Chapter 2

The Cyclic Graph of Semigroups

Abdollahi and Hassanabadi [2007] introduced the noncyclic graph of a group G. The noncyclic graph of a group G is the simple undirected graph whose vertex set is $G \backslash T$, where $T = \{x \in G :$ the subgroup generated by x, y is cyclic for all $y \in G\}$ and two distinct vertices x, y are adjacent if the subgroup generated by x, y is not cyclic. The complement of noncyclic graph of G is called the cyclic graph of G. Further, the notion of cyclic graph was slightly modified by Ma et al. [2013] as follows. The cyclic graph $\Gamma(G)$ is the simple undirected graph whose vertex set is G and two distinct vertices x and y are adjacent if the subgroup generated by x, y is cyclic. Ma et al. [2013] studied the graph-theoretic properties of cyclic graph of a finite group namely, bipartite, diameter, clique number etc. Particularly, they have investigated the graph invariants of the cyclic graph $\Gamma(G)$ when G is dihedral group D_{2n} or dicyclic group Q_{4n}. Also, Ma et al. [2013] proved that $\mathrm{Aut}(G) = \mathrm{Aut}(\Gamma(G))$ if and only if G is a Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$. The definition of cyclic graph has been extended on semigroups by Afkhami et al. [2014]. The cyclic graph $\Gamma(S)$ is the simple undirected graph whose vertex set is the semigroup S and two distinct vertices x and y are adjacent if $\langle x, y \rangle$ is a monogenic subsemigroup of S. Afkhami et al. [2014] provided the structure of $\Gamma(S)$, where S is a finite semigroup and discussed the graph-theoretic properties of $\Gamma(S)$, viz. planarity, genus, girth etc. This chapter concerns to study the graph-theoretic properties of $\Gamma(S)$ as well as structural properties of $\Gamma(S)$. This chapter is arranged as follows. In Section 2.1, first we provide the structure of $\Gamma(S)$ for an arbitrary semigroup S. Then the basic properties of $\Gamma(S)$, namely: completeness, bipartite, tree and regularity etc. are discussed. Section 2.2 is devoted to the study of chromatic number of $\Gamma(S)$ and it is proved that the chromatic number of $\Gamma(S)$, where S is an infinite semigroup, is at most countable. In Section 2.3 and Section 2.4, we obtain the clique number and independence number of $\Gamma(S)$, respectively. The content of Section 2.2 is published in SCIE journal *"Graphs and Combinatorics"*, 36, 109-113, 2020, Springer.

2.1 Basic Properties of $\Gamma(S)$

In order to study some basic properties of $\Gamma(S)$, first we discuss the structure of $\Gamma(S)$ for an arbitrary semigroup S. For $x \in S$ and $m, n \in \mathbb{N}$, we define

$$S(x, m, n) = \{y \in S : \ x^m = y^n\}$$

and we write $C(x) = \bigcup\limits_{m,n \in \mathbb{N}} S(x, m, n)$. The following proposition describes the structure of $\Gamma(S)$.

Proposition 2.1.1. *The set $C(x)$ is a connected component of $\Gamma(S)$. Moreover, the components of the graph $\Gamma(S)$ are precisely $\{C(x) \mid x \in S\}$.*

Proof. Let $y, z \in C(x)$. Then $y \in S(x, m, n)$ and $z \in S(x, p, q)$. It follows that $x^m = y^n$ and $x^p = z^q$. Note that $y \sim x^{mp} \sim z$. Thus, $C(x)$ is connected in $\Gamma(S)$. Now suppose that the element z of S is adjacent to a vertex y in $C(x)$. Since $y \sim z$ implies $\langle y, z\rangle = \langle t\rangle$ for some $t \in S$. For $y \in C(x)$, we have $x^m = y^n$ for some $m, n \in \mathbb{N}$. It follows that $y = t^\alpha$, $z = t^\beta$ and $z^{n\alpha} = x^{m\beta}$. Thus, $z \in C(x)$. Hence, $C(x)$ is a connected component of $\Gamma(S)$. □

In view of Equation 1.2, we have the following corollary.

Corollary 2.1.2. *For $f \in E(S)$, we have $C(f) = S_f$. Moreover, if $x \in S$ such that $o(x)$ is finite then $x \in S_f$ for some $f \in E(S)$.*

Let S be a semigroup of exponent n. Each connected component of $\Gamma(S)$ is of the form S_f for some $f \in E(S)$ (see Remark 1.1.26 and Corollary 2.1.2). Since S_f is a connected component and $f \sim x$ for all $x \in S_f \setminus \{f\}$. Thus, $E(S)$ is a dominating set of $\Gamma(S)$. Consequently, we have the following Lemma.

Lemma 2.1.3. *Let S be a semigroup of exponent n. Then the dominance number of $\Gamma(S)$ is equals to the number of idempotents in S.*

Let $x \in S$ such that $x \neq x^2$. Then $x \sim x^2$ in $\Gamma(S)$. Consequently, we have the following corollary.

Corollary 2.1.4. *For any semigroup S, $\Gamma(S)$ is a null graph if and only if S is a band.*

Theorem 2.1.5. *Let S be a semigroup of exponent n. Then $\Gamma(S)$ is complete if and only if one of the following holds:*

(i) $S = \langle a : a^{1+r} = a \rangle$

(ii) $S = \langle a : a^{2+r} = a^2 \rangle$

(iii) $S = \langle a : a^{3+r} = a^3 \rangle$, *where r is odd.*

Proof. Suppose that $\Gamma(S)$ is complete. Since $o(x) \leq 2n$ for all $x \in S$. Now choose $x \in S$ such that $o(x)$ is maximum in $\pi(S)$. Let $x, y \in S$. Then $x \sim y$ implies $\langle x, y \rangle = \langle z \rangle$ for some $z \in S$. Therefore, $y \in \langle z \rangle = \langle x \rangle$. It follows that $S = \langle x \rangle$. Consider $S = M(m, r)$ for some $m, r \in \mathbb{N}$. On contrary, suppose that S is not of the form given in (i), (ii) and (iii). Then either $S = M(3, r)$ such that r is even or $S = M(m, r)$, where $m \geq 4$.

If $S = M(3, r)$ such that r is even, then clearly $3 + r - 1$ and $3 + r + 1$ are even. Since $a^{3+r} = a^3$ implies $a^{4+r} = a^4$ so that $\langle a^2 \rangle = \{a^2, a^4, \dots, a^{2+r}\}$. Note that $a^3 \notin \langle a^2 \rangle$. If $a^2 \in \langle a^3 \rangle$, then $a^2 = a^{3k}$ for some $k \in \mathbb{N}$. Thus $m \leq 2$; a contradiction for $m = 3$. Consequently, $a^2 \notin \langle a^3 \rangle$. Let if possible $\langle a^2, a^3 \rangle = \langle a^t \rangle$ for some $a^t \in S$. We now show that no such $t \in \mathbb{N}$ exists. If $t = 1$, then $\langle a^2, a^3 \rangle = \langle a \rangle$ so that $a = a^l$, where $l \geq 2$. Thus, $m = 1$; a contradiction. If $t \in \{2, 3\}$, then either $a^2 \in \langle a^3 \rangle$ or $a^3 \in \langle a^2 \rangle$; again a contradiction. Thus, we have $\langle a^2, a^3 \rangle = \langle a^t \rangle$ such that $t > 3$. Since $a^2 \in \langle a^2, a^3 \rangle = \langle a^t \rangle$ so that $a^2 = (a^t)^k$ for some $k \in \mathbb{N}$. Consequently, $m \leq 2$; a contradiction. Thus, $\langle a^2, a^3 \rangle$ is not a monogenic subsemigroup of S implies a^2 is not adjacent to a^3 in $\Gamma(S)$ so that $\Gamma(S)$ is not complete which is a contradiction.
We may now suppose $S = M(m, r)$, where $m \geq 4$. In this case, first note that a, a^2, a^3, a^4 all are distinct elements of S. Now, we show that $\langle a^2, a^3 \rangle$ is not a monogenic subsemigroup of S so that a^2 and a^3 are not adjacent in $\Gamma(S)$, which is a contradiction of the fact $\Gamma(S)$ is complete. If possible, let $\langle a^2, a^3 \rangle = \langle a^i \rangle$ for some $a^i \in S$. If $i = 1$, then $a \in \langle a^2, a^3 \rangle$. Thus, $a = a^t$, where $t \geq 5$ so that $m = 1$; a contradiction. For $i = 2$, note that $a^3 \in \langle a^2 \rangle$ gives $a^3 = a^{2k}$ for some $k \geq 3$. Thus $m \leq 3$; a contradiction. If $i \geq 3$, then $a^2 \in \langle a^i \rangle$, which implies that $a^2 = a^{ik}$ for some $k \in \mathbb{N}$. Thus $m \leq 2$; again a contradiction.

Conversely, suppose S is one of the form given in (i), (ii) and (iii). Thus, we have the following cases.

Case 1: $S = M(1, r)$ i.e. $S = \{a, a^2, \dots, a^r\}$. Since S is a cyclic group, for any two distinct $x, y \in S$, note that $\langle x, y \rangle$ is a cyclic subgroup of S. Consequently, $\langle x, y \rangle$ is a monogenic subsemigroup of S (cf. Lemma 1.1.5) so that $\langle x, y \rangle = \langle z \rangle$ for some $z \in S$. Thus, $x \sim y$ in $\Gamma(S)$. Hence, $\Gamma(S)$ is complete.

Case 2: $S = M(2, r)$ i.e. $S = \{a, a^2, \dots, a^{r+1}\}$ with $a^{2+r} = a^2$. Clearly, $\mathcal{K}_a = \{a^2, a^3, \dots, a^{r+1}\}$. For $2 \leq i \leq r + 1$, we have $a^i \in \langle a \rangle$ so that $\langle a^i, a \rangle = \langle a \rangle$. Thus $a \sim a^i$ in $\Gamma(S)$. Since $\mathcal{K}_a$ is a cyclic subgroup of S and for any $a^i, a^j \in \mathcal{K}_a$, the subsemigroup $\langle a^i, a^j \rangle$ is monogenic in S so that $a^i \sim a^j$ in $\Gamma(S)$. Thus, $\Gamma(S)$ is complete.

Case 3: $S = M(3, r)$ such that r is odd. Clearly, $S = \{a, a^2, a^3, \dots, a^{2+r}\}$ with $a^{3+r} = a^3$. By the similar argument used in **Case 2**, note that for $2 \leq i \leq 2 + r$, we have $a \sim a^i$ in $\Gamma(S)$. Since $3 + r$ is even implies $3 + r = 2k$ for some $k \in \mathbb{N}$. Thus, $a^3 = a^{3+r} = a^{2k} = (a^2)^k$ so that $a^3 \in \langle a^2 \rangle$. Consequently, $a^5, a^7, \cdots, a^{2+r} \in \langle a^2 \rangle$. For $i > 2$, note that $\langle a^2, a^i \rangle = \langle a^2 \rangle$ and it gives $a^2 \sim a^i$ in $\Gamma(S)$. Now, $\mathcal{K}_a = \{a^3, a^4, \dots, a^{2+r}\}$ is a cyclic subgroup of S and for any $a^i, a^j \in \mathcal{K}_a$ note that $\langle a^i, a^j \rangle$ is a monogenic subsemigroup of S. Thus $a^i \sim a^j$ in $\Gamma(S)$. Hence, $\Gamma(S)$ is complete. □

Corollary 2.1.6. [Ma et al., 2013, Theorem 9] *Let G be a finite group. Then the cyclic graph $\Gamma(G)$ is complete if and only if G is a cyclic group.*

Theorem 2.1.7. *Let S be a semigroup. Then the following statements are equivalent:*

(i) $\pi(S) \subseteq \{1, 2\}$

(ii) $\Gamma(S)$ *is acyclic*

(iii) $\Gamma(S)$ *is bipartite.*

Proof. (i) ⇒ (ii) Suppose $\pi(S) \subseteq \{1, 2\}$. Clearly, S is of bounded exponent. Let if possible, there exists a cycle $C : a_0 \sim a_1 \sim \cdots \sim a_k \sim a_0$, in $\Gamma(S)$. Then $C \subseteq S_f$ for some $f \in E(S)$. If none of the vertices of C are idempotents, then $a_0, a_1 \in \langle z_1 \rangle$ for some $z_1 \in S$. Consequently, $\text{o}(z_1) \geq 3$; a contradiction. If one of the vertex of C is idempotent, then note that there exist two non idempotent elements a_i, a_j such that $a_i \sim a_j$. Thus, $a_i, a_j \in \langle z \rangle$ for some $z \in S$. Since $\langle z \rangle$ contains an idempotent element also, we get $\text{o}(z) \geq 3$; again a contradiction.
(ii) ⇒ (iii) Since $\Gamma(S)$ is acyclic graph so that it does not contain any cycle. By Theorem 1.2.2, $\Gamma(S)$ is bipartite.

(iii) $\Rightarrow$ (i) Suppose $\Gamma(S)$ is a bipartite graph. By Theorem 1.2.2, $\Gamma(S)$ does not contain any odd cycle. To prove $\pi(S) \subseteq \{1, 2\}$. Let if possible, there exist $a \in S$ such that $o(a) \geq 3$. If $o(a)$ is infinite then clearly $a \sim a^2 \sim a^4 \sim a$ is a triangle in $\Gamma(S)$; a contradiction (see Theorem 1.2.2). Now we assume that $o(a)$ is finite. Note that there exist $x, f \in S$ such that $f \in E(S) \cap \langle a \rangle$ and $x \in \langle a \rangle \setminus \{a, f\}$. As a consequence, the vertices x, a and f forms a triangle; again a contradiction of the fact that $\Gamma(S)$ is bipartite. □

Corollary 2.1.8. *Let S be a semigroup of bounded exponent. Then $\Gamma(S)$ is a tree if and only if $|E(S)| = 1$ and $\pi(S) \subseteq \{1, 2\}$.*

Corollary 2.1.9. *Let G be a finite group. Then the following statements are equivalent:*

(i) $\pi(G) \subseteq \{1, 2\}$

(ii) $\Gamma(G)$ *is acyclic graph*

(iii) $\Gamma(G)$ *is bipartite*

(iv) $\Gamma(G)$ *is a tree*

(v) $\Gamma(G)$ *is a star graph*

(vi) $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.

Theorem 2.1.10. *Let S be a semigroup of bounded exponent. Then S is completely regular semigroup if and only if all the connected components of $\Gamma(S)$ forms a group.*

Proof. Suppose S is completely regular semigroup. Then every $\mathcal{H}$-class of S is a group (see Proposition 1.1.23). To prove that each connected component $\Gamma(S)$ forms a group, by Corollary 2.1.2, we show that $S_f = H_f$ for each $f \in E(S)$. Let $a \in H_f$. Then $a^n = f$ for some $n \in \mathbb{N}$ so that $a \in S_f$. On the other hand, suppose

$a \in S_f$. If $a \in H_{f'}$ for some $f' \neq f \in E(S)$, then $a \in S_{f'}$; a contradiction. Thus $H_f = S_f$.

Conversely suppose that every connected component of $\Gamma(S)$ forms a group. To prove S is completely regular, we show that every $\mathcal{H}$-class forms a group (see Proposition 1.1.23). Let $a \in S$. Then $a \in S_f$ for some $f \in E(S)$ as S is of bounded exponent. We claim that $H_a = S_f$. Suppose $b \in S_f$. By Remark 1.1.21, $(b, f) \in \mathcal{H}$. Also, we have $(a, f) \in \mathcal{H}$ so that $(a, b) \in \mathcal{H}$. It follows that $S_f \subseteq H_a$. On the other hand, let $b \in H_a$. Then $a \in S_f$ and it implies that $b \in H_f$. Since H_f contains an idempotent so that H_f forms a group (see Corollary 1.1.22). It follows that $b^m = f$ for some $m \in \mathbb{N}$. Hence, $H_a = S_f$ for some $f \in E(S)$. □

Theorem 2.1.11. *Let S be a semigroup of bounded exponent. Then the cyclic graph $\Gamma(S)$ is regular if and only if $|S_f| = |S_{f'}|$ for all $f, f' \in E(S)$ and for $f \in E(S)$ there exists $a \in S$ with one of the following holds:*

(i) $S_f = \langle a : a^{1+r} = a \rangle$

(ii) $S_f = \langle a : a^{2+r} = a^2 \rangle$

(iii) $S_f = \langle a : a^{3+r} = a^3 \rangle$, *where r is odd.*

Proof. First suppose that $\Gamma(S)$ is regular and $f, f' \in E(S)$. Then $\deg(f) = |S_f| - 1 = |S_{f'}| - 1 = \deg(f')$. Thus $|S_f| = |S_{f'}|$. Let $f \in E(S)$ and $x, y \in S_f$. Consequently, S_f forms a clique as $\Gamma(S)$ is a regular graph. Then $\langle x, y \rangle = \langle z \rangle$ for some $z \in S$. Observe that $\langle z \rangle \subseteq S_f$. It follows that S_f forms a subsemigroup of S. Since $\Gamma(S_f)$ is complete, by Theorem 2.1.5, S_f satisfies one of the given condition.

Conversely, suppose $x \in S_f$ and $y \in S_{f'}$ for some $f, f' \in E(S)$. By the given hypothesis, note that $\Gamma(S_f)$ and $\Gamma(S_{f'})$ are complete (see Theorem 2.1.5). Then $|S_f| = |S_{f'}|$ follows that $\deg(x) = \deg(y)$. Thus, $\Gamma(S)$ is regular. □

2.2 Chromatic Number of $\Gamma(S)$

Aalipour et al. [2017] proved that the clique number of the power graph of any group is at most countable. In fact Shitov [2017] has shown that the chromatic number of the power graph of any semigroup is at most countable. In this section, we show that the chromatic number of the cyclic graph of any semigroup is at most countable. To prove our main result of this section, first we develop various lemmas.

Lemma 2.2.1. *Let $a, b \in S$ such that $m_a = m_b = 1$ and $o(a) = o(b) = n$. Then $a \sim b$ in $\Gamma(S)$ if and only if $a \sim b$ in $\mathcal{P}(S)$.*

Proof. Suppose that $a \sim b$ in $\Gamma(S)$. Then $\langle a, b\rangle = \langle c\rangle$ for some $c \in S$. Since $m_a = m_b = 1$ so that $a, b \in \mathcal{K}_c$ which is a cyclic subgroup of S. Without loss of generality, we assume that $m_c = 1$. Since a cyclic group contains exactly one subgroup of a particular order, we have $\langle a\rangle = \langle b\rangle$ so that $a \sim b$ in $\mathcal{P}(S)$. The converse is straightforward as $\mathcal{P}(S)$ is the spanning subgraph of $\Gamma(S)$. □

Lemma 2.2.2. *For $n \in \mathbb{N}$, let $H_n = \{s \in S \ : \ m_s = 1$ and $r_s = n\}$. Then $\chi(\Gamma(H_n)) = \phi(n)$, where ϕ is the Euler's totient function.*

Proof. Let $\equiv$ be a relation on H_n such that $x \equiv y$ if and only if x is a power of y. Clearly, this relation is reflexive and transitive. In order to show that the relation $\equiv$ is symmetric, we assume that $x \equiv y$. Then $x = y^k$ for some $k \in \mathbb{N}$. Since $o(x) = o(y) = n$ and $x \in \langle y\rangle$ implies $\langle x\rangle = \langle y\rangle$. Therefore, we have $\equiv$ is an equivalence relation. By Lemma 2.2.1, $x \sim y$ in $\Gamma(H_n) \iff x \equiv y$. Since the equivalence class of x forms a clique and it is the set of all the generators of $\langle x\rangle$, we have $\chi(\Gamma(H_n)) = \phi(n)$. □

Lemma 2.2.3. *For $k > 1$, the set $I_k = \{x \in S \ : \ m_x = k\}$ is independent in $\Gamma(S)$.*

Proof. On contrary, if $x, y \in I_k$ such that $x \sim y$ in $\Gamma(S)$, then $\langle x, y\rangle = \langle z\rangle$ for some $z \in S$. Note that $x = z^i$, $y = z^j$ and $z = x^{t_1}y^{t_2}$ implies $z = z^{it_1+jt_2}$. As a consequence, we get either $z \in \{x, y\}$ or $m_z = 1$. If $m_z = 1$, then $m_x = m_y = 1$; a contradiction. Otherwise, $x \sim y$ in $\mathcal{P}(S)$. Without loss of generality, we may assume that $x = y^t$ for some $t > 1$. Clearly, $t < k$. By division algorithm, we can write $k = tq + r$, where $0 \leq r \leq t - 1$. If $r = 0$, then we have $m_x = m_{y^t} = q$ so that $k = q$. Consequently, $k = kt$; a contradiction as $t > 1$. If $r > 0$, then $m_{y^t} = q + 1 = k$. Thus, $k = t(k-1) + r > t(k-1)$ implies $t < \frac{k}{k-1}$; a contradiction. □

Lemma 2.2.4. *Let $x \in S$ be an element of infinite order. Then the set $S(x, m, n)$ is an independent subset of $\Gamma(S)$.*

Proof. If possible, let $y, z \in S(x, m, n)$ such that $y \sim z$ in $\Gamma(S)$. Then $\langle y, z\rangle = \langle t\rangle$ for some $t \in S$. Clearly, $y = t^\alpha$, $z = t^\beta$ for some $\alpha, \beta \in \mathbb{N}$. For $y, z \in S(x, m, n)$, we get $x^m = y^n = z^n$ implies $t^{n\alpha} = y^n = z^n = t^{n\beta}$. Note that $\alpha \neq \beta$. Consequently, $o(t)$ is finite and so is $o(x)$; a contradiction. □

Theorem 2.2.5. *The chromatic number of the cyclic graph of any semigroup is at most countable.*

Proof. First note that if $x, y \in S$ such that $o(x)$ is finite and $o(y)$ is infinite, then x is not adjacent with y. As a consequence, order of all the vertices in a connected component of $\Gamma(S)$ will be either finite or infinite. Now suppose that $x \in S$ such that $o(x)$ is infinite, then $C(x)$ is the union of countable number of independent sets (see Lemma 2.2.4). Thus, the chromatic number of the subgraph induced by $C(x)$ is at most countable. If $o(x)$ is finite, then $C(x) = A \cup B$, where $A = \{y \in C(x) : m_y > 1\}$ and $B = \{y \in C(x) : m_y = 1\}$. By Lemma 2.2.3, A is the union of countable number of independent sets so that $\chi(\Gamma(A))$ is at most countable. Since $B = \bigcup\limits_{n \in \mathbb{N}} H_n$, by Lemma 2.2.2, $\chi(\Gamma(B))$ is at most countable. Hence, the chromatic number of $C(x)$ is at most countable and so is of $\Gamma(S)$. □

Corollary 2.2.6. *The clique number of the cyclic graph of any semigroup is at most countable.*

Now we provide an upper bound for $\chi(\Gamma(S))$ where S is of exponent n, in the following theorem.

Theorem 2.2.7. *Let S be a semigroup with exponent n and let $M = \max\{m_a : a \in S, m_a > 1\}$ and $R = \sup\{r_a : a \in S,\ m_a = 1\}$. Then*

$$\chi(\Gamma(S)) \leq (M-1) + \sum_{k=1}^{R} \phi(k),$$

where ϕ is the Euler's totient function.

Proof. Consider the sets $I_j = \{x \in S:\ m_x = j > 1\}$ and $H_k = \{x \in S:\ m_x = 1,\ r_x = k\}$. By Lemma 2.2.3, I_j is an independent set. Note that $\chi(\Gamma(H_k)) = \phi(k)$ (see Lemma 2.2.2). Since $S = \left(\bigcup_{j=2}^{M} I_j\right) \cup \left(\bigcup_{k=1}^{R} H_k\right)$, we have $\chi(\Gamma(S)) \leq (M-1) + \sum_{k=1}^{R} \phi(k)$. □

It is also proved that if S is of unbounded exponent, then $\chi(\Gamma(S))$ is countably infinite (see Corollary 2.3.7).

2.3 Clique Number of $\Gamma(S)$

In this section, we obtain the clique number of $\Gamma(S)$. The following lemma is useful in the sequel.

Lemma 2.3.1. *For $m > 1$, let $S = M(m, r) = \langle a \rangle$ be a monogenic semigroup such that $i < j$ and $i, j < m$. Then the followings are equivalent:*

(i) $a^i \sim a^j$

(ii) $a^j \in \langle a^i \rangle$

(iii) $i \mid j$.

Proof. (i) $\Rightarrow$ (ii) First suppose that $a^i \sim a^j$ in $\Gamma(S)$. Then $\langle a^i, a^j\rangle = \langle a^k\rangle$ for some $k \in \mathbb{N}$. It follows that $a^i = a^{kt}$, $a^j = a^{ks}$ and $a^k = a^{ui+vj}$ for some $s, t \in \mathbb{N}$ and $u, v \in \mathbb{N}_0$. First note that $a^i \neq a^j$. If $a^i = a^j$ then $m \leq i$ which is not possible. From $a^k = a^{ui+vj}$, we get $a^k = a^{(tu+sv)k}$. Also, observe that either $s > 1$ or $t > 1$. Otherwise, $a^i = a^j$; a contradiction. Now if $v = 0$ then (ii) holds. We may now suppose that $v \neq 0$. If $u = 0$, then $a^k \in \langle a^j\rangle$ implies $a^i \in \langle a^j\rangle$. Thus, $a^i = a^{jl}$ for some $l \in \mathbb{N}$. Since $i < j$, we have $m \leq i$ but given that $i < m$. Therefore, $u \neq 0$. Consequently we have $u \neq 0$ and $v \neq 0$. Since $(su + tv)k > k$, we get $m \leq k$. Now consider $a^i = a^{kt}$. If $kt \neq i$, then we get $m \leq i$; a contradiction. Thus, $kt = i$. If $t \neq 1$, then $k < i$ so $m \leq k < i$; again a contradiction. It follows that $t = 1$ and so $i = k$. Therefore, $a^j \in \langle a^i\rangle$.

(ii) $\Rightarrow$ (iii) Suppose $a^j \in \langle a^i\rangle$ so that $a^j = a^{it}$ for some $t \in \mathbb{N}$. If $j \neq it$ then $m \leq j$, which is not possible. Thus $j = it$ and so $i \mid j$.

(iii) $\Rightarrow$ (i) Suppose $i \mid j$. Then $j = ki$ for some $k \in \mathbb{N}$. It follows that $\langle a^i, a^j\rangle = \langle a^i\rangle$ and hence $a^i \sim a^j$. □

For a positive integer k such that $k = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, we denote $\nu(k) = \sum_{i=1}^{r} \alpha_i$ as number of terms in its prime factorization. Now, in the following proposition, we obtain the clique number of $\Gamma(S)$, where S is a monogenic semigroup.

Proposition 2.3.2. *Let $S = M(m, r) = \langle a\rangle$ be a monogenic semigroup. For $m > 1$, we have*

$$\omega(\Gamma(S)) = \max\{\mu_k : \ 1 \leq k < m\},$$

$\mu_1 = 1 + r$ *and for* $k \geq 2$, $\mu_k = 1 + \nu(k) + \frac{r}{(r,\, k)}$.

Proof. Let C be an arbitrary maximal clique of $\Gamma(S)$. To prove the result, we show that the number of elements of index more than one in C is $1 + \nu(k)$ and $|C \cap \mathcal{K}_a| = \frac{r}{(k,\, r)}$ for some k. First note that $\mathrm{N}[a] = S$ and so $a \in C$. Without loss of generality, we assume that $a, a^{i_1}, a^{i_2}, \dots, a^{i_s}$ are in C such that $1 < i_1 < i_2 < \cdots < i_s$ and index of each of these elements is more than one. We claim that i_1 is a prime. If

i_1 is not a prime, then there exists a prime p such that $p \mid i_1$. By Lemma 2.3.1, we get $a^p \sim a^j$ for all $j \in \{1, i_1, i_2, \dots, i_s\}$. Let $a^q \in C \cap \mathcal{K}_a$. Then by the proof of Lemma 2.3.1((i) $\Rightarrow$ (ii)), we get $a^q \in \langle a^{i_s} \rangle$. Again by Lemma 2.3.1, we get $\langle a^q \rangle \subseteq \langle a^p \rangle$. It follows that $\langle a^q, a^p \rangle = \langle a^p \rangle$ and so $a^q \sim a^p$. Consequently, $C \cup \{a^p\}$ forms a clique; a contradiction to the maximality of C. Thus, i_1 is a prime and we write $i_1 = p_1$. Now $a^{p_1} \sim a^{i_2}$ implies $p_1 \mid i_2$ (see Lemma 2.3.1). We get $i_2 = p_1 t$ for some $t \in \mathbb{N} \setminus \{1\}$. Note that t is a prime. Otherwise, there exists a prime p' such that $p' \mid t$. Since $p_1 p' \mid p_1 t = i_2$, by the similar argument as used above, $C \cup \{a^{p_1 p'}\}$ forms a clique and again we get a contradiction. As a result, t is a prime and we write $t = p_2$ so that $i_2 = p_1 p_2$. On continuing similar process, we get $i_s = p_1 p_2 \dots p_s$.

Thus, there are $1 + \nu(i_s)$ elements of index more than one in C. Now we count the number of elements of index one in C. We show that $C \cap \mathcal{K}_a = \mathcal{K}_{a^{i_s}}$ so that $|C \cap \mathcal{K}_a| = \frac{r}{(i_s, r)}$ (see Proposition 1.1.10).

To prove $C \cap \mathcal{K}_a = \mathcal{K}_{a^{i_s}}$, consider $a^t \in \mathcal{K}_{a^{i_s}}$. Since $\mathcal{K}_{a^{i_s}} = \mathcal{K}_a \cap \langle a^{i_s} \rangle$ (cf. Remark 1.1.2), we get $a^t \in \langle a^{i_s} \rangle \subseteq \langle a^j \rangle$ for all j, where $j \in \{1, i_1, i_2, \dots, i_s\}$. Therefore, $a^t \sim a^j$. Since the subgraph induced by $\mathcal{K}_a$ is complete and $a^t \in \mathcal{K}_a$, we get a^t is adjacent with all the elements of C whose index is one. If $a^t \notin C$, then $C \cup \{a^t\}$ forms a clique; a contradiction to the maximality of C. Consequently $a^t \in C \cap \mathcal{K}_a$ so that $\mathcal{K}_{a^{i_s}} \subseteq \mathcal{K}_a \cap C$. Let $a^l \in \mathcal{K}_a \cap C$. Then $a^l \sim a^{i_s}$. By the proof of Lemma 2.3.1((i) $\Rightarrow$ (ii)), we get $a^l \in \langle a^{i_s} \rangle$. Thus, $a^l \in \mathcal{K}_a \cap \langle a^{i_s} \rangle = \mathcal{K}_{a^{i_s}}$ so that $\mathcal{K}_a \cap \langle a^{i_s} \rangle = \mathcal{K}_{a^{i_s}}$.

Now, for each k such that $1 \leq k < m$, we provide a clique of size μ_k. For $k = 1$, $\mathcal{K}_a \cup \{a\}$ forms a clique of size $1 + r$. For $k > 1$, we have $k = p_1 p_2 \dots p_{s'}$. Then by Lemma 2.3.1, note that $\{a, a^{p_1}, a^{p_1 p_2}, \dots, a^k\} \cup \mathcal{K}_{a^k}$ forms a clique. Because if $x \in \mathcal{K}_{a^k} = \langle a^k \rangle \cap \mathcal{K}_a$, then $x \in \langle a^k \rangle \subseteq \langle a^j \rangle$ for all $j \in \{p_1, p_1 p_2, \dots, k\}$. Consequently, $x \sim a^j$ for all j. This completes our proof. □

The following lemma will be useful in the sequel.

Lemma 2.3.3 ([Aalipour et al., 2017, Lemma 32]). *Let G be a group and suppose $x, y, z \in G$ such that $\langle x, y \rangle$, $\langle y, z \rangle$ and $\langle x, z \rangle$ are the cyclic subgroup of G. Then*

$\langle x, y, z\rangle$ is cyclic subgroup of G.

Proposition 2.3.4. *Let S be a semigroup with exponent n and C be a maximal clique in $\Gamma(S)$. Then $C \subseteq \langle a\rangle$ for some $a \in S$.*

Proof. Let $x \in C$. Then $x^n = f$ for some $f \in E(S)$. In view of Corollary 2.1.2, $x \in S_f$ and S_f is a connected component of $\Gamma(S)$. Since C is a clique, we get $C \subseteq S_f$. Now consider $\mathsf{M} = \max\{m_x : x \in C\}$. We prove our result through the following cases:

Case 1: $\mathsf{M} = 1$. First we prove that $\langle C\rangle$ is a subgroup of S. For that, suppose $x \in C \subseteq S_f$. Then $x^n = f$ for some $n \in \mathbb{N}$. Since $m_x = 1$, we get $\langle x\rangle$ is a cyclic subgroup of S. It follows that $xf = fx = x$. Further note that for any $x, y \in C$, we have $xy = yx$. Consequently, for $a \in \langle C\rangle$, we have $a = c_1^{k_1}c_2^{k_2}\ldots c_n^{k_n}$, where $c_i \in C$ and $k_i \in \mathbb{N}$. Observe that $af = a$ so that $\langle C\rangle$ forms a monoid with the identity element f. Since $a = c_1^{k_1}c_2^{k_2}\ldots c_n^{k_n}$, we have $ab = ba = f$, where $b = (c_n^{k_n})^{-1}\ldots(c_1^{k_1})^{-1}$. Thus, $\langle C\rangle$ is a subgroup of S. Now we show that C is a cyclic subgroup of S and let $x, y, z \in C$. By Lemma 2.3.3, $\langle x, y, z\rangle$ is a cyclic subgroup of $\langle C\rangle$. Consequently, $x^iy^j \sim z$ for each $i, j \in \mathbb{N}$. It follows that $C \cup \langle x, y\rangle$ is a clique of $\Gamma(S)$. Since C is a maximal clique, we must have $\langle x, y\rangle \subseteq C$. Therefore, $a \in C$ so that $\langle C\rangle \subseteq C$ gives $\langle C\rangle = C$. Thus C is a subgroup of S. In view of Lemma 1.1.25, $o(x) \leq 2n$ for all $x \in C$. Choose $x \in C$ such that $o(x) \geq o(y)$ for all $y \in C$. In order to prove $C \subseteq \langle x\rangle$, let $y \in C$. Then $\langle x, y\rangle = \langle z\rangle$ for some $z \in C$ implies $y \in \langle z\rangle = \langle x\rangle$. Thus the result holds.

Case 2: $\mathsf{M} > 1$. By Lemma 1.1.25, $o(x) \leq 2n$ for all $x \in C$. Now choose $x \in C$ with $m_x = \mathsf{M}$ and $o(x) \geq o(y)$ for all $y \in C$ such that $m_y = \mathsf{M}$. We show that $C \subseteq \langle x\rangle$. Let $y \in C$. Then $\langle x, y\rangle = \langle z\rangle$. It follows that $x = z^i$, $y = z^j$ and $z = x^uy^v$ for some $i, j \in \mathbb{N}$ and $u, v \in \mathbb{N}_0$. If either $v = 0$ or $i = 1$, then observe that $y \in \langle z\rangle \subseteq \langle x\rangle$. Therefore, $C \subseteq \langle x\rangle$. We may now suppose that $v \neq 0$ and $i > 1$. If $u \neq 0$, then $z = z^{ui+vj}$ gives $m_z = 1$ and so $m_x = 1$; a contradiction because $m_x = \mathsf{M} > 1$. Consequently, we get $u = 0$ and so $x \in \langle y\rangle$. By Lemma 1.1.8, $m_x \leq m_y$. Thus,

$m_x = m_y = \mathsf{M}$ gives $o(x) \geq o(y)$. Since $x \in \langle y \rangle$, we obtain $x = y^l$ for some $l \in \mathbb{N}$. If $l > 1$, then $o(x) < o(y)$ (cf. Lemma 1.1.8) which is not possible. Thus, $x = y$ and hence $C \subseteq \langle x \rangle$. □

Theorem 2.3.5. *Let S be a semigroup with exponent n. Then*
$\omega(\Gamma(S)) = \max\left(\{r_a : \ m_a = 1 \text{ and } a \in \mathcal{M}\} \cup \{\mu_k^a : \ m_a > 1, \ 1 \leq k < m_a \text{ and } a \in \mathcal{M}\}\right)$,
$\mu_1^a = 1 + r_a$ *and for* $k \geq 2$, $\mu_k^a = 1 + \nu(k) + \frac{r_a}{(r_a,\, k)}$.

Proof. Let C be a clique of maximum size in $\Gamma(S)$. Consider the sets
$A = \{r_a : \ a \in \mathcal{M}, \ m_a = 1\}$ and $B = \{\mu_k^a : \ a \in \mathcal{M}, \ m_a > 1, \ 1 \leq k < m_a\}$. We claim that $|C| \in A \cup B$. By Proposition 2.3.4, $C \subseteq \langle a' \rangle$ for some $a' \in S$ and $a' \in \langle a \rangle$ for some $a \in \mathcal{M}$ (cf. Lemma 1.1.25). Then $C \subseteq \langle a \rangle$. If $m_a = 1$, then $\langle a \rangle$ is a cyclic subgroup of S and so $\Gamma(\langle a \rangle)$ is complete (see Corollary 2.1.6). Since C is a clique of maximum size in $\Gamma(S)$ so it is of maximum size in $\Gamma(\langle a \rangle)$ also. It follows that $C = \Gamma(\langle a \rangle)$. Consequently, we get $|C| = |\langle a \rangle| = r_a \in A$. Now let $m_a > 1$. Then C is again a clique of maximum size in $\Gamma(\langle a \rangle)$. By Proposition 2.3.2, $|C| = \mu_k^a \in B$ for some k, where $1 \leq k < m_a$. Next we provide a clique of size t for each $t \in A \cup B$. If $t \in A$, then there exists $a \in \mathcal{M}$ such that $t = r_a$ and $m_a = 1$. Thus, $\langle a \rangle$ is a cyclic subgroup of S and the subgraph induced by $\langle a \rangle$ is complete (cf. Corollary 2.1.6). We get a clique of size $V(\Gamma(\langle a \rangle)) = o(a) = r_a = t$. If $t \in B$, then $t = \mu_k^a$ of some $a \in \mathcal{M}$ such that $m_a > 1$ and $1 \leq k < m_a$. Since the prime factorization of k is $p_1 p_2 \ldots p_s$ and by the proof of Proposition 2.3.2, the set $\{a, a^{p_1}, a^{p_1 p_2}, \ldots, a^k\} \cup \mathcal{K}_{a^k}$ forms a clique of size $t = \mu_k^a$. This completes the proof. □

Theorem 2.3.6. *Let S be a semigroup of unbounded exponent. Then $\omega(\Gamma(S))$ is countably infinite.*

Proof. In view of Corollary 2.2.6, to prove the result, we show that for $k \in \mathbb{N}$ there exists a clique of size $\lfloor \log_2 k \rfloor + 1$. We claim that: there exists $a \in S$ such that $a, a^2, \ldots, a^k$ are non idempotent elements of S. Let, if possible, there exists $i_a \leq k$ such that $a^{i_a} = f$ for some $f \in E(S)$. Now choose $n = k!$. Note that $a^n =$

$(a^{i_a})^{23\cdots(i_a-1)(i_a+1)\cdots k} = f$. Thus, S is of bounded exponent; a contradiction. This proves the claim. By Proposition 2.3.1, note that the sets $\{a, a^2, a^4, \ldots, a^{2^{\lfloor \log_2 k \rfloor}}\}$ forms a clique of size $\lfloor \log_2 k \rfloor + 1$. This completes our proof. □

In view of Theorem 2.2.5, we have the following corollary.

Corollary 2.3.7. *Let S be a semigroup of unbounded exponent. Then $\chi(\Gamma(S))$ is countably infinite.*

2.4 Independence Number of $\Gamma(S)$

In this section, we investigate the independence number of $\Gamma(S)$. First we obtain $\alpha(\Gamma(S))$ for a monogenic semigroup S in the following theorem.

Theorem 2.4.1. *Let $S = \langle a \rangle$ be a monogenic semigroup. Then the independence number of $\Gamma(S)$ is given below:*

$$\alpha(\Gamma(S)) = \begin{cases} \infty & \text{if } S \text{ is infinite;} \\ 1 & \text{if } S \text{ is finite and } m_a = 1; \\ \left\lfloor \frac{m_a}{2} \right\rfloor + 1 & \text{if } m_a > 1,\ (i, r_a) > 1 \text{ for all } i, \text{ where } \left\lceil \frac{m_a}{2} \right\rceil \le i \le m_a - 1; \\ \left\lfloor \frac{m_a}{2} \right\rfloor & \text{if } m_a > 1,\ (i, r_a) = 1 \text{ for some } i, \text{where } \left\lceil \frac{m_a}{2} \right\rceil \le i \le m_a - 1. \end{cases}$$

Proof. Suppose that S is an infinite semigroup. We first claim that for $i < j$, we have $a^i \sim a^j$ in $\Gamma(S)$ if and only if $i \mid j$. If $i \mid j$, then clearly $a^j \in \langle a^i \rangle$ so that $a^i \sim a^j$ in $\Gamma(S)$. On the other hand, if $a^i \sim a^j$, then $\langle a^i, a^j \rangle = \langle a^k \rangle$ for some $a^k \in S$. Thus, $a^i = a^{tk}$, $a^j = a^{t'k}$ and $a^k = a^{ui} + a^{vj}$ for some $t, t' \in \mathbb{N}$ and $u, v \in \mathbb{N}_0$. Therefore, $a^k = a^{ui+vj} = a^{(tu+t'v)k}$. If $tu + t'v \neq 1$, then $m_a \le k$ which is not possible as S is an infinite semigroup. Thus, $tu + t'v = 1$. Further note that both t, t' can not be 1. Otherwise, $a^i = a^j$ which is not possible. It follows that either $t > 1$ or $t' > 1$. If $t > 1$ then $u = 0$ as $tu + t'v = 1$. We get $a^i \in \langle a^j \rangle$ so that $a^i = a^{jl}$ for some $l \in \mathbb{N}$. For $i \neq jl$, we have $m_a \le i$; a contradiction. Consequently, $i = jl$ and so $j \mid i$ which is not possible as $i < j$. If $t' > 1$, then by the similar argument used above, we get

$j = il'$ for some $l' \in \mathbb{N}$. Thus, $i \mid j$. Consequently, the set $\{a^p : p \text{ is a prime}\}$ is an independent in $\Gamma(S)$ so that $\alpha(\Gamma(S)) = \infty$.

Now we prove our result, when S is finite. If $m_a = 1$, then $S = \langle a \rangle$ is a cyclic group and therefore by Corollary 2.1.6, $\Gamma(S)$ is complete. It follows that $\alpha(\Gamma(S)) = 1$. We now assume that $m_a > 1$. Consider the set

$$\mathcal{I} = \{a^i : \left\lceil \frac{m_a}{2} \right\rceil \leq i \leq m_a - 1\}.$$

By Lemma 2.3.1, $\mathcal{I}$ is an independent set of size $\left\lfloor \frac{m_a}{2} \right\rfloor$. Further, we split our proof in two cases:

Case 1: $(i, r_a) > 1$ for all i, where $\left\lceil \frac{m_a}{2} \right\rceil \leq i \leq m_a - 1$. Then $|K_{a^i}| < r_a$ (cf. Proposition 1.1.10). Since $\mathcal{K}_a = \langle a^{m_a+g} \rangle$ for some g, where $0 \leq g \leq r_a - 1$ and $m_a + g \equiv 1 (\text{mod } r_a)$. Note that $\mathcal{I} \cup \{a^{m_a+g}\}$ is an independent set. If $a^{m_a+g} \sim a^i$ for some i, where $\left\lceil \frac{m_a}{2} \right\rceil \leq i \leq m_a - 1$, then $a^{m_a+g} \in \langle a^i \rangle$ (see proof of Lemma 2.3.1 (i) $\Rightarrow$ (ii)). Thus, $\mathcal{K}_a \subseteq \mathcal{K}_{a^i}$ and so $\mathcal{K}_a = \mathcal{K}_{a^i}$, which is a contradiction of $|\mathcal{K}_{a^i}| < r_a$. To prove our result, in this case, we show that if $\mathcal{I}'$ is an arbitrary independent set in $\Gamma(S)$ then $|\mathcal{I}'| \leq \left\lfloor \frac{m_a}{2} \right\rfloor + 1$. Since the subgraph induced by $\mathcal{K}_a$ is complete, we get $|\mathcal{K}_a \cap \mathcal{I}'| \leq 1$. Without loss of generality, consider the set

$$\mathcal{I}' \cap (S \setminus \mathcal{K}_a) = \{a^{i_1}, a^{i_2}, \ldots, a^{i_t}, a^{i_{t+1}}, \ldots, a^{i_l}\},$$

where $i_1 < i_2 < \cdots < i_t < \frac{m_a}{2}$ and $\frac{m_a}{2} \leq i_{t+1} < i_{t+2} < \cdots < i_l < m_a$. For each $i_s \in \{i_1, i_2, \ldots, i_t\}$, we have $2i_s < m_a$. Now choose the smallest natural number α_s such that $i_s 2^{\alpha_s+1} \geq m_a$. Then $\frac{m_a}{2} \leq i_s 2^{\alpha_s} < m_a$. We claim that if $i_{s_1} \neq i_{s_2}$ then $i_{s_1} 2^{\alpha_{s_1}} \neq i_{s_2} 2^{\alpha_{s_2}}$. If $i_{s_1} 2^{\alpha_{s_1}} = i_{s_2} 2^{\alpha_{s_2}}$, then clearly $\alpha_{s_1} \neq \alpha_{s_2}$. Without loss of generality, we assume that $\alpha_{s_1} > \alpha_{s_2}$. Thus, $i_{s_1}(2^{\alpha_{s_1} - \alpha_{s_2}}) = i_{s_2}$ implies $i_{s_1} \mid i_{s_2}$. By Lemma 2.3.1, $a^{i_{s_1}} \sim a^{i_{s_2}}$; a contradiction of the fact that $\mathcal{I}'$ is an independent set. Moreover, for each $i_s \in \{i_1, i_2, \ldots, i_t\}$, $a^{i_s} \sim a^{i_s 2^{\alpha_s}}$ and $a^{i_s 2^{\alpha_s}} \in \mathcal{I}$ but $a^{i_s 2^{\alpha_s}}$ can not be in $\mathcal{I}'$. Thus, we have $|\mathcal{I}' \cap (S \setminus \mathcal{K}_a)| \leq t + \left\lfloor \frac{m_a}{2} \right\rfloor - t = \left\lfloor \frac{m_a}{2} \right\rfloor$.

Case 2: $(i, r_a) = 1$ for some i, where $\left\lceil \frac{m_a}{2} \right\rceil \leq i \leq m_a - 1$. Then $|\mathcal{K}_{a^i}| = r_a = |\mathcal{K}_a|$ and so $\mathcal{K}_{a^i} = \mathcal{K}_a$. Thus, $\mathcal{K}_a \subseteq \langle a^i \rangle$. It follows that $a^i \sim x$ for all $x \in \mathcal{K}_a$. Now let

$j < \frac{m_a}{2}$. Then by the similar argument used in **Case 1**, we get $a^j \sim a^{j2^{\alpha_j}}$, where $a^{j2^{\alpha_j}} \in \mathcal{I}$. Consequently, $\mathcal{I}$ is a maximal independent set. Now to prove $\mathcal{I}$ is an independent set of maximum size, we assume that $\mathcal{I}''$ is an independent set different from $\mathcal{I}$. Since the subgraph induced by $\mathcal{K}_a$ is complete, we get $|\mathcal{I}'' \cap \mathcal{K}_a| \leq 1$. Also, by the similar argument used in **Case 1**, we get $|\mathcal{I}'' \cap (S \setminus \mathcal{K}_a)| \leq \lfloor \frac{m_a}{2} \rfloor$. If $|\mathcal{I}'' \cap \mathcal{K}_a| = 0$, then $|\mathcal{I}''| \leq \lfloor \frac{m_a}{2} \rfloor$. If $|\mathcal{I}'' \cap \mathcal{K}_a| = 1$, then there exists $a^j \in \mathcal{I}'' \cap \mathcal{K}_a$. Since $\mathcal{K}_a = \mathcal{K}_{a^i}$, we get $a^j \in \mathcal{K}_{a^i} = \langle a^i \rangle \cap \mathcal{K}_a$. It follows that $a^j \sim a^i$ and so $a^i \in \mathcal{I}$ but $a^i \notin \mathcal{I}''$. Again by the similar argument used in **Case 1**, we get $|\mathcal{I}''| \leq \lfloor \frac{m_a}{2} \rfloor$. Thus, $\mathcal{I}$ becomes an independent set of maximum size $\lfloor \frac{m_a}{2} \rfloor$. This completes our proof. □

Now we determine a lower and upper bound of $\alpha(\Gamma(S))$, where S is a semigroup of exponent n. Consider a relation τ on S defined by $x \ \tau \ y$ if and only if $\langle x \rangle = \langle y \rangle$. Clearly τ is an equivalence relation. Let X be a complete set of distinct representative elements for τ. Now, let $I_2 = \{x \in S : \ m_x = 2\}$ and
$J_2 = \{a \in \overline{\mathcal{M}} \cap X : \ a \notin \langle x \rangle \text{ for any } x \in I_2\}$.

Theorem 2.4.2. *Let S be a semigroup with exponent n and $\mathcal{M}$ is finite. Then*

$$|I_2| + |J_2| \leq \alpha(\Gamma(S)) \leq |J_2| + \sum_{a \in \mathcal{M}} \left\lfloor \frac{m_a}{2} \right\rfloor .$$

Proof. To find a lower bound of $\alpha(\Gamma(S))$, we show that $I_2 \cup J_2$ is an independent set of $\Gamma(S)$. By Lemma 2.2.3, I_2 is an independent set. If $a, b \in J_2$ such that $a \sim b$, then $\langle a, b \rangle = \langle z \rangle$ for some $z \in S$. Since $o(z) \leq 2n$ (see Lemma 1.1.25) and $m_a = m_b = 1$, we get $\langle a, b \rangle = \langle z \rangle$ is a cyclic subgroup of S. The maximality of $\langle a \rangle$ and $\langle b \rangle$ follows that $\langle z \rangle = \langle a \rangle = \langle b \rangle$, which is not possible. Consequently, J_2 is an independent set. Further, to show $I_2 \cup J_2$ is an independent set, let $x \in I_2$ and $y \in J_2$ such that $x \sim y$. By Lemma 1.1.7 and by the proof Lemma 2.3.1, we have $y \in \langle x \rangle$; a contradiction of $y \in J_2$. Thus, $I_2 \cup J_2$ is an independent set. Since $I_2 \cap J_2 = \varnothing$ we get $\alpha(\Gamma(S)) \geq |I_2| + |J_2|$.

Now we obtain an upper bound for $\alpha(\Gamma(S))$. Note that the sets

$$A = \left\{a \in \mathcal{M} : \ m_a > 1, \ (i, r_a) > 1 \text{ for all } i, \ \left\lceil \frac{m_a}{2} \right\rceil \le i \le m_a - 1\right\},$$

$$B = \left\{a \in \mathcal{M} : \ m_a > 1, \ (i, r_a) = 1 \text{ for some } i, \ \left\lceil \frac{m_a}{2} \right\rceil \le i \le m_a - 1\right\},$$

and $C = \{a \in \mathcal{M} : \ m_a = 1\}$ forms a partition of $\mathcal{M}$. Since S is of exponent n, by Lemma 1.1.25, note that $S = \left(\underset{a\in A}{\cup}\langle a\rangle\right) \cup \left(\underset{b\in B}{\cup}\langle b\rangle\right) \cup \left(\underset{c\in C}{\cup}\langle c\rangle\right)$. Let $\mathcal{I}$ be any independent set in $\Gamma(S)$. We assume that $x_1, x_2, \ldots, x_l, y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_n$ are the elements of index one belongs to $\mathcal{I}$ such that

- for each i, where $1 \le i \le l$, $x_i \in \langle a\rangle$ for some $a \in A$ and $x_i \notin \langle b\rangle$ for any $b \in B$. (2.1)
- for each j, where $1 \le j \le m$, $y_j \in \langle b\rangle$ for some $b \in B$. (2.2)
- for each k, where $1 \le k \le n$, $z_k \in \langle c\rangle$ for some $c \in C$ and $z_k \notin \langle b\rangle$ for any $b \in B$. (2.3)

Now consider the set $J = \{x \in \mathcal{I} : \ m_x > 1\}$. To prove our result, it is sufficient to show $|J| + m \le \sum\limits_{a\in\mathcal{M}} \left\lfloor \frac{m_a}{2} \right\rfloor$ and $l + n \le |J_2|$. By Theorem 2.4.1, $|\mathcal{I} \cap \langle b\rangle| \le \left\lfloor \frac{m_b}{2} \right\rfloor$ for all $b \in B$ and by the proof of Theorem 2.4.1, $|\mathcal{I} \cap (\langle a\rangle \setminus \mathcal{K}_a)| \le \left\lfloor \frac{m_a}{2} \right\rfloor$ for all $a \in A$. For $x \in S$ such that $m_x > 1$, we get $x \in \left(\underset{a\in A}{\cup}\langle a\rangle\right) \cup \left(\underset{b\in B}{\cup}\langle b\rangle\right)$. It follows that $|J| + m \le \sum\limits_{a\in A\cup B} \left\lfloor \frac{m_a}{2} \right\rfloor = \sum\limits_{a\in\mathcal{M}} \left\lfloor \frac{m_a}{2} \right\rfloor$.

Next we show that $l + m \le |J_2|$. We establish a one-one map from the set $\mathcal{O} = \{x_1, x_2, \ldots, x_l, z_1, z_2, \ldots, z_m\}$ to some subset of J_2. In view of this, for each $p \in \mathcal{O}$, first we provide an element of J_2 corresponding to p. Let $p \in \mathcal{O}$ such that $p = z_k$ for some k, where $1 \le k \le n$. Since $z_k \in \langle c\rangle$ for some $c \in C \subseteq \mathcal{M}$, we have $m_c = 1$ and so $\langle c\rangle$ is a maximal cyclic subgroup of S. Choose $v \in X$ such that $\langle c\rangle = \langle v\rangle$. Then clearly, $v \in \overline{\mathcal{M}}$. If $v \in \langle x\rangle$ for some $x \in I_2$, then $\langle v\rangle = \langle x\rangle$ because $\langle v\rangle$ is a maximal monogenic subsemigroup of S. But $\langle v\rangle = \langle x\rangle$ is not possible because $m_v = 1$ and $m_x = 2$. It follows that for $p = z_k$ we have $v \in J_2$ such that $z_k \in \langle v \rangle$. We may now assume that $p' \in \mathcal{O}$ such that $p' = x_i$ for some i, where $1 \leq i \leq l$. Then $p' = x_i \in \langle a \rangle$ for some $a \in A$. Since $m_{x_i} = 1$, we get $x_i \in \mathcal{K}_a$. By the similar argument used in proof of Lemma 1.1.25 (part (ii)), we get $\mathcal{K}_a \subseteq \langle d \rangle$ for some $d \in \overline{\mathcal{M}}$. If $d \in \langle x \rangle$ for some $x \in I_2$, then $x_i \in \langle x \rangle$. Since $m_x = 2$, we have either $x \in \langle a \rangle$ for some $a \in A$ or $x \in \langle b \rangle$ for some $b \in B$. If $x \in \langle a \rangle$, then $x = a^j$ for some j. Clearly, $\langle d \rangle \subseteq \langle a^j \rangle$ and $\langle d \rangle = \mathcal{K}_a$ as $d \in \overline{\mathcal{M}}$. Then $\mathcal{K}_a \subseteq \langle a^j \rangle$ follows that $|K_{a^j}| = |\mathcal{K}_a| = \frac{r_a}{(j,\, r_a)} = r_a$ and so $(j,\ r_a) = 1$. Since $m_x = 2$ and $x = a^j$, by Lemma 1.1.7, we get $\lceil \frac{m_a}{2} \rceil \leq j \leq m_a - 1$. Therefore, $(j, r_a) > 1$ which is not possible. If $x \in \langle b \rangle$ for some $b \in B$, then $x_i \in \langle b \rangle$; a contradiction of (2.1). Now choose $w \in X$ such that $\langle w \rangle = \langle d \rangle$. Thus, for $p' = x_i$ there exits $w \in J_2$ such that $x_i \in \langle w \rangle$. For each $p \in \mathcal{O}$, choose exactly one $s \in J_2$ such that $p \in \langle s \rangle$ for some $s \in J_2$. In view of this the assignment $f : p \mapsto s$ is a map from $\mathcal{O}$ to some subset D of J_2. In fact, this map is one-one. For instance, if $p, q \in \mathcal{O}$ such that $pf = qf = s$ then $p, q \in \langle s \rangle$. It follows that $p \sim q$ (see Corollary 2.1.6) which is a contradiction as $p, q \in I$. Consequently, from above $l + n = |\mathcal{O}| \leq |J_2|$. □

In view of Lemma 1.1.13, note that $I_2 = \{(i, a, \lambda) :\ a \in G \text{ and } p_{\lambda i} = 0\}$. Note that $I_2 \subseteq \mathcal{M}$. If $x = (i, a, \lambda) \in \mathfrak{M}^0[G, I, \Lambda, P] \setminus I_2$ then $m_x = 1$. Observe that $|I_2| = \sum\limits_{a \in I_2} \left\lfloor \frac{m_a}{2} \right\rfloor$. By Theorem 2.4.2, we get the independence number of $\Gamma(S)$, where S is a finite completely 0-simple semigroup, in the following corollary.

Corollary 2.4.3. *Let S be a finite completely 0-simple semigroup. Then*

$$\alpha(\Gamma(S)) = |J_2| + \sum_{a \in \mathcal{M}} \left\lfloor \frac{m_a}{2} \right\rfloor.$$

Theorem 2.4.4. *Let S be a semigroup such that it satisfies one of the following condition*

(i) *there exists $a \in S$ such that $o(a)$ is infinite*

(ii) *$M = \sup\{m_a : a \in S\}$ is infinite*

(iii) *the set* $E(S)$ *is infinite*

(iv) *the set* $\mathcal{M} = \{a \in S : \langle a \rangle$ is a maximal monogenic subsemigroup of $S\}$ *is infinite.*

Then the independence number of $\Gamma(S)$ *is infinite.*

Proof. If there exists $a \in S$ such that $o(a)$ is infinite, then by the similar argument used in the proof of Lemma 2.3.1, the set $\{a^p : p \text{ is prime}\}$ is an independent set of $\Gamma(S)$. Now suppose that $M = \sup\{m_a : a \in S\}$ is infinite. For $k \in \mathbb{N}$, we establish an independent set of size $\eta(k)$, where $\eta(k)$ is the number of primes less than k. Note that there exists $a \in S$ such that $m_a \geq k$. By Lemma 2.3.1, the set $\{a^p : p \text{ is a prime and } p < k\}$ is an independent set of size $\eta(k)$. Thus, $\alpha(\Gamma(S))$ is infinite. If $E(S)$ is infinite, then $\alpha(\Gamma(S))$ must be infinite becuase any two idempotent elements are not adjacent in $\Gamma(S)$. Further, we assume that the set $\mathcal{M}$ is infinite. Then S contains infinitely many maximal monogenic subsemigroup of S. Let $a, b \in \mathcal{M}$ such that $\langle a \rangle \neq \langle b \rangle$. Clearly, $a \nsim b$. It follows that $\alpha(\Gamma(S))$ is infinite. □