Numerical Solution for Cauchy and Hypersingular Integral Equations by Using Legendre Polynomials

THESIS

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Under the Supervision of

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Certificate

This is to certify that the thesis entitled **'Numerical Solution for Cauchy and Hypersingular Integral Equations by Using Legendre Polynomials'** and submitted by **VAISHALI SHARMA**, ID.No. **2012PHXF0006G** for award of Ph.D. of the Institute embodies original work done by her under my supervision.

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I, Vaishali Sharma, declare that this thesis titled, 'Numerical Solution for Cauchy and Hypersingular Integral Equations by Using Legendre Polynomials', submitted by me under the supervision of Dr. Amit Setia is a bonafide research work. I also declare that it has not been submitted previously in part or in full to this University or any other University or Institution for award of any degree.

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Abstract

Singular integral equations have various applications in several basic fields of engineering mechanics like elasticity, plasticity, and aerodynamics etc. Many crack problems occurring in the field of fracture mechanics such as thermoelastic stress problems around an arbitrary number of arbitrarily-located planar cracks are reducible into singular integral equations or their system. In this thesis, we consider the numerical solution of two kind of singular integral equations. Firstly, the singular integral equations of first kind with Cauchy kernel and the system of these equations. Secondly, the hypersingular integral equations of first kind and their system.

Singular integral equations (SIEs) with Cauchy kernel play a vital role in studying many problems of aerodynamics, fracture mechanics, neutron transport, wave propagation etc. System of Cauchy singular integral equations also have great importance as various problems occur in the field of aerodynamics, queuing system analysis, electrocardiology, elasticity theory etc., can be formulated as system of Cauchy singular integral equations. The analytic solution of such equations as well as for their system, are known when these equations are dominant equations. But these analytic solutions are of limited use as it is a nontrivial task to use it practically due to the presence of singularity in the known solutions itself. Therefore, there is a necessity to find their approximate solutions.

Analogous to Cauchy singular integral equations, the hypersingular integral equations as well as their system are equally important. Several problems occurring in the field of aerodynamics, aeronautics, interference or interaction problems such as wing-tail surfaces problem etc., are reducible into hypersingular integral equations or their system. Similar to Cauchy singular integral equations, in case of hypersingular integral equations also, the analytical solution of these equations and their system are known only for dominant equations. Further, there are many real world problems such as crack problems occurring in the field of fracture mechanics which may not be always reducible as dominant equations. This is one of the reason why there is a need to develop numerical methods. Although, various methods are available to find the approximate solution of Cauchy and hypersingular integral equations. However, search for numerical methods which are better than the available methods in some sense, is always there. Hence, we propose numerical methods to find the approximate solution of Cauchy singular integral equations, hypersingular integral equations and their systems. The proposed methods converts the singular integral equations into a system of linear algebraic equations which can be solved easily. The convergence of sequence of approximate solutions is proved for both kind of singular integral equations considered in this thesis. The derived convergence helps to obtain theoretical error bound for the error between the exact and approximate solutions. Hadamard conditions of well-posedness are also established for each of the system of linear algebraic equations which is obtained as a result of approximation of corresponding singular integral equations and their systems. Finally, all the derived theoretical results are validated with the help of numerical examples.

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List of Acronyms and Symbols

List of Acronyms

SIEs : Singular integral equations

CSIEs : Cauchy singular integral equations

HSIEs : Hypersingular integral equations

CPV : Cauchy principal value

HFP : Hadamard finite part

List of Symbols

$\forall$: For all

$\in$: Belongs to

$\sum_{i=0}^{i}$: Sum over i

$\det(\,\cdot\,)$: Determinant of a matrix

$[\,\cdot\,]^T$: Transpose of matrix

$⨍$: Integral with Cauchy kernel

$⨎$: Integral with hyper kernel

n : Integer

Γ_j : Smooth simple arc in the complex plane with no common points and each is of finite length

Δ_j : Parametrization for each arc Γ_j

$C^{\,0,\,\alpha}(-1,1)$: Space of functions which are Hölder continuous on the interval $(-1,1)$ with the exponent $0 < \alpha \leq 1$

$C^{1,\,\alpha}(-1,1)$: Space of functions whose first derivatives are Hölder continuous on the interval $(-1,1)$ with the exponent $0 < \alpha \leq 1$

$\mathbb{R}^{n+1}$: Real vector space having $(n+1)$-tuples of real numbers

$\mathbb{C}^{n+1}$: Complex vector space having $(n+1)$-tuples of complex numbers

$\mathbb{R}$: Set of real numbers

$\mathbb{C}$: Set of complex numbers

$L^2[-1,1]$: Hilbert Space of all square integrable functions over $[-1,1]$

$\|\cdot\|^2_{L^2}$: Norm with respect to L^2 space

$\langle\cdot,\cdot\rangle_{L^2}$: Inner product with respect to L^2 space

$M^{[c]}$: Hilbert space (In case of CSIEs over $[-1,1]$)

$\|\cdot\|_{M^{[c]}}$: Norm with respect to $M^{[c]}$ space

$\langle\cdot,\cdot\rangle_{M^{[c]}}$: Inner product with respect to $M^{[c]}$ space

$M^{[h]}$: Hilbert space (In case of HSIEs)

$\|\cdot\|_{M^{[h]}}$: Norm with respect to $M^{[h]}$ space

$\langle\cdot,\cdot\rangle_{M^{[h]}}$: Inner product with respect to $M^{[h]}$ space

$M^{[cs]}$: Hilbert space (In case of system of CSIEs for single CSIE)

$\|\cdot\|_{M^{[cs]}}$: Norm with respect to $M^{[cs]}$ space

$\langle\cdot,\cdot\rangle_{M^{[cs]}}$: Inner product with respect to $M^{[cs]}$ space

$M^{[hs]}$: Hilbert space (In case of system of HSIEs for single HSIE)

$\|\cdot\|_{M^{[hs]}}$: Norm with respect to $M^{[hs]}$ space

$\langle\cdot,\cdot\rangle_{M^{[hs]}}$: Inner product with respect to $M^{[hs]}$ space

$M_N^{[cs]}$: Hilbert space (In case of system of CSIEs)

$\|\cdot\|_{M_N^{[cs]}}$: Norm with respect to $M_N^{[cs]}$ space

$\langle\cdot,\cdot\rangle_{M_N^{[cs]}}$: Inner product with respect to $M_N^{[cs]}$ space

${L_N^2}^{[cs]}$: Hilbert space (In case of system of CSIEs)

$\|\cdot\|_{{L_N^2}^{[cs]}}$: Norm with respect to ${L_N^2}^{[cs]}$ space

$\langle\cdot,\cdot\rangle_{{L_N^2}^{[cs]}}$: Inner product with respect to ${L_N^2}^{[cs]}$ space

$M_N^{[hs]}$: Hilbert space (In case of system of HSIEs)

$\|\cdot\|_{M_N^{[hs]}}$: Norm with respect to $M_N^{[hs]}$ space

$\langle\cdot,\cdot\rangle_{M_N^{[hs]}}$: Inner product with respect to $M_N^{[hs]}$ space

${L_N^2}^{[hs]}$: Hilbert space (In case of system of HSIEs)

$\|\cdot\|_{{L_N^2}^{[hs]}}$: Norm with respect to ${L_N^2}^{[hs]}$ space

$\langle\cdot,\cdot\rangle_{{L_N^2}^{[hs]}}$: Inner product with respect to ${L_N^2}^{[hs]}$ space

$\mathbb{Z}_N^{[cs]}, \mathbb{X}_{[cs]}^{N\times(n+1)}$: Vector space in case of system of CSIEs

$\mathbb{Z}_N^{[hs]}, \mathbb{X}_{[hs]}^{N\times(n+1)}$: Vector space in case of system of HSIEs

$\chi^{[c]}(t), \xi^{[c]}(t)$: Unknown functions over $[-1,1]$ for CSIEs

$\hat{\chi}^{[c]}(t), \hat{\xi}^{[c]}(t)$: Unknown functions over $[0,\infty)$ for CSIEs

$\chi^{[h]}(t), \xi^{[h]}(t)$: Unknown functions over $[-1,1]$ for HSIEs

${\xi_n^*}^{[c]}(t), {\hat{\xi}_n^*}{}^{[c]}(t)$: Approximate solutions in case of CSIEs

${\chi^*}^{[h]}(t), {\xi^*}^{[h]}(t)$: Approximate solutions in case of HSIEs

$\varphi^{[cs]}(x) = (\varphi_1^{[cs]}(x), \varphi_2^{[cs]}(x), \ldots, \varphi_N^{[cs]}(x))^T$, $\psi^{[cs]}(x) = ({\psi_1}^{[cs]}(x), {\psi_2}^{[cs]}(x), \ldots, {\psi_N}^{[cs]}(x))^T$: Matrix of unknown function in case system of CSIEs

$\psi^{[cs]}(t) \approx {\psi_n^*}^{[cs]}(t) = ({\psi_{1n}^*}^{[cs]}(t), {\psi_{2n}^*}^{[cs]}(t), \ldots, {\psi_{Nn}^*}^{[cs]}(t))^T$,
$\phi^{[cs]}(t) \approx {\phi_n^*}^{[cs]}(t) = ({\phi_{1n}^*}^{[cs]}(t), {\phi_{2n}^*}^{[cs]}(t), \ldots, {\phi_{Nn}^*}^{[cs]}(t))^T$: Matrix of unknown function in case system of CSIEs

$\varphi^{[hs]}(x) = (\varphi_1^{[hs]}(x), \varphi_2^{[hs]}(x), \ldots, \varphi_N^{[hs]}(x))^T$, $\psi^{[hs]}(t) = ({\psi_1}^{[hs]}(t), {\psi_2}^{[hs]}(t), \ldots, {\psi_N}^{[hs]}(t))^T$: Matrix of unknown function in case system of HSIEs

$\psi^{[hs]}(t) \approx \psi_n^{*[hs]}(t) = (\psi_{1n}^{*\,[hs]}(t), \psi_{2n}^{*\,[hs]}(t), \ldots, \psi_{Nn}^{*\,[hs]}(t))^T$,
$\phi^{[hs]}(t) \approx \phi_n^{*[hs]}(t) = (\phi_{1n}^{*\,[hs]}(t), \phi_{2n}^{*\,[hs]}(t), \ldots, \phi_{Nn}^{*\,[hs]}(t))^T$: Matrix of unknown function in case system of HSIEs

$k^{[c]}(x,t),\ \hat{k}^{[c]}(x,t),\ k^{[h]}(x,t)$: Kernel in CSIEs over $[-1,1]$, $[0,\infty)$ and HSIEs respectively

$g^{[c]}(x)$, $g^{[h]}(x)$: Known function in CSIEs and HSIEs respectively

$\eta_j^{[cs]}(x)$, $g_j^{[cs]}(x)$, $g_j^{[hs]}(x)$, $\hat{g}_{jr}^{[hs]}$, $k_{ij}^{[hs]}(x,t)$, $p_{ij}^{[hs]}(x,t)$: Known function in system of CSIEs and HSIEs respectively

$D_j^{[cs]}(t)$: $N \times N$ matrix of functions where N is number of equations in system of CSIEs

$g^{[cs]}(x) = (g_1^{[cs]}(x), g_2^{[cs]}(x), \ldots, g_N^{[cs]}(x))^T$: $N \times 1$ matrix of known functions where N is number of equations in system of CSIEs

$g^{[hs]}(x) = (g_1^{[hs]}(x), g_2^{[hs]}(x), \ldots, g_N^{[hs]}(x))^T$, $f^{[hs]}(x) = (f_1^{[hs]}(x), f_2^{[hs]}(x), \ldots, f_N^{[hs]}(x))^T$: $N \times 1$ matrices of known functions where N is number of equations in system of HSIEs

$a_j^{[c]}$, $\hat{a}_j^{[c]}$, $a_j^{[h]}$ ($j = 0, 1, 2, \cdots, n)$: Unknown constant coefficient in CSIEs ovre $[-1,1]$, $[0,\infty)$ and HSIEs respectively

$a_{jk}^{[cs]}$, $a_{jk}^{[hs]}$: Unknown constant coefficient in case of system of CSIEs and and in case of system of HSIEs respectively

$\mathcal{R}^{[c]}$, $\mathcal{R}^{[h]}$, $\mathcal{R}^{[cs]}$, $\mathcal{R}^{[hs]}$: Residual error in CSIEs, HSIEs, system of CSIEs and system of HSIEs respectively

$d_i^{[c]}$, $b_{rq}^{[c]}$, $g_q^{[c]}$, $\hat{h}_i^{\ [c]}$, $\hat{b}_{rq}^{[c]}$, $\hat{g}_q^{[c]}$: Constants in case of CSIEs

$d_i^{[h]}$, $b_{rq}^{[h]}$, $g_q^{[h]}$: Constants in case of HSIEs

$d_i^{[cs]}$, $b_{rq}^{[cs]}$, $g_q^{[cs]}$: Constants in case of system of CSIEs

$d_i^{[hs]}$, $b_{rq}^{[hs]}$, $g_q^{[hs]}$: Constants in case of system of HSIEs

$S^{[c]}, K^{[c]}, Q_n{}^{[c]}, P_n{}^{[c]}, R_n{}^{[c]}$: Operators (In case of CSIEs $[-1,1]$)

$S^{[cs]}, Q_n{}^{[cs]}, P_n{}^{[cs]}, R_n{}^{[cs]}, P_N^{n\,[cs]}, R_N^{n\,[cs]}, Q_N^{n\,[cs]}$: Operators (In case of system of CSIEs $[-1,1]$)

$S^{[h]}, K^{[h]}, Q_n{}^{[h]}, P_n{}^{[h]}, R_n{}^{[h]}$: Operators (In case of HSIEs)

$\mathbb{K}^{[hs]}, K_{ij}^{[hs]}, \hat{K}_{ji}^{[hs]}, \hat{K}^{[hs]}, \mathbb{S}^{[hs]}, S^{[hs]}, Q_n{}^{[hs]}, P_n{}^{[hs]}, R_n{}^{[hs]}, P_N^{n\,[hs]}, R_N^{n\,[hs]}, Q_N^{n\,[hs]}$: Operators (In case of system of HSIEs)

$A^{[c]}$, $B^{[c]}$, $B_1^{[c]}$, $G^{[c]}$, $A^{[cs]}$, $\beta^{[cs]}$, $\hat{D}^{[cs]}$, $\hat{C}^{[cs]}$, $\hat{E}^{[cs]}$, $A^{[h]}$, $B^{[h]}$, $B_1^{[h]}$, $G^{[h]}$, $\hat{D}^{[hs]}$, $\hat{C}^{[hs]}$, $\hat{E}^{[hs]}$, $G^{[hs]}$, $A^{[hs]}$: Matrices

x, y, z, t : Independent variables

$e_j(t);\ j = 0, 1, \ldots, n$: Orthonormalized Legendre polynomial of degree j

$E = span\{e_j(t)\}_{j=0}^{n}$

$u(t), v(t),\ w_1(t),\ w_2(t),\ \hat{\theta}(z)\ \hat{h}(z,y),\ \hat{f}(y),\ v_k^{[c]}(x),\ \hat{v}_k^{[c]}(x),\ v_k^{[h]}(x),\ {v_k}^{[cs]}(x),\ {v_k}^{[hs]}(x),\ v(y),\ h(x),$ $\hat{A}_j(x,t)$: Functions

λ, ${c_i}^{[c]}$, ${\hat{c}_i}^{[c]}$, ${c_i}^{[h]}$, ${c_i}^{[cs]}$, ${c_i}^{[hs]}$: Constants

M_1, M_2 : Mediums

ν_1, ν_2 : The Poisson's ratios for mediums M_1 and M_2 respectively

Chapter 1

Introduction

1.1 Background and objective

For more than 20 years, a substantial demand has been increased for the use of singular integral equations in various fields of science [25, 39, 92, 95, 96, 120, 128] and engineering like fluid dynamics [10, 56], elasticity [27, 93], electric field [110], acoustic waves [53], fracture mechanics [16], aeronautics [41], electromagnetic [133] etc. The singular integral equations act as a convenient tool for the solution of most of the problems arising in above mentioned fields such as electromagnetic waves reflection problem [50] through irregular medium interface, interference or interaction problems [7], stationary linear problem of ideal fluid flow around a finite span wing [10], problem of finding the distribution of stresses around a Griffith crack [83], electromagnetic scattering problems [107] through an orthotropic medium etc.

Various type of singular integral equations are available in literature such as logarithmic singular integral equation [13], Abel's singular integral equations [121], Cauchy singular integral equations [42], hypersingular integral equations [83] etc. The singular integral equations with Abel's kernel were first encountered by Abel [83] in 1825 during the determination of the shape of the curve at the given end points along which a particle moves under the influence of gravity alone in a given interval of time. In 1963, Peters [105] has used the solution of Abel's integral equation, to derive the analytic solution of Cauchy singular integral equations (CSIEs) for the case when the equations contains only singular integral, that is, $k(x, t) = 0$ in equation (1.1). In this thesis, we focus exclusively on Cauchy and hypersingular Fredholm integral equations of the form respectively

$$\fint_{-1}^{1} \frac{u(t)}{(t-x)} dt + \int_{-1}^{1} k(x,t)u(t)dt = g(x), \quad | x |< 1, \tag{1.1}$$

and

$$\fint_{-1}^{1} \frac{u(t)}{(t-x)^2} dt + \int_{-1}^{1} k(x,t)u(t)dt = g(x), \quad | x |< 1, \tag{1.2}$$

where $k(x,t)$ is a known Hölder continuous function over $[-1,1] \times [-1,1]$, $g(x)$ is a known Hölder continuous function over the interval $[-1,1]$ and $u(t)$ is an unknown function to be determined.

If the kernel $k(x,t) = 0$ in equations (1.1) and (1.2), then these are known as dominant [45] equations. In equation (1.1), the integral appearing on left hand side is having a singularity at $t = x$ and it does not exist as a Riemann integral since the integrand becomes unbounded at $t = x$. Further, this integral does not exist as an improper integral too, but if we consider in the sense of Cauchy principal value (CPV), the integral does exist.

The investigations of Cauchy singular integral equations by various mathematicians such as Gakhov [39], Muskhelishvili [92], Vekua [127], Ivanov [54], Pogorzelski [106], Elliott [29] etc., have had a huge impact on the further development of the general theory of singular integral equations. These equations have many applications in various fields of science [87, 92, 93] and engineering such as aerodynamics [70], fracture mechanics [32], potential theory [18], electromagnetic [107] etc. The system of these equations is equally important due to their occurrence during the formulation of many boundary value problems of science and engineering [31, 32, 70, 92]. A complete treatment of the analytical methods for the solution of CSIEs is given by Gakhov [39] and Muskhelishvilli [92]. Although, an analytic solution for CSIEs of first kind having only dominant part is derived by Peters [105] in 1963, with the aid of solution of Abel's integral equation, but it is not always attainable due to the presence of singularity in the solution itself. Therefore, researchers have been investigating various numerical methods to find their approximate solution. For instance, Eshkuvatov et al. [33] have proposed an efficient approximate method for solving the Cauchy type singular integral equation of the first kind over a finite interval with the help of Chebyshev polynomials of the first kind, second kind, third kind and fourth kind. Kim used inverse method [63] to find approximate solution of CSIEs. Gong has used Galerkin method [46], Mandal [81] and Junghanns [57] have used collocation method etc., to find the approximate solution of CSIEs. Despite of the availability of various numerical methods, construction and justification of new numerical schemes to find the approximate solution of CSIEs, is still a topic of considerable interest. Also, in last few years, researchers have proposed various new methods to find the approximate solution of CSIEs such as fast multipole method [130], reproducing kernel Hilbert space (RKHS) method [28], collocation technique based on Bernstein polynomials [115] etc., to get numerical solution of Cauchy singular integral equations. For system of Cauchy type singular integral equations, Bonis and Laurita [26] have proposed a quadrature method and Turhan [125] has used Chebyshev polynomial based method for the solution of the system of Cauchy type singular integral equations.

The hypersingular integral equations (HSIEs) which are defined in equation (1.2) are as important as CSIEs due to their significant role in different areas of applied science and engineering [70]. In equation (1.2), the definition of CPV for the integral, having the factor $\frac{1}{(t-x)^2}$, fails to exist due to the higher singularity. Hence, to handle this kind of singularity, in 1952, the French mathematician Hadamard [48] introduced the concept of finite-part integral. The Hadamard finite part integral (HFP) is used to define those integrals where even CPV integral does not exist. The concept of finite-part integrals was firstly introduced by Hadamard [48], but after many years later, for the evaluation of finite-part singular integrals, in 1975, Kutt [69] has proposed some numerical formulas. Later in 1983, Golberg [43] has studied the convergence of several algorithms for solving finite-part singular integrals. Kaya and Erdogan [60, 61] have explored complicated problems of elasticity as well as fracture mechanics which are reducible to the solution of hypersingular integral equations and their system. The hypersingular integral equations are becoming increasingly important due to their applications in various fields such as fluid mechanics [103], fracture mechanics [86] and acoustics [77], etc. The HSIEs also play an important role in studying many problems. For instance, the problem of circulation distribution of a finite span wing [8, 88] which arises in the theory of incompressible flow and the problem of electromagnetic scattering by three-dimensional anisotropic media [114] etc., are formulated as HSIEs. Further, many boundary value problems, in particular, the reformulation of the exterior Neumann problem in scattering theory [3] leads to hypersingular equation or their system. Martin [85] has given a plenty of application areas such as acoustics, potential theory, elastostatics and hydrodynamics where the hypersingular integral equations quite naturally occur. These HSIEs arise during the formulation of various problems appearing in the field of applied mathematics, mathematical physics, engineering [5, 20, 22] etc., have been explored by many researchers using different analytical methods and approximation techniques. For instance, complex variable function method [14], boundary element method [64], polynomial approximation [99], modified Adomian decomposition [79], etc. But search for the methods to find their approximate solution which give better results in some sense as compare to already exist methods, is still a topic of exposition for many researchers. For last few years, various new numerical methods have been proposed such as Legendre multiwavelets [104] method, modified homotopy perturbation method [34], best mean-square approximation method [2], piecewise linear approximations [118] etc., to find the approximate solution of HSIEs. Recently, Kostenko [65] has proposed a numerical method which includes regularization of operators, interpolation polynomials and quadrature formulas to find the approximate solution of system of HSIEs of second kind. Similar to Cauchy singular integral equations, the analytic solution of hypersingular integral equations is also available for the case when the equations are having only dominant part, that is, when the kernel $k(x,t)$ is identically zero. However, this available analytic solution practically is of limited use since the operations involved in calculations of analytic solution can rarely be carried out exactly due to the presence of singularity. Also, there are many physical problems which are formulated as singular integral equations [19, 21, 36] in which the kernel is not identically zero. Therefore, it is required to find their approximate solution. Although, as we have mentioned above that various methods are available to find the approximate solution of hypersingular integral equations. However, search for better methods than available methods in some sense, is still an interesting as well as a challenging topic to explore. Also, the literature on numerical methods to find the approximate solution for the system of singular integral equations of first kind is still scarce. In this thesis, we propose residual based Galerkin method with Legendre polynomial as basis function to find the approximate solution of CSIEs, HSIEs and their systems. We show the convergence of sequence of approximate solutions, which are obtained with the aid of our proposed method, to the exact solution. Moreover, we also validate all the derived theoretical results with the help of numerical examples.

1.1.1 Basic definitions and theorem

In this section, we define some useful definitions and important results [49, 73] which are essential in order to understand the error analysis of the proposed method of solution.

Definition 1.1.1. Legendre polynomial: The n^{th} degree Legendre polynomials over the interval $[-1,1]$ are defined as

$$p_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n, \tag{1.3}$$

where n is a non-negative integer.

Definition 1.1.2. Orthogonal property of Legendre polynomials: The orthogonal property of Legendre polynomials over the interval $[-1,1]$ makes them very useful and it can be stated as follows: let $p_m(t)$ and $p_n(t)$ be Legendre polynomials of degree m and n respectively, then we have

$$\int_{-1}^{1} p_m(t)p_n(t)dt = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{2}{2n+1} & \text{if } m = n, \end{cases} \tag{1.4}$$

where m, n are non-negative integers.

Definition 1.1.3. Hölder continuous function: A Hölder continous [92] function $u(t)$ is the function which satisfies the following condition:

$$\mid u(t) - u(x) \mid \leq c \mid t - x \mid^{\alpha}, \forall\ \ t, x \in \Omega(u), \tag{1.5}$$

where c denotes a non-negative real constant, α is an exponent of Hölder condition such that $0 < \alpha \leq 1$ and $\Omega(u)$ stands for the domain of the function u.

Definition 1.1.4. Cauchy principal value (CPV): If $u(t) \in C^{0,\alpha}(-1,1)$, then

$$CPV\int_{-1}^{1}\frac{u(t)}{t-x}dt = ⨍_{-1}^{1}\frac{u(t)}{t-x}dt = \lim_{\epsilon\to 0^+}\left(\int_{-1}^{x-\epsilon}\frac{u(t)}{t-x}dt + \int_{x+\epsilon}^{1}\frac{u(t)}{t-x}dt\right), \tag{1.6}$$

where $|x| < 1$ and $C^{0,\alpha}(-1,1)$ is the space of functions which are Hölder continuous on the interval $(-1,1)$ with the exponent $0 < \alpha \leq 1$.

The function $u(t)$ needs to be a Hölder continuous function over the interval $(-1,1)$. This requirement of Hölder continuity is necessary in order to ensure the existence of Cauchy principal value [17] of integral equation (1.6).

Definition 1.1.5. Hadamard finite-part integral (HFP): The HFP definition defined below is given by Monogato [88] which is a simplified form of the definition of Hadamard finite part integral [48]. For the function $u(t) \in C^{1,\alpha}(-1,1)$, the Hadamard finite part integral is defined as

$$\text{HFP} \int_{-1}^{1} \frac{u(t)}{(t-x)^2} dt = \int_{-1}^{1} \frac{u(t) - (t-x)u^{'}(x) - u(x)}{(t-x)^2} dt \\ - u^{'}(x) \log \left| \frac{1-x}{1+x} \right| - u(x) \Big[\frac{1}{1-x} + \frac{1}{1+x} \Big], \quad | x | < 1, \tag{1.7}$$

where $C^{1,\alpha}(-1,1)$ stands for the space of functions whose first derivatives are Hölder continuous on the interval $(-1,1)$ with the exponent $0 < \alpha \leq 1$. The requirement that the function $u(t) \in C^{1,\alpha}(-1,1)$, is necessary in order to regularized hypersingular integrals [17] appearing in equation (1.7). Few more properties of finite-part integral can be found in [89].

Definition 1.1.6. Well-posedness conditions by Hadmard: A mathematical problem is called well-posed (in the sense of Hadmard [48, 122]) if it satisfies the following conditions:

1. it has a solution (existence).
2. the solution is unique (uniqueness).
3. the solution depends continuously on given input data (stability).

Theorem 1.1.1. ***Bounded inverse theorem:*** *Let H_1, H_2 be two Hilbert spaces and $\mathcal{T} : H_1 \to H_2$ be a bijective bounded linear operator. Then the inverse linear map $\mathcal{T}^{-1} : H_2 \to H_1$ is also bounded (for proof of the theorem, please refer [68]).*

1.2 Outline of the thesis

The thesis is structured as follows: In **Chapter 2**, we propose a residual based Galerkin method to find the approximate solution of Cauchy singular integral equations with index zero over the finite interval $[-1,1]$. The test examples are given for illustration of proposed numerical method. The convergence of sequence of approximate solutions of CSIEs, is shown. Further, the error bound is derived as well as verified in all the numerical examples.

In **Chapter 3**, the problem of finding numerical solution for a system of Cauchy type singular integral equations of first kind with index zero, is considered. A residual based Galerkin method is proposed with Legendre polynomials as basis functions to find its numerical solution. The theoretical error bound is derived which can be used to obtain any desired accuracy in the approximate solution of system of Cauchy singular integral equations. The convergence of sequence of approximate solutions to the exact solution of system of CSIEs is proved. The derived theoretical error bound is also validated with the help of numerical examples.

Chapter 4 proposes a residual based Galerkin method with Legendre polynomials as a basis functions to find the approximate solution of hypersingular integral equations. The convergence of sequence of approximate solutions is proved and error bound is obtained theoretically. The validation of all derived theoretical results and implementation of proposed method is also shown with the aid of numerical illustrations.

In **Chapter 5**, we propose a residual based Galerkin method to calculate approximate solution for system of hypersingular integral equations which occurs quite naturally during the formulation of many problems of applied science and engineering. The convergence of sequence of the approximate solutions is proved which helps to derive theoretical error bound for the error between the exact and approximate solution. An application of the proposed method in finding numerical solution of hypersingular integral equation over curves is also shown. Finally, the derived theoretical error bound is numerically calculated and validated with the help of numerical examples.

Chapter 6 concludes my research work described in this thesis as well as gives the direction for future work in the field of singular integral equations.

We have used *Wolfram Mathematica 11.0* for all the numerical calculations.

Chapter 2

Numerical solution of Cauchy singular integral equations

2.1 Introduction

Singular integral equations (SIEs) with Cauchy kernel occur frequently in mixed boundary value problems for partial differential equations. The stationary linear problem of ideal fluid flow around a finite span wing is reducible to Cauchy singular integral equation (see [10] and the references therein). An interesting and comprehensive survey of applications of SIEs can be found in [25, 128]. It is worth noting that the methods of exact and approximate solutions of SIEs have been still a challenging problem for the research community. In this chapter, we consider the numerical solution of Cauchy singular integral equations (CSIEs) of first kind

$$\fint_{-1}^{1} \frac{\chi^{[c]}(t)}{t-x}dt - \int_{-1}^{1} k^{[c]}(x,t)\chi^{[c]}(t)dt = g^{[c]}(x), \mid x \mid < 1, \tag{2.1}$$

with the boundary conditions

$$\chi^{[c]}(t) = \begin{cases} 0 & \text{if } t = 1, \\ \text{unbounded} & \text{if } t = -1, \end{cases} \tag{2.2}$$

where $\chi^{[c]}(t)$ is unknown function. The functions $g^{[c]}(x)$ and $k^{[c]}(x,t)$ are known real valued Hölder continuous over the interval $[-1,1]$ and $[-1,1]\times[-1,1]$ respectively. The first integral in equation(2.1) is understood to be exist in the sense of CPV. Also, the function $\chi^{[c]}(t)$ is assumed to be a Hölder continuous in order to ensure the existence [17] of Cauchy principal value. The Cauchy singular integral equations have many applications in the field of aerodynamics [70], fracture mechanics [32], neutron transport [87] etc. CSIEs are also widely used in many areas of mathematical physics such as potential theory [18], elasticity problems as well as electromagnetic scattering [134]. The numerical methods which are developed for one-dimensional Cauchy type singular integral equations include: Galerkin's method [9, 116], collocation method [81], quadrature method [62, 123], inverse method [63], Sinc approximations [1] etc. In this chapter, we propose a residual based Galerkin's method using Legendre polynomial as basis function to find the numerical solution of equation (2.1).

2.2 Method of solution for CSIEs over $[-1, 1]$

In order to find the approximate solution of Cauchy singular integral equation (2.1), we write the unknown function [42] as

$$\chi^{[c]}(t) = \sqrt{\frac{1-t}{1+t}}\xi^{[c]}(t), \tag{2.3}$$

where $\xi^{[c]}(t)$ is an unknown function of $t \in [-1, 1]$. Now using equation (2.3) in equation (2.1), we obtain

$$\fint_{-1}^{1} \sqrt{\frac{1-t}{1+t}}\frac{\xi^{[c]}(t)}{t-x}dt - \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}}k^{[c]}(x,t)\xi^{[c]}(t)dt = g^{[c]}(x), | x < 1. \tag{2.4}$$

We approximate the function $\xi^{[c]}(t)$ by orthonormalized Legendre polynomials as follows:

$$\xi^{[c]}(t) \approx \xi_n^{*[c]}(t) = \sum_{j=0}^{n} a_j^{[c]} e_j(t), \tag{2.5}$$

where $\{e_j(t)\}_{j=0}^n$ denotes the set of $(n+1)$ orthonormalized Legendre polynomials on $[-1, 1]$ and $a_j^{[c]}; j = 0, 1, 2, \ldots, n$ are unknown constant coefficients.

To get the the values of unknown coefficients $a_j^{[c]}$, we use residual based Galerkin's method. On using the above approximation for $\xi^{[c]}(t)$ in equation (2.4), the residual error $\mathcal{R}^{[c]}(x, a_0^{[c]}, a_1^{[c]}, a_2^{[c]}, ..., a_n^{[c]})$ will be

$$\mathcal{R}^{[c]}(x, a_0^{[c]}, a_1^{[c]}, a_2^{[c]}, ..., a_n^{[c]}) = \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}}\frac{\xi_n^{*[c]}(t)}{t-x}dt - \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}}k^{[c]}(x,t)\xi_n^{*[c]}(t)dt - g^{[c]}(x), | x |< 1. \tag{2.6}$$

In Galerkin's method, this residual error $\mathcal{R}^{[c]}(x, a_0^{[c]}, a_1^{[c]}, a_2^{[c]}, ..., a_n^{[c]})$ is assumed to be orthogonal to the space spanned by orthonormal polynomials $\{e_j(x)\}_{j=0}^n$, that is, we have

$$\langle \mathcal{R}^{[c]}(x, a_0^{[c]}, a_1^{[c]}, a_2^{[c]}, ..., a_n^{[c]}), e_j \rangle_{L^2} = 0, \ \forall j = 0, 1, 2, \cdots, n. \tag{2.7}$$

From equation (2.7), we get a system of $(n+1)$ linear algebraic equations in $(n+1)$ unknowns. The explicit form of equation (2.7) is as follows:

$$\begin{aligned}\sum_{r=0}^{n} a_r^{[c]} \Big(\int_{-1}^{1} \fint_{-1}^{1} \sqrt{\frac{1-t}{1+t}}\frac{1}{t-x}e_r(t)e_q(x)dtdx - \int_{-1}^{1} \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}}k^{[c]}(x,t)e_r(t)e_q(x)dtdx \Big) \\ = \int_{-1}^{1} g^{[c]}(x)e_q(x)dx, \qquad q = 0, 1, 2, \ldots, n,\end{aligned} \tag{2.8}$$

where

$$b_{rq}^{[c]} = \int_{-1}^{1} \fint_{-1}^{1} \sqrt{\frac{1-t}{1+t}} \frac{1}{x-t} e_r(x) e_q(t) dt dx - \int_{-1}^{1} \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}} k^{[c]}(x,t) e_r(t) e_q(x) dt dx, \ r, q = 0, 1, 2, \ldots, n,$$

$$g_q^{[c]} = \int_{-1}^{1} g^{[c]}(x) e_q(x) dx, \quad q = 0, 1, 2, \ldots, n.$$

Finally, the system (2.8) can be written in matrix form as

$$B^{[c]T} A^{[c]} = B_1^{[c]} A^{[c]} = G^{[c]}, \tag{2.9}$$

where

$$B_1^{[c]} = B^{[c]T}, \ B^{[c]} = \begin{pmatrix} b_{00}^{[c]} & b_{01}^{[c]} & \ldots & b_{0n}^{[c]} \\ b_{10}^{[c]} & b_{11}^{[c]} & \ldots & b_{1n}^{[c]} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n0}^{[c]} & b_{n1}^{[c]} & \ldots & b_{nn}^{[c]} \end{pmatrix}, \ A^{[c]} = \begin{pmatrix} a_0^{[c]} \\ a_1^{[c]} \\ \vdots \\ a_n^{[c]} \end{pmatrix}, \ G^{[c]} = \begin{pmatrix} g_0^{[c]} \\ g_1^{[c]} \\ \vdots \\ g_n^{[c]} \end{pmatrix}. \tag{2.10}$$

After solving the system (2.9) which is obtained as a result of approximation of equation (2.4), we get the values of $a_j; \ j = 0, 1, 2, \ldots, n$. Now, finally substituting the values of $a_j; \ j = 0, 1, 2, \ldots, n$, in equation (2.5), we get the approximate solution of equation (2.4) and hence, for equation (2.1).

This completes the description of proposed method which we use to find the approximate solution of equation (2.1).

2.3 Error analysis

In this section, we show the convergence of sequence of approximate solutions in L^2 space and we also derive the error bound for the error between the exact and approximate solution. We write the equation (2.4) in operator form

$$(S^{[c]} - K^{[c]})\xi^{[c]}(x) = g^{[c]}(x), \quad | x |< 1. \tag{2.11}$$

In the above equation (2.11), the operators $S^{[c]}$ and $K^{[c]}$ are defined as

$$S^{[c]}\xi^{[c]}(x) = \fint_{-1}^{1} \sqrt{\frac{1-t}{1+t}} \frac{\xi^{[c]}(t)}{t-x} dt, \tag{2.12}$$

$$K^{[c]}\xi^{[c]}(x) = \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}} k^{[c]}(x,t)\xi^{[c]}(t) dt. \tag{2.13}$$

We assume that

$$\int_{-1}^{1} \int_{-1}^{1} \Big(\sqrt{\frac{1-t}{1+t}} k^{[c]}(x,t)\Big)^2 dt dx < \infty. \tag{2.14}$$

Now we define the function space $L^2[-1, 1]$ as

$$L^2[-1,1] = \left\{u : [-1,1] \to \mathbb{R} : \int_{-1}^{1} (u(t))^2 dt < \infty\right\}. \tag{2.15}$$

The space defined above is a Hilbert space of all square integrable functions over the interval $[-1,1]$ with the following norm $\|.\|_{L^2}^2$ and inner product $\langle ., . \rangle_{L^2}$

$$\|u\|_{L^2} = \left(\int_{-1}^{1} \Big(u(t)\Big)^2 dt\right)^{1/2}, \text{ for } u(t) \in L^2, \tag{2.16}$$

$$\langle u, v\rangle_{L^2} = \int_{-1}^{1} u(t)v(t)dt, \text{ for } u(t), v(t) \in L^2. \tag{2.17}$$

We define another function space say, $M^{[c]}$ such that

$$M^{[c]} = \{u(t) \in L^2 : \sum_{i=0}^{\infty} (d_i^{[c]})^2 \langle u, e_i\rangle_{L^2}^2 < \infty\}. \tag{2.18}$$

where $d_i^{[c]} = \sqrt{\sum_{j=0}^{\infty} \langle S^{[c]} e_i, e_j\rangle_{L^2}^2}$. Following results [12], the function $S^{[c]} e_i(x)$ is a polynomial of degree at the most of i and therefore, $d_i^{[c]}\ \ \forall\, i$ will be a finite number. This space $M^{[c]}$ is a subspace of L^2 space which is made into a Hilbert space with respect to the following norm $\|\cdot\|_{M^{[c]}}$ and inner product $\langle \cdot, \cdot\rangle_{M^{[c]}}$

$$\| u \|_{M^{[c]}}^2 = \sum_{i=0}^{\infty} (d_i^{[c]})^2 \langle u, e_i\rangle_{L^2}^2, \text{ for } u(t) \in M^{[c]}, \tag{2.19}$$

$$\langle u, v\rangle_{M^{[c]}} = \sum_{i=0}^{\infty} (d_i^{[c]})^2 \langle u, e_i\rangle_{L^2} \langle v, e_i\rangle_{L^2}, \text{ for } u(t),\ v(t) \in M^{[c]}. \tag{2.20}$$

Let $v_k^{[c]}(x) = \frac{e_k(x)}{d_k^{[c]}}$, then $\| v_k^{[c]} \|_{M^{[c]}} = 1$.

This set $\{v_k^{[c]}(x)\}_{k=0}^{\infty}$ forms complete orthonormal basis for the Hilbert space $M^{[c]}$, that is, if $u \in M^{[c]}$, then we have

$$u(x) = \sum_{k=0}^{\infty} \langle u, v_k^{[c]}\rangle_{M^{[c]}} v_k^{[c]}(x). \tag{2.21}$$

Now with the aid of results in [12], we obtain

$$S^{[c]} e_n(x) = \sum_{i=0}^{n} \alpha_i^{[c]} e_i(x), \tag{2.22}$$

where the coefficients $\alpha_i^{[c]} = \langle S^{[c]} e_n, e_i\rangle_{L^2},\ \ i = 0, 1, 2, \ldots, n$. Using the above result (2.22), the operator

$S^{[c]} : M^{[c]} \to L^2$, can be extended as a bounded linear operator and defined as

$$S^{[c]}\xi^{[c]}(x) = \sum_{i=0}^{\infty} \langle \xi^{[c]}, e_i \rangle_{L^2} \sum_{j=0}^{i} \langle S e_i^{[c]}, e_j \rangle_{L^2} e_j(x) \in L^2[-1,1]. \tag{2.23}$$

Using the orthogonal property of Legendre polynomial, we find the norm of operator $S^{[c]}$

$$\| S^{[c]}\xi^{[c]} \|_{L^2}^2 = \sum_{i=0}^{\infty} (d_i^{[c]})^2 \langle \xi^{[c]}, e_i \rangle_{L^2}^2 = \| \xi \|_{M^{[c]}}^2 . \tag{2.24}$$

Therefore, using equation (2.24), we obtain

$$\| S^{[c]} \| = 1. \tag{2.25}$$

Also, the operator $S^{[c]}$ from $M^{[c]} \to L^2$ is one-one and onto [42]. Hence, the operator $(S^{[c]})^{-1} : L^2 \to M^{[c]}$ exists as a bounded linear operator by using *Bounded Inverse Theorem* [68]. This operator $(S^{[c]})^{-1}$ is defined as

$$(S^{[c]})^{-1}\xi^{[c]}(x) = \sum_{i=0}^{\infty} \frac{\langle \xi^{[c]}(x), e_i(x) \rangle_{L^2}}{d_i^{[c]}} e_i(x). \tag{2.26}$$

Using the above definition (2.26), we find the norm of linear operator S^{-1}

$$\| (S^{[c]})^{-1} \| = 1. \tag{2.27}$$

The equation (2.11) has a unique solution if and only if the operator $(S^{[c]} - K^{[c]})$ has a bounded inverse. We assume that this condition holds on from now on. We consider the mapping $Q_n^{[c]} : L^2 \to L^2$ which is defined as

$$Q_n^{[c]}\xi^{[c]}(x) = \sum_{i=0}^{n} \langle \xi^{[c]}, e_i \rangle_{L^2} e_i(x), \tag{2.28}$$

where $Q_n^{[c]}$ is the operator of orthogonal projection and n is the degree of orthonormalized Legendre polynomial. With the aid of equation (2.7), we obtain

$$Q_n^{[c]}\Big((S^{[c]} - K^{[c]})\xi_n^{*[c]}(x) - g^{[c]}(x) \Big) = 0. \tag{2.29}$$

Since the function $\xi^{*[c]}(x)$ defined in equation (2.5), is a polynomial. Therefore, with the help of the formulas given in [12], the function $S^{[c]}\xi_n^{*[c]}(x)$ will be a polynomial and by the definition of operator $Q_n^{[c]}$, we obtain

$$Q_n^{[c]} S^{[c]} \xi_n^{*[c]}(x) = S^{[c]} \xi_n^{*[c]}(x), \tag{2.30}$$

and hence, the equation (2.29) becomes

$$S^{[c]}\xi_n^{*[c]}(x) - Q_n^{[c]}K^{[c]}\xi_n^{*[c]}(x) = Q_n^{[c]}g^{[c]}(x). \tag{2.31}$$

Since the operator $(S^{[c]})^{-1}$ is bounded and the operator $K^{[c]}$ is compact due to the condition defined in equation (2.14) and therefore, for all $n \geq n_0$, the operator $(S^{[c]} - Q_n^{[c]}K^{[c]})^{-1}$ exists as a bounded linear operator [45]. Hence, the equation (2.31) has a unique solution which is defined as

$$\xi_n^{*[c]}(x) = (S^{[c]} - Q_n^{[c]}K^{[c]})^{-1}Q_n^{[c]}g^{[c]}(x). \tag{2.32}$$

From equations (2.11) and (2.32), for all $n \geq n_0$, we have

$$\xi^{[c]}(x) - \xi_n^{*[c]}(x) = (S^{[c]} - Q_n^{[c]}K^{[c]})^{-1}\Big(g^{[c]}(x) - Q_n^{[c]}g^{[c]}(x) + K^{[c]}\xi^{[c]}(x) - Q_n^{[c]}K^{[c]}\xi^{[c]}(x)\Big). \tag{2.33}$$

Now taking $M^{[c]}$ norm on both the sides of equation (2.33), we obtain

$$\begin{aligned}\| \xi^{[c]} - \xi_n^{*[c]} \|_{M^{[c]}} \leq& \| (S^{[c]} - Q_n{}^{[c]}K^{[c]})^{-1} \|\| g^{[c]} - Q_n^{[c]}g^{[c]} \|_{L^2} \\ &+ \| (S^{[c]} - Q_n{}^{[c]}K^{[c]})^{-1} \|\| K^{[c]}\xi^{[c]}(x) - Q_n^{[c]}K^{[c]}\xi^{[c]}(x) \|_{L^2} .\end{aligned} \tag{2.34}$$

Due the compactness of operator $K^{[c]}$, we have $\| K^{[c]} - Q_n^{[c]}K^{[c]} \|_{L^2} \to 0$ as $n \to \infty$ [45]. Also, $\| g^{[c]} - Q_n^{[c]}g^{[c]} \|_{L^2} \to 0$ as $n \to \infty$. Therefore $\| \xi^{[c]} - \xi_n^{*[c]} \|_M^{[c]} \to 0$ as $n \to \infty$. Further, it is noticed that if $\xi^{[c]} \in M^{[c]}$, then we have

$$\| \xi^{[c]} \|_{L^2} \leq \| \xi^{[c]} \|_{M^{[c]}} . \tag{2.35}$$

On using equation (2.35) in equation (2.34), we finally obtain

$$\begin{aligned}\| \xi^{[c]} - \xi_n^{*[c]} \|_{L^2} \leq& \| (S^{[c]} - Q_n{}^{[c]}K^{[c]})^{-1} \|\| g^{[c]} - Q_n^{[c]}g^{[c]} \|_{L^2} \\ &+ \| (S^{[c]} - Q_n{}^{[c]}K^{[c]})^{-1} \|\| K^{[c]}\xi^{[c]}(x) - Q_n^{[c]}K^{[c]}\xi^{[c]}(x) \|_{L^2} .\end{aligned} \tag{2.36}$$

Hence, the convergence of sequence of approximate solutions $\{\xi_n^{*[c]}(x)\}_{n=0}^{\infty}$ to the exact solution $\xi^{[c]}(x)$ in L^2 space is proved. Also, the right hand side of equation (2.36) is an error bound for the error between the exact solution $\xi^{[c]}(x)$ and the approximate solution $\xi_n^{*[c]}(x)$.

2.3.1 Well-posedness

In this subsection, we verify the Hadamard well-posedness of problem (2.31). The existence the operator $(S^{[c]} - Q_n{}^{[c]}K^{[c]})^{-1}$ which is already shown above, implies that the problem (2.31) has a solution. We now show the uniqueness of the solution to the problem (2.31) with the aid of principle of contradiction.

Let us assume that the system (2.31) has two distinct solutions say y_1 and y_2. Then we have

$$S^{[c]}y_1(x) - Q_n{}^{[c]}K^{[c]}y_1(x) = Q_n{}^{[c]}g^{[c]}(x), \tag{2.37}$$

and

$$S^{[c]}y_2(x) - Q_n^{[c]}K^{[c]}y_2(x) = Q_n^{[c]}g^{[c]}(x). \tag{2.38}$$

Taking the difference of equations (2.37) and (2.38), we obtain

$$(S^{[c]} - Q_n^{[c]}K^{[c]})(y_1(x) - y_2(x)) = 0. \tag{2.39}$$

In Section 2.3, it is already proved that the operator $(S^{[c]} - Q_n^{[c]}K^{[c]})^{-1}$ exists as a bounded linear operator. Therefore on applying the inverse operator $(S^{[c]} - Q_n^{[c]}K^{[c]})^{-1}$ on both the sides of equation (2.39), we get

$$y_1(x) = y_2(x), \quad | x |< 1. \tag{2.40}$$

The above equation (2.40) contradicts our assumption. Hence, the problem (2.31) has a unique solution. Also, the boundedness of the inverse operator $(S^{[c]} - Q_n^{[c]}K^{[c]})^{-1}$ implies the continuity of $(S^{[c]} - Q_n^{[c]}K^{[c]})^{-1}$. This means that a small change in the given data will lead to a small change in the solution. Hence, the boundedness of the operator $(S^{[c]} - Q_n^{[c]}K^{[c]})^{-1}$ proves that the solution depends continuously on given data. Since the problem (2.31) satisfies all the conditions of well-posedness therefore it is a well-posed problem.

2.3.2 Existence and uniqueness of solution for the linear system

In this subsection, we show the existence and uniqueness of solution for linear system (2.9) which is obtained after applying the proposed method to the equation (2.1). We start the proof by defining the prolongation operator [45] $P_n^{[c]} : \mathbb{R}^{n+1} \to E$ as follows:

$$P_n^{[c]}G^{[c]} = \sum_{j=0}^{n} \langle g^{[c]}, e_j \rangle_{L^2} e_j(x) \in E, \tag{2.41}$$

where $\mathbb{R}^{n+1}$ is a real vector space [67] having $(n+1)$-tuples of real numbers as its vectors, $E = span\{e_j(x)\}_{j=0}^{n}$ and $G^{[c]}$ is same as defined in equation (2.10). Now using the definition of orthogonal projection Q_n, we obtain

$$Q_n^{[c]}g^{[c]}(x) = \sum_{j=0}^{n} \langle g^{[c]}, e_j \rangle_{L^2} e_j(x). \tag{2.42}$$

From equations (2.41) and (2.42), we have

$$P_n^{[c]}G^{[c]} = Q_n^{[c]}g^{[c]}(x), \quad g^{[c]}(x) \in L^2, \quad G^{[c]} \in \mathbb{R}^{n+1}, \quad | x |< 1. \tag{2.43}$$

We further define a restriction operator [45] $R_n^{[c]} : E \to \mathbb{R}^{n+1}$ as follows:

$$R_n^{[c]}\xi_n^{*[c]}(x) = (\langle \xi_n^{*[c]}, e_0 \rangle_{L^2}, \langle \xi_n^{*[c]}, e_1 \rangle_{L^2}, \dots \langle \xi_n^{*[c]}, e_n \rangle_{L^2})^T \in \mathbb{R}^{n+1}, \tag{2.44}$$

where the function $\xi_n^{*[c]}(x)$ is same as defined in equation (2.5). On using the orthogonal property of Legendre polynomials in equation (2.5), we get

$$a_j^{[c]} = \langle \xi_n^{*[c]}, e_j \rangle_{L^2}, \; j = 0, 1, \ldots, n. \tag{2.45}$$

Therefore, from equations (2.44) and (2.45), we obtain

$$R_n^{[c]} \xi_n^{*[c]}(x) = A^{[c]}, \tag{2.46}$$

where the matrix $A^{[c]}$ is same as defined in equation (2.10). Since system (2.32) has a unique solution $\xi_n^{*[c]}(x)$ due to the existence of bounded linear operator $(S^{[c]} - Q_n^{[c]} K^{[c]})^{-1}$. Therefore, from equation (2.46), the solution $A^{[c]}$ of system (2.9) exists uniquely. This completes the proof of existence and uniqueness of solution for system of linear algebraic equation (2.9).

2.4 Illustrative examples

In this section, we find the approximate solution of numerical examples by using the proposed method discussed in Section 2.2.

Example 2.1 Consider the Cauchy singular integral equation

$$\fint_{-1}^{1} \sqrt{\frac{1-t}{1+t}} \frac{\xi^{[c]}(t)}{t-x} dt - \frac{1}{2} \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}} (x - xt^2) \xi^{[c]}(t) dt = \frac{1}{\pi} \Big(xe^x - J_1(1) x^2 + 7x^5 \Big), \tag{2.47}$$

where $J_1(1)$ is Bessel function of first kind of order one. The exact solution of the above problem is not known. We find its approximate solution by using the method described in Section2.2. It is clear from Figure 2.1 that as n is increasing, the approximate solutions are coming closer to each other this shows that the error between the approximate solution and the exact solutions is keep on decreasing.

Table 2.1: The theoretical error bound in case of Example 2.1 for different values of n

Degree of Legendre polynomial	Error bound for $\|\xi^{[c]} - \xi_n^{*[c]}\|_{L^2}$
$n = 1$	1.21436
$n = 2$	1.19832
$n = 3$	2.51361×10^{-1}
$n = 4$	2.51042×10^{-1}
$n = 5$	1.5630×10^{-4}
$n = 6$	1.2961×10^{-5}
$n = 7$	9.22537×10^{-7}
$n = 8$	5.75069×10^{-8}

Further, it can be seen from Table 2.1. that the error bound is decreasing as n is increasing which indicates that the sequence of approximate solutions is converging to the exact solution.

Table 2.2: The actual error and theoretical error bound in case of Example 2.2 for different values of n

Degree of Legendre polynomial	Actual error $\|\xi^{[c]} - \xi_n^{*[c]}\|_{L^2}$	Error bound for $\|\xi - \xi^{*[c]}\|_{L^2}$
$n = 1$	1.00668	3.58307
$n = 2$	2.40175×10^{-1}	1.62093
$n = 3$	2.10905×10^{-1}	1.02621
$n = 4$	0	0

Table 2.3: The theoretical error bound in case of Example 2.3 for different values of n

Degree of Legendre polynomial	Error bound for $\|\xi^{[c]} - \xi_n^{*[c]}\|_{L^2}$
$n = 1$	1.66512
$n = 2$	6.33927×10^{-1}
$n = 3$	3.47631×10^{-1}
$n = 4$	3.05371×10^{-2}
$n = 5$	1.75190×10^{-2}
$n = 6$	2.09064×10^{-4}
$n = 7$	1.19283×10^{-5}
$n = 8$	1.00000×10^{-8}

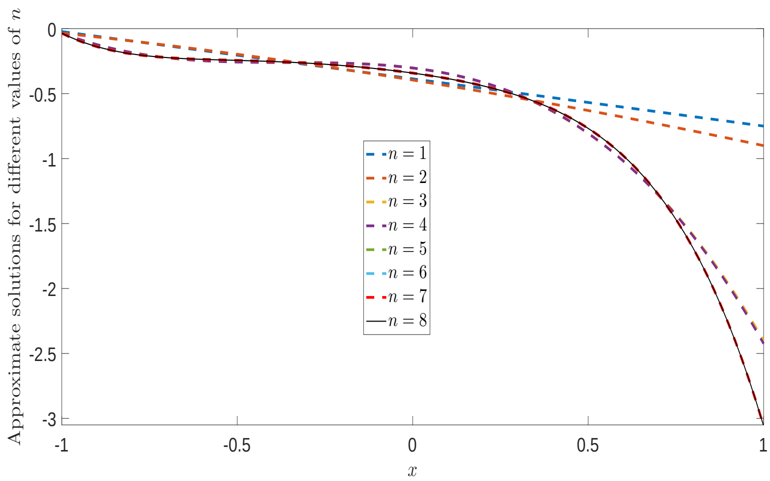


Figure 2.1: Comparison of approximate solution for different values of n in case of Example 2.1

Example 2.2 Consider the singular integral equation with Cauchy kernel

$$\int_{-1}^{1} \sqrt{\frac{1-t}{1+t}} \frac{\xi^{[c]}(t)}{t-x} dt - \frac{1}{5} \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}} (1+x^4)(1+t)^2 \xi^{[c]}(t) dt = \frac{1}{\pi} \left(\frac{5179}{40} + 4x - 5x^2 + 7x^3 + \frac{97x^4}{20} \right). \quad (2.48)$$

The exact solution of this problem is $\chi^{[c]}(x) = \frac{1}{\pi^2}\sqrt{\frac{1-x}{1+x}}\Big(-120 + x + \frac{3x^2}{2} + 7x^4\Big)$. Figure 2.2 shows that the approximate solution is the exact solution for $n = 4$. It is also shown in Table 2.2 that actual error satisfies the error bound which is calculated by using equation (2.36).

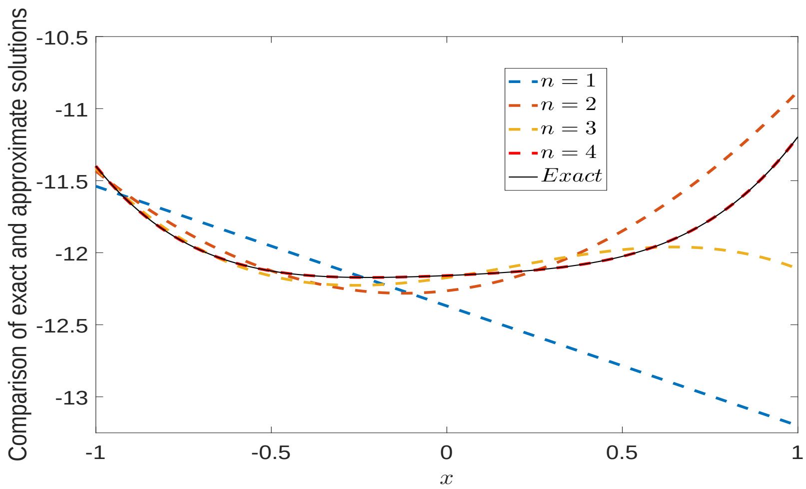


Figure 2.2: Comparison of exact solution and approximate solutions in case of Example 2.2

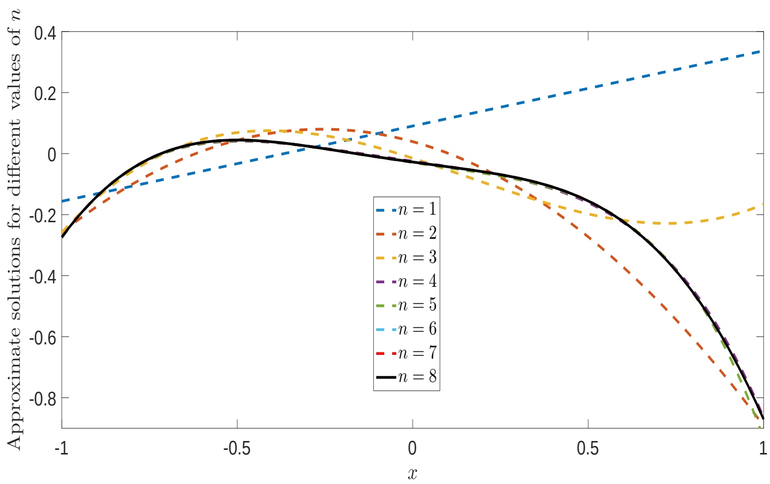


Figure 2.3: Comparison of approximate solution for different values of n in case of Example 2.3

Example 2.3 Consider a Cauchy singular integral equation

$$\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\frac{\xi^{[c]}(t)}{t-x}dt - \frac{1}{2}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}x(1+t)\xi^{[c]}(t)dt = \frac{1}{\pi}\Big(x - 5x^2 + 7x^3\Big)\sin x. \tag{2.49}$$

The exact solution of Example 2.3 is not known. We find its approximate solution by proposed method and the results are detailed in Table 2.3. Figure 2.3 shows that the approximate solutions are getting closer to each other as n is increasing which verifies the results (2.36). Table 2.3 implies that the sequence of approximate solutions is converging to the exact solution .

2.5 CSIEs over the half-line

In this section, we show that the numerical method described in Section 2.2 can be used to find the approximate solution of Cauchy singular integral equations over the half-line. These kind of equations occur quite naturally in the field of fracture mechanics [51, 91]. Also, the singular integral equation with Cauchy kernel over the half-line arises during the formulation of boundary value problems such as the problems occurring in the field of fracture mechanics [112], quantum mechanics [100], aerodynamics [124], flow and heat transfer theory [126] etc. The general theory related to these kind of singular integral equations can be found in [35, 39]. Our main objective in this section is to determine an approximate solution for a Cauchy singular integral equation of the form

$$\fint_0^\infty \frac{\hat{\theta}(z)}{z-y}dz - \int_0^\infty \hat{h}(z,y)\hat{\theta}(z)dz = \hat{f}(y),\ 0<y<\infty, \tag{2.50}$$

where $\hat{\theta}(z)$ is Hölder continuous [39] unknown function which vanishes at infinity and becomes unbounded at zero. The functions $\hat{f}(y)$ and $\hat{h}(z,y)$ are known real valued Hölder continuous over the interval $[0,\infty)$ and $[0,\infty)\times[0,\infty)$ respectively. The first integral in equation (2.50) is understood in the sense of Cauchy principal value [35] (CPV) which is defined as

$$CPV\int_0^\infty \frac{\hat{\theta}(z)}{z-y}dz = \fint_0^\infty \frac{\hat{\theta}(z)}{z-y}dz = \lim_{\epsilon\to 0^+}\left(\int_0^{y-\epsilon}\frac{\hat{\theta}(z)}{z-y}dz + \int_{y+\epsilon}^\infty \frac{\hat{\theta}(z)}{z-y}dz\right),\ 0<y<\infty. \tag{2.51}$$

In order to ensure the existence of CPV, the unknown function is assumed to be Hölder Continuous [39] function over the interval $(0,\infty)$. The analytic solution of equation (2.50) can be obtained (for details see chapter 14 of Mushkelishvili [92]). However, the analytic solution even if it is derived, will be of limited practical value, since the operations involved in such calculations, can rarely be carried out exactly. Hence, it is required to find its approximate solution. Also, various numerical methods to find the approximate solution of Cauchy singular integral equation over finite interval are available such as Galerkin's method [117, 129], collocation method [58], quadrature method [55], inverse method [63], real variable method [15], Sinc approximations [1] etc. However, the numerical methods to find the approximate solution of Cauchy singular integral equation over the half-line is still scarce. To the best of our knowledge only few methods are available which use different kind of quadrature rules such as interpolation quadrature on Chebyshev nodes [108, 109], Gauss-Jacobi [91], Gauss and Radau-Laguerre quadrature rules [52] to find the approximate solution of Cauchy singular integral equation over the half-line. The quadrature methods, in principle, are confined to the second kind equations [66]. Therefore, we propose a numerical method which can be used to find the approximate solution of singular integral equations of the first kind. This method first transform the singular integral equation over the half-line into a singular integral equation with Cauchy kernel over a finite interval $[-1,1]$. Then we use the numerical method explained in Section 2.2 to find the approximate solution of equation (2.50).

2.6 Method of solution for CSIEs over $[0,\infty)$

In order to find the approximate solution of equation (2.50), we start with the following transformation

$$\frac{1}{z-y}=\frac{1+y}{(1+z)(z-y)}+\frac{1}{1+z},\quad z=\frac{1+t}{1-t},\ y=\frac{1+x}{1-x}. \tag{2.52}$$

We use the above transformation (2.52) in equation (2.50) and write the unknown function $\hat{\chi}^{[c]}(t)=\hat{\theta}\Big(\frac{1+t}{1-t}\Big)$ which vanishes at $t=1$ and becomes unbounded at $t=-1$, as

$$\hat{\chi}^{[c]}(t)=\sqrt{\frac{1-t}{1+t}}\hat{\xi}^{[c]}(t), \tag{2.53}$$

we obtain

$$\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\frac{\hat{\xi}^{[c]}(t)}{t-x}dt-\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\,\hat{k}^{[c]}(x,t)\hat{\xi}^{[c]}(t)dt=\hat{g}^{[c]}(x)-\int_{-1}^{1}\frac{\hat{\xi}^{[c]}(t)}{\sqrt{1-t^2}}dt,\mid x\mid<1. \tag{2.54}$$

In the above equation (2.54), $\hat{\xi}^{[c]}(t)$ is an unknown function, $\hat{k}^{[c]}(x,t)=\dfrac{2\hat{h}(\frac{1+t}{1-t},\frac{1+x}{1-x})}{(1-t)^2}$ and $\hat{g}(x)=\hat{f}\Big(\frac{1+x}{1-x}\Big)$.

Finally, we transformed the Cauchy singular integral equation (2.50) over the half-line into the Cauchy singular integral equation (2.54) over the interval $[-1,1]$. We assume that

$$\int_{-1}^{1}\int_{-1}^{1}\Big(\sqrt{\frac{1-t}{1+t}}\,\hat{k}^{[c]}(x,t)\Big)^2dtdx<\infty. \tag{2.55}$$

We approximate the function $\hat{\xi}^{[c]}(t)$ by orthonormalized Legendre polynomials as follows:

$$\hat{\xi}^{[c]}(t)\approx\hat{\xi}_n^{*[c]}(t)=\sum_{j=0}^{n}\hat{a}_j^{[c]}e_j(s), \tag{2.56}$$

where $\{e_j(s)\}_{j=0}^{n}$ denotes the set of $(n+1)$ orthonormalized Legendre polynomials on $[-1,1]$ and $\hat{a}_j^{[c]};\ j=0,1,2,\ldots,n$ are unknown constant coefficients. Now in order to get the approximate solution of equation (2.54), we proceed in the same way as in Section 2.2 with the residual error

$$\hat{R}^{[c]}(x,\hat{a}_0^{[c]},\hat{a}_1^{[c]},\hat{a}_2^{[c]},...,\hat{a}_n^{[c]})=\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\frac{\hat{\xi}_n^{*[c]}(t)}{t-x}dt+\int_{-1}^{1}\frac{\hat{\xi}^{[c]}(t)}{\sqrt{1-t^2}}dt-\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\hat{k}^{[c]}(x,t)\hat{\xi}_n^{*[c]}(t)dt \\ -\hat{g}^{[c]}(x),\quad |x|\leq 1. \tag{2.57}$$

Also, since we are using residual based Galerkin's method which is described in Section 2.2, we have

$$\langle \hat{R}^{[c]}(x, \hat{a}_0^{[c]}, \hat{a}_1^{[c]}, \hat{a}_2^{[c]}, ..., \hat{a}_n^{[c]}), e_j\rangle_{L^2} = 0, \ \forall j = 0, 1, 2, \dots, n. \tag{2.58}$$

This equation (2.58) gives a system of linear algebraic equations of order $(n+1) \times (n+1)$. The matrix form of (2.58) is

$$\hat{B}_1^{[c]} \hat{A}^{[c]} = \hat{G}^{[c]}, \tag{2.59}$$

where

$$\hat{B}_1^{[c]} = \hat{B}^{[c]T}, \quad \hat{B}_1^{[c]} = \begin{pmatrix} \hat{b}^{[c]}_{00} & \hat{b}^{[c]}_{01} & \dots & \hat{b}^{[c]}_{0n} \\ \hat{b}^{[c]}_{10} & \hat{b}^{[c]}_{11} & \dots & \hat{b}^{[c]}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{b}^{[c]}_{n0} & \hat{b}^{[c]}_{n1} & \dots & \hat{b}^{[c]}_{nn} \end{pmatrix}, \ \hat{A}^{[c]} = \begin{pmatrix} \hat{a}_0^{[c]} \\ \hat{a}_1^{[c]} \\ \vdots \\ \hat{a}_n^{[c]} \end{pmatrix}, \ \hat{G}^{[c]} = \begin{pmatrix} \hat{g}_0^{[c]} \\ \hat{g}_1^{[c]} \\ \vdots \\ \hat{g}_n^{[c]} \end{pmatrix}, \tag{2.60}$$

$$\hat{b}_{rq}^{[c]} = \int_{-1}^{1} \fint_{-1}^{1} \sqrt{\frac{1-t}{1+t}} \frac{e_r(t)e_q(x)}{t-x} dt dx$$
$$- \int_{-1}^{1}\int_{-1}^{1} \sqrt{\frac{1-t}{1+t}} \hat{k}^{[c]}(x,t) e_r(t) e_q(x) dt dx + \int_{-1}^{1}\int_{-1}^{1} \frac{e_r(t)e_q(x)}{\sqrt{1-t^2}} dt dx, \quad r, q = 0, 1, 2, \dots, n,$$
$$\hat{g}_q = \int_{-1}^{1} \hat{g}^{[c]}(x) e_q(x) dx, \quad q = 0, 1, 2, \dots, n.$$

2.7 Application of the method to an antiplane shear crack

In this problem, there are two elastic mediums M_1 and M_2 shown with florescent green and blue color in the Figure 2.4. The semi-infinite crack is lying in the medium M_1 and the crack is perpendicular to the interface of mediums M_1 and M_2. Further, the crack is at the distance h from the interface. The constants μ_1, μ_2 represent the shear moduli and while ν_1, ν_2 represent the Poisson's ratios for mediums M_1 and M_2 respectively.

This problem [30, 51] results finally in the form of singular integral equation

$$\frac{1}{\pi} \fint_{h}^{\infty} \frac{\hat{\theta}(z_1)}{z_1 - y_1} dz_1 + \int_{h}^{\infty} \frac{\lambda \hat{\theta}(z_1)}{z_1 + y_1} dz_1 = \hat{f}_1(y_1), \ \ h < y_1 < \infty, \tag{2.61}$$

where h is the distance of one crack end from the interface, $\hat{f}_1(y_1)$ represents the known shear stress along the crack edges and λ is a constant given by $\lambda = \dfrac{1-m}{1+m}, m = \dfrac{\mu_2}{\mu_1}$. In problem (2.61), we take $h = 1$ and $\hat{f}_1(y_1) = \dfrac{-1}{y_1^3}$. We now transform the interval $[1, \infty]$ to the interval $[0, \infty]$ by substituting $z_1 = z + 1$,

$y_1 = y + 1$, in equation (2.61) and we obtain

$$\frac{1}{\pi}\int_0^{\infty} \frac{\hat{\chi}(z)}{z-y}dz + \int_0^{\infty} \frac{\lambda\hat{\chi}(z)}{z+y+2}dz = \frac{-1}{(1+y)^3}, \quad 0 < y < \infty, \tag{2.62}$$

where $\hat{\chi}(z) = \hat{\theta}(z+1)$.

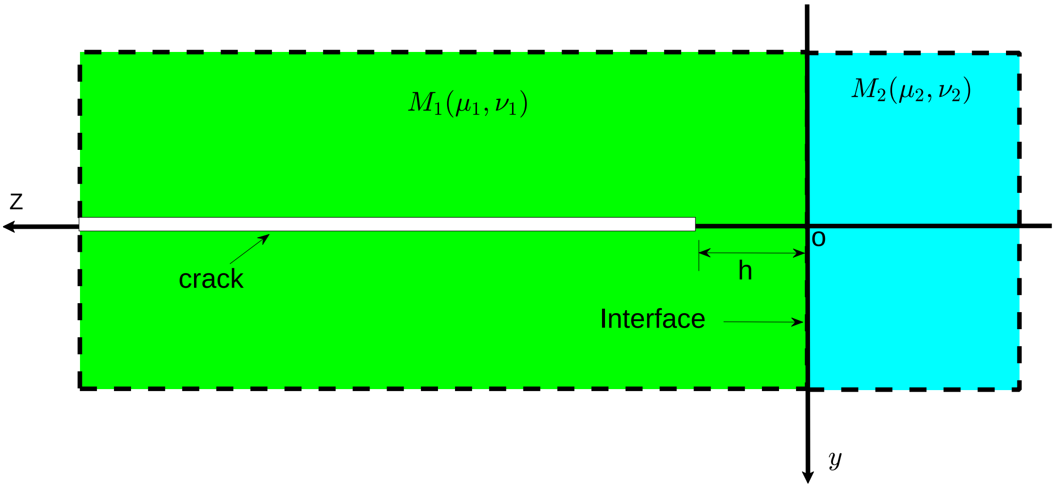


Figure 2.4: A semi-infinite straight crack perpendicular to the bimaterial interface

Table 2.4: Stress intensity factor for different materials

M_1	M_2	m	k^* from Chapter 2	k^* from Ref. [51]
Aluminum	Aluminum	1	0.531	0.531
Rigid	Aluminum	0	0.634	0.632
Epoxy	Aluminum	23.077	0.503	0.503
Aluminum	Epoxy	$\frac{1}{23.077}$	0.615	0.618

Now we implement the proposed method described in Section 2.6 to solve the above singular integral equation (2.62) and calculate the stress intensity factor at the crack tip $(0, 0)$. The stress intensity factor [51] can be calculated by the formula $k* = \sqrt{2}\hat{\chi}(0)$, where $\hat{\chi}(0)$ is the solution of the singular integral equation (2.62) at $z = 0$. We obtained the stress intensity factors given in Table 2.4, when we approximate $\hat{\chi}(z)$ by the Legendre polynomial of degree 3 while the results shown in the last column of the table have been obtained by using Radu-Chebyshev method [51] with at least 5 node points.

Conclusion

In this chapter, we have considered the Cauchy singular integral equations over the interval $[-1, 1]$. A residual based Galerkin's method using Legendre polynomial as a basis function is proposed in order to get the numerical solution of CSIEs over the finite interval. This method converts the singular integral equation into a system of linear equations which is easily solvable. The existence and uniqueness of solution for the system of linear algebraic equations, which is obtained as a result of approximation of Cauchy singular integral equation over the intervals, are shown. The convergence of sequence of approximate solutions to the exact solution is proved and the error bound is also obtained. The derived theoretical error bound is also verified with the help of numerical examples which indicates the good behavior of the proposed method.

Chapter 3

Numerical solution of system of Cauchy singular integral equations

3.1 Introduction

System of Cauchy singular integral equations occur naturally in physics and engineering during the formulation of many boundary value problems containing different geometric singularities. Many crack problems in the field of fracture mechanics are formulated as system of Cauchy singular integral equations using Green's function [97]. For instance, in T-stress problem [71] near the tips of a cruciform crack with unequal arms, the system of Cauchy type singular integral equations arise naturally. Also, it can be obtained by the decomposition of two dimensional Cauchy singular integral equation over a curve in complex plane [94]. Problems in the field of aerodynamics [70, 73], queuing system analysis [24], electrocardiology [38], elasticity theory [74, 93] are modeled as system of CSIEs.

In this chapter, a numerical method to solve the following system of Cauchy singular integral equations (3.1)

$$\int_{-1}^{1} \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1N} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{N1} & \beta_{N2} & \cdots & \beta_{NN} \end{pmatrix} \begin{pmatrix} \varphi_1^{[cs]}(t) \\ \varphi_2^{[cs]}(t) \\ \vdots \\ \varphi_N^{[cs]}(t) \end{pmatrix} \frac{1}{t-x} dt = \begin{pmatrix} g_1^{[cs]}(x) \\ g_2^{[cs]}(x) \\ \vdots \\ g_N^{[cs]}(x) \end{pmatrix}, \quad | x | < 1, \tag{3.1}$$

is proposed. In system (3.1), the coefficients $\beta_{ij};\ i,\ j = 1, 2, \ldots, N$ are known real constants and $g_j^{[cs]}(x);\ j = 1, 2, \ldots, N$, are known complex valued Hölder continuous functions over the interval $[-1, 1]$. The functions $\varphi_j^{[cs]}(t);\ j = 1, 2, \ldots, N$, are unknown complex valued functions which vanish at $t = 1$ and become unbounded at $t = -1$. The singular integral appearing in each equation of system (3.1) is understood in the sense of Cauchy principal value (CPV). Also, each unknown function $\varphi_j^{[cs]}(t);\ j = 1, 2, \ldots, N$, is assumed to be Hölder continuous in order to ensure the [17] existence of CPV. In system (3.1), each Cauchy equation is of index zero [42]. These equations are well studied [42, 59, 73, 93]. They play a vital role in the study of mixed boundary value problems for partial differential equations [93] and have many applications in the field of aerodynamics [70], fracture mechanics [32], neutron transport [87] etc. Singular integral equations can also be used to determine the existence of travelling-wave solutions [131, 131, 132] of particular reaction-convection-diffusion equations [40]. The numerical methods developed for these kind of singular integral equations include: Galerkin's method [11, 116, 117, 129], collocation method [58, 111], quadrature-collocation method [123], Nyström method [37], inverse method [63], real variable method [15], Sinc approximations [1]. However, the literature on numerical methods to find the solution of system of Cauchy type singular integral equations is still scarce. Although, the basic work related to this kind of system can be found in [93, 127]. Bonis and Laurita [26] proposed a quadrature method for system of Cauchy type singular integral equations. Turhan [125] used Chebyshev polynomial based method for the solution of the system of Cauchy type singular integral equations of the first kind. The analytic solution (3.3) for one-dimensional Cauchy type singular integral (3.2) is well known [42]. We use this solution to derive the analytic solution for system (3.1) in Section 3.2. But still it is of limited use in practical situations. Since it is not possible to solve the singular integral on the right side of equation (3.4) for every choice of $\eta_j^{[cs]}(t)$ due to the presence of singularity. Therefore, it is required to go for numerical solution for system (3.1). Hence, we propose a residual based Galerkin's method for solving the system (3.1) of Cauchy type singular integral equations. The error analysis of the proposed numerical method is also derived and validated through numerical examples.

3.2 Method for solution of the problem

For one-dimensional case of Cauchy type singular integral equation of first kind

$$\fint_{-1}^{1} \frac{\varphi_1^{[cs]}(t)}{t-x} dt = g_1^{[cs]}(x), |x| < 1, \tag{3.2}$$

with boundary conditions: $\varphi_1^{[cs]}(1) = 0$ and $\varphi_1^{[cs]}(-1)$ is unbounded. The analytical solution [42] of equation (3.2) is given by

$$\varphi_1^{[cs]}(x) = -\frac{1}{\pi^2}\sqrt{\frac{1-x}{1+x}} \fint_{-1}^{1} \sqrt{\frac{1+t}{1-t}} \frac{g_1^{[cs]}(t)}{t-x} dt. \tag{3.3}$$

Extending the above analytic solution for system of Cauchy type singular integral equations of first kind (3.1), we obtain

$$\varphi^{[cs]}(x) = \begin{pmatrix} \varphi_1^{[cs]}(x) \\ \varphi_2^{[cs]}(x) \\ \vdots \\ \varphi_N^{[cs]}(x) \end{pmatrix} = -\frac{1}{\pi^2}\sqrt{\frac{1-x}{1+x}} \begin{pmatrix} \fint_{-1}^{1} \sqrt{\frac{1+t}{1-t}} \frac{\eta_1(t)}{t-x} dt \\ \fint_{-1}^{1} \sqrt{\frac{1+t}{1-t}} \frac{\eta_2(t)}{t-x} dt \\ \vdots \\ \fint_{-1}^{1} \sqrt{\frac{1+t}{1-t}} \frac{\eta_N(t)}{t-x} dt \end{pmatrix}, \tag{3.4}$$

where $\eta_j^{[cs]}(t) = \frac{det(D_j^{[cs]}(t))}{det(\beta^{[cs]})}$; $j = 1, 2, \ldots, N$, $\det(\cdot)$ denotes determinant of a matrix, $D_j^{[cs]}(t)$ is an $N \times N$ matrix which is obtained from matrix

$$\beta^{[cs]} = \begin{pmatrix} \beta_{11} & \beta_{12} & \ldots & \beta_{1N} \\ \beta_{21} & \beta_{22} & \ldots & \beta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{N1} & \beta_{N2} & \ldots & \beta_{NN} \end{pmatrix}, \tag{3.5}$$

by replacing the j^{th} column of $\beta^{[cs]}$ with $(g_1^{[cs]}(x), g_2^{[cs]}(x), \ldots, g_N^{[cs]}(x))^T$ and the superscript T is used to denote the transpose of a matrix throughout the chapter. Now since the analytical solution (3.4) is of limited use as discussed in Section 3.1, therefore we propose a numerical method to find the approximate solution of system (3.1). But we can still take benefit of the known form of analytical solution (3.4) in finding the numerical solution. That is, each of the unknown functions $\varphi_j^{[cs]}(t)$, $j = 1, 2, \ldots, N$, in the system (3.1) can be written [42] as

$$\varphi_j^{[cs]}(t) = \sqrt{\frac{1-t}{1+t}}\psi_j^{[cs]}(t),\ j = 1, 2, \ldots, N, \tag{3.6}$$

where each $\psi_j^{[cs]}(t)$, $j = 1, 2, \ldots, N$, is an unknown function of $t \in [-1, 1]$. Using equation (3.6), the system (3.1) becomes

$$\fint_{-1}^{1} \begin{pmatrix} \beta_{11} & \beta_{12} & \ldots & \beta_{1N} \\ \beta_{21} & \beta_{22} & \ldots & \beta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{N1} & \beta_{N2} & \ldots & \beta_{NN} \end{pmatrix} \begin{pmatrix} \psi_1^{[cs]}(t) \\ \psi_2^{[cs]}(t) \\ \vdots \\ \psi_N^{[cs]}(t) \end{pmatrix} \sqrt{\frac{1-t}{1+t}}\frac{1}{t-x}dt = \begin{pmatrix} g_1^{[cs]}(x) \\ g_2^{[cs]}(x) \\ \vdots \\ g_N^{[cs]}(x) \end{pmatrix}. \tag{3.7}$$

In operator form, the system (3.7) becomes

$$S^{[cs]}\beta^{[cs]}\psi^{[cs]}(x) = g^{[cs]}(x),\ |x| < 1, \tag{3.8}$$

where $S^{[cs]}$ is a linear integral operator defined by

$$S^{[cs]}\psi^{[cs]}(x) = \fint_{-1}^{1} \sqrt{\frac{1-t}{1+t}}\frac{\psi^{[cs]}(t)}{t-x}dt, \tag{3.9}$$

the matrix $\beta^{[cs]}$ is already defined in equation (3.5) and $g^{[cs]}(x) = (g_1^{[cs]}(x), g_2^{[cs]}(x), \ldots, g_N^{[cs]}(x))^T$ is an $N \times 1$ matrix. Now the unknown function $\psi^{[cs]}(t) = (\psi_1{}^{[cs]}(t), \psi_2{}^{[cs]}(t), \ldots, \psi_N{}^{[cs]}(t))^T$ in system (3.7) is approximated as

$$\psi^{[cs]}(t) \approx \psi_n^{*[cs]}(t) = (\psi_{1n}^{*\ [cs]}(t), \psi_{2n}^{*\ [cs]}(t), \ldots, \psi_{Nn}^{*\ [cs]}(t))^T, \tag{3.10}$$

where

$$\psi^{*\ [cs]}_{jn}(t) = \sum_{k=0}^{n} a^{[cs]}_{jk} e_k(t),\ \forall j = 1, 2, \ldots, N, \tag{3.11}$$

and $\{e_k(t)\}_{k=0}^{n}$ denotes the set of $n+1$ orthonormalized Legendre polynomials on $[-1, 1]$. In order to solve the system (3.1) or its equivalent form (3.7), a residual based Galerkin's method is used. The residual error is defined as

$$\mathcal{R}^{[cs]}(x; a^{[cs]}_{10}, a^{[cs]}_{11}, \ldots, a^{[cs]}_{1n}; a^{[cs]}_{20}, a^{[cs]}_{21}, \ldots, a^{[cs]}_{2n}; \ldots; a^{[cs]}_{N0}, a^{[cs]}_{N1}, \ldots, a^{[cs]}_{Nn}) =$$

$$\begin{pmatrix} \mathcal{R}^{[cs]}_{1}(x; a^{[cs]}_{10}, a^{[cs]}_{11}, \ldots, a^{[cs]}_{1n}; a^{[cs]}_{20}, a^{[cs]}_{21}, \ldots, a^{[cs]}_{2n}; \ldots; a^{[cs]}_{N0}, a^{[cs]}_{N1}, \ldots, a^{[cs]}_{Nn}) \\ \mathcal{R}^{[cs]}_{2}(x; a^{[cs]}_{10}, a^{[cs]}_{11}, \ldots, a^{[cs]}_{1n}; a^{[cs]}_{20}, a^{[cs]}_{21}, \ldots, a^{[cs]}_{2n}; \ldots; a^{[cs]}_{N0}, a^{[cs]}_{N1}, \ldots, a^{[cs]}_{Nn}) \\ \vdots \\ \mathcal{R}^{[cs]}_{N}(x; a^{[cs]}_{10}, a^{[cs]}_{11}, \ldots, a^{[cs]}_{1n}; a^{[cs]}_{20}, a^{[cs]}_{21}, \ldots, a^{[cs]}_{2n}; \ldots; a^{[cs]}_{N0}, a^{[cs]}_{N1}, \ldots, a^{[cs]}_{Nn}) \end{pmatrix}, \tag{3.12}$$

where

$$\begin{aligned} \mathcal{R}^{[cs]}_{j}(x; a^{[cs]}_{10}, a^{[cs]}_{11}, \ldots, a^{[cs]}_{1n}; a^{[cs]}_{20}, a^{[cs]}_{21}, \ldots, a^{[cs]}_{2n}; \ldots; a^{[cs]}_{N0}, a^{[cs]}_{N1}, \ldots, a^{[cs]}_{Nn}) \\ = \sum_{i=1}^{N} \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}} \frac{\beta_{ji}\psi^{*\ [cs]}_{in}(t)}{t-x} dt - g^{[cs]}_{j}(x), \end{aligned} \tag{3.13}$$

for $j = 1, 2, \ldots, N$, is assumed to be orthogonal to the finite dimensional vector space $E =$ span $\{e_k(t)\}_{k=0}^{n}$. Therefore, we have

$$\begin{aligned} &\langle \mathcal{R}^{[cs]}_{j}(x; a^{[cs]}_{10}, a^{[cs]}_{11}, \ldots, a^{[cs]}_{1n}; a^{[cs]}_{20}, a^{[cs]}_{21}, \ldots, a^{[cs]}_{2n}; \ldots; a^{[cs]}_{N0}, a^{[cs]}_{N1}, \ldots, a^{[cs]}_{Nn}) \\ &= \sum_{i=1}^{N} \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}} \frac{\beta_{ji}\psi^{*\ [cs]}_{in}(t)}{t-x} dt - g^{[cs]}_{j}(x), e_k(x)\rangle_{L^2} = 0, \forall\ k = 0, 1, \ldots, n;\ \forall\ j = 1, 2, \ldots, N, \end{aligned} \tag{3.14}$$

where $\langle \cdot, \cdot \rangle_{L^2}$ represents the inner product in $L^2[-1, 1]$ space. This will result into a linear system of $N \times (n+1)$ equations in $N \times (n+1)$ unknowns $a^{[cs]}_{jk}$; $j = 1, 2, \ldots, N$; $k = 0, 1, \ldots, n$. The explicit expression for the system of linear algebraic equations is given by

$$\begin{aligned} &\sum_{k=0}^{n} \sum_{j=1}^{N} a^{[cs]}_{jk} \beta_{ij} \int_{-1}^{1} \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}} \frac{e_k(t) e_r(x)}{t-x} dt dx \\ &= \int_{-1}^{1} g^{[cs]}_{i}(x) e_r(x) dx, r = 0, 1, 2, \ldots, n, i = 1, 2, 3, \ldots, N. \end{aligned} \tag{3.15}$$

Now we define

$$b_{kr}^{[cs]} = \int_{-1}^{1}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\frac{e_k(t)e_r(x)}{t-x}dtdx, \quad k = 0,1,2,\ldots,n, \quad r = 0,1,2,\ldots,n,$$

$$g_{ir}^{[cs]} = \langle g_i^{[cs]}, e_r\rangle_{L^2}, \quad i = 1,2,3,\ldots,N, \quad r = 0,1,2,3,\ldots,n.$$

Finally, the above system in matrix form becomes

$$\beta^{[cs]} \otimes B^{[cs]^T} vec\, A^{[cs]} = vec\, G^{[cs]}, \tag{3.16}$$

where

$$A^{[cs]} = \begin{pmatrix} a_{10}^{[cs]} & a_{11}^{[cs]} & \ldots & a_{1n}^{[cs]} \\ a_{20}^{[cs]} & a_{21}^{[cs]} & \ldots & a_{2n}^{[cs]} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N0}^{[cs]} & a_{N1}^{[cs]} & \ldots & a_{Nn}^{[cs]} \end{pmatrix}, \; G^{[cs]} = \begin{pmatrix} g_{10}^{[cs]} & g_{11}^{[cs]} & \ldots & g_{1n}^{[cs]} \\ g_{20}^{[cs]} & g_{21}^{[cs]} & \ldots & g_{2n}^{[cs]} \\ \vdots & \vdots & \ddots & \vdots \\ g_{N0}^{[cs]} & g_{N1}^{[cs]} & \ldots & g^{[cs]}\hat{g}_{Nn} \end{pmatrix}, \; B^{[cs]} = \begin{pmatrix} b_{00}^{[cs]} & b_{01}^{[cs]} & \ldots & b_{0n}^{[cs]} \\ b_{10}^{[cs]} & b_{11}^{[cs]} & \ldots & b_{1n}^{[cs]} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n0}^{[cs]} & b_{n1}^{[cs]} & \ldots & b_{nn}^{[cs]} \end{pmatrix}$$

and the matrix $\beta^{[cs]}$ is same as defined in equation (3.5). In equation (3.16), $vec\; A^{[cs]}$ and $vec\; G^{[cs]}$ are column vectors [47] of order $N \times (n+1)$ which are defined as

$$vec\, A^{[cs]} = \begin{pmatrix} A_1^{[cs]} \\ A_2^{[cs]} \\ \vdots \\ A_N^{[cs]} \end{pmatrix}, \; vec\, G^{[cs]} = \begin{pmatrix} G_1^{[cs]} \\ G_2^{[cs]} \\ \vdots \\ G_N^{[cs]} \end{pmatrix}, \tag{3.17}$$

where $A_j^{[cs]}$, $G_j^{[cs]}$, $j = 1,2,\ldots,N$, stands for the j^{th} column of matrices $A^{[cs]}$ and $G^{[cs]}$ respectively. We can further write equation (3.16) as

$$\hat{D}^{[cs]}\hat{C}^{[cs]} = \hat{E}^{[cs]}, \tag{3.18}$$

where $\hat{D}^{[cs]} = \beta^{[cs]} \otimes B^{[cs]^T}$, $\hat{C}^{[cs]} = vec\; A^{[cs]}$, $\hat{E}^{[cs]} = vec\; G^{[cs]}$ and symbol $\otimes$ stands for the Kronecker product [47] of matrices $\beta^{[cs]}$ and $B^{[cs]^T}$. The above linear system (3.18) of algebraic equations can be solved easily.

3.3 Error Analysis

In this section, firstly we define the suitable function spaces which will be used for error analysis of our proposed numerical method. Then we prove the well-posedness in the sense of Hadamard for the problem (3.1) as well as for the system of linear algebraic equations which is obtained as a result of approximation of problem (3.1). We also show the convergence of sequence of approximate solutions to the exact solution and obtain the rate of convergence in $L_N^{2\,[cs]}$ norm. Finally, we derive the error bound for the error $\|\psi^{[cs]}(x) - \psi_n^{*\,[cs]}\|_{L_N^{2\,[cs]}}$ which occurs as a result of approximation of solution of system (3.1). Also, we show that whenever the known function in system (3.1) is in the form of a polynomial, the approximate solution is an exact solution.

3.3.1 Function spaces

In this subsection, we initialize the error analysis by defining the space $L_N^{2\,[cs]}$

$$L_N^{2\,[cs]} = \{u(x) = (u_1(x), u_2(x), ..., u_N(x))^T : u_j(x) \in L^2[-1,1];\ j = 1, 2, \ldots, N\}, \tag{3.19}$$

which is a Hilbert space equipped with the following norm $\|\cdot\|_{L_N^{2\,[cs]}}$ and inner product $\langle\cdot,\cdot\rangle_{L_N^{2\,[cs]}}$

$$\| u \|^2_{L_N^{2\,[cs]}} = \frac{1}{N}\sum_{j=1}^{N} \| u_j \|^2_{L^2} \ \text{ for } u(x) \in L_N^{2\,[cs]}, \tag{3.20}$$

$$\langle u, v\rangle_{L_N^{2\,[cs]}} = \frac{1}{N}\sum_{j=1}^{N} \langle u_j, v_j\rangle_{L_N^{2\,[cs]}} \text{ for } u(x), v(x) \in L_N^{2\,[cs]}, \tag{3.21}$$

where $L^2[-1,1] = \{u_j : [-1,1] \to \mathbb{C} : \int_{-1}^{1} u_j(x)\overline{u_j(x)}dx < \infty,\ j = 1, 2, \ldots, N\}$ is a Hilbert Space of all complex valued functions which are square integrable in the interval $[-1,1]$ equipped with the norm $\|\cdot\|^2_{L^2}$ and inner product $\langle.,.\rangle_{L_N^{2\,[cs]}}$ defined as

$$\|u_j\|^2_{L^2} = \int_{-1}^{1} |u_j(x)|^2 dx \text{ for } u_j(x) \in L^2;\ j = 1, 2, \ldots, N, \tag{3.22}$$

$$\langle u_j, v_j\rangle_{L^2} = \int_{-1}^{1} u_j(x)\overline{v_j(x)}dx \ \text{ for } u_j(x),\ v_j(x) \in L^2;\ j = 1, 2, \ldots, N. \tag{3.23}$$

Now let us consider the set of functions

$$M_N^{[cs]} = \{u(x) \in L_N^{2\,[cs]} : \frac{1}{N}\sum_{j=1}^{N}\sum_{k=0}^{\infty}(d_k^{[cs]})^2 |\langle u_j, e_k\rangle_{L^2}|^2 < \infty\}, \tag{3.24}$$

where

$$u(x) = (u_1(x), u_2(x), ..., u_N(x))^T \in L_N^{2\,[cs]} \ \text{ for } u_j(x) \in L^2[-1,1], \tag{3.25}$$

$$(d_k^{[cs]})^2 = \sum_{i=0}^{\infty} |\langle S^{[cs]} e_k, e_i\rangle_{L^2}|^2. \tag{3.26}$$

The set $M_N^{[cs]}$ is a subspace of L_N^2 which is made into a Hilbert space with the following norm $\| \cdot \|_{M_N^{[cs]}}$ and inner product $\langle \cdot, \cdot \rangle_{M_N^{[cs]}}$ defined as

$$\| u \|^2_{M_N^{[cs]}} = \frac{1}{N} \sum_{j=1}^{N} \| uj \|^2_{M^{[cs]}} \text{ for } u(x) \in M_N^{[cs]}, \tag{3.27}$$

$$\langle u, v \rangle_{M_N^{[cs]}} = \frac{1}{N} \sum_{j=1}^{N} \langle u_j, v_j \rangle_{M^{[cs]}} \text{ for } u(x), v(x) \in M_N^{[cs]}, \tag{3.28}$$

where $M^{[cs]}$ denotes a subspace of $L^2[-1, 1]$ such that

$$M^{[cs]} = \{u_j(x) \in L^2 : \sum_{k=0}^{\infty} (d_k^{[cs]})^2 |\langle u_j, e_k \rangle_{L^2}|^2 < \infty\}, \tag{3.29}$$

and $(d_k^{[cs]})^2$ is defined in equation (3.26). This subspace $M^{[cs]}$ is a Hilbert space with respect to the norm $\| \cdot \|_{M^{[cs]}}$ and inner product $\langle \cdot, \cdot \rangle_{M^{[cs]}}$ which are defined as

$$\| u_j \|^2_{M^{[cs]}} = \sum_{k=0}^{\infty} (d_k^{[cs]})^2 |\langle u_j, e_k \rangle_{L^2}|^2 \text{ for } u_j(x) \in M^{[cs]}, \tag{3.30}$$

$$\langle u_j, v_j \rangle_{M^{[cs]}} = \sum_{k=0}^{\infty} (d_k^{[cs]})^2 \langle u_j, e_k \rangle_{L^2} \overline{\langle v_j, e_k \rangle}_{L^2} \text{ for } u_j(x), v_j(x) \in M^{[cs]}. \tag{3.31}$$

3.3.2 Well-posedness of the problems

In this subsection, we show that system of Cauchy singular integral equations (3.1) as well as the system of linear algebraic equations obtained from its approximation are well-posed in the spaces defined above. Now operating the integral operator $S^{[cs]}$ (defined in equation (3.9)) on the orthonormalized Legendre polynomials of degree $0, 1, \ldots, n$, we get [12] the following results:

$$S^{[cs]} e_0(x) = -\pi e_0(x),$$
$$S^{[cs]} e_1(x) = \pi \Big[\frac{\| p_0 \|}{\| p_1 \|} e_0(x) - e_1(x) \Big],$$
$$S^{[cs]} e_2(x) = \pi \Big[-\frac{3}{4} \frac{\| p_0 \|}{\| p_2 \|} e_0(x) + \frac{3}{2} \frac{\| p_1 \|}{\| p_2 \|} e_1(x) - e_2(x) \Big],$$
....
$$S^{[cs]} e_n(x) = \sum_{i=0}^{n} c_i^{[cs]} e_i(x); \text{ where the coefficients } c_i^{[cs]} = \langle S^{[cs]} e_n, e_i \rangle_{L^2},\ i = 0, 1, 2, \ldots, n,$$

where $p_k(x)$, $e_k(x)$, $k = 0, 1, \ldots, n$, are Legendre polynomial and orthonormalized Legendre polynomials respectively. We define $v_k^{[cs]}(x) = \frac{e_k(x)}{d_k^{[cs]}}$, $\| v_k^{[cs]} \|_{M^{[cs]}} = 1$, these $\{v_k^{[cs]}\}_{k=0}^{\infty}$ form complete orthonormal basis for the space $M^{[cs]}$, i.e. if $u_j \in M^{[cs]}$, then we have

$$u_j = \sum_{k=0}^{\infty} \langle u_j, v_k^{[cs]} \rangle_{M^{[cs]}} v_k^{[cs]}(x). \tag{3.32}$$

With the help of equation (3.32), we can extend the operator $S^{[cs]} : M_N^{[cs]} \to L_N^{2^{[cs]}}$ as a bounded linear operator which can be defined as

$$S^{[cs]} \beta^{[cs]} \psi^{[cs]}(x) = S^{[cs]} \phi^{[cs]}(x), \tag{3.33}$$

where

$$\beta^{[cs]} \psi^{[cs]}(x) = \phi^{[cs]}(x); \phi^{[cs]}(x) = (\phi_1^{[cs]}(x), \phi_2^{[cs]}(x), \ldots, \phi_N^{[cs]}(x))^T \in M_N^{[cs]}, \tag{3.34}$$

$$S^{[cs]} \phi^{[cs]}(x) = (S^{[cs]} \phi_1^{[cs]}(x), S^{[cs]} \phi_2^{[cs]}(x), \ldots, S^{[cs]} \phi_N^{[cs]}(x))^T \in L_N^{2^{[cs]}}, \tag{3.35}$$

$$S^{[cs]} \phi_j^{[cs]}(x) = \sum_{k=0}^{\infty} \langle \phi_j^{[cs]}, e_k \rangle_{L^2} \sum_{i=0}^{k} \langle S^{[cs]} e_k, e_i \rangle_{L^2} e_i(x) \in L^2[-1, 1]. \tag{3.36}$$

Now using equation (3.36), the norm of the bounded linear operator $S^{[cs]}$ is

$$\| S^{[cs]} \phi_j^{[cs]} \|_{L^2}^2 = \sum_{k=0}^{\infty} (d_k^{[cs]})^2 |\langle \phi_j^{[cs]}, e_k \rangle_{L^2}|^2 = \| \phi_j^{[cs]} \|_{M^{[cs]}}^2, \text{ for } j = 1, 2, \ldots, N. \tag{3.37}$$

Hence, using equation (3.37), we obtain

$$\| S^{[cs]} \| = 1. \tag{3.38}$$

Also, the mapping $S^{[cs]}$ from $M_N^{[cs]} \to L_N^{2^{[cs]}}$ is one-one and onto [42].
Hence, by using Theorem 1.1.1, the operator $(S^{[cs]})^{-1} : L_N^{2^{[cs]}} \to M_N^{[cs]}$ exists as a bounded linear operator and can be defined as

$$(S^{[cs]})^{-1} \phi^{[cs]}(x) = ((S^{[cs]})^{-1} \phi_1^{[cs]}(x), (S^{[cs]})^{-1} \phi_2^{[cs]}(x), \ldots, (S^{[cs]})^{-1} \phi_N^{[cs]}(x))^T \in M_N^{[cs]},$$

where

$$(S^{[cs]})^{-1} \phi_j^{[cs]}(x) = \sum_{k=0}^{\infty} \frac{\langle \phi_j^{[cs]}, e_k \rangle_{L^2}}{d_k^{[cs]}} e_k(x), \tag{3.39}$$

$$\| (S^{[cs]})^{-1}\phi_j^{[cs]} \|^2_{M^{[cs]}} = \sum_{l=0}^{\infty}(d_l^{[cs]})^2|\langle (S^{[cs]})^{-1}\phi_j^{[cs]}, e_l\rangle_{L^2}|^2, \text{ for } j = 1, 2, \ldots, N. \tag{3.40}$$

Using equation (3.39) in equation (3.40), we obtain

$$\| (S^{[cs]})^{-1}\phi_j^{[cs]} \|^2_{M^{[cs]}} = \sum_{l=0}^{\infty}|\langle \phi_j^{[cs]}, e_l\rangle_{L^2}|^2 = \| \phi_j^{[cs]} \|^2_{L^2} \text{ for } j = 1, 2, \ldots, N. \tag{3.41}$$

Hence, using the above equation (3.41), the norm for the bounded linear operator $(S^{[cs]})^{-1}$ is

$$\| (S^{[cs]})^{-1} \| = 1. \tag{3.42}$$

We assume that the matrix $\beta^{[cs]}$ in system (3.8) is a non-singular and well-conditioned matrix. Hence, the existence of the operator $(S^{[cs]})^{-1}$ and the matrix $(\beta^{[cs]})^{-1}$ show the existence of solution to the system (3.8). The solution of the system (3.8) is

$$\psi^{[cs]}(x) = (\beta^{[cs]})^{-1}(S^{[cs]})^{-1}g^{[cs]}(x), | x |< 1. \tag{3.43}$$

Now we will show the uniqueness of solution of the problem (3.8) by the principle of contradiction. If possible, suppose the problem (3.8) has two solutions $w_1(x)$ and $w_2(x)$, then both should satisfy equation (3.8) i.e.

$$S^{[cs]}\beta^{[cs]}w_1(x) = g^{[cs]}(x), | x |< 1, \tag{3.44}$$

$$S^{[cs]}\beta^{[cs]}w_2(x) = g^{[cs]}(x), | x |< 1. \tag{3.45}$$

Now from equations (3.44) and (3.45), we obtain

$$S^{[cs]}\beta^{[cs]}(w_1(x) - w_2(x)) = 0, | x |< 1. \tag{3.46}$$

Operating $(\beta^{[cs]})^{-1}(S^{[cs]})^{-1}$ on both sides of equation (3.46), we get

$$w_1(x) = w_2(x), | x |< 1. \tag{3.47}$$

Hence, the equation (3.47) proves the uniqueness of solution of equation (3.8). Also, the inverse operator $(S^{[cs]})^{-1}$ is a bounded linear operator and the matrix $\beta^{[cs]}$ is assumed to exist as nonsingular well-conditioned matrix, which show the stability [66] of the problem. Since the problem (3.8) satisfies all the three conditions, therefore we have proved that the problem (3.8), is a well-posed problem.

Consider the mapping $Q_N^{n\,[cs]} : L_N^{2\,[cs]} \to L_N^{2\,[cs]}$, where $Q_N^{n\,[cs]}$ is an operator of orthogonal projection defined as follows:

$$Q_N^{n\,[cs]}\phi^{[cs]}(x) = (Q_n^{[cs]}\phi_1^{[cs]}(x), Q_n^{[cs]}\phi_2^{[cs]}(x), ..., Q_n^{[cs]}\phi_N^{[cs]}(x))^T, \tag{3.48}$$

where

$$Q_n^{[cs]} \phi_j^{[cs]}(x) = \sum_{k=0}^{n} \langle \phi_j^{[cs]}, e_k \rangle_{L^2} e_k(x), \; j = 1, 2, \ldots, N, \tag{3.49}$$

and n is the degree of orthonormalized Legendre polynomial by which $\phi_j(x), \; j = 1, 2, ..., N$ is approximated. Substituting equation (3.33) in equation (3.8), we get

$$S^{[cs]} \phi^{[cs]}(x) = g^{[cs]}(x); \text{ where } \phi^{[cs]}(x) \in M_N^{[cs]}, \;\; g^{[cs]}(x) \in L_N^{2\,[cs]}. \tag{3.50}$$

Then it follows from equation (3.14)

$$Q_N^{n\,[cs]} (S^{[cs]} \phi_n^{*\,[cs]}(x) - g^{[cs]}(x)) = 0, \tag{3.51}$$

where

$$\begin{aligned} S^{[cs]}(x) &\approx \phi_n^{*\,[cs]}(x) = (\phi_{1n}^{*\;[cs]}(x), \phi_{2n}^{*\;[cs]}(x), ..., \phi_{Nn}^{*\;[cs]}(x))^T, \\ \phi_{jn}^{*\;[cs]}(x) &= \sum_{i=1}^{N} \beta_{ji} \psi_{in}^{*\;[cs]}(x), \; j = 1, 2, \ldots, N, \\ \psi_{in}^{*\;[cs]}(x)&; \; i = 1, 2, \ldots, N, \text{ are defined in equation(3.11)}. \end{aligned} \tag{3.52}$$

Using the fact that

$$Q_N^{n\,[cs]} S^{[cs]} \phi_n^{*\,[cs]}(x) = S^{[cs]} \phi_n^{*\,[cs]}(x), \tag{3.53}$$

equation (3.51) becomes

$$S^{[cs]} \phi_n^{*\,[cs]}(x) = Q_N^{n\,[cs]} g^{[cs]}(x). \tag{3.54}$$

Now using the existence of operator $(S^{[cs]})^{-1}$ which is defined in equation (3.39), we can say that the system (3.54) has a unique solution which is

$$\phi_n^{*\,[cs]}(x) = (S^{[cs]})^{-1} Q_N^{n\,[cs]} g^{[cs]}(x). \tag{3.55}$$

Using equation (3.33), we obtain

$$\beta^{[cs]} \psi_n^{*\,[cs]}(x) = (S^{[cs]})^{-1} Q_N^{n\,[cs]} g^{[cs]}(x). \tag{3.56}$$

Further, the matrix $(\beta^{[cs]})^{-1}$ is assumed to exist, therefore we obtain

$$\psi_n^{*\,[cs]}(x) = (\beta^{[cs]})^{-1} (S^{[cs]})^{-1} Q_N^{n\,[cs]} g^{[cs]}(x). \tag{3.57}$$

Also, operator $(S^{[cs]})^{-1}$ exist as bounded linear operator, this fact ensures the stability [66] of system (3.54). Hence, the problem (3.54) is also a well-posed problem.

3.3.3 Convergence analysis and error bound

In this subsection, we show the convergence of sequence of approximate solutions. We also derive the error bound for the error between approximate and exact solution of system (3.1). From equations (3.50) and (3.54), we get

$$S^{[cs]}(\psi^{[cs]}(x) - \psi_n^{*[cs]}(x)) = g^{[cs]}(x) - Q_N^{n\,[cs]} g^{[cs]}(x). \tag{3.58}$$

Using equations (3.33) and (3.52) in equation (3.58), we obtain

$$S^{[cs]}\beta^{[cs]}(\psi^{[cs]}(x) - \psi_n^{*[cs]}(x)) = g^{[cs]}(x) - Q_N^{n\,[cs]} g^{[cs]}(x). \tag{3.59}$$

Let $(\beta^{[cs]})^{-1}$ exists, then equation (3.59), becomes

$$\psi^{[cs]}(x) - \psi_n^{*[cs]}(x) = (\beta^{[cs]})^{-1}(S^{[cs]})^{-1}(g^{[cs]}(x) - Q_N^{n\,[cs]} g^{[cs]}(x)). \tag{3.60}$$

Taking norm on both the sides of equation (3.60), we get

$$\| \psi^{[cs]} - \psi_n^{*[cs]} \|_{M_N^{[cs]}} \leq \| (\beta^{[cs]})^{-1} \| \| (S^{[cs]})^{-1} \| \| g^{[cs]} - Q_N^{n\,[cs]} g^{[cs]} \|_{L_N^{2\,[cs]}} . \tag{3.61}$$

Here $\|(\beta^{[cs]})^{-1}\|$ denotes the matrix norm which can be calculated by using $[\rho((\beta^{[cs]})^{-1})^T(\beta^{[cs]})^{-1})]^{1/2}$, where $\rho((\beta^{[cs]})^{-1}) = \max|\lambda|$ defines the spectral radius [45] of the matrix $(\beta^{[cs]})^{-1}$ such that λ is an eigenvalue of the matrix $(\beta^{[cs]})^{-1}$. Using equation (3.42), the equation (3.61) becomes

$$\| \psi^{[cs]} - \psi_n^{*[cs]} \|_{M_N^{[cs]}} \leq \| (\beta^{[cs]})^{-1} \| \| g^{[cs]} - Q_N^{n\,[cs]} g^{[cs]} \|_{L_N^{2\,[cs]}} . \tag{3.62}$$

Since $\|(\beta^{[cs]})^{-1}\| < \infty$ and $\|g^{[cs]} - Q_N^{n\,[cs]} g^{[cs]}\|_{L_N^{2\,[cs]}}$ converges to zero as $n \to \infty$, we get

$$\|\psi^{[cs]} - \psi_n^{*[cs]}\|_{M_N^{[cs]}} \to 0 \text{ as } n \to \infty. \tag{3.63}$$

In fact, for all $\psi^{[cs]} \in M_N^{[cs]}$, we have $\|\psi^{[cs]}\|_{L_N^{2\,[cs]}} \leq \|\psi^{[cs]}\|_{M_N^{[cs]}}$, therefore the equations (3.62) and (3.63) become

$$\|\psi^{[cs]} - \psi_n^{*[cs]}\|_{L_N^{2\,[cs]}} \leq \| \psi^{[cs]} - \psi_n^{*[cs]} \|_{M_N^{[cs]}} \leq \| (\beta^{[cs]})^{-1} \| \| g - Q_N^{n\,[cs]} g \|_{L_N^{2\,[cs]}}, \tag{3.64}$$

$$\|\psi^{[cs]} - \psi_n^{*[cs]}\|_{L_N^{2\,[cs]}} \leq \|\psi^{[cs]} - \psi_n^{*[cs]}\|_{M_N^{[cs]}} \to 0 \text{ as } n \to \infty. \tag{3.65}$$

Hence, the sequence of approximate solutions $\{\psi_n^{*[cs]}\}_{n=0}^{\infty}$ converges to the exact solution $\psi^{[cs]}$ in $L_N^{2\,[cs]}$ norm.

3.3.4 Invertibility of coefficient matrix of system of linear algebraic equations

In this subsection, we show that the linear system (3.18), which is obtained after applying the proposed method described in Section 3.2 to system of Cauchy singular integral equation (3.1), has a unique solution. For this, we first define the vector space as

$$\mathbb{X}_{[cs]}^{N\times(n+1)} = \{vec\, G^{[cs]} = (G_1^{[cs]}, G_2^{[cs]}, ..., G_N^{[cs]})^T : G_j^{[cs]} \in \mathbb{C}^{n+1},\ j = 1, 2, \ldots, N\}, \tag{3.66}$$

where $\mathbb{C}^{n+1}$ denotes a complex vector space [67] having $(n+1)$-tuples of complex numbers as its vectors, $vec\, G^{[cs]}$ is same as defined in equation (3.17) and $G_j^{[cs]} = (\langle g_j^{[cs]}, e_0\rangle_{L^2}, \langle g_j^{[cs]}, e_1\rangle_{L^2}, \ldots, \langle g_j^{[cs]}, e_n\rangle_{L^2})^T$, $j = 1, 2, \ldots, N$.

We further define another vector space $\mathbb{Z}_N^{[cs]}$ as

$$\mathbb{Z}_N^{[cs]} = \{z^{[cs]} = (z_1^{[cs]}, z_2^{[cs]}, ..., z_N^{[cs]})^T : z_j^{[cs]} \in E,\ j = 1, 2, \ldots, N\}, \tag{3.67}$$

where $E = span\{e_i(x)\}_{i=0}^n$. Now consider the operator $P_N^{n\,[cs]} : \mathbb{X}_{[cs]}^{N\times(n+1)} \to \mathbb{Z}_N^{[cs]}$ as

$$P_N^{n\,[cs]}(vec\, G^{[cs]}) = (P_n^{[cs]} G_1^{[cs]}, P_n^{[cs]} G_2^{[cs]}, \ldots, P_n^{[cs]} G_N^{[cs]})^T, \tag{3.68}$$

where the operator $P_n^{[cs]} : \mathbb{C}^{n+1} \to E$ denotes the prolongation operator [45] and defined as

$$P_n^{[cs]} G_j^{[cs]} = \sum_{i=0}^{n} \langle g_j^{[cs]}, e_i\rangle_{L^2} e_i(x) \in E, \quad j = 0, 1, \ldots, N. \tag{3.69}$$

We now take the orthogonal projection of function $g^{[cs]}(x) \in L_N^2$

$$Q_N^{n\,[cs]} g^{[cs]}(x) = (Q_n^{[cs]} g_1^{[cs]}(x), Q_n^{[cs]} g_2^{[cs]}(x), \ldots, Q_n^{[cs]} g_N^{[cs]}(x))^T. \tag{3.70}$$

In equation (3.70), the operator $Q_n^{[cs]}$ denotes the orthogonal projection onto the finite dimensional vector space $E = span\{e_i(x)\}_{i=0}^n$ and defined as

$$Q_n^{[cs]} g_j^{[cs]}(x) = \sum_{i=0}^{n} \langle g_j^{[cs]}, e_i\rangle_{L^2} e_i(x). \tag{3.71}$$

With the aid of equations (3.69) and (3.71), we obtain

$$Q_n^{[cs]} g_j^{[cs]}(x) = P_n^{[cs]} G_j^{[cs]},\ j = 1, 2, \ldots, N. \tag{3.72}$$

Also, from equations (3.68), (3.70) and (3.72), we have

$$Q_N^{n\,[cs]} g^{[cs]}(x) = P_N^{n\,[cs]}(vec\, G^{[cs]}). \tag{3.73}$$

Now we further define an operator $R_N^{n\,[cs]} : \mathbb{Z}_N^{[cs]} \rightarrow \mathbb{X}_{[cs]}^{N\times(n+1)}$ as

$$R_N^{n\,[cs]} \psi_n^{*\,[cs]} = (R_n^{[cs]} \psi_{1n}^{*\;[cs]}, R_n^{[cs]} \psi_{2n}^{*\;[cs]}, \ldots, R_n^{[cs]} \psi_{Nn}^{*\;[cs]})^T, \tag{3.74}$$

where the operator $R_n^{[cs]} : E \rightarrow \mathbb{C}^{n+1}$ denotes restriction operator [45] and defined as

$$R_n^{[cs]} \psi_{jn}^{*\;[cs]} = (\langle \psi_{jn}^{*\;[cs]}, e_0\rangle_{L^2}, \langle \psi_{jn}^{*\;[cs]}, e_1\rangle_{L^2}, \ldots \langle \psi_{jn}^{*\;[cs]}, e_n\rangle_{L^2})^T \in \mathbb{C}^{n+1}, \tag{3.75}$$

where $\psi_{jn}^{*\;[cs]}$ is same as defined in equation (3.11). Using orthogonal property of Legendre polynomials in equation (3.11), we get

$$a_{ji}^{[cs]} = \langle \psi_{jn}^{*\;[cs]}, e_i\rangle_{L^2}, \; i = 0, 1, \ldots, n, \quad j = 1, 2, \ldots, N. \tag{3.76}$$

Using the above equation (3.76), the value of $A_j^{[cs]}$ (j^{th} column of matrix $A^{[cs]}$) is

$$A_j^{[cs]} = (\langle \psi_{jn}^{*\;[cs]}, e_0\rangle_{L^2}, \langle \psi_{jn}^{*\;[cs]}, e_1\rangle_{L^2}, \ldots, \langle \psi_{jn}^{*\;[cs]}, e_n\rangle_{L^2})^T, \;\; j = 1, 2, \ldots, N. \tag{3.77}$$

From equations (3.75) and (3.77), we obtain

$$vec\, A^{[cs]} = R_N^{n\,[cs]} \psi_n^{*\,[cs]}, \tag{3.78}$$

where $vec\, A^{[cs]}$ and $R_N^{n\,[cs]} \psi_n^{*\,[cs]}$ are already defined in equations (3.17) and (3.74) respectively. Since the existence and uniqueness of the solution $\psi_n^{*\,[cs]}$ of system (3.57) are already shown in subsection 3.3.2. Hence, the equation (3.78) implies that the solution $vec\, A^{[cs]}$ of system (3.16) also exists uniquely. Now using the value of $\psi_n^{*\,[cs]}$ from equation (3.57) in equation (3.78), we obtain

$$vec\, A^{[cs]} = R_N^{n\,[cs]} (\beta^{[cs]})^{-1} (S^{[cs]})^{-1} Q_N^{n\,[cs]} g^{[cs]}(x). \tag{3.79}$$

On using equations (3.73) and (3.79), we get

$$vec\, A^{[cs]} = R_N^{n\,[cs]} (\beta^{[cs]})^{-1} (S^{[cs]})^{-1} P_N^{n\,[cs]} (vec\, G^{[cs]}). \tag{3.80}$$

We can rewrite equation (3.80) as

$$\hat{C}^{[cs]} = R_N^{n\,[cs]} (\beta^{[cs]})^{-1} (S^{[cs]})^{-1} P_N^{n\,[cs]} (vec\, G^{[cs]}). \tag{3.81}$$

equation (3.81) implies that the solution $\hat{C}^{[cs]}$ of the system $\hat{D}^{[cs]}\hat{C}^{[cs]} = \hat{E}^{[cs]}$ defined in equation (3.18) exists uniquely for every given $\hat{E}^{[cs]} \in \mathbb{X}_{[cs]}^{N\times(n+1)}$. Hence, the matrix $\hat{D}^{[cs]}$ is invertible [119].

Theorem 3.3.1. *Consider a system of Cauchy type singular integral equations*

$$S^{[cs]}\beta^{[cs]}\psi^{[cs]}(x) = \hat{g}^{[cs]}(x), \; |x| < 1, \; \det(\beta^{[cs]}) \neq 0, \tag{3.82}$$

where $\beta^{[cs]}$ and $S^{[cs]}\psi^{[cs]}(x)$ are defined in equations (3.5) and (3.9) respectively, $\hat{g}^{[cs]}(x) = (\hat{g}_1^{[cs]}(x), \hat{g}_2^{[cs]}(x), \ldots, \hat{g}_N^{[cs]}(x))^T$ such that each $\hat{g}_j^{[cs]}(x)$ is a polynomial of degree m_j; $j = 1, 2, \ldots, N$. If each unknown function $\psi_j^{[cs]}(x)$; $j = 1, 2, \cdot, N$, is approximated as in equation (3.11) by the orthonormalized Legendre polynomials of degree $n = \max\{m_1, m_2, \ldots, m_N\}$, then the approximate solution of equation (3.82) is an exact solution.

Proof. The residual error in equation (3.82) is

$$S^{[cs]}\beta^{[cs]}\psi_n^{*[cs]}(x) - \hat{g}^{[cs]}(x),$$

where

$$\psi^{[cs]}(x) \approx \psi_n^{*[cs]}(x) = \psi_{1n}^{*\ [cs]}(x), \psi_{1n}^{*\ [cs]}(x), \ldots, \psi_{1n}^{*\ [cs]}(x))^T. \tag{3.83}$$

Now with the aid of equation (3.14), its orthogonal projection $Q_N^{n\ [cs]}$ which is defined in equation (3.48), we get

$$Q_N^{n\ [cs]}(S^{[cs]}\beta^{[cs]}\psi_n^{*[cs]}(x) - \hat{g}^{[cs]}(x)) = 0. \tag{3.84}$$

Now using equations (3.82) and (3.84), we obtain

$$S^{[cs]}\beta^{[cs]}\psi^{[cs]}(x) - Q_N^{n\ [cs]}(S^{[cs]}\beta^{[cs]}\psi_n^{*[cs]}(x)) = \hat{g}^{[cs]}(x) - Q_N^{n\ [cs]}\hat{g}^{[cs]}(x)). \tag{3.85}$$

Since functions $S^{[cs]}\beta^{[cs]}\psi_n^{*[cs]}(x)$ and $\hat{g}^{[cs]}(x)$ both are polynomials, therefore, we have

$$\begin{aligned} Q_N^{n\ [cs]}(S^{[cs]}\beta^{[cs]}\psi_n^{*[cs]}(x)) &= S^{[cs]}\beta^{[cs]}\psi_n^{*[cs]}(x), \\ Q_N^{n\ [cs]}\hat{g}^{[cs]}(x) &= \hat{g}^{[cs]}(x). \end{aligned} \tag{3.86}$$

From the above results mentioned in equation (3.86), equation (3.85) becomes

$$S^{[cs]}\beta^{[cs]}(\psi^{[cs]}(x) - \psi_n^{*[cs]}(x)) = 0. \tag{3.87}$$

Now using the existence of the operator $(S^{[cs]})^{-1}$ and the matrix $(\beta^{[cs]})^{-1}$, equation (3.87) becomes

$$\psi^{[cs]}(x) - \psi_n^{*[cs]}(x) = 0. \tag{3.88}$$

Hence, the approximate solution is the exact solution. □

3.4 Illustrative examples

In this section, numerical solution of some test examples using method of solution is discussed.

Example 3.1 Consider the following system of Cauchy type singular integral equations

$$\begin{aligned} \frac{1000}{\pi}\int_{-1}^{1}\frac{\varphi_{_1}^{[cs]}(t)}{t-x}dt+\frac{10}{\pi}\int_{-1}^{1}\frac{\varphi_{_2}^{[cs]}(t)}{t-x}dt=g_{_1}^{[cs]}(x),\\ \frac{500}{\pi}\int_{-1}^{1}\frac{\varphi_{_1}^{[cs]}(t)}{t-x}dt+\frac{200}{\pi}\int_{-1}^{1}\frac{\varphi_{_2}^{[cs]}(t)}{t-x}dt=g_{_2}^{[cs]}(x), \end{aligned} \tag{3.89}$$

where

$$\begin{aligned} g_{_1}^{[cs]}(x) &= -990x^8+1089x^7+937x^6-\frac{26704x^5}{25}-\frac{349161x^4}{1000}+\frac{792327x^3}{2000}+\frac{1761x^2}{250}-\frac{53511x}{4000}\\ &\quad-\frac{53929}{2000}+i\Big(990x^8-1189x^7-\frac{8971x^6}{10}+\frac{119047x^5}{100}+\frac{163961x^4}{500}-\frac{279198x^3}{625}-\frac{30873x^2}{10000}\\ &\quad+\frac{69533x}{4000}+\frac{1130501}{40000}\Big),\\ g_{_2}^{[cs]}(x) &= -300x^8+330x^7+215x^6-\frac{2447x^5}{10}-\frac{8607x^4}{100}+\frac{735x^3}{8}-\frac{17253x^2}{2000}+\frac{29541x}{4000}-\frac{14701}{1000}\\ &\quad+i\Big(300x^8-380x^7-197x^6+\frac{1462x^5}{5}+\frac{9419x^4}{100}-\frac{27549x^3}{250}-\frac{183x^2}{400}-\frac{7957x}{2000}+\frac{57853}{4000}\Big). \end{aligned}$$

Using equation (3.4), the exact solution in this case is given by

$$\begin{aligned} \varphi_{_1}^{[cs]}(x) &= \sqrt{\frac{1-x}{1+x}}\Big[x^8-\frac{x^7}{10}-\frac{31x^6}{20}+\frac{83x^5}{1000}+\frac{7867x^4}{10000}-\frac{1891x^3}{100000}-\frac{147177x^2}{1000000}+\frac{441x}{400000}+\frac{3969}{500000}\\ &\quad+i\Big(-x^8+\frac{x^7}{5}+\frac{161x^6}{100}-\frac{49x^5}{250}-\frac{343x^4}{400}+\frac{2401x^3}{50000}+\frac{165663x^2}{1000000}-\frac{9x}{3125}-\frac{567}{62500}\Big)\Big],\\ \varphi_{_2}^{[cs]}(x) &= \sqrt{\frac{1-x}{1+x}}\Big[-x^8+\frac{x^7}{10}+\frac{19x^6}{10}-\frac{67x^5}{500}-\frac{11129x^4}{10000}+\frac{29x^3}{800}+\frac{451x^2}{2000}-\frac{9x}{4000}-\frac{63}{5000}\\ &\quad+i\Big(x^8-\frac{x^7}{10}-\frac{189x^6}{100}+\frac{213x^5}{1000}+\frac{2727x^4}{2500}-\frac{3213x^3}{25000}-\frac{1339x^2}{6250}+\frac{97x}{6250}+\frac{42}{3125}\Big)\Big]. \end{aligned}$$

On applying the method of solution described in Section 3.2, we obtain the approximate solutions $\psi_n^{*[cs]}(x)$ for $n=1,2,\ldots,8$. The actual error $\|\psi^{[cs]}-\psi_n^{*[cs]}\|_{L_N^{2\,[cs]}}$ and the corresponding theoretical error bound are calculated by using equation (3.64) for different values of n. The results detailed in Table 3.1 show the good behavior of our proposed numerical method as the actual error in the approximate solution satisfies the derived theoretical error bound for different values of n. It is also observed from Table 3.1 that $\|\psi^{[cs]}-\psi_n^{*[cs]}\|_{L_N^{2\,[cs]}}\to 0$ as $n\to\infty$ which verifies our theoretical claim of convergence mentioned in equation (3.65).

Table 3.1: The actual error and theoretical error bound in case of Example 3.1 for different values of n

Degree of Legendre polynomial	Actual error $\|\psi^{[cs]} - \psi_n^{*[cs]}\|_{L_N^{2\,[cs]}}$	Error bound for $\|\psi^{[cs]} - \psi_n^{*[cs]}\|_{L_N^{2\,[cs]}}$
$n = 1$	0.14219	2.63021
$n = 2$	0.14211	1.96229
$n = 3$	0.13718	1.34228
$n = 4$	0.09658	0.85279
$n = 5$	0.08276	0.52128
$n = 6$	0.05990	0.28566
$n = 7$	0.05104	0.11371
$n = 8$	0	0

Table 3.2: The theoretical error bound in case of Example 3.2 for different values of n

Degree of Legendre polynomial	Error bound for $\|\psi^{[cs]} - \psi_n^{*[cs]}\|_{L_N^{2\,[cs]}}$
$n = 1$	6.92480
$n = 2$	5.18346
$n = 3$	3.58536
$n = 4$	2.19992
$n = 5$	1.25032
$n = 6$	0.63041
$n = 7$	0.28904
$n = 8$	0.11840
$n = 9$	0.04362

Figures 3.1.(a), 3.1.(b), show the comparison of real and imaginary part of the exact solution of $\psi_1^{[cs]}(x)$ with corresponding real and imaginary part of approximate solution $\psi_{1n}^{*\,[cs]}(x)$ for $n = 1, 2, \ldots, 8$. And, the comparison of real and imaginary part of the exact solution of $\psi_2^{[cs]}(x)$ with corresponding real and imaginary part of approximate solution $\psi_{2n}^{*\,[cs]}(x)$ for $n = 1, 2, \ldots, 8$, is shown in Figures 3.1.(c), 3.1.(d). Further, it can be observed in all the Figures 3.1.(a), 3.1.(b), 3.1.(c), 3.1.(d) as the value of n increases, the sequence of approximate solutions converges to the exact solution with respect to $L_N^{2\,[cs]}$ norm. This example also verifies the Theorem 3.3.1.

Example 3.2 Consider the following system of Cauchy type singular integral equations

$$
\begin{aligned}
6\fint_{-1}^{1}\frac{\varphi_1^{[cs]}(t)}{t-x}dt+10\fint_{-1}^{1}\frac{\varphi_2^{[cs]}(t)}{t-x}dt+40\fint_{-1}^{1}\frac{\varphi_3^{[cs]}(t)}{t-x}dt&=g_1^{[cs]}(x),\\
10\fint_{-1}^{1}\frac{\varphi_1^{[cs]}(t)}{t-x}dt+50\fint_{-1}^{1}\frac{\varphi_2^{[cs]}(t)}{t-x}dt+30\fint_{-1}^{1}\frac{\varphi_3^{[cs]}(t)}{t-x}dt&=g_2^{[cs]}(x),\\
20\fint_{-1}^{1}\frac{\varphi_1^{[cs]}(t)}{t-x}dt+10\fint_{-1}^{1}\frac{\varphi_2^{[cs]}(t)}{t-x}dt+25\fint_{-1}^{1}\frac{\varphi_3^{[cs]}(t)}{t-x}dt&=g_3^{[cs]}(x),
\end{aligned}
\tag{3.90}
$$

where

$$
\begin{aligned}
g_1^{[cs]}(x)&=e^{5x}+i\cos(5x),\\
g_2^{[cs]}(x)&=i\left(7x^5+1\right)e^{5x},\\
g_3^{[cs]}(x)&=\left(90x^8+45x^6+1\right)+ix^{10}.
\end{aligned}
$$

Using equation (3.4), the analytical solution for the above problem is given by

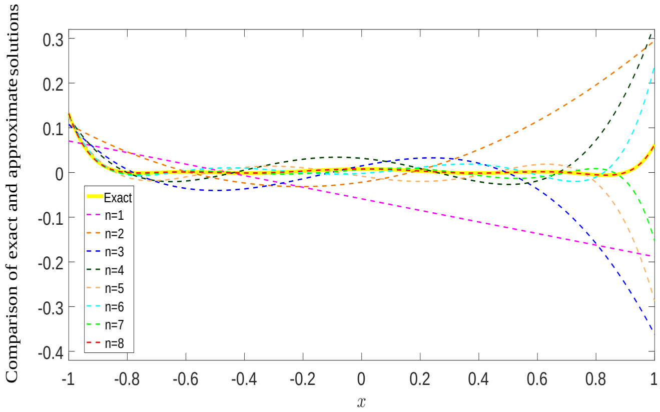
$$
\begin{pmatrix}\varphi_1^{[cs]}(x)\\ \varphi_2^{[cs]}(x)\\ \varphi_3^{[cs]}(x)\end{pmatrix}=-\frac{1}{\pi^2}\sqrt{\frac{1-x}{1+x}}\begin{pmatrix}\fint_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\frac{\eta_1^{[cs]}(t)}{t-x}dt\\ \fint_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\frac{\eta_2^{[cs]}(t)}{t-x}dt\\ \fint_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\frac{\eta_3^{[cs]}(t)}{t-x}dt\end{pmatrix},
\tag{3.91}
$$

where

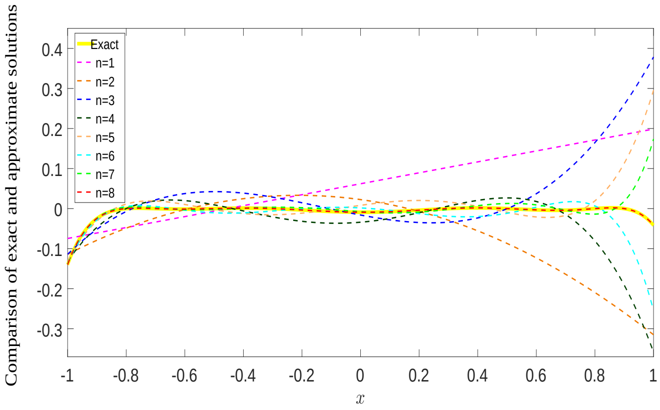
$$
\begin{aligned}
\eta_1^{[cs]}(t)&=\frac{765t^8}{134}+\frac{765t^6}{268}-\frac{19e^{5t}}{536}+\frac{17}{268}+i\left(\frac{17t^{10}}{268}-\frac{21}{536}e^{5t}t^5-\frac{3e^{5t}}{536}-\frac{19}{536}\cos(5t)\right),\\
\eta_2^{[cs]}(t)&=-\frac{99t^8}{134}-\frac{99t^6}{268}-\frac{7e^{5t}}{536}-\frac{11}{1340}+i\left(-\frac{11t^{10}}{1340}+\frac{91}{536}e^{5t}t^5+\frac{13e^{5t}}{536}-\frac{7}{536}\cos(5t)\right),\\
\eta_3^{[cs]}(t)&=-\frac{45t^8}{67}-\frac{45t^6}{134}+\frac{9e^{5t}}{268}-\frac{1}{134}+i\left(-\frac{t^{10}}{134}-\frac{49}{1340}e^{5t}t^5+\frac{9}{268}\cos(5t)-\frac{7e^{5t}}{1340}\right).
\end{aligned}
$$

For the Example 3.2, although the analytic solution is known but the above analytical solution is of limited use as described in Section 3.1. Therefore, we solve Example 3.2 by the numerical method described in Section 3.2. However, in this case it is not possible to calculate the actual error between the approximate and exact solution, since the available solution (3.91) has no explicit form. But the error bound can be calculated by using the equation (3.64) for different values of n which is of great importance. As the error bound decreases to zero with the increase in value of n, it implies that the sequence of approximate solutions is converging to the exact solution and the same is reflected from the results detailed in Table 3.2

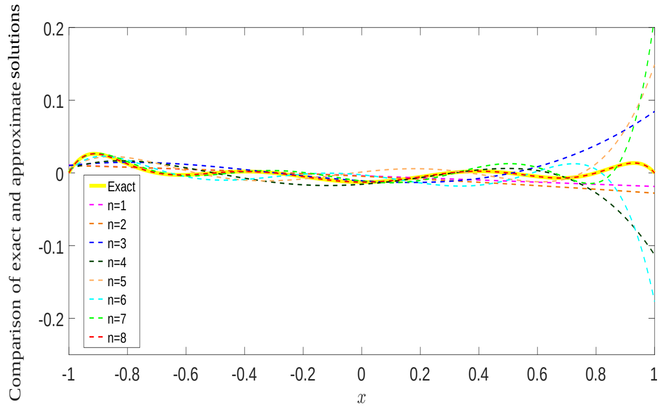
up to $n = 9$.



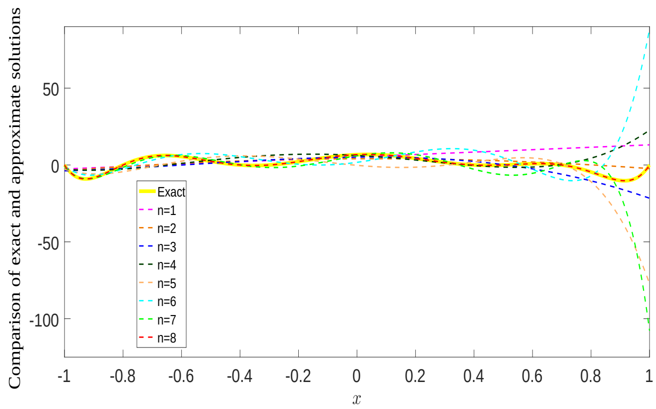
(a) Comparison of $Re[\psi_1{}^{[cs]}(x)]$ with $Re[\psi^*_{1n}{}^{[cs]}(x)]$



(b) Comparison of $Im[\psi_1{}^{[cs]}(x)]$ with $Im[\psi^*_{1n}{}^{[cs]}(x)]$

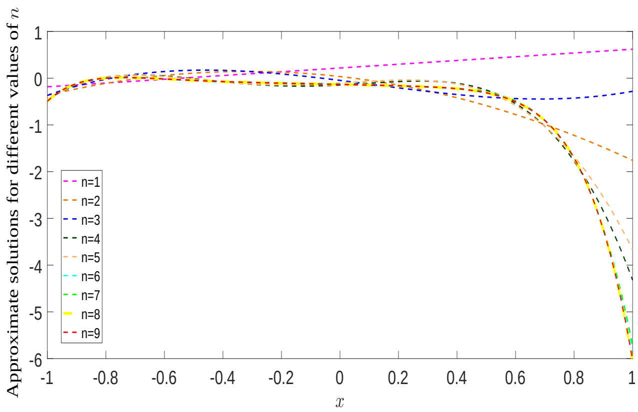


(c) Comparison of $Re[\psi_2{}^{[cs]}(x)]$ with $Re[\psi^*_{2n}{}^{[cs]}(x)]$

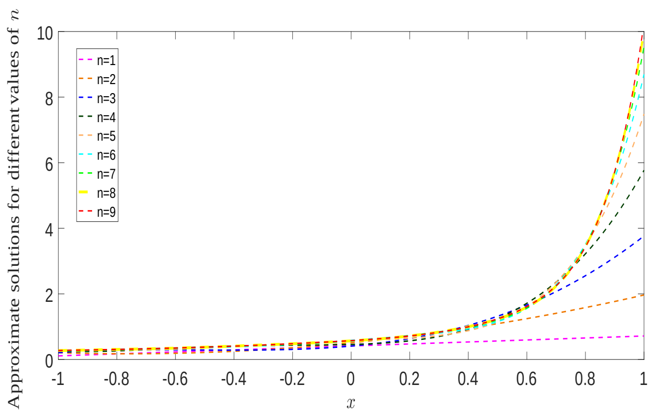


(d) Comparison of $Im[\psi_2{}^{[cs]}(x)]$ with $Im[\psi^*_{2n}{}^{[cs]}(x)]$

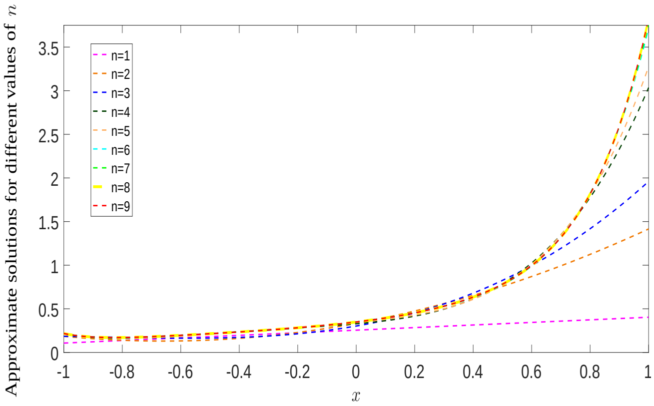
Figure 3.1: Comparison of exact solution with approximate solutions of Example 3.1.



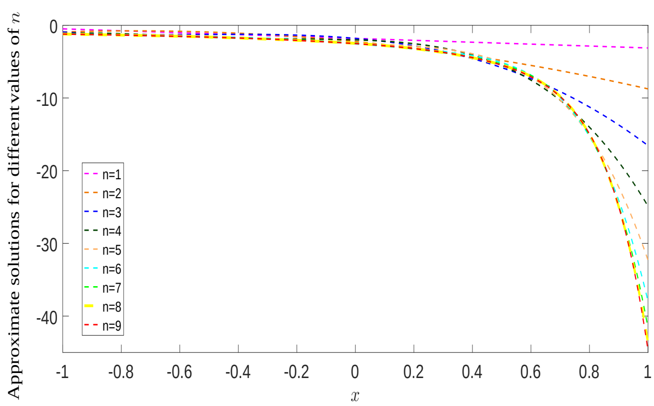
(a) Comparison of approximate solutions $Re[\psi^*_{1n}{}^{[cs]}(x)]$



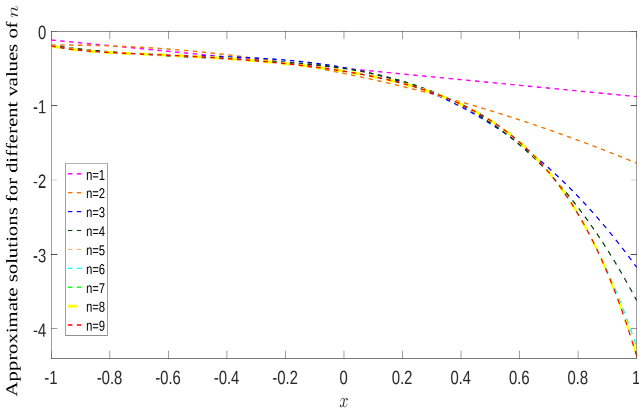
(b) Comparison of approximate solutions $Im[\psi^*_{1n}{}^{[cs]}(x)]$



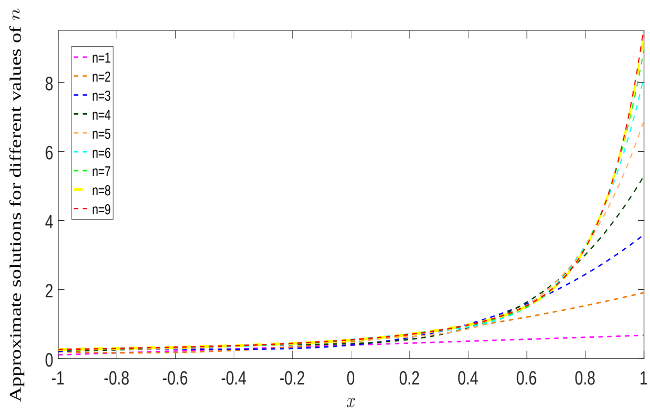
(c) Comparison of approximate solutions $Re[\psi^{*}_{2n}{}^{[cs]}(x)]$



(d) Comparison of approximate solutions $Im[\psi^{*}_{2n}{}^{[cs]}(x)]$



(e) Comparison of approximate solutions $Re[\psi^{*}_{3n}{}^{[cs]}(x)]$



(f) Comparison of approximate solutions $Im[\psi^{*}_{3n}{}^{[cs]}(x)]$

Figure 3.2: Comparison of approximate solutions for different values of n in case of Example 3.2.

The comparison of real and imaginary part of the approximate solutions of $\psi^{*}_{1n}{}^{[cs]}(x)$ for $n = 1, 2, \ldots, 9$, is shown in Figures 3.2.(a), 3.2.(b). Similarly, Figures 3.2.(c), 3.2.(d); 3.2.(e), 3.2.(f) show the comparison of real and imaginary part of the approximate solutions of $\psi^{*}_{2n}{}^{[cs]}(x)$, $\psi^{*}_{3n}{}^{[cs]}(x)$ respectively for $n = 1, 2, \ldots, 9$.

Conclusion

In this chapter, we have proposed a Legendre polynomial based numerical method to find the approximate solution of system of Cauchy type singular integral equations of first kind with zero index. The proposed method converts the system of integral equations into a linear system of algebraic equations. It is shown theoretically and numerically that the proposed method gives the exact solution for the systems having known functions in the form of polynomial. The well-posedness conditions in the sense of Hadamard are verified for the system of Cauchy singular integral equations as well as for the system of linear algebraic equations which is obtained after applying the proposed method to the system of singular integral equations with Cauchy kernel. Further, the derived theoretical error bound is verified with the aid of numerical examples which implies the good behavior of our proposed numerical method.

Chapter 4

Numerical solution of hypersingular integral equations

4.1 Introduction

Hypersingular integral equations has great importance in the field of aeronautics [7, 70, 72, 84]. These singular integral equations occur during the formulation of interference or interaction problems such as wing and tail surfaces problem, pairs or collections of wings (biplanes or cascades) problems [7]. These hypersingular integral equations also appeared during the mathematical modeling of vortex wakes behind aircraft at altitude, near the ground at the time of takeoff and landing operations [41]. Many two dimensional problems of aerodynamics can be modeled as singular integral equation such as for an inviscid incompressible fluid flow past a rectangular airfoil problem reduces into a hypersingular integral equation [10]. Apart from problems of aeronautics, the problems of electromagnetic scattering [41], acoustics [53], fluid dynamics [113], electromagnetic diffraction [135], elasticity [27] and fracture mechanics [16] are modeled as hypersingular integral equations. In early nineties, Parsons and Martin [102] used hypersingular integral equation to study the problem of water wave scattering. Further, these equations for crack problems in the field of fracture mechanics [5, 20, 22] have been explored by many researchers. Many analytical methods and numerical methods such as complex variable function method [14], boundary element method [64], polynomial approximation method [16, 78, 80], reproducing kernel method [23], piecewise linear approximations on a nonuniform grid [118] for solving singular integral equations have been already explored. However, in order to solve hypersingular integral equations, search for a method which is easy to understand, easy to implement, providing high accuracy, converging fast and numerically stable is always there. In this article, we propose a residual based Galerkin's method with Legendre polynomial as basis function to find the approximate solution of hypersingular integral equations. The hypersingular integral equations occur during the formulation of many boundary value problems of practical interest are of the form

$$\neq\!\!\!\!\!\!\int_{-1}^{1} \frac{\chi^{[h]}(t)}{(t-x)^2}dt - \int_{-1}^{1} k^{[h]}(x,t)\chi^{[h]}(t)dt = g^{[h]}(x),\ |x|<1, \tag{4.1}$$

with $\chi^{[h]}(\pm 1) = 0$. The functions $g^{[h]}(x)$ and $k^{[h]}(x,t)$ are known real valued Hölder continuous over the interval $[-1,1]$ and $[-1,1]\times[-1,1]$ respectively. $\chi^{[h]}(x)$ is an unknown function defined over the interval $[-1,1]$. The first integral in equation (4.1) is understood to be exist in the sense of Hadamard finite part integral. Also, the unknown function $\chi^{[h]}(x)$ is assumed to have the Hölder continuous derivative of first order which is required in order to ensure the existence of finite-part integral [85].

4.2 Method of solution to the problem

A function $\chi^{[h]}(t)$ defined over the interval [-1,1] in equation (4.1) with $\chi^{[h]}(\pm 1) = 0$ can be represented [44] as follows:

$$\chi^{[h]}(t) = \sqrt{1-t^2}\,\xi^{[h]}(t), \tag{4.2}$$

where $\xi^{[h]}(t)$ is an unknown function defined over the interval $\in [-1,1]$. Using equation (4.2) in equation (4.1), we obtain

$$\neq\!\!\!\!\!\!\int_{-1}^{1} \frac{\xi^{[h]}(t)\sqrt{1-t^2}}{(t-x)^2}dt - \int_{-1}^{1} \sqrt{1-t^2}k^{[h]}(x,t)\xi^{[h]}(t)dt = g^{[h]}(x),\ |x|<1. \tag{4.3}$$

Now we approximate the function $\xi^{[h]}(t)$ by orthonormalized Legendre polynomials as follows:

$$\xi^{[h]}(t) \approx \xi_n^{*[h]}(t) = \sum_{j=0}^{n} a_j^{[h]} e_j(t), \tag{4.4}$$

where $\{e_j(t)\}_{j=0}^n$ denotes the set of $(n+1)$ orthonormalized Legendre polynomials on $[-1,1]$ and $a_j^{[h]}$; $j = 1,2,\ldots,n$, are unknown constant coefficients.

On using the approximation defined in equation (4.4) for $\xi^{[h]}(t)$ in equation (4.3), the residual error $\mathcal{R}^{[h]}(x, a_0^{[h]}, a_1^{[h]}, a_2^{[h]}, \ldots, a_n^{[h]})$ is as follows:

$$\mathcal{R}^{[h]}(x, a_0^{[h]}, a_1^{[h]}, a_2^{[h]}, \ldots, a_n^{[h]}) = \neq\!\!\!\!\!\!\int_{-1}^{1} \frac{\xi_n^{*[h]}(t)\sqrt{1-t^2}}{(t-x)^2}dt - \int_{-1}^{1} \sqrt{1-t^2}k^{[h]}(x,t)\xi_n^{*[h]}(t)dt - g^{[h]}(x),\ |x|<1. \tag{4.5}$$

In Galerkin's method, this residual error $\mathcal{R}^{[h]}(x, a_0^{[h]}, a_1^{[h]}, a_2^{[h]}, \ldots, a_n^{[h]})$ is assumed to be orthogonal to the space spanned by orthonormal polynomials, say $E = span\{e_j(x)\}_{j=0}^n$, that is, we have

$$\langle \mathcal{R}^{[h]}(x, a_0^{[h]}, a_1^{[h]}, a_2^{[h]}, \ldots, a_n^{[h]}), e_j \rangle_{L^2} = 0,\ \forall j = 0,1,2,\ldots,n. \tag{4.6}$$

Using equation (4.5) for $j = 0,1,2,\ldots,n$, equation (4.6) becomes

$$\left\langle \neq\!\!\!\!\!\!\int_{-1}^{1} \frac{\xi_n^{*[h]}(t)\sqrt{1-t^2}}{(t-x)^2}dt - \int_{-1}^{1} k^{[h]}(x,t)\xi_n^{*[h]}(t)\sqrt{1-t^2}dt - g^{[h]}(x), e_0 \right\rangle_{L^2} = 0,$$

$$\left\langle \fint_{-1}^{1} \frac{\xi_n^{*^{[h]}}(t)\sqrt{1-t^2}}{(t-x)^2}dt - \int_{-1}^{1} k^{^{[h]}}(x,t)\xi_n^{*^{[h]}}(t)\sqrt{1-t^2}dt - g^{^{[h]}}(x), e_1 \right\rangle_{L^2} = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \tag{4.7}$$
$$\left\langle \fint_{-1}^{1} \frac{\xi_n^{*^{[h]}}(t)\sqrt{1-t^2}}{(t-x)^2}dt - \int_{-1}^{1} k^{^{[h]}}(x,t)\xi_{sn}^{*^{[h]}}(t)\sqrt{1-t^2}dt - g^{^{[h]}}(x), e_n \right\rangle_{L^2} = 0.$$

In order to evaluate singular integral in each integral equation of system (4.7), we use the results of [61] (see formula (35) of reference [61]) and we get a linear system of order $(n+1)\times(n+1)$. The above system (4.7) can be written in matrix form as

$$B^{[h]^T} A^{^{[h]}} = B_1^{^{[h]}} A^{^{[h]}} = G^{^{[h]}}, \tag{4.8}$$

where

$$B_1^{[h]} = B^{[h]^T}, \quad B^{[h]} = \begin{pmatrix} b_{00}^{[h]} & b_{01}^{[h]} & \dots & b_{0n}^{[h]} \\ b_{10}^{[h]} & b_{11}^{[h]} & \dots & b_{1n}^{[h]} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n0}^{[h]} & b_{n1}^{[h]} & \dots & b_{nn}^{[h]} \end{pmatrix}, \; A^{[h]} = \begin{pmatrix} a_0^{[h]} \\ a_1^{[h]} \\ \vdots \\ a_n^{[h]} \end{pmatrix}, \; G^{[h]} = \begin{pmatrix} g_0^{[h]} \\ g_1^{[h]} \\ \vdots \\ g_n^{[h]} \end{pmatrix}, \tag{4.9}$$

$$b^{[h]}{}_{rq} = \int_{-1}^{1}\Big(\fint_{-1}^{1}\Big(\frac{\sqrt{1-t^2}e_r(t)}{(t-x)^2}dt - \int_{-1}^{1} k^{^{[h]}}(x,t)e_r(t)\sqrt{1-t^2}dt\Big)e_q(x)dx, \;\; r, \;\; q = 0,1,2,\dots,n,$$

$$g_q^{^{[h]}} = \int_{-1}^{1} g^{^{[h]}}(x)e_q(x)dx, \;\; q = 0,1,2,\dots,n.$$

Now we solve the linear system (4.8) which gives the value of unknown coefficients $a_j^{^{[h]}}$; $j = 0,1,2,\dots,n$. The substitution of these $a_j^{^{[h]}}$ values in equation (4.4) provides the approximate solution of equation (4.3) and hence, for equation (4.1). This completes the description of proposed method use to find the approximate solution of equation (4.1).

4.3 Convergence analysis and Error bound

In this section, we show that sequence $\{\xi_n^{*^{[h]}}\}_{n=0}^{\infty}$ converges to the exact solution $\xi_n^{*^{[h]}}(x)$ in L^2 space and we derive the error bound.

4.3.1 Function Spaces

We initialize this subsection by defining function spaces in which the error analysis analysis of numerical method takes place. $L^2[-1,1] = \{u(t) : [-1,1] \to \mathbb{R} : \int_{-1}^{1}(u(t))^2dt < \infty\}$ is a Hilbert space of all square integrable real functions over the interval $[-1,1]$, equipped with the norm $\|\cdot\|^2_{L^2}$ and inner product $\langle .,.\rangle_{L^2}$ defined as

$$\|u(t)\|_{L^2} = \Big(\int_{-1}^{1}(u(t))^2dt\Big)^{1/2} \text{ for } u(t)\in L^2[-1,1], \tag{4.10}$$

$$\langle u,v\rangle_{L^2} = \int_{-1}^{1}u(t)v(t)dt \text{ for } u(t),v(t)\in L^2[-1,1]. \tag{4.11}$$

Now we define the set of functions

$$M^{[h]} = \{u(t)\in L^2 : \sum_{j=0}^{\infty}(d_j^{[h]})^2\langle u,e_j\rangle^2_{L^2} < \infty\}, \tag{4.12}$$

where

$$(d_j^{[h]})^2 = \| S^{[h]}e_j \|^2_{L^2}, \tag{4.13}$$

$$S^{[h]}e_j(x) = \neq\!\!\!\!\!\!\int_{-1}^{1}\frac{\sqrt{1-t^2}}{(t-x)^2}e_j(t)dt. \tag{4.14}$$

The set $M^{[h]}$ is a subspace of L^2 space which is made into a Hilbert space with the following norm $\|\cdot\|_{M^{[h]}}$ and inner product $\langle .,.\rangle_{M^{[h]}}$

$$\| u \|^2_{M^{[h]}} = \sum_{j=0}^{\infty}(d_j^{[h]})^2\langle u,e_j\rangle^2_{L^2} \text{ for } u(t)\in M^{[h]}, \tag{4.15}$$

$$\langle u,v\rangle_{M^{[h]}} = \sum_{j=0}^{\infty}(d_j^{[h]})^2\langle u,e_j\rangle_{L^2}\langle v,e_j\rangle_{L^2} \text{ for } u(t),v(t)\in M^{[h]}, \tag{4.16}$$

where $d_j^{[h]}$ is same as defined in equation (4.13). Let $v_k^{[h]}(x) = \frac{e_k(x)}{d_k^{[h]}}$, then $\| v_k^{[h]} \|_{M^{[h]}} = 1$. This set $\{v_k^{[h]}(x)\}_{k=0}^{\infty}$ forms complete orthonormal basis for the Hilbert space $M^{[h]}$, that is, if $u\in M^{[h]}$, then we have

$$u(x) = \sum_{k=0}^{\infty}\langle u,v_k^{[h]}\rangle_{M^{[h]}}v_k^{[h]}(x). \tag{4.17}$$

Now operating the operator $S^{[h]}$, defined in equation (4.14) on orthonormalized Legendre polynomials $e_j(x); j=0,1,2,\dots,n$ and using the results of [61] (see formula (35) of reference [61]), we obtain:

$$S^{[h]}e_0(x) = -\pi e_0(x),$$

$$S^{[h]}e_1(x) = -2\pi e_1(x),$$
$$S^{[h]}e_2(x) = -\pi\left[\frac{\sqrt{5}}{4\sqrt{3}}e_0(x) + 3e_2(x)\right],$$
....
$$S^{[h]}e_n(x) = \sum_{i=0}^{n} c_i^{[h]}e_i(x); \text{ where } c_i^{[h]} = \langle S^{[h]}e_n, e_i\rangle_{L^2}, i = 0, 1, 2, \ldots, n. \tag{4.18}$$

4.3.2 Error Bound

With the help of equation (4.18), we can extend the operator $S^{[h]} : M^{[h]} \to L^2$ as a bounded linear operator defined as

$$S^{[h]}\xi^{[h]}(x) = \sum_{j=0}^{\infty}\langle \xi^{[h]}, e_j\rangle_{L^2}\sum_{i=0}^{j}\langle S^{[h]}e_j, e_i\rangle_{L^2}e_i(x) \in L^2[-1, 1]. \tag{4.19}$$

Using equation (4.19), we find the norm of bounded linear operator $S^{[h]}$

$$\| S^{[h]}\xi^{[h]} \|_{L^2}^2 = \sum_{j=0}^{\infty}(d_j^{[h]})^2\langle \xi^{[h]}, e_j\rangle_{L^2}^2 = \| \xi^{[h]} \|_{M^{[h]}}^2 . \tag{4.20}$$

Hence, using equation (4.20), we obtain

$$\| S^{[h]} \| = 1. \tag{4.21}$$

Moreover, the mapping $S^{[h]} : M^{[h]} \to L^2$ is one-one and onto. Therefore, following *Bounded Inverse Theorem* defined in Theorem 1.1.1, the operator $(S^{[h]})^{-1} : L^2 \to M^{[h]}$ exists as a bounded linear operator which is defined as

$$(S^{[h]})^{-1}\xi^{[h]}(x) = \sum_{j=0}^{\infty}\frac{\langle \xi^{[h]}(x), e_j(x)\rangle_{L^2}}{d_j}e_j(x). \tag{4.22}$$

Now, with the aid of equation (4.22), we calculate the norm for linear operator $(S^{[h]})^{-1}$

$$\| (S^{[h]})^{-1}\xi^{[h]}(x) \|_{M^{[h]}} = \| \xi^{[h]}(x) \|_{L^2} . \tag{4.23}$$

Finally, using the above equation (4.23), the norm of bounded operator $(S^{[h]})^{-1}$ is

$$\| (S^{[h]})^{-1} \| = 1. \tag{4.24}$$

Now we consider the mapping ${Q_n}^{[h]} : L^2 \to L^2$, where ${Q_n}^{[h]}$ is an orthogonal projection operator which is defined as

$${Q_n}^{[h]}\xi^{[h]}(x) = \sum_{j=0}^{n}\langle \xi^{[h]}, e_j\rangle_{L^2}e_j(x), \tag{4.25}$$

where n is the degree of orthonormalized Legendre polynomial by which $\xi^{[h]}(x)$ is approximated. After defining all the operators and function spaces, we can finally estimate the error bound for the error which occurs in approximating the exact solution of equation (4.1) by taking its projection from Hilbert space onto a vector space spanned by the orthonormalized Legendre polynomials, say $E = span\{e_j(x)\}_{j=0}^{n}$. Writing equation (4.3) in an operator equation from the spaces $M^{[h]}$ to L^2

$$(S^{[h]} - K^{[h]})\xi^{[h]}(x) = g^{[h]}(x),\ g^{[h]}(x) \in L^2,\ \xi^{[h]}(x) \in M^{[h]}, \tag{4.26}$$

where the operator $S^{[h]}$ is same as defined in equation (4.14) and we define the operator $K^{[h]} : M^{[h]} \to L^2$ is as follows:

$$K^{[h]}\xi^{[h]}(x) = \int_{-1}^{1} \sqrt{1-t^2}\, k^{[h]}(x,t)\xi^{[h]}(t)dt. \tag{4.27}$$

The operator $K^{[h]} : M^{[h]} \to L^2$ defined in equation (4.27) will be a compact operator with the following assumption:

$$\int_{-1}^{1}\int_{-1}^{1} (\sqrt{1-t^2}\, k^{[h]}(x,t))^2 dtdx < \infty. \tag{4.28}$$

The equation (4.26) has a unique solution if and only if the inverse of the operator $(S^{[h]} - K^{[h]})$ exists as a bounded linear operator. We assume that the bounded linear operator $(S^{[h]} - K^{[h]})^{-1}$ exists. From equation (4.6), we have

$$Q_n^{[h]}\Big((S^{[h]} - K^{[h]})\xi_n^{*[h]}(x) - g^{[h]}(x)\Big) = 0. \tag{4.29}$$

Since the function $S^{[h]}\xi_n^{*[h]}(x)$ is a polynomial therefore following the definition of operator $Q_n^{[h]}$, we get

$$Q_n^{[h]}S^{[h]}\xi_n^{*[h]}(x) = S^{[h]}\xi_n^{*[h]}(x). \tag{4.30}$$

Using the above fact, equation (4.29) becomes

$$S^{[h]}\xi_n^{*[h]}(x) - Q_n^{[h]}K^{[h]}\xi_n^{*[h]}(x) = Q_n^{[h]}g^{[h]}(x). \tag{4.31}$$

Since the operator $S^{[h]}$ has a bounded inverse and the operator $K^{[h]}$ is compact due to the condition (4.28), therefore, for all n arbitrarily large, say $n > n_0$, $(S^{[h]} - Q_n^{[h]}K^{[h]})^{-1}$ exists as a bounded linear operator [45]. Hence, equation (4.31) has a unique solution, which is as follows:

$$\xi_n^{*[h]}(x) = (S^{[h]} - Q_n^{[h]}K^{[h]})^{-1}Q_n^{[h]}g^{[h]}(x). \tag{4.32}$$

Now from equations (4.26) and (4.32), we have

$$\xi^{[h]}(x) - \xi_n^{*[h]}(x) = (S^{[h]} - Q_n^{[h]} K^{[h]})^{-1}\Big(g^{[h]}(x) - Q_n^{[h]} g^{[h]}(x) + K^{[h]}\xi^{[h]}(x) - Q_n^{[h]} K^{[h]}\xi^{[h]}(x)\Big). \quad (4.33)$$

Taking norm of both the sides of equation (4.33), we obtain

$$\begin{aligned}\| \xi^{[h]} - \xi_n^{*[h]} \|_{M^{[h]}} \leq & \| (S^{[h]} - Q_n^{[h]} K^{[h]})^{-1} \| \| g^{[h]} - Q_n^{[h]} g^{[h]} \|_{L^2} \\ & + \| (S^{[h]} - Q_n^{[h]} K^{[h]})^{-1} \| \| K^{[h]}\xi^{[h]}(x) - Q_n^{[h]} K^{[h]}\xi^{[h]}(x) \|_{L^2} . \end{aligned} \quad (4.34)$$

Due to the assumption defined in equation (4.28), the operator $S^{[h]}$ is a Hilbert-Schmidt operator [45] and hence, $\| K^{[h]} - Q_n^{[h]} K^{[h]} \|_{L^2} \to 0$ as $n \to \infty$. Also, we have $\| g^{[h]} - Q_n^{[h]} g^{[h]} \|_{L^2} \to 0$ as $n \to \infty$. Thus, we get $\| \xi^{[h]} - \xi_n^{*[h]} \|_{M^{[h]}} \to 0$ as $n \to \infty$.

Further, due to the fact that if $\xi^{[h]} \in M^{[h]}$, then we have

$$\| \xi^{[h]} \|_{L^2} \leq \| \xi^{[h]} \|_{M^{[h]}} . \quad (4.35)$$

On using equation (4.35), the equation (4.34) can be written as follows:

$$\begin{aligned}\| \xi^{[h]} - \xi_n^{*[h]} \|_{L^2} \leq & \| (S^{[h]} - Q_n^{[h]} K^{[h]})^{-1} \| \| g^{[h]} - Q_n g^{[h]} \|_{L^2} \\ & + \| (S^{[h]} - Q_n^{[h]} K^{[h]})^{-1} \| \| K\xi^{[h]}(x) - Q_n^{[h]} K^{[h]}\xi^{[h]}(x) \|_{L^2} . \end{aligned} \quad (4.36)$$

Also, the sequence $\{\xi_n^{*[h]}\}_{n=0}^{\infty}$ converges to the exact solution in L^2 space

$$\| \xi^{[h]} - \xi_n^{*[h]} \|_{L^2} \to 0 \text{ as } n \to \infty. \quad (4.37)$$

Hence, the convergence of the sequence $\{\xi_n^{*[h]}\}_{n=0}^{\infty}$ is shown.

4.3.3 Well-posedness of linear system

In this subsection, we show that the problem (4.32) is a well-posed problem in the sense of Hadamard. The problem (4.32) has a solution, this is due to the existence of inverse operator $(S^{[h]} - Q_n^{[h]} K^{[h]})^{-1}$ which is already proved in previous section. In order to prove the uniqueness of solution to the problem (4.32), we use principle of contradiction. If possible suppose that problem (4.32) has two distinct solutions say y_1 and y_2. Then we obtain

$$S^{[h]} y_1(x) - Q_n^{[h]} K^{[h]} y_1(x) = Q_n^{[h]} g^{[h]}(x), \quad (4.38)$$

and

$$S^{[h]} y_2(x) - Q_n^{[h]} K^{[h]} y_2(x) = Q_n^{[h]} g^{[h]}(x). \quad (4.39)$$

From equations (4.38) and (4.39), we get

$$(S^{[h]} - Q_n^{[h]} K^{[h]})(y_1(x) - y_2(x)) = 0. \tag{4.40}$$

Since the operator $(S^{[h]} - Q_n^{[h]} K^{[h]})^{-1}$ exists as a bounded linear operator, then from equation (4.40), we get

$$y_1(x) = y_2(x), \quad | x |< 1. \tag{4.41}$$

Equation (4.41) contradicts our assumption. Hence, we have proved that solution to the problem (4.32) exists uniquely. Moreover, the continuity of the inverse operator $(S^{[h]} - Q_n^{[h]} K^{[h]})^{-1}$ indicates that a small change in the given data will produce a small change in the solution. As it shown that the problem (4.32) satisfies all the well-posedness conditions therefore it is a well-posed problem.

4.3.4 Existence and uniqueness of solution to linear system

This subsection shows that the solution of the linear system (4.8) which is obtained after using the method described in Section 4.2, to the equation (4.1), has a unique solution. We start the proof by defining the prolongation operator [45] $\mathcal{P}_n^{[h]} : \mathbb{R}^{n+1} \to E$ as follows:

$$\mathcal{P}_n^{[h]} G^{[h]} = \sum_{j=0}^{n} \langle g^{[h]}, e_j \rangle_{L^2} e_j(x) \in E, \tag{4.42}$$

where $\mathbb{R}^{n+1}$ is a real vector space [67] whose elements are $(n+1)$-tuples of real numbers, $E = span\{e_j(t)\}_{j=0}^{n}$ and $G^{[h]}$ is already defined in equation (4.9). Now from the definition of orthogonal projection $Q_n^{[h]}$, we get

$$Q_n^{[h]} g^{[h]}(x) = \sum_{j=0}^{n} \langle g^{[h]}, e_j \rangle_{L^2} e_j(x). \tag{4.43}$$

Following equations (4.42) and (4.43), we have

$$\mathcal{P}_n^{[h]} G^{[h]} = Q_n^{[h]} g^{[h]}(x), \quad g^{[h]}(x) \in L^2, \quad G^{[h]} \in \mathbb{R}^{n+1}, \quad | x |< 1. \tag{4.44}$$

Further, we define a restriction operator [45] $R_n^{[h]} : E \to \mathbb{R}^{n+1}$ as follows:

$$R_n^{[h]} \xi_n^{*[h]}(x) = (\langle \xi_n^{*[h]}(x), e_0 \rangle_{L^2}, \langle \xi_n^{*[h]}(x), e_1 \rangle_{L^2}, \ldots \langle \xi_n^{*[h]}(x), e_n \rangle_{L^2})^T \in \mathbb{R}^{n+1}, \tag{4.45}$$

where the function $\xi_n^{*[h]}(x)$ is already defined in equation (4.4). On using the orthogonal property of Legendre polynomials in equation (4.4), we get

$$a_j^{[h]} = \langle \xi_n^{*[h]}(x), e_j \rangle_{L^2}, \; j = 0, 1, \ldots, n. \tag{4.46}$$

Therefore, from equations (4.45) and (4.46), we obtain

$$R_n{}^{[h]}\xi_n^{*[h]}(x) = A^{[h]}, \tag{4.47}$$

where the matrix $A^{[h]}$ is already defined in equation (4.9). Since bounded linear operator $(S^{[h]} - Q_n^{[h]}K^{[h]})^{-1}$ exists which is already shown in Section 4.3. This existence of the operator $(S^{[h]} - Q_n^{[h]}K^{[h]})^{-1}$ implies that $\xi_n^{*[h]}(x)$ also exists uniquely. Therefore, from equation (4.47), the solution $A^{[h]}$ of system (4.8) exists uniquely for every given $G^{[h]}$. Also, the inverse of matrix $B_1^{[h]}$ exist [119].

4.4 Illustrative examples

This section shows the efficiency of our proposed numerical method and verification of the theoretical results obtained in Section 4.3, with the help of numerical illustrations.

Example 4.1 Consider the following equation [23]:

$$\fint_{-1}^{1} \frac{\xi^{[h]}(t)\sqrt{1-t^2}}{(t-x)^2}dt + \int_{-1}^{1} xt\xi^{[h]}(t)\sqrt{1-t^2}dt = \pi\Big(-8x^3 + \frac{17}{8}x - 1\Big),\ |x| < 1. \tag{4.48}$$

This problem has an exact solution $\xi^{[h]}(x) = 1 + 2x^3$. We find the approximate solution with the aid of proposed method.

Table 4.1: Details of obtained numerical results for different n in case of Example 4.1

n	Actual Error (In L^2 norm)	Error bound
1	0.46595	9.29106
2	0.46595	9.29106
3	0	0

It can been seen from Table 4.1 that the approximate solution obtained by implementing the method discussed in Section 4.2 is identical to the exact solution at just $n = 3$.

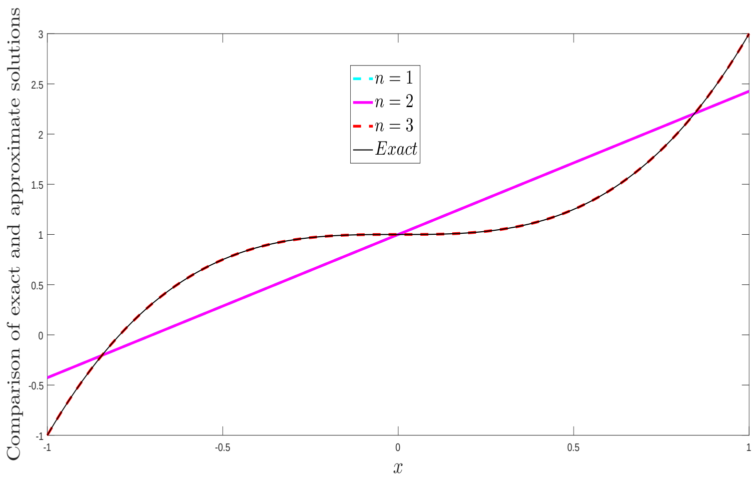


Figure 4.1: Comparison of exact solution with approximate solutions of Example 4.1

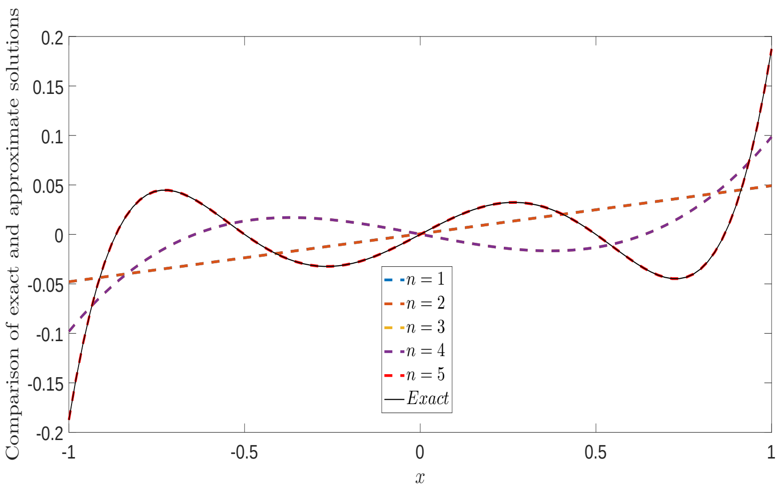


Figure 4.2: Comparison of exact solution with approximate solutions of Example 4.2

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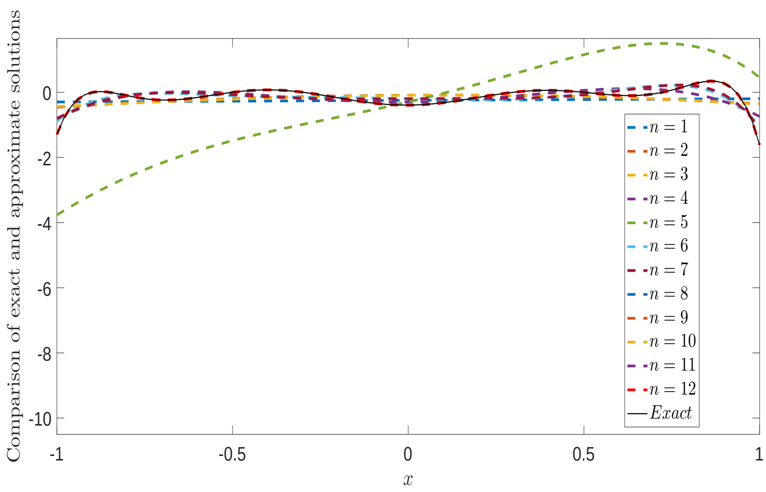


Figure 4.3: Comparison of exact solution with approximate solutions of Example 4.3

Although Chen [23] also solved this problem up to $n = 25$ by using method of reproducing kernel, but his method did not give the exact solution. In Figure 4.1, the exact solution is compared with the approximate solutions for different values of n. Further, it can be seen from the Figure 4.1 that the exact and approximate solutions coincides. The actual error is also calculated for Example 4.1 with respect to L^2 norm and detailed in Table 4.1. These actual error lies under the error bound which is obtained by using equation (4.34).

Example 4.2 We consider another singular integral equation:

$$\int_{-1}^{1} \frac{\xi^{[h]}(t)\sqrt{1-t^2}}{(t-x)^2}dt + \int_{-1}^{1} \frac{\xi^{[h]}(t)\sqrt{1-t^2}\exp(t+x)}{12}dt = \pi g^{[h]}(x), \ |x| < 1, \tag{4.49}$$

where $g^{[h]}(x) = -\frac{9}{8}x + 6x^3 - 6x^5 + \frac{81}{64}\exp(x)I_2(1) - \frac{31}{4}\exp(x)I_3(1)$, and I_2, I_3 are modified Bessel functions of first kind of order two and three respectively. $\xi^{[h]}(x) = \frac{3x}{16} - x^3 + x^5$ is the exact solution.

Table 4.2: Details of obtained numerical results for different n in case of Example 4.2

n	Actual Error (In L^2 norm)	Error bound
1	0.06481	1.43228
2	0.06480	1.43183
3	0.05688	1.31111
4	0.05687	1.31110
5	2.19155×10^{-14}	1.19141×10^{-10}

The numerical results for actual error and error bound are detailed in Table 4.2. For $n = 1, 2, \ldots, 5$, the comparison between approximate solutions and exact solution is shown in Figure 4.2. Further the figure shows the convergence of sequence of approximate solutions to the exact one.

Table 4.3: Details of obtained numerical results for different n in case of Example 4.3

n	Actual Error (In L^2 norm)	Error bound
1	0.38064	8.84521
2	0.37161	8.74921
3	0.37160	8.74689
4	0.28343	7.76302
5	0.27503	7.64867
6	0.25688	7.24213
7	0.25274	7.15094
8	0.00570	0.19899
9	0.00564	0.19739
10	0.00054	0.02196
11	0.00049	0.02053
12	5.95198×10^{-16}	1.07677×10^{-10}

Example 4.3 Consider one more hypersingular integral equation:

$$\neq\!\!\!\!\!\!\int_{-1}^{1} \frac{\xi^{[h]}(t)\sqrt{1-t^2}}{(t-x)^2}dt + \int_{-1}^{1} \frac{(x+x^2)\xi^{[h]}(t)\sqrt{1-t^2}}{36+12s}dt = \pi g^{[h]}(x), \ |x| < 1, \tag{4.50}$$

where

$$\begin{aligned} g^{[h]}(x) =& \frac{1326099}{655360} - \frac{1469711672063x}{7864320} + \frac{84573531x}{320\sqrt{2}} - \frac{1470155415887x^2}{7864320} + \frac{84573531x^2}{320\sqrt{2}} + \frac{115527x^3}{10240} \\ &+ \frac{4953727x^4}{16384} - \frac{88851x^5}{2560} - \frac{5394557x^6}{10240} + \frac{7571x^7}{320} + \frac{1453239x^8}{5120} + \frac{327x^9}{64} - \frac{1793x^{10}}{256} \\ &- \frac{45x^{11}}{16} + \frac{1885x^{12}}{128}. \end{aligned}$$

The exact solution of this example is

$$\xi^{[h]}(x) = \frac{1}{640}\Big(-252 + 45x + 4510x^2 - 725x^3 - 22258x^4 + 2680x^5 + 38000x^6 - 2000x^7 - 20252x^8 - 252x^9 + 45x^{10} + 150x^{11} - 725x^{12}\Big).$$

Table 4.3 shows all the obtained numerical results for Example 4.3. It is clear from Table 4.3 that the error is decreasing with the increase in the value of n which verifies the result (4.37). The approximate solutions for $n = 1, 2, \ldots, 12$ are compared with the exact solution in Figure 4.3. And, it can be seen from the Figure 4.3, that the exact solution coincides with the approximate solution. The actual error is also calculated for Example 4.3 with respect to norm in L^2 and it is shown in Table 4.3 that the actual error is lying with in the error bound which follows from our result defined by equation (4.36).

Conclusion

A residual based Galerkin's method is proposed to find the numerical solution of hypersingular integral equation of first kind. The method converts the singular integral equation with hyper kernel into a linear system. This system can be solved easily. The existence and uniqueness of solution of linear system which is obtained as a result of approximation of the exact solution of equation (4.1), are shown. The convergence of sequence of approximate solutions to the exact solution, is proved in L^2 space. The error bound for the error in approximate and exact solution of equation (4.1) is also derived. Moreover, the good behavior of the proposed method can be seen as all numerical illustrations verify our theoretical claim such as actual error lies within the theoretical error bound and sequence of approximate solutions is convergent.

Chapter 5

Numerical solution of system of hypersingular integral equations

5.1 Introduction

Many boundary value problems of applied mathematics, mathematical physics, engineering etc., can be modeled as system of hypersingular integral equation (HSIE). For instance, crack problems arise in the field of fracture mechanics [6, 76], thermoelastic stress problems around an arbitrary number of arbitrarily-located planar cracks [4] are formulated as system of HSIEs. Also, two-dimensional hypersingular integral equations over a curve in complex plane like water scattering problems [75], curved crack problems [19, 21] can be decomposed into a system of HSIEs. The hypersingular integral equations in one dimensional occurs frequently in the field of electromagnetic scattering [90], acoustics [53], aerodynamics [70], elasticity [27] and fracture mechanics [16]. The numerical methods to find the approximate solution of these equations are extensively available in literature such as reproducing kernel method [23], quadrature method [101], complex variable function method [14], boundary element method [64], polynomial approximation [78, 80, 82]. However, the literature on numerical methods to find the approximate solution of system of hypersingular integral equations of first kind is still scarce. Therefore, there is a great need to develop more numerical methods. In this chapter, we propose residual based Galerkin's method with Legendre polynomial as basis function to find the approximate solution of system of HSIEs. In many known physical problems of practical interest [6, 19, 36] the system of hypersingular integral equations (HSIEs) of first kind occurs in the following form:

$$\oint_{-1}^{1} \frac{1}{(t-x)^2} \begin{pmatrix} \varphi_1^{[hs]}(t) \\ \varphi_2^{[hs]}(t) \\ \vdots \\ \varphi_N^{[hs]}(t) \end{pmatrix} dt \; - \int_{-1}^{1} \begin{pmatrix} k_{11}^{[hs]}(x,t) & k_{12}^{[hs]}(x,t) & \dots & k_{1N}^{[hs]}(x,t) \\ k_{21}^{[hs]}(x,t) & k_{22}^{[hs]}(x,t) & \dots & k_{2N}^{[hs]}(x,t) \\ \vdots & \vdots & \ddots & \vdots \\ k_{N1}^{[hs]}(x,t) & k_{N2}^{[hs]}(x,t) & \dots & k_{NN}^{[hs]}(x,t) \end{pmatrix} \begin{pmatrix} \varphi_1^{[hs]}(t) \\ \varphi_2^{[hs]}(t) \\ \vdots \\ \varphi_N^{[hs]}(t) \end{pmatrix} dt$$
$$= \begin{pmatrix} g_1^{[hs]}(x) \\ g_2^{[hs]}(x) \\ \vdots \\ g_N^{[hs]}(x) \end{pmatrix}, \tag{5.1}$$

$\mid x \mid < 1$, with boundary condition $\varphi_j^{[hs]}(\pm 1) = 0, j = 1, 2, \dots, N$. $\varphi_j^{[hs]}(x)$, $j = 1, 2, \dots, N$, is an unknown complex valued functions defined over the interval $[-1, 1]$. In system (5.1), the functions $k_{ij}^{[hs]}(x,t)$, $j, i = 1, 2, \dots, N$, $g_j^{[hs]}$, $j = 1, 2, \dots, N$, are known complex valued Hölder continuous functions defined over the interval $[-1,1] \times [-1,1]$ and $[-1,1]$ respectively. In the above system (5.1) each HSIE is understood in the sense of Hadamard finite-part integral (HFP). Also, in order to ensure the existence of HFP each unknown function $\varphi_j^{[hs]}(x)$, $j = 1, 2, \dots, N$, is assumed to have the first derivative to be Hölder continuous [85]. The analytical solution of system (5.1) for the case when the functions $k_{ij}^{[hs]}(x,t) = 0$, $\forall j, i = 1, 2, \dots, N$, can be obtained by extending the analytic solution of HISE in one dimension [83] as follows:

$$\begin{pmatrix} \varphi_1^{[hs]}(x) \\ \varphi_2^{[hs]}(x) \\ \vdots \\ \varphi_N^{[hs]}(x) \end{pmatrix} = \begin{pmatrix} \frac{1}{\pi^2} \int_{-1}^{1} g_1^{[hs]}(t) \ln \left| \frac{t-x}{1-xt+\sqrt{(1-t^2)(1-x^2)}} \right| dt \\ \frac{1}{\pi^2} \int_{-1}^{1} g_2^{[hs]}(t) \ln \left| \frac{t-x}{1-xt+\sqrt{(1-t^2)(1-x^2)}} \right| dt \\ \vdots \\ \frac{1}{\pi^2} \int_{-1}^{1} g_N^{[hs]}(t) \ln \left| \frac{t-x}{1-xt+\sqrt{(1-t^2)(1-x^2)}} \right| dt \end{pmatrix}. \tag{5.2}$$

Firstly, the above analytic solution is for a particular case of more general form (5.1). Secondly, it is of limited use as it is not possible to solve equation (5.2) for every choice of $g_j^{[hs]}(t)$ due to the presence of singularity in the solution. Further, the analytical solution of more general form (5.1) where $k_{ij}^{[hs]}(x,t) \neq 0$, $\forall\, j, i = 1, 2, \dots, N$, is not known. Therefore, in this chapter we propose a numerical method to find its approximate solution.

5.2 Numerical method

In this section, we describe the numerical method to find an approximate solution of the system (5.1). In order to find the approximate solution, we replace each [44] unknown function $\varphi_j^{[hs]}(t)$, $j = 1, 2, \dots, N$, in the system (5.1) as follows:

$$\varphi_j^{[hs]}(t) = \sqrt{1-t^2}\,\psi_j^{[hs]}(t), \; j = 1, 2, \dots, N, \tag{5.3}$$

where each $\psi_j^{[hs]}(t),\ j = 1, 2, \ldots, N$, is an unknown function of $t \in [-1, 1]$. Using equation (5.3), the system (5.1) becomes

$$\begin{aligned}&\oint_{-1}^{1} \frac{\sqrt{1-t^2}}{(t-x)^2} \begin{pmatrix} \psi_1^{[hs]}(t) \\ \psi_2^{[hs]}(t) \\ \vdots \\ \psi_N^{[hs]}(t) \end{pmatrix} dt \\ &- \int_{-1}^{1} \sqrt{1-t^2} \begin{pmatrix} k_{11}^{[hs]}(x,t) & k_{12}^{[hs]}(x,t) & \ldots & k_{1N}^{[hs]}(x,t) \\ k_{21}^{[hs]}(x,t) & k_{22}^{[hs]}(x,t) & \ldots & k_{2N}^{[hs]}(x,t) \\ \vdots & \vdots & \ddots & \vdots \\ k_{N1}^{[hs]}(x,t) & k_{N2}^{[hs]}(x,t) & \ldots & k_{NN}^{[hs]}(x,t) \end{pmatrix} \begin{pmatrix} \psi_1^{[hs]}(t) \\ \psi_2^{[hs]}(t) \\ \vdots \\ \psi_{1n}^{*\ [hs]}(t) \end{pmatrix} dt = \begin{pmatrix} g_1^{[hs]}(x) \\ g_2^{[hs]}(x) \\ \vdots \\ g_N^{[hs]}(x) \end{pmatrix}.\end{aligned} \tag{5.4}$$

In operator form equation (5.4) can be written as

$$(\mathbb{S}^{[hs]} - \mathbb{K}^{[hs]})\, \psi^{[hs]}(x) = g^{[hs]}(x), \tag{5.5}$$

where

$$\psi^{[hs]}(x) = \begin{pmatrix} \psi_1^{[hs]}(x) \\ \psi_2^{[hs]}(x) \\ \vdots \\ \psi_{1n}^{*\ [hs]}(x) \end{pmatrix},\ g^{[hs]}(x) = \begin{pmatrix} g_1^{[hs]}(x) \\ g_2^{[hs]}(x) \\ \vdots \\ g_N^{[hs]}(x) \end{pmatrix}, \tag{5.6}$$

$$\mathbb{S}^{[hs]} = \begin{pmatrix} S^{[hs]} & O & \ldots & O \\ O & S^{[hs]} & \ldots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \ldots & S^{[hs]} \end{pmatrix}, \quad \mathbb{K}^{[hs]} = \begin{pmatrix} K_{11}^{[hs]} & K_{12}^{[hs]} & \ldots & K_{1N}^{[hs]} \\ K_{21}^{[hs]} & K_{22}^{[hs]} & \ldots & K_{2N}^{[hs]} \\ \vdots & \vdots & \ddots & \vdots \\ K_{N1}^{[hs]} & K_{N2}^{[hs]} & \ldots & K_{NN}^{[hs]} \end{pmatrix}, \tag{5.7}$$

the symbol O denotes the zero operator and $S^{[hs]}$, $K_{ij}^{[hs]}$ are linear integral operators defined as

$$S^{[hs]}\psi_j^{[hs]}(x) = \oint_{-1}^{1} \frac{\sqrt{1-t^2}\psi_j^{[hs]}(t)}{(t-x)^2} dt,\ \ j = 1, 2, \ldots, N, \tag{5.8}$$

$$K_{ji}^{[hs]}\psi_j^{[hs]}(x) = \int_{-1}^{1} \sqrt{1-t^2} k_{ij}^{[hs]}(x,t)\psi_j^{[hs]}(t)dt,\ \ j, i = 1, 2, \ldots, N. \tag{5.9}$$

Now we approximate the unknown function $\psi_j^{[hs]}(x)$ in system (5.5) as

$$\psi^{[hs]}(x) \approx \psi_n^{*[hs]}(x) = (\psi_{1n}^{*\ [hs]}(x), \psi_{2n}^{*\ [hs]}(x), \ldots, \psi_{Nn}^{*\ [hs]}(x))^T, \tag{5.10}$$

where

$$\psi_j^{[hs]}(x) \approx \psi_{jn}^{*\,[hs]}(x) = \sum_{i=0}^{n} a_{ji}^{[hs]} e_i(x),\ \forall j = 1,2,\ldots,N, \tag{5.11}$$

and $\{e_i(x)\}_{i=0}^{n}$ denotes the set of $(n+1)$ orthonormalized Legendre polynomials on $[-1,1]$. In order to solve the system (5.1) or its equivalent system (5.5), we use residual based Galerkin's method. We define the residual error as

$$\mathcal{R}^{[hs]}(x; a_{10}^{[hs]}, a_{11}^{[hs]}, \ldots, a_{1n}^{[hs]}; a_{20}^{[hs]}, a_{21}^{[hs]}, \ldots, a_{2n}^{[hs]}; \ldots; a_{N0}^{[hs]}, a_{N1}^{[hs]}, \ldots, a_{Nn}^{[hs]}) =$$

$$\begin{pmatrix} \mathcal{R}_1^{[hs]}(x; a_{10}^{[hs]}, a_{11}^{[hs]}, \ldots, a_{1n}^{[hs]}; a_{20}^{[hs]}, a_{21}^{[hs]}, \ldots, a_{2n}^{[hs]}; \ldots; a_{N0}^{[hs]}, a_{N1}^{[hs]}, \ldots, a_{Nn}^{[hs]}) \\ \mathcal{R}_2^{[hs]}(x; a_{10}^{[hs]}, a_{11}^{[hs]}, \ldots, a_{1n}^{[hs]}; a_{20}^{[hs]}, a_{21}^{[hs]}, \ldots, a_{2n}^{[hs]}; \ldots; a_{N0}^{[hs]}, a_{N1}^{[hs]}, \ldots, a_{Nn}^{[hs]}) \\ \vdots \\ \mathcal{R}_N^{[hs]}(x; a_{10}^{[hs]}, a_{11}^{[hs]}, \ldots, a_{1n}^{[hs]}; a_{20}^{[hs]}, a_{21}^{[hs]}, \ldots, a_{2n}^{[hs]}; \ldots; a_{N0}^{[hs]}, a_{N1}^{[hs]}, \ldots, a_{Nn}^{[hs]}) \end{pmatrix}, \tag{5.12}$$

where

$$\begin{aligned} \mathcal{R}_j^{[hs]}(x; a_{10}^{[hs]}, a_{11}^{[hs]}, \ldots, a_{1n}^{[hs]}; a_{20}^{[hs]}, a_{21}^{[hs]}, \ldots, a_{2n}^{[hs]}; \ldots; a_{N0}^{[hs]}, a_{N1}^{[hs]}, \ldots, a_{Nn}^{[hs]}) &= \fint_{-1}^{1} \frac{\sqrt{1-t^2}\,\psi_{jn}^{*\,[hs]}(t)}{(t-x)^2} dt \\ - \sum_{i=1}^{N} \int_{-1}^{1} \sqrt{1-t^2}\, k_{ij}^{[hs]}(x,t) \psi_{in}^{*\,[hs]}(t) dt - g_j^{[hs]}(x),\ j = 1,2,\ldots,N, \end{aligned} \tag{5.13}$$

is assumed to be orthogonal to the vector space $E = span\{e_k(x)\}_{k=0}^{n}$, which is of finite dimension. Therefore, we have

$$\begin{aligned} &\langle \mathcal{R}_j^{[hs]}(x; a_{10}^{[hs]}, a_{11}^{[hs]}, \ldots, a_{1n}^{[hs]}; a_{20}^{[hs]}, a_{21}^{[hs]}, \ldots, a_{2n}^{[hs]}; \ldots; a_{N0}^{[hs]}, a_{N1}^{[hs]}, \ldots, a_{Nn}^{[hs]}), e_k(x) \rangle_{L^2} = 0, \\ &\forall\, k = 0,1,\ldots,n, \quad \forall\, j = 1,2,\ldots,N, \end{aligned} \tag{5.14}$$

where $\langle \cdot, \cdot \rangle_{L^2}$ represents the inner product in $L^2[-1,1]$ space. To solve the singular integrals appeared on the right hand side of equation (5.13), we use the Hadamard finite-part integral formulas (see equation (35) in reference [61]). Finally, the system of integral equations results into a system of $N \times (n+1)$ linear algebraic equations in $N \times (n+1)$ unknowns which can be solved easily. The explicit expression for the system of linear algebraic equations is as follows:

$$\begin{aligned} &\sum_{q=0}^{n} a_{jq}^{[hs]} \int_{-1}^{1} \fint_{-1}^{1} \frac{\sqrt{1-t^2}\, e_q(t) e_r(x)}{(t-x)^2} dt dx - \sum_{q=0}^{n} \sum_{i=1}^{N} \int_{-1}^{1} \int_{-1}^{1} \sqrt{1-t^2}\, k_{ij}^{[hs]}(x,t) a_{iq}^{[hs]} e_q(t) e_r(x) \\ &= \int_{-1}^{1} g_j^{[hs]}(x) e_r(x) dx, \quad r = 0,1,2,\ldots,n,\ \ j = 1,2,3,\ldots,N. \end{aligned}$$

In order to write the above system (5.15) in matrix form, we define

$$\hat{e}_{qr} = e_q(t)e_r(x), \quad q = 0, 1, 2, \ldots, n, \quad r = 0, 1, 2, \ldots, n,$$
$$\hat{g}_{jr}^{[hs]} = \langle g_j^{[hs]}, e_r \rangle_{L^2}, \quad j = 1, 2, 3, \ldots, N, \quad r = 0, 1, 2, 3, \ldots, n,$$

and

$$\hat{K}_{ji}^{[hs]} \hat{e}_{qr} = \begin{cases} \displaystyle\int_{-1}^{1}\int_{-1}^{1} \Big(-\sqrt{1-t^2} k_{ij}^{[hs]}(x,t)\hat{e}_{qr} \Big) dt dx, & i \neq j, \\ \displaystyle\int_{-1}^{1} \Big(\fint_{-1}^{1} \frac{\sqrt{1-t^2}}{(t-x)^2} - \int_{-1}^{1} \sqrt{1-t^2} k_{ij}^{[hs]}(x,t) \Big) \hat{e}_{qr} dt dx, & i = j, \quad i,\ j = 1, 2, 3, \ldots, N. \end{cases}$$

Therefore, the matrix form of system (5.15) is given by

$$\hat{K}^{[hs]} \otimes {E^{[hs]}}^T \; vec\, A^{[hs]} = vec\, G^{[hs]}, \tag{5.15}$$

where

$$\hat{K}^{[hs]} = \begin{pmatrix} \hat{K}_{11}^{[hs]} & \hat{K}_{12}^{[hs]} & \ldots & \hat{K}_{1N}^{[hs]} \\ \hat{K}_{21}^{[hs]} & \hat{K}_{22}^{[hs]} & \ldots & \hat{K}_{2N}^{[hs]} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{K}_{N1}^{[hs]} & \hat{K}_{N2}^{[hs]} & \ldots & \hat{K}_{NN}^{[hs]} \end{pmatrix}, \; G^{[hs]} = \begin{pmatrix} g_{10}^{[hs]} & g_{11}^{[hs]} & \ldots & g_{1n}^{[hs]} \\ g_{20}^{[hs]} & g_{21}^{[hs]} & \ldots & g_{2n}^{[hs]} \\ \vdots & \vdots & \ddots & \vdots \\ g_{N0}^{[hs]} & g_{N1}^{[hs]} & \ldots & g_{Nn}^{[hs]} \end{pmatrix},$$

$$E^{[hs]} = \begin{pmatrix} \hat{e}_{00} & \hat{e}_{01} & \ldots & \hat{e}_{0n} \\ \hat{e}_{10} & \hat{e}_{11} & \ldots & \hat{e}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{e}_{n0} & \hat{e}_{n1} & \ldots & \hat{e}_{nn} \end{pmatrix}, \; A^{[hs]} = \begin{pmatrix} a_{10}^{[hs]} & a_{11}^{[hs]} & \ldots & a_{1n}^{[hs]} \\ a_{20}^{[hs]} & a_{21}^{[hs]} & \ldots & a_{2n}^{[hs]} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N0}^{[hs]} & a_{N1}^{[hs]} & \ldots & a_{Nn}^{[hs]} \end{pmatrix}.$$

In equation (5.15), $vec\ A$ and $vec\ \hat{G}$ are vectors [47] of order $N \times (n+1)$ which are defined as

$$vec\, A^{[hs]} = \begin{pmatrix} A_1^{[hs]} \\ A_2^{[hs]} \\ \vdots \\ A_N^{[hs]} \end{pmatrix}, \; vec\, G^{[hs]} = \begin{pmatrix} G_1^{[hs]} \\ G_2^{[hs]} \\ \vdots \\ G_N^{[hs]} \end{pmatrix}, \tag{5.16}$$

where $A_j^{[hs]}$, $G_j^{[hs]}$, $j = 1, 2, \ldots, N$, denote the j^{th} column of matrices $A^{[hs]}$ and $G^{[hs]}$ respectively. The equation (5.15) can be further written as

$$\hat{D}^{[hs]} \hat{C}^{[hs]} = \hat{E}^{[hs]}, \tag{5.17}$$

where $\hat{D}^{[hs]} = \hat{K}^{[hs]} \otimes E^{[hs]^T}$, $\hat{C}^{[hs]} = vec\ A^{[hs]}$, $\hat{E}^{[hs]} = vec\ G^{[hs]}$ and symbol $\otimes$ denotes the kronecker [47] of matrices $\hat{K}^{[hs]}$ and $E^{[hs]^T}$. This linear system (5.17) can be solved easily.

5.3 Error analysis

In this section, we initiate the error analysis by defining suitable function spaces which are essential for deriving error bound of error and showing the convergence of sequence of the approximate solutions of system (5.1). We further show the well-posedness of system of linear algebraic equations obtained as a result of approximation of system (5.1). Finally, we prove that under certain condition, the approximate solution of (5.5) is an exact solution. Further, it should be noted that the elements of the function space ${L^2_N}^{[hs]}$ and $M_N^{[hs]}$ are in vectorial form.

5.3.1 Function spaces

The function space ${L^2_N}^{[hs]}$ can be defined as

$$ {L^2_N}^{[hs]} = \{u(x) = (u_1(x), u_2(x), ..., u_N(x))^T : u_j(x) \in L^2[-1,1];\ j = 1, 2, \ldots, N\}, \tag{5.18}$$

which is a Hilbert space with respect to the norm $\|\cdot\|^{[hs]}_{L^2_N}$

$$\| u \|^2_{{L^2_N}^{[hs]}} = \frac{1}{N} \sum_{j=1}^{N} \| u_j \|^2_{L^2}, \text{ for } u(x) \in {L^2_N}^{[hs]}, \tag{5.19}$$

generated by the inner product $\langle \cdot, \cdot \rangle_{L^2_N}$

$$\langle u, v \rangle_{L^2_N} = \frac{1}{N} \sum_{j=1}^{N} \langle u_j, v_j \rangle_{L^2}, \text{ for } u(x), v(x) \in {L^2_N}^{[hs]}, \tag{5.20}$$

where $L^2[-1,1] = \{u_j : [-1,1] \to \mathbb{C} : \int_{-1}^{1} u_j(x)\overline{u_j(x)}dx < \infty, j = 1, 2, \ldots, N\}$ is a Hilbert Space of all complex valued functions which are Lebesgue square-integrable in the interval $[-1,1]$ with respect to the norm $\|\cdot\|^2_{L^2}$

$$\|u_j\|_{L^2} = \left(\int_{-1}^{1} |u_j(x)|^2 dx \right)^{1/2}, \text{ for } u_j(x) \in L^2,\ \ j = 1, 2, \ldots, N, \tag{5.21}$$

induced by the inner product

$$\langle u_j, v_j \rangle_{L^2} = \int_{-1}^{1} (u_j(x)\overline{v_j(x)})dx, \text{ for } u_j(x),\ v_j(x) \in L^2,\ j = 1, 2, \ldots, N. \tag{5.22}$$

Now we consider the set of functions

$$M_N^{[hs]} = \{u(x) \in {L^2_N}^{[hs]} : \frac{1}{N} \sum_{j=1}^{N} \sum_{k=0}^{\infty} (d_k^{[hs]})^2 |\langle u_j, e_k \rangle_{L^2}|^2 < \infty\}, \tag{5.23}$$

where

$$u(x) = (u_1(x), u_2(x)(x), \dots, u_N(x))^T \in L_N^{2\,[hs]}, \text{ for } u_j(x) \in L^2, \;\; j = 1, 2, \dots, N, \tag{5.24}$$

and

$$(d_k^{[hs]})^2 = \sum_{i=0}^{\infty} |\langle S^{[hs]} e_k, e_i \rangle_{L^2}|^2. \tag{5.25}$$

The set $M_N^{[hs]}$ is a subspace of L_N^2 which is a Hilbert space with respect to the norm $\| \cdot \|_{M_N^{[hs]}}$

$$\| u \|^2_{M_N^{[hs]}} = \frac{1}{N} \sum_{j=1}^{N} \| u_j \|^2_{M^{[hs]}}, \text{ for } u(x) \in M_N^{[hs]}, \tag{5.26}$$

induced by inner product $\langle \cdot, \cdot \rangle_{M_N^{[hs]}}$

$$\langle u, v \rangle_{M_N^{[hs]}} = \frac{1}{N} \sum_{j=1}^{N} \langle u_j, v_j \rangle_{M^{[hs]}}, \text{ for } u(x), \; v(x) \in M_N^{[hs]}, \tag{5.27}$$

where $M^{[hs]}$ is a subspace of $L^2[-1, 1]$ and can be define in a similar manner as in [44, 98]

$$M^{[hs]} = \{u_j(x) \in L^2 : \sum_{k=0}^{\infty} (d_k^{[hs]})^2 |\langle u_j, e_k \rangle_{L^2}|^2 < \infty\}. \tag{5.28}$$

In the above equation each $e_k(t)$ is a polynomial of degree k and l_k is same as defined in equation (5.25). This subspace $M^{[hs]}$ is a Hilbert space with respect to the norm $\| \cdot \|_{M^{[hs]}}$

$$\| u_j \|^2_{M^{[hs]}} = \sum_{k=0}^{\infty} (d_k^{[hs]})^2 |\langle u_j, e_k \rangle_{L^2}|^2, \text{ for } u_j(x) \in M^{[hs]}, \tag{5.29}$$

induced by inner product $\langle \cdot, \cdot \rangle_{M^{[hs]}}$

$$\langle u_j, v_j \rangle_{M^{[hs]}} = \sum_{k=0}^{\infty} (d_k^{[hs]})^2 \langle u_j, e_k \rangle_{L^2} \overline{\langle v_j, e_k \rangle}_{L^2}, \text{ for } u_j(x), \; v_j(x) \in M^{[hs]}. \tag{5.30}$$

5.3.2 Convergence analysis

In this subsection, we show that the sequence of approximate solutions of system (5.1) converges to the exact solution in L_N^2 space. With the aid of Hadamard finite-part integral formula (see equation (35) in reference [61]), we obtain a polynomial

$$S^{[hs]} e_k(x) = \sum_{i=0}^{k} c_i^{[hs]} e_i(x), \tag{5.31}$$

where the coefficients $c_i^{[hs]} = \langle S^{[hs]} e_k, e_i \rangle_{L^2}$, $i = 0, 1, 2, \ldots, k$, $k = 0, 1, 2, \ldots, n$. If we define $v_k^{[hs]}(x) = \dfrac{e_k(x)}{l_k}$, then $\| v_k \|_B = 1$, these $\{v_k^{[hs]}\}_{k=0}^{\infty}$ form complete orthonormal basis for the space B, that is, if $\psi_j{}^{[hs]} \in M^{[hs]}$, then we have

$$\psi_j{}^{[hs]}(x) = \sum_{k=0}^{\infty} \langle \psi_j{}^{[hs]}, v_k^{[hs]} \rangle_{M^{[hs]}} v_k^{[hs]}(x). \tag{5.32}$$

Using equation (5.31), we extend the operator $\mathbb{S}^{[hs]} : M_N^{[hs]} \to L_N^{2\ [hs]}$ as a bounded linear operator

$$\mathbb{S}^{[hs]} \psi^{[hs]}(x) = (S^{[hs]} \psi_1{}^{[hs]}(x), S^{[hs]} \psi_2{}^{[hs]}(x), \ldots, S^{[hs]} \psi_N{}^{[hs]}(x))^T \in L_N^{2\ [hs]}, \tag{5.33}$$

where

$$S^{[hs]} \psi_j{}^{[hs]}(x) = \sum_{k=0}^{\infty} \langle \psi_j{}^{[hs]}, e_k \rangle_{L^2} \sum_{i=0}^{k} \langle S^{[hs]} e_k, e_i \rangle_{L^2} e_i(x) \in L^2[-1, 1], \quad j = 0, 1, \ldots, N. \tag{5.34}$$

Since we have

$$\| S^{[hs]} \psi_j{}^{[hs]} \|_{L^2}^2 = \sum_{k=0}^{\infty} l_k^2 |\langle \psi_j{}^{[hs]}, e_k \rangle_{L^2}|^2 = \| \psi_j{}^{[hs]} \|_{M^{[hs]}}^2, \text{ for } j = 1, 2, \ldots, N, \tag{5.35}$$

therefore, the norm of bounded linear operator $\mathbb{S}^{[hs]}$ is

$$\| \mathbb{S}^{[hs]} \| = 1. \tag{5.36}$$

Also, the mapping $\mathbb{S}^{[hs]} : M_N^{[hs]} \to L_N^{2\ [hs]}$ is bijective. Using Theorem 1.1.1, the inverse operator $(\mathbb{S}^{[hs]})^{-1} : L_N^{2\ [hs]} \to M_N^{[hs]}$ exists as a bounded linear operator and defined as

$$(\mathbb{S}^{[hs]})^{-1} \psi^{[hs]}(x) = \Big((S^{[hs]})^{-1} \psi_1{}^{[hs]}(x), (S^{[hs]})^{-1} \psi_2{}^{[hs]}(x), \ldots, (S^{[hs]})^{-1} \psi_N{}^{[hs]}(x) \Big)^T,$$

where

$$(S^{[hs]})^{-1} \psi_j{}^{[hs]}(x) = \sum_{k=0}^{\infty} \frac{\langle \psi_j{}^{[hs]}(x), e_k(x) \rangle_{L^2}}{l_k} e_k(x). \tag{5.37}$$

Therefore, we obtain the norm for the bounded linear operator $(\mathbb{S}^{[hs]})^{-1}$ as

$$\| (\mathbb{S}^{[hs]})^{-1} \| = 1. \tag{5.38}$$

Now we assume that $\sqrt{1 - t^2}\, k_{ji}(x, t)$, $j, i = 0, 1, 2, \ldots, N$, are Lebesgue square-integrable functions of x and t

$$\int_{-1}^{1}\int_{-1}^{1}|\sqrt{1-t^2}\, k_{ij}^{[hs]}(x,t)|^2 dtdx < \infty, \tag{5.39}$$

and therefore, the operator $\mathbb{K}^{[hs]} : M_N^{[hs]} \to L_N^{2\,[hs]}$ is a compact operator. Further, with the help of suitable spaces defined in subsection 5.3.1, reconsider the operator equation (5.5) from $M_N^{[hs]}$ to $L_N^{2\,[hs]}$

$$(\mathbb{S}^{[hs]} - \mathbb{K}^{[hs]})\psi^{[hs]}(x) = g^{[hs]}(x),\ g^{[hs]}(x) \in L_N^{2\,[hs]},\ \psi^{[hs]}(x) \in M_N^{[hs]}. \tag{5.40}$$

We also assume that the operator $(\mathbb{S}^{[hs]} - \mathbb{K}^{[hs]})$ has a bounded inverse and hence, the system (5.1) has a unique solution.

We now show the convergence of sequence of approximate solutions $\psi_n^{*[hs]}(x)$ to the exact solution $\psi^{[hs]}(x)$ with respect to $L_N^{2\,[hs]}$ norm.

From the equation (5.14), we have

$$Q_N^{n\,[hs]}((\mathbb{S}^{[hs]} - \mathbb{K}^{[hs]})\psi_n^{*[hs]}(x) - g^{[hs]}(x)) = 0, \tag{5.41}$$

where $Q_N^{n\,[hs]}$ is an orthogonal projection from $L_N^{2\,[hs]}$ to $L_N^{2\,[hs]}$

$$Q_N^{n\,[hs]}\psi^{[hs]}(x) = (Q_n^{\,[hs]}\psi_1^{\,[hs]}(x), Q_n^{\,[hs]}\psi_2^{\,[hs]}(x), \ldots, Q_n^{\,[hs]}\psi_N^{\,[hs]}(x))^T, \tag{5.42}$$

such that $Q_n^{\,[hs]}$ is the orthogonal projection onto the finite dimensional vector space $E = span\{e_i(x)\}_{i=0}^{n}$ and defined as

$$Q_n^{\,[hs]}\psi_j^{\,[hs]}(x) = \sum_{k=0}^{n}\langle\psi_j^{\,[hs]}, e_k\rangle_{L^2} e_k(x). \tag{5.43}$$

Since the function $\psi_n^{*[hs]}(x)$ defined in equation (5.10), is a polynomial therefore with the aid of equation (5.31), the function $\mathbb{S}^{[hs]}\psi_n^{*[hs]}(x)$ is also a polynomial.

Hence, by the definition of $Q_N^{n\,[hs]}$, we have

$$Q_N^{n\,[hs]}\mathbb{S}^{[hs]}\psi_n^{*[hs]}(x) = \mathbb{S}^{[hs]}\psi_n^{*[hs]}(x). \tag{5.44}$$

Now using the equation (5.44) in equation (5.41), we get

$$\mathbb{S}^{[hs]}\psi_n^{*}(x) - Q_N^{n\,[hs]}\mathbb{K}^{[hs]}\psi_n^{*[hs]}(x) = Q_N^{n\,[hs]}g^{[hs]}(x). \tag{5.45}$$

Since the operator $\mathbb{S}^{[hs]}$ has a bounded inverse and the operator $\mathbb{K}^{[hs]}$ is compact, it is easy to see that for all $n \geq n_0$, $(\mathbb{S}^{[hs]} - Q_N^{n\,[hs]}\mathbb{K}^{[hs]})^{-1}$ exists as a bounded linear operator [45]. Hence, the equation (5.45) has a unique solution which is given by

$$\psi_n^{*[hs]}(x) = (\mathbb{S}^{[hs]} - Q_N^{n\,[hs]}\mathbb{K}^{[hs]})^{-1}Q_N^{n\,[hs]}g^{[hs]}(x). \tag{5.46}$$

Now using equations (5.40) and (5.45), for all $n \geq n_0$ we have

$$\psi^{[hs]}(x) - \psi_n^{*[hs]}(x) = (\mathbb{S}^{[hs]} - Q_N^{n\,[hs]}\mathbb{K}^{[hs]})^{-1}\Big(g^{[hs]}(x) - Q_N^{n\,[hs]}g^{[hs]}(x) + \mathbb{K}^{[hs]}\psi^{[hs]}(x) - Q_N^{n\,[hs]}\mathbb{K}^{[hs]}\psi^{[hs]}(x)\Big). \tag{5.47}$$

Applying norm on both the sides of equation (5.47) with respect to $M_N^{[hs]}$ space, we obtain

$$\begin{aligned} \| \psi^{[hs]} - \psi_n^{*[hs]} \|_{M_N^{[hs]}} \leq & \| (\mathbb{S}^{[hs]} - Q_N^{n\,[hs]}\mathbb{K}^{[hs]})^{-1} \| \| g^{[hs]} - Q_N^{n\,[hs]}g^{[hs]} \|_{L_N^{2\,[hs]}} \\ & + \| (\mathbb{S}^{[hs]} - Q_N^{n\,[hs]}\mathbb{K}^{[hs]})^{-1} \| \| \mathbb{K}^{[hs]}\varphi(x) - Q_N^{n\,[hs]}\mathbb{K}^{[hs]}\varphi(x) \|_{L_N^{2\,[hs]}} . \end{aligned} \tag{5.48}$$

Also, it is noticed that if $\psi(x) \in M_N^{[hs]}$, we have the following relation

$$\| \psi^{[hs]} - \psi_n^{*[hs]} \|_{L_N^{2\,[hs]}} \leq \| \psi^{[hs]} - \psi_n^{*[hs]} \|_{M_N^{[hs]}} . \tag{5.49}$$

Using the above relation defined in equation (5.49), the equation (5.48) becomes

$$\begin{aligned} \| \psi^{[hs]} - \psi_n^{*[hs]} \|_{L_N^{2\,[hs]}} \leq & \| (\mathbb{S}^{[hs]} - Q_N^{n\,[hs]}\mathbb{K}^{[hs]})^{-1} \| \| g^{[hs]} - Q_N^{n\,[hs]}g^{[hs]} \|_{L_N^{2\,[hs]}} \\ & + \| (\mathbb{S}^{[hs]} - Q_N^{n\,[hs]}\mathbb{K}^{[hs]})^{-1} \| (\mathbb{K}^{[hs]} - Q_N^{n\,[hs]}\mathbb{K}^{[hs]})\psi^{[hs]} \|_{L_N^{2\,[hs]}} . \end{aligned} \tag{5.50}$$

Due to the assumption given in equation (5.39), the operator $\mathbb{K}^{[hs]}$ is an Hilbert-Schmidt operator [45]. Therefore, we obtain $\| \mathbb{K}^{[hs]} - Q_N^{n\,[hs]}\mathbb{K}^{[hs]} \|_{L_N^{2\,[hs]}} \to 0$ as $n \to \infty$. Also, we have $\| g^{[hs]} - Q_N^{n\,[hs]}g^{[hs]} \|_{L_N^{2\,[hs]}} \to 0$ as $n \to \infty$. Thus, from equation (5.50), we get $\| \psi^{[hs]} - \psi_n^{*[hs]} \|_{L_N^{2\,[hs]}} \to 0$ as $n \to \infty$. Now using equation (5.3), we define

$$\varphi^{[hs]}(x) = \sqrt{1-x^2}\,\psi^{[hs]}(x), \tag{5.51}$$

where
$$\varphi^{[hs]}(x) = (\varphi_1^{[hs]}(x), \varphi_2^{[hs]}(x), \ldots, \varphi_N^{[hs]}(x))^T, \tag{5.52}$$

and the function $\psi^{[hs]}(x)$ is same as defined in equation (5.6). Similarly, we define

$$\varphi_n^{*[hs]}(x) = \sqrt{1-x^2}\,\psi_n^{*[hs]}(x), \tag{5.53}$$

where
$$\varphi^{[hs]}(x) \approx \varphi_n^{*[hs]}(x) = (\varphi_{1n}^{*\,[hs]}(x), \varphi_{2n}^{*\,[hs]}(x), \ldots, \varphi_{Nn}^{*\,[hs]}(x))^T, \tag{5.54}$$

and the function ${\psi_n^*}^{[hs]}$ is already defined in equation (5.10). Now with the help of equations (5.51) and (5.53) we have

$$\| \varphi^{[hs]} - {\varphi_n^*}^{[hs]} \|_{{L_N^2}^{[hs]}} \leq \| \sqrt{1-x^2} \|_{{L_N^2}^{[hs]}} \| \psi^{[hs]} - {\psi_n^*}^{[hs]} \|_{{L_N^2}^{[hs]}} . \tag{5.55}$$

Also, the above equation (5.55) and the fact that $\| \psi^{[hs]} - {\psi_n^*}^{[hs]} \|_{{L_N^2}^{[hs]}} \to 0$ as $n \to \infty$ proves that $\| \varphi^{[hs]} - {\varphi_n^*}^{[hs]} \|_{{L_N^2}^{[hs]}} \to 0$ as $n \to \infty$. Hence, the convergence of sequence of approximate solutions of system (5.1) is shown.

Moreover, using equations (5.50) and (5.55), we obtain

$$\| \varphi - \varphi_n^* \|_{{L_N^2}^{[hs]}} \leq \| \sqrt{1-x^2} \|_{{L_N^2}^{[hs]}} \| (\mathbb{S}^{[hs]} - {Q_N^n}^{[hs]} \mathbb{K}^{[hs]})^{-1} \| \gamma^{[hs]}, \tag{5.56}$$

where $\gamma^{[hs]} = (\| g^{[hs]} - {Q_N^n}^{[hs]} g^{[hs]} \|_{{L_N^2}^{[hs]}} + \| (\mathbb{K}^{[hs]} - {Q_N^n}^{[hs]} \mathbb{K}^{[hs]}) \psi^{[hs]} \|_{{L_N^2}^{[hs]}})$. In equation (5.56), the operator $(\mathbb{S}^{[hs]} - {Q_N^n}^{[hs]} \mathbb{K}^{[hs]})^{-1}$ is bounded, $\| \sqrt{1-x^2} \|_{{L_N^2}^{[hs]}} = 1.1547$ and the sum of norms $\Big(\| g^{[hs]} - {Q_N^n}^{[hs]} g^{[hs]} \|_{{L_N^2}^{[hs]}} + \| (\mathbb{K}^{[hs]} - {Q_N^n}^{[hs]} \mathbb{K}^{[hs]}) \psi^{[hs]} \|_{{L_N^2}^{[hs]}} \Big)$ appearing on right hand side of equation (5.56) is converging to 0 as $n \to \infty$. Therefore, the right hand side of equation (5.56) will be a finite real number and hence, it will be the error bound for the error $\| \varphi^{[hs]} - {\varphi_n^*}^{[hs]} \|_{{L_N^2}^{[hs]}}$.

5.3.3 Problem is well-posed

In this subsection, we show that the well-posedness of system of linear algebraic equations (5.45) obtained as a result of approximation of system (5.1).

With reference to subsection 5.3.2, the inverse operator $(\mathbb{S}^{[hs]} - {Q_N^n}^{[hs]} \mathbb{K}^{[hs]})^{-1}$ exists which implies that the solution to the system (5.45) exists. Now we prove the uniqueness of solution of system (5.45) by the principle of contradiction. If possible, let w_1 and w_2 be the two distinct solutions of system (5.45). Now since $w_1(x)$ and $w_2(x)$ both are its solution, therefore we have

$$\mathbb{S}^{[hs]} w_1(x) - {Q_N^n}^{[hs]} \mathbb{K}^{[hs]} w_1(x) = {Q_N^n}^{[hs]} g^{[hs]}(x), \tag{5.57}$$

$$\mathbb{S}^{[hs]} w_2(x) - {Q_N^n}^{[hs]} \mathbb{K}^{[hs]} w_2(x) = {Q_N^n}^{[hs]} g^{[hs]}(x). \tag{5.58}$$

From equations (5.57) and (5.58), we obtain

$$(\mathbb{S}^{[hs]} - {Q_N^n}^{[hs]} \mathbb{K}^{[hs]})(w_1(x) - w_2(x)) = 0, | x |< 1. \tag{5.59}$$

Applying the operator $(\mathbb{S}^{[hs]} - {Q_N^n}^{[hs]} \mathbb{K}^{[hs]})^{-1}$ on both sides of equation (5.59), we get

$$w_1(x) = w_2(x), | x |< 1. \tag{5.60}$$

The above equation (5.60) proves the uniqueness of the solution of system (5.45). Also, the inverse operator $(\mathbb{S}^{[hs]} - Q_N^{n\,[hs]}\mathbb{K}^{[hs]})^{-1}$ exists as a bounded linear operator which means that the operator $(\mathbb{S}^{[hs]} - Q_N^{n\,[hs]}\mathbb{K}^{[hs]})^{-1}$ is continuous. Therefore, a small change in the given data reflects a small change in the solution. Hence, the boundness of the operator $(\mathbb{S}^{[hs]} - Q_N^{n\,[hs]}\mathbb{K}^{[hs]})^{-1}$ confirms the stability [66] of the problem. Hence, the problem (5.45) is a well-posed problem.

5.3.4 Solvability of system of linear algebraic equations

In this subsection, we show the linear system (5.17) is solvable (i.e. inverse of the matrix $\hat{B}$ exists). In order to show the solvability of system, we define the vector space as

$$\mathbb{X}_{[hs]}^{N\times(n+1)} = \{vec\, G^{[hs]} = (G_1^{[hs]}, G_2^{[hs]}, ..., G_N^{[hs]})^T : G_j^{[hs]} \in \mathbb{C}^{n+1},\ j = 1, 2, \ldots, N\}, \tag{5.61}$$

where $\mathbb{C}^{n+1}$ is a complex vector space [67] having $(n+1)$-tuple of complex numbers as its vectors, $vec\, G^{[hs]}$ is same as defined in equation (5.16) and $G_j^{[hs]} = (\langle g_j^{[hs]}, e_0\rangle_{L^2}, \langle g_j^{[hs]}, e_1\rangle_{L^2}, \ldots, \langle g_j^{[hs]}, e_n\rangle_{L^2})^T$ $j = 1, 2, \ldots, N$.

We define another vector space $\mathbb{Z}_N^{[hs]}$ as

$$\mathbb{Z}_N^{[hs]} = \{z^{[hs]} = (z_1^{[hs]}, z_2^{[hs]}, ..., z_N^{[hs]})^T : z_j^{[hs]} \in E,\ j = 1, 2, \ldots, N\},, \tag{5.62}$$

where $E = span\{e_i(x)\}_{i=0}^n$. Now consider the operator $P_n^{[hs]} : \mathbb{X}_{[hs]}^{N\times(n+1)} \to \mathbb{Z}_N^{[hs]}$ as

$$P_N^{n\,[hs]}(vec\, G^{[hs]}) = (P_n^{[hs]} G_1^{[hs]}, P_n^{[hs]} G_2^{[hs]}, ..., P_n^{[hs]} G_N^{[hs]})^T, \tag{5.63}$$

such that $P_n^{[hs]} : \mathbb{C}^{n+1} \to E$ is a prolongation operator [45] and defined as

$$P_n^{[hs]} G_j^{[hs]} = \sum_{i=0}^{n} \langle g_j^{[hs]}, e_i\rangle_{L^2} e_i(x) \in E,\ \ j = 0, 1, \ldots, N. \tag{5.64}$$

We take the orthogonal projection of function $g^{[hs]}(x) \in L_N^2$

$$Q_N^{n\,[hs]} g^{[hs]}(x) = (Q_n^{[hs]} g_1^{[hs]}(x), Q_n^{[hs]} g_2^{[hs]}(x), \ldots, Q_n^{[hs]} g_N^{[hs]}(x))^T, \tag{5.65}$$

such that $Q_n^{[hs]}$ is the orthogonal projection onto the finite dimensional vector space $E = span\{e_i(x)\}_{i=0}^n$ and defined as

$$Q_n^{[hs]} g_j^{[hs]}(x) = \sum_{k=0}^{n} \langle g_j^{[hs]}, e_k\rangle_{L^2} e_k(x). \tag{5.66}$$

Using equation (5.64) and the above equation (5.66), we obtain

$$Q_n^{[hs]} g_j^{[hs]}(x) = P_n^{[hs]} G_j^{[hs]},\ j = 1, 2, \ldots, N. \tag{5.67}$$

From equations (5.63), (5.65) and (5.67), we have

$$Q_N^{n\,[hs]} g^{[hs]}(x) = P_N^{n\,[hs]}(vec\, G^{[hs]}). \tag{5.68}$$

We now define operator $R_N^{n\,[hs]} : \mathbb{Z}_N^{[hs]} \to \mathbb{C}^{N(n+1)}$ as

$$R_N^{n\,[hs]} \psi_n^{*\,[hs]} = (R_n^{\,[hs]} \psi_{1n}^{*\,\,[hs]}, R_n^{\,[hs]} \psi_{2n}^{*\,\,[hs]}, \ldots, R_n^{\,[hs]} \psi_{Nn}^{*\,\,[hs]})^T, \tag{5.69}$$

such that $R_n^{\,[hs]} : E \to \mathbb{C}^{n+1}$ is a restriction operator [45] which is as follows:

$$R_n^{\,[hs]} \psi_{jn}^{*\,\,[hs]} = (\langle \psi_{jn}^{*\,\,[hs]}, e_0 \rangle_{L^2}, \langle \psi_{jn}^{*\,\,[hs]}, e_1 \rangle_{L^2}, \ldots \langle \psi_{jn}^{*\,\,[hs]}, e_n \rangle_{L^2})^T \in \mathbb{C}^{n+1}, \tag{5.70}$$

where $\psi_{jn}^{*\,\,[hs]}$ is same as defined in equation (5.10). Further, on applying the orthogonal property of Legendre polynomials in equation (5.11), we get

$$a_{ji}^{[hs]} = \langle \psi_{jn}^{*\,\,[hs]}, e_i \rangle_{L^2}, \ i = 0, 1, \ldots, n, \quad j = 1, 2, \ldots, N. \tag{5.71}$$

Using the above equation (5.71), the value of $A_j^{[hs]}$ (j^{th} column of matrix $A^{[hs]}$) is

$$A_j^{\,[hs]} = (\langle \psi_{jn}^{*\,\,[hs]}, e_0 \rangle_{L^2}, \langle \psi_{jn}^{*\,\,[hs]}, e_1 \rangle_{L^2}, \ldots, \langle \psi_{jn}^{*\,\,[hs]}, e_n \rangle_{L^2})^T, \ \ j = 1, 2, \ldots, N. \tag{5.72}$$

From equations (5.70) and (5.72), we obtain

$$vec\, A^{[hs]} = R_N^{n\,[hs]} \psi_n^{*\,[hs]}, \tag{5.73}$$

where $vec\, A^{[hs]}$ and $R_N^{n\,[hs]} \psi_n^{*\,[hs]}$ are defined in equations (5.16) and (5.69) respectively.

Since the solution $\psi_n^{*\,[hs]}$ of system (5.45) exist uniquely which is already proved in subsection 5.3.3. Therefore, from equation (5.73), the solution $vec\, A^{[hs]}$ of system (5.17) also exists uniquely.

Now substituting the value of $\psi_n^{*\,[hs]}$ from equation (5.46) in system (5.73), we obtain

$$vec\, A^{[hs]} = R_N^{n\,[hs]} (\mathbb{S}^{[hs]} - Q_N^{n\,[hs]} \mathbb{K}^{[hs]})^{-1} Q_N^{n\,[hs]} g^{[hs]}(x). \tag{5.74}$$

On using equations (5.68) and (5.74), we get

$$vec\, A^{[hs]} = R_N^{n\,[hs]} (\mathbb{S}^{[hs]} - Q_N^{n\,[hs]} \mathbb{K}^{[hs]})^{-1} P_N^{n\,[hs]} (vec\, G^{[hs]}). \tag{5.75}$$

The above equation (5.75) can be rewritten as

$$\hat{C}^{[hs]} = R_N^{n\,[hs]} (\mathbb{S}^{[hs]} - Q_N^{n\,[hs]} \mathbb{K}^{[hs]})^{-1} P_N^{n\,[hs]} \hat{E}. \tag{5.76}$$

From equation (5.76), we can say that the unique solution $\hat{C}^{[hs]}$ of the system $\hat{D}^{[hs]}\hat{C}^{[hs]} = \hat{E}^{[hs]}$ defined in equation (5.17) exists for every given $\hat{E}^{[hs]} \in \mathbb{X}_{[hs]}^{N\times(n+1)}$. Hence, the matrix $\hat{D}^{[hs]}$ is invertible [119].

Theorem 5.3.1. *Consider the system of hypersingular integral equations*

$$(\mathbb{S}^{[hs]} - \mathbb{K}^{[hs]})\psi^{[hs]}(x) = f^{[hs]}(x), |x| < 1, \tag{5.77}$$

where the operators $\mathbb{S}^{[hs]}$ and $\mathbb{K}^{[hs]}$ are defined in equation (5.7). In the system (5.77) if $k_{ij}^{[hs]}(x,t)$, $\forall j, i = 1, 2, \ldots, N$, are polynomial functions in x, t and $f^{[hs]}(x) = (f_1^{[hs]}(x), f_2^{[hs]}(x), \ldots, f_N^{[hs]}(x))^T$ is such that each $f_j^{[hs]}(x)$ is a polynomial of degree r_j, $j = 1, 2, \ldots, N$. Then the approximate solution of equation (5.77) is an exact solution if we approximate each unknown function $\psi_j^{[hs]}(x)$, $j = 1, 2, \cdots, N$, as in equation (5.11) by the orthonormalized Legendre polynomials of degree $n = \max\{r_1, r_2, \ldots, r_N\}$ and if the degree of polynomial $K_{ji}^{[hs]}\psi_j^{*[hs]}(x) \leq n, \forall j, i = 1, 2, \ldots, N$.

Proof. The residual error for system (5.77) is defined as

$$(\mathbb{S}^{[hs]} - \mathbb{K}^{[hs]})\psi_n^{*[hs]}(x) - f^{[hs]}(x), \tag{5.78}$$

where the function $\psi_n^{*[hs]}(x)$ is same as defined in equation (5.10). Using equation (5.14), we obtain

$$Q_N^{n\,[hs]}((\mathbb{S}^{[hs]} - \mathbb{K}^{[hs]})\,\psi_n^{*[hs]}(x) - f^{[hs]}(x)) = 0. \tag{5.79}$$

From equations (5.77) and (5.79), we obtain

$$(\mathbb{S}^{[hs]} - \mathbb{K}^{[hs]})\psi^{[hs]}(x) - Q_N^{n\,[hs]}((\mathbb{S}^{[hs]} - \mathbb{K}^{[hs]})\,\psi_n^{*[hs]}(x)) = f^{[hs]}(x) - Q_N^{n\,[hs]}f^{[hs]}(x). \tag{5.80}$$

since the function $\psi_n^{*[hs]}(x)$ is a polynomial therefore with the help of Hadamard finite-part integral formulas (see equation (35) in reference [61]), the function $\mathbb{S}^{[hs]}\psi_n^{*[hs]}(x)$ will be a polynomial.

Also, the functions $k_{ij}^{[hs]}(x,t)$, $\forall j, i = 1, 2, \ldots, N$ and $\psi_{jn}^{*\,[hs]}(x)$ are polynomial, therefore the product $k_{ij}^{[hs]}(x,t)\psi_{jn}^{*\,[hs]}(x)$ will be a polynomial. Hence, $\mathbb{K}^{[hs]}\psi_n^{*[hs]}(x)$ is also a polynomial. Applying the definition of operator $Q_N^{n\,[hs]}$ as defined in equation (5.42), we obtain

$$Q_N^{n\,[hs]}(\mathbb{S}^{[hs]} - \mathbb{K}^{[hs]})\psi_n^{*[hs]}(x) = (\mathbb{S}^{[hs]} - \mathbb{K}^{[hs]})\psi_n^{*[hs]}(x). \tag{5.81}$$

Since the function $f(x)$ is also a polynomial, therefore we have

$$Q_N^{n\,[hs]}f^{[hs]}(x) = f^{[hs]}(x). \tag{5.82}$$

Therefore, using equations (5.81) and (5.82), equation (5.80) becomes

$$(\mathbb{S}^{[hs]} - \mathbb{K}^{[hs]})(\psi^{[hs]}(x) - \psi_n^{*[hs]}(x)) = 0. \tag{5.83}$$

Since the operator $(\mathbb{S}^{[hs]} - \mathbb{K}^{[hs]})^{-1}$ exists, equation (5.83) becomes

$$\psi^{[hs]}(x) - \psi_n^{*[hs]}(x) = 0. \tag{5.84}$$

Therefore, the approximate solution is same as the exact solution. □

5.4 Application

In this section, we show an application of the proposed method for the solution of some hypersingular integral equations on two-dimensional curves in the complex plane. This kind of hypersingluar integral equation occurs quite naturally in crack problems in the field of fracture mechanics [19, 21] where the two dimensional curves are representing the shape of various cracks existing in the given material. Now we consider the following hypersingular integral equation

$$\neq\!\!\!\!\!\!\int_{\Gamma} \frac{v(y)}{(y-z)^2} dy = h(z), \ z \in \Gamma, \tag{5.85}$$

where the integral is understood in the sense of HFP and is considered over the curve $\Gamma = \bigcup_{j=1}^{N} \Gamma_j$ where each Γ_j denotes smooth simple arc in the complex plane with no common points and each is of finite length. In the equation (5.85), the functions $h(z)$ and $v(y)$ are complex valued known and unknown functions defined on the curve Γ respectively. We further use the notations $v_j(z)$ and $h_j(z)$ if $z \in \Gamma_j, \ j = 1, 2, \ldots, N$. Now we rewrite the equation (5.85) as

$$\sum_{i=1}^{N} \neq\!\!\!\!\!\!\int_{\Gamma_i} \frac{v_i(y)}{(y-z)^2} dy = h_j(z), \quad z \in \Gamma_j, \quad j = 1, 2, \ldots, N. \tag{5.86}$$

On introducing the parametrization Δ_j for each arc Γ_j

$$\Gamma_j : y = \Delta_j(t), \ |t| < 1, \ \ j = 1, 2, \ldots, N,$$

the equation (5.86) becomes

$$\sum_{i=1}^{N} \neq\!\!\!\!\!\!\int_{-1}^{1} \frac{\Delta_i'(t) v_i(\Delta_i(t))}{(\Delta_i(t) - \Delta_j(x))^2} dt = h_j(\Delta_j(x)), \ \ |x| < 1. \tag{5.87}$$

On rewriting the last equation in a simplified form as

$$\neq\!\!\!\!\!\!\int_{-1}^{1} \frac{\varphi_j^{[hs]}(t)}{(t-x)^2} dt + \sum_{i=1}^{N} \int_{-1}^{1} k_{ji}^{[hs]}(x,t) \varphi_j^{[hs]}(t) dt = g_j^{[hs]}(x), \ \ |x| < 1, \tag{5.88}$$

where for $j, i = 1, 2, \ldots, N$,

$$k_{ji}^{[hs]}(x,t) = \begin{cases} \dfrac{(\hat{A}_j(x,t) - \hat{A}_j(x,x))}{\hat{A}_j(x,x)(t-x)^2}, & j = i, \\ \dfrac{\Delta_i'(t)}{\hat{A}_j(x,x)(\Delta_i(t) - \Delta_j(x))^2}, & j \neq i, \end{cases}$$
$$\hat{A}_j(x,t) = \frac{(t-x)^2 \Delta_j'(t)}{(\Delta_j(t) - \Delta_j(x))^2}, \quad \Delta_j(t) \neq \Delta_j(x),$$
$$\hat{A}_j(x,x) \neq 0,$$
$$\varphi_j^{[hs]}(t) = v_j(\Delta_i(t)),$$
$$g_j^{[hs]}(x) = \frac{h_j(\Delta_j(x))}{\hat{A}_j(x,x)}.$$

If the function $g^{[hs]}(x) = (g_1^{[hs]}(x), g_2^{[hs]}(x), \ldots, g_N^{[hs]}(x))^T$ satisfies the condition $\| g^{[hs]} - Q_N^{n\,[hs]} g^{[hs]} \|_{\mathcal{L}^2} \rightarrow 0$ as $n \rightarrow \infty$ and $k_{ij}^{[hs]}(x,t)$ satisfy the assumption (5.39), then the system of HSIEs obtain from equation (5.88) for $j = 1, 2, \ldots, N$, can be solved using the method proposed in Section 5.2.

5.5 Illustrative numerical examples

In this section, our aim is to validate the theoretical results derived in Section 5.3 with the help of test examples.

Example 5.1 Let us consider the hypersingular system of integral equations

$$\begin{aligned} &\oint_{-1}^{1} \frac{\varphi_1^{[hs]}(t)}{(t-x)^2} dt + \int_{-1}^{1} \frac{(\frac{1}{2} + t)\varphi_1^{[hs]}(t)}{16} dt + \int_{-1}^{1} \frac{(\frac{1}{4} + x)\varphi_2^{[hs]}(t)}{9} dt = g_1(x), \mid x \mid < 1, \\ &\int_{-1}^{1} \frac{(\frac{1}{3} + xt)\varphi_1^{[hs]}(t)}{16} dt + \oint \frac{\varphi_2^{[hs]}(t)}{(t-x)^2} dt + \int_{-1}^{1} \frac{(\frac{1}{4} - t^2)\varphi_2^{[hs]}(t)}{9} dt = g_2(x), \mid x \mid < 1, \end{aligned} \tag{5.89}$$

where

$$\begin{aligned} g_1^{[hs]}(x) = \pi\Big(& -2x^8 - 4x^7 + \frac{7x^6}{9} - \frac{9x^5}{2} - \frac{175x^4}{36} + \frac{9x^3}{4} + \frac{19x^2}{24} - \frac{6049x}{9216} - \frac{52169}{73728} \\ &+ i\Big(-2x^8 - 8x^7 - \frac{49x^6}{18} + 3x^5 - \frac{5x^4}{6} - \frac{x^3}{2} - \frac{101x^2}{48} - \frac{2387x}{1536} + \frac{18749}{73728}\Big)\Big), \\ g_2^{[hs]}(x) = \pi\Big(& -\frac{27x^8}{4} + \frac{7x^6}{8} - 6x^5 - \frac{125x^4}{32} + 2x^3 + \frac{79x^2}{64} - \frac{6085x}{8192} - \frac{4003}{55296} \\ &+ i\Big(-\frac{9x^8}{2} - 2x^7 + \frac{7x^6}{4} - \frac{69x^5}{4} + \frac{5x^4}{16} + \frac{213x^3}{40} - \frac{93x^2}{32} + \frac{10071x}{20480} - \frac{13677}{28672}\Big)\Big). \end{aligned}$$

The exact solution of above problem is given by

$$\varphi_1^{[hs]}(x) = \sqrt{1-x^2}\Big(1 + \frac{x}{2} + \frac{x^2}{4} + x^4 + x^5 + \frac{x^7}{2} + \frac{2x^8}{9} + i\Big(\frac{8}{21} + x + x^2 + \frac{x^3}{4} + \frac{4x^4}{9} + \frac{x^6}{2} + x^7 + \frac{2x^8}{9}\Big)\Big),$$

$$\varphi_2^{[hs]}(x) = \sqrt{1-x^2}\Big(\frac{1}{3} + \frac{x}{2} + \frac{x^2}{6} + x^4 + x^5 + \frac{x^6}{4} + \frac{3x^8}{4} + i\Big(1 + \frac{x}{4} + x^2 + \frac{x^3}{5} + 3x^5 + \frac{x^7}{4} + \frac{x^8}{2}\Big)\Big).$$

First we apply the proposed numerical method described in Section 5.2 to the system (5.89) having $N = 2$ coupled hypersingular equations, we obtain a sequence of approximated vector functions

$$\varphi_n^{*[hs]}(x) = \begin{pmatrix} \varphi_{1n}^{*\,[hs]}(x) \\ \varphi_{2n}^{*\,[hs]}(x) \end{pmatrix} = \begin{pmatrix} Re[\varphi_{1n}^{*\,[hs]}(x)] + iIm[\varphi_{1n}^{*\,[hs]}(x)] \\ Re[\varphi_{2n}^{*\,[hs]}(x)] + iIm[\varphi_{2n}^{*\,[hs]}(x)] \end{pmatrix}, \quad n = 1, 2, \ldots, 8,$$

with $N = 2$ components, where n is the degree of a Legendre polynomial used for approximation. Further, we will show that the above sequence of vector functions $\varphi_n^{*[hs]}(x)$ converges to the exact solution

$$\varphi^{[hs]}(x) = \begin{pmatrix} \varphi_1^{[hs]}(x) \\ \varphi_2^{[hs]}(x) \end{pmatrix} = \begin{pmatrix} Re[\varphi_1^{[hs]}(x)] + iIm[\varphi_1^{[hs]}(x)] \\ Re[\varphi_2^{[hs]}(x)] + iIm[\varphi_2^{[hs]}(x)] \end{pmatrix},$$

of system (5.89) by showing each $\varphi_{jn}^{*\,[hs]}(x)$ converges to $\varphi_j^{[hs]}(x)$ respectively, $j = 1, 2$. Since each $\varphi_{jn}^{*\,[hs]}(x)$ is a complex valued function of a real variable x, therefore the sequence $\varphi_{jn}^{*\,[hs]}(x)$ is convergent to $\varphi_j^{[hs]}(x)$ if and only if the sequence of its corresponding real part $Re[\varphi_{jn}^{*\,[hs]}(x)] \to Re[\varphi_j^{[hs]}(x)]$ and imaginary part $Im[\varphi_{jn}^{*\,[hs]}(x)] \to Im[\varphi_j^{[hs]}(x)]$, $j = 1, 2$. The above can be summarized as follows:

As $n \to \infty$, we get

$$\varphi_n^{*[hs]}(x) \to \varphi^{[hs]}(x) \Longleftarrow \begin{cases} \varphi_{1n}^{*\,[hs]}(x) \to \varphi_1^{[hs]}(x) \Longleftarrow \begin{cases} Re[\varphi_{1n}^{*\,[hs]}(x)] \to Re[\varphi_1^{[hs]}(x)], \\ Im[\varphi_{1n}^{*}(x)] \to Im[\varphi_1^{[hs]}(x)], \end{cases} \\ \varphi_{2n}^{*\,[hs]}(x) \to \varphi_2^{[hs]}(x) \Longleftarrow \begin{cases} Re[\varphi_{2n}^{*}(x)] \to Re[\varphi_2^{[hs]}(x)], \\ Im[\varphi_{2n}^{*}(x)] \to Im[\varphi_2^{[hs]}(x)]. \end{cases} \end{cases} \tag{5.90}$$

We computed the actual error $\|\varphi^{[hs]} - \varphi_n^{*[hs]}\|_{L_N^{2\,[hs]}}$ and the corresponding theoretical error bound by using equation (5.56) for different values of n and the details are shown in Table 5.1 for Example 5.1. The actual error can never exceed the error bound and this fact is validated in Table 5.1. Further, theoretically we proved in equation (5.56) that $\|\varphi^{[hs]} - \varphi_n^{*[hs]}\|_{L_N^{2\,[hs]}} \to 0$ as $n \to \infty$ and the same is observed in Table 5.1. The comparison of real and imaginary part of the exact solution of $\varphi_1^{[hs]}(x)$ with corresponding real and imaginary part of approximate solution $\varphi_{1n}^{*\,[hs]}(x)$ for $n = 0, 1, 2, \ldots, 8$, is shown in Figures 5.1(a), 5.1(b). Similarly, Figures 5.1(c), 5.1(d) show the comparison of real and imaginary part of the exact solution of $\varphi_2^{[hs]}(x)$ with corresponding real and imaginary part of approximate solution $\varphi_{2n}^{*\,[hs]}(x)$ for

$n = 0, 1, 2, \ldots, 8$. Further, for each component $\varphi_j^{[hs]}(x), j = 1, 2$, it can be seen in all the Figures 5.1(a), 5.1(b), 5.1(c), 5.1(d) that as the value of n increases from 0 to 7, the real and imaginary part of approximate solutions are getting closer to the respective real and imaginary part of exact solution. And, as soon as the value of n becomes 8, the approximate solution overlaps the exact solution. Hence, due to the fact mentioned in equation (5.90), Figure 5.1 shows that the sequence of approximate solutions $\varphi_n^{*[hs]}(x)$ converges to the exact solution $\varphi^{[hs]}(x)$.

Table 5.1: The actual error and theoretical error bound for various values of n for Example 5.1

Degree of Legendre polynomial	Actual error $\|\varphi^{[hs]} - \varphi_n^{*[hs]}\|_{L^2_N{}^{[hs]}}$	Error bound for $\|\varphi^{[hs]} - \varphi_n^{*[hs]}\|_{L^2_N{}^{[hs]}}$
$n = 1$	1.21590	17.96830
$n = 2$	0.67989	13.32240
$n = 3$	0.26737	6.62679
$n = 4$	0.14942	4.20855
$n = 5$	0.03684	1.24737
$n = 6$	0.01028	0.39426
$n = 7$	0.00393	0.16759
$n = 8$	0	0

Table 5.2: The theoretical error bound for various values of n for Example 5.2

Degree of Legendre polynomial	Error bound for $\|\varphi^{[hs]} - \varphi_n^{*[hs]}\|_{L^2_N{}^{[hs]}}$
$n = 1$	8.09776
$n = 2$	5.38558
$n = 3$	1.82725
$n = 4$	0.25418
$n = 5$	0.23657
$n = 6$	0.08312
$n = 7$	0.01463
$n = 8$	0.00292
$n = 9$	0.00148

Example 5.2 Consider the following hypersingular integral equation over the curve

$$\int\!\!\!\!\!\!=_{\Gamma} \frac{v(y)}{(y-z)^2} dy = i \cos(iz), \; z \in \Gamma, \tag{5.91}$$

where $\Gamma = \Gamma_1 \bigcup \Gamma_2$ such that Γ_1 and Γ_2 are two disjoint parabolas with the following parameterizations:

$$\begin{aligned}\Gamma_1 : y &= \Delta_1(t) = t + i(t^2-1),\ |t|<1,\\ \Gamma_2 : y &= \Delta_2(t) = \frac{1}{60}\Big(t + \frac{i(t^2-1)}{5}\Big),\ |t|<1.\end{aligned}$$

On following the procedure discussed in Section 5.4, we obtain the following system of HSIEs:

$$\begin{aligned}\fint_{-1}^{1}\frac{\varphi_1^{[hs]}(t)}{(t-x)^2}dt + \int_{-1}^{1}k_{11}^{[hs]}(x,t)\varphi_1^{[hs]}(t)dt + \int_{-1}^{1}k_{12}^{[hs]}(x,t)\varphi_2^{[hs]}(t)dt &= g_1^{[hs]}(x),\\ \int_{-1}^{1}k_{21}^{[hs]}(x,t)\varphi_1^{[hs]}(t)dt + \fint_{-1}^{1}\frac{\varphi_2^{[hs]}(t)}{(t-x)^2}dt + \int_{-1}^{1}k_{22}^{[hs]}(x,t)\varphi_2^{[hs]}(t)dt &= g_2^{[hs]}(x),\end{aligned} \tag{5.92}$$

where

$$\begin{aligned}k_{11}^{[hs]}(x,t) &= \frac{1}{1-x^2-2xt-t^2+i(2x+2t)},\\ k_{12}^{[hs]}(x,t) &= \frac{\frac{1}{60}+i(\frac{x}{30}+\frac{t}{150})-\frac{xt}{75}}{p_{12}},\\ k_{21}^{[hs]}(x,t) &= \frac{\frac{1}{60}-\frac{xt}{75}+i(\frac{x}{150}+\frac{t}{30})}{p_{21}},\\ k_{22}^{[hs]}(x,t) &= \frac{1}{25-x^2-2xt-t^2+i(10x+10t)},\end{aligned}$$

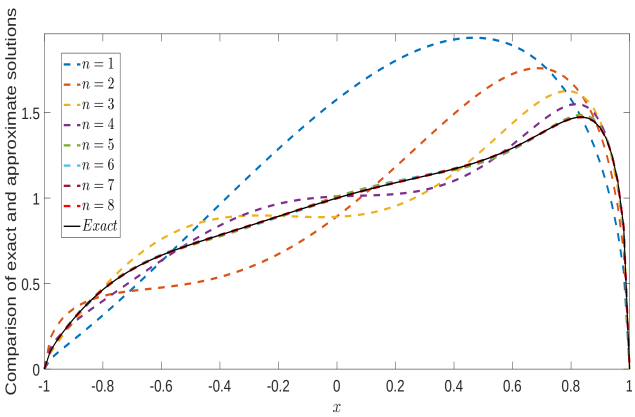
$$\begin{aligned}p_{12}^{[hs]}(x,t) &= -\frac{89401}{90000}+\frac{449x^2}{150}-x^4-\frac{xt}{30}-\frac{191t^2}{30000}+\frac{x^2t^2}{150}-\frac{t^4}{90000}+i\Big(-\frac{299x}{150}+2x^3+\frac{299t}{9000}-\frac{x^2t}{30}-\frac{xt^2}{150}+\frac{t^3}{9000}\Big),\\ p_{21}^{[hs]}(x,t) &= -\frac{89401}{90000}-\frac{191x^2}{30000}-\frac{x^4}{90000}-\frac{xt}{30}+\frac{449t^2}{150}+\frac{x^2t^2}{150}-t^4+i\Big(\frac{299x}{9000}+\frac{x^3}{9000}-\frac{299t}{150}-\frac{x^2t}{150}-\frac{xt^2}{30}+2t^3\Big),\end{aligned}$$

$$\begin{aligned}g_1^{[hs]}(x) &= (i-2x)\cos(ix-x^2+1),\\ g_2^{[hs]}(x) &= \frac{i}{60}\Big(1+\frac{2ix}{5}\Big)\cos\Big(\frac{ix}{60}+\frac{(1-x^2)}{300}\Big).\end{aligned}$$

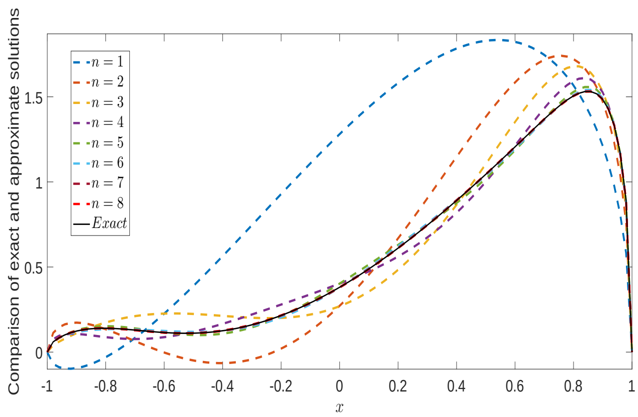
Now we solve the system (5.92) obtain from equation (5.91) by using the method described in Section 5.2. Similar to Example 5.1, we can obtain a sequence of approximate solutions $\varphi_n^{*[hs]}(x), n = 0,1,2,3,\ldots$. In contrast to Example 5.1, the exact solution is not known for Example 5.2. Therefore, we cannot calculate the actual error which we calculated in Example 5.1. Hence, in order to show the convergence of sequence of approximate solutions to exact solution, we numerically computed error bound for system (5.92) by using equation (5.56) and the details are tabulated in Table 5.2. It can be noticed that the error bound decreases with the increase in value of n which validates $\|\varphi^{[hs]} - \varphi_n^{*[hs]}\|_{L_N^{2\,[hs]}} \to 0$ as $n \to \infty$. Figures 5.2(a)

Table 5.3: The theoretical error bound for various values of n for Example 5.3

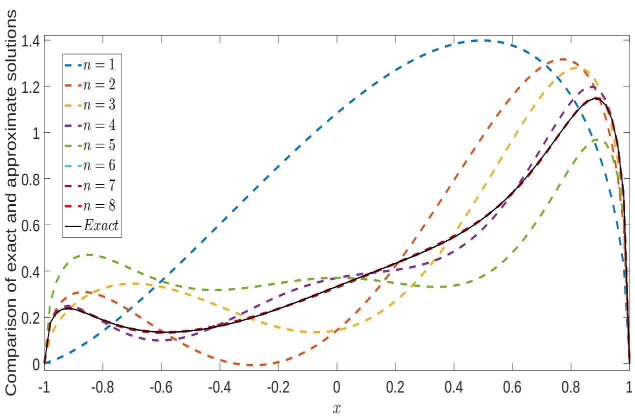
Degree of Legendre polynomial	Error bound for $\|\varphi^{[hs]} - \varphi_n^{*[hs]}\|_{L_N^{2\,[hs]}}$
$n = 1$	4.17322
$n = 2$	1.89030
$n = 3$	0.73065
$n = 4$	0.24891
$n = 5$	0.07662
$n = 6$	0.02166
$n = 7$	0.00569
$n = 8$	0.00140
$n = 9$	0.00033



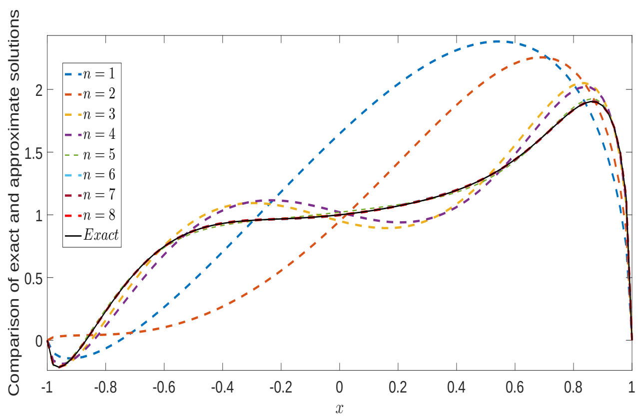
(a) Comparison of $Re[\varphi_{1n}^{*\,[hs]}(x)]$ with $Re[\varphi_1^{[hs]}(x)]$



(b) Comparison of $Im[\varphi_{1n}^{*\,[hs]}(x)]$ with $Im[\varphi_1^{[hs]}(x)]$



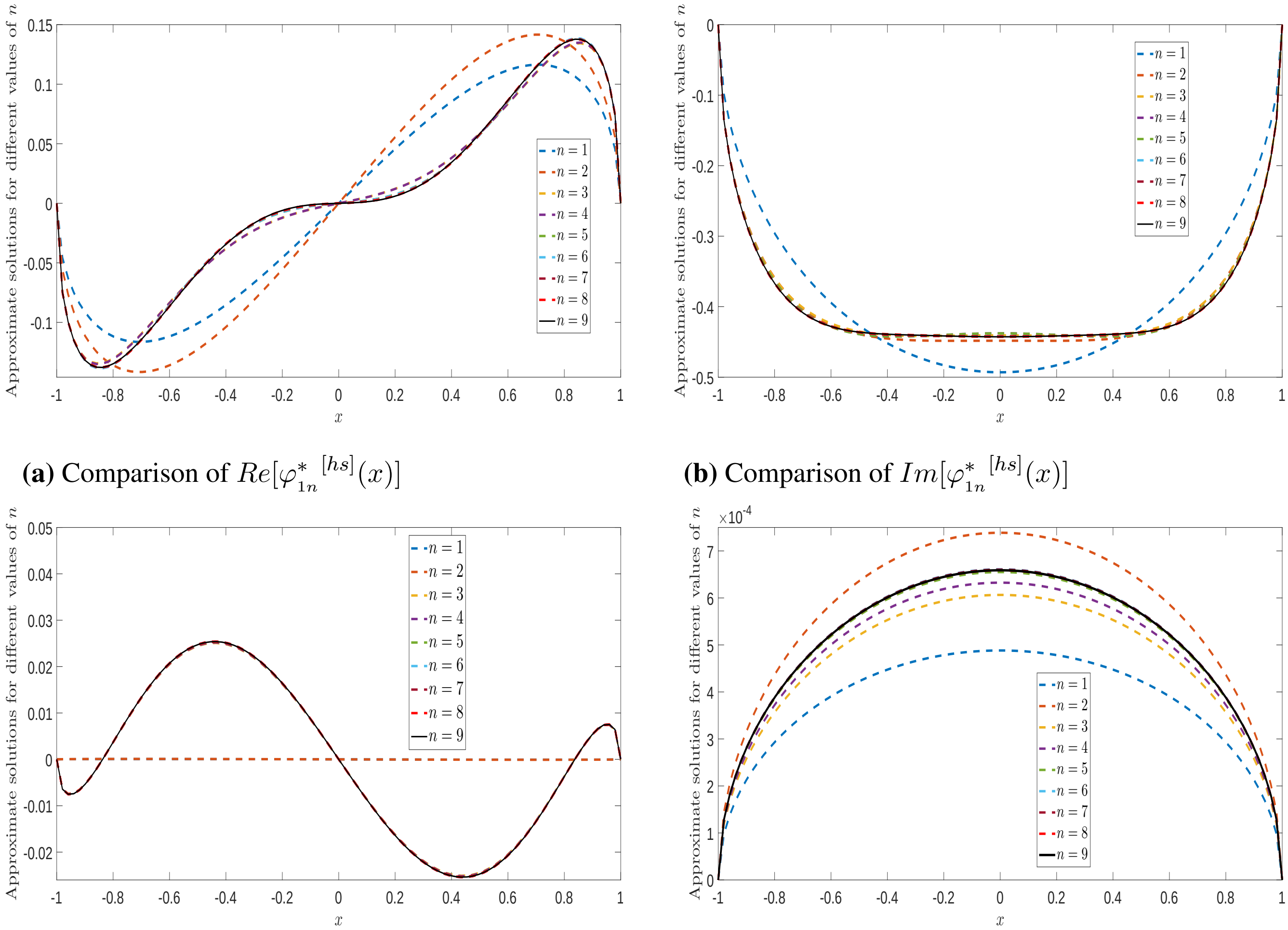
(c) Comparison of $Re[\varphi_{2n}^{*\,[hs]}(x)]$ with $Re[\varphi_2^{[hs]}(x)]$



(d) Comparison of $Im[\varphi_{2n}^{*\,[hs]}(x)]$ with $Im[\varphi_2^{[hs]}(x)]$

Figure 5.1: Comparison of exact solution with approximate solutions of Example 5.1

and 5.2(b) show the comparison of $Re[{\varphi^*_{1n}}^{[hs]}(x)]$ and $Im[\varphi^*_{1n}(x)]$ respectively for $n = 0, 1, 2, \ldots, 9$. Similarly, Figures 5.2(c) and 5.2(d) show the comparison of $Re[{\varphi^*_{2n}}^{[hs]}(x)]$ and $Im[{\varphi^*_{2n}}^{[hs]}(x)]$ respectively $n = 0, 1, 2, \ldots, 9$. Further, it is noticed that the approximate solutions are coming closer to each other as the value of n increases.



(a) Comparison of $Re[{\varphi^*_{1n}}^{[hs]}(x)]$

(b) Comparison of $Im[{\varphi^*_{1n}}^{[hs]}(x)]$

(c) Comparison of $Re[{\varphi^*_{2n}}^{[hs]}(x)]$

(d) Comparison of $Im[{\varphi^*_{2n}}^{[hs]}(x)]$

Figure 5.2: Comparison of approximate solutions for different values of n in case of Example 5.2

Example 5.3 Consider the following hypersingular integral equation over the curve:

$$\neq\!\!\!\!\!\!\int_\Gamma \frac{v(y)}{(y-z)^2} dy = ie^{(iz+i\frac{z^2}{2})}, \; z \in \Gamma. \tag{5.93}$$

Let us assume that $\Gamma = \Gamma_1 \bigcup \Gamma_2 \bigcup \Gamma_3$, where Γ_1, Γ_2 and Γ_3 are disjoint arcs and have the following parameterizations:

$$\Delta_1 : t \in [-1, 1] \to \Delta_1(t) = \frac{1}{18}\Big(\frac{t^2}{2} + it\Big) \in \Gamma_1,$$

$$\Delta_2 : t \in [-1, 1] \to \Delta_2(t) = t + i\Big(\frac{t^2}{18} - \frac{3}{2}\Big) \in \Gamma_2,$$

$$\Delta_3 : t \in [-1,1] \to \Delta_3(t) = t + i\Big(\frac{t^2}{12} + \frac{7}{2}\Big) \in \Gamma_3.$$

Following the procedure discussed in Section 5.4, we obtain the following system of HSIEs:

$$\begin{aligned}
&\fint_{-1}^{1} \frac{{\varphi_1}^{[hs]}(t)}{(t-x)^2 dt} + \int_{-1}^{1} k_{11}^{[hs]}(x,t){\varphi_1}^{[hs]}(t)dt + \int_{-1}^{1} k_{12}^{[hs]}(x,t){\varphi_2}^{[hs]}(t)dt + \int_{-1}^{1} k_{13}^{[hs]}(x,t){\varphi_3}^{[hs]}(t)dt = g_1^{[hs]}(x),\\
&\int_{-1}^{1} k_{21}^{[hs]}(x,t){\varphi_1}^{[hs]}(t)dt + \fint_{-1}^{1} \frac{{\varphi_2}^{[hs]}(t)}{(t-x)^2}dt + \int_{-1}^{1} k_{22}^{[hs]}(x,t)\varphi_2(t)dt + \int_{-1}^{1} k_{23}^{[hs]}(x,t){\varphi_3}^{[hs]}(t)dt = g_2^{[hs]}(x),\\
&\int_{-1}^{1} k_{31}^{[hs]}(x,t){\varphi_1}^{[hs]}(t)dt + \int_{-1}^{1} k_{32}^{[hs]}(x,t){\varphi_2}^{[hs]}(t)dt + \int_{-1}^{1} k_{33}^{[hs]}(x,t){\varphi_3}^{[hs]}(t)dt + \fint_{-1}^{1} \frac{{\varphi_3}^{[hs]}(t)}{(t-x)^2}dt = g_3^{[hs]}(x),
\end{aligned} \tag{5.94}$$

where

$$\begin{aligned}
k_{11}^{[hs]}(x,t) &= \frac{1}{-4 + x^2 + 2xt + t^2 + i(4x+4t)},\\
k_{12}^{[hs]}(x,t) &= \frac{\frac{x}{18} + i(\frac{1}{18} + \frac{xt}{162}) - \frac{t}{162}}{p_{12}^{[hs]}},\\
k_{13}^{[hs]}(x,t) &= \frac{\frac{x}{18} - \frac{t}{108} + i(\frac{1}{18} + \frac{xt}{108})}{p_{13}^{[hs]}},\\
k_{21}^{[hs]}(x,t) &= \frac{-\frac{x}{162} + \frac{t}{18} + i(\frac{1}{18} + \frac{xt}{162})}{p_{21}^{[hs]}},\\
k_{22}^{[hs]}(x,t) &= \frac{1}{324 - x^2 - 2xt - t^2 + i(36x + 36t)},\\
k_{23}^{[hs]}(x,t) &= \frac{1 - \frac{xt}{54} + i(\frac{x}{9} + \frac{t}{6})}{p_{23}^{[hs]}},\\
k_{31}^{[hs]}(x,t) &= \frac{-\frac{x}{108} + \frac{t}{18} + i(\frac{1}{18} + \frac{xt}{108})}{p_{31}^{[hs]}},\\
k_{32}^{[hs]}(x,t) &= \frac{1 - \frac{xt}{54} + i(\frac{x}{6} + \frac{t}{9})}{p_{32}^{[hs]}},\\
k_{33}^{[hs]}(x,t) &= \frac{1}{144 - x^2 - 2xt - t^2 + i(24x + 24t)},
\end{aligned}$$

$$\begin{aligned}
p_{12}^{[hs]} &= -\frac{9}{4} - \frac{x}{6} - \frac{x^2}{324} + \frac{x^4}{1296} - \frac{x^2 t}{18} + \frac{7t^2}{6} + \frac{xt^2}{162} - \frac{t^4}{324} + i\Big(\frac{x^2}{12} + \frac{x^3}{324} - 3t - \frac{xt}{9} - \frac{x^2t^2}{324} + \frac{t^3}{9}\Big),\\
p_{13}^{[hs]} &= -\frac{49}{4} + \frac{7x}{18} - \frac{x^2}{324} + \frac{x^4}{1296} - \frac{x^2 t}{18} + \frac{5t^2}{12} + \frac{xt^2}{108} - \frac{t^4}{144} + i\Big(-\frac{7x^2}{36} + \frac{x^3}{324} + 7t - \frac{xt}{9} - \frac{x^2t^2}{216} + \frac{t^3}{6}\Big),\\
p_{21}^{[hs]} &= -\frac{-9}{4} + \frac{7x^2}{6} - \frac{x^4}{324} - \frac{t}{6} + \frac{x^2 t}{162} - \frac{t^2}{324} - \frac{st^2}{18} + \frac{t^4}{1296} + i\Big(-3x + \frac{x^3}{9} - \frac{xt}{9} + \frac{t^2}{12} - \frac{x^2t^2}{324} + \frac{t^3}{324}\Big),\\
p_{23}^{[hs]} &= -25 + \frac{14x^2}{9} - \frac{x^4}{324} - 2xt + \frac{t^2}{6} + \frac{x^2t^2}{108} - \frac{t^4}{144} + i\Big(-10s + \frac{x^3}{9} - 10t - \frac{x^2 t}{9} - \frac{xt^2}{6} + \frac{t^3}{6}\Big),
\end{aligned}$$

$$p_{31}^{[hs]} = -\frac{49}{4} + \frac{5x^2}{12} - \frac{x^4}{144} + \frac{7t}{18} + \frac{x^2t}{108} - \frac{t^2}{324} - \frac{xt^2}{18} + \frac{t^4}{1296} + i\Big(7x + \frac{x^3}{6} - \frac{xt}{9} - \frac{7t^2}{36} - \frac{x^2t^2}{216} + \frac{t^3}{324}\Big),$$

$$p_{32}^{[hs]} = -25 + \frac{x^2}{6} - \frac{x^4}{144} - 2xt + \frac{14t^2}{9} + \frac{x^2t^2}{108} - \frac{t^4}{324} + i\Big(10x + \frac{x^3}{6} - 10t - \frac{x^2t}{6} - \frac{xt^2}{9} + \frac{t^3}{9}\Big),$$

$$\begin{aligned} g_1^{[hs]}(x) &= \Big(\frac{ix-1}{18}\Big)e^{(\frac{ix^2}{36} - \frac{x}{18} + i\frac{(\frac{x^2}{36}+\frac{ix}{18})^2}{2})}, \\ g_2^{[hs]}(x) &= \Big(i - \frac{x}{9}\Big)e^{(ix - \frac{x^2}{18} + \frac{3}{2} + i\frac{(x+i(\frac{x^2}{18}-\frac{3}{2}))^2}{2})}, \\ g_3^{[hs]}(x) &= \Big(i - \frac{x}{6}\Big)e^{(is - \frac{x^2}{12} - \frac{7}{2} + i\frac{(x+i(\frac{x^2}{12}+\frac{7}{2}))^2}{2})}. \end{aligned}$$

First we apply the proposed numerical method described in Section 5.2 to the system (5.94) having $N = 3$ coupled hypersingular equations to obtain a sequence of approximated vector functions $\varphi_n^{*[hs]}(x)$ which we need to show that it converges to the exact solution $\varphi^{[hs]}(x)$, that is, as $n \to \infty$, we have

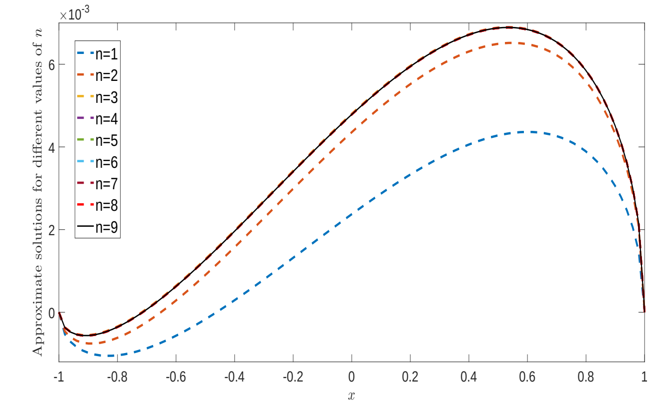
$$\varphi_n^{*[hs]}(x) \to \varphi^{[hs]}(x) \Longleftarrow \begin{cases} \varphi_{1n}^{*[hs]}(x) \to \varphi_1^{[hs]}(x) \Longleftarrow \begin{cases} Re[\varphi_{1n}^{*[hs]}(x)] \to Re[\varphi_1^{[hs]}(x)], \\ Im[\varphi_{1n}^{*[hs]}(x)] \to Im[\varphi_1^{[hs]}(x)], \end{cases} \\ \varphi_{2n}^{*[hs]}(x) \to \varphi_2^{[hs]}(x) \Longleftarrow \begin{cases} Re[\varphi_{2n}^{*[hs]}(x)] \to Re[\varphi_2^{[hs]}(x)], \\ Im[\varphi_{2n}^{*[hs]}(x)] \to Im[\varphi_2^{[hs]}(x)], \end{cases} \\ \varphi_{3n}^{*[hs]}(x) \to \varphi_3^{[hs]}(x) \Longleftarrow \begin{cases} Re[\varphi_{3n}^{*[hs]}(x)] \to Re[\varphi_3^{[hs]}(x)], \\ Im[\varphi_{3n}^{*[hs]}(x)] \to Im[\varphi_3^{[hs]}(x)], \end{cases} \end{cases}$$

where

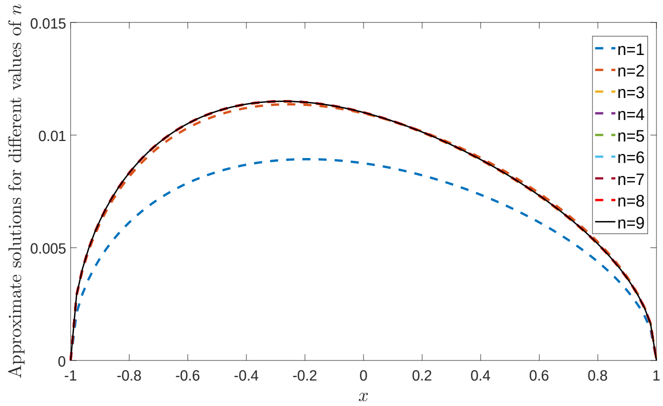
$$\varphi_n^{*[hs]}(x) = \begin{pmatrix} \varphi_{1n}^{*[hs]}(x) \\ \varphi_{2n}^{*[hs]}(x) \\ \varphi_{3n}^{*[hs]}(x) \end{pmatrix} = \begin{pmatrix} Re[\varphi_{1n}^{*[hs]}(x) + iIm[\varphi_{1n}^{*[hs]}(x)] \\ Re[\varphi_{2n}^{*[hs]}(x) + iIm[\varphi_{2n}^{*[hs]}(x)] \\ Re[\varphi_{3n}^{*[hs]}(x) + iIm[\varphi_{3n}^{*[hs]}(x)] \end{pmatrix}, \quad n = 0, 1, 2, 3, \ldots,$$

$$\varphi^{[hs]}(x) = \begin{pmatrix} \varphi_1^{[hs]}(x) \\ \varphi_2^{[hs]}(x) \\ \varphi_3^{[hs]}(x) \end{pmatrix} = \begin{pmatrix} Re[\varphi_1^{[hs]}(x)] + iIm[\varphi_1^{[hs]}(x)] \\ Re[\varphi_2^{[hs]}(x)] + iIm[\varphi_2^{[hs]}(x)] \\ Re[\varphi_3^{[hs]}(x)] + iIm[\varphi_3^{[hs]}(x)] \end{pmatrix}.$$

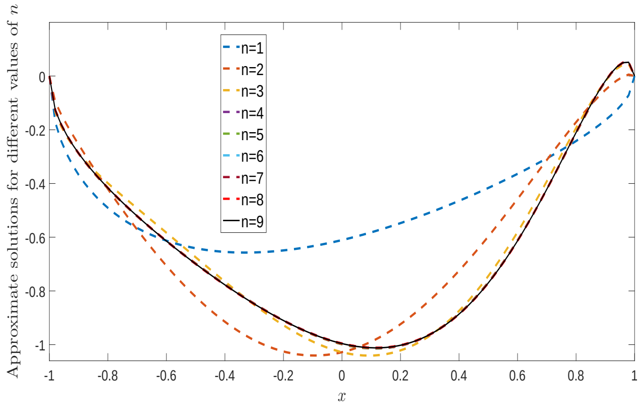
In contrast to Example 5.1 and Example 5.2, here we consider the resultant system (5.94) with $N = 3$ hypersingular coupled equations. Similar to Example 5.2, we calculated the error bound by using equation (5.56) and tabulated the details in Table 5.3. It can be observed from Table 5.3 that as the value of n increases, the error bound decreases which further implies $\|\varphi^{[hs]} - \varphi_n^{*[hs]}\|_{L_N^{2}{}^{[hs]}} \to 0$ as $n \to \infty$. The comparison of $Re[\varphi_{1n}^{*[hs]}(x)]$, $Im[\varphi_{1n}^{*[hs]}(x)]$, $Re(\varphi_{2n}^{*[hs]}(x))$, $Im[\varphi_{2n}^{*[hs]}(x)]$, $Re[\varphi_{3n}^{*[hs]}(x)]$ and $Im[\varphi_{3n}^{*[hs]}(x)]$ is shown in Figures 5.3(a), 5.3(b), 5.3(c), 5.3(d), 5.3(e) and 5.3(f) respectively for $n = 0, 1, 2, \ldots, 9$. Also, it is observed that as the value of n increases the approximate solutions are getting closer to each other.



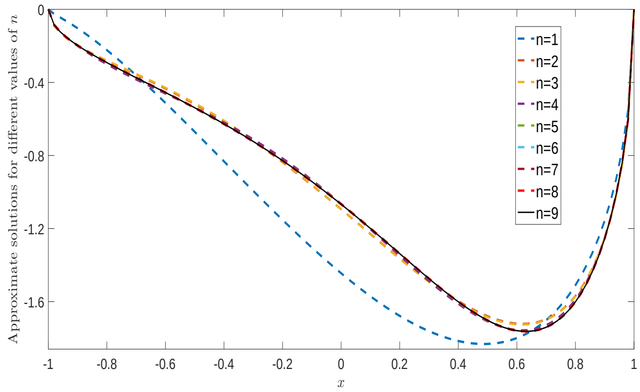
(a) Comparison of $Re[\varphi^{*}_{1_n}{}^{[hs]}(x)]$



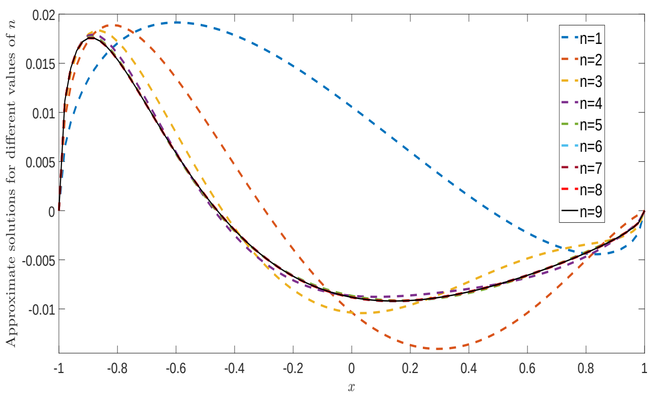
(b) Comparison of $Im[\varphi^{*}_{1_n}{}^{[hs]}(x)]$



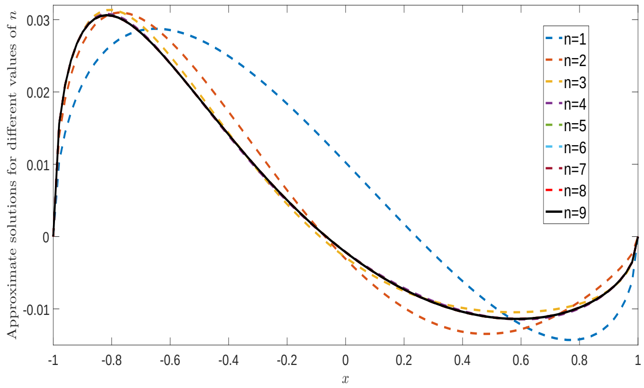
(c) Comparison of $Re[\varphi^{*}_{2_n}{}^{[hs]}(x)]$



(d) Comparison of $Im[\varphi^{*}_{2_n}{}^{[hs]}(x)]$



(e) Comparison of $Re[\varphi^{*}_{3_n}{}^{[hs]}(x)]$



(f) Comparison of $Im[\varphi^{*}_{3_n}{}^{[hs]}(x)]$

Figure 5.3: Comparison of approximate solutions for different values of n in case of Example 5.3

Conclusion

The Legendre polynomials based numerical method has been proposed to find an approximate solution for the system of hypersingular integral equations of first kind. The proposed method reduces system of

HSIEs to a system of linear algebraic equations for which the well-posedness conditions are proved in the sense of Hadamard. The theoretically error bound is derived and convergence of sequence of approximate solutions is proved. The error bound has been calculated numerically and convergence is shown for all the test examples considered. The specific conditions under which proposed method provides exact solution are also shown. Finally, the application of the proposed method is shown by finding the approximate solution of hypersingular integral equation over a two-dimensional curve in complex plane.

Chapter 6

Conclusion and future work

Now we summarize the work presented in this thesis and will address some future possibilities in the field of singular integral equations.

6.1 Conclusion

In this thesis, we have addressed two kinds of singular integral equations, they are Cauchy singular integral equations and hypersingular integral equations. In **Chapter 1**, we have given the background and the importance of singular integral equations in various areas which motivate us to do research on these equations. We have also stated some basic definitions and theorems which we have used to obtain theoretical results such as convergence analysis, well-posedness and error analysis of the proposed method. In **Chapter 2**, we have found the approximate solution for Cauchy singular integral equations over the finite interval $[-1, 1]$ as well as over the half-line by residual based Galerkin's method. An application of the proposed method is also shown in this chapter by solving an antiplane shear crack problem. After this, in **Chapter 3**, the same method is applied to find the approximate solution for system of Cauchy singular integral equations of first kind. The error bound is obtained and the convergence of sequence of approximate solutions for CSIEs as well as their system is also shown in their respective chapters. We have proposed a residual based Galerkin's method to find the approximate solution of hypersingular integral equations in **Chapter 4**. The well-posedness for the system of linear algebraic equations which is obtained after applying the proposed method to the HSIEs, is shown. The convergence of sequence of approximate solutions is proved and the error bound is derived. In **Chapter 5**, we solve the system of HSIEs with the aid of residual based Galerkin's method. The application of the method is also shown by decomposing a hypersingular integral equation over the curve in a complex plane into a system of HSIEs. The well-posedness for the system of linear algebraic equations and convergence of sequence of approximate solutions, are shown. The error bound is also obtained. In all chapters, the validation of derived theoretical results is shown with the aid of various numerical examples.

6.2 Future work

There are some fair possibilities to carry forward the research work presented in this thesis in the following directions:

- In our research, we have proposed residual Galerkin's method to find the approximate solution for system of Cauchy singular integral equations having each equation of index zero. The same method can be proposed to solve the system of Cauchy singular integral equations of different indices such as for index -1, 1 or mixed kind. Further, proving the convergence of sequence of approximate solutions of system of CSIEs of different indices would be an interesting as well as a challenging future task.

- In this thesis, we have used our proposed method to find the approximate solution for system of CSIEs and HSIEs over the finite interval. This work can be extended by proposing the residual Galerkin's method to find the approximate solution for system of CSIEs as well as for system of HSIEs over the half-line or even over the plane.

- Further, obtaining the error bound, proving the convergence of sequence of approximate solutions for system of CSIEs and for system of HSIEs over the half-line or even over the plane would be a very interesting problem to be consider as a future work.

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List of published and communicated work related to thesis

Journal

1. Vaishali Sharma and Amit Setia, "Numerical solution and its analysis for a system of hypersingular integral equations", Journal of Computational and Applied Mathematics.{Accepted, 2018 }{SCI, Scopus}

2. Vaishali Sharma, Amit Setia and Ravi P. Agarwal, "Numerical solution for system of Cauchy type singular integral equations with its error analysis in complex plane", Applied Mathematics and Computation 328 (2018), 338–352. {SCI, Scopus}

3. Amit Setia, Vaishali Sharma and Yucheng Liu, "Numerical solution of Cauchy singular integral equation with an application to a crack problem", Neural, Parallel and Scientific Computations 23 (2015) 387-392.{Scopus}

4. Amit Setia, Vaishali Sharma and Ravi P. Agarwal, "Numerical solution of Cauchy singular integral equation over the half-line with application to a crack problem in a bimaterial'. {**under review** }

5. Vaishali Sharma and Amit Setia, "Approximate solution and its convergence analysis for hypersingular integral equations". {**under review** }

Conference proceedings

1. Amit Setia, Vaishali Sharma and Yucheng Liu "Numerical method to solve Cauchy type singular integral equation with error bounds" AIP Conference Proceedings. Vol. 1798. No. 1. AIP Publishing, 2017. {Scopus}

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List of publications

Journal

1. Vaishali Sharma and Amit Setia, "Numerical solution and its analysis for a system of hypersingular integral equations", Journal of Computational and Applied Mathematics.{**Accepted, 2018**}{SCI, Scopus}

2. Vaishali Sharma, Amit Setia, Ravi P. Agarwal, "Numerical solution for system of Cauchy type singular integral equations with its error analysis in complex plane", Applied Mathematics and Computation 328, (2018), 338–352.{SCI, Scopus}

3. Bijil Prakash, Amit Setia, Deepak Alapatt "Numerical solution of nonlinear fractional SEIR epidemic model by using Haar wavelets" Journal of Computational Science 22, (2017) 109-118. {SCIE, Scopus}

4. Bijil Prakash, Amit Setia, Shourya Bose "Numerical solution for a system of fractional differential equations with applications in Fluid dynamics and Chemical Engineering" International Journal of Chemical Reactor Engineering. 15(5)(2017). {SCIE, Scopus}

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6. Amit Setia, Vaishali Sharma, Yucheng Liu "Numerical solution of Cauchy singular integral equation with an application to a crack problem" Neural, Parallel and Scientific Computations 23 (2015) 387-392.{Scopus}

7. Amit Setia "Solution of various cases of Cauchy type singular integral equation" Applied Mathematics and Computation 230, (2014) 200-207. {SCI, Scopus}

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