|u|^\beta dx,\ \forall\, t>1, \end{aligned}$$

this implies that $J(tu) \to -\infty$ as $n \to \infty$, since $\beta > N\theta$. Now, by setting $v_0 = t_0 u$ for sufficiently large $t_0 > 1$, we get the desired result. □

By Lemmas 5.3.3 and 5.3.4, the geometric conditions of the mountain pass theorem are satisfied for the functional J. Hence, by the version of the mountain pass theorem without (PS) condition, $\exists$ a sequence $\{u_n\} \subseteq W_0^{1,\mathscr{H}}(\Omega)$ such that $J(u_n) \to c_M$ and $J'(u_n) \to 0$ as $n \to \infty$, where

$$c_M = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)) > 0,$$

and

$$\Gamma = \{\gamma \in C([0,1], W_0^{1,\mathscr{H}}(\Omega)) : \gamma(0) = 0,\ \gamma(1) < 0\}.$$

In this section, we gather some information about $(PS)_{c_M}$ sequence and prove the $(PS)_{c_M}$ condition holds for the functional J.

Lemma 5.3.5. *The $(PS)_{c_M}$ sequence is bounded in $W_0^{1,\mathscr{H}}(\Omega)$. Moreover, there exists $u_0 \in W_0^{1,\mathscr{H}}(\Omega)$ such that, up to a subsequence, we have $u_n \rightharpoonup u_0$ weakly in $W_0^{1,\mathscr{H}}(\Omega)$ and $u_n(x) \to u_0(x)$ a.e. $x \in \Omega$.*

Proof. If $\psi = \min\{\chi, \beta, (p^*)^-\}$, we have

$$0 < \psi F(x,t) \leq f(x,t)t,\ \forall (x,t) \in \Omega \times (\mathbb{R}\backslash\{0\}). \tag{5.10}$$

Since $\{u_n\}$ is a $(PS)_{c_M}$ sequence for J, we have $J(u_n) \to c_M$ and $J'(u_n) \to 0$ as $n \to \infty$, i.e.,

$$A(m(u_n)) - \int_\Omega F(x,u_n)\,dx = c_M + \delta_n, \tag{5.11}$$

where $\delta_n \to 0$ as $n \to \infty$ and

$$\left| a(m(u_n)) \int_\Omega h(x,|\nabla u_n|)\,\nabla u_n\,\nabla v dx - \int_\Omega f(x,u_n)v \right| \leq \varepsilon_n \|v\|, \tag{5.12}$$

$\forall\, v \in W_0^{1,\mathscr{H}}(\Omega)$, where $\varepsilon_n \to 0$ as $n \to \infty$. On taking $v = u_n$, by using (5.10), (5.11) and (5.12), we obtain

$$A\left(m(u_n)\right) - \frac{1}{\psi} a\left(m(u_n)\right) \int_\Omega h(x, |\nabla u_n|)\, |\nabla u_n|^2 \\ \leq c_9(1 + \|u_n\|),$$

for some $c_9 > 0$. It follows from (a_1) that

$$a_0 \left(1 - \frac{q^+}{\psi}\right) m(u_n) \leq c_9(1 + \|u_n\|).$$

If $\|u\| \geq 1$, by Proposition 2.2.15, we obtain

$$a_0 \left(1 - \frac{q^+}{\psi}\right) \|u_n\|^{p^-} \leq c_9(1 + \|u_n\|).$$

This implies that $\{u_n\}$ is bounded in $W_0^{1,\mathscr{H}}(\Omega)$. As $W_0^{1,\mathscr{H}}(\Omega)$ is a reflexive space, $\exists\, u_0 \in W_0^{1,\mathscr{H}}(\Omega)$ such that up to a subsequence, we have $u_n \rightharpoonup u_0$ weakly in $W_0^{1,\mathscr{H}}(\Omega)$. Further, we have $u_n(x) \to u_0(x)$ a.e. $x \in \Omega$. □

Due to the lack of compactness, we are not able to prove that (PS) condition holds for J and we need some additional information about the mountain pass level c_M.

Lemma 5.3.6. *There exists $\lambda_1 > 1$ such that for each $\lambda \geq \lambda_1$, we have*

$$c_M < a_0 \left(1 - \frac{q^+}{\psi}\right) \min\left\{ \frac{1}{N} \left(\frac{\alpha_N}{2^{\frac{N\alpha}{N-1}}}\right)^{N-1}, \frac{a_{\min}}{p^+} S^N \right\},$$

where $\psi = \min\{\chi, \beta, (p^*)^-\}$ and $a_{\min} = \min_{x\in\Omega} a_0^{(N/p(x))-1}$. Moreover, for any $(PS)_{c_M}$ sequence $\{u_n\}$, we have

$$\limsup_{n\to\infty} \|\nabla u_n\|_{L^N(\Omega_N)}^{N/(N-1)} < \frac{\alpha_N}{2^{\frac{N\alpha}{N-1}}}.$$

Proof. If $0 \leq s \leq t_1 = \max\{t_0^{q^+}, t_0^{p^-}\}$, then $A(s) \leq a(t_1)s$, where t_0 is defined in the proof of the Lemma 5.3.4 . Let $v_0 \in C_c^\infty(\Omega_N \backslash \overline{(\Omega_q)_\delta}) \backslash \{0\}$ be as in the Lemma 5.3.4 and $0 \leq t \leq 1$. By using $(\mathscr{H}_6)$, (5.9) and Proposition 2.2.15, we have

$$J(t v_0) = A\left(m(t v_0)\right) - \int_\Omega F(x, t v_0) dx \\ \leq a(t_1)\left(m(t v_0)\right) - \frac{\lambda |t|^\beta}{\beta} \int_{\Omega_N} |v_0|^\beta dx,$$

$$\leq \frac{a(t_1)t^N C_1}{N}\left(\int_{\Omega_N}|\nabla v_0|^N\right) - \frac{\lambda|t|^\beta}{\beta}\int_{\Omega_N}|v_0|^\beta dx.$$

Further, we get

$$\max_{0\leq t\leq 1} J(tv_0) \leq \frac{1}{\lambda^{\frac{N}{\beta-N}}}\left(\frac{1}{N}-\frac{1}{\beta}\right)\frac{\left(a(t_1)C_1\|\nabla v_0\|^N_{L^N(\Omega_N)}\right)^{\frac{\beta}{\beta-N}}}{\left(\|v_0\|^\beta_{L^\beta(\Omega_N)}\right)^{\frac{N}{\beta-N}}}.$$

On taking, $\gamma = tv_0$ for $0 \leq t \leq 1$, we have

$$c_M \leq \max_{0\leq t\leq 1} J(tv_0) \leq \frac{1}{\lambda^{\frac{N}{\beta-N}}}\left(\frac{1}{N}-\frac{1}{\beta}\right)\frac{\left(a(t_1)C_1\|\nabla v_0\|^N_{L^N(\Omega_N)}\right)^{\frac{\beta}{\beta-N}}}{\left(\|v_0\|^\beta_{L^\beta(\Omega_N)}\right)^{\frac{N}{\beta-N}}}.$$

Now, choosing $\lambda_1 > 1$ in such a way that $\forall\ \lambda \geq \lambda_1$, we have

$$\frac{1}{\lambda^{\frac{N}{\beta-N}}}\left(\frac{1}{N}-\frac{1}{\beta}\right)\frac{\left(a(t_1)C_1\|\nabla v_0\|^N_{L^N(\Omega_N)}\right)^{\frac{\beta}{\beta-N}}}{\left(\|v_0\|^\beta_{L^\beta(\Omega_N)}\right)^{\frac{N}{\beta-N}}} < a_0\left(1-\frac{q^+}{\psi}\right)\min\left\{\frac{1}{N}\left(\frac{\alpha_N}{2^{\frac{N\alpha}{N-1}}}\right)^{N-1}, \frac{a_{\min}}{p^+}S^N\right\}.$$

Therefore,

$$c_M < a_0\left(1-\frac{q^+}{\psi}\right)\min\left\{\frac{1}{N}\left(\frac{\alpha_N}{2^{\frac{N\alpha}{N-1}}}\right)^{N-1}, \frac{a_{\min}}{p^+}S^N\right\},\ \forall\lambda \geq \lambda_1. \tag{5.13}$$

Moreover, since $\{u_n\}$ is a $(PS)_{c_M}$ sequence for J, we have $J(u_n) \to c_M$ and $J'(u_n) \to 0$ as $n \to \infty$. By (5.11) and (5.12), we get

$$A\,(m(u_n)) - \frac{1}{\psi}a\,(m(u_n))\int_\Omega h(x,|\nabla u_n|)\,|\nabla u_n|^2 \leq \delta_n + c_M + \varepsilon_n\|u_n\|.$$

It follows from (a_1) that

$$a_0\left(1-\frac{q^+}{\psi}\right)m(u_n) \leq \delta_n + c_M + \varepsilon_n\|u_n\|.$$

By $(\mathscr{H}_6)$ and (5.13), we get

$$\limsup_{n\to\infty} \|\nabla u_n\|_{L^N(\Omega_N)}^{N/(N-1)} \leq c_M < \frac{\alpha_N}{2^{\frac{N\alpha}{N-1}}}.$$

□

Lemma 5.3.7. *The functional J satisfies the $(PS)_{c_M}$ condition.*

Proof. Define

$$P_n = a(m(u_n)) \int_\Omega h(x, |\nabla u_n|)\, \nabla u_n\, \nabla(u_n - u).$$

Then

$$P_n = J'(u_n)u_u + \int_\Omega f(x,u_n)u_n dx - J'(u_n)u - \int_\Omega f(x,u_n)udx.$$

Using the definition of f, P_n can be rewritten as

$$\begin{aligned} P_n = & \int_\Omega \widetilde{\psi}_q(x)\varphi(x,u_n)u_n dx + \int_{\Omega_N} \lambda |u_n|^\beta e^{\alpha|u_n|^{\frac{N}{N-1}}} dx + \int_{\Omega\setminus\Omega_N} \lambda \psi_N(x)|u_n|^\beta e^{\alpha|u_n|^{\frac{N}{N-1}}} dx \\ & + \int_{\Omega_p} |u_n|^{p^*(x)} dx + \int_{\Omega\setminus\Omega_p} \psi_p(x)|u_n|^{p^*(x)} dx - \int_\Omega \widetilde{\psi}_q(x)\varphi(x,u_n)udx \\ & - \int_{\Omega_N} \lambda |u_n|^{\beta-2}u_n e^{\alpha|u_n|^{\frac{N}{N-1}}} udx - \int_{\Omega\setminus\Omega_N} \lambda \psi_N(x)|u_n|^{\beta-2}u_n e^{\alpha|u_n|^{\frac{N}{N-1}}} udx \\ & - \int_{\Omega_p} |u_n|^{p^*(x)-2}u_n udx - \int_{\Omega\setminus\Omega_p} \psi_p(x)|u_n|^{p^*(x)-2}u_n udx + o_n(1). \end{aligned}$$

From the embedding results, we have

$$\begin{aligned} P_n = & \lambda \int_{\Omega_N} |u_n|^\beta e^{\alpha|u_n|^{\frac{N}{N-1}}} dx + \int_{\Omega_p} |u_n|^{p^*(x)} dx \\ & - \lambda \int_{\Omega_N} |u_n|^{\beta-2}u_n e^{\alpha|u_n|^{\frac{N}{N-1}}} udx - \int_{\Omega_p} |u_n|^{p^*(x)-2}u_n udx + o_n(1). \end{aligned}$$

As proved in the [12, Lemma 3.13], we have

$$P_n = \int_{\Omega_p} |u_n|^{p^*(x)} dx - \int_{\Omega_p} |u_n|^{p^*(x)-2}u_n udx + o_n(1).$$

By [52, Theorem 1.14] and [85, Theorem 3.1], one gets

$$P_n = \int_{\Omega_p} |u_n|^{p^*(x)} dx - \int_{\Omega_p} |u|^{p^*(x)} dx + o_n(1).$$

Next, we will apply the Lemma 2.5.3 to the sequence $\{u_n\} \subset W_0^{1,p(x)}(\Omega_p)$ and will prove that

$$\int_{\Omega_p} |u_n|^{p^*(x)} dx \to \int_{\Omega_p} |u|^{p^*(x)} dx. \tag{5.14}$$

Since, the $W^{1,\mathscr{H}}(\Omega) \hookrightarrow C(\overline{(\Omega_q)_\delta})$ is compact and $\{u_n\}$ is bounded in $W^{1,\mathscr{H}}(\Omega)$, we get $u_n \to u$ in $L^{p^*(x)}((\Omega_q)_\delta)$, which implies that $x_i \in \overline{\Omega_p} \backslash (\Omega_q)_\delta$ for each $i \in I$. To prove (5.14), it is suffices to prove that I is finite. Further, the set I can be partitioned as $I = I_1 \cup I_2$, where $I_1 = \{i \in I : x_i \in \Omega_p \cap \partial(\Omega_q)_\delta\}$ and $I_2 = \{i \in I : x_i \in \overline{\Omega_p} \backslash \overline{(\Omega_q)_\delta}\}$. First, we show that I_1 is finite. Choose a cutoff function $v_0 \in C_c^\infty(\mathbb{R}^N)$ such that

$$v_0 \equiv 1 \text{ on } B(0,1), \ v_0 \equiv 0 \text{ on } B(0,2)^c.$$

Now, for each $\varepsilon > 0$, define $v(x) = v_0((x - x_i)/\varepsilon)$ $\forall x \in \mathbb{R}^N$. As $\{u_n\}$ is a $(PS)_{c_M}$ sequence, we have

$$\begin{aligned} a(m(u_n)) \int_\Omega h(x, |\nabla u_n|) \nabla u_n \nabla(v u_n) &= \int_\Omega \widetilde{\psi}_q(x) \varphi(x, u_n) v u_n dx \\ &+ \int_\Omega \psi_p(x) |u_n|^{p^*(x)} v dx + o_n(1). \end{aligned}$$

It follows from (a_1) that

$$a_0 \int_\Omega h(x, |\nabla u_n|) \nabla u_n \nabla(v u_n) dx \le \int_\Omega \widetilde{\psi}_q(x) \varphi(x, u_n) v u_n dx + \int_\Omega \psi_p(x) |u_n|^{p^*(x)} v dx + o_n(1),$$

which implies that

$$\begin{aligned} a_0 \int_\Omega h(x, |\nabla u_n|) u_n \nabla u_n \nabla v dx &\le \int_\Omega \widetilde{\psi}_q(x) \varphi(x, u_n) v u_n dx \\ &+ \int_\Omega \psi_p(x) |u_n|^{p^*(x)} v dx - a_0 \int_\Omega h(x, |\nabla u_n|) |\nabla u_n|^2 v dx + o_n(1). \end{aligned} \tag{5.15}$$

Next, by using (2.3), Δ_2-condition and Young's inequality, we get

$$\int_\Omega |h(x, |\nabla u_n|)| |\nabla u_n| |u_n| |\nabla v| dx \le \zeta m(u_n) + C_\zeta \int_\Omega \mathscr{H}(x, |u_n| |\nabla v|) dx. \tag{5.16}$$

On using $(\mathscr{H}_8)$, one gets

$$\int_\Omega \mathscr{H}(x, |u_n| |\nabla v|) dx \le c_{10} \left(\int_\Omega \eta_2(x) |u_n|^{q(x)} |\nabla v|^{q(x)} dx + \int_\Omega |u_n|^{p(x)} |\nabla v|^{p(x)} dx \right),$$

for some $c_{10} > 0$. Using generalized Hölder's inequality 2.2.10, we get

$$\int_{\Omega} \mathscr{H}(x, |u_n|\ |\nabla v|)dx \le c_{11} \| |\nabla v|^{q(x)} \|_{L^{\frac{p^*(x)}{p^*(x)-q(x)}}(\Omega)} \| |u_n|^{q(x)} \|_{L^{\frac{p^*(x)}{q(x)}}(\Omega)} + c_{12} \| |\nabla v|^{p(x)} \|_{L^{\frac{N}{p(x)}}(\Omega)} \| |u_n|^{p(x)} \|_{L^{\frac{N}{N-p(x)}}(\Omega)},$$

for some $c_{11}, c_{12} > 0$. Further, by Proposition 2.2.11 and Proposition 2.2.12, we have

$$\begin{aligned}\int_{\Omega} \mathscr{H}(x, |u_n|\ |\nabla v|)dx \le{}& c_{11} \max\left\{ \left(\int_{B(x_i,2\varepsilon)} |\nabla v|^{\frac{q(x)p^*(x)}{p^*(x)-q(x)}} \right)^{p_1}, \left(\int_{B(x_i,2\varepsilon)} |\nabla v|^{\frac{q(x)p^*(x)}{p^*(x)-q(x)}} \right)^{p_2} \right\} \\ & \max\left\{ \|u_n\|^{q^-}_{L^{p^*(x)}(\Omega)}, \|u_n\|^{q^+}_{L^{p^*(x)}(\Omega)} \right\} \\ & + c_{12} \max\left\{ \left(\int_{B(x_i,2\varepsilon)} |\nabla v|^N \right)^{p^-/N}, \left(\int_{B(x_i,2\varepsilon)} |\nabla v|^N \right)^{p^+/N} \right\} \\ & \max\left\{ \|u_n\|^{p^-}_{L^{p^*(x)}(\Omega)}, \|u_n\|^{p^+}_{L^{p^*(x)}(\Omega)} \right\},\end{aligned}$$

where $p_1 = \min_{x\in\Omega} \left\{ \frac{p^*(x)-q(x)}{p^*(x)} \right\}$ and $p_2 = \max_{x\in\Omega} \left\{ \frac{p^*(x)-q(x)}{p^*(x)} \right\}$ · Hence, by using Proposition 5.3.1(b) together with the fact that $\{u_n\}$ is bounded, we have

$$\lim_{\varepsilon\to 0} \int_{\Omega} \mathscr{H}(x, |u_n|\ |\nabla v|) = 0. \tag{5.17}$$

By using (5.16), (5.17), Proposition 2.2.15 and using the boundedness of $\{u_n\}$, we get

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \int_{\Omega} |h(x, |\nabla u_n|)|\ |\nabla u_n||u_n|\ |\nabla v| dx \le c_{12}\zeta,$$

for some $c_{12} > 0$. As ζ is arbitrary, one get

$$\lim_{\varepsilon\to 0} \left(\lim_{n\to\infty} \int_{\Omega} |h(x, |\nabla u_n|)|\ |\nabla u_n||u_n|\ |\nabla v| dx \right) = 0. \tag{5.18}$$

Consequently, by $(\mathscr{H}_8)$, (5.15) and (5.18), we get

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \left(\int_{\Omega} \widetilde{\psi}_q(x)\varphi(x, u_n) v u_n dx + \int_{\Omega} \psi_p(x)|u_n|^{p^*(x)} v dx - a_0 \int_{\Omega_p} |\nabla u_n|^{p(x)}\ v dx \right) \ge 0,$$

which is

$$\nu_i - a_0\mu_i = \lim_{\varepsilon\to 0} \left(\lim_{n\to\infty} \int_{\Omega_p} |u_n|^{p^*(x)} v dx - a_0 \lim_{n\to\infty} \int_{\Omega_p} |\nabla u_n|^{p(x)}\ v dx \right) \ge 0.$$

By Lemma 2.5.3, we get $\nu_i \ge a_0 S^{p(x_i)} \nu_i^{p(x_i)/p^*(x_i)}$, consequently either $\nu_i = 0$ or $\nu_i \ge a_0^{\frac{N}{p(x_i)}} S^N$.

Next, we will prove that $\nu_i \geq a_0^{\frac{N}{p(x_i)}} S^N$ is not possible. Let suppose $\nu_i \geq a_0^{\frac{N}{p(x_i)}} S^N$, then by Lemma 2.5.3, we get $\mu_i \geq S^N a_0^{\frac{N}{p(x_i)}-1}$. Also, since $|\nabla u_n|^{p(x)}$ converges weakly to a measure μ,

$$\liminf_{n\to\infty} \int_{\Omega_p} |\nabla u_n|^{p(x)} dx \geq \mu_i,$$

and hence,

$$\liminf_{n\to\infty} \int_{\Omega_p} |\nabla u_n|^{p(x)} dx \geq S^N a_0^{\frac{N}{p(x_i)}-1} \geq S^N a_{\min}, \tag{5.19}$$

where, $a_{\min} = \min_{x\in\Omega} a_0^{(N/p(x_i))-1}$.

Since, $\{u_n\}$ is a $(PS)_{c_M}$ sequence for J, we have $J(u_n) \to c_M$ and $J'(u_n) \to 0$ as $n \to \infty$. By (5.10), (5.11) and (5.12), we get

$$A(m(u_n)) - \frac{1}{\psi} a(m(u_n)) \int_\Omega h(x, |\nabla u_n|)\, |\nabla u_n|^2 dx \\ \leq \delta_n + c_M + \varepsilon_n \|u_n\|.$$

It follows from (a_1) that

$$a_0 \left(1 - \frac{q^+}{\psi}\right) m(u_n) \leq \delta_n + c_M + \varepsilon_n \|u_n\|.$$

By $(\mathscr{H}_8)$ and (5.13), we obtain

$$\frac{a_0}{p^+}\left(1 - \frac{q^+}{\psi}\right) \liminf_{n\to\infty} \int_{\Omega_p} |\nabla u_n|^{p(x)} dx \leq c_M < \frac{a_0}{p^+}\left(1 - \frac{q^+}{\psi}\right) S^N a_{\min},$$

which is a contradiction to (5.19). Hence, I_1 is an empty set.

By using $(\mathscr{H}_8)$ and proceeding as above, one can show that $I_2 = \emptyset$.

Therefore, we get $P_n = o_n(1)$, and so

$$\lim_{n\to\infty} a(m(u_n)) \int_\Omega h(x, |\nabla u_n|)\, \nabla u_n\, \nabla(u_n - u) = 0,$$

from which we have

$$\lim_{n\to\infty} \int_\Omega h(x, |\nabla u_n|)\, \nabla u_n\, \nabla(u_n - u) \leq 0.$$

By Lemma 5.3.2, we have $u_n \to u$ in $W_0^{1,\mathscr{H}}(\Omega)$. □

Now, we are ready to prove the Theorem 5.2.3.

Proof of the Theorem 5.2.3. By Lemmas 5.3.3 and 5.3.4, the geometric conditions of the mountain pass theorem are satisfied for the functional J and by Lemma 5.3.6, $(PS)_{c_M}$ condition is satisfied. Hence, by the mountain pass theorem, $\exists$ a critical point u_M of J with level c_M, i.e., $J'(u_M) = 0$ and $J(u_M) = c_M$. Thus, u_M is the weak solution of the problem (5.1). $\square$

5.4 Examples

Example 5.4.1. *A typical example of a function a satisfying the conditions* $(a_1) - (a_2)$ *is given by*

$$a(t) = 1 + t^b$$

where $b > 0$ *and* $b + 1 < \frac{\beta}{N}$ *(By condition* (f_1) *and using the fact that* $N \geq 2$*, we have* $\frac{\beta}{N} > 1$*). Clearly,* $a(t) \geq 1$ *and* $a'(t) = bt^{b-1} > 0, \forall t > 0$ *which implies that* (a_1) *is satisfied. Consider,*

$$\begin{aligned} \frac{d}{dt}\left(\frac{1+t^b}{t^{\theta-1}}\right) &= \frac{t^{\theta-1}bt^{b-1} - (\theta-1)(1+t^b)t^{\theta-2}}{t^{2(\theta-1)}} \\ &= \frac{bt^b - (\theta-1)(1+t^b)}{t^{\theta}} \leq 0 \end{aligned}$$

for all $t > 0$ *if we choose* θ *in such a way that* $\theta \in \left(b+1, \frac{\beta}{N}\right)$*. This implies* $\frac{a(t)}{t^{\theta-1}} = \frac{1+t^b}{t^{\theta-1}}$ *is non-increasing for* $\theta \in \left(b+1, \frac{\beta}{N}\right)$*, hence* (a_2) *is satisfied.*

Example 5.4.2. *The model function for* $\mathscr{H}$ *satisfying the conditions* $(\mathscr{H}_1) - (\mathscr{H}_8)$ *is given by*

$$\mathscr{H}(x,t) = \frac{\psi_q(x)}{q}|t|^q + \frac{\psi_p(x)}{p}|t|^p + \frac{\psi_N(x)}{N}|t|^N.$$

Chapter 6

Generalized Choquard Schrödinger equation with vanishing potential

This chapter[1] deals with Hardy-Littlewood-Sobolev type inequality for Lebesgue Mus-ielak spaces and their applications to the class of generalized Choquard Schrödinger equations in homogeneous fractional Musielak-Sobolev space.

In Section 6.1, we introduce the homogeneous fractional Musielak-Sobolev space and investigate their properties. Section 6.2 deals with the existence of a solution to the class of generalized Choquard Schrödinger equations with vanishing potential. Finally, in Section 6.3, we prove the existence of a ground-state solution.

6.1 Homogeneous fractional Musielak-Sobolev space

This section introduces fractional homogeneous Musielak-Sobolev space and investigates its properties. Let $\mathscr{H}$ be any locally integrable generalized N-function such that

$$\mathscr{H}(x,y,t) = \int_0^{|t|} h(x,y,s)s\, ds,$$

where $h : \mathbb{R}^N \times \mathbb{R}^N \times [0,\infty) \to [0,\infty)$.

To know more about the generalized N-function $\mathscr{H}$, $[\cdot]_{s,\mathscr{H}}$ and $\mathscr{H}^*$, please refer Section 2.2 of Chapter 2.

We consider the following assumptions on the functions $\mathscr{H}$ and h :

$(\mathscr{H}_1)$ $h(x,y,\cdot) \in C^1$ in $(0,\infty)$, $\forall x,y \in \mathbb{R}^N$.

[1] Shilpa Gupta, Gaurav Dwivedi, Ground state solution for a generalized Choquard Schrödinger equation with vanishing potential in homogeneous fractional Musielak Sobolev spaces (Communicated).

$(\mathscr{H}_2)$ $h_1 \leq \frac{h(x,y,|t|)|t|^2}{\mathscr{H}(x,y,|t|)} \leq h_2 < N$ for all $x, y \in \mathbb{R}^N$ and $t \neq 0$ for some $1 \leq h_1 < h_2 < h_1^*$, where $h_1^* = \frac{Nh_1}{N-sh_1} \leq h_2^* = \frac{Nh_2}{N-sh_2}$.

$(\mathscr{H}_3)$ $\inf_{x,y\in\mathbb{R}^N} \mathscr{H}(x,y,1) = b_1$ and $\sup_{x,y\in\mathbb{R}^N} \mathscr{H}(x,y,1) = b_2$ for some $b_1, b_2 > 0$.

$(\mathscr{H}_4)$ $\int_a^\infty \left(\frac{t}{\mathscr{H}(t)}\right)^{\frac{s}{N-s}} dt = \infty$ and $\int_0^b \left(\frac{t}{\mathscr{H}(t)}\right)^{\frac{s}{N-s}} dt < \infty$, for some $a, b > 0$.

Fractional Musielak-Sobolev spaces are not sufficient to study the Problem (2.10), as $\inf V(x)$ can be zero. In this section, we introduce the suitable space to study Problem (2.10) which we called homogeneous fractional Musielak-Sobolev space and investigate their properties.

One can verify that the space $C_c^\infty(\mathbb{R}^N)$ is normed space with the norm $[\cdot]_{s,\mathscr{H}}$. However, the normed space $(C_c^\infty(\mathbb{R}^N), [\cdot]_{s,\mathscr{H}})$ is not complete. Further, we define the completion $D^{s,\mathscr{H}}(\mathbb{R}^N)$ of $(C_c^\infty(\mathbb{R}^N), [\cdot]_{s,\mathscr{H}})$ in the standard way. More precisely,

$$D^{s,\mathscr{H}}(\mathbb{R}^N) = \left\{[u_n] : \{u_n\} \subseteq C_c^\infty(\mathbb{R}^N) \text{ is a Cauchy sequence under the norm } [\cdot]_{s,\mathscr{H}}\right\},$$

where $[u_n]$ is the equivalence class of the Cauchy sequence $\{u_n\}$ with the equivalence relation $' \sim'_{s,\mathscr{H}}$, which is defined as $\{u_n\} \sim_{s,\mathscr{H}} \{v_n\}$ iff $\lim_{n\to\infty} [u_n - v_n]_{s,\mathscr{H}} = 0$.

$D^{s,\mathscr{H}}(\mathbb{R}^N)$ is the Banach space with the norm $\|[u_n]\|_{D^{s,\mathscr{H}}(\mathbb{R}^N)} = \lim_{n\to\infty} [u_n]_{s,\mathscr{H}}$.

6.1.1 Characterization

Next, we prove the characterization of the normed space $(D^{s,\mathscr{H}}(\mathbb{R}^N), \|\cdot\|_{D^{s,\mathscr{H}}(\mathbb{R}^N)})$. Consider the space

$$\mathring{W}^{s,\mathscr{H}}(\mathbb{R}^N) = \left\{u \in L^{\mathscr{H}^*}(\mathbb{R}^N) : [u]_{s,\mathscr{H}} < \infty\right\}.$$

$\mathring{W}^{s,\mathscr{H}}(\mathbb{R}^N)$ is a normed space with the norm $\|u\|_{\mathring{W}^{s,\mathscr{H}}(\mathbb{R}^N)} = [u]_{s,\mathscr{H}}$.

Theorem 6.1.1. *Let $\mathscr{H}$ be a generalized N-function and $s \in (0,1)$. Then $C_c^\infty(\mathbb{R}^N)$ is the dense subspace of $\mathring{W}^{s,\mathscr{H}}(\mathbb{R}^N)$. Moreover, there exists an linear isomorphism between $\mathring{W}^{s,\mathscr{H}}(\mathbb{R}^N)$ and $D^{s,\mathscr{H}}(\mathbb{R}^N)$. In other words, the space $D^{s,\mathscr{H}}(\mathbb{R}^N)$ can be identified as $\mathring{W}^{s,\mathscr{H}}(\mathbb{R}^N)$ and $\|\cdot\|_{D^{s,\mathscr{H}}(\mathbb{R}^N)} = \|\cdot\|_{\mathring{W}^{s,\mathscr{H}}(\mathbb{R}^N)} = [\cdot]_{s,\mathscr{H}}$.*

Proof. We present the proof of the theorem in three steps:

Step 1: In this step, we will prove that $C_c^\infty(\mathbb{R}^N)$ is dense in $\mathring{W}^{s,\mathscr{H}}(\mathbb{R}^N)$, i.e., for any $u \in \mathring{W}^{s,\mathscr{H}}(\mathbb{R}^N)$ there exists a sequence in $(C_c^\infty(\mathbb{R}^N), [\cdot]_{s,\mathscr{H}})$ which converges to u in $\mathring{W}^{s,\mathscr{H}}(\mathbb{R}^N)$.

Let $\rho \in C_c^\infty(\mathbb{R}^N)$ be the standard mollifier with support inside $B_1(0)$. Define, $\rho_\varepsilon(x) = \varepsilon^{-n}\rho\left(\frac{x}{\varepsilon}\right)$. It can be seen that $\rho_\varepsilon(x) \in C_c^\infty(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} \rho_\varepsilon(x)dx = 1$ and support of ρ_ε belongs to $B_\varepsilon(0)$.

Let $u \in \mathring{W}^{s,\mathscr{H}}(\mathbb{R}^N)$ be any arbitrary element. Then $u_\varepsilon = \rho_\varepsilon * u \in C^\infty(\mathbb{R}^N)$. Next, we claim that $[u_\varepsilon - u]_{s,\mathscr{H}} \to 0$ as $\varepsilon \to 0$.

By using Proposition 2.2.5, Remarks 2.2.7, 2.2.16 and the properties of mollifiers, we have

$$\begin{aligned}
[u_\varepsilon - u]_{s,\mathscr{H}} &= \left\| \frac{(u_\varepsilon(x) - u(x)) - (u_\varepsilon(y) - u(y))}{|x-y|^s} \right\|_{L^{\mathscr{H}}(\mathbb{R}^N \times \mathbb{R}^N; d\mu)} \\
&= \sup_{\|v\|_{L^{\widehat{\mathscr{H}}}(\mathbb{R}^N \times \mathbb{R}^N; d\mu)} \le 1} \left\{ \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_\varepsilon(x) - u(x)) - (u_\varepsilon(y) - u(y))}{|x-y|^s} v(x,y) d\mu \right| \right\} \\
&\le \sup_{\|v\|_{L^{\widehat{\mathscr{H}}}(\mathbb{R}^N \times \mathbb{R}^N; d\mu)} \le 1} 2\|v\|_{L^{\widehat{\mathscr{H}}}(\mathbb{R}^N \times \mathbb{R}^N; d\mu)} \\
&\left\{ \int_{|\xi|<1} \rho(\xi) d\xi \left\| \frac{(u(x-\varepsilon\xi) - u(y-\varepsilon\xi)) - (u(x) - u(y))}{|x-y|^s} \right\|_{L^{\mathscr{H}}(\mathbb{R}^N \times \mathbb{R}^N; d\mu)} \right\} \\
&= 2\int_{|\xi|<1} \rho(\xi) \left\| \frac{(u(x-\varepsilon\xi) - u(y-\varepsilon\xi)) - (u(x) - u(y))}{|x-y|^s} \right\|_{L^{\mathscr{H}}(\mathbb{R}^N \times \mathbb{R}^N; d\mu)} d\xi.
\end{aligned}$$

As we know that, $w(x,y) = \dfrac{|u(x) - u(y)|}{|x-y|^s} \in L^{\mathscr{H}}(\mathbb{R}^N \times \mathbb{R}^N; d\mu)$ and $C_c^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ is dense in $L^{\mathscr{H}}(\mathbb{R}^N \times \mathbb{R}^N; d\mu)$, hence, for a given $\varepsilon > 0$ there exists $k(x,y) \in C_c^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ such that $\|w - k\|_{L^{\mathscr{H}}(\mathbb{R}^N \times \mathbb{R}^N; d\mu)} \le \frac{\varepsilon}{4}$. Further, we have

$$\|w(x-\varepsilon\xi, y-\varepsilon\xi) - k(x-\varepsilon\xi, y-\varepsilon\xi)\|_{L^{\mathscr{H}}(\mathbb{R}^N \times \mathbb{R}^N; d\mu)} \le \frac{\varepsilon}{4}$$

and

$$\|K(x-\varepsilon\xi, y-\varepsilon\xi) - k(x,y)\|_{L^{\mathscr{H}}(\mathbb{R}^N \times \mathbb{R}^N; d\mu)} \le \frac{\varepsilon}{4}$$

for sufficiently small ε and for all $|\xi| \le 1$. Therefore, we get $[u_\varepsilon - u]_{s,\mathscr{H}} \le \varepsilon$. As ε was arbitrary, we get $[u_\varepsilon - u]_{s,\mathscr{H}} \to 0$ as $\varepsilon \to 0$. We conclude the claim by using Theorem 2.2.4.

Step 2: Let $\{u_n\} \subseteq (C_c^\infty(\mathbb{R}^N), [\cdot]_{s,\mathscr{H}})$ be a Cauchy sequence.

Claim: There exists $u \in \mathring{W}^{s,\mathscr{H}}(\mathbb{R}^N)$ such that $u_n \to u$ in $\mathring{W}^{s,\mathscr{H}}(\mathbb{R}^N)$.

By Theorem 2.2.19, we have

$$\|u_n\|_{L^{\mathscr{H}^*}(\mathbb{R}^N)} \le c[u_n]_{s,\mathscr{H}} < \infty, \ \forall n.$$

Hence, $\{u_n\} \subseteq L^{\mathscr{H}^*}(\mathbb{R}^N)$ and $\{u_n\}$ is Cauchy sequence in $L^{\mathscr{H}^*}(\mathbb{R}^N)$. As we know that $L^{\mathscr{H}^*}(\mathbb{R}^N)$ is a Banach space, thus there exists $u \in L^{\mathscr{H}^*}(\mathbb{R}^N)$ such that $u_n \to u$ in $L^{\mathscr{H}^*}(\mathbb{R}^N)$. This implies that, $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$. By the continuity of $\mathscr{H}$, we have $\mathscr{H}\left(x, y, \dfrac{|u_n(x) - u_n(y)|}{|x-y|^s}\right) \to \mathscr{H}\left(x, y, \dfrac{|u(x) - u(y)|}{|x-y|^s}\right)$ a.e. in $\mathbb{R}^N$.

Thanks to the Fatou's lemma,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|u(x)-u(y)|}{|x-y|^s}\right)d\mu \leq \liminf_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)d\mu<\infty,$$

which implies $u\in \mathring{W}^{s,\mathscr{H}}(\mathbb{R}^N)$.

Next, we will prove that $[u_n-u]_{s,\mathscr{H}}\to 0$ as $n\to\infty$.

As $\{u_n\}\subseteq (C_c^\infty(\mathbb{R}^N),[\cdot]_{s,\mathscr{H}})$, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)d\mu=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}<\infty$$

for each n. Thus $\dfrac{|u_n(x)-u_n(y)|}{|x-y|^s}\in L^{\mathscr{H}}(\mathbb{R}^N\times\mathbb{R}^N;d\mu)$.

Let

$$z_n(x,y)=\frac{|u_n(x)-u_n(y)|}{|x-y|^s}.$$

It can be also seen that $\{z_n(x,y)\}$ is a Cauchy sequence in $L^{\mathscr{H}}(\mathbb{R}^N\times\mathbb{R}^N;d\mu)$. As $L^{\mathscr{H}}(\mathbb{R}^N\times\mathbb{R}^N;d\mu)$ is a Banach space, there exists $z(x,y)\in L^{\mathscr{H}}(\mathbb{R}^N\times\mathbb{R}^N;d\mu)$ such that $z_n\to z$ in $L^{\mathscr{H}}(\mathbb{R}^N\times\mathbb{R}^N;d\mu)$. Further, by uniqueness of the limit, we have $z(x,y)=\dfrac{|u(x)-u(y)|}{|x-y|^s}$. Hence, $[u_n-u]_{s,\mathscr{H}}\to 0$ as $n\to\infty$, which proves our claim.

Step 3: Let $[u_n]\in D^{s,\mathscr{H}}(\mathbb{R}^N)$, i.e. $[u_n]$ is an equivalence class of the Cauchy sequence $\{u_n\}\subseteq (C_c^\infty(\mathbb{R}^N),[\cdot]_{s,\mathscr{H}})$. By Step 2, there exists $u\in \mathring{W}^{s,\mathscr{H}}(\mathbb{R}^N)$ such that $[u_n-u]_{s,\mathscr{H}}\to 0$ as $n\to\infty$. Define a function, $\Bbbk: D^{s,\mathscr{H}}(\mathbb{R}^N)\to \mathring{W}^{s,\mathscr{H}}(\mathbb{R}^N)$ such that $\Bbbk([u_n])=u$. It can be seen that $\Bbbk$ is well-defined one-one, onto and isometry by Step 1 and Step 2, which completes the proof of the theorem. □

6.2 Existence of solution for generalized Choquard Schrödinger equation

In this section, we establish the existence of a weak solution to the following problem:

$$(-\Delta)^s_{\mathscr{H}}u(x)+V(x)h(x,x,|u|)u(x)=\left(\int_{\mathbb{R}^N}\frac{K(y)F(u(y))}{|x-y|^\lambda}dy\right)K(x)f(u(x))\ \text{ in }\ \mathbb{R}^N, \tag{6.1}$$

where $N\geq 1$, $s\in(0,1),\lambda\in(0,N)$,

$$\mathscr{H}(x,y,t)=\int_0^{|t|}h(x,y,r)r\,dr,$$

and $h:\mathbb{R}^N\times\mathbb{R}^N\times[0,\infty)\to[0,\infty)$ is a generalized N-function. The functions $V,K:\mathbb{R}^N\to(0,\infty)$, nonlinear function $f:\mathbb{R}\to\mathbb{R}$ are continuous and $F(t)=\int_0^t f(r)dr$.

Due to the presence of the Choquard type nonlinearity, Problem (6.1) is known as a Choquard equation. One of the main tools to deal with such types of equations is Hardy-Littlewood-Sobolev [97] inequality which is stated below.

Proposition 6.2.1. *[97] Let $t_1,t_2>1$ and $0<\lambda<N$ with $1/t_1+1/t_2+\lambda/N=2$, $f\in L^{t_1}(\mathbb{R}^N)$ and $g\in L^{t_2}(\mathbb{R}^N)$. Then there exists a sharp constant C independent of f and g such that*

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)g(y)}{|x-y|^\lambda}dxdy\right|\le C\|f\|_{L^{t_1}(\mathbb{R}^N)}\|g\|_{L^{t_2}(\mathbb{R}^N)}.$$

Problem (6.1) involves the potential term which vanishes at infinity, such types of equations are studied widely by many researchers. In 2013, Alves-Souto [14] proved the existence result for the equation

$$-\Delta u+V(x)u=K(x)f(u)\text{ in }\mathbb{R}^N, \tag{6.2}$$

where $N\ge 3$. They assumed that $V,K:\mathbb{R}^N\to(0,\infty)$ are continuous functions and satisfy the following conditions:

(K_1') $K\in L^\infty(\mathbb{R}^N)$ and if $\{A_n\}$ is a sequence of Borel sets such that $\sup_n|A_n|<\infty$ then

$$\lim_{s\to\infty}\int_{A_n\cap B_s(0)^c}K(x)=0\text{ uniformly in }n\in\mathbb{N}.$$

(K_2') One of the following condition is true:

(K_{21}) $\frac{K}{V}\in L^\infty(\mathbb{R}^N)$.

(K_{22}) $\dfrac{K(x)}{[V(x)]^{\frac{2^*-p}{2^*-2}}}\to 0$ as $|x|\to\infty$ for some $p\in(2,2^*)$.

If V,K satisfies $(K_1')-(K_2')$ then we say $(V,K)\in\mathbb{K}$.

Further, Chen-Yuan [34], considered the problem:

$$-\Delta u+V(x)u=\left(\int_{\mathbb{R}^N}\frac{K(y)F(u(y))}{|x-y|^\lambda}dy\right)K(x)f(u(x))\text{ in }\mathbb{R}^N, \tag{6.3}$$

where they assumed that $(V,K)\in\mathbb{K}$ but the conditions (K_1') and (K_{22}) are replaced by the conditions (K_1) and (K_{23}), respectively. (K_1) and (K_{23}) are as follows:

(K_1) $K \in L^\infty(\mathbb{R}^N)$ and if $\{A_n\}$ is a sequence of Borel sets such that $\sup_n |A_n| < \infty$ then

$$\lim_{s\to\infty} \int_{A_n \cap B_s(0)^c} |K(x)|^{\frac{2N}{2N-\lambda}} = 0 \text{ uniformly in } n \in \mathbb{N}.$$

(K_{23}) $\dfrac{|K(x)|^{\frac{2N}{2N-\lambda}}}{[V(x)]^{\frac{2^*-p}{2^*-2}}} \to 0$ as $|x| \to \infty$ for some $p \in (2, 2^*)$.

In this sequence, Li-Teng-Wu [93] studied the following fractional Schrödinger equation:

$$(-\Delta)^s u + V(x)u = |u|^{2_s^*-2}u + \lambda K(x) f(u) \text{ in } \mathbb{R}^N, \tag{6.4}$$

where $\lambda > 0$, $s \in (0,1)$, $2_s^* = \frac{2N}{N-2s}$ and $(-\Delta)^s$ is the fractional Laplace operator of order s. They assumed that $(V,K) \in \mathbb{K}$ but in the condition (K_{22}), 2^* is replaced by 2_s^*.

Luo-Li-Li [105], considered the fractional Choquard equation:

$$(-\Delta)^s u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{K(y)F(u(y))}{|x-y|^\lambda} dy \right) K(x) f(u(x)) \text{ in } \mathbb{R}^N, \tag{6.5}$$

in which they assumed that the conditions (K_1) and

(K_{24}) $\dfrac{|K(x)|^{\frac{2N}{2N-\lambda}}}{[V(x)]^{\frac{2_s^*-p}{2_s^*-2}}} \to 0$ as $|x| \to \infty$ for some $p \in (2, 2_s^*)$

are satisfied.

After that, many researchers studied the nonlinear equations involving vanishing potential with different types of operators and different conditions on the nonlinearity, we refer to, Deng-Li-Shuai [47] (p-Laplace operator), Perera-Squassina-Yang [123] (fractional p-Laplacian), Isernia [79] (Fractional $p\&q$-Laplacian), Isernia-Repovš [80] (Double-phase operator). Recently, Silva-Souto [25] developed the existence result for generalized Schrödinger equation in Orlicz-Sobolev spaces.

Existence results for Choquard-type equations with vanishing potential have been obtained by Chen-Yuan [34], Alves-Figueiredo-Yang [11] (for Laplace operator), Albu-querque-Silva-Sousa [6](fractional coupled Choquard-type systems).

We assume that $V, K : \mathbb{R}^N \to (0, \infty)$ are continuous functions and satisfies (K_1). Moreover, we assume that

(K_2) V, K satisfies one of the following conditions:

(K_{2a}) $\frac{K}{V} \in L^\infty(\mathbb{R}^N)$.

(K_{2b}) $\dfrac{|K(x)|^{\frac{2N}{2N-\lambda}}}{L(x)} \to 0$ as $|x| \to \infty$, where $L(x) = \min_{t>0}\left\{V(x)\frac{\mathscr{H}(x,x,t)}{\Psi(x,x,t)}\right\}$.

We assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following conditions:

(f_1) There exist a generalized N-function $\Psi : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to [0,\infty)$ such that $h_2 < \psi_1 < \psi_2 < \psi_2 l < h_1^*$, $\psi_1 \le \frac{\psi(x,y,t)|t|^2}{\Psi(x,y,t)} \le \psi_2$, $\forall (x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ and $t \neq 0$ and

$$\lim_{t\to 0} \frac{f(t)}{\psi(x,x,t)t} = 0,\ \forall\, x \in \mathbb{R}^N,$$

where $\Psi(x,y,t) = \int_0^{|t|} \psi(x,y,r)r\, dr$ and $\frac{2N}{2N-\lambda} = l$.

(f_2) $\lim_{t\to\infty} \dfrac{F(t)}{(\mathscr{H}^*(x,x,t))^{1/l}} = 0,\ \forall\, x \in \mathbb{R}^N$, where $\mathscr{H}^*$ is define in (2.8).

(f_3) For $i \in \{1,2\}$, $\lim_{t\to\infty} \dfrac{f(t)}{(\mathscr{H}^*(x,x,t))^{\frac{b-1}{h_i^*}}} = 0,\ \forall\, x \in \mathbb{R}^N$ for some $bl \in (h_2, h_1^*)$.

(f_4) There exists $\sigma > h_2/2$ such that

$$0 < \sigma F(t) = \sigma \int_0^t f(s)ds \le 2tf(t),$$

for all $t > 0,\ x \in \mathbb{R}^N$.

6.2.1 Weighted homogeneous fractional Musielak-Sobolev space

Weighted fractional Musielak-Sobolev spaces are not sufficient to study the Problem (6.1), as $\inf V(x)$ can be zero. In this section, we introduce the suitable space to study Problem (6.1), which we called weighted homogeneous fractional Musielak-Sobolev space and investigate their properties.

Let $\mathscr{H}$ be any generalized N-function satisfying $(\mathscr{H}_2)-(\mathscr{H}_3)$. Then we define weighted Lebesgue-Musielak space $L_V^{\mathscr{H}_x}(\mathbb{R}^N)$ as:

$$L_V^{\mathscr{H}_x}(\mathbb{R}^N) = \left\{u : \mathbb{R}^N \to \mathbb{R} \text{ is measurable} \,\middle|\, \int_{\mathbb{R}^N} V(x)\mathscr{H}_x(x,\tau|u|)\,dx < \infty, \text{ for some } \tau > 0\right\}.$$

$L_V^{\mathscr{H}_x}(\mathbb{R}^N)$ is a normed space [117] with the Luxemburg norm

$$\|u\|_{V,\mathscr{H}} = \inf\left\{\tau > 0 \,\middle|\, \int_{\mathbb{R}^N} V(x)\mathscr{H}_x(x,\tau|u|)\,dx \le 1\right\}.$$

Proposition 6.2.2. *Let $\mathscr{H}$ be any generalized N-function satisfying $(\mathscr{H}_2)-(\mathscr{H}_3)$. Assume that $u \in L_V^{\mathscr{H}_x}(\mathbb{R}^N)$. Then, we have*

$$\min\left\{\|u\|_{V,\mathscr{H}}^{h_1}, \|u\|_{V,\mathscr{H}}^{h_2}\right\} \leq \int_{\mathbb{R}^N} V(x)\mathscr{H}_x(x,|u|)dx \leq \max\left\{\|u\|_{V,\mathscr{H}}^{h_1}, \|u\|_{V,\mathscr{H}}^{h_2}\right\}.$$

Due to the presence of potential term V in the Problem (6.1), we consider the following weighted space:

$$W = \left\{u \in D^{s,\mathscr{H}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)\mathscr{H}_x(x,|u|)\,dx < \infty\right\}$$

which is a normed space with the norm

$$\|u\|_W = \|u\|_{D^{s,\mathscr{H}}(\mathbb{R}^N)} + \|u\|_{V,\mathscr{H}}.$$

For the sake of simplicity, we denote $\|\cdot\|_W$ as $\|\cdot\|$.

Next, we have the following lemma from the definition of the space W.

Lemma 6.2.3. *The space W is compactly embedded in $L^{\mathscr{H}}_{loc}(\mathbb{R}^N)$. Also, W is continuously embedded in $L^{\mathscr{H}^*}(\mathbb{R}^N)$.*

Next, we will state some results which are used to prove our main result. Define the function $D^{s,\mathscr{H}}(\mathbb{R}^N) \to \mathbb{R}$ as

$$m(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}.$$

Proposition 6.2.4. *[19] For all $u \in D^{s,\mathscr{H}}(\mathbb{R}^N)$ we have*

1. *If $\|u\| > 1$ then $\|u\|^{h_1} \leq m(u) \leq \|u\|^{h_2}$.*
2. *If $\|u\| < 1$ then $\|u\|^{h_2} \leq m(u) \leq \|u\|^{h_1}$.*

In particular, $m(u) = 1$ iff $\|u\| = 1$. Moreover, if $\{u_n\} \subset D^{s,\mathscr{H}}(\mathbb{R}^N)$ then $\|u_n\| \to 0$ iff $m(u_n) \to 0$.

Theorem 6.2.5. *The space W is continuously embedded in $L_Q^P(\mathbb{R}^N)$, where $Q(x) = |K(x)|^l$ and $p : \mathbb{R}^N \times \mathbb{R}^N \times [0,\infty) \to [0,\infty)$, $P(x,y,t) = \int_0^{|t|} p(x,y,r)r\,dr$ is a generalized N-function such that*

$$p_1 \leq \frac{p(x,y,|t|)|t|^2}{P(x,y,|t|)} \leq p_2,\ \forall (x,y) \in \mathbb{R}^N \times \mathbb{R}^N \text{ and } t \neq 0$$

for some $p_1, p_2 \in (h_2, h_1^*)$.

Proof. The proof is similar to [25, Lemma 5.1]. □

6.2.2 Hardy-Littlewood-Sobolev type inequality in Lebesgue Musielak spaces

Hardy-Littlewood-Sobolev inequality is the primary tool for dealing with the Choquard-type non linearity in the context of variational methods. So far, we do not have Hardy-Littlewood-Sobolev inequality for Lebesgue Musielak spaces. By taking advantage of the condition $(\mathscr{H}_2)$ and using Proposition 6.2.1, we prove and use the following version of Hardy-Littlewood-Sobolev inequality in Lebesgue Musielak spaces.

Proposition 6.2.6. *For any $u \in W$, we have $K(x)F(u(x)) \in L^l(\mathbb{R}^N)$. Moreover, for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that*

$$\begin{aligned}\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(u(x))F(u(y))}{|x-y|^\lambda}dxdy\right| \\ &\leq C_1 \max\left\{\varepsilon^2\|u\|^{2\psi_1} + c_\varepsilon^2\|u\|^{2h_1^*/l}, \varepsilon^2\|u\|^{2\psi_2} + c_\varepsilon^2\|u\|^{2h_2^*/l}\right\} \\ &\leq C_2(\|u\|^{2\psi_1} + \|u\|^{2h_1^*/l} + \|u\|^{2\psi_2} + \|u\|^{2h_2^*/l})\end{aligned}$$

for some $C_1, C_2 > 0$.

Proof. Let $u \in W$. It follows, from $(f_1)-(f_2)$ that, for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$|F(t)| \leq \varepsilon\Psi_x(x,t) + c_\varepsilon(\mathscr{H}^*(x,t))^{1/l}, \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

By Proposition 2.2.8, we have

$$\begin{aligned}\int_{\mathbb{R}^N}|K(x)F(u(x))|^l dx &\leq 2^{l-1}\int_{\mathbb{R}^N}(\varepsilon(\Psi_x(x,u(x)))^l + c_\varepsilon^l\mathscr{H}^*(x,u(x)))dx \\ &\leq c_1\varepsilon^l \max\left\{\|u\|^{\psi_1 l}_{L^{\psi_1 l}(\mathbb{R}^N)}, \|u\|^{\psi_2 l}_{L^{\psi_2 l}(\mathbb{R}^N)}\right\} + c_2 c_\varepsilon^l \max\left\{\|u\|^{h_1^*}_{L^{\mathscr{H}^*}(\mathbb{R}^N)}, \|u\|^{h_2^*}_{L^{\mathscr{H}^*}(\mathbb{R}^N)}\right\},\end{aligned}$$

for some $c_1, c_2 > 0$. Further, by Theorem 6.2.5 and Lemma 6.2.3, one gets

$$\int_{\mathbb{R}^N}|K(x)F(u(x))|^l dx \leq c_3\varepsilon^l \max\left\{\|u\|^{\psi_1 l}, \|u\|^{\psi_2 l}\right\} + c_4 c_\varepsilon^l \max\left\{\|u\|^{h_1^*}, \|u\|^{h_2^*}\right\} < \infty, \quad (6.6)$$

which implies, $K(x)F(u(x)) \in L^l(\mathbb{R}^N)$, for some $c_3, c_4 > 0$.

By Proposition 6.2.1 and (6.6), we get

$$\begin{aligned}\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(u(x))F(u(y))}{|x-y|^\lambda}dxdy\right| \\ &\leq \left(c_3\varepsilon^l \max\left\{\|u\|^{\psi_1 l}, \|u\|^{\psi_2 l}\right\} + c_4 c_\varepsilon^l \max\left\{\|u\|^{h_1^*}, \|u\|^{h_2^*}\right\}\right)^{2/l} \\ &\leq C_1 \max\left\{\varepsilon^2\|u\|^{2\psi_1} + c_\varepsilon^2\|u\|^{2h_1^*/l}, \varepsilon^2\|u\|^{2\psi_2} + c_\varepsilon^2\|u\|^{2h_2^*/l}\right\}\end{aligned}$$

$$\leq C_2(\|u\|^{2\psi_1} + \|u\|^{2h_1^*/l} + \|u\|^{2\psi_2} + \|u\|^{2h_2^*/l})$$

for some $C_1, C_2 > 0$. □

6.2.3 Functional setting and main result

First, we define a weak solution to (6.1) and the corresponding energy functional.

Definition 6.2.7. We say that $u \in W$ is a weak solution of (6.1) if the following holds:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} h\left(x,y,\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dx\,dy$$
$$+\int_{\mathbb{R}^N} V(x)h_x(x,|u|)uvdx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u(x))f(u(y))v(y)}{|x-y|^{\lambda}}dxdy,\ \forall v \in W.$$

Thus, the energy functional $I : W \to \mathbb{R}$ corresponding to (6.1) is given by

$$I(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N} + \int_{\mathbb{R}^N} V(x)\mathscr{H}_x(x,|u|)dx$$
$$-\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u(x))F(u(y))}{|x-y|^{\lambda}}dxdy.$$

It can be seen that I is well defined by Proposition 6.2.6, C^1 [13, Lemma 3.2] and the derivative of I at any point $u \in W$ is given by

$$I'(u)(v) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} h\left(x,y,\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dx\,dy$$
$$+\int_{\mathbb{R}^N} V(x)h_x(x,|u|)uvdx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u(x))f(u(y))v(y)}{|x-y|^{\lambda}}dxdy,\ \forall v \in W.$$

Moreover, the critical points of I are the weak solutions to (6.1). Let $J : W \to \mathbb{R}$ such that

$$J(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N} + \int_{\mathbb{R}^N} V(x)\mathscr{H}_x(x,|u|)dx.$$

Remark 6.2.8. *The functional J is convex since $\mathscr{H}$ is convex. Consequently, J is weakly lower semicontinuous, i.e., if $\{u_n\} \rightharpoonup u$ in $W_0^{s,\mathscr{H}}(\mathbb{R}^N)$ then $J(u) \leq \liminf_{n\to\infty} J(u_n)$.*

Lemma 6.2.9. *[6, Theorem 3.14] Let $\mathscr{H}$ be a generalized N-function and $s \in (0,1)$. Assume that the sequence $\{u_n\}$ converges weakly to u in W and*

$$\limsup_{n\to\infty}\langle J'(u_n), u_n - u\rangle \leq 0.$$

Then $\{u_n\}$ *converges strongly to* u *in* W.

The main existence results of this chapter is as follows:

Theorem 6.2.10. *Suppose that the conditions* $(f_1)-(f_4)$, $(K_1)-(K_2)$ *and* $(\mathscr{H}_1)-(\mathscr{H}_4)$ *are satisfied. Then the Problem* (6.1) *has a non-trivial weak solution.*

To prove the existence of a ground-state solution, we need the following additional assumption on f:

(GS) The map $t \mapsto \frac{f(t)}{t|t|^{\frac{h_2}{2}-2}}$ is strictly increasing for $t > 0$.

Theorem 6.2.11. *If* $(f_1)-(f_4)$, (GS), $(K_1)-(K_2)$ *and* $(\mathscr{H}_1)-(\mathscr{H}_4)$ *are satisfied, then the solution obtained through Theorem 6.2.10 is a ground-state solution.*

6.2.4 Proof of the Theorem 6.2.11

To prove the Theorem 6.2.11, we first establish a series of lemmas.

Lemma 6.2.12. *There exist positive real numbers* α *and* ρ *such that*

$$I(u) \geq \alpha, \ \ \forall u \in W : \|u\| = \rho.$$

Proof. By using the Proposition 6.2.2 and Proposition 6.2.4, we have

$$\begin{aligned} I(u) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N} + \int_{\mathbb{R}^N} V(x)\mathscr{H}_x(x,|u|)dx \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(u(x))F(u(y))}{|x-y|^\lambda}dxdy \\ &\geq \min\left\{[u]^{h_1}_{s,\mathscr{H}}, [u]^{h_2}_{s,\mathscr{H}}\right\} + \min\left\{\|u\|^{h_1}_{V,\mathscr{H}}, \|u\|^{h_2}_{V,\mathscr{H}}\right\} \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(u(x))F(u(y))}{|x-y|^\lambda}dxdy\cdot \end{aligned}$$

If $\|u\| < 1$, Proposition 6.2.6 implies

$$\begin{aligned} I(u) &\geq \|u\|^{h_2} - (C_1\varepsilon^2\|u\|^{2\psi_1} + C_1c_\varepsilon^2\|u\|^{2h_1^*/l}) \\ &\geq \|u\|^{h_2}\left(1 - \frac{C_1\varepsilon^2}{\|u\|^{h_2-2\psi_1}}\right) - C_1c_\varepsilon^2\|u\|^{2h_1^*/l}. \end{aligned}$$

We conclude the result by choosing ρ and ε sufficiently small and using the fact that $(2h_1^*/l) > h_2$. □

Lemma 6.2.13. *There exist $v_0 \in W$ and $\beta > 0$ such that*

$$I(v_0) < 0 \ \textit{and} \ \|v_0\| > \beta.$$

Proof. By (f_4), there exist $m_1, m_2 > 0$ such that

$$F(s) \geq m_1 s^{\sigma} - m_2, \ \ \forall\, s \in [0, \infty). \tag{6.7}$$

Let $u \in W \backslash \{0\}$ and $u \geq 0$ with compact support $K \subseteq \mathbb{R}^N$. For $t > 1$, by proposition 6.2.2 and 6.2.4, we have

$$\begin{aligned} I(tu) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x, y, \frac{|tu(x) - tu(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N} + \int_{\mathbb{R}^N} V(x)\mathscr{H}_x(x, |tu|)dx \\ &\quad - \frac{1}{2}\int_K\int_K \frac{K(x)K(y)F(tu(x))F(tu(y))}{|x-y|^{\lambda}}dxdy \\ &\leq t^{h_2}\left(\max\left\{[u]^{h_1}_{s,\mathscr{H}}, [u]^{h_2}_{s,\mathscr{H}}\right\} + \max\left\{\|u\|^{h_1}_{V,\mathscr{H}}, \|u\|^{h_2}_{V,\mathscr{H}}\right\}\right) \\ &\quad - \frac{1}{2}\int_K\int_K \frac{K(x)K(y)(m_1 t^{\sigma}(u(x))^{\sigma} - m_2)(m_1 t^{\sigma}(u(y))^{\sigma} - m_2)}{|x-y|^{\lambda}}dxdy \end{aligned}$$

this implies that $I(tu) \to -\infty$ as $t \to \infty$, since $2\sigma > h_2$. Now, by setting $v_0 = tu$ for sufficiently large t, we get the desired result. □

By Lemmas 6.2.12 and 6.2.13, the geometric conditions of the mountain pass theorem are satisfied for the functional I. Hence, by the version of the mountain pass theorem without (PS) condition, there exists a sequence $\{u_n\} \subseteq W$ such that $I(u_n) \to c_M$ and $I'(u_n) \to 0$ as $n \to \infty$, where

$$0 < c_M = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) > 0,$$

and

$$\Gamma = \{\gamma \in C([0,1], W) : \gamma(0) = 0, \ \gamma(1) < 0\}\cdot$$

Lemma 6.2.14. *The $(PS)_{c_M}$ sequence is bounded in W. Moreover, there exists $u \in W$ such that, up to a subsequence, we have $u_n \rightharpoonup u$ weakly in W.*

Proof. Since $\{u_n\}$ is a $(PS)_{c_M}$ sequence of I, we have $I(u_n) \to c_M$ and $I'(u_n) \to 0$ as $n \to \infty$, i.e.,

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(\frac{|u_n(x) - u_n(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N} + \int_{\mathbb{R}^N} V(x)\mathscr{H}_x(x, |u_n|)dx \\ &\qquad - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(u_n(x))F(u_n(y))}{|x-y|^{\lambda}}dxdy = c_M + \delta_n, \end{aligned} \tag{6.8}$$

where $\delta_n \to 0$ as $n \to \infty$ and

$$\left| \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} h\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{(u_n(x)-u_n(y))(v(x)-v(y))}{|x-y|^{N+2s}}dx\,dy \right.$$
$$\left. + \int_{\mathbb{R}^N} V(x)h_x(x,|u_n|)u_n v dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u_n(x))f(u_n(y))v(y)}{|x-y|^{\lambda}}dxdy\right| \leq \varepsilon_n\|v\|, \quad (6.9)$$

$\forall v \in W$, where $\varepsilon_n \to 0$ as $n \to \infty$. On taking $v = u_n$, by (6.8), (6.9) and using (f_4), we obtain

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}\right.$$
$$\left. -\frac{1}{\sigma}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} h\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{(u_n(x)-u_n(y))^2}{|x-y|^{N+2s}}dxdy\right)$$
$$+\int_{\mathbb{R}^N}(V(x)\mathscr{H}_x(x,|u|) - \frac{1}{\sigma}V(x)h_x(x,|u|)u^2)dx \leq c_5(1+\|u_n\|),$$

for some $c_5 > 0$. It follows from $(\mathscr{H}_2)$ that

$$\left(1-\frac{h_2}{\sigma}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}$$
$$+\left(1-\frac{h_2}{\sigma}\right)\int_{\mathbb{R}^N} V(x)\mathscr{H}_x(x,|u_n|)dx \leq c_5(1+\|u_n\|).$$

If $\|u_n\| > 1$, by propositions 6.2.2 6.2.4, we have

$$\left(1-\frac{h_2}{\sigma}\right)([u_n]^{h_1}_{s,\mathscr{H}} + \|u_n\|^{h_1}_{V,\mathscr{H}}) \leq c_5(1+\|u_n\|)$$
$$\left(1-\frac{h_2}{\sigma}\right)\|u_n\|^{h_1} \leq c_5(1+\|u_n\|).$$

Consequently, $\|u_n\| \leq c_6$ for some $c_6 > 0$. Thus $\{u_n\}$ is bounded in W. As W is a reflexive Banach space, $\exists\, u \in W$ such that up to a subsequence, we have $u_n \rightharpoonup u$ weakly in W. □

Lemma 6.2.15. *Let $\{u_n\}$ is bounded in W such that $u_n \rightharpoonup u$ weakly in W. Then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|K(x)f(u_n(x))(u_n(x)-u(x))|^l dx = 0.$$

Proof. Let $\{u_n\}$ is bounded in W such that $u_n \rightharpoonup u$ weakly in W. By Lemma 6.2.3, we have $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$.

Define $Q(x)=|K(x)|^l$. It follows from (f_1) and (f_3) that, for all $\varepsilon>0$ there exist $t_0,t_1,c_\varepsilon>0$ such that

$$f(t)\le\varepsilon\left(\psi_x(x,t)t+(\mathscr{H}^*(x,t))^{\frac{b-1}{h_i^*}}\right)+c_\varepsilon(\mathscr{H}^*(x,t))^{\frac{b-1}{h_i^*}}\chi_{[t_0,t_1]}(t),\forall(x,t)\in\mathbb{R}^N\times\mathbb{R}. \tag{6.10}$$

Further, by (6.10), we have

$$\begin{aligned}K(x)f(u_n(x))\le\varepsilon K(x)&\left(\psi_x(x,u_n(x))u_n(x)+(\mathscr{H}^*(x,u_n(x)))^{\frac{b-1}{h_i^*}}\right)\\&+c_\varepsilon K(x)(\mathscr{H}^*(x,u_n(x)))^{\frac{b-1}{h_i^*}}\chi_{[t_0,t_1]}(u_n(x)),\forall(x,t)\in\mathbb{R}^N\times\mathbb{R}.\end{aligned}$$

Consider,

$$\begin{aligned}&\int_{\mathbb{R}^N}|K(x)f(u_n(x))(u_n(x)-u(x))|^l dx\\&\le 2^{l-1}\varepsilon^l\int_{\mathbb{R}^N}Q(x)\left|\left\{\psi_x(x,u_n(x))u_n(x)+(\mathscr{H}^*(x,u_n(x)))^{\frac{b-1}{h_i^*}}\right\}(u_n(x)-u(x))\right|^l dx\\&+2^{l-1}c_\varepsilon^l\int_{\mathbb{R}^N}Q(x)\left|(\mathscr{H}^*(x,u_n(x)))^{\frac{b-1}{h_i^*}}\chi_{[t_0,t_1]}(u_n(x))(u_n(x)-u(x))\right|^l dx.\end{aligned} \tag{6.11}$$

Now, define the set $A_n=\{x\in\mathbb{R}^N:|u_n(x)|\ge t_0\}$. Thus, $(\mathscr{H}_3)$ and definition of $\mathscr{H}^*$ implies

$$c_7|A_n|\le\int_{A_n}\mathscr{H}^*(x,t_0)dx\le\int_{A_n}\mathscr{H}^*(x,u_n(x))dx\le\int_{\mathbb{R}^N}\mathscr{H}^*(x,u_n(x))dx<c_8,$$

since $\{u_n\}$ is bounded in W, for some $c_7,c_8>0$.

Therefore, we have $\sup_{n\in\mathbb{N}}|A_n|<\infty$. Using (K_1), we get

$$\lim_{d\to\infty}\int_{A_n\cap B_d(0)^c}|K(x)|^{\frac{2N}{2N-\lambda}}dx=0 \text{ uniformly in } n\in\mathbb{N}$$

consequently, for a given $\varepsilon>0$ there exists $d_0>0$ such that

$$\int_{A_n\cap B_{d_0}(0)^c}|K(x)|^{\frac{2N}{2N-\lambda}}dx<\varepsilon^{\frac{b}{(b-1)}} \text{ for each } n. \tag{6.12}$$

Using Hölder's inequality and Proposition 2.2.20, we have

$$\begin{aligned}&\int_{B_{d_0}(0)^c}Q(x)\left|(\mathscr{H}^*(x,u_n(x)))^{\frac{b-1}{h_i^*}}\chi_{[t_0,t_1]}(u_n(x))(u_n(x)-u(x))\right|^l dx\\&\qquad\le c_9\int_{A_n\cap B_{d_0}(0)^c}Q(x)|u_n(x)|^{(b-1)l}\chi_{[t_0,t_1]}(|u_n(x)|)|u_n(x)-u(x)|^l dx\end{aligned}$$

$$\leq c_9 \max_{i\in\{1,2\}} \left\{ \left(\int_{A_n\cap B_{d_0}(0)^c} Q(x)|u_n(x)|^{bl} \chi_{[t_0,t_1]}(u_n(x))dx \right)^{\frac{(b-1)}{b}} \right.$$
$$\left. \left(\int_{A_n\cap B_{d_0}(0)^c} Q(x)|u_n(x)-u(x)|^{bl} dx \right)^{\frac{1}{b}} \right\}$$
$$\leq c_{10} t_1^{(b-1)l} \left(\int_{A_n\cap B_{d_0}(0)^c} Q(x) \right)^{\frac{(b-1)}{b}}$$

for some $c_9, c_{10} > 0$.

Further, by (6.12), we obtain

$$\int_{B_{d_0}(0)^c} Q(x) \left| (\mathscr{H}^*(x,u_n(x)))^{\frac{b-1}{h_i^*}} \chi_{[t_0,t_1]}(u_n(x))(u_n(x)-u(x)) \right|^l dx \leq c_{11}\varepsilon, \tag{6.13}$$

for some $c_{11} > 0$.

By Propositions 2.2.8, 2.2.20, Lemma 6.2.3, Theorem 6.2.5 and Hölder's inequality, we have

$$\int_{B_{d_0}(0)^c} Q(x) \left| \left\{ \psi_x(x,u_n(x))u_n(x) + (\mathscr{H}^*(x,u_n(x)))^{\frac{b-1}{h_i^*}} \right\} (u_n(x)-u(x)) \right|^l dx$$
$$\leq c_{12} \max_{i\in\{1,2\}} \left\{ \int_{\mathbb{R}^N} Q(x) \left(|u_n(x)|^{(\psi_i-1)l} + |u_n(x)|^{(b-1)l} \right) |u_n(x)-u(x)|^l dx \right\}$$
$$\leq c_{12} \max_{i\in\{1,2\}} \left\{ \left(\int_{\mathbb{R}^N} Q(x)|u_n(x)|^{\psi_i l} dx \right)^{\frac{(\psi_i-1)}{\psi_i}} \left(\int_{\mathbb{R}^N} Q(x)|u_n(x)-u(x)|^{\psi_i l} dx \right)^{\frac{1}{\psi_i}} \right\}$$
$$+ c_{12} \max_{i\in\{1,2\}} \left\{ \left(\int_{\mathbb{R}^N} Q(x)|u_n(x)|^{bl} dx \right)^{\frac{(b-1)}{b}} \left(\int_{\mathbb{R}^N} Q(x)|u_n(x)-u(x)|^{bl} dx \right)^{\frac{1}{b}} \right\}$$
$$\leq c_{13} \max_{i\in\{1,2\}} \left\{ \|u_n(x)-u(x)\|^l_{L_Q^{\psi_i l}(\mathbb{R}^N)} + \|u_n(x)-u(x)\|^l_{L_Q^{bl}(\mathbb{R}^N)} \right\}$$

for some $c_{12}, c_{13} > 0$.

Therefore, by Theorem 6.2.5, we obtain

$$\int_{B_{d_0}(0)^c} Q(x) \left| \left\{ \psi_y(x,u_n(x))u_n(x) + (\mathscr{H}^*(x,u_n(x)))^{\frac{b-1}{h_i^*}} \right\} (u_n(x)-u(x)) \right|^l dx \leq c_{14}, \tag{6.14}$$

for some $c_{14} > 0$.

Consequently, (6.11), (6.13) and (6.14) implies

$$\int_{B_{d_0}(0)^c} |K(x)f(u_n(x))(u_n(x)-u(x))|^l dx \leq 2^{l-1}\varepsilon^l c_{14} + 2^{l-1}c_\varepsilon^l c_{11}\varepsilon \to 0$$

as ε was arbitrary.

On the other side, by (f_3) and Strauss compactness lemma [23, Theorem A.I], we have

$$\lim_{n\to\infty}\int_{B_{d_0}(0)} |K(x)f(u_n(x))(u_n(x)-u(x))|^l dx = 0,$$

which completes the proof. □

Proof of the Theorem 6.2.10. Both the geometric conditions of the mountain pass theorem follows from Lemmas 6.2.12 and 6.2.13. Next, we will prove that the functional I satisfies the $(PS)_{c_M}$ condition.

Let $\{u_n\} \subseteq W$ be any Palais-Smale sequence, i.e., $I(u_n) \to c_M$ and $I'(u_n) \to 0$ in dual space of W. By Lemma 6.2.14, we conclude that $\{u_n\}$ is bounded in W and $u_n \rightharpoonup u$ weakly in W. As a consequence, $I'(u_n)(u_n - u) \to 0$ as $n \to \infty$, i.e.,

$$J'(u_n)(u_n-u) - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(u_n(x))f(u_n(y))(u_n(y)-u(y))}{|x-y|^\lambda}dxdy \to 0 \tag{6.15}$$

as $n \to 0$. Next, we claim that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(u_n(x))f(u_n(y))(u_n(y)-u(y))}{|x-y|^\lambda}dxdy \to 0 \text{ as } n \to 0. \tag{6.16}$$

Let $|K(y)|^l = Q(y)$. By (6.6), we have

$$\|K(x)F(u_n(x))\|_{L^l(\mathbb{R}^N)} \leq c_{15}, \tag{6.17}$$

for some $c_{15} > 0$ since $\{u_n\}$ is bounded in W. By Lemma 6.2.15, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |K(x)f(u_n(x))(u_n(x)-u(x))|^l dx = 0. \tag{6.18}$$

By (6.17), (6.18) and Proposition 6.2.1, the claim in the (6.16) is proved. Hence, by Lemma 6.2.9, we have $u_n \to u$. Thus, $(PS)_{c_M}$ condition is satisfied for the functional I.

Hence, by the mountain pass theorem, there exists critical point u_M of I with level c_M, i.e., $I'(u_M) = 0$ and $I(u_M) = c_M > 0$. Thus, u_M is the non-trivial weak solution of the Problem (6.1). □

6.3 Ground State Solution

In this section, we prove that the solution obtained through Theorem 6.2.10 is a ground-state solution. In fact, we prove the Theorem 6.2.11.

Define

$$\aleph = \{u \in W\backslash\{0\} | I'(u)u = 0\} \text{ and } b = \inf_{u\in\aleph} I(u). \tag{6.19}$$

The set $\aleph$ is called the Nehari manifold.

For $u \in W$, define the function, $h_u : [0,\infty) \to \mathbb{R}$ such that $h_u(t) = I(tu)$, i.e.,

$$h_u(t) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x, y, \frac{|tu(x)-tu(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N} + \int_{\mathbb{R}^N} V(x)\mathscr{H}_x(x,|tu(x)|)dx \\ -\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(tu(x))F(tu(y))}{|x-y|^\lambda}dxdy.$$

Lemma 6.3.1. *Let $(f_1)-(f_4)$, (GS), $(K_1)-(K_2)$ and $(\mathscr{H}_1)-(\mathscr{H}_4)$ hold. If $u \in W\backslash\{0\}$, then there exists unique $t_u > 0$ such that $t_u u \in \aleph$. Moreover, $\max_{t\in[0,\infty]} h_u(t) = h_u(t_u) = I(ut_u)$.*

Proof. We observe that $h'_u(t) = 0$ if and only if $tu \in \aleph$. Lemma 6.2.12 and Lemma 6.2.13 imply that $h_u(t) > 0$ for sufficiently small t and $h_u(t) < 0$ for sufficiently large t. Thus, $\exists\, t_u \in (0,\infty)$ such that $\max_{t\in[0,\infty]} h_u(t) = h_u(t_u) = I(ut_u)$. Consequently, $h'_u(t_u) = 0$ and $t_u u \in \aleph$. Next, we will prove the uniqueness of t_u. If t is the critical point of h_u, then we have

$$h'_u(t) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} h\left(x, y, \frac{|tu(x)-tu(y)|}{|x-y|^s}\right)\frac{(tu(x)-tu(y))^2}{t|x-y|^{N+2s}}dx\,dy \\ + \int_{\mathbb{R}^N}\frac{V(x)h_x(x,|tu(x)|)(tu(x))^2dx}{t} - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u(x))f(tu(y))u(y)}{|x-y|^\lambda}dxdy = 0,$$

which implies that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} h\left(x, y, \frac{|tu(x)-tu(y)|}{|x-y|^s}\right)\frac{(tu(x)-tu(y))^2}{t^{h_2}|x-y|^{N+2s}}dx\,dy + \int_{\mathbb{R}^N}\frac{V(x)h_x(x,|tu(x)|)(tu(x))^2dx}{t^{h_2}} \\ = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(tu(x))f(tu(y))tu(y)}{t^{h_2}|x-y|^\lambda}dxdy. \tag{6.20}$$

On proceeding as [136, Lemma 4.3], one can check that the right-hand side of (6.20) is decreasing for $t > 0$. Consider,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(tu(x))f(tu(y))tu(y)}{t^{h_2}|x-y|^\lambda}dxdy \\ = \int_{\mathbb{R}^N} K(y)\left(\int_{\mathbb{R}^N}\frac{K(x)F(tu(x))dx}{|x-y|^\lambda}\right)\frac{f(tu(y))tu(y)}{t^{h_2}}dy$$

$$= \int_{\mathbb{R}^N} K(y) \left(\int_{\mathbb{R}^N} \frac{K(x)F(tu(x))dx}{t^{\frac{h_2}{2}}|x-y|^{\lambda}} \right) \frac{f(tu(y))|u(y)|^{\frac{h_2}{2}}}{|tu(y)|^{\frac{h_2}{2}-2}(tu(y))} dy$$

which implies the left hand side of (6.20) is increasing strictly for $t > 0$ by (GS) and (f_4). Therefore, t_u is a unique critical point of h_u. □

Proof of the Theorem 6.2.11 It is enough to prove that, $c_M = b = r$, where b and r as are defined in (2.12) and (6.19), respectively.

By using the fact that $S \subseteq \aleph$, we have $b \leq r$. Also, it can be seen $r \leq c_M$. It will be sufficient to prove that $b \geq c_M$.

If $v \in \aleph$, then $h_v'(1) = 0$. By Lemma 6.3.1, we have $\max\limits_{t\in[0,\infty]} h_v(t) = h_v(1) = I(v)$.

Choose a function $\gamma : [0,1] \to W$ such that $\gamma(t) = tt_0 v$, where $t_0 > 0$ such that $I(t_0 v) < 0$, which implies that $\gamma \in \Gamma$. Therefore, we have

$$c_M \leq \max_{t\in[0,1]} I(\gamma(t)) = \max_{t\in[0,1]} I(tt_0 v) \leq \max_{t\geq 0} I(tv) = I(v),$$

which is true for every element $v \in \aleph$. Hence, $b \geq c_M$, which completes the proof. □

Chapter 7

Generalized concentration compactness principle and their applications

In this chapter, we prove the concentration compactness principle for fractional Musielak Sobolev spaces for studying the class of generalized fractional problems with critical growth in $\mathbb{R}^N$. Section 7.1 recalls the definition of homogeneous fractional Musielak-Sobolev spaces and provides some useful results that are used in the subsequent sections. The proof of concentration compactness principle for fractional Musielak Sobolev space and its variant at infinity are provided in the 7.2. Finally, in Section 7.3, we discuss the existence result for the generalized fractional problem with critical growth in $\mathbb{R}^N$.

7.1 Introduction

Let

$$\mathscr{H}(x,y,t) = \int_0^{|t|} h(x,y,r)r\,dr,$$

and $: \mathbb{R}^N \times \mathbb{R}^N \times [0,\infty) \to [0,\infty)$ be any locally integrable generalized N-function which satisfies the following conditions:

(h_1) $h_1 \leq \frac{h(x,y,|t|)|t|^2}{\mathscr{H}(x,y,|t|)} \leq h_2 < N$ for all $x,y \in \mathbb{R}^N$ and $t \neq 0$, for some $1 < h_1 < h_2 < h_1^*$, where $h_1^* = \frac{Nh_1}{N-sh_1} \leq h_2^* = \frac{Nh_2}{N-sh_2}$ and $s \in (0,1)$.

(h_2) $\inf\limits_{x,y\in\mathbb{R}^N} \mathscr{H}(x,y,1) = b_1$ and $\sup\limits_{x,y\in\mathbb{R}^N} \mathscr{H}(x,y,1) = b_2$ for some $b_1, b_2 > 0$.

(h_3) $\int_a^\infty \left(\frac{t}{\mathscr{H}_x(x,t)}\right)^{\frac{s}{N-s}} dt = \infty$ and $\int_0^b \left(\frac{t}{\mathscr{H}_x(x,t)}\right)^{\frac{s}{N-s}} dt < \infty$, for some $a,b > 0$ and $\forall x \in \mathbb{R}^N$.

Next, we recall the definition of homogeneous fractional Musielak-Sobolev spaces introduced in Chapter 6.

The homogeneous fractional Musielak-Sobolev space denoted by $D^{s,\mathscr{H}}(\mathbb{R}^N)$ is defined as:

$$D^{s,\mathscr{H}}(\mathbb{R}^N) = \left\{u \in L^{\mathscr{H}^*}(\mathbb{R}^N) : [u]_{s,\mathscr{H}} < \infty\right\}.$$

$D^{s,\mathscr{H}}(\mathbb{R}^N)$ is a normed space with the norm $\|u\|_{D^{s,\mathscr{H}}(\mathbb{R}^N)} = [u]_{s,\mathscr{H}}$.

Next, we have the following lemma from the definition of the space $D^{s,\mathscr{H}}(\mathbb{R}^N)$.

Lemma 7.1.1. *The space $D^{s,\mathscr{H}}(\mathbb{R}^N)$ is compactly embedded in $L^{\mathscr{H}}_{loc}(\mathbb{R}^N)$. Also, $D^{s,\mathscr{H}}(\mathbb{R}^N)$ is continuously embedded in $L^{\mathscr{H}^*}(\mathbb{R}^N)$. Moreover, there exists $S_0 > 0$ such that $\|u\|_{L^{\mathscr{H}^*}(\mathbb{R}^N)} \leq S_0[u]_{s,\mathscr{H}}$.*

The following embedding theorem is available in [10, 49].

Theorem 7.1.2. *The space $D^{s,\mathscr{H}}(\mathbb{R}^N)$ is continuously embedded in $L^P(\mathbb{R}^N)$, where $p : \mathbb{R}^N \times \mathbb{R}^N \times [0,\infty) \to [0,\infty)$, $P(x,y,t) = \int_0^{|t|} p(x,y,r)r\,dr$ is a generalized N-function such that*

$$p_1 \leq \frac{p(x,y,|t|)|t|^2}{P(x,y,|t|)} \leq p_2,\ \forall (x,y) \in \mathbb{R}^N \times \mathbb{R}^N \text{ and } t \neq 0$$

for some $p_1, p_2 \in (h_2, h_1^*)$.

Lemma 7.1.3. *(Brezis-Lieb Lemma, [26]) Let $\mathscr{H}_x$ be a generalized N-function such that $f_n \to f$ a.e. and u_n converges weakly to u in $L^{\mathscr{H}_x}(\Omega)$ then we have*

$$\lim_{n\to\infty}\left(\int_\Omega \mathscr{H}_x(x,|u_n|)v dx - \int_\Omega \mathscr{H}_x(x,|u_n-u|)v dx\right) = \int_\Omega \mathscr{H}_x(x,|u|)v dx$$

for all $v \in L^\infty(\Omega)$.

For a given generalized N-function $\mathscr{H}$, we define Matuszewska-Orlicz function $Q_{\mathscr{H}}$ and $\mathscr{H}_1$ such that

$$Q_{\mathscr{H}}(x,t) = \lim_{r\to\infty} \frac{\mathscr{H}(x,x,rt)}{\mathscr{H}(x,x,r)}$$

and

$$\mathscr{H}_1(x,t) = \max\{t^{h_1}, t^{h_2}\},$$

respectively. It is known that [26] $Q_{\mathscr{H}}$ and $\mathscr{H}_1$ are also a generalized N-functions and $Q_{\mathscr{H}}$ is essentially larger than $\mathscr{H}_1$, i.e., $\lim\limits_{t\to\infty} \frac{Q_{\mathscr{H}}(x,ct)}{\mathscr{H}_1(x,t)} = 0$ for any $c > 0$ and $\forall (x,y) \in \Omega \times \Omega$.

7.2 Concentration compactness principle

Theorem 7.2.1. *Let $\{u_n\}$ be a bounded sequence in $D^{s,\mathscr{H}}(\mathbb{R}^N)$ which converges weakly to limit u such that $D^s\mathscr{H}(|u_n|)$ converges weakly to a measure μ and $\mathscr{H}^*(x,|u_n|)$ converges weakly to* *a measure* ν*, where* μ *and* ν *are bounded measures on* $\mathbb{R}^N$*. Then there exist at most countable index set J and families* $(x_j)_{j\in J}\in\mathbb{R}^N$*,* $(\nu_j)_{j\in J}>0$ *and* $(\mu_j)_{j\in J}>0$ *such that*

(1) $\nu=\mathscr{H}^*(x,|u|)+\sum\limits_{j\in J}\nu_j\delta_{x_j}$

(2) $\mu\geq D^s\mathscr{H}(|u|)+\sum\limits_{j\in J}\mu_j\delta_{x_j}$

with

$$\nu_j\leq\max\left\{S_0^{h_1^*}(\delta_1\mu_j)^{h_1^*/h_1},S_0^{h_2^*}(\delta_1\mu_j)^{h_2^*/h_1},S_0^{h_1^*}(\delta_1\mu_j)^{h_1^*/h_2},S_0^{h_2^*}(\delta_1\mu_j)^{h_2^*/h_2}\right\}\ \forall j\in J. \tag{7.1}$$

where

$$D^s\mathscr{H}(|u(x)|)=\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dy}{|x-y|^N}.$$

Theorem 7.2.2. *Let the assumptions of the Theorem 7.2.1 be true. If*

$$\mu_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R}D^s\mathscr{H}(|u_n(x)|)dx$$

and

$$\nu_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R}\mathscr{H}^*(x,|u_n(x)|)dx$$

then

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}D^s\mathscr{H}(|u_n(x)|)dx=\mu(\mathbb{R}^N)+\mu_\infty \tag{7.2}$$

and

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\mathscr{H}^*(x,|u_n(x)|)dx=\nu(\mathbb{R}^N)+\nu_\infty \tag{7.3}$$

with

$$\nu_\infty\leq\max\left\{S_0^{h_1^*}(\delta_1\mu_\infty)^{h_1^*/h_1},S_0^{h_2^*}(\delta_1\mu_\infty)^{h_2^*/h_1},S_0^{h_1^*}(\delta_1\mu_\infty)^{h_1^*/h_2},S_0^{h_2^*}(\delta_1\mu_\infty)^{h_2^*/h_2}\right\}.$$

Lemma 7.2.3. *Let* $v\in C_c^\infty(\mathbb{R}^N)$*. Then*

$$\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|(v(x)-v(y))|}{|x-y|^s}\right)\frac{dy}{|x-y|^N}\in L^\infty(\mathbb{R}^N).$$

Moreover,

$$\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|(v(x)-v(y))|}{|x-y|^s}\right)\frac{dy}{|x-y|^N}\leq c_1\min\left\{1,\max_{i\in\{1,2\}}\left\{|x|^{-(h_is+N)}\right\}\right\}$$

for some $c_1 > 0$. In particular,

$$\max_{i\in\{1,2\}}\left\{\int_{\mathbb{R}^N}\frac{|(v(x)-v(y))|^{h_i}}{|x-y|^{h_is+N}}dy\right\} < c_1\min\left\{1,\max_{i\in\{1,2\}}\left\{|x|^{-(h_is+N)}\right\}\right\}.$$

Proof. Let $v\in C_c^\infty(\mathbb{R}^N)$ and support of v is contained in $B(0,R)$ for some $R>0$. By Corollary 2.2.9, we get

$$\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|(v(x)-v(y))|}{|x-y|^s}\right)\frac{dy}{|x-y|^N}\leq b_2\max_{i\in\{1,2\}}\left\{\int_{\mathbb{R}^N}\frac{|(v(x)-v(y))|^{h_i}}{|x-y|^{h_is+N}}dy\right\}.$$

For $i\in\{1,2\}$, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{|(v(x)-v(y))|^{h_i}}{|x-y|^{h_is+N}}dy &= \int_{|y|\geq R}\frac{|(v(x)-v(y))|^{h_i}}{|x-y|^{h_is+N}}dy+\int_{|y|<R}\frac{|(v(x)-v(y))|^{h_i}}{|x-y|^{h_is+N}}dy\\ &= I_1+I_2.\end{aligned}$$

Consider,

$$\begin{aligned}I_1=\int_{|y|\geq R}\frac{|(v(x)-v(y))|^{h_i}}{|x-y|^{h_is+N}}dy &\leq 2^{h_i}\|v\|^{h_i}_{L^\infty(\mathbb{R}^N)}\int_{|h|\geq R}\frac{dh}{|h|^{h_is+N}}\\ &\leq 2^{h_i}N\omega_{N-1}\|v\|^{h_i}_{L^\infty(\mathbb{R}^N)}\int_R^\infty\frac{dr}{r^{h_is+1}}\\ &\leq\frac{2^{h_i}\|v\|^{h_i}_{L^\infty(\mathbb{R}^N)}N\omega_{N-1}}{h_isR^{h_is}}.\end{aligned}$$

Consider,

$$\begin{aligned}I_2=\int_{|y|<R}\frac{|(v(x)-v(y))|^{h_i}}{|x-y|^{h_is+N}}dy &\leq \|\nabla v\|^{h_i}_{L^\infty(\mathbb{R}^N)}\int_{|h|<R}\frac{dh}{|h|^{h_i(s-1)+N}}\\ &\leq N\omega_{N-1}\|\nabla v\|^{h_i}_{L^\infty(\mathbb{R}^N)}\int_0^R\frac{dr}{r^{h_i(s-1)+1}}\\ &\leq\frac{\|\nabla v\|^{h_i}_{L^\infty(\mathbb{R}^N)}N\omega_{N-1}R^{h_i(1-s)}}{h_i(1-s)}.\end{aligned}$$

By I_1 and I_2, we have $\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|(v(x)-v(y))|}{|x-y|^s}\right)\frac{dy}{|x-y|^N}\in L^\infty(\mathbb{R}^N)$.

Also, If $|x|>2R$ then

$$\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|(v(x)-v(y))|}{|x-y|^s}\right)\frac{dy}{|x-y|^N}\leq\int_{|y|\leq R}\mathscr{H}\left(x,y,\frac{|(v(y))|}{|x-y|^s}\right)\frac{dy}{|x-y|^N}.$$

If $|x|>2R$ and $|y|\le R$ then $|x-y|\ge |x|-|y|\ge |x|-R\ge \frac{|x|}{2}$, we have

$$\int_{|y|\le R}\mathscr{H}\left(x,y,\frac{|(v(y))|}{|x-y|^s}\right)\frac{dy}{|x-y|^N}\le b_2\max_{i\in\{1,2\}}\left\{\int_{|y|\le R}\frac{|(v(y))|^{h_i}}{|x-y|^{h_is+N}}dy\right\}$$
$$\le b_2\max_{i\in\{1,2\}}\left\{\frac{2^{h_is+N}}{|x|^{h_is+N}}\|v\|^{h_i}_{L^\infty(\mathbb{R}^N)}|B(0,R)|\right\}.$$

Further, by combining all the above calculations, we have

$$\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|(v(x)-v(y))|}{|x-y|^s}\right)\frac{dy}{|x-y|^N}\le c_1\min\left\{1,\max_{i\in\{1,2\}}\left\{|x|^{-(h_is+N)}\right\}\right\}$$

for some $c_1>0$. □

Lemma 7.2.4. *Let $\{u_n\}$ be a sequence in $D^{s,\mathscr{H}}(\mathbb{R}^N)$ which converges weakly to limit 0. Then, for every $v\in C_c^\infty(\mathbb{R}^N)$ we have*

$$\|v\|_{Q_{\mathscr{H}^*},\nu}\le \|v\|_{\mathscr{H}_1,\mu},$$

where $Q_{\mathscr{H}^*}(t)$ is the Matuszewska-Orlicz function and $\mathscr{H}_1(x,y,t)=\max\{t^{h_1},t^{h_2}\}$.

Proof. By Theorem 7.1.1, the embedding $D^{s,\mathscr{H}}(\mathbb{R}^N)\hookrightarrow L^{\mathscr{H}^*}(\mathbb{R}^N)$ is continuous, i.e.,

$$\|u\|_{L^{\mathscr{H}^*}(\mathbb{R}^N)}\le S_0\|u\|,\ \forall u\in D^{s,\mathscr{H}}(\mathbb{R}^N). \tag{7.4}$$

If $u\in D^{s,\mathscr{H}}(\mathbb{R}^N)$ and $v\in C_c^\infty(\mathbb{R}^N)$ then $uv\in D^{s,\mathscr{H}}(\mathbb{R}^N)$. Now, by applying (7.4) to the functions vu_n, we get

$$\|vu_n\|_{L^{\mathscr{H}^*}(\mathbb{R}^N)}\le S_0\|vu_n\|,\ \text{for each } n.$$

As proved in the [26, Lemma 4.4], we have

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N}\mathscr{H}^*(x,|vu_n|)dx\ge\int_{\mathbb{R}^N}Q_{\mathscr{H}^*}(x,|v|)dx,$$

which implies

$$\liminf_{n\to\infty}\|vu_n\|_{L^{\mathscr{H}^*}(\mathbb{R}^N)}\ge\|v\|_{Q_{\mathscr{H}^*},\nu}. \tag{7.5}$$

By (2.2), we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|v(x)u_n(x)-v(y)u_n(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}$$
$$=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|v(x)u_n(x)-v(y)u_n(x)+v(y)u_n(x)-v(y)u_n(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}$$
$$\le\delta_1\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|v(y)(u_n(x)-u_n(y))|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}$$

$$+\delta_1 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|u_n(x)(v(x)-v(y))|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}. \tag{7.6}$$

Next, we will prove that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|u_n(x)(v(x)-v(y))|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}\to 0. \tag{7.7}$$

For any $k>0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|u_n(x)(v(x)-v(y))|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}\\
&\quad=\left(\int_{B(0,k)}+\int_{B(0,k)^C}\right)\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|u_n(x)(v(x)-v(y))|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}\\
&\quad=I_1+I_2.
\end{aligned}$$

By Corollary 2.2.9, Lemmas 7.1.1 and 7.2.3, we obtain

$$\begin{aligned}
I_1&=\int_{B(0,k)}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|u_n(x)(v(x)-v(y))|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}\\
&\le b_2\int_{B(0,k)}(|u_n(x)|^{h_1}+|u_n(x)|^{h_2})\max_{i\in\{1,2\}}\left\{\int_{\mathbb{R}^N}\frac{|(v(x)-v(y))|^{h_i}}{|x-y|^{h_is+N}}dy\right\}dx\\
&\le b_2c_1\int_{B(0,k)}(|u_n(x)|^{h_1}+|u_n(x)|^{h_2})dx\\
&\le b_2c_1|B(0,k)|\left(\int_{B(0,k)}|u_n(x)|^{h_2}\right)^{\frac{h_1}{h_2}}+b_2c_1\int_{B(0,k)}|u_n(x)|^{h_2}dx\to 0,
\end{aligned}$$

since, $u_n\rightharpoonup 0$ in $D^{s,\mathscr{H}}(\mathbb{R}^N)$.

By Lemma 7.2.3 and Corollary 2.2.9, we obtain

$$\begin{aligned}
I_2&=\int_{B(0,k)^C}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|u_n(x)(v(x)-v(y))|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}\\
&\le b_2c_1\int_{B(0,k)^C}\max_{i\in\{1,2\}}\left\{\frac{|u_n(x)|^{h_i}}{|x|^{(h_is+N)}}\right\}dx.
\end{aligned}$$

For $i\in\{1,2\}$, using Hölder's inequality (with exponents $\dfrac{b}{h_i}$ and $\dfrac{b}{b-h_i}$, where $b\in(h_2,h_1^*)$) and Theorem 7.1.2, one gets

$$\int_{B(0,k)^C}|u_n(x)|^{h_i}\frac{dx}{|x|^{(h_is+N)}}\le\left(\int_{B(0,k)^C}|u_n(x)|^b dx\right)^{\frac{h_i}{b}}\left(\int_{B(0,k)^C}\frac{dx}{|x|^{(h_is+N)b/(b-h_i)}}\right)^{\frac{b-h_i}{b}}$$

$$\leq cN\omega_{N-1}\left(\frac{1}{k^{\frac{(h_is+N)b}{(b-h_i)}-N}}\right)^{(b-h_i)/b} \to 0 \text{ as } k\to\infty$$

since, $\frac{(h_is+N)b}{(b-h_i)} > N$. This implies that $I_2 \to 0$ as $k\to\infty$. Hence, (7.7) is true.

Thus, (7.6) and (7.7) imply

$$\begin{aligned}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|v(x)u_n(x)-v(y)u_n(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}\\ \leq \delta_1\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|v(y)(u_n(x)-u_n(y))|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}\\ \leq \delta_1\int_{\mathbb{R}^N}\mathscr{H}_1(x,|v|)d\mu\end{aligned}$$

which implies

$$\limsup_{n\to\infty}\|vu_n\| \leq \|v\|_{\mathscr{H}_1,\mu}. \tag{7.8}$$

By (7.5) and (7.8), we have

$$\|v\|_{Q_{\mathscr{H}^*},\nu} \leq \liminf_{n\to\infty}\|vu_n\|_{L^{\mathscr{H}^*}(\mathbb{R}^N)} \leq \limsup_{n\to\infty}\|vu_n\| \leq \|v\|_{\mathscr{H}_1,\mu}.$$

Therefore, we have

$$\|v\|_{Q_{\mathscr{H}^*},\nu} \leq C\|v\|_{\mathscr{H}_1,\mu},$$

which proves the lemma. □

Lemma 7.2.5. *[26] Let τ be a bounded measure and M_1, M_2 be two generalized N-functions such that M_2 is essentially larger than M_1, i.e., $\lim_{t\to\infty}\frac{M_1(ct)}{M_2(t)} = 0$ for any $c>0$ and $\forall (x,y)\in\Omega\times\Omega$. Let*

$$\|v\|_{M_2,\tau} \leq c_3\|v\|_{M_1,\tau}$$

for some $c_3>0$ and for all $v\in C_c^\infty(\mathbb{R}^N)$. Then, there exist atmost countable index set J and two families $(x_j)_{j\in J}\in\mathbb{R}^N$ and $(\delta_j)_{j\in J}>0$ such that

$$\tau = \sum_{j\in J} v_j\delta_{x_j}.$$

Proof of the Theorem 7.2.1. If $u=0$, by the help of the Lemmas 7.2.4, 7.2.5 and [26, Lemma 4.7], there exist atmost countable index set J and two families $(x_j)_{j\in J}\in\mathbb{R}^N$ and $(\delta_j)_{j\in J}>0$ such that

$$\nu = \sum_{j\in J} v_j\delta_{x_j}.$$

Let $u \neq 0$. Define $v_n = u_n - u$. Now, v_n converges weakly to 0 in $D^{s,\mathscr{H}}(\mathbb{R}^N)$. Thus, there exist atmost countable index set J and two families $(x_j)_{j\in J} \in \mathbb{R}^N$ and $(\delta_j)_{j\in J} > 0$ such that $\mathscr{H}^*(x,|v_n|) \rightharpoonup \nu_1 = \sum_{j\in J} \nu_j \delta_{x_j}$. By Brezis-Lieb Lemma 7.1.3, we obtain

$$\lim_{n\to\infty}\left(\int_\Omega \mathscr{H}^*(x,|u_n|)\varphi dx - \int_\Omega \mathscr{H}^*(x,|v_n|)\varphi dx\right) = \int_\Omega \mathscr{H}^*(x,|u|)\varphi dx$$

for all $v \in C_c^\infty(\Omega)$. Hence, we have

$$\mathscr{H}^*(x,|u_n|) \rightharpoonup \nu = \mathscr{H}^*(x,|u|) + \mathscr{H}^*(x,|v_n|) = \mathscr{H}^*(x,|u|) + \nu_1 = \mathscr{H}^*(x,|u|) + \sum_{j\in J} \nu_j \delta_{x_j}$$

which proves (1).

Next, we will prove the relation in μ_j and ν_j. Let $v \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le |v| \le 1$.

By propositions 2.2.8, 2.2.20 and Lemma 7.1.1, we get

$$\begin{aligned}
\int_{\mathbb{R}^N} \mathscr{H}^*(x,|vu_n|)dx &\le \max\left\{\|vu_n\|^{h_1^*}_{L^{\mathscr{H}^*}(\mathbb{R}^N)}, \|vu_n\|^{h_2^*}_{L^{\mathscr{H}^*}(\mathbb{R}^N)}\right\}\\
&\le \max\left\{S_0^{h_1^*}\|vu_n\|^{h_1^*}, S_0^{h_2^*}\|vu_n\|^{h_2^*}\right\}\\
&\le \max\left\{S_0^{h_1^*}M(vu_n)^{h_1^*/h_1}, S_0^{h_2^*}M(vu_n)^{h_2^*/h_1},\right.\\
&\left. S_0^{h_1^*}M(vu_n)^{h_1^*/h_2}, S_0^{h_2^*}M(vu_n)^{h_2^*/h_2}\right\}
\end{aligned}$$

where

$$M(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}.$$

By (7.7), we have

$$\begin{aligned}
\lim_{n\to\infty} M(vu_n) &\le \delta_1 \lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|v(y)(u_n(x)-u_n(y))|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}\\
&+ \delta_1 \lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|u_n(x)(v(x)-v(y))|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}\\
&\le \delta_1 \lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|v(y)(u_n(x)-u_n(y))|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}\\
&\le \delta_1 \int_{\mathbb{R}^N} |v|^{h_1} d\mu.
\end{aligned}$$

Hence, we have

$$\int_{\mathbb{R}^N} |v|^{h_2^*} d\nu \le \max\left\{S_0^{h_1^*}\left(\delta_1\int_{\mathbb{R}^N}|v|^{h_1}d\mu\right)^{h_1^*/h_1}, S_0^{h_2^*}\left(\delta_1\int_{\mathbb{R}^N}|v|^{h_1}d\mu\right)^{h_2^*/h_1},\right.$$

$$S_0^{h_1^*}\left(\delta_1\int_{\mathbb{R}^N}|v|^{h_1}d\mu\right)^{h_1^*/h_2}, S_0^{h_2^*}\left(\delta_1\int_{\mathbb{R}^N}|v|^{h_1}d\mu\right)^{h_2^*/h_2}\Bigg\}$$

which implies

$$v(B)\leq \max\left\{S_0^{h_1^*}(\delta_1\mu(B))^{h_1^*/h_1}, S_0^{h_2^*}(\delta_1\mu(B))^{h_2^*/h_1}, S_0^{h_1^*}(\delta_1\mu(B))^{h_1^*/h_2}, S_0^{h_2^*}(\delta_1\mu(B))^{h_2^*/h_2}\right\}$$

for any Borel set $B\subseteq\mathbb{R}^N$. In particular, for any $j\in J$, taking $B=B(x_j,\varepsilon)$ and taking $\varepsilon\to 0$, we obtain

$$v_j\leq \max\left\{S_0^{h_1^*}(\delta_1\mu_j)^{h_1^*/h_1}, S_0^{h_2^*}(\delta_1\mu_j)^{h_2^*/h_1}, S_0^{h_1^*}(\delta_1\mu_j)^{h_1^*/h_2}, S_0^{h_2^*}(\delta_1\mu_j)^{h_2^*/h_2}\right\}\ \forall j\in J.$$

Proof of the Theorem 7.2.2. Choose a function $v\in C^\infty(\mathbb{R}^N)$ such that $0\leq v\leq 1$

$$v\equiv 0 \text{ on } B(0,1),\ v\equiv 1 \text{ on } B(0,2)^C.$$

Now, for each $R>0$, define $v_R(x)=\frac{v(x)}{R}\ \forall x\in\mathbb{R}^N$ then

$$v_R\equiv 0 \text{ on } B(0,R),\ v_R\equiv 1 \text{ on } B(0,2R)^C.$$

Consider,

$$\int_{\mathbb{R}^N}D^s\mathscr{H}(|u_n(x)|)dx=\int_{\mathbb{R}^N}D^s\mathscr{H}(|u_n(x)|)v_R(x)dx+\int_{\mathbb{R}^N}D^s\mathscr{H}(|u_n(x)|)(1-v_R(x))dx. \tag{7.9}$$

As we can observe that

$$\int_{|x|>2R}D^s\mathscr{H}(|u_n(x)|)v_R(x)dx\leq\int_{\mathbb{R}^N}D^s\mathscr{H}(|u_n(x)|)v_R(x)dx\leq\int_{|x|>R}D^s\mathscr{H}(|u_n(x)|)dx,$$

therefore, we have

$$\mu_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}D^s\mathscr{H}(|u_n(x)|)v_R(x)dx. \tag{7.10}$$

By the definition of weak convergence together with the fact $\lim\limits_{R\to\infty}v_R(x)=0$ a.e. $x\in\mathbb{R}^N$, we have

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}D^s\mathscr{H}(|u_n(x)|)(1-v_R(x))dx=\int_{\mathbb{R}^N}(1-v_R(x))d\mu$$

after that, applying Dominated Convergence Theorem, we get

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}D^s\mathscr{H}(|u_n(x)|)(1-v_R(x))dx=\mu(\mathbb{R}^N). \tag{7.11}$$

By (7.9), (7.10) and (7.11), one gets

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N} D^s\mathscr{H}(|u_n(x)|)dx = \mu(\mathbb{R}^N)+\mu_\infty$$

which proves (7.2). Similarly, we can prove (7.3). To prove the relation between μ_∞ and ν_∞, we will use the same approach as we have used to prove (7.1).

By propositions 2.2.8, 2.2.20 and Lemma7.1.1, we get

$$\begin{aligned}\int_{\mathbb{R}^N}\mathscr{H}^*(|v_R u_n|)dx &\le \max\left\{\|v_R u_n\|^{h_1^*}_{L^{\mathscr{H}^*}(\mathbb{R}^N)}, \|v_R u_n\|^{h_2^*}_{L^{\mathscr{H}^*}(\mathbb{R}^N)}\right\}\\ &\le \max\left\{S_0^{h_1^*}\|v_R u_n\|^{h_1^*}, S_0^{h_2^*}\|v_R u_n\|^{h_2^*}\right\}\\ &\le \max\left\{S_0^{h_1^*}M(v_R u_n)^{h_1^*/h_1}, S_0^{h_2^*}M(v_R u_n)^{h_2^*/h_1},\right.\\ &\left. S_0^{h_1^*}M(v_R u_n)^{h_1^*/h_2}, S_0^{h_2^*}M(v_R u_n)^{h_2^*/h_2}\right\}. \end{aligned} \tag{7.12}$$

Next, (2.2) implies

$$\begin{aligned} &M(v_R u_n) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|v_R(x)u_n(x)-v_R(y)u_n(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}\\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|v_R(x)u_n(x)-v_R(y)u_n(x)+v_R(y)u_n(x)-v_R(y)u_n(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}\\ &\le \delta_1\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|v_R(y)(u_n(x)-u_n(y))|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}\\ &+\delta_1\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|u_n(x)(v_R(x)-v_R(y))|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}. \end{aligned} \tag{7.13}$$

$$\text{Claim: } \lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|u_n(x)(v_R(x)-v_R(y))|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}\to 0. \tag{7.14}$$

Consider,

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|u_n(x)(v_R(x)-v_R(y))|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}\\ &\quad=\left(\int_{B(0,2R)}\int_{\mathbb{R}^N}+\int_{B(0,2R)^C}\int_{B(0,2R)^C}+\int_{B(0,2R)^C}\int_{B(0,2R)}\right)\\ &\quad\mathscr{H}\left(x,y,\frac{|u_n(x)(v_R(x)-v_R(y))|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}\\ &\quad= I_1+I_2+I_3. \end{aligned}$$

By Corollary 2.2.9, Lemmas 7.1.1 and 7.2.3, we obtain

$$\begin{aligned}
I_1 &= \int_{B(0,2R)} \int_{\mathbb{R}^N} \mathscr{H}\left(x, y, \frac{|u_n(x)(v_R(x) - v_R(y))|}{|x-y|^s}\right) \frac{dx\,dy}{|x-y|^N} \\
&\leq b_2 \int_{B(0,2R)} (|u_n(x)|^{h_1} + |u_n(x)|^{h_2}) \max_{i\in\{1,2\}} \left\{ \int_{\mathbb{R}^N} \frac{|(v_R(x) - v_R(y))|^{h_i}}{|x-y|^{h_i s+N}} dy \right\} dx \\
&\leq b_2 c_1 |B(0,2R)| \left(\int_{B(0,2R)} |u_n(x)|^{h_2} \right)^{\frac{h_1}{h_2}} + b_2 c_1 \int_{B(0,2R)} |u_n(x)|^{h_2} dx \to 0,
\end{aligned}$$

since, $u_n \rightharpoonup 0$ in $D^{s,\mathscr{H}}(\mathbb{R}^N)$.

$I_2 = 0$, since $v_R = 1$ in $B(0,2R)^C$.

For any $k > 4$, we have

$$\begin{aligned}
I_3 &= \left(\int_{B(0,kR)^C} \int_{B(0,2R)} + \int_{B(0,kR)\setminus B(0,2R)} \int_{B(0,2R)} \right) \mathscr{H}\left(x, y, \frac{|u_n(x)(v_R(x) - v_R(y))|}{|x-y|^s}\right) \frac{dx\,dy}{|x-y|^N} \\
&= I_3' + I_3''.
\end{aligned}$$

If $|x| \geq kR$ and $|y| < 2R$ then $|x-y| \geq |x| - |y| \geq |x| - 2R \geq \dfrac{|x|}{2}$, we have

$$\begin{aligned}
I_3' &= \int_{B(0,kR)^C} \int_{B(0,2R)} \mathscr{H}\left(x, y, \frac{|u_n(x)(v_R(x) - v_R(y))|}{|x-y|^s}\right) \frac{dx\,dy}{|x-y|^N} \\
&\leq b_2 \max_{i\in\{1,2\}} \left\{ \int_{B(0,kR)^C} \int_{B(0,2R)} \frac{|(u_n(x))|^{h_i}}{|x|^{h_i s+N}} dx dy \right\} \\
&\leq b_2 |B(0,2R)| \max_{i\in\{1,2\}} \left\{ \int_{B(0,kR)^C} \frac{|(u_n(x))|^{h_i}}{|x|^{h_i s+N}} dx \right\}.
\end{aligned}$$

For any $i \in \{1,2\}$, using Hölder's inequality (with exponents $\dfrac{b}{h_i}$ and $\dfrac{b}{b-h_i}$, where $b \in (h_2, h_1^*)$) and Theorem 7.1.2, one gets

$$\begin{aligned}
\int_{B(0,kR)^C} |u_n(x)|^{h_i} \frac{dx}{|x|^{(h_i s+N)}} &\leq \left(\int_{B(0,kR)^C} |u_n(x)|^b dx \right)^{h_i/b} \left(\int_{B(0,kR)^C} \frac{dx}{|x|^{(h_i s+N)b/(b-h_i)}} \right)^{(b-h_i)/b} \\
&\leq c_2 N \omega_{N-1} \left(\frac{1}{(kR)^{\frac{(h_i s+N)b}{(b-h_i)} - N}} \right)^{(b-h_i)/b} \to 0 \text{ as } k \to \infty \text{ and } R \to \infty
\end{aligned}$$

which implies $I_3' \to 0$ as $k \to \infty$. By Corollary 2.2.9, Lemmas 7.1.1 and 7.2.3, we obtain

$$I_3'' = \int_{B(0,kR)\setminus B(0,2R)} \int_{B(0,2R)} \mathscr{H}\left(x, y, \frac{|u_n(x)(v_R(x) - v_R(y))|}{|x-y|^s}\right) \frac{dx\,dy}{|x-y|^N}$$

$$
\begin{aligned}
&\le b_2 \int_{B(0,kR)\setminus B(0,2R)} (|u_n(x)|^{h_1} + |u_n(x)|^{h_2}) \max_{i\in\{1,2\}} \left\{ \int_{B(0,2R)} \frac{|(v_R(x)-v_R(y))|^{h_i}}{|x-y|^{h_i s+N}} dy \right\} dx \\
&\le b_2 c_1 \int_{B(0,kR)\setminus B(0,2R)} (|u_n(x)|^{h_1} + |u_n(x)|^{h_2}) \\
&\le c_3 \left(\int_{B(0,kR)\setminus B(0,2R)} |u_n(x)|^{h_2} \right)^{\frac{h_1}{h_2}} + c_3 \int_{B(0,kR)\setminus B(0,2R)} |u_n(x)|^{h_2} dx \to 0,
\end{aligned}
$$

since, $u_n \rightharpoonup 0$ in $D^{s,\mathscr{H}}(\mathbb{R}^N)$.
Gathering all the above calculations, our claim (7.14) is proved.
By (7.13) and (7.14), we have

$$
\begin{aligned}
&\lim_{R\to\infty}\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|v_R(x)u_n(x)-v_R(y)u_n(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N} \\
&\qquad \le \delta_1 \lim_{R\to\infty}\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|v_R(y)(u_n(x)-u_n(y))|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N} \\
&\qquad \le \delta_1 \lim_{R\to\infty}\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v_R(y)|^{h_1} \mathscr{H}\left(x,y,\frac{|(u_n(x)-u_n(y))|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N} \\
&\qquad \le \delta_1 \mu_\infty. \qquad (7.15)
\end{aligned}
$$

(7.12) and (7.15) implies

$$
\nu_\infty \le \max\left\{ S_0^{h_1^*}(\delta_1\mu_\infty)^{h_1^*/h_1}, S_0^{h_2^*}(\delta_1\mu_\infty)^{h_2^*/h_1}, S_0^{h_1^*}(\delta_1\mu_\infty)^{h_1^*/h_2}, S_0^{h_2^*}(\delta_1\mu_\infty)^{h_2^*/h_2} \right\}.
$$

□

7.3 Applications

In this section, we establish the existence of a weak solution to the following problem:

$$
(-\Delta)^s_{\mathscr{H}} u(x) = \lambda g(x,|u|)u + f(x,u) \ \text{ in } \ \mathbb{R}^N, \qquad (7.16)
$$

where $N \ge 1$, $s \in (0,1)$,

$$
\mathscr{H}(x,y,t) = \int_0^{|t|} h(x,y,r) r\, dr,
$$

and $h : \mathbb{R}^N \times \mathbb{R}^N \times [0,\infty) \to [0,\infty)$ is a generalized N-function. The functions $g : \mathbb{R}^N \times \mathbb{R} \to [0,\infty)$, $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are continuous, $\lambda > 0$ is sufficiently small parameter and $(-\Delta)^s_{\mathscr{H}}$ is the generalized fractional Laplace operator.

7.3.1 Hypotheses

We assume that $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following conditions:

(f_1) There exist generalized N-function $\Psi : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to [0,\infty)$ and $\psi_1, \psi_2(h_2, h_1^*)$ such that $\psi_1 \leq \frac{\psi(x,y,|t|)|t|^2}{\Psi(x,y,|t|)} \leq \psi_2$, $\forall (x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ and $t \neq 0$ and

$$\lim_{t\to 0} \frac{f(x,t)}{\psi(x,x,|t|)|t|} = 0,\ \forall\, x \in \mathbb{R}^N,$$

where $\Psi(x,y,t) = \int_0^{|t|} \psi(x,y,r)r\, dr$.

(f_2) $\lim_{t\to\infty} \frac{f(x,t)}{h^*(x,x,|t|)|t|} = 0$, $\forall\, x \in \mathbb{R}^N$, where h^* is define in (2.9).

(f_3) There exists $\sigma \in (h_2, h_1^*)$ such that

$$0 < \sigma F(x,t) = \sigma \int_0^t f(x,s)ds \leq t f(x,t),$$

for all $t > 0,\ x \in \mathbb{R}^N$.

We assume that $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is C^1 and satisfy the following conditions:

(g_1) $h_1^* \leq \frac{g(x,|t|)|t|^2}{G(x,|t|)} \leq h_2^*$ for all $x \in \mathbb{R}^N$ and $t \neq 0$, where $G(x,t) = \int_0^{|t|} g(x,r)dr$.

(g_2) There exist $g_1, g_2 > 0$ such that $g_1 \leq \frac{G(x,t)}{\mathscr{H}^*(x,t)} \leq g_2$ for all $t > 0$.

7.3.2 Functional setting and main result

First, we define a weak solution for the Problem (7.16) and its associated energy functional.

Definition 7.3.1. We say $u \in D^{s,\mathscr{H}}(\mathbb{R}^N)$ is a weak solution of (7.16) if the following holds:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} h\left(x,y,\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dx\,dy$$
$$= \lambda \int_{\mathbb{R}^N} g(x,|u|)uv dx + \int_{\mathbb{R}^N} f(x,u)v dx,\ \forall v \in D^{s,\mathscr{H}}(\mathbb{R}^N).$$

Thus, the energy functional $I : D^{s,\mathscr{H}}(\mathbb{R}^N) \to \mathbb{R}$ corresponding to (7.16) is given by

$$I(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N} - \lambda \int_{\mathbb{R}^N} G(x,|u|)dx - \int_{\mathbb{R}^N} F(x,u)dx.$$

It can be seen that I is well-defined and C^1 functional. The derivative of I at any point $u \in D^{s,\mathscr{H}}(\mathbb{R}^N)$ is given by

$$I'(u)(v) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} h\left(x,y,\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dx\,dy$$
$$-\lambda\int_{\mathbb{R}^N} g(x,|u|)uvdx - \int_{\mathbb{R}^N} f(x,u)vdx,\ \forall v\in D^{s,\mathscr{H}}(\mathbb{R}^N).$$

Moreover, the critical points of I are the weak solutions to (7.16). Let $M : D^{s,\mathscr{H}}(\mathbb{R}^N) \to \mathbb{R}$ such that

$$M(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}.$$

Remark 7.3.2. *The functional M is convex, since $\mathscr{H}$ is convex. Consequently, M is weakly lower semicontinuous, i.e., if $\{u_n\} \rightharpoonup u$ in $D^{s,\mathscr{H}}(\mathbb{R}^N)$ then $M(u) \le \liminf_{n\to\infty} M(u_n)$.*

The main existence result of this chapter is as follows:

Theorem 7.3.3. *Suppose that the conditions $(f_1)-(f_4)$, $(g_1)-(g_2)$ and $(h_1)-(h_3)$ are satisfied. Then there exists $\lambda_1 > 0$ such that for each $0 < \lambda \le \lambda_1$, Problem (7.16) has a non-trivial weak solution.*

7.3.3 Proof of the Theorem 7.3.3

To prove the Theorem 7.3.3, we first establish a series of lemmas.

Lemma 7.3.4. *There exist positive real numbers α and ρ such that*

$$I(u) \ge \alpha,\ \ \forall u \in D^{s,\mathscr{H}}(\mathbb{R}^N) : \|u\| = \rho.$$

Proof. Using assumptions $(f_1)-(f_2)$ there exist $c_1, c_2 > 0$ such that

$$|F(x,t)| \le c_4\Psi(x,x,|t|) + c_5\mathscr{H}^*(x,x,|t|), \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{7.17}$$

By (g_2), Propositions 2.2.8, 2.2.20, 6.2.4 and (7.17), we have

$$I(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N} - \lambda\int_{\mathbb{R}^N} G(x,|u|)dx - \int_{\mathbb{R}^N} F(x,u)dx$$
$$\ge \min\left\{\|u\|^{h_1}, \|u\|^{h_2}\right\} - \lambda g_2\int_{\mathbb{R}^N}\mathscr{H}^*(x,|u|)dx - c_4\int_{\mathbb{R}^N}\Psi(x,x,|u|)dx$$
$$-c_5\int_{\mathbb{R}^N}\mathscr{H}^*(x,x,|u|)dx$$
$$\ge \min\left\{\|u\|^{h_1}, \|u\|^{h_2}\right\} - \lambda g_2 \max\left\{\|u\|^{h_1^*}_{L^{\mathscr{H}^*}(\mathbb{R}^N)}, \|u\|^{h_2^*}_{L^{\mathscr{H}^*}(\mathbb{R}^N)}\right\}$$
$$-c_5 \max\left\{\|u\|^{h_1^*}_{L^{\mathscr{H}^*}(\mathbb{R}^N)}, \|u\|^{h_2^*}_{L^{\mathscr{H}^*}(\mathbb{R}^N)}\right\} - c_4\max\left\{\|u\|^{\psi_1}_{L^{\Psi_x}(\mathbb{R}^N)}, \|u\|^{\psi_2}_{L^{\Psi_x}(\mathbb{R}^N)}\right\}.$$

If $\|u\| < 1$, Lemma 7.1.1 and Theorem 7.1.2 implies

$$I(u) \geq \|u\|^{h_2} - S_0(g_2\lambda + c_5)\|u\|^{h_1^*} - c_6\|u\|^{\psi_1}.$$

We conclude the result by choosing ρ sufficiently small and using the fact that $h_1^* > h_2$ and $\psi_1 > h_2$. □

Lemma 7.3.5. *There exist $v_0 \in D^{s,\mathscr{H}}(\mathbb{R}^N)$ and $\beta > 0$ such that*

$$I(v_0) < 0 \ \text{ and } \|v_0\| > \beta.$$

Proof. By (f_3), there exist $m_1, m_2 > 0$ such that

$$F(x,s) \geq m_1 s^{\sigma} - m_2, \ \ \forall\, (x,s) \in \mathbb{R}^N \times [0,\infty). \tag{7.18}$$

Let $u \in D^{s,\mathscr{H}}(\mathbb{R}^N)\backslash\{0\}$ and $u \geq 0$ with compact support $K \subseteq \mathbb{R}^N$. For $t > 1$, by (g_2), Propositions 2.2.20, 6.2.4 and (7.18), we have

$$\begin{aligned}
I(tu) = & \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|tu(x)-tu(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N} - \lambda\int_{\mathbb{R}^N} G(x,|tu|)dx \\
& - \int_{\mathbb{R}^N} F(x,tu)dx \\
& \leq t^{h_2}\max\left\{\|u\|^{h_1}, \|u\|^{h_2}\right\} - \lambda g_1 \int_{\mathbb{R}^N} \mathscr{H}^*(x,|tu|)dx - m_1 t^{\sigma}\int_K |u|^{\sigma} + m_2|K| \\
& \leq t^{h_2}\max\left\{\|u\|^{h_1}, \|u\|^{h_2}\right\} - \lambda g_1 t^{h_1^*}\min\left\{\|u\|^{h_1^*}_{L^{\mathscr{H}^*}(\mathbb{R}^N)}, \|u\|^{h_2^*}_{L^{\mathscr{H}^*}(\mathbb{R}^N)}\right\} \\
& - m_1 t^{\sigma}\int_K |u|^{\sigma} + m_2|K|
\end{aligned}$$

this implies that $I(tu) \to -\infty$ as $t \to \infty$, since $\sigma > h_2$. Now, by setting $v_0 = tu$ for sufficiently large t, we get the desired result. □

By Lemmas 7.3.4 and 7.3.5, the geometric conditions of the mountain pass theorem are satisfied for the functional I. Hence, by the version of the mountain pass theorem without (PS) condition, there exists a sequence $\{u_n\} \subseteq D^{s,\mathscr{H}}(\mathbb{R}^N)$ such that $I(u_n) \to c_M$ and $I'(u_n) \to 0$ as $n \to \infty$, where

$$0 < c_M = \inf_{\gamma \in \Gamma}\max_{t\in[0,1]} I(\gamma(t)) > 0,$$

and

$$\Gamma = \{\gamma \in C([0,1], D^{s,\mathscr{H}}(\mathbb{R}^N)) : \gamma(0) = 0, \ \gamma(1) < 0\}\cdot$$

Lemma 7.3.6. *The $(PS)_{c_M}$ sequence is bounded in $D^{s,\mathscr{H}}(\mathbb{R}^N)$. Moreover, there exists $u \in D^{s,\mathscr{H}}(\mathbb{R}^N)$ such that, up to a subsequence, we have $u_n \rightharpoonup u$ weakly in $D^{s,\mathscr{H}}(\mathbb{R}^N)$.*

Proof. Since $\{u_n\}$ is a $(PS)_{c_M}$ sequence of I, we have $I(u_n) \to c_M$ and $I'(u_n) \to 0$ as $n \to \infty$, i.e.,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N} - \lambda\int_{\mathbb{R}^N} G(x,|u_n|)dx - \int_{\mathbb{R}^N} F(x,u_n)dx = c_M + \delta_n, \tag{7.19}$$

where $\delta_n \to 0$ as $n \to \infty$ and

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} h\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{(u_n(x)-u_n(y))(v(x)-v(y))}{|x-y|^{N+2s}}dx\,dy - \lambda\int_{\mathbb{R}^N} g(x,|u_n|)u_n v dx - \int_{\mathbb{R}^N} f(x,u_n)v dx\right| \le \varepsilon_n\|v\|, \tag{7.20}$$

$\forall v \in D^{s,\mathscr{H}}(\mathbb{R}^N)$, where $\varepsilon_n \to 0$ as $n \to \infty$. On taking $v = u_n$, by (7.19), (7.20) and using (f_3), we obtain

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N} - \frac{1}{\sigma}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} h\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{(u_n(x)-u_n(y))^2}{|x-y|^{N+2s}}dxdy\right) - \lambda\int_{\mathbb{R}^N}\left(G(x,|u_n|) - \frac{1}{\sigma}g(x,|u_n|)u_n^2\right)dx \le c_7(1+\|u_n\|).$$

It follows from (h_1) and $(g_1)-(g_2)$ that

$$\left(1-\frac{h_2}{\sigma}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N} + g_1\lambda\left(\frac{h_1^*}{\sigma}-1\right)\int_{\mathbb{R}^N}\mathscr{H}^*(x,|u_n|)dx \le c_7(1+\|u_n\|).$$

If $\|u_n\| > 1$, we have

$$\left(1-\frac{h_2}{\sigma}\right)\|u_n\|^{h_1} \le c_7(1+\|u_n\|)$$

since, $\sigma \in (h_2, h_1^*)$. Consequently, $\|u_n\| \le c_8$. Thus $\{u_n\}$ is bounded in $D^{s,\mathscr{H}}(\mathbb{R}^N)$. As $D^{s,\mathscr{H}}(\mathbb{R}^N)$ is a reflexive Banach space, $\exists\ u \in D^{s,\mathscr{H}}(\mathbb{R}^N)$ such that up to a subsequence, we have $u_n \rightharpoonup u$ weakly in $D^{s,\mathscr{H}}(\mathbb{R}^N)$. □

Let $\{u_n\}$ be any $(PS)_{c_M}$ sequence of I then by Lemma 7.3.6, it is bounded in $D^{s,\mathscr{H}}(\mathbb{R}^N)$ and there exists $u \in D^{s,\mathscr{H}}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ weakly in $D^{s,\mathscr{H}}(\mathbb{R}^N)$. By Theorem 7.1.1, we get

$\{u_n\}$ is also bounded in $L^{\mathscr{H}^*}(\mathbb{R}^N)$; thus $u_n \rightharpoonup u$ weakly in $L^{\mathscr{H}^*}(\mathbb{R}^N)$. This implies, $D^s\mathscr{H}(|u_n|)$ converges weakly to a measure μ and $\mathscr{H}^*(x,|u_n|)$ converges weakly to a measure ν, where μ and ν are some bounded measures on $\mathbb{R}^N$.

Then by Theorem 7.2.1, there exist atmost countable index set J and families $(x_j)_{j\in J} \in \mathbb{R}^N$, $(\nu_j)_{j\in J} > 0$ and $(\mu_j)_{j\in J} > 0$ such that

$$\nu = \mathscr{H}^*(x,|u|) + \sum_{j\in J} \nu_j \delta_{x_j} \tag{7.21}$$

$$\mu \geq D^s\mathscr{H}(|u|) + \sum_{j\in J} \mu_j \delta_{x_j}$$

with

$$\nu_j \leq \max\left\{ S_0^{h_1^*}(\delta_1\mu_j)^{h_1^*/h_1}, S_0^{h_2^*}(\delta_1\mu_j)^{h_2^*/h_1}, S_0^{h_1^*}(\delta_1\mu_j)^{h_1^*/h_2}, S_0^{h_2^*}(\delta_1\mu_j)^{h_2^*/h_2} \right\} \ \forall j \in J. \tag{7.22}$$

Also, by Theorem 7.2.2, we have

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} D^s\mathscr{H}(|u_n(x)|)dx = \mu(\mathbb{R}^N) + \mu_\infty \tag{7.23}$$

and

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \mathscr{H}^*(x,|u_n(x)|)dx = \nu(\mathbb{R}^N) + \nu_\infty \tag{7.24}$$

with

$$\nu_\infty \leq \max\left\{ S_0^{h_1^*}(\delta_1\mu_\infty)^{h_1^*/h_1}, S_0^{h_2^*}(\delta_1\mu_\infty)^{h_2^*/h_1}, S_0^{h_1^*}(\delta_1\mu_\infty)^{h_1^*/h_2}, S_0^{h_2^*}(\delta_1\mu_\infty)^{h_2^*/h_2} \right\}. \tag{7.25}$$

Lemma 7.3.7. *Let $\{u_n\}$ be a $(PS)_{c_M}$ sequence. Then there exists $\lambda_1 > 0$ such that for each $0 < \lambda \leq \lambda_1$, $\nu_j = 0$ for each $j \in J$ and $\nu_\infty = 0$.*

Proof. Choose a cutoff function $v \in C_c^\infty(\mathbb{R}^N)$ such that

$$v \equiv 1 \text{ on } B(0,1),\ v \equiv 0 \text{ on } B(0,2)^c.$$

Now, for each $\varepsilon > 0$, define $v_\varepsilon(x) = v((x-x_j)/\varepsilon)\ \forall x \in \mathbb{R}^N$ then

$$v_\varepsilon \equiv 1 \text{ on } B(x_j,\varepsilon),\ v_\varepsilon \equiv 0 \text{ on } B(x_j,2\varepsilon)^c.$$

Now, $J'(u_n)(u_n v_\varepsilon) = o_n(1)$, since $\{u_n\}$ is $(PS)_{c_M}$ sequence. Therefore, using (h_1) and (2.2), we have

$$
\begin{aligned}
J'(u_n)(u_n v_\varepsilon) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} h\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{(u_n(x)-u_n(y))(u_n(x)v_\varepsilon(x)-u_n(y)v_\varepsilon(y))}{|x-y|^{N+2s}}dx\,dy \\
&\quad -\lambda\int_{\mathbb{R}^N} g(x,|u_n|)|u_n|^2 v_\varepsilon dx - \int_{\mathbb{R}^N} f(x,u_n)u_n v_\varepsilon dx = o_n(1) \\
&\le \delta_1 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} h\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{|u_n(x)-u_n(y)|^2 v_\varepsilon(x)}{|x-y|^{N+2s}}dx\,dy \\
&\quad +\delta_1 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} h\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{(u_n(x)-u_n(y))(v_\varepsilon(x)-v_\varepsilon(y))u_n(y)}{|x-y|^{N+2s}}dx\,dy \\
&\quad -\lambda\int_{\mathbb{R}^N} g(x,|u_n|)|u_n|^2 v_\varepsilon dx - \int_{\mathbb{R}^N} f(x,u_n)u_n v_\varepsilon dx = o_n(1),
\end{aligned}
$$

this implies

$$
\begin{aligned}
& h_1 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{v_\varepsilon(x)dx\,dy}{|x-y|^N} \\
& + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} h\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{(u_n(x)-u_n(y))(v_\varepsilon(x)-v_\varepsilon(y))u_n(y)}{|x-y|^{N+2s}}dx\,dy \\
& \le \lambda\int_{\mathbb{R}^N} g(x,|u_n|)|u_n|^2 v_\varepsilon dx + \int_{\mathbb{R}^N} f(x,u_n)u_n v_\varepsilon dx + o_n(1). \qquad (7.26)
\end{aligned}
$$

Next, we prove that

$$
\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} h\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{(u_n(x)-u_n(y))(v_\varepsilon(x)-v_\varepsilon(y))u_n(y)}{|x-y|^{N+2s}}dx\,dy = 0. \qquad (7.27)
$$

By using (2.4), we have

$$
\begin{aligned}
& \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \widehat{\mathscr{H}}\left(h\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{(u_n(x)-u_n(y))}{|x-y|^s}\right)\frac{dxdy}{|x-y|^N} \\
& \qquad \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{2|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^N},
\end{aligned}
$$

therefore, Lemma 7.3.6 implies

$$
\left\{h\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{(u_n(x)-u_n(y))}{|x-y|^s}\right\} \subset L^{\widehat{\mathscr{H}}}(\mathbb{R}^N\times\mathbb{R}^N; d\chi)
$$

is a bounded sequence, where $d\chi = \dfrac{dxdy}{|x-y|^N}\cdot$

As $L^{\widehat{\mathscr{H}}}(\mathbb{R}^N \times \mathbb{R}^N; d\chi)$ is a reflexive space, there exists $w_1 \in L^{\widehat{\mathscr{H}}}(\mathbb{R}^N \times \mathbb{R}^N; d\chi)$ such that

$$h\left(x, y, \frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{(u_n(x)-u_n(y))}{|x-y|^s} \rightharpoonup w_1 \tag{7.28}$$

weakly in $L^{\widehat{\mathscr{H}}}(\mathbb{R}^N \times \mathbb{R}^N; d\chi)$. Define $v_n = u_n - u$. We can observe that $v_n \rightharpoonup 0$ weakly in $D^{s,\mathscr{H}}(\mathbb{R}^N)$. As we have proved in the Lemma 7.2.4, for fixed ε, we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x, y, \frac{(v_\varepsilon(x)-v_\varepsilon(y))v_n(y)}{|x-y|^s}\right) d\chi \to 0$$

as $n \to \infty$, which implies

$$\frac{(v_\varepsilon(x)-v_\varepsilon(y))u_n(y)}{|x-y|^s} \to \frac{(v_\varepsilon(x)-v_\varepsilon(y))u(y)}{|x-y|^s} \tag{7.29}$$

in $L^{\mathscr{H}}(\mathbb{R}^N \times \mathbb{R}^N; d\chi)$.

Definition of weak convergence, (7.28), (7.29) and Proposition 2.2.6 imply that

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} h\left(x, y, \frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{(u_n(x)-u_n(y))}{|x-y|^s}\frac{(v_\varepsilon(x)-v_\varepsilon(y))u_n(y)}{|x-y|^s} d\chi \\ \to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} w_1(x,y)\frac{(v_\varepsilon(x)-v_\varepsilon(y))u(y)}{|x-y|^s} d\chi \end{aligned} \tag{7.30}$$

as $n \to \infty$.

On the other side, as $J'(u_n)\varphi = o_n(1)$ for all $\varphi \in D^{s,\mathscr{H}}(\mathbb{R}^N)$, i.e.,

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} h\left(x, y, \frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} dx\, dy \\ &- \lambda \int_{\mathbb{R}^N} g(x,|u_n|)u_n\varphi dx - \int_{\mathbb{R}^N} f(x,u_n)\varphi dx = o_n(1),\end{aligned}$$

by Lemma 2.2.6, Remark 2.2.16 and (7.28), we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} w_1(x,y)\frac{(\varphi(x)-\varphi(y))}{|x-y|^s} d\chi = \lim_{n\to\infty}\int_{\mathbb{R}^N} \lambda g(x,|u_n|)u_n\varphi + f(x,u_n)\varphi dx \tag{7.31}$$

Claim: $\lambda g(x,|u_n|)u_n + f(x,u_n) \subset L^{\widehat{\mathscr{H}^*}}(\mathbb{R}^N)$ is a bounded sequence.

By $(g_1)-(g_2)$, (2.5), Lemmas 7.1.1 and 7.3.6, we get

$$\int_{\mathbb{R}^N} \widehat{\mathscr{H}^*}(x, g(x,|u_n|)u_n) dx \le \int_{\mathbb{R}^N} \widehat{\mathscr{H}^*}\left(x, h_2^* \frac{G(x,|u_n|)}{|u_n|}\right) dx$$

$$\begin{aligned}
&\leq c_9 \int_{\mathbb{R}^N} \widehat{\mathscr{H}^*}\left(x, \frac{\mathscr{H}^*(x,|u_n|)}{|u_n|}\right) dx \\
&\leq c_9 \int_{\mathbb{R}^N} \mathscr{H}^*(x,|u_n|) dx \\
&\leq c_{10} \text{ (independent of } n),
\end{aligned}$$

which implies that $\{g(x,|u_n|)u_n\} \subset L^{\widehat{\mathscr{H}^*}}(\mathbb{R}^N)$ is a bounded sequence.

Using $\psi_1, \psi_2 \in (h_2, h_1^*)$, (h_1), $(f_1)-(f_2)$, (2.5), Propositions 2.2.8, 2.2.20 and 2.2.21, we get

$$\begin{aligned}
\int_{\mathbb{R}^N} \widehat{\mathscr{H}^*}(x, f(x,|u_n|)u_n) dx &\leq \int_{\mathbb{R}^N} \widehat{\mathscr{H}^*}\left(c_{11}\psi(x,x,|u_n|)|u_n| + c_{12}h^*(x,|u_n|)|u_n|\right) dx \\
&\leq c_{13} \int_{\mathbb{R}^N} \widehat{\mathscr{H}^*}(x,|u_n|)|u_n| dx \\
&\leq c_{14} \int_{\mathbb{R}^N} \widehat{\mathscr{H}^*}\left(\frac{\Psi(x,|u_n|)}{|u_n|}\right) dx + c_{15} \int_{\mathbb{R}^N} \widehat{\mathscr{H}^*}\left(\frac{\mathscr{H}^*(x,|u_n|)}{|u_n|}\right) dx \\
&\leq c_{16} \left(\int_{\Omega_1} + \int_{\Omega_2}\right) \widehat{\mathscr{H}^*}\left(\frac{\Psi(x,|u_n|)}{|u_n|}\right) dx + c_{17} \int_{\mathbb{R}^N} \mathscr{H}^*(x,|u_n|) dx \\
&\leq c_{18} + c_{19} \int_{\Omega_1} \widehat{\mathscr{H}^*}(|u_n|^{\psi_1 - 1}) dx + c_{20} \int_{\Omega_2} \widehat{\mathscr{H}^*}(|u_n|^{\psi_2 - 1}) dx \\
&\leq c_{18} + c_{19} \int_{\Omega_1} \widehat{\mathscr{H}^*}(|u_n|^{h_2^* - 1}) dx + c_{20} \int_{\Omega_2} \widehat{\mathscr{H}^*}(|u_n|^{h_1^* - 1}) dx \\
&\leq c_{18} + c_{21} \int_{\Omega_1} |u_n|^{h_2^*} dx + c_{22} \int_{\Omega_2} |u_n|^{h_1^*} dx \\
&\leq c_{18} + c_{23} \int_{\Omega_1} \mathscr{H}^*(x,|u_n|) dx + c_{24} \int_{\Omega_2} \mathscr{H}^*(x,|u_n|) dx \\
&\leq c_{25} \text{ (independent of } n),
\end{aligned}$$

where $\Omega_1 = \{x \in \mathbb{R}^N : 0 \leq |u_n| \leq 1\}$ and $\Omega_2 = \{x \in \mathbb{R}^N : |u_n| > 1\}$. By Combining the above calculations, we obtain $\lambda g(x,|u_n|)u_n + f(x,u_n) \subset L^{\widehat{\mathscr{H}^*}}(\mathbb{R}^N)$ is a bounded sequence. By reflexivity of $L^{\widehat{\mathscr{H}^*}}(\mathbb{R}^N)$, we have

$$\lambda g(x,|u_n|)u_n + f(x,u_n) \rightharpoonup w_2 \tag{7.32}$$

for some $w_2 \in L^{\widehat{\mathscr{H}^*}}(\mathbb{R}^N)$. By using (7.31) and (7.32), we have

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} w_1(x,y) \frac{(\varphi(x) - \varphi(y))}{|x-y|^s} d\chi = \int_{\mathbb{R}^N} w_2(x)\varphi dx, \tag{7.33}$$

for all $\varphi \in D^{s,\mathscr{H}}(\mathbb{R}^N)$. In particular taking $\varphi = uv_\varepsilon$ in (7.33), one gets

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} w_1(x,y) \frac{(u(x)v_\varepsilon(x) - u(y)v_\varepsilon(y))}{|x-y|^s} d\chi = \int_{\mathbb{R}^N} w_2(x) u v_\varepsilon dx$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} w_1(x,y)\frac{(v_\varepsilon(x)-v_\varepsilon(y))u(y)}{|x-y|^s}d\chi = \int_{\mathbb{R}^N} w_2(x)uv_\varepsilon dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} w_1(x,y)\frac{(u(x)-u(y))v_\varepsilon(x)}{|x-y|^s}d\chi.$$

As $w_2 \in L^{\widehat{\mathscr{H}^*}}(\mathbb{R}^N)$ and $u \in L^{\mathscr{H}^*}(\mathbb{R}^N)$ then by Proposition 2.2.5, $w_2u \in L^1(\mathbb{R}^N)$. Also, as $w_1 \in L^{\widehat{\mathscr{H}}}(\mathbb{R}^N\times\mathbb{R}^N;d\chi)$ and $\dfrac{(u(x)-u(y))}{|x-y|^s} \in L^{\mathscr{H}}(\mathbb{R}^N\times\mathbb{R}^N;d\chi)$ then again by Proposition 2.2.5, $w_1\dfrac{(u(x)-u(y))}{|x-y|^s} \in L^1(\mathbb{R}^N)$. One can observe that, $v_\varepsilon \to 0$ as $\varepsilon \to 0$. Thus by Dominated Convergence Theorem, we have

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} w_1(x,y)\frac{(v_\varepsilon(x)-v_\varepsilon(y))u(y)}{|x-y|^s}d\chi = 0 \tag{7.34}$$

Further, (7.30) and (7.34) implies that (7.27) is true. Hence, by $(g_1)-(g_2)$, (7.26) and (7.27), we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{n\to\infty} h_1\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{v_\varepsilon(x)dx\,dy}{|x-y|^N} &\le \lambda\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} g(x,|u_n|)|u_n|^2 v_\varepsilon dx + \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_n)u_n v_\varepsilon dx.\\ &\le \lambda\lim_{\varepsilon\to 0}\lim_{n\to\infty} h_2^* g_2\int_{\mathbb{R}^N}\mathscr{H}^*(x,|u_n|)v_\varepsilon dx + \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_n)u_n v_\varepsilon dx.\end{aligned}$$

Therefore, we have

$$h_1\lim_{\varepsilon\to 0}\int_{B(x_j,2\varepsilon)} v_\varepsilon d\mu \le \lambda h_2^* g_2\int_{B(x_j,2\varepsilon)} v_\varepsilon dv + \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_n)u_n v_\varepsilon dx. \tag{7.35}$$

Next, we will prove that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_n)u_n v_\varepsilon dx = 0. \tag{7.36}$$

By (f_2) and Strauss compactness lemma [23, Theorem A.I], we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_n)u_n v_\varepsilon dx = \int_{\mathbb{R}^N} f(x,u)uv_\varepsilon dx.$$

Also,

$$\begin{aligned}\int_{\mathbb{R}^N} f(x,u)uv_\varepsilon dx &\le c_{26}\int_{\mathbb{R}^N}(\psi(|u|)|u|^2 + h^*(|u|)|u|^2)v_\varepsilon dx\\ &\le c_{27}\int_{B(x_j,2\varepsilon)}(\Psi(|u|)+\mathscr{H}^*(|u|))v_\varepsilon dx \to 0,\end{aligned}$$

as $\varepsilon \to 0$ which proves (7.36). By (7.35) and (7.36), we have

$$h_1 \mu_j \leq \lambda h_2^* g_2 \nu_j$$

then using (7.22), we have

$$\nu_j \left(\nu_j^{h_3 - 1} - \left(\frac{h_1}{\lambda g_2 h_2^* \delta_1} \right)^{h_3} S_0^{-h_4} \right) \geq 0,$$

for some $h_3 \in \left\{ \frac{h_1^*}{h_1}, \frac{h_2^*}{h_1}, \frac{h_1^*}{h_2}, \frac{h_2^*}{h_2} \right\}$ and $h_4 \in \{h_1^*, h_2^*\}\cdot$ Consequently, either

$$\nu_j = 0 \text{ or } \nu_j \geq \left(\frac{h_1}{\lambda g_2 h_2^* \delta_1} \right)^{\frac{h_3}{h_3 - 1}} S_0^{\frac{-h_4}{h_3 - 1}}.$$

Next, we will prove that $\nu_j \geq \left(\frac{h_1}{\lambda g_2 h_2^* \delta_1} \right)^{\frac{h_3}{h_3 - 1}} S_0^{\frac{-h_4}{h_3 - 1}}$ can not be possible, i.e., $\nu_j = 0$. Let on the contrary that

$$\nu_j \geq \left(\frac{h_1}{\lambda g_2 h_2^* \delta_1} \right)^{\frac{h_3}{h_3 - 1}} S_0^{\frac{-h_4}{h_3 - 1}}. \tag{7.37}$$

Since $\{u_n\}$ is a $(PS)_{c_M}$ sequence of I, we have $I(u_n) \to c_M$ and $I'(u_n) \to 0$ as $n \to \infty$, i.e.,

$$\begin{aligned}
c_M &= \lim_{n\to\infty} I(u_n) = \lim_{n\to\infty} (I(u_n) - \frac{1}{\sigma} I'(u_n)(u_n)) \\
&= \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \mathscr{H}\left(x, y, \frac{|u_n(x) - u_n(y)|}{|x-y|^s} \right) \frac{dx\, dy}{|x-y|^N} - \lambda \int_{\mathbb{R}^N} G(x, |u_n|) dx \right. \\
&\left. - \int_{\mathbb{R}^N} F(x, u_n) dx \right) - \frac{1}{\sigma} \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} h\left(x, y, \frac{|u_n(x) - u_n(y)|}{|x-y|^s} \right) \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}} dx\, dy \right. \\
&\left. - \lambda \int_{\mathbb{R}^N} g(x, |u_n|) |u_n|^2 dx - \int_{\mathbb{R}^N} f(x, u_n) u_n dx \right).
\end{aligned}$$

Using (f_3), we obtain

$$\begin{aligned}
\lim_{n\to\infty} &\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \mathscr{H}\left(x, y, \frac{|u_n(x) - u_n(y)|}{|x-y|^s} \right) \frac{dx\, dy}{|x-y|^N} \right. \\
&\left. - \frac{1}{\sigma} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} h\left(x, y, \frac{|u_n(x) - u_n(y)|}{|x-y|^s} \right) \frac{(u_n(x) - u_n(y))^2}{|x-y|^{N+2s}} dx dy \right) \\
&- \lambda \lim_{n\to\infty} \int_{\mathbb{R}^N} (G(x, |u_n|) - \frac{1}{\sigma} g(x, |u_n|) u_n^2) dx \leq c_M.
\end{aligned}$$

It follows from (h_1) and $(g_1)-(g_2)$ that

$$\lim_{n\to\infty}\left(1-\frac{h_2}{\sigma}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N}$$
$$+g_1\lambda\left(\frac{h_1^*}{\sigma}-1\right)\lim_{n\to\infty}\int_{\mathbb{R}^N}\mathscr{H}^*(x,|u_n|)dx\le c_M.$$

we have

$$g_1\lambda\left(\frac{h_1^*}{\sigma}-1\right)v_j\le g_1\lambda\left(\frac{h_1^*}{\sigma}-1\right)\lim_{n\to\infty}\int_{\mathbb{R}^N}\mathscr{H}^*(x,|u_n|)dx\le c_M.$$

$$v_j\le\frac{1}{\lambda}c_Mg_1^{-1}\left(\frac{h_1^*}{\sigma}-1\right)^{-1}.$$

Now, choosing $\lambda_2>0$ (sufficiently small) in such a way that for all $0<\lambda\le\lambda_2$, we have

$$v_i<\left(\frac{h_1}{\lambda g_2h_2^*\delta_1}\right)^{\frac{h_3}{h_3-1}}S_0^{\frac{-h_4}{h_3-1}} \tag{7.38}$$

for some $h_3\in\left\{\frac{h_1^*}{h_1},\frac{h_2^*}{h_1},\frac{h_1^*}{h_2},\frac{h_2^*}{h_2}\right\}$ and $h_4\in\{h_1^*,h_2^*\}\cdot$ From (7.37) and (7.38), we get a contradiction. Hence, $v_j=0$ for all $j\in J$.

Next, we will prove that $v_\infty=0$.

Choose a function, $v\in C^\infty(\mathbb{R}^N)$ such that $0\le v\le 1$

$$v\equiv 0 \text{ on } B(0,1),\ v\equiv 1 \text{ on } B(0,2)^C.$$

Now, for each $R>0$, define $v_R(x)=\frac{v(x)}{R}\ \forall x\in\mathbb{R}^N$ then

$$v_R\equiv 0 \text{ on } B(0,R),\ v_R\equiv 1 \text{ on } B(0,2R)^C.$$

We can observe that $\lim_{R\to\infty}v_R(x)=0$ a.e. $x\in\mathbb{R}^N$.

Similarly as proved above, we have

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}h\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{(u_n(x)-u_n(y))(v_R(x)-v_R(y))u_n(y)}{|x-y|^{N+2s}}dx\,dy=0 \tag{7.39}$$

and

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}f(x,u_n)u_nv_Rdx=0. \tag{7.40}$$

Using (7.39) and (7.40) together with the fact that $J'(u_n)(u_n v_\varepsilon) = o_n(1)$, we get

$$\begin{aligned}\lim_{R\to\infty}\lim_{n\to\infty} h_1 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathscr{H}\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{v_R(x)dx\,dy}{|x-y|^N} \\ \le \lambda \lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N} g(x,|u_n|)|u_n|^2 v_R dx \\ \le \lambda \lim_{R\to\infty}\lim_{n\to\infty} h_2^* g_2 \int_{\mathbb{R}^N}\mathscr{H}^*(x,|u_n|)v_R dx.\end{aligned}$$

By using (7.10), we have

$$h_1\mu_\infty \le \lambda h_2^* g_2 \nu_\infty$$

then using (7.25), we have

$$\nu_\infty\left(\nu_\infty^{h_3-1} - \left(\frac{h_1}{\lambda g_2 h_2^*\delta_1}\right)^{h_3} S_0^{-h_4}\right) \ge 0,$$

for some $h_3 \in \left\{\frac{h_1^*}{h_1}, \frac{h_2^*}{h_1}, \frac{h_1^*}{h_2}, \frac{h_2^*}{h_2}\right\}$ and $h_4 \in \{h_1^*, h_2^*\}\cdot$ Consequently, either

$$\nu_\infty = 0 \text{ or } \nu_\infty \ge \left(\frac{h_1}{\lambda g_2 h_2^*\delta_1}\right)^{\frac{h_3}{h_3-1}} S_0^{\frac{-h_4}{h_3-1}}.$$

Next, we will prove that $\nu_\infty \ge \left(\frac{h_1}{\lambda g_2 h_2^*\delta_1}\right)^{\frac{h_3}{h_3-1}} S_0^{\frac{-h_4}{h_3-1}}$ can not be possible, i.e., $\nu_\infty = 0$. Let, on the contrary, that

$$\nu_\infty \ge \left(\frac{h_1}{\lambda g_2 h_2^*\delta_1}\right)^{\frac{h_3}{h_3-1}} S_0^{\frac{-h_4}{h_3-1}}. \tag{7.41}$$

Since $\{u_n\}$ is a $(PS)_{c_M}$ sequence of I, we have $I(u_n) \to c_M$ and $I'(u_n) \to 0$ as $n \to \infty$, i.e.,

$$\begin{aligned}\lim_{n\to\infty} I(u_n) = \lim_{n\to\infty}\left(I(u_n) - \frac{1}{\sigma}I'(u_n)(u_n)\right) = c_M \\ \lim_{n\to\infty}\left(1-\frac{h_2}{\sigma}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mathscr{H}\left(x,y,\frac{|u_n(x)-u_n(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^N} \\ + g_1\lambda\left(\frac{h_1^*}{\sigma}-1\right)\lim_{n\to\infty}\int_{\mathbb{R}^N}\mathscr{H}^*(x,|u_n|)dx \le c_M\end{aligned}$$

consequently,

$$g_1\lambda\left(\frac{h_1^*}{\sigma}-1\right)\nu_\infty \le c_M$$

$$v_\infty \le \frac{1}{\lambda} c_M g_1^{-1} \left(\frac{h_1^*}{\sigma} - 1 \right)^{-1}.$$

Now, choosing $\lambda_3 > 0$ (sufficiently small) in such a way that $\forall\, \lambda \le \lambda_3$, we have

$$v_\infty < \left(\frac{h_1}{\lambda g_2 h_2^* \delta_1} \right)^{\frac{h_3}{h_3-1}} S_0^{\frac{-h_4}{h_3-1}} \tag{7.42}$$

for some $h_3 \in \left\{ \frac{h_1^*}{h_1}, \frac{h_2^*}{h_1}, \frac{h_1^*}{h_2}, \frac{h_2^*}{h_2} \right\}$ and $h_4 \in \{h_1^*, h_2^*\}\cdot$ From (7.41) and (7.42), we get a contradiction. Hence, $v_\infty = 0$. Choosing, $\lambda_1 = \min\{\lambda_2, \lambda_3\}$, we conclude the result. □

Lemma 7.3.8. *The functional I satisfies the $(PS)_{c_M}$ condition for all $0 < \lambda \le \lambda_1$, where λ_1 as in the Lemma 7.3.7.*

Proof. Let $\{u_n\}$ be any $(PS)_{c_M}$ sequence. Then, by (7.21), (7.24) and Lemma 7.3.7, we have

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \mathscr{H}^*(x, |u_n(x)|)dx = \int_{\mathbb{R}^N} \mathscr{H}^*(x, |u(x)|)dx. \tag{7.43}$$

Taking the advantage of Brezis-Lieb Lemma 7.1.3 and (7.43), we have

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \mathscr{H}^*(x, |u_n - u|)dx = 0,$$

witch implies $u_n \to u$ in $L^{\mathscr{H}^*}(\mathbb{R}^N)$. Define

$$P_n = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} h\left(x, y, \frac{|u_n(x) - u_n(y)|}{|x-y|^s}\right) \frac{(u_n(x) - u_n(y))((u_n - u)(x) - (u_n - u)(y))}{|x-y|^{N+2s}} dx\, dy.$$

Then

$$\begin{aligned} P_n = J'(u_n)u_u + \lambda \int_{\mathbb{R}^N} g(x, |u_n|)|u_n|^2 dx + \int_{\mathbb{R}^N} f(x, u_n)u_n dx \\ - J'(u_n)u - \lambda \int_{\mathbb{R}^N} g(x, |u_n|)u_n u dx - \int_{\mathbb{R}^N} f(x, u_n)u dx. \end{aligned} \tag{7.44}$$

In Lemma (7.3.7), we have proved that $T_n = \lambda g(x, |u_n|)u_n + f(x, u_n) \subset L^{\widehat{\mathscr{H}^*}}(\mathbb{R}^N)$ is a bounded sequence. Thus, by Lemmas 7.1.1, 7.3.6, and Proposition 2.2.5, we get

$$0 \le \int_{\mathbb{R}^N} |(\lambda g(x, |u_n|)u_n + f(x, u_n))(u_n - u)|dx \le 2\|T_n\|_{L^{\widehat{\mathscr{H}^*}}(\mathbb{R}^N)} \|u_n - u\|_{L^{\mathscr{H}^*}(\mathbb{R}^N)} \to 0,$$

as $n \to \infty$, which implies

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (\lambda g(x, |u_n|)u_n + f(x, u_n))(u_n - u)dx = 0. \tag{7.45}$$

Further, (7.44) and (7.45) implies, $\lim_{n\to\infty} P_n = 0$, consequently, by Lemma 6.2.9

$$u_n \to u \text{ in } D^{s,\mathscr{H}}(\mathbb{R}^N).$$

Hence, $(PS)_{c_M}$ condition holds for all $0 < \lambda \leq \lambda_1$. □

Proof of the Theorem 7.3.3. Both the geometric conditions of the mountain pass theorem follows from Lemmas 7.3.4 and 7.3.5.

Also, $(PS)_{c_M}$ condition is holds for the functional I by Lemmas 7.3.7 and 7.3.8.

Hence, by the mountain pass theorem, there exists a critical point, say, u of I with level c_M, i.e., $I'(u) = 0$ and $I(u) = c_M > 0$. Thus, u is the non-trivial weak solution of the Problem (7.16). □

Conclusions and future directions

Conclusions

The primary objective of the thesis was stated in Chapter 1 as the investigation of non-local equations with non-standard growth conditions using variational methods. We carried out all the research objectives with great attention. The concluding remarks for this thesis are as follows:

- **In Chapter 3,** we have proved the existence of a positive weak solution to the Kirchhoff-type problem with gradient nonlinearity. The main difficulty in studying the problem was due to the presence of a gradient in the reaction term, which makes the problem non-variational; hence, one can not apply variational methods directly. To prove the existence result, first, we considered the corresponding auxiliary variational problem by freezing the gradient term and proved the existence of the solution for the frozen problem by mountain pass theorem. Later, with the help of the solutions of the frozen problem and truncation techniques, we proved the existence of a positive, weak solution for our original problem. We have also covered the borderline case, i.e., $N = 2$, where nonlinearity assumes the exponential growth in the second variable and the polynomial growth in the third variable. Another significant aspect of the problem is that we have considered the perturbation term without any growth conditions, which has been carefully handled by truncation techniques.

- **Chapter 4** is divided into two major sections. In the first section, we have proved the existence of at least two weak solutions to N-Kirchhoff equation with critical exponential growth. In addition, we have considered the perturbation term and the singularity with the reaction term. Due to the critical exponential growth, we faced the lack of compactness issue. To deal with the possible loss of compactness, we have illustrated the additional information for the mountain pass critical level and used the singular version of Moser-Trudinger inequality. We have used the mountain pass theorem and Ekeland variational principle to demonstrate the main results. In addition, the existence result is also demonstrated without using the Ambrosetti-Rabinowitz condition, and the existence of the ground state solution is proved for the N-Kirchhoff equation with critical exponential growth in the bounded domain. Without Ambrosetti-Rabinowitz condition it is not possible to prove the Palais-Smale sequence for the associated functional. To overcome this issue we used Cerami sequence and achieved the compactness result.

In the last section of this chapter, we have proved the existence of at least two weak solutions to N-Kirchhoff equation with critical exponential growth. The main highlight of the problem is that here we have considered Schrödinger's equations in the entire $\mathbb{R}^N$; consequently, we have experienced a double loss of compactness.

- **In Chapter 5,** we have studied the generalized Kirchhoff problem. The salient feature of the problem is that it covers the more general class of Kirchhoff equations; for example, Kirchhoff problems involving p-Laplacian, (p,q)-Laplacian, weighted Laplacian and double-phase Laplace operators are the particular cases of our problem. The main difficulty that arises while studying our problem is that the nonlinear function assumes variable exponent critical growth on some parts of the domain and exponential growth on the other parts, i.e., the nonlinearity and the associated operator change their behaviour from one domain to another. Utilizing concentration compactness principle for variable exponent Sobolev spaces and a variant of the Moser-Trudinger inequality, these issues were resolved.

- **In Chapter 6,** we defined fractional homogeneous Musielak-Sobolev spaces and then demonstrated their characterization. The Hardy-Littlewood-Sobolev type inequality for Lebesgue Musielak spaces and their applications to the class of generalized Choquard Schrödinger equations in homogeneous fractional Musi-elak Sobolev space have also been discussed. The key aspect of the problem is that it considers the vanishing potential at infinity. In addition, it involves the generalized fractional Laplace operator, which covers a broader class of fractional problems; for instance, fractional p-Laplacian, fractional (p,q)-Laplacian, fractional weighted Laplacian, and fractional double-phase Laplace operators are particular cases of our problem. Using the method of Nehari manifold, we have also proved the existence of a ground-state solution.

- **In Chapter 7,** we demonstrated that a non-trivial weak solution exists for the generalized fractional problem with critical growth in $\mathbb{R}^N$. Due to the critical growth, it faces the loss of compactness. To address the compactness issue, we established and implemented the concentration compactness principle and its variant at infinity. The existence of a non-trivial weak solution is proved by using mountain pass theorem.

Future Directions

The study of non-local equations is an active area of research, and there are several avenues for future work that can be explored. Some potential directions for future research on non-local equations are:

- It would be interesting to study multiplicity result for the generalized Choquard Schrödinger equation with vanishing potential. Also, the zero mass case for such types of problems, is yet to explored.
- The study of generalized multivalued fractional problems with critical growth is an open task that is worth work on.
- Fractional Musielak Sobolev spaces are new in the literature; hence, various things are yet to explore in these spaces such as well known density results and regularity results for the problems in these spaces. The well-known Hardy-Littlewood-Sobolev inequality can be explored for these spaces.

Bibliography

[1] S. Adachi and K. Tanaka, “Trudinger type inequalities in $\mathbb{R}^N$ and their best exponents,” *Proceedings of the American Mathematical Society*, vol. 128, no. 7, pp. 2051–2057, 2000.

[2] R. A. Adams and J. J. Fournier, “Sobolev spaces,” 2003.

[3] A. Adimurthi, “Existence of positive solutions of the semilinear dirichlet problem with critical growth for the n-Laplacian,” *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie IV*, vol. 17, no. 3, pp. 393–413, 1990.

[4] A. Alberico, A. Cianchi, L. Pick, and L. Slavíková, “Fractional Orlicz-Sobolev embeddings,” *Journal de Mathématiques Pures et Appliquées*, vol. 149, pp. 216–253, 2021.

[5] F. S. Albuquerque, A. Bahrouni, and U. B. Severo, “Existence of solutions for a nonhomogeneous Kirchhoff-Schrödinger type equation in $\mathbb{R}^2$ involving unbounded or decaying potentials,” *Topological Methods in Nonlinear Analysis*, vol. 56, pp. 263–281, 2020.

[6] J. C. de Albuquerque, K. Silva, and S. M. de Sousa, “Existence of solutions for a class of fractional coupled Choquard–type systems with potential vanishing at infinity,” *Journal of Mathematical Analysis and Applications*, vol. 507, no. 2, p. 125 848, 2022.

[7] C. O. Alves and T. Boudjeriou, “Existence of solution for a class of nonlocal problem via dynamical methods,” *Rendiconti del Circolo Matematico di Palermo Series 2*, vol. 71, pp. 611–632, 2022.

[8] C. O. Alves and F. Corrêa, “On existence of solutions for a class of problem involving a nonlinear operator,” *Communications on Applied Nonlinear Analysis*, vol. 8, no. 2, pp. 43–56, 2001.

[9] C. O. Alves, F. Corrêa, and T. F. Ma, “Positive solutions for a quasilinear elliptic equation of Kirchhoff type,” *Computers & Mathematics with Applications*, vol. 49, no. 1, pp. 85–93, 2005.

[10] C. O. Alves, G. M. Figueiredo, and J. A. Santos, "Strauss and Lions type results for a class of Orlicz-Sobolev spaces and applications," *Topological Methods in Nonlinear Analysis*, pp. 435–456, 2014.

[11] C. O. Alves, G. M. Figueiredo, and M. Yang, "Existence of solutions for a nonlinear Choquard equation with potential vanishing at infinity," *Advances in Nonlinear Analysis*, vol. 5, no. 4, pp. 331–345, 2016.

[12] C. O. Alves, P. Garain, and V. D. Rădulescu, "High perturbations of quasilinear problems with double criticality," *Mathematische Zeitschrift*, vol. 299, pp. 1875–1895, 2021.

[13] C. O. Alves, V. D. Rădulescu, and L. S. Tavares, "Generalized Choquard equations driven by nonhomogeneous operators," *Mediterranean Journal of Mathematics*, vol. 16, no. 1, p. 20, 2019.

[14] C. O. Alves and M. A. Souto, "Existence of solutions for a class of nonlinear Schrödinger equations with potential vanishing at infinity," *Journal of Differential Equations*, vol. 254, no. 4, pp. 1977–1991, 2013.

[15] A. Ambrosetti, "On the existence of multiple solutions for a class of nonlinear boundary value problems," *Rendiconti del Seminario Matematico della Università di Padova*, vol. 49, pp. 195–204, 1973.

[16] A. Ambrosetti and P. H. Rabinowitz, "Dual variational methods in critical point theory and applications," *Journal of Functional Analysis*, vol. 14, no. 4, pp. 349 –381, 1973.

[17] S. Aouaoui, "Existence of multiple solutions to elliptic problems of Kirchhoff type with critical exponential growth," *Electronic Journal of Differential Equations*, vol. 107, pp. 1–12, 2014.

[18] G. Astarita and G. Marrucci, *Principles of non-Newtonian fluid mechanics*. McGraw Hill, 1974.

[19] E Azroul, A Benkirane, M Shimi, and M Srati, "On a class of nonlocal problems in new fractional Musielak-Sobolev spaces," *Applicable Analysis*, vol. 101, no. 6, pp. 1933–1952, 2022.

[20] S. Bahrouni and H. Ounaies, "Embedding theorems in the fractional orlicz-sobolev space and applications to non-local problems," *Discrete and Continuous Dynamical Systems*, vol. 40, no. 5, pp. 2917–2944, 2020.

[21] S. Bahrouni, H. Ounaies, and L. S. Tavares, "Basic results of fractional Orlicz-Sobolev space and applications to non-local problems," *Topological Methods in Nonlinear Analysis*, pp. 681–695, 2020.

[22] P. Belchior, H. Bueno, O. Miyagaki, and G. Pereira, "Remarks about a fractional Choquard equation: Ground state, regularity and polynomial decay," *Nonlinear Analysis*, vol. 164, pp. 38–53, 2017.

[23] H. Berestycki and P. L. Lions, "Nonlinear scalar field equations, 1 existence of a ground state," *Archive for Rational Mechanics and Analysis*, vol. 82, no. 4, pp. 313–345, 1983, ISSN: 1432-0673.

[24] J. F. Bonder, N. Saintier, and A. Silva, "The concentration-compactness principle for fractional order Sobolev spaces in unbounded domains and applications to the generalized fractional Brezis–Nirenberg problem," *Nonlinear Differential Equations and Applications NoDEA*, vol. 25, pp. 1–25, 2018.

[25] J. F. Bonder and A. Silva, "The concentration-compactness principle for variable exponent spaces and applications," *Electronic Journal of Differential Equations*, vol. 2010, pp. 1–30, 2010.

[26] J. F. Bonder and A. Silva, "The concentration-compactness principle for Orlicz spaces and applications," *arXiv preprint arXiv:2111.13199*, 2021.

[27] H. Brézis and E. Lieb, "A relation between pointwise convergence of functions and convergence of functionals," *Proceedings of the American Mathematical Society*, vol. 88, no. 3, pp. 486–490, 1983.

[28] D. Cao, "Nontrivial solution of semilinear elliptic equations with critical exponent in $\mathbb{R}$," *Communications in Partial Differential Equations*, vol. 17, no. 3-4, pp. 407–435, 1992.

[29] X. Cao and S. Wen, "Existence and multiplicity of solutions for a quasilinear nonlocal problem," *Computers & Mathematics with Applications*, vol. 78, no. 8, pp. 2575–2583, 2019.

[30] G. Cerami, "An existence criterion for the critical points on unbounded manifolds," *Istituto Lombardo Accademia di Scienze e Lettere A*, vol. 112, no. 2, pp. 332–336, 1978.

[31] G. Cerami, "On the existence of eigenvalues for a nonlinear boundary value problem," *Annali di Matematica Pura ed Applicata*, vol. 124, no. 4, pp. 161–179, 1980.

[32] J. Chabrowski, "Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents," *Calculus of Variations and Partial Differential Equations*, vol. 3, pp. 493–512, 1995.

[33] M. M. Chaharlang and A. Razani, "Existence of infinitely many solutions for a class of nonlocal problems with dirichlet boundary condition," *Communications of the Korean Mathematical Society*, vol. 34, no. 1, pp. 155–167, 2019.

[34] S. Chen and S. Yuan, "Ground state solutions for a class of Choquard equations with potential vanishing at infinity," *Journal of Mathematical Analysis and Applications*, vol. 463, no. 2, pp. 880–894, 2018.

[35] W. Chen and F. Yu, "On a nonhomogeneous Kirchhoff-type elliptic problem with critical exponential in dimension two," *Applicable Analysis*, vol. 101, no. 2, pp. 421–436, 2022.

[36] Y. Chen, S. Levine, and M. Rao, "Variable exponent, linear growth functionals in image restoration," *SIAM journal on Applied Mathematics*, vol. 66, no. 4, pp. 1383–1406, 2006.

[37] L. Cherfils and Y. Il'yasov, "On the stationary solutions of generalized reaction diffusion equations with $p\&q$-Laplacian," *Communications on Pure and Applied Analysis*, vol. 4, no. 1, pp. 9–22, 2005.

[38] M. Chipot and B. Lovat, "Some remarks on non local elliptic and parabolic problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 30, no. 7, pp. 4619–4627, 1997.

[39] I. Chlebicka, "A pocket guide to nonlinear differential equations in Musielak–Orlicz spaces," *Nonlinear Analysis*, vol. 175, pp. 1–27, 2018.

[40] N. Chung, "Multiplicity results for a class of $p(x)$-Kirchhoff type equations with combined nonlinearities," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 2012, no. 42, pp. 1–13, 2012.

[41] F. Colasuonno and P. Pucci, "Multiplicity of solutions for $p(x)$-polyharmonic elliptic Kirchhoff equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 17, pp. 5962–5974, 2011.

[42] D. G. Costa and O. H. Miyagaki, "Nontrivial solutions for perturbations of the p-Laplacian on unbounded domains," *Journal of Mathematical Analysis and Applications*, vol. 193, no. 3, pp. 737–755, 1995.

[43] D. V. Cruz Uribe and A. Fiorenza, *Variable Lebesgue spaces: Foundations and harmonic analysis*. Springer Science & Business Media, 2013.

[44] G. Dai and R. Hao, "Existence of solutions for a $p(x)$-Kirchhoff-type equation," *Journal of Mathematical Analysis and Applications*, vol. 359, no. 1, pp. 275–284, 2009.

[45] G. Dai and D. Liu, "Infinitely many positive solutions for a $p(x)$-Kirchhoff-type equation," *Journal of Mathematical Analysis and Applications*, vol. 359, no. 2, pp. 704–710, 2009.

[46] P. d'Avenia, G. Siciliano, and M. Squassina, "On fractional Choquard equations," *Mathematical Models and Methods in Applied Sciences*, vol. 25, no. 08, pp. 1447–1476, 2015.

[47] Y. Deng, Y. Li, and W. Shuai, "Existence of solutions for a class of p-Laplacian type equation with critical growth and potential vanishing at infinity," *Discrete and Continuous Dynamical Systems*, vol. 36, no. 2, pp. 683–699, 2015.

[48] J. I. Diaz and D. T. Francois, "On a nonlinear parabolic problem arising in some models related to turbulent flows," *SIAM Journal on Mathematical Analysis*, vol. 25, no. 4, pp. 1085–1111, 1994.

[49] T. K. Donaldson and N. S. Trudinger, "Orlicz-Sobolev spaces and imbedding theorems," *Journal of Functional Analysis*, vol. 8, no. 1, pp. 52–75, 1971.

[50] L. C. Evans, *Partial differential equations*. American Mathematical Society, 2022.

[51] X. Fan, "Differential equations of divergence form in Musielak–Sobolev spaces and a sub-supersolution method," *Journal of Mathematical Analysis and Applications*, vol. 386, no. 2, pp. 593–604, 2012.

[52] X. Fan and D. Zhao, "On the spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$," *Journal of Mathematical Analysis and Applications*, vol. 263, no. 2, pp. 424–446, 2001.

[53] L. Faria, O. Miyagaki, D Motreanu, and M Tanaka, "Existence results for nonlinear elliptic equations with Leray–Lions operator and dependence on the gradient," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 96, pp. 154–166, 2014.

[54] G. M. Figueiredo, "Quasilinear equations with dependence on the gradient via mountain pass techniques in $\mathbb{R}^N$," *Applied Mathematics and Computation*, vol. 203, no. 1, pp. 14–18, 2008.

[55] G. M. Figueiredo, "Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument," *Journal of Mathematical Analysis and Applications*, vol. 401, no. 2, pp. 706–713, 2013.

[56] G. M. Figueiredo and F. B. M. Nunes, "Existence of positive solutions for a class of quasilinear elliptic problems with exponential growth via the nehari manifold method," *Revista Matemática Complutense*, vol. 32, pp. 1–18, 2019.

[57] G. M. Figueiredo and U. B. Severo, "Ground state solution for a Kirchhoff problem with exponential critical growth," *Milan Journal of Mathematics*, vol. 84, pp. 23–39, 2016.

[58] L. R. de Freitas, "Multiplicity of solutions for a class of quasilinear equations with exponential critical growth," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 95, pp. 607–624, 2014.

[59] N. Fukagai, M. Ito, and K. Narukawa, "Positive solutions of quasilinear elliptic equations with critical Orlicz–Sobolev nonlinearity on $\mathbb{R}^N$," *Funkcialaj Ekvacioj*, vol. 49, no. 2, pp. 235–267, 2006.

[60] M. F. Furtado, L. D. de Oliveira, and J. P. P. da Silva, "Multiple solutions for a Kirchhoff equation with critical growth," *Zeitschrift für angewandte Mathematik und Physik*, vol. 70, pp. 1–15, 2019.

[61] M. F. Furtado, E. D. Silva, and U. B. Severo, "Kirchhoff elliptic problems with asymptotically linear or superlinear nonlinearities," *Zeitschrift für angewandte Mathematik und Physik*, vol. 71, pp. 1–18, 2020.

[62] M. F. Furtado and H. R. Zanata, "Kirchhoff–Schrödinger equations in $\mathbb{R}^2$ with critical exponential growth and indefinite potential," *Communications in Contemporary Mathematics*, vol. 23, no. 07, p. 126 252, 2021.

[63] M. Girardi and M. Matzeu, "Positive and negative solutions of a quasi–linear elliptic equation by a mountain pass method and truncature techniques," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 59, no. 1-2, pp. 199–210, 2004.

[64] J. P. Gossez, "Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients," *Transactions of the American Mathematical Society*, vol. 190, pp. 163–205, 1974.

[65] S. Goyal, P. K. Mishra, and K. Sreenadh, "n-Kirchhoff type equations with exponential nonlinearities," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 110, pp. 219–245, 2016.

[66] S. Gupta and G. Dwivedi, "An existence result for p-Laplace equation with gradient nonlinearity in $\mathbb{R}^N$," *Communications in Mathematics*, vol. 30, 2022.

[67] S. Gupta and G. Dwivedi, "Existence and multiplicity of solutions to N-Kirchhoff equations with critical exponential growth and a perturbation term," *Complex Variables and Elliptic Equations*, pp. 1–29, 2022.

[68] M. K. Hamdani, A. Harrabi, F. Mtiri, and D. D. Repovš, "Existence and multiplicity results for a new $p(x)$-Kirchhoff problem," *Nonlinear Analysis*, vol. 190, p. 111 598, 2020.

[69] A. Hamydy, M. Massar, and N. Tsouli, "Existence of solutions for p-Kirchhoff type problems with critical exponent," *Electronic Journal of Differential Equations*, vol. 105, pp. 1–8, 2011.

[70] J. A. Hempel and D. Gilbarg, "Multiple solutions for a class of nonlinear boundary value problems," *Indiana University Mathematics Journal*, vol. 20, no. 11, pp. 983–996, 1971.

[71] J. Hempel, G. Morris, and N. Trudinger, "On the sharpness of a limiting case of the sobolev imbedding theorem," *Bulletin of the Australian Mathematical Society*, vol. 3, no. 3, pp. 369–373, 1970.

[72] K. Ho and Y. H. Kim, "The concentration-compactness principles for $W^{s,p(\cdot,\cdot)}(\mathbb{R}^N)$ and application," *Advances in Nonlinear Analysis*, vol. 10, no. 1, pp. 816–848, 2020.

[73] K. Ho and I. Sim, "On degenerate $p(x)$-Laplace equations involving critical growth with two parameters," *Nonlinear Analysis*, vol. 132, pp. 95–114, 2016.

[74] J. Huang, "Existence and multiplicity of solutions for a p-Kirchhoff equation on $\mathbb{R}^N$," *Boundary Value Problems*, vol. 2018, pp. 1–12, 2018.

[75] T. Huang and S. Deng, "Existence of ground state solutions for Kirchhoff type problem without the Ambrosetti–Rabinowitz condition," *Applied Mathematics Letters*, vol. 113, p. 106 866, 2021.

[76] H. Hudzik, "On generalized Orlicz-Sobolev space," *Functiones et Approximatio Commentarii Mathematici*, vol. 4, pp. 37–51, 1976.

[77] H. Hudzik, "The problem of separability, duality, reflexivity and of comparison for generalized Orlicz-Sobolev spaces $W_M^k(\Omega)$," *Commentationes Mathematicae*, vol. 21, no. 2, 1979.

[78] N. Ikoma, "Existence of ground state solutions to the nonlinear Kirchhoff type equations with potentials," *Discrete and Continuous Dynamical Systems*, vol. 35, no. 3, pp. 943–966, 2014.

[79] T. Isernia, "Fractional $p\&q$-Laplacian problems with potentials vanishing at infinity," *Opuscula Mathematica*, vol. 40, no. 1, 2020.

[80] T. Isernia and D. D. Repovš, "Nodal solutions for double phase Kirchhoff problems with vanishing potentials," *Asymptotic Analysis*, vol. 124, no. 3-4, pp. 371–396, 2021.

[81] J. Jost, *Partial differential equations*. Springer Science & Business Media, 2012, vol. 214.

[82] F. Júlio, S. Corrêa, and G. M. Figueiredo, "On an elliptic equation of p-Kirchhoff type via variational methods," *Bulletin of the Australian Mathematical Society*, vol. 74, no. 2, pp. 263–277, 2006.

[83] J. R. S. Júnior and G. Siciliano, "Positive solutions for a Kirchhoff problem with vanishing nonlocal term," *Journal of Differential Equations*, vol. 265, no. 5, pp. 2034–2043, 2018.

[84] R. Kajikiya, "Positive solutions of semilinear elliptic equations with small perturbations," *Proceedings of the American Mathematical Society*, vol. 141, no. 4, pp. 1335–1342, 2013.

[85] Y. Kaya, "A weakly convergence result on $L^{p(x)}$ spaces," *Mathematical and Computational Applications*, vol. 20, no. 2, pp. 106–110, 2015.

[86] S. Kesavan, *Topics in functional analysis and applications*. New Age International, 1989.

[87] G. Kirchhoff and K. Hensel, *Vorlesungen über mathematische Physik*. Druck und Verlag von BG Teubner, 1883, vol. 1.

[88] O Kovacik and J Rakosnik, "On spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$," *Czechoslovak Math, J*, vol. 41, no. 116, pp. 592–618, 1991.

[89] N. Lam and G. Lu, "Existence and multiplicity of solutions to equations of N-Laplacian type with critical exponential growth in $\mathbb{R}^N$," *Journal of Functional Analysis*, vol. 262, no. 3, pp. 1132–1165, 2012.

[90] N. Lam and G. Lu, "Elliptic equations and systems with subcritical and critical exponential growth without the Ambrosetti–Rabinowitz condition," *Journal of Geometric Analysis*, vol. 24, pp. 118–143, 2014.

[91] P. Le, N. V. Huynh, and V. Ho, "Positive solutions of the p-Kirchhoff problem with degenerate and sign-changing nonlocal term," *Zeitschrift für angewandte Mathematik und Physik*, vol. 70, pp. 1–12, 2019.

[92] J. Lee, J. M. Kim, and Y. H. Kim, "Existence and multiplicity of solutions for Kirchhoff–Schrödinger type equations involving $p(x)$-Laplacian on the entire space $\mathbb{R}^N$," *Nonlinear Analysis: Real World Applications*, vol. 45, pp. 620–649, 2019.

[93] Q. Li, K. Teng, and X. Wu, "Existence of positive solutions for a class of critical fractional Schrödinger equations with potential vanishing at infinity," *Mediterranean Journal of Mathematics*, vol. 14, pp. 1–14, 2017.

[94] S. Li, Z. Shen, and M. Yang, "Multiplicity of solutions for a nonlocal nonhomogeneous elliptic equation with critical exponential growth," *Journal of Mathematical Analysis and Applications*, vol. 475, no. 2, pp. 1685–1713, 2019.

[95] Y. Li, M. Mei, and K. Zhang, "Existence of multiple nontrivial solutions for a p-kirchhoff type elliptic problem involving sign-changing weight functions," *Discrete & Continuous Dynamical Systems-Series B*, vol. 21, no. 3, pp. 883–908, 2016.

[96] Y. Li and B. Ruf, "A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^N$," *Indiana University mathematics journal*, no. 57, pp. 451–480, 2008.

[97] E. H. Lieb and M. Loss, *Analysis*. American Mathematical Society, 2001, vol. 14.

[98] J. L. Lions, "On some questions in boundary value problems of mathematical physics," in *North-Holland Mathematics Studies*, vol. 30, 1978, pp. 284–346.

[99] P. L. Lions, "The concentration-compactness principle in the calculus of variations. the limit case, part 1," *Revista Matemática Iberoamericana*, vol. 1, no. 1, pp. 145–201, 1985.

[100] D. Liu and P. Zhao, "Multiple nontrivial solutions to a p-Kirchhoff equation," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 13, pp. 5032–5038, 2012.

[101] D. Liu and P. Zhao, "Solutions for a quasilinear elliptic equation in Musielak–Sobolev spaces," *Nonlinear Analysis: Real World Applications*, vol. 26, pp. 315–329, 2015.

[102] G. Liu, S. Shi, and Y. Wei, "Semilinear elliptic equations with dependence on the gradient," *Electronic Journal of Differential Equations*, vol. 2012, no. 139, pp. 1–9, 2012.

[103] G. Liu, S. Shi, and Y. Wei, "Sign-changing solutions for semilinear elliptic equations with dependence on the gradient via the nehari method," *Mediterranean Journal of Mathematics*, vol. 14, pp. 1–12, 2017.

[104] J. Liu, L. Wang, and P. Zhao, "Positive solutions for a nonlocal problem with a convection term and small perturbations," *Mathematical Methods in the Applied Sciences*, vol. 40, no. 3, pp. 720–728, 2017.

[105] H. Luo, "Ground state solutions of pohozaev type for fractional Choquard equations with general nonlinearities," *Computers & Mathematics with Applications*, vol. 77, no. 3, pp. 877–887, 2019.

[106] S. Maatouk and A. El Hachimi, "Quasilinear elliptic problem without Ambrosetti–Rabinowitz condition involving a potential in Musielak–Sobolev spaces setting," *Complex Variables and Elliptic Equations*, vol. 66, no. 12, pp. 2028–2054, 2021.

[107] M. Matzeu and R. Servadei, “A variational approach to a class of quasilinear elliptic equations not in divergence form,” *Discrete and Continuous Dynamical Systems - S*, vol. 5, no. 4, pp. 819–831, 2012.

[108] J. B. de Mendonça Xavier, “Some existence theorems for equations of the form,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 15, no. 1, pp. 59–67, 1990.

[109] H. Missaoui and H. Ounaies, “Ground state and nodal solutions for fractional Orlicz problems with lack of regularity and without the Ambrosetti-Rabinowitz condition,” *Journal of Mathematical Analysis and Applications*, vol. 519, no. 2, p. 126 833, 2023.

[110] O. H. Miyagaki, L. Paes Leme, and B. M. Rodrigues, “Multiplicity of positive solutions for the Kirchhoff-type equations with critical exponent in $\mathbb{R}^N$,” *Computers & Mathematics with Applications*, vol. 75, no. 9, pp. 3201–3212, 2018.

[111] V. Moroz and J. Van Schaftingen, “Groundstates of nonlinear Choquard equations: Existence, qualitative properties and decay asymptotics,” *Journal of Functional Analysis*, vol. 265, no. 2, pp. 153–184, 2013.

[112] V. Moroz and J. Van Schaftingen, “Existence of groundstates for a class of nonlinear Choquard equations,” *Transactions of the American Mathematical Society*, vol. 367, no. 9, pp. 6557–6579, 2015.

[113] V. Moroz and J. Van Schaftingen, “A guide to the Choquard equation,” *Journal of Fixed Point Theory and Applications*, vol. 19, pp. 773–813, 2017.

[114] J. Moser, “A sharp form of an inequality by N. Trudinger,” *Indiana University Mathematics Journal*, vol. 20, no. 11, pp. 1077–1092, 1971.

[115] D. Motreanu, V. V. Motreanu, and N. S. Papageorgiou, *Topological and variational methods with applications to nonlinear boundary value problems*. Springer, 2014, vol. 1.

[116] T. Mukherjee and K. Sreenadh, “Fractional Choquard equation with critical nonlinearities,” *Nonlinear Differential Equations and Applications NoDEA*, vol. 24, pp. 1–34, 2017.

[117] J. Musielak, *Orlicz spaces and modular spaces*. Springer, 2006, vol. 1034.

[118] T. C. Nguyen, “Three solutions for a class of nonlocal problems in Orlicz-Sobolev spaces,” *Journal of the Korean Mathematical Society*, vol. 50, no. 6, pp. 1257–1269, 2013.

[119] J. M. B. do Ó, “Semilinear dirichlet problems for the N-Laplacian in $\mathbb{R}^N$ with nonlinearities in the critical growth range,” *Differential and Integral Equations*, vol. 9, no. 5, pp. 967–979, 1996.

[120] J. M. B. do Ó, "N-Laplacian equations in $\mathbb{R}^N$ with critical growth," *Abstract and Applied Analysis*, vol. 2, pp. 301–315, 1997.

[121] J. M. B. do Ó, E. Medeiros, and U. Severo, "On a quasilinear nonhomogeneous elliptic equation with critical growth in $\mathbb{R}^N$," *Journal of Differential Equations*, vol. 246, no. 4, pp. 1363–1386, 2009.

[122] R. Panda, "Nontrivial solution of a quasilinear elliptic equation with critical growth in $\mathbb{R}^n$," in *Proceedings of the Indian Academy of Sciences-Mathematical Sciences*, Springer India, vol. 105, 1995, pp. 425–444.

[123] K. Perera, M. Squassina, and Y. Yang, "Critical fractional p-Laplacian problems with possibly vanishing potentials," *Journal of Mathematical Analysis and Applications*, vol. 433, no. 2, pp. 818–831, 2016.

[124] M. T. O. Pimenta and R. Servadei, "Some existence results for variational inequalities with nonlocal fractional operators," *Nonlinear Analysis*, vol. 189, p. 111 561, 2019.

[125] P. Pucci, M. Xiang, and B. Zhang, "Existence results for Schrödinger–Choquard–Kirchhoff equations involving the fractional p-Laplacian," *Advances in Calculus of Variations*, vol. 12, no. 3, pp. 253–275, 2019.

[126] P. H. Rabinowitz, "On a class of nonlinear Schrödinger equations," *Zeitschrift für angewandte Mathematik und Physik ZAMP*, vol. 43, no. 2, pp. 270–291, 1992.

[127] P. H. Rabinowitz and J. K. Moser, "Pairs of positive solutions of nonlinear elliptic partial differential equations," *Indiana University Mathematics Journal*, vol. 23, no. 2, pp. 173–186, 1973.

[128] P. H. Rabinowitz and J. K. Moser, "Variational methods for nonlinear elliptic eigenvalue problems," *Indiana University Mathematics Journal*, vol. 23, no. 8, pp. 729–754, 1974.

[129] V. D. Radulescu and D. D. Repovs, *Partial differential equations with variable exponents: variational methods and qualitative analysis*. CRC press, 2015.

[130] B. Ruf, "A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^2$," *Journal of Functional Analysis*, vol. 219, no. 2, pp. 340–367, 2005.

[131] D. Ruiz, "A priori estimates and existence of positive solutions for strongly nonlinear problems," *Journal of Differential Equations*, vol. 199, no. 1, pp. 96–114, 2004.

[132] K Sandeep, "A singular Moser-Trudinger embedding and its applications," *Nonlinear Differential Equations and Applications NoDEA*, vol. 13, no. 5-6, pp. 585–603, 2007.

[133] R. Servadei, "A semilinear elliptic pde not in divergence form via variational methods," *Journal of Mathematical Analysis and Applications*, vol. 383, no. 1, pp. 190–199, 2011.

[134] Z. Shi and S. Wu, "Existence of solutions for Kirchhoff type problems in Musielak–Orlicz–Sobolev spaces," *Journal of Mathematical Analysis and Applications*, vol. 436, no. 2, pp. 1002–1016, 2016.

[135] R. E. Showalter and N. J. Walkington, "Diffusion of fluid in a fissured medium with microstructure," *SIAM Journal on Mathematical Analysis*, vol. 22, no. 6, pp. 1702–1722, 1991.

[136] E. D. Silva, M. Carvalho, J. C. de Albuquerque, and S. Bahrouni, "Compact embedding theorems and a Lions' type lemma for fractional Orlicz–Sobolev spaces," *Journal of Differential Equations*, vol. 300, pp. 487–512, 2021.

[137] M. de Souza, "On a singular elliptic problem involving critical growth in $\mathbb{R}^N$," *Nonlinear Differential Equations and Applications NoDEA*, vol. 18, no. 2, pp. 199–215, 2011.

[138] M. de Souza, E. de Medeiros, and U. Severo, "An improvement for the Trudinger–Moser inequality and applications," *Journal of Differential Equations*, vol. 256, no. 4, pp. 1317–1349, 2014.

[139] M. de Souza, J. M. B. do Ó, and T. da Silva, "On a class quasilinear Schrödinger equations in $\mathbb{R}^N$," *Applicable Analysis*, vol. 95, no. 2, pp. 323–340, 2016.

[140] J. Sun, Y. Ji, and T. F. Wu, "Infinitely many solutions for Kirchhoff-type problems depending on a parameter," *Electronic Journal of Differential Equations*, vol. 224, pp. 1–10, 2016.

[141] J. Sun and T. f. Wu, "Existence and multiplicity of solutions for an indefinite Kirchhoff-type equation in bounded domains," *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, vol. 146, no. 2, pp. 435–448, 2016.

[142] J. Sun and T. f. Wu, "The effect of nonlocal term on the superlinear elliptic equations in $\mathbb{R}^N$," *Communications on Pure & Applied Analysis*, vol. 18, no. 6, 2019.

[143] A. Szulkin and T. Weth, "The method of nehari manifold, handbook of nonconvex analysis and applications," *International Press, Somerville, MA*, vol. 3, pp. 597–632, 2010.

[144] N. S. Trudinger, "On imbeddings into Orlicz spaces and some applications," *Journal of Mathematics and Mechanics*, vol. 17, no. 5, pp. 473–483, 1967.

[145] C. F. Vasconcellos, "On a nonlinear stationary problem in unbounded domains," *Revista Matematica Complutense*, vol. 5, no. 2-3, pp. 309–318, 1992.

[146] F. Wang, M. Avci, and Y. An, "Existence of solutions for fourth order elliptic equations of Kirchhoff type," *Journal of Mathematical Analysis and Applications*, vol. 409, no. 1, pp. 140–146, 2014.

[147] F. Wang, M. Avci, and Y. An, "Existence of solutions for fourth order elliptic equations of Kirchhoff type," *Journal of Mathematical Analysis and Applications*, vol. 409, no. 1, pp. 140–146, 2014.

[148] L. Wang, K. Xie, and B. Zhang, "Existence and multiplicity of solutions for critical Kirchhoff-type p-Laplacian problems," *Journal of Mathematical Analysis and Applications*, vol. 458, no. 1, pp. 361–378, 2018.

[149] Y. Wei and J. Tian, "Asymptotically linear and superlinear elliptic equations with gradient terms," *Electronic Journal of Differential Equations*, vol. 2020, no. 99, pp. 1–14, 2020.

[150] M. Willem, *Minimax theorems*. Springer Science & Business Media, 1997, vol. 24.

[151] M. Xiang, B. Zhang, and V. D. Rădulescu, "Superlinear Schrödinger–Kirchhoff type problems involving the fractional p-Laplacian and critical exponent," *Advances in Nonlinear Analysis*, vol. 9, no. 1, pp. 690–709, 2019.

[152] W. Xie and H. Chen, "Multiple positive solutions for the critical Kirchhoff type problems involving sign-changing weight functions," *Journal of Mathematical Analysis and Applications*, vol. 479, no. 1, pp. 135–161, 2019.

[153] X. Xie, T. Wang, and W. Zhang, "Existence of solutions for the (p,q)-Laplacian equation with nonlocal Choquard reaction," *Applied Mathematics Letters*, vol. 135, p. 108 418, 2023.

[154] B. Yan, D. O'Regan, and R. P. Agarwal, "Infinite number of solutions for some elliptic eigenvalue problems of Kirchhoff-type with non-homogeneous material," *Boundary Value Problems*, vol. 2021, no. 1, pp. 1–15, 2021.

[155] Y. Yang, "An interpolation of hardy inequality and Trudinger–Moser inequality in $\mathbb{R}^N$ and its applications," *International Mathematics Research Notices*, vol. 2010, no. 13, pp. 2394–2426, 2010.

[156] A. Youssfi and Y. Ahmida, "Some approximation results in musielak-orlicz spaces," *Czechoslovak Mathematical Journal*, vol. 70, pp. 453–471, 2020.

[157] H. S. Yu, *Foundations of the Theory of Plasticity*. Springer, 2006.

[158] C. Zhang and L. Chen, "Concentration-compactness principle of singular Trudinger-Moser inequalities in $\mathbb{R}^n$ and n-Laplace equations," *Advanced Nonlinear Studies*, vol. 18, no. 3, pp. 567–585, 2018.

[159] Q. G. Zhang, H. R. Sun, and J. J. Nieto, "Positive solution for a superlinear Kirchhoff type problem with a parameter," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 95, pp. 333–338, 2014.

List of publications

Journal Publications

1. **Shilpa Gupta**, Gaurav Dwivedi, Ground state solution for N-Kirchhoff equation with critical exponential growth without Ambrosetti-Rabinowitz condition, Rendiconti del Circolo Matematico di Palermo Series 2, 2023.
   `https://doi.org/10.1007/s12215-023-00902-7`

2. **Shilpa Gupta**, Gaurav Dwivedi, Kirchhoff type elliptic equations with double criticality in Musielak-Sobolev spaces, Mathematical Methods in the Applied Sciences, 46 (2023), 8463-8477.
   `https://doi.org/10.1002/mma.8991`

3. **Shilpa Gupta**, Gaurav Dwivedi, Existence and multiplicity of solutions to N-Kirchhoff equations with critical exponential growth and perturbation term, *Complex Variables and Elliptic Equations*, 2022.
   `https://doi.org/10.1080/17476933.2022.2048297`

4. Gaurav Dwivedi, **Shilpa Gupta**, Existence of solution to Kirchhoff type problem with gradient nonlinearity and a perturbation term, *Journal of Elliptic and Parabolic Equations*, 8 (2022), 533–553.
   `https://doi.org/10.1007/s41808-022-00164-x`

5. **Shilpa Gupta**, Gaurav Dwivedi, Existence of solution to a nonlocal biharmonic problem with dependence on gradient and Laplacian, *Journal of Applied Analysis*, 28 (2022), 211-218 .
   `https://doi.org/10.1515/jaa-2021-2073`

6. **Shilpa Gupta**, Gaurav Dwivedi, An existence result for p-Laplace equation with gradient nonlinearity in $\mathbb{R}^n$, *Communications in Mathematics*, 30 (2022), 149–159.
   `https://doi.org/10.46298/cm.9316`

7. **Shilpa Gupta**, Gaurav Dwivedi, Ground state solution for a generalized Choquard Schrödinger equation with vanishing potential in homogeneous fractional Musielak Sobolev spaces (Communicated).

Conferences/ Workshops/ Schools attended

Conferences

1. Presented a research paper in the **International Conference on Advances in Mechanics, Modelling, Computing and Statistics** organized by BITS Pilani, Pilani Campus, Rajasthan, India. (19-21 March, 2022).

2. Presented a research paper in the national conference **Recent Advances in Mathematical Analysis and Applications** held at Banaras Hindu University, Varanasi, India. (7-8 May, 2022).

3. Presented a Poster in the International Conference on **Trends in Calculus of Variations and PDEs**, jointly organized by University of Sussex, UK and the Analysis & PDE Center of Ghent University, Belgium, (18-20 May, 2022).

4. Presented a research paper in **O. A. Ladyzhenskaya Centennial Conference on PDEs (ICM 2022 Satellite Conference)** held at V. A. Steklov Institute of Mathematics, St. Petersburg, Russia. (16-22 July, 2022).

5. Presented a research paper in **The 37th Annual Conference of the Ramanujan Mathematical Society** held at Sri Sivasubramaniya Nadar College of Engineering, Chennai, India. (06-08 December, 2022).

6. Presented a research paper in **The 13th AIMS Conference on Dynamical Systems, Differential Equations and Applications** held at Wilmington, NC USA. (May 31-June 04, 2023).

7. Presented a research paper in **2023 UNC Greensboro PDE Conference** held online at UNC Greensboro, NC USA. (June 09-11, 2023).

Workshops/Schools

1. Participated in the **Seven-day Workshop on Academic Writing** organized in BITS Pilani, Pilani Campus, Rajasthan, India. (5-11 April, 2019).

2. Participated in the **Annual Foundation School (AFS-1)**, organized in Harish Chandra Research Institute, Prayagraj, Uttar Pradesh India. (May 06-June 01, 2019).

3. Participated in the Advanced Instructional School on **Geometric Analysis**, organized at Indian Institute of Technology, Bombay, Maharashtra, India. (09- 28 December, 2019).

4. Participated in the International **Workshop in Nonlinear PDE'S, Functional and Geometric Analysis** held Online at Universidade Federal da Paraíba, Paraíba, Brazil. (06-11 December, 2021).

Brief biography of the candidate

Ms. Shilpa received her B.Sc., M.Sc. and M.phil degree in Mathematics from Banasthali Vidyapith, India in 2015, 2017 and 2018, respectively. In 2019, she joined as a full-time Ph.D. scholar in the Department of Mathematics at Birla Institute of Technology and Science (BITS) Pilani, Pilani Campus under the supervision of Dr. Gaurav Dwivedi. Her research interests include generalized spaces (variable exponent Sobolev spaces, Orlicz spaces, Musielak Sobolev spaces, etc.), Kirchhoff problems, and nonlinear problems involving non-standard growth nonlinearities such as critical growth, exponential critical growth, gradient dependent growth, and Choquard type growth. She has served as a teaching assistant for the various courses of mathematics at BITS Pilani, Pilani Campus. So far, she has published/accepted six articles in peer-reviewed international journals and presented seven papers at national and international conferences. More information about her research contributions can be found at-
`https://www.researchgate.net/profile/Shilpa_Gupta12`.
Contact her at `p20180442@pilani.bits-pilani.ac.in`.

Brief biography of the supervisor

Dr. Gaurav Dwivedi is an Assistant professor in the Department of Mathematics at Birla Institute of Technology and Science Pilani, Pilani campus, India. He completed his Bachelor's degree from CSJM University, Kanpur, India in 2008. He received his Master's degree from the Department of Mathematics, Indian Institute of Technology Delhi, India in 2010. He earned his Ph.D. degree from the Department of Mathematics, Indian Institute of Technology Gandhinagar, India in 2017. His research interests include the existence and other qualitative properties of elliptic partial differential equations. He has several research publications in refereed journals.

More information about his research contributions can be found at-
`https://www.researchgate.net/profile/Gaurav-Dwivedi-3`.
Contact him at `gaurav.dwivedi@pilani.bits-pilani.ac.in`.