Study of Dynamics of Automorphisms on Solenoids and other Compact Groups

Thesis

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CERTIFICATE

This is to certify that the thesis entitled, "*Study of Dynamics of Automorphisms on Solenoids and other Compact Groups*" submitted by **FAIZ IMAM**, ID No. 2018PHXF0029H in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY embodies the work done by him under my supervision.

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Date:

All praises and thanks to "The One" who taught me through the pen and taught me what I didn't know.

Dedicated

to

My Parents &

My Grand-Parents

(who mean everything to me).

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Abstract

In recent decades, the study of various aspects of topological dynamical systems has been a major research focus. In order to provide a more detailed and in-depth explanation of their long-term behaviour scenario, researchers have used a variety of mathematical techniques to discuss and explain it. However, some spaces are yet to be thoroughly explored, and there is a need to understand their dynamics in depth. As a result, the purpose of this thesis is to look into some related problems in order to find an accurate explanation of the dynamics of these systems.

Before discussing about the investigated problems, some preliminaries and basic definitions are discussed in Chapter-1. In Chapter-2, we describe the sets of periodic points of automorphisms of a one dimensional solenoid, considering it as the inverse limit of a sequence of maps on the unit circle. Later, we study the periodic points for a class of automorphisms on certain higher dimensional solenoids that are inverse limits of sequences of maps on n-dimensional torus. Chapter-3 extends the previous work about the periodic points characterization of automorphisms of some solenoids, whose duals are subgroups of algebraic number fields. Chapter-4 aims to investigate the dynamical properties like periodicity, topological mixing, topological transitivity, distality and proximality of real projective transformations from a topological viewpoint. Then, Chapter-5 interprets a non-autonomous dynamical system (NDS) as a discrete switch dynamical system (SDS) and describes how the dynamics of a non-autonomous dynamical system can be better understood using the notion of switch.

Keywords: Topological Dynamics; Solenoids; Inverse Limits; Periodic Points; Pontryagin Dual; Algebraic Dynamics; Adeles; Projective Transformations; Topological Transitivity; Topological Mixing; Distality; Discrete Switch Dynamical Systems; Non-Autonomous Discrete Dynamical System.

Contents

List of Symbols

$\mathbb{R}$	: real numbers
$\mathbb{Q}$	: rationals
$\mathbb{Z}$	: integers
$\mathbb{N}$	: positive integers
$\mathbb{N}_0$	: non-negative integers
$GL_n(\mathbb{R})$	: set of all invertible $n \times n$ matrices with real entries
$f^k(x)$	: $f \circ f \circ ... \circ f(x)$ (k times) for $k \in \mathbb{N}$ and $f^0(x) = x$
$P(f)$	: $\{x \in X : x$ is a periodic point of $f\}$
$Per(f)$	: $\{n \in \mathbb{N} : f$ has a periodic point of least period $n\}$
$Fix(f)$	: set of all fixed points of f
$\omega(x)$	: set of all $\omega-$limit points of x
S^1	: unit circle i.e. $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ or $\mathbb{R}/\mathbb{Z}$
D^2	: solid unit disk i.e. $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$
$\mathbb{T}^n$	: $n-$dimensional torus i.e. $\mathbb{R}^n/\mathbb{Z}^n$
Σ	: solenoid
$\widehat{\Sigma}$	: Pontryagin dual of Σ
$\lim\limits_{\overleftarrow{k}}(X_k, f_k)$	: inverse limit of the sequence of maps (f_k)
P	: set of prime numbers
$\mathbb{Q}_p$	: $p-$adic rationals
$\mathbb{Z}_p$	: $p-$adic integers
$\mathbb{Z}_p^*$	: $\{x \in \mathbb{Z}_p : \|x\|_p = 1\}$
$P^{\mathbb{Q}}$	: set of places of $\mathbb{Q}$
$\mathbb{Q}_{\mathbb{A}}$	: ring of adeles of $\mathbb{Q}$

$\mathbb{K}$	: algebraic number field
$P^{\mathbb{K}}$	: set of places of $\mathbb{K}$
$P_f^{\mathbb{K}}$	: set of all finite places of $\mathbb{K}$
$P_\infty^{\mathbb{K}}$	: set of all infinite places of $\mathbb{K}$
$\mathbb{K}_v$	: completion of $\mathbb{K}$ with respect to the place v
$\Re_v$	: $\{x \in \mathbb{K}_v : \; \|x\|_v \leq 1\}$
$\Re_v^*$	: $\{x \in \Re_v : \; \|x\|_v = 1\}$
$\mathbb{K}_{\mathbb{A}}$	: ring of adeles of $\mathbb{K}$
$\mathbb{P}_n(\mathbb{R})$	: $n-$dimensional projective space

Chapter 1

Introduction

For many important classes of dynamical systems, it turns out that the family of their sets of periods, is elegantly describable and in fact is often totally unexpected. There is a lot of variety in the answers and the methods used to arrive at them. - [29] V. Kannan, *Sets of periods of dynamical systems*, Indian Journal of Pure and Applied Mathematics **41 (1)** (2010), 225-240.

This thesis titled **Study of Dynamics of Automorphisms on Solenoids and other Compact Groups** has been focused on investigating the dynamics of the automorphisms of compact groups. We will be mainly considering two classes of compact groups, namely the solenoids and real projective spaces. Later we also discuss various dynamical notions about discrete switch dynamical systems.

In this introductory chapter, we start with a section covering various basic notions of topological dynamics followed by sections on solenoids, projective transformations and discrete switch dynamical systems.

1.1 Topological Dynamics

Topological Dynamics is a branch of dynamical system theory that investigates the qualitative and asymptotic properties of dynamical systems from the standpoint of general topology. The theory of dynamical systems is essentially the study of eventual behavior of evolving systems.

By definition, a *topological discrete dynamical system* (briefly, a dynamical system) is a pair (X, f), where X is a topological space and f is a continuous self map of X. The *trajectory* of $x \in X$ is defined as the sequence $(x, f(x), f^2(x), f^3(x), ...)$, where $f^k(x) = f \circ f \circ ... \circ f(x)$ (k times) for $k \in \mathbb{N}$ and $f^0(x) = x$. The *forward orbit* of x is defined as the set $\{f^k(x) : k$ is a non-negative integer$\}$. The study of dynamics is primarily concerned with the behavior of trajectories in the long run.

A point $x \in X$ is said to be periodic with a period n if there is an $n \in \mathbb{N}$ such that $f^n(x) = x$. The smallest such value of "n" for which x becomes a periodic point is known as the least period of x. A point is defined as a fixed point when $f(x) = x$ (period one point). The problems of characterizing the sets of periods and periodic points of a family of dynamical systems have been well-studied in the literature. To put formally, we define the problem in the following manner:

If $\mathfrak{F}$ is a family of self maps on a space X, then give a characterization of the collections:

$\{Per(f) : f \in \mathfrak{F}\}$ where, $Per(f) = \{n \in \mathbb{N} : f$ has a periodic point of least period $n\}$, and

$\{P(f) : f \in \mathfrak{F}\}$ where , $P(f) = \{x \in X : x$ is a periodic point of $f\}$.

The sets of periods and periodic points of a family of dynamical systems are well studied in literature. See for instance, [4], [10], [11], [15], [22], [47] and [48].

If x is a periodic point, then it is easy to see that, x is a limit point of its trajectory. For a general element we have the following notion of ω-limit point.

Definition. A point $y \in X$ is said to be an ω-limit point of a point $x \in X$ if there is a sequence of natural numbers $(n_k) \to \infty$ (as $k \to \infty$) such that $(f^{n_k}(x)) \to y$. The $\omega-$limit set of x is the set of all $\omega-$limit points of x, denoted by $\omega(x)$.

A point $x \in X$ approaches arbitrarily close to each of its ω-limit points. If it approaches arbitrarily close to itself, it is called a recurrent point. The precise definition is as follows.

Definition. A point x is called recurrent if $x \in \omega(x)$. Equivalently, $(f^{n_k}(x)) \to x$ for some sequence of natural numbers $(n_k) \to \infty$.

All the above notions are about the trajectories of a single point. We now consider the kind of dynamics that can happen between two different points. A pair of points is said to be proximal, if their trajectories come arbitrarily close to each other and those which are not proximal are called distal.

Definition. Let X be a compact Hausdorff space and $f : X \to X$ be a homeomorphism. $x,\ y \in X$ are said to be proximal if the closure $\overline{\{(f^n(x), f^n(y)) : n \in \mathbb{Z}\}}$ of the full orbit of (x, y) under $f \times f$ intersects the diagonal $\Delta = \{(z, z) \in X \times X : z \in X\}$. Points which are not proximal are called distal points.

If (X, d) is a compact metric space, then $x,\ y \in X$ are proximal if there is a sequence (n_k) in $\mathbb{Z}$ such that $(d(f^{n_k}(x), f^{n_k}(y))) \to 0$ as $k \to \infty$. Equivalently, $x, y \in X$ are distal if there is an $\epsilon > 0$ such that $d(f^n(x), f^n(y)) > \epsilon$ for all $n \in \mathbb{Z}$.

A peculiar behaviour a pair of points can exhibit is that they come arbitrarily close to each other and still maintain a minimum positive distance at infinitely many times. This behavior leads to the concept of scrambledness.

Definition. A subset $Y \subseteq X$ is said to be scrambled if for any two distinct points x and y in Y, $\liminf d(f^n(x), f^n(y)) = 0$ and $\limsup d(f^n(x), f^n(y)) > 0$.

A subset $A \subseteq X$ is said to be forward $f-invariant$ if $f(A) \subseteq A$. If a subset $A \subseteq X$ is forward $f-invariant$, then the dynamics can be restricted to A. Thus we have the following definition.

Definition. Let X be a compact space. A closed, non-empty, forward $f-$invariant subset $Y \subseteq X$ is called a subsystem of (X, f). A subsystem is a minimal system if it contains no proper subsystem.

Note that a subsystem Y is minimal if and only if the forward orbit of every point in Y is dense in Y.

Definition. A system is said to be transitive if it has a dense forward orbit.

The word transitive looks more intuitive, owing to the following equivalent notion. If X is a locally compact Hausdorff space and if for any two non-empty open sets U and V of X, there is $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \varphi$ then f is topologically transitive (See [12]).

We now mention two stronger notions of transitivity. For a system (X, f), if (X, f^n) is transitive for every $n \in \mathbb{N}$, then (X, f) is called *totally transitive.* Another stronger version of transitivity is that, if for any two non-empty open sets U and V, there is $n \in \mathbb{N}$ such that $f^k(U) \cap V \neq \varphi$ for every $k \geq n$, then (X, f) is called *topologically mixing.*

Coming to the equivalence of systems, we have the following notion called topological conjugacy, defined as follows.

Definition. If (X, f) and (Y, g) are two dynamical systems and $\phi : X \to Y$ is a surjective continuous map such that $\phi \circ f = g \circ \phi$, then ϕ is called a *topological semiconjugacy* from f to g and (Y, g) is called a *factor* of (X, f). If ϕ is a homeomorphism, then ϕ is called a *topological conjugacy* (or briefly a conjugacy); in this case (X, f) and (Y, g) are said to be *topologically conjugate* (or briefly conjugate).

1.2 Solenoids

The solenoid was first known to topologists and it was first introduced by L. Vietoris in 1927 and was also later introduced as an example in dynamical systems by Stephen

Smale in 1967. It was then picked up by R.F. Williams who developed the theory of one-dimensional expanding attractors in 1967 and 1974.

A solenoid can be formally defined with the following approaches, in the light of topological as well as algebraic notions (see [32], [46]).

1. A solenoid is defined as a compact connected finite dimensional abelian group.

2. Solenoids are also described using inverse limits. For an integer $n > 1$, let $\pi^n : \mathbb{R}^n \to \mathbb{T}^n$ be the homomorphism defined as $\pi^n((x_1, x_2, ..., x_n)) = (x_1 \ (mod\ 1), x_2 \ (mod\ 1), ..$ $.., x_n \ (mod\ 1))$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the unit circle. Let $\overline{M} = (M_k)_{k=1}^{\infty} = (M_1, M_2, ...)$ be a sequence of $n \times n$ matrices with integer entries and non-zero determinant. Then, the $n-$dimensional solenoid $\sum_{\overline{M}}$ is defined as $\sum_{\overline{M}} = \{(\mathbf{x}_k) \in (\mathbb{T}^n)^{\mathbb{N}_0} : \pi^n(M_k \mathbf{x}_k) = \mathbf{x}_{k-1}$ for every $k \in \mathbb{N}\}$.

3. An $n-$dimensional solenoid Σ is an abelian group whose Pontryagin dual $\widehat{\Sigma}$ (group of all characters of Σ) is an (additive) subgroup of $\mathbb{Q}^n$ and contains $\mathbb{Z}^n$. If $\widehat{\Sigma} = \mathbb{Z}^n$, then Σ is the $n-$dimensional torus and on the other hand when $\widehat{\Sigma} = \mathbb{Q}^n$, we call it a full solenoid.

4. The book [26] defines a solenoidal group in the following way. Let G be a topological group. If G contains a dense cyclic subgroup, then G is said to be monothetic. Let τ be a continuous homomorphism of $\mathbb{R}$ into G. Then $\tau(\mathbb{R})$ is called a one-parameter subgroup of G. If G contains a dense one-parameter subgroup, then G is said to be solenoidal.

5. The book [46] discusses about the dynamical systems defined with the help of $R-$modules, where $R = \mathbb{Z}[u^{\pm 1}]$, i.e., the ring of Laurent polynomials in the variable u with coefficients from $\mathbb{Z}$. It is proven that the dual of the $R-$module, $R/(f)$, where (f) is a prime ideal in R, is isomorphic to a $k-$dimensional solenoid, provided that there is an $n \in \mathbb{Z}$ such that $u^n f(u) = c_0 + c_1 u + ... + c_k u^k$ with $|c_0 c_k| > 1$. Evidently, if α is an automorphism of a compact group X, then the dynamics of α, when viewed as a $\mathbb{Z}$-action on X provides us with an R-module. If this R-module is $R/\mathfrak{P}$, where $\mathfrak{P}$ is a prime ideal, then X is a solenoid if and only if $\mathfrak{P} = \{f \in R | f(c) = 0\}$,

for some $c \in \mathbb{Q} - \{0\}$. Furthermore, the $\mathbb{Z}$-action on X is transformed into the shift action on a subgroup of $\mathbb{T}^{\mathbb{Z}}$.

In this thesis, our aim is to describe the sets of periodic points of solenoidal automorphisms. In the process, we first describe a solenoid in suitable terms, for instance as an inverse limit in Chapter 2 and in terms of adeles in Chapter 3 and then describe the sets of periodic points of certain automorphisms on them. Since we work with only these two notions, we will elaborate on these aspects in this section and also mention few main results that we prove in subsequent chapters. The statements of most of these results involve some notations that are defined in the respective chapters but we do not define them here again for brevity. Before all these, we start with a particular one-dimensional solenoid, namely the dyadic solenoid which is easy to visualize and also becomes a prototype for other solenoids.

1.2.1 Dyadic Solenoids

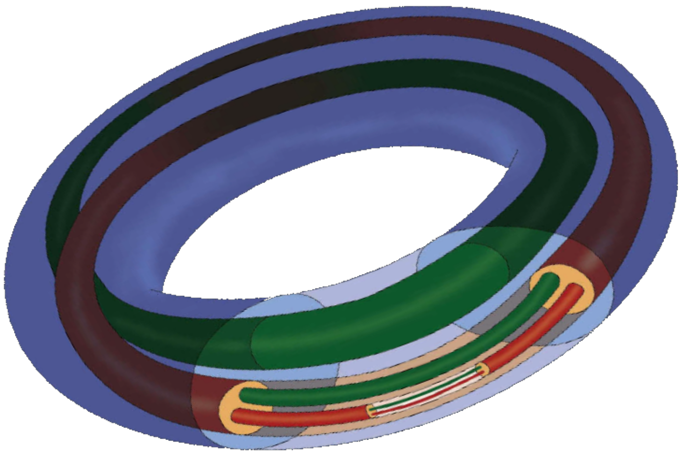

Image Source: [12]

We will consider the unit circle S^1 as the quotient space $[0,1]/\sim$, where 0 and 1 are identified. Let $D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ describe the solid unit disk. Consider the solid torus : $T = S^1 \times D^2 = \{(\phi, x, y) |\ 0 \le \phi < 1,\ x^2 + y^2 \le 1\}$. Fix a real $\lambda \in \left(0, \frac{1}{2}\right)$. Define $F : T \to T$ such that $F(\phi, x, y) = \left(2\phi\,(mod\,1)\,, \lambda x + \frac{1}{2}\cos 2\pi\phi, \lambda y + \frac{1}{2}\sin 2\pi\phi\right)$.

The map stretches the solid torus and then wraps the stretched torus inside the original.

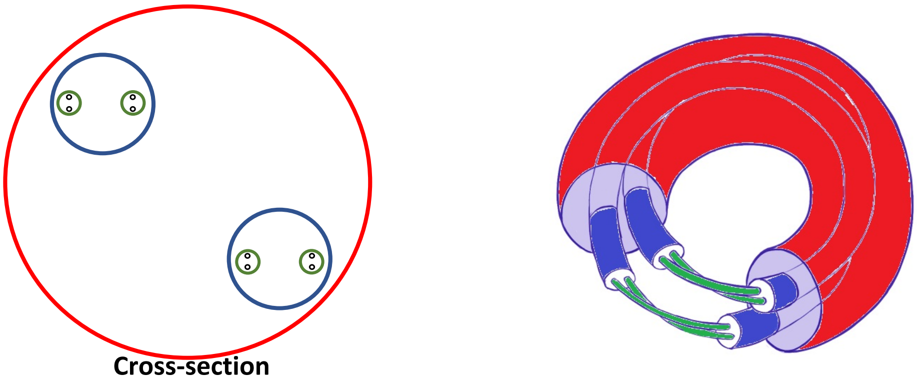

Image Source : Internet

The map F stretches the solid torus by a factor of 2 in the S^1-direction and contracts by a factor of λ in the D^2-direction. The map F wraps the image (stretched torus) twice inside the original torus T and the image $F^{n+1}(T)$ is contained inside $int(F^n(T))$. The intersection of a non-empty, nested sequence of compact sets is non-empty and compact and so we obtain the set $S = \cap_{n=0}^{\infty} F^n(T)$ known as the *dyadic solenoid*.

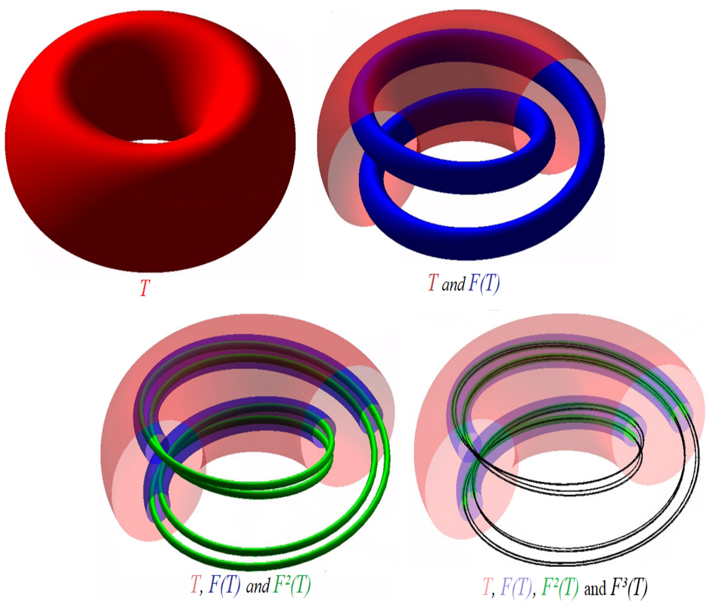

Image Source : Wikipedia

1.2.2 One-dimensional Solenoids as Inverse Limits

The dyadic solenoid defined above can also be viewed as a special subspace of a countable product of circles. To understand this, we first define the notion of an inverse limit.

Definition. Let X_k be a topological space for each $k \in \mathbb{N}_0$ and $f_k : X_k \to X_{k-1}$ be a continuous map for each $k \in \mathbb{N}$. Then the subspace of $\prod_{k=0}^{\infty} X_k$ defined as $\varprojlim_k (X_k, f_k) = \{(x_k) \in \prod_{k=0}^{\infty} X_k : x_{k-1} = f_k(x_k), \forall k \in \mathbb{N}\}$ is called the *inverse limit* of the sequence of maps (f_k).

Now, the dyadic solenoid is the inverse limit $\prod_{k=0}^{\infty} X_k$, where each X_k is S^1 and each f_k is given by $f_k(x) = 2x \ (mod\ 1)$. In a similar manner, a general one-dimensional solenoid is obtained by defining f_k to be the multiplication with an integer larger than one. Hence, we have the following definition.

Definition. Let $A = (a_k)_{k=1}^{\infty}$ be a sequence of positive integers such that $a_k \geq 2$ for every $k \in \mathbb{N}$. The *solenoid* corresponding to the sequence A, denoted by Σ_A, is defined as $\Sigma_A = \{(x_k) \in (S^1)^{(\mathbb{N}_0)} : x_{k-1} = a_k x_k \,(mod\, 1) \text{ for every } k \in \mathbb{N}\}$.

We now list our main results, namely the description of periodic points (Theorem 2.6) and the number of periodic points (Theorem 2.8) of an automorphism of a one dimensional solenoid.

Theorem. Let ϕ be an automorphism of a one dimensional solenoid Σ_A induced by $\frac{\alpha}{\beta}$, where $A = (\beta b_k)$, each b_k being co-prime to β. For each $l \in \mathbb{N}$, define $U_l = \bigcap_{p \in P} \left(\frac{1}{p^{e_{p,l}}} \mathbb{Z}_p \cap \mathbb{Q} \cap S^1 \right)$, where $p^{e_{p,l}} = \frac{1}{|\alpha^l - \beta^l|_p}$. If $\gamma_{k,l} : U_l \to U_l$ is the map defined as $\gamma_{k,l}(x) = \beta b_k x (mod\ 1)$ for each $k \in \mathbb{N}$ and $l \in \mathbb{N}$, then $P(\phi) = \bigcup_{l=1}^{\infty} \varprojlim_k (U_l, \gamma_{k,l})$.

Theorem. Let ϕ be an automorphism of a one dimensional solenoid Σ_A induced by $\frac{\alpha}{\beta}$ and for every $l \in \mathbb{N}$, let $e_{p,l} = \frac{1}{|\alpha^l - \beta^l|_p}$. Then the number of periodic points of ϕ with a period l is $\prod_{p \notin D_{\infty}^{(S)}} p^{e_{p,l}}$.

1.2.3 Higher dimensional Solenoids as Inverse Limits

For a positive integer $n > 1$, let $\pi^n : \mathbb{R}^n \to \mathbb{T}^n$ be the homomorphism defined as $\pi^n((x_1, x_2, ..., x_n)) = (x_1 \ (mod\ 1), x_2 \ (mod\ 1), ..., x_n \ (mod\ 1))$. Let $\overline{M} = (M_k)_{k=1}^{\infty} = (M_1, M_2, ...)$ be a sequence of $n \times n$ matrices with integer entries and non-zero determinant. Then, the $n-$dimensional solenoid $\sum_{\overline{M}}$ is defined as $\sum_{\overline{M}} = \{(\mathbf{x}_k) \in (\mathbb{T}^n)^{\mathbb{N}_0} : \pi^n(M_k \mathbf{x}_k) = \mathbf{x}_{k-1}$ for every $k \in \mathbb{N}\}$. In other words, $\sum_{\overline{M}} = \lim\limits_{\overleftarrow{k}} (\mathbb{T}^n, \delta_k)$, where $\delta_k : \mathbb{T}^n \to \mathbb{T}^n$ is defined as $\delta_k(\mathbf{x}) = \pi^n(M_k \mathbf{x})$.

If ϕ is an automorphism of $\sum_{\overline{M}}$, then there is a matrix $L \in GL(n, \mathbb{Q})$ such that $\phi((\mathbf{x}_k)) = (\pi^n(L\mathbf{x}_k))$, for each $\mathbf{x}_k \in \mathbb{T}^n$. We say that ϕ is induced by the matrix L. Now, consider n sequences of positive integers $A_1 = (a_1^1, a_2^1, ...)$, $A_2 = (a_1^2, a_2^2, ...)$, ... $A_n = (a_1^n, a_2^n, ...)$. Then define the sequence $\overline{M} = (M_k)$ of matrices as $M_k = diag[a_k^1, a_k^2, ..., a_k^n]$. These sequences of positive integers and matrices give rise to n one-dimensional solenoids and an $n-$dimensional solenoid.

We state the following main result (Theorem 2.11) about its periodic points.

Theorem. For each $l \in \mathbb{N}$, define $V_l = \prod_{i=1}^n \left(\bigcap\limits_{p \in P} \left(\frac{1}{p^{e_{p,l,i}}} \mathbb{Z}_p \cap \mathbb{Q} \cap S^1 \right) \right)$, where $p^{e_{p,l,i}} = \frac{1}{|\alpha_i^l - \beta_i^l|_p}$. If $\delta_{k,l} : V_l \to V_l$ is the map defined as $\delta_{k,l}(\mathbf{x}) = \pi^n(M_k \mathbf{x})$ for each $k \in \mathbb{N}$ and $l \in \mathbb{N}$, then $P(\phi) = \bigcup\limits_{l=1}^{\infty} \lim\limits_{\overleftarrow{k}} (V_l, \delta_{k,l})$.

1.2.4 Solenoids in terms of Adeles

The work of characterizing the sets of periodic points for automorphisms on one-dimensional solenoids in terms of adeles was completed by Sharan and Raja in [22]. A one dimensional solenoid was described as a quotient of $\mathbb{Q}_{\mathbb{A}}$, the ring of adeles and then the set of periodic points were described based on that.

The characterization was entirely based on description of subgroups of $\mathbb{Q}$ as given by [5]. Here, given a subgroup S of $\mathbb{Q}$, for every $x \in S$ and for every prime number p, we associate a non-negative integer or the symbol ∞, called as the p-height of x in S, and denoted by $h_p^{(S)}(x)$. These prime heights play a very crucial role in describing the solenoids in the above mentioned paper [22].

However, these results cannot be extended to higher dimensional solenoids, due to a simple reason that there is no such description available for subgroups of $\mathbb{Q}^n$. In fact, [31] implies that there is probably "no reasonably simple classification" of these groups.

In this thesis, we overcome this hurdle by considering the adeles of an algebraic number field $\mathbb{K}$, instead of $\mathbb{Q}$ as done above; this may be justified in the following way. In the earlier case, since the dual $\widehat{\Sigma}$ of a one dimensional solenoid Σ is a subgroup of $\mathbb{Q}$, we have $\Sigma(\cong \widehat{\widehat{\Sigma}})$ is a quotient of $\widehat{\mathbb{Q}}$ which is in turn a quotient of $\mathbb{Q}_{\mathbb{A}}$. In case of higher dimensional solenoids Σ, we have $\widehat{\Sigma}$ to be a subgroup of $\mathbb{Q}^n$ and we consider $\mathbb{Q}^n$ as n-dimensional vector space over $\mathbb{Q}$. Since this general case cannot be considered on the same lines, we considered those cases where $\widehat{\Sigma}$ is a subgroup of $\mathbb{K}$, where $\mathbb{K}$ is a finite algebraic extension of $\mathbb{Q}$ (not just a vector space over $\mathbb{Q}$), which is in other words, called an algebraic number field.

We now state our main results Theorem 3.5 and Theorem 3.7, leaving most of the details about notations to Chapter 3. In the following statements, $P^{\mathbb{Q}}$ denotes the set of all places of $\mathbb{Q}$ and D is a particular subset of $P^{\mathbb{Q}}$. $\mathbb{K}_{\mathbb{A}}$ denotes the ring of adeles of $\mathbb{K}$ and $i : \mathbb{K} \to \mathbb{K}_{\mathbb{A}}$ is the diagonal inclusion map and finally $V = i(\mathbb{K}) + \prod_{p \in P^{\mathbb{Q}}} U_p$, where

$$U_p = \begin{cases} (0) & \text{for } p \in D \cup \{\infty\} \\ \{x \ \in \prod_{v|_p} \mathbb{K}_v : |x^{(j)}|_p \leq n_p^{(j)} \text{ for every } j\} & \text{for } p \notin D \cup \{\infty\} \end{cases},$$

where $x^{(1)}, x^{(2)}, \cdots, x^{(n)}$ are $\mathbb{Q}_p-$coordinates of x.

Theorem. Σ is isomorphic to $\mathbb{K}_{\mathbb{A}}/V$.

For a fixed element $(d) = (d^{(1)}, d^{(2)}, \cdots, d^{(n)}) \in \mathbb{Q}^n$ satisfying certain conditions, we get an automorphism M_d of $\mathbb{K}_{\mathbb{A}}$ under which V is an invariant subgroup and thus inducing an automorphism $\overline{M_d}$ on Σ. The following Theorem 3.7 describes the set of periodic points of $\overline{M_d}$.

Theorem. The set of periodic points of $\overline{M_d}$, where $d^{(j)} \neq \pm 1$ for every $1 \leq j \leq n$, is given by $P(\overline{M_d}) = \frac{i(\mathbb{K}) + \prod' \mathbb{K}_v}{V}$, where $\prod' \mathbb{K}_v = \Big\{x \in \mathbb{K}_{\mathbb{A}} : \textit{for every}\, 1 \leq j \leq n, x_p^{(j)} = 0 \ \textit{whenever}$ $p \in D \cup \{\infty\}$ and $|x_p^{(j)}|_p \leq n_p^{(j)}$ for all but finitely many $p \notin D \cup \{\infty\}\Big\}$.

1.3 Projective Transformations

The dynamical system that we consider in Chapter 4 is $\left(\mathbb{P}_n(\mathbb{R}), \widetilde{T}\right)$, where $\mathbb{P}_n(\mathbb{R})$ and $\widetilde{T}$ are defined as follows. For $x, y \in \mathbb{R}^{n+1} \setminus \{\bar{0}\}$, if there exists a non-zero $\lambda \in \mathbb{R}$ such that $x = \lambda y$, then define $x \sim y$. The quotient space $\mathbb{R}^{n+1} \setminus \{\bar{0}\} / \sim$, denoted by $\mathbb{P}_n(\mathbb{R})$ is called the n-dimensional *real projective space*. The quotient map is denoted by π and for an $x \in \mathbb{R}^{n+1} \setminus \{\bar{0}\}, \pi(x)$ is also denoted as $[x]$. It is well known that $\mathbb{P}_n(\mathbb{R})$ is compact and connected. Given a linear map $T \in GL_{n+1}(\mathbb{R})$, its associated projective transformation denoted by $\widetilde{T}$, is defined as $\widetilde{T}(\pi(x)) = \pi(Tx)$, for every $x \in \mathbb{R}^{n+1} \setminus \{\bar{0}\}$. It can be easily observed that $\left(\mathbb{P}_n(\mathbb{R}), \widetilde{T}\right)$ is a factor of $(\mathbb{R}^{n+1} \setminus \{\bar{0}\}, T)$. We use the metric $d([x],[y]) = min\left\{\left|\left|\frac{x}{||x||} - \frac{y}{||y||}\right|\right|, \left|\left|\frac{x}{||x||} + \frac{y}{||y||}\right|\right|\right\}$ which induces the topology of $\mathbb{P}_n(\mathbb{R})$; where $||x||$ is the Euclidean norm of x.

Our main goal here is to understand the various dynamical aspects of projective transformations. There has been an extensive literature in this area. See for instance [18], [23] and [35]. In the present work, we consider periodicity, transitivity, mixing, distality and proximality. A characterization of sets of least periods of these transformations is obtained in Theorem 4.4. On the other hand, the sets of periodic points can be found easily; this was also discussed in the introduction of the Chapter 4.

Regarding transitivity, we use the results of Herzog [25] who proved that $\mathbb{R}^n$ does not admit linear operators with hypercyclic vectors unless $n \in \{0, 1, 2\}$. We first prove that the existence of a supercyclic vector for T is equivalent to the transitivity of $\widetilde{T}$, thus proving that $\mathbb{P}_n(\mathbb{R})$ admits a transitive projective transformation only in the case $n = 1$. However, we also prove that $\mathbb{P}_n(\mathbb{R})$ does not admit a mixing projective transformation for any $n \in \mathbb{N}$. Coming to distality, we prove that isometries are the only projective transformations that are distal.

1.4 Discrete Switch Dynamical Systems

The concept of a discrete switch dynamical system (SDS) is an interpretation of the idea of non-autonomous discrete dynamical system (NDS) in a way that it makes more easy to understand the dynamics. The theory of NDS started with the paper [34]. By definition, an NDS is a topological space X along with a sequence $(f_n)_{n=1}^{\infty}$ of continuous self maps on X and the orbit of a point $x \in X$ is given by $(x_n)_{n=0}^{\infty}$, where $x_n = f_n(x_{n-1})$ for $n \geq 1$ and $x_0 = x$. Note that this coincides with the usual dynamical system (X, f), if each $f_n = f$. Following this paper, several people have studied dynamical properties like entropy and stability in general for NDS.

On the other hand, a similar notion exists in literature for continuous dynamical system with the name switch continuous dynamical system. In this thesis, we adopt this notion of switch to discrete dynamical systems, thus giving rise to the discrete switch dynamical system (SDS). Though SDS is similar to NDS, this new way of looking at this makes it more easy to understand the underlying dynamics. We also study some properties that are not verified for NDS like transitivity and recurrence. To the best of our knowledge, the literature on NDS is more focused on sensitivity properties like entropy and chaos.

In Chapter 5, we introduce the definition of SDS and various dynamical notions in this new setup. This will be followed by some results in periodicity, transitivity and recurrent points. We then have a section on switching rotations on the unit circle. We also gave some examples to show the significance of the idea of SDS. We do not claim that there are non-trivial results in this chapter as many of the notions are already explored for NDS. However, this study was made to get an insight into how facts about NDS can be explained using the idea of switch.

Chapter 2

Solenoids as Inverse Limits

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In this chapter, we describe the sets of periodic points of automorphisms of a one dimensional solenoid Σ, considering it as the inverse limit, $\lim\limits_{\overleftarrow{k}}(S^1, \gamma_k)$ of a sequence (γ_k) of maps on the circle S^1, where $\mathbb{Z} \subsetneqq \widehat{\Sigma} \subseteq \mathbb{Q}$. Later, in section 2.3, we discuss the periodic points for a class of automorphisms on certain higher dimensional solenoids that are inverse limits of sequences of maps on n-dimensional torus, $\mathbb{T}^n$ with $n > 1$.

In all these cases, we show that the set of periodic points of a given period is the inverse limit of the same maps (that define the solenoid) restricted to a subgroup of $\mathbb{T}^n$.

2.1 Introduction

Solenoids are extensively studied in literature. Some of the papers consider solenoids as inverse limits of certain maps on $\mathbb{T}^n$. The paper [53] shows that an ergodic automorphism of a solenoid is measure theoretically isomorphic to a Bernoulli shift.

A shift space is the dynamical system (X_m, σ), for a positive integer $m > 1$, where X_m is the set of infinite two-sided sequences of symbols from the set of $\{1, 2, \cdots, m\}$ and the shift map $\sigma : X_m \to X_m$ is defined as $\sigma(x)_i = x_{i+1}$. The set $C^{n_1,n_2,\cdots,n_k}_{j_1,j_2,\cdots,j_k} = \{x = (x_l) : x_{n_i} = j_i, i = 1, 2, \cdots, k\}$, where $n_1 < n_2 < \cdots < n_k$ are indices in $\mathbb{Z}$ and $j_i \in \{1, 2, \cdots, m\}$ is called a cylinder. Now, let A be an $m \times m$ matrix with non negative entries such that all the rows of A are same, say $(q_1, q_2, \cdots, q_m)$ and $q_1 + q_2 + \cdots + q_m = 1$. For a cylinder C^n_j of length one in X_m, define $P(C^n_j) = q_j$ and let $P(C^{n,n+1,n+2,\cdots,n+k}_{j_0,j_1,j_2,\cdots,j_k}) = \prod_{i=0}^{k} q_{j_i}$. It can then be extended to a shift invariant measure defined on the completion of the Borel sigma-algebra generated by cylinders. The shift space with this measure is called a Bernoulli shift.

The papers [1], [7] and [41] discuss about the structure of a solenoid, whereas [32] describes the structure of group of automorphisms of a solenoid. [2] and [19] calculate the entropy and the zeta function respectively, for an automorphism of a solenoid. The papers [16] and [17] consider the flows on higher dimensional solenoids. We use results from [16] to describe the sets of periodic points of some automorphisms on certain higher dimensional solenoids.

There are articles on counting the number of periodic points of a dynamical system; this forms a crucial part in defining the zeta function. The number of periodic points of any given period for some continuous homomorphisms of a one dimensional solenoid was discussed in [42]. Our description of periodic points of one dimensional solenoidal automorphisms is in accordance with this result.

2.2 One Dimensional Solenoids

As seen in the introduction, a topological group Σ is an $n-$dimensional solenoid if and only if its Pontryagin dual $\widehat{\Sigma}$ is (isomorphic to) a subgroup of the discrete additive group $\mathbb{Q}^n$ and contains $\mathbb{Z}^n$; so $\mathbb{Z}^n \leq \widehat{\Sigma} \leq \mathbb{Q}^n$. We now recall the definition of a one-dimensional solenoid as an inverse limit as this forms the central part of this chapter.

Definition 2.1. Let $A = (a_k)_{k=1}^{\infty}$ be a sequence of positive integers such that $a_k \geq 2$ for every $k \in \mathbb{N}$. The *solenoid* corresponding to the sequence A, denoted by Σ_A, is defined as $\Sigma_A = \{(x_k) \in (S^1)^{(\mathbb{N}_0)} : x_{k-1} = a_k x_k \,(mod\, 1)$ for every $k \in \mathbb{N}\}$.

In other words, the one dimensional solenoid Σ_A is the inverse limit, $\varprojlim_k (S^1, \gamma_k)$, where $\gamma_k : S^1 \to S^1$ is defined as $\gamma_k(x) = a_k x \,(mod\, 1)$.

Definition 2.1 has been taken from the article [41]. The dyadic solenoid described earlier in 1.2.1 is an example of the above definition, viewed as the inverse limit $\prod_{k=0}^{\infty} X_k$, where each X_k is S^1 and each f_k is given by $f_k(x) = 2x \ (mod\ 1)$.

The descriptions of a one dimensional solenoid as an inverse limit and as the dual group of a subgroup of $\mathbb{Q}$ are very closely related. The dual of a one dimensional solenoid Σ_A, where $A = (a_k)$ is isomorphic to the subgroup of $\mathbb{Q}$ generated by $\{\frac{1}{a_1 a_2 \cdots a_k} : k \in \mathbb{N}\}$. Thus, a one dimensional solenoid is a topological group whose dual is a subgroup of rationals and strictly contains integers : $\mathbb{Z} \subsetneq \widehat{\Sigma} \subseteq \mathbb{Q}$.

Now, a subgroup of $\mathbb{Q}$ is characterized by a sequence, called the *height sequence*, indexed by prime numbers and with values in $\mathbb{N}_0 \cup \{\infty\}$. We will now discuss about this sequence and establish a relation between the terms of this sequence and the integers a_k's. One may refer to [5] for more details about the structure of subgroups of $\mathbb{Q}$.

Let $S \subseteq \mathbb{Q}$ and $x \in S$. For a $p \in P$, the $p-$height of x with respect to S, denoted by $h_p^{(S)}(x)$ is defined as the largest non-negative integer n, if it exists, such that $\frac{x}{p^n} \in S$; otherwise, define $h_p^{(S)}(x) = \infty$. Thus, we have a sequence $(h_p^{(S)}(x))$, p ranging over prime numbers in the usual order, with values in $\mathbb{N}_0 \cup \{\infty\}$. We call such sequences as *height sequences*. If (u_p) and (v_p) are two height sequences such that $u_p = v_p$ for all but finitely many primes and $u_p = \infty \Leftrightarrow v_p = \infty$, then they are said to be equivalent. If S is a subgroup of $\mathbb{Q}$, then there is a unique height sequence (up to equivalence) associated to all non-zero elements of S. Also, two subgroups of $\mathbb{Q}$ are isomorphic if and only if their associated height sequences are equivalent.

Given a subgroup S of $\mathbb{Q}$, for every $p \in P$, we assign an element $n_p^{(S)}$ of $\mathbb{N}_0 \cup \{\infty\}$ as follows. Let $\mathbb{Q}_p$ and $\mathbb{Z}_p$ denote the field of $p-$adic numbers and the ring of $p-$adic integers respectively and $|u|_p$ denote the $p-$adic norm of $u \in \mathbb{Q}_p$. Then define $n_p^{(S)} = sup\{h_p^{(S)}(x) : x \in S \cap \mathbb{Z}_p^*\}$, where $\mathbb{Z}_p^*$ is the multiplicative group $\{x \in \mathbb{Z}_p : |x|_p = 1\}$. Now, the information whether $n_p^{(S)}$ is finite or not, for a given p, is going to play a crucial role in our discussion. So, we define $D_\infty^{(S)} = \{p \in P : n_p^{(S)} = \infty\}$. We will use the notations $n_p^{(S)}$ and $D_\infty^{(S)}$, as defined here, throughout this chapter. We now have the following relation between $(n_p^{(S)})$ and A, where S is the dual of Σ_A.

Proposition 2.2. *Let Σ_A be a one dimensional solenoid and $S = \widehat{\Sigma}_A$, where $A = (a_k)$. Let $p \in P$ and $n_p = n_p^{(S)}$. Then,*

1. *$p \in D_\infty^{(S)}$ if and only if for every $j \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ such that $p^j | a_1 a_2 \cdots a_k$.*
2. *If $p \notin D_\infty^{(S)}$, then n_p is the largest integer such that $p^{n_p} | a_1 a_2 \cdots a_k$ for some k.*

Proof. 1. Suppose $p \in D_\infty^{(S)}$. Since $n_p = \infty$, for any $j \in \mathbb{N}$, there exists an $x \in S \cap \mathbb{Z}_p^*$ with $h_p^{(S)}(x) > j$. Now, $x \in \mathbb{Z}_p^*$ implies that $x = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $(a,p) = (b,p) = 1$. Also, $h_p^{(S)}(x) > j$ implies that $\frac{x}{p^j} = \frac{a}{p^j b} \in S$. But, $S = \left\{\frac{i}{a_1 a_2 \cdots a_k} : i \in \mathbb{Z}, k \in \mathbb{N}\right\}$. Thus, $\frac{a}{p^j b} = \frac{i}{a_1 a_2 \cdots a_k}$ for some $i \in \mathbb{Z}$ and $k \in \mathbb{N}$. Then, we have $a a_1 a_2 \cdots a_k = i p^j b$ implying that $p^j | a_1 a_2 \cdots a_k$.

For the converse, let $j \in \mathbb{N}$. Then, there exists a $k \in \mathbb{N}$ such that $a_1a_2\cdots a_k = p^j i$ for some $i \in \mathbb{N}$. This implies that $\frac{1}{p^j} = \frac{i}{a_1a_2\cdots a_k} \in S$ and thus $h_p^{(S)}(1) \geq j$. Since j is chosen arbitrarily and $1 \in S \cap \mathbb{Z}_p^*$, we get, $n_p = \infty$ i.e., $p \in D_\infty^{(S)}$.

2. Suppose $p \notin D_\infty^{(S)}$. Then, $n_p = max\{h_p^{(S)}(x) : x \in S \cap \mathbb{Z}_p^*\}$. Say $n_p = h_p^{(S)}(x_0)$ for some $x_0 \in S \cap \mathbb{Z}_p^*$ i.e., $\frac{x_0}{p^{n_p}} \in S$. Let $x_0 = \frac{u_0}{v_0}$, for some $u_0, v_0 \in \mathbb{Z}$. Then $(u_0, p) = (v_0, p) = 1$. Now, $\frac{x_0}{p^{n_p}} \in S$ implies that $\frac{u_0}{p^{n_p}v_0} = \frac{i}{a_1a_2\cdots a_k}$ for some $i \in \mathbb{Z}$ and $k \in \mathbb{N}$ i.e, $u_0a_1a_2\cdots a_k = ip^{n_p}v_0$ and hence $p^{n_p}|a_1a_2\cdots a_k$.

   If possible, let $l > n_p$ such that $p^l|a_1a_2\cdots a_j$ for some j. But then, $a_1a_2\cdots a_j = p^l i'$ for some $i' \in \mathbb{N}$ implying that $\frac{1}{p^l} = \frac{i'}{a_1a_2\cdots a_j} \in S$ and thus $h_p^{(S)}(1) \geq l > n_p$ which is a contradiction. Therefore, n_p is the largest integer such that $p^{n_p}|a_1a_2\cdots a_k$ for some k.

□

The following corollary follows from the above proposition.

Corollary 2.3. *Let Σ_A, S and $D_\infty^{(S)}$ be defined as above. Then, for a $p \in P$, $p \in D_\infty^{(S)}$ if and only if p divides infinitely many a_k's.*

If f is an automorphism of a one dimensional solenoid Σ, then its dual is an automorphism of a subgroup of $\mathbb{Q}$ and thus, it is multiplication by a non-zero rational number, say $\frac{\alpha}{\beta}$ and for any $(x_k) \in \Sigma$, $f((x_k)) = (\frac{\alpha}{\beta}x_k(mod\, 1))$. We say that f is induced by $\frac{\alpha}{\beta}$. It is known that f is ergodic if and only if $\frac{\alpha}{\beta} \neq \pm 1$. Further, we can assume that $A = (\beta b_k)$, where each b_k is a positive integer coprime to β. In this case, we can write $f((x_k)) = (\alpha b_1 x_1, \alpha b_2 x_2, ...)$ for each $(x_k) \in \Sigma_{(\beta b_k)}$. See [53] for all these details about automorphisms.

We now state and prove our main results, namely the description of periodic points (Theorem 2.6) and the number of periodic points (Theorem 2.8) of an automorphism of a one dimensional solenoid. Before that, the following proposition describes the elements of a one dimensional solenoid with rational coordinates, in terms of the prime factors of a_k's and the succeeding proposition shows that a periodic point should have only rational coordinates.

Proposition 2.4. *Let Σ_A be a one dimensional solenoid where $A = (a_k)$ and $(x_k) = (\frac{u_k}{v_k}) \in \Sigma_A \cap \mathbb{Q}^{\mathbb{N}_0}$, where $u_k, v_k \in \mathbb{Z}$ such that $(u_k, v_k) = 1$. For a $p \in P$, denote $|v_k|_p = \frac{1}{p^{c_k}}$, for every $k \geq 0$ and let $|a_k|_p = \frac{1}{p^{d_k}}$, for every $k \geq 1$. If h is the least integer such that $c_h > 0$, then $c_k = c_h + d_{h+1} + d_{h+2} + \cdots + d_k$, for every $k > h$.*

Proof. It follows from the definition of a one dimensional solenoid that $\frac{u_h}{v_h} = a_{h+1}a_{h+2}\cdots \cdots a_k \frac{u_k}{v_k} + j$ for some $j \in \mathbb{Z}$. Since $c_h > 0$, it follows that $(u_h, p) = (u_k, p) = 1$. Then, we can find positive integers $a'_{h+1}, a'_{h+2}, \cdots, a'_k, v'_k$ and v'_h, each of which is coprime to p, such that

$$\frac{u_h}{p^{c_h}v_h^{'}} = \frac{p^{d_{h+1}+d_{h+2}+\cdots d_k}a_{h+1}^{'}a_{h+2}^{'}\cdots a_k^{'}u_k}{p^{c_k}v_k^{'}} + j$$
$$\Rightarrow p^{c_k}v_k^{'}u_h = p^{c_h+d_{h+1}+\cdots d_k}v_h^{'}a_{h+1}^{'}\cdots a_k^{'}u_k + jp^{c_k+c_h}v_k^{'}v_h^{'}$$
$$\Rightarrow p^{c_k}\left(v_k^{'}u_h - p^{c_h}jv_k^{'}v_h^{'}\right) = p^{c_h+d_{h+1}+\cdots d_k}v_h^{'}a_{h+1}^{'}\cdots a_k^{'}u_k$$

Now, since $c_h > 0$, p does not divide $\left(v_k^{'}u_h - p^{c_h}jv_k^{'}v_h^{'}\right)$. Thus, $c_k = c_h + d_{h+1} + \cdots d_k$ for every $k > h$. □

Proposition 2.5. *Let Σ_A be a one dimensional solenoid and $S = \widehat{\Sigma}_A$, where $A = (a_k)$. If (x_k) is periodic in (Σ_A, ϕ), where ϕ is an automorphism of Σ_A induced by $\frac{\alpha}{\beta}$, then $x_k \in \mathbb{Q}$ for every $k \in \mathbb{N}_0$. Further, for any $p \in D_\infty^{(S)}$, we have $|x_k|_p \leq 1$ for every $k \in \mathbb{N}_0$.*

Proof. Say $\phi^l((x_k)) = (x_k)$ for some $l \in \mathbb{N}$. Then, for any $k \in \mathbb{N}_0$, $\frac{\alpha^l}{\beta^l}x_k = x_k + j_k$ for some $j_k \in \mathbb{Z}$ and thus $x_k \in \mathbb{Q}$. Let $x_k = \frac{u_k}{v_k}$, where $u_k,\ v_k \in \mathbb{Z}$ and $(u_k, v_k) = 1$. Then, $(\alpha^l - \beta^l)u_k = \beta^l v_k j_k$ for every $k \geq 0$. For a prime number p, let us now denote $|v_k|_p = \frac{1}{p^{c_k}}$, for every $k \geq 0$ and $|a_k|_p = \frac{1}{p^{d_k}}$, for every $k \geq 1$.

Let $p \in D_\infty^{(S)}$. Then, by Corollary 2.3, $p|a_k$ for infinitely many k and thus $d_k > 0$ for infinitely many k. Suppose there exists an $r \in \mathbb{N}_0$ such that $p|v_r$. Then, $c_r > 0$ and $(\alpha^l - \beta^l)u_r = \beta^l v_r j_r$ implies that $p^{c_r}|(\alpha^l - \beta^l)$. Now from Proposition 2.4, $c_{r+k} = c_h + d_{h+1} + \cdots + d_r + d_{r+1} + \cdots + d_{r+k}$, where h is the least integer such that $c_h > 0$. Then, $h \leq r$ and $c_{r+k} = c_r + d_{r+1} + d_{r+2} + \cdots + d_{r+k}$.
Again, since $(\alpha^l - \beta^l)u_{r+k} = \alpha^l v_{r+k}j_{r+k}$ for every $k \geq 0$, we get $p^{c_r+d_{r+1}+\cdots+d_{r+k}}|(\alpha^l - \beta^l)$. This is a contradiction, as infinitely many of $d_{r+1}, d_{r+2}, \cdots$ are non-zero. Hence, $p \nmid v_k$ for any k. Therefore, $|x_k|_p \leq 1$ for every $k \geq 0$. □

In the following theorem about the set of periodic points of the dynamical system$(\Sigma_{(a_k)}, \frac{\alpha}{\beta})$, we assume that $a_k = \beta b_k$, where each b_k is a positive integer coprime to β. As noted already, there is no loss of generality in assuming this (see [53]).

Theorem 2.6. *Let ϕ be an automorphism of a one dimensional solenoid Σ_A induced by $\frac{\alpha}{\beta}$, where $A = (\beta b_k)$, each b_k being co-prime to β. For each $l \in \mathbb{N}$, define $U_l = \bigcap_{p \in P} \left(\frac{1}{p^{e_{p,l}}} \mathbb{Z}_p \cap \mathbb{Q} \cap S^1 \right)$, where $p^{e_{p,l}} = \frac{1}{|\alpha^l - \beta^l|_p}$. If $\gamma_{k,l} : U_l \to U_l$ is the map defined as $\gamma_{k,l}(x) = \beta b_k x (mod\ 1)$ for each $k \in \mathbb{N}$ and $l \in \mathbb{N}$, then $P(\phi) = \bigcup_{l=1}^{\infty} \varprojlim_k (U_l, \gamma_{k,l})$.*

Proof. Let (x_k) be a periodic point with a period l. Then, $x_k \in \mathbb{Q}$ for every $k \geq 0$; say $x_k = \frac{u_k}{v_k}$, where $u_k,\ v_k \in \mathbb{Z}$ such that $(u_k, v_k) = 1$. Again, for every prime p, let $|v_k|_p = \frac{1}{p^{c_k}}$, for every $k \geq 0$. Now, $\phi^l\left((x_k)\right) = (x_k)$ implies that $(\alpha^l - \beta^l)u_k = n^l v_k j_k$ for some $j_k \in \mathbb{Z}$. Since $p^{c_k} | v_k$, it follows that $p^{c_k} | (\alpha^l - \beta^l)$ and thus $c_k \leq e_{p,l}$. We can now write $x_k = \frac{1}{p^{e_{p,l}}} . \frac{p^{e_{p,l} - c_k} . u_k}{v_k'}$, for some $v_k' \in \mathbb{Z}$ such that $(v_k', p) = 1$. It then follows that $x_k \in \frac{1}{p^{e_{p,l}}} \mathbb{Z}_p$, because $|\frac{p^{e_{p,l} - c_k} . u_k}{v_k'}|_p \leq \frac{1}{p^{e_{p,l} - c_k}} \leq 1$. Since p was chosen arbitrarily, we conclude that $x_k \in U_l$, for every $k \geq 0$.

On the other hand, let $(x_k) \in \varprojlim_k (U_l, \gamma_{k,l})$ for some $l \in \mathbb{N}$. Say $x_k = \frac{u_k}{v_k}$, where $u_k,\ v_k \in \mathbb{Z}$ such that $(u_k, v_k) = 1$. Write $v_k = \prod_{p | v_k} p^{c_p}$, for some $c_p \in \mathbb{N}$. Then, for any $p | v_k$, $|x_k|_p = p^{c_p}$. Also, $|x_k|_p \leq p^{e_{p,l}}$, for any $p \in P$. Thus, $c_p \leq e_{p,l}$ and hence $v_k | (\alpha^l - \beta^l)$. Therefore, $\frac{\alpha^l - \beta^l}{v_k} \in \mathbb{Z}$, for every k. Then, $\phi^l((x_k)) - (x_k) = \left((\alpha^l - \beta^l) b_{k+1} b_{k+2} ... b_{k+l} x_{k+l}\right) = (0)$ implying that (x_k) is periodic. □

Remark 2.7. The set of periodic points of period l is equal to $\varprojlim_k (U_l, \gamma_{k,l})$. Here U_l is a subgroup of S^1 and the map $\gamma_{k,l}$ is the restriction of γ_k to U_l, where γ_k is a map on S^1 such that $\Sigma_{(nb_k)} = \varprojlim_k (S^1, \gamma_k)$.

The following theorem about the number of periodic points, which follows from the above description, is in accordance with a similar result in [42].

Theorem 2.8. *Let ϕ be an automorphism of a one dimensional solenoid Σ_A induced by $\frac{\alpha}{\beta}$ and for every $l \in \mathbb{N}$, let $e_{p,l} = \frac{1}{|\alpha^l - \beta^l|_p}$. Then the number of periodic points of ϕ with a period l is $\prod_{p \notin D_\infty^{(S)}} p^{e_{p,l}}$.*

Proof. Since $\alpha^l - \beta^l \in \mathbb{Z}$, $e_{p,l}$ is positive only for finitely many primes. Thus, there is a finite subset F of $P \setminus D_\infty^{(S)}$ such that for a $p \notin D_\infty^{(S)}$, $e_{p,l} \neq 0$ if and only if $p \in F$. Therefore $\prod\limits_{p \notin D_\infty^{(S)}} p^{e_{p,l}} = \prod\limits_{p \in F} p^{e_{p,l}}$.

We first claim that (x_k) is periodic with a period l if and only if for every $k \in \mathbb{N}_0$, $x_k = \frac{u_k}{v_k}$, where $u_k,\ v_k \in \mathbb{Z}$, $0 \leq u_k < v_k$ and $v_k = \prod\limits_{p \in F} p^{f_{p,k}}$ with $0 \leq f_{p,k} \leq e_{p,l}$.

If $\phi^l\left((x_k)\right) = (x_k)$, then for every $k \in \mathbb{N}_0$, $x_k \in \frac{1}{p^{e_{p,l}}}\mathbb{Z}_p \cap \mathbb{Q}$, for every $p \in P$. Let $x_k = \frac{u_k}{v_k}$ for some $u_k,\ v_k \in \mathbb{Z}$ such that $(u_k, v_k) = 1$. Now, $x_k \in \frac{1}{p^{e_{p,l}}}\mathbb{Z}_p$ implies that $|x_k|_p \leq p^{e_{p,l}}$, for every p. From Proposition 2.5, if $p \in D_\infty^{(S)}$, then $p \nmid v_k$. Also, for a prime p not in F, $e_{p,l} = 0$ implies that $p \nmid v_k$. Thus, the prime factorisation of $v_k = \prod\limits_{p \in F} p^{f_{p,k}}$ for some $0 \leq f_{p,k} \leq e_{p,l}$. Since $x_k \in [0,1)$, we conclude that $0 \leq u_k < v_k$.

Conversely, if $x_k = \frac{u_k}{v_k}$, where u_k and v_k satisfy the given conditions, then $|x_k|_p \leq 1$, for $p \notin F$ and $|x_k|_p \leq p^{f_{p,k}}$ for $p \in F$. In any case $|x_k|_p \leq p^{e_{p,l}}$ and thus $x_k \in U_l$. Hence the claim follows.

For a $p \in F$, let $|a_k|_p = \frac{1}{p^{d_k}}$, for every $k \in \mathbb{N}$. As this d_k depends on p we will denote $d_k = d_k^{(p)}$. Again, there are at most finitely many $k \in \mathbb{N}$ for which $d_k^{(p)} > 0$, as $F \subseteq P \setminus D_\infty^{(S)}$; let these positive integers be denoted by $d_{k_1}^{(p)}, d_{k_2}^{(p)}, ..., d_{k_{\alpha(p)}}^{(p)}$, where $\alpha(p) \in \mathbb{N}_0$. Further, assume that $k_1 < k_2 < ... < k_{\alpha(p)}$. Let $K = max\{k_{\alpha(p)} : p \in F\}$, if $k_{\alpha(p)} > 0$ for at least some $p \in F$; otherwise, define $K = 0$. Then, $d_k^{(p)} = 0$ for every $k > K$ and for every $p \in F$.

Let $(x_k) \in \Sigma_A$ be periodic; say $x_k = \frac{u_k}{v_k}$, where $u_k,\ v_k \in \mathbb{Z}$ such that $(u_k, v_k) = 1$. We have $x_K = \frac{u_K}{v_K}$, where $0 \leq u_K < v_K$ and $v_K = \prod\limits_{p \in F} p^{f_{p,K}}$ with $0 \leq f_{p,K} \leq e_{p,l}$. For any $k < K$, the value of x_k is uniquely determined by x_K, as $x_k = a_{k+1}a_{k+2}...a_K x_K \,(mod\ 1)$.

Now, let $k > K$. It follows from Proposition 2.4 that $v_k = v_K$.
Also, $x_K = a_{K+1}...a_k x_k \,(mod\ 1)$ i.e., $\frac{u_K}{v_K} = a_{K+1}...a_k \frac{u_k}{v_k} + j$ for some $j \in \mathbb{Z}$. By denoting $a_{K+1}...a_k = g_k$ and using the fact that $v_k = v_K$, we have $\frac{u_K}{v_K} = g_k \frac{u_k}{v_K} + j$. Since $d_k^{(p)} = 0$ for any $k > K$ and every $p \in F$, it follows that $p \nmid g_k$ for any $p \in F$. Having defined $\frac{u_K}{v_K}$, the distinct possible values for $\frac{u_k}{v_K}$ are $\frac{u_k}{v_K} = \frac{u_K}{g_k v_K} - \frac{j}{g_k}$, where $j \in \{0, 1, ..., g_k - 1\}$. Consider two such values, say $\frac{u_k^{(1)}}{v_K} = \frac{u_K}{g_k v_K} - \frac{j_1}{g_k}$ and $\frac{u_k^{(2)}}{v_K} = \frac{u_K}{g_k v_K} - \frac{j_2}{g_k}$ for some $j_1,\ j_2 \in \{0, 1, ...g_k - 1\}$. Then, $\frac{u_k^{(1)} - u_k^{(2)}}{v_K} = \frac{j_2 - j_1}{g_k}$ and thus $g_k\left(u_k^{(1)} - u_k^{(2)}\right) = v_K\,(j_2 - j_1)$. Now, if $j_1 \neq j_2$, then

$|j_2 - j_1| < g_k$ and thus $g_k \not| (j_2 - j_1)$. But then, there will be a prime p such that $p \mid g_k$ and $p \mid v_K$. On one hand, $p \mid g_k$ implies that $p \notin F$. On the other hand, $p \mid v_K$ implies that $p \mid m^l - n^l$ and also $p \notin D_\infty^{(S)}$, which means that $p \in F$ leading to a contradiction. Hence, $j_1 = j_2$ i.e., $u_k^{(1)} = u_k^{(2)}$. Thus, there is only one possible value for x_k. Thus, a periodic point (x_k) is uniquely determined by the coordinate x_K. Now, since $0 \leq f_{p,K} \leq e_{p,l}$, the possible values of x_K are $\frac{j}{\prod_{p\in F} p^{e_{p,l}}}$, where $0 \leq j < \prod_{p\in F} p^{e_{p,l}}$. Thus, the theorem follows. □

2.3 n-dimensional solenoids

We now extend our result about periodic points to some automorphisms of certain higher dimensional solenoids. Though this seems to be a small class, the reason for considering it is that the result follows immediately from what we have shown for one dimensional case. The higher dimensional solenoids that we are going to consider are isomorphic to products of one dimensional solenoids, as described in [16]. We mention here some notations, definitions and results from this paper that are needed to discuss our result.

For a positive integer $n > 1$, let $\pi^n : \mathbb{R}^n \to \mathbb{T}^n$ be the homomorphism defined as $\pi^n((x_1, x_2, ..., x_n)) = (x_1 \ (mod\ 1), x_2 \ (mod\ 1), ..., x_n \ (mod\ 1))$. Let $\overline{M} = (M_k)_{k=1}^{\infty} = (M_1, M_2, ...)$ be a sequence of $n \times n$ matrices with integer entries and non-zero determinant. Then, the $n-$dimensional solenoid $\sum_{\overline{M}}$ is defined as $\sum_{\overline{M}} = \{(\mathbf{x}_k) \in (\mathbb{T}^n)^{\mathbb{N}_0} : \pi^n(M_k\mathbf{x}_k) = \mathbf{x}_{k-1}$ for every $k \in \mathbb{N}\}$. In other words, $\sum_{\overline{M}} = \varprojlim_k (\mathbb{T}^n, \delta_k)$, where $\delta_k : \mathbb{T}^n \to \mathbb{T}^n$ is defined as $\delta_k(\mathbf{x}) = \pi^n(M_k\mathbf{x})$.

If ϕ is an automorphism of $\sum_{\overline{M}}$, then there is a matrix $L \in GL(n, \mathbb{Q})$ such that $\phi((\mathbf{x}_k)) = (\pi^n(L\mathbf{x}_k))$. We say that ϕ is induced by the matrix L. Now, consider n sequences of positive integers $A_1 = (a_1^1, a_2^1, ...)$, $A_2 = (a_1^2, a_2^2, ...)$, ... $A_n = (a_1^n, a_2^n, ...)$. Then define the sequence $\overline{M} = (M_k)$ of matrices as $M_k = diag[a_k^1, a_k^2, ..., a_k^n]$. These sequences of positive integers and matrices give rise to n one-dimensional solenoids and an $n-$dimensional solenoid. The following lemma from [16] gives a connection between these.

Lemma 2.9. *The map* $\eta : \prod_{i=1}^n \sum_{A_i} \to \sum_{\overline{M}}$ *given by* $\eta((x_k^1)_{k=1}^\infty, (x_k^2)_{k=1}^\infty, ..., (x_k^n)_{k=1}^\infty)$ $= ((x_1^1, x_1^2, ..., x_1^n), (x_2^1, x_2^2, ..., x_2^n), ..., (x_k^1, x_k^2, ..., x_k^n), ...)$ *is a topological isomorphism.*

We reserve these symbols A_i, $i = 1, 2, ..., n$ for the sequences of positive integers and M_k, $k \in \mathbb{N}$ for the corresponding diagonal matrices as described above. Now, let ϕ be an automorphism of $\sum_{\overline{M}}$ induced by a diagonal matrix, say $D = diag[\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}, ..., \frac{\alpha_n}{\beta_n}]$. Then for each i, $\frac{\alpha_i}{\beta_i}$ induces an automorphism of the one dimensional solenoid $\sum_{A_i}$, say ψ_i. Again, by following [53], we assume that $A_i = (\beta_i b_k^i)$ for some suitable sequence (b_k^i) of positive integers. Then, the map $\psi : ((x_k^1)_{k=1}^{\infty}, (x_k^2)_{k=1}^{\infty}, ..., (x_k^n)_{k=1}^{\infty}) \mapsto (\psi_1((x_k^1)_{k=1}^{\infty}), \psi_2((x_k^2)_{k=1}^{\infty}), ..., \psi_n((x_k^n)_{k=1}^{\infty}))$ is an automorphism of $\prod_{i=1}^{n} \sum_{A_i}$. It is easy to see that $\eta \circ \psi = \phi \circ \eta$. Thus, we have the following proposition.

Proposition 2.10. *$(\prod_{i=1}^{n} \sum_{A_i}, \psi)$ is conjugate to $(\sum_{\overline{M}}, \phi)$.*

We now state and prove a theorem regarding the periodic points.

Theorem 2.11. *For each $l \in \mathbb{N}$, define $V_l = \prod_{i=1}^{n} \left(\bigcap_{p \in P} \left(\frac{1}{p^{e_{p,l,i}}} \mathbb{Z}_p \cap \mathbb{Q} \cap S^1 \right) \right)$, where $p^{e_{p,l,i}} = \frac{1}{|\alpha_i^l - \beta_i^l|_p}$. If $\delta_{k,l} : V_l \to V_l$ is the map defined as $\delta_{k,l}(\mathbf{x}) = \pi^n(M_k \mathbf{x})$ for each $k \in \mathbb{N}$ and $l \in \mathbb{N}$, then $P(\phi) = \bigcup_{l=1}^{\infty} \varprojlim_k (V_l, \delta_{k,l})$, where ϕ is the automorphism of $\sum_{\overline{M}}$ induced by a diagonal matrix, defined as above.*

Proof. Let $P_l(\phi)$ and $P_l(\psi)$ be the sets of periodic points of ϕ and ψ respectively, with a period $l \in \mathbb{N}$. Since η is a conjugacy from $(\prod_{i=1}^{n} \sum_{A_i}, \psi)$ to $(\sum_{\overline{M}}, \phi)$, it follows that $P_l(\phi) = \eta(P_l(\psi))$. But $P_l(\psi) = \prod_{i=1}^{n} P_l(\psi_i)$, where ψ_i is the automorphism of $\sum_{A_i}$ induced by $\frac{\alpha_i}{\beta_i}$. Thus by Theorem 2.6,
$P_l(\psi) = \prod_{i=1}^{n} \left\{ (x_k^i)_{k=1}^{\infty} \in \sum_{A_i} : x_k^i \in \mathbb{Q} \text{ and } |x_k^i|_p \leq \frac{1}{p^{e_{p,l,i}}} \text{ for every } p \in P \right\}$.
Then, $P_l(\phi) = \left\{ ((x_k^1, x_k^2, \cdots x_k^n))_{k=1}^{\infty} \in \sum_{\overline{M}} : x_k^i \in \mathbb{Q} \text{ and } |x_k^i|_p \leq \frac{1}{p^{e_{p,l,i}}} \text{ for every } p \in P \right\} = \varprojlim_k (V_l, \delta_{k,l})$. Thus, $P(\phi) = \bigcup_{l=1}^{\infty} \varprojlim_k (V_l, \delta_{k,l})$. □

Remark 2.12. The set of periodic points of ϕ with a period l is equal to $\varprojlim_k (V_l, \delta_{k,l})$. Here, V_l is a subgroup of $\mathbb{T}^n$ and $\delta_{k,l}$ is the restriction of δ_k to V_l, where each δ_k is a map on $\mathbb{T}^n$ such that $\sum_{\overline{M}} = \varprojlim_k (\mathbb{T}^n, \delta_k)$.

2.4 Future Scope and Conclusion

The periodic points of an automorphism of a one dimensional solenoid are described in Section 2.2 of this chapter. There are papers that discuss the number of periodic points or in general the zeta function of such automorphisms, whereas this chapter gives an explicit description of these points. Then, we have extended this result to certain automorphisms of higher dimensional solenoids also. Hence, the present description in terms of inverse limits may be helpful in the open problem of periodic point characterization for the more general case of an $n-$dimensional arbitrary solenoid Σ, which is an abelian group whose Pontryagin dual $\widehat{\Sigma}$ is an (additive) subgroup of $\mathbb{Q}^n$ and contains $\mathbb{Z}^n$.

Chapter 3

Solenoids in terms of Adeles

* The work in this chapter is covered by the following article:

Faiz Imam, Sharan Gopal, *Periodic points of solenoidal automorphisms in terms of adeles*, (In Communication).

This chapter is about the characterization of periodic points of automorphisms of some solenoids, whose duals are subgroups of algebraic number fields. We use a concept from algebraic number theory namely adeles, to describe a solenoid and the periodic points of its automorphisms. The work done in this chapter is consistent with previous characterizations of other types of solenoids.

In the upcoming section, we define the key terms used in this chapter and discuss the prior characterizations of specific solenoids and the periodic points of their automorphisms. The main results are then stated and proved in the subsequent section. The first theorem describes a solenoid using adeles, and then the second theorem characterizes the set of periodic points of some automorphisms on it.

As mentioned in the introduction to the previous chapter, solenoids are considered by several people in literature. Regarding their algebraic structure, solenoids are discussed in [32], [33], [46], etc. Richard Miles [42] found the zeta function for solenoidal automorphisms independently using algebraic techniques. In this chapter, we study solenoids using adeles, a number theoretic concept. We rely on [30] and [52] for terminology and a variety of results about adeles.

3.1 Introduction

A finite extension of the field of rational numbers $\mathbb{Q}$ is defined as an algebraic number field $\mathbb{K}$. In other words, the dimension of $\mathbb{K}$ as a vector space over $\mathbb{Q}$ is finite.

Definition 3.1. A map $\rho : \mathbb{K} \to \mathbb{R}$ is defined as a *valuation* if it satisfies the following properties for all $x, y \in \mathbb{K}$:
(1) $\rho(x) \geq 0$ and $\rho(x) = 0 \Leftrightarrow x = 0$,
(2) $\rho(x.y) = \rho(x)\rho(y)$,
(3) $\rho(x + y) \leq \rho(x) + \rho(y)$.

If a valuation satisfies the stronger property $\rho(x + y) \leq max(\rho(x), \rho(y))$, for all x and y, then ρ is called a non-Archimedean valuation, and otherwise an Archimedean valuation. Two valuations ρ_1 and ρ_2 are defined to be equivalent if there exists a positive real number

s such that $\rho_1(r) = \rho_2(r)^s$ for every $r \in \mathbb{K}$. Moreover, if ρ_1 is equivalent to ρ_2, then either both of them are Archimedian or both of them are non-Archimedian. An equivalence class of valuations of $\mathbb{K}$ is called a place of $\mathbb{K}$ and the set of all places of $\mathbb{K}$ is denoted by $P^{\mathbb{K}}$. A place is called finite if it contains a non-Archimedian valuation and infinite otherwise. The collection of finite places will be denoted by $P_f^{\mathbb{K}}$ whereas $P_\infty^{\mathbb{K}}$ denotes the set of infinite places. It may be noted that $P_\infty^{\mathbb{K}}$ is a finite set for any $\mathbb{K}$.

For each $v \in P^{\mathbb{K}}$, $\mathbb{K}_v$ denotes the completion of $\mathbb{K}$ with respect to v and $\Re_v = \{x \in \mathbb{K}_v : |x|_v \leq 1\}$. $\Re_v$ is always a compact subset of $\mathbb{K}_v$ and when $v \in P_f^{\mathbb{K}}$, $\Re_v$ is an open, unique maximal compact subring of $\mathbb{K}_v$. We also consider $\Re_v^* := \{x \in \Re_v : |x|_v = 1\}$ in our discussion. The adele ring of $\mathbb{K}$, denoted by $\mathbb{K}_{\mathbb{A}}$ is then defined as $\mathbb{K}_{\mathbb{A}} = \left\{(x_v) \in \prod_{v \in \mathbb{P}^{\mathbb{K}}} \mathbb{K}_v / \ x_v \in \Re_v \text{ for all but finitely many } v \in P_f^{\mathbb{K}} \ \right\}$.

For the field $\mathbb{Q}$, every finite valuation is equivalent to a $p-$adic valuation where p is a rational prime and an infinite valuation is equivalent to the usual absolute value. Thus we can view $P^{\mathbb{Q}} = \{p \ : \ p \text{ is a rational prime}\} \cup \{\infty\}$, where $| \ |_\infty$ is the usual absolute value. Now, if $\mathbb{K}$ is an algebraic number field, then for each $p \in P^{\mathbb{Q}}$, there exists finitely many $v \in P^{\mathbb{K}}$ such that v lies above p (denoted as $v|_p$), i.e. v restricted to $\mathbb{Q}$ is equal to p.

If Σ is a one-dimensional solenoid, then we have $\mathbb{Z} \subsetneq \widehat{\Sigma} \subseteq \mathbb{Q}$. As discussed in the previous chapter, for any subgroup of rationals, there is a unique height sequence (up to the equivalence relation) corresponding to all the non-zero elements of the subgroup. Also, two corresponding height sequences are equivalent if and only if there exists an isomorphism between these two subgroups.

In [22][1], the solenoid Σ and the set of periodic points of an automorphism are described using $\mathbb{Q}_{\mathbb{A}}$ in the following way. In these statements, $h_p(x)$ denotes $h_p^{(S)}(x)$ for brevity.

Theorem 3.2 (Sharan, Raja; 2017). *Let Σ be a one-dimensional solenoid.*
Let $n_p := sup\{h_p(x) : x \in \widehat{\Sigma} \cap \mathbb{Z}_p^*\}$ and $D_\infty = \{p \in P : n_p = \infty\}$.
Then $\Sigma = \frac{\mathbb{Q}_{\mathbb{A}}}{i(\mathbb{Q})+L}$, where $L = \prod_{p \leq \infty} U_p$ and $U_p = \begin{cases} (0) & \text{if } p \in D_\infty \cup \{\infty\} \\ p^{n_p}\mathbb{Z}_p & \text{if } p \notin D_\infty \cup \{\infty\} \end{cases}$,
where $i : \mathbb{Q} \to \mathbb{Q}_{\mathbb{A}}$ is the diagonal inclusion given by a constant adele sequence.

[1]Both the theorems mentioned here are the corrigendum version of results from the article [22]. A simple observation that the collection D(defined in [22]) is in fact the set of all primes P, was inadvertently missed in that paper.

Theorem 3.3 (Sharan, Raja; 2017)**.** *Let Σ, L and D_∞ be defined as above. If α is an ergodic automorphism of Σ, then $P(\alpha) = \frac{i(\mathbb{Q})+\prod' \mathbb{Q}_p}{i(\mathbb{Q})+L}$, where $\prod' \mathbb{Q}_p := \{x \in \mathbb{Q}_\mathbb{A} : x_p = 0$ for every $p \in D_\infty \cup \{\infty\}$ and $x_p \in p^{n_p}\mathbb{Z}_p$ for all but finitely many p in $P \setminus D_\infty\}$.*

In this chapter, we consider solenoids of any arbitrary dimension n whose duals are additive subgroups of algebraic number fields. In the rest of this chapter, we denote a solenoid by Σ and an algebraic number field by $\mathbb{K}$. Now consider $\mathbb{K}_\mathbb{A}$, the ring of adeles of $\mathbb{K}$. For any $p \in P^\mathbb{Q}$, $\mathbb{Z}_p$ can be considered as a subring of $\mathbb{Q}_\mathbb{A}$ by identifying $c \in \mathbb{Z}_p$ with $x \in \mathbb{Q}_\mathbb{A}$, where $x_p = c$ and $x_q = 0$ for $q \neq p$. Similarly, $\prod_{v|_p} \mathbb{K}_v$ can be considered as a subring of $\mathbb{K}_\mathbb{A}$ by identifying $\prod_{v|_p} a_v \in \prod_{v|_p} \mathbb{K}_v$ with $b \in \mathbb{K}_\mathbb{A}$, where $b_v = a_v$ for $v|_p$ and $b_w = 0$ otherwise. From Lemma 6.101 of [30], it follows that there is an isomorphism (of topological groups) $\alpha : \mathbb{K}_\mathbb{A} \to (\mathbb{Q}_\mathbb{A})^n$ such that $\alpha\left(\prod_{v|_p} \Re_v\right)$ is equal to $(\mathbb{Z}_p)^n$ for almost all finite p. We further assume that $\alpha\left(\prod_{v|_p} \Re_v\right) = (\mathbb{Z}_p)^n$ for all the finite places.

We write $\alpha(x) = \left(x^{(1)}, x^{(2)}, \cdots, x^{(n)}\right) \in (\mathbb{Q}_\mathbb{A})^n$, for each $x \in \mathbb{K}_\mathbb{A}$ and write $x^{(j)} = \left(x_p^{(j)}\right)_{p \in P^\mathbb{Q}}$, for each $x^{(j)} \in \mathbb{Q}_\mathbb{A}$. For every $r \in \mathbb{K}$, we write $\beta(r) = \left(r^{(1)}, r^{(2)}, \cdots, r^{(n)}\right) \in \mathbb{Q}^n$ where $r = \sum_{i=1}^n r^{(i)}\alpha_i$ and $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is a $\mathbb{Q}$-basis for $\mathbb{K}$. Then, β is an isomorphism from $\mathbb{K}$ to $\mathbb{Q}^n$. We further assume that $\beta(\widehat{\Sigma})$ is a $\mathbb{Z}^n$-module and also $\mathbb{Z}^n \subseteq \beta(\widehat{\Sigma})$.

For $a = (a_v)_{v \in P^\mathbb{K}} \in \mathbb{K}_\mathbb{A}$, let $\bar{a}_p = \prod_{v|_p} a_v \in \prod_{v|_p} \mathbb{K}_v$, for every $p \in P^\mathbb{Q}$. We know that $\prod_{v|_p} \mathbb{K}_v$ is a vector space over $\mathbb{Q}_p$. It follows from Lemma 6.69 and 6.101 of [30] that the $\mathbb{Q}_p$-coordinates of $\bar{a}_p$ are same as $\left(a_p^{(1)}, a_p^{(2)}, \cdots, a_p^{(n)}\right)$, where $\left(a^{(1)}, a^{(2)}, \cdots, a^{(n)}\right) = \alpha(a)$ and $a^{(j)} = \left(a_q^{(j)}\right)_{q \in P^\mathbb{Q}}$.

3.2 Main Results

Consider the map $\eta : \mathbb{Q}_\mathbb{A} \to \widehat{\mathbb{Q}}$ given by $\eta(x) = \eta_x$, where $\eta_x : \mathbb{Q} \to S^1$ is defined as $\eta_x(r) = e^{-2\pi i x_\infty r} \cdot \prod\limits_{p<\infty} e^{2\pi i \{x_p r\}_p}$ and $x = (x_p)_{p \in P^\mathbb{Q}}$. It is known that this map η is a surjective homomorphism. Now, consider the map $\xi : (\mathbb{Q}_\mathbb{A})^n \to \widehat{\mathbb{Q}^n}$ given by $\xi(\bar{x}) = \xi_{\bar{x}}$, where $\xi_{\bar{x}} : \mathbb{Q}^n \to S^1$ is defined as $\xi_{\bar{x}}(\bar{r}) = \eta_{x^{(1)}}(r^{(1)}) \cdot \eta_{x^{(2)}}(r^{(2)}) \cdots \eta_{x^{(n)}}(r^{(n)})$, where $\bar{x} = (x^{(1)}, x^{(2)}, \cdots, x^{(n)}) \in (\mathbb{Q}_\mathbb{A})^n$ and $\bar{r} = (r^{(1)}, r^{(2)}, \cdots, r^{(n)}) \in \mathbb{Q}^n$. Observe that ξ is a homomorphism. Note that $\xi_{(\bar{x})}(\bar{r}) = e^{-2\pi i \sum_{j=1}^{n} x_\infty^{(j)} r^{(j)}} \cdot \prod_{p<\infty} e^{2\pi i \sum_{j=1}^{n} \{x_p^{(j)} r^{(j)}\}_p}$. Now, define $\omega : \mathbb{K}_\mathbb{A} \to \widehat{\mathbb{Q}^n}$ as $\omega(a) = \omega_a$, where $\omega_a = \xi \circ \alpha(a)$; in other words, if $a \in \mathbb{K}_\mathbb{A}$ and $\alpha(a) = (a^{(1)}, a^{(2)}, \cdots, a^{(n)})$, then $\omega_a(\bar{r}) = e^{-2\pi i \sum_{j=1}^{n} a_\infty^{(j)} r^{(j)}} \cdot \prod_{p<\infty} e^{2\pi i \sum_{j=1}^{n} \{a_p^{(j)} r^{(j)}\}_p}$. Since ξ and α are homomorphisms, ω is also a homomorphism. Finally, define $\psi : \mathbb{K}_\mathbb{A} \to \widehat{\mathbb{K}}$ as $\psi(a) = \psi_a$, for every $a \in \mathbb{K}_\mathbb{A}$, where $\psi_a : \mathbb{K} \to S^1$ is given by $\psi_a(r) = \omega_a \circ \beta(r)$, for every $r \in \mathbb{K}$. Note that if $\alpha(a) = (a^{(1)}, a^{(2)}, \cdots, a^{(n)}) \in (\mathbb{Q}_\mathbb{A})^n$ and $\beta(r) = (r^{(1)}, r^{(2)}, \cdots, r^{(n)}) \in \mathbb{Q}^n$,

$$\begin{aligned}
\text{then } \psi_a(r) &= w_a \circ \beta(r) \\
&= w_a(r^{(1)}, r^{(2)}, \ldots, r^{(n)}) \\
&= \xi_{\alpha(a)}(r^{(1)}, r^{(2)}, \ldots, r^{(n)}) \\
&= \xi_{(a^{(1)}, a^{(2)}, \ldots, a^{(n)})}(r^{(1)}, r^{(2)}, \ldots, r^{(n)}) \\
&= e^{-2\pi i \sum_{j=1}^{n} a_\infty^{(j)} r^{(j)}} \cdot \prod_{p<\infty} e^{2\pi i \sum_{j=1}^{n} \{a_p^{(j)} r^{(j)}\}_p}.
\end{aligned}$$

Note that ψ is a homomorphism.

Proposition 3.4. *ψ is a surjective homomorphism that is trivial on $i(\mathbb{K})$.*

Proof : Let $\chi \in \widehat{\mathbb{K}}$. For each $1 \leq j \leq n$, define $\eta^{(j)} : \mathbb{Q} \to S^1$ as $\eta^{(j)}(b) = \chi\Big(\beta^{-1}(0, \cdots, b, \cdots, 0)\Big)$ where b is in j^{th} position. Then $\eta^{(j)} \in \widehat{\mathbb{Q}}$ and hence there exists $x^{(j)} = (x_p^{(j)}) \in \mathbb{Q}_\mathbb{A}$, such that $\eta^{(j)} = \eta_{x^{(j)}}$.

Then, for every $r \in \mathbb{K}$,

$$\begin{aligned}
&\psi\Big(\alpha^{-1}\big(x^{(1)}, x^{(2)}, \cdots, x^{(n)}\big)\Big)\Big(r\Big) \\
&= \xi_{(x^{(1)}, x^{(2)}, \cdots, x^{(n)})}\Big(r^{(1)}, r^{(2)}, \cdots, r^{(n)}\Big) \\
&= \eta_{x^{(1)}}\Big(r^{(1)}\Big).\eta_{x^{(2)}}\Big(r^{(2)}\Big) \cdots \eta_{x^{(n)}}\Big(r^{(n)}\Big) \\
&= \chi\Big(\beta^{-1}\Big(r^{(1)}, 0, 0, \cdots, 0\Big)\Big).\chi\Big(\beta^{-1}\Big(0, r^{(2)}, 0, 0, \cdots, 0\Big)\Big) \cdots \chi\Big(\beta^{-1}\Big(0, 0, \cdots, 0, r^{(n)}\Big)\Big) \\
&= \chi\Big(\beta^{-1}\Big(r^{(1)}, 0, 0, \cdots, 0\Big) + \beta^{-1}\Big(0, r^{(2)}, 0, 0, \cdots, 0\Big) + \cdots + \beta^{-1}\Big(0, 0, \cdots, 0, r^{(n)}\Big)\Big) \\
&= \chi\Big(\beta^{-1}\Big(r^{(1)}, r^{(2)}, \cdots, r^{(n)}\Big)\Big) \\
&= \chi\Big(r\Big).
\end{aligned}$$

Hence, ψ is surjective.

We now claim that ψ is trivial on $i(\mathbb{K})$. If $a = (a_v) \in i(\mathbb{K})$, then there exists an $x \in \mathbb{K}$ such that $a_v = x$ for every $v \in P^{\mathbb{K}}$. Now, $\bar{a}_p = (x, x, \cdots, x) \in \prod_{v|p} \mathbb{K}_v$ and the $\mathbb{Q}_p-$coordinates of $\bar{a}_p$ are $a_p^{(1)}, a_p^{(2)}, \cdots, a_p^{(n)}$. Since $\bar{a}_p$ is same for all values of p, each sequence $(a_p^{(j)})_{p\in P^{\mathbb{Q}}}$ is a constant sequence and thus $a^{(j)} \in i(\mathbb{Q})$. Say $(a_p^{(1)}) = (\delta^{(1)})$, $(a_p^{(2)}) = (\delta^{(2)})$ $, \cdots,$ $(a_p^{(n)}) = (\delta^{(n)})$, where $\delta^{(j)} \in \mathbb{Q}$. Then, for any $r \in \mathbb{K}$ with $\beta(r) = (r^{(1)}, r^{(2)}, \cdots, r^{(n)}) \in \mathbb{Q}^n$, we have $\psi_{(a)}(r) = \xi_{((\delta^{(1)}),(\delta^{(2)}),\cdots,(\delta^{(n)}))}(r^{(1)}, r^{(2)}, \cdots$ $\cdots, r^{(n)}) = \prod_{j=1}^{n} \eta_{(\delta^{(j)})}(r^{(j)})$. But, $\eta_{(\delta^{(j)})}(r^{(j)}) = e^{-2\pi i\delta^{(j)}r^{(j)}} \cdot \prod_{p<\infty} e^{2\pi i\{\delta^{(j)}r^{(j)}\}} = 1,$ for every j. Therefore, $\psi_{(a)} = 1,$ for every $a \in i(\mathbb{K})$. Hence the claim. $\square$

Since $\widehat{\Sigma}$ is a subgroup of $\mathbb{K}$, we have $\widehat{\widehat{\Sigma}} = \widehat{\mathbb{K}}/ann(\widehat{\Sigma})$ and thus, $\Sigma = \widehat{\mathbb{K}}/ann(\widehat{\Sigma})$ (it is well known that if H is a subgroup of G, then $\widehat{H} = \widehat{G}/ann(H)$, where $ann(H)$ is the annihilator of H defined by $ann(H) = \{g \in \widehat{G} \ / \ g(h) = 0, for\ all\ h \in H\}$). Define $\psi' : \mathbb{K}_{\mathbb{A}} \to \Sigma$ as $\psi' = \pi \circ \psi$, where $\pi : \widehat{\mathbb{K}} \to \Sigma$ is the quotient map. Since π and ψ are surjective, ψ' is surjective. We will now find Ker ψ' and thus obtain Σ as a quotient of $\mathbb{K}_{\mathbb{A}}$.

For every $p \in P_f^{\mathbb{Q}}$ and $1 \le j \le n$, define $m_p^{(j)} = \sup\{|r^{(j)}|_p : r \in \widehat{\Sigma}\}$, where $\beta(r) = \left(r^{(1)}, r^{(2)}, \cdots, r^{(n)}\right)$. Since $\mathbb{Z}^n \subset \beta(\widehat{\Sigma})$, we have $r = \beta^{-1}(0, \cdots, p, \cdots, 0) \in \widehat{\Sigma}$ and thus $|r^{(j)}|_p = |p|_p = \frac{1}{p} \neq 0$ concluding that $m_p^{(j)} \neq 0$. Let $n_p^{(j)} = \begin{cases} \frac{1}{m_p^{(j)}} & \text{if } m_p^{(j)} < \infty \\ 0 & \text{if } m_p^{(j)} = \infty \end{cases}$ and

$D = \{p \in P_f^{\mathbb{Q}} : m_p^{(j)} = \infty \text{ for every } 1 \le j \le n\}$. Now, define a subgroup U_p of $\prod_{v|p} \mathbb{K}_v$ for every $p \in P^{\mathbb{Q}}$ as $U_p = \begin{cases} (0) & \text{for } p \in D \cup \{\infty\} \\ \{x \in \prod_{v|p} \mathbb{K}_v : |x^{(j)}|_p \le n_p^{(j)} \text{ for every } j\} & \text{for } p \notin D \cup \{\infty\} \end{cases}$, where $x^{(1)}, x^{(2)}, \cdots, x^{(n)}$ are $\mathbb{Q}_p-$coordinates of x. Finally, define $V = i(\mathbb{K}) + \prod_{p\in P^{\mathbb{Q}}} U_p$. Note that D is similar to the set D_∞ of [22].

Theorem 3.5. *Σ is isomorphic to $\mathbb{K}_{\mathbb{A}}/V$.*

Proof : We prove that Ker $\psi' = V$ so that the required result follows. If $a \in V$, then $\psi'(a) = \pi(\psi(a))$; so we need to prove that $\psi(a) \in ann(\widehat{\Sigma})$, that is $\psi(a)(r) = 1$ for every $r \in \widehat{\Sigma}$. Now, $a \in V$ implies that $a = (\delta) + l$ where $(\delta) \in i(\mathbb{K})$ and $l \in \prod U_p$. So, we have the equality $\psi(a) = \psi(\delta).\psi(l) = \psi(l)$. Further, if $p \in D \cup \{\infty\}$, then $\bar{l}_p = 0$, where $\bar{l}_p = \prod_{v|p} l_v \in \prod_{v|p} \mathbb{K}_v$ which implies that $l_p^{(j)} = 0$ for all $1 \le j \le n$. So, $\psi(l)(r) = \prod_{p\notin D\cup\{\infty\}} e^{2\pi i \sum_{j=1}^{n}\{l_p^{(j)}r^{(j)}\}_p}$, for all $r \in \widehat{\Sigma}$.

Even for a $p \notin D \cup \{\infty\}$, $\bar{l_p} \in U_p$

$\Rightarrow |l_p^{(j)}|_p \leq n_p^{(j)} = \dfrac{1}{m_p^{(j)}}$ *or* $|l_p^{(j)}|_p = 0$, for every $1 \leq j \leq n$

$\Rightarrow |l_p^{(j)} r^{(j)}|_p \leq |l_p^{(j)}||r^{(j)}|_p \leq \dfrac{1}{m_p^{(j)}}.m_p^{(j)} = 1$ *or* $|l_p^{(j)} r^{(j)}|_p = 0$, for every $r \in \widehat{\Sigma}$ *and* $1 \leq j \leq n$.

$\Rightarrow |l_p^{(j)} r^{(j)}|_p \leq 1$, for all $r \in \widehat{\Sigma}$ *and* $1 \leq j \leq n$.

$\Rightarrow \{l_p^{(j)} r^{(j)}\}_p = 0$, for all $r \in \widehat{\Sigma}$ *and* $1 \leq j \leq n$.

$\Rightarrow e^{-2\pi i \sum_{j=1}^{n} \{l_p^{(j)} r^{(j)}\}_p} = 1$

Hence, $\psi(a)(r) = \psi(l)(r) = 1$, for every $r \in \widehat{\Sigma}$. Therefore, $V \subset Ker(\psi')$.

For the converse, let $a \in Ker(\psi')$. Following Proposition 6 of Chapter V in [52], we can write $a = (\delta) + l$, where $(\delta) \in i(\mathbb{K})$, $l_v \in \Re_v$ for every $v \in P^{\mathbb{K}}$ and $l_\infty^{(j)} \in [0,1)$ for all $1 \leq j \leq n$. So, it is enough to prove that $l \in \prod U_p$, or equivalently $\bar{l_p} \in U_p$, where $\bar{l_p} = \prod_{v|p} l_v \in \prod_{v|p} \mathbb{K}_v$. Again, $\psi(a) = \psi(l)$. Note that for any $p < \infty$, $\bar{l_p} \in \prod_{v|p} \Re_v$ implies that $l_p^{(j)} \in \mathbb{Z}_p$ for all $1 \leq j \leq n$ and thus $\{l_p^{(j)}\}_p = 0$ for all $1 \leq j \leq n$. Choose $r \in \mathbb{K}$ for each $1 \leq j \leq n$, such that $r^{(j)} = 1$ and $r^{(i)} = 0$ for $i \neq j$. Since $\mathbb{Z}^n \subset \beta(\widehat{\Sigma})$, it follows that $r \in \widehat{\Sigma}$ and $\psi(l)(r) = e^{-2\pi i l_\infty^{(j)}}$. Now, since $\psi(a) \in ann(\widehat{\Sigma})$, we have $e^{-2\pi i l_\infty^{(j)}} = 1$ implying that $l_\infty^{(j)} = 0$. Hence, $\psi(l)(r) = \prod_{p<\infty} e^{-2\pi i \sum_{j=1}^{n} \{l_p^{(j)} r^{(j)}\}_p}$, for all $1 \leq j \leq n$ and for every $r \in \widehat{\Sigma}$.

Fix a $j \in \{1, 2, \cdots, n\}$. Suppose $m_p^{(j)} = \infty$. Therefore, for arbitrary $k \in \mathbb{N}$, we can choose $r \in \widehat{\Sigma}$ such that $|r^{(j)}|_p > p^k$. Choose $\delta \in \mathbb{K}$ such that $\delta = (\delta^{(1)}, \delta^{(2)}, \cdots, \delta^{(n)}) \in \mathbb{Z}^n$, where $|\delta^{(j)}|_p = 1$, $|\delta^{(j)} r^{(j)}|_q \leq 1$ for all $q \neq p$ and $\delta^{(m)} r^{(m)} \in \mathbb{Z}$ for all $m \neq j$. Since $\beta(\widehat{\Sigma})$ is a $\mathbb{Z}^n-$module, the element $t = \beta^{-1}(r^{(1)}\delta^{(1)}, r^{(2)}\delta^{(2)}, \cdots, r^{(n)}\delta^{(n)}) \in \widehat{\Sigma}$. Say $\beta(t) = (t^{(1)}, t^{(2)}, \cdots, t^{(n)})$. Then, $|t^{(j)}|_p = |r^{(j)}|_p > p^k$, $|t^{(j)}|_q \leq 1$ for every $q \neq p$ and $|t^{(m)}|_q = |r^{(m)}\delta^{(m)}|_q \leq 1$ for every q and for every $m \neq j$. Therefore, $\psi(l)(t) = e^{-2\pi i \{l_p^{(j)} r^{(j)}\}_p}$. So, $\psi(l)(t) = 1$, implies $\{l_p^{(j)} r^{(j)}\}_p = 0$, giving us the inequality $|l_p^{(j)}|_p \leq \frac{1}{|r^{(j)}|_p} < \frac{1}{p^k}$. Since k is arbitrary, it follows that $l_p^{(j)} = 0$.

Now, consider the case $m_p^{(j)} < \infty$. If possible, suppose $|l_p^{(j)}|_p > n_p^{(j)} = \frac{1}{m_p^{(j)}}$. So we have $m_p^{(j)} > \frac{1}{|l_p^{(j)}|_p}$ and thus there exists $r \in \widehat{\Sigma}$ such that $\frac{1}{|l_p^{(j)}|_p} < |r^{(j)}|_p \leq m_p^{(j)}$ which gives $|l^{(j)}r^{(j)}|_p > 1$. As done above, choose $\delta \in \mathbb{K}$ such that $|\delta^{(j)}r^{(j)}|_p = |r^{(j)}|_p$, also $|\delta^{(j)}r^{(j)}|_q \leq 1$ for every $q \neq p$ and $\delta^{(m)}r^{(m)} \in \mathbb{Z}$ for every $m \neq j$. Then, $t = \beta^{-1}\Big(\delta^{(1)}r^{(1)}, \delta^{(2)}r^{(2)}, \cdots \cdots, \delta^{(n)}r^{(n)}\Big) \in \widehat{\Sigma}$ with $|l_p^{(j)}t^{(j)}|_p = |l_p^{(j)}\delta^{(j)}r^{(j)}|_p = |l_p^{(j)}r^{(j)}|_p > 1$, $|l_q^{(j)}t^{(j)}|_q \leq 1$ for every $q \neq p$ and $|l_q^{(m)}t^{(m)}|_q = |l_q^{(m)}\delta^{(m)}r^{(m)}|_q \leq 1$ for every q and for every $m \neq j$. This implies that $\psi(l)(t) = e^{-2\pi i \{l_p^{(j)}t^{(j)}\}_p} \neq 1$ giving us a contradiction. Hence $|l_p^{(j)}|_p \leq \eta_p^{(j)}$.

Finally, for any $p \in D$, $m_p^{(j)} = \infty$ for every j implies that $l_p^{(j)} = 0$ and thus $\bar{l}_p \in U_p$. Even for $p \notin D$, either $l_p^{(j)} = 0$ or $|l_p^{(j)}|_p \leq \eta_p^{(j)}$ and again $\bar{l}_p \in U_p$. So, for every prime p, $\bar{l}_p \in U_p$ and thus $a \in V$. □

We now describe the periodic points of some automorphisms of Σ. Fix an element $d = (d^{(1)}, d^{(2)}, \cdots, d^{(n)}) \in \mathbb{Q}^n$ such that for every j, $|d^{(j)}| \neq 0$ and $|d^{(j)}|_p = 1$ for $p \notin D \cup \{\infty\}$. Define a map $M_d : \mathbb{K}_{\mathbb{A}} \to \mathbb{K}_{\mathbb{A}}$ as $\alpha^{-1} \circ m_d \circ \alpha$, where $m_d : (\mathbb{Q}_{\mathbb{A}})^n \to (\mathbb{Q}_{\mathbb{A}})^n$ is given by $m_d(a^{(1)}, a^{(2)}, \cdots, a^{(n)}) = ((d^{(1)}a_p^{(1)})_p, (d^{(2)}a_p^{(2)})_p, \cdots, (d^{(n)}a_p^{(n)})_p)$. Note that $(d^{(j)}a^{(j)}p)_p = (d^{(j)}a_\infty^{(j)}, d^{(j)}a_2^{(j)}, d^{(j)}a_3^{(j)}, \cdots)$. It can be observed easily that m_d is an isomorphism and thus M_d is an automorphism of $\mathbb{K}_{\mathbb{A}}$.

Proposition 3.6. *$M_d(V) = V$.*

Proof : Let $(\delta) + l \in V$, where $(\delta) \in i(\mathbb{K})$ and $l = \big(\bar{l_p}\big) \in \prod U_p$. Then, $\alpha\big((\delta)\big) = \big((\delta^{(1)}), (\delta^{(2)}), \cdots, (\delta^{(n)})\big)$, where $\delta^{(j)} \in \mathbb{Q}$. Then, $M_d\big((\delta)\big) = \alpha^{-1} \circ m_d \circ \alpha\big((\delta)\big) = \alpha^{-1} \circ m_d\left((\delta^{(1)}), (\delta^{(2)}), \cdots, (\delta^{(n)})\right) = \alpha^{-1}\left((d^{(1)}\delta^{(1)}), (d^{(2)}\delta^{(2)}), \cdots, (d^{(n)}\delta^{(n)})\right) \in i(\mathbb{K})$.

Now, $M_d(l) = \alpha^{-1} \circ m_d \circ \alpha\big((l)\big) = \alpha^{-1} \circ m_d(l^{(1)}, l^{(2)}, \cdots, l^{(n)}) = \alpha^{-1}((d^{(1)}l_p^{(1)}), (d^{(2)}l_p^{(2)}), , \cdots \cdots, (d^{(n)}l_p^{(n)}))$. For $p \in D \cup \{\infty\}$, we have $l_p^{(1)} = l_p^{(2)} = \cdots = l_p^{(n)} = 0$ and thus $d^{(1)}l_p^{(1)} = d^{(2)}l_p^{(2)} = \cdots = d^{(n)}l_p^{(n)} = 0$. On the other hand, $p \notin D \cup \{\infty\}$ implies that $|l_p^{(j)}|_p \leq n_p^{(j)}$; so, $|d^{(j)}l_p^{(j)}|_p = |l_p^{(j)}|_p \leq n_p^{(j)}$. Therefore, $M_d(l) \in \prod U_p$ and thus $M_d(V) \subset V$. Similarly, $M_d^{-1}\left((\delta) + l\right) = \alpha^{-1}((\frac{\delta^{(1)}}{d^{(1)}}), (\frac{\delta^{(2)}}{d^{(2)}}), \cdots, (\frac{\delta^{(n)}}{d^{(n)}})_p) + \alpha^{-1}((\frac{l_p^{(1)}}{d^{(1)}}), (\frac{l_p^{(2)}}{d^{(2)}}), \cdots, (\frac{l_p^{(n)}}{d^{(n)}})_p) \in i(\mathbb{K}) + \prod U_p$ for every $(\delta) \in i(\mathbb{K})$ and $l = \big(\bar{l_p}\big) \in \prod U_p$. □

Since M_d is an automorphism of $\mathbb{K}_{\mathbb{A}}$ and V is an M_d-invariant subgroup of $\mathbb{K}_{\mathbb{A}}$, M_d induces an automorphism of Σ, say $\overline{M_d}$.

Theorem 3.7. *The set of periodic points of $\overline{M_d}$, where $d^{(j)} \neq \pm 1$ for every $1 \leq j \leq n$, is given by $P(\overline{M_d}) = \frac{i(\mathbb{K})+\prod' \mathbb{K}_v}{V}$, where $\prod' \mathbb{K}_v = \Big\{x \in \mathbb{K}_{\mathbb{A}} : for\ every\, 1 \leq j \leq n, x_p^{(j)} = 0$ whenever $p \in D \cup \{\infty\}$ and $|x_p^{(j)}|_p \leq n_p^{(j)}$ for all but finitely many $p \notin D \cup \{\infty\}\Big\}$.*

Proof : Let $\bar{a} = a + V$ be periodic in Σ. Then, $m_d^k \circ \alpha(a) - \alpha(a) \in \alpha(V)$. Say, $m_d^k \circ \alpha(a) - \alpha(a) = \alpha((\delta) + l)$, where $\delta \in \mathbb{K}$ and $\bar{l_p} \in \prod U_p$. Now, $(\delta) \in i(\mathbb{K})$ implies that $\alpha((\delta)) = (\delta^{(1)}, \delta^{(2)}, \cdots, \delta^{(n)})$, where $(\delta^{(j)}) \in i(\mathbb{Q})$. Also, $\bar{l_p} \in U_p$ implies that for every $1 \leq j \leq n$, $l_p^{(j)} = 0$ for $p \in D \cup \{\infty\}$ and $|l_p^{(j)}|_p \leq n_p^{(j)}$ for $p \notin D \cup \{\infty\}$. Now, for any $p \in P^{\mathbb{Q}}, ({d^{(j)}}^k a_p^{(j)})_p - (a_p^{(j)})_p = (\delta^{(j)})_p + (l_p^{(j)})_p$ for all $1 \leq j \leq n$ and for every $k \in \mathbb{N}$ implies that $({d^{(j)}}^k - 1)a_p^{(j)} = \delta^{(j)} + l_p^{(j)}$ and thus $a_p^{(j)} = \frac{\delta^{(j)}}{{d^{(j)}}^k - 1} + \frac{l_p^{(j)}}{{d^{(j)}}^k - 1}$. Now, for all $1 \leq j \leq n$, $|\frac{l_p^{(j)}}{{d^{(j)}}^k - 1}|_p \leq |l_p^{(j)}|$ for all but finitely many primes p, since ${d^{(j)}}^k - 1 \in \mathbb{Q}$. Therefore, $\alpha(a) = \big((a_p^{(1)}), (a_p^{(2)}), \cdots, (a_p^{(n)})\big) = \big((\frac{\delta^{(1)}}{{d^{(1)}}^k - 1}), (\frac{\delta^{(2)}}{{d^{(2)}}^k - 1}), \cdots, (\frac{\delta^{(n)}}{{d^{(n)}}^k - 1})\big) + \big((\frac{l_p^{(1)}}{{d^{(1)}}^k - 1}), (\frac{l_p^{(2)}}{{d^{(2)}}^k - 1}), \cdots \cdots, (\frac{l_p^{(n)}}{{d^{(n)}}^k - 1})\big) \in \alpha(i(\mathbb{K})) + \alpha(\prod{}' \mathbb{K}_v)$ and thus $a \in i(\mathbb{K}) + \prod' \mathbb{K}_v$.

For the converse, let $\pi(x) \in \frac{i(\mathbb{K})+\prod' \mathbb{K}_v}{V}$; then, $x = (\delta) + l$, where $(\delta) \in i(\mathbb{K})$ and $l \in \prod' \mathbb{K}_v$. To prove that $\pi(x)$ is periodic, we need to find a $k \in \mathbb{N}$ such that $m_d^k \circ \alpha(x) - \alpha(x) \in \alpha(V)$. Note that, for any $k \in \mathbb{N}$, $(({d^{(1)}}^k - 1)x^{(1)}, ({d^{(2)}}^k - 1)x^{(2)}, \cdots, ({d^{(n)}}^k - 1)x^{(n)}) = \big(((({d^{(1)}}^k - 1)\delta^{(1)}), (({d^{(2)}}^k - 1)\delta^{(2)}), \cdots, (({d^{(n)}}^k - 1)\delta^{(n)})\big) + \big((({d^{(1)}}^k - 1)l_p^{(1)})_p, (({d^{(2)}}^k - 1)l_p^{(2)})_p, \cdots, (({d^{(n)}}^k - 1)l_p^{(n)})_p\big)$. So, it remains to prove that $\big((({d^{(1)}}^k - 1)l_p^{(1)})_p, (({d^{(2)}}^k - 1)l_p^{(2)})_p, \cdots, (({d^{(n)}}^k - 1)l_p^{(n)})_p\big) \in \alpha(\prod U_p)$ for some $k \in \mathbb{N}$. However, if $p \in D \cup \{\infty\}$, then $({d^{(j)}}^k - 1)l_p^{(j)} = 0$ for all $1 \leq j \leq n$ and for any $k \in \mathbb{N}$. Now, let $G = \{p \notin D \cup \{\infty\} |\ \ |l_p^{(j)}|_p > n_p^{(j)}$ for some $1 \leq j \leq n\}$. Then, G is a finite set; say $G = \{p_1, p_2, \cdots, p_M\}$. Also, $|{d^{(j)}}^k - 1|_p \leq max\{|{d^{(j)}}^k|_p, |1|_p\} \leq 1$ for every $k \in \mathbb{N}$. So, if $p \notin G$, then $|({d^{(j)}}^k - 1)l_p^{(j)}|_p \leq |l_p^{(j)}|_p \leq n_p^{(j)}$ for every $1 \leq j \leq n$. So, it remains to find a $k \in \mathbb{N}$ such that $|({d^{(j)}}^k - 1)l_p^{(j)}|_p \leq n_p^{(j)}$ for all $1 \leq j \leq n$ and for every $p \in G$.

For any $p \in G$, since $p \notin D$, we have $|d^{(j)}|_p = 1$ and thus $d^{(j)}$ is a unit in $\mathbb{Z}_p$ for every $1 \leq j \leq n$. Fix a $j \in \{1, 2, \cdots, n\}$. The element $\overline{d^{(j)}} = (d^{(j)}, d^{(j)}, \cdots, d^{(j)}) \in \prod_{i=1}^M \mathbb{Z}_{p_i}^*$, which is a compact multiplicative group. On the other hand, the set $\{\overline{(d^{(j)})}^N | N \in \mathbb{N}\}$ is a semigroup and hence its closure is a subgroup of $\prod_{i=1}^M \mathbb{Z}_{p_i}^*$, which implies that $(1, 1, \cdots, 1) \in \overline{\{\overline{(d^{(j)})}^N | N \in \mathbb{N}\}}$. Thus, $((d^{(j)})^{N_y})$ converges to 1 in $\mathbb{Z}_{p_i}^*$, for a subsequence (N_y) of positive integers, for all $1 \leq i \leq M$; so choose $N^{(j)}$ such that $|(d^{(j)})^N - 1|_p <$

$\frac{n_p^{(j)}}{|l_p^{(j)}|_p}$ for every $N \geq N^{(j)}$. Then, for all $1 \leq j \leq n$ and for $k = max\{N^{(1)}, N^{(2)}, \cdots, N^{(n)}\}$, we have $|((d^{(j)})^k - 1)l_p^{(j)}|_p < n_p^{(j)}$. □

3.3 Future Scope and Conclusion

This chapter describes the periodic points of automorphisms on n-dimensional solenoids whose duals are subgroups of algebraic number fields. Using the concept of adeles, it gives a description of such solenoids as well as a characterization of the periodic points of their automorphisms. This work is a development of the previous work done in [21] and [22].

We conclude this chapter by posing the same (open) problem as mentioned in the previous chapter for the more general case of any arbitrary solenoid Σ, where $\mathbb{Z}^n \subseteq \widehat{\Sigma} \subseteq \mathbb{Q}^n$. There is another class of dynamical systems, namely $S-$integer dynamical systems that can be considered as generalization of solenoidal automorphisms. We hope that the algebraic techniques that we have used to study solenoids can be extended to $S-$integer dynamical systems also.

Chapter 4

Dynamics of Real Projective Transformations

* The work in this chapter is covered by the following article:

Faiz Imam, Pabitra Narayan Mandal, Sharan Gopal, *Periodicity, Transitivity and Distality of Real Projective Transformations*, (In Communication).

This chapter investigates the dynamical properties of real projective transformations from a topological viewpoint. We study properties like periodicity, topological mixing, topological transitivity, distality and proximality. Regarding periodicity, we give a complete characterisation of the sets of least periods. We show that projective transformations are not topologically mixing and that it is only the isometries among them that are distal.

The literature on the dynamics of projective transformations is extensive. See for example [18], [23] and [35]. In the present article, we investigate some dynamical properties of projective transformations. In the next section, we define these properties followed by related known results in some cases and then our main results. We refer to [12] for most of the definitions.

4.1 Projective Transformations

Our dynamical system is $\left(\mathbb{P}_n(\mathbb{R}), \widetilde{T}\right)$, for an $n \in \mathbb{N}$, where $\mathbb{P}_n(\mathbb{R})$ and $\widetilde{T}$ are defined as follows. For $x, y \in \mathbb{R}^{n+1} \setminus \{\bar{0}\}$, if there exists a non-zero $\lambda \in \mathbb{R}$ such that $x = \lambda y$, then define $x \sim y$. Then the quotient space $\mathbb{R}^{n+1} \setminus \{\bar{0}\} /\sim$, denoted by $\mathbb{P}_n(\mathbb{R})$ is called the n-dimensional *real projective space*. Under this equivalence relation, the antipodal points are identified. The quotient map is denoted by π and for an $x \in \mathbb{R}^{n+1} \setminus \{\bar{0}\}, \pi(x)$ is also denoted as $[x]$. It is well known that $\mathbb{P}_n(\mathbb{R})$ is compact and connected. Besides, note that any open subset of $\mathbb{R}^{n+1} \setminus \{\bar{0}\}$ is open in $\mathbb{R}^{n+1}$ as well. Given a linear map $T \in GL_{n+1}(\mathbb{R})$, its associated projective transformation denoted by $\widetilde{T}$, is defined as $\widetilde{T}(\pi(x)) = \pi(Tx)$, for every $x \in \mathbb{R}^{n+1} \setminus \{\bar{0}\}$. It can be easily observed that $\left(\mathbb{P}_n(\mathbb{R}), \widetilde{T}\right)$ is a factor of $(\mathbb{R}^{n+1} \setminus \{\bar{0}\}, T)$.

Before proceeding to the results, we will now define a metric d on $\mathbb{P}_n(\mathbb{R})$. A metric on $\mathbb{P}_n(\mathbb{R})$ may be already well known but we will define a metric that is convenient for our calculations and show that it does induce the topology of $\mathbb{P}_n(\mathbb{R})$. We finally mention some notations and terms that we are going to use. The cardinality of any set A is denoted by $|A|$. T denotes an invertible linear transformation of $\mathbb{R}^{n+1}$ and $\widetilde{T}$, its associated projective transformation on $\mathbb{P}_n(\mathbb{R})$ for any non-negative integer n. We also identify T with the matrix associated to it. By an eigenvector of T, we mean an eigenvector corresponding to a real eigenvalue, unless otherwise mentioned. We use $||x||$ to denote the Euclidean norm of $x \in \mathbb{R}^n$ for any $n \in \mathbb{N}$.

Definition 4.1. For any $[x], [y] \in \mathbb{P}_n(\mathbb{R})$,
define $d([x],[y]) = min\left\{\left\|\frac{x}{||x||} - \frac{y}{||y||}\right\|, \left\|\frac{x}{||x||} + \frac{y}{||y||}\right\|\right\}$.

Proposition 4.2. *d is a metric on $\mathbb{P}_n(\mathbb{R})$.*

Proof. If $d([x],[y]) = 0$, then either $\frac{x}{||x||} = \frac{y}{||y||}$ or $\frac{x}{||x||} = -\frac{y}{||y||}$ and in either case $[x] = [y]$. Also, if $[x] = [y]$, then $x = \lambda y$ for some non-zero $\lambda \in \mathbb{R}$ and thus $\frac{x}{||x||} = \pm\frac{y}{||y||}$; hence $d([x],[y]) = 0$. Obviously, for any $[x],[y] \in \mathbb{P}_n(\mathbb{R})$, we have $d([x],[y]) = d([y],[x])$. So, it remains to verify the triangle inequality. For any $[x],[y],[z] \in \mathbb{P}_n(\mathbb{R})$, since there are two possibilities for each of the values of $d([x],[y])$ and $d([y],[z])$, we have four possibilities for the sum $d([x],[y])+d([y],[z])$. It can be easily verified that each of them is atleast the value of either $\left\|\frac{x}{||x||} - \frac{z}{||z||}\right\|$, or $\left\|\frac{x}{||x||} + \frac{z}{||z||}\right\|$ and hence $d([x],[z]) \leq d([x],[y]) + d([y],[z])$. □

Proposition 4.3. *d induces the topology of $\mathbb{P}_n(\mathbb{R})$.*

Proof. To avoid ambiguity, we refer to the topology of $\mathbb{P}_n(\mathbb{R})$ as the quotient topology, as it is a quotient space of $\mathbb{R}^{n+1} \setminus \{\bar{0}\}$ and the topology induced by d as metric topology. Let U be an open set in $\mathbb{P}_n(\mathbb{R})$ with respect to the quotient topology and $[x] \in U$. Then $\pi^{-1}(U)$ is open in $\mathbb{R}^{n+1} \setminus \{\bar{0}\}$ and $\{\lambda x \mid \lambda \in \mathbb{R} \setminus \{0\}\} \subset \pi^{-1}(U)$; in particular, $\frac{x}{||x||} \in \pi^{-1}(U)$. Choose an $\epsilon > 0$ such that the Euclidean open ball $B_E(\frac{x}{||x||}, \epsilon)$ centered at $\frac{x}{||x||}$ with radius ϵ is contained in $\pi^{-1}(U)$. Now, consider $B_d([x],\epsilon)$, the open ball in $\mathbb{P}_n(\mathbb{R})$, centered at $[x]$ and radius ϵ with respect to the metric d. If $[y] \in B_d([x],\epsilon)$, then either $\left\|\frac{x}{||x||} - \frac{y}{||y||}\right\| < \epsilon$ or $\left\|\frac{x}{||x||} + \frac{y}{||y||}\right\| < \epsilon$. Then $\frac{y}{||y||} \in B_d(\frac{x}{||x||}, \epsilon) \subset \pi^{-1}(U)$ or $-\frac{y}{||y||} \in B_d(\frac{x}{||x||}, \epsilon) \subset \pi^{-1}(U)$ and in either case $[y] \in U$. Hence U is open in metric topology.

Conversely, consider $B_d([x],\epsilon)$, the open ball in $\mathbb{P}_n(\mathbb{R})$ centered at $[x]$ with radius ϵ. Now, $\pi^{-1}(B_d([x],\epsilon) = \phi^{-1}\left(B_E(\frac{x}{||x||}, \epsilon)\right) \cup \phi^{-1}\left(B_E(\frac{-x}{||x||}, \epsilon)\right)$, where $\phi : \mathbb{R}^{n+1} \setminus \{\bar{0}\} \to S^n$ is the map given by $\phi(z) = \frac{z}{||z||}$. Since ϕ is continuous, the set $\pi^{-1}(B_d([x],\epsilon))$ is open in $\mathbb{R}^{n+1} \setminus \{\bar{0}\}$ and thus $B_d([x],\epsilon)$ is open in the quotient topology. □

4.2 Main Results

4.2.1 Periodicity

In this section, we are concerned with the characterization of the sets of periodic points and the sets of least periods of projective transformations. The set $P(\widetilde{T})$ can be easily found as described in one of the following paragraphs and $Per(\widetilde{T})$ is described in Theorem 4.4. Beside these characterisations, an another well studied notion is a dynamical invariant, called the zeta function. If the number of fixed points of f^k, denoted by $|Fix(f^k)|$ is finite for every $k \in \mathbb{N}$ in a dynamical system (X, f), we define the zeta function $\zeta_f(z)$ of f as the formal power series $\zeta_f(z) = exp(\sum_{k=1}^{\infty} \frac{1}{k}|Fix(f^k)|z^k)$. The dynamical zeta function for a projective transformation was found in [23].

We will now describe the periodic points of $\left(\mathbb{P}_n(\mathbb{R}), \widetilde{T}\right)$. If $v \in \mathbb{R}^{n+1} \setminus \{\bar{0}\}$ is an eigenvector of T with eigenvalue λ, then $\widetilde{T}([v]) = [Tv] = [\lambda v] = [v]$, and therefore $[v]$ is a fixed point. Conversely, if $[v]$ is a periodic point with period k, it is a fixed point of $\widetilde{T}^k$, and therefore $[T^k v] = [v]$, i.e. $T^k v = \lambda' v$ for some scalar $\lambda' \in \mathbb{R} \setminus \{0\}$. As a result, v is an eigenvector of T^k. To summarize, $[v]$ is periodic if and only if v is an eigenvector of T^k for some $k \in \mathbb{N}$.

We now state and prove our theorem about the sets of least periods. We introduce the following notation to make the statement of theorem simpler. For an $n \in \mathbb{N}$, $\Im_n = \left\{A \subset \mathbb{N} \mid |A| \leq \frac{n}{2}\right\}$, if n is even and $\Im_n = \left\{A \subset \mathbb{N} \mid 1 \in A \text{ and } |A| \leq \frac{n+1}{2}\right\}$, if n is odd.

Theorem 4.4. *$\{Per(\widetilde{T}) \mid \widetilde{T}$ is a projective transformation on $\mathbb{P}_n(\mathbb{R})\} = \Im_n$, for any $n \in \mathbb{N}$.*

Proof. If $[x]$ is a periodic point of $\widetilde{T}$ with least period k, then x is an eigenvector of T^k. Also, $T^l(x) = \lambda x$ for some non-zero $\lambda \in \mathbb{R}$ will imply that $\widetilde{T}^l([x]) = [x]$. Hence, $k \in Per(\widetilde{T})$ if and only if T^k has an eigenvector x such that x is not an eigenvector of T^l for any $l < k$.

If $\mu \in \mathbb{C}$ is a complex eigenvalue of T and $\mu^k \in \mathbb{R}$ for some $k \in \mathbb{N}$, then denote by k_μ to be the least positive integer such that $\mu^{k_\mu} \in \mathbb{R}$. Note that $k_\mu = 1$ if and only if $\mu \in \mathbb{R}$. By the above argument, it follows that $k_\mu \in Per(\widetilde{T})$. Conversely, if $k \in Per(\widetilde{T})$, then $T^k x = \lambda x$

for some non-zero $\lambda \in \mathbb{R}$. It is very well known that $\sqrt[k]{\lambda}$ is a complex eigenvalue of T and hence $k = k_\mu$, where $\mu = \sqrt[k]{\lambda}$. Therefore, $Per(\widetilde{T}) = \{k_\mu \mid \mu \text{ is a complex eigenvalue of } T\}$.

Since T has atmost $\frac{n}{2}$ or $\frac{n-1}{2}$ complex eigenvalues which are not conjugates of each other, depending on whether n is even or odd respectively, we have $|Per(\widetilde{T})\ |\leq \frac{n}{2}$, when n is even and $|Per(\widetilde{T})| \leq \frac{n+1}{2}$, when n is odd. In case n is odd, T has at least one real eigenvalue; so $1 \in Per(\widetilde{T})$. Hence $Per(\widetilde{T}) \in \Im_n$.

Conversely, for any $A \in \Im_n$, say $A \setminus \{1\} = \{m_1, m_2, \cdots, m_l\} \subset \mathbb{N}$. Define $\mu_j = e^{i\frac{\pi}{m_j}}$, where $1 \leq j \leq l$. Let $R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and T be the block diagonal matrix with the diagonal blocks as $R_{\frac{\pi}{m_1}}, R_{\frac{\pi}{m_2}}, \cdots, R_{\frac{\pi}{m_l}}$ if $l = \frac{n}{2}$ and $R_{\frac{\pi}{m_1}}, R_{\frac{\pi}{m_2}}, \cdots, R_{\frac{\pi}{m_l}}, I_{n-2l}$ if $l < \frac{n}{2}$, where I_{n-2l} is the identity matrix of size $n - 2l$. Then, the set of eigenvalues of T is $\{\mu_1, \overline{\mu_1}, \mu_2, \overline{\mu_2}, \cdots, \mu_l, \overline{\mu_l}\} \cup U$, where $U = \phi$ or $U = \{1\}$. Note that $m_j = k_{\mu_j}$ and hence $Per(\widetilde{T}) \setminus \{1\} = \{k_\mu \mid \mu \text{ is a non-real eigenvalue of } T\} = \{m_j | 1 \leq j \leq l\}$. Therefore, $Per(\widetilde{T}) = A$. □

4.2.2 Transitivity and Mixing

In this section, we will consider topological transitivity and topological mixing. A dynamical system (X, f) is said to be *topologically transitive* if it has an element whose forward orbit is dense in X. Equivalently, a continuous self map f on a locally compact Hausdorff topological space X is topologically transitive, if for any pair of non-empty open sets U and V in X, there exists a non-negative integer n for which $T^n(U) \cap V \neq \phi$. In addition, if there exists an integer $N > 0$ with $T^n(U) \cap V \neq \phi$ for every $n \geq N$, then (X, f) is called *topologically mixing*. In the contrapositive sense, no topological transitivity ensures no topological mixing. Note that a factor of a mixing system is also mixing (see [12]).

There are several papers in literature on these aspects also, particularly [25] and [40] are related to the current problem. In fact, the author in [40] hinted that the methods in that paper may help in discussing topological transitivity for projective transformations. Though the paper [25] does not mention the term transitivity explicitly, the concept of supercyclic vectors discussed in it is closely related to the topological transitivity of a projective transformation. We will be using that here and hence quote the necessary results. Let X be a real Banach space and $B(X)$ be the set of linear continuous mappings from X onto itself. A vector $x \in X$ is called a supercyclic vector of $T \in B(X)$ if $\overline{\{\lambda T^k(x) \mid \lambda \in \mathbb{R} \text{ and } k \in \mathbb{N}_0\}} = X$. It is proved in Theorem 1 of [25] that there exist operators in $B(X)$ having supercyclic vectors if and only if dim $X \in \{0, 1, 2\}$ or dim $X = \infty$. We now state and prove our result about the relation between the existence of supercyclic vectors for T and the transitivity of $\widetilde{T}$.

Proposition 4.5. *Let $T \in GL_{n+1}(\mathbb{R})$. T has a supercyclic vector if and only if $\widetilde{T}$ is transitive on $\mathbb{P}_n(\mathbb{R})$.*

Proof. Assume that $\widetilde{T}$ has a supercyclic vector, say x i.e., $\overline{\{\lambda T^k(x)|\lambda \in \mathbb{R} \text{ and } k \in \mathbb{N}_0\}} = \mathbb{R}^{n+1}$. Let U be a non-empty open set in $\mathbb{P}_n(\mathbb{R})$. Then, $\pi^{-1}(U)$ is open in $\mathbb{R}^{n+1} \setminus \{\bar{0}\}$. So, $\lambda T^k(x) \in \pi^{-1}(U)$ for some $\lambda \in \mathbb{R}$ and for some $k \in \mathbb{N}_0$. Thus, $\widetilde{T}^k([x]) \in U$.

For the converse, let $[x] \in \mathbb{P}_n(\mathbb{R})$ whose forward orbit is dense in $\mathbb{P}_n(\mathbb{R})$ and let V be a non-empty open set in $\mathbb{R}^{n+1}$. Choose $y \in V$ and an Euclidean ball $B_1 = B_E(y, \epsilon)$ such that $B_1 \subset V$. Define $W = \{tz | t \in \mathbb{R} \setminus \{0\}, z \in B_1\}$. The map $\phi_t : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ for any $t \neq 0$, defined by $\phi_t(u) = \frac{1}{t}u$ is continuous and thus the set $W_t := \{tz | z \in B_1\}$, being the pre-image of B_1 under ϕ_t is open. Since $W = \cup_{t\neq 0} W_t$, W is open. Also, W is saturated with respect to the map π i.e if $\pi^{-1}([u]) \cap W \neq \phi$ for some $[u] \in \mathbb{P}_n(\mathbb{R})$ then $\pi^{-1}([u]) \subset W$. Hence, $\pi(W)$ is open in $\mathbb{P}_n(\mathbb{R})$. Then, $\widetilde{T}^k([x]) \in \pi(W)$ for some k, implying that $\lambda T^k(x) \in W$ for every non-zero $\lambda \in \mathbb{R}$; in particular $T^k(x) \in W$ and thus $T^k(x) = tz$ for some non-zero $t \in \mathbb{R}$ and $z \in B_1$. It then follows that $\frac{1}{t}T^k(x) = z \in B_1 \subset V$ and hence $\{\lambda T^k(x)| \lambda \in \mathbb{R} \text{ and } k \in \mathbb{N}_0\}$ is dense in $\mathbb{R}^{n+1}$. □

Corollary 4.6. *$\mathbb{P}_n(\mathbb{R})$ admits a transitive projective transformation if and only if $n = 1$.*

The proof of the corollary follows from Theorem 1 of [25] and the above Proposition 4.5.

Since every topologically mixing system is topologically transitive, it is enough to check the existence of topological mixing maps only on $\mathbb{P}_1(\mathbb{R})$. We prove in Theorem 4.7 that there exist no projective transformations on $\mathbb{P}_1(\mathbb{R})$ that are topologically mixing; hence

$\mathbb{P}_n(\mathbb{R})$ does not admit a topologically mixing projective transformation for any $n \in \mathbb{N}$. However, Example 4.1 is of some interest, because it is a continous map of $\mathbb{P}_1(\mathbb{R})$ which is mixing; but is not a projective transformation i.e., not induced by a linear transformation of $\mathbb{R}^2$.

Theorem 4.7. $\mathbb{P}_1(\mathbb{R})$ *does not admit a topologically mixing projective transformation.*

Proof. Let $T \in GL_2(\mathbb{R})$. We can assume that T is equal to one of the following matrices:
(i) $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where a and b are distinct real eigenvalues of T.
(ii) $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ or $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$, where a is a real eigenvalue of T.
(iii) aR_θ where $a \in \mathbb{R} \setminus \{0\}$ and $R_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$ for some $\theta \in \mathbb{R}$.
Case (i): When $T = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, let $U' = \{(x, y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y > 0\}$ and consider the open set $U = \pi(U')$ in $\mathbb{P}_1(\mathbb{R})$. If $ab > 0$, then for any $[(x, y)] \in U$, $\widetilde{T}^k([(x, y)]) = \big[(a^k x, b^k y)\big] \in U$ for every $k \in \mathbb{N}$. If $V = \pi(V')$, where $V' = \{(x, y) \in \mathbb{R}^2 \mid x < 0$ and $y > 0\}$, then V is a non-empty open set such that $\widetilde{T}^k(U) \cap V = \phi$ for every $k \in \mathbb{N}$. Thus $\widetilde{T}$ is not mixing. If $ab < 0$, then for any even $k, \widetilde{T}^k([(x, y)]) \in U$ and thus again $\widetilde{T}$ is not mixing.
Case (ii): If $T = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ then $\widetilde{T}([x]) = [ax] = [x]$, i.e. $\widetilde{T}$ is the identity map and hence not mixing. If $T = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ then $\widetilde{T}^k([(x, y)]) = \big[(a^k x + na^{k-1} y, a^k y)\big]$. Consider the open sets $U = \pi(U')$ and $V = \pi(V')$, where $U' = \{(x, y) \mid x > 0 \text{ and } y > 0\}$ and $V' = \{(x, y) \mid x < 0 \text{ and } y > 0\}$. If $a > 0$, then $\widetilde{T}^k(U) \cap V = \phi$, for any $k \in \mathbb{N}$ and if $a < 0$, then $\widetilde{T}^k(U) \cap V = \phi$ for large enough odd values of k. Hence $\widetilde{T}$ is not mixing.

Case (iii): In this case, $\widetilde{T}$ is an isometry and hence it is not mixing. □

Since every topologically mixing transformation is topologically transitive, the following corollary follows from Corollary 4.6 and Theorem 4.7.

Corollary 4.8. $\mathbb{P}_n(\mathbb{R})$ *does not admit a topologically mixing projective transformation for any* $n \in \mathbb{N}$.

Though there are no projective transformations on $\mathbb{P}_1(\mathbb{R})$ that are mixing, we can still have a mixing continuous map, as shown in the following example.

Example 4.1. *Consider the expanding endomorphism* $E_3 : S^1 \to S^1$ *given by* $E_3(e^{i\theta}) = e^{i3\theta}$. *Since* E_3 *is mixing in* S^1 *(refer to [12]),* $\widetilde{E_3}$ *being a factor of* E_3 *is also mixing.*

4.2.3 Distality and Proximality

We finally consider distality and proximality which are asymptotic dynamical attributes based on the distance between comparable positions on pairs of orbits. They are also dichotomic in nature. Let X be a compact Hausdorff topological space with a homeomorphism $f : X \to X$ and x, y be any two points of X. We define the diagonal set in $X \times X$ as $\Delta = \{(z, z) \in X \times X : z \in X\}$ and the orbit of (x, y) under $f \times f$ is denoted by $\mathcal{O}(x, y)$. A pair of points $x, y \in X$ are called proximal if their orbit closure i.e. $\overline{\mathcal{O}(x, y)}$ has a non-empty intersection with the diagonal set Δ, else they are known as distal. A homeomorphism on a space X is called distal if any two distinct points $x, y \in X$ are distal. If d is a metric on X, then $x, y \in X$ are proximal if and only if there exists a sequence n_k of integers such that $d(f^{n_k}(x), f^{n_k}(y))$ goes to zero as k tends to infinity. Note that an isometry is distal. We will also need the fact that a factor of a distal homeomorphism of a compact Hausdorff space is also distal (See Corollary 2.7.7, [12]).

Let T be an invertible linear transformation on $\mathbb{R}^{n+1}$. If $\widetilde{T}$ is an isometry on $\mathbb{P}_n(\mathbb{R})$, then it is obviously distal. We now prove in the following theorem that $\widetilde{T}$ is not distal in all other cases. We continue to assume that $T \in GL_{n+1}(\mathbb{R})$ and also use the following notations in the next theorem and its proof. A denotes an arbitary matrix of an appropriate order, I_2 stands for the identity matrix of order 2×2.

Theorem 4.9. $\widetilde{T}$ *is distal on* $\mathbb{P}_n(\mathbb{R})$ *if and only if* $\widetilde{T}$ *is an isometry.*

Proof. An isometry is obviously distal; so, we now assume that $\widetilde{T}$ is distal and show that it is an isometry. We first claim that T is of the form $T = \bigoplus_{l=1}^{k} \alpha_l T_l$, where each $\alpha_l \in \mathbb{R}, |\alpha_i| = |\alpha_j|$ for any $i, j \in \{1, 2, \cdots, k\}$ and each T_l is an isometry of either $\mathbb{R}$ or $\mathbb{R}^2$.

In case T is not of this form, we can assume that T is equal to one of the following :

(i) $\begin{pmatrix} J & I_2 & O \\ O & J & \cdots \\ O & O & A \end{pmatrix}$, where $J = \alpha.R_\theta$ for some $\alpha \in \mathbb{R} \setminus \{0\}$ and $\theta \in \mathbb{R}$.

(ii) $\begin{pmatrix} \lambda & 1 & O \\ 0 & \lambda & \cdots \\ O & O & A \end{pmatrix}$, where $\lambda \in \mathbb{R} \setminus \{0\}$.

(iii) $\begin{pmatrix} a & 0 & \cdots \\ 0 & b & \cdots \\ O & O & A \end{pmatrix}$, where $a, b \in \mathbb{R} \setminus \{0\}$, with $|a| \neq |b|$.

(iv) $\begin{pmatrix} J_1 & O & \cdots \\ O & J_2 & \cdots \\ O & O & A \end{pmatrix}$, where each $J_i = \alpha_i.R_{\theta_i}$, $\theta_i \in \mathbb{R}$ and $\alpha_i \in \mathbb{R} \setminus \{0\}$ such that $|\alpha_1| \neq |\alpha_2|$.

(v) $\begin{pmatrix} \lambda & O & \cdots \\ O & J & \cdots \\ O & O & A \end{pmatrix}$, where $\lambda \in \mathbb{R} \setminus \{0\}$, $J = \alpha.R_\theta$ such that $\alpha \in \mathbb{R} \setminus \{0\}$, $\theta \in \mathbb{R}$ and $|\lambda| \neq |\alpha|$.

In the first case, where $T = \begin{pmatrix} J & I_2 & O \\ O & J & \cdots \\ O & O & A \end{pmatrix}$, consider an element $(\overline{x}, \overline{y}, 0, \cdots, 0) \in \mathbb{R}^{n+1}$, such that $\overline{x}, \overline{y} \in \mathbb{R}^2 \setminus \{(0,0)\}$.

Note that $T^n (\overline{x}, \overline{y}, 0, \cdots, 0) = \left(\alpha^n R_\theta^n \overline{x} + n\alpha^{n-1} R_\theta^{n-1} \overline{y}, \alpha^n R_\theta^n \overline{y}, 0, \cdots, 0\right)$
and $T^n \left(\frac{R_\theta^{-1}\overline{y}}{\left\|R_\theta^{-1}\overline{y}\right\|}, 0, \cdots, 0 \right) = \left(\frac{\alpha^n R_\theta^n R_\theta^{-1}\overline{y}}{\left\|R_\theta^{-1}\overline{y}\right\|}, 0, \cdots, 0 \right)$. Then,

$$\widetilde{T}^n \left[(\overline{x}, \overline{y}, 0, \cdots, 0)\right] = \left[\left(\frac{\frac{\alpha}{n} R_\theta^n \overline{x} + R_\theta^{n-1}\overline{y}}{\sqrt{\left\|\frac{\alpha}{n} R_\theta^n \overline{x} + R_\theta^{n-1}\overline{y}\right\|^2 + \left\|\frac{\alpha}{n} R_\theta^n \overline{y}\right\|^2}}, \frac{\alpha R_\theta^n \overline{y}}{\sqrt{\left\|\alpha R_\theta^n \overline{x} + nR_\theta^{n-1}\overline{y}\right\|^2 + \left\|\alpha R_\theta^n \overline{y}\right\|^2}}, 0, \cdots, 0 \right) \right]$$

and $\widetilde{T}^n \left[\left(\frac{R_\theta^{-1}\overline{y}}{\left\|R_\theta^{-1}\overline{y}\right\|}, 0, \cdots, 0 \right) \right] = \left[\left(\frac{R_\theta^{n-1}\overline{y}}{\left\|R_\theta^{n-1}\overline{y}\right\|}, 0, \cdots, 0 \right) \right]$.

Note that, as $n \to \infty$,

$$\left\| \frac{\frac{\alpha}{n} R_\theta^n \overline{x} + R_\theta^{n-1}\overline{y}}{\sqrt{\left\|\frac{\alpha}{n} R_\theta^n \overline{x} + R_\theta^{n-1}\overline{y}\right\|^2 + \left\|\frac{\alpha}{n} R_\theta^n \overline{y}\right\|^2}} - \frac{R_\theta^{n-1}\overline{y}}{\left\|R_\theta^{n-1}\overline{y}\right\|} \right\| \to 0$$

and

$$\left\| \frac{\alpha R_\theta^n \overline{y}}{\sqrt{\left\|\alpha R_\theta^n \overline{x} + n R_\theta^{n-1}\overline{y}\right\|^2 + \left\|\alpha R_\theta^n \overline{y}\right\|^2}} \right\| \to 0.$$

Hence, $d\left(\widetilde{T}^n\left[(\overline{x}, \overline{y}, 0, \cdots, 0)\right],\ \widetilde{T}^n\left[\left(\frac{R_\theta^{-1}\overline{y}}{\left\|R_\theta^{-1}\overline{y}\right\|}, 0, \cdots, 0\right)\right]\right) \to 0$ and therefore $\widetilde{T}$ is not distal.

For the second case, where $T = \begin{pmatrix} \lambda & 1 & O \\ O & \lambda & \cdots \\ O & O & A \end{pmatrix}$, let $(x, y, 0, \cdots, 0) \in \mathbb{R}^{n+1}$, such that $x, y \in \mathbb{R} \setminus \{0\}$. Then, $\widetilde{T}^n\left[(x, y, 0, \cdots, 0)\right] = \left[\frac{(\lambda x + ny,\ \lambda y,\ 0, \cdots, 0)}{\sqrt{(\lambda x+ny)^2 + (\lambda y)^2}}\right]$
and $\widetilde{T}^n\left[\left(\frac{y}{\|y\|}, 0, 0, \cdots, 0\right)\right] = [(1, 0, 0, \cdots, 0)]$.

$$\text{Now, } \left\|\widetilde{T}^n\left[(x, y, 0, \cdots, 0)\right] - \widetilde{T}^n\left[\left(\frac{y}{\|y\|}, 0, 0, \cdots, 0\right)\right]\right\|$$
$$= \sqrt{\left(\frac{\lambda x + ny}{\sqrt{(\lambda x + ny)^2 + (\lambda y)^2}} - 1\right)^2 + \left(\frac{\lambda y}{\sqrt{(\lambda x + ny)^2 + (\lambda y)^2}}\right)^2} \to 0 \text{ as } n \to \infty.$$

Hence, $\widetilde{T}$ is not distal in this case also.

In the remaining cases, T is of the form $T = \bigoplus_{l=1}^k \alpha_l T_l$, where each T_l is an isometry of either $\mathbb{R}$ or $\mathbb{R}^2$ and $|\alpha_i| \neq |\alpha_j|$ for some i and j. Without loss of generality, we assume that $i < j$ and $|\alpha_i| < |\alpha_j|$. Let η_l be the projection of $\mathbb{R}^{n+1}$ on to the domain of T_l for each $l \in \{1, 2, \cdots, k\}$; note that the range of each η_l is either either $\mathbb{R}$ or $\mathbb{R}^2$.

Take two elements $x, x' \in \mathbb{R}^{n+1}$ such that $\eta_l(x) = 0$ for every $l \notin \{i, j\}$, $\eta_l(x') = 0$ for every $l \neq j$, $\eta_i(x) \neq 0 \neq \eta_j(x)$ and finally $\eta_j(x') = \eta_j(x)$. Say $\eta_i(x) = \overline{x_i}$ and $\eta_j(x) = \eta_j(x') = \overline{x_j}$.

$$\text{Then, } \widetilde{T}^n([x]) = \left[\frac{\left(0, \cdots, 0,\ \alpha_i^n T_i^n \overline{x_i},\ 0, \cdots, 0,\ \alpha_j^n T_j^n \overline{x_j}, 0,\ \cdots, 0\right)}{\sqrt{\left\|\alpha_i^n T_i^n \overline{x_i}\right\|^2 + \left\|\alpha_j^n T_j^n \overline{x_j}\right\|^2}}\right]$$
$$\text{and } \widetilde{T}^n([x']) = \left[\left(0, \cdots, 0,\ \frac{T_j^n \overline{x_j}}{\left\|T_j^n \overline{x_j}\right\|}, 0, \cdots, 0\right)\right].$$

Note that, as $n \to \infty$, $\left\|\frac{\alpha_i^n T_i^n \overline{x_i}}{\sqrt{\left\|\alpha_i^n T_i^n \overline{x_i}\right\|^2 + \left\|\alpha_j^n T_j^n \overline{x_j}\right\|^2}}\right\| \to 0$ and $\left\|\frac{\alpha_j^n T_j^n \overline{x_j}}{\sqrt{\left\|\alpha_i^n T_i^n \overline{x_i}\right\|^2 + \left\|\alpha_j^n T_j^n \overline{x_j}\right\|^2}} - \frac{T_j^n \overline{x_j}}{\left\|T_j^n \overline{x_j}\right\|}\right\|$ $\to 0$.

Hence $d(\widetilde{T}^n([x]), \widetilde{T}^n([x'])) \to 0$ and thus $\widetilde{T}$ is not distal.

Therefore, $T = \bigoplus_{l=1}^{k} \alpha_l T_l$, where each $\alpha_l \in \mathbb{R}$, each T_l is an isometry of either $\mathbb{R}$ or $\mathbb{R}^2$ and $|\alpha_i| = |\alpha_j| = |\alpha|$ (say) for every $i, j \in \{1, 2, \cdots, k\}$. If $x = (\overline{x_1}, \overline{x_2}, \cdots, \overline{x_k})$ and $y = (\overline{y_1}, \overline{y_2}, \cdots, \overline{y_k})$ are in $\mathbb{R}^{n+1}$, with $\|x\| = \|y\| = 1$ and $\overline{x_l}, \overline{y_l}$ belong to the domain of T_l for each l, then $\frac{Tx}{\|Tx\|} = \frac{1}{|\alpha|}\left(\alpha_1 T_1 \overline{x_1}, \alpha_2 T_2 \overline{x_2}, \cdots, \alpha_k T_k \overline{x_k}\right)$
and $\frac{Ty}{\|Ty\|} = \frac{1}{|\alpha|}\left(\alpha_1 T_1 \overline{y_1}, \alpha_2 T_2 \overline{y_2}, \cdots, \alpha_k T_k \overline{y_k}\right)$.

$$\begin{aligned}
\text{Thus },\ & \left\|\frac{Tx}{\|Tx\|} \pm \frac{Ty}{\|Ty\|}\right\| \\
&= \frac{1}{|\alpha|}\left\|\alpha_1 T_1\left(\overline{x_1} \pm \overline{y_1}\right), \alpha_2 T_2\left(\overline{x_1} \pm \overline{y_2}\right), \cdots, \alpha_k T_k\left(\overline{x_k} \pm \overline{y_k}\right)\right\| \\
&= \frac{1}{|\alpha|}\sqrt{|\alpha_1|^2 \left\|\overline{x_1} \pm \overline{y_1}\right\|^2 + |\alpha_2|^2 \left\|\overline{x_2} \pm \overline{y_2}\right\|^2 + \cdots + |\alpha_k|^2 \left\|\overline{x_k} \pm \overline{y_k}\right\|^2} \\
&= \|x \pm y\|.
\end{aligned}$$

Hence $d\left(\widetilde{T}[x], \widetilde{T}[y]\right) = d([x], [y])$ and therefore $\widetilde{T}$ is an isometry. □

4.3 Future Scope and Conclusion

This chapter explores some dynamical properties namely periodicity, topological mixing, topological transitivity, distality and proximality of real projective transformations. A complete characterisation of the sets of least periods was obtained. It was also shown that only $\mathbb{P}_1(\mathbb{R})$ admits a transitive projective transformation whereas projective transformations on $\mathbb{P}_n(\mathbb{R})$, for any $n \in \mathbb{N}$ are not mixing. Finally it is proved that the isometries are the only projective transformation that are distal. This work may be further extended to a general normed vector space V(over field F) of infinite dimension, where a similar coordinate wise λ-multiple equivalence relation exists between two infinite sequences $(x) = (x_n)$ and $(y) = (y_n) \in V \setminus (0)$, i.e. $(x) \sim (y)$, if there exists a non-zero $\lambda \in F$ such that $x_i = \lambda y_i, \forall i \in \mathbb{N}$.

Chapter 5

Discrete Switch Dynamical Systems

* The work presented in this chapter is based on the following article :

Faiz Imam, Sharan Gopal, *Topological Aspects of Discrete Switch Dynamical Systems*, accepted for publication in The Australian Journal of Mathematical Analysis and Applications.

The notion of non-autonomous discrete dynamical systems (NDS) is well studied in the literature. On the other hand, a similar idea exists in literature for continuous dynamical systems with the name continuous switch dynamical systems. In this chapter, we interpret an NDS as a switch system and describe how the dynamics of an NDS can be better understood using the notion of switch. We do not claim that there are non-trivial results in this chapter as many of the notions are already explored for NDS. However, this study was made to get an insight into how facts about NDS can be explained using the idea of switch.

A continuous dynamical system on a topological space X, is the action of the group $\{f^t : t \in \mathbb{R}\}$ (or the semigroup $\{f^t : t \in \mathbb{R}$ and $t \geq 0\}$) on X, where each f^t is a self map on X such that f^0 is the identity map and $f^{t+s} = f^t \circ f^s$, for every $t, s \in \mathbb{R}$. Now, instead of considering this one-parameter family of maps, if we have more than one such family, say $\{f_i^t : t \in \mathbb{R}\}$, $i \in \{1, 2, ..., k\}$ and consider the action of different families at different instances of time, then we obtain a new notion of dynamics, called a continuous switch dynamical system. This idea of "action of different functions at different instances" is explained more precisely using a "switch function", which will be discussed more in the following paragraphs. In a usual dynamical system on X obtained by the action of $\{f^t : t \in \mathbb{R}\}$, the trajectory of a point $x \in X$ is $\{f^t(x) : t \in \mathbb{R}\}$, whereas in the switch system as described above, the trajectory of x would be (x_t), where $x_t = f^t_{\sigma(t)}(x)$, where $\sigma(t)$ is the switch function, which will be defined elaborately in the terminology section.

In this chapter, we define an analogous switch system for discrete dynamical systems. Before we formally define discrete switch dynamical systems in the next section, continuous switch systems will be discussed in detail. Most of the terminology and ideas discussed about switch dynamical systems are from [36]. Then follows a discussion about an another similar notion, namely free semigroup action.

5.1 Introduction

A continuous dynamical system arises naturally from the first order autonomous system of ordinary differential equations. Consider the following autonomous system of differential equations,

$$\begin{aligned} \dot{x}(t) &= h(x), t \in \mathbb{R}, \\ x(0) &= x_0, \end{aligned} \tag{5.1}$$

where $h : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable map (C^1-map) with $||Dh(x)||_\infty < \infty$, here D is the derivative operator.

From Picard's theorem, it follows that system (5.1) has a unique solution for any $x_0 \in \mathbb{R}^n$, say $\phi(t, x_0)$, defined for every $t \in \mathbb{R}$. Moreover, we have $\phi(0, x_0) = x_0$ and $\phi(t + s, x_0) = \phi(t, \phi(s, x_0))$. Thus, we have a one-parameter family of maps $\{f^t : t \in \mathbb{R}\}$ with $f^t(x) = \phi(t, x)$ such that f^0 is the identity and $f^{t+s} = f^t \circ f^s$ for every $t, s \in \mathbb{R}$.

We now introduce the concept of continuous switch dynamical system using an example. Let $h_i : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-map with bounded derivative for each $i \in \{1, 2, ..., k\}$. Consider the initial value problems

$$\begin{aligned} \frac{d}{dt}x^{(i)}(t) &= h_i(x^{(i)}), t \in \mathbb{R}, \\ x^{(i)}(0) &= x_0. \end{aligned} \tag{5.2}$$

As we mentioned earlier, for each $i \in \{1, 2, ..., k\}$, the solution map $\phi_i(t, x_0)$ of (5.2) is a continuous dynamical system.

Let $\sigma : \mathbb{R} \to \{1, 2, ..., k\}$ be a piecewise constant function i.e., σ has finitely many discontinuities in any bounded interval and on the interval between any two consecutive discontinuities, σ is constant. The function σ is called a *switch function* and its discontinuities are called *switches*. For each $i \in \{1, 2, ..., k\}$, let $J_i = \sigma^{-1}(i)$, which is a union of intervals, the endpoints of each of which are the consecutive switches of σ. Note that $\mathbb{R}$ is the disjoint union of J_i's.

Now, consider the following system.

$$\begin{aligned} \dot{x}(t) &= h_{\sigma(t)}(x), \ \ t \in \mathbb{R}, \\ x(0) &= x_0. \end{aligned} \tag{5.3}$$

By a solution $\phi(t, x_0)$ of this system (5.3), we mean $\phi(t, x_0) = \phi_i(t, x_0)$, where $i \in \{1, 2, ..., k\}$ is the unique index such that $t \in J_i$ and $\phi_i(t, x_0)$ is the solution of (5.2) for the respective value of i. These solutions of (5.3) corresponding to all values of x_0 give rise to a continuous switch dynamical system in the following way.

For each $i \in \{1, 2, ..., k\}$, consider the one-parameter family of functions, $\mathcal{F}_i = \{f_i^t : t \in \mathbb{R}\}$, where $f_i^t(x) = \phi_i(t, x)$. The triplet $(\mathbb{R}^n, \{\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_k\}, \sigma)$ is called a continuous switch dynamical system. The trajectory of a point $x \in \mathbb{R}^n$ in this system is given by $(x_t)_{t\in\mathbb{R}}$, where $x_t = f^t_{\sigma(t)}(x)$. Continuous switch dynamical systems are studied by many people in literature with main focus on stability (see, for instance [3], [20], [24] and [36]).

We extend this idea to discrete systems. Consider $(X, \mathfrak{F}, \sigma)$ with $\mathfrak{F} = \{f_1, f_2, ..., f_k\}$ being a collection of continuous self maps on a topological space X and $\sigma : \mathbb{N} \to \{1, 2, ..., k\}$, any function. In this case, the trajectory of a point $x \in X$, which is denoted by $(x_n)_{n\in\mathbb{N}_0}$, is defined as $x_0 = x$, and $x_n = f_{\sigma(n)}(x_{n-1})$, for every $n \geq 1$. There is an analogous idea of dynamics, namely the action of the free semigroup generated by a family $\{f_1, f_2, ..., f_k\}$ of self maps on X. Semigroup actions are well studied in literature; see, for example, [6], [8], [9], [13], [14], [37], [38], [39], [45], [50], [51]. We now briefly introduce semigroup actions.

Consider a topological space X and a family $\{f_1, f_2, ..., f_k\}$ of continuous self maps on X. Let G be the free semigroup generated by these maps i.e., a typical element of G is $f_{i_1} \circ f_{i_2} \circ ...f_{i_n}$ (hereafter written simply as $f_{i_1} f_{i_2} ...f_{i_n}$) for some $n \in \mathbb{N}$ and $i_j \in \{1, 2, ..., k\}$ for each j. For a concise notation, we consider the sequence space $\sum_k := \{(w_n)_{n\in\mathbb{N}_0} : w_n \in \{1, .., k\}\}$. A trajectory of a point $x \in X$ is defined as the sequence $(f^n_w(x))_{n\in\mathbb{N}_0}$, where $w = (w_n)_{n\in\mathbb{N}_0} \in \sum_k$ and $f^n_w(x) := f_{w_{n-1}} f_{w_{n-2}} ...f_{w_0}(x)$. It is easy to see that a point $x \in X$ has many different trajectories. In fact, every $w \in \sum_k$ gives a trajectory of x. In fact, many notions of topological dynamics are introduced and studied for these semigroup actions. For instance, entropy is studied in [6], [9], [13], [37], [39], [50], [51]. Rodrigues et al. [45] introduced the specification property whereas Bahabadi [6] introduced the shadowing and average shadowing properties for semigroup actions. Carbalho et al. [14] study the action of semigroups generated by expanding maps. There is a recent paper by Huihui and Ma [28] which introduced the notions of weakly mixing and total transitivity for free semigroup actions. In fact, the well studied theory of iterated functions systems (IFS) is also a particular form of semigroup actions.

The discrete switch dynamical systems that we are interested to study now are more rigid than the free semigroup actions, in particular IFS, in the following sense. In a discrete switch dynamical system, every point will have a unique trajectory, whereas there are several trajectories for a point in a free semigroup action. In fact, in a free semigroup action, every $w \in \sum_k$ acts as a switch function. Thus, the dynamics of a switch system are entirely different from the dynamics in a free semigroup action.

Throughout this chapter, in most of the instances, a discrete switch dynamical system will be referred to as a *switch system* whereas a discrete dynamical system will be called a *usual system*. This chapter is organized as follows. In the next section, we introduce the terminology required to study switch dynamical systems. We prove some results on transitivity, periodicity and recurrent points in Section 5.3. Then, in Section 5.4, we study switch systems of circle rotations. Here again, we discuss the periodicity of these switching rotations on the circle. Finally, the chapter is concluded with Section 5.5 with discussions on significance of switch systems.

5.2 Terminology

We now develop terminology for switch systems. Most of the concepts defined here for switch systems are generalizations of the corresponding notions in usual systems. However, there are instances, where a switch system $(X, \mathfrak{F}, \sigma)$ has a particular property with no usual individual system (X, f) for $f \in \mathfrak{F}$ has it. For instance, Example 5.1 gives a switch system $(X, \{f_1, f_2\}, \sigma)$ with $X = [0, 2] \cup [4, 6]$, in which the point $x = 1$ is recurrent but it is recurrent neither in (X, f_1) nor in (X, f_2). We now begin with definitions of a switch system and the trajectory of a point in it.

Definition 5.1. Let X be a topological space, $\mathfrak{F} = \{f_1, f_2, ..., f_k\}$ be a family of continuous self maps on X and $\sigma : \mathbb{N} \to \{1, 2, ..., k\}$ be any map. The triplet $(X, \mathfrak{F}, \sigma)$ is called a *topological discrete switch dynamical system.* For each point x in X, the *trajectory* $(x_n)_{n\geq 0}$

of x is defined as $x_0 = x$ and $x_n = f_{\sigma(n)}(x_{n-1})$ for every $n \in \mathbb{N}$. The map σ is called the *switch function* or simply as a *switch* of the system $(X, \mathfrak{F}, \sigma)$.

In a usual system (X, f), the trajectory of a point $x \in X$ is defined as $(f^n(x))_{n \in \mathbb{N}_0}$ i.e, the n^{th} term in the trajectory of x is given by $f^n(x)$. In a switch system, the n^{th} term of the trajectory of x is given by $x_n = f_{(\sigma(n))} f_{(\sigma(n-1))} ... f_{(\sigma(1))}(x)$ i.e., the switch σ specifies which function to apply at the n^{th} time.

Assumptions:

1. We assume that, in all the switch systems that are considered in this chapter, the switch σ is surjective and $\sigma^{-1}(i)$ is an infinite set for every $i \in \{1, 2, ..., k\}$, i.e., each f_i in $\mathfrak{F}$ occurs infinitely many times in every trajectory.
2. As a convention, we define $\sigma(0) = 0$ and $f_{\sigma(0)}(x) = x$, for every $x \in X$.

Definition 5.2. A switch function σ is said to be a periodic function with a period u, if $\sigma(nu + l) = \sigma(l)$ for every $1 \leq l \leq u$ and for every $n \in \mathbb{N}$.

We will now define various dynamical notions for switch systems.

Definition 5.3. Let $(X, \mathfrak{F}, \sigma)$ be a switch system.

1. A point $x \in X$ is called a periodic point if there is an $m \in \mathbb{N}$ such that $x_{nm+l} = x_l$ for every $0 \leq l < m$ and for every $n \in \mathbb{N}$. The least such m is called the least period of x.
2. A point $x \in X$ is called a fixed point if $x_n = x$ for every $n \in \mathbb{N}$.
3. Let $x \in X$. An element $y \in X$ is called an ω-limit point of x if there is a sequence (n_m) of positive integers such that $(n_m) \to \infty$ and $(x_{n_m}) \to y$. The set of all ω-limit points of x is denoted by $\omega(x)$. Further, if $x \in \omega(x)$, then x is called a recurrent point.
4. $(X, \mathfrak{F}, \sigma)$ is called topologically transitive, if there is an $x \in X$ such that $\overline{\{x_n : n \in \mathbb{N}_0\}} = X$.

5. A subset $V \subseteq X$ is called invariant, if for every $x \in V$, $x_n \in V$ for every $n \in \mathbb{N}$.

6. A closed non-empty invariant subset V of X is called a minimal set, if V does not contain a proper closed, non-empty and invariant set. If X itself is a minimal set, then $(X, \mathfrak{F}, \sigma)$ is a called a minimal switch system.

We now define the notion of topological conjugacy. In usual dynamical systems, (X, f) is said to be topologically conjugate to (Y, g) if there is a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$. It follows that for every term $(f^n(x))$ in the trajectory of a point $x \in X$, we have $h(f^n(x)) = g^n(h(x))$. However, for switch systems, we need to ensure this as a part of the definition. After the following definition, we will simply use the words *conjugacy* and *conjugate* instead of *topological conjugacy* and *topologically conjugate* respectively.

Definition 5.4. Let $(X, \mathfrak{F}_1, \sigma_1)$ and $(Y, \mathfrak{F}_2, \sigma_2)$ be two switch dynamical systems. If there is a homeomorphism $h : X \to Y$ such that for any $x \in X$, $h(x_n) = (h(x))_n$ for every $n \in \mathbb{N}$, where (x_n) and $((h(x))_n)$ are the trajectories of x and $h(x)$ in $(X, \mathfrak{F}_1, \sigma_1)$ and $(Y, \mathfrak{F}_2, \sigma_2)$ respectively, then h is called a topological conjugacy from $(X, \mathfrak{F}_1, \sigma_1)$ to $(Y, \mathfrak{F}_2, \sigma_2)$. In such a case, the two switch systems are said to be topologically conjugate.

5.3 Main Results

In this section, we state and prove some results about the periodicity, transitivity and ω-limit points. The following theorem gives a sufficient condition for existence of periodic points in a switch system. In a usual system (X, f), for a point $x \in X$, if $f^m(x) = x$ for some $m \in \mathbb{N}$, then x is a periodic point, but in a switch system $(X, \mathfrak{F}, \sigma)$, $x_m = x$ does not imply that x is periodic. However, if σ is a periodic function with the same integer m as a period, then we prove in the following theorem that x is a periodic point.

Theorem 5.5. *Let $(X, \mathfrak{F}, \sigma)$ be a switch dynamical system, where σ is a periodic function with a period $m \in \mathbb{N}$. If $x \in X$ such that $x_m = x$, then x is periodic in $(X, \mathfrak{F}, \sigma)$.*

Proof. It is enough to prove that $x_n = x_{n(mod\ m)}$ for every $n \in \mathbb{N}$. If $m = 1$, then $\sigma(n) = \sigma(1)$ for every $n \in \mathbb{N}$ i.e., σ is a constant function. Then $\mathfrak{F}$ consists of only one function, say f and we have a usual dynamical system (X, f). Then $x_1 = x$ is same as saying that $f(x) = x$ and hence x is periodic (in fact, a fixed point).

Now, consider the case where $m > 1$. It is obvious that $x_1 = x_{1(mod\ m)}$. Suppose that $x_n = x_{n(mod\ m)}$ for some $n \in \mathbb{N}$. We now claim that $x_{n+1} = x_{n+1(mod\ m)}$. We have $n = rm + l$ for some $0 \leq l < m$, so that $x_n = x_l$. Then, $n + 1 = rm + l + 1$ would imply that $\sigma(n + 1) = \sigma(l + 1)$ and thus, $x_{n+1} = f_{\sigma(n+1)}(x_n) = f_{\sigma(l+1)}(x_l) = x_{l+1}$. If $0 \leq l \leq m - 2$, then $l + 1 = n + 1(mod\ m)$ and thus we have $x_{n+1} = x_{n+1(mod\ m)}$. In case $l = m - 1$, we have $x_{n+1} = x_{l+1} = x_m$. Since it is given that $x_m = x$, it follows that $x_{n+1} = x = x_0 = x_{n+1(mod\ m)}$. Hence, by induction, we conclude that x is periodic in $(X, \mathfrak{F}, \sigma)$. □

In the following proposition, we characterize the fixed points of a switch system $(X, \mathfrak{F}, \sigma)$ in terms of fixed points of the usual individual systems, (X, f_i), where $\mathfrak{F} = \{f_1, f_2, ..., f_k\}$.

Proposition 5.6. *Let $(X, \mathfrak{F}, \sigma)$ be a switch system, where $\mathfrak{F} = \{f_1, f_2, ..., f_k\}$. An element $x \in X$ is a fixed point in $(X, \mathfrak{F}, \sigma)$ if and only if x is a fixed point in (X, f_i) for every $1 \leq i \leq k$.*

Proof. Suppose x is a fixed point in (X, f_i) for each i. Then $x_1 = f_{\sigma(1)}(x) = x$. Further, if $x_n = x$ for some $n \in \mathbb{N}$, then $x_{n+1} = f_{\sigma(n+1)}(x_n) = f_{\sigma(n+1)}(x) = x$. Thus, by induction, $x_n = x$ for every $n \in \mathbb{N}$ and hence x is a fixed point in $(X, \mathfrak{F}, \sigma)$.

Now, assume that x is a fixed point in $(X, \mathfrak{F}, \sigma)$. Fix an $f_i \in \mathfrak{F}$ for some $1 \leq i \leq k$. We know that $\sigma^{-1}(i)$ is an infinite subset of $\mathbb{N}$. Choose an $r \in \sigma^{-1}(i)$. Then, using the hypothesis that $x_n = x$ for every $n \in \mathbb{N}$, we get $f_i(x) = f_{\sigma(r)}(x) = f_{\sigma(r)}(x_{r-1}) = x_r = x$.

Thus, x is a fixed point of f_i. □

In the literature, a usual dynamical system is said to be topologically transitive, if it has a dense forward orbit. Notice that we have adopted the same definition for topological transitivity to switch systems. Under certain mild conditions on a usual dynamical system (X, f), it can be proved that, if for any two non-empty open sets U and V in X, there exists $x \in U$ with $f^n(x) \in V$ for some $n \in \mathbb{N}$, then (X, f) is topologically transitive (See

[12], Proposition 2.2.1). Here we prove a similar result for switch systems assuming the same conditions. The proof given here uses the ideas, similar to those used in proving the above mentioned result for usual dynamical systems in [12].

Theorem 5.7. *Let X be a second countable locally compact Hausdorff space. If for any two non-empty open sets U and V in X, there exists an $x \in U$ such that $x_n \in V$ for some $n \in \mathbb{N}$, then $(X, \mathfrak{F}, \sigma)$ is topologically transitive.*

Proof. Fix an open set V in X. Define $V' = \bigcup_{n=1}^{\infty} f_{\sigma(1)}^{-1} f_{\sigma(2)}^{-1} ... f_{\sigma(n)}^{-1}(V)$.
If U is any non-empty open set in X, then it is given that there is an $x \in U$ with $x_n \in V$ for some $n \in \mathbb{N}$. Now, $x_n \in V$ implies that $f_{\sigma(n)}(f_{\sigma(n-1)}(...(f_{\sigma(1)}(x))...)) \in V$ and thus $x \in V' \cap U$. Since this is true for any non-empty open set U, it follows that V' is dense in X.

Choose a countable basis for X, say $\{V_i : i \in \mathbb{N}\}$. For each V_i,
define $V_i' = \bigcup_{n=1}^{\infty} f_{\sigma(1)}^{-1} f_{\sigma(2)}^{-1} ... f_{\sigma(n)}^{-1}(V_i)$. It follows from the above discussion that V_i' is dense in X for each i. Thus, the set $Y = \bigcap_{i=1}^{\infty} V_i'$ is intersection of countably many open dense sets in X. Since X is locally compact and Hausdorff, it is a Baire space. Therefore $Y \neq \emptyset$.

Now, choose $y \in Y$. Then $y \in V_i'$ for each $i \in \mathbb{N}$. This implies that $y \in f_{\sigma(1)}^{-1} f_{\sigma(2)}^{-1} ... f_{\sigma(n)}^{-1}(V_i)$ and thus $y_n \in V_i$ for some $n \in \mathbb{N}$. Thus, $\{y_n : n \in \mathbb{N}_0\} \cap V_i \neq \emptyset$ for each $i \in \mathbb{N}$. Since $\{V_i : i \in \mathbb{N}\}$ is a basis for X, we get $\overline{\{y_n : n \in \mathbb{N}_0\}} = X$. Thus, $(X, \mathfrak{F}, \sigma)$ is transitive. □

In a usual system (X, f), the converse of the above theorem is also true. Because, if U and V are any two non empty open sets in X and the orbit of a point $x \in X$ is dense in X, then $f^m(x) \in U$ and $f^n(x) \in V$ for some $m, n \in \mathbb{N}$. Without loss of generality, we can assume that $m < n$; in that case, $f^{n-m}(U) \cap V \neq \emptyset$. However, this argument doesn't work for a switch system. Hence, we ask the question, if the converse of the above theorem is true for a switch system.

We now turn our attention towards the study of recurrent points and ω-limit points. The following example shows that a point $x \in X$ can be a recurrent point in $(X, \mathfrak{F}, \sigma)$ without being a recurrent point in any individual usual system (X, f_i) for $i \in \{1, 2, ..., k\}$. In other words, $\omega_{\mathfrak{F}}(x) \not\subset \bigcup_{i=1}^{k} \omega_{f_i}(x)$, where $\omega_{\mathfrak{F}}(x)$ and $\omega_{f_i}(x)$ are the $\omega-$ limit sets of x in $(X, \mathfrak{F}, \sigma)$ and (X, f_i) respectively.

Example 5.1. *Let $X = [0,2] \cup [4,6]$. Define $f_1, f_2 : X \to X$ as*

$$f_1(x) = \begin{cases} x+4, & \text{if } x \in [0,2], \\ \frac{x-4}{2}+4, & \text{if } x \in [4,6], \end{cases} \quad \text{and } f_2(x) = \begin{cases} \frac{x}{2}, & \text{if } x \in [0,2], \\ x-4, & \text{if } x \in [4,6]. \end{cases}$$

Consider the switch system $(X, \mathfrak{F}, \sigma)$, where $\mathfrak{F} = \{f_1, f_2\}$ and $\sigma : \mathbb{N} \to \{1,2\}$, defined as

$$\sigma(n) = \begin{cases} 1, & \text{if } n \text{ is odd }, \\ 2, & \text{if } n \text{ is even }. \end{cases}$$

Let $x = 1$. Then the trajectory of x is given by

$$x_n = \begin{cases} 1, & \text{if } n = 0 \text{ or } n \text{ is even}, \\ 5, & \text{if } n \text{ is odd}. \end{cases} .$$

Thus, $(x_{n_{2m}}) \to 1$ and hence, $x = 1$ is a recurrent point in $(X, \mathfrak{F}, \sigma)$. However, $x = 1$ is a recurrent point neither in (X, f_1) nor in (X, f_2), because $\omega_{f_1}(1) = \{4\}$ and $\omega_{f_2}(1) = \{0\}$.

However, we can ensure that, if $y \in \omega(x)$ in $(X, \mathfrak{F}, \sigma)$, then $y \in \overline{R(f_i)}$ for some $i \in \{1, 2, ..., k\}$, where $R(f_i)$ is the range of f_i. This is proved in the following theorem.

Theorem 5.8. *If $y \in \omega(x)$ in $(X, \mathfrak{F}, \sigma)$, then $y \in \overline{R(f_i)}$ for some $i \in \{1, 2, ..., k\}$, where $R(f_i)$ is the range of f_i.*

Proof. By definition, for every $m \in \mathbb{N}$, there is an $n_m \in \mathbb{N}$ such that $x_{n_m} \in B(y, \frac{1}{m})$ and $(n_m) \to \infty$. In other words, $f_{\sigma(n_m)}(x_{n_m - 1}) \in B(y, \frac{1}{m})$ and thus $R(f_{\sigma(n_m)}) \cap B(y, \frac{1}{m}) \neq \emptyset$ for every $m \in \mathbb{N}$. Since σ can take only finitely many values, there is an $i \in \{1, 2, ..., k\}$ such that $\sigma(n_m) = i$ for infinitely many m. We now claim that $y \in \overline{R(f_i)}$. For any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $B(y, \frac{1}{N}) \subseteq B(y, \epsilon)$. Since $\sigma(n_m) = i$ for infinitely many m, there is a $K \in \mathbb{N}$ such that $K > N$ and $\sigma(n_K) = i$. Then $B(y, \frac{1}{K}) \subseteq B(y, \epsilon)$ and thus $R(f_i) \cap B(y, \epsilon) = R(f_{\sigma(n_K)}) \cap B(y, \epsilon) \supset R(f_{\sigma(n_K)}) \cap B(y, \frac{1}{K}) \neq \emptyset$. Hence the claim. $\square$

The following example is another instance to show the difference between usual and switch dynamical systems. Here, the switch systems $(X, \mathfrak{F}_1, \sigma_1)$ and $(Y, \mathfrak{F}_2, \sigma_2)$ are conjugate but the map $f_2 \in \mathfrak{F}_1$ is not conjugate to any of the maps in $\mathfrak{F}_2 = \{g_1, g_2\}$.

Example 5.2. *Let $X = Y = \mathbb{R}$. Define maps $f_1,\ f_2,\ g_1$ and g_2 on $\mathbb{R}$ as follows*

$$f_1(x) = \begin{cases} x+1, & \text{if } x \geq 0 \\ 1-x, & \text{if } x < 0 \end{cases}, \qquad f_2(x) = \begin{cases} x+4, & \text{if } x \geq 0 \\ 4, & \text{if } x < 0 \end{cases}$$
$$g_1(x) = \begin{cases} x+\frac{1}{2}, & \text{if } x \geq 0 \\ \frac{1}{2}-x, & \text{if } x < 0 \end{cases}, \qquad g_2(x) = \begin{cases} x+2, & \text{if } x \geq 0 \\ 2-x, & \text{if } x < 0 \end{cases}.$$

Consider the switch systems $(X, \mathfrak{F}_1, \sigma_1)$ and $(Y, \mathfrak{F}_2, \sigma_2)$, where $\mathfrak{F}_1 = \{f_1,\ f_2\}$, $\mathfrak{F}_2 = \{g_1, g_2\}$ and $\sigma_1(n) = \sigma_2(n) = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 2, & \text{if } n \text{ is even} \end{cases}$. Let us denote $\sigma = \sigma_1 = \sigma_2$.

Now, define $h : X \to Y$ as $h(x) = \frac{x}{2}$. We now show that h is a conjugacy from $(X, \mathfrak{F}_1, \sigma_1)$ to $(Y, \mathfrak{F}_2, \sigma_2)$.

For any $x \in X$, note that

$$h(f_1(x)) = g_1(h(x)) = \begin{cases} \frac{x}{2}+\frac{1}{2}, & \text{if } x \geq 0 \\ \frac{1}{2}-\frac{x}{2}, & \text{if } x < 0 \end{cases} \tag{5.4}$$

and

$$h(f_2(f_1(x))) = g_2(g_1(h(x))) = \begin{cases} \frac{x}{2}+\frac{5}{2}, & \text{if } x \geq 0 \\ \frac{5}{2}-\frac{x}{2}, & \text{if } x < 0 \end{cases} \tag{5.5}$$

We now prove that $h(x_n) = (h(x))_n$ for every $n \in \mathbb{N}$ using induction. The above calculations show that the statement is true for $n = 1$ and $n = 2$. Assume now that $h(x_k) = (h(x))_k$ for every $k \leq n$.

If n is even, then

$$\begin{aligned} h(x_{n+1}) &= h(f_{\sigma(n+1)}(x_n)) \\ &= h(f_1(x_n)) \\ &= g_1(h(x_n)) \\ &= g_1((h(x))_n) \\ &= g_{\sigma(n+1)}((h(x))_n) \\ &= (h(x))_{n+1}. \end{aligned}$$

If n is odd, then

$$\begin{aligned} h(x_{n+1}) &= h(f_{\sigma(n+1)}(x_n)) \\ &= h(f_{\sigma(n+1)} \circ f_{\sigma(n)}(x_{n-1})) \\ &= h(f_2 \circ f_1(x_{n-1})) \\ &= g_2 \circ g_1(h(x_{n-1})) \\ &= g_2 \circ g_1((h(x))_{n-1}) \\ &= g_{\sigma(n+1)} \circ g_{\sigma(n)}((h(x))_{n-1}) \\ &= (h(x))_{n+1}. \end{aligned}$$

Hence, h is a conjugacy from $(X, \mathfrak{F}_1, \sigma_1)$ to $(Y, \mathfrak{F}_2, \sigma_2)$.

However, f_2 is not conjugate to any of the maps g_1 and g_2. For, if $\alpha_i : (X, f_2) \to (Y, g_i)$, i=1,2 is a conjugacy, then $f_2(-1) = f_2(-2) = f_2(-3) = 4$ would imply that $g_i(\alpha_i(-1)) = g_i(\alpha_i(-2)) = g_i(\alpha_i(-3)) = \alpha_i(4)$, which is a contradiction because a point in Y has atmost two pre-images under any of the maps g_1 and g_2.

However, it can be observed that if h is a conjugacy from $(X, \mathfrak{F}_1, \sigma_1)$ to $(Y, \mathfrak{F}_2, \sigma_2)$, then for any $x \in X$, $h \circ f_{\sigma_1(1)}(x) = h(x_1) = (h(x))_1 = g_{\sigma_2(1)} \circ h(x)$. Thus, $(X, f_{\sigma_1(1)})$ and $(Y, g_{\sigma_2(1)})$ are conjugate. Hence, we have the following proposition.

Proposition 5.9. *If $(X, \mathfrak{F}_1, \sigma_1)$ and $(Y, \mathfrak{F}_2, \sigma_2)$ are two conjugate switch systems, then $(X, f_{\sigma_1(1)})$ and $(Y, g_{\sigma_2(1)})$ are conjugate (usual) dynamical systems.*

The following theorem can be easily proved using the definition of conjugacy. So, we state it without giving an explicit proof.

Theorem 5.10. *Let h be a conjugacy from $(X, \mathfrak{F}_1, \sigma_1)$ to $(Y, \mathfrak{F}_2, \sigma_2)$ and let $x \in X$. Then,*
(i) x is periodic if and only if $h(x)$ is periodic.
(ii) x is recurrent if and only if $h(x)$ is recurrent.
(iii) $(X, \mathfrak{F}_1, \sigma_1)$ is transitive if and only if $(Y, \mathfrak{F}_2, \sigma_2)$ is transitive.
(iii) $(X, \mathfrak{F}_1, \sigma_1)$ is minimal if and only if $(Y, \mathfrak{F}_2, \sigma_2)$ is minimal.

Finally, we have the following theorem, which ensures the existence of a minimal set in a switch system on a compact space. The same is true for a usual system also (see

Proposition 2.1.2, [12]). In fact, the proof given in [12] for usual systems, also holds for the following theorem. So, we simply state the theorem and omit the proof.

Theorem 5.11. *Let $(X, \mathfrak{F}, \sigma)$ be a switch system. If X is compact, then X contains a minimal set.*

5.4 Switching rotations on S^1

This section deals with a switch dynamical system $(X, \mathfrak{F}, \sigma)$, where $X = S^1$ and $\mathfrak{F}$ is a family of rotations on S^1. We consider the circle S^1 as $[0, 1]/_\sim$, where only the end points 0 and 1 are identified under the equivalence relation $\sim$. In a usual dynamical system (S^1, R_α), where R_α denotes the rotation $x \mapsto x + \alpha(mod\ 1)$, the set of periodic points is either empty or the entire space S^1, depending upon whether α is irrational or rational respectively. We prove a similar result for switch system of rotations also.

Theorem 5.12. *Let $k \in \mathbb{N}$ and for each $1 \leq i \leq k$, let $\alpha_i \in \mathbb{R}$ and define $f_i : S^1 \to S^1$ as $f_i(x) = x + \alpha_i(mod\ 1)$. Let $\mathfrak{F} = \{f_1, f_2, ..., f_k\}$ and σ be any switch function. Then in the switch system $(S^1, \mathfrak{F}, \sigma)$, the set of periodic points, $P(\mathfrak{F})$ is either empty or S^1. If $P(\mathfrak{F}) = S^1$, then σ is a periodic function and there exists $r_i \in \mathbb{Z}$ for each $1 \leq i \leq k$ such that $\sum_{i=1}^{k} r_i\alpha_i \in \mathbb{Z}$.*

Proof. Suppose $P(\mathfrak{F}) \neq \emptyset$ and $x \in S^1$ is periodic with period $m \in \mathbb{N}$. Then $x_{nm+l} = x_l$ for every $0 \leq l < m$ and $n \in \mathbb{N}$. In particular, $x_{nm} = x_0$ for every $n \in \mathbb{N}$. This is the same as $x_0 + \sum_{i=1}^{nm} \alpha_{\sigma(i)}(mod\ 1) = x_0$ and then it follows that

$$\sum_{i=1}^{nm} \alpha_{\sigma(i)} \in \mathbb{Z} \tag{5.6}$$

for every $n \in \mathbb{N}$.

Proceeding along the same lines, for any $0 \leq l < m$, we have $x_{nm+l} = x_l$, or
$(x_0 + \sum_{i=1}^{nm} \alpha_{\sigma(i)} + \sum_{i=nm+1}^{nm+l} \alpha_{\sigma(i)})(mod\ 1) = (x_0 + \sum_{i=1}^{l} \alpha_{\sigma(i)})(mod\ 1)$.
Hence we have $\sum_{i=nm+1}^{nm+l} \alpha_{\sigma(i)} - \sum_{i=1}^{l} \alpha_{\sigma(i)} \in \mathbb{Z}$, or $\sum_{i=1}^{l}(\alpha_{\sigma(nm+i)} - \alpha_{\sigma(i)}) \in \mathbb{Z}$.

Since this is true for any $0 \leq l < m$, it is easy to show by induction that $\alpha_{\sigma(nm+l)} - \alpha_{\sigma(l)} \in \mathbb{Z}$, for every $0 \leq l < m$. This implies that $f_{\sigma(nm+l)} = f_{\sigma(l)}$ and thus $\alpha_{\sigma(nm+l)} = \alpha_{\sigma(l)}$ for every $0 \leq l < m$. Therefore, σ is a periodic function with a period m.

In view of (5.6), we obtain $\sum_{i=1}^{nm} \alpha_{\sigma(i)} \in \mathbb{Z}$ for every $n \in \mathbb{N}$. Moreover, it follows that, for any $y \in S^1$, $n \in \mathbb{N}$ and $0 \leq l < m$,

$$\begin{aligned} y_{nm+l} &= (y_0 + \sum_{i=1}^{nm} \alpha_{\sigma(i)} + \sum_{i=nm+1}^{nm+l} \alpha_{\sigma(i)})(mod\ 1) \\ &= (y_0 + \sum_{i=nm+1}^{nm+l} \alpha_{\sigma(i)}(mod\ 1)) \\ &= (y_0 + \sum_{i=1}^{l} \alpha_{\sigma(i)})(mod\ 1) \\ &= y_l. \end{aligned}$$

The last equality follows because, σ is periodic.

Thus, y is periodic and hence $P(\mathfrak{F}) = S^1$.

Finally, the expression $\sum_{i=1}^{nm} \alpha_{\sigma(i)} \in \mathbb{Z}$ can be written as $\sum_{i=1}^{k} r_i \alpha_i \in \mathbb{Z}$, by making some rearrangements and also taking some r_i's to be 0, if necessary. □

Theorem 5.13. *Let $\mathfrak{F}$ be a family of rotations on S^1 as described in the above theorem. If σ is a periodic function, then $(S^1, \mathfrak{F}, \sigma)$ is either minimal or every point in it is a periodic point.*

Proof. Let $\sigma(nm + l) = \sigma(l)$ for every $n \in \mathbb{N}$ and for every $1 \leq l \leq m$ and $x \in S^1$. Then, for any $n \in \mathbb{N}$, $x_{nm} = x_0 + n\beta$, where $\beta = \sum_{i=1}^{m} \alpha_{\sigma(i)}$.

If $\beta \in \mathbb{Q}$, then $q\beta \in \mathbb{Z}$ for some $q \in \mathbb{N}$ and thus $x_{qm} = x_0$. Since qm is also a period for σ, it follows from Theorem 5.5, that x is a periodic point. Thus, by Theorem 5.12, every point in S^1 is periodic.

If $\beta \in \mathbb{R} \setminus \mathbb{Q}$, then $\overline{\{x_0 + n\beta(mod 1) : n \in \mathbb{N}\}} = S^1$ and thus $\overline{\{x_n : n \in \mathbb{N}\}} = S^1$. Hence, $(S^1, \mathfrak{F}, \sigma)$ is a minimal system. □

5.5 Future Scope and Conclusion

In this chapter, we have introduced the notion of a discrete switch dynamical system. The terminology for switch systems is developed analogous to the theory of topological dynamics. In a usual system (X, f), the trajectory of a point $x \in X$ is determined by only one function f, whereas in a system $(X, \mathfrak{F}, \sigma)$, all the maps $f_1, f_2, ..., f_k \in \mathfrak{F}$ determine the trajectory and σ plays a major role by specifying the function that has to be applied at a given instant of time. After explaining the origin of the idea of a switch system, we related it to the continuous switch systems and the free semigroup actions. Then, we have introduced various notions which are natural generalizations of the corresponding notions in usual systems. This is followed by a section on proving certain results on periodicity, transitivity and ω-limit points and then a section devoted to the rotation maps on S^1.

Though the idea of a switch system and the further notions that are introduced in this chapter are generalizations of notions in a usual system, there are several differences between them. It is evident from Example 5.1 that there can be striking differences between the trajectories followed by a point in a switch system and a usual system. This variety makes the study of switch systems more interesting and significant. There are various other dynamical notions which have not been covered in this chapter. They can be studied in light of this setup.

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List of Articles/Talks/Conferences

Publications

1. Sharan Gopal, **Faiz Imam**, *Periodic points of solenoidal automorphisms in terms of inverse limits*, **Applied General Topology, 22 (2)** (2021), 321-330 . (Published by Universitat Politècnica de València.)

2. **Faiz Imam**, Sharan Gopal, *Topological Aspects of Discrete Switch Dynamical Systems*, accepted for publication in **The Australian Journal of Mathematical Analysis and Applications**.

Preprints

1. **Faiz Imam**, Sharan Gopal, *Periodic points of solenoidal automorphisms in terms of adeles*, **(In Communication)**.

2. **Faiz Imam**, Pabitra Narayan Mandal, Sharan Gopal, *Periodicity, Transitivity and Distality of Real Projective Transformations*, **(In Communication)**.

Talks

1. Contributed talk (In-Person) on "Periodicity of solenoidal automorphisms", on July 29, 2022 , at the Symposia on General Topology and its Relations to Modern Analysis and Algebra held at Institute of Mathematics of the Czech Academy of Sciences, Prague, Czech Republic.

2. Online expository talk on "Solenoidal automorphisms & their periodic properties, on April 5, 2022 , at the Graduate Talks in Geometry and Topology, Australia : GT^2 (supported by MATRIX and Australian Mathematical Sciences Institute).

3. Paper Presentation (Remotely) on "Periodicity of automorphisms on one dimensional solenoids", on March 10, 2022 , in the Special session on Dynamical Systems at the 55th Spring Topology and Dynamics Conference, Baylor University, Waco, Texas, USA.

Conferences/Workshops

1. Weekly Research Paper Writing Workshop (Saturday) organized by Student Alumni Relations Cell, BITS-Pilani, Hyderabad Campus during 24 September - 22 October, 2022 (online).

2. Prague Symposia on General Topology and its Relations to Modern Analysis and Algebra organized under the auspices of the Faculty of Mathematics and Physics of the Charles University, the Institute of Mathematics of the Czech Academy of Sciences and the Faculty of Information Technology of the Czech Technical University during July 25-29, 2022, Prague, Czech Republic.

3. The 39th Annual (Online) Workshop in Geometric Topology from June 6 - 8, 2022 organized by University of Wisconsin-Milwaukee.

4. International Symposium on Differential Geometry and its Applications organized by MANUU, Hyderabad during 30-31 March, 2022 (online).

5. 55th Spring Topology and Dynamics Conference organized by Baylor University, Waco, Texas, USA from 9-12 March, 2022 (online).

6. 11th Tech Topology Conference organized by Georgia Institute of Technology, USA from December 10-12, 2021 (online).

7. IWCME-2021 organized by Mizoram University, India and University of Guadalajara, Mexico from 15-20 November 2021 (online).

8. International Conference on Discrete Groups, Geometry and Arithmetic, August 9-11, 2021 (online) organized by National Centre for Mathematics.

9. Weekend NCM workshop (Thu to Sun) on Finite Groups of Lie Type, 31 July - 3 October 2021 (online) organized by National Centre for Mathematics.

10. Indian National Young Academy of Sciences (INYAS) workshop on Academic & Professional Development for Young Scientists held at University of Hyderabad on 15 February, 2020.

11. International Workshop on Topology and Topological Data Analysis held at Rajagiri School of Engineering & Technology, Kochi during 25-26 October, 2019.

12. Annual Foundation School, Level-I held at University of Hyderabad during May 6-June 1, 2019.

13. International Workshop and Conference on Topology & Applications held at Rajagiri School of Engineering & Technology, Kochi during 5-11 December, 2018.

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