Class of Operators on Variable Exponent Sequence Spaces and their Corresponding Ergodic Version

Thesis

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Certificate

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I, **A. Sri Sakti Swarup**, declare that this thesis titled, '**Class of Operators on Variable Exponent Sequence Spaces and their Corresponding Ergodic Version**' and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.

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Abstract

In this thesis we study a class of operators namely discrete maximal operator, discrete singular operator and commutator of discrete singular operator and their corresponding ergodic versions. We consider centred, non centred, dyadic and sharp maximal operators. The non-centred maximal operator is also known as Hardy-Littlewood maximal operator. We prove inequalities for these discrete operators on weighted $\ell^p_w(\mathbb{Z})$ spaces. Using these inequalities and Rubio de Francia method, we prove the corresponding inequalities on variable exponent $\ell^{p(\cdot)}(\mathbb{Z})$ spaces.

Using Calderón-Coifman-Wiess transference principle, we prove the inequalities for the ergodic maximal operator, ergodic singular operator and commutator of ergodic singular operator on weighted $L^p_w(X, \mathcal{B}, \mu)$ spaces, where $(X, \mathcal{B}, \mu)$ is a probability space equipped with an invertible measure preserving transformation U. With the assumption the ergodic maximal operator is bounded on $L^{p(\cdot)}(X, \mathcal{B}, \mu)$, we prove that the corresponding inequalities for ergodic singular operator and commutator of ergodic singular operator on variable $L^{p(\cdot)}(X, \mathcal{B}, \mu)$ spaces.

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List of Symbols

H	Hilbert Transform
M	Hardy-Littlewood Maximal Operator
M_J	Truncated Hardy-Littlewood Maximal Operator
$\tilde{M}$	Maximal Ergodic Operator
$\tilde{M}_J$	Truncated Maximal Ergodic Operator
$\tilde{H}$	Ergodic Hilbert Transform
T_ϕ	Singular Operator
$T_\phi^\star$	Maximal Singular Operator
$[b, T_\phi]$	Commutator of $b \in BMO$ and a singular operator T_ϕ
$[b, T_\phi]^\star$	Commutator of Maximal Ergodic Singular Operator
$[b, \tilde{T}_\phi]_J^\star$	Commutator of truncated maximal ergodic singular operator
$\rho_{p(\cdot)}(u)$	Modular functional
$\|u\|_{p(\cdot)}(\mathbb{Z})$	Variable Lebesgue norm
$LH_\infty(\mathbb{Z})$	Log-Hölder condition at Infinity
$LH_0(\mathbb{Z})$	Log-Hölder condition at Zero

Chapter 1

Introduction

1.1 Introduction

In this thesis, we study a class of operators namely discrete maximal operator, discrete singular operator, commutator of discrete singular operator and their corresponding ergodic versions. We consider centered, non-centered, dyadic versions of maximal operator and sharp maximal operator. The non-centered version of discrete maximal operator is usually known as Hardy-Littlewood maximal operator which is denoted by M. The generalization of this operator is fractional Hardy-Littlewood maximal operator which is denoted by M_α. We derive inequalities for M_α on $\ell^{p(\cdot)}(\mathbb{Z})$ using Calderón-Zygmund theorem for sequences and by the use of Log Hölder continuity at infinity. When we take $\alpha = 0$, M_α is nothing but the Hardy-Littlewood maximal operator. Using good-Lambda inequalities, relations between maximal operator, we also derive inequalities for Hardy-Littlewood maximal operator M on weighted $\ell^p_w(\mathbb{Z})$.

Using boundedness of Hardy-Littlewood maximal operator M on weighted $\ell^p_w(\mathbb{Z})$ spaces, we derive strong type and weak type inequalities for discrete singular operator T_ϕ and discrete maximal singular operator $T^\star_\phi$ on $\ell^p_w(\mathbb{Z})$. Further, using boundedness of $T^\star_\phi$ on $\ell^p_w(\mathbb{Z})$, boundedness of M on $\ell^{p(\cdot)}(\mathbb{Z})$ and Rubio de Francia method, we derive boundedness of $T^\star_\phi$ on variable exponent sequence spaces $\ell^{p(\cdot)}(\mathbb{Z})$. We extend the same procedure for deriving inequalities for the commutator of discrete singular operators and obtain the corresponding results.

We also consider ergodic versions of these class of operators, i.e maximal ergodic operator, maximal ergodic singular operator and commutator of maximal ergodic singular operator which are denoted by $\tilde{M}, \tilde{T}^\star_\phi, [b, \tilde{T}_\phi]^\star$ respectively. For these operators, we first prove inequalities for these class of operators on weighted $L^p_w(X, \mathcal{B}, \mu)$ spaces, where $(X, \mathcal{B}, \mu)$ is a probability space equipped with an invertible measure preserving transformation U. Assuming the fact that ergodic maximal operator is bounded on $L^{p(\cdot)}(X, \mathcal{B}, \mu)$, we prove boundedness of ergodic singular operator and commutator of ergodic singular operator on $L^{p(\cdot)}(X, \mathcal{B}, \mu)$.

The classical example of the ergodic singular operator is ergodic Hilbert transform which is defined as

$$\tilde{H}f(x) = \lim_{N\to\infty} \sum_{k=-N}^{N}{}' \frac{f(U^{-k}x)}{k}.$$

where the prime in the summation means exclusion of the term $k = 0$. In 1955, Cotlar proved the almost everywhere convergence of the above series for $f \in L^p(X), 1 \leq p < \infty$ and that the operator $\tilde{H}$ is bounded on $L^p(X)$ for $1 < p < \infty$ and is of weak type (1,1). A proof of Cotlar's result using Calderón-Coifman-Weiss principle of transference was given by Petersen in 1983. This proof consists in proving ℓ^p inequalities for the maximal discrete Hilbert transform on sequence spaces, that is, for the operator

$$H^{\star}a(n) = \sup_N \left| \sum_{k=-N}^{N}{}' \frac{a(n-k)}{k} \right|, a \in \ell^p(\mathbb{Z}), \quad 1 \leq p < \infty.$$

Then, inequalities of the maximal ergodic Hilbert transform

$$\tilde{H}^{\star}f(x) = \sup_N \left| \sum_{k=-N}^{N}{}' \frac{f(U^{-k}x)}{k} \right|.$$

are obtained by transference.

For the discrete singular operators and commutators of discrete singular operators we study in this thesis, we prove inequalities for the corresponding maximal operators. These inequalities will ensure the existence of these operators due to the following theorem which is known as Banach principle.

Theorem 1.1.1 (Banach Principle [10])**.** *Let $1 \leq p < \infty$. Let (X, μ) be a measure space and B a Banach space. Let (T_n) be a sequence of operators defined on $L^p_B(X)$. Let*

$$T^{\star}f(x) = \sup_{n\geq 1} \|T_n f(x)\|.$$

If there exists a positive decreasing function $C(\lambda)$ on $(0, \infty)$ which tends to zero as $\lambda \to \infty$ such that

$$\mu(\left\{x \in X : T^{\star}f(x) > \lambda \|f\|_p\right\}) \leq C(\lambda).$$

Then the set $\left\{f \in L^p_B(X) : T_n f(x) \text{ converges a.e}\right\}$ is closed in $L^p_B(X)$.

In Chapter 2, we give the notation and standard or known results that are needed in later chapters. We present the properties of A_p weights on sequences, ergodic A_p weights. We also discuss about the reverse-Hölder inequality and its corollaries. For each result we give a suitable reference for the proof. Also properties of variable exponent sequence spaces and ergodic variable exponent $L^{p(\cdot)}(X, \mathcal{B}, \mu)$ are given. The definition of various maximal operators are defined which are required in later chapters. The definition of BMO is given in this chapter. This definition is required in the theory of commutators of singular operator(both discrete and ergodic versions). We also define ergodic rectangles that will be used to derive some results for commutators of ergodic singular operators.

In Chapter 3, we define norm and present various results on sequence spaces with variable exponents that are needed in later chapters. Properties of modular functional, norm on variable exponent sequence spaces, Log Hölder continuity, variable Hölder inequality, Rubio de Francia method are presented in detail in this chapter.

In Chapter 4, we study several types of discrete maximal operators on real valued sequence spaces and present relationships between them. Mainly, we use Calderón-Zygmund decomposition for sequences, discrete version of good-Lambda inequality which will be used to study relations between these maximal operators. We also derive weighted good-Lambda estimate using reverse Holder inequality. With the use of weighted good-Lambda estimate, we derive weighted version of relationship between sharp maximal operator and dyadic maximal operator. We also discuss weighted inequalities for Hardy-Littlewood maximal operator on $\ell^p_w(\mathbb{Z})$ spaces. Also, we present strong type $(p(\cdot), p(\cdot))$, weak type $(p(\cdot), p(\cdot))$ inequalities for the generalization of Hardy-Littlewood maximal operator M which is also known as fractional Hardy-Littlewood maximal operator M_α on $\ell^{p(\cdot)}(\mathbb{Z})$. For these results, we use Calderón-Zygmund decomposition for sequences, boundedness of Hardy-Littlewood maximal operator on $\ell^p(\mathbb{Z})$ spaces. As a corollary, we obtain strong type $(p(\cdot), p(\cdot))$, weak type $(p(\cdot), p(\cdot))$ inequalities for Hardy-Littlewood maximal operator.

Using Calderón-Coifman-Weiss transference principle, we study the ergodic maximal operator on $L^p_w(X, \mathcal{B}, \mu)$ spaces. We also prove strong type and weak type inequalities for ergodic maximal operator on weighted $L^p_w(X, \mathcal{B}, \mu)$ spaces. Finally, we discuss strong type $(p(\cdot), p(\cdot))$, weak type $(p(\cdot), p(\cdot))$ inequalities for ergodic maximal operator on $L^{p(\cdot)}_w(X, \mathcal{B}, \mu)$ spaces.

In Chapter 5, we define and study discrete singular operators of Calderón-Zygmund kernel type. We derive strong type and weak type inequalities for these operators on $\ell^p_w(\mathbb{Z})$ spaces. Using these results and Rubio de Francia method, we obtain corresponding inequalities for these operators on $\ell^{p(\cdot)}(\mathbb{Z})$. We also derive inequalities for maximal singular operators by transferring the corresponding results from real line. We also derive similar results for ergodic versions of these operators. Finally, we discuss boundedness of ergodic maximal singular operator on $L^{p(\cdot)}(X, \mathcal{B}, \mu)$.

In Chapter 6, we define and study the commutator of discrete singular operator. We derive strong type and weak type inequalities for these operators on $\ell^p_w(\mathbb{Z})$ spaces. Using these results and Rubio de Francia method, we obtain corresponding inequalities for these operators on $\ell^{p(\cdot)}(\mathbb{Z})$. We also derive similar results for ergodic version of these operators.

Chapter 2

Notation and Preliminaries

In this chapter, we specify the notation and state standard and known results which are needed in later chapters. For each such result we have given a suitable reference.

Throughout $\mathbb{Z}$ denotes the group of integers with the counting measure and $\mathbb{R}$ denotes the real line with the Lebesgue measure dx.

For a subset A of $\mathbb{R}$ or $\mathbb{Z}$, $|A|$ denotes the Lebesgue measure of A or counting measure of A respectively. For any set $A \subseteq \mathbb{Z}, \complement A$ denotes complement of A in $\mathbb{Z}$. Let $\mathcal{M}(\mathbb{R})$ denote set of measurable functions on $\mathbb{R}$. The space $L^p(\mathbb{R})$ is defined as $\left\{f \in \mathcal{M}(\mathbb{R}) : \int_{\mathbb{R}} |f(x)|^p dx < \infty\right\}$ and norm in $L^p(\mathbb{R})$ is defined as

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

The space $\ell^p(\mathbb{Z})$ is defined as the set of all sequences $\{a(n) : n \in \mathbb{Z}\}$ such that

$$\sum_{k\in\mathbb{Z}} |a(k)|^p < \infty.$$

Norm in $\ell^p(\mathbb{Z})$ is given by

$$\|a\|_{\ell^p(\mathbb{Z})} = \left(\sum_{k\in\mathbb{Z}} |a(k)|^p\right)^{\frac{1}{p}}.$$

For $\{a(n) : n \in \mathbb{Z}\} \in \ell^p(\mathbb{Z})$, norm in $\ell^p(\mathbb{Z})$ (refer to [7]) can be computed using distribution function as follows

$$\|a\|_{\ell^p(\mathbb{Z})} = \int_0^\infty p\lambda^{p-1} |\{m \in \mathbb{Z} : |a(m)| > \lambda\}| d\lambda.$$

An operator T is bounded on $\ell^p(\mathbb{Z})$ if $\forall a \in \ell^p(\mathbb{Z})$

$$\|Ta\|_{\ell^p(\mathbb{Z})} \leq C_p \|a\|_{\ell^p(\mathbb{Z})}.$$

An operator T is of weak type (1,1) on $\ell^p(\mathbb{Z})$ if for each $a \in \ell_1(\mathbb{Z})$

$$|\{m \in \mathbb{Z} : |Ta(m)| > \lambda\}| \leq \frac{C}{\lambda} \|a\|_1 .$$

2.1 Intervals in integers

Throughout this thesis, $\mathbb{Z}$ denotes the set of all integers and $\mathbb{Z}_+$ denotes the set of all positive integers. For a given interval I in $\mathbb{Z}$ (we always mean finite interval of integers) , $|I|$ always denotes the cardinality of I. For each positive integer N, consider collection of disjoint intervals of cardinality 2^N,

$$\{I_{N,j}\}_{j\in\mathbb{Z}} = \left\{[(j-1)2^N + 1, \dots, j2^N]\right\}_{j\in\mathbb{Z}} .$$

The set of intervals which are of the form $I_{N,j}$ where $N \in \mathbb{Z}_+$ and $j \in \mathbb{Z}$ are called dyadic intervals. For fixed N, $I_{N,j}$ are disjoint. Given a dyadic interval $I = \left\{[(j-1)2^N + 1, \dots, j2^N]\right\}_{j\in\mathbb{Z}}$ and a positive integer m, we define

$$\begin{aligned}
2LI &= [(j-2)2^N + 1, \dots, j2^N] \\
4LI &= [(j-4)2^N + 1, \dots, j2^N] \\
2RI &= [(j-1)2^N + 1, \dots, (j+1)2^N] \\
4RI &= [(j-1)2^N + 1, \dots, (j+3)2^N] \\
3I &= 2LI \cup 2RI \\
5I &= 4LI \cup 4RI
\end{aligned}$$

Let I be an interval in $\mathbb{Z}$ with center j_0. If I is an interval of cardinality $2N$, by center we mean

$$I = [j_0 - N - 1, j_0, j_0 + 1, \dots, j_0 + N].$$

If I is an interval of cardinality $2N + 1$, by center we mean

$$I = [j_0 - N, \dots, j_0, \dots, j_0 + N].$$

If j_0 is a center of an interval I, $2I$ denotes an interval $[j_0 - 2N, \dots, j_0 + 2N]$.
In general, if $I = [m, m+1, \dots n]$ is any interval in $\mathbb{Z}$, then the left doubling of I denoted as $2LI$ and right doubling interval of I denoted as $2RI$ are as shown below

$$2LI = [2m - n - 1, 2m - n, \dots, m, m + 1, \dots n].$$

and

$$2RI = [m, m + 1, \dots n, n + 1, n + 2, \dots, 2n - m + 1].$$

Then $3I = 2LI \cup 2RI$.

2.2 Weights on sequence spaces and Ergodic weights

Throughout this thesis, weight sequences $\{w(n) : n \in \mathbb{Z}\}$ considered are non-zero.

Definition 2.2.1. For a fixed p, $1 < p < \infty$, we say that a positive sequence $\{w(n) : n \in \mathbb{Z}\}$ belongs to class A_p if there is a constant C such that, for all intervals I in $\mathbb{Z}$, we have

$$\left(\frac{1}{|I|}\sum_{k\in I} w(k)\right)\left(\frac{1}{|I|}\sum_{k\in I}\{w(k)\}^{-\frac{1}{p-1}}\right)^{p-1} \leq C.$$

Infimum of all such constants C is called A_p constant. We say that $\{w(m) : m \in \mathbb{Z}\}$ belongs to class A_1 if there a constant C such that, for all intervals I in $\mathbb{Z}$,

$$\frac{1}{|I|}\sum_{k\in I} w(k) \leq Cw(m),$$

for all $m \in I$. Infimum of all such constants C is called A_1 constant.
Let $1 \leq p < \infty$ and $\{w(n) : n \in \mathbb{Z}\} \in A_p(\mathbb{Z})$. We say that a sequence $\{a(n) : n \in \mathbb{Z}\}$ is in $\ell^p_w(\mathbb{Z})$ if

$$\sum_{n\in\mathbb{Z}} |a(n)|^p w(n) < \infty.$$

We define norm in $\ell^p_w(\mathbb{Z})$ by

$$\|a\|_{\ell^p_w(\mathbb{Z})} = \left(\sum_{k\in\mathbb{Z}} |a(k)|^p w(k)\right)^{\frac{1}{p}}.$$

For a subset A of $\mathbb{Z}$, $w(A)$ denotes $\sum_{k\in A} w(k)$.
For a given sequence $\{a(n) : n \in \mathbb{Z}\} \in \ell^p_w(\mathbb{Z})$, the weighted weak(p,p) inequality for a positive weight sequence $\{w(n) : n \in \mathbb{Z}\}$ is as follows:

$$w(\{m \in \mathbb{Z} : Ma(m) > \lambda\}) \leq \frac{C}{\lambda^p}\sum_{m\in\mathbb{Z}} |a(m)|^p w(m).$$

A_p conditions which are due to Muckenhoupt[14] are characterized by $w \in A_p$ is equivalent to boundedness of M on $\ell^p(\mathbb{Z})$.

Now, we give definition of ergodic weights [5].

Definition 2.2.2. Let $(X, \mathbf{B}, \mu)$ be a probability space and U an invertible measure preserving transformation on X. Suppose $1 < p < \infty$ and $w : X \to \mathbb{R}$ be a positive integrable function. The function w is said to satisfy ergodic A_p condition if

$$\operatorname*{ess\,sup}_{x\in X} \sup_{N\geq 1} \left(\frac{1}{2N+1}\sum_{k=-N}^{N} w(U^k x)\right)\left(\frac{1}{2N+1}\sum_{k=-N}^{N} w(U^k x)^{\frac{-1}{p-1}}\right)^{p-1} \leq C.$$

The function w is said to satisfy ergodic A_1 condition if

$$\operatorname*{ess\,sup}_{x\in X} \sup_{N\geq 1} \frac{1}{2N+1}\sum_{k=-N}^{N} w(U^k x) \leq C w(U^m x).$$

for $m = -N, -N+1, \ldots, N$.

Definition 2.2.3. Let $0 < p < \infty$. We say that a function $f \in L^p_w(X)$ if

$$\int_X |f(x)|^p w(x) d\mu(x) < \infty.$$

For $f \in L^p_w(X)$, define norm in $L^p_w(X)$ as follows

$$\|f\|_{L^p_w(X)} = \left(\int_X |f(x)|^p w(x) d\mu(x)\right)^{\frac{1}{p}}.$$

We will also require Young's inequality in this thesis. For any real $a, b \in \mathbb{R}$, and any two real numbers p, q such that $\frac{1}{p} + \frac{1}{q} = 1$, Young's inequality [9] is as follows:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

2.3 Properties of A_p weights on sequence spaces

In the following lemmas, we state and prove the properties of A_p weights on sequence spaces on an interval I.

Lemma 2.3.1. *For $1 \leq p < q$, weight classes A_p and A_q have the following property: $A_p \subset A_q, 1 \leq p < q$.*

Proof. Assume $p = 1$. Let $D = \sup_{k\in I} w(k)^{1-q'}$. Then

$$\begin{aligned}\left(\frac{1}{|I|}\sum_{m\in I} w(m)^{1-q'}\right)^{q-1} &\leq \left(\frac{1}{|I|}\sum_{k\in I} D\right)^{q-1} \\ &\leq \left(\frac{D|I|}{|I|}\right)^{q-1} \leq D^{q-1} = \left(\sup_{k\in I} w(k)^{1-q'}\right)^{q-1} \leq \sup w(m)^{-1}. \qquad (2.3.1)\end{aligned}$$

Hence,

$$\left(\frac{1}{|I|}\sum_{m\in I} w(m)^{1-q'}\right)^{q-1} \le \sup_{m\in I} w(m)^{-1} = \left(\inf_{m\in I} w(m)\right)^{-1} \le C\left(\frac{w(I)}{|I|}\right)^{-1}.$$

For $p > 1$, use Hölder inequality with the following exponents

$$\frac{p-1}{q-1} + \frac{1}{r} = 1,$$

where $\frac{1}{r} = \frac{q-p}{q-1}$.
Let $w(m) \in A_p, m \in I$. Then

$$\begin{aligned}
\left(\frac{1}{|I|}\sum_{m\in\mathbb{Z}} w(m)^{\frac{-1}{q-1}}\right)^{q-1} &\le \frac{1}{|I|^{q-1}}\left(\sum_{m\in\mathbb{Z}} w(m)^{\frac{-1}{p-1}}\right)^{\frac{p-1}{q-1}(q-1)}\left(\sum_{m\in\mathbb{Z}} 1^r\right)^{\frac{1}{r}(q-1)} \\
&= \frac{1}{|I|^{q-1}}\left(\sum_{m\in\mathbb{Z}} w(m)^{\frac{-1}{p-1}}\right)^{p-1}\left(|I|\right)^{q-p} \\
&= \left(\frac{1}{|I|}\sum_{m\in\mathbb{Z}} w(m)^{\frac{-1}{p-1}}\right)^{p-1} \\
&\le C\left(\frac{1}{|I|}\sum_{m\in\mathbb{Z}} w(m)\right)^{-1}.
\end{aligned}$$

Thus, $A_p \subset A_q, 1 \le p < q$. □

Lemma 2.3.2. *For weights $w_0, w_1 \in A_1$ we have the following property: $w_0, w_1 \in A_1 \implies w_0 {w_1}^{1-p} \in A_p$.*

Proof. We need to prove that

$$\left(\frac{1}{|I|}\sum_{m\in I} w_0(m)w_1(m)^{1-p}\right)\left(\frac{1}{|I|}\sum_{m\in I} (w_0(m)w_1(m)^{1-p})^{1-p'}\right)^{p-1} \le C.$$

But this is,

$$\left(\frac{1}{|I|}\sum_{m\in I} w_0(m)w_1(m)^{1-p}\right)\left(\frac{1}{|I|} w_0(m)^{1-p'}w_1(m)\right)^{p-1} \le C.$$

Since $w_0, w_1 \in A_1$ we have $\frac{w_0(I)}{|I|} \le Cw_0(m)$ and $\frac{w_0(I)}{|I|} \le Cw_1(m)$ for all $m \in I$. Further, note that

$$w_0(m)^{-1} \le \sup_{m\in I} w_0(m)^{-1} = \left(\inf_{m\in I} w_0(m)\right)^{-1} \le C\left(\frac{w_0(I)}{|I|}\right)^{-1}.$$

$$w_1(m)^{-1} \le \sup_{m\in I} w_1(m)^{-1} = \left(\inf_{m\in I} w_1(m)\right)^{-1} \le C\left(\frac{w_1(I)}{|I|}\right)^{-1}.$$

Applying this relation for weights with negative exponents, we get

$$\begin{aligned}
&\left(\frac{1}{|I|}\sum_{m\in\mathbb{Z}} w_0(m)w_1(m)^{1-p}\right)\left(\frac{1}{|I|}\sum_{m\in\mathbb{Z}} w_0(m)^{1-p'}w_1(m)\right)^{p-1}\\
&\leq \left(\frac{1}{|I|}\sum_{m\in\mathbb{Z}} w_0(m)\left(\frac{w_1(m)}{|I|}\right)^{1-p}\right)\left(\frac{1}{|I|}\sum_{m\in\mathbb{Z}}\left(\frac{w_0(I)}{|I|}\right)^{1-p'} w_1(m)\right)^{p-1} \leq C.
\end{aligned}$$

‘

□

Lemma 2.3.3. *Let p' be conjugate to p. Then, we have the following property for weights: $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$.*

Proof. Let $w(m) \in A_p, m \in I$. Note that $A_{p'}$ condition for $w^{1-p'}$ is

$$\left(\frac{1}{|I|}\sum_{m\in I} w(m)^{1-p'}\right)\left(\frac{1}{|I|}\sum_{m\in I} w(m)^{(1-p)(1-p')}\right)^{p'-1} \leq C.$$

This equation, when raised to power $p-1$, gives

$$\left(\frac{1}{|I|}\sum_{m\in I} w(m)^{1-p'}\right)^{p-1}\left(\frac{1}{|I|}\sum_{m\in I} w(m)\right) \leq C.$$

Now let $w(m)^{1-p'} \in A_{p'}, m \in I$. From the LHS of A_p condition i.e

$$\begin{aligned}
&\left(\frac{1}{|I|}\sum_{m\in\mathbb{Z}} w(m)^{1-p'}\right)^{p-1}\left(\frac{1}{|I|}\sum_{m\in\mathbb{Z}} w(m)\right)\\
&= \left(\frac{1}{|I|}\sum_{m\in\mathbb{Z}} w(m)^{1-p'}\right)^{p-1}\left(\frac{1}{|I|}\sum_{m\in\mathbb{Z}} w(m)^{(1-p)(1-p')}\right)\\
&= \left(\left(\frac{1}{|I|}\sum_{m\in\mathbb{Z}} w(m)^{1-p'}\right)\left(\frac{1}{|I|}\sum_{m\in\mathbb{Z}} w(m)^{(1-p)(1-p')}\right)^{\frac{1}{p-1}}\right)^{p-1} \leq C.
\end{aligned}$$

□

The proof of the following theorem is similar to the proof of corresponding result in continuous version [7]. We state here without proof.

Theorem 2.3.4. *Let $\{a(n) : n \in \mathbb{Z}\}$ be a positive sequence and $\{w(n) : n \in \mathbb{Z}\} \in A_p, 1 \leq p < \infty$ be a positive weight sequence. Let I be an interval such that $a(m) > 0$ for some $m \in I$. Then,*

1.

$$w(I)\left(\frac{a(I)}{|I|}\right)^p \leq C\sum_{m\in I}|a(m)|^p w(m). \qquad (2.3.4[\text{A}])$$

2. *Given a finite set $S \subset I$,*

$$w(I)\left(\frac{|S|}{|I|}\right)^p \leq Cw(S). \qquad (2.3.4[\mathrm{B}])$$

2.3.4[A] follows from Hölder's inequality and the A_p condition. 2.3.4[B] follows by taking $a = \chi_S$ in 2.3.4[A].

2.4 Reverse Hölder inequality

Here, we state and prove reverse Hölder inequality for weighted sequence spaces. For continuous version of these proofs, refer to [7].

Theorem 2.4.1. *Let $w \in A_p, 1 \leq p < \infty$. Then for every $\alpha, 0 < \alpha < 1$, there exists $\beta, 0 < \beta < 1$, such that given an interval I and $S \subset I$ with $|S| \leq \alpha|I|, w(S) \leq \beta w(I)$.*

Proof. If we replace S by $I \setminus S$ in inequality 2.3.4[B], we get

$$w(I)\left(1 - \frac{|S|}{|I|}\right)^p \leq C(w(I) - w(S)).$$

If $|S| \leq \alpha|I|$, then

$$w(S) \leq \frac{C - (1-\alpha)^p}{C} w(I),$$

which gives us the desired result with $\beta = 1 - C^{-1}(1-\alpha)^p$. □

Theorem 2.4.2. *[Reverse Hölder Inequality] Let $w \in A_p, 1 \leq p < \infty$. Then there exists constants C and $\epsilon > 0$, depending only on p and the A_p constant of w, such that for any interval I,*

$$\left(\frac{1}{|I|}\sum_{m \in I} w(m)^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}} \leq \frac{C}{|I|}\sum_{m \in I} w(m).$$

Proof. Fix an interval I and consider the following increasing sequence

$$\frac{w(I)}{|I|} = \lambda_0 < \lambda_1 \cdots < \lambda_k < \ldots.$$

For each λ_k, apply the Calderón-Zygmund decomposition at height λ_k. We get a family of disjoint cubes $\{I_{k,j}\}$

$$\lambda_k < \frac{1}{I_{k,j}} \sum_{m \in I_{k,j}} w(m) \leq 2\lambda_k.$$

and if

$$m \notin \Omega_k = \cup_j I_{k,j}, \frac{1}{|I_{k,j}|} \sum_{m \in I_{k,j}} w(m) \leq \lambda, \forall j.$$

Note from construction $\Omega_{k+1} \subseteq \Omega_k$. If we fix I_{k,j_0} from Calderón Zygmund decomposition at height λ_k, then $I_{k,j_0} \cap \Omega_{k+1}$ is union of intervals $I_{k+1,i}$ Therefore,

$$\begin{aligned}
|I_{k,j_0} \cap \Omega_{k+1}| &= \sum_i |I_{k+1,i}| \\
&\leq \frac{1}{\lambda_{k+1}} \sum_i \sum_{m \in I_{k+1,i}} w(m) \\
&\leq \frac{1}{\lambda_{k+1}} \sum_{m \in I_{k,j_0}} w(m) \\
&\leq \frac{2\lambda_k}{\lambda_{k+1}} |I_{k,j_0}|.
\end{aligned}$$

Fix $\alpha < 1$, choose λ_k such that $\frac{2\lambda_k}{\lambda_{k+1}} = \alpha$. i.e

$$\lambda_k = (2\alpha^{-1})^k \frac{w(I)}{|I|}.$$

Then $|I_{k,j_0} \cap \Omega_{k+1}| \leq \alpha |I_{k,j_0}|$ By Lemma 2.4.1 , there exists $\beta < 1$ such that

$$w(I_{k,j_0} \cap \Omega_{k+1}) \leq \beta w(I_{k,j_0}).$$

Now sum over all intervals in decomposition at height λ_k. Then get $w(\Omega_{k+1}) \leq \beta w(\Omega_k)$. Iterating $w(\Omega_k) \leq \beta^k w(\Omega_0)$. Similarly $|\Omega_k| \leq \alpha^k |\Omega_0|$. Hence, $|\cap_k \Omega_k| = \lim_{k\to\infty} |\Omega_k| = 0$. Therefore,

$$\begin{aligned}
\frac{1}{|I|} \sum_I w^{1+\epsilon} &= \frac{1}{|I|} \sum_{I \setminus \Omega_0} w^{1+\epsilon} + \frac{1}{|I|} \sum_{k=0}^{\infty} \sum_{(\Omega_k \setminus \Omega_{k+1}) \cap I} w^{1+\epsilon} \\
&\leq \lambda_0{}^{\epsilon} \frac{w(I)}{|I|} + \frac{1}{|I|} \sum_{k=0}^{\infty} \lambda_{k+1}^{\epsilon} w(I) \\
&\leq \lambda_0{}^{\epsilon} \frac{w(I)}{|I|} + \frac{1}{|I|} \sum_{k=0}^{\infty} (2\alpha^{-1})^{(k+1)\epsilon} \lambda_0{}^{\epsilon} \beta^k w(I).
\end{aligned}$$

Fix $\epsilon > 0$ such that $(2\alpha^{-1})^{\epsilon}\beta < 1$, then series converges and last term is bounded by $c\frac{\lambda_0{}^{\epsilon} w(I)}{|I|}$. Since $\lambda_0 = \frac{w(I)}{|I|}$, result follows. □

Corollary 2.4.3.

$$A_p = \cup_{q<p} A_q, 1 < p < \infty.$$

Proof. If $w(m) \in A_p, m \in I$, then by Lemma[2.3.2], $w(m)^{1-p'} \in A_{p'}, m \in I$. This means that

$$\left(\frac{1}{|I|} \sum_{m \in I} w(m)^{1-p'} \right) \left(\frac{1}{|I|} \sum_{m \in I} w(m)^{(1-p')(1-p)} \right)^{p'-1} \leq C.$$

But $(1-p')(1-p) = 1$. So,

$$\left(\frac{1}{|I|}\sum_{m\in I} w(m)^{1-p'}\right)\left(\frac{1}{|I|}\sum_{m\in I} w(m)\right)^{p'-1} \leq C.$$

This gives

$$\left(\frac{1}{|I|}\sum_{I} w(m)^{1-p'}\right)^{1+\epsilon} \leq C\left(\frac{1}{|I|}\sum_{m\in I} w(m)\right)^{(1-p')(1+\epsilon)}.$$

Now, fix q such that $q'-1 = (p'-1)(1+\epsilon)$ where $\frac{1}{q}+\frac{1}{q'} = 1$. Since $q' > p'$, $q < p$. Therefore,

$$\left(\frac{1}{|I|}\sum_{m\in I} w^{1-p'}\right) \leq C\left(\frac{1}{|I|}\sum_{m\in I} w(m)\right)^{\frac{1-q'}{1+\epsilon}}. \tag{2.4.1}$$

Also note, $w^{1-p'} \in A_{p'}$ satisfies reverse Hölder inequality for some $\epsilon > 0$. This gives,

$$\left(\frac{1}{|I|}\sum_{m\in I} w(m)^{(1-p')(1+\epsilon)}\right)^{\frac{1}{1+\epsilon}} \leq \frac{c}{|I|}\sum_{m\in I} w(m)^{1-p'}. \tag{2.4.2}$$

that is using 2.4.1

$$\left(\frac{1}{|I|}\sum_{m\in I} w(m)^{1-q'}\right) \leq \left(\frac{C}{|I|}\sum_{m\in I} w(m)^{1-p'}\right)^{1+\epsilon} \leq C\left(\frac{1}{|I|}\sum_{m\in I} w(m)\right)^{1-q'}.$$

It follows that

$$\left(\frac{1}{|I|}\sum_{m\in I} w(m)^{1-q'}\right)^{q-1} \leq C\left(\frac{1}{|I|}\sum_{m\in I} w(m)\right)^{-1},$$

which shows that $w \in A_q$. Hence $w \in \cup_{q<p} A_q$.
For the converse, if $w \in \cup_{q<p} A_q$, then by Lemma[2.3.1], $w \in A_p$. □

Corollary 2.4.4. *If $w \in A_p, 1 \leq p < \infty$, then there exists $\epsilon > 0$ such that $w^{1+\epsilon} \in A_p$*

Proof. Let $w(m) \in A_p, m \in I$, Then

$$\left(\frac{1}{|I|}\sum_{m\in I} w(m)\right)\left(\frac{1}{|I|}\sum_{m\in I} w(m)^{1-p'}\right)^{p-1} \leq C.$$

Additionally, we will show that $w(m)^{1+\epsilon} \in A_p, m \in I$.

$$\left(\frac{1}{|I|}\sum_{m\in I} w(m)^{1+\epsilon}\right)\left(\frac{1}{|I|}\sum_{m\in I} w(m)^{(1-p')(1+\epsilon)}\right)^{p-1}$$
$$\leq \left(\frac{C}{|I|}\sum_{m\in I} w(m)\right)^{1+\epsilon}\left(\frac{C}{|I|}\sum_{m\in I} w(m)^{1-p'}\right)^{(1+\epsilon)(p-1)} \text{Using Reverse Hölder Inequality.}$$

$$\leq \left(\left(\frac{C}{|I|}\sum_{m\in I} w(m)\right)\left(\frac{C}{|I|}\sum_{m\in I} w(m)^{1-p'}\right)^{p-1}\right)^{1+\epsilon} \leq C^{1+\epsilon} \leq C.$$

Hence $w^{1+\epsilon} \in A_p$. □

Corollary 2.4.5. *If $w \in A_p, 1 \leq p < \infty$, then there exists $\delta > 0$ such that given an interval I and $S \subset I$,*

$$\frac{w(S)}{w(I)} \leq C\left(\frac{|S|}{|I|}\right)^{\delta}$$

Proof. Fix $S \subset I$ and suppose w satisfies the reverse Hölder inequality with exponent $1+\epsilon$. Then

$$w(S) = \sum_{m\in I} \chi_S w(m) \leq \left(\sum_{m\in I} w(m)^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}} |S|^{\frac{\epsilon}{1+\epsilon}} \leq Cw(I)\left(\frac{|S|}{|I|}\right)^{\frac{\epsilon}{1+\epsilon}}.$$

□

2.5 Variable exponent sequence spaces and Ergodic $L^{p(\cdot)}$ spaces

In this thesis, the following notation are used. Given a bounded sequence $\{p(n) : n \in \mathbb{Z}\}$ which takes values in $[1, \infty)$, define $\ell^{p(\cdot)}(\mathbb{Z})$ to be set of all sequences $\{a(n) : n \in \mathbb{Z}\}$ such that for some $\lambda > 0$, $\sum_{k\in\mathbb{Z}}(\frac{|a(k)|}{\lambda})^{p(k)} < \infty$.

Throughout this thesis, $\{p(n) : n \in \mathbb{Z}\}$ denotes a bounded sequence, which takes values in $[1, \infty)$. Define $p_- = \inf\{p(n) : n \in \mathbb{Z}\}, p_+ = \sup\{p(n) : n \in \mathbb{Z}\}$.
Let $\mathcal{S}$ denote the set of all bounded sequences which takes values in $[1, \infty)$.

Definition 2.5.1. We define 3 canonical subsets of $\Omega \in \mathbb{Z}$ as follows.

$$\Omega_\infty^{p(\cdot)} = \{n \in \mathbb{Z} : p(n) = \infty\}.$$
$$\Omega_1^{p(\cdot)} = \{n \in \mathbb{Z} : p(n) = 1\}.$$
$$\Omega_\star^{p(\cdot)} = \{n \in \mathbb{Z} : 1 < p(n) < \infty\}.$$

Then define modular functional associated with $p(\cdot)$ as

$$\rho_{p(\cdot)}(a) = \sum_{k\in\mathbb{Z}\setminus\Omega_\infty} |a(k)|^{p(k)} + \|a\|_{\ell^\infty(\Omega_\infty)}.$$

Note, when $p_+ < \infty$, modular functional becomes

$$\rho_{p(\cdot)}(a) = \sum_{k\in\mathbb{Z}} |a(k)|^{p(k)}$$

Throughout this thesis, we assume $p_+ < \infty$. Further for a given sequence $\{a(k) : k \in \mathbb{Z}\}$, define norm in $\ell^{p(\cdot)}(\mathbb{Z})$ as

$$\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})} = \inf\left\{\lambda > 0 : \rho_{p(\cdot)}(\frac{a}{\lambda}) \leq 1\right\}.$$

$\| \quad \|_{\ell^{p(\cdot)}(\mathbb{Z})}$ is a norm [12]. A similar norm on $\ell^{p(\cdot)}(\mathbb{R}^n)$ is defined in [8] and there it is proved it is a norm.

Further, details about variable exponent sequence spaces are given in Chapter 3.

2.6 Variable exponent $L^{p(\cdot)}(X, \mathcal{B}, \mu)$ spaces

In this thesis, we derive inequalities for ergodic maximal operator, ergodic singular operator and commutator of ergodic singular operator on $L^{p(\cdot)}(X, \mathcal{B}, \mu)$ spaces.

Let $(X, \mathcal{B}, \mu)$ be a σ-finite, complete measure space. Then by $L^0(X, \mu)$, we denote the space of all real valued, μ measurable functions on X. For the following details refer [13].

The variable exponent Lebesgue space $L^{p(\cdot)}(X, \mu)$ is

$$L^{p(\cdot)}(X, \mu) = \left\{f \in L^0(X, \mu) : \rho_{L^{p(\cdot)}(X)}(\frac{f}{\lambda}) < \infty \quad \text{for some} \quad \lambda > 0\right\},$$

equipped with the norm

$$\|f\|_{L^{p(\cdot)}(X,\mu)} = \inf\left\{\lambda > 0 : \rho_{L^{p(\cdot)}(X)}(\frac{f}{\lambda}) \leq 1\right\}.$$

2.7 Maximal operators on sequence spaces and Ergodic maximal operators

Let $\{a(n) : n \in \mathbb{Z}\}$ be a sequence. We define the following types of Hardy-Littlewood maximal operators as follows:

Definition 2.7.1. If I_r is the interval $\{-r, -r+1, \ldots, 0, 1, 2, \ldots, r-1, r\}$, define centered Hardy-Littlewood maximal operator

$$M'a(m) = \sup_{r>0} \frac{1}{(2r+1)} \sum_{n \in I_r} |a(m-n)|.$$

For any positive integer J, define truncated centered Hardy-Littlewood maximal operator as

$$(M'_J a)(m) = \sup_{J>r>0} \frac{1}{(2r+1)} \sum_{n \in I_r} |a(m-n)|.$$

Definition 2.7.2 (Hardy-Littlewood Maximal Operator)**.** We define Hardy-Littlewood maximal operator as follows

$$Ma(m) = \sup_{m \in I} \frac{1}{|I|} \sum_{n \in I} |a(n)|,$$

where the supremum is taken over all intervals containing m. For any positive integer J, define truncated Hardy-Littlewood maximal operator as

$$M_J a(m) = \sup_{J \geq m \in I} \frac{1}{|I|} \sum_{n \in I} |a(n)|.$$

Definition 2.7.3 (Dyadic Hardy-Littlewood Maximal Operator)**.** We define dyadic Hardy-Littlewood maximal operator as follows:

$$M_d a(m) = \sup_{m \in I} \frac{1}{|I|} \sum_{k \in I} |a(k)|,$$

where supremum is taken over all dyadic intervals containing m.

Definition 2.7.4 (Sharp Maximal Operator)**.** Given a sequence $\{a(n) : n \in \mathbb{Z}\}$ and an interval I, let a_I denote average of $\{a(n) : n \in \mathbb{Z}\}$ on I. Let, $a_I = \frac{1}{|I|} \sum_{m \in I} a(m)$. Define the sharp maximal operator $M^{\#}$ as follows

$$M^{\#} a(m) = \sup_{m \in I} \frac{1}{|I|} \sum_{n \in I} |a(n) - a_I|,$$

where the supremum is taken over all intervals I containing m.

Definition 2.7.5 (Fractional Hardy-Littlewood Maximal Operator)**.** Given a non-negative sequence of real numbers, $\{a(n) : n \in \mathbb{Z}\}$, $0 \leq \alpha < 1$, define fractional Hardy-Littlewood Maximal operator as follows:

$$M_\alpha a(n) = \sup_{n \in I} |I|^{\alpha - 1} \sum_{k \in I} |a(k)|,$$

where supremum is taken over all intervals of integers which contain n. When $\alpha = 0$, fractional Hardy-Littlewood maximal operator becomes Hardy-Littlewood maximal operator.

Here, we define a sequence space called sequences of bounded mean oscillation which is denoted by BMO($\mathbb{Z}$). This space plays an important role in commutators which are presented in Chapter 6.

Definition 2.7.6 (BMO)**.** We say that sequence $\{a(n) : n \in \mathbb{Z}\}$ has bounded mean oscillation if the sequence $M^{\#}a$ is bounded. The space of sequences with this property is denoted by BMO($\mathbb{Z}$). We define a norm in BMO($\mathbb{Z}$) by $\|a\|_\star = \left\|M^{\#}a\right\|_\infty$. The space BMO($\mathbb{Z}$) is studied in [2].

2.8 Boundedness of Hardy-Littlewood maximal operator

Here, we state the inequalities for Hardy-Littlewood maximal operator on sequence spaces [4]. The following well-known theorem is proved exactly as in the case of $\mathbb{R}$ (See [7]).

Theorem 2.8.1. *Let $1 \leq p \leq \infty$. There exist constants $C_p > 0$ such that*

1. *(i) If $1 < p \leq \infty$ then*
$$\|Ma\|_p \leq C_p \|a\|_p\,, \quad \forall a \in \ell^p(\mathbb{Z}).$$

2. *(ii)*
$$\Big|\{n : Ma(n) > \lambda\}\Big| \leq \frac{C_1}{\lambda}\|a\|_1\,, \quad \forall a \in \ell^1(\mathbb{Z}) \quad \textit{and} \quad \forall \lambda > 0.$$

2.9 Ergodic rectangle

Let $(X, \mathbf{B}, \mu)$ be a probability space, U an ergodic invertible measure preserving transformation on X.

Definition 2.9.1 (Ergodic Rectangle)**.** [2] Let E be a subset of X with positive measure and let $K \geq 1$ be such that $U^iE \cap U^jE = \phi$ if $i \neq j$ and $-K \leq i, j \leq K$. Then the set $R = \cup_{i=-K}^{K} U^iE$ is called ergodic rectangle of length $2K + 1$ with base E.

Lemma 2.9.2. *Let $(X, \boldsymbol{B}, \mu)$ be a probability space, U an ergodic invertible measure preserving transformation on X and K a positive integer.*

1. *If $F \subseteq X$ is a set of positive measure then there exists a subset $E \subseteq F$ of positive measure such that E is base of an ergodic rectangle of length $2K + 1$.*

2. *There exists a countable family $\{E_j\}$ of bases of ergodic rectangles of length $2K + 1$ such that $X = \cup_j E_j$.*

Chapter 3

Variable Exponent Sequence Spaces

In this chapter, we state and prove various results on variable exponent sequence spaces. Most of the proofs are similar to corresponding proofs for the real line. We give the proofs for the sake of completeness.

3.1 Variable exponent sequence spaces

Definition 3.1.1. Given a sequence $p(\cdot) \in \mathcal{S}$, we say that $p(\cdot)$ is locally log-Hölder continuous, if there exists positive real constants C_0 such that for all $m, n \in \mathbb{Z}$,

$$|p(m) - p(n)| \leq \frac{C_0}{-\log(|m-n|)} \quad \forall m, n \in \mathbb{Z}.$$

In this case, we write $p(\cdot) \in LH_0(\mathbb{Z})$.

Definition 3.1.2. Given a sequence $p(\cdot) \in \mathcal{S}$, we say that $p(\cdot)$ is log-Hölder continuous at infinity, if there exists positive real constants C_∞, p_∞ such that for all $n \in \mathbb{Z}$,

$$|p(n) - p_\infty| \leq \frac{C_\infty}{\log(e + |n|)}, n \in \mathbb{Z},$$

and e is exponential number. In this case, we write $p(\cdot) \in LH_\infty(\mathbb{Z})$.

Remark 3.1.3. One can easily observe that local Log Hölder continuity automatically follows for sequence spaces.

Definition 3.1.4. Let $p(\cdot) \in \mathcal{S}$. Then $\{q(n) : n \in \mathbb{Z}\}$ is called conjugate sequence of $\{p(n) : n \in \mathbb{Z}\}$ which satisfies

$$\frac{1}{p(n)} + \frac{1}{q(n)} = 1 \quad \forall n \in \mathbb{Z}.$$

Recall that, we define 3 canonical subsets of $\Omega \in \mathbb{Z}$ as follows.

Definition 3.1.5. We define.

$$\begin{aligned}
\Omega_{\infty}^{p(\cdot)} &= \{n \in \mathbb{Z} : p(x) = \infty\} \\
\Omega_{1}^{p(\cdot)} &= \{n \in \mathbb{Z} : p(x) = 1\} \\
\Omega_{\star}^{p(\cdot)} &= \{n \in \mathbb{Z} : 1 < p(x) < \infty\}.
\end{aligned}$$

Lemma 3.1.6. *Let $\{q(n)\}$ be the sequence which satisfies $\frac{1}{p(n)} + \frac{1}{q(n)} = 1, \forall n \in \mathbb{Z}$. Let $1 \leq p_- \leq p(n) \leq p_+ < \infty$, $\forall n \in \mathbb{Z}$. The following are equivalent:*

1. $p(\cdot) \in LH_\infty(\mathbb{Z})$
2. $\frac{1}{p(\cdot)} \in LH_\infty(\mathbb{Z})$
3. $\frac{1}{q(\cdot)} \in LH_\infty(\mathbb{Z})$
4. $q(\cdot) \in LH_\infty(\mathbb{Z})$.

Proof. (a) We shall prove (1) $\implies$ (2). Let $p(\cdot) \in LH_\infty(\mathbb{Z})$. Note, that when $p_+ < \infty$, $\forall n \in \mathbb{Z}$,

$$|\frac{1}{p(n)} - \frac{1}{(p_\infty)}| = |\frac{p(n) - (p_\infty)}{p(n)(p_\infty)}| \leq \frac{|p(n) - (p_\infty)|}{(p_-)(p_\infty)} \leq \frac{\frac{C_\infty}{(p_-)(p_\infty)}}{\log(e + |n|)},$$

for some LH_∞ constant, which is $k_\infty = \frac{C_\infty}{(p_-)(p_\infty)}$.

(b) We shall prove (2) $\implies$ (1). Let $\frac{1}{p(\cdot)} \in LH_\infty(\mathbb{Z})$. Then, $\forall n \in \mathbb{Z}$

$$\begin{aligned}
|p(n) - (p_\infty)| = |p(n)(p_\infty)\left(\frac{1}{p(n)} - \frac{1}{(p_\infty)}\right)| &\leq |p(n)(p_\infty)|\frac{C_\infty}{\log(e + |n|)} \\
&\leq (p_+)(p_\infty)\frac{C_\infty}{\log(e + |n|)} \leq \frac{k_\infty}{\log(e + |n|)},
\end{aligned}$$

for some LH_∞ constant which is $k_\infty = (p_+)(p_\infty)C_\infty$.

(c) We shall prove that (1) $\implies$ (3). Let $p(\cdot) \in LH_\infty(\mathbb{Z})$. Note, that when $p_+ < \infty$, for $n \in \mathbb{Z}$,

$$\begin{aligned}|\frac{1}{q(n)} - \frac{1}{q_\infty}| = |\frac{1}{p(n)} - \frac{1}{(p_\infty)}| = |\frac{p(n) - (p_\infty)}{p(n)(p_\infty)}| &\leq \frac{|p(n) - (p_\infty)|}{(p_-)(p_\infty)} \\ &\leq \frac{\frac{C_\infty}{(p_-)(p_\infty)}}{\log(e + |n|)},\end{aligned}$$

for some LH_∞ constant, which is $k_\infty = \frac{C_\infty}{(p_-)(p_\infty)}$.

(d) We shall prove (3) $\implies$ (4) and (4) $\implies$ (1)
Let $\frac{1}{q(\cdot)} \in LH_\infty(\mathbb{Z})$. Then $\frac{1}{\frac{1}{q(\cdot)}} \in LH_\infty(\mathbb{Z})$, which implies that $q(\cdot) \in LH_\infty(\mathbb{Z})$. This shows (3) $\implies$ (4).
(4) $\implies$ (1) follows same argument as (c) .

$\square$

Lemma 3.1.7. *Let $q(\cdot)$ be conjugate sequence to $p(\cdot)$. Let $1 \leq p_- \leq p(n) \leq p_+ < \infty$, $\forall n \in \mathbb{Z}$. Then, $p(\cdot) \in LH_\infty(\mathbb{Z})$ implies $p_\infty \geq p_-$.*

Proof. Given that $p(\cdot) \in LH_\infty(\mathbb{Z})$ implies $\frac{1}{p(\cdot)} \in LH_\infty(\mathbb{Z})$ using Lemma[3.1.6] and since $p_- \geq 1$, it follows that

$$\frac{1}{p_\infty} = |\frac{1}{p_\infty}| = |\frac{1}{p(n)} + \frac{1}{p_\infty} - \frac{1}{p(n)}| \leq |\frac{1}{p(n)}| + |\frac{1}{p_\infty} - \frac{1}{p(n)}| \leq \frac{1}{p_-} + \frac{C_\infty}{\log(e + |n|)}.$$

Since this is true for every n, $p_\infty \geq p_-$. $\square$

We state below some properties of modular function [4],[8]. Recall that $\rho_{p(\cdot)}$ denotes the modular functional associated with the variable exponent $p(\cdot)$ norm which is defined as

$$\rho_{p(\cdot)} = \sum_{k \in \mathbb{Z}} |a(k)|^{p(k)},$$

and $\mathcal{S}$ denote the set of all bounded sequences which takes values in $[1, \infty)$. Also, the norm in $\ell^{p(\cdot)}(\mathbb{Z})$ is defined as

$$\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})} = \inf\left\{\lambda > 0 : \rho_{p(\cdot)}(\frac{a}{\lambda}) \leq 1\right\}.$$

Lemma 3.1.8. *Let $\{u(n)\}$ be a non-negative sequence of real numbers. Let $p(\cdot) \in \mathcal{S}$. Then*

1. *For all u, $\rho_{p(\cdot)}(u) \geq 0$ and $\rho_{p(\cdot)}(|u|) = \rho_{p(\cdot)}(u)$.*

2. *$\rho_{p(\cdot)}(u) = 0$ if and only if $u(k) = 0$ for all $k \in \mathbb{Z}$.*

3. *If $\rho_{p(\cdot)}(u) < \infty$, then $u(k) < \infty$ for all $k \in \mathbb{Z}$.*

4. *$\rho_{p(\cdot)}$ is convex: Given $\alpha, \beta \geq 0, \alpha + \beta = 1, \rho_{p(\cdot)}(\alpha u + \beta v) \leq \alpha \rho_{p(\cdot)}(u) + \beta \rho_{p(\cdot)}(v)$.*

5. *If for every k, $|u(k)| \geq |v(k)|$, then $\rho_{p(\cdot)}(u) \geq \rho_{p(\cdot)}(v)$.*

6. *If for some $\delta > 0, \rho_{p(\cdot)}(\frac{u}{\delta}) < \infty$, then the function $\lambda \to \rho_{p(\cdot)}(\frac{u}{\lambda})$ is continuous and decreasing on $[\delta, \infty)$. Further $\rho_{p(\cdot)}(\frac{u}{\lambda}) \to 0$ as $\lambda \to \infty$.*

Lemma 3.1.9. *Let $\{a(n), n \in \mathbb{Z}\}$ be a non-negative sequence of real numbers such that $a \in \ell^{p(\cdot)}(\mathbb{Z})$ and let $p(\cdot) \in \mathcal{S}$.*

1. *For all $\lambda \geq 1$,*

$$\lambda^{p-} \rho_{p(\cdot)}(a) \leq \rho_{p(\cdot)}(\lambda a) \leq \lambda^{p+} \rho_{p(\cdot)}(a).$$

2. *When $0 < \lambda < 1$, the reverse inequalities are true.*

$$\lambda^{p+} \rho_{p(\cdot)}(a) \leq \rho_{p(\cdot)}(\lambda a) \leq \lambda^{p-} \rho_{p(\cdot)}(a).$$

Proof. For $\lambda \geq 1$,

$$\rho_{p(\cdot)}(\lambda a) = \sum_{n=1}^{\infty} \Big(\lambda a(n)\Big)^{p(n)} = \sum_{n=1}^{\infty} \lambda^{p(n)} a(n)^{p(n)} \leq \lambda^{p+} \rho_{p(\cdot)}(a).$$

Further,

$$\lambda^{p-} \rho_{p(\cdot)}(a) = \sum_{n=1}^{\infty} \lambda^{p-} a(n)^{p(n)} \leq \sum_{n=1}^{\infty} \Big(\lambda a(n)\Big)^{p(n)} = \rho_{p(\cdot)}(\lambda a).$$

Similar proof can be used for the case $0 < \lambda < 1$. □

The following lemma gives the connection between modular functional and $\ell^{p(\cdot)}(\mathbb{Z})$ norm.

Lemma 3.1.10. *Let $\{u(n), n \in \mathbb{Z}\}$ be a non-negative sequence of real numbers. Let $p(\cdot) \in \mathcal{S}$. Then $u \in \ell^{p(\cdot)}(\mathbb{Z})$ if and only if*

$$\rho_{p(\cdot)}(u) = \sum_{k \in \mathbb{Z}} |u(k)|^{p(k)} < \infty.$$

Proof. If $\rho_{p(\cdot)}(u) < \infty$, then by definition of norm $u \in \ell^{p(\cdot)}(\mathbb{Z})$. Conversely since $u \in \ell^{p(\cdot)}(\mathbb{Z})$ by definition of norm, $\rho_{p(\cdot)}(\frac{u}{\lambda}) < \infty$ for some $\lambda > 0$. Further by (6) of modular functional properties, we have $\rho_{p(\cdot)}(\frac{u}{\lambda}) < \infty$ for some $\lambda > 1$. So, it follows that

$$\rho_{p(\cdot)}(u) = \sum_{k\in\mathbb{Z}} \left(\frac{|u(k)|\lambda}{\lambda}\right)^{p(k)} \leq \lambda^{p_+}\rho_{p(\cdot)}(\frac{u}{\lambda}) < \infty.$$

□

Proof of (4) follows from the fact that when$p_- \geq 1$, the mapping $x \to |x|^p$ is convex.

Lemma 3.1.11. $\| \|_{p(\cdot)}$ *defined is a norm on the linear space* $\ell^{p(\cdot)}(\mathbb{Z})$ *[12].*

Proof. If $u \in \ell^{p(\cdot)}(\mathbb{Z})$, then the set $\left\{\lambda > 0 : \rho_{p(\cdot)}(\frac{u}{\lambda}) > 1\right\}$ is bounded below by zero. Since $\mathbb{R}$ has greatest lower bound property(as such every set which is bounded below has infimum) $\inf\left\{\lambda > 0 : \rho_{p(\cdot)}(\frac{u}{\lambda}) \leq 1\right\}$ exists. Suppose $u \in \ell^{p(\cdot)}(\mathbb{Z})$. Then the set $\left\{\lambda > 0 : \rho_{p(\cdot)}(\frac{u}{\lambda}) \leq 1\right\}$ is nonempty and bounded below. Hence by the greatest lower bound property of real line, we have

$$\inf\left\{\lambda > 0 : \rho(\frac{u}{\lambda}) \leq 1\right\}.$$

exists. Next, we check the required properties of norm as below.

1. Positive Definiteness: If $u = 0$, then $u(i) = 0$ for each $i \in \mathbb{Z}$ and thus $\frac{u(i)}{\lambda} = 0$ for each $i \in \mathbb{Z}$. As a result $\rho(\frac{u}{\lambda}) = 0$ for each $\lambda > 0$, from which it follows that $\|u\|_{p(\cdot)} = 0$.
   On the other hand, if $\|u\|_{p(\cdot)} = 0$, then there is positive sequence $\lambda_n \to 0^+$ such that $\rho_{p(\cdot)}(\frac{u}{\lambda_n}) \leq 1$. Suppose (for contradiction) that $u \neq 0$, then there exists $i \in \mathbb{Z}$ such that $u(i) \neq 0$. It then follows that $\frac{|u(i)|^{p(i)}}{\lambda_n} \leq \rho_{p(\cdot)}(\frac{u}{\lambda_n}) \leq 1$ and hence $0 < |u(i)|^{p(i)} \leq \lambda_n \to 0^+$ as $n \to \infty$, which contradicts with $u \neq 0$.

2. Scalar Multiplication: For any $u \in \ell^{p(\cdot)}(\mathbb{Z})$ and $\alpha \in \mathbb{R}, \alpha u \in \ell^{p(\cdot)}(\mathbb{Z})$ and in addition

$$\begin{aligned}
\|\alpha u\|_{p(\cdot)} &= \inf\left\{\lambda > 0 : \rho_{p(\cdot)}(\frac{\alpha u}{\lambda}) \leq 1\right\} \\
&= \inf\left\{\lambda = \mu|\alpha| > 0 : \rho_{p(\cdot)}(\frac{\alpha u}{\lambda}) \leq 1\right\} \\
&= |\alpha| \inf\left\{\mu > 0 : \rho_{p(\cdot)}(\frac{u}{\mu}) \leq 1\right\} \\
&= |\alpha| \, \|u\|_p .
\end{aligned}$$

3. Triangular Inequality: For any $u, v \in \ell^{p(\cdot)}, u + v \in \ell^{p(\cdot)}$. Given an arbitrary $\epsilon > 0$ by properties of infimum, there exists $\lambda_u, \lambda_v \in \left\{\lambda > 0 : \rho(\frac{u}{\lambda}) \leq 1\right\}$such that $\lambda_u < \|u\|_{p(\cdot)} + \epsilon$ and $\lambda_v < \|v\|_{p(\cdot)} + \epsilon$ with $\rho(\frac{u}{\lambda_u}) \leq 1, \rho(\frac{u}{\lambda_v}) \leq 1$. Let $\theta = \frac{\lambda_u}{\lambda_u+\lambda_v}$, then by the convexity of $\rho_{p(\cdot)}$, we have

$$\rho_{p(\cdot)}(\frac{u+v}{\lambda_u+\lambda_v}) \leq \theta\rho_{p(\cdot)}(\frac{u}{\lambda_u+\lambda_v}) + (1-\theta)\rho_{p(\cdot)}(\frac{u}{\lambda_u+\lambda_v}) \leq \theta\rho_{p(\cdot)}(\frac{u}{\lambda_u}) + (1-\theta)\rho_{p(\cdot)}(\frac{u}{\lambda_v}) \leq 1.$$

It then follows from the definition of $\|\|_{p(\cdot)}$ that

$$\|u+v\|_{p(\cdot)} \leq \lambda_u + \lambda_v < \|u\|_{p(\cdot)} + \|v\|_{p(\cdot)} + 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, then

$$\|u+v\|_{p(\cdot)} \leq \|u\|_{p(\cdot)} + \|v\|_{p(\cdot)}.$$

$\square$

Lemma 3.1.12. *[Fatou Property of the Norm]. Let $u \in \ell^{p(\cdot)}(\mathbb{Z})$ be a sequence of non-negative real numbers and $p(\cdot) \in \mathcal{S}$. Further, let $\{u_k\} \subset \ell^{p(\cdot)}(\mathbb{Z})$ be a non-negative sequences of real numbers such that u_k increases to the sequence u pointwise. Then $\|u_k\|_{\ell^{p(\cdot)}(\mathbb{Z})} \to \|u\|_{\ell^{p(\cdot)}(\mathbb{Z})}$.*

Proof. Since for every n, $\frac{u_k(n)}{\lambda} \leq \frac{u_{k+1}(n)}{\lambda}$, by property (5) of modular functional, $\rho_{p(\cdot)}(\frac{u_k}{\lambda}) \leq \rho_{p(\cdot)}(\frac{u_{k+1}}{\lambda})$ and hence $\|u_k\|_{\ell^{p(\cdot)}(\mathbb{Z})} \leq \|u_{k+1}\|_{\ell^{p(\cdot)}(\mathbb{Z})}$. Therefore, $\{u_k\}$ is an increasing sequence, so is $\left\{\|u_k\|_{\ell^{p(\cdot)}(\mathbb{Z})}\right\}$ and so this increasing sequence either converges to a finite limit or diverges to ∞.
It is required to prove $\lim_{k\to\infty} \|u_k\|_{\ell^{p(\cdot)}(\mathbb{Z})} = \|u\|_{\ell^{p(\cdot)}(\mathbb{Z})}$. Note that $\|u_k\|_{\ell^{p(\cdot)}(\mathbb{Z})}$ is increasing and $\|u_k\|_{\ell^{p(\cdot)}(\mathbb{Z})} \leq \|u\|_{\ell^{p(\cdot)}(\mathbb{Z})}$. Hence $\lim_{k\to\infty} \|u_k\|_{\ell^{p(\cdot)}(\mathbb{Z})} \leq \|u\|_{\ell^{p(\cdot)}(\mathbb{Z})}$.

Take $\lambda > 0, \|u\|_{\ell^{p(\cdot)}(\mathbb{Z})} > \lambda$. We shall prove that if $\|u\|_{\ell^{p(\cdot)}(\mathbb{Z})} > \lambda$ then for sufficiently large values of k, $\|u_k\|_{\ell^{p(\cdot)}(\mathbb{Z})} > \lambda$. Since $\rho_{p(\cdot)}(\frac{u}{\lambda}) > 1$ and using Monotone convergence theorem,

$$\begin{aligned}\rho_{p(\cdot)}(\frac{u}{\lambda}) &= \sum_{m\in\mathbb{Z}} \left(\frac{u(m)}{\lambda}\right)^{p(m)} = \sum_{m\in\mathbb{Z}} \left(\frac{\lim_{k\to\infty} u_k(m)}{\lambda}\right)^{p(m)} \\ &= \lim_{k\to\infty} \left(\sum_{m\in\mathbb{Z}} \left(\frac{u_k(m)}{\lambda}\right)^{p(m)}\right) = \lim_{k\to\infty} \rho_{p(\cdot)}(\frac{u_k}{\lambda}).\end{aligned}$$

So, $\rho_{p(\cdot)}(\frac{u_k}{\lambda}) > 1$ for sufficiently large values of k. Let $A_k = \left\{\lambda > 0 : \rho_{p(\cdot)}(\frac{u_k}{\lambda}) \leq 1\right\}$ and $B = \left\{\lambda > 0 : \rho_{p(\cdot)}(\frac{u}{\lambda}) \leq 1\right\}$. From above discussion, $B^{\complement} \subseteq A_k^{\complement}$ for sufficiently large values of k. Therefore $A_k \subseteq B$ for sufficiently large values of k. Hence $\inf A_k \geq \inf B$ for sufficiently large values of k. Therefore $\|u_k\|_{\ell^{p(\cdot)}(\mathbb{Z})} \geq \|u\|_{\ell^{p(\cdot)}(\mathbb{Z})}$ for sufficiently large values of k. and hence $\lim_{k\to\infty} \|u_k\|_{\ell^{p(\cdot)}(\mathbb{Z})} \geq \|u_k\|_{\ell^{p(\cdot)}(\mathbb{Z})}$. $\square$

Lemma 3.1.13. *[Fatou's lemma for sequences]. Let $\{u_k\}$ be a non-negative sequence of real numbers such that $\{u_k\} \in \ell^{p(\cdot)}(\mathbb{Z})$. Let $p(\cdot) \in \mathcal{S}$, suppose the sequence $\{u_k\} \in \ell^{p(\cdot)}(\mathbb{Z})$ such that $u_k(n) \to u(n)$ for every n. If*

$$\liminf_{k\to\infty} \|u_k\|_{\ell^{p(\cdot)}(\mathbb{Z})} < \infty.$$

then $u \in \ell^{p(\cdot)}(\mathbb{Z})$ *and*

$$\|u\|_{\ell^{p(\cdot)}(\mathbb{Z})} \leq \liminf_{k\to\infty} \|u_k\|_{\ell^{p(\cdot)}(\mathbb{Z})}.$$

Proof. Define

$$v_k(i) = \inf_{m\geq k} u_m(i).$$

Then for all $m \geq k, v_k(i) \leq u_m(i)$ and this shows that $v_k \in \ell^{p(\cdot)}(\mathbb{Z})$. Since $\{v_k\}$ is an increasing sequence and

$$\lim_{k\to\infty} v_k(i) = \liminf_{m\to\infty} u_m(i) = u(i), \quad i \in \mathbb{Z}.$$

Also,

$$\|u\|_{\ell^{p(\cdot)}(\mathbb{Z})} = \left\|\lim_{k\to\infty} v_k\right\|_{\ell^{p(\cdot)}(\mathbb{Z})} = \lim_{k\to\infty} \|v_k\|_{\ell^{p(\cdot)}(\mathbb{Z})} = \liminf_{k\to\infty} \|v_k\|_{\ell^{p(\cdot)}(\mathbb{Z})}.$$

Therefore, by Fatou's norm property[3.1.12] for sequences

$$\|u\|_{\ell^{p(\cdot)}(\mathbb{Z})} = \lim_{k\to\infty} \|v_k\|_{\ell^{p(\cdot)}(\mathbb{Z})} \leq \lim_{k\to\infty} (\inf_{m\geq k} \|u_m\|_{\ell^{p(\cdot)}(\mathbb{Z})}) = \liminf_{k\to\infty} \|u_k\|_{\ell^{p(\cdot)}(\mathbb{Z})}.$$

So, if $\liminf_{k\to\infty} \|u_k\|_{\ell^{p(\cdot)}(\mathbb{Z})} < \infty$, then $\|u\|_{\ell^{p(\cdot)}(\mathbb{Z})} < \infty$, which implies $u \in \ell^{p(\cdot)}(\mathbb{Z})$. □

The following lemma shows normalizing a sequence gives $\rho_{p(\cdot)}(\frac{a}{\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}}) \leq 1$. Additionally, if $p_+ < \infty$, $\rho_{p(\cdot)}(\frac{a}{\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}}) = 1$.

Lemma 3.1.14. *Let* $\{a(k)\}$ *be a non-negative sequence of real numbers and* $p(\cdot) \in \mathcal{S}$*,*

1. *If* $a \in \ell^{p(\cdot)}(\mathbb{Z})$ *and* $\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})} > 0$*, then* $\rho_{p(\cdot)}(\frac{a}{\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}}) \leq 1$.

2. *If* $p_+ < \infty$*, then* $\rho_{p(\cdot)}(\frac{a}{\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}}) = 1$ *for all nontrivial* $a \in \ell^{p(\cdot)}(\mathbb{Z})$.

Proof. (1) By definition

$$\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})} = \inf\left\{\lambda > 0 : \rho_{p(\cdot)}(a/\lambda) \leq 1\right\}$$

and $\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})} + \frac{1}{n}$ is not an infimum. Therefore, there exists a λ_n such that $\lambda_n \leq \|a\|_{\ell^{p(\cdot)}(\mathbb{Z})} + \frac{1}{n}$ and $\rho_{p(\cdot)}(\frac{a}{\lambda_n}) \leq 1$. Fix such a decreasing sequence $\{\lambda_n\}$ such that $\{\lambda_n\} \to \|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}$. Then by

Fatou's lemma and the definition of modular functional,

$$\begin{aligned}
\rho_{p(\cdot)}(\frac{a}{\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}}) &= \sum_{k\in\mathbb{Z}}\left(\frac{|a(k)|}{\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}}\right)^{p(k)} \\
&= \sum_{k\in\mathbb{Z}}\lim_{n\to\infty}\left(\frac{|a(k)|}{\lambda_n}\right)^{p(k)} \\
&\leq \liminf_{n\to\infty}\sum_{k\in\mathbb{Z}}\left(\frac{|a(k)|}{\lambda_n}\right)^{p(k)} \\
&= \liminf_{n\to\infty}\rho_{p(\cdot)}(\frac{a}{\lambda_n}) \leq 1.
\end{aligned}$$

(2). Assume $p_+ < \infty$ but $\rho(\frac{a}{\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}}) < 1$. Then $\forall\lambda$ such that $0 < \lambda < \|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}$, by Lemma 3.1.9

$$\rho_{\ell^{p(\cdot)}(\mathbb{Z})}(\frac{a}{\lambda}) = \rho_{p(\cdot)}(\frac{\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}}{\lambda}\frac{a}{\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}}) \leq \left(\frac{\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}}{\lambda}\right)^{p+}\rho_{p(\cdot)}(\frac{a}{\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}}).$$

Now, choose λ close to $\|a\|_{p(\cdot)}$ such that

$$\left(\frac{\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}}{\lambda}\right)^{p+} < \frac{1}{\rho_{p(\cdot)}\left(\frac{a}{\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}}\right)}.$$

For this λ, $\rho_{p(\cdot)}(\frac{a}{\lambda}) < 1$. This will contradict the fact that $\lambda < \|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}$.

□

Using Lemma 3.1.9, properties of modular functional and homogeneity of the norm, following lemma can be proved. Continuous version of Lemma 3.1.15 can be found in [8].

Lemma 3.1.15. *Let $p(\cdot) \in \mathcal{S}$ and $\{a(k)\}$ be a non-negative sequence of real numbers such that $\{a(k)\} \in \ell^{p(\cdot)}(\mathbb{Z})$*

$$\begin{aligned}
&\textit{If} \quad \|a\|_{\ell^{p(\cdot)}(\mathbb{Z})} \leq 1, \textit{then} \qquad \rho_{p(\cdot)}(a) \leq \|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}. \\
&\textit{If} \quad \|a\|_{\ell^{p(\cdot)}(\mathbb{Z})} \geq 1, \textit{then} \qquad \rho_{p(\cdot)}(a) \geq \|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}.
\end{aligned}$$

Proof. If $\|a\|_{p(\cdot)} = 0$, then $a = 0$ and so $\rho(a) = 0$.
If $0 < \|a\|_{p(\cdot)} \leq 1$, then by the convexity of the modular

$$\rho_{p(\cdot)}(a) = \rho_{p(\cdot)}(\|a\|_{p(\cdot)}\frac{a}{\|a\|_{p(\cdot)}}) \leq \|a\|_{p(\cdot)}\rho_{p(\cdot)}(\frac{a}{\|a\|_{p(\cdot)}}) \leq \|a\|_{p(\cdot)}.$$

If $\|a\|_{p(\cdot)} > 1$, then $\rho_{p(\cdot)}(a) > 1$, for if $\rho_{p(\cdot)}(a) \leq 1$, then by the definition of the norm we would have $\|a\|_{p(\cdot)} \leq 1$. But then

$$\begin{aligned}\rho_{p(\cdot)}(\frac{a}{\rho_{p(\cdot)}(a)}) &= \sum_{k\in\Omega}\left|\frac{a(k)}{\rho_{p(\cdot)}(a)}\right|^{p(k)}\\ &\leq \frac{1}{\rho_{p(\cdot)}(a)}\left(\sum_{k\in\Omega}|a(k)|^{p(k)}\right) = 1.\end{aligned}$$

It follows that $\|a\|_{p(\cdot)} \leq \rho_{p(\cdot)}(a)$. □

Using Lemma 3.1.9, we can prove the following lemma below. Continuous version of Lemma 3.1.16 can be found in [8]. Same line of proof works here.

Lemma 3.1.16. *Let $\{a(k)\}$ be a non-negative sequence of real numbers such that $\{a(k)\} \in \ell^{p(\cdot)}(\mathbb{Z})$ and $p(\cdot) \in \mathcal{S}$. Then*

1. *If $\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})} > 1$, then $\rho_{p(\cdot)}(a)^{1/p_+} \leq \|a\|_{\ell^{p(\cdot)}(\mathbb{Z})} \leq \rho_{p(\cdot)}(a)^{1/p_-}$.*

2. *If $0 < \|a\|_{\ell^{p(\cdot)}(\mathbb{Z})} \leq 1$, then $\rho_{p(\cdot)}(a)^{1/p_-} \leq \|a\|_{\ell^{p(\cdot)}(\mathbb{Z})} \leq \rho_{p(\cdot)}(a)^{1/p_+}$.*

Proof. Since $p_+ < \infty$, by Lemma 3.1.9 and using the fact that $\rho_{p(\cdot)}(\frac{a}{\|a\|_{p(\cdot)}}) = 1$

$$\frac{\rho_{p(\cdot)}(a)}{\|a\|_{p(\cdot)}^{p_+}} \leq \rho_{p(\cdot)}(\frac{a}{\|a\|_{p(\cdot)}}) \leq \frac{\rho_{\ell^{p(\cdot)}}(a)}{\|a\|_{p(\cdot)}^{p_-}}.$$

□

3.2 Variable Hölder inequality

Theorem 3.2.1. *Let $\{a(n) : n \in \mathbb{Z}\} \in \ell^{p(\cdot)}(\mathbb{Z}), \{b(n) : n \in \mathbb{Z}\} \in \ell^{q(\cdot)}(\mathbb{Z})$, then*

$$\sum_{k\in\mathbb{Z}}|a(k)b(k)| \leq C\,\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}\,\|b\|_{\ell^{q(\cdot)}(\mathbb{Z})}$$

where constant C depends on $p(\cdot)$.

Proof. First, we will show that

$$q(\cdot)_- = {p_+}'.$$

We use the fact that $\sup(\frac{1}{A}) = \frac{1}{\inf A} \quad \forall A > 0$, $\inf(\frac{1}{A}) = \frac{1}{\sup A}, \quad \forall A > 0$ and using $\sup(A-B) = \sup A - \inf B$

$$\frac{1}{q(\cdot)_-} = \frac{1}{\inf q(x)} = \sup(\frac{1}{q(k)}) = \sup\left(\frac{p(k)-1}{p(k)}\right) = 1 - \inf(\frac{1}{p(k)}) = 1 - \frac{1}{\sup p(k)} = 1 - \frac{1}{p_+}.$$

Hence

$$\frac{1}{q(\cdot)_-} = \frac{1}{p_+{}'} = 1 - \frac{1}{p_+}.$$

We use 3 canonical subsets of $\Omega \in \mathbb{Z}$ as follows.

$$\begin{aligned} \Omega_\infty^{p(\cdot)} &= \{n \in \mathbb{Z} : p(n) = \infty\} \\ \Omega_1^{p(\cdot)} &= \{n \in \mathbb{Z} : p(n) = 1\} \\ \Omega_\star^{p(\cdot)} &= \{n \in \mathbb{Z} : 1 < p(n) < \infty\}\,. \end{aligned}$$

On Ω_∞ using Holder inequality, we have

$$\sum_{\Omega_\infty} |a(k)b(k)| \le \|a(k)\chi_{\Omega_\infty}\|_\infty \|b(k)\chi_{\Omega_\infty}\|_1 = \|a\chi_{\Omega_\infty}\|_{\ell^{p(\cdot)}(\mathbb{Z})} \|b\chi_{\Omega_\infty}\|_{\ell^{q(\cdot)}(\mathbb{Z})} \le \|a\|_{\ell^{p(\cdot)}(\mathbb{Z})} \|b\|_{\ell^{q(\cdot)}(\mathbb{Z})}\,.$$

A similar result is valid on Ω_1 by reversing roles of $p(\cdot), q(\cdot)$. So, on Ω_1 using Holder inequality, we have

$$\sum_{\Omega_1} |a(k)b(k)| \le \|a(k)\chi_{\Omega_1}\|_\infty \|b(k)\chi_{\Omega_1}\|_1 = \|a\chi_{\Omega_1}\|_{\ell^{p(\cdot)}(\mathbb{Z})} \|b\chi_{\Omega_1}\|_{\ell^{q(\cdot)}(\mathbb{Z})} \le \|a\|_{\ell^{p(\cdot)}(\mathbb{Z})} \|b\|_{\ell^{q(\cdot)}(\mathbb{Z})}\,.$$

Now on $\Omega_\star$ we have

$$\begin{aligned} &\sum_{k \in \Omega_\star} \frac{|a(k)b(k)|}{\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})} \|b\|_{\ell^{q(\cdot)}(\mathbb{Z})}} \le \sum_{k \in \Omega_\star} \frac{1}{p(k)} \left(\frac{|a(k)|}{\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}}\right)^{p(k)} + \sum_{k \in \Omega_\star} \frac{1}{q(k)} \left(\frac{|b(k)|}{\|b\|_{\ell^{q(\cdot)}(\mathbb{Z})}}\right)^{q(k)} \\ &\le \frac{1}{p_-} \rho_{p(\cdot)(\mathbb{Z})} \left(\frac{a}{\|a\|_{p(\cdot)(\mathbb{Z})}}\right) + \frac{1}{q(\cdot)_-} \rho_{q(\cdot)(\mathbb{Z})} \left(\frac{b}{\|b\|_{q(\cdot)(\mathbb{Z})}}\right). \end{aligned}$$

Using

$$\frac{1}{q(\cdot)_-} = \frac{1}{p_+{}'} = 1 - \frac{1}{p_+}.$$

and norm-homogeneity, which is

$$\rho_{p(\cdot)(\mathbb{Z})} \left(\frac{a}{\|a\|_{p(\cdot)(\mathbb{Z})}}\right) \le 1$$

and

$$\rho_{p(\cdot)(\mathbb{Z})}\left(\frac{b}{\|b\|_{p(\cdot)(\mathbb{Z})}}\right) \le 1.$$

we have that

$$\sum_{k\in\Omega_\star} \frac{|a(k)|b(k)}{\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}\|b\|_{\ell^{q(\cdot)}(\mathbb{Z})}} \le \frac{1}{p_-} + 1 - \frac{1}{p_+}.$$

combining the above statements, and using the fact that each is needed precisely when the ℓ^∞ norm of the corresponding characteristic function equals 1, we have

$$\sum_{k\in\mathbb{Z}}|a(k)b(k)| \le \left(\|\chi_{\Omega_\infty}\|_\infty + \|\chi_{\Omega_1}\|_\infty + (\frac{1}{p_-} + 1 - \frac{1}{p_+})\|\chi_{\Omega_\star}\|_\infty\right)\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}\|b\|_{\ell^{q(\cdot)}(\mathbb{Z})}.$$

□

Lemma 3.2.2. *Given a sequence $\{a(n) : n \in \mathbb{Z}\}$, and $p(\cdot) \in \mathcal{S}$, then for all s, $\frac{1}{p_-} \le s < \infty$,*

$$\||a|^s\|_{p(\cdot)} = \|a\|^s_{sp(\cdot)}.$$

Proof. This follows at once from the defintion of the norm, if we assume $\mu = \lambda^{\frac{1}{s}}$.

$$\begin{aligned}\||a|^s\|_{sp(\cdot)} &= \inf\left\{\lambda > 0 : \sum_{m\in\mathbb{Z}}\left(\frac{|a(m)|^s}{\lambda}\right)^{p(m)} \le 1\right\}\\ &= \inf\left\{\mu^s > 0 : \sum_{m\in\mathbb{Z}}\left(\frac{|a(m)|}{\mu}\right)^{sp(m)} \le 1\right\} = \|a\|^s_{sp(\cdot)}.\end{aligned}$$

□

3.3 Rubio de Francia method

In later chapters, we use a method called Rubio de Francia method to derive inequalities for variable sequence spaces from the corresponding inequalities for weighted sequence spaces. Here we define the operator which is used in this method. We state and prove some properties of this operator as follows.

Lemma 3.3.1. *Given $p(\cdot)$ such that M is bounded on $\ell^{p(\cdot)}(\mathbb{Z})$, for each $h \in \ell^{p(\cdot)}(\mathbb{Z})$, define*

$$Rh(m) = \sum_{k=0}^{\infty} \frac{M^k h(m)}{2^k\|M\|^k_{B(\ell^{p(\cdot)}(\mathbb{Z}))}}.$$

where for $k \geq 1$, $M^k = M \circ \ldots M$ where $\circ$ denotes composition operator acting k times and $M^0 = |I|$, I being identity operator. Then
(a) For all $m \in \mathbb{Z}, |h(m)| \leq Rh(m)$.
(b) R is bounded on $\ell^{p(\cdot)}(\mathbb{Z})$ and $\|Rh\|_{\ell^{p(\cdot)}(\mathbb{Z})} \leq 2\,\|h\|_{\ell^{p(\cdot)}(\mathbb{Z})}$.
(c) $Rh \in A_1$ and $[Rh]_{A_1} \leq 2\,\|M\|_{B(\ell^{p(\cdot)}(\mathbb{Z}))}$.

Proof. (a) Proof of (a) is obvious.
(b) Using subadditivity of norm and that $\left\|M^k\right\| \leq \|M\|^k$, it follows that

$$\begin{aligned}
\|Rh\|_{p(\cdot)} &\leq \sum_{k=0}^{\infty} \frac{\left\|M^k\right\|_{p(\cdot)}}{2^k \|M\|^k_{B(\ell^{p(\cdot)}(\mathbb{Z}))}} \\
&\leq \|h\|_{p(\cdot)} \sum_{k=0}^{\infty} \left\|M^k\right\|_{p(\cdot)} \frac{1}{2^k \|M\|^k_{B(\ell^{p(\cdot)}(\mathbb{Z}))}} \\
&\leq \|h\|_{p(\cdot)} \sum_{k=0}^{\infty} \|M\|^k_{p(\cdot)} \frac{1}{2^k \|M\|^k_{B(\ell^{p(\cdot)}(\mathbb{Z}))}}. \\
&\leq \|h\|_{p(\cdot)} \sum_{k=0}^{\infty} 2^{-k} = 2\,\|h\|_{p(\cdot)}.
\end{aligned}$$

(c) Using subadditivity and homogeneity of the maximal operator, it follows that

$$\begin{aligned}
M(Rh)(m) &\leq \sum_{k=0}^{\infty} \frac{M^{k+1}h(m)}{2^k \|M\|^k_{B(\ell^{p(\cdot)}(\mathbb{Z}))}} \\
&\leq 2\|M\|_{B(\ell^{p(\cdot)}(\mathbb{Z}))} \sum_{k=0}^{\infty} \frac{M^{k+1}h(m)}{2^{k+1} \|M\|^{k+1}_{B(\ell^{p(\cdot)}(\mathbb{Z}))}} \\
&\leq 2\,\|M\|_{B(\ell^{p(\cdot)}(\mathbb{Z}))}\, Rh(m).
\end{aligned}$$

$\square$

Chapter 4

Maximal Operators

4.1 Introduction

In this chapter [1] [2], we study several types of maximal operators on sequence spaces and ergodic spaces. Some contents of this chapter appeared for publications mentioned in footnote[1], [2]. Several such operators are defined in continuous case and studied in standard literature in harmonic analysis [7]. In this regard, we discuss them namely, Hardy-Littlewood maximal operator (both centered and non-centered), dyadic maximal operator, sharp maximal operator and fractional Hardy-Littlewood maximal operator. We also present truncated versions of these maximal operators that are required in the proofs of results in this chapter. For details of these maximal operators on real line, refer to [7].

The method of Calderón-Zygmund decomposition on sequence spaces plays an important role in studying the relationship between these operators [3]. In the discrete case, Calderón-Zygmund decomposition uses dyadic intervals. When we study the relation between the maximal operators, we are required to double the intervals that destroys the dyadic nature of the intervals. This challenge is not there in the case of real line [7].

A good-λ inequality is presented which relates dyadic maximal operator and sharp maximal operator.

[1] Anupindi Sri Sakti Swarup and A. Michael Alphonse, *Relations Between Discrete Maximal Operators in Harmonic Analysis.* Proceedings of the Ninth International Conference on Mathematics and Computing. ICMC 2023. Lecture Notes in Networks and Systems, vol 697. Springer, Singapore. `https://doi.org/10.1007/978-981-99-3080-7_30`.

[2] Anupindi Sri Sakti Swarup and A. Michael Alphonse, *The boundedness of Fractional Hardy-Littlewood maximal operator on variable lp(Z) spaces using Calderon-Zygmund decomposition.* Accepted for publication in "The Journal of Indian Mathematical Society (2022), Vol 89", Acceptance Letter Dated: May 9 2023.

The following lemma will be used as a variation of Hölder's inequality. Proof of this lemma in continuous version can be found in [8]. Same line of proof works here. For completeness sake, we provide the proof below.

Lemma 4.1.1. *Let $0 \le \alpha < 1$, and p, q such that $1 < p < \frac{1}{\alpha}, \frac{1}{p} - \frac{1}{q} = \alpha$. For every interval I in $\mathbb{Z}$ and non-negative sequence $\{a(n) : n \in \mathbb{Z}\}$*

$$|I|^{\alpha-1} \sum_{k \in I} a(k) \le \left(\sum_{k \in I} a(k)^p \right)^{\frac{1}{p} - \frac{1}{q}} \left(\frac{1}{|I|} \sum_{k \in I} a(k) \right)^{\frac{p}{q}}.$$

Proof. Suppose p' is conjugate exponent of p. Then

$$\frac{1}{p'} + \frac{1}{p} = 1.$$

Since

$$\frac{1}{p} - \frac{1}{q} = \alpha,$$

it follows that

$$\alpha - 1 + \frac{\alpha p}{p'} = -\frac{p}{q}.$$

So,

$$\begin{aligned}
|I|^{\alpha-1} \sum_{k \in I} a(k) &= |I|^{\alpha-1} \left(\sum_{k \in I} a(k) 1 \right)^{\alpha p} \left(\sum_{k \in I} a(k) \right)^{1-\alpha p} \\
&\le |I|^{\alpha-1} \left(\sum_{k \in I} a(k)^p \right)^{\frac{1}{p}\alpha p} \left(\sum_{k \in I} 1^{p'} \right)^{\frac{\alpha p}{p'}} \left(\sum_{k \in I} a(k) \right)^{1-\alpha p} \\
&\le |I|^{\alpha-1} |I|^{\frac{\alpha p}{p'}} \left(\sum_{k \in I} a(k)^p \right)^{\alpha} \left(\sum_{k \in I} a(k) \right)^{\frac{p}{q}} \\
&\le \left(\sum_{k \in I} a(k)^p \right)^{\frac{1}{p} - \frac{1}{q}} \left(\frac{1}{|I|} \sum_{k \in I} a(k) \right)^{\frac{p}{q}}.
\end{aligned}$$

This completes the proof. □

4.2 Calderón-Zygmund decomposition theorem for sequences

The Calderón-Zygmund decomposition theorem for sequences [3] is as follows.

Theorem 4.2.1. *Let $0 \leq \alpha < 1$. Take a real number p such that $1 \leq p < \frac{1}{\alpha}$ (If $\alpha = 0$ then $1 \leq p < \infty$). Let $\{a(n) : n \in \mathbb{Z}\} \in \ell^p(\mathbb{Z})$. Then, for every $t > 0$, there exists a sequence of disjoint dyadic intervals $\left\{I_j^t\right\}$ such that*

$$(i) \quad t < \frac{1}{|I_j^t|^{1-\alpha}} \sum_{k \in I_j^t} |a(k)| \leq 2t, \forall j \in \mathbb{Z}$$

$$(ii) \quad \forall n \notin \cup_j I_j^t, \quad |a(n| \leq t$$

$$(iii) \quad \textit{If} \quad t_1 > t_2, \quad \textit{then each} \quad I_j^{t_1} \quad \textit{is subinterval of some} \quad I_m^{t_2}, \quad \forall j, m \in \mathbb{Z}.$$

Proof. For each positive integer N, consider the collection of disjoint intervals of cardinality 2^N,

$$\{I_{N,j}\} = \left\{[(j-1)2^N + 1, \ldots, j2^N]\right\}, j \in \mathbb{Z}.$$

For each $t > 0$, let $N = N_t$ be the smallest positive integer such that

$$\frac{1}{|I_{N_t,j}|^{1-\alpha}} \sum_{k \in I_{N_t,j}} |a(k)| \leq t.$$

Such N_t is possible as $\{a(n) : n \in \mathbb{Z}\} \in \ell^p(\mathbb{Z})$ and $p < \frac{1}{\alpha}$. Now consider collection $\{I_{N_t,j}\}$ and subdivide each of these intervals into two intervals of equal cardinality. If I is one of these intervals either

$$(A) \quad \frac{1}{|I|^{1-\alpha}} \sum_{k \in I} |a(k)| > t.$$

$$\text{or} \quad (B) \quad \frac{1}{|I|^{1-\alpha}} \sum_{k \in I} |a(k)| \leq t.$$

In case (A) we select this interval and include it in a collection $\{I_{r,j}\}$.

In case (B) we subdivide I once again unless I is a singleton and select intervals as above. Now the elements which are not included in $\{I_{r,j}\}$ form a set S such that for every $n \in S$, $|a(n)| \leq t$. This proves (i).

Also, from the choice of $\{I_{r,j}\}$, note that $\{I_{r,j}\}$ are disjoint and satisfy

$$\frac{1}{|I_{r,j}|^{1-\alpha}} \sum_{k \in I_{r,j}} |a(k)| > t.$$

Since each $I_{r,j}$ is contained in an interval J_0 with card $J_0 = |2I_{r,j}|^{1-\alpha}$, which is not selected in the previous step, we have

$$\frac{1}{|I_{r,j}|^{1-\alpha}} \sum_{K \in I_{r,j}} |a(k)| \leq \frac{2}{|J_0|} \sum_{k \in J_0} |a(k)| \leq 2t.$$

This proves (ii). It remains to prove (iii).
If $t_1 > t_2$ then $N_{t_1} \leq N_{t_2}$. So each $I_{N_{t_1},j}$ is contained in some $I_{N_{t_2},j}$. In the subdivision and the selecting process for t_1 we have

$$\frac{1}{|I_{N_{t_1},j}|^{1-\alpha}} \sum_{k \in I_{N_{t_1},j}} |a(k)| > t_1 > t_2.$$

So, if $I_j^{t_1}$ is not one of the intervals $I_m^{t_2}$, then it must be subinterval of some $I_m^{t_2}$ selected in an earlier step. This completes proof. □

4.3 Relations between maximal operators

Theorem 4.3.1. *Given a sequence $\{a(m) : m \in \mathbb{Z}\}$, the following relation holds:*

$$M'a(m) \leq Ma(m) \leq 3M'a(m).$$

Proof. First inequality is obvious as $M'a$ considers supremum over centered intervals, while M considers supremum over all intervals. For second inequality, let $I = [m - r_1, m - r_1 + 1, \ldots, m + r_2 - 1, m + r_2]$ be an interval containing m. Let $r = max\,\{r_1, r_2\}$. Consider $I_1 = [m - r, m - r + 1, \ldots, m + r - 1, m + r]$ containing m. Note that $|I_1| = 2r + 1, |I| = r_1 + r_2 + 1$. Then

$$|I| = r_2 + r_1 + 1 \geq r = \frac{1}{3}3r \geq \frac{1}{3}(2r + 1) = \frac{1}{3}|I_1|.$$

This gives

$$\frac{1}{|I|} \sum_{k \in I} |a(k)| \leq \frac{3}{|I_1|} \sum_{k \in I_1} a(k) \leq 3M'a(m).$$

□

Theorem 4.3.2. *If $\boldsymbol{a} = \{a(k) : k \in \mathbb{Z}\}$ is a sequence with $\boldsymbol{a} \in \ell_1(\mathbb{Z})$, then*

$$|\big\{m \in \mathbb{Z} : M'a(m) > 4\lambda\big\}| \leq 3|\{m \in \mathbb{Z} : M_d a(m) > \lambda\}|.$$

Proof. Using Calderón Zygmund decomposition at height λ, we obtain a collection of disjoint dyadic intervals $\{I_j : j \in \mathbb{Z}^+\}$ such that

$$\lambda < \frac{1}{|I_j|} \sum_{k \in I_j} |a(k)| \leq 2\lambda$$

Then

$$\cup_j I_j \subseteq \{m \in \mathbb{Z} : M_d a(m) > \lambda\}.$$

It suffices to show that

$$\{m \in \mathbb{Z} : M'a(m) > 4\lambda\} \subset \cup_j 3I_j.$$

Let $m \notin \cup_j 3I_j$. We shall prove $m \notin \{k \in \mathbb{Z} : M'a(k) > 4\lambda\}$. Let I be any interval centered at m. Choose $N \in \mathbb{Z}_+$ such that $2^{N-1} \leq |I| < 2^N$. Then I intersects exactly 2 dyadic intervals in $\mathcal{I}_N$ say R_1, R_2. Assume R_1 intersects I on the left and R_2 intersects I on the right. Since $m \notin \cup_{j=1}^{\infty} 3I_j$, $m \notin 2RI_j, j = 1, 2, 3 \ldots$ and $m \notin 2LI_j, j = 1, 2, 3, \ldots$. But $m \in 2RR_1$ and $m \in 2LR_2$.

Therefore, both R_1 and R_2 cannot be any one of I_j.

Hence, the average of $\{a(n) : n \in \mathbb{Z}\}$ on each $R_i, i = 1, 2$ is at most λ. Further note that $\frac{|R_1|}{|I|} \leq 2, \frac{|R_2|}{|I|} \leq 2$. So

$$\begin{aligned}
\frac{1}{|I|} \sum_{m \in I} |a(m)| &\leq \frac{1}{|I|} \Bigg(\sum_{k \in R_1} |a(k)| + \sum_{k \in R_2} |a(k)| \Bigg) \\
&= \Bigg(\frac{1}{|R_1|} \frac{|R_1|}{|I|} \sum_{k \in R_1} |a(k)| + \frac{1}{|R_2|} \frac{|R_2|}{|I|} \sum_{k \in R_2} |a(k)| \Bigg) \\
&\leq 2 \Bigg(\frac{1}{|R_1|} \sum_{k \in R_1} |a(k)| + \frac{1}{|R_2|} \sum_{k \in R_2} |a(k)| \Bigg) = 2(\lambda + \lambda) = 4\lambda.
\end{aligned}$$

□

Using Theorem 4.3.2, we have the following corollary.

Corollary 4.3.3. *For a sequence $\{a(n) : n \in \mathbb{Z}\}$, if $M_d a \in \ell^p(\mathbb{Z}), 1 < p < \infty$, then*

$$\|M'a\|_{\ell^p(\mathbb{Z})} \leq C \|M_d a\|_{\ell^p(\mathbb{Z})}.$$

Proof.

$$\begin{aligned}
\|M'a\|_{\ell^p(\mathbb{Z})} &= \int_0^\infty p\lambda^{p-1} |\{m : M'a(m) > \lambda\}| d\lambda \\
&\leq 3(4)^{p-1} \int_0^\infty p(\frac{\lambda}{4})^{p-1} \Big|\Big\{m : M_d a(m) > \frac{\lambda}{4}\Big\}\Big| d\lambda
\end{aligned}$$

$$\leq 3(4)^p \int_0^\infty pu^{p-1} |\{m : M_d a(m) > u\}| du.$$
$$\leq 3(4)^p \|M_d a\|_{\ell^p(\mathbb{Z})}$$

□

In the following lemma, we see that in the norm of BMO($\mathbb{Z}$) space, we can replace the average a_I of $\{a(n) : n \in \mathbb{Z}\}$ by a constant b. The proof is similar to the proof in continuous version [7]. We provide the proof for the sake of completeness.

Lemma 4.3.4. *Consider a non-negative sequence $\{a(n) : n \in \mathbb{Z}\}$. Then the following are valid.*

1. $\frac{1}{2} \|a\|_\star \leq \sup_{m \in I} \inf_{b \in \mathbb{R}} \frac{1}{|I|} |a(m) - b| \leq \|a\|_\star$.
2. $M^\#(|a|)(i) \leq M^\# a(i), i \in \mathbb{Z}$.

Proof. For first inequality, note for all $b \in \mathbb{R}$,

$$\sum_{m \in I} |a(m) - a_I| \leq \sum_{m \in I} |a(m) - b| + \sum_{m \in I} |b - a_I| = A + B \quad (say).$$

Now

$$B = \sum_{m \in I} |b - a_I| = |I||b - a_I| = |I| \left| b - \frac{1}{|I|} \sum_{k \in I} |a(k)| \right|$$
$$= |I| \left| \frac{1}{|I|} \left(\sum_{k \in I} (b - a(k)) \right) \right| \leq \sum_{k \in I} |b - a(k)|.$$

So,

$$\sum_{m \in I} |a(m) - a_I| \leq \sum_{m \in I} |a(m) - b| + \sum_{m \in I} |b - a_I| \leq 2 \sum_{m \in I} |a(m) - b|.$$

Now, divide both sides by $|I|$, and take infimum over all b followed by, supremum over all I. This proves

$$\frac{1}{2} \|a\|_\star \leq \sup_{m \in I} \inf_{b \in \mathbb{R}} \frac{1}{|I|} |a(m) - b|.$$

The proof for second inequality

$$\sup_{m \in I} \inf_{b \in \mathbb{R}} \frac{1}{|I|} |a(m) - b| \leq \|a\|_\star.$$

is obvious.

The proof of (2) follows from the fact that $||a| - |b|| \leq |a| - |b|$ for any $a, b \in \mathbb{R}$. □

4.4 Good lambda estimate

The following is the good- λ inequality for the unweighted case.

Lemma 4.4.1. *If $a \in \ell^{p_0}(\mathbb{Z})$ for some p_0, $1 \le p_0 < \infty$, then for all $\gamma > 0$ and $\lambda > 0$*

$$\left|\left\{n \in \mathbb{Z} : M_d a(n) > 2\lambda, M^{\#}a(n) \le \gamma\lambda\right\}\right| \le 2\gamma|\{n \in \mathbb{Z} : M_d a(n) > \lambda\}|.$$

Proof. Perform Calderón-Zygmumd decomposition for the sequence $\{a(n) : n \in \mathbb{Z}\}$ at height λ, which gives collection of intervals $\{I_j\}$ such that for each j,

$$\lambda \le \frac{1}{|I_j|}\sum_{k\in I_j}|a(k)| \le 2\lambda.$$

Let I be one of the interval in the collection $\{I_j\}$. In Calderón-Zygmund decomposition, there exists interval $\tilde{I}$ such that $\tilde{I}$ is either $2RI$ or $2LI$ and

$$\frac{1}{|\tilde{I}|}\sum_{k\in\tilde{I}}|a(k)| \le \lambda.$$

We shall show that $\forall m \in I$, $M_d a(m) > 2\lambda$ implies $M_d(a\chi_I)(m) > 2\lambda$. Given $M_d a(m) > 2\lambda$, by definition of M_d, there exists a dyadic interval J containing m such that

$$\frac{1}{|J|}\sum_{k\in J}|a(k)| > 2\lambda. \tag{4.4.1}$$

Consider two possibilities:
Case (1): $I \subseteq J$ and $I \ne J$. From 4.4.1, since

$$\frac{1}{|J|}\sum_{k\in J}|a(k)| > 2\lambda > \lambda.$$

the dyadic interval J must have been chosen in the collection of intervals in Calderón-Zygmund decomposition instead of I. So this case is not possible.

Case (2): $J \subseteq I$. Here $|J| \le |I|$ implies $\frac{1}{|J|} \ge \frac{1}{|I|}$. So from 4.4.1,

$$\frac{1}{|J|}\sum_{k\in J}|a(k)\chi_I(k)| \ge \frac{1}{|J|}\sum_{k\in J}|a(k)| > 2\lambda.$$

From above cases, we conclude that the only possibility is that $M_d(a\chi_I)(m) > 2\lambda$. Write $a\chi_I = a_1 + a_2$ where

$$a_1 = (a - a_{\tilde{I}})\chi_I$$

$$a_2 = a_{\tilde{I}}\chi_I.$$

Since M_d is sublinear and using the fact that M_d is sublinear and $M_d(a\chi_I)(m) \leq a_{\tilde{I}}$, we have

$$\begin{aligned} M_d(a_1 + a_2) &\leq M_d((a - a_{\tilde{I}})\chi_I) + M_d(a_{\tilde{I}}\chi_I). \\ &\leq M_d((a - a_{\tilde{I}})\chi_I) + (a_{\tilde{I}}). \end{aligned}$$

Since $M_d(a_{\tilde{I}}\chi_I)(k) \leq a_{\tilde{I}} \quad \forall k$, it follows that

$$M_d(a_1 + a_2) = M_d(a\chi_I) \leq M_d((a - a_{\tilde{I}})\chi_I) + (a_{\tilde{I}}).$$

Hence for every, $k \in I$, it follows that

$$M_d((a - a_{\tilde{I}})\chi_I)(k) \geq M_d(a\chi_I)(k) - a_{\tilde{I}}.$$

So, for those $k's$,

$$M_d((a - a_{\tilde{I}})\chi_I)(k) \geq M_d(a\chi_I)(k) - a_{\tilde{I}} > 2\lambda - \lambda = \lambda.$$

Using $weak(1,1)$ inequality for M_d

$$\begin{aligned} &\left| \left\{ k \in I : M_d[(a - a_{\tilde{I}})](k) > 2\lambda \right\} \right| \leq \left| \left\{ k \in I : M_d[(a - a_{\tilde{I}})\chi_I](k) > 2\lambda \right\} \right| \\ &\leq \left| \left\{ k \in \mathbb{Z} : M_d[(a - a_{\tilde{I}})\chi_I](k) > \lambda \right\} \right| \leq \frac{C}{\lambda} \sum_{k \in I} |a(k) - a_{\tilde{I}}| \\ &\leq \frac{2}{\lambda} |I| \frac{C}{|\tilde{I}|} \sum_{k \in \tilde{I}} |a(k) - a_{\tilde{I}}| \\ &\leq \frac{2C}{\lambda} |I| M^{\#}a(m), \end{aligned}$$

for every $m \in I$.
Now,

$$\left| \left\{ n \in I : M_d a(n) > 2\lambda, M^{\#}a(n) \leq \gamma\lambda \right\} \right| \leq \frac{2C}{\lambda} |I| \left(\inf_{n \in I} M^{\#}a(n) \right) \leq \frac{2C}{\lambda} |I| \gamma\lambda = 2C|I|\lambda.$$

Therefore,

$$\begin{aligned} &\left| \left\{ n \in \mathbb{Z} : M_d a(n) > 2\lambda, M^{\#}a(n) \leq \gamma\lambda \right\} \right| = \left| \cup_j \left\{ n \in I_j : M_d a(n) > 2\lambda, M^{\#}a(n) \leq \gamma\lambda \right\} \right| \\ &\leq 2C\gamma \sum_{j \in I_j} |I_j| \leq 2\gamma \left| \{ n \in \mathbb{Z} : M_d a(n) > \lambda \} \right|. \end{aligned}$$

$\square$

As a consequence of good-λ inequality, we prove the following theorem.

Theorem 4.4.2. *Let $\{a(n) : n \in \mathbb{Z}\}$ be a non negative sequence in $\ell^p(\mathbb{Z}), 1 < p < \infty$. Then*

$$\sum_{m\in\mathbb{Z}} |M_d a(m)|^p \leq C \sum_{m\in\mathbb{Z}} |M^{\#} a(m)|^p.$$

where M_d is the dyadic maximal operator and $M^{\#}$ is the sharp maximal operator, whenever, the left hand side is finite.

Proof. For a positive integer $N > 0$, let

$$I_N = \int_0^N p\lambda^{p-1} |\{m \in \mathbb{Z} : M_d a(m) > \lambda\}| d\lambda.$$

I_N is finite, since $a \in \ell^p(\mathbb{Z})$ implies $M_d a \in \ell^p(\mathbb{Z})$

$$\begin{aligned}
I_N &= \int_0^N p\lambda^{p-1} |\{m \in \mathbb{Z} : M_d a(m) > \lambda\}| d\lambda \\
&= 2^p \int_0^{\frac{N}{2}} p\lambda^{p-1} |\{m \in \mathbb{Z} : M_d a(m) > 2\lambda\}| d\lambda \\
&\leq 2^p \int_0^{\frac{N}{2}} p\lambda^{p-1} \left|\left\{m \in \mathbb{Z} : M_d a(m) > 2\lambda, M^{\#} a(m) \leq \gamma\lambda\right\}\right| d\lambda + \\
&2^p \int_0^{\frac{N}{2}} p\lambda^{p-1} \left|\left\{m \in \mathbb{Z} : M_d a(m) > 2\lambda, M^{\#} a(m) > \gamma\lambda\right\}\right| d\lambda \\
&\leq 2^p \int_0^{\frac{N}{2}} p\lambda^{p-1} C\gamma |\{m \in \mathbb{Z} : M_d a(m) > \lambda\}| d\lambda + \\
&2^p \int_0^{\frac{N}{2}} p\lambda^{p-1} \left|\left\{m \in \mathbb{Z} : M^{\#} a(m) > \gamma\lambda\right\}\right| d\lambda \\
&\leq 2^p C\gamma \int_0^N p\lambda^{p-1} |\{m \in \mathbb{Z} : M_d a(m) > \lambda\}| d\lambda \quad + \\
&2^p \int_0^{\frac{N}{2}} p\lambda^{p-1} \left|\left\{m \in \mathbb{Z} : M^{\#} a(m) > \gamma\lambda\right\}\right| d\lambda.
\end{aligned}$$

It follows that

$$(1 - 2^p C\gamma) I_N \leq 2^p \int_0^{\frac{N}{2}} p\lambda^{p-1} \left|\left\{m \in \mathbb{Z} : M^{\#} a(m) > \gamma\lambda\right\}\right| d\lambda.$$

Now choose $\gamma = \frac{1}{C2^{p+1}}$ such that $(1 - 2^p C\gamma) = \frac{1}{2}$. Then,

$$\begin{aligned}
\frac{1}{2} I_N &\leq 2^p \int_0^{\frac{N}{2}} p\lambda^{p-1} \left|\left\{m \in \mathbb{Z} : M^{\#} a(m) > \gamma\lambda\right\}\right| d\lambda \\
&\leq \frac{2^p}{\gamma^p} \int_0^{\frac{N\gamma}{2}} p\lambda^{p-1} \left|\left\{m \in \mathbb{Z} : M^{\#} a(m) > \lambda\right\}\right| d\lambda.
\end{aligned}$$

Now, take $N \to \infty$, we get

$$\sum_{m\in\mathbb{Z}} M_d a(m)^p \leq C \sum_{m\in\mathbb{Z}} M^{\#}a(m)^p.$$

$\square$

4.5 Weighted good lambda estimate

Lemma 4.5.1. *Let $\{a(n) : n \in \mathbb{Z}\}$ be a non-negative sequence in $\ell^p_w(\mathbb{Z})$. Let $w \in A_p, 1 \leq p_0 \leq p < \infty$. If $\{a(n) : n \in \mathbb{Z}\}$ is such that $M_d a \in \ell^{p_0}_w(\mathbb{Z})$, then*

$$\sum_{m\in\mathbb{Z}} |M_d a(m)|^p w(m) \leq C \sum_{m\in\mathbb{Z}} |M^{\#}a(m)|^p w(m).$$

where M_d is the dyadic maximal operator and $M^{\#}$ is the sharp maximal operator, whenever, the left hand side is finite.

Proof. In order to prove Lemma 4.4.2, first we prove the weighted good-λ inequality, which is as follows: We know

$$\left\{m \in \mathbb{Z} : M_d a(m) > 2\lambda, M^{\#}a(m) \leq \gamma\lambda\right\} \subseteq \{m \in \mathbb{Z} : M_d a(m) > \lambda\}.$$

Also from Theorem 2.3.4 we have

$$\frac{w(S)}{w(I)} \leq C\left(\frac{|S|}{|I|}\right)^{\delta} = C\gamma^{\delta}.$$

Hence,

$$w(\left\{m \in \mathbb{Z} : m_d a(m) > 2\lambda, M^{\#}a(m) \leq \gamma\lambda\right\}) = w(S) = C\gamma^{\delta} w(I) = C\gamma^{\delta} w(\{M_d a(m) > \lambda\}).$$

For some $\delta > 0$,

$$w(\left\{m \in \mathbb{Z} : M_d a(m) > 2\lambda, M^{\#}a(m) \leq \gamma\lambda\right\}) \leq C\gamma^{\delta} w(\{m \in \mathbb{Z} : M_d a(m) > \lambda\}).$$

Since $\{m \in \mathbb{Z} : M_d a(m) > \lambda\}$ can be decomposed into disjoint dyadic intervals, it is enough to show that for each such interval I,

$$w(\left\{m \in I : M_d a(m) > 2\lambda, M^{\#}a(m) \leq \gamma\lambda\right\}) \leq C\gamma^{\delta} w(I).$$

Using this weighted good-λ inequality, along the same line of proof of Theorem 4.4.2, we can prove the weighted version of Theorem 4.4.2. $\square$

Theorem 4.5.2. *Let $0 \leq \alpha < 1$. Take a real number p such that $1 \leq p < \frac{1}{\alpha}$ (If $\alpha = 0$ then $1 \leq p < \infty$). Let $\{a(n) : n \in \mathbb{Z}\} \in \ell^p(\mathbb{Z})$. Let $\left\{I_j^t\right\}$ be intervals obtained from Calderón-Zygmund decomposition at height t. Then*

$$\{n : M_\alpha a(n) > 9t\} \subseteq \cup_j 2I_j^t.$$

Proof. Since I_j^t are the intervals obtained from Calderón-Zygmund decomposition at height t, for each j, we have

$$\frac{1}{|I_j^t|^{1-\alpha}} \sum_{k \in I_j^t} |a(k)| > t.$$

Therefore, $\cup_j I_j^t \subseteq \{n : M_\alpha a(n) > t\}$. Let $n \notin \cup_j 2I_j^t$ and I be any interval which contains n. Then

$$\sum_{k \in I} |a(k)| = \sum_{k \in I \cap (\cup_j I_j^t)} |a(k)| + \sum_{I \cap (\cup_j I_j^t)^{\complement}} |a(k)| = S_1 + S_2.$$

To estimate S_1, we observe a simple geometric fact. If $I \cap I_j^t$ is non-empty and I is not contained in $2I_j^t$, then $I_j^t \subset 4I$. Since $n \in I$ and $n \notin 2I_j^t$, for each j, I is not contained in $2I_j^t$ for each j. Also, note that $S_2 \leq t|I|$. Therefore,

$$\begin{aligned}
S_1 &\leq \sum_{\{j : I_j^t \subseteq 4I\}} \sum_{k \in I_j^t} |a(k)|. \\
&\leq \sum_{\{j : I_j^t \subseteq 4I\}} 2t|I_j^t|. \\
&\leq 2t|4I| \\
&\leq 8t|I|.
\end{aligned}$$

Hence, $\sum_{k \in I} |a(k)| \leq S_1 + S_2 \leq 9t|I|$.

Since I was an arbitrary interval containing n, we have $M_\alpha a(n) \leq 9t$. Therefore

$$(\cup_j 2I_j^t)^{\complement} \subseteq \{n : M_\alpha a(n) \leq 9t\}.$$

$\square$

Lemma 4.5.3. *Let $0 \leq \alpha < 1$. Take a real number p such that $1 \leq p < \frac{1}{\alpha}$ (If $\alpha = 0$ then $1 \leq p < \infty$). Let $\{a(n) : n \in \mathbb{Z}\} \in \ell^p(\mathbb{Z})$ and $q(\cdot) \in \mathcal{S}$. Let $\left\{I_j^k\right\}$ be intervals obtained from Calderón-Zygmund decomposition at height $(9t)^{k-1}$ where $k \in \mathbb{Z}$ and $0 < t < \frac{1}{9}$. Define*

$$\Omega_k = \left\{i \in \mathbb{Z} : M_\alpha a(i) > 9A^k = 9(9t)^k\right\}.$$

and

$$\Omega_{k+1} \setminus \Omega_k = \cup_j E_j^k.$$

where E_j^k are pairwise disjoint for all $j, k \in \mathbb{Z}$. Then the fractional maximal operator satisfies

$$\sum_{i\in\mathbb{Z}} M_\alpha a(i)^{q(i)} \leq A^{q_+} 18^{q_+(1-\alpha)} \sum_{k,j} \sum_{i\in E_j^k} \left(\frac{1}{|2I_j^k|^{1-\alpha}} \sum_{m\in 2I_j^k} |a(m)| \right)^{q(i)}.$$

where A is chosen based on t.

Proof. Take $A = 9t < 1$. Define $\Omega_k = \left\{i \in \mathbb{Z} : M_\alpha a(i) > 9A^k = 9(9t)^k\right\}$. For each integer k, apply Theorem 4.2.1 Calderón-Zygmund decomposition for sequence $\{a(i)\}$, at height $t = A^{k+1}$ to get pairwise disjoint cubes $\left\{I_j^k\right\}$ such that

$$\Omega_{k+1} \subset \cup_j 2I_j^k. \tag{4.5.1}$$

$$\frac{1}{|I_j^k|^{1-\alpha}} \sum_{i\in I_j^k} a(i) > A^{k+1}. \tag{4.5.2}$$

Multiply both sides of (4.5.2) by $\frac{1}{2^{1-\alpha}}$, we get

$$\frac{1}{|2I_j^k|^{1-\alpha}} \sum_{i\in 2I_j^k} a(i) > \frac{1}{2^{1-\alpha}} A^{k+1}.$$

Define sets inductively as follows:

$$\begin{aligned}
E_1^k &= \Big(\Omega_{k+1} \setminus \Omega_k\Big) \cap 2I_1^k \\
E_2^k &= \Big(\Big(\Omega_{k+1} \setminus \Omega_k\Big) \cap 2I_2^k\Big) \setminus E_1^k \\
E_3^k &= \Big(\Big(\Big(\Omega_{k+1} \setminus \Omega_k\Big) \cap 2I_3^k\Big) \setminus (E_1^k \cup E_2^k)\Big) \\
\dots E_m^k &= \Big(\Big(\Big(\Omega_{k+1} \setminus \Omega_k\Big) \cap 2I_m^k\Big) \setminus (E_1^k \cup E_2^k \dots E_{m-1}^k)\Big).
\end{aligned}$$

Then, sets E_j^k are pairwise disjoint for all j and k and satisfy for every k

$$\Omega_{k+1} \setminus \Omega_k = \bigcup_j E_j^k.$$

So, $\mathbb{Z} = \bigcup_k \Big(\Omega_{k+1} \setminus \Omega_k\Big) = \bigcup_k \bigcup_j E_j^k$. Further, note $\Omega_{k+1} \setminus \Omega_k = \big\{9A^{k+1} < M_\alpha a(i) \leq 9A^k\big\}$. We now estimate $M_\alpha a$ as follows, noting $A < 1$,

$$\begin{aligned}
\sum_{i\in\mathbb{Z}} M_\alpha a(i)^{q(i)} &= \sum_k \sum_{i\in\Omega_{k+1}\setminus\Omega_k} M_\alpha a(i)^{q(i)} \\
&\leq \sum_k \sum_{i\in\Omega_{k+1}\setminus\Omega_k} [9A^k]^{q(i)} \\
&\leq A^{q+} 18^{q+(1-\alpha)} \sum_{k,j} \sum_{i\in E_j^k} \left(\frac{1}{|2I_j^k|^{1-\alpha}} \sum_{m\in 2I_j^k} |a(m)|\right)^{q(i)}.
\end{aligned}$$

This completes the proof. □

4.6 Weighted classical results for maximal operators

In this section, for a given sequence $\{a(n) : n \in \mathbb{Z}\}$ in $\ell^p_w(\mathbb{Z})$, we prove weighted weak(p,p) inequality with respect to the weight $\{w(n) : n \in \mathbb{Z}\} \in A_p$ which is as follows:

$$w(\{m \in \mathbb{Z} : Ma(m) > \lambda\}) \leq \frac{C}{\lambda^p} \sum_{m\in\mathbb{Z}} |a(m)|^p w(m). \tag{4.6.1}$$

Inequality $A4$ will be proved via several theorems, Theorem 4.6.1 to Theorem 4.6.2.

The proof of Theorem 4.6.1 uses Calderón-Zygmund decomposition. For the proofs of the corresponding results for the continuous version, we refer [7]. The proofs of Theorem 4.6.1 and Theorem 4.6.2 are same as the proof for the continuous versions of the corresponding results apart from the fact that the constants obtained here are slightly different from the constants obtained for the continuous version due to the nature of dyadic intervals in $\mathbb{Z}$ (see [4]). So, we give here the proof of Theorem 4.6.1 and Theorem 4.6.2 for the sake of completeness.

Theorem 4.6.1. *If $\{w(n) : n \in \mathbb{Z}\} \in A_p$, $1 \leq p < \infty$ and $\{a(n) : n \in \mathbb{Z}\}$ is a sequence in $\ell^p(\mathbb{Z})$, then for $p > 1$, there exists a constant C_p such that*

$$\sum_{m\in\mathbb{Z}} Ma(m)^p w(m) \leq C_p \sum_{m\in\mathbb{Z}} |a(m)|^p Mw(m). \tag{4.6.1[A]}$$

Furthermore, for $p = 1$, there exists a constant C_1 such that

$$\sum_{\{m\in\mathbb{Z}:Ma(m)>\lambda\}} w(m) \leq \frac{C_1}{\lambda} \sum_{m\in\mathbb{Z}} |a(m)| Mw(m). \tag{4.6.1[B]}$$

Proof. We will show that $\|Ma\|_{\ell^\infty_w(\mathbb{Z})} \leq \|a\|_{\ell^\infty_{Mw}(\mathbb{Z})}$ and that weak(1,1) inequality holds; the strong(p,p) inequality then follows from the Marcinkiewicz interpolation theorem.
If $Mw(m) = 0$ for any $m \in \mathbb{Z}$, then $w(m) = 0$ and there is nothing to prove
If $Mw(m) > 0$ for some m, then $w(m) > 0$ for those m. But since $Mw(m)$ is evaluated over all intervals I containing m, $Mw(m) > 0, \forall m \in \mathbb{Z}$. Therefore, we assume that for every $m \in \mathbb{Z}$, $Mw(m) > 0$. Take $C > \|a\|_{\ell^\infty_{Mw}(\mathbb{Z})}$. Then

$$\sum_{\{n\in\mathbb{Z}:|a(n)|>C\}} Mw(n) = 0.$$

which shows that

$$\{n \in \mathbb{Z} : |a(n)| > C\} = \emptyset.$$

Hence, $|a(n)| \leq C, \forall n \in \mathbb{Z}$ which implies that $|Ma(n)| \leq C, \forall n \in \mathbb{Z}$. Therefore

$$\{n \in \mathbb{Z} : |Ma(n)| > C\} = \emptyset.$$

So,

$$\sum_{\{n\in\mathbb{Z}:|Ma(n)|>C\}} w(n) = 0.$$

which gives $w(\{n \in \mathbb{Z} : |Ma(n)| > C\}) = 0$.
Therefore, $\|Ma\|_{\ell^\infty_w(\mathbb{Z})} \leq C$. Taking $\inf \Big\{C : \|a\|_{\ell^\infty_{Mw}(\mathbb{Z})} < C\Big\}$, we get

$$\|Ma\|_{\ell^\infty_w(\mathbb{Z})} \leq \|a\|_{\ell^\infty_{Mw}(\mathbb{Z})}.$$

To prove the weak(1,1) inequality we may assume that $\{a(n) : n \in \mathbb{Z}\} \in \ell^1(\mathbb{Z})$. Form Calderón-Zygmund decomposition of sequence $\{a(n) : n \in \mathbb{Z}\}$ at height $\frac{\lambda}{4} > 0$. Then we get a sequence $\{I_j\}$ of dyadic intervals in $\mathbb{Z}$ such that

$$\frac{1}{|I_j|} \sum_{k\in I_j} |a(k)| > \frac{\lambda}{4}.$$

Further as we showed in the proof of Lemma 4.3.2(*See* [3]),

$$\{m \in \mathbb{Z} : M'a(m) > \lambda\} \subset \cup_j 3I_j.$$

It follows that

$$\begin{aligned}
\sum_{\{m:M'a(m)>\lambda\}} w(m) &\leq \sum_j \sum_{m\in 3I_j} w(m) \\
&= \sum_j 3|I_j| \frac{1}{|3I_j|} \sum_{m\in 3I_j} w(m) \\
&\leq \frac{12}{\lambda} \sum_j \left(\left(\sum_{n\in I_j} |a(n)| \right) \left(\frac{1}{|3I_j|} \sum_{m\in 3I_j} w(m) \right) \right) \\
&\leq \frac{12C}{\lambda} \sum_{n\in\mathbb{Z}} |a(n)| Mw(n).
\end{aligned}$$

Since by Lemma 4.3.1, $\{m : Ma(m) > \lambda\} \subseteq \{m : M'a(m) > \frac{\lambda}{3}\}$, it follows that

$$\sum_{\{m:Ma(m)>\lambda\}} w(m) \leq \sum_{\{m:M'a(m)>\frac{\lambda}{3}\}} w(m) \leq \frac{36C}{\lambda} \sum_{n\in\mathbb{Z}} |a(n)| Mw(n).$$

□

Theorem 4.6.2. *Assume* $\{w(n) : n \in \mathbb{Z}\} \in A_p$. *Given a non-negative sequence* $\{a(n) : n \in \mathbb{Z}\} \in \ell^p_w(\mathbb{Z})$, *for* $1 \leq p < \infty$, *the weighted weak(p,p) inequality holds:*

$$w(\{m \in \mathbb{Z} : Ma(m) > \lambda\}) \leq \frac{C}{\lambda^p} \sum_{m\in\mathbb{Z}} |a(m)|^p w(m).$$

Proof. Let $\{a(n) : n \in \mathbb{Z}\} \in \ell^p_w(\mathbb{Z})$. Form the Calderón-Zygmund decomposition of $\{a(n) : n \in \mathbb{Z}\}$ at height $\frac{\lambda}{12}$ to get a collection of disjoint intervals $\{I_j\}$ such that

$$\sum_{i\in I_j} \frac{1}{|I_j|} a(i) > \frac{\lambda}{12}.$$

By the proof of Lemma 4.3.2 in [3] and Lemma 4.3.1 , we have

$$\{m \in \mathbb{Z} : Ma(m) > \lambda\} \subseteq \left\{m \in \mathbb{Z} : M'a(m) > \frac{\lambda}{3}\right\} \subseteq \cup_j 3I_j.$$

Therefore, using Theorem 2.3.4, we have

$$w(\{m \in \mathbb{Z} : Ma(m) > \lambda\}) \leq \sum_j w(3I_j) \leq C3^p \sum_j w(I_j)$$

$$\leq C3^p \sum_j \left(\frac{|I_j|}{a(I_j)}\right)^p \sum_{m\in I_j} |a(m)|^p w(m)$$
$$\leq C3^p \left(\frac{12}{\lambda}\right)^p \sum_{m\in\mathbb{Z}} |a(m)|^p w(m).$$

□

Theorem 4.6.3. *If $w \in A_p, 1 < p < \infty$, then M is bounded on $\ell^p_w(\mathbb{Z})$.*

The proof follows from Theorem 4.6.2 and Marcinkiewicz interpolation theorem.

4.7 Fractional Hardy-Littlewood maximal operator

Lemma 4.7.1. *Define sets*

$$E = E_1 \cup E_2$$
$$E_1 = \{x \in E : F(m) < R(m)\}\,; m \in \mathbb{Z}$$
$$E_2 = \{x \in E : R(m) \leq F(m)\}\,; m \in \mathbb{Z}.$$

Let $p(\cdot) : \mathbb{Z} \to [0, \infty)$ be such that $p(\cdot) \in LH_\infty(\mathbb{Z})$ and $0 < p_\infty < \infty$. Let $R(k) = (e+|k|)^{-N}, N > \frac{1}{p_\infty}$. Then there exists a real constant C depending on N and $LH_\infty(\mathbb{Z})$ constant of $p(\cdot)$ such that given any set E and any function F with $0 \leq F(m) \leq 1$ for $m \in E$

$$\sum_E F(m)^{p(m)} \leq \sum_E F(m)^{p_\infty} + C\sum_E R(m)^{p_\infty}. \tag{4.7.1}$$
$$\sum_E F(m)^{p_\infty} \leq C\sum_E F(m)^{p(m)} + C\sum_E R(m)^{p_\infty}. \tag{4.7.2}$$

Proof. For $m \in \mathbb{Z}$, let $R(m) = (e + |m|)^{-N}$.
Since $(e + |m|) \geq e \geq 1, \forall m \in \mathbb{Z}$, we have $(e + |m|)^{-N} \leq 1$ which implies that $R(m) \leq 1 \forall m \in \mathbb{Z}$. Further since $p(\cdot) \in LH_\infty : |p(k) - p_\infty| \leq \frac{C_\infty}{\log(e+|k|)}; k \in \mathbb{Z}$, $|p(m) - p_\infty| \log(e + |m|) \leq c_\infty$. So, it follows that

$$\begin{aligned} R(m)^{-|p(m)-p_\infty|} &= (e + |m|)^{(-N)(-|p(m)-p_\infty|)} \\ &= (e + |m|)^{N|p(m)-p_\infty|} \\ &= \exp(N \log(e + |m|)|p(m) - p_\infty|) \leq \exp(NC_\infty). \end{aligned}$$

Also, $p(m) - p_\infty \geq -|p(m) - p_\infty|, m \in \mathbb{Z}$. Since $R(m) \leq 1, \forall m \in \mathbb{Z}$

$$R(m)^{p(m)-p_\infty} \leq R(m)^{-|p(m)-p_\infty|}.$$

Therefore,

$$\begin{aligned}
\sum_{E_1} F(m)^{p(m)} &\leq \sum_{E_1} R(m)^{p(m)} \\
&\leq \sum_{E_1} R(m)^{p_\infty} R(m)^{-|p(m)-p_\infty|} \\
&\leq \exp(NC_\infty) \sum_{E_1} R(m)^{p_\infty}.
\end{aligned}$$

Also $F(m) \leq 1$, and on E_2, $R(y) \leq F(y) \leq 1$ hence,

$$\begin{aligned}
\sum_{E_2} F(m)^{p(m)} &\leq \sum_{E_2} F(m)^{p_\infty} F(m)^{-|p(m)-p_\infty|} \\
&\leq \sum_{E_2} F(m)^{p_\infty} R(m)^{-|p(m)-p_\infty|} \\
&\leq \exp(NC_\infty) \sum_{E_2} F(m)^{p_\infty}.
\end{aligned}$$

Combining both estimates, above

$$\sum_{E} F(m)^{p(m)} \leq C \sum_{E} F(m)^{p_\infty} + C \sum_{E} R(m)^{p_\infty}.$$

This completes the proof of (4.7.1) .

Proof of (4.7.2):
Note, from $p(y) - p_\infty \leq |p(y) - p_\infty|$, we get $p_\infty \geq p(y) - |p(y) - p_\infty|$.
On $E_2 = \{F(y) > R(y); y \in \mathbb{Z}\}$ and since $F(y) \leq 1$

$$\begin{aligned}
\sum_{E_2} F(y)^{p_\infty} &\leq \sum_{y\in E_2} F(y)^{p(y)} F^{-|p(y)-p_\infty|} \\
&\leq \sum_{y\in E_2} F(y)^{p(y)} R(y)^{-|p(y)-p_\infty|} \\
&\leq \exp\left(NC_\infty\right) \sum_{y\in E_2} F(y)^{p(y)}. \qquad (4.7.3)
\end{aligned}$$

On $E_1 = \{F(y) < R(y); y \in \mathbb{Z}\}$

$$\sum_{y\in E_1} F(y)^{p_\infty} \leq \sum_{y\in E_1} R^{p_\infty}. \tag{4.7.4}$$

From (4.7.4), (4.7.3) we conclude that

$$\sum_{E} F(y)^{p_\infty} \leq C \sum_{E} F(y)^{p(y)} + \sum_{E} R(y)^{p_\infty}.$$

□

4.7.1 Strong $(p(\cdot), p(\cdot))$ inequality for Fractional Hardy-Littlewood maximal operator

In this section, we derive strong $(p(\cdot), p(\cdot))$ inequality for fractional Hardy-Littlewood maximal Operator. In order to prove this theorem, we use Lemma 4.5.3.

Theorem 4.7.2 (Strong $(p(\cdot), p(\cdot))$ inequality for Fractional Hardy-Littlewood maximal operator)**.** *Let $\{a(n) : n \in \mathbb{Z}\}$ be a non-negative sequence in $\ell^{p(\cdot)}(\mathbb{Z})$ where $\{p(n) : n \in \mathbb{Z}\}$ is a sequence in $\mathcal{S}$ that belongs to $LH_\infty(\mathbb{Z})$. Let $\{a(n) : n \in \mathbb{Z}\}$ be a sequence which satisfies*

$$\frac{1}{p(n)} - \frac{1}{q(n)} = \alpha, n \in \mathbb{Z}$$

and $p(\cdot) \in LH_\infty, 1 < p_- \leq p_+ < \frac{1}{\alpha}$. Further assume that $q(\cdot) \in LH_\infty$.

Then

$$\|M_\alpha a\|_{\ell^{q(\cdot)}(\mathbb{Z})} \leq C \|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}.$$

Proof. Now, we are going to prove $\|M_\alpha a\|_{\ell^{q(\cdot)}(\mathbb{Z})} \leq C \|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}$. We may assume without loss of generality that $\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})} = 1$. We will show that there exist a constant $\lambda_2 = \lambda_2(p(\cdot)) > 0$ such that $\rho_{q(\cdot)}(M_\alpha \frac{a}{\lambda_2}) \leq 1$. For this it suffices to prove $\rho_{q(\cdot)}(\alpha_2\beta_2\gamma_2\delta_2 M_\alpha a) \leq \frac{1}{2}$ form some non-negative real numbers $\alpha_2, \beta_2, \gamma_2, \delta_2$.
Let ${\lambda_2}^{-1} = \alpha_2\beta_2\gamma_2\delta_2$. To estimate this term we perform Calderón-Zygmund decomposition for sequences $\{a(k)\}$ at height $(9t)^{k+1}$ with $0 < \alpha < \frac{1}{9}$ and obtain disjoint dyadic intervals $\left\{I_j^k\right\}$ Then we use Lemma 4.5.3 for the fractional Hardy-Littlewood maximal operator M_α. We will show that $\rho_{q(\cdot)}(\alpha_2\beta_2\gamma_2\delta_2 M_\alpha a) \leq \frac{1}{2}$ for suitable choices of $\alpha_2, \beta_2, \gamma_2, \delta_2$.
Then

$$\rho_{q(\cdot)}(\alpha_2\beta_2\gamma_2\delta_2 M_\alpha a) = \sum_{m\in\mathbb{Z}} [\alpha_2\beta_2\gamma_2\delta_2 M_\alpha a(m)]^{q(m)}$$

$$\leq \alpha_2^{q-}\beta_2^{q-}\gamma_2^{q-}\delta_2^{q-}\sum_{m\in\mathbb{Z}} M_\alpha a(m)^{q(m)}$$

$$\leq \alpha_2^{q-}\beta_2^{q-}\gamma_2^{q-}\delta_2^{q-}A^{q+}18^{q+(1-\alpha)}\sum_{k,j}\sum_{m\in E_j^k}\left(\frac{1}{|2I_j^k|^{1-\alpha}}\sum_{r\in 2I_j^k}|a(r)|\right)^{q(m)}.$$

Now choose

$$\alpha_2^{q-} = \frac{1}{A^{q+}18^{q+(1-\alpha)}}.$$

Then

$$LHS \leq \beta^{q-}\gamma^{q-}\delta^{q-}\sum_{k,j}\sum_{m\in E_j^k}\left(\frac{1}{|2I_j^k|^{1-\alpha}}\sum_{r\in 2I_j^k}|a(r)|\right)^{q(m)}.$$

Hence

$$\sum_{m\in\mathbb{Z}}[\alpha_2\beta_2\gamma_2\delta_2 M_\alpha a(m)]^{q(m)} \leq \beta^{q-}\gamma^{q-}\delta^{q-}\sum_{k,j}\sum_{m\in E_j^k}\left(\frac{1}{|2I_j^k|^{1-\alpha}}\sum_{r\in 2I_j^k}|a(r)|\right)^{q(m)}. \tag{4.7.5}$$

We have to estimate right hand side of equation (4.7.5). At this point, we note that $q_\infty < \infty$. Let $g_2(r) = a(r)^{p(r)}$, then equation (4.7.5) becomes,

$$\sum_{k,j}\sum_{m\in E_j^k}\left(\beta_2\gamma_2\delta_2|2I_j^k|^{\alpha-1}\sum_{r\in 2I_j^k} g_2(r)^{\frac{1}{p(r)}}\right)^{q(m)}.$$

Since $\frac{1}{q(\cdot)} \in LH_\infty$ from the definition we have

$$\left|\frac{1}{q(m)} - \frac{1}{q(\infty)}\right| \leq \frac{C_\infty}{\log(e+|m|)} \forall m \in \mathbb{Z}.$$

Also,

$$\frac{1}{p(m)} - \frac{1}{q(m)} = 1 \quad \forall m \in \mathbb{Z}.$$

and

$$\frac{1}{p(m)} - \frac{1}{q(m)} = \alpha \quad 0 \leq \alpha < 1.$$

So,

$$\left|\frac{1}{q(m)} - \frac{1}{q(\infty)}\right| = \left|\frac{1}{p(m)} - \alpha - \frac{1}{q(\infty)}\right| \leq \frac{C_\infty}{\log(e+|m|)} \quad m \in \mathbb{Z}.$$

which shows $\frac{1}{q(\infty)}$ is the LH_∞ constant with value $\frac{1}{p(\infty)} + \alpha$. Hence we conclude

$$\frac{1}{p(\infty)} - \frac{1}{q(\infty)} = \alpha.$$

Note, since $\frac{1}{p(\cdot)} \in LH_\infty(\mathbb{Z})$, it follows that the exponents p_∞, q_∞ satisfy $\frac{1}{p_\infty} - \frac{1}{q_\infty} = \alpha$. Hence, using Lemma 4.1.1 with exponents p_∞, q_∞.

$$|2I_j^k|^{\alpha-1}\sum_{r\in 2I_j^k} g_2(r)^{\frac{1}{p(r)}} \quad \le \left(\sum_{r\in 2I_j^k} g_2(r)^{\frac{p_\infty}{p(r)}}\right)^{\frac{1}{p_\infty}-\frac{1}{q_\infty}}\left(\frac{1}{|2I_j^k|}\sum_{r\in 2I_j^k} g_2(r)^{\frac{1}{p(r)}}\right)^{\frac{p_\infty}{q_\infty}}. \tag{4.7.6}$$

Further, note the following estimates.

1. $g_2(r)^{p_\infty} \le 1$. since $g_2(r)^{p_\infty} = a(r)^{p(r)p_\infty} \le 1$ since $a(r) \le 1$.

2. For $N > 1$, define $R(k) = (e+|k|)^{-N}$. Since $N > \frac{1}{p_\infty}$, $\sum_{k\in\mathbb{Z}} R(k)^{Np_\infty} = \sum_{k\in\mathbb{Z}}(\frac{1}{e+|k|})^{Np_\infty}$ converges and can be bounded by some constant. So, the second integral is a constant depending only on p_∞ by taking sufficiently large $N > \frac{1}{p_\infty}$. By taking N large enough, it follows that

$$\sum_{m\in\mathbb{Z}} R(m)^{\frac{1}{p_\infty}} \le \sum_{m\in\mathbb{Z}} R(m)^{\frac{1}{q_\infty}} \le 1.$$

3. Also, note that by the property of variable norm

$$\sum_{r\in 2I_j^k} g_2(r) = \sum_{r\in 2I_j^k} a(r)^{p(r)} \le \sum_{r\in\mathbb{Z}} a(r)^{p(r)} \le \|a\|_{\ell^{p(\cdot)}(\mathbb{Z})} = 1.$$

4. Let $F(k) = g_2(k)^{p_\infty}$. Since $g_2(k)^{p_\infty} \le 1 \forall k, F(k) \le 1 \forall k$. Hence, we use following form of Lemma 4.7.1 , where $\frac{1}{r(\cdot)}$ is taken as LH_∞ constant.

$$\sum_{m\in 2I_j^k} F(m)^{\frac{1}{p(m)}} \le C\sum_{m\in 2I_j^k} F(m)^{\frac{1}{p_\infty}} + C\sum_{m\in 2I_j^k} R(m)^{\frac{1}{p_\infty}}.$$

Since $\frac{1}{p(\cdot)} \in LH_\infty(\mathbb{Z})$ and using Lemma 4.7.1 with exponents p_∞, q_∞ with $F = g_2(r)^{p_\infty} \le 1$, based on estimates [1-4], it follows that

$$\begin{aligned}\sum_{m\in 2I_j^k}\left(g_2(k)^{p_\infty}\right)^{\frac{1}{p(m)}} &\le C\sum_{m\in\mathbb{Z}}\left(g_2(m)^{p_\infty}\right)^{\frac{1}{p_\infty}} + C\sum_{m\in\mathbb{Z}} R(m)^{\frac{1}{p_\infty}}\\ &= C\sum_{m\in\mathbb{Z}} g_2(m) + C\sum_{m\in\mathbb{Z}} R(m)^{\frac{1}{p_\infty}} \le C.\end{aligned}$$

Therefore equation (4.7.6) is

$$|2I_j^k|^{\alpha-1}\sum_{r\in 2I_j^k} g_2(r)^{\frac{1}{p(r)}} \le \left(\sum_{r\in 2I_j^k} g_2(r)^{\frac{p_\infty}{p(r)}}\right)^{\frac{1}{p_\infty}-\frac{1}{q_\infty}}\left(\frac{1}{|2I_j^k|}\sum_{r\in 2I_j^k} g_2(r)^{\frac{1}{p(r)}}\right)^{\frac{p_\infty}{q_\infty}}$$
$$\le C^\alpha\left(\frac{1}{|2I_j^k|}\sum_{r\in 2I_j^k} g_2(r)^{\frac{1}{p(r)}}\right)^{\frac{p_\infty}{q_\infty}}.$$

Therefore, using estimates [1-4], we can now choose constant $\beta_2 > 0, \beta_2 = \frac{1}{C^\alpha}$ such that

$$\beta_2^{q-}\gamma_2^{q-}\delta_2^{q-}\sum_{k,j}\sum_{E_j^k}\left(|2I_j|^{\alpha-1}g_2(r)^{\frac{1}{p(r)}}\right)^{q(m)}$$
$$\le \gamma_2^{q-}\delta_2^{q-}\sum_{k,j}\sum_{E_j^k}\left(\left(\frac{1}{|2I_j^k|}\sum_{r\in 2I_j^k} g_2(r)^{\frac{1}{p(r)}}\right)^{p_\infty}\right)^{\frac{q(m)}{q_\infty}}.$$

Note,

$$\frac{1}{|2I_j^k|}\sum_{r\in 2I_j^k} g_2(r)^{\frac{p_\infty}{p(r)}} = \frac{1}{|2I_j^k|}\sum_{r\in 2I_j^k} a_2(r)^{p_\infty} \le 1.$$

So, let $F(r) = \left(g_2(r)^{\frac{1}{p(r)}}\right)^{q(r)p_\infty} \le \left(\sum_{r\in\mathbb{Z}} g_2(r)^{\frac{1}{p(r)}}\right)^{q(r)p_\infty} \le 1.$
Using Lemma 4.7.1 and with $\frac{1}{q(\cdot)} \in LH_\infty(\mathbb{Z})$, we get

$$\sum_{r\in\mathbb{Z}} F(r)^{\frac{1}{q_\infty}} \le C_1\sum_{r\in\mathbb{Z}} F(r)^{\frac{1}{q(r)}} + C_2\sum_{r\in\mathbb{Z}} R(r)^{\frac{1}{q_\infty}}.$$

Therefore,

$$\sum_{k\in\mathbb{Z}}\left(g_2(k)^{\frac{1}{p(k)}}\right)^{q(k)\frac{p_\infty}{q_\infty}} \le C_1\sum_{k\in\mathbb{Z}}\left(g_2(k)^{\frac{1}{p(k)}}\right)^{p_\infty} + C_2\sum_{k\in\mathbb{Z}} R(k)^{\frac{1}{q_\infty}}.$$

and so,

$$\sum_{k,j}\sum_{E_j^k}\gamma_2\delta_2\left(\left(\frac{1}{|2I_j^k|}\sum_{r\in 2I_j^k} g_2(r)^{\frac{1}{p(r)}}\right)^{p_\infty}\right)^{\frac{q(m)}{q_\infty}}$$
$$\leq \delta_2\left(C_1\left(\frac{1}{|2I_j^k|}\sum_{r\in 2I_j^k} g_2(r)^{\frac{1}{p(r)}}\right)^{p_\infty} + C_2\sum_{k\in\mathbb{Z}} R(k)^{\frac{1}{q_\infty}}.\right.$$

Take $\gamma_2 > 0$ such that $C_1 = 1$. Then

$$\gamma_2^{q_-}\delta_2^{q_-}\sum_{k,j}\sum_{E_j^k}\left(\left(\frac{1}{|2I_j^k|}\sum_{r\in 2I_j^k} g_2(r)^{\frac{1}{p(r)}}\right)^{p_\infty}\right)^{\frac{q(m)}{q_\infty}}$$
$$\leq \delta_2^{q_-}\sum_{k,j}\sum_{E_j^k}\left(\left(\frac{1}{|2I_j^k|}\sum_{r\in 2I_j^k} g_2(r)^{\frac{1}{p(r)}}\right)^{p_\infty}\right) + C_2\sum_{k\in\mathbb{Z}} R(k)^{\frac{1}{q_\infty}}$$
$$\leq \delta_2^{q_-}\sum_{\mathbb{Z}}\left(Mg_2(\cdot)^{\frac{1}{p(\cdot)}}\right)(k)^{p_\infty} + C.$$

Note that the maximal operator is bounded on $\ell^{p_\infty}(\mathbb{Z})$, since $p_\infty \geq p_- > 1$. Again apply Lemma 4.7.1 to get

$$\sum_{k\in\mathbb{Z}}\left(M(g_2(\cdot)^{\frac{1}{p(\cdot)}})\right)(k)^{p_\infty} \leq C\sum_{k\in\mathbb{Z}} g_2(k)^{\frac{p_\infty}{p(k)}} \leq C\sum_{k\in\mathbb{Z}} g_2(k) + C\sum_{k\in\mathbb{Z}} R(k)^{\frac{1}{p_\infty}} \leq C.$$

Finally, note $a(\cdot)\in\ell^{p_\infty}(\mathbb{Z})$, $g_2(\cdot)^{\frac{1}{p(\cdot)}}\in\ell^{p_\infty}(\mathbb{Z})$. So, we can choose $\delta_2 > 0$ such that

$$\delta_2^{q_-}\sum_{k\in\mathbb{Z}}\left(M(g_2(\cdot)^{\frac{1}{p(\cdot)}})\right)(k)^{p_\infty} + \frac{1}{6} \leq \frac{1}{3} + \frac{1}{6} = \frac{1}{2}.$$

□

4.7.2 Weak $(p(\cdot), p(\cdot))$ inequality for Fractional Hardy-Littlewood maximal operator

Theorem 4.7.3 (Weak $(p(\cdot), p(\cdot))$ inequality for Fractional Hardy-Littlewood maximal operator)**.** *Given a non-negative sequence* $\{a(i)\}\in\ell^{p(\cdot)}(\mathbb{Z})$*, let* $p(\cdot)\in\mathcal{S}, p_+ < \infty, p_- = 1, 1 \leq p_- \leq p_+ <$

$\frac{1}{\alpha}$ *and* $p(\cdot) \in LH_\infty(\mathbb{Z})$*. Define sets* Ω*,* $E_j, j \in \mathbb{Z}$ *as follows.*

$$\Omega = \{k \in \mathbb{Z} : M_\alpha a(k) > 9t\} = \cup_j E_j.$$

where E_j *are disjoint for all* $j \in \mathbb{Z}$*. Then*

$$\sup_{t>0} \left\| t\chi_{\{M_\alpha a(k)>9t\}} \right\|_{\ell^{q(\cdot)}(\mathbb{Z})} \leq C \left\| a \right\|_{\ell^{p(\cdot)}(\mathbb{Z})}.$$

Proof. Using Theorem 4.5.2 for sequence $\{a(n) : n \in \mathbb{Z}\}$, we get for fixed $t > 0$, disjoint intervals $\left\{ I_k^j \right\}$

$$\{k \in \mathbb{Z} : M_\alpha a(k) > 9t\} = \cup_j 2I_j.$$

Define disjoint sets E_j such that $E_j \subset 2I_j$ inductively as follows:

$$\begin{aligned} E_1 &= 2I_1 \\ E_2 &= 2I_2 \setminus E_1 \\ E_3 &= 2I_3 \setminus E_2 \\ \ldots E_m &= 2I_m \setminus E_{m-1}. \end{aligned}$$

Now $\Omega = \cup_j E_j$. To prove the weak inequality, it will suffice to show that for each $k \in \Omega$, $\|t\chi_\Omega(k)\|_{q(\cdot)} \leq C$ and in turn it will suffice to show that for some $\alpha_2 > 0$,

$$\rho_{q(\cdot)}(\alpha_2 t\chi_\Omega) = \sum_{k\in\Omega} [\alpha_2 t]^{q(m)} \leq 1.$$

We will show that each term on the right is bounded by $\frac{1}{2}$ for suitable choice of α_2.
To estimate $\sum_{k\in\Omega} [\alpha_2 t]^{q(k)}$, we note from Lemma 4.5.3 and results from previous theorem, we have

$$\begin{aligned} &\rho_{q(\cdot)}(\alpha_2 t\chi_\Omega) = \sum_{k\in\mathbb{Z}} [\alpha_2 t\chi_\Omega(k)]^{q(k)} \\ &\leq \alpha_2{}^{q-} \sum_{k\in\mathbb{Z}} [t\chi_\Omega(k)]^{q(k)} = \alpha_2{}^{q-} \sum_{k\in\Omega} [t]^{q(k)} \\ &= (\frac{\alpha_2}{9})^{q-} \sum_{k\in\Omega} \Big(M_\alpha a(k) \Big)^{q(k)} \\ &\leq C^{q-} \alpha_2{}^{q-} \sum_j \sum_{k\in E_j} (M_\alpha a(k))^{q(k} \\ &\leq C^{q-} \left(\sum_j \sum_{k\in E_j} \left(\frac{1}{|2I_j|^{1-\alpha}} \sum_{r\in 2I_j} a(r) \right)^{q(k)} \right) \end{aligned}$$

$$\leq C^{q_-}\delta_2^{q_-}\left(\sum_j \sum_{E_j}\left(\frac{1}{|2I_j|}\sum_{k\in 2I_j} g_2(k)^{\frac{1}{p(k)}}\right)^{p_\infty} + \frac{1}{6}\right)$$

$$\leq C^{q_-}\delta_2^{q_-}\left(\sum_j \sum_{E_j}\left(\frac{1}{|2I_j|}\sum_{k\in 2I_j} g_2(k)^{\frac{p_\infty}{p(k)}}\right) + \frac{1}{6}\right)$$

$$\leq C^{q_-}\delta_2^{q_-}\left(\sum_{k\in\Omega} g_2(k)^{\frac{p_\infty}{p(k)}} + \frac{1}{6}\right).$$

Now, choose $\delta_2 > 0$ such that right hand side is bounded by $\frac{1}{2}$. □

4.8 Hardy-Littlewood maximal operator

In this section, we prove boundedness of Hardy-Littlewood maximal operator for $\ell^{p(\cdot)}(\mathbb{Z})$ spaces where $p_- > 1$. The proof is based on boundedness of Hardy-Littlewood maximal operator on $\ell^p(\mathbb{Z})$, where p is a fixed number, $1 < p < \infty$.

Remark 4.8.1. Note that when $\alpha = 0$ the fractional Hardy-Littlewood maximal operator is nothing but Hardy-Littlewood maximal operator. However we can prove strong type, weak type inequalities for Hardy-Littlewood maximal operator on $\ell^{p(\cdot)}(\mathbb{Z})$ directly from the corresponding results for fixed $\ell^p(\mathbb{Z})$ spaces, $1 < p < \infty$. The key point of the proofs is Lemma 4.7.1. The proof of continuous version of Lemma 4.7.1 can be found in [8]. Same line of proof works here.

4.8.1 Strong $(p(\cdot), p(\cdot))$ inequality

Theorem 4.8.2 (Strong $(p(\cdot), p(\cdot))$ inequality)**.** *Given a non-negative sequence* $\{a(i)\} \in \ell^{p(\cdot)}(\mathbb{Z}), p(\cdot) \in \mathcal{S}, p_+ < \infty, p_- > 1$ *, then*

$$\|Ma\|_{\ell^{p(\cdot)}(\mathbb{Z})} \leq C\,\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}\,.$$

Proof. By homogeneity, it is enough to prove the above result with the assumption $\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})} = 1.$. By Lemma 3.1.15, $\sum_{i\in\mathbb{Z}}|a(i)|^{p(i)} \leq 1$. So, it is enough to prove that

$$\sum_{i\in\mathbb{Z}}|Ma(i)|^{p(i)} \leq C.$$

Given that $0 \leq a(k) \leq 1$, it follows that $0 \leq Ma(k) \leq 1$. To prove boundedness of $\{Ma\}$, we start with Lemma 4.7.1 as follows:

$$\sum_{k\in\mathbb{Z}} Ma(k)^{p(k)} \leq C\sum_{k\in\mathbb{Z}} Ma(k)^{p_\infty} + C\sum_{k\in\mathbb{Z}} R(k)^{p_\infty}.$$

Since $N > \frac{1}{p_\infty}$, $\sum_{k\in\mathbb{Z}} R(k)^{Np_\infty} = \sum_{k\in\mathbb{Z}}(\frac{1}{e+|k|})^{Np_\infty}$ converges and can be bounded by some constant. So, the second integral is a constant depending only on p_∞ by taking sufficiently large $N > \frac{1}{p_\infty}$.

To bound the first integral, note that $1 < p_- \leq p_\infty$. Since $p_\infty > 1$, M is bounded on $\ell^{p_\infty}(\mathbb{Z})$ and by using strong (p,p) inequality valid for classical Lebesgue spaces with index p_∞, we get using Lemma 4.7.1 and equation (4.7.1) ,

$$\sum_{k\in\mathbb{Z}} Ma(k)^{p_\infty} \leq C\sum_{k\in\mathbb{Z}} a(k)^{p_\infty} \leq C\sum_{k\in\mathbb{Z}} a(k)^{p(k)} + C\sum_{k\in\mathbb{Z}} R(k)^{p_\infty} \leq C\left\|a\right\|_{p(\cdot)} + C\sum_{k\in\mathbb{Z}} R(k)^{p_\infty} \leq C.$$

Like previous case, the term involving summation of $R(k)$ is bounded by a constant depending only on p_∞ by taking sufficiently large $N > \frac{1}{p_\infty}$.
Therefore, using above results,

$$\rho_{p(\cdot)}(Ma) = \sum_{k\in\mathbb{Z}} Ma(k)^{p(k)} \leq C.$$

□

4.9 Maximal ergodic operator

Let $(X, \mathbf{B}, \mu)$ be a probability space and U an invertible measure preserving transformation on X. We define maximal ergodic operator as

$$\tilde{M}f(x) = \sup_{J\geq 1} \frac{1}{2J+1}\sum_{k=-J}^{J} |f(U^{-k}x)|.$$

For any positive integer J, we also define truncated maximal ergodic operator as

$$\tilde{M}_J f(x) = \sup_{1\leq n\leq J} \frac{1}{2n+1}\sum_{k=-n}^{n} |f(U^{-k}x)|.$$

In the following theorem using transference, we prove that the maximal ergodic operator is bounded on weighted $L^p_w(X, \mathcal{B}, \mu), 1 < p < \infty$ where w is ergodic A_p weight and the maximal ergodic operator satisfies weak type (1,1) inequality on $L^1_w(X, \mathbf{B}, \mu)$ space. For the definition of ergodic A_p weights refer Chapter 2.

Theorem 4.9.1. *Let $(X, \boldsymbol{B}, \mu)$ be a probability space and U an invertible measure preserving transformation on X.*

1. *If w is an ergodic A_p weight, $1 < p < \infty$ and $f \in L^p_w(X, \mathcal{B}, \mu)$, then the maximal ergodic operator*
$$\left\|\tilde{M}f(x)\right\|_{L^p_w(X)} \leq C_p \|f\|_{L^p_w(X)} \quad \textit{if} \quad 1 < p < \infty.$$
2. *If w is an ergodic A_1 weight and $f \in L^1_w(X, \mathcal{B}, \mu)$, then*
$$\int_{\left\{x \in X : |\tilde{M}f(x)| > \lambda\right\}} w(x) d\mu(x) \leq \frac{C}{\lambda} \int_X |f(x)| w(x) d\mu(x).$$

Proof. Take $p, 1 \leq p < \infty$ and take a function $f \in L^p_w(X)$. For a positive integer$J > 0$ define

$$\tilde{M}_J f(x) = \sup_{1 \leq n \leq J} \frac{1}{2n+1} \sum_{k=-n}^{n} |f(U^{-k}x)|.$$

It is enough to prove that $\tilde{M}_J$ satisfies (1) and (2) with constants not depending on J. Let $\lambda > 0$ and put

$$E_\lambda = \left\{x \in X : |\tilde{M}_J f(x)| > \lambda\right\}.$$

For x lying outside a μ null set and a positive integer L, define sequences

$$a_x(k) = \begin{cases} f(U^{-k}x) & if \quad |k| \leq L + J \\ 0 \quad otherwise \end{cases}$$

$$w_x(k) = \begin{cases} w(U^{-k}x) & if \quad |k| \leq L + J \\ 0 \quad otherwise \end{cases}$$

Using Theorem 4.3.1, observe that for an integer m with $|m| \leq L$ we have

$$\tilde{M}_J f(U^{-m}x) = \sup_{1 \leq n \leq J} \frac{1}{2n+1} \sum_{k=-n}^{n} |f(U^{-k-m}x)|$$
$$= \sup_{1 \leq n \leq J} \frac{1}{2n+1} \sum_{k=-n}^{n} |a_x(m+k)| = M'_J a_x(m) \leq M a_x(m)$$

Therefore,

$$
\begin{aligned}
& w(\Big\{x \in X : |\tilde{M}_J f(x)| > \lambda\Big\}) = \int_{E_\lambda} w(x) d\mu(x) = \frac{1}{\lambda^p} \int_{E_\lambda} \lambda^p w(x) d\mu(x) \\
& \leq \frac{1}{\lambda^p} \int_{E_\lambda} |\tilde{M}_J f(x)|^p w(x) d\mu(x) \\
& \leq \frac{1}{\lambda^p} \int_X |\tilde{M}_J f(x)|^p w(x) d\mu(x) \\
& = \frac{1}{\lambda^p} \frac{1}{2L+1} \sum_{m=-L}^{L} \int_X |\tilde{M}_J f(U^{-m}x)|^p w(U^{-m}x) d\mu(x) \\
& \leq \frac{1}{\lambda^p} \frac{1}{2L+1} \sum_{m=-L}^{L} \int_X |M a_x(m)|^p w_x(m) d\mu(x) \\
& = \frac{1}{\lambda^p} \frac{1}{2L+1} \int_X \sum_{m=-L}^{L} |M a_x(m)|^p w_x(m) d\mu(x) \\
& \leq \frac{1}{\lambda^p} \frac{1}{2L+1} \int_X \sum_{m=-\infty}^{\infty} |M a_x(m)|^p w_x(m) d\mu(x) \\
& \leq \frac{C}{\lambda^p} \frac{1}{2L+1} \int_X \sum_{m=-\infty}^{\infty} |a_x(m)|^p w_x(m) d\mu(x) \\
& = \frac{C}{\lambda^p} \frac{1}{2L+1} \int_X \sum_{m=-(L+J)}^{(L+J)} |a_x(m)|^p w_x(m) d\mu(x) \\
& = \frac{C}{\lambda^p} \frac{1}{2L+1} \int_X \sum_{m=-(L+J)}^{(L+J)} |f(U^{-m}x)|^p w(U^{-m}x) d\mu(x) \\
& \leq \frac{C}{\lambda^p} \frac{1}{2L+1} \sum_{m=-(L+J)}^{(L+J)} \int_X |f(U^{-m}x)|^p w(U^{-m}x) d\mu(x) \\
& = \frac{C}{\lambda^p} \frac{1}{2L+1} \sum_{m=-(L+J)}^{(L+J)} \int_X |f(x)|^p w(x) d\mu(x) \\
& \leq \frac{C}{\lambda^p} \frac{1}{2L+1} (2(L+J)+1) \, \|f\|^p_{L^p_w(X)} \\
& \leq \frac{C}{\lambda^p} (\frac{2L}{2L+1} + \frac{2J+1}{2L+1}) \, \|f\|^p_{L^p_w(X)} \\
& \leq \frac{C}{\lambda^p} \, \|f\|^p_{L^p_w(X)} .
\end{aligned}
$$

by choosing L appropriately. Conclusion (1) of the theorem now follows by using the Marcinkiewicz interpolation theorem. □

Now, we prove the converse of Theorem 4.9.1 for $p > 1$ with the additional assumptions (1) $(X, \mathcal{B}, \mu)$ is a probability space and (2) U is ergodic measure preserving transformation. Using transference method, we prove the converse of Theorem 4.9.1. A direct proof can be seen in [5]. For this we require the concept of ergodic rectangles [5].

The following lemma, whose proof is obvious is used in the proof of below theorem.

Lemma 4.9.2. *Suppose $\{w(n) : n \in \mathbb{Z}\}$ is a sequence in $A_p(\mathbb{Z}), 1 \leq p < \infty$. Put*

$$w'(x) = \begin{cases} w(j) & if \quad x \in [j - \frac{1}{4}, j + \frac{1}{4}], \quad j \in \mathbb{Z} \\ 0 & otherwise \end{cases}$$

If $w \in A_p(\mathbb{Z})$, then $w' \in A_p(\mathbb{R}), 1 \leq p < \infty$.
If $w \in A_1(\mathbb{Z})$, then $w' \in A_1(\mathbb{R})$.

Theorem 4.9.3. *Let $(X, \mathcal{B}, \mu)$ be a probability space, U an invertible ergodic measure preserving transformation on X. If $\tilde{M}f$ is bounded on $L^p_w(X)$ for some $1 < p < \infty$, then $w \in A_P(X)$.*

Proof. For the given function w on X, for a.e $x \in X$ define the sequence $w_x(k) = w(U^{-k}x)$. We shall prove that

$$esssup_{x\in X}\left(\frac{1}{|I|}\sum_{k\in I}|w_x(k)|\right)\left(\frac{1}{|I|}\sum_{k\in I}|w_x(k)|^{p'-1}\right)^{p-1} \leq C.$$

This will prove that $w \in A_p(X)$. In order to prove this, we shall prove that the Hardy-Littlewood maximal operator M is bounded on $\ell^p_{w_x}(\mathbb{Z})$ and

$$\|Ma\|_{\ell^p_{w_x}(\mathbb{Z})} \leq C_p \|a\|_{\ell^p_{w_x}(\mathbb{Z})}$$

where C_p is independent of x. In order to prove the above inequality, take a sequence $\{a(n) : n \in \mathbb{Z}\} \in \ell^p_{w_x}(\mathbb{Z})$.

Let $R = \cup_{k=-2J}^{2J} U^k E$ be an ergodic rectangle of length $4J + 1$ with base E. Let F be any measurable subset of E. Then F is also base of an ergodic rectangle of length $4J + 1$. Let $R' = \cup_{k=-2J}^{2J} U^k F$. Define function f and w as follows.

$$f(U^{-k}x) = \begin{cases} a(k) & if \quad x \in F \quad \text{and} - J \leq k \leq J \\ 0 & otherwise. \end{cases}$$

Then,

$$\begin{aligned} \|f\|^p_{L^p_w(X)} &= \int_X |f(x)|^p w(x) d\mu(x) = \int_{R'} |f(x)|^p w(x) d\mu(x) \\ &= \sum_{k=-J}^{J} \int_{U^k F} |f(x)|^p w(x) d\mu(x) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=-J}^{J} \int_F |f(U^{-k}x)|^p w(U^{-k}x) d\mu(x) \\
&= \sum_{k=-J}^{J} \int_F |a(k)|^p w_x(k) d\mu(x) \\
&= \int_F \Big(\sum_{k=-J}^{J} |a(k)|^p w_x(k) \Big) d\mu(x) \\
&\leq \|a\|_{\ell^p_{w_x}(\mathbb{Z})} \mu(F).
\end{aligned}
$$

Using Lemma 4.3.1, it is easy to observe that for $-J \leq m \leq J$ and $x \in F$

$$\tilde{M_J} f(U^{-m}x) = M'_J a(m) \geq \frac{1}{3} M_J a(m).$$

Now,

$$
\begin{aligned}
C \|f\|^p_{L^p_w(X)} &\geq \int_X |\tilde{M_J} f(x)|^p w(x) d\mu(x) \\
&= \int_{R'} |\tilde{M_J} f(x)|^p w(x) d\mu(x) \\
&= \sum_{k=-J}^{J} \int_{U^k F} |\tilde{M_J} f(x)|^p w(x) d\mu(x) \\
&= \sum_{k=-J}^{J} \int_F |\tilde{M_J} f(U^{-k}x)|^p w(U^{-k}x) d\mu(x) \\
&= \sum_{k=-J}^{J} \int_F |M'_J a(k)|^p w_x(k) d\mu(x) \\
&= \int_F \sum_{k=-J}^{J} |M'_J a(k)|^p w_x(k) d\mu(x) \\
&\geq \frac{1}{3} \int_F \sum_{k=-J}^{J} |M_J a(k)|^p w_x(k) d\mu(x).
\end{aligned}
$$

So, from the above estimates

$$\frac{1}{\mu(F)} \int_F \sum_{k=-J}^{J} |M_J a(k)|^p w_x(k) d\mu(x) \leq C \|a\|_{\ell^p_{w_x}(\mathbb{Z})}.$$

Since F was an arbitrary subset of E, we get

$$\sum_{k=-J}^{J} |M_J a(k)|^p w_x(k) \leq C \|a\|_{\ell^p_{w_x}(\mathbb{Z})}.$$

a.e $x \in E$. Since U is ergodic, X can be written as countable union of bases of ergodic rectangles of length $4J+1$. Therefore for a.e $x \in X$,

$$\sum_{k=-J}^{J} |M_J a(k)|^p w_x(k) \leq C \|a\|_{\ell^p_{w_x}(\mathbb{Z})} .$$

Since C is independent of J, a.e $x \in X$,

$$\sum_{k \in \mathbb{Z}} |Ma(k)|^p w_x(k) \leq C \|a\|_{\ell^p_{w_x}(\mathbb{Z})} .$$

It follows that the sequence $\{w_x(n) : n \in \mathbb{Z}\}$ as defined by $w_x(k) = w(U^k x)$ belongs to $A_p(\mathbb{Z})$ a.e $x \in X$ and A_p weight constant for w_x is independent of x so that $w \in A_p(X)$. $\square$

Chapter 5

Singular Operators

In this chapter, we study Singular operator of Calderón-Zygmund kernel type on weighted $\ell_w^{p(\cdot)}(\mathbb{Z})$ spaces. This operator on $\ell^p(\mathbb{Z})$ spaces is studied in [1]. Here we prove the strong type (p,p) inequality on $\ell_w^p(\mathbb{Z})$ and weak(1,1) inequality on $\ell_w^1(\mathbb{Z})$ for both singular operator and maximal singular operator. The inequalities for singular operators are obtained from the corresponding inequalities for Hardy-Littlewood maximal operators. The inequalities for maximal singular operator are proved using transference method by transferring the corresponding results on real line and the using the observation that the linear extension of singular kernel on $\mathbb{Z}$ is a singular kernel on $\mathbb{R}$.

5.1 Calderón-Zygmund singular operator

The continuous version of singular operator are studied in [15]. The singular kernel $K(x)$ is defined as follows.

Definition 5.1.1. A locally integrable function $K(x)$ defined on $\mathbb{R}$ is said to be Calderón-Zygmund singular kernel if the following conditions are satisfied:

$$\int_{\epsilon<|x|<\frac{1}{\epsilon}} K(x) \quad dx \quad \text{converges as } \epsilon \to 0. \tag{K1}$$

$$|K(x)| \leq \frac{C}{|x|}. \tag{K2}$$

$$|K(x) - K(x-y)| \leq \frac{C|y|}{x^2} \text{ for} |x| > 2|y|. \tag{K3}$$

The principal value integral

$$T_K f(x) = \lim_{\epsilon > 0} \int_{|x-y|>\epsilon} K(x-y)f(y)dy,$$

and the maximal singular integral operator

$$T_K^{\star} f(x) = \sup_{\epsilon > 0} |\int_{|x-y|>\epsilon} K(x-y)f(y)dy|,$$

are of strong type (p,p) for $1 < p < \infty$ and weak type (1,1) [1].

Definition 5.1.2. A sequence $\{\phi(n)\}$ is said to be a singular kernel if there exist constants C_1 and $C_2 > 0$ such that

(S1): $\sum_{n=-N}^{N} \phi(n)$ converges as $N \to \infty$.
(S2): $\phi(0) = 0$ and $|\phi(n)| \leq \frac{C_1}{|n|}, n \neq 0$
(S3): $|\phi(n+1) - \phi(n)| \leq \frac{C_2}{n^2}, \quad n \neq 0.$

If $\phi = \{\phi(n)\}$ is a singular kernel and $\{a(n) : n \in \mathbb{Z}\} \in \ell^p(\mathbb{Z}), 1 \leq p < \infty$, define

$$T_\phi a(n) = (\phi \star a)(n) = \sum_{k\in\mathbb{Z}} \phi(n-k)a(k).$$

Since S2 implies that $\phi \in \ell^r$ for all $1 < r \leq \infty$, the above convolution is defined. The operator T_ϕ defined above is called discrete singular operator.

The maximal discrete singular operator corresponding to this discrete singular operator is defined as

$$T_\phi^{\star} a(n) = \sup_N \left| \sum_{k=-N}^{N} \phi(k)a(n-k) \right|.$$

The proof of following theorems can be found in [1]. The following theorems state that the discrete maximal singular operator and discrete singular operator are bounded on $\ell^p(\mathbb{Z}), 1 < p < \infty$ and they satisfy weak(1,1) inequality.

Theorem 5.1.3. *Let $\phi = \{\phi(n)\}$ be a singular kernel on $\mathbb{Z}$. Then there exists constant $C_p > 0$ such that*

1. *If $1 < p < \infty, \|T_\phi a\|_p \leq C_p \|a\|_p, \forall a \in \ell^p(\mathbb{Z})$.*

2. $\left| \{n : |T_\phi a(n)| > \lambda\} \right| \leq \frac{C_1}{\lambda} \|a\|_1, \forall a \in \ell^1(\mathbb{Z})$ *and* $\lambda > 0$.

Theorem 5.1.4 ([1]). *Let ϕ be a singular kernel and $1 \leq p < \infty$. Then there exists a constant $C_p > 0$ such that (i)*

$$\left\|T_\phi^\star a\right\|_p \leq C_p \left\|a\right\|_p \quad \forall a \in \ell^p(\mathbb{Z}), \quad \textit{if} \quad 1 < p < \infty.$$

(ii)

$$|\{j \in \mathbb{Z} : T_\phi^\star a(j) > \lambda\}| \leq \frac{C_1}{\lambda} \left\|a\right\|_1 \quad \forall \lambda > 0 \quad \textit{and} \quad a \in \ell^1(\mathbb{Z}).$$

In the following lemma, we observe that if ϕ is singular kernel on $\mathbb{Z}$, then its linear extension is singular kernel on $\mathbb{R}$.

Lemma 5.1.5. *Let ϕ be a singular kernel. Let K be the linear extension of ϕ defined as*

$$K(x) = (1-t)\phi(j) + t\phi(j+1) \quad \textit{if} \quad x = j + t, 0 \leq t \leq 1.$$

Then K is a singular integral kernel on $\mathbb{R}$

Proof. For $x \in \mathbb{R}$, $x = \lfloor x \rfloor + t, \quad 0 < t \leq 1$. Note that for $x \neq 0$

$$|K(x)| = |(1-t)\phi(\lfloor x \rfloor) + t\phi(\lfloor x \rfloor + 1)|.$$

Hence K satisfies **K2**. Since $\phi(0) = 0, \lim_{\epsilon \to 0} \int_\epsilon^1 K(x)dx$ exists.
Now

$$\int_{-R}^{R} K(x)dx = \phi(\lfloor R \rfloor)/2 + \phi(-\lfloor R \rfloor)/2 + \sum_{k=-(\lfloor R \rfloor - 1)}^{(\lfloor R \rfloor - 1)} \phi(k) + \int_{\lfloor R \rfloor < |x| \leq R} k(x)dx.$$

But

$$\frac{\phi([R])}{2} + \frac{\phi(-[R])}{2} + \int_{[R] < |x| \leq R} K(x)dx \leq \frac{c}{[R]} \to 0$$

as $R \to \infty$. Therefore, by S1, S2, and K2 , we have **K1.**

For **K3:**
Let $x, y \in \mathbb{R}$ such that $|x| > 2|y|$.
Let $x = n + t, y = j + s$, where $t \geq 0, s \leq 1$ and $n, j \in \mathbb{Z}$
We consider two cases, namely, $|y| < 1$ and $|y| \geq 1$.
For the first case there are 2 possibilities namely: (1) $y \geq 0$ and (2) $y < 0$.

Since (1) and (2) are dealt in a similar fashion we only discuss (1)
Now $x - y = n + (t - y)$. Let us assume $t - y \geq 0$.

(If $t-y<0$, we write $x-y=(n-1)+[1-(y-t)]$ and proceed.)
Now

$$|K(x-y)-K(x)| = \Big|[1-(t-y)]\phi(n)+(t-y)\phi(n+1)-(1-t)\phi(n)-t\phi(n+1)\Big|$$
$$\le |y||\phi(n)-\phi(n+1)|$$
$$\le C|y|\min\{1/n^2, 1/(n+1)^2\}$$
$$\le C|y|/x^2$$

For the cases $|y|\ge 1$. first we assume $t-s\ge 0$
(If $t-s<0$, we write

$$x-y=(n-j-1)+[1-(s-t)]$$

and proceed).
Then

$$|K(x-y)-K(x)| \le |K(x-y)-K(n-j)|+|\phi(n-j)-\phi(n)|+|K(n)-K(x)|.$$

By mean value theorem and the fact that the slope of the line joining the points $\phi(k),\phi(k+1)$ is less that or equal to $\frac{C}{k^2}$ we have

$$|K(x-y)-K(x)|$$
$$\le C(t-s)/(n-j)^2+C|j|/(n-j)^2+Ct/n^2$$
$$\le C|y|/n-j^2+C|y|/n^2(\text{since}|y|>1)$$
$$\le C|y|/(x-y)^2+c|y|/x^2$$
$$\le C|y|/x^2(\text{since}|x|>2|y|).$$

□

5.2 Results on Calderón-Zygmund singular operators on weighted $\ell^p_w(\mathbb{Z})$ spaces

In this section, we provide some results for Calderón-Zygmund singular operators on weighted $\ell^{p(\cdot)}_w(\mathbb{Z})$ spaces.

For this we require the following lemmas.

Lemma 5.2.1. *Let ϕ be a singular kernel. Given an interval I which contains integers m,n, then for $r \notin 5I$,*

$$|\phi(m-r) - \phi(n-r)| \leq \frac{C|I|}{|n-r|^2}.$$

Proof. If $m > n$, then

$$\begin{aligned}
&|\phi(n-r) - \phi(m-r)| \\
&\leq |\phi(n-r) - \phi(n-r+1) + \phi(n-r+1) - \phi(n-r+2) \cdots + \\
&+ \phi(n-r+m-n-1) - \phi(n-r+m-n)| \\
&\leq |\phi(n-r) - \phi(n-r+1)| + |\phi(n-r+1) - \phi(n-r+2)| \cdots + \\
&+ |\phi(n-r+m-n-1) - \phi(n-r+m-n)| \\
&\leq \frac{C}{|n-r|^2} + \frac{C}{|n-r+1|^2} + \cdots + \frac{C}{|m-r-1|^2} \\
&\leq C\frac{|n-m|}{|n-r|^2} \\
&\leq C\frac{|I|}{|n-r|^2}.
\end{aligned}$$

By the same argument, if $n > m$, then

$$|\phi(m-r) - \phi(n-r)| \leq \frac{C|I|}{|m-r|^2}.$$

Also

$$|m-r| = |(m-n) + (n-r)| \geq |n-r| - |m-n| \geq |n-r| - |I|.$$

Since $r \in \mathbb{Z} \setminus 5I$, we have $|n-r| \geq 2|I|$. Hence for $r \in \mathbb{Z} \setminus 5I$. $|m-r| \geq |n-r| - \frac{|n-r|}{2} \geq \frac{|n-r|}{2}$ i.e $\frac{1}{|m-r|} \leq \frac{2}{|n-r|}$. Therefore, in this case also, $|\phi(m-r) - \phi(n-r)| \leq \frac{C|I|}{(n-r)^2}$. $\square$

Lemma 5.2.2. *If T_ϕ is a singular operator, then for each $s > 1$, there exists a constant $C_s > 0$ such that*

$$M^{\#}(T_\phi a(m)) \leq C_s \Big[M(|a|^s)(m)\Big]^{\frac{1}{s}}.$$

for each integer $m \in \mathbb{Z}$.

Proof. Fix $s > 1$. Given an integer m and an interval I which contains m, by Lemma 4.3.4 , it is enough to find a constant h such that

$$\frac{1}{|I|}\sum_{n\in I}|T_\phi a(n) - h| \leq CM(|a|^s)(m)^{\frac{1}{s}}.$$

Decompose $a = a_1 + a_2$, where $a_1 = a\chi_{5I}, a_2 = a - a_1$. Now let $h = T_\phi a(m)$, then

$$\frac{1}{|I|}\sum_{n\in I}|T_\phi a(n) - h| \leq \frac{1}{|I|}\sum_{n\in I}|T_\phi a_1(n)| + \frac{1}{|I|}\sum_{n\in I}|T_\phi a_2(n) - T_\phi a_2(m)|.$$

Since $s > 1$, T_ϕ is bounded on $\ell^s(\mathbb{Z})$. Therefore,

$$\begin{aligned}
&\frac{1}{|I|}\sum_{n\in I}|T_\phi a_1(n)| \leq \left(\frac{1}{|I|}\sum_{n\in I}|T_\phi a_1(n)|^s\right)^{\frac{1}{s}}\\
&\leq C\left(\frac{1}{|I|}\sum_{n\in \mathbb{Z}}|a_1(n)|^s\right)^{\frac{1}{s}}\\
&\leq C\left(\frac{5}{|5I|}\sum_{n\in 5I}|a(n)|^s\right)^{\frac{1}{s}}\\
&\leq 5^{\frac{1}{s}}C\left[M(|a|^s)(m)\right]^{\frac{1}{s}}.
\end{aligned}$$

To deal with a_2, we require the estimate from Lemma 5.2.1.

Now, we estimate the second term as follows.

$$\begin{aligned}
&\frac{1}{|I|}\sum_{n\in I}|T_\phi a_2(n) - T_\phi a_2(m)|\\
&\leq \frac{1}{|I|}\sum_{n\in I}\Big|\sum_{r\in\mathbb{Z}\setminus 5I}\Big(\phi(n-r) - \phi(m-r)\Big)a(r)\Big|\\
&\leq \frac{1}{|I|}\sum_{n\in I}\sum_{r\in\mathbb{Z}\setminus 5I}|\phi(n-r) - \phi(m-r)||a(r)|\\
&\leq C\frac{1}{|I|}\sum_{n\in I}\sum_{r\in\mathbb{Z}\setminus 5I}\frac{|I|}{|n-r|^2}|a(r)|\\
&\leq C\frac{1}{|I|}\sum_{n\in I}\sum_{k=1}^{\infty}\sum_{2^k|I|<|n-r|\leq 2^{k+1}|I|}\frac{|I|}{|n-r|^2}|a(r)|\\
&\leq C\frac{1}{|I|}\sum_{n\in I}\sum_{k=1}^{\infty}\frac{|I|}{2^{2k}|I|^2}\sum_{|n-r|\leq 2^{k+1}|I|}|a(r)|\\
&\leq C\frac{1}{|I|}\sum_{n\in I}\sum_{k=1}^{\infty}\frac{1}{2^{2k}|I|}\sum_{|n-r|\leq 2^{k+1}|I|}|a(r)|\\
&\leq C\frac{1}{|I|}\sum_{n\in I}\sum_{k=1}^{\infty}\frac{2}{2^k 2^{k+1}|I|}\sum_{|n-r|\leq 2^{k+1}|I|}|a(r)|
\end{aligned}$$

$$\leq 2CMa(m)\frac{1}{|I|}\sum_{n\in I}\sum_{k=1}^{\infty}\frac{1}{2^k}$$
$$\leq CMa(m)\frac{1}{|I|}\sum_{n\in I}1$$
$$= CMa(m) \leq CM(|a|^s)(m)^{\frac{1}{s}}.$$

□

Theorem 5.2.3. *If T_ϕ is a singular operator, then for any $w \in A_p, 1 < p < \infty$, T_ϕ is bounded on $\ell^p_w(\mathbb{Z})$.*

Proof. Let $w \in A_p$. Since $A_p = \cup_{q<p} A_q$, we can find s such that $p > s > 1$ and $w \in A_{\frac{p}{s}}$. Consider a sequence $\{a(n) : n \in \mathbb{Z}\}$ such that $a(n) = 0$ outside the interval $[-R, -R+1, \ldots, R]$. Therefore,

$$\sum_{m\in\mathbb{Z}}|T_\phi a(m)|^p w(m)$$
$$\leq \sum_{m\in\mathbb{Z}}\left[M_d\left[T_\phi a(m)\right]\right]^p w(m) \qquad \textit{Lemma} \quad 4.4.2$$
$$\leq C\sum_{m\in\mathbb{Z}}\left[M^{\#}\left[T_\phi a(m)\right]\right]^p w(m) \qquad \textit{Theorem} \quad 5.2.2$$
$$\leq C\sum_{m\in\mathbb{Z}}\left[M(|a(m)|^s)\right]^{\frac{p}{s}} w(m)$$
$$\leq C\sum_{m\in\mathbb{Z}}|a(m)|^p w(m).$$

In the second step, we use Lemma 4.4.2(Weighted Good -Lambda estimate) provided

$$\sum_{m\in\mathbb{Z}}\left[M_d(T_\phi a(m))\right]^p w(m).$$

is finite. To show this it is enough to show that $T_\phi a \in \ell^p_w(\mathbb{Z})$.
We have to prove

$$\sum_{m\in\mathbb{Z}}\left(T_\phi a(m)\right)^p w(m) < \infty.$$

To show that this is finite, we split this sum as

$$\sum_{m\leq 2R}\left(T_\phi a(m)\right)^p w(m).$$

and

$$\sum_{m>2R}\left(T_\phi a(m)\right)^p w(m).$$

The former sum

$$\sum_{m\leq 2R}\Big(T_\phi a(m)\Big)^p w(m) < \infty.$$

is trivial as shown below.
For $|m| \leq 2R$,

$$|T_\phi a(m)| \leq C \sum_{|n|\leq 2R, m\neq n} |a(n)|\frac{C}{|m-n|} \leq C\,\|a\|_\infty 4R < \infty. \tag{A4}$$

For $|m| > 2R$,

$$|T_\phi a(m)| = |\sum_{n\in\mathbb{Z}} a(n)\phi(m-n)| \leq C \sum_{|n|<R, m\neq n} \frac{|a(n)|}{|m-n|} \leq C\frac{\|a\|_\infty}{|m|}.$$

Further, $I(0,2R) \subset I(0,2^{k+1}R)$ and $w(I(0,2R))$ is a constant independent of m. Also, since $w \in A_p$, by Lemma 2.4.3, there exists $q < p$ such that $w \in A_q$. Then by Lemma 2.3.4

$$w(I(0,2^{k+1}R)) \leq Cw(I(0,2R))\left(\frac{|2^{k+1}R|}{|2R|}\right)^q \leq Cw(I(0,2R))(2^k)^q \leq C(w,R)2^{kq}.$$

So,

$$\begin{aligned}
&\sum_{|m|>2R} |T_\phi a(m)|^p w(m) \leq C\sum_{k=1}^{\infty}\sum_{2^kR<|m|\leq 2^{k+1}R}\frac{w(m)}{|m|^p}\\
&\leq C\sum_{k=1}^{\infty}(2^kR)^{-p}\sum_{|m|\leq 2^{k+1}R} w(m)\\
&\leq C\sum_{k=1}^{\infty}(2^kR)^{-p}C(w,R)2^{kq}\\
&= C(w,R)\sum_{k=1}^{\infty}2^{k(q-p)} = C(w,R)\sum_{k=1}^{\infty}\left(\frac{1}{2^{p-q}}\right)^k < \infty.
\end{aligned}$$

Combining both results, $T_\phi a \in \ell^p_w(\mathbb{Z})$.

□

Theorem 5.2.4. *Let T_ϕ be a Calderón-Zygmund operator and let $w \in A_1$. Then for any $\{a(n) : n \in \mathbb{Z}\} \in \ell^1_w(\mathbb{Z})$,*

$$w(\{m \in \mathbb{Z} : |T_\phi a(m)| > \lambda\}) \leq \frac{C}{\lambda}\sum_{m\in\mathbb{Z}} |a(m)|w(m).$$

Proof. Perform Calderón-Zygmund decomposition of sequence $\{a(n) : n \in \mathbb{Z}\}$ at height λ and obtain disjoint dyadic intervals $\{I_j\}$ which satisfy

$$\lambda \leq \frac{1}{|I_j|} \sum_{m \in I_j} |a(m)| \leq 2\lambda.$$

Decompose $a(m) = g(m) + b(m), m \in \mathbb{Z}$

$$g(m) = \begin{cases} a(m) & \text{if} \quad m \notin \Omega, \\ \frac{1}{|I_j|} \sum_{k \in I_j} a(k) & \text{if} \quad m \in I_j. \end{cases}$$

where $\Omega = \cup_j I_j$

$$b(m) = \sum_{j=1}^{\infty} b_j(m)$$

where

$$b_j(m) = \left(a(m) - \frac{1}{|I_j|} \sum_{k \in I_j} a(k) \right) \chi_{I_j}(m).$$

Write

$$\begin{aligned} & w(\{m \in \mathbb{Z} : |T_\phi a(m)| > \lambda\}) \\ & \leq w(\left\{ m \in \mathbb{Z} : |T_\phi g(m)| > \frac{\lambda}{2} \right\}) + w(\left\{ m \in \mathbb{Z} : |Tb(m)| > \frac{\lambda}{2} \right\}). \end{aligned}$$

To estimate the first term, note that $w \in A_1$ implies $w \in A_2$. Further, since T_ϕ is bounded on $\ell^2_w(\mathbb{Z})$, it follows that

$$\begin{aligned} & w(\left\{ m \in \mathbb{Z} : |T_\phi g(m)| > \frac{\lambda}{2} \right\}) \\ & \leq \frac{4}{\lambda^2} \sum_{m \in \mathbb{Z}} |T_\phi g(m)|^2 w(m) \\ & \leq \frac{C}{\lambda^2} \sum_{m \in \mathbb{Z}} |g(m)|^2 w(m) \\ & = \frac{C}{\lambda^2} \left(\sum_{m \in \Omega^c} |g(m)|^2 w(m) + \sum_{m \in \Omega} |g(m)|^2 w(m) \right). \end{aligned}$$

Now,

$$\begin{aligned} & \sum_{m \in \Omega^c} |g(m)|^2 w(m) \\ & \leq \lambda \sum_{m \in \Omega^c} |g(m)| w(m) \leq \lambda \sum_{m \in \Omega^c} |a(m)| w(m). \end{aligned}$$

Note $w \in A_1$ implies $\frac{w(I)}{|I|} \leq Cw(m) \quad \forall m \in I$. So on Ω,

$$
\begin{aligned}
&\sum_{m\in\Omega} |g(m)|^2 w(m) \leq 4\lambda^2 \sum_{m\in\Omega} w(m) \\
&= 4\lambda \sum_j \left(\left(\frac{1}{|I_j|} \sum_{k\in I_j} |a(k)| \right) \left(\sum_{m\in I_j} w(m) \right) \right. \\
&= 4\lambda \sum_j \left(\left(\frac{1}{|I_j|} \sum_{k\in I_j} |a(k)| \right) \left(w(k)|I_j| \right) \right. \\
&= 4\lambda \sum_j \left(\left(\sum_{k\in I_j} |a(k)| \right) \left(w(k) \right) \right) \\
&\leq 4C\lambda \sum_j \left(\sum_{m\in I_j} |a(m)| w(m) \right) \\
&\leq 4C\lambda \sum_{m\in\mathbb{Z}} |a(m)| w(m).
\end{aligned}
$$

From above estimates we get

$$w(\left\{ m \in \mathbb{Z} : |T_\phi g(m)| > \frac{\lambda}{2} \right\}) \leq \frac{C}{\lambda} |a(m)| w(m).$$

Consider,

$$w(\left\{ m \in \mathbb{Z} : Tb(m) > \frac{\lambda}{2} \right\}) \leq w(\cup_j 3I_j) + w(\left\{ m \in \mathbb{Z} \setminus \cup_j 3I_j : |Tb(m)| > \frac{\lambda}{2} \right\}).$$

For the second estimate, by Lemma 2.3.4

$$
\begin{aligned}
&w(\cup_j 3I_j) \leq C \sum_j w(I_j) \leq C \sum_j \frac{w(I_j)}{|I_j|} |I_j| \\
&\leq C \sum_j \frac{w(I_j)}{|I_j|} \frac{C}{\lambda} \left(\sum_{k\in I_j} |a(k)| \right) \\
&\leq \frac{C}{\lambda} \sum_j \left(\sum_{k\in I_j} |a(k)| \frac{w(I_j)}{|I_j|} \right) \\
&\leq \frac{C}{\lambda} \left(\sum_{k\in I_j} |a(k)| w(k) \right) \\
&\leq \frac{C}{\lambda} \sum_{k\in\Omega} |a(k)| w(k) \\
&\leq \frac{C}{\lambda} \sum_{k\in\mathbb{Z}} |a(k)| w(k).
\end{aligned}
$$

Now let c_j be center of I_j. Then, since b_j has zero average on I_j.

$$\begin{aligned}
&w(\left\{m \in \mathbb{Z} \setminus \cup_j 3I_j : |T_\phi b(m)| > \frac{\lambda}{2}\right\}) \\
&\leq \frac{C}{\lambda} \sum_{m \in \mathbb{Z} \setminus \cup_j 3I_j} |T_\phi b(m)| w(m) \\
&= \frac{C}{\lambda} \sum_{m \in \mathbb{Z} \setminus \cup_j 3I_j} |\sum_{n \in \mathbb{Z}} \phi(m-n) b_j(n)| w(m) \\
&\leq \frac{C}{\lambda} \sum_{m \in \mathbb{Z} \setminus \cup_j 3I_j} |\sum_{j} \sum_{n \in I_j} \phi(m-n) b_j(n)| w(m) \\
&\leq \frac{C}{\lambda} \sum_{m \in \mathbb{Z} \setminus \cup_j 3I_j} |\sum_{j} \sum_{n \in I_j} [\phi(m-n) - \phi(c_j - m)] b_j(n)| w(m).
\end{aligned}$$

If $m \in \mathbb{Z} \setminus \cup_j 3I_j$ and $n \in I_j$ then $|m-n| \geq |I_j| \quad \forall j$. So, it follows that $\forall j \in \mathbb{Z}$, from Lemma 5.2.1 $|\phi(m-n) - \phi(c_j - m)| \leq C \frac{|I_j|}{|m-n|^2}$. It follows that,

$$\begin{aligned}
&w(\{m \in \mathbb{Z} \setminus \cup_j 3I_j : |Tb(m)| > \lambda\}) \\
&\leq \frac{C}{\lambda} \sum_{m \in \mathbb{Z} \setminus 3I_j} \sum_{j} \sum_{n \in I_j} \left(\frac{C|I_j|}{|m-n|^2} w(m) \right) |b_j(n)| \\
&\leq \frac{C}{\lambda} \sum_{j} \sum_{n \in I_j} \sum_{m \in \mathbb{Z} \setminus 3I_j} \left(\frac{|I_j|}{|m-n|^2} w(m) \right) |b_j(n)| \\
&\leq \frac{C}{\lambda} \sum_{j} \sum_{n \in I_j} \sum_{s=0}^{\infty} \sum_{2^s|I_j| < |m-n| \leq 2^{s+1}|I_j|} \left(\frac{|I_j|}{|m-n|^2} w(m) \right) |b_j(n)| \\
&\leq \frac{C}{\lambda} \sum_{j} \sum_{n \in I_j} \sum_{s=0}^{\infty} \frac{I_j}{2^{2s}|I_j|^2} \sum_{2^s|I_j| < |m-n| \leq 2^{s+1}|I_j|} w(m) |b_j(n)| \\
&\leq \frac{C}{\lambda} \sum_{j} \sum_{n \in I_j} Mw(n) |b_j(n)| \\
&\leq \frac{C}{\lambda} \sum_{j} \sum_{n \in I_j} w(n) |b_j(n)| \\
&\leq \frac{C}{\lambda} \sum_{j} \sum_{n \in I_j} \Big(|a_j(n)| + |g_j(n)| \Big) \chi_{I_j}(n) w(n) \\
&\leq \frac{C}{\lambda} \sum_{j} \sum_{n \in I_j} |a_j(n)| w(n) + \frac{C}{\lambda} \sum_{j} \sum_{n \in I_j} |g_j(n)| w(n) \\
&\leq \frac{C}{\lambda} \sum_{n \in \mathbb{Z}} |a(n)| w(n) + \frac{C}{\lambda} \sum_{n \in \mathbb{Z}} |g(n)| w(n) \\
&\leq \frac{C}{\lambda} \sum_{m \in \mathbb{Z}} |a(n)| w(n).
\end{aligned}$$

Combining both estimates for $T_\phi g, T_\phi b$, we get desired result. $\square$

Now, we prove the weak and strong type inequalities for the maximal singular operator $T_\phi^\star$ operator on $l_w^p(\mathbb{Z})$ spaces. Here, we use transference principle of the corresponding results on $\mathbb{R}$.

The following lemma, whose proof is obvious, is used in the proof of Theorem 5.2.6.

Lemma 5.2.5. *Suppose $\{w(n) : n \in \mathbb{Z}\}$ is a sequence in $A_p(\mathbb{Z}), 1 \le p < \infty$. Put*

$$w'(x) = \begin{cases} w(j) \quad if \quad x \in [j - \frac{1}{4}, j + \frac{1}{4}], \quad j \in \mathbb{Z} \\ 0 \quad otherwise \end{cases}$$

If $w \in A_p(\mathbb{Z})$, then $w' \in A_p(\mathbb{R}), 1 \le p < \infty$.
If $w \in A_1(\mathbb{Z})$, then $w' \in A_1(\mathbb{R})$.

Theorem 5.2.6. *If T_ϕ is a singular kernel operator, then for $1 < p < \infty$, $T_\phi^\star$ is bounded on $\ell_w^p(\mathbb{Z})$ if $w \in A_p$ and $T_\phi^\star$ is weak (1,1) with respect to w if $w \in A_1$.*

Proof. Let $K(x)$ be the linear extension of ϕ. Also, for a given sequence $\{a(n) : n \in \mathbb{Z}\}$, we define a function $f(x) = \sum_{m\in\mathbb{Z}} a(m)\chi_{I_m}(x)$ where $I_m = (m - \frac{1}{4}, m + \frac{1}{4})$. The following inequality which gives the relation between the maximal singular operator on $\mathbb{Z}$ and the maximal singular integral operator on $\mathbb{R}$ is proved in [1].

$$T_\phi^\star a(m) \le C(T_K^\star f(x) + Sf(x)), \quad x \in I_m \tag{A5}$$

where

$$\begin{aligned} Sf(x) &= \int_{|x-y|>\frac{1}{2}} \frac{|f(y)|}{(x-y)^2} dy \\ &= \sum_{k=0}^{\infty} \int_{2^k \ge |x-y| > 2^{k-1}} \frac{f(y)}{(x-y)^2} dy \\ &\le \sum_{k=0}^{\infty} \frac{4}{2^{2k}} \int_{|x-y|\le 2^k} |f(y)| dy \\ &\le CMf(x). \end{aligned}$$

Now for $1 \le p < \infty$,

$$\begin{aligned} &\|f\|_{L_{w'}^p(\mathbb{R})}^p = \int_{\mathbb{R}} |f(x)|^p w'(x) dx \\ &= \sum_{m\in\mathbb{Z}} \int_{I_m} |a(m)|^p w(m) dx = \sum_{m\in\mathbb{Z}} \frac{1}{2} |a(m)|^p w(m) = \frac{1}{2} \|a\|_{\ell_w^p(\mathbb{Z})}. \end{aligned} \tag{A6}$$

and

$$\|Sf\|_{L_{w'}^p(\mathbb{R})} = \left(\int_{\mathbb{R}} | \int_{|x-y|>\frac{1}{2}} \frac{|f(y)|}{(x-y)^2} dy |^p w'(x) dx \right)^{\frac{1}{p}}.$$

Therefore, using Lemma 5.2.5

$$
\begin{aligned}
\left\|T_\phi^\star a\right\|_{l_w^p(\mathbb{Z})} &= \left(\sum_{m\in\mathbb{Z}} |T_\phi^\star a(m)|^p w(m)\right) \\
&\le \sum_{m\in\mathbb{Z}} 2\int_{I_m} |T_\phi^\star a(m)|^p w(m)dx \\
&\le \left(2C\sum_{m\in\mathbb{Z}}\int_{I_m}\left[T_k^\star f(x)+Sf(x)\right]^p w'(x)dx\right)^{\frac{1}{p}} \\
&= C\left\|T_K^\star f + Sf\right\|_{L^p_{w'}(\mathbb{R})} \\
&\le C\left\|T^\star f\right\|_{L^p_{w'}(\mathbb{R})} + C\left\|Sf\right\|_{L^p_{w'}(\mathbb{R})} \\
&\le C\left\|T^\star f\right\|_{L^p_{w'}(\mathbb{R})} + C\left\|Mf\right\|_{L^p_{w'}(\mathbb{R})} \\
&\le C\left\|f\right\|_{L^p_{w'}(\mathbb{R})} \\
&= C\left\|a\right\|_{\ell^p_w(\mathbb{Z})}.
\end{aligned}
$$

The last inequality follows from that fact $T_K^\star$ and M are bounded on $L^p_{w'}(\mathbb{R})$. Refer [1]. It follows that $T_\phi^\star$ is strong type (p,p) on $\ell^p_w(\mathbb{Z})$.

Now, we shall prove the weak type (1,1) inequality. From [*A*5], we have

$$
\begin{aligned}
&\left\{m\in\mathbb{Z} : T_\phi^\star a(m) > \lambda\right\} \\
&\subseteq \left\{x\in I_m : T_K^\star f(x) > \frac{\lambda}{2C}\right\} \cup \left\{x\in I_m : Sf(x) > \frac{\lambda}{2C}\right\} \\
&\subseteq \left\{x\in I_m : T_K^\star f(x) > \frac{\lambda}{2C}\right\} \cup \left\{x\in I_m : Mf(x) > \frac{\lambda}{2C}\right\}.
\end{aligned}
$$

Therefore, the weighted analogue would give for each $x \in I_m$,

$$
\begin{aligned}
&w(\left\{m\in\mathbb{Z} : T_\phi^\star a(m) > \lambda\right\}) \\
&\le w(\left\{x\in I_m : T_K^\star f(x) > \frac{\lambda}{2C}\right\}) + w(\left\{m\in\mathbb{Z} : Mf(x) > \frac{\lambda}{2C}\right\}) \\
&\le \frac{C}{\lambda}\left\|f\right\|_{L^1_{w'}(\mathbb{R})} + \frac{C}{\lambda}\left\|f\right\|_{L^1_{w'}(\mathbb{R})} \le \frac{C}{\lambda}\left\|a\right\|_{\ell^1_w(\mathbb{Z})}.
\end{aligned}
$$

Hence, $T_K^\star$ is of weak type (1,1) and M is also of weak type (1,1) on $L^1_w(\mathbb{Z})$. Refer [1]. This gives $T_\phi^\star$ is weak type (1,1) on $\ell^1_w(\mathbb{Z})$. □

Now, we want to prove that if $H^\star$ is bounded on $\ell^p_w(\mathbb{Z}), 1<p<\infty$, then $w \in A_p(\mathbb{Z})$. The methodology used in our proof is given in [6]. Observe that for $1<p<\infty$, if $H^\star$ is bounded on $\ell^p_w(\mathbb{Z})$ then H is bounded on $\ell^p_w(\mathbb{Z})$,

Theorem 5.2.7. *If for $1 < p < \infty$ and any positive sequence $\{w(n) : n \in \mathbb{Z}\}$*

$$\sum_{m\in\mathbb{Z}} |Ha(m)|^p w(m) \leq C \sum_{m\in\mathbb{Z}} |a(m)|^p w(m) \quad \forall \{a(n) : n \in \mathbb{Z}\}$$

then w satisfies the discrete A_p condition which is as follows

$$\left(\frac{1}{|I|}\sum_{m\in I} w(m)\right)\left(\frac{1}{|I|}\sum_{m\in I} w(m)^{\frac{-1}{p-1}}\right)^{p-1} \leq C.$$

for any interval I in $\mathbb{Z}$.

Proof. Let $I_1 = [m, m+1, \dots n]$ be any interval in $\mathbb{Z}$. Consider the right doubling interval of I_1 as

$$2RI = [m, m+1, \dots n, n+1, n+2, \dots 2n-m+1].$$

Let

$$I_2 = [n+1, n+2 \dots 2n-m+1].$$

Take a non-negative sequence $\{a(n) : n \in \mathbb{Z}\}$ supported in I_1. Observe that

$$|Ha(m)| = |\sum_{n\in I_1} \frac{a(n)}{m-n}| = \sum_{n\in I_1} \frac{a(n)}{|m-n|}.$$

So, for $m \in I_2$ we get

$$|Ha(m)| \geq \frac{1}{2}\left(\frac{1}{|I_1|}\sum_{n\in I_1} a(n)\right)\chi_{I_2}(m) \quad \forall n \in I_1.$$

Now, using boundedness of H on $\ell^p_w(\mathbb{Z})$ i.e,

$$\sum_{m\in\mathbb{Z}} |Ha(m)|^p w(m) \leq C \sum_{m\in\mathbb{Z}} |a(m)|^p w(m).$$

Since support of $\{a(n) : n \in \mathbb{Z}\}$ is in I_1, we have,

$$\left(\frac{1}{|I_1|}\sum_{n\in I_1} a(n)\right)^p \left(\sum_{m\in I_2} w(m)\right) \leq \sum_{m\in\mathbb{Z}}\left(\left(\frac{1}{|I_1|}\sum_{n\in I_1} a(n)\right)^p \chi_{I_2}(m) w(m)\right)$$
$$\leq \sum_{m\in\mathbb{Z}} \chi_{I_2}(m)|Ha(m)|^p w(m) \leq C\sum_{m\in\mathbb{Z}} |a(m)|^p w(m) = C\sum_{m\in I_1} |a(m)|^p w(m).$$

It follows that

$$\left(\frac{1}{|I_1|}\sum_{n\in I_1} a(n)\right)^p \left(\sum_{m\in I_2} w(m)\right) \leq C\sum_{m\in I_1} |a(m)|^p w(m). \tag{A7}$$

Take $a(n) = 1, \quad \forall n \in \mathbb{Z}$ in [A7] and by interchanging I_1 and I_2, we have the following two inequalities.

$$\sum_{m \in I_2} w(m) \leq C \sum_{m \in I_1} w(m). \tag{A8}$$

$$\sum_{m \in I_1} w(m) \leq C \sum_{m \in I_2} w(m). \tag{A9}$$

Likewise, take $a(n) = w(n)^{\frac{-1}{p-1}}, \quad \forall n \in \mathbb{Z}$ in [A7] to get

$$\left(\sum_{m \in I_2} w(m)\right)\left(\frac{1}{|I_1|}\sum_{m \in I_1} w(m)^{\frac{-1}{p-1}}\right)^p \leq C \sum_{m \in I_1} w(m)^{\frac{-p}{p-1}} w(m).$$

So,

$$\left(\sum_{m \in I_2} w(m)\right)\left(\frac{1}{|I_1|}\sum_{m \in I_1} w(m)^{\frac{-1}{p-1}}\right)^{p-1} \leq C.$$

Therefore,

$$\begin{aligned}
&\left(\frac{1}{|I_1|}\sum_{m \in I_1} w(m)\right)\left(\frac{1}{|I_1|}\sum_{m \in I_1} w(m)^{\frac{-1}{p-1}}\right)^{p-1} \\
&\leq \left(\frac{C}{|I_1|}\sum_{m \in I_2} w(m)\right)\left(\frac{1}{|I_1|}\sum_{m \in I_1} w(m)^{\frac{-1}{p-1}}\right)^{p-1} \leq C.
\end{aligned}$$

It follows that $w \in A_p(\mathbb{Z})$. □

5.3 Maximal singular operator on variable sequence spaces $\ell^{p(\cdot)}(\mathbb{Z})$

In this section, we prove weak type, and strong type inequalities for the maximal singular operator on $\ell^{p(\cdot)}(\mathbb{Z})$ spaces, $1 \leq p < \infty$, using Rubio de Francia extrapolation method given in Chapter3.

Theorem 5.3.1. *Given a sequence $\{a(n) : n \in \mathbb{Z}\}$, suppose $p(\cdot) \in \mathcal{S}$ such that $p_- > 1$. Let $T^\star_\phi$ be a maximal singular operator.*
Then,

$$\left\|T^\star_\phi a\right\|_{\ell^{p(\cdot)}(\mathbb{Z})} \leq C \left\|a\right\|_{\ell^{p(\cdot)}(\mathbb{Z})}.$$

If $p_- = 1$, then for all $t > 0$

$$\left\|t\chi_{\{n : |T^\star_\phi a(n)| > t\}}\right\|_{\ell^{p(\cdot)}(\mathbb{Z})} \leq C \left\|a\right\|_{\ell^{p(\cdot)}(\mathbb{Z})}.$$

Proof. We will prove strong type inequality when $p_- > 1$. Take p_0 such that $1 \leq p_0 \leq p_- \leq p_+ < \infty$. Therefore by Lemma 3.2.2

$$\begin{aligned}
\left\|(T_\phi^\star a)\right\|_{\ell^{p(\cdot)}(\mathbb{Z})}^{p_0} &= \left\|(T_\phi^\star a)^{p_0}\right\|_{\ell^{\frac{p(\cdot)}{p_0}}(\mathbb{Z})} \\
&= \sup_{h\in \ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z}), \|h\|_{\ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z})}=1} \sum_{k\in\mathbb{Z}} |T_\phi^\star a(k)|^{p_0}|h(k)| \\
&\leq \sup_{h\in \ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z}), \|h\|_{\ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z})}=1} \sum_{k\in\mathbb{Z}} |T_\phi^\star a(k)|^{p_0}Rh(k) \\
&\leq \sup_{h\in \ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z}), \|h\|_{\ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z})}=1} \sum_{k\in\mathbb{Z}} |a(k)|^{p_0}Rh(k) \\
&\leq C \sup_{h\in \ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z}), \|h\|_{\ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z})}=1} \left\||a|^{p_0}\right\|_{\ell^{\frac{p(\cdot)}{p_0}}(\mathbb{Z})} \|Rh\|_{\ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z})} \\
&= C \sup_{h\in \ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z}), \|h\|_{\ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z})}=1} \||a|\|_{\ell^{p(\cdot)}(\mathbb{Z})}^{p_0} \|Rh\|_{\ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z})} \\
&\leq 2C\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})}^{p_0}.
\end{aligned}$$

Now we are going to prove type weak type (1,1) inequality stated in the theorem.

Let $A = \left\{m \in \mathbb{Z} : |T_\phi^\star a(m)| > t\right\}$. Then,

$$\begin{aligned}
&\left\|(t\chi_{\left\{m\in\mathbb{Z}:|T_\phi^\star a(m)|>t\right\}})\right\|_{p(\cdot)} \\
&\leq \sup_{h\in \ell^{p(\cdot)'}(\mathbb{Z}), \|h\|_{\ell^{p(\cdot)'}(\mathbb{Z})}=1} \sum_{k\in\mathbb{Z}} |t\chi_{\left\{m\in\mathbb{Z}:|T_\phi^\star a(m)|>t\right\}}(k)||h(k)| \\
&= \sup_{h\in \ell^{p(\cdot)'}(\mathbb{Z}), \|h\|_{\ell^{p(\cdot)'}(\mathbb{Z})}=1} \sum_{k\in\mathbb{Z}} t\chi_A(k)Rh(k) \\
&= \sup_{h\in \ell^{p(\cdot)'}(\mathbb{Z}), \|h\|_{\ell^{p(\cdot)'}(\mathbb{Z})}=1} tRh(A) \\
&= \sup_{h\in \ell^{p(\cdot)'}(\mathbb{Z}), \|h\|_{\ell^{p(\cdot)'}(\mathbb{Z})}=1} t\frac{C}{t}\sum_{k\in\mathbb{Z}} |a(k)|Rh(k) \\
&\leq \sup_{h\in \ell^{p(\cdot)'}(\mathbb{Z}), \|h\|_{\ell^{p(\cdot)'}(\mathbb{Z})}=1} C\sum_{k\in\mathbb{Z}} |a(k)|Rh(k) \\
&= C\sum_{k\in\mathbb{Z}} |a(k)|Rh(k) \\
&\leq C\|a\|_{\ell^{p(\cdot)}(\mathbb{Z})} \|Rh\|_{\ell^{p(\cdot)'}(\mathbb{Z})} \leq 2C\|a\|_{p(\cdot)}.
\end{aligned}$$

□

5.4 Maximal Ergodic singular operator

Let $(X, \mathbf{B}, \mu)$ be a probability space and U an invertible measure preserving transformation on X. We define the truncated maximal ergodic singular operator and maximal ergodic singular operator as follows:

$$\tilde{T}^{\star}_{\phi,N}f(x) = \sup_{1\leq n\leq N} |\sum_{k=-n}^{n} f(U^{-k}x)\phi(k)|.$$

$$\tilde{T}^{\star}_{\phi}f(x) = \sup_{n} |\sum_{k=-n}^{n} f(U^{-k}x)\phi(k)|.$$

We define discrete Hilbert transform and maximal discrete Hilbert transform for a sequence $\{a(n) : n \in \mathbb{Z}\} \in \ell^p(\mathbb{Z})$ as follows:

$$Ha(m) = \sum_{n\in\mathbb{Z}} \left|\frac{a(n)}{m-n}\right|.$$

maximal discrete Hilbert transform

$$H^{\star}a(n) = \sup_{N} \left|\sum_{k=-N}^{N} \frac{a(n-k)}{k}\right|, a \in \ell^p, \quad 1 \leq p < \infty.$$

We define maximal ergodic Hilbert transform and truncated maximal ergodic Hilbert transform for a function $f \in L^p(\mathbb{R}), 1 \leq p < \infty$ as follows:

$$\tilde{H}^{\star}f(x) = \sup_{N} \left|\sum_{k=-N}^{N} \frac{f(U^{-k}x)}{k}\right|$$

$$\tilde{H}_J f(x) = \sup_{1\leq n\leq J} \left|\sum_{k=-n}^{n} \frac{f(U^{-k}x)}{k}\right|.$$

Now, we prove the strong type, weak type inequalities for the maximal ergodic singular operator on weighted $L^p_w(X, \mathcal{B}, \mu)$ spaces.

Theorem 5.4.1. *Let $(X, \boldsymbol{B}, \mu)$ be a probability space and U an invertible measure preserving transformation on X. If w is an ergodic A_p weight, $1 < p < \infty$, then the maximal ergodic singular operator satisfies*

1.

$$\left\|\tilde{T}^{\star}_{\phi}f\right\|_{L^p_w(X)} \leq C_p \left\|f\right\|_{L^p_w(X)} \quad \text{if} \quad 1 < p < \infty.$$

where C_p is independent of N.

2. *If $w \in A_1$, then*

$$\int_{\{x \in X : |\tilde{T}^{\star}_{\phi} f(x)| > \lambda\}} w(x) d\mu(x) \leq \frac{C}{\lambda} \int_X |f(x)| w(x) d\mu(x).$$

where C_1 is independent of N.

Proof. Fix $N > 0$ and take a function $f \in L^p_w(X)$.

$$\tilde{T}^{\star}_{\phi,N} f(x) = \sup_{1 \leq n \leq N} |\sum_{k=-n}^{n} f(U^{-k}x)\phi(k)|.$$

It is enough to prove that $\tilde{T}^{\star}_{\phi,N}$ satisfies (1) and (2) with constants not depending on N. Let $\lambda > 0$ and put

$$E_\lambda = \left\{ x \in X : |\tilde{T}^{\star}_{\phi,N} f(x)| > \lambda \right\}.$$

For x lying outside a μ null set and a positive integer L, define sequences

$$a_x(k) = \begin{cases} f(U^{-k}x) & if \quad |k| \leq L + N \\ 0 & otherwise. \end{cases}$$

$$w_x(k) = \begin{cases} w(U^{-k}x) & if \quad |k| \leq L + N \\ 0 & otherwise. \end{cases}$$

Therefore,

$$\begin{aligned}
& w(\left\{ x \in X : |\tilde{T}^{\star}_{\phi,N} f(x)| > \lambda \right\}) = \int_{E_\lambda} w(x) d\mu(x) = \frac{1}{\lambda^p} \int_{E_\lambda} \lambda^p w(x) d\mu(x) \\
& \leq \frac{1}{\lambda^p} \int_{E_\lambda} |\tilde{T}^{\star}_{\phi,N} f(x)|^p w(x) d\mu(x) \\
& \leq \frac{1}{\lambda^p} \int_X |\tilde{T}^{\star}_{\phi,N} f(x)|^p w(x) d\mu(x) \\
& = \frac{1}{\lambda^p} \frac{1}{2L+1} \sum_{m=-L}^{L} \int_X |\tilde{T}^{\star}_{\phi,N} f(U^{-m}x)|^p w(U^{-m}x) d\mu(x) \\
& \leq \frac{1}{\lambda^p} \frac{1}{2L+1} \int_X \sum_{m=-L}^{L} |\tilde{T}^{\star}_{\phi,N} f(U^{-m}x)|^p w(U^{-m}x) d\mu(x) \\
& \leq \frac{1}{\lambda^p} \frac{1}{2L+1} \int_X \sum_{m=-L}^{L} |T^{\star}_{\phi,N} a_x(m)|^p w_x(m) d\mu(x)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\lambda^p}\frac{1}{2L+1}\int_X \sum_{m=-\infty}^{\infty} |T^{\star}_{\phi,N} a_x(m)|^p w_x(m) d\mu(x) \\
&\leq C\frac{1}{\lambda^p}\frac{1}{2L+1}\int_X \sum_{m=-\infty}^{\infty} |a_x(m)|^p w_x(m) d\mu(x) \\
&= C\frac{1}{\lambda^p}\frac{1}{2L+1}\int_X \sum_{m=-(L+N)}^{(L+N)} |a_x(m)|^p w_x(m) d\mu(x) \\
&= C\frac{1}{\lambda^p}\frac{1}{2L+1}\int_X \sum_{m=-(L+N)}^{(L+N)} |f(U^{-m}x)|^p w(U^{-m}x) d\mu(x) \\
&\leq \frac{C}{\lambda^p}\frac{1}{2L+1}\sum_{m=-(L+N)}^{(L+N)} \int_X |f(U^{-m}x)|^p w(U^{-m}x) d\mu(x) \\
&= \frac{C}{\lambda^p}\frac{1}{2L+1}\sum_{m=-(L+N)}^{(L+N)} \int_X |f(x)|^p w(x) d\mu(x) \\
&\leq \frac{C}{\lambda^p}\frac{1}{2L+1}(2(L+N)+1)\,\|f\|^p_{L^p_w(X)} \\
&\leq \frac{C}{\lambda^p}\,\|f\|^p_{L^p_w(X)}.
\end{aligned}
$$

by choosing L appropriately. Conclusion (1) of the theorem now follows by using the Marcinkiewicz interpolation theorem. □

Remark 5.4.2. Using the same transference argument, we can prove that the singular operator T_ϕ is bounded on $L^p_w(X)$ and satisfies weak type (1,1) inequality in $L^p_w(X)$.

Now, we prove the converse of the above theorem when $\tilde{T}^{\star}_{\phi,N}$ with singular kernel as

$$
\phi(k) = \begin{cases} \frac{1}{k} & \text{if} \quad k \neq 0 \\ 0 & k = 0. \end{cases}
$$

The singular operator associated with this particular singular kernel is known as the maximal ergodic Hilbert transform and is denoted by $\tilde{H}^{\star}$. Here, we further assume that the associated measure preserving transformation is ergodic.

Definition 5.4.3 (Ergodic Rectangle)**.** Let E be a subset of X with positive measure and let $K \geq 1$ be such that $U^iE \cap U^jE = \phi \quad \text{if} \quad i \neq j$ and $-K \leq i, j \leq K$. Then the set $R = \cup_{i=-K}^{K} U^iE$ is called ergodic rectangle of length $2K+1$ with base E.

For the proof of following lemma, refer[5].

Lemma 5.4.4. *Let $(X, \boldsymbol{B}, \mu)$ be a probability space, U an ergodic invertible measure preserving transformation on X and K a positive integer.*

1. *If $F \subseteq X$ is a set of positive measure then there exists a subset $E \subseteq F$ of positive measure such that E is base of an ergodic rectangle of length $2K+1$.*

2. *There exists a countable family $\{E_j\}$ of bases of ergodic rectangles of length $2K+1$ such that $X = \cup_j E_j$.*

Theorem 5.4.5. *Let $(X, \mathcal{B}, \mu)$ be a probability space, U an invertible ergodic measure preserving transformation on X. If $\tilde{H}^\star f$ is bounded on $L^p_w(X)$ for some $1 < p < \infty$, then $w \in A_P(X)$.*

Proof. For the given function w on X, for a.e $x \in X$ define the sequence $w_x(k) = w(U^{-k}x)$. We shall prove that

$$esssup_{x\in X} \sup_{N\geq 1}\left(\frac{1}{2N+1}\sum_{k=-N}^{N}|w_x(k)|\right)\left(\frac{1}{2N+1}\sum_{k-N}^{N}|w_x(k)|^{p'-1}\right)^{p-1} \leq C.$$

This will prove that $w \in A_p(X)$. In order to prove this, we shall prove that the maximal Hilbert transform $H^\star$ is bounded on $\ell^p_{w_x}(\mathbb{Z})$ and

$$\|H^\star a\|_{\ell^p_{w_x}(\mathbb{Z})} \leq C_p \|a\|_{\ell^p_{w_x}(\mathbb{Z})}.$$

where C_p is independent of x. In order to prove the above inequality, take a sequence $\{a(n) : n \in \mathbb{Z}\} \in \ell^p_{w_x}(\mathbb{Z})$.

Let $R = \cup_{k=-2J}^{2J} U^k E$ be an ergodic rectangle of length $4J+1$ with base E. Let F be any measurable subset of E. Then F is also base of an ergodic rectangle of length $4J+1$. Let $R' = \cup_{k=-2J}^{2J} U^k F$. Define function f and w as follows.

$$f(U^{-k}x) = \begin{cases} a(k) \quad if \quad x \in F \quad \text{and} -2J \leq k \leq 2J, \\ 0 \quad otherwise \end{cases}$$

Then as shown in the proof of Theorem 4.9.3

$$\|f\|^p_{L^p_w(X)} = \|a\|_{\ell^p_{w_x}(\mathbb{Z})}\, \mu(F).$$

It follows that for $-J \leq m \leq J$ and $x \in F$

$$\tilde{H}^\star_J f(U^{-m}x) = \sup_{1\leq N\leq J}\left|\sum_{k=-N}^{N}\frac{f(U^{-k-m}x)}{k}\right| = \sup_{1\leq N\leq J}\left|\sum_{k=-N}^{N}\frac{a(m+k)}{k}\right| = H^\star_J a(m).$$

Now,

$$
\begin{aligned}
C\|f\|^p_{L^p_w(X)} &\geq \int_X |\tilde{H}^\star_J f(x)|^p w(x) d\mu(x) \\
&\geq \int_{R'} |\tilde{H}^\star_J f(x)|^p w(x) d\mu(x) \\
&= \sum_{k=-J}^{J} \int_{U^k F} |\tilde{H}^\star_J f(x)|^p w(x) d\mu(x) \\
&= \sum_{k=-J}^{J} \int_{F} |\tilde{H}^\star_J f(U^{-k}x)|^p w(U^{-k}x) d\mu(x) \\
&= \sum_{k=-J}^{J} \int_{F} |H^\star_J a(k)|^p w_x(k) d\mu(x) \\
&= \int_{F} \sum_{k=-J}^{J} |H^\star_J a(k)|^p w_x(k) d\mu(x).
\end{aligned}
$$

So from the above estimates

$$\frac{1}{\mu(F)} \int_F \sum_{k=-J}^{J} |H^\star_J a(k)|^p w_x(k) d\mu(x) \leq C\|a\|_{\ell^p_{w_x}(\mathbb{Z})}.$$

Since F was an arbitrary subset of E, we get

$$\sum_{k=-J}^{J} |H^\star_J a(k)|^p w_x(k) \leq C\|a\|_{\ell^p_{w_x}(\mathbb{Z})}.$$

a.e $x \in E$. Since U is ergodic, X can be written as countable union of bases of ergodic rectangles of length $4J+1$. Therefore for a.e $x \in X$,

$$\sum_{k=-J}^{J} |H^\star_J a(k)|^p w_x(k) \leq C\|a\|_{\ell^p_{w_x}(\mathbb{Z})}.$$

Since C is independent of J, a.e $x \in X$,

$$\sum_{k\in\mathbb{Z}} |H^\star a(k)|^p w_x(k) \leq C\|a\|_{\ell^p_{w_x}(\mathbb{Z})}.$$

It follows that the sequence $\{w_x(n) : n \in \mathbb{Z}\}$ as defined by $w_x(k) = w(U^k x)$ belongs to $A_p(\mathbb{Z})$ a.e $x \in X$ and A_p weight constant for w_x is independent of x so that $w \in A_p(X)$. $\square$

Remark 5.4.6. We shall prove that the ergodic singular operator T_ϕ is bounded on variable $L^{p(\cdot)}(X, \mathcal{B}, \mu)$ when $p_- > 1$ and satisfies weak type (1,1) on variable $L^{p(\cdot)}(X)$ when $p_- \geq 1$ with the assumption that the ergodic maximal operator is bounded on variable $L^{p(\cdot)}(X, \mathcal{B}, \mu)$. We follow the method of Rubio de Francia to achieve this result.

5.5 Rubio de Francia Method for $L^{p(\cdot)}(X, \mathcal{B}, \mu)$

Lemma 5.5.1. *Given $p(\cdot)$ such that M is bounded on $L^{p(\cdot)}(X)$, for each $h \in L^{p(\cdot)}(X)$, define*

$$Rh(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \|M\|^k_{B(L^{p(\cdot)}(X))}}.$$

where for $k \geq 1$, $M^k = M \circ \ldots M$ where $\circ$ denotes composition operator acting k times and $M^0 = |I|$, I being identity operator. Then
(a) For all $x \in X, |h(x)| \leq Rh(x)$
(b) R is bounded on $L^{p(\cdot)}(X)$ and $\|Rh\|_{L^{p(\cdot)}} \leq 2\,\|h\|_{p(\cdot)}$
(c) $Rh \in A_1$ and $[Rh]_{A_1} \leq 2\,\|M\|_{B(L^{p(\cdot)}(X))}$, where $[w]_{A_1} = esssup_{x \in X} \frac{Mw(x)}{w(x)} < \infty$.

Proof. Let $B = \|M\|_{B(L^{p(\cdot)}(X))}$. We proceed as follows.

(a)

$$Rh(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k B^k} = h(x) + \frac{Mh(x)}{2B} + \frac{M^2 h(x)}{2^2 B^2} + \cdots .$$

So $h(x) \leq Rh(x)$.

(b) Using subadditivity of norm and that $\left\|M^k\right\| \leq \|M\|^k$, it follows that

$$\begin{aligned}
\|Rh\|_{p(\cdot)} &= \left\| \sum_{k=0}^{\infty} \frac{M^k h}{2^k \|M\|^k_{B(L^{p(\cdot)}(X))}} \right\| \\
&\leq \sum_{k=0}^{\infty} \frac{\left\|M^k h\right\|_{p(\cdot)}}{2^k \|M\|^k_{B(L^{p(\cdot)}(X))}} \\
&\leq \|h\|_{p(\cdot)} \sum_{k=0}^{\infty} \left\|M^k\right\|_{p(\cdot)} \frac{1}{2^k \|M\|^k_{B(L^{p(\cdot)}(X))}} \\
&\leq \|h\|_{p(\cdot)} \sum_{k=0}^{\infty} \|M\|^k_{p(\cdot)} \frac{1}{2^k \|M\|^k_{B(L^{p(\cdot)}(X))}} \\
&\leq \|h\|_{p(\cdot)} \sum_{k=0}^{\infty} 2^{-k} = 2\,\|h\|_{p(\cdot)} .
\end{aligned}$$

(c) Using subadditivity and homogeneity of the maximal operator, it follows that

$$M(Rh)(m) \leq \sum_{k=0}^{\infty} \frac{M^{k+1} h(x)}{2^k \|M\|^k_{B(L^{p(\cdot)}(X))}}$$

$$\leq 2\|M\|_{B(L^{p(\cdot)}(X))} \sum_{k=0}^{\infty} \frac{M^{k+1}h(x)}{2^{k+1}\|M\|^{k+1}_{B(L^{p(\cdot)}(X))}}$$
$$\leq 2\,\|M\|_{B(L^{p(\cdot)}(X))}\, Rh(x).$$

□

Remark 5.5.2. It is well known that the ergodic maximal operator $\tilde{M}$ is bounded on $L^{p(\cdot)}(X, \mathcal{B}, \mu)$ when $p(\cdot)$ is invariant under U [11]. But, the boundedness of $\tilde{M}$ on $L^{p(\cdot)}(X, \mathcal{B}, \mu)$ when $p(\cdot)$ satisfies LH_∞ condition remains open.

Chapter 6

Commutator of singular operator on sequence spaces

In this chapter we study the commutator of the operator of pointwise multiplication by a sequence $b = \{b(n) : n \in \mathbb{Z}\}$ and a singular operator. Also we study the commutator of ergodic singular operator on the spaces $L^p(X)$ where $(X, \mathcal{B}, \mu)$ is a probability space equipped with an invertible measure preserving transfromation U. For the commutator on the spaces $L^p(\mathbb{R}), 1 < p < \infty$, we refer to [15].

6.1 Commutator on weighted sequence spaces $\ell^p_w(\mathbb{Z})$

Let $\{a(n) : n \in \mathbb{Z}\} \in \ell^p(\mathbb{Z})$ and $f \in L^p(\mathbb{R}), 1 \leq p < \infty$.

Definition 6.1.1. We define commutator of singular operator as the operator of pointwise multiplication by a sequence $b = \{b(n) : n \in \mathbb{Z}\}$ and a singular operator T_ϕ on $\ell^p(\mathbb{Z})$. More precisely, we consider the operators given

$$([b, T_\phi]a)(n) = b(n)T_\phi a(n) - T_\phi(ba)(n) = \sum_{k=-\infty}^{\infty} \phi(k)[b(n) - b(n-k)]a(n-k).$$

and its maximal version $T^\star_\phi$ on $\ell^p(\mathbb{Z})$ which is defined as

$$([b, T_\phi]^\star a)(n) = \sup_N \left| \sum_{k=-N}^{N} \phi(k)[b(n) - b(n-k)]a(n-k) \right|.$$

We will use the word commutator instead of commutator of singular operator and the maximal commutator instead of maximal commutator of singular operator.

Definition 6.1.2. We define commutator of maximal ergodic singular operator $\tilde{T}_\phi$

$$[b,\tilde{T}_\phi]^\star f(x) = \sup_{N\geq 1}\left|\sum_{k=-N}^{N}\phi(k)[b(x)-b(U^{-k}x)]f(U^{-k}x))\right|.$$

and its truncated version corresponding to commutator of maximal ergodic singular operator $\tilde{T}_\phi^\star$ as follows: For $J \geq 1$

$$[b,\tilde{T}_\phi]_J^\star f(x) = \sup_{N\leq J}\left|\sum_{k=-N}^{N}[b(x)-b(U^{-k}x)]f(U^{-k}x)\phi(k)\right|.$$

We will use the word ergodic maximal commutator instead of commutator of maximal ergodic singular operator.

Definition 6.1.3. We define discrete Hilbert transform and maximal discrete Hilbert transform for a sequence $\{a(n) : n \in \mathbb{Z}\} \in \ell^p(\mathbb{Z})$ as follows:

$$Ha(m) = \sum_{n\in\mathbb{Z}}\left|\frac{a(n)}{m-n}\right|.$$

$$H^\star a(n) = \sup_N\left|\sum_{k=-N}^{N}\frac{a(n-k)}{k}\right|, a \in \ell^p, \quad 1 \leq p < \infty.$$

We define maximal ergodic Hilbert transform and truncated maximal ergodic Hilbert transform for a function $f \in L^p(\mathbb{R}), 1 \leq p < \infty$ as follows:

$$\tilde{H}^\star f(x) = \sup_N\left|\sum_{k=-N}^{N}\frac{f(U^{-k}x)}{k}\right|.$$

$$\tilde{H}_J f(x) = \sup_{1\leq n\leq J}\left|\sum_{k=-n}^{n}\frac{f(U^{-k}x)}{k}\right|.$$

6.2 Some results on BMO($\mathbb{Z}$)

In order to prove the boundedness of commutator on weighted sequence spaces, we require the properties of BMO($\mathbb{Z}$) which we state and prove in the following lemmas.

One of the most important results about BMO is the John-Nirenberg inequality. As a consequence we get a family of equivalent norms on BMO($\mathbb{Z}$).

Lemma 6.2.1. *Let $b \in BMO(\mathbb{Z})$. Then there exists consstants $C_1, C_2 > 0$ such that, for every finite interval I in $\mathbb{Z}$ and $\lambda > 0$,*

$$\frac{|\{n \in I : |b(n) - b_I| > \lambda\}|}{|I|} \leq C_1 e^{\frac{-C_2\lambda}{\|b\|_\star}}.$$

Proof. The key to the proof of the theorem is the Calderón-Zygmund decomposition restricted to an interval in $\mathbb{Z}$. Proof is same as the result in case of $\mathbb{R}$. For details refer [7]. □

John-Nirenberg theorem has an interesting corollary, namely, the reverse Hölder's inequality.

Corollary 6.2.2. *Let $b \in BMO(\mathbb{Z})$. Then for every finite $p > 1$,*

$$\sup_I \left(\frac{1}{|I|} \sum_{k \in I} |b(k) - b_I|^p \right)^{\frac{1}{p}} \leq C_p \|b\|_\star.$$

Proof.

$$\begin{aligned}
&\left(\frac{1}{|I|} \sum_{k \in I} |b(k) - b_I|^p \right) \\
&= p \int_0^\infty \frac{\lambda^{p-1} |\{n \in I : |b(n) - b_I| > \lambda\}|}{|I|} d\lambda \\
&\leq C_1 p \int_0^\infty \lambda^{p-1} e^{-C_2 \frac{\lambda}{\|b\|_\star}} d\lambda \\
&\leq \frac{C_1}{C_2^p} p \|b\|_\star^p \int_0^\infty \lambda^{p-1} e^{-\lambda} d\lambda = \frac{C_1}{C_2^p} p \Gamma_p \|b\|_\star^p = C_p \|b\|_\star^p.
\end{aligned}$$

□

Lemma 6.2.3. *Let $b \in BMO(\mathbb{Z})$ and I, J be two finite intervals in $\mathbb{Z}, I \subset J$.*
(a) If $|J| \leq 2|I|$, then

$$|b_I - b_J| \leq 2 \|b\|_\star.$$

(b) If $|J| > 2|I|$, then

$$|b_I - b_J| \leq 2 \log[\frac{|J|}{|I|}] \|b\|_\star.$$

Proof.

(a)

$$|b_I - b_J| \leq \left| \frac{1}{|I|} \sum_{k \in I} [b(k) - b_J] \right| \leq \frac{2}{|J|} \sum_{k \in J} |b(k) - b_J| \leq 2 \|b\|_\star.$$

(b) Let $I = I_1 \subset I_2 \subset \ldots I_n = J$. where $I_1, \ldots I_n$ are intervals in $\mathbb{Z}$ such that $|I_{K+1}| \leq 2|I_k|$ and where $n \leq C \log\left(\frac{|J|}{|I|}\right)$. Repeated applications of (a) yields (b). □

Lemma 6.2.4. *Let $b \in BMO(\mathbb{Z})$, I any interval in $\mathbb{Z}$ and n_0 the centre of I. Then for each $r > 1$, there exists a constant C_r such that*

$$\left(\sum_{n \notin 3I} \frac{|b(n) - b_I|^r}{|n_0 - n|^r}\right)^{\frac{1}{r}} \leq \frac{C_r \, \|b\|_\star}{|I|^{\frac{1}{r'}}},$$

where $\frac{1}{r} + \frac{1}{r'} = 1$.

Proof. Recall that $3I = 2LI \cup 2RI$. For $k = 1, 2 \ldots,$ let $I_k = 3^k I$ and $J_k = I_{k+1} \setminus I_k$. Now

$$\begin{aligned}
&\left(\sum_{n \notin 3I} \frac{|b(n) - b_I|^r}{|n_0 - n|^r}\right)^{\frac{1}{r}} = \left(\sum_{k=1}^{\infty} \sum_{n \in J_k} \left(\frac{|b(n) - b_I|^r}{|n_0 - n|^r}\right)^{\frac{1}{r}}\right) \\
&\leq \left(\sum_{k=0}^{\infty} \sum_{n \in J_k} \frac{|b(n) - b_I|^r}{2^{kr}|I|^r}\right)^{\frac{1}{r}} \\
&\leq \left(\sum_{k=0}^{\infty} \sum_{n \in J_k} \frac{|b(n) - b_{I_{k+1}}|^r}{2^{kr}|I|^r}\right)^{\frac{1}{r}} + \left(\sum_{k=0}^{\infty} \sum_{n \in J_k} \frac{|b_I - b_{I_{k+1}}|^r}{2^{kr}|I|^r}\right)^{\frac{1}{r}} \\
&= A_1 + A_2.
\end{aligned}$$

Then using Corollary 6.2.2, we have

$$\begin{aligned}
A_1 &\leq \left(\sum_{k=1}^{\infty} \frac{4}{2^{(r-1)k}|I|^{r-1}} \frac{1}{|I_{k+1}|} \sum_{n \in I_{k+1}} |b(n) - b_{I_{k+1}}|^r\right)^{\frac{1}{r}} \\
&\leq \left(\sum_{k=1}^{\infty} \frac{4}{2^{(r-1)k}|I|^{r-1}} \|b\|_\star{}^r\right)^{\frac{1}{r}} \leq \frac{C}{|I|^{\frac{1}{r'}}} \|b\|_\star .
\end{aligned}$$

To estimate A_2, we use (b) part of Lemma 6.2.3

$$\begin{aligned}
A_2 &\leq \left(\sum_{k=1}^{\infty} \sum_{n \in I_{k+1} \setminus I_k} \frac{[\log(\frac{|I_{k+1}|}{|I|})]^r \, \|b\|_\star^r}{2^{rk}|I|^r}\right)^{\frac{1}{r}} \\
&\leq C\left(\sum_{k=1}^{\infty} \frac{2^{k+2}|I|(\log 2^{k+1})^r \, \|b\|_\star^r}{2^{rk}|I|^r}\right)^{\frac{1}{r}} \\
&\leq C\frac{\|b\|_\star 4^{\frac{1}{r}}}{|I|^{\frac{1}{r'}}}\left(\sum_{k=1}^{\infty} \frac{(k+1)^r}{2^{(r-1)k}}\right)^{\frac{1}{r}} \leq \frac{C\,\|b\|_\star}{|I|^{\frac{1}{r'}}}.
\end{aligned}$$

□

6.3 Weighted sharp maximal sequence theorem

The proof of weighted sharp maximal sequence theorem uses Calderón-Zygmund decomposition theorem for sequences, relations between maximal operators and weighted good-Lambda estimate which were proved in Chapter 4.

Theorem 6.3.1 (Weighted sharp maximal sequence theorem). *Let $1 \le p < \infty$ and $w \in A_p(\mathbb{Z})$. Then there exists a constant $C_{p,w} > 0$ such that*

$$\|Ma\|_{p,w} \le C_{p,w} \left\|a^{\#}\right\|_{p,w}, \quad \forall a \in \ell^p_w(\mathbb{Z}).$$

6.4 Weighted maximal commutator theorem

In this section we prove weighted strong type inequalities for the discrete maximal commutator. the strong type inequalities without weights for the discrete maximal commutators can be found in [2].

Definition 6.4.1. For a sequence $b = \{b(n)\} \in \mathbf{BMO}(\mathbb{Z})$, define maximal commutator of singular integral operator as follows

$$[b, T_\phi]^\star a(n) = \sup_N \Big| \sum_{k=-N}^{N} \phi(k)[b(n) - b(n-k)]a(n-k) \Big|.$$

We want to prove that the maximal commutator is bounded on $\ell^p_w(\mathbb{Z})$, where $1 < p < \infty$ which we state it as the following theorem.

Theorem 6.4.2 (Weighted maximal commutator theorem). *Let $1 < p < \infty$, $b \in BMO(\mathbb{Z})$. Then there exists a constant $C_P > 0$ such that*

$$\|[b, T_\phi]^\star a\|_{\ell^p_w(\mathbb{Z})} \le C_p \|a\|_{\ell^p_w(\mathbb{Z})}.$$

Condition $(S3)$ in the definition of singular kernel plays a crucial role in this proof. Let ϕ_N denote truncation of ϕ which is defined as follows

$$\phi_N(k) = \begin{cases} \phi(k) & \text{if} \quad |k| \le N, \\ 0 & \text{if} |k| > n. \end{cases} \tag{6.4.1}$$

The proof of the maximal commutator inequalities would have been simpler if the ϕ_N's satisfied $(S3)$ uniformly in N. However this is not true even for $\phi(n) = \frac{1}{n}$. To overcome this difficulty we dominate $[b, T_\phi]^\star$ by a sum of two operators $[b, T_\nu]^\star$ and $[b, T_{|\psi|}]^\star$, whose truncated kernels ν_N, ψ_N satisfy $S3$ uniformly in N. Then we prove the boundedness of the corresponding maximal operators on $\ell^p_w(\mathbb{Z})$. We define the kernels $\{\nu\}, \{\psi\}$ and their truncation$\{\nu_N\}, \{\psi_N\}$ as follows.

Definition 6.4.3. Consider the differentiable function ν and ψ defined on $(0, \infty)$ by

$$\nu(t) = \begin{cases} 1 & \text{if} \quad 0 < t \leq \frac{1}{2} \\ \frac{1}{2}[1 - \cos 2\pi t] & \text{if} \quad \frac{1}{2} < t \leq 1 \\ 0 & \text{if} \quad t > 1 \end{cases} \tag{6.4.2}$$

$$\psi(t) = \begin{cases} 0 & \text{if} \quad 0 < t \leq \frac{1}{2} \\ \frac{1}{2}[1 + \cos 2\pi t] & \text{if} \quad \frac{1}{2} < t \leq 1 \\ 1 & \text{if} \quad 1 < t < 2 \\ \frac{1}{2}[1 - \cos \frac{\pi t}{2}] & \text{if} \quad 2 \leq t \leq 4 \\ 0 & \text{if} \quad t > 4 \end{cases} \tag{6.4.3}$$

Observe that

$$|\chi_{[0,1)}(t) - \nu(t)| \leq \psi(t), \quad t \in (0, \infty). \tag{6.4.4}$$

For $j \in Z$, let

$$\nu_N(j) = \phi(j)\nu(\frac{|j|}{N})$$
$$\psi_N(j) = \phi(j)\psi(\frac{|j|}{N}).$$

Using the kernels $\{\nu_N\}$ and $\{\psi_N\}$, we define the operators $[b, T_\nu]^\star$ and $[b, T_{|\psi|}]^\star$ as

$$[b, T_\nu]^\star a(n) = \sup_{N \geq 1} \left| \sum_{j=-\infty}^{\infty} [b(n) - b(j)]\nu_N(n-j)a(j) \right|$$
$$[b, T_{|\psi|}]^\star a(n), = \sup_{N \geq 1} \sum_{j=-\infty}^{\infty} |[b(n) - b(j)]||\psi_N(n-j)||a(j)|.$$

Because of inequality 6.4.4, we can prove the following lemma.

Lemma 6.4.4. *For each $n \in \mathbb{Z}$*

$$[b, T_\phi]^\star a(n) \leq [b, T_\nu]^\star a(n) + [b, T_{|\psi|}]^\star a(n).$$

As we mentioned earlier in the following lemma, we prove both the truncated kernels $\{\nu_N\}, \{\psi_N\}$ satisfy $(S3)$ uniformly in N.

Lemma 6.4.5. *There exists a constant $C > 0$ such that*

$$|\nu_N(n-j) - \nu_N(n)| \leq \frac{C|j|}{(n-j)^2} \quad \textit{for} \quad |n| > 2|j| \quad \textit{and} \quad \forall N \geq 1. \tag{6.4.5[A]}$$

$$|\psi_N(n-j) - \psi_N(n)| \leq \frac{C|j|}{(n-j)^2} \quad \textit{for} \quad |n| > 2|j| \quad \textit{and} \quad \forall N \geq 1. \tag{6.4.5[B]}$$

Proof. We will prove first of the inequalities 6.4.5$[A]$. The proof for second inequality 6.4.5$[A]$ is similar.

Consider the kernel $\{\nu_N\}$. Let $|n| > 2|j|$. Then as in [4] we can show that

$$\begin{aligned}
|\nu_N(n-j) - \nu_N(n)| &= \left|\phi(n-j)\nu(\frac{|n-j|}{N}) - \phi(n)\nu(\frac{|n|}{N})\right| \\
&\leq |\phi(n-j) - \phi(n)||\nu(\frac{|n|}{N})| + |\phi(n-j)||\nu(\frac{|n|}{N}) - \nu(\frac{|n-j|}{N})| \\
&\leq C\frac{|j|}{(n-j)^2} + C\frac{1}{|n-j|}\left|\nu(\frac{|n|}{N}) - \nu(\frac{|n-j|}{N})\right|.
\end{aligned}$$

Since supp $\nu \subseteq (0,1]$ and $|n| > 2|j|$,

$$|\nu(\frac{|n-j|}{N}) - \nu(\frac{|n|}{N})| = 0 \quad \text{if} \quad \frac{|n-j|}{N} > 2.$$

If $\frac{|n-j|}{N} \leq 2$, applying the mean value theorem, we get

$$\left|\nu(\frac{|n-j|}{N}) - \nu(\frac{|n|}{N})\right| \leq \frac{|j|}{N}\nu'(t_0).$$

where t_0 is a point between $\frac{|n-j|}{N}$ and $\frac{|n|}{N}$. But $|\nu'(t)| \leq \pi, \forall t \in (0,\infty)$. Therefore,

$$|\nu(\frac{|n-j|}{N}) - \nu(\frac{|n|}{N})| \leq \frac{|j|\pi}{N} \leq \frac{2\pi|j|}{|n-j|}.$$

Hence, the kernels $\{\nu_N\}$ satisfy condition S3 uniformly. $\square$

For proving the boundedness of the operators $[b, T_\nu]^\star$ and $[b, T_{|\psi|}]^\star$ on $\ell^p_w(\mathbb{Z})$, we need to consider the maximal operators $T^\star_\nu$ and $T^\star_{|\psi|}$ defined as:

$$T^\star_\nu = \sup_{N\geq 1}|\sum_{k=-\infty}^{\infty} \nu_N(n-k)a(k)|$$

$$T^\star_{|\psi|} = \sup_{N\geq 1}\sum_{k=-\infty}^{\infty} |\psi_N(n-k)||a(k)|.$$

Lemma 6.4.6. *Let $1 < p < \infty$. Then there exists a constant $C_p > 0$ such that*

$$\|T_\nu^\star a\|_{\ell_w^p(\mathbb{Z})} \leq c_p \|a\|_{\ell_w^p(\mathbb{Z})} \quad \forall a \in \ell_w^p(\mathbb{Z})$$
$$\left\|T_{|\psi|}^\star a\right\|_{\ell_w^p(\mathbb{Z})} \leq c_p \|a\|_{\ell_w^p(\mathbb{Z})} \quad \forall a \in \ell_w^p(\mathbb{Z}).$$

Proof. For a non negative real number α. let $[\alpha]$ denote the greatest integer less than or equal to α. Then

$$\begin{aligned}
&|\sum_{|n-j|\leq N} \phi(n-j)\nu(\frac{|n-j|}{N})a(j)| \\
&\leq |\sum_{|n-j|\leq[N/2]} \phi(n-j)a(j)| + \sum_{N\geq|n-j|>[N/2]} |\phi(n-j)||a(j)| \\
&\leq |\sum_{|n-j|\leq[N/2]} \phi(n-j)a(j)| + C_2 \sum_{N\geq|n-j|>[N/2]} \frac{|a(j)|}{|n-j|}. \\
&\leq C[T_\phi^\star a(n) + Ma(n)]
\end{aligned}$$

where Ma is the Hardy-Littlewood maximal sequence of $\{a(n) : n \in \mathbb{Z}\}$. Therefore,

$$T_\nu^\star a(n) \leq C[T_\phi^\star a(n) + Ma(n)].$$

Since we already prove that M and $T_\phi^\star$ are bounded on $\ell_w^p(\mathbb{Z})$ in Chapter4 and Chapter5 respectively , it follows that $T_\nu^\star$ is also bounded on $\ell_w^p(\mathbb{Z})$.
For the proof of second inequality, fix N and consider

$$\begin{aligned}
&\sum_{|n-j|\leq 4N} |\phi(n-j)|\psi(\frac{|n-j|}{N})|a(j)| \\
&\leq \sum_{4N\geq|n-j|>N/2} |\phi(n-j)||a(j)| \leq C_2 \sum_{4N\geq|n-j|>N/2} \frac{|a(j)|}{|n-j|} \\
&\leq \frac{C}{8N+1} \sum_{|n-j|\leq 4N} |a(j)| \leq CMa(n).
\end{aligned}$$

Therefore, $T_{|\psi|}^\star a(n) \leq CMa(n)$. It follows that $T_{|\psi|}^\star$ is bounded on $\ell_w^p(\mathbb{Z})$. ☐

Theorem 6.4.7. *Let $1 < p < \infty$ and $B \in BMO(\mathbb{Z})$. Then there exists a constant $C_p > 0$ such that*

$$\|[b, T_\nu]^\star a\|_{\ell_w^p(\mathbb{Z})} \leq c_p \|a\|_{\ell_w^p(\mathbb{Z})} \quad \forall a \in \ell_w^p(\mathbb{Z}).$$
$$\left\|[b, T_{|\psi|}]^\star a\right\|_{\ell_w^p(\mathbb{Z})} \leq c_p \|a\|_{\ell_w^p(\mathbb{Z})} \quad \forall a \in \ell_w^p(\mathbb{Z}).$$

Proof. For J=1,2,3... define

$$V_J a(n) = \sup_{N \leq J} \left| \sum_{j=n-N}^{n+N} [b(n) - b(j)] \nu_N(n-j) a(j) \right|.$$

$$W_J a(n) = \sup_{N \leq J} \sum_{j=n-4N}^{n+4N} |b(n) - b(j)| |\psi_N(n-j)| |a(j)|.$$

Then

$$[b, T_\nu]^\star a(n) = \sup_J V_J a(n).$$

$$[b, T_{|\psi|}]^\star a(n) = \sup_J W_J a(n).$$

We will prove that

$$\|V_J a\|_{\ell^p_w(\mathbb{Z})} \leq C \|a\|_{\ell^p_w(\mathbb{Z})}.$$

$$\|W_J a\|_{\ell^p_w(\mathbb{Z})} \leq C \|a\|_{\ell^p_w(\mathbb{Z})}.$$

For the proof of above inequalities, we first obtain estimates for the corresponding sharp maximal sequences. Then we will prove Theorem 6.4.7 using weighted sharp maximal sequence theorem i.e Theorem 6.3.1. These estimates are proved in the following lemma. □

Lemma 6.4.8. *Let $r > 1$ and $\{a(n) : n \in \mathbb{Z}\}$ be a sequence. then there exist constants C and C_1 such that*

$$\sup_J (V_J a)^{\#}(n) \leq C \|b\|_\star [M(T_\nu^\star a)^r(n)]^{\frac{1}{r}} + [M(a)^r(n)]^{\frac{1}{r}} \tag{6.4.5}$$

$$\sup_J (W_J a)^{\#}(n) \leq C_1 \|b\|_\star [M(T_{|\psi|}^\star a)^r(n)]^{\frac{1}{r}} + [M(a)^r(n)]^{\frac{1}{r}}. \tag{6.4.6}$$

Proof. Fix $J \geq 1$ and $n \in \mathbb{Z}$. Then if I is an interval containing n, put

$$C_I = \sup_{N \leq J} \left| \sum_{i=-\infty}^{\infty} [b(i) - b_I] \nu_N(j_o - i) a \chi_{Z \setminus 3I}(i) \right|.$$

where j_o is centre of I. Then, for $j \in I$

$$|V_J a(j) - C_I| =$$

$$\left| \sup_{N \leq J} \left| \sum_{i=-\infty}^{\infty} [b(j) - b(i)] \nu_N(j-i) a(i) \right| - \sup_{N \leq J} \left| \sum_{i=-\infty}^{\infty} [b_I - b(i)] \nu_N(j_o - i) a \chi_{Z \setminus 3I}(i) \right| \right|$$

$$\leq \sup_{N \leq J} \left| \sum_{i=-\infty}^{\infty} [b(j) - b(i)] \nu_N(j-i) a(i) - \sum_{i=-\infty}^{\infty} [b_I - b(i)] \nu_N(j_o - i) a \chi_{Z \setminus 3I}(i) \right|$$

$$\begin{aligned}
&\leq \sup_{N\leq J}\left|\sum_{i=-\infty}^{\infty}[b(j)-b_I]\nu_N(j-i)a(i)\right| \\
&+\sup_{N\leq J}\left|\sum_{i=-\infty}^{\infty}[b(i)-b_I]\nu_N(j_o-i)a\chi_{3I}(i)\right| \\
&+\sup_{N\leq J}\left|\sum_{i=-\infty}^{\infty}[b(i)-b_I][\nu_N(j-i)-\nu_N(j_o-i)]a\chi_{Z\setminus 3I}(i)\right| \\
&= A_1(j)+A_2(j)+A_3(j).
\end{aligned}$$

For the first term, we have, with $\frac{1}{r}+\frac{1}{r'}=1$

$$\begin{aligned}
\frac{1}{|I|}\sum_{j\in I}A_1(j) &\leq \frac{1}{|I|}\sum_{j\in I}|b(j)-b_I|T_\nu^\star a(j) \\
&\leq \left(\frac{1}{|I|}\sum_{j\in I}|b(j)-b_I|^{r'}\right)^{\frac{1}{r'}}\left(\frac{1}{|I|}\sum_{j\in I}|T_\nu^\star a(j)|^r\right)^{\frac{1}{r}} \\
&\leq \|b\|_\star\left[M(T_\nu^\star a)^r(n)\right]^{\frac{1}{r}}.
\end{aligned}$$

Now consider

$$\begin{aligned}
\frac{1}{|I|}\sum_{j\in I}A_2(j) &\leq \frac{1}{|I|}\sum_{j\in I}|T_\nu^\star[(b-b_I)a\chi_{3I}](j)| \\
&\leq \left\{\frac{1}{|I|}\sum_{j\in I}|T_\nu^\star[(b-b_I)a\chi_{2I}](j)|^s\right\}^{\frac{1}{s}},
\end{aligned}$$

where $s>1$. We can further replace the above summation over I by a summation over $\mathbb{Z}$. Then using the boundedness of $T_\nu^\star$ on l^s, we get

$$\begin{aligned}
\frac{1}{|I|}\sum_{j\in I}A_2(j) &\leq C\left\{\frac{1}{|I|}\sum_{j\in 3I}|b(j)-b_I|^s|a(j)|^s\right\}^{\frac{1}{s}} \\
&\leq C\left\{\frac{1}{|3I|}\sum_{j\in 3I}|b(j)-b_I|^{sq}\right\}^{\frac{1}{sq}}\left\{\frac{1}{|3I|}\sum_{j\in 2I}|a(j)|^{sq'}\right\}^{\frac{1}{sq'}},
\end{aligned}$$

where $q>1$ and $\frac{1}{q}+\frac{1}{q'}=1$

Now,

$$\left\{\frac{1}{|3I|}\sum_{j\in 3I}|b(j)-b_I|^{sq}\right\}^{\frac{1}{sq}} \leq C\left(\frac{1}{|3I|}\sum_{j\in 3I}|b(j)-b_{2I}|^{sq}\right)^{\frac{1}{sq}}+|b_{2I}-b_I|\leq C\|b\|_\star.$$

Therefore,

$$\frac{1}{|I|}\sum_{j\in I} A_2(j) \leq c\,\|b\|_{\star}\,[M(|a|^r)(n)]^{\frac{1}{r}}.$$

provided we choose s and q' so that $sq' = r$. It remains to estimate $A_3(j), j \in I$.

$$A_3(j) \leq \sup_{N\leq J}\sum_{i=-\infty}^{\infty} |b(i)-b_I||\nu_N(j-i)-\nu_N(j_0-i)||a\chi_{\mathbb{Z}\setminus 2I}(i)|.$$

But since $i \notin 3I$ and $j \in I$, $|j_0 - i| > 2|j-j_0|$

Therefore,

$$|\nu_N(j-i)-\nu_N(j_0-i)| \leq \frac{C|j-j_0|}{(j-i)^2} \leq \frac{C|j-j_0|}{(j_o-i)^2}.$$

Therefore,

$$\begin{aligned} A_3(j) &\leq C\sup_{N\leq J}\sum_{i=-\infty}^{\infty} |b(i)-b_I|\frac{|j-j_0|}{(j_0-i)^2}|a\chi_{\mathbb{Z}\setminus 3I}(i)| \\ &\leq C(|I|)\left(\sum_{i\notin 3I}\frac{|b(i)-b_I|^{r'}}{|j_0-i|^{r'}}\right)^{\frac{1}{r'}}\left(\sum_{i\notin 3I}\frac{|a(i)|^r}{|j_0-i|^r}\right)^{\frac{1}{r}}. \end{aligned}$$

Now by Lemma 6.2.4

$$\left(\sum_{i\notin 3I}\frac{|b(i)-b_I|^{r'}}{|j_0-i|^{r'}}\right)^{\frac{1}{r'}} \leq \frac{C\,\|b\|_{\star}}{(|I|)^{\frac{1}{r}}}.$$

Now let $I_k = 3^k I$, then by standard techniques, we get

$$\left(\sum_{i\notin 3I}\frac{|a(i)|^r}{|j_0-i|^r}\right)^{\frac{1}{r}}. \leq \frac{C[M(|a|^r)(n)]^{\frac{1}{r}}}{|I|^{\frac{1}{r'}}}.$$

Therefore,

$$A_3(j) \leq C\,\|b\|_{\star}\,[M(|a|^r)(n)]^{1/r}.$$

and so,

$$\frac{1}{|I|}\sum_{j\in I} A_3(j) \leq C\,\|b\|_{\star}\,[M(|a|^r)(n)]^{1/r}.$$

For the proof of second inequality in Lemma 6.4.8, we proceed as follows. For $n \in \mathbb{Z}$ and any interval I containing n, we choose

$$C_I = \sup_{N\leq J}\sum_{i=-\infty}^{\infty} |b(i)-b_I||\psi_N(j_0-i)||a\chi_{\mathbb{Z}\setminus 3I}(i)|.$$

where j_0 is centre of I. Then

$$
\begin{aligned}
&|W_j a(j) - C_I| = \\
&\left| \sup_{N \le J} \sum_{i=-\infty}^{\infty} |b(j) - b(i)||\psi_N(j-i)||a(i)| \right. \\
&\left. - \sup_{N \le j} \sum_{i=-\infty}^{\infty} |b(i) - b_I||\psi_N(j_0 - i)||a\chi_{\mathbb{Z}\setminus 3I}(i)| \right| \\
&\le \sup_{N \le J} \sum_{i=-\infty}^{\infty} \left| |b(j) - b(i)||\psi_N(j-i)||a(i)| - \Big(|b(i) - b_I||\psi_N(j_0 - i)||a\chi_{Z\setminus 3I}(i)| \Big) \right| \\
&\le \sup_{N \le J} \sum_{i=-\infty}^{\infty} \left| \left[\left| \Big((b(j) - b_I)|\psi_N(j-i)||a(i)| + (b_I - b(i))|\psi_N(j-i)||a\chi_{3I}(i)| \Big) \right. \right. \right. \\
&\left. \left. \left. + (b_I - b(i))|\psi_N(j-i)||a\chi_{Z\setminus 3I}(i)| \right| \right] - \Big(|b_I - b(i)||\psi_N(j_0 - i)||a\chi_{Z\setminus 3I}| \Big) \right|.
\end{aligned}
$$

Now

$$
\Big| |x+y| - |z| \Big| \le |x| + \Big| |y| - |z| \Big|, \forall x, y, z \in \mathbb{C}. \tag{6.4.7}
$$

We have

$$
\begin{aligned}
&|W_j a(j) - C_I| \\
&\le \sup_{N \le J} \sum_{i=-\infty}^{\infty} \Big| |b(j) - b_I||\psi_N(j-i)||a(i)| + (b_I - b(i))|\psi_N(j-i)||a\chi_{3I}(i)| \Big| \\
&+ \Big| |b(i) - b_I||\psi_N(j-i)||a\chi_{Z\setminus 3I(i)}| - |b(i) - b_I||\psi_N(j_0 - i)||a\chi_{\mathbb{Z}\setminus 3I(i)}| \Big| \\
&\le \sup_{N \le J} \sum_{i=-\infty}^{\infty} |b(j) - b_I||\psi_N(j-i)||a(i)| \\
&+ \sup_{N \le J} \sum_{i=-\infty}^{\infty} |b(i) - b_I||\psi_N(j-i)||a\chi_{3I}(i)| \\
&+ \sup_{N \le J} \sum_{i=-\infty}^{\infty} |b(i) - b_I||\psi_N(j-i) - \psi_N(j_0 - i)||a\chi_{\mathbb{Z}\setminus 3I(i)}| \\
&= B_1(j) + B_2(j) + B_3(j).
\end{aligned}
$$

The estimates for each of these terms are obtained exactly as in the previous case by replacing ν by ψ . This concludes proof of Lemma 6.4.8. □

Here we prove the boundedness of sharp maximal sequences $\{(V_J a)^{\#}\}, \{(W_J a)^{\#}\}$ on $\ell^p_w(\mathbb{Z})$.

Theorem 6.4.9. *For* $1 < p < \infty$,

$$\left\|(V_J a)^{\#}\right\|_{\ell^p_w(\mathbb{Z})} \leq C \left\|a\right\|_{\ell^p_w(\mathbb{Z})}, \quad \forall a \in \ell^p_w(\mathbb{Z}).$$

$$\left\|(W_J a)^{\#}\right\|_{\ell^p_w(\mathbb{Z})} \leq C_1 \left\|a\right\|_{\ell^p_w(\mathbb{Z})}, \quad \forall a \in \ell^p_w(\mathbb{Z}).$$

where C, C_1 *are independent of* J

Proof.

$$\begin{aligned}
&\left\|(V_J a)^{\#}\right\|_{\ell^p_w(\mathbb{Z})} \\
&= \left\{\sum_{n=-\infty}^{\infty} |(V_J a)^{\#}(n)|^p w(n)\right\}^{\frac{1}{p}} \\
&\leq C\left\{\sum_{n=-\infty}^{\infty} M(T^{\star}_{\nu} a)^r(n) w(n)]^{\frac{p}{r}}\right\}^{\frac{1}{p}} + \left\{\sum_{n=-\infty}^{\infty} [M(|a|^r(n)]^{\frac{p}{r}} w(n)\right\}^{\frac{1}{p}} \\
&\leq C\left\{\sum_{n=-\infty}^{\infty} [T^{\star}_{\nu} a(n)]^p w(n)\right\}^{\frac{1}{p}} + \left\{\sum_{n=-\infty}^{\infty} [a(n)]^p w(n)\right\}^{\frac{1}{p}} \\
&\leq C \left\|a\right\|_{\ell^p_w(\mathbb{Z})}.
\end{aligned}$$

By a similar argument

$$\|(W_J a)^{\star}\|_{\ell^p_w(\mathbb{Z})} \leq C \left\|a\right\|_{\ell^p_w(\mathbb{Z})}.$$

□

It remains to prove $V_J a, W_J a \in \ell^p_w(\mathbb{Z})$. Then the boundedness of sequences $\{(V_J a)\}, \{(W_J a)\}$ on $\ell^p_w(\mathbb{Z})$ hold by the weighted sharp maximal sequence theorem i.e Theorem 6.3.1 as follows

$$\|(V_J a)\|_p \leq \|M(V_J a)\|_p \leq C_p \left\|(V_J a)^{\#}\right\|_p, \quad \forall \quad a \in \ell^p_w(\mathbb{Z}).$$

$$\|(V_J a)\|_p \leq \|M(V_J a)\|_p \leq C_p \left\|(V_J a)^{\#}\right\|_p, \quad \forall \quad \in \ell^p_w(\mathbb{Z}).$$

The $\ell^p_w(\mathbb{Z})$ norms of $\{(V_J a)\}, \{(W_J a)\}$(may depend on J).
Alternatively, we claim that

$$V_J a(n) \leq C_J \left\|b\right\|_{\star} (M(|a|^r)(n))^{1/r}, \quad 1 < r < \infty.$$

and

$$W_J a(n) \leq C'_J \left\|b\right\|_{\star} T^{\star}_{|\psi|} a(n).$$

We have

$$V_J a(n) = \sup_{N\le J} |\sum_{i=n-N}^{n+N} [b(n)-b(i)]\nu_N(n-i)a(i)|$$
$$= \sup_{N\le J} |\sum_{i=-\infty}^{\infty} [b(n)-b(i)]\nu_N(n-i)a\chi_{I_J}(i)|.$$

where $I_J = [n-J, n+J]$. We estimate this exactly as we estimated the term A_2 in Lemma 6.4.8 and we have

$$V_J a(n) \le (2J+1)\{C\|b\|_\star + (\log J)\|b\|_\star\}\{M(|a|^r)(n)\}^{1/r}$$
$$\le C_J \|b\|_\star \{M(|a|^r)(n)\}^{1/r}.$$

Therefore, choosing $r < p$ we see that $V_J a \in \ell^p_w(\mathbb{Z})$ for $a \in \ell^p_w(\mathbb{Z})$.

Next let $n \in \mathbb{Z}$ and $I_{4J} = [n-4J, n+4J]$. Then for $i \in I_{4J}$

$$|b(n)-b(i)| \le |b(n)-b_{I_{4J}}| + |b_{I_{4J}} - b(i)| \le 2(8J+1)\|b\|_\star.$$

Therefore,

$$W_J a(n) = \sup_{N\le J} \sum_{i=n-4N}^{n+4N} |b(n)-b(i)||\psi_N(n-i)||a(i)| \le C_J \|b\|_\star T^\star_{|\psi|}a(n).$$

So $W_J a \in \ell^p_w(\mathbb{Z}), \forall a \in \ell^p_w(\mathbb{Z})$.

Hence we conclude the boundedness of operators V_J, W_J on $\ell^p_w(\mathbb{Z})$. Since the constants obtained in the inequalities stated in Theorem 6.4.9 are independent of J, boundedness of $[b,T_\nu]^\star, [b,T_{|\psi|}]^\star$ on $\ell^p_w(\mathbb{Z}), 1 < p < \infty$ follow immediately using weighted sharp maximal sequence theorem.

Corollary 6.4.10. *Let $1 < p < \infty$. If $b \in BMO(\mathbb{Z})$, then the commutator of the singular operator $[b, T_\phi]a$ exists for every $a \in \ell^p_w(\mathbb{Z})$.*

Proof. Note that finite sequences are dense in $\ell^p_w(\mathbb{Z})$ [15] and $[b,T_\phi]a$ exists for every finite sequence $\{a(n) : n \in \mathbb{Z}\}$. Since $[b,T_\phi]^\star$ is bounded on $\ell^p_w(\mathbb{Z})$, the proof follows. □

6.5 Maximal commutator of singular operator on variable sequence spaces $\ell^{\ell^{p(\cdot)}}(\mathbb{Z})$

In this section, we prove strong type inequality for the maximal commutator of singular operator on $\ell^{p(\cdot)}(\mathbb{Z}), 1 < p_- \le p_+ < \infty$, $1 \le p < \infty$, using Rubio de Francia extrapolation method given in [8].

Theorem 6.5.1. *Given a sequence $\{a(n) : n \in \mathbb{Z}\}$, suppose $p(\cdot) \in \mathcal{S}$ such that $p_- > 1$. Let $[b, T_\phi]^\star$ be a maximal commutator of singular operator.*
Then,

$$\|[b, T_\phi]^\star a\|_{\ell^{p(\cdot)}(\mathbb{Z})} \leq C \, \|a\|_{\ell^{p(\cdot)}(\mathbb{Z})} .$$

Proof. Take p_0 such that $1 < p_0 \leq p_- \leq p_+ < \infty$. Therefore by Lemma 3.2.2

$$\begin{aligned}
\|([b, T_\phi]^\star a)\|^{p_0}_{\ell^{p(\cdot)}(\mathbb{Z})} &= \|([b, T_\phi]^\star a)^{p_0}\|_{\ell^{\frac{p(\cdot)}{p_0}}}(\mathbb{Z}) \\
&= \sup_{h \in \ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z}), \|h\|_{\ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z})} = 1} \sum_{k \in \mathbb{Z}} |[b, T_\phi]^\star a(k)|^{p_0} |h(k)| \\
&\leq \sup_{h \in \ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z}), \|h\|_{\ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z})} = 1} \sum_{k \in \mathbb{Z}} |[b, T_\phi]^\star a(k)|^{p_0} Rh(k) \\
&\leq C \sup_{h \in \ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z}), \|h\|_{\ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z})} = 1} \sum_{k \in \mathbb{Z}} |a(k)|^{p_0} Rh(k) \\
&\leq C \sup_{h \in \ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z}), \|h\|_{\ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z})} = 1} \||a|^{p_0}\|_{\ell^{\frac{p(\cdot)}{p_0}}(\mathbb{Z})} \|Rh\|_{\ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z})} \\
&= C \sup_{h \in \ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z}), \|h\|_{\ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z})} = 1} \||a|\|^{p_0}_{\ell^{p(\cdot)}(\mathbb{Z})} \|Rh\|_{\ell^{(\frac{p(\cdot)}{p_0})'}(\mathbb{Z})} \\
&\leq 2C \|a\|^{p_0}_{\ell^{p(\cdot)}(\mathbb{Z})}.
\end{aligned}$$

□

6.6 Ergodic maximal commutator of singular operator

Let $(X, \mathbf{B}, \mu)$ be a probability space and U an invertible measure preserving transformation on X. We define the commutator of the truncated ergodic maximal singular operator as follows:

$$[b, \tilde{T_\phi}]^\star_J f(x) = \sup_{N \leq J} |\sum_{k=-N}^{N} \phi(k)[b(x) - b(U^{-k}x)] f(U^{-k}x)|.$$

Definition 6.6.1 (BMO(X))**.** For a probability space $(X, \mathcal{B}, \mu)$ and an invertible measure preserving transformation on X, the space BMO(X) is defined as the space of those functions

$b \in L^1(X)$ satisfying

$$esssup_{x\in X}\left(\sup_{N\geq 1}\frac{1}{2N+1}\sum_{k=-N}^{N}|b(U^k x)-\frac{1}{2N+1}\sum_{j=-N}^{N}b(U^j x)|\right)=\|b\|_{\star}<\infty.$$

Now, we prove the strong type inequality for the ergodic maximal commutator of singular operator on $L^p_w(X,\mathcal{B},\mu)$ spaces.

Theorem 6.6.2. *Let $(X,\mathcal{B},\mu)$ be a probability space and U an invertible measure preserving transformation on X. If w is an ergodic A_p weight, $1<p<\infty$ and $b \in BMO(X)$, then there exists a constant $C_p>0$ such that*

$$\left\|[b,\tilde{T}_\phi]^{\star}f\right\|_{L^p_w(X)}\leq C_p\|f\|_{L^p_w(X)} \quad \forall f\in L^p_w(X,\mathcal{B},\mu).$$

Proof. Observe that if $b \in BMO(X)$, then $b\in L^1(X)$ and for a.e the sequence b_x given by $b_x(n)=b(U^n x)$ is in BMO($\mathbb{Z}$), with $\|b_x\|_\star\leq C$, where C is independent of x. For $J\geq 1$, let

$$[b,\tilde{T}_\phi]^{\star}_J f(x)=\sup_{N\leq J}\left|\sum_{k=-N}^{N}[b(x)-b(U^{-k}x)]f(U^{-k}x)\phi(k)\right|.$$

We will prove that

$$\left\|[b,\tilde{T}_\phi]^{\star}_J f\right\|_{L^p_w(X)}\leq C\|f\|_{L^p_w(X)} \quad \forall f\in L^p_w(X,\mathcal{B},\mu),$$

where the constant C is independent of f and J. Then the theorem will follow by monotone convergence theorem. For a.e $x\in X$ let

$$[b_x,T_\phi]^{\star}_J a(n)=\sup_{N\leq J}\left|\sum_{k=-N}^{N}[b_x(n)-b_x(n-k)]a(n-k)\phi(k)\right|.$$

It is easy to see that $[b_x,T_\phi]^{\star}_J$ is sub linear. Let L be any integer greater than or equal to J. Observe that if $\{a(n):n\in\mathbb{Z}\}$ is a sequence such that $a(n)=0$ for $|n|\leq L$ then

$$[b,T_\phi]^{\star}_J a(n)=0 \quad \text{for} \quad |n|\leq L-J.$$

For a.e $x\in X$, let $f_x(n)=f(U^n x)$ and $w_x(n)=w(U^n x)$. For $k\in\mathbb{Z}_+$, define

$$f_x^K(n)=\begin{cases} f_x(n) & \text{if} \quad |n|\leq K,\\ 0 & \text{if} \quad |n|>K.\end{cases}$$

Since $[b,T_\phi]^\star_J$ is sub linear, for $K, L \in \mathbb{Z}$ and $n \in \mathbb{Z}$, we have

$$[b_x, T_\phi]^\star_J f_x(n) \le [b_x, T_\phi]^\star_J f_x^{K+L}(n) + [b,T_\phi]^\star_J (f_x - f_x^{K+L})(n).$$

We can choose L large enough so that

$$[[b_x, T_\phi]^\star_J (f_x - f_x^{K+L})](n) = 0 \quad \text{if} \quad |n| \le K.$$

Note that L depends only on J and not on K. Therefore,

$$[b_x, T_\phi]^\star_J f_x(n) \le [b_x, T_\phi]^\star_J f_x^{K+L}(n)$$

for $|n| \le K$.

Also, for a.e $x \in X$and $j \in \mathbb{Z}$, we have

$$\begin{aligned}
&[b, \tilde{T_\phi}]^\star_J f(U^j x) = \sup_{N \le J} |\sum_{k=-N}^{N} [b(U^j x) - b(U^{j-k})] f(U^{j-k}) \phi(k)| \\
&= \sup_{N \le J} |\sum_{k=-N}^{N} [b_x(j) - b_x(j-k)] f_x(j-k) \phi(k)| \\
&= [b_x, T_\phi]^\star_J f_x(j).
\end{aligned}$$

Then

$$\begin{aligned}
&\int_X ([b, \tilde{T_\phi}]^\star_J f(x))^p w(x) d\mu = \frac{1}{2K+1} \sum_{j=-K}^{K} \int_X ([b, \tilde{T_\phi}]^\star_J f(U^j x))^p w(U^j x) d\mu \\
&= \frac{1}{2K+1} \sum_{j=-K}^{K} \int_X ([b_x, T_\phi]^\star_J f_x(j))^p w_x(j) d\mu \\
&\le \frac{1}{2K+1} \sum_{j=-K}^{K} \int_X ([b_x, T_\phi]^\star_J f_x^{K+L}(j))^p w_x(j) d\mu \\
&\le \frac{C}{2K+1} \int_X \sum_{j=-(K+L)}^{K+L} |f_x(j)|^p w_x(j) d\mu \\
&= \frac{C}{2K+1} \sum_{j=-(K+L)}^{K+L} \int_X |f(U^j x)|^p w(U^j x) d\mu \\
&\le \frac{C[2(K+L)+1] \, \|f\|^p_{L^p_w(X)}}{2K+1}.
\end{aligned}$$

Choosing K sufficiently large, we get

$$\left\|[b,\tilde{T}_\phi]_J^\star f\right\|_{L^p_w(X)} \leq C_p \|f\|_{L^p_w(X)}.$$

□

Remark 6.6.3. Using the boundedness of maximal ergodic commutator of singular operator and Rubio de Francia method, we can prove that the maximal ergodic commutator of singular operator is bounded on variable $L^{p(\cdot)}(X, \mathcal{B}, \mu)$ spaces. But Rubio de Francia method assumes maximal ergodic operator is bounded on the variable $L^{p(\cdot)}(X, \mathcal{B}, \mu)$ spaces. With this assumption we can prove the boundedness of maximal ergodic commutator singular operator to variable $L^{p(\cdot)}(X, \mathcal{B}, \mu)$ spaces.

Chapter 7

Concluding remarks and future Perspectives

1. Using the boundedness of ergodic maximal commutator and Rubio de Francia method, we can prove that the maximal ergodic discrete singular operator and commutator of maximal ergodic discrete singular operator are bounded on variable $L^{p(\cdot)}(X,\mathcal{B},\mu)$ spaces. But Rubio de Francia method assumes maximal ergodic operator is bounded on the variable $L^{p(\cdot)}(X,\mathcal{B},\mu)$ spaces. With this assumption we can prove the boundedness of maximal ergodic commutator singular operator to variable $L^{p(\cdot)}(X,\mathcal{B},\mu)$ spaces.

2. In [11] maximal ergodic theorem is proved on $L^{p(\cdot)}(X,\mathcal{B},\mu)$ using the condition that the variable exponent $p(\cdot)$ is invariant under U, where $(X,\mathcal{B},\mu)$ is a probability space equipped with an invertible measure preserving transformation U. One has to explore that the maximal ergodic theorem holds under the condition that $p(\cdot)$ satisfies log Holder LH_∞ condition.

3. Results obtained for boundedness of various operators on $\ell^p_w(\mathbb{Z})$ mentioned in this thesis may be extended to Lorentz spaces, Sobolev spaces.

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[4] Anupindi Sri Sakti Swarup and A. Michael Alphonse, *The boundedness of Fractional Hardy-Littlewood maximal operator on variable lp(Z) spaces using Calderon-Zygmund decomposition,* Accepted for publication in "The Journal of Indian Mathematical Society (2022), Vol 89", Acceptance Letter Dated: May 9 2023.

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List of Publications

Thesis Publications

1. Anupindi Sri Sakti Swarup and A. Michael Alphonse, *Relations Between Discrete Maximal Operators in Harmonic Analysis.* Proceedings of the Ninth International Conference on Mathematics and Computing. ICMC 2023. **Lecture Notes in Networks and Systems, vol 697. Springer, Singapore**. `https://doi.org/10.1007/978-981-99-3080-7_30`.

2. Anupindi Sri Sakti Swarup and A. Michael Alphonse, *The boundedness of Fractional Hardy-Littlewood maximal operator on variable lp(Z) spaces using Calderon-Zygmund decomposition* - **The Journal of Indian Mathematical Society), Vol 91, Issue 1-2, Jan-June 2024.** `https://doi.org/10.18311/jims/2024/31327`.

3. Anupindi Sri Sakti Swarup and A. Michael Alphonse, *Maximal Ergodic Theorem On Weighted* $L^p_w(X)$ *spaces* -**Accepted for publication (The Journal of Indian Mathematical Society)**.

4. Anupindi Sri Sakti Swarup and A. Michael Alphonse, *Maximal Singular Operators on Variable Exponent Sequence Spaces and their corresponding Ergodic version* **(Submitted to "The Australian Journal of Mathematical Analysis and Applications")**.

5. Anupindi Sri Sakti Swarup and A. Michael Alphonse, *Commutator of Singular Operator on Variable Exponent Sequence Spaces and their corresponding Ergodic version* -**Accepted for publication (The Australian Journal of Mathematical Analysis and Apppli-cations)**.

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