Chapter 5

Flexural Study of Beams: Analytical Modelling

5.1 Introduction

This chapter deals with the determination of flexural characteristics of FRP I-beams with and without stiffening elements. There are no code provisions for designing of FRP beams with stiffening elements. Hence, equation to calculate the deflection of beams with different stiffening elements is derived using Castigliano's theorem. In order to get the better understanding of failure of FRP beams, various formulae for beams without stiffening elements available in design manuals are incorporated in the analytical model for prediction of failure load and mode. The flexural response of the beams in terms of deflection, critical local and lateral buckling loads, crippling load, and delamination is predicted using self-derived equations and those available in codes. Accuracy of the different analytical formulae available in codes is verified by comparing with experimental results. In this analytical model, material nonlinearity is not incorporated, because the load-deflection response of the beams obtained from experimental investigation is linear till the failure of specimens. The results obtained from analytical model give the good comparison of results with experimental investigation. Furthermore, a failure criterion is recommended for prediction of failure load of beams with and without stiffening elements. In this chapter, analytical equations to find the flexural response of the beams are presented, and the comparison of analytical response with experimental and numerical responses is presented in Chapter 6.

5.2 Analytical approach

In this study, flexural behavior of the FRP I-beams with and without stiffening elements is investigated under three-point bending. Timoshenko’s first order shear deformation beam theory is used to determine the deflection of beams. Critical buckling load of flange, web and beam, and failure criterion based on strength are used to calculate the failure load of beams. Fig. 5.1 shows the flow chart of different steps of the computational model. An integrated computer program is developed to simulate the flexural behavior of FRP I-beams. Step-by-step procedure to determine the response of beams using analytical equations are discussed in the following sections.

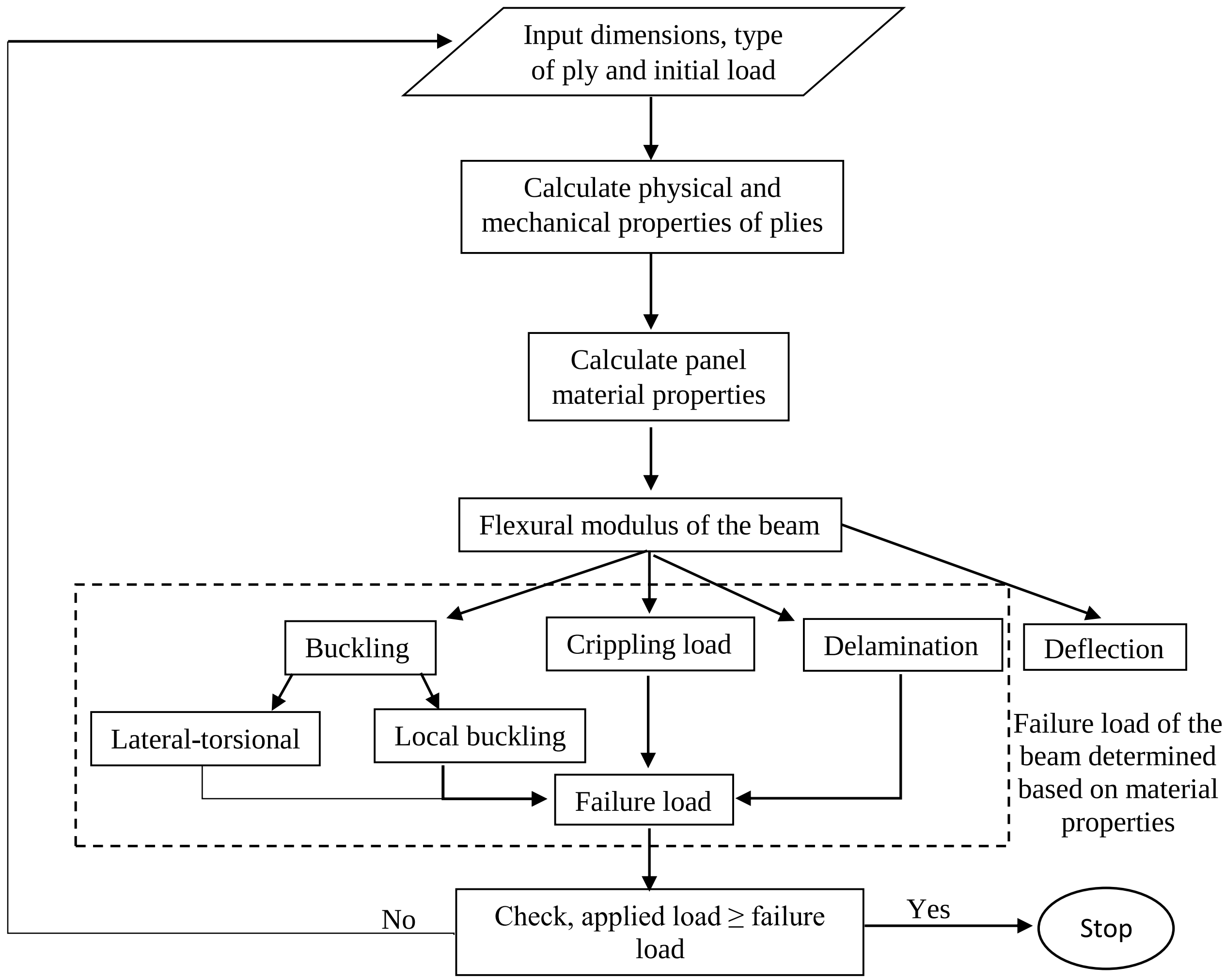


Fig. 5.1. Flow chart of the computational model of FRP I-beam.

5.3 Kinematic formulation

A cartesian coordinate system (x, y and z) is considered for representation of the beam as shown in Fig. 5.2. The x-axis is parallel to the beam, while y and z-axis coincides with principal axis of the beam. Displacement of the beam along the x, y and z coordinates is expressed as

$$u = u_0 + z\phi_x \tag{5.1}$$

$$v = v_0 + z\phi_y \tag{5.2}$$

$$w = w_0 \tag{5.3}$$

where, u_o, v_o and w_o are the displacements of the mid-plane of the laminate. The rotation of transverse normal about x and y-axis is represented by ϕ_y and ϕ_x, respectively. The strain-displacement relationship is given by

$$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} + z\frac{\partial \phi_x}{\partial x} \tag{5.4}$$

$$\varepsilon_{yy} = \frac{\partial v_0}{\partial y} + z\frac{\partial \phi_y}{\partial y} \tag{5.5}$$

$$\gamma_{xy} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + z\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) \tag{5.6}$$

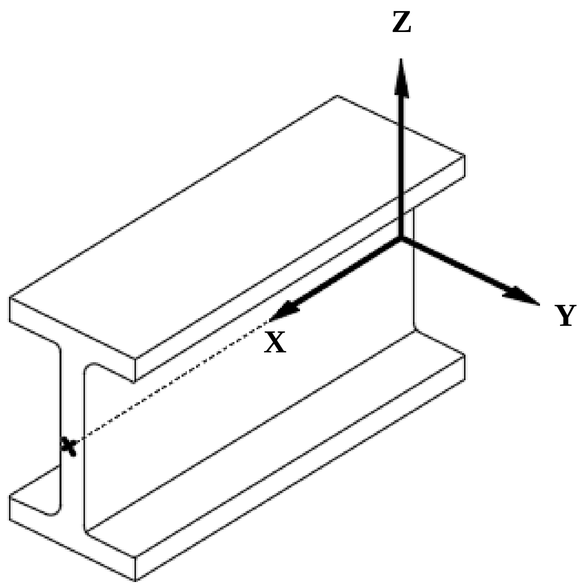


Fig. 5.2. Coordinate system of I-beam.

According to the Timoshenko's beam theory, cross-section of the beam does not remain perpendicular to neutral axis after deformation. It is because of transverse shear strains produces in the cross-section. Hence, the slope of the neutral plane $\left(\frac{\partial w_o}{\partial x}\right)$ is not equal to the rotation of cross-section (ϕ_x) as shown in Fig. 5.3. The difference of the slope of neutral axis and rotation of the cross-section gives the shear strain as given next.

$$\gamma_{xz} = \frac{\partial w_0}{\partial x} - \phi_x \tag{5.7}$$

Similarly, transverse shear strains produced along the y-axis is given by Eq. (5.8).

$$\gamma_{yz} = \frac{\partial w_0}{\partial y} - \phi_y \tag{5.8}$$

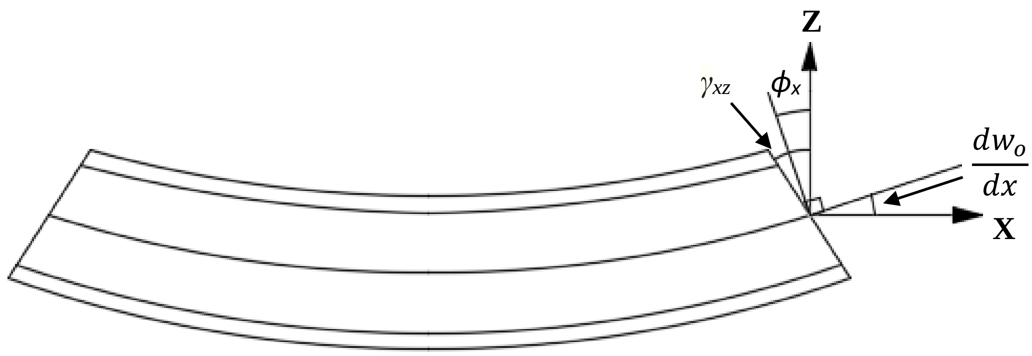


Fig. 5.3. Shear deformation of I-beam.

For a beam loaded with transverse loading, the resultant moments (M_{xx}, M_{yy} and M_{xy}) and shear forces (Q_x and Q_y) produced in a symmetric laminate are given by Eqs. (5.9) and (5.10), respectively (Reddy, 2003 and Jones, 1975).

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} \end{Bmatrix} \tag{5.9}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K_z \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial w_0}{\partial y} - \phi_y \\ \dfrac{\partial w_0}{\partial x} - \phi_x \end{Bmatrix} \tag{5.10}$$

where, D_{ij} is the coefficient of bending stiffness matrix, A_{ij} is the coefficient of shear stiffness matrix and K_z is the shear correction factor (0.83). The inverse form of the Eqs. (5.9) and (5.10) can be written as

$$\begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} \end{Bmatrix} = \begin{bmatrix} D'_{11} & D'_{12} & D'_{16} \\ D'_{12} & D'_{22} & D'_{26} \\ D'_{16} & D'_{26} & D'_{66} \end{bmatrix} \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} \tag{5.11}$$

$$\begin{Bmatrix} \dfrac{\partial w_0}{\partial y} - \phi_y \\ \dfrac{\partial w_0}{\partial x} - \phi_x \end{Bmatrix} = \frac{1}{K_z} \begin{bmatrix} A'_{44} & A'_{45} \\ A'_{45} & A'_{55} \end{bmatrix} \begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} \tag{5.12}$$

In case of in-plane bending of the beams, $M_{yy} = M_{xy} = Q_y = \phi_y = 0$. As a result of this assumption, the relation between curvature and bending moment of a particular laminate is defined as

$$\frac{\partial \phi_x}{\partial x} = D'_{11} M_{xx} \tag{5.13}$$

Here D'_{11} is the bending stiffness of a panel. In order to use Eq. (5.13) for I-beams, bending stiffness (D_y) of I-section is determined by adding the stiffnesses of each panels. Mechanics of laminated beam theory (Davalos et al., 1996) is used to predict the bending stiffness of I-beams. The bending stiffness of beam (D_y) having symmetrical layup and geometry about minor axis is calculated by Eq. (5.14).

$$D_y = 2b_f \left[D_f + A_f z^2 \right] + A_w \frac{h_w^2}{12} \tag{5.14}$$

where, z is the distance from the center of the flange to the centroid of the cross-section (see Fig. 5.4), b_f and h_w is the flange width and depth of web of I-section, respectively. While $A_f = [A_{11}]_f$ and $D_f = [D_{11}]_f$ are elements of extensional and bending stiffness matrix of flange, respectively. Similarly, $A_w = [A_{11}]_w$ is the element of extensional stiffness matrix of web panel. The shear force resultant over the thickness of laminate is given by

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz} dz \tag{5.15}$$

where, τ_{xz} is the shear stress over the laminate and is given by $\tau_{xz} = G_{xz}\gamma_{xz}$. Putting the expression of shear stress in Eq. (5.15) and integrating it with respect to thickness, yields

$$Q_x = K_z h_w G_{xz} \left(\frac{dw}{dx} - \phi_x \right) \tag{5.16}$$

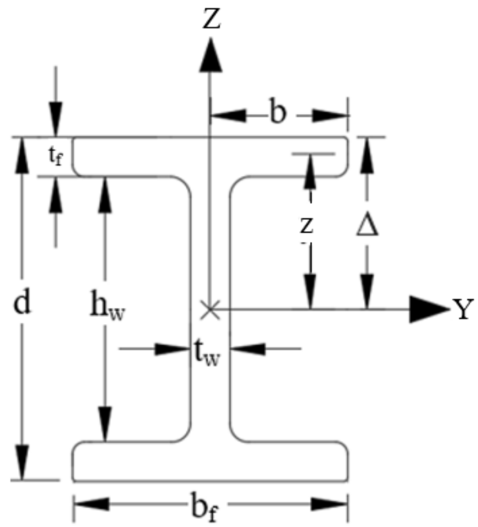


Fig. 5.4. Typical cross-section of I-beam

Using the beam theory, shear force can be written in terms of bending moment as

$$Q_x = \frac{dM_{xx}}{dx} \tag{5.17}$$

Combining Eqs. (5.16) and (5.17), and rearranging it yields

$$\frac{dw}{dx} = \phi_x + \frac{1}{h_w G_{xz} K_z} \frac{dM_{xx}}{dx} \tag{5.18}$$

$$\delta = \int_0^L \left(\frac{1}{h_w G_{xz} K_z} \frac{dM_{xx}}{dx} + \phi_x \right) dx \tag{5.19}$$

5.3.1 Beam with bearing plate

For accurate measurement of flexural response, it is essential to consider loading from bearing plate as uniformly distributed load (UDL) instead of concentrated load. Hence, deflection equation of beam with bearing plate is derived by considering the UDL (q) of length 'a' (length of bearing plate) applied on the mid-span of the beam. Bending moment equation for simply supported beam loaded by bearing plate is given by

$$M_{xx} = q\left(ax - \left(x - \left(\frac{L-a}{2} \right) \right)^2 \right) \quad \& \quad q = \frac{P}{a} \tag{5.20}$$

where, P is the load applied on bearing plate. Substituting for M_{xx} in Eq. (5.13) and integrating with respect to x, ϕ_x can be written as

$$\phi_x = \frac{q}{2E_{xx}I_{yy}}\left[\frac{a}{8}\left(x^2 - L^2\right) - \frac{1}{3}\left(\frac{a^3}{8} - \left(x - \left(\frac{L-a}{2}\right)\right)^3\right)\right] \quad (5.21)$$

where, $E_{xx} = \frac{D_y}{I_{yy}}$ Substituting for ϕ_x in Eq. (5.19) and integrating with respect to x, deflection of the beam at distance x from the left support can be determined by Eq. (5.22).

$$w = \frac{q}{2E_{xx}I_{yy}}\left(\frac{ax^3}{6} - \frac{aL^2x}{8} + \frac{a^3}{24}x - \frac{1}{12}\left(x - \left(\frac{L-a}{2}\right)\right)^4 + \frac{1}{12}\left(\frac{L-a}{2}\right)^4\right) + \frac{q}{2GK_zA}\left(ax - \left(x - \left(\frac{L-a}{2}\right)\right)^2 + \left(\frac{L-a}{2}\right)^2\right) \quad (5.22)$$

where, G is the shear modulus of the beam.

5.3.2 Beam with bearing plate and stiffening element

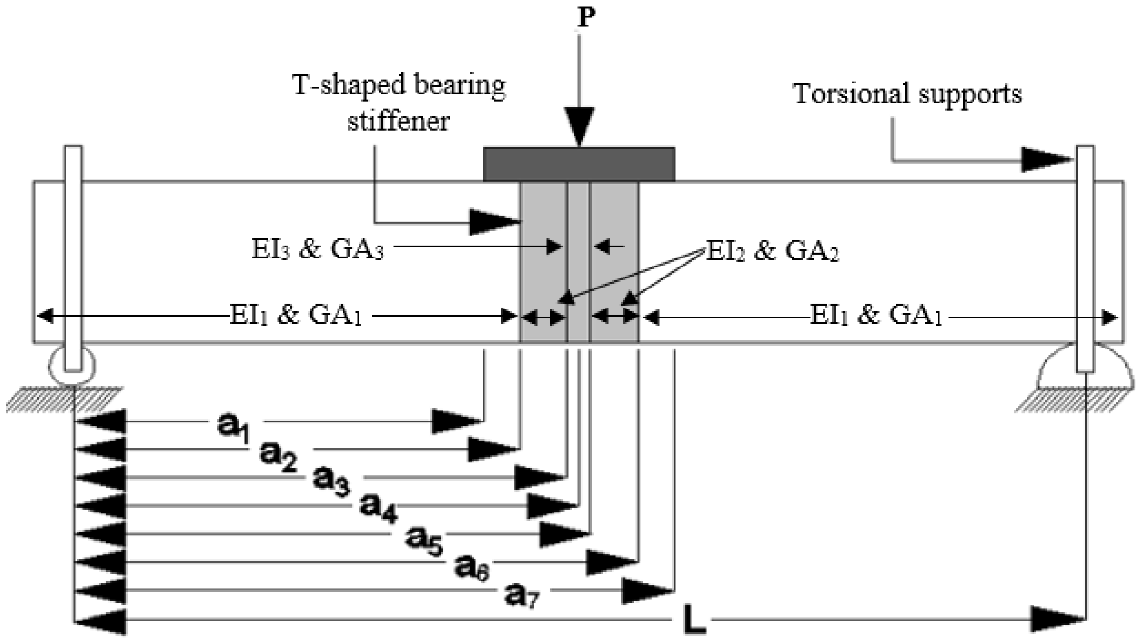
Stiffening elements not only increase the crushing strength of web-flange junction but also add stiffness to the beams. Therefore, the sections where bearing stiffeners is provided, beam have different flexural rigidities than other portion of the beam. In other words, flexural rigidity is not constant on each section of the beam as shown in the Fig. 5.5. Eq. (5.22) is based on the constant flexural rigidity throughout the length of beam, i.e., prismatic beam. Hence, the Castigliano's theorem is used to derive the deflection equation of beam by considering the different flexural rigidity at different sections of beam. Flexural rigidity of composite section (beam and stiffening element) is determined by using transformed area method, i.e., dimensions of stiffening element are transformed by multiplying it with modular ratio. Further, parallel axis theorem is used to evaluate the combined flexural rigidity of the composite section.

The strain energy (U) stored in the beam due to bending moment (M) and shear force (V) is expressed as

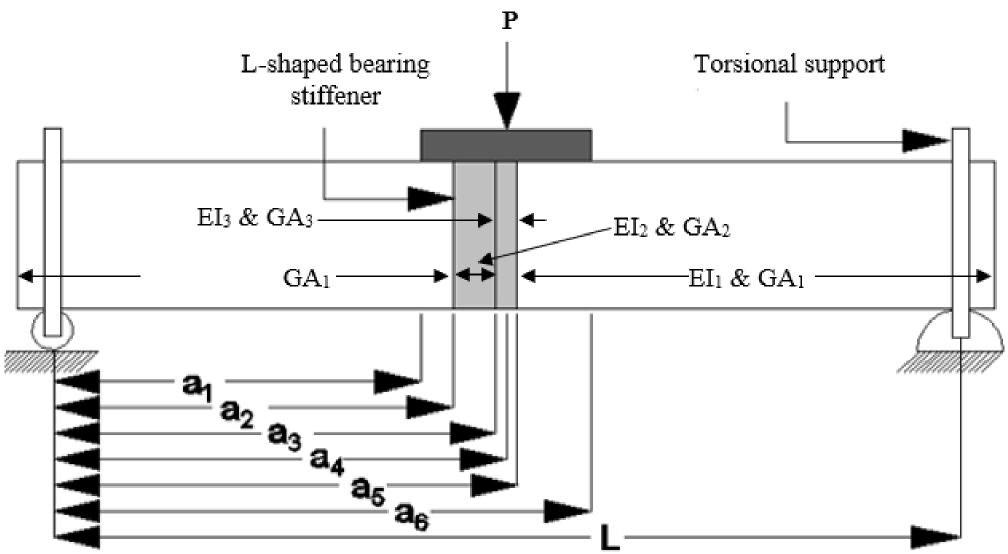
$$U = \int_0^L \left(\frac{M^2}{2EI} + \frac{V^2}{GK_zA}\right)dx \quad (5.23)$$

Deflection of beam (w) under the point of application of loading 'P' is determined by differentiating the strain energy (U) w.r.t. 'P' and it is evaluated as

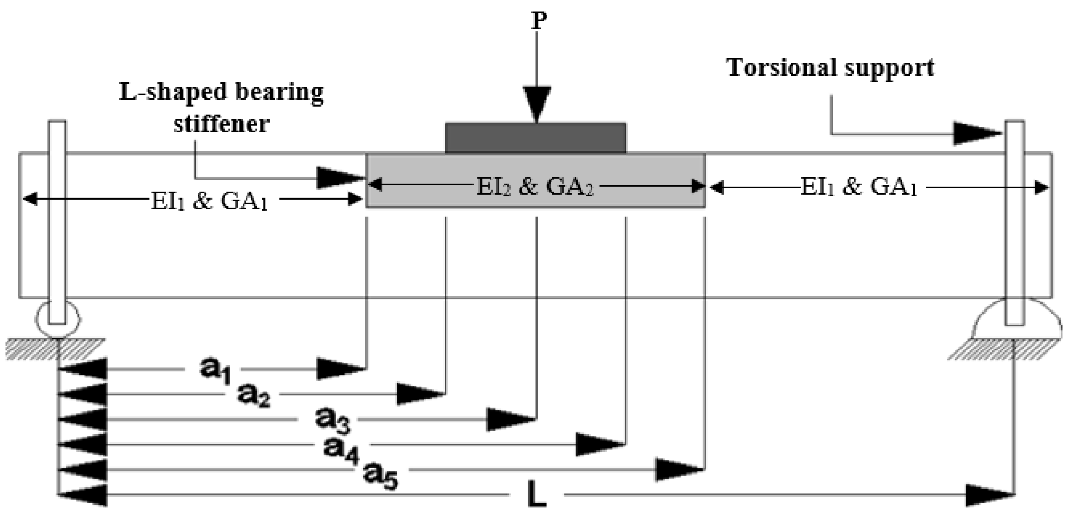
$$w = \frac{\partial U}{\partial P} = \int_0^L \left(\frac{M}{EI}\left(\frac{\partial M}{\partial P}\right) + \frac{V}{GK_z A}\left(\frac{\partial V}{\partial P}\right) \right) dx \tag{5.24}$$



(a) T-shaped bearing stiffener



(b) L-shaped bearing stiffener



(c) Longitudinal stiffening element

Fig. 5.5. Schematic of beams with stiffening elements.

5.3.3 Beam with bearing stiffeners

The flange of T- or L-shaped bearing stiffener increases the flexural stiffness of beam. Bearing stiffeners over supports have negligible effect on enhancing the stiffness of the beam. Hence, the deflection equation of beam is derived with consideration of flexural rigidity of load bearing stiffeners in span only. Using Eq. (5.23), the strain energy of I-beam with T-shaped bearing stiffener at web-flange junction under the bearing plate is given by

$$
\begin{aligned}
U = & \int_0^{a_1} \left(\frac{qax}{2EI_1}\left(\frac{x}{2}\right) + \frac{qa}{4GK_zA_1} \right) dx + \int_{a_1}^{a_2} \left(\frac{q}{2EI_1}\left(ax-(x-a_1)^2\right)\left(\frac{x}{2}\right) + \frac{q}{2GK_zA_1}\left(\frac{a}{2}-(x-a_1)\right)\left(\frac{1}{2}\right) \right) dx + \\
& \int_{a_1}^{a_2} \left(\frac{q}{2EI_1}\left(ax-(x-a_1)^2\right)\left(\frac{x}{2}\right) + \frac{q}{2GK_zA_1}\left(\frac{a}{2}-(x-a_1)\right)\left(\frac{1}{2}\right) \right) dx + \int_{a_3}^{a_4} (\frac{q}{2EI_3}\left(ax-(x-a_1)^2\right)\left(\frac{x}{2}\right) + \frac{q}{2GK_zA_3} \\
& \left(\frac{a}{2}-(x-a_1)\right)\left(\frac{1}{2}\right)) dx + \int_{a_4}^{a_5} \left(\frac{q}{2EI_3}\left(\left(ax-(x-a_1)^2\right)\left(a_4-\frac{x}{2}\right)\right) + \frac{q}{2GK_zA_3}\left(-\frac{a}{2}+(x-a_1)\right)\left(\frac{1}{2}\right) \right) dx + \\
& \int_{a_5}^{a_6} \left(\frac{q}{2EI_2}\left(\left(ax-(x-a_1)^2\right)\left(a_4-\frac{x}{2}\right)\right) + \frac{q}{2GK_zA_2}\left(-\frac{a}{2}+(x-a_1)\right)\left(\frac{1}{2}\right) \right) dx + \int_{a_6}^{a_7} (\frac{q}{2EI_2}(\left(ax-(x-a_1)^2\right) \\
& \left(a_4-\frac{x}{2}\right)) + \frac{q}{2GK_zA_2}\left(-\frac{a}{2}+(x-a_1)\right)) dx + \int_{a_7}^{L} (\frac{q}{EI_1}\left(\left(\frac{ax}{2}-a\left(\frac{a}{2}+(x-a_7)\right)\right)\left(a_4-\frac{x}{2}\right)\right) + \frac{q}{2GK_zA_1}(-\frac{a}{2}+ \\
& a)) dx
\end{aligned}
\tag{5.25}
$$

where, x is the distance from left support, while the flexural rigidities (EI_1 to EI_3) and shear rigidities (GA_1 & GA_2) are the rigidities at different sections of the beam, i.e., at locations where beam is stiffened and unstiffened as shown in Fig. 5.5(a). After integration of Eq. (5.25) and putting the updated expression of strain energy into Eq. (5.24), the equation to determine the maximum deflection (at mid-span) of the beam with T-shaped bearing stiffener is obtained as

$$w=\frac{q\left(a^4-3b_{tf}{}^4-4L\left(a^3-b_{tf}{}^3\right)+8aL^3-6a^2b_{tf}{}^2+4Lb_{tf}{}^2\left(b_{tf}+3a\right)+12aLb_{tf}\left(a-2L\right)\right)}{384EI_1}+\frac{3q\left(\left(b_{tf}{}^4-t_{tw}{}^4\right)-2a^2t_{tw}^2\right)}{384EI_2}+$$
$$\frac{q\left(-4L\left(b_{tf}{}^3-t_{tw}{}^3\right)+\left(6ab_{tf}{}^2-12aLb_{tf}\right)\left(a-2L\right)+12aLt_{tw}\left(t_{tw}+a-2L\right)\right)}{384EI_2}-\frac{qLt_{tw}\left(t_{tw}{}^2+3a^2-6aL\right)}{96EI_3}-\frac{q}{8GK_zA_1}\left(a^2+b_{tf}{}^2\right.$$
$$\left.-2La\right)+\frac{q\left(b_{tf}{}^2-t_{tw}{}^2\right)}{8GK_zA_2}+\frac{qt_{tw}{}^2}{8GK_zA_3} \tag{5.26}$$

where, b_{tf} and t_{tw} are the flange width and web thickness of T-shaped bearing stiffener, respectively. The maximum deflection (mid-span) of the beam with L-shaped bearing stiffener under the loading is calculated by Eq. (5.27).

$$w=\frac{q\left(3b_{lf}{}^4-3t_{lw}{}^4-4L\left(b_{lf}{}^3-t_{lw}{}^3\right)+6a^2\left(b_{lf}{}^2-t_{lw}{}^2\right)-12b_{lf}aL\left(b_{lf}+a-2L\right)+12at_{lw}L\left(t_{lw}+a-2L\right)\right)}{768EI_2}-\frac{qt_{lw}L}{96EI_3}\left(t_{lw}{}^2\right.$$
$$\left.+3a^2-6aL\right)-\frac{q\left(2a^2+b_{lf}{}^2+t_{lw}{}^2-4aL\right)}{16GK_zA_1}+\frac{q\left(b_{lf}{}^2-t_{lw}{}^2\right)}{16GK_zA_2}+\frac{qt_{lw}{}^2}{8GK_zA_3} \tag{5.27}$$

where, b_{lf} and t_{lw} are the width of connecting leg and the thickness of outstanding leg of angle section, respectively. The flexural rigidities (EI_1 to EI_3) and shear rigidities (GK_zA_1 and GK_zA_2) of stiffened and un-stiffened section of the beam are illustrated in Fig. 5.5(b).

5.3.4 Beam with longitudinal stiffening elements

Aforementioned Eqs. (5.26) and (5.27) are used to determine the structural response of beams loaded by bearing plate and stiffened by vertical stiffeners under the loading and over supports. Stiffening elements installed along the length of the beams (longitudinal stiffeners or cover plate) increases the flexural strength and stiffness of beams. Therefore, load-deflection equations are derived by calculating the effective stiffness at the location where beam is stiffened and unstiffened. The maximum deflection of beam, i.e., at the mid-section with stiffening element and bearing plate is derived as

$$w = \frac{qa}{16EI_1}\left(\frac{L^3}{3} - L_sL^2 + LL_s^2 - \frac{L_s^3}{3}\right) + \frac{qa}{16EI_2}\left(\frac{L_s^3}{3} + L_sL^2 - LL_s^2 - \frac{La^2}{12} - \frac{9a^3}{48}\right) + \frac{qa}{4GK_zA_1}\left(L - L_s\right) + \frac{qa}{4GK_zA_2}\left(L_s - \frac{a}{2}\right) \tag{5.28}$$

where, L_s denotes the length of bearing stiffener, flexural rigidities (EI_1 and EI_2) and shear rigidities (GK_zA_1 and GK_zA_2) at different sections of the beam are shown in Fig. 5.5(c).

5.4 Failure load

From the experimental investigation, it is observed that the failure mode of the beams with bearing plate was crushing of web below the web-flange junction under the load bearing plate. The crushing of the web may be due to the lateral-torsional buckling of beams, local buckling of the web or crippling of the web. As per Borowicz and Bank (2011), the web-flange junction failure of the beam having L/d ratio 3 with bearing plate is due to the local buckling of web. Therefore, it is necessary to determine the critical lateral-torsional buckling or local buckling load of the beam to study the failure of beam with bearing plate under three-point bending. In case of beams with stiffening elements, failure load observed from experimental investigation is local buckling of compression flange or tearing of the web-flange junction. Correia et al. (2011) reported that the beam having L/d ratio 7, failed by local buckling of the compression flange and the failure mode is different for different L/d ratios. Hence, it is necessary to include the local and lateral-torsional buckling equations in analytical model for predicting the actual mode of failure and failure load of the beam. Here in this study, buckling equations given in various standards (Clarke, 1996; Pultrusions, 2013; CNR DT-205, 2007; ASCE, 2011).) and in other literature are used to determine the failure load of beams and are compared with the experimental results. This study is also helpful in checking the accuracy of proposed formulae given in literature.

5.4.1 Buckling of beams

There are two types of buckling which are possible in beams, i.e., lateral-torsional and local buckling of beams. Equations to determine critical buckling load from different design manuals and other equations suggested by researchers are explained next:

5.4.1.1 European code

Buckling of compression flange

Clarke (1996) determined the critical buckling stress of the compression flange under flexural loading by considering the flange as a long plate. The longitudinal free edge of the flange of I-beam is considered as the free edge of the plate, while other longitudinal edges connected to web is assumed as pinned. The critical buckling stress is determined from flexural rigidity of the flange in longitudinal direction (D_{11}) and in-plane shear rigidity of the flange (D_{66}) and it is given as

$$\sigma_{x,cr} = \frac{\pi^2}{t_f b_f^2}\left\{\left(D_{11}\left(\frac{b_f}{L}\right)^2\right)+\left(\frac{12D_{66}}{\pi^2}\right)\right\} \tag{5.29}$$

Author (Clarke, 1996) has considered the laminate (web or flange) as single orthotropic layer. Therefore, the local flexural rigidities of the beam are given as

$$D_{11} = \frac{E_{xx}t_f^3}{12\left(1-v_{xy}v_{yx}\right)},\ D_{12} = v_{xy}D_{22},\ D_{22} = \frac{E_{yy}t_f^3}{12\left(1-v_{xy}v_{yx}\right)}\ \&\ D_{66} = \frac{G_{xy}t_f^3}{12} \tag{5.30}$$

Critical buckling flexural stress of web under in-plane bending

The in-plane bending of the beam produces flexural stresses in the web. Therefore, web of I-beam is prone to buckling during flexural deformation. Hence, the depth of web is considered as a column, therefore effective length is taken as depth of the web and critical buckling stress is given by Eq. (5.31).

$$\sigma_{x,cr,b} = \frac{k'\pi^2 D_{11}}{h_w^2 t_w} \tag{5.31}$$

If web is assumed as clamped with flange, k'= 50 otherwise k' = 20.

Critical shear stress in web

Like flexural stresses, in-plane shear stresses are also produced in the web of I-beam, during three-point bending. The critical in-plane buckling shear stress is calculated by Eq. (5.32).

$$\tau_{cr} = \frac{32\left(D_{11}D_{22}^3\right)^{0.25}}{h_w^2 t_w} \tag{5.32}$$

Lateral-torsional buckling load of the beam

The lateral-torsional buckling equation (Eq. 5.33) of FRP I-beams given in European code (Clarke, 1996) is the buckling equation of the isotropic beams. This equation includes the effect of warping of the cross-section and shear rigidity of the beam and is expressed as

$$M_b = \frac{P_{ey}}{C_1}\left[k\frac{I_w}{I_{zz}} + \frac{G_{xy}J}{P_{ey}} \right]^{0.25} \tag{5.33}$$

where, J is the torsion constant, k is the effective length factor, I_{zz} is the polar moment of inertia, I_w is the warping moment of inertia, C_1 is the factor depends on the loading and support conditions. Euler buckling load (P_{ey}) is computed by

$$P_{ey} = \frac{\pi^2 E_{zz} I_{zz}}{(kL)^2} \tag{5.34}$$

5.4.1.2 Italian code (CNR DT-205, 2007)

Critical buckling stress of compression flange

The critical buckling stress of flanges presented in Italian code (CNR DT-205, 2007) is based upon the study of Kollar and Springer (2003). In this code (CNR DT-205, 2007), the connection of flange with web is considered as rotational springs. The critical buckling stress of flanges is determined by considering the rotational constraints provided by web on flanges and is given by Eqs. (5.35) and (5.36).

$$(\sigma_{x,cr})_f = \frac{\sqrt{(D_{11})_f (D_{22})_f}}{t_f \left(\frac{b_f}{2}\right)^2} \left\{ K\left[15\eta\sqrt{1-\rho} + 6(1-\rho)(1-\eta)\right] + \frac{7(1-K)}{\sqrt{1+4.12\zeta}} \right\}, \text{ for } K \le 1 \tag{5.35}$$

$$(\sigma_{x,cr})_f = \frac{\sqrt{(D_{11})_f (D_{22})_f}}{t_f \left(\frac{b_f}{2}\right)^2} \left[15\eta\sqrt{1-\rho} + 6(1-\rho)(K-\eta)\right], \text{ for } K > 1 \tag{5.36}$$

The parameters (K, ζ, ρ, & η) used to determine the critical buckling load is defined as

$$K = \frac{2(D_{66})_f + (D_{12})_f}{\sqrt{(D_{11})_f (D_{22})_f}}, \ \rho = \frac{(D_{12})_f}{2(D_{66})_f + (D_{12})_f}, \ \eta = \frac{1}{\sqrt{1+(7.22-3.55\rho)\zeta}} \ \& \ \zeta = \frac{(D_{22})_f}{\tilde{k}\frac{b_f}{2}} \tag{5.37}$$

The rotation stiffness constraint ($\tilde{k}$) of the web to flange is determined by

$$\tilde{k} = \frac{(D_{22})_w}{h_w}\left[1 - \frac{t_f \sigma^{ss}_{cr,f} \dfrac{1}{(E_{xx})_f t_f}}{t_w \sigma^{ss}_{cr,w} \dfrac{1}{(E_{xx})_w t_w}}\right] \tag{5.38}$$

In Eq. (5.38), $\sigma^{ss}_{cr,f}$ and $\sigma^{ss}_{cr,w}$ are the critical buckling stresses of flanges and web, respectively. These critical stresses are determined separately, by assuming flange and web as simply supported plates. Critical buckling stress of flange and web plates assuming simply supported is given as

$$\sigma^{ss}_{cr,f} = \frac{12(D_{66})_f}{t_f\left(\dfrac{b_f}{2}\right)^2} \tag{5.39}$$

$$\sigma^{ss}_{cr,w} = \frac{\pi^2}{t_w h_w^2}\left\{2\sqrt{(D_{11})_w (D_{22})_w} + 2\left[(D_{12})_w + 2(D_{66})_w\right]\right\} \tag{5.40}$$

Buckling stress in web

In order to calculate the critical buckling stress of web, Italian code has considered the web of beam as a short column, having length equal to depth of the column. The critical buckling stress of web of I-section under flexural loading is given by Eq. (5.41).

$$\sigma_w = \sigma^{ss}_w = k_f \frac{\pi^2 E_{xx} t_w^2}{12(1 - \nu_{xz}\nu_{zx}) h_w^2} \tag{5.41}$$

$$k_f = 13.9\sqrt{\frac{E_{zz}}{E_{xx}}} + 22.2\frac{G_{xz}}{E_{xx}}\left(1 - \nu_{xz}^2 \frac{E_{zz}}{E_{xx}}\right) + 11.1\left(\nu_{xz} \frac{E_{zz}}{E_{xx}}\right) \tag{5.42}$$

Lateral-torsional buckling of beam

Alike the buckling equation of European code, Italian code also considers the effect of torsional and warping deformation of the beam. Along with, Italian code also includes the effect of shear deformation, which is missing in equation of European code and Pultex design manual (Pultrusions, 2013). As per the Italian code, the equation to determine the lateral-torsional buckling moment of beam has the following form (Bulson and Allen, 1980).

$$M_{cr} = 1.35N_{cr,z}\left(0.55\Delta \pm \sqrt{(0.55\Delta)^2 + \left(\frac{N_{cr,\psi}i_\omega^2}{N_{cr,z}}\right)}\right) \tag{5.43}$$

where, Δ is the distance between load and centroid of I-beam (see Fig. 5.4), $N_{cr,z}$ is the critical buckling load of beam, when beam buckles in the x-z plane as shown in Fig. 5.6(a). The critical torsional warping load of the beam (see Fig. 5.6(b)) is represented by $N_{cr,\omega}$ and is given by Eq. (5.45).

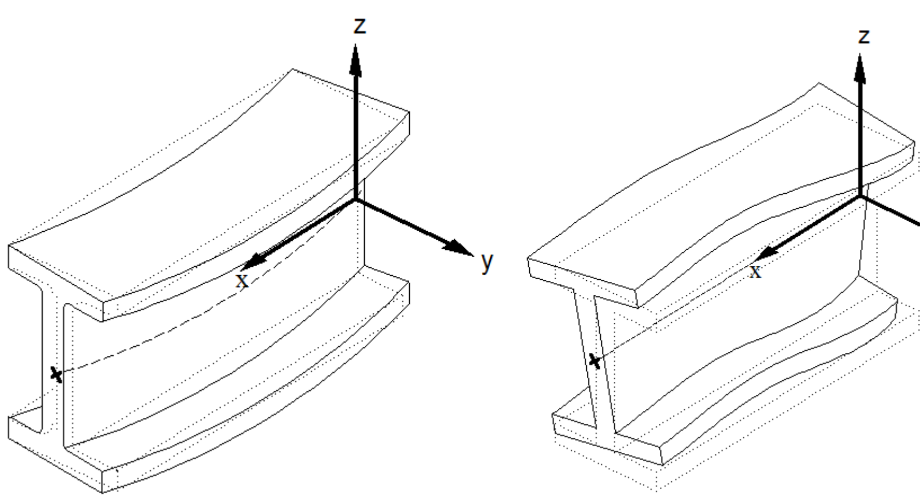
$$N_{cr,z} = \left(\frac{k^2L^2}{\pi^2 EI_{yy}} + \frac{6d^2y^2}{a_{33}y^2\left(d+0.5b_f\right)}\right)^{-1} \;\&\; y = 1 + \frac{d}{6b} \tag{5.44}$$

$$N_{cr,\psi} = \left(\frac{\left(i_w kL\right)^2}{\pi^2 EI_w} + \frac{i_w^2 bd^2}{2.4b + a_{33}d^2}\right)^{-1} + \frac{GI_t}{t_w^2} \;\&\; i_\omega = \sqrt{\frac{EI_{yy}+EI_{zz}}{EA}} \tag{5.45}$$

In the Italian code, the flexural (EI_{yy} and EI_{zz}) and torsional stiffness (EA) is determined from Eq. (5.46).

$$EA = \frac{2b_f}{\left(a_{11}\right)_f} + \frac{h_w}{\left(a_{11}\right)_w},\; EI_{yy} = \frac{b_f}{\left(a_{11}\right)_f} - \frac{2b_f yd}{\left(a_{11}\right)_f} + \frac{b_f y^2}{\left(a_{11}\right)_f} + \frac{2b_f}{\left(d_{11}\right)_f} + \frac{h_{w1}^3 + h_{w2}^3}{3\left(a_{11}\right)_w} \;\&\; EI_{zz} = \frac{h_w}{\left(d_{11}\right)_w} + \frac{b_f^3}{6\left(a_{11}\right)_f} \tag{5.46}$$

where, $(a_{11})_f$ and $(d_{11})_f$ are the elements of compliance matrix of flange. While $(a_{11})_w$ and $(d_{11})_w$ are the elements of compliance matrix of web.



(a) Buckling in x-y plane (b) Torsional about x-axis

Fig. 5.6. Modes of failure of I-section.

5.4.1.3 Pultex design manual

Local buckling stress of compression flange

In the Pultex design manual, the equation to determine the critical buckling stress of the flange is expressed as

$$\left(\sigma_{x,cr}\right)_f = \frac{\pi^2}{12}\left(\frac{t_f}{0.5b_f}\right)^2\left[\sqrt{q^{'}}\left(2\sqrt{\left(E_{xx}\right)_f\left(E_{yy}\right)_f}\right)+p\left(\left(E_{yy}\right)_f\left(v_{xy}\right)_f+2\left(G_{xy}\right)_f\right)\right] \tag{5.47}$$

The longitudinal, transverse, shear modulus are represented by E_{xx} , E_{yy} and G_{xy}, respectively. Poisson's ratio of the beam is denoted by v_{xy}. The constants $q^{'}$ and p are based upon the coefficients of restraints of web-flange junction and are introduced in the following expressions

$$p = 0.3+\left(\frac{0.004}{\zeta-0.5}\right); q^{'} = 0.025+\left(\frac{0.065}{\zeta+0.4}\right) \ \& \ \zeta = \frac{2b_f\left(E_y\right)_f}{h_w\left(E_y\right)_w} \tag{5.48}$$

Lateral-torsional buckling of the beam

The buckling Eq. (5.49) is derived based on the assumption that beam is adequately laterally supported. The code considered the effect of different supports and loading condition provided on the beam by multiplying with a coefficient C_b. The critical torsional buckling moment of the beam is given by

$$M_{cr} = C_b\frac{\pi}{L}\sqrt{\left(\frac{\pi E_z}{L}\right)^2 C_w I_z + EI_{zz}GJ} \tag{5.49}$$

$$C_w = \frac{h^2 I_z}{4} \ \& \ J = \frac{1}{3}\left(2b_f t_f^3 + b_w t_w^3\right) \tag{5.50}$$

5.4.1.4 ASCE code

Local buckling of the compression flange

As per the ASCE code, local buckling of FRP pultruded beams bent about strong axis is given by

$$f_{cr} = \frac{t_f^2}{b_f^2}\left(\frac{7}{12}\sqrt{\frac{E_{xx}E_{yy}}{1+4.1\xi}}+G_{xy}\right) \tag{5.51}$$

$$\xi = \frac{E_{xx}t_f^3}{6b_f k_r} \ \& \ k_r = \frac{E_{yy,w}t_w^3}{6h}\left(1-\left[\left(\frac{48t_f^2h^2E_{xx,\mathrm{w}}}{11.1\pi^2t_w^2b_f^2E_{xx,f}}\right)\left(\frac{G_{xy}}{1.25\sqrt{E_{xx}E_{yy}}+E_{yy}v_{xy}+2G_{xy}}\right)\right]\right) \quad (5.52)$$

Local buckling of web

As per the ASCE code, the local buckling stress of web is given as

$$\left(\sigma_{x,cr}\right)_f = \frac{11.1\pi^2t_{\mathrm{w}}^2}{12h^2}\left[1.25\sqrt{E_{xx,\mathrm{w}}E_{yy,\mathrm{w}}}+v_{xy}E_{yy,\mathrm{w}}+2G_{xy}\right] \quad (5.53)$$

5.4.2 Failure load based on strength

During the bending of beams, flexural, shear and bearing stresses are produced. Failure other than buckling occurs when the stresses produced from loading exceeds the strength of the beams. European and American code have analytical equations to determine the crushing or crippling strength of the beams.

5.4.2.1 European code

The beams without bearing stiffener have chances to fail by crushing of web. According to the European code, the crushing of beams occurs when the applied compressive stresses exceed the compressive strength. Therefore crushing failure load (V_y) depends on the bearing length and transverse compressive strength (S_{tc}) of the web and is determined as

$$V_y = \left(a+h_w\right)t_wS_{tc} \quad (5.54)$$

5.4.2.2 ASCE code

Like crushing, crippling also occurs in the beams due to transverse loading and in the absence of bearing stiffeners. The local crippling failure load (R_n) of beams is given by

$$R_n = 0.7Ht_wS_{xz}\left(1+\frac{\left(2k+6t_{bp}+a\right)}{h_w}\right) \quad (5.55)$$

where, S_{xz} is the interlaminar shear strength of the members and t_{bp} is the thickness of bearing plate.

5.5 Delamination failure criterion

Under the flexural loading, FRP I-beams deforms in-plane and/or out-of-plane. After a certain deformation, beam have chances to fail by delamination of web-flange joint, crushing of web, ply failure or buckling of compression flange or web. A quadratic delamination criterion proposed by Brewer and Lagace (1988) is used to determine the failure load of stiffened and un-stiffened beams. This criterion is also used by Bai and Wu (2013) to predict the web-flange junction failure of FRP beams. From experimental investigation (Chapter 4), it is observed that response of the beam is linear until failure; it means that not a single ply failed just before the ultimate failure of the beam. It is also noted that the beam without bearing stiffener failed due to the interlaminar shear failure which leads to the crushing of the web under the bearing plate, while beams with stiffening element failed due to the delamination of fiber layers under web-flange junction due to bearing stress produced by bearing plate. In each unstiffened beam, high bearing stresses are produced by bearing plate and is resisted by the junction of web-flange. Therefore, the transverse compressive strength as well as shear strength of web plays the vital role in resisting the bearing stress of the bearing plate. It is also observed that the strength of beams increases with decrease in the L/d ratio. Hence, failure criterion includes the longitudinal compressive stress, transverse compressive stress and shear stresses produced due to the bending of the beam and is given by Eq. (5.56).

$$\left(\frac{\sigma_{lb}}{S_{lb}}\right)^2 + \left(\frac{\sigma_{tc}}{S_{tc}}\right)^2 + \left(\frac{\tau_{xz}}{S_{xz}}\right)^2 \geq 1 \tag{5.56}$$

where, σ_{lb}, σ_{tc} and τ_{xz} are the longitudinal compressive bending stress, transverse compressive stress (bearing stress) and shear stress, respectively. Corresponding strengths such as bending compressive, transverse compressive and shear strengths are represented by S_{tb}, S_{tc} and S_{xz}, respectively. The crushing of web is due to the diagonal bearing stresses produced from the bearing plate. Therefore, like stiffened steel beams, bearing stresses are determined at vertical distance of "t_f+R" from bottom of flange and at slope of 1:2.5 as shown in Fig. 5.7. Here *R* is the radius of curvature of web-flange junction. In Eq. (5.56), bearing stresses produced in beams by bearing plate are determined by Eq. (5.57).

$$\sigma_{tc} = \frac{P}{A_{ef}} \text{ where } A_{ef} = t_w\left(a + 2.5\left(t_f + R\right)\right) \tag{5.57}$$

For the stiffened beams, the effective area (A_{ef}) depends on the cross-section of the stiffening element under compression flange as shown in Fig. 5.8. Using the elementary theory of solid mechanics, the shear stress at web-flange junction of I-beam is given by

$$\tau_{xz} = \frac{2Vb_f t_f}{8t_w I_{yy}} \tag{5.58}$$

where, V is the shear force in the beam and I_{yy} is the moment of inertia of the beam about y-axis.

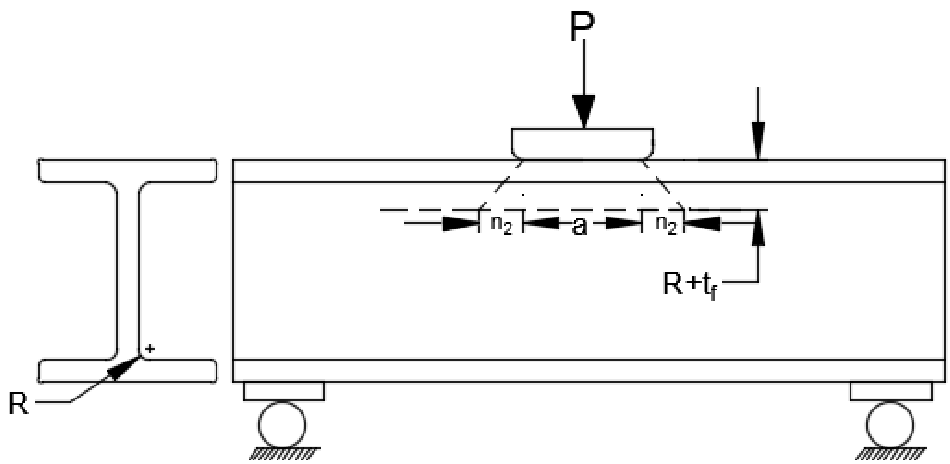
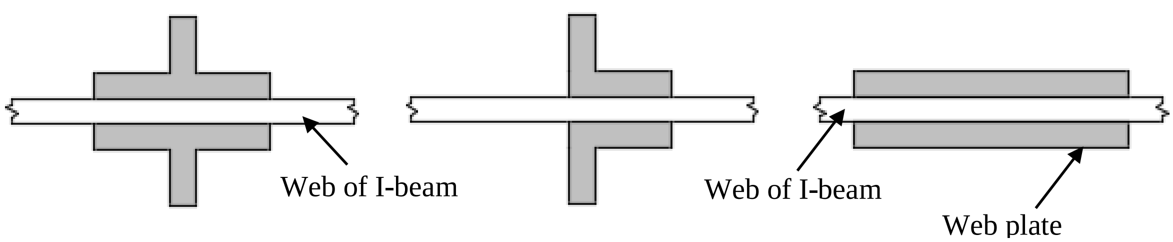


Fig. 5.7. Distribution of bearing stress in beam.



(a) T-shaped stiffener (b) L-shaped stiffener (c) web of cover angle or web plate

Fig. 5.8. Bearing area of beam with different stiffening elements. (Top view of I-beam without flanges)