Chapter 1

Background

In this chapter, we recall necessary definitions and results that are useful for the upcoming chapters. In Section 1.1, we provide essential definitions and results related to the theory of semigroups and groups. Also, we proved required results related to semigroups that are used in the thesis. Section 1.2 is devoted to the notion of graph theory. Also, we recall some fundamental results on graphs which will be useful in the thesis. This chapter also fixes various notations used in the thesis.

1.1 Semigroups and Groups

In this section, we recall the necessary definitions and results of semigroup theory from Howie [1995], and group theory from Hungerford [1974]. Also, we derive some results related to monogenic semigroups which will be useful for further chapters of the thesis.

A *semigroup* is a non-empty set S together with an associative binary operation on S. We say S to be a *monoid* if it contains an identity element e. A monoid S is said to be a *group* if for each x there exists $x^{-1} \in S$ such that $xx^{-1} = x^{-1}x = e$.

A *subsemigroup* of a semigroup is a subset that is also a semigroup under the same operation. A subsemigroup of S which is a group with respect to the operation inherited from S will be called *subgroup*. A non-empty subset I of a semigroup S is said to be an *ideal* of S if $SIS \subseteq I$. A semigroup S is said to be *commutative* if $xy = yx$ for all $x, y \in S$. An element a of a semigroup S is *idempotent* if $a^2 = a$ and the set of all idempotents in S is denoted by $E(S)$. A *band* is a semigroup in which every element is idempotent. A semigroup S is said to be an *inverse semigroup* if for each $x \in S$ there is a unique element $x^{-1} \in S$ satisfying $x = xx^{-1}x$ and $x^{-1} = x^{-1}xx^{-1}$. A semigroup S is said to be *regular* if for each $a \in S$ there exists $x \in S$ such that $axa = a$. If a semigroup S with at least two elements contains an element 0 such that for all $x \in S$, $0x = x0 = 0$ then $0 \in S$ is called the *zero* of S and in this case, S is known as a *semigroup with zero*. If S does not contain zero, then we say that S is a *semigroup without zero*.

Example 1.1.1. (i) Let X be a non-empty set and $\mathcal{T}_X$ be the set of all mappings on X. Then $\mathcal{T}_X$ forms a semigroup under the composition of mappings is called *full transformation semigroup*.

(ii) The set $I(X)$ of partial injective mappings on X forms a semigroup under the composition of relations and it is known as *symmetric inverse semigroup*.

(iii) Given a finite group G and a natural number n, write $[n] = \{1, 2, \ldots, n\}$ and $B_n(G) = ([n] \times G \times [n]) \cup \{0\}$. Define a binary operation '$\cdot$' on $B_n(G)$ by

$$(i, a, j) \cdot (k, b, l) = \begin{cases} (i, ab, l) & \text{if } j = k; \\ 0 & \text{if } j \neq k \end{cases}$$

and, for all $\alpha \in B_n(G), \alpha \cdot 0 = 0 \cdot \alpha = 0$, is known as *Brandt semigroup*. When G is trivial group, the Brandt semigroup $B(\{e\}, n)$ is denoted by B_n. Instead of writing the identity element $e \in G$ in the triplets of elements of B_n, we use the following description in the definition of B_n. For any integer $n \geq 1$, let

$[n] = \{1, 2, \ldots, n\}$. The semigroup $(B_n, \cdot)$, where $B_n = ([n] \times [n]) \cup \{0\}$ and the operation '$\cdot$' is given by

$$(i,j) \cdot (k,l) = \begin{cases} (i,l) & \text{if } j = k; \\ 0 & \text{if } j \neq k; \end{cases}$$

and, $\alpha \cdot 0 = 0 \cdot \alpha = 0$ for all $\alpha \in B_n$.

For a subset X of a semigroup S, the subsemigroup generated by X, denoted by $\langle X \rangle$, is the intersection of all the subsemigroup of S containing X and it is the smallest subsemigroup of S containing X. The subsemigroup $\langle X \rangle$ is the set of all the elements in S that can be written as finite product of elements of X. If X is finite then $\langle X \rangle$ is called finitely generated subsemigroup of S. A semigroup S is called *monogenic* if there exists $a \in S$ such that $S = \langle a \rangle$. Clearly, $\langle a \rangle = \{a^m \ : \ m \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of positive integers. A subgroup generated by X can be defined analogously. If $X = \{a\}$, then the subgroup generated by X is called *cyclic*. Note that the cyclic subgroup generated by a is $\langle a \rangle = \{a^m : \ m \in \mathbb{Z}\}$.

For $X \subseteq S$, the number of elements in X is called the order (or size) of X and it is denoted by $|X|$. The *order* of an element $a \in S$, denoted by $o(a)$, is defined as $|\langle a \rangle|$. The set $\pi(S)$ consists order of all the elements of a semigroup S. In case of finite monogenic semigroup, there are repetitions among the powers of a. Then the set

$$\{x \in \mathbb{N} : (\exists\, y \in \mathbb{N}) a^x = a^y, x \neq y\}$$

is non-empty and so has a least element. Let us denote this least element by m and call it the *index* of the element a. Then the set

$$\{x \in \mathbb{N} \ : \ a^{m+x} = a^m\}$$

is non-empty and so it too has a least element r, which we call the *period* of a. Let a be an element with index m and period r. Thus, $a^m = a^{m+r}$. It follows that $a^m = a^{m+qr}$ for all $q \in \mathbb{N}$. By the minimality of m and r we may deduce that the

powers $a, a^2, \ldots, a^m, a^{m+1}, \ldots, a^{m+r-1}$ are all distinct. For every $s \geq m$, by division algorithm we can write $s = m + qr + u$, where $q \geq 0$ and $0 \leq u \leq r-1$. then it follows that

$$a^s = a^{m+qr}a^u = a^m a^u = a^{m+u}.$$

Thus, $\langle a \rangle = \{a, a^2, \ldots, a^{m+r-1}\}$ and $o(a) = m + r - 1$. The subset

$$\mathcal{K}_a = \{a^m, a^{m+1}, \ldots, a^{m+r-1}\}$$

is a subsemigroup of $\langle a \rangle$. Moreover, there exists $g \in \mathbb{N}$ such that $0 \leq g \leq r-1$ and $m + g \equiv 0 (\text{mod } r)$. Note that a^{m+g} is the idempotent element and so it is the identity element of $\mathcal{K}_a$. Because $a^{(m+g)^2} = a^{2m+2g} = a^{m+(m+g)+g} = a^{m+tr+g}$ as $m + g \equiv 0 (\text{mod } r)$ which gives $a^{(m+g)^2} = a^{m+g}$. If we choose $g' \in \mathbb{N}$ such that

$$0 \leq g' \leq r-1 \text{ and } m + g' \equiv 1 (\text{mod } r),$$

then $k(m + g') \equiv k (\text{mod } r)$ for all $k \in \mathbb{N}$, and so the powers $(a^{m+g'})^k$ of $a^{m+g'}$ for $k = 1, 2, \ldots, r$, deplete $\mathcal{K}_a$. Thus, $\mathcal{K}_a$ is the cyclic subgroup of order r, generated by $a^{m+g'}$. Let a be an element of a semigroup S with index m and period r. Then the monogenic semigroup $\langle a \rangle$ is denoted by $M(m, r)$. Also, sometimes $M(m, r)$ shall be written as $\langle a : a^m = a^{m+r} \rangle$. The notations m_x and r_x denotes the index and period of x in a monogenic semigroup S, respectively. It is easy to observe that index of every element in a finite group G is one. Consequently, for $a \in G$, we have $\langle a \rangle$ is the cyclic subgroup of G. The following results are useful in the sequel.

Remark 1.1.2. Let $S = M(m, r) = \langle a \rangle$ be a monogenic semigroup. Then $\mathcal{K}_{a^i} = \langle a^i \rangle \cap \mathcal{K}_a$.

Proposition 1.1.3 ([Howie, 1995, Proposition 1.2.3])**.** *Every finite semigroup contains at least one idempotent.*

Lemma 1.1.4. *Let a be an element of a finite semigroup S. Then the subsemigroup $\langle a \rangle$ contains exactly one idempotent.*

Proof. Let m and r be the index and period of a, respectively. Then, $\langle a\rangle = \{a, a^2, \ldots, a^{m+r-1}\}$. The subgroup $\mathcal{K}_a = \{a^m, a^{m+1}, \ldots, a^{m+r-1}\}$ of $\langle a\rangle$ contains exactly one idempotent. If for $1 \leq i < m$, a^i is an idempotent, then we have $a^{2i} = a^i$. Consequently, $m \leq i$; a contradiction. Hence, the idempotent element of $\mathcal{K}_a$ is the only idempotent in $\langle a\rangle$. □

Lemma 1.1.5. *Every finite cyclic subgroup of a semigroup S is a monogenic subsemigroup of S.*

Proof. Let H be a cyclic subgroup of S. Then $H = \langle a\rangle$ for some $a \in S$. Since H is finite so that $o(a) = n$ for some $n \in \mathbb{N}$. Thus $a^n = e$, where e is the identity element of H. Consequently, $a^{-1} = a^{n-1}$. Now for any non-negative integer k, we get $a^{-k} = a^{k(n-1)}$. Thus, every element of H is a positive power of a. Hence, H is a monogenic subsemigroup of S. □

Lemma 1.1.6. *Let $S = M(m, r) = \langle a\rangle$ be a monogenic semigroup. Then $\mathcal{K}_{a^i} = \langle a^{m+g+i}\rangle$ for some g such that $0 \leq g \leq r-1$.*

Proof. For $m, r \in \mathbb{N}$, there exists $g \in \mathbb{N}$ such that $0 \leq g \leq r-1$ and $m + g \equiv 0 (\mathrm{mod}\, r)$. Since a^{m+g} is an idempotent element of S and so is the identity element of $\mathcal{K}_{a^i}$. First we show that $\mathcal{K}_{a^i} = \langle a^{m+g+i}\rangle$. Clearly, $a^{m+g}x = x$ for all $x \in \mathcal{K}_{a^i}$. To prove that $\mathcal{K}_{a^i} \subseteq \langle a^{m+g+i}\rangle$, consider $x \in \mathcal{K}_{a^i}$. Then $x = (a^i)^t$ for some $t \in \mathbb{N}$. Consequently, we get $x = a^{m+g}a^{it} = (a^{m+g+i})^t \subseteq \langle a^{m+g+i}\rangle$. Now assume that $y \in \langle a^{m+g+i}\rangle$ and so $y = (a^{m+g+i})^s$ for some $s \in \mathbb{N}$. It follows that $y = a^{m+g+si} = (a^i)^t a^{si} \in \langle a^i\rangle$. Thus $y \in \langle a^i\rangle \cap \mathcal{K}_a = \mathcal{K}_{a^i}$. Therefore $\mathcal{K}_{a^i} = \langle a^{m+g+i}\rangle$. □

Lemma 1.1.7. *Let $a^i \in \langle a\rangle = M(m, r)$. Then*

(i) *$m_{a^i} = 1$ if and only if $i \geq m$.*

(ii) *$m_{a^i} = 2$ if and only if $\lceil \frac{m}{2} \rceil \leq i \leq m-1$.*

Proof. (i) Let $a^i \in \langle a \rangle = M(m,r)$. If $i \geq m$, then $a^i \in \mathcal{K}_a$. Since $\mathcal{K}_a$ is a finite group of order r so $(a^i)^{r+1} = a^i$. Consequently, we get $m_{a^i} = 1$. To prove the converse part, we show that $m_{a^i} > 1$ for all $i < m$. On contrary, we assume that $i < m$ such that $m_{a^i} = 1$. Clearly, $m > 1$. Then $(a^i)^t = a^i$ for some $t > 1$ gives $m \leq i$; a contradiction.

(ii) For $\left\lceil \frac{m}{2} \right\rceil \leq i \leq m-1$, we have $(a^i)^2 \in \mathcal{K}_a$. Therefore, $((a^i)^2)^{r+1} = (a^i)^2$ gives $m_{a^i} \leq 2$. If $m_{a^i} = 1$, then we must have $m \leq i$; a contradiction. It follows that $m_{a^i} = 2$. For the converse part, we assume that $m_{a^i} = 2$. By (i), note that $i < m$. Since $m_{a^i} = 2$ then we get $m \leq 2i$. If $1 \leq i < \left\lceil \frac{m}{2} \right\rceil$, then $2i < m$ which is not possible. Thus, we have $\left\lceil \frac{m}{2} \right\rceil \leq i \leq m-1$. □

Lemma 1.1.8. *Let $S = \langle a \rangle = M(m,r)$ be a monogenic semigroup and $x \neq a \in \langle a \rangle$. If $m_a > 1$, then $m_x \leq m_a$ and $o(x) < m + r - 1$.*

Proof. Let $x \neq a \in \langle a \rangle$. Then $x = a^i$ for some $i > 1$. Clearly, $\langle a^i \rangle \subsetneq \langle a \rangle$ so that $o(x) < m+r-1$. Otherwise, $\langle a^i \rangle = \langle a \rangle$ gives $a = (a^i)^k$ gives $m_a = 1$; a contradiction. For $i \geq m_a$, we have $a^i \in \mathcal{K}_a$ and so $m_x = 1 \leq m_a$ (cf. Lemma 1.1.7). If $i < m_a$, then $(a^i)^{m_a} \in \mathcal{K}_a \cap \langle a^i \rangle = \mathcal{K}_{a^i}$ implies $m_x \leq m_a$. □

An *involution* (if exists) in a semigroup S is an element of order 2. A *maximal monogenic subsemigroup* of S is a monogenic subsemigroup of S that is not properly contained in any other monogenic subsemigroup of S. We shall denote $\mathcal{M}$ by the set of all elements of S that generates maximal monogenic subsemigroup of S i.e.

$$\mathcal{M} = \{a \in S : \langle a \rangle \text{ is a maximal monogenic subsemigroup of } S\}.$$

Similarly, a *maximal cyclic subgroup* of S is a cyclic subgroup of S that is not properly contained in any other cyclic subgroup of S. We shall denote $\overline{\mathcal{M}}$ by the set of all elements of S that generates maximal cyclic subgroup of S i.e.

$$\overline{\mathcal{M}} = \{a \in S : \langle a \rangle \text{ is a maximal cyclic subgroup of } S\}.$$

For integers a and b, we denote their greatest common divisor by (a, b) and the symbol $a \mid b$ denotes a divides b. The following result about the Euler's totient function ϕ is well known (for instance, see Burton [2006]).

Lemma 1.1.9. *For any integer $n \geq 3$, $\phi(n)$ is even.*

For any $n \in \mathbb{N}$, $\mathbb{Z}_n$ is the group of integers modulo n. We denote by $\mathbb{Z}_n^m$ the group obtained by taking direct product of m copies of $\mathbb{Z}_n$.

Proposition 1.1.10. *Let $S = M(m, r) = \langle a \rangle$ be a monogenic semigroup. Then $|\mathcal{K}_{a^i}| = \frac{r}{(i, r)}$, where $1 \leq i \leq m + r - 1$.*

Proof. In view of Lemma 1.1.6, we have $\mathcal{K}_{a^j} = \langle a^{m+g+j} \rangle$ for some g, where $0 \leq g \leq r - 1$, $1 \leq j \leq m + r - 1$ and $m + g \equiv 0 (\text{mod } r)$. We prove the result through the following cases:

Case 1: $g = r-1$. Then clearly $\mathcal{K}_a = \langle a^m \rangle$. Note that the map $\psi : \mathcal{K}_a \to \mathbb{Z}_r$ defined by $\psi(a^{m+j-1}) = \bar{j}$, where $1 \leq j \leq r$ is a group isomorphism. Then $\psi(a^{m+i-1}) = \bar{i}$. By division algorithm, we get x_i and i' such that $i = rx_i + i'$ where $0 \leq i' < r$. Also note that $(i, r) = (i', r)$. By Lemma 1.1.6, $\mathcal{K}_{a^i} = \langle a^{m+r+i-1} \rangle = \langle a^{m+i-1} \rangle = \langle a^{m+rx_i+i'-1} \rangle = \langle a^{m+i'-1} \rangle$. Therefore $|\mathcal{K}_{a^i}| = o(a^{m+i'-1}) = o(\bar{i'})$. It follows that $|\mathcal{K}_{a^i}| = \frac{r}{(i', r)} = \frac{r}{(i, r)}$.

Case 2: $0 \leq g < r-1$. Then clearly $\mathcal{K}_a = \langle a^{m+g+1} \rangle$. Note that the map $\phi : \mathcal{K}_a \to \mathbb{Z}_r$ defined by $\phi(a^{m+g+j}) = \bar{j}$ for $g \leq g + j \leq r - 1$ and $\phi(a^{m+j}) = \overline{(r - g) + j}$ for $0 \leq j < g$ is a group isomorphism. Now if $g + i < r$ then $|\mathcal{K}_{a^i}| = o(a^{m+g+i}) = o(\bar{i})$ in $\mathbb{Z}_r$. Thus $|\mathcal{K}_{a^i}| = \frac{r}{(i, r)}$. If $g + i \geq r$, then by division algorithm we get $g + i = lr + i'$ where $0 \leq i' < r$. We prove the result through the following subcases:

Subcase 1: $g < i'$. Then $i' = g + i''$ for some natural number i''. Since $\mathcal{K}_{a^i} = \langle a^{m+g+i} \rangle$

$= \langle a^{m+lr+i'} \rangle = \langle a^{m+i'} \rangle = \langle a^{m+g+i''} \rangle$. Therefore, we have

$$\begin{aligned} |\mathcal{K}_{a^i}| &= o(\overline{i''}) \text{ in } \mathbb{Z}_r \text{ because } g + i'' < r \\ &= \frac{r}{(i'',\ r)} = \frac{r}{(i' - g,\ r)} = \frac{r}{(g + i - lr - g,\ r)} \\ &= \frac{r}{(i - lr,\ r)} = \frac{r}{(i,\ r)}. \end{aligned}$$

Subcase 2: $g \geq i'$. Then

$$|\mathcal{K}_{a^i}| = o(a^{m+g+i}) = o(a^{m+lr+i'}) = o(a^{m+i'}) = o(\overline{r - g + i'}) \text{ in } \mathbb{Z}_r.$$

It follows that

$$\begin{aligned} |\mathcal{K}_{a^i}| &= \frac{r}{(r - g + i',\ r)} = \frac{r}{(r - g + g + i - lr,\ r)} = \frac{r}{(r(1 - l) + i,\ r)} \\ &= \frac{r}{(i,\ r)}. \end{aligned}$$

□

A semigroup S without zero is said to be *simple* if it has no proper ideals. A semigroup S with zero is called *0-simple* if (i) $\{0\}$ and S are its only ideals and (ii) $S^2 \neq \{0\}$.

A nonzero idempotent in a semigroup S is said to be *primitive* if it is a minimal element in $E(S) \setminus \{0\}$ with respect to the partial order relation $\leq$ on $E(S)$ defined by, for $a, b \in E(S), a \leq b \iff ab = ba = a$.

A semigroup S is said to be *completely* 0*-simple* if it is 0-simple and has a primitive idempotent. Let G be a group and let I, Λ be non-empty sets. Let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in $G^0 (= G \cup \{0\})$, and suppose P is *regular*, in the sense that no row or column of P consists entirely of zeros. Let $S = \mathfrak{M}^0[G, I, \Lambda, P] = (I \times G \times \Lambda) \cup \{0\}$, and define a composition on S by

$$(i, a, \lambda)(j, b, \mu) = \begin{cases} (i, ap_{\lambda j}b, \mu) & \text{if } p_{\lambda j} \neq 0 \\ 0 & \text{if } p_{\lambda j} = 0 \end{cases} \tag{1.1}$$

$$(i, a, \lambda)0 = 0(i, a, \lambda) = 0.$$

Theorem 1.1.11 ([Howie, 1995, Theorem 3.2.3]). *Let S be a semigroup. Then S is a completely 0-simple semigroup if and only if $S \cong \mathfrak{M}^0[G, I, \Lambda, P]$ for some non-empty index sets I, Λ, regular matrix P and a group G.*

Theorem 1.1.12 ([Howie, 1995, Theorem 5.1.8]). *A finite semigroup S is both completely 0-simple and an inverse semigroup if and only if S is isomorphic to the semigroup $B_n(G)$ for some group G.*

Lemma 1.1.13. *Let $x = (i, a, \lambda) \in \mathfrak{M}^0[G, I, \Lambda, P]$ such that $o(x)$ is finite.*

(i) *If $p_{\lambda i} \neq 0$, then $m_x = 1$.*

(ii) *If $p_{\lambda i} = 0$, then $m_x = 2$. Moreover, $x \in \mathcal{M}$.*

Proof. Suppose $x = (i, a, \lambda) \in \mathfrak{M}^0[G, I, \Lambda, P]$ such that $o(x)$ is finite and let $p_{\lambda i} \neq 0$. Then $(i, a, \lambda)^n = (i, (ap_{i\lambda})^{n-1}a, \lambda)$. Choose n such that $n-1$ is the order of $ap_{i\lambda}$, we get $(i, a, \lambda)^n = (i, a, \lambda)$. Consequently, $m_x = 1$. Now to prove (ii), we assume that $p_{\lambda i} = 0$. Then $x^2 = (i, a, \lambda)^2 = 0$ implies $m_x = 2$. Let if possible, $\langle x \rangle \subset \langle y \rangle$ for some $y = (j, b, \mu) \in \mathfrak{M}^0[G, I, \Lambda, P]$. Since $o(x)$ is finite so that $o(y)$ is finite. Note that $p_{\mu j} = 0$. Then $y^2 = 0$ gives $x = y$; a contradiction. □

For any relation R on a set X, we define R^{-1} by

$$R^{-1} = \{(x, y) \in X \times X : (y, x) \in R\}$$

and 1_X denote the identity relation on X. Now we define another relation

$$R^{\infty} = \bigcup \{R^n : n \geq 1\},$$

where R^n is the n time composition of R. We denote the relation R^e is the smallest equivalence relation containing R.

Proposition 1.1.14 ([Howie, 1995, Proposition 1.4.9]). *For every relation R on a set X, we have $R^e = [R \cup R^{-1} \cup 1_X]^{\infty}$.*

Proposition 1.1.15 ([Howie, 1995, Proposition 1.4.10]). *If R is a relation on a set X and R^e is the smallest equivalence relation on X containing R, then $(x, y) \in R^e$ if and only if either $x = y$ or, for some $n \in \mathbb{N}$, there is a sequence of transitions*

$$x = z_1 \to z_2 \to \cdots \to z_n = y$$

in which, for each i in $\{1, 2, \ldots, n-1\}$, either $(z_i, z_{i+1}) \in R$ or $(z_{i+1}, z_i) \in R$.

Definition 1.1.16. Let S be a semigroup. A relation R on S is *left compatible* if

$$(\forall a, s, t \in S) \ \ (s,t) \in R \ \ \Rightarrow (as, at) \in R,$$

and *right compatible* if

$$(\forall a, s, t \in S) \ \ (s,t) \in R \ \ \Rightarrow (sa, ta) \in R.$$

It is called *compatible* if

$$(\forall s, s', t, t' \in S) \ \ (s,s') \in R \ \text{ and } \ (t,t') \in R \ \ \Rightarrow (ss', tt') \in R.$$

A left [right] compatible equivalence relation is called *left [right] congruence*. A compatible equivalence relation is called *congruence*.

Proposition 1.1.17 ([Howie, 1995, Proposition 1.5.1]). *A relation ρ on a semigroup S is a congruence if and only if it is both left and right congruence.*

Theorem 1.1.18 ([Howie, 1995, Theorem 1.5.4]). *Let S be semigroup and let ρ be a congruence on S. Then $S/\rho = \{a\rho : \ a \in S\}$ is a semigroup with respect to the operation is defined by $(a\rho)(b\rho) = (ab)\rho$.*

The smallest congruence relation containing R is denoted by $R^{\#}$. By S^1 we shall mean the monoid obtained from S by adjoining an identity element (if S does not already have such an element). Now we define another relation R^c by

$$R^c = \{(xay, xby) : x, y \in S^1 \ \ , (a,b) \in R\}.$$

Lemma 1.1.19 ([Howie, 1995, Proposition 1.5.5]). *The relation R^c is the smallest left and right compatible containing R.*

Proposition 1.1.20 ([Howie, 1995, Proposition 1.5.8]). *For every relation R on a semigroup S, we have $R^{\#} = (R^c)^e$.*

Green's relations were introduced by J.A Green in 1951 that characterize the elements of S in terms of principals ideal. They become a standard tool for investigating the structure of semigroup. These relations are defined by

1. $x \mathcal{L} y$ if and only if $S^1x = S^1y$.
2. $x \mathcal{R} y$ if and only if $xS^1 = yS^1$.
3. $x \mathcal{J} y$ if and only if $S^1xS^1 = S^1yS^1$.
4. $x \mathcal{H} y$ if and only if $x \mathcal{L} y$ and $x \mathcal{R} y$.
5. $x \mathcal{D} y$ if and only if $x \mathcal{L} z$ and $z \mathcal{R} y$ for some $z \in S$.

Remark 1.1.21 ([Howie, 1995, p. 46]). Let G be a group. Then

$$\mathcal{L} = \mathcal{R} = \mathcal{H} = \mathcal{D} = \mathcal{J} = G \times G.$$

Corollary 1.1.22. *Let S be a semigroup and f be an idempotent element of S. Then the $\mathcal{H}$-class H_f containing f is a subgroup of S.*

A semigroup is said to be *completely regular* if every element a of S lies in a subgroup of S. Further, we have the following characterization of completely regular semigroup.

Proposition 1.1.23 ([Howie, 1995, Proposition 4.1.1]). *Let S be a semigroup. Then the following statements are equivalent:*

(i) *S is completely regular*

(ii) *every $\mathcal{H}$-class in S is a group.*

Definition 1.1.24. A semigroup S is said to be of *bounded exponent* if there exists a positive integer n such that for all $x \in S$, $x^n = f$ for some $f \in E(S)$.

If S is of bounded exponent then the *exponent* of S is the least n such that for each $x \in S$, $x^n = f$ for some $f \in E(S)$.

Lemma 1.1.25. *Let S be a semigroup with exponent n. Then for $x \in S$, we have*

(i) *$o(x) \leq 2n$ for all $x \in S$.*

(ii) *the subsemigroup $\langle x \rangle$ is contained in some maximal monogenic subsemigroup of S.*

Proof. (i) Since $x^n = f$ for some $f \in E(S)$, we have $\langle x \rangle = M(m, r)$ for some $m, r \in \mathbb{N}$. There exists g with $0 \leq g < r$ and $m + g \equiv 0 (\mathrm{mod}\, r)$ such that $x^{m+g} = f$. Clearly, $n \geq m$ as x^n is the idempotent element of $\langle x \rangle$. Let if possible $n < m + g$. For $m \leq i \neq j \leq m + r - 1$, we have $x^i \neq x^j$ in $\langle x \rangle$. Therefore, $x^n \neq x^{m+g}$, which is not true because $x^n = x^{m+g} = f$. Thus, $n \geq m + g$. Also $r \mid m + g$ and $m \leq n$ gives $r \leq n$. It follows that $o(x) \leq m + r \leq 2n$.
(ii) If $\langle x \rangle$ is a maximal monogenic subsemigroup, then the result holds. Otherwise, $\langle x \rangle \subsetneq \langle x_1 \rangle$. If $\langle x_1 \rangle$ is maximal monogenic then this completes our proof. By (i), since $o(x) \leq 2n$, we get a finite chain such that $\langle x \rangle \subsetneq \langle x_1 \rangle \cdots \subsetneq \langle x_{k-1} \rangle \subsetneq \langle x_k \rangle$, where $k \leq 2n$ and $\langle x_k \rangle$ is a maximal monogenic subsemigroup of S. This complete the proof. □

We often use the following fundamental properties of semigroups without referring to it explicitly.

Let S be a semigroup of bounded exponent. For $f \in E(S)$, we define

$$S_f = \{a \in S \ : \ a^m = f \text{ for some } m \in \mathbb{N}\}. \tag{1.2}$$

Remark 1.1.26. $S = \bigcup_{f \in E(S)} S_f$ and for distinct $f,\ f' \in E(S)$, we have $S_f \bigcap S_{f'} = \emptyset$.

1.2 Graphs

In this section, we recall necessary definitions, results and notations of graph theory from West [1996]. A graph $\mathcal{G}$ is a pair $\mathcal{G} = (V, E)$, where $V = V(\mathcal{G})$ and $E = E(\mathcal{G})$ are the set of vertices and edges of $\mathcal{G}$, respectively. We say that two different vertices a, b are *adjacent*, denoted by $a \sim b$, if there is an edge between a and b. We are considering simple graphs, i.e. undirected graphs with no loops or repeated edges. If a and b are not adjacent, then we write $a \nsim b$. The *neighbourhood* $\mathrm{N}(x)$ of a vertex x is the set of all vertices adjacent to x in $\mathcal{G}$. Additionally, we denote $\mathrm{N}[x] = \mathrm{N}(x) \cup \{x\}$. A *subgraph* of a graph $\mathcal{G}$ is a graph $\mathcal{G}'$ such that $V(\mathcal{G}') \subseteq V(\mathcal{G})$ and $E(\mathcal{G}') \subseteq E(\mathcal{G})$. A subgraph $\mathcal{G}'$ of graph $\mathcal{G}$ is said to be a *spanning subgraph* of $\mathcal{G}$ if $V(\mathcal{G}) = V(\mathcal{G}')$ and we shall write it as $\mathcal{G}' \preceq \mathcal{G}$. A *walk* λ in $\mathcal{G}$ from the vertex u to the vertex w is a sequence of vertices $u = v_1, v_2, \ldots, v_m = w$ $(m > 1)$ such that $v_i \sim v_{i+1}$ for every $i \in \{1, 2, \ldots, m-1\}$. If no edge is repeated in λ, then it is called a *trail* in $\mathcal{G}$. A trail whose initial and end vertices are identical is called a *closed trail*. A walk is said to be a *path* if no vertex is repeated. The length of a path is the number of edges it contains. If $U \subseteq V(\mathcal{G})$, then the subgraph of $\mathcal{G}$ induced by U, denoted by $\mathcal{G}(U)$, is the graph $\mathcal{G}'$ with vertex set U, and two vertices are adjacent in $\mathcal{G}'$ if and only if they are adjacent in $\mathcal{G}$. A graph $\mathcal{G}$ is said to be *connected* if there is a path between every pair of vertex. A graph $\mathcal{G}$ is said to be *complete* if any two distinct vertices are adjacent. We denote K_n, is the complete graph of n vertices. A path that begins and ends on the same vertex is called a *cycle*. A graph $\mathcal{G}$ is said to be *acyclic* if $\mathcal{G}$ does not contain any cycle. A cycle in a graph $\mathcal{G}$ that includes every vertex of $\mathcal{G}$ is called a *Hamiltonian cycle* of $\mathcal{G}$. If $\mathcal{G}$ contains a Hamiltonian cycle, then $\mathcal{G}$ is called a *Hamiltonian graph*.

Also, recall that the *girth* of a graph $\mathcal{G}$ is the length of the shortest cycle in $\mathcal{G}$, if $\mathcal{G}$ has a cycle; otherwise we say the *girth* of $\mathcal{G}$ is ∞. The *distance* between vertices u and w, denoted by $d(u, w)$, is the length of a minimal path from u to w. If there is no path from u to w, we say that the distance between u and w is ∞. The *diameter*

of a connected graph $\mathcal{G}$ is the maximum distance between two vertices and it is denoted by $diam(\mathcal{G})$. The *degree* of a vertex v is the number of edges incident to v and it is denoted as $\deg(v)$. The smallest degree among the vertices of $\mathcal{G}$ is called the *minimum degree* of $\mathcal{G}$ and it is denoted by $\delta(\mathcal{G})$. The *chromatic number* $\chi(\mathcal{G})$ of a graph $\mathcal{G}$ is the smallest positive integer k such that the vertices of $\mathcal{G}$ can be colored in k colors so that no two adjacent vertices share the same color. A graph $\mathcal{G}$ is *Eulerian* if $\mathcal{G}$ is both connected and has a closed trail (walk with no repeated edge) containing all the edges of a graph.

Theorem 1.2.1 ([West, 1996, Theorem 1.2.26])**.** *A connected graph is Eulerian if and only if its every vertex is of even degree.*

A *clique* of a graph $\mathcal{G}$ is a complete subgraph of $\mathcal{G}$ and the number of vertices in a clique of maximum size is called the *clique number* of $\mathcal{G}$ and it is denoted by $\omega(\mathcal{G})$. An *independent set* of a graph $\mathcal{G}$ is a subset of $V(\mathcal{G})$ such that no two vertices in the subset are adjacent in $\mathcal{G}$. The *independence number* of $\mathcal{G}$ is the maximum size of an independent set, it is denoted by $\alpha(\mathcal{G})$. A graph $\mathcal{G}$ is said to be *bipartite* if $V(\mathcal{G})$ is the union of two disjoint independent sets. A graph $\mathcal{G}$ is called *complete bipartite* if $\mathcal{G}$ is bipartite with $V(\mathcal{G}) = A \cup B$, where A and B are disjoint independent sets such that $x \sim y$ if and only if $x \in A$ and $y \in B$. We shall denote it by $K_{n,m}$, where $|A| = n$ and $|B| = m$. A graph $\mathcal{G}$ is said to be *star graph* if $\mathcal{G} = K_{1,n}$ for some $n \in \mathbb{N}$.

Theorem 1.2.2 ([West, 1996, Theorem 1.2.18])**.** *For a graph $\mathcal{G}$, we have $\mathcal{G}$ is bipartite if and only if $\mathcal{G}$ does not contain an odd cycle.*

Also, recall that a dominating set D of a graph $\mathcal{G}$ is a subset of the vertex set such that every vertex not in D is adjacent to some vertex in D and the number of vertices in a smallest dominating set of $\mathcal{G}$ is called the *dominance number* of $\mathcal{G}$. A vertex v in $\mathcal{G}$ is said to be *dominating* if $\mathrm{N}[v] = V(\mathcal{G})$. The identity element e is

a dominating vertex of every graph $\mathcal{G}$, where $\mathcal{G} \in \{\Gamma(G), \mathcal{P}_e(G), \Delta(G)\}$. We call a graph $\mathcal{G}$ dominatable if it has a dominating vertex other than e.

The following lemma will be useful in the sequel.

Lemma 1.2.3 ([West, 1996, Lemma 3.1.33]). *A set of vertices in a graph is an independent dominating set if and only if it is a maximal independent set.*

The graph $\mathcal{G}$ is *perfect* if $\omega(\mathcal{G}') = \chi(\mathcal{G}')$ for every induced subgraph $\mathcal{G}'$ of $\mathcal{G}$. Recall that the *complement* $\overline{\mathcal{G}}$ of $\mathcal{G}$ is a graph with same vertex set as $\mathcal{G}$ and distinct vertices u, v are adjacent in $\overline{\mathcal{G}}$ if they are not adjacent in $\mathcal{G}$. A subgraph $\mathcal{G}'$ of $\mathcal{G}$ is called a *hole* if $\mathcal{G}'$ is a cycle as an induced subgraph, and $\mathcal{G}'$ is called an *antihole* of $\mathcal{G}$ if $\overline{\mathcal{G}'}$ is a hole in $\overline{\mathcal{G}}$.

Theorem 1.2.4 (Chudnovsky et al. [2006]). *A finite graph $\mathcal{G}$ is perfect if and only if it does not contain hole or antihole of odd length at least* 5.

The following remarks will be usefull in the sequel.

Remark 1.2.5. Let $\mathcal{G}$ be any graph and x be a dominating vertex of $\mathcal{G}$. Then x does not belong to the vertex set of any hole of length greater than 3, or any antihole of $\mathcal{G}$.

Remark 1.2.6. In a graph $\mathcal{G}$, any vertex of degree one does not belongs to any hole or antihole of length greater than 4.

A *planar graph* is a graph that can be embedded in the plane, i.e. it can be drawn on the plane in such a way that its edges intersect only at their endpoints. The following theorem will be useful in the sequel.

Theorem 1.2.7 ([West, 1996, Theorem 6.2.2]). *A graph $\mathcal{G}$ is planar if and only if it does not contain a subdivison of K_5 or $K_{3,3}$.*

A *vertex (edge) cut-set* in a connected graph $\mathcal{G}$ is a set X of vertices (edges) such that the remaining subgraph $\mathcal{G} \setminus X$, by removing the set X, is disconnected or has

only one vertex. The *vertex connectivity* (*edge connectivity*) of a connected graph $\mathcal{G}$ is the minimum size of a vertex (edge) cut-set and it is denoted by $\kappa(\mathcal{G})$ ($\kappa'(\mathcal{G})$). For $k \geq 1$, graph $\mathcal{G}$ is *k-connected* if $\kappa(\mathcal{G}) \geq k$. The following results will be useful in the sequel.

Theorem 1.2.8 (Plesník [1975])**.** *If the diameter of any graph is at most* 2*, then its edge connectivity and minimum degree are equal.*

Theorem 1.2.9 ([West, 1996, Theorem 4.1.9])**.** *If $\mathcal{G}$ is a simple graph, then*

$$\kappa(\mathcal{G}) \leq \kappa'(\mathcal{G}) \leq \delta(\mathcal{G}).$$

An *edge cover* in a graph $\mathcal{G}$ without isolated vertices is a set L of edges such that every vertex of $\mathcal{G}$ is incident to some edge of L. The minimum cardinality of an edge cover in $\mathcal{G}$ is called the *edge covering number*, it is denoted by $\beta'(\mathcal{G})$. A *vertex cover* of a graph $\mathcal{G}$ is a set Q of vertices such that it contains at least one endpoint of every edge of $\mathcal{G}$. The minimum cardinality of a vertex cover in $\mathcal{G}$ is called the *vertex covering number*, it is denoted by $\beta(\mathcal{G})$. A *matching* in a graph $\mathcal{G}$ is a set of edges with no share endpoints and the maximum cardinality of a matching is called the *matching number* and it is denoted by $\alpha'(\mathcal{G})$. We have the following equalities involving the above parameters.

Lemma 1.2.10. *Consider a graph $\mathcal{G}$.*

(i) $\alpha(\mathcal{G}) + \beta(\mathcal{G}) = |V(\mathcal{G})|$.

(ii) *If $\mathcal{G}$ has no isolated vertices,* $\alpha'(\mathcal{G}) + \beta'(\mathcal{G}) = |V(\mathcal{G})|$.

The concept of detour distance was introduced by Chartrand et al. [1993]. The *detour distance*, $d_D(u, v)$, between two vertices u and v in a graph $\mathcal{G}$ is the length of longest $u - v$ path in $\mathcal{G}$. The (*detour*) *eccentricity* of a vertex u, denoted by ($ecc_D(u)$) $ecc(u)$, is the maximum (detour) distance between u and any vertex of $\mathcal{G}$. The minimum (detour) eccentricity among the vertices of $\mathcal{G}$ is called the (*detour*)

radius of $\mathcal{G}$, it is denoted by $(rad_D(\mathcal{G}))$ $rad(\mathcal{G})$. The *detour diameter* of a graph $\mathcal{G}$ is the maximum detour eccentricity in $\mathcal{G}$, denoted by $diam_D(G)$. A vertex v is said to be *eccentric vertex* for u if $d(u,v) = ecc(u)$. A vertex v is said to be an eccentric vertex of the graph $\mathcal{G}$ if v is an eccentric vertex for some vertex of $\mathcal{G}$. A graph $\mathcal{G}$ is said to be an *eccentric graph* if every vertex of $\mathcal{G}$ is an eccentric vertex. The *centre* of $\mathcal{G}$ is a subgraph of $\mathcal{G}$ induced by the vertices having minimum eccentricity and it is denoted by $Cen(\mathcal{G})$. The *closure* of a graph $\mathcal{G}$ of order n is the graph obtained from $\mathcal{G}$ by recursively joining pairs of non-adjacent vertices whose sum of degree is at least n until no such pair remains and it is denoted by $Cl(\mathcal{G})$. The graph $\mathcal{G}$ is said to be *closed* if $\mathcal{G} = Cl(\mathcal{G})$ (cf. Chartrand and Zhang [2004]).

A vertex v in a graph $\mathcal{G}$ is a *boundary vertex* of a vertex u if $d(u,w) \leq d(u,v)$ for $w \in \mathrm{N}(v)$, while a vertex v is a boundary vertex of a graph $\mathcal{G}$ if v is a boundary vertex of some vertex of $\mathcal{G}$. The subgraph $\mathcal{G}$ induced by its boundary vertices is the *boundary* $\partial(\mathcal{G})$ of $\mathcal{G}$. A vertex v is said to be a *complete vertex* if the subgraph induced by the neighbours of v is complete. A vertex v is said to be an *interior vertex* of a graph $\mathcal{G}$ if for each $u \neq v$, there exists a vertex w and a path $u - w$ such that v lies in that path at the same distance from both u and w. A subgraph induced by the interior vertices of $\mathcal{G}$ is called *interior* of $\mathcal{G}$ and it is denoted by $Int(\mathcal{G})$.

Theorem 1.2.11 ([Chartrand and Zhang, 2004, p.337]). *Let $\mathcal{G}$ be a connected graph and $v \in V(\mathcal{G})$. Then v is a complete vertex of $\mathcal{G}$ if and only if v is a boundary vertex of x for all $x \in V(\mathcal{G}) \setminus \{v\}$.*

Theorem 1.2.12 ([Chartrand and Zhang, 2004, p.339]). *Let $\mathcal{G}$ be a connected graph and $v \in V(\mathcal{G})$. Then v is a boundary vertex of $\mathcal{G}$ if and only if v is not an interior vertex of $\mathcal{G}$.*

Slater [1975] introduced the concept of *metric dimension* and then separately studied by in Harary et al. [1976]. For z in $\mathcal{G}$, we say that z *resolves* u and v if

$d(z,u) \neq d(z,v)$. A subset U of $V(\mathcal{G})$ is a *resolving set* of $\mathcal{G}$ if every pair of vertices of $\mathcal{G}$ is resolved by some vertex of U. The least cardinality of a resolving set of $\mathcal{G}$ is called the metric dimension of $\mathcal{G}$ and is denoted by $\dim(\mathcal{G})$. An *i-subset* of $V(\mathcal{G})$ is a subset of $V(\mathcal{G})$ of cardinality i. Let $\mathcal{R}(\mathcal{G}, i)$ be the family of resolving sets which are i-subsets and $r_i = |\mathcal{R}(\mathcal{G}, i)|$. Then we define the *resolving polynomial* of a graph $\mathcal{G}$ of order n, denoted by $\beta(\mathcal{G}, x)$ as $\beta(\mathcal{G}, x) = \sum_{i=dim(\mathcal{G})}^{n} r_i x^i$. The sequence $(r_{dim(\mathcal{G})}, r_{dim(\mathcal{G})+1}, \ldots, r_n)$ of coefficients of $\beta(\mathcal{G}, x)$ is called the *resolving sequence*. Two distinct vertices u and v are said to be *true twins* if $\mathrm{N}[u] = \mathrm{N}[v]$. Two distinct vertices u and v are said to be *false twins* if $\mathrm{N}(u) = \mathrm{N}(v)$. If u and v are true twins or false twins then u and v are *twins*. For more details on twin vertices, we refer the reader to Hernando et al. [2010]. A set $U \subseteq V(\mathcal{G})$ is said to be a *twin-set* in $\mathcal{G}$ if u, v are twins for every pair of distinct pair of vertices $u, v \in U$. In order to obtain the resolving polynomial $\beta(\Delta(SD_{8n}), x)$, the following results will be useful.

Remark 1.2.13 ([Ali et al., 2016, Remark 3.3])**.** If U is twin-set in a connected graph $\mathcal{G}$ of order n with $|U| = l \geq 2$, then every resolving set for $\mathcal{G}$ contains at least $l - 1$ vertices of U.

Proposition 1.2.14 ([Ali et al., 2016, Proposition 3.5])**.** *Let $\mathcal{G}$ be a connected graph of order n. Then the only resolving set of cardinality n is the set $V(\mathcal{G})$ and a resolving set of cardinality $n-1$ can be chosen n possible different ways.*

The concept of *strong metric dimension* has been introduced by Sebő and Tannier [2004]. For vertices u and v in a graph $\mathcal{G}$, we say that z *strongly resolves* u and v if there exists a shortest path from z to u containing v, or a shortest path from z to v containing u. A subset U of $V(\mathcal{G})$ is a *strong resolving set* of $\mathcal{G}$ if every pair of vertices of $\mathcal{G}$ is strongly resolved by some vertex of U. The least cardinality of a strong resolving set of $\mathcal{G}$ is called the strong metric dimension of $\mathcal{G}$ and is denoted by $\mathrm{sdim}(\mathcal{G})$. For vertices u and v in a graph $\mathcal{G}$, we write $u \equiv v$ if $\mathrm{N}[u] = \mathrm{N}[v]$. Notice that that $\equiv$ is an equivalence relation on $V(\mathcal{G})$. We denote by $\widehat{v}$ the $\equiv$-class

containing a vertex v of $\mathcal{G}$. Consider a graph $\widehat{\mathcal{G}}$ whose vertex set is the set of all $\equiv$-classes, and vertices $\widehat{u}$ and $\widehat{v}$ are adjacent if u and v are adjacent in $\mathcal{G}$. This graph is well-defined because in $\mathcal{G}$, $w \sim v$ for all $w \in \widehat{u}$ if and only if $u \sim v$. We observe that $\widehat{\mathcal{G}}$ is isomorphic to the subgraph $\mathcal{R}_{\mathcal{G}}$ of $\mathcal{G}$ induced by a set of vertices consisting of exactly one element from each $\equiv$-class. Subsequently, we have the following result of Ma, Feng and Wang [2018] with $\omega(\mathcal{R}_{\mathcal{G}})$ replaced by $\omega(\widehat{\mathcal{G}})$.

Theorem 1.2.15 ([Ma, Feng and Wang, 2018, Theorem 2.2])**.** *Let $\mathcal{G}$ be a graph with diameter 2. Then $sdim(\mathcal{G}) = |V(\mathcal{G})| - \omega(\widehat{\mathcal{G}})$.*

For a finite simple undirected graph $\mathcal{G}$ with vertex set $V(\mathcal{G}) = \{v_1, v_2, \ldots, v_n\}$, the *adjacency matrix* $A(\mathcal{G})$ is the $n \times n$ matrix with $(i,j)th$ entry is 1 if v_i and v_j are adjacent and 0 otherwise. We denote the diagonal matrix $D(\mathcal{G}) = \text{diag}(d_1, d_2, \ldots, d_n)$ where d_i is the degree of the vertex v_i of $\mathcal{G}$, $i = 1, 2, \ldots, n$. The *Laplacian matrix* $L(\mathcal{G})$ of $\mathcal{G}$ is the matrix $D(\mathcal{G}) - A(\mathcal{G})$. The matrix $L(\mathcal{G})$ is symmetric and positive semidefinite, so that its eigenvalues are real and non-negative. Furthermore, the sum of each row (column) of $L(\mathcal{G})$ is zero. Recall that the *characteristic polynomial* of $L(\mathcal{G})$ is denoted by $\Phi(L(\mathcal{G}), x)$. The eigenvalues of $L(\mathcal{G})$ are called the *Laplacian eigenvalues* of $\mathcal{G}$ and it is denoted by $\lambda_1(\mathcal{G}) \geq \lambda_2(\mathcal{G}) \geq \cdots \geq \lambda_n(\mathcal{G}) = 0$. Now let $\lambda_{n_1}(\mathcal{G}) \geq \lambda_{n_2}(\mathcal{G}) \geq \cdots \geq \lambda_{n_r}(\mathcal{G}) = 0$ be the distinct eigenvalues of $\mathcal{G}$ with multiplicities $m_1, m_2, \ldots, m_r$, respectively. The *Laplacian spectrum* of $\mathcal{G}$, that is, the spectrum of $L(\mathcal{G})$ is represented as $\begin{pmatrix} \lambda_{n_1}(\mathcal{G}) & \lambda_{n_2}(\mathcal{G}) & \cdots & \lambda_{n_r}(\mathcal{G}) \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}$. We denote the matrix J_n as the square matrix of order n having all the entries as 1 and I_n is the identity matrix of order n.

An *automorphism* of a graph $\mathcal{G}$ is a permutation f on $V(\mathcal{G})$ with the property that, for any vertices u and v, we have $uf \sim vf$ if and only if $u \sim v$. The set $\text{Aut}(\mathcal{G})$ of all graph automorphisms of a graph $\mathcal{G}$ forms a group with respect to composition of mappings.

We conclude this chapter with the following lemma which provides a relation

between the graphs considered in this thesis.

Lemma 1.2.16. *For a semigroup S, we have $\mathcal{P}(S) \preceq \Gamma(S) \preceq \mathcal{P}_e(S) \preceq \Delta(S)$. Furthermore, for a finite group G, $\mathcal{P}_e(G) = \Gamma(G)$.*

Proof. By [Afkhami et al., 2014, Theoren 3.13], note that $\mathcal{P}(S) \preceq \Gamma(S)$. Now, suppose $a \sim b$ in $\Gamma(S)$. Then, for some $c \in S$, we have $\langle a, b\rangle = \langle c\rangle$. Consequently, $a, b \in \langle c\rangle$ so that $a \sim b$ in $\mathcal{P}_e(S)$. Thus, $\Gamma(S) \preceq \mathcal{P}_e(S)$. Further, by the definition of the enhanced power graph of a group, we have $\mathcal{P}_e(G) = \Gamma(G)$. □