Chapter 3

The Enhanced Power Graphs

In order to see how close the power graph is to the commuting graph, Aalipour et al. [2017] introduced the notion of enhanced power graph of a group. The enhanced power graph $\mathcal{P}_e(G)$ of a group G is a simple undirected graph with vertex set G and two vertices are adjacent if they belong to the same cyclic subgroup. A significant number of publications devoted to enhanced power graphs associated with algebraic structures. Bera and Bhuniya [2017] classified the abelian groups and the non abelian p-groups having dominatable enhanced power graphs. Ma and She [2020] investigated the metric dimension of an enhanced power graph of finite groups. Zahirović et al. [2020] supplied a characterization of finite nilpotent groups whose enhanced power graphs are perfect. Recently, Bera et al. [2021] achieved an upper bound for the vertex connectivity of the enhanced power graphs of an abelian group. Additionally, they characterized the class of an abelian group such that their proper enhanced power graph is connected. The enhanced power graph of semigroup can be defined analogously.

This chapter concerns the study of enhanced power graph associated with groups and semigroups. This chapter is arranged as follows. In Section 3.1, we study various graph invariants, viz. minimum degree, independence number and matching number for $\mathcal{P}_e(G)$, where G is any finite group and then determine them when G is a finite abelian p-group, the dihedral group D_{2n}, the semidihedral group SD_{8n}, the dicyclic group Q_{4n}, U_{6n} or V_{8n}. In Section 3.2, first we ascertain the structure of enhanced power graph of a semigroup S. Then we classify the semigroup S such that $\mathcal{P}_e(S)$ is bipartite, acyclic, planar and complete etc. Finally, we obtain the minimum degree and independence number of $\mathcal{P}_e(S)$. The content of Section 3.1 has been partitioned into two parts and these parts are published in SCIE journal *"Communications in Algebra"*, 49(4), 1697-1716, 2021 and in SCOPUS indexed journal *"Discrete Mathematics, Algorithms and Applications"*, 13(1) : 205009, 2021, respectively.

3.1 The Enhanced Power Graph of Groups

The enhanced power graph $\mathcal{P}_e(G)$ of a group G is a simple undirected graph with vertex set G and two vertices are adjacent if they belong to the same cyclic subgroup. In this section, we study the enhanced power graph of various groups. In this connection, first we consider the minimum degree, independence number and matching number of enhanced power graph of finite groups. Then we determine them when G is a finite abelian p-group, the dihedral group D_{2n}, the semidihedral group SD_{8n}, the dicyclic group Q_{4n}, $U_{6n} = \langle a, b : a^{2n} = b^3 = e, ba = ab^{-1}\rangle$ or $V_{8n} = \langle a, b \ : \ a^{2n} = b^4 = e, \ ba = a^{-1}b^{-1}, b^{-1}a = a^{-1}b\rangle$ in respective subsections. If G is any of these groups, we prove that $\mathcal{P}_e(G)$ is perfect and then obtain its strong metric dimension. Additionally, we give an expression for the independence number of $\mathcal{P}_e(G)$ for any finite abelian group G. These results along with certain known equalities yield the edge connectivity, vertex covering number and edge covering number of enhanced power graphs of the respective groups as well.

We begin with minimum degree and edge connectivity of enhanced power graphs of groups. In view of Theorem 1.2.8, we have the following lemma.

Lemma 3.1.1. *For any finite group G, the edge connectivity and minimum degree of $\mathcal{P}_e(G)$ coincide.*

In light of Lemma 3.1.1, the following determines the edge connectivity of $\mathcal{P}_e(G)$ as well.

Theorem 3.1.2. *For any finite group G, the minimum degree $\delta(\mathcal{P}_e(G)) = m - 1$, where m is the order of a smallest maximal cyclic subgroup of G.*

Proof. Let $x \in G$. Then x belongs to some maximal cyclic subgroup, say $\langle y \rangle$, of G. Since $\langle y \rangle$ induces a clique in $\mathcal{P}_e(G)$, we have $\deg(x) \geq o(y) - 1$. Moreover, as $\langle y \rangle$ is a maximal cyclic subgroup of G, $\mathrm{N}(y) = \langle y \rangle \setminus \{y\}$. This implies $\deg(y) = o(y) - 1$, so that $\deg(x) \geq \deg(y)$. Now let M be a maximal cyclic subgroup of G of least order. Then $\deg(x) \geq |M| - 1$ for any $x \in G$, and that $\deg(z) = |M| - 1$ for any z generating M. Accordingly, the proof follows. □

Recall that the following theorem will be useful in the sequel

Lemma 3.1.3 ([Robinson, 1996, Theorem 5.2.4])**.** *A finite group G is nilpotent if and only if G is isomorphic to a direct product of its Sylow subgroups.*

Now we study the independence number of enhanced power graphs.

Theorem 3.1.4. *For any finite group G, the independence number of $\mathcal{P}_e(G)$ coincides with the number of maximal cyclic subgroups of G. Furthermore, if G is nilpotent and the prime factors of $|G|$ are $p_1, p_2, \ldots, p_r$, then the independence number*

$$\alpha(\mathcal{P}_e(G)) = m_1 m_2 \cdots m_r,$$

where m_i is the number of maximal subgroups of a Sylow-p_i subgroup.

Proof. Let $\mu(G)$ denote the number of maximal cyclic subgroups of G. If x and y are two elements generating two different maximal cyclic subgroups of G, then they are non-adjacent in $\mathcal{P}_e(G)$. As a result, $\alpha(\mathcal{P}_e(G)) \geq \mu(G)$. Now consider an independent set S in $\mathcal{P}_e(G)$. Recall that any group can be written as the union of its maximal cyclic subgroups. Since a maximal cyclic subgroup induces a clique in $\mathcal{P}_e(G)$, no two members of an independent set in $\mathcal{P}_e(G)$ belong to the same maximal cyclic subgroup. As a result, $\alpha(\mathcal{P}_e(G)) \leq \mu(G)$. Thus we conclude that $\alpha(\mathcal{P}_e(G)) = \mu(G)$.

Next let G be a nilpotent group and P_i be a Sylow-p_i subgroup of G for $1 \leq i \leq r$. By Lemma 3.1.3, we have $G = P_1P_2 \cdots P_r$. Then H is a maximal cyclic subgroup of G if and only if $H = H_1H_2 \cdots H_r$, where H_i is a maximal cyclic subgroup of P_i for $1 \leq i \leq r$ (see Chattopadhyay et al. [2020b]).

Let H_i, H_i' are maximal cyclic subgroups of P_i for $1 \leq i \leq r$. If $H_k \neq H_k'$ for any $1 \leq k \leq r$, then $H_1H_2 \cdots H_r \neq H_1'H_2' \cdots H_r'$. This is because the generators of H_k belong to $H_1H_2 \cdots H_r$, but not to $H_1'H_2' \cdots H_r'$. Therefore, if the number of maximal subgroups of P_i is m_i, then $\alpha(\mathcal{P}_e(G)) = m_1m_2 \cdots m_r$. □

In view of Lemma 1.2.10(i), we have the following consequence of Theorem 3.1.4.

Corollary 3.1.5. *For any finite group G, the vertex covering number $\beta(\mathcal{P}_e(G)) = |G| - \mu(G)$, where $\mu(G)$ is the number of maximal cyclic subgroups of G.*

We next compute values and bounds of the matching number of enhanced power graphs of finite groups. First we recall some necessary definition and notations. For any $x \in G$, $\langle x \rangle$ is the cyclic subgroup of G generated by x. For any $x, y \in G$, we write $x \approx y$ if $\langle x \rangle = \langle y \rangle$. Observe that $\approx$ is an equivalence relation on G. We denote by $[x]$ the $\approx$-class containing x. Note that $[x]$ is precisely the set of generators of $\langle x \rangle$. For any $x \in G$ with $o(x) \geq 3$, the set $[x]$ of vertices is a clique in $\mathcal{P}_e(G)$. Since $|[x]| = \phi(o(x))$, we have a matching, denoted by M_x, of order $\frac{\phi(o(x))}{2}$ consisting of edges with ends in $[x]$.

Theorem 3.1.6. *Let G be a finite group. If G is of odd order, then $\alpha'(\mathcal{P}_e(G)) = \frac{|G|-1}{2}$. If G is of even order, then*

$$\frac{|G| - (t-1)}{2} \leq \alpha'(\mathcal{P}_e(G)) \leq \frac{|G|}{2},$$

where t is the number of involutions in G.

Proof. First let G be of odd order. Observe that for distinct $x_1, x_2 \in G\setminus\{e\}$, either $[x_1] = [x_2]$ or $[x_1] \cap [x_2] = \emptyset$. Accordingly, either $M_{x_1} = M_{x_2}$ or $M_{x_1} \cap M_{x_2} = \emptyset$. Hence $M := \cup_{x\in G\setminus\{e\}} M_x$ is a matching of order $\frac{|G|-1}{2}$ in $\mathcal{P}_e(G)$. On the other hand, the order of a largest matching in a graph of order n is $\left\lfloor \frac{n}{2} \right\rfloor$. Hence we get $\alpha'(\mathcal{P}_e(G)) = \frac{|G|-1}{2}$.

Now suppose G is of even order. Then it has at least one involution, say y. We denote the edge with ends e and y by ϵ. Then $M = \{\epsilon\} \bigcup \cup_{x\in G, o(x)\geq 3} M_x$ is a matching of order $\frac{|G|-(t-1)}{2}$ in $\mathcal{P}_e(G)$, where t is the number of involutions in G. Additionally, as $\alpha'(\mathcal{P}_e(G)) \leq \frac{|G|}{2}$ holds trivially, we get the desired inequality when G is of even order. $\square$

Considering Lemma 1.2.10(ii), we have the following corollary of Theorem 3.1.6.

Corollary 3.1.7. *Let G be a finite group. If G is of odd order, then $\beta'(\mathcal{P}_e(G)) = \frac{|G|+1}{2}$. If G is of even order, then*

$$\frac{|G|}{2} \leq \beta'(\mathcal{P}_e(G)) \leq \frac{|G|+(t-1)}{2},$$

where t is the number of involutions in G.

From the two preceding results, for any finite group G with a unique involution, $\alpha'(\mathcal{P}_e(G)) = \beta'(\mathcal{P}_e(G)) = \frac{|G|}{2}$. Recall that for any prime p, a finite p-group G has exactly one subgroup of order p if and only if G is cyclic, or $p = 2$ and G is generalized quaternion (see Robinson [1996]). These facts along with Lemma 3.1.3 yield the following corollary.

Corollary 3.1.8. *If G is a nilpotent group with a cyclic or generalized quaternion Sylow-2 subgroup, then $\alpha'(\mathcal{P}_e(G)) = \beta'(\mathcal{P}_e(G)) = \frac{|G|}{2}$.*

Now we investigate various structural properties of enhanced power graphs of the groups under consideration.

3.1.1 Finite p-Group.

Throughout this subsection, p denotes a prime number. The following lemmas will be useful in the sequel.

Lemma 3.1.9 ([Aalipour et al., 2017, Theorem 28]). *For any finite group G, $\mathcal{P}_e(G) = \mathcal{P}(G)$ if and only if every cyclic subgroup of G has prime power order.*

Lemma 3.1.10 ([Panda and Krishna, 2018a, Proposition 3.2]). *For any prime p and finite p-group G, each component of $\mathcal{P}(G\backslash\{e\})$ has exactly $p-1$ elements of order p.*

Lemma 3.1.11 ([Cameron, 2010, Proposition 4]). *Let G be a finite group, and S be the set of vertices of $\mathcal{P}(G)$ that are adjacent to all other vertices. If $|S| > 1$, then one of the following occurs.*

(i) *G is cyclic of prime power order,and $S = G$*

(ii) *G is cyclic of non-prime-power order n and S consists of the identity and the generators of G*

(iii) *G is generalized quaternion and S contains the identity and the unique involution.*

In view of Lemma 3.1.9, we have the following straightforward remark.

Remark 3.1.12. For any finite p-group G, the graphs $\mathcal{P}(G)$ and $\mathcal{P}_e(G)$ are equal.

Accordingly, we consider power graphs of p-groups in this subsection. Let G be a finite p-group. Then $\mathcal{P}(G)$ is perfect, since more generally, the power graph of any finite group is perfect (see Doostabadi et al. [2015]). It is known that a finite abelian group G is isomorphic to a unique direct product of cyclic groups of prime power order. In this product, let $\tau(G)$ be the order of the smallest cyclic group. Then the following is a consequence of Theorem 3.1.2.

Theorem 3.1.13 ([Panda and Krishna, 2018b, Theorem 5.1])**.** *For any finite abelian p-group G, the minimum degree of $\mathcal{P}(G)$ is $\tau(G) - 1$.*

Before proceeding further, we are required to fix some notations.

Notation 3.1.14. Consider a prime p and positive integers $\alpha_1 > \cdots > \alpha_s$ and $m_1, \ldots, m_s$. For any $1 \leq j \leq s$, we denote $n_j = p^{\sum_{i=1}^{j} m_i \alpha_i}$, $r_j = \sum_{i=1}^{j} m_i$, and that $n = n_s$ and $r = r_s$. Additionally, we write $n_0 = 1$ and $r_0 = 0$ for consistency.

Lemma 3.1.15 ([Ma, Fu and Lu, 2018, Corollary 2.11])**.** *If G is a finite p-group, then $\alpha(\mathcal{P}(G)) = |\mathcal{M}(G)|$.*

Note that Theorem 3.1.4 and Lemma 3.1.9 together yield the above lemma as well. In the next theorem, we compute $|\mathcal{M}(G)|$ and thus $\alpha(\mathcal{P}(G))$ for any finite abelian p-group G. To state as well as to prove it, we follow Notation 3.1.14 throughout. Moreover, as every finite abelian group is a direct product of cyclic groups of prime power order, assume in the following that $G \cong \mathbb{Z}_{p^{\alpha_1}}^{m_1} \times \cdots \times \mathbb{Z}_{p^{\alpha_s}}^{m_s}$.

Theorem 3.1.16. *For any finite abelian p-group G,*

$$\alpha(\mathcal{P}(G)) = \sum_{t=1}^{r} \frac{n}{p^{t-1}} \left\{ \frac{p^{(r_k-1)\alpha_k}}{n_k} + \mu_k \right\},$$

where

$$\mu_k = \begin{cases} \sum_{j=1}^{k-1} \dfrac{p^{r_j} - 1}{n_j} \sum_{\beta=\alpha_{j+1}}^{\alpha_j - 1} p^{(r_j-1)\beta} & \textit{if } k > 1, \\ 0 \ \textit{if } k = 1, \end{cases}$$

and k is such that $r_{k-1} < t \leq r_k$.

Proof. It is enough to prove the theorem for $G = \mathbb{Z}_{p^{\alpha_1}}^{m_1} \times \cdots \times \mathbb{Z}_{p^{\alpha_s}}^{m_s}$. Notice that $|G| = n$ and G is direct a product of r cyclic subgroups. If G is cyclic, then $\alpha(\mathcal{P}(G)) = 1$ since $\mathcal{P}(G)$ is complete. Now let G be noncyclic, that is, $r \geq 2$.

By Lemma 3.1.15, the objective is to compute $|\mathcal{M}(G)|$. For that, we shall partition $\mathcal{M}(G)$ as follows. For any $x \in G$, we denote the ith component of x by x_i.

For any maximal cyclic subgroup $\langle x\rangle$ of G, observe that $(x_i, p) = 1$ for at least one $1 \leq i \leq r$. For any $1 \leq t \leq r$, we define $\mathcal{M}_t = \{\langle x\rangle \in \mathcal{M}(G) : (x_t, p) = 1 \text{ and } (x_1, p) \neq 1, \ldots, (x_{t-1}, p) \neq 1 \text{ if } t > 1\}$. Thus $\alpha(\mathcal{P}(G)) = \sum_{t=1}^{r} |\mathcal{M}_t|$.

Now we fix $1 \leq t \leq r$ and compute $|\mathcal{M}_t|$. We have $t = r_{k-1} + m$ for some $1 \leq k \leq s$ and $1 \leq m \leq m_k$. So that if $\langle x\rangle \in \mathcal{M}_t$, then $o(x) \geq p^{\alpha_k}$. Next for any $\beta \geq \alpha_k$, we define $\mathcal{M}_{t,p^\beta} = \{\langle x\rangle \in \mathcal{M}_t : o(x) = p^\beta\}$. Then we have,

$$
\begin{aligned}
&|\{x : \langle x\rangle \in \mathcal{M}_{t,p^{\alpha_k}}\}| \\
&= p^{r_{k-1}\alpha_k} \cdot p^{(m-1)(\alpha_k-1)} \cdot \phi(p^{\alpha_k}) \cdot \frac{n}{n_{k-1} \cdot p^{m\alpha_k}} \\
&= \frac{n\phi(p^{\alpha_k})}{p^{\{\sum_{i=1}^{k} m_i(\alpha_i-\alpha_k)\}+\alpha_k+m-1}}.
\end{aligned}
$$

Hence,

$$
|\mathcal{M}_{t,p^{\alpha_k}}| = \frac{n}{p^{\{\sum_{i=1}^{k} m_i(\alpha_i-\alpha_k)\}+\alpha_k+m-1}}. \tag{3.1}
$$

Suppose that $t \leq m_1$. Then $k = 1$ and that $t = m$. Moreover, $o(x) = p^{\alpha_1}$ for any $\langle x\rangle \in \mathcal{M}_t$, so that $\mathcal{M}_t = \mathcal{M}_{t,p^{\alpha_k}}$. Thus we have

$$
|\mathcal{M}_t| = \frac{n}{p^{\alpha_1+t-1}} = \frac{n}{p^{t-1}}\left\{\frac{p^{(r_1-1)\alpha_1}}{n_1}\right\}.
$$

Consequently, the proof follows for $t \leq m_1$.

For remaining of the proof, we take $t > m_1$, that is, $k > 1$. We observe that $p^{\alpha_k} \leq o(x) \leq p^{\alpha_1-1}$ for any $\langle x\rangle \in \mathcal{M}_t$. Let $\beta > \alpha_k$ be such that $\alpha_{l+1} \leq \beta \leq \alpha_l - 1$

for some $1 \leq l \leq k-1$. Then

$$\begin{aligned}
&|\{x : \langle x \rangle \in \mathcal{M}_{t,p^\beta}\}| \\
&= \{\phi(p^\beta)(p^\beta)^{m_1+\cdots+m_l-1} + p^{\beta-1}\phi(p^\beta)(p^\beta)^{m_1+\cdots+m_l-2} + \cdots + (p^{\beta-1})^{m_1+\cdots+m_l-1}\phi(p^\beta)\} \\
&\qquad \cdot p^{m_{l+1}(\alpha_{l+1}-1)} \cdots p^{m_{k-1}(\alpha_{k-1}-1)} \cdot \frac{n \cdot p^{(m-1)(\alpha_k-1)} \cdot \phi(p^{\alpha_k})}{p^{m_1\alpha_1} \cdots p^{m_{k-1}\alpha_{k-1}} \cdot p^{m\alpha_k}} \\
&= p^{(m_1+\cdots+m_l-1)(\beta-1)} \left(\frac{p^{m_1+\cdots+m_l}-1}{p-1}\right) \\
&\qquad \cdot p^{m_{l+1}(\alpha_{l+1}-1)} \cdots p^{m_{k-1}(\alpha_{k-1}-1)} \cdot \frac{n \cdot p^{(m-1)(\alpha_k-1)} \cdot \phi(p^{\alpha_k}) \cdot \phi(p^\beta)}{p^{m_1\alpha_1} \cdots p^{m_{k-1}\alpha_{k-1}} \cdot p^{m\alpha_k}} \\
&= p^{(m_1+\cdots+m_l-1)(\beta-1)} \left(p^{m_1+\cdots+m_l}-1\right) \\
&\qquad \cdot p^{m_{l+1}(\alpha_{l+1}-1)} \cdots p^{m_{k-1}(\alpha_{k-1}-1)} \cdot \frac{n \cdot p^{m(\alpha_k-1)} \cdot \phi(p^\beta)}{p^{m_1\alpha_1} \cdots p^{m_{k-1}\alpha_{k-1}} \cdot p^{m\alpha_k}} \\
&= \frac{n\phi(p^\beta)\left(p^{\sum_{i=1}^{l} m_i}-1\right)}{p^{\{\sum_{i=1}^{l} m_i(\alpha_i-\beta)\}+\sum_{i=1}^{k} m_i - m_k+\beta+m-1}}.
\end{aligned}$$

Hence,

$$|\mathcal{M}_{t,p^\beta}| = \frac{n\left(p^{\sum_{i=1}^{l} m_i}-1\right)}{p^{\{\sum_{i=1}^{l} m_i(\alpha_i-\beta)\}+\sum_{i=1}^{k} m_i - m_k+\beta+m-1}}.$$

From (3.1), we get

$$|\mathcal{M}_{t,p^{\alpha_k}}| = \frac{n}{p^{\sum_{i=1}^{k} m_i(\alpha_i-\alpha_k)+\sum_{i=1}^{k-1} m_i+\alpha_k+m-1}} + \frac{n\left(p^{\sum_{i=1}^{k-1} m_i}-1\right)}{p^{\sum_{i=1}^{k} m_i(\alpha_i-\alpha_k)+\sum_{i=1}^{k-1} m_i+\alpha_k+m-1}}.$$

Additionally,

$$\begin{aligned}
&\sum_{j=1}^{k-1} \sum_{\beta=\alpha_{j+1}}^{\alpha_j-1} \frac{n\left(p^{\sum_{i=1}^{j} m_i}-1\right)}{p^{\sum_{i=1}^{j} m_i(\alpha_i-\beta)+\sum_{i=1}^{k-1} m_i+\beta+m-1}} \\
&= \sum_{j=1}^{k-1} \frac{n\left(p^{\sum_{i=1}^{j} m_i}-1\right)}{p^{\sum_{i=1}^{j} m_i\alpha_i+\sum_{i=1}^{k-1} m_i+m-1}} \sum_{\beta=\alpha_{j+1}}^{\alpha_j-1} p^{(\sum_{i=1}^{j} m_i-1)\beta} \\
&= \frac{n}{p^{\sum_{i=1}^{k-1} m_i+m-1}} \sum_{j=1}^{k-1} \frac{p^{\sum_{i=1}^{j} m_i}-1}{p^{\sum_{i=1}^{j} m_i\alpha_i}} \sum_{\beta=\alpha_{j+1}}^{\alpha_j-1} p^{(\sum_{i=1}^{j} m_i-1)\beta}.
\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
|\mathcal{M}_t| &= \sum_{j=1}^{k-1} \sum_{\beta=\alpha_{j+1}}^{\alpha_j - 1} |\mathcal{M}_{t,p^\beta}| \\
&= \frac{n}{p^{\sum_{i=1}^{k} m_i(\alpha_i - \alpha_k) + \sum_{i=1}^{k-1} m_i + \alpha_k + m - 1}} + \sum_{j=1}^{k-1} \sum_{\beta=\alpha_{j+1}}^{\alpha_j - 1} \frac{n\left(p^{\sum_{i=1}^{j} m_i} - 1\right)}{p^{\sum_{i=1}^{j} m_i(\alpha_i - \beta) + \sum_{i=1}^{k-1} m_i + \beta + m - 1}} \\
&= \frac{n}{p^{\sum_{i=1}^{k-1} m_i + m - 1}} \left\{ \frac{1}{p^{\sum_{i=1}^{k} m_i(\alpha_i - \alpha_k) + \alpha_k}} + \sum_{j=1}^{k-1} \frac{p^{\sum_{i=1}^{j} m_i} - 1}{p^{\sum_{i=1}^{j} m_i \alpha_i}} \sum_{\beta=\alpha_{j+1}}^{\alpha_j - 1} p^{(\sum_{i=1}^{j} m_i - 1)\beta} \right\} \\
&= \sum_{t=1}^{r} \frac{n}{p^{t-1}} \left\{ \frac{p^{(r_k - 1)\alpha_k}}{n_k} + \sum_{j=1}^{k-1} \frac{p^{r_j} - 1}{n_j} \sum_{\beta=\alpha_{j+1}}^{\alpha_j - 1} p^{(r_j - 1)\beta} \right\}.
\end{aligned}
$$

This concludes the proof of the theorem. □

Since every abelian group is nilpotent, from Lemma 3.1.3 and Theorem 3.1.16, we have the following corollary.

Corollary 3.1.17. *For any finite abelian group G with the Sylow subgroups $P_1, P_2, \ldots, P_r$ of G, the independence number $\alpha(\mathcal{P}_e(G)) = \prod_{i=1}^{r} \mu(P_i)$, where $\mu(P_i)$ can be computed using Theorem 3.1.16 for $1 \le i \le r$.*

The following theorem computes the matching number of enhanced power graphs of finite groups.

Theorem 3.1.18. *For any finite p-group G,*

$$
\alpha'(\mathcal{P}(G)) = \begin{cases} \dfrac{|G| - 1}{2} & p > 2, \\ \dfrac{|G| - (t-1)}{2} & p = 2, \end{cases}
$$

where t is the number of involutions in G.

Proof. For $p > 2$, the result follows from Theorem 3.1.6. So for the rest of the proof, we take $p = 2$.

Observe that the only common vertex between any two distinct blocks in $\mathcal{P}(G)$ is e. Thus the number of blocks in $\mathcal{P}(G)$ coincides with the number of components of $\mathcal{P}(G\backslash\{e\})$. Following Lemma 3.1.10, every block in $\mathcal{P}(G)$ has exactly one vertex of order two. So that the number of blocks in $\mathcal{P}(G)$ coincides with the number of involutions, say t, in G. Moreover, in light of Lemma 1.1.9, the number of vertices in any block in $\mathcal{P}(G)$ is even.

Consider a matching M in $\mathcal{P}(G)$. Let M' be the subset of M containing all elements of M whose endpoints are non identity elements of G. Let B be any block in $\mathcal{P}(G)$. Since $|V(B)|$ is even, $E(B)$ contains at most $\dfrac{V(B)-2}{2}$ elements of M'. From this and the fact that e is a vertex in every block in $\mathcal{P}(G)$, we have $|M'| \leq \dfrac{|G|-(t+1)}{2}$. Additionally, M contains at most one edge with e as an endpoint. Thus we have $|M| \leq |M'|+1 \leq \dfrac{|G|-(t-1)}{2}$.

Therefore, to prove the theorem, it is enough to produce a matching of cardinality $\dfrac{|G|-(t-1)}{2}$. Let ϵ be an edge with e as one end and the other an involution. Recall the definition of M_x for any $x \in G$. We consider $M_G := \{\epsilon\} \bigcup \cup_{x\in S,\, o(x)>2} M_x$, where S is a subset of G containing exactly one element from each $\approx$-class. Then all elements of M_G, except ϵ, have both ends in same $\approx$-class, and ϵ does not have common endpoints with any other element M_G. Hence M_G is a matching in $\mathcal{P}(G)$. Finally, as $|M_G| = \dfrac{|G|-(t+1)}{2} + 1 = \dfrac{|G|-(t-1)}{2}$, the proof follows. □

Theorem 3.1.19. *Let G be any finite abelian p-group with exponent p^α. Then the strong metric dimension of $\mathcal{P}_e(G)$ is*

(i) $|G|-(\alpha+1)$ *if G is non-cyclic,*

(ii) $|G|-1$ *if G is cyclic.*

Proof. The proof is straightforward when G is cyclic. Now let G be noncyclic. Then $G \cong C_1 \times \cdots \times C_r$ for some cyclic p-groups $C_1, \ldots, C_r$, $r \geq 2$. Consider two distinct elements $x = (x_1, x_2, \ldots, x_r)$ and $y = (y_1, y_2, \ldots, y_r)$ in G with $o(x) \geq o(y)$. Suppose that $\mathrm{N}[x] = \mathrm{N}[y]$. Clearly, $x \sim y$.

If $y = e$, we have $\mathrm{N}[x] = \mathrm{N}[y] = G$. However, this is impossible in view of Lemma

3.1.11. Analogous situation occurs when $x = e$. Hence both x and y are nonidentity elements.

If possible, let $o(x) = 2$. Then $o(y) = 2$, since $o(x) \geq o(y)$. As $x \sim y$, we thus have $\langle x \rangle = \langle y \rangle$. That is, $x = y$, which is a contradiction.

So that $o(x) \geq 3$. Now suppose $o(x) > o(y)$. Then as $x \sim y$, there exists an integer t with $p \mid t$ such that $y = x^t$. We have the following cases.

Case 1: $x_i = e$ for some fixed $1 \leq i \leq r$. We define an element $z = (z_1, z_2, \dots, z_r)$ such that z_i is an element of order p in C_i and $z_j = x_j$ for $1 \leq j \leq r$, $j \neq i$. Then notice that $o(z) = o(x)$ and $\langle z \rangle \neq \langle x \rangle$. As a result, $z \nsim x$. However, as $y = z^t$, we have $z \sim y$. This contradicts our assumption that $\mathrm{N}[x] = \mathrm{N}[y]$.

Case 2: $x_j \neq e$ for all $1 \leq j \leq r$. Let $1 \leq k \leq r$ be such that $o(x_k) \leq o(x_j)$ for all $1 \leq j \leq r$. We define an element $w = (w_1, w_2, \dots, w_r)$ such that $w_k = {x_k}^{p^{\beta-1}+1}$, where $o(x_k) = p^\beta$, and $w_j = x_j$ for $1 \leq j \leq r$, $j \neq k$. Then $y = w^t$, so that $w \sim y$. Moreover, as $o(w) = o(x)$, we have $w = x^s$ for some $(s, p) = 1$. Accordingly, $o(x_i) \mid (s-1)$ for all $1 \leq j \leq r$, $j \neq k$. Since $o(x_k) \leq o(x_j)$ for all $1 \leq j \leq r$, we thus get $p^\beta \mid (s-1)$. This implies ${x_k}^{p^{\beta-1}+1} = x_k$, which is not possible. As a result, $w \nsim x$. This again results in a contradiction.

Consequently, $o(x) = o(y)$. Hence as $x \sim y$, we have $\langle x \rangle = \langle y \rangle$.

Since converse is trivial, we therefore conclude that $\mathrm{N}[x] = \mathrm{N}[y]$ if and only if $o(x) \geq 3$ and $\langle x \rangle = \langle y \rangle$. From Lemma 3.1.11, we get $\widehat{e} = \{e\}$. Moreover, if $p = 2$, then $\widehat{x} = \{x\}$ for every element x of order 2 in G. Hence the $\equiv$-classes and $\approx$-classes coincide for every $x \in G$.

Now consider a clique C in $\widehat{\mathcal{P}}(G)$ with at least two vertices. Then for any pair of distinct vertices $\widehat{x}$, $\widehat{y}$ in C, we have $o(x) \neq o(y)$. Additionally, for any $x \in G$, we have $o(x) = p^i$ for some $0 \leq i \leq \alpha$. Thus $\omega(\widehat{\mathcal{P}}(G)) \leq \alpha + 1$. Since G is p-group of exponent p^α, there exists $z \in G$ of order p^α. We observe that $\{\widehat{e}\} \cup \{\widehat{z^{p^i}} : 0 \leq i \leq \alpha - 1\}$ is a clique in $\widehat{\mathcal{P}}(G)$. Therefore, we get $\omega(\widehat{\mathcal{P}}(G)) = \alpha + 1$ and subsequently, $\mathrm{sdim}(\mathcal{P}(G)) = |G| - (\alpha + 1)$, by Theorem 1.2.15. □

3.1.2 The Group U_{6n}

For $n \geq 1$, the group U_{6n} of order $6n$ is defined in James and Liebeck [1993] as the group generated by the elements a and b such that $a^{2n} = b^3 = e, ba = ab^{-1}$ i.e.

$$U_{6n} = \langle a, b : a^{2n} = b^3 = e, ba = ab^{-1} \rangle.$$

Further, in order to discuss the existence of an orthogonal basis associated with U_{6n}, the conjugacy classes and the characters of U_{6n} have been obtained in Darafsheh and Poursalavati [2001]. To investigate the graph invariants of $\mathcal{P}_e(U_{6n})$ first we study the structure of U_{6n}.

Remark 3.1.20. The group U_{6n} is of order $6n$ if and only if $b \notin \langle a \rangle$.

Since $ba = ab^{-1}$, for any $0 \leq i \leq 2n-1$, we have

$$ba^i = \begin{cases} a^i b & \text{if } i \text{ is even,} \\ a^i b^2 & \text{if } i \text{ is odd,} \end{cases} \quad \text{and} \quad b^2 a^i = \begin{cases} a^i b^2 & \text{if } i \text{ is even,} \\ a^i b & \text{if } i \text{ is odd.} \end{cases}$$

Thus every element of $U_{6n} \backslash \langle a \rangle$ is of the form $a^i b^j$ for some $0 \leq i \leq 2n-1$ and $1 \leq j \leq 2$.

Moreover, for any $0 \leq i \leq 2n-1$,

$$(ab)^i = \begin{cases} a^i & \text{if } i \text{ is even,} \\ a^i b & \text{if } i \text{ is odd,} \end{cases} \quad \text{and} \quad (ab^2)^i = \begin{cases} a^i & \text{if } i \text{ is even,} \\ a^i b^2 & \text{if } i \text{ is odd.} \end{cases} \tag{3.2}$$

Consequently, we have the following remarks.

Remark 3.1.21. For $x \in U_{6n}$, we have $x = a^{2s+1}b$ for some $0 \leq s \leq n-1$ if and only if $x \in \langle ab \rangle \backslash \langle a \rangle$.

Remark 3.1.22. For $x \in U_{6n}$, we have $x = a^{2s+1}b^2$ for some $0 \leq s \leq n-1$ if and only if $x \in \langle ab^2 \rangle \backslash \langle a \rangle$.

Remark 3.1.23. Every element of $U_{6n} \backslash (\langle a \rangle \cup \langle ab \rangle \cup \langle ab^2 \rangle)$ is of the form $a^{2i}b$ and $a^{2j}b^2$ for some i, j.

From the presentation of U_{6n} and by mathematical induction, we have

$$(a^{2\cdot3^i}b)^j = \begin{cases} a^{2\cdot3^i\cdot j} & \text{if } j \equiv 0 (\text{mod } 3), \\ a^{2\cdot3^i\cdot j}b & \text{if } j \equiv 1 (\text{mod } 3), \\ a^{2\cdot3^i\cdot j}b^2 & \text{if } j \equiv 2 (\text{mod } 3), \end{cases} \quad \text{and} \quad (a^{2\cdot3^i}b^2)^j = \begin{cases} a^{2\cdot3^i\cdot j} & \text{if } j \equiv 0 (\text{mod } 3), \\ a^{2\cdot3^i\cdot j}b^2 & \text{if } j \equiv 1 (\text{mod } 3), \\ a^{2\cdot3^i\cdot j}b & \text{if } j \equiv 2 (\text{mod } 3). \end{cases} \tag{3.3}$$

In the rest of this subsection, we shall write $n = 3^k t$ for integers $k \geq 0$ and $t > 0$ such that $3 \nmid t$.

By Remark 3.1.23 and (3.3), we have the following lemma.

Lemma 3.1.24. *For $n \geq 1$, we have* $U_{6n} = \langle a \rangle \cup \langle ab \rangle \cup \langle ab^2 \rangle \cup \bigcup_{i=0}^{k} \left(\langle a^{2\cdot3^i}b \rangle \cup \langle a^{2\cdot3^i}b^2 \rangle \right).$

For the remainder of this subsection, we shall denote $P_i = \langle a^{2\cdot3^i}b \rangle$, $Q_i = \langle a^{2\cdot3^i}b^2 \rangle$ for $0 \leq i \leq k$, and $P_{k+1} = \langle ab \rangle$, $Q_{k+1} = \langle ab^2 \rangle$.

Thus we have

$$U_{6n} = \langle a \rangle \cup \bigcup_{i=0}^{k} (P_i \cup Q_i) \cup (P_{k+1} \cup Q_{k+1}). \tag{3.4}$$

We observe that that $\{b, a^{2\cdot3^k}\} \subset \langle a^{2\cdot3^k}b \rangle \cap \langle a^{2\cdot3^k}b^2 \rangle$. As consequences, we have the following remarks.

Remark 3.1.25. For $i = k$, we have $P_i = Q_i$.

Remark 3.1.26. For $0 \leq i \leq k-1$, we have $\langle a^{2\cdot3^{i+1}} \rangle \subset \langle a^{2\cdot3^i} \rangle$.

Remark 3.1.27. For $i = k$, we have $\langle a^{2\cdot3^{i+1}} \rangle = \langle a^{2\cdot3^i} \rangle$.

Since $o(a) = 2n$, we have the following remark.

Remark 3.1.28. In U_{6n}, we have $o(a^{2\cdot3^k}) = t$.

In view of (3.2) and (3.3), we have the following lemma.

Lemma 3.1.29. *For $i \leq k$, we have $P_i \cap \langle a \rangle = Q_i \cap \langle a \rangle = \langle a^{2\cdot 3^{i+1}} \rangle$. Moreover, $P_k \cap \langle a \rangle = \langle a^{2\cdot 3^k} \rangle$.*

Proof. By (3.3), we have $(a^{2\cdot 3^i}b)^j \in P_i \cap \langle a \rangle$ if and only if $j = 3l$ for some l. Then $P_i \cap \langle a \rangle = \langle (a^{2\cdot 3^i}b)^3 \rangle = \langle a^{2\cdot 3^{i+1}} \rangle$. Similarly, we have $Q_i \cap \langle a \rangle = \langle a^{2\cdot 3^{i+1}} \rangle$. For $i = k$, we have $P_k \cap \langle a \rangle = \langle a^{2\cdot 3^{k+1}} \rangle$. Since $a^{2\cdot 3^{k+1}} \in \langle a^{2\cdot 3^k} \rangle$, and

$$a^{2\cdot 3^k} = \begin{cases} \left(a^{2\cdot 3^{k+1}}\right)^{\frac{2t+1}{3}} & \text{if } t \equiv 1 \text{mod } 3, \\ \left(a^{2\cdot 3^{k+1}}\right)^{\frac{t+1}{3}} & \text{if } t \equiv 2 \bmod 3, \end{cases}$$

we get $P_k \cap \langle a \rangle = \langle a^{2\cdot 3^k} \rangle$. □

Lemma 3.1.30. *For the group U_{6n}, we have*

(i) $|P_i| = |Q_i| = 3^{k-i}t$, *where* $0 \leq i \leq k-1$. *Moreover,* $|P_i \backslash \langle a \rangle| = |Q_i \backslash \langle a \rangle| = 2 \cdot 3^{k-i-1}t$

(ii) $|P_k| = |Q_k| = 3t$. *Moreover,* $|P_k \backslash \langle a \rangle| = |Q_k \backslash \langle a \rangle| = 2t$.

(iii) $|P_{k+1}| = |Q_{k+1}| = 2n$. *Moreover,* $|P_{k+1} \backslash \langle a \rangle| = |Q_{k+1} \backslash \langle a \rangle| = n$.

Proof. (i) Since $P_i = \langle a^{2\cdot 3^i}b \rangle$, we have $(a^{2\cdot 3^i}b)^{3^{k-i}t} = e$. If $l < 3^{k-i}t$, then by (3.3) and Remark 3.1.20, $(a^{2\cdot 3^i}b)^l \neq e$. So that $|P_i| = 3^{k-i}t$. Now we observe that $o(a^{2\cdot 3^{i+1}}) = 3^{k-i-1}t$. Thus we get $|P_i \backslash \langle a \rangle| = 2 \cdot 3^{k-i-1}t$, by Lemma 3.1.29. Similarly, $|Q_i \backslash \langle a \rangle| = 2 \cdot 3^{k-i-1}t$.

(ii) Since $P_k = \langle a^{2\cdot 3^k}b \rangle$, by (3.3), we have $a^{2\cdot 3^k} \in \{(a^{2\cdot 3^k}b)^{2t+1}, (a^{2\cdot 3^k}b)^{t+1}\}$ and $(a^{2\cdot 3^k}b)^t \in \{b, b^2\}$. Consequently, $a^{2\cdot 3^k}, b \in \langle a^{2\cdot 3^k}b \rangle$. By Remark 3.1.28 and as $o(a^{2\cdot 3^k}) = t$, it follows that $(a^{2\cdot 3^k})^i b, (a^{2\cdot 3^k})^j b^2$ and $(a^{2\cdot 3^k})^l$ are all distinct elements in $\langle a^{2\cdot 3^k}b \rangle$, where $1 \leq i, j, l \leq t$. As a result, $o(a^{2\cdot 3^k}b) \geq 3t$. Since $(a^{2\cdot 3^k}b)^{3t} = e$, we obtain $o(a^{2\cdot 3^k}b) = 3t = |P_k|$. Accordingly, by Lemma 3.1.29 and Remark 3.1.28, $|P_k \backslash \langle a \rangle| = 2t$. The proof for Q_k is similar.

(iii) The proof is straightforward following (3.2). □

The next result describes structure of U_{6n} further.

Proposition 3.1.31. *For $0 \leq i \leq k+1$, the following hold:*

(i) *If $x \in P_i \backslash \langle a \rangle$, then $x \notin \left(\bigcup_{j=0, j \neq i}^{k+1} P_j \right) \cup \left(\bigcup_{j=0}^{k+1} Q_j \right)$*

(ii) *If $x \in Q_i \backslash \langle a \rangle$, then $x \notin \left(\bigcup_{j=0}^{k+1} P_j \right) \cup \left(\bigcup_{j=0, j \neq i}^{k+1} Q_j \right)$.*

Proof. (i) If possible, let $x \in U_{6n} \backslash P_i$. Then in view of (3.4), we have the followings.

Case 1: $x \in Q_i$. For $i = k+1$, we have $x \in \langle ab \rangle \cap \langle ab^2 \rangle$. By Remark 3.1.21 and Remark 3.1.22, we have $x = a^{2r+1}b$ and $x = a^{2s+1}b^2$ for some r, s. Consequently, $b \in \langle a \rangle$, a contradiction of Remark 3.1.20. Suppose $0 \leq i \leq k$. Then by (3.4), $x = (a^{2 \cdot 3^i} b)^p$ and $x = (a^{2 \cdot 3^i} b^2)^q$, where $p, q \leq 3^{k-i}t$ (see Lemma 3.1.30(i)). Clearly, $3 \nmid p$ and $3 \nmid q$. Otherwise, $x \in \langle a \rangle$, a contradiction. If $p \equiv 1 (\text{mod } 3)$ and $q \equiv 2 (\text{mod } 3)$, then by (3.3), $x = a^{2 \cdot 3^i \cdot p} b$ and $x = a^{2 \cdot 3^i \cdot q} b$. Since $p, q \leq 3^{k-i}t$, we have $2 \cdot 3^i \cdot p \leq 2n$ and $2 \cdot 3^i \cdot q \leq 2n$. Consequently, $a^{2 \cdot 3^i \cdot p} b = a^{2 \cdot 3^i \cdot q} b$ gives $p = q$, which is not possible. Similarly, we get $p = q$ for the case $p \equiv 2 (\text{mod } 3)$ and $q \equiv 1 (\text{mod } 3)$, again a contradiction. For the case $p \equiv 1 (\text{mod } 3)$ and $q \equiv 1 (\text{mod } 3)$, we have $x = a^{2 \cdot 3^i \cdot p} b$ and $x = a^{2 \cdot 3^i \cdot q} b^2$. Consequently, $b \in \langle a \rangle$, a contradiction. We get a similar contradiction for the case $p \equiv 2 (\text{mod } 3)$ and $q \equiv 2 (\text{mod } 3)$.

Case 2: $x \in P_j$ with $j \neq i$. If $j = k+1$, then by Remark 3.1.21, we have $x = a^l b$ for some odd l. Since $j \neq i$, we get either $x = a^{l'} b$ or $x = a^{l'} b^2$ for some even l', a contradiction. Similarly, we get a contradiction when $i = k+1$. Let $0 \leq i, j \leq k$. Then $x = (a^{2 \cdot 3^i} b)^u$ and $x = (a^{2 \cdot 3^j} b)^v$ for some $u \leq 3^{k-i}t$ and $v \leq 3^{k-j}t$ (see Lemma 3.1.30 (i)). Clearly, u, v are not divisible by 3. Otherwise, $x \in \langle a \rangle$, a contradiction. If $u \equiv 1 (\text{mod } 3)$ and $v \equiv 2 (\text{mod } 3)$, then by (3.3), we get $x = a^{2 \cdot 3^i \cdot u} b$ and $x = a^{2 \cdot 3^j \cdot v} b^2$. Consequently, $b \in \langle a \rangle$, a contradiction of Lemma 3.1.20. Similarly, we get a contradiction if $u \equiv 2 (\text{mod } 3)$ and $v \equiv 1 (\text{mod } 3)$. For the case $u \equiv 1 (\text{mod } 3)$ and $v \equiv 1 (\text{mod } 3)$, we have $x = a^{2 \cdot 3^i \cdot u} b$ and $x = a^{2 \cdot 3^j \cdot v} b$. Since $u \leq 3^{k-i}t$ and

$v \leq 3^{k-j}t$, we have $2 \cdot 3^i \cdot u \leq 2n$ and $2 \cdot 3^j \cdot v \leq 2n$. Consequently, $a^{2\cdot 3^i \cdot u}b = a^{2\cdot 3^j \cdot v}b$ gives $2 \cdot 3^i \cdot u = 2 \cdot 3^j \cdot v$. Without loss of generality, we assume that $i < j$. Now, we get $u = 3^{j-i}v$ implies $3 \,|\, u$, a contradiction. Similarly, we arrive at a contradiction if $u \equiv 2(\text{mod } 3)$ and $v \equiv 2(\text{mod } 3)$.

Case 3: $x \in Q_j$ with $j \neq i$. If $j = k+1$, then by Remark 3.1.22, we get $x = a^m b^2$ for some odd m. Since $i \neq j$, we have either $x = a^{m'}b$ or $x = a^{m'}b^2$ for some even m' which is not possible. Similarly, we have a contradiction if $i = k+1$. So, we assume that $0 \leq i, j \leq k$. Then $x = (a^{2\cdot 3^i}b)^{r'}$ and $x = (a^{2\cdot 3^j}b^2)^{s'}$ for some $r' \leq 3^{k-i}t$ and $s' \leq 3^{k-j}t$ (see Lemma 3.1.30 (i)). Clearly, r', s' are not divisible by 3. Otherwise, $x \in \langle a \rangle$, a contradiction. For the case $r' \equiv 1(\text{mod } 3)$ and $s' \equiv 2(\text{mod } 3)$, we have $x = a^{2\cdot 3^i \cdot r'}b$ and $x = a^{2\cdot 3^j \cdot s'}b$. Since $r' \leq 3^{k-i}t$ and $s' \leq 3^{k-j}t$, we have $2 \cdot 3^i \cdot r' \leq 2n$ and $2 \cdot 3^j \cdot s' \leq 2n$. Consequently $a^{2\cdot 3^i \cdot r'}b = a^{2\cdot 3^j \cdot s'}b$ gives $2 \cdot 3^i \cdot r' = 2 \cdot 3^j \cdot s'$. Without loss of generality, we assume that $i < j$. Now, we get $r' = 3^{j-i}s'$ implies $3 \,|\, r'$, a contradiction. Similarly, we get $3 \,|\, r'$ if $r' \equiv 2(\text{mod } 3)$ and $s' \equiv 1(\text{mod } 3)$. For $r' \equiv 1(\text{mod } 3)$ and $s' \equiv 1(\text{mod } 3)$, we have $x = a^{2\cdot 3^i \cdot r'}b$ and $x = a^{2\cdot 3^j \cdot s'}b^2$. Consequently, $b^2 \in \langle a \rangle$, again a contradiction. We get a similar contradiction when $r' \equiv 2(\text{mod } 3)$ and $s' \equiv 2(\text{mod } 3)$.

(ii) The proof is similar to that of (i). $\square$

As a consequence of Proposition 3.1.31, we have the following lemma.

Lemma 3.1.32. *Let $x \in P_i \backslash \langle a \rangle$, where $0 \leq i \leq k+1$. Then $x \sim y$ in $\mathcal{P}_e(U_{6n})$ if and only if $y \in P_i$.*

Proof. Suppose $x \sim y$ for some $y \in U_{6n} \backslash P_i$. Then $x, y \in \langle z \rangle$ for some $z \in U_{6n}$. Note that $z \notin P_i$ and so $x \in U_{6n} \backslash P_i$, a contradiction of Proposition 3.1.31 (i). Since P_i is a cyclic subgroup and $x, y \in P_i$, the converse holds. $\square$

The proof of the following lemma is similar to the proof of Lemma 3.1.32.

Lemma 3.1.33. *Let $x \in Q_i \backslash \langle a \rangle$, where $0 \leq i \leq k+1$. Then $x \sim y$ in $\mathcal{P}_e(U_{6n})$ if and only if $y \in Q_i$.*

The following proposition determines neighbourhoods of vertices of $\mathcal{P}_e(U_{6n})$.

Proposition 3.1.34. *For the graph $\mathcal{P}_e(U_{6n})$, we have*

(i) $\mathrm{N}[x] = P_i$ *if and only if* $x \in P_i \backslash \langle a \rangle$, *where* $0 \le i \le k+1$.

(ii) $\mathrm{N}[x] = Q_i$ *if and only if* $x \in Q_i \backslash \langle a \rangle$, *where* $0 \le i \le k+1$.

(iii) $\mathrm{N}[x] = \langle a \rangle$ *if and only if* $x = a^i$ *for some odd* i.

(iv) $\mathrm{N}[x] = U_{6n}$ *if and only if* $x \in P_k \cap \langle a \rangle = \langle a^{2\cdot 3^k} \rangle$.

(v) $\mathrm{N}[x] = \bigcup_{j=0}^{i-1} (P_j \cup Q_j) \cup P_{k+1} \cup Q_{k+1} \cup \langle a \rangle$ *if and only if* $x \in \langle a^{2\cdot 3^i} \rangle \backslash \langle a^{2\cdot 3^{i+1}} \rangle$, *where* $0 \le i \le k-1$.

Proof. (i) If $x \in P_i \backslash \langle a \rangle$, then by Lemma 3.1.32, we have $\mathrm{N}[x] = P_i$. Conversely, suppose that $x \notin P_i \backslash \langle a \rangle$. Then either $x \in P_i \cap \langle a \rangle$ or $x \in U_{6n} \backslash P_i$. If $x \in P_i \cap \langle a \rangle$, then clearly $x \sim a$. But $a \notin P_i$, so that $\mathrm{N}[x] \neq P_i$, a contradiction. On the other hand, if $x \in U_{6n} \backslash P_i$, then by Lemma 3.1.32, we have $\mathrm{N}[x] \neq P_i$, again a contradiction. This proves (i).

(ii) The proof is similar to that of (i).

(iii) Let $x = a^i$ for some odd i. By (3.2), (3.3) and (3.4), we have $a^i \notin P_j \cup Q_j$ for all $0 \le j \le k+1$. Consequently, $a^i \sim x$ if and only if $x \in \langle a \rangle$. Thus $\mathrm{N}[x] = \langle a \rangle$.

Now let $x \in U_{6n}$ be such that $\mathrm{N}[x] = \langle a \rangle$. If $x = a^{2i}$ for some i, then by (3.2), $x \in \langle ab \rangle$. This implies $x \sim ab$, which is a contradiction. Moreover, from (i) and (ii), we have $x \notin (P_i \cup Q_i) \backslash \langle a \rangle$. Thus $x = a^i$ for some odd i.

(iv) Let $x \in P_k \cap \langle a \rangle = \langle a^{2\cdot 3^k} \rangle$. By Remark 3.1.26, $\langle a^{2\cdot 3^k} \rangle \subseteq \langle a^{2\cdot 3^i} \rangle$ for all $0 \le i \le k-1$. By Lemma 3.1.29, since $P_i \cap \langle a \rangle = \langle a^{2\cdot 3^{i+1}} \rangle = Q_i \cap \langle a \rangle$, we have $a^{2\cdot 3^k} \in P_i \cap Q_i$ for all $0 \le i \le k$. Accordingly, $x \in P_i \cap Q_i$ for any $0 \le i \le k$. Since $P_{k+1} \cap \langle a \rangle = \langle a^2 \rangle = Q_{k+1} \cap \langle a \rangle$, we get $x \in P_{k+1} \cap Q_{k+1}$. Thus $x \in \langle a \rangle \cap (P_i \cap Q_i)$ for any i, $0 \le i \le k+1$. For any $y \in U_{6n}$, by (3.4), we have $x \sim y$. Consequently, x is a dominating vertex of $\mathcal{P}_e(U_{6n})$.

Conversely, suppose x is a dominating vertex. Then x is adjacent with every element of P_i and Q_i for all $0 \leq i \leq k+1$. Consequently, $x \in P_i \cap Q_i \cap \langle a \rangle = \langle a^{2\cdot 3^{i+1}} \rangle$ for all $0 \leq i \leq k$. Hence $x \in P_k \cap \langle a \rangle = \langle a^{2\cdot 3^k} \rangle$.

(v) Let $x \in \langle a^{2\cdot 3^i} \rangle \backslash \langle a^{2\cdot 3^{i+1}} \rangle$. Clearly, $i < k$. We prove that

$$x \sim y \text{ if and only if } y \in \bigcup_{j=0}^{i-1} (P_j \cup Q_j) \cup P_{k+1} \cup Q_{k+1} \cup \langle a \rangle.$$

In order to prove this, by (3.2), we have $x \in P_{k+1} \cap Q_{k+1}$. Clearly, $x \in \langle a \rangle$. By Lemma 3.1.29, we have $P_j \cap \langle a \rangle = \langle a^{2\cdot 3^{j+1}} \rangle$. Moreover, by Remark 3.1.26, $x \in \langle a^{2\cdot 3^{j+1}} \rangle$ for all $0 \leq j \leq i-1$. Thus it follows that $x \in P_j$. Similarly, one can observe $x \in Q_j$. Consequently, for any $y \in \bigcup_{j=0}^{i-1} (P_j \cup Q_j) \cup P_{k+1} \cup Q_{k+1} \cup \langle a \rangle$, we have $y \sim x$. If possible, let $y \notin \bigcup_{j=0}^{i-1} (P_j \cup Q_j) \cup P_{k+1} \cup Q_{k+1} \cup \langle a \rangle$. Then in view of (3.4), we have either $y \in P_j \backslash \langle a \rangle$ or $y \in Q_j \backslash \langle a \rangle$ for some j, where $i \leq j \leq k$. If $x \sim y$, then by Lemmas 3.1.32 and 3.1.33, we have either $x \in P_j$ or $x \in Q_j$. If $x \in P_j$, then clearly $x \in P_j \cap \langle a \rangle = \langle a^{2\cdot 3^{j+1}} \rangle$. Since $j \geq i$, by Remark 3.1.26, we get $x \in \langle a^{2\cdot 3^{i+1}} \rangle$, a contradiction. Similarly, we get a contradiction when $x \in Q_j$. Consequently, $x \nsim y$. Hence if $x \sim y$, then $y \in \bigcup_{j=0}^{i-1} (P_j \cup Q_j) \cup P_{k+1} \cup Q_{k+1} \cup \langle a \rangle$. □

In view of Theorem 3.1.34, the enhanced power graph of U_{6n} shown in Figure 3.1, where $T = \{a^{2i+1} : 0 \leq i \leq n-1\}$, $T' = \{a^{2i} : 1 \leq i \leq n-1\}$, $T_i = \langle a^{2\cdot 3^i} \rangle \setminus \langle a^{2\cdot 3^{i+1}} \rangle$ and $0 \leq j \leq i-1$.

Applying preceding results of this subsection, we now show that the enhanced power graph of U_{6n} is prefect.

Theorem 3.1.35. *The enhanced power graph of U_{6n} is perfect.*

Proof. In view of Theorem 1.2.4, it is enough to show that $\mathcal{P}_e(U_{6n})$ does not contain a hole or antihole of odd length greater than or equal to five. Suppose first that $\mathcal{P}_e(U_{6n})$ contains a hole C given by $x_1 \sim x_2 \sim \cdots \sim x_l \sim x_1$, where $l \geq 5$. Then we have the following two cases and each of them ends up contradicting the fact that C is a hole.

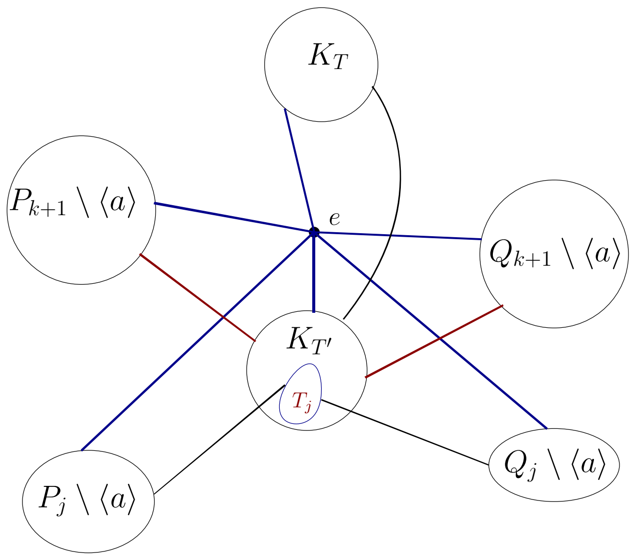


FIGURE 3.1: The enhanced power graph of U_{6n}

Case 1: $x_i \notin \langle a \rangle$ for all i. In view of (3.4), we obtain either $x_1 \in P_j \backslash \langle a \rangle$ or $x_1 \in Q_j \backslash \langle a \rangle$ for some j. Without loss of generality, we suppose that $x_1 \in P_j \backslash \langle a \rangle$ for some j. Since $x_l \sim x_1 \sim x_2$, by Lemma 3.1.32, we have $x_l, x_2 \in P_j$. Consequently, $x_l \sim x_2$.

Case 2: $x_i \in \langle a \rangle$ for some i. Without loss of generality, we assume that $x_1 \in \langle a \rangle$. Since x_4 is not adjacent with x_1 so by (3.4), either $x_4 \in P_j \backslash \langle a \rangle$ or $x_4 \in Q_j \backslash \langle a \rangle$ for some j. Also, $x_3 \sim x_4 \sim x_5$, by Lemmas 3.1.32-3.1.33, we have either $x_3, x_5 \in P_j$ or $x_3, x_5 \in Q_j$. We obtain $x_3 \sim x_5$.

Now suppose C' is an antihole of length at least 5 in $\mathcal{P}_e(U_{6n})$, that is, we have a hole $y_1 \sim y_2 \sim \cdots \sim y_l \sim y_1$, where $l \geq 5$, in $\overline{\mathcal{P}_e(U_{6n})}$. Then again we arrive at contradiction in each of the following cases.

Case 1: $y_i \notin \langle a \rangle$ for all i. Since $y_1 \notin \langle a \rangle$, by (3.4), either $y_1 \in P_j \backslash \langle a \rangle$ or $y_1 \in Q_j \backslash \langle a \rangle$ for some j. Since $y_1 \sim y_3$ and $y_1 \sim y_4$ in $\mathcal{P}_e(U_{6n})$. By Lemma 3.1.32 and Lemma 3.1.33, we get either $y_3, y_4 \in P_j$ or $y_3, y_4 \in Q_j$. Thus we have $y_3 \sim y_4$ in $\mathcal{P}_e(U_{6n})$.

Case 2: $y_i \in \langle a \rangle$ for some i. Without loss of generality, we assume that $y_1 \in \langle a \rangle$. Notice that we have $y_1 \nsim y_2$ in $\mathcal{P}_e(U_{6n})$. Consequently, either $y_2 \in P_j \backslash \langle a \rangle$ or

$y_2 \in Q_j \backslash \langle a \rangle$. Moreover, $y_2 \sim y_4$ and $y_2 \sim y_5$ in $\mathcal{P}_e(U_{6n})$, as C' is an antihole in $\mathcal{P}_e(U_{6n})$. Thus either $y_4,\ y_5 \in P_j$ or $y_4, y_5 \in Q_j$. As a result, $x_4 \sim y_5$ in $\mathcal{P}_e(U_{6n})$. □

In the following theorem, we compute various graph invariants under consideration for $\mathcal{P}_e(U_{6n})$.

Theorem 3.1.36. *For $n \geq 1$, the following hold:*

(i) *The minimum degree of $\mathcal{P}_e(U_{6n})$ is*

$$\delta(\mathcal{P}_e(U_{6n})) = \begin{cases} 2t-1 & \text{if } k = 0, \\ 3t-1 & \text{if } k > 0 \end{cases}$$

(ii) *The independence number of $\mathcal{P}_e(U_{6n})$ is $2k+4$*

(iii) *For $n > 1$, the matching number $\alpha'(\mathcal{P}_e(U_{6n}))$ is $3n$ and $\alpha'(\mathcal{P}_e(U_6)) = 2$*

(iv) *The strong metric dimension of $\mathcal{P}_e(U_{6n})$ is $6n-k-2$.*

Proof. (i) By Proposition 3.1.34, we have

(a) $\deg(x) = 2n-1$ for all $x \in (P_{k+1} \cup Q_{k+1}) \backslash \langle a \rangle$.

(b) $\deg(x) = 3^{k-i}t - 1$ for all $x \in (P_i \cup Q_i) \backslash \langle a \rangle$, where $0 \leq i \leq k-1$.

(c) $\deg(x) = 3t-1$ for all $x \in P_k \backslash \langle a \rangle$.

(d) $\deg(a^i) = 2n-1$, where i is odd.

(e) $\deg(x) = 6n-1$, where $x \in P_k \cap \langle a \rangle$.

Moreover, for $0 \leq i \leq k-1$ and $x \in \langle a^{2 \cdot 3^i} \rangle \backslash \langle a^{2 \cdot 3^{i+1}} \rangle$, we get

$$\mathrm{N}[x] = \bigcup_{j=0}^{i-1} (P_j \cup Q_j) \cup P_{k+1} \cup Q_{k+1} \cup \langle a \rangle.$$

As a result,

$$\begin{aligned}\deg(x) &= |\langle a\rangle| + \sum_{j=0}^{i-1}(|P_j\backslash\langle a\rangle| + |Q_j\backslash\langle a\rangle|) + |P_{k+1}\backslash\langle a\rangle| + |Q_{k+1}\backslash\langle a\rangle| - 1\\ &= 4n + \sum_{j=0}^{i-1} 2|P_j\backslash\langle a\rangle| - 1\\ &= 4n + 2t3^{k-i}(3^i-1) - 1.\end{aligned}$$

Let $x \in U_{6n}$. If $k = 0$, then clearly $n = t$. Then in view of (3.4) and from above, $\deg(x) \in \{2n-1, 3n-1, 6n-1\}$. Thus $\delta(\mathcal{P}_e(U_{6n})) = 2n-1 = 2t-1$.

Now take $k > 0$. From above we observe that $\deg(x) \geq 3t-1$, and when $x \in P_k\backslash\langle a\rangle$, we have $\deg(x) = 3t-1$. Hence $\delta(\mathcal{P}_e(U_{6n})) = 3t-1$.

(ii) Consider the set $I = \{a, ab, ab^2, a^{2\cdot 3^k}b\} \cup \{a^{2\cdot 3^i}b : 0 \leq i \leq k-1\} \cup \{a^{2\cdot 3^i}b^2 : 0 \leq i \leq k-1\}$. Then by Lemma 3.1.32 and Lemma 3.1.33, I is an independent set of size $2k+4$. If there exists another independent set I' such that $|I'| > 2k+4$, then there exist $x, y \in I'$ with the following possibilities: (a) $x, y \in P_i$ for some $0 \leq i \leq k+1$, or (b) $x, y \in Q_j$ for some $0 \leq j \leq k+1$, or (c) $x, y \in \langle a\rangle$. For each case, we have $x \sim y$, which is a contradiction. So that $\alpha(\mathcal{P}_e(U_{6n})) = 2k+4$.

(iii) The result is straightforward for $n = 1$. Now let $n > 1$. In order to prove that $\alpha'(\mathcal{P}_e(U_{6n}))$ is $3n$, we provide a partition of $V(\mathcal{P}_e(U_{6n}))$ into subsets of even size such that the subgraph induced by each subset is complete. Note that (3.4) can be written as

$$U_{6n} = \langle a\rangle \cup \bigcup_{i=0}^{k-1}(P_i\backslash\langle a\rangle \cup Q_i\backslash\langle a\rangle) \cup (P_k\backslash\langle a\rangle) \cup (P_{k+1}\backslash\langle a\rangle) \cup (Q_{k+1}\backslash\langle a\rangle).$$

These sets on the the right hand side of above expression forms a partition of $V(\mathcal{P}_e(U_{6n}))$. For $0 \leq i \leq k$, the subsets $P_i\backslash\langle a\rangle$ and $Q_i\backslash\langle a\rangle$ are of even cardinality (cf. Lemma 3.1.30 (i)). If n is even, then by Lemma 3.1.30, the subsets $P_{k+1}\backslash\langle a\rangle$ and $Q_{k+1}\backslash\langle a\rangle$ are of even size. Also, the subgraph induced by each subset of the given partition is complete. Consequently, $\alpha'(\mathcal{P}_e(U_{6n})) = 3n$.

Also, notice that the subsets on the right hand side of the following expression

$$U_{6n} = \big(\langle a\rangle\backslash\{e,a^2\}\big) \cup \bigcup_{i=0}^{k-1} (P_i\backslash\langle a\rangle \cup Q_i\backslash\langle a\rangle)$$
$$\cup\, (P_k\backslash\langle a\rangle) \cup ((P_{k+1}\backslash\langle a\rangle)\cup\{e\}) \cup \big((Q_{k+1}\backslash\langle a\rangle)\cup\{a^2\}\big)$$

forms a partition of $V(\mathcal{P}_e(U_{6n}))$. If n is odd, then by Lemma 3.1.30, size of each subset of this partition is even. Thus we have $\alpha'(\mathcal{P}_e(U_{6n})) = 3n$.

(iv) We denote

$$V_1 = \{\widehat{ab}, \widehat{ab^2}\} \cup \{\widehat{a^{2\cdot 3^i}b} : 0 \le i \le k\} \cup \{\widehat{a^{2\cdot 3^i}b^2} : 0 \le i \le k-1\}$$

and

$$V_2 = \{\widehat{e}, \widehat{a}\} \cup \{\widehat{a^{2\cdot 3^j}} : 0 \le j \le k-1\}.$$

Then in view of the properties of U_{6n} derived earlier and Proposition 3.1.34, $\widehat{U_{6n}} = V_1 \cup V_2$. For any $\widehat{x} \in V_2$, we have $x \in \langle a\rangle$. So that V_2 is a clique in $\widehat{\mathcal{P}}_e(U_{6n})$, and thus $\omega(\widehat{\mathcal{P}}_e(\mathcal{R}_{U_{6n}})) \ge k+2$. If possible, let C be another clique in $\widehat{\mathcal{P}}_e(U_{6n})$ with $|C| > k+2$.

By Lemma 3.1.32 and Lemma 3.1.33, $x \not\sim y$ for every pair of distinct elements $\widehat{x}$, $\widehat{y}$ in V_1. Accordingly, V_1 is an independent set in $\widehat{\mathcal{P}}_e(U_{6n})$. Then $|V_1 \cap C| \le 1$, so that $V_2 \subset C$. In fact, comparing the cardinalities, $|V_1 \cap C| = 1$. We denote $V_1 \cap C = \{\widehat{x}\}$. Consequently, $x \in P_i\backslash\langle a\rangle$ or $x \in Q_i\backslash\langle a\rangle$ for some $0 \le i \le k+1$. Moreover, $\widehat{a} \in C$, so that $x \sim a$ in $\mathcal{P}_e(U_{6n})$. Then by Lemma 3.1.32 and Lemma 3.1.33, $a \in P_i$ or $a \in Q_i$. Since this is a contradiction, we get $\omega(\widehat{\mathcal{P}}_e(U_{6n})) = k+2$. Hence $\mathrm{sdim}(\mathcal{P}_e(U_{6n})) = 6n - k - 2$, by Theorem 1.2.15. □

3.1.3 The Dihedral Group

For $n \ge 3$, the *dihedral group* D_{2n} of order $2n$ is defined in terms of generators and relations as

$$D_{2n} = \langle a, b \ : \ a^n = b^2 = e,\ ab = ba^{-1}\rangle.$$

It is known that every element of $D_{2n} \setminus \langle a \rangle$ is of the form $a^i b$ for some $0 \leq i \leq n-1$, and that $\langle a^i b \rangle = \{e, a^i b\}$. In particular,

$$D_{2n} = \langle a \rangle \cup \bigcup_{i=0}^{n-1} \langle a^i b \rangle.$$

Theorem 3.1.37. *The enhanced power graph of D_{2n} is perfect.*

Proof. We apply Theorem 1.2.4 to prove the theorem. Let C be a hole of $\mathcal{P}_e(D_{2n})$. We have $\deg(a^i b) = 1$, so that $a^i b \notin C$ for all $0 \leq i \leq n-1$. Thus the vertices of C belong to $\langle a \rangle$. Since the subgraph induced by $\langle a \rangle$ is complete, C is a cycle of length three.

Now let C' be an antihole of length at least 5 of $\mathcal{P}_e(D_{2n})$. If possible, suppose that $V(C') \cap \langle a \rangle \neq \emptyset$. Then there exists $x_1 \in C' \cap \langle a \rangle$ such that $x_1 \sim x_2$ in $\overline{\mathcal{P}_e(D_{2n})}$ for some vertex x_2 of C'. Equivalently, $x_1 \nsim x_2$ in $\mathcal{P}_e(D_{2n})$. Thus $x_2 = a^i b$ for some i. As $|V(C')| \geq 5$, there exists $x_3 \in V(C') \setminus \{x_1, x_2\}$ such that $x_1 \sim x_3$ and $x_2 \nsim x_3$ in $\mathcal{P}_e(D_{2n})$. This implies $x_3 \in \langle a \rangle$. Similarly, there exists $x_4 \in V(C') \setminus \{x_1, x_2, x_3\}$ such that $x_3 \nsim x_4$, $x_1 \sim x_4$ and $x_2 \sim x_4$ in $\mathcal{P}_e(D_{2n})$. Note that none of these $x_i's$ are e. As $x_1 \sim x_4$, we get $x_1 \in \langle a \rangle$, whereas, $x_3 \nsim x_4$ yield $x_1 \notin \langle a \rangle$. Since this is impossible, $V(C') \cap \langle a \rangle = \emptyset$. That is, every element of $V(C')$ is of the form $a^i b$. However, the subgraph of $\overline{\mathcal{P}_e(D_{2n})}$ induced by the set $\{a^i b : 0 \leq i \leq n-1\}$ is complete. This contradicts the fact that the length of C' is at least 5.

Consequently, the proof follows from Theorem 1.2.4. □

Theorem 3.1.38. *For $n \geq 2$, we have the following results:*

(i) *The minimum degree of $\mathcal{P}_e(D_{2n})$ is* 1

(ii) *The independence number of $\mathcal{P}_e(D_{2n})$ is $n+1$*

(iii) *The matching number of $\mathcal{P}_e(D_{2n})$ is $\lceil \frac{n}{2} \rceil$*

(iv) *The strong metric dimension of $\mathcal{P}_e(D_{2n})$ is $2(n-1)$.*

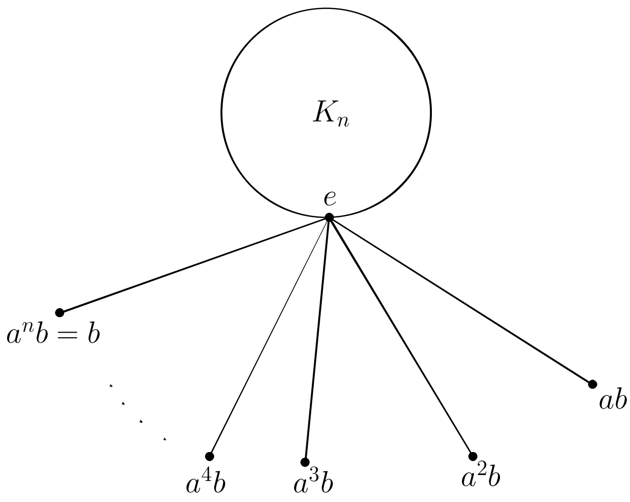


FIGURE 3.2: The enhanced power graph of D_{2n}

Proof. (i) The proof follows from Figure 3.2.

(ii) From Figure 3.2, observe that the set $I = \{a\} \cup \{a^i b : 0 \leq i \leq n-1\}$ is an independent set, and thus $\alpha(\mathcal{P}_e(D_{2n})) \geq n+1$. If there exists an independent set I' such that $|I'| > n+1$, then we must have $x, y \in I'$ such that $x, y \in \langle a \rangle$. However, this results in $x \sim y$, which is a contradiction. As a result, $\alpha(\mathcal{P}_e(D_{2n})) = n+1$.

(iii) If n is even, then observe that the size of maximum matching is $\frac{n}{2}$ which can be constructed from the complete graph induced by $\langle a \rangle$. If n is odd, then the size of maximum matching is $\lceil \frac{n}{2} \rceil$ which can be constructed $\frac{n-1}{2}$ edges of $\langle a \rangle \backslash \{e\}$ and one edge (ab, e) of H_1. Therefore, $\alpha'(\mathcal{P}_e(D_{2n})) = \lceil \frac{n}{2} \rceil$.

(iv) The $\equiv$-classes in D_{2n} are $\widehat{e}$, $\widehat{a}$, $\widehat{b}$, $\widehat{ab}$, $\ldots$, $\widehat{a^{n-1}b}$, where $\widehat{e} = \{e\}$, $\widehat{a} = \langle a \rangle \backslash \{e\}$ and $\widehat{a^i b} = \{a^i b\}$. Then in view of the adjacency relation of elements of these classes in $\mathcal{P}_e(D_{2n})$, we have $\widehat{\mathcal{P}}_e(\mathcal{R}_{D_{2n}}) \cong K_{1,n+1}$. So that $\omega(\widehat{\mathcal{P}}_e(\mathcal{R}_{D_{2n}})) = 2$. Hence $\text{sdim}(\mathcal{P}_e(D_{2n})) = 2(n-1)$, by Theorem 1.2.15. $\square$

3.1.4 The Semidihedral Group

For $n \geq 2$, the *semidihedral group* SD_{8n} is a group of order $8n$ is defined in terms of generators and relations as

$$SD_{8n} = \langle a, b : a^{4n} = b^2 = e, ba = a^{2n-1}b \rangle.$$

We have

$$ba^i = \begin{cases} a^{4n-i}b & \text{if } i \text{ is even}, \\ a^{2n-i}b & \text{if } i \text{ is odd}, \end{cases}$$

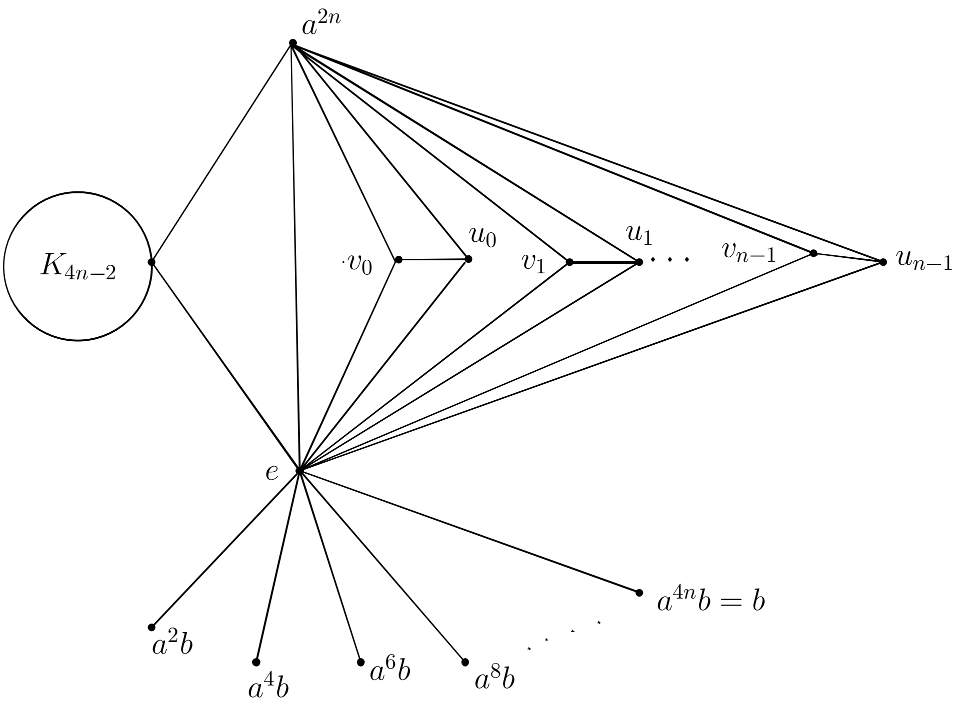
so that every element of $SD_{8n} \backslash \langle a \rangle$ is of the form $a^i b$ for some $0 \leq i \leq 4n-1$. We denote the subgroups $H_i = \langle a^{2i}b \rangle = \{e, a^{2i}b\}$ and $T_j = \langle a^{2j+1}b \rangle = \{e, a^{2n}, a^{2j+1}b, a^{2n+2j+1}b\}$. Then we have

$$SD_{8n} = \langle a \rangle \cup \left(\bigcup_{i=0}^{2n-1} H_i \right) \cup \left(\bigcup_{j=0}^{n-1} T_j \right).$$

Theorem 3.1.39. *The enhanced power graph of SD_{8n} is perfect.*

Proof. We utilize the notion of hole or antihole once again to prove the theorem. First suppose C is a hole of $\mathcal{P}_e(SD_{8n})$. For any $0 \leq i \leq 2n-1$, we notice $\deg(a^{2i}b) = 1$, so that $a^{2i}b \notin V(C)$. Since $a^{2n} \sim x$ for all $x \in \langle a \rangle \cup T_j$, we have $a^{2n} \notin V(C)$. Then we observe that $\mathrm{N}[x] = \langle a \rangle$ if and only if $x \in \langle a \rangle \backslash \{e, a^{2n}\}$, and that $\mathrm{N}[x] = T_j$ if and only if $x \in T_j \backslash \{e, a^{2n}\}$. Additionally, $e \notin V(C)$ as well. Thus all vertices of C either belong to $\langle a \rangle \backslash \{e, a^{2n}\}$ or $T_j \backslash \{e, a^{2n}\}$. As $|T_j \backslash \{e, a^{2n}\}| = 2$, we have $V(C) \not\subseteq T_j \backslash \{e, a^{2n}\}$. Accordingly, $V(C) \subseteq \langle a \rangle \backslash \{e, a^{2n}\}$. Hence the length of C is 3, as $\langle a \rangle$ induces a complete subgraph in $\mathcal{P}_e(SD_{8n})$.

Next, if possible, let C' be an antihole of $\mathcal{P}_e(SD_{8n})$ of length at least five. Since $a^{2i}b$ is adjacent with every element except e in $\overline{\mathcal{P}_e(SD_{8n})}$, and $e \notin V(C')$, we have $a^{2i}b \notin V(C')$. Now suppose $V(C') \cap \langle a \rangle \neq \emptyset$. Then there exists $x_1 \in V(C') \cap \langle a \rangle$ such that $x_1 \sim x_2$ in $\overline{\mathcal{P}_e(SD_{8n})}$ for some $x_2 \in V(C')$. Then $x_1 \nsim x_2$ in $\mathcal{P}_e(SD_{8n})$, so that $x_2 \in T_j \backslash \{e, a^{2n}\}$ for some j. Since $|V(C')| \geq 5$, there exists $x_3 \in V(C') \backslash \{x_1, x_2\}$ such that $x_1 \sim x_3$ and $x_2 \nsim x_3$ in $\mathcal{P}_e(SD_{8n})$. As a result, $x_3 \in \langle a \rangle \backslash \{e, a^{2n}\}$.



(where $u_i = a^{2i+1}b,\ v_i = a^{2n+2i+1}b$ for $0 \le i \le n-1$)

Figure 3.3: The enhanced power graph of SD_{8n}

Furthermore, there exists $x_4 \in V(C')\backslash\{x_1, x_2, x_3\}$ such that $x_3 \nsim x_4$, $x_1 \sim x_4$ and $x_2 \sim x_4$ in $\mathcal{P}_e(SD_{8n})$. Then $x_1 \sim x_4$ and $x_3 \nsim x_4$ imply, respectively, that $x_3 \in \langle a \rangle$ and $x_3 \notin \langle a \rangle$. Since this is impossible, $V(C') \cap \langle a \rangle = \emptyset$. Consequently, every vertex of C' is of the form $a^{2i+1}b$. However, for any $0 \le i \le 2n-1$, $a^{2i+1}b$ is adjacent to every vertex in $\{a^{2j+1}b : 0 \le j \le 2n-1\}\backslash\{a^{2i+1}b, a^{2n+2i+1}b\}$ in $\overline{\mathcal{P}_e(SD_{8n})}$. This contradicts our assumption that C' is an antihole of $\mathcal{P}_e(SD_{8n})$ of length at least five.

Following Theorem 1.2.4, we therefore conclude that $\mathcal{P}_e(SD_{8n})$ is perfect. $\square$

Now we investigate graph invariants of $\mathcal{P}_e(SD_{8n})$ in the following theorem.

Theorem 3.1.40. *For $n \ge 1$, we have the following results:*

(i) *The minimum degree of $\mathcal{P}_e(SD_{8n})$ is* 1

(ii) *The independence number of $\mathcal{P}_e(SD_{8n})$ is* $3n + 1$

(iii) *The matching number of* $\mathcal{P}_e(SD_{8n})$ *is* $3n$

(iv) *The strong metric dimension of* $\mathcal{P}_e(SD_{8n})$ *is* $8n-3$.

Proof. (i) The proof follows from Figure 3.3.

(ii) Note that the set $I = \{a\} \cup \{a^{2i}b : 0 \le i \le 2n-1\} \cup \{a^{2j+1}b : 0 \le j \le n-1\}$ is an independent set in $\mathcal{P}_e(SD_{8n})$ (see Figure 3.3) so $\alpha(\mathcal{P}_e(SD_{8n})) \ge 3n+1$. If possible, suppose there exists an independent set I' such that $|I'| > 3n+1$. Then there exist $x, y \in I'$ such that $x, y \in \langle a \rangle$, $x, y \in H_i$ for some i or $x, y \in T_j$ for some j. Since subgraphs induced by $\langle a \rangle$, H_i and T_j, respectively forms a clique, we have $x \sim y$ for each of the possibility, a contradiction. Accordingly, $\alpha(\mathcal{P}_e(SD_{8n})) = 3n+1$.

(iii) Let M be a matching in $\mathcal{P}_e(SD_{8n})$. Consider the set U of endpoints of edges in M. We observe that $a^{2i}b \sim x$ if and only if $x = e$. As a result, $|U| \le 6n+1$. However, as M is a matching, $|U|$ is even. Then $|U| \le 6n$ and thus $|M| \le 3n$. Now let ϵ_i be the edge with endpoints a^i, a^{2n+i}, and ϵ'_j be the edge with endpoints $a^{2j+1}b, a^{2n+2j+1}b$. Then the set $M' = \{\epsilon_i : 0 \le i \le 2n-1\} \cup \{\epsilon'_j : 0 \le j \le n-1\}$ is a matching of size $3n$ in $\mathcal{P}_e(SD_{8n})$. Hence we get $\alpha'(\mathcal{P}_e(SD_{8n})) = 3n$.

(iv) From the structure of SD_{8n}, we have $\widehat{SD_{8n}} = \{\widehat{e}, \widehat{a^{2n}}, \widehat{a}\} \cup \{\widehat{a^{2i}b} : 0 \le i \le 2n-1\} \cup \{\widehat{a^{2j+1}b} : 0 \le j \le n-1\}$, where $\widehat{e} = \{e\}$, $\widehat{a^{2n}} = \{a^{2n}\}$, $\widehat{a} = \langle a \rangle \setminus \{e, a^{2n}\}$, $\widehat{a^{2i}b} = \{a^{2i}b\}$ and $\widehat{a^{2j+1}b} = \{a^{2j+1}b, a^{2n+2j+1}b\}$. Furthermore, $\widehat{SD_{8n}} \setminus \{\widehat{e}, \widehat{a^{2n}}\}$ is an independent set, and each of $\widehat{e}$ and $\widehat{a^{2n}}$ are adjacent to the rest of the vertices in $\widehat{\mathcal{P}}_e(\mathcal{R}_{SD_{8n}})$. Thus we get $\omega(\widehat{\mathcal{P}}_e(\mathcal{R}_{SD_{8n}}) = 3$. Finally by Theorem 1.2.15, $\mathrm{sdim}(\mathcal{P}_e(SD_{8n})) = 8n-3$. □

3.1.5 The Group V_{8n}

For positive integer n, the group V_{8n} (see Mahmiani [2016]) of order $8n$ is defined in terms of generators and relations as

$$V_{8n} = \langle a, b \;:\; a^{2n} = b^4 = e,\; ba = a^{-1}b^{-1}, b^{-1}a = a^{-1}b \rangle.$$

Since $ba = a^{-1}b^{-1}$ and $b^{-1}a = a^{-1}b$, we have

$$b^2a^i = a^ib^2, \quad ba^i = \begin{cases} a^{2n-i}b & \text{if } i \text{ is even ;} \\ a^{2n-i}b^3 & \text{if } i \text{ is odd,} \end{cases} \quad \text{and} \quad b^3a^i = \begin{cases} a^{2n-i}b^3 & \text{if } i \text{ is even ;} \\ a^{2n-i}b & \text{if } i \text{ is odd.} \end{cases} \tag{3.5}$$

Thus, every element of $V_{8n} \setminus \langle a \rangle$ is of the form a^ib^j for some i, j, where $1 \leq i \leq 2n$ and $1 \leq j \leq 3$.

Remark 3.1.41. The group V_{8n} is of order $8n$ if and only if $\langle b \rangle \cap \langle a \rangle = \{e\}$.

Lemma 3.1.42. *For $1 \leq i \leq n$, we have*

(i) $\langle a^{2i+1}b \rangle = \{e, a^{2i+1}b\}$.

(ii) $\langle a^{2i+1}b^3 \rangle = \{e, a^{2i+1}b^3\}$.

(iii) $\langle a^{2i}b^3 \rangle = \langle a^{2i}b \rangle = \{e, b^2, a^{2i}b, a^{2i}b^3\}$.

Proof. In view of Eq. 3.5 , the proof is straightforward. □

From above presentation of V_{8n} and by mathematical induction, we get

$$(a^{2^j}b^2)^i = \begin{cases} a^{2^j \cdot i} & \text{if } i \text{ is even ;} \\ a^{2^j \cdot i}b^2 & \text{if } i \text{ is odd.} \end{cases} \tag{3.6}$$

In the remaining part of this subsection, for some non negative integer k and a positive integer t, we shall write $n = 2^kt$ such that $2 \nmid t$.

Lemma 3.1.43. *For $x \in V_{8n} \setminus \langle a \rangle$ and $1 \leq s \leq n$, we have*

(i) *$x = a^{2s}b^2$ if and only if $x \in \langle a^{2^i}b^2 \rangle$ for some i, where $1 \leq i \leq k+1$.*

(ii) *$x = a^{2s+1}b^2$ if and only if $x \in \langle ab^2 \rangle$.*

Proof. (i) Suppose $x = a^{2s}b^2$. Write $s = 2^{i-1}r$ for some $i \geq 1$ and r is a positive integer such that $2 \nmid r$. By Eq. 3.6, note that $x = a^{2s}b^2 = (a^{2^i}b^2)^r$. Consequently, $x \in \langle a^{2^i}b^2 \rangle$. If $i \leq k+1$, the result holds. For $i > k+1$, observe that $(a^{2^{k+1}}b^2)^t = b^2$

and $(a^{2^{k+1}}b^2)^{t+1} = a^{2^{k+1}}$. Thus, $b^2, a^{2^{k+1}} \in \langle a^{2^{k+1}}b^2\rangle$. Since $a^{2s}b^2 = (a^{2^{k+1}})^{2^{i-k-1}\cdot r}b^2$, we have $x = a^{2s}b^2 \in \langle a^{2^{k+1}}b^2\rangle$. Conversely, suppose that $x \in \langle a^{2^i}b^2\rangle$ for some i, where $i \geq 1$. Since $x \notin \langle a\rangle$, by Eq. 3.6, we have $x = a^{2s}b^2$ for some s, where $1 \leq s \leq n$.

(ii) Let $x = a^{2s+1}b^2$. Then by Eq. 3.6, we have $x = (ab^2)^{2s+1}$ and so $x \in \langle ab^2\rangle$. Converse is straightforward by Eq. 3.6. $\square$

For $X_i = \langle a^{2i+1}b\rangle \cup \langle a^{2i+1}b^3\rangle \cup \langle a^{2i}b\rangle$ and $Y_i = \langle a^{2^i}b^2\rangle$, we have

$$V_{8n} = \langle a\rangle \cup \left(\bigcup_{i=1}^{n} X_i\right) \cup \left(\bigcup_{i=0}^{k+1} Y_i\right). \tag{3.7}$$

Lemma 3.1.44. *In the group V_{8n},*

(i) *$o(a^{2^i}) = 2^{k+1-i}t$, where $0 \leq i \leq k+1$.*

(ii) *for $0 \leq i \leq k$, we have $\langle a^{2^{i+1}}\rangle \subsetneq \langle a^{2^i}\rangle$ and $\langle a^{2^{k+1}}\rangle = \langle a^{2^{k+2}}\rangle$.*

(iii) *for $0 \leq i \leq k$, we have $\langle a^{2^i}b^2\rangle \cap \langle a\rangle = \langle a^{2^{i+1}}\rangle$ and $\langle a^{2^{k+1}}b^2\rangle \cap \langle a\rangle = \langle a^{2^{k+1}}\rangle$.*

(iv) *for $0 \leq i \leq k$, we have $o(a^{2^i}b^2) = 2^{k+1-i}t$. Moreover, $|\langle a^{2^i}b^2\rangle \setminus \langle a^{2^{i+1}}\rangle| = 2^{k-i}t$.*

(v) *$o(a^{2^{k+1}}b^2) = 2t$. Moreover, $|\langle a^{2^{k+1}}b^2\rangle \setminus \langle a^{2^{k+1}}\rangle| = t$.*

Proof. (i) Since $\left(a^{2^i}\right)^{2^{k+1-i}t} = e$ and for $l < 2^{k+1-i}t$, we have $(a^{2^i})^l \neq e$. Thus, the result holds.

(ii) Clearly, $\langle a^{2^{i+1}}\rangle \subseteq \langle a^{2^i}\rangle$ for all $i \geq 0$. For $i \leq k$, from part (i) note that $o(a^{2^{i+1}}) < o(a^{2^i})$. Thus, $\langle a^{2^{i+1}}\rangle \subsetneq \langle a^{2^i}\rangle$. Since $\langle a^{2^{k+2}}\rangle \subseteq \langle a^{2^{k+1}}\rangle$ and $a^{2^{k+1}} = \left(a^{2^{k+2}}\right)^{\frac{t+1}{2}}$, we obtain $\langle a^{2^{k+1}}\rangle = \langle a^{2^{k+2}}\rangle$.

(iii) Note that $a^{2^{i+1}} = (a^{2^i}b^2)^2$. Consequently, $\langle a^{2^{i+1}}\rangle \subseteq \langle a^{2^i}b^2\rangle \cap \langle a\rangle$. If $x \in \langle a^{2^i}b^2\rangle \cap \langle a\rangle$, then by Eq. 3.6, $x = (a^{2^i}b^2)^{2p}$ for some $p \geq 0$. Thus, $x \in \langle a^{2^{i+1}}\rangle$.

(iv) By Eq. 3.6, we get $\left(a^{2^i}b^2\right)^{2^{k+1-i}t} = e$. If $l < 2^{k+1-i}t$ then $2^il < 2n$. For even l, we have $(a^{2^i}b^2)^l = a^{2^il} \neq e$ as $o(a) = 2n$. For odd l, $(a^{2^i}b^2)^l = a^{2^il}b^2 \neq e$. Otherwise,

we get a contradiction of Remark 3.1.41. Thus, $o(a^{2^i}b^2) = 2^{k+1-i}t$. By part (i), we have $o(a^{2^{i+1}}) = 2^{k-i}t$. Also, $\langle a^{2^i}b^2\rangle \cap \langle a\rangle = \langle a^{2^{i+1}}\rangle$ follows that $|\langle a^{2^i}b^2\rangle \setminus \langle a\rangle| = 2^{k-i}t$.
(v) From the proof of Lemma 3.1.43 (i), we have $b^2, a^{2^{k+1}} \in \langle a^{2^{k+1}}b^2\rangle$. By part (i), $o(a^{2^{k+1}}) = t$ follows that $(a^{2^{k+1}})^i$ and $(a^{2^{k+1}})^jb^2$ are all distinct elements in $\langle a^{2^{k+1}}b^2\rangle$, where $1 \leq i, j \leq t$. Thus $o(a^{2^{k+1}}b^2) \geq 2t$. Since $(a^{2^{k+1}}b^2)^{2t} = e$, we obtain $o(a^{2^{k+1}}b^2) = 2t$. By part (i) and part (iii), we have $|\langle a^{2^{k+1}}b^2\rangle \setminus \langle a^{2^{k+1}}\rangle| = t$. □

Proposition 3.1.45. *Let $x \in V_{8n} \setminus \{b^2\}$ such that $x \notin \langle a\rangle$. Then the following result hold:*

(i) *If $x \in X_i$ for some i, then*

$$x \notin \langle a\rangle \cup \left(\bigcup_{j=1, j\neq i}^{n} X_j\right) \cup \left(\bigcup_{j=0}^{k+1} Y_j\right)$$

(ii) *If $x \in Y_i$ for some i, then*

$$x \notin \langle a\rangle \cup \left(\bigcup_{j=1}^{n} X_j\right) \cup \left(\bigcup_{j=0, j\neq i}^{k+1} Y_j\right).$$

Proof. (i) In view of Lemma 3.1.42, we get $x \in \{a^{2i+1}b, a^{2i+1}b^3, a^{2i}b, a^{2i}b^3\}$. Also, note that $x \notin X_j$ for any $j \neq i$. For instance, if $x \in X_j$, then we have either $o(a) < 2n$ or $\langle b\rangle \cap \langle a\rangle \neq \{e\}$; a contradiction. By Lemma 3.1.43, $x \notin Y_j$ for any j, where $0 \leq j \leq k+1$. Thus,

$$x \notin \langle a\rangle \cup \left(\bigcup_{j=1, j\neq i}^{n} X_j\right) \cup \left(\bigcup_{j=0}^{k+1} Y_j\right).$$

(ii) In view of Lemma 3.1.43, we have $x = a^rb^2$ for some r, where $1 \leq r < 2n$. By Lemma 3.1.42, $x \notin X_j$ for any j. If $x \in Y_j$ for some $j \neq i$ follows that $x \in \langle a^{2^j}b^2\rangle \cap \langle a^{2^i}b^2\rangle$. Further, by Eq. 3.6, $x = a^{2^ip}b^2 = a^{2^jq}b^2$ for some odd integers p and q such that $p < o(a^{2^i}b^2) = 2^{k+1-i}t$ and $q < o(a^{2^j}b^2) = 2^{k+1-j}t$. Consequently, $p2^i < 2n$ and $q2^j < 2n$. If $p2^i \neq q2^j$ then $o(a) < 2n$; a contradiction. Thus, we have $p2^i = q2^j$. Since $i \neq j$, we get either p or q is even; a contradiction. □

In the following theorem, we obtain the neighbours of x for each $x \in \mathcal{P}_e(V_{8n})$.

Theorem 3.1.46. *In the enhanced power graph of V_{8n}, we have the following results:*

(i) $\mathrm{N}[x] = \langle a \rangle$ *if and only if* $x = a^{2i+1}$. *Moreover,* $\deg(a^{2i+1}) = 2n - 1$

(ii) $\mathrm{N}[x] = \langle a^{2i+1}b \rangle$ *if and only if* $x = a^{2i+1}b$. *Moreover,* $\deg(a^{2i+1}b) = 1$

(iii) $\mathrm{N}[x] = \langle a^{2i+1}b^3 \rangle$ *if and only if* $x = a^{2i+1}b^3$. *Moreover,* $\deg(a^{2i+1}b^3) = 1$

(iv) $\mathrm{N}[x] = V_{8n}$ *if and only if* $x = e$. *Moreover,* $\deg(e) = 8n - 1$

(v) $\mathrm{N}[x] = \langle a^{2i}b \rangle$ *if and only if* $x \in \{a^{2i}b, a^{2i}b^3\}$. *Moreover,* $\deg(a^{2i}b) = \deg(a^{2i}b^3) = 3$

(vi) *For* $0 \leq i \leq k$, *we have* $\mathrm{N}[x] = Y_i$ *if and only if* $x \in Y_i \setminus \langle a \rangle$. *Moreover,* $\deg(x) = 2^{k+1-i}t - 1$

(vii) $\mathrm{N}[x] = Y_{k+1}$ *if and only if* $x \in Y_{k+1} \setminus (\langle a \rangle \cup \{b^2\})$. *Moreover,* $\deg(x) = 2t - 1$

(viii) $\mathrm{N}[x] = \left(\bigcup_{i=1}^{n} \langle a^{2i}b \rangle\right) \cup Y_{k+1}$ *if and only if* $x = b^2$. *Moreover,* $\deg(b^2) = 2n+2t-1$

(ix) $\mathrm{N}[x] = \langle a \rangle \cup \left(\bigcup_{j=0}^{i-1} Y_j\right)$ *if and only if* $x \in \langle a^{2^i} \rangle \setminus \langle a^{2^{i+1}} \rangle$, *where* $1 \leq i \leq k$. *Moreover,* $\deg(x) = 4n - 2^{k+1-i}t - 1$

(x) $\mathrm{N}[x] = \langle a \rangle \cup \left(\bigcup_{i=0}^{k+1} Y_i\right)$ *if and only if* $x \in \langle a^{2^{k+1}} \rangle \setminus \{e\}$. *Moreover,* $\deg(x) = 4n - 1$.

Proof. (i) Let $\mathrm{N}[x] = \langle a \rangle$. Then $x \in \langle a \rangle$. If $x = a^{2i}$ for some i, then by Eq. 3.6, $x = (ab^2)^{2i}$. It follows that $x \sim ab^2$ and so $ab^2 \in \langle a \rangle$. Consequently, $\langle a \rangle \cap \langle b \rangle \neq \{e\}$; a contradiction. Then $x = a^{2i+1}$ for some i. Conversely suppose $x = a^{2i+1}$ for some i. Clearly, $\langle a \rangle \subseteq \mathrm{N}[x]$. Let, if possible $x \sim y$ for some $y \in V_{8n} \setminus \langle a \rangle$. Then $x, y \in \langle z \rangle$ for some $z \in V_{8n}$. Clearly $z \notin \langle a \rangle$, otherwise $y \in \langle a \rangle$. By Eq. 3.7, we must have either $z \in X_r$ or $z \in Y_j$. If $z \in X_r$ for some r, where $1 \leq r \leq n$, then

$z \in \langle a^{2r+1}b\rangle \cup \langle a^{2r+1}b^3\rangle \cup \langle a^{2r}b\rangle$. Consequently, $x \in \langle a^{2r+1}b\rangle \cup \langle a^{2r+1}b^3\rangle \cup \langle a^{2r}b\rangle$; a contradiction (see Lemma 3.1.42). If $z \in Y_j$ for some j, then $x \in Y_j$. By Eq. 3.6, we obtain $x = a^q$ for some even q, which is not possible as $x = a^{2i+1}$.

(ii) For $x = a^{2i+1}b$, we have $\langle a^{2i+1}b\rangle \subseteq \mathrm{N}[x]$. Let, if possible $y \in \mathrm{N}[x]$ such that $y \notin \{e, a^{2i+1}b\}$. Since $y \in \mathrm{N}[x]$ so there exists $z \in V_{8n}$ such that $x, y \in \langle z\rangle$. By Eq. 3.7, the possibilities of z are (i) $z \in \langle a\rangle$ (ii) $z \in X_j$ (iii) $z \in Y_l$. If $z \in \langle a\rangle$, then $x \in \langle a\rangle$; a contradiction. If $z \in X_j = \{e, b^2, a^{2j+1}b, a^{2j+1}b^3, a^{2j}b, a^{2j}b^3\}$, then we have either $o(a) < 2n$ or $\langle b\rangle \cap \langle a\rangle \neq \{e\}$; a contradiction. If $z \in Y_l$, we must have $x \in Y_l$ which is not possible (see Eq. 3.6).

(iii) Proof is similar to part (ii).

(iv) Since for any $x \in V_{8n} \setminus \{e\}$, we have $e, x \in \langle x\rangle$. Consequently, $\mathrm{N}[e] = V_{8n}$. Conversely, suppose $\mathrm{N}[x] = V_{8n}$. By part (ii), we must have $x = e$.

(v) Suppose $\mathrm{N}[x] = \langle a^{2i}b\rangle$. Then $x \in \langle a^{2i}b\rangle$. Clearly, $x \neq e$. If $x = b^2$, then $x \in \langle a^{2^{k+1}}b^2\rangle$ as $x = (a^{2^{k+1}}b^2)^t$. It follows that $a^{2^{k+1}}b^2 \in \mathrm{N}[x] = \langle a^{2i}b\rangle$ which is not possible by Lemma 3.1.42 (iii). Consequently, $x \in \{a^{2i}b, a^{2i}b^3\}$. Conversely suppose $x \in \{a^{2i}b, a^{2i}b^3\}$. Clearly, $\langle a^{2i}b\rangle \subseteq \mathrm{N}[x]$. If $y \in V_{8n}$ such that $y \sim x$ then $x, y \in \langle z\rangle$ for some $z \in V_{8n}$. By Eq. 3.7 and Proposition 3.1.45(ii), we have $z \in X_i$ and so $y \in X_i$. In view of part (ii) and part (iii), $a^{2i+1}b, a^{2i+1}b^3 \notin \mathrm{N}[x]$. Since $y \in X_i \setminus \{a^{2i+1}b, a^{2i+1}b^3\}$, we have $y \in \langle a^{2i}b\rangle$. Consequently, $\mathrm{N}[x] = \langle a^{2i}b\rangle$.

(vi) Let $x \in Y_i \setminus \langle a\rangle$. Then $Y_i \subseteq \mathrm{N}[x]$. If $x \sim y$ for some $y \in V_{8n}$, then there exists $z \in V_{8n}$ such that $x, y \in \langle z\rangle$. By Eq. 3.7 and Proposition 3.1.45(ii), we have $z \in Y_i \setminus \langle a\rangle$. Thus $y \in Y_i$ so that $\mathrm{N}[x] = Y_i$. Conversely suppose that $\mathrm{N}[x] = Y_i$. Clearly $x \in Y_i$. If $x \in \langle a\rangle$, then $x \sim a$ so that $a \in \mathrm{N}[x] = Y_i$ which is not possible (cf. Eq. 3.6). Thus $x \in Y_i \setminus \langle a\rangle$.

(vii) If $x \in Y_{k+1} \setminus (\langle a\rangle \cup \{b^2\})$, then by the similar argument used in part (vi), we get $\mathrm{N}[x] = Y_{k+1}$. Conversely, suppose $\mathrm{N}[x] = Y_{k+1}$. Clearly, $x \in Y_{k+1}$. If $x = b^2 \in \langle a^2b\rangle$, then $x \sim a^2b$ so that $a^2b \in \mathrm{N}[x] = Y_{k+1}$; a contradiction (see Eq. 3.6). Also $x \in \langle a\rangle$ is not possible, otherwise $a \in Y_{k+1}$; a contradiction of Eq. 3.6.

(viii) Let $x = b^2$. Then by Lemma 3.1.42 (iii), $x \in \langle a^{2i}b\rangle$ for all i and $x = (a^{2^{k+1}}b^2)^t \in \langle a^{2^{k+1}}b^2\rangle = Y_{k+1}$. We get $\left(\bigcup_{i=1}^{n}\langle a^{2i}b\rangle\right) \cup Y_{k+1} \subseteq \mathrm{N}[x]$. If $x \sim y$ for some $y \in V_{8n} \setminus \left(\bigcup_{i=1}^{n}\langle a^{2i}b\rangle \cup Y_{k+1}\right)$, then by Eq. 3.7, we get the following possibilities of y (i) $y = a^{2i+1}$ for some i (ii) $y = a^{2i+1}b$ (iii) $y = a^{2i+1}b^3$ (iv) $y \in Y_j \setminus \langle a\rangle$ for some j; $0 \leq j \leq k$ (v) $y = a^{2i}$ for some i. By the above parts of this theorem, the possibilities of y in (i) - (iv) is not possible. Thus, $y = a^{2i}$ for some i. Since $x \sim y$ there exists $z \in V_{8n}$ such that $x, y \in \langle z\rangle$. By Lemma 3.1.42 and Eq. 3.6, we have $z \in Y_r$ for some $0 \leq r \leq k+1$. Consequently, $x, y \in Y_r$. If $r < k+1$, then again by Eq. 3.6, $x \notin Y_r$; a contradiction. Thus, $y \in Y_{k+1}$; again a contradiction. It follows that $\mathrm{N}[x] = \left(\bigcup_{i=1}^{n}\langle a^{2i}b\rangle\right) \cup Y_{k+1}$. Conversely, suppose $\mathrm{N}[x] = \left(\bigcup_{i=1}^{n}\langle a^{2i}b\rangle\right) \cup Y_{k+1}$, we get $x \in \left(\bigcup_{i=1}^{n}\langle a^{2i}b\rangle\right) \cup Y_{k+1}$. By part (iv), (v) and (vii), $x \in \langle a\rangle \cup \{b^2\}$. Also, by part (i), $x \in \langle a^2\rangle \cup \{b^2\}$. If $x = a^{2i}$ for some i, $1 \leq i \leq n$, then $x \sim a$ so that $a \in \mathrm{N}[x]$; a contradiction as $a \notin \left(\bigcup_{i=1}^{n}\langle a^{2i}b\rangle\right) \cup Y_{k+1}$. Thus, $x = b^2$.

(ix) Suppose $x \in \langle a^{2^i}\rangle \setminus \langle a^{2^{i+1}}\rangle$. Then by Lemma 3.1.44 (ii), $x \in \langle a^{2^{j+1}}\rangle$ for all j, where $0 \leq j \leq i-1$. Consequently, by Lemma 3.1.44 (iii), $x \in \langle a^{2^j}b^2\rangle \cap \langle a\rangle$. Thus, $\left(\bigcup_{j=0}^{i-1} Y_j\right) \cup \langle a\rangle \subseteq \mathrm{N}[x]$. If $x \sim y$, then there exists $z \in V_{8n}$ such that $x, y \in \langle z\rangle$. Clearly, $z \notin X_i$ for any i, $1 \leq i \leq n$. Otherwise, $x \in X_i$ which is not possible by Lemma 3.1.42. If $z \in Y_j$ for some $j \geq i$ then $x \in Y_j \cap \langle a\rangle = \langle a^{2^{j+1}}\rangle$. Since $j \geq i$, $x \in \langle a^{2^{j+1}}\rangle \subseteq \langle a^{2^{i+1}}\rangle$; a contradiction. Thus, $z \in \langle a\rangle \cup \left(\bigcup_{j=0}^{i-1} Y_j\right)$ and so $y \in \langle a\rangle \cup \left(\bigcup_{j=0}^{i-1} Y_j\right)$. Consequently, $\mathrm{N}[x] = \langle a\rangle \cup \left(\bigcup_{j=0}^{i-1} Y_j\right)$. Conversely suppose $\mathrm{N}[x] = \langle a\rangle \cup \left(\bigcup_{j=0}^{i-1} Y_j\right)$. Then $x \in \langle a\rangle \cup \left(\bigcup_{j=0}^{i-1} Y_j\right)$. If $x \in Y_j \setminus \langle a\rangle$ for some j, where $0 \leq j \leq i-1$, then by part (vi) $\mathrm{N}[x] = Y_j$; a contradiction. Consequently, $x \in \langle a\rangle$. If $x = a^{2i+1}$ for some i, then by part (i) $\mathrm{N}[x] = \langle a\rangle$; again a contradiction. Thus,

$x = a^{2l}$ for some l. If $x \in \langle a^{2^j} \rangle$ for some $j > i$, then by Lemma 3.1.44 (iii), $x \in Y_{j-1}$ where $j-1 > i-1$ so that $Y_{j-1} \subseteq \mathrm{N}[x]$; which is not possible. Thus $x \notin \langle a^{2^{i+1}} \rangle$. Now we show that $x \in \langle a^{2^i} \rangle$. Since $Y_{i-1} \subseteq \mathrm{N}[x]$ and $a^{2^{i-1}}b^2 \in Y_{i-1}$, we have $x \sim a^{2^{i-1}}b^2$. By part (vi), $x \in Y_{i-1} \cap \langle a \rangle = \langle a^{2^i} \rangle$.

(x) Let $x \in \langle a^{2^{k+1}} \rangle \setminus \{e\}$. Then by Lemma 3.1.44 (ii), $x \in \langle a^{2^i} \rangle$ for all i, where $1 \leq i \leq k+2$. By Lemma 3.1.44 (iii), follows that $x \in Y_{i-1} \cap \langle a \rangle$. Consequently, $\langle a \rangle \cup \left(\bigcup_{i=0}^{k+1} Y_i \right) \subseteq \mathrm{N}[x]$. If $x \sim y$ for some $y \in V_{8n} \setminus \left(\langle a \rangle \cup \left(\bigcup_{i=0}^{k+1} Y_i \right) \right)$, then by Eq. 3.7, $y \in X_i$ for some i, $1 \leq i \leq n$. In fact $y \in \{a^{2i+1}b, a^{2i+1}b^3, a^{2i}b, a^{2i}b^3\}$ for some i. By part (ii), (iii) and (v) $x \nsim y$; a contradiction. Consequently, $\mathrm{N}[x] = \left(\bigcup_{i=0}^{k+1} Y_i \right) \cup \langle a \rangle$. Conversely, suppose $\mathrm{N}[x] = \left(\bigcup_{i=0}^{k+1} Y_i \right) \cup \langle a \rangle$. Then clearly, $x \neq e$ (see Theorem 3.1.46 (iv)). Since $x \sim a^{2^{k+1}}b^2$, we get $x \in \mathrm{N}[a^{2^{k+1}}b^2] = Y_{k+1}$. Also $x \sim a$ so that $x \in \mathrm{N}[a] = \langle a \rangle$. Consequently, $x \in Y_{k+1} \cap \langle a \rangle = \langle a^{2^{k+1}} \rangle$. Hence $x \in \langle a^{2^{k+1}} \rangle \setminus \{e\}$. □

In view of Theorem 3.1.46, the enhanced power graph of V_{8n} is as follows, where $T = \{a^{2i+1}b^2 : 0 \leq i \leq n-1\}$, $T' = \{a^{2i+1} : 0 \leq i \leq n-1\}$ and $U = \left(\bigcup_{i=1}^{k+1} Y_i \cup \langle a^2 \rangle \right) \setminus \{e, b^2\}$. Moreover, $\mathcal{P}_e(U)$ denotes the subgraph of $\mathcal{P}_e(V_{8n})$ induced by U.

Since $\kappa(\mathcal{G}) \leq \kappa'(\mathcal{G}) \leq \delta(\mathcal{G})$, we have the following consequences of the Theorem 3.1.46.

Corollary 3.1.47. *For $n \geq 1$, we have $\delta(\mathcal{P}_e(V_{8n})) = \kappa(\mathcal{P}_e(V_{8n})) = \kappa'(\mathcal{P}_e(V_{8n})) = 1$.*

Corollary 3.1.48. *The graph $\mathcal{P}_e(V_{8n})$ is not dominatable.*

Corollary 3.1.49. *The graph $\mathcal{P}_e(V_{8n})$ is not Hamiltonian.*

Theorem 3.1.50. *The matching number of $\mathcal{P}_e(V_{8n})$ is given below:*

$$\alpha'(\mathcal{P}_e(V_{8n})) = \begin{cases} 3n & \text{if } t > 1 \ ; \\ 3n-1 & \text{otherwise.} \end{cases}$$

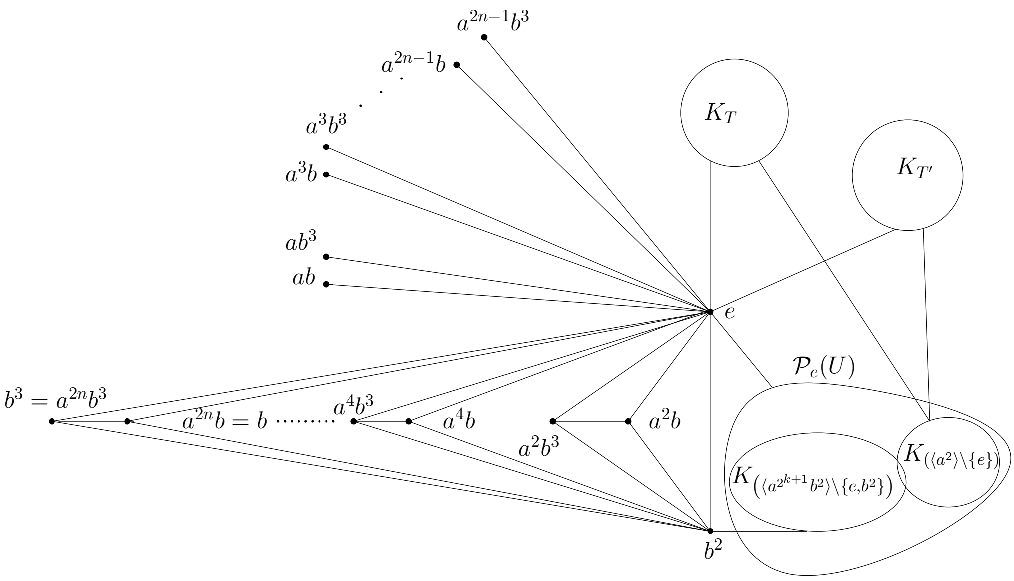


FIGURE 3.4: The enhanced power graph of V_{8n}

Proof. Let M be an arbitrary matching set of $\mathcal{P}_e(V_{8n})$ and $t > 1$. Clearly, M contains at most one edge of the form $x \sim e$ for some $x \in V_{8n} \setminus \{e\}$. For any i $(1 \leq i \leq n)$, the vertices $a^{2i+1}b$ and $a^{2i+1}b^3$ are adjacent only with e (see Figure 3.4). Thus, M contains at most one edge whose end vertices belongs to the set $K = \{a^{2i+1}b, a^{2i+1}b^3 : 1 \leq i \leq n\}$. Since $|K| = 2n$ it follows that at most $8n - (2n-1) = 6n + 1$ vertices can be used for a construction of M. Consequently, we get $|M| \leq 3n$. Now in the following, we give a construction of a matching set M' such that $|M'| = 3n$. For $0 \leq i \leq k-1$ and $1 \leq j \leq n$, consider the subsets $(Y_k \setminus \langle a\rangle) \cup \{a^{2^{k+1}}\}$, $(Y_{k+1} \setminus \langle a\rangle) \cup \{e\}, \langle a\rangle \setminus \{e, a^{2^{k+1}}\}$, $\{a^{2j}b, a^{2j}b^3\}$ and $Y_i \setminus \langle a\rangle$. Each of these subset is of even cardinality (see Lemma 3.1.44) and these subsets are mutually disjoint with each other. Also, note that the subgraph induced by each of these subset is complete. Thus, the required edges for the construction of M' can be collected from each of these complete subgraph and the number of such edges is exactly half of the total number of elements in these subsets. Thus, in view of Lemma 3.1.44, we have

$$\begin{aligned}
2|M'| &= |(Y_k \setminus \langle a\rangle) \cup \{a^{2^{k+1}}\}| + |(Y_{k+1} \setminus \langle a\rangle) \cup \{e\}| + |\langle a\rangle \setminus \{e, a^{2^{k+1}}\}| + \sum_{i=0}^{k-1} |Y_i \setminus \langle a\rangle| \\
&+ \sum_{i=1}^{n} |\langle a^{2i}b\rangle \setminus \{e, b^2\}| \\
&= 2(t+1) + (2n-2) + \sum_{i=0}^{k-1} 2^{k-i}t + 2n \\
&= 6n
\end{aligned}$$

and hence $\alpha'(\mathcal{P}_e(V_{8n})) = 3n$.

We may now suppose $t = 1$. In the following cases, for an arbitrary matching set M of $\mathcal{P}_e(V_{8n})$, we show that $|M| \leq 3n - 1$.

Case 1: $(x, e) \in M$ for some $x \in \bigcup_{i=1}^{n}(\langle a^{2i}b\rangle \setminus \{e\})$. Then for any i, where $1 \leq i \leq n$, the edges $(a^{2i+1}b, e), (a^{2i+1}b^3, e), (a^{2^k}b^2, e) \notin M$. By Figure 3.4, we get $\deg(a^{2i+1}b) = \deg(a^{2i+1}b^3) = \deg(a^{2^k}b^2) = 1$. Consequently, M does not contain the edges whose one of end point belongs to $\{a^{2i+1}b, a^{2i+1}b^3 : 1 \leq i \leq n\} \cup \{a^{2^k}b^2\}$. Thus, the maximum number of vertices can be used for the construction of M is $8n-(2n+1) = 6n - 1$. It follows that $|M| \leq 3n - 1$.

Case 2: $(x, e) \notin M$ for all $x \in \bigcup_{i=1}^{n}(\langle a^{2i}b\rangle \setminus \{e\})$. By the similar argument used in **Case 1**, note that at most one vertex from the set $\{a^{2i+1}b, a^{2i+1}b^3 : 1 \leq i \leq n\} \cup \{a^{2^k}b^2\}$ can be used for the construction of M. Thus $|M| \leq 3n$. Now we provide one more vertex $x \in V_{8n}$ such that it is not an end point of any edge belongs to M. Consequently, $|M| \leq 3n - 1$. Since $t = 1$, $Y_{k+1} \subseteq \bigcup_{i=1}^{n}\langle a^{2i}b\rangle$, again by Theorem 3.1.46, $\mathrm{N}[b^2] = \bigcup_{i=1}^{n}\langle a^{2i}b\rangle$. For all $1 \leq i \leq n$, by Figure 3.4, we have $\mathrm{N}[a^{2i}b] = \{e, b^2, a^{2i}b, a^{2i}b^3\} = \mathrm{N}[a^{2i}b^3]$. Thus for any i, M contains at most one edge from the set $\{(a^{2i}b, b^2), (a^{2i}b^3, b^2), (a^{2i}b, a^{2i}b^3)\}$ of edges. Consequently, there exists a vertex $x \in \langle a^{2i}b\rangle$ for some i such that it is not an end point of any edge lies in M.

Now in the following, we give a construction of a matching M' such that $|M'| = 3n - 1$. For $0 \leq i \leq k-1$ and $1 \leq j \leq n$, consider the subsets $\langle a\rangle$, $\{a^{2j}b, a^{2j}b^3\}$ and

$Y_i \setminus \langle a \rangle$. Each of these subset is of even cardinality (see Lemma 3.1.44) and these subsets are mutually disjoint with each other. Also note that the subgraph induced by each of these subset is complete. Thus, the required edges for the construction of M' can be collected from each of these complete subgraph and the number of such edges is exactly half of the total number of elements in these subsets. Thus, in view of Lemma 3.1.44, we have

$$\begin{aligned} 2|M'| &= |\langle a \rangle| + \sum_{i=0}^{k-1} |Y_i \setminus \langle a \rangle| + \sum_{i=1}^{n} |\langle a^{2i}b \rangle \setminus \{e, b^2\}| \\ &= 2n + \sum_{i=0}^{k-1} 2^{k-i}t + 2n \\ &= 6n - 2 \end{aligned}$$

and hence $\alpha'(\mathcal{P}_e(V_{8n})) = 3n - 1$. □

In view of Lemma 1.2.10, we have the following corollary.

Corollary 3.1.51. *The edge covering number is given below:*

$$\beta'(\mathcal{P}_e(V_{8n})) = \begin{cases} 5n & \text{if } t > 1 \text{ ;} \\ 5n + 1 & \text{otherwise.} \end{cases}$$

Theorem 3.1.52. *The independence number of $\mathcal{P}_e(V_{8n})$ is $3n + k + 3$.*

Proof. In view of Lemma 3.1.42 and Proposition 3.1.45, the maximal cyclic subgroups of V_{8n} are $\langle a \rangle, \langle a^{2i+1}b \rangle$, $\langle a^{2i+1}b^3 \rangle$, $\langle a^{2i}b \rangle$, $\langle a^{2^j}b^2 \rangle$, where $1 \le i \le n$ and $0 \le j \le k+1$. Therefore, it follows by Theorem 3.1.4, we have $\alpha(\mathcal{P}_e(V_{8n})) = 3n + k + 3$. □

In view of Lemma 1.2.10, we have the following corollary.

Corollary 3.1.53. *The vertex covering number of $\mathcal{P}_e(V_{8n})$ is $5n - k - 3$.*

Now we show that the enhanced power graph of V_{8n} is perfect (see Theorem 3.1.56). For this purpose, the following two lemmas will be useful.

Lemma 3.1.54. *Let C be a hole of odd length at least 5 in $\mathcal{P}_e(V_{8n})$. Then $b^2 \notin C$.*

Proof. Let $C : x_1 \sim x_2 \sim \cdots \sim x_{2l} \sim x_{2l+1}$, where $l \geq 2$, be a hole in $\mathcal{P}_e(V_{8n})$. By Remark 1.2.5, we get $x_r \neq e$ for any r, where $1 \leq r \leq 2l+1$. On contrary, assume that $x_r = b^2$ for some r. Without loss of generality, let $x_1 = b^2$. Then by Theorem 3.1.46 (viii), $\mathrm{N}[x_1] = \left(\bigcup_{i=1}^{n} \langle a^{2i}b \rangle\right) \cup Y_{k+1}$. Since $x_2 \sim x_1$, we get either $x_2 \in \langle a^{2i}b \rangle$ for some i or $x_2 \in Y_{k+1}$. If $x_2 \in \langle a^{2i}b \rangle$ for some i, then again by Theorem 3.1.46 (v), we obtain $x_3 \in \langle a^{2i}b \rangle$ as $x_2 \sim x_3$. Then $x_1 = b^2 \in \langle a^{2i}b \rangle$ and $x_3 \in \langle a^{2i}b \rangle$ follows that $x_1 \sim x_3$; a contradiction. Consequently, $x_2 \in Y_{k+1}$. Also, for $x_{2l+1} \sim x_1$, we have either $x_{2l+1} \in \langle a^{2i}b \rangle$ for some i or $x_{2l+1} \in Y_{k+1}$. If $x_{2l+1} \in \langle a^{2i}b \rangle$ for some i, then $x_{2l} \in \langle a^{2i}b \rangle$. We get $x_1 \sim x_{2l}$; a contradiction. As a result, we get $x_{2l+1} \in Y_{k+1}$. It follows that $x_2 \sim x_{2l+1}$; again a contradiction. Hence, $b^2 \notin C$. □

Lemma 3.1.55. *Let C be an antihole of odd length at least 5 in $\mathcal{P}_e(V_{8n})$. Then $b^2 \notin C$.*

Proof. Let C be an antihole in $\mathcal{P}_e(V_{8n})$ of length $2l+1$, where $l \geq 2$. Then we have $\overline{C} : x_1 \sim x_2 \sim \cdots \sim x_{2l} \sim x_{2l+1}$ is a hole in $\overline{\mathcal{P}_e(V_{8n})}$. By Remark 1.2.5, $x_r \neq e$ for any r, where $1 \leq r \leq 2l+1$. Let, if possible $x_r = b^2$ for some r. Without loss of generality, let $x_1 = b^2$. Then by Theorem 3.1.46 (viii), $\mathrm{N}[x_1] = \left(\bigcup_{i=1}^{n} \langle a^{2i}b \rangle\right) \cup Y_{k+1}$ in $\mathcal{P}_e(V_{8n})$. Since $x_3 \sim x_1$ and $x_4 \sim x_1$ in $\mathcal{P}_e(V_{8n})$, we get $x_3, x_4 \in \mathrm{N}[x_1]$. If $x_3 \in \langle a^{2i}b \rangle$ for some i, then again by Theorem 3.1.46 (vi), we have $\mathrm{N}[x_3] = \langle a^{2i}b \rangle$. Consequently, $x_{2l+1} \in \langle a^{2i}b \rangle$ as $x_3 \sim x_{2l+1}$ in $\mathcal{P}_e(V_{8n})$. Since $x_1 = b^2 \in \langle a^{2i}b \rangle$, we have $x_1 \sim x_{2l+1}$ in $\mathcal{P}_e(V_{8n})$; a contradiction. Note that $x_2 \sim x_4$ in $\mathcal{P}_e(V_{8n})$. If $x_4 \in \langle a^{2i}b \rangle$, then $x_1 \sim x_2$ in $\mathcal{P}_e(V_{8n})$; a contradiction. Consequently, $x_3, x_4 \in Y_{k+1}$ so that $x_3 \sim x_4$ in $\mathcal{P}_e(V_{8n})$; again a contradiction. Hence, $b^2 \notin C$. □

Theorem 3.1.56. *The enhanced power graph of V_{8n} is perfect.*

Proof. By Figure 3.4, $\deg(a^{2i+1}b) = \deg(a^{2i+1}b^3) = 1$ and $\deg(e) = 8n-1$ in $\mathcal{P}_e(V_{8n})$. By Remarks 1.2.5 and 1.2.6, it follows that the vertices $a^{2i+1}b, a^{2i+1}b^3$ and e are neither belongs to a hole nor to an antihole.

In order to prove $\mathcal{P}_e(V_{8n})$ is perfect, we shall show that $\mathcal{P}_e(V_{8n})$ does not contain a hole or an antihole of odd length at least five. On contrary, assume that $\mathcal{P}_e(V_{8n})$ contains a hole $C : x_1 \sim x_2 \sim \cdots \sim x_{2l+1}$, where $l \geq 2$. We have the following cases. In each case, we show that such C is not possible.

Case 1(a): $x_r \notin \langle a \rangle$ for all r, where $1 \leq r \leq 2l+1$. First we show that for any r, we have $x_r \in \bigcup_{i=1}^{n} X_i$. Let if possible, for some r and j, where $0 \leq j \leq k+1$, $x_r \in Y_j$. Without loss of generality, let $x_1 \in Y_j$. By Theorem 3.1.46 and Lemma 3.1.54, $\mathrm{N}[x_1] = Y_j$. It follows that $x_2, x_{2l+1} \in Y_j$ and so $x_2 \sim x_{2l+1}$ for $l \geq 2$; a contradiction. As a result, for each r, we get $x_r \in X_i$ for some i. Since $X_i = \{e, b^2, a^{2i+1}b, a^{2i+1}b^3, a^{2i}b, a^{2i}b^3\}$, from above and Lemma 3.1.54, we have $x_r \in \{a^{2i}b, a^{2i}b^3\}$. In particular, $x_1 \in \{a^{2i}b, a^{2i}b^3\} \subseteq \langle a^{2i}b \rangle$ for some i. Since $x_1 \sim x_2$ and $x_1 \sim x_{2l+1}$, by Theorem 3.1.46(v), we get $x_1, x_2, x_{2l+1} \in \{a^{2i}b, a^{2i}b^3\}$; a contradiction.
Case 2(a): $x_r \in \langle a \rangle$ for some r. Without loss of generality, let $x_1 \in \langle a \rangle$. Since $x_1 \nsim x_3$, we get either $x_3 \in X_i$ for some i, $1 \leq i \leq n$ or $x_3 \in Y_j \setminus \langle a \rangle$ for some j, where $0 \leq j \leq k+1$. If $x_3 \in X_i$, for some i, then by the similar argument used in **Case 1(a)**, $x_3 \in \{a^{2i}b, a^{2i}b^3\}$ for some i. Now $x_2 \sim x_3$ and $x_3 \sim x_4$ gives $x_2, x_3, x_4 \in \{a^{2i}b, a^{2i}b^3\}$; a contradiction. If $x_3 \in Y_j \setminus \langle a \rangle$ for some j, then by Theorem 3.1.46 and Lemma 3.1.54, we have $x_2, x_4 \in Y_j$. Consequently, $x_2 \sim x_4$; a contradiction.

Next, suppose that $\mathcal{P}_e(V_{8n})$ contains an antihole C of length at least five. Then $\overline{C} : x_1 \sim x_2 \sim \cdots \sim x_{2l} \sim x_{2l+1}$, where $l \geq 2$, is a hole in $\overline{\mathcal{P}_e(V_{8n})}$. We show that such C is not possible. We have the following cases:

Case 1(b): $x_r \notin \langle a \rangle$ for all r, where $1 \leq r \leq 2l+1$. Let if possible for some r and j, where $0 \leq j \leq k+1$, $x_r \in Y_j \setminus \langle a \rangle$. Without loss of generality $x_1 \in Y_j \setminus \langle a \rangle$. By Theorem 3.1.46 and Lemma 3.1.55, $\mathrm{N}[x_1] = Y_j$. Since $x_1 \sim x_3$ and $x_1 \sim x_4$ in $\mathcal{P}_e(V_{8n})$, we get $x_3 \sim x_4$ in $\mathcal{P}_e(V_{8n})$; a contradiction. Consequently, $x_r \in \bigcup_{i=1}^{n} X_i$. Now by the similar argument used in **Case 1(a)**, one can get, $x_1 \in \langle a^{2i}b \rangle$. Now for $x_1 \sim x_3$ and $x_1 \sim x_4$ in $\mathcal{P}_e(V_{8n})$, we have $x_3 \sim x_4$ in $\mathcal{P}_e(V_{8n})$ which is not possible as $x_3 \sim x_4$ in $\overline{C}$.

Case 2(b): $x_r \in \langle a \rangle$ for some r. Without loss of generality, let $x_1 \in \langle a \rangle$. Since $x_1 \nsim x_2$ in $\mathcal{P}_e(V_{8n})$, we get either $x_2 \in X_i$ for some i or $x_2 \in Y_j \setminus \langle a \rangle$ for some j. If $x_2 \in X_i$ for some i, then we have $x_2 \in \{a^{2i}b, a^{2i}b^3\}$. Consequently, for $x_2 \sim x_4$ and $x_2 \sim x_5$ in $\mathcal{P}_e(V_{8n})$, we get $x_2, x_4, x_5 \in \{a^{2i}b, a^{2i}b^3\}$ which is not possible. If $x_2 \in Y_j \setminus \langle a \rangle$ for some j, then we obtain $x_4, x_5 \in Y_j$ so that $x_4 \sim x_5$ in $\mathcal{P}_e(V_{8n})$; a contradiction.

Thus, $\mathcal{P}_e(V_{8n})$ does not contain a hole or an antihole of odd length at least 5. Hence, $\mathcal{P}_e(V_{8n})$ is perfect. □

Theorem 3.1.57. *The strong metric dimension of $\mathcal{P}_e(V_{8n})$ is given below:*

$$\mathrm{sdim}(\mathcal{P}_e(V_{8n})) = \begin{cases} 8n-k-3 & \text{if } k>0; \\ 8n-4 & \text{if } k=0. \end{cases}$$

Proof. In view of Theorem 3.1.46, we have the partition of $\widehat{V_{8n}}$ into V_1, V_2, V_3, V_4, V_5 and V_6, where $V_1 = \{\widehat{e},\ \widehat{a},\ \widehat{b^2}\}$, $V_2 = \{\widehat{a^{2i+1}b} : 1 \le i \le n\}, V_3 = \{\widehat{a^{2i+1}b^3} : 1 \le i \le n\}, V_4 = \{\widehat{a^{2i}b} : 1 \le i \le n\}, V_5 = \{\widehat{a^{2^i}b^2} : 0 \le i \le k+1\}$, and $V_6 = \{\widehat{a^{2^i}} : 1 \le i \le k+1\}$. Now we show that $\omega(\widehat{\mathcal{P}}_e(V_{8n})) = k+3$. For any $\widehat{x} \in \{\widehat{e},\ \widehat{a}\} \cup V_6$, we have $x \in \langle a \rangle$. So that $\{\widehat{e},\ \widehat{a}\} \cup V_6$ forms a clique in $\widehat{\mathcal{P}}_e(V_{8n})$ and thus $\omega(\widehat{\mathcal{P}}_e(V_{8n})) \ge k+3$.

Suppose C is another clique in $\widehat{\mathcal{P}}_e(V_{8n})$ with $|C| > k+3$. By Theorem 3.1.46, note that $V_2 \cup V_3 \cup V_4 \cup V_5$ is an independent set in $\widehat{\mathcal{P}}_e(V_{8n})$. Consequently, $|\,(V_2 \cup V_3 \cup V_4 \cup V_5) \cap C| \le 1$. Thus, $|C| \le k+5$. Further note that $\widehat{b^2} \notin C$. Otherwise, $\widehat{a^{2^k}}, \widehat{a} \notin C$ as $a \nsim b^2$ and $a^{2^k} \nsim b^2$ so that $|C| \le k+3$; a contradiction. As a result, we get $|C| \le k+4$. If some $\widehat{x} \in (V_2 \cup V_3 \cup V_4 \cup V_5) \cap C$, then $C = V_6 \cup \{\widehat{x}, \widehat{e}, \widehat{a}\}$. Consequently, $x \in \mathrm{N}[a] = \langle a \rangle$; again a contradiction. Thus, $V_6 \cup \{\widehat{e}, \widehat{a}\} = C$ implies $|C| = k+3$ which is not possible. Hence, we get $\omega(\widehat{\mathcal{P}}_e(V_{8n})) = k+3$ and by Theorem 1.2.15, $\mathrm{sdim}(\widehat{\mathcal{P}}_e(V_{8n})) = 8n-k-3$.

Now we prove the result for $k=0$. In this case, we have the partition of $\widehat{V_{8n}}$ into V_1, V_2, V_3, V_4, V_5 and V_6, where $V_1 = \{\widehat{e}, \widehat{a}, \widehat{a^2}\}$, $V_2 = \{\widehat{a^{2i+1}b} : 1 \le i \le n\}, V_3 =$

$\{\widehat{a^{2i+1}b^3} \; : \; 1 \leq i \leq n\}, V_4 = \{\widehat{a^{2i}b} \; : \; 1 \leq i \leq n\}, \; V_5 = \{\widehat{b^2}, \widehat{ab^2}, \widehat{a^2b^2}\}$. Note that $\{\widehat{e}, \widehat{a^2}, \widehat{a^2b^2}, \widehat{b^2}\}$ forms a clique. It follows that $\omega(\widehat{\mathcal{P}}_e(V_{8n})) \geq 4$. Let, if possible there exists a clique C in $\widehat{\mathcal{P}}_e(V_{8n})$ such that $|C| > 4$. Since $V_2 \cup V_3 \cup V_4 \cup \{\widehat{a^2b}, \widehat{a^2b^2}\}$ is an independent set in $\widehat{\mathcal{P}}_e(V_{8n})$, we have $|C| \leq 5$. But $b^2 \nsim a$ gives $|C| \leq 4$; a contradiction. It follows that $\omega(\widehat{\mathcal{P}}_e(V_{8n})) = 4$ and hence by Theorem 1.2.15, $\mathrm{sdim}(\mathcal{P}_e(V_{8n})) = 8n - 4$. □

3.1.6 The Dicyclic Group Q_{4n}

The dicyclic group Q_{4n} of order $4n$ is defined in terms of generators and relations as

$$Q_{4n} = \langle a, b \; : \; a^{2n} = e, \; b^2 = a^n, \; ab = ba^{-1} \rangle.$$

Note that every element in $Q_{4n} \setminus \langle a \rangle$ can be written as $a^i b$ for some $1 \leq i \leq 2n-1$. Also, $\langle a^i b \rangle = \langle a^{n+i} b \rangle = \{e, a^i b, a^n, a^{n+i} b\}$ for all $0 \leq i \leq n-1$. We thus have

$$Q_{4n} = \langle a \rangle \bigcup_{i=0}^{n-1} \langle a^i b \rangle. \tag{3.8}$$

For $n = 1$, Q_{4n} is a cyclic group of order 4 so that $\mathcal{P}_e(Q_{4n})$ is complete.

Theorem 3.1.58. *For $n \geq 2$, the enhanced power graph of Q_{4n} is perfect.*

Proof. In order to prove $\mathcal{P}_e(Q_{4n})$ is perfect, we shall show that $\mathcal{P}_e(Q_{4n})$ does not contain a hole or an antihole of odd length at least 5. Suppose C is a hole of odd length at least 5. Since e, a^n are the dominating vertex of $\mathcal{P}_e(Q_{4n})$ we have $a^n, e \notin C$. From Figure 3.5, note that $\mathrm{N}[x] = \langle a \rangle$ if and only if $x \in \langle a \rangle \setminus \{e, a^n\}$. Also, we have $\mathrm{N}[x] = \langle a^i b \rangle$ if and only if $x \in \langle a^i b \rangle \setminus \{e, a^n\}$. Consequently all the elements in C are either from $\langle a \rangle \setminus \{e, a^n\}$ or $\langle a^i b \rangle \setminus \{e, a^n\}$. In both the cases C is not an induced cycle of odd length at least 5.

Now we show that $\mathcal{P}_e(Q_{4n})$ does not contain an antihole of length $2k+1$, where $k \geq 2$. Let if possible C be an antihole of length $2k+1$, where $k \geq 2$. Then $\overline{C}$ is a hole in $\overline{\mathcal{P}_e(Q_{4n})}$. Clearly, dominating vertices e and a^n are not the element in

C. Now suppose $C \cap \langle a \rangle \neq \emptyset$. Then there exists $x_1 \in C \cap \langle a \rangle$ such that $x_1 \sim x_2$ in $\overline{\mathcal{P}_e(Q_{4n})}$, where $x_2 \in C$. Consequently, $x_1 \nsim x_2$ in $\mathcal{P}_e(Q_{4n})$. We must have $x_2 \in \langle a^i b \rangle \setminus \{e, a^n\}$ for some i. Since $|C| \geq 5$ so that there exists $x_3 \in C \setminus \{x_1, x_2\}$ such that $x_1 \sim x_3$ and $x_2 \nsim x_3$ in $\mathcal{P}_e(Q_{4n})$. Thus $x_3 \in \langle a \rangle$. Also for $|C| \geq 5$ there exists $x_4 \in C \setminus \{x_1, x_2, x_3\}$ such that $x_3 \nsim x_4$, $x_1 \sim x_4$ and $x_2 \sim x_4$ in $\mathcal{P}_e(Q_{4n})$. Consequently, $x_4 \in (\langle a \rangle \cap \langle a^i b \rangle) \setminus \{e, a^n\}$; a contradiction. Thus we have $C \cap \langle a \rangle = \emptyset$. Now, let x_1, x_2, x_3, x_4 be consecutive vertices in C. Then $x_1 \sim x_3$ and $x_1 \sim x_4$ in $\mathcal{P}_e(Q_{4n})$ so that $x_1, x_3, x_4 \in \langle a^i b \rangle \setminus \{e, a^n\}$ for some i; a contradiction. Hence $\mathcal{P}_e(Q_{4n})$ is perfect. □

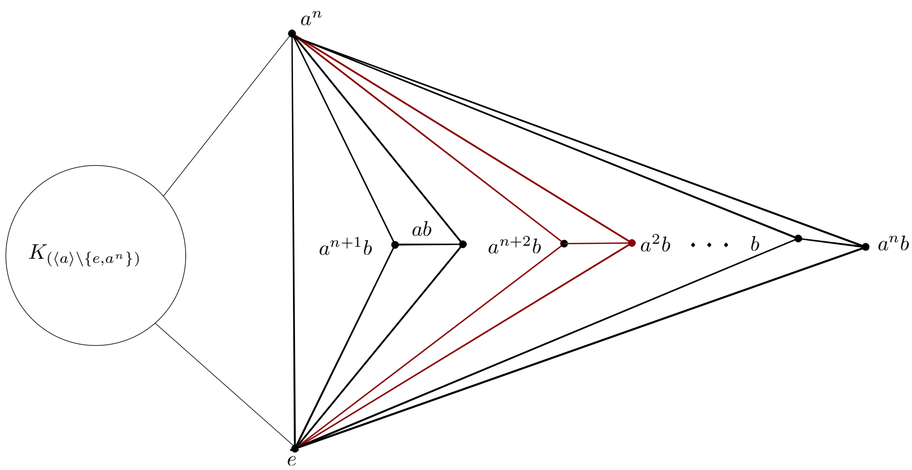


Figure 3.5: The enhanced power graph of Q_{4n}

In the following theorem, we investigate various parameters of $\mathcal{P}_e(Q_{4n})$.

Theorem 3.1.59. *For $n \geq 2$, we have the following results:*

(i) *The edge connectivity of $\mathcal{P}_e(Q_{4n})$ is* 3

(ii) *The independence number of $\mathcal{P}_e(Q_{4n})$ is $n + 1$*

(iii) *The matching number of $\mathcal{P}_e(Q_{4n})$ is $2n$*

(iv) *The vertex connectivity of $\mathcal{P}_e(Q_{4n})$ is* 2.

Proof. (i) In view of [Ma et al., 2013, Proposition 34], $\delta(\mathcal{P}_e(Q_{4n})) = 3$. Since $\kappa'(\mathcal{P}_e(Q_{4n})) \leq \delta(\mathcal{P}_e(Q_{4n}))$, we have $\kappa'(\mathcal{P}_e(Q_{4n})) \leq 3$. By Figure 3.5, we observe that every edge contained in some clique of size at least 4. Also, note that e and a^n are dominating vertex in $\mathcal{P}_e(Q_{4n})$. We conclude that $\kappa'(\mathcal{P}_e(Q_{4n})) \geq 3$ and hence $\kappa'(\mathcal{P}_e(Q_{4n})) = 3$.

(ii) Considering the structure of Q_{4n}, each of $\langle a \rangle$ and $\langle a^i b \rangle$ is a maximal cyclic subgroup of Q_{4n}. Therefore, it follows from Theorem 3.1.4 and Eq. 3.8 that $\alpha(\mathcal{P}_e(Q_{4n})) = n + 1$.

(iii) Consider the edge set $M = \{(a^i, a^{n+i}) \ : \ 1 \leq i \leq n\} \cup \{(a^i b, a^{n+i} b) \ : \ 1 \leq i \leq n\}$. Note that M is a matching set of size $2n$. Consequently, $\alpha'(\mathcal{P}_e(Q_{4n})) \geq 2n$. It is well known that the matching number $\alpha'(\mathcal{G})$ of a graph $\mathcal{G}$ is at most $\frac{1}{2}|V(\mathcal{G})|$. Thus $\alpha'(\mathcal{P}_e(Q_{4n})) \leq 2n$. Hence result holds.

(iv) From Figure 3.5, we observe that for any $1 \leq i \leq 2n$, we have $\mathrm{N}[a^i b] = \{e, a^i b, a^{n+i} b, a^n\}$ follows the subgraph induced by the vertices of $Q_{4n} \setminus \{e, a^n\}$ is disconnected. As a consequence, $\kappa(\mathcal{P}_e(Q_{4n})) \leq 2$. On the other hand, e, a^n are the dominating vertex so that $\kappa(\mathcal{P}_e(Q_{4n})) > 1$ and hence $\kappa(\mathcal{P}_e(Q_{4n})) = 2$. □

In view of Lemma 1.2.10, we have the following corollaries.

Corollary 3.1.60. *The vertex covering number of* $\mathcal{P}_e(Q_{4n})$ *is* $3n - 1$.

Corollary 3.1.61. *The edge covering number of* $\mathcal{P}_e(Q_{4n})$ *is* $2n$.

In the following, we calculate the strong metric dimension of $\mathcal{P}_e(Q_{4n})$.

Theorem 3.1.62. *The strong metric dimension of* $\mathcal{P}_e(Q_{4n})$ *is* $4n - 2$.

Proof. We observe that the partition of $\widehat{Q}_{4n}$ into V_1 and V_2, where

$$V_1 = \{\widehat{e},\ \widehat{a},\ \widehat{b}\},\ V_2 = \{\widehat{a^i b} \ : \ 1 \leq i \leq n-1\}.$$

From the adjacency relation in $\mathcal{P}_e(Q_{4n})$, we have $\widehat{\mathcal{P}}_e(Q_{4n}) \cong K_{1,n+1}$, and hence $\omega(\widehat{\mathcal{P}}_e(Q_{4n})) = 2$. Finally, by Theorem 1.2.15, we have $\mathrm{sdim}(\mathcal{P}_e(Q_{4n})) = 2(2n - 1)$. □

3.2 The Enhanced Power Graph of Semigroups

In this section, we study the enhanced power graph of a semigroup S. First we classify the structure of the enhanced power graph of a semigroup S. We categorized the semigroup S such that $\mathcal{P}_e(S)$ is complete, connected, bipartite, tree, regular and planar, respectively. Finally, we obtain the independence number and the minimum degree of $\mathcal{P}_e(S)$.

We begin with the classification of connected components of $\mathcal{P}_e(S)$ in the following theorem.

Theorem 3.2.1. *Let S be a semigroup of bounded exponent. Then S_f is a connected component of $\mathcal{P}_e(S)$ with unique idempotent f. Moreover, the connected components of $\mathcal{P}_e(S)$ are precisely $\{S_f \ : \ f \in E(S)\}$ and the number of connected components of $\mathcal{P}_e(S)$ is equal to $|E(S)|$.*

Proof. Suppose $a, b \in S_f$. If any one of them is f, then $a \sim b$ in $\mathcal{P}_e(S)$. If $a, b \in S_f \setminus \{f\}$, then there is a path $a \sim f \sim b$ in $\mathcal{P}_e(S)$. Let if possible $x \in S \setminus S_f$ such that $x \sim b$ for some $b \in S_f$ so that $x, b \in \langle c \rangle$ for some $c \in S$. Since $b^n = f$ for some $n \in \mathbb{N}$ so that $f \in \langle c \rangle$. Consequently we get $f \in \langle y \rangle$ for all $y \in \langle c \rangle$ (see Lemma 1.1.4). It follows that $x \in S_f$; a contradiction. Hence, S_f is a connected component of $\mathcal{P}_e(S)$ and by Remark 1.1.26, the connected components of $\mathcal{P}_e(S)$ are $\{S_f \ : \ f \in E(S)\}$. □

Corollary 3.2.2. *For any semigroup S, $\mathcal{P}_e(S)$ is a null graph if and only if S is a band.*

Corollary 3.2.3. *Let S be a semigroup of bounded exponent. Then $\mathcal{P}_e(S)$ is connected if and only if S contains exactly one idempotent. In this case,* $\text{diam}(\mathcal{P}_e(S)) \leq 2$.

Unless stated otherwise, hereafter S_f always denotes the connected component of $\mathcal{P}_e(S)$ containing the idempotent f.

Theorem 3.2.4. *Let S be a semigroup with exponent n. Then $\mathcal{P}_e(S)$ is complete if and only if S is a monogenic semigroup.*

Proof. Let S be a monogenic semigroup. Then there exists $a \in S$ such that $S = \langle a \rangle$. For any $x, y \in \mathcal{P}_e(S)$, we have $x, y \in \langle a \rangle$. Thus, by definition, $\mathcal{P}_e(S)$ is complete. Conversely, suppose that $\mathcal{P}_e(S)$ is complete. By Lemma 1.1.25, $o(x) \leq 2n$ for all $x \in S$. Now choose an element $x \in S$ such that $o(x)$ is maximum. In order to prove that S is monogenic, we show that $S = H$, where $H = \langle x \rangle$. If $S \neq H$, then there exists $y \in S$ but $y \notin H$. Since $\mathcal{P}_e(S)$ is complete, $x, y \in \langle z \rangle$ for some $z \in S$. Also note that $\langle z \rangle = \langle x \rangle$. Consequently, $y \in \langle x \rangle$; a contradiction. Thus, $S = \langle x \rangle$. Hence, S is a monogenic semigroup. □

Theorem 3.2.5. *Let S be a semigroup. Then the following statements are equivalent:*

(i) *The set $\pi(S) \subseteq \{1, 2\}$*

(ii) *$\mathcal{P}_e(S)$ is acyclic graph*

(iii) *$\mathcal{P}_e(S)$ is bipartite.*

Proof. (i)⇒(ii) Suppose $\pi(S) \subseteq \{1, 2\}$. Let if possible, there exists a cycle, viz. $a_0 \sim a_1 \sim \cdots \sim a_k \sim a_0$, in $\mathcal{P}_e(S)$. Since this cycle must belongs to some connected component of $\mathcal{P}_e(S)$. As $o(a) \leq 2$ for all $a \in S$, we get S is of bounded exponent. By Theorem 3.2.1, there exists $f \in E(S)$ such that $a_i \in S_f$ for all i, where $0 \leq i \leq k$. Consequently, at most one i such that a_i is an idempotent element. If none of the vertices of this cycle are idempotents, then $a_0, a_1, f \in \langle z_1 \rangle$ for some $z_1 \in S$ and $f \in E(S)$. Consequently, $\mathrm{o}(z_1) \geq 3$; a contradiction. If one of the vertex of the cycle $a_0 \sim a_1 \sim \cdots \sim a_k \sim a_0$ is idempotent, then note that there exist two non idempotent elements a_i, a_j such that $a_i \sim a_j$. Thus, $a_i, a_j, f \in \langle z \rangle$ for some $z \in S$ which is not possible as $o(z) \leq 2$.

(ii)⇒(iii) Since $\mathcal{P}_e(S)$ is acyclic graph so that it does not contain any cycle. By Theorem 1.2.2, $\mathcal{P}_e(S)$ is bipartite.

(iii)⇒(i) Suppose $\mathcal{P}_e(S)$ is a bipartite graph. By Theorem 1.2.2, $\mathcal{P}_e(S)$ does not contain any odd cycle. To prove $\pi(S) \subseteq \{1, 2\}$. Let if possible there exists $a \in S$ such that $\mathrm{o}(a) \geq 3$, then $a \sim a^2 \sim a^3 \sim a$ is an odd cycle in $\mathcal{P}_e(S)$; a contradiction of the fact that $\mathcal{P}_e(S)$ is bipartite. □

In view of Corollary 3.2.3 and Theorem 3.2.5, we have the following corollary.

Corollary 3.2.6. *The enhanced power graph $\mathcal{P}_e(S)$ is a tree if and only if $|E(S)| = 1$ and $\pi(S) \subseteq \{1, 2\}$.*

Theorem 3.2.7. *The enhanced power graph $\mathcal{P}_e(S)$ is k-regular if and only if S is the union of mutually disjoint monogenic subsemigroups of S of size $k + 1$.*

Proof. Suppose $\mathcal{P}_e(S)$ is k-regular. Note that the order of each element is at most $k+1$. Otherwise there exists $x \in S$ such that $\deg(x) \neq k$; a contradiction of the fact that $\mathcal{P}_e(S)$ is k-regular. For each $x \in S$, there exist $m_x, g_x \in \mathbb{N}_0$ such that $m_x + g_x \leq k + 1$ and $a^{m_x+g_x} \leq k + 1$ is an idempotent element. Choose $n = (k + 1)!$ gives a^n is an idempotent element for all $a \in S$. Then S is of bounded exponent. By Remark 1.1.26, $S = \bigcup\limits_{f \in E(S)} S_f$. For $f, f' \in E(S)$, we get $|S_f| - 1 = \deg(f) = \deg(f') = |S_{f'}| - 1$. It follows that $|S_f| = |S_{f'}| = k + 1$. Since $\mathcal{P}_e(S)$ is regular, for each $a \in S_f$, we have $\deg(a) = |S_f| - 1$. Consequently, for each $f \in E(S)$, the subgraph induced by S_f is complete. Now we show that S_f is a subsemigroup of S. Let $x, y \in S_f$. Then there exist $m, n \in \mathbb{N}$ such that $x^m = y^n = f$. Since $x \sim y$ as S_f is complete so $x, y \in \langle z \rangle$ for some $z \in S$. Consequently, $\langle z \rangle \subseteq S_f$ so that $xy \in S_f$. By Theorem 3.2.4, S_f is a monogenic subsemigroup of S.

Conversely, suppose S is the union of mutually disjoint monogenic subsemigroup S_i of S of size $k + 1$ where $i \in \Lambda$ and Λ is an index set. For our convenient, we assume that $S_i = \langle a_i \rangle$, where $i \in \Lambda$. Note that $a_i, a_j \notin S_f$ for some $f \in E(S)$. For instance if $a_i, a_j \in S_f$, then $f \in \langle a_i \rangle \cap \langle a_j \rangle$; a contradiction. Also, $f \in E(S) \subseteq S$

implies $f \in \langle a_i \rangle$ for some i. Consequently, $a_i \in S_f$ and so $\langle a_i \rangle \subseteq S_f$. If $x \in S_f \setminus \langle a_i \rangle$ then $x \in \langle a_j \rangle$ for some $j \neq i$. Therefore, $x \in S_{f'}$ for some $f' \neq f$; a contradiction of Remark 1.1.26. Thus, for each $f \in E(S)$, we have $S_f = \langle a_i \rangle$ for some i. By Theorem 3.2.4, for each $f \in E(S)$, the subgraph induced by S_f is complete and by hypothesis the graph $\mathcal{P}_e(S)$ is regular. □

By the similar lines of the proof of Theorem 3.2.7, the following theorem on the completeness of the connected components of $\mathcal{P}_e(S)$.

Theorem 3.2.8. *Let S be a semigroup of bounded exponent. Then the connected components of the enhanced power graph $\mathcal{P}_e(S)$ are complete if and only if S is the union of mutually disjoint monogenic subsemigroup of S.*

Theorem 3.2.9. *A semigroup S with exponent n is completely regular semigroup if and only if each connected component of $\mathcal{P}_e(S)$ forms a group.*

Proof. Suppose S is completely regular semigroup. Then every $\mathcal{H}$-class of S is a group (see Proposition 1.1.23). In view of Theorem 3.2.1, each connected component of $\mathcal{P}_e(S)$ is of the form S_f for some $f \in E(S)$. To prove that each connected component $\mathcal{P}_e(S)$ forms a group, we show that $S_f = H_f$ for each $f \in E(S)$. Let $a \in H_f$. Then $a^n = f$ for some $n \in \mathbb{N}$ as S is of exponent n so that $a \in S_f$. On the other hand, suppose $a \in S_f$. If $a \in H_{f'}$ for some $f' \neq f \in E(S)$, then $a \in S_{f'}$; a contradiction. Thus $H_f = S_f$.

Conversely, suppose every connected component of $\mathcal{P}_e(S)$ forms a group. To prove S is completely regular, we show that every $\mathcal{H}$-class forms a group (see Proposition 1.1.23). Let $a \in S$. Then $a \in S_f$ for some $f \in E(S)$. We claim that $H_a = S_f$. Suppose $b \in S_f$. By Remark 1.1.21, $(b, f) \in \mathcal{H}$. Also, we have $(a, f) \in \mathcal{H}$ so that $(a, b) \in \mathcal{H}$. It follows that $S_f \subseteq H_a$. On the other hand let $b \in H_a$. Then $a \in S_f$ implies that $b \in H_f$. Since H_f contains an idempotent so that H_f forms a group (see Corollary 1.1.22). It follows that $b^m = f$ for some $m \in \mathbb{N}$. Hence, $H_a = S_f$ for some $f \in E(S)$. □

Proposition 3.2.10. *An element a of an arbitrary semigroup S is an isolated vertex in $\mathcal{P}_e(S)$ if and only if*

(i) *a is an idempotent in S.*

(ii) $H_a = \{a\}$.

(iii) *$m_x = 1$ for each $x \in S_a$.*

Proof. Let a be an isolated vertex in $\mathcal{P}_e(S)$. Clearly $a \in E(S)$. Otherwise $a \sim a^2$. Consequently, H_a forms a group with the identity element a. Thus, every vertex of the enhanced power graph induced by H_a will be adjacent with a. Let if possible, $x \in H_a \setminus \{a\}$ then $x \sim a$; a contradiction. If $b \neq a \in S_a$ then $b^m = a$ for some $m \in \mathbb{N}$. It follows that $b \sim a$; a contradiction. Thus, $S_a = \{a\}$ and $m_a = 1$.

Conversely suppose for $a \in S$ satisfy (i), (ii) and (iii). Let if possible, $a \sim x$ for some $x \in S$. Then $a, x \in \langle b \rangle$ for some $b \in S$. Consequently, $\langle x \rangle \subseteq \langle b \rangle$ and $a \in \langle x \rangle$. Note that $r_x = 1$. If $r_x > 1$, then $a = x^p$ for some $p > m_x$. Consequently, $(a, x^q) \in \mathcal{H}$ for some $m \leq q \neq p$ and so $|H_a| > 1$; a contradiction. Thus $r_x = 1$ and $m_x = 1$ implies $x = a$; a contradiction. □

Now we discuss the planarity of $\mathcal{P}_e(S)$. We begin with the following proposition which ensures the non-planarity of $\mathcal{P}_e(S)$.

Proposition 3.2.11. *Let $\mathcal{P}_e(S)$ be a planar graph. Then $o(a) < 5$ for all $a \in S$.*

Proof. Let if possible, there exists an element $a \in S$ such that $o(a) \geq 5$. Then the subgraph induced by $\langle a \rangle$ contains K_5. By Theorem 1.2.7, $\mathcal{P}_e(S)$ is non-planar; a contradiction. □

Theorem 3.2.12. *Let S be a semigroup such that the index of every element of order four is either one or two. Then $\mathcal{P}_e(S)$ is planar if and only if the following condition holds*

(i) *For $a \in S$, we have $o(a) \leq 4$*

(ii) *S does not contain $a, b, c \in S$ such that $o(a)$ =$o(b)$ =$o(c) = 4$, $m_a = m_b = m_c = 2$ and $|\langle a\rangle \cap \langle b\rangle \cap \langle c\rangle| = 3$.*

Proof. First suppose that S satisfies the conditions (i) and (ii). In order to prove $\mathcal{P}_e(S)$ is planar, it is sufficient to show that every connected component of $\mathcal{P}_e(S)$ is planar. Since $o(a) \le 4$ for all $a \in S$ so that S is of bounded exponent. In view of Theorem 3.2.1, each connected component of $\mathcal{P}_e(S)$ is of the form S_f, where $f \in E(S)$. Now we establish a planar drawing for S_f. Consider the set $A_4 = \{a \in S_f : \mathrm{o}(a) = 4\}$. In view of given hypothesis, note that the sets $A = \{a \in A_4 : m_a = 1\}$ and $B = \{a \in A_4 : m_a = 2\}$ forms a partition of A_4 i.e. $A_4 = A \cup B$. Let $a \in A_4$. Observe that $a \sim b$ if and only if $b \in \langle a\rangle$. For instance, if $a \sim b$ for some $b \in S \setminus \langle a\rangle$, then $a, b \in \langle c\rangle$ for some $c \in S$. Since $o(a) = 4$ and $o(c) \le 4$, we have $b \in \langle a\rangle$; a contradiction. Now we prove that the subgraph induced by the elements of $\mathrm{N}(A_4)$ is planar though the following claims:

Claim 3.2.13. *The subgraph induced by* $\mathrm{N}[B]$ *is planar.*

Proof of claim: First suppose that $a \in B$. Then $m_a = 2$ so that $\langle a\rangle = \{a, a^2, a^3, a^4 : a^5 = a^2\}$ and $\langle a^2\rangle = \langle a^4\rangle$. It follows that $a^2 \sim x$ if and only if $a^4 \sim x$ for all $x \in S$. Further, if $a^2 \sim x$ for some $x \in S \setminus \langle a\rangle$, then $a^2, x \in \langle y\rangle$ for some $y \in S$. By (i), $o(y) \le 4$ and $x \in \{a^2, a^3, a^4\}$ gives $x = y$ and $o(x) = 4$. By hypothesis, we get either $m_x = 2$ or $m_x = 1$. Since $o(a^2) = 3$ and $a^2 \in \langle x\rangle$ implies that $m_x = 2$. If possible, let $a^2 \sim y$ for some $y \in S \setminus (\langle a\rangle \cup \{x\})$. Then by using the similar argument, we get $\langle y\rangle = \{y, a^2, a^3, a^4\}$ and $m_y = 2$. Therefore, we have $|\langle a\rangle \cap \langle x\rangle \cap \langle y\rangle| = 3$ and $m_a = m_x = m_y = 2$; a contradiction of (ii). It follows that either $\mathrm{N}[\langle a\rangle] = \langle a\rangle$ or $\mathrm{N}[\langle a\rangle] = \langle a\rangle \cup \{x\}$ for some $x \in B$. Note that $\mathrm{N}[x] = \langle x\rangle = \{x, a^2, a^3, a^4\}$ as $x \in A_4$. Thus, we can draw the subgraph induced by all the vertices of $\mathrm{N}(B)$ in a plane without cutting an edge shown in Figure 3.6.

Claim 3.2.14. *The subgraph induced by* $\mathrm{N}[A]$ *is planar.*

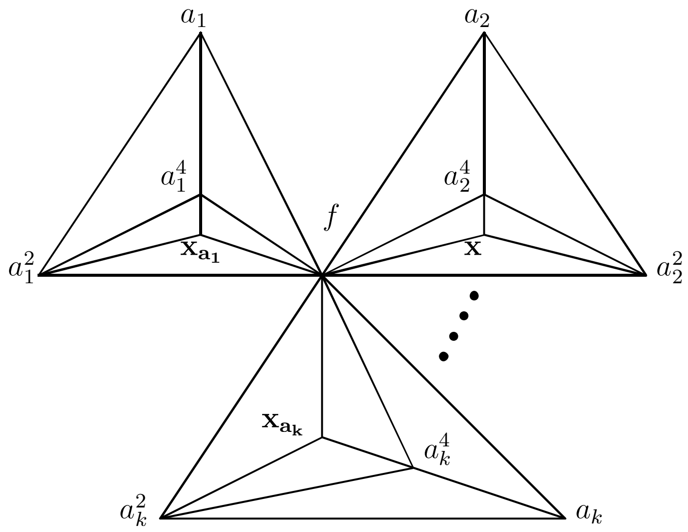


Figure 3.6: Planar drawing of $\mathcal{P}_e(\mathrm{N}[B])$

Proof of claim: Let $a \in A$. Then $\langle a \rangle$ is a cyclic subgroup of order 4 and $\mathrm{N}[a] = \mathrm{N}[a^3] = \langle a \rangle$. If $a^2 \sim z$ for some $z \in S \setminus \langle a \rangle$, then we prove that $\mathrm{N}[z] = \langle z \rangle$. Since $a^2, z \in \langle t \rangle$ for some $t \in S$, we get $3 \le o(t) \le 4$ as a^2, f, z are distinct element and $o(t) \le 4$ (by (ii)). If $o(t) = 3$, then we must have $t = z$. Let if possible, $z \sim s$ for some $s \in S \setminus \langle z \rangle$, then $z, s \in \langle q \rangle$ for some $q \in S$. Since $o(z) = 3$ and $o(q) \le 4$ (by (i)) implies $q = s$ and $o(s) = 4$. As a result, $m_s = 2$. Also $a^2 \in \langle s \rangle$ and $o(a^2) = 2$ which is not possible because $\langle s \rangle$ does not contain an element of order 2. Therefore, $\mathrm{N}[z] = \langle z \rangle = \{f, z, a^2\}$. We may now assume that $o(t) = 4$. Note that $m_t \in \{1, 2\}$ as $t \in A_4$. For $o(a^2) = 2$ and $a^2, z \in \langle t \rangle$, we get $m_t = 1$ and $\langle t \rangle = \langle z \rangle$. It follows that $\mathrm{N}[z] = \langle z \rangle = \{f, z, a^2, z^3 = t\}$. Therefore, we have $\mathrm{N}[\langle a \rangle] = \langle a \rangle \cup P \cup Q$, where $P = \{z \in S_f : \ \mathrm{N}[z] = \{z, a^2, f\} = \langle z \rangle\}$ and $Q = \{z \in S_f : \ \mathrm{N}[z] = \{z, a^2, z^3, f\} = \langle z \rangle\}$. Thus the subgraph induced by the vertices of $\mathrm{N}[\mathrm{N}[A] \setminus \{f\}]$ can be drawn in a plane without cutting an edge shown in Figure 3.7.

Moreover, we observed that if $x \in \mathrm{N}[\mathrm{N}[A] \setminus \{f\}]$, then $\mathrm{N}[x] \subseteq \mathrm{N}[\mathrm{N}[A] \setminus \{f\}]$.

Claim 3.2.15. *The subgraph induced by the neighbours of* $A_3 = \{x \in S_f \setminus \mathrm{N}(A_4) \ :$

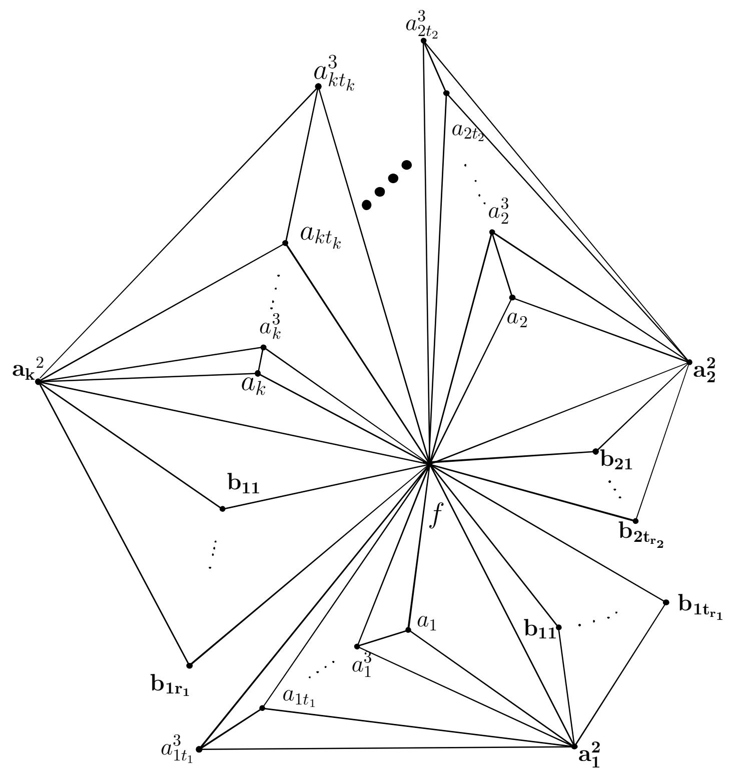


FIGURE 3.7: Planar drawing of $\mathcal{P}_e(\mathrm{N}[A])$

$o(x) = 3\}$ *is planar.*

Proof of claim: Let $a \in A_3$ and $a \sim x$ for some $x \in S_f \setminus \langle a \rangle$. Note that $x \notin \mathrm{N}[A_4]$. Since $a, x \in \langle t \rangle$ for some $t \in S$ gives $a \in \langle x \rangle$ and $o(x) = 4$; a contradiction. If $m_a = 1$, then $\mathrm{N}[\langle a \rangle] = \langle a \rangle$. Otherwise, there exists $y_a \in \langle a \rangle$ such that $o(y_a) = 2$. Now we define $A_{y_a} = \{t \in A_3 \ : \ y_a \in \langle t \rangle\}$. For $t \in A_{y_a}$, clearly, we have $t \in A_3$. Thus the subgraph induced by $\mathrm{N}[A_3]$ can be drawn in a plane without cutting an edge.

Additionally, we can conclude that the drawing of the subgraph induced by $\mathrm{N}[A_3] \cup \mathrm{N}[\mathrm{N}[A] \setminus \{f\}]$ is planar. Now the set A_2 consists the remaining vertices is left in the subgraph induced by $S_f \setminus \mathrm{N}[A_3] \cup \mathrm{N}[\mathrm{N}[A] \setminus \{f\}]$. For $x \in A_2$, we must have $o(x) = 2$ and $\mathrm{N}(x) = \{f\}$. Thus, the result follows.

Conversely, suppose $\mathcal{P}_e(S)$ is planar. Then by Proposition 3.2.11, $o(a) \leq 4$ for all $a \in S$. On contrary, we assume S does not satisfies the condition (ii). Then there exists $a, b, c \in S$ such that $o(a) = o(b) = o(c) = 4$, $m_a = m_b = m_c = 2$ and $|\langle a\rangle \cap \langle b\rangle \cap \langle c\rangle| = 3$. Therefore, we have $\langle a\rangle \cap \langle b\rangle \cap \langle c\rangle = \{x, y, z\}$. Consequently, $\mathcal{P}_e(S)$ contains a subgraph $K_{3,3}$ whose partitions sets are $X = \{a, b, c\}$ and $Y = \{x, y, z\}$; a contradiction of the fact that $\mathcal{P}_e(S)$ is planar (see Theorem 1.2.7). □

In view of Theorem 1.1.23, we have the following corollaries of Theorem 3.2.12.

Corollary 3.2.16. *Let S be a completely regular semigroup. Then $\mathcal{P}_e(S)$ is planar if and only if $o(a) \leq 4$ for all $a \in S$.*

Corollary 3.2.17. [Bera and Bhuniya, 2017, Theorem 2.6] *Let G be a finite group. Then $\mathcal{P}_e(G)$ is planar if and only if $\pi(G) \subseteq \{1, 2, 3, 4\}$.*

Now we construct a semigroup S which do not satisfying condition (ii) of Theorem 3.2.12 such that $\mathcal{P}_e(S)$ is non-planar.

Example 3.2.18. Assume that $S = \{a, x, y, z, b, c : a^5 = a^2\}$ is a semigroup shown in Figure 3.8 and the enhanced power graph of a semigroup S given in Figure 3.9. Clearly, it contains a subgraph $K_{3,3}$ with the partitioned sets $A = \{a, b, c\}$ and $B = \{x, y, z\}$. So by Theorem 1.2.7, the enhanced power graph of S is non-planar.

Now we obtain the minimum degree and the independence number of $\mathcal{P}_e(S)$.

Theorem 3.2.19. *Let S be a semigroup with exponent n. Then*

(i) *$\delta(\mathcal{P}_e(S)) = m - 1$, where $m = \min\{o(x) : x \in \mathcal{M}\}$.*

(ii) *$\alpha(\mathcal{P}_e(S))$ is the number of maximal monogenic subsemigroup of S.*

Proof. (i) Let $x \in S$. Then $x \in \langle y\rangle$ for some y, where $y \in \mathcal{M}$. Since the subgraph induced by $\langle y\rangle$ forms a clique, we get $\deg(x) \geq m - 1$. Now choose $z \in S$ such that $z \in \mathcal{M}$ and $o(z) = m$. Then $\deg(z) = m - 1$. Thus, we have the result.

Table

	a	x	y	z	b	c
a	x	y	z	x	x	x
x	y	z	x	y	y	y
y	z	x	y	z	z	z
z	x	y	z	x	x	x
b	x	y	z	x	x	x
c	x	y	z	x	x	x

FIGURE 3.8: Cayley table

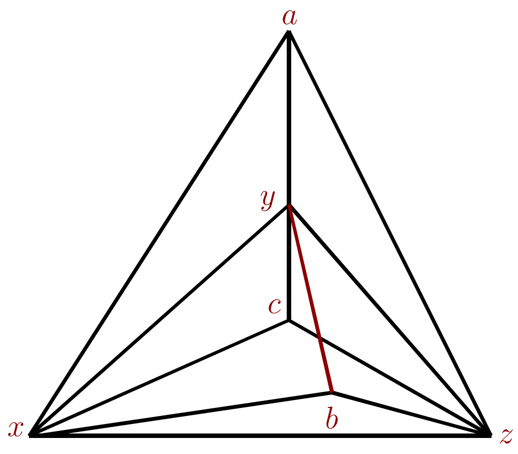


FIGURE 3.9: $\mathcal{P}_e(S)$

(ii) First note that for $x, y \in \mathcal{M}$ such that $\langle x \rangle \neq \langle y \rangle$, we have $x \nsim y$. For instance, if $x \sim y$ then $x, y \in \langle z \rangle$ for some $z \in S$. Consequently, $\langle x \rangle = \langle y \rangle = \langle z \rangle$; a contradiction. Thus, $\alpha(\mathcal{P}_e(S))$ is more than or equal to number of maximal monogenic subsemigroup of S. Further, observed that S is the union of maximal monogenic subsemigroup of S and the subgraph induced by maximal monogenic subsemigroups forms a clique in $\mathcal{P}_e(S)$. It follows that $\alpha(\mathcal{P}_e(S))$ is less than or equal to number of maximal monogenic subsemigroup of S. Hence, the result holds . $\square$

3.2.1 Chromatic Number of $\mathcal{P}_e(S)$

Shitov [2017], proved that the chromatic number of power graph of an arbitrary semigroup is at most countable. In Chapter 2, we have shown that the chromatic number of cyclic graph of an arbitrary semigroup is at most countable. But this result need not hold in the case of enhanced power graph associated with semigroups. In the following example, we construct a semigroup S such that $\chi(\mathcal{P}_e(S))$ is uncountable.

Example 3.2.20. Consider the sets $B = \{(i, j) \in [1, 2] \times [1, 2] : \ j < i\}$ and

$A = B \cup [1, 2]$. Now we define a relation ρ on A by

$$(i, j)^2 \rho\, i \text{ and } (i, j)^3 \rho\, j \text{ for all } (i, j) \in B.$$

We prove that the chromatic number of the enhanced power graph of the semigroup $A^*/\rho^\#$ (see Theorem 1.1.18) is uncountable. First we claim that: $i\rho^\# \neq j\rho^\#$ for $i, j \in [1, 2]$. On contrary, we assume that $i\rho^\# = j\rho^\#$ for some $i, j \in [1, 2]$. Then $i\ \rho^\# j$. Since $\rho^\# = (\rho^c)^e$ (see Proposition 1.1.20) so that there exist $z_1, z_2, \ldots, z_n \in A^*$ such that $z_1 = i$ and $z_n = j$ with $z_k\ \rho^c\ z_{k+1}$ for all k, where $1 \leq k \leq n - 1$. Consequently, we get $z_k = x_k a_k y_k$ and $z_{k+1} = x_k b_k y_k$ for some $x_k, y_k, a_k, b_k \in A^*$ and $a_k\ \rho\ b_k$. Observe that $a_k, b_k \in \{u, (v, w)^2, (v', w')^3\}$ for some $u, v, w, v', w' \in [1, 2]$. For $i = z_1 = x_1 a_1 y_1$, we have $x_1 = y_1 = \epsilon$ and $a_1 = i$. Then $z_2 = b_1$ because $z_2 = x_1 b_1 y_1$. For $z_2 = b_1\ \rho\ i$, we get either $z_2 = (i, t)^2$ or $z_2 = (t, i)^3$, where $t \in [1, 2]$. Suppose $z_2 = (i, t)^2$. Since $z_2 = x_2 a_2 y_2$ and so $x_2 = y_2 = \epsilon$ and $a_2 = (i, t)^2$. On continuing this process we obtain either $z_n = i$ or $z_n = (i, t)^2$ which is not possible because $j \neq i$ and $j \neq (i, t)^2$. Similarly we get a contradiction when $z_2 = (t, i)^3$. This completes the proof of claim.

Let $i, j \in [1, 2]$. Without loss of generality, we assume that $j < i$. Then $i\rho^\# \neq j\rho^\#$. Since $\rho^\#$ is a congruence so that $(i, j)^m \rho^\# = ((i, j)\rho^\#)^m$ for all $m \in \mathbb{N}$. Therefore, we have $i\rho^\# = (i, j)^2 \rho^\# = ((i, j)\rho^\#)^2$ and $j\rho^\# = (i, j)^3 \rho^\# = ((i, j)\rho^\#)^3$. This implies that $i\rho^\#, j\rho^\# \in \langle (i, j)\rho^\# \rangle$ gives $i\rho^\# \sim j\rho^\#$ in $\mathcal{P}_e(A^*/\rho^\#)$. Thus the uncountable set $C = \{i\rho^\# : \ i \in [1, 2]\}$ forms a clique in $\mathcal{P}_e(A^*/\rho^\#)$. By $\omega(\mathcal{P}_e(A^*/\rho^\#)) \leq \chi(\mathcal{P}_e(A^*/\rho^\#))$, the result holds. $\square$