Chapter 5

Equality of Graphs

Aalipour et al. [2017] characterise finite groups such that an arbitrary pair of graphs $\mathcal{P}(G), \mathcal{P}_e(G)$ and $\Delta(G)$ are equal. In order to extend their results from groups to semigroups, in this chapter, we classify finite semigroups such that the pair of graphs, viz. $\mathcal{P}(S), \Gamma(S), \mathcal{P}_e(S)$ and $\Delta(S)$ are equal. In this chapter by $\Delta(S)$ we mean $\Delta(S, \Omega)$ with $\Omega = S$.

We begin with an example of a semigroup whose cyclic graph and enhanced power graphs are not equal.

Example 5.0.1. Let $S = M(3, 2) = \{a, a^2, a^3, a^4\}$, where $a^5 = a^3$. Note that $\mathcal{P}_e(S)$ is complete (cf. Theorem 3.2.4) but $\Gamma(S)$ is not complete (cf. Theorem 2.1.5). Then $\mathcal{P}_e(S) \neq \Gamma(S)$. See Figure 5.1.

Theorem 5.0.2. *The enhanced power graph $\mathcal{P}_e(S)$ is equal to $\Gamma(S)$ if and only if for each $a \in S$, we have one of the following form:*

(i) $\langle a \rangle = \langle a : a^{1+r} = a \rangle$

(ii) $\langle a \rangle = \langle a : a^{2+r} = a^2 \rangle$

(iii) $\langle a \rangle = \langle a : a^{3+r} = a^3 \rangle$ *where r is odd.*

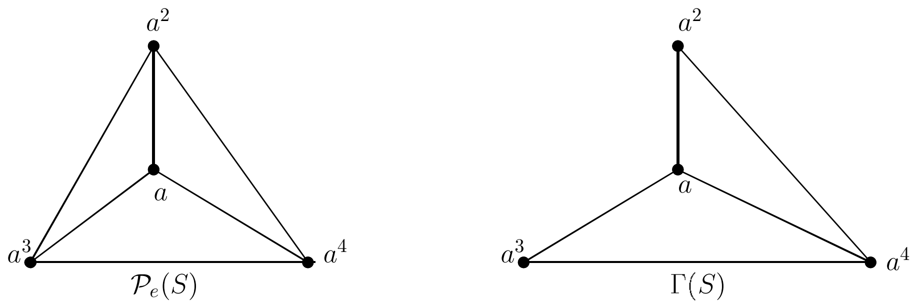

FIGURE 5.1: $\mathcal{P}_e(S)$ and $\Gamma(S)$ of the semigroup $S = M(3,2)$

Proof. First suppose that $\mathcal{P}_e(S) = \Gamma(S)$. On contrary, suppose that there exists $a \in S$ such that $\langle a \rangle$ is not of the form given in (i), (ii) and (iii). Then either $\langle a \rangle = M(3,r)$ with r is even or $\langle a \rangle = M(m,r)$, where $m \geq 4$. If $\langle a \rangle = M(3,r)$ with r is even, then by the similar argument used in the proof of Theorem 2.1.5, note that a^2 and a^3 are not adjacent in $\Gamma(S)$. Since $a^2, a^3 \in \langle a \rangle$, we have $a^2 \sim a^3$ in $\mathcal{P}_e(S)$. Thus $\mathcal{P}_e(S) \neq \Gamma(S)$; a contradiction. If $\langle a \rangle = M(m,r)$ with $m \geq 4$, then again by the proof of Theorem 2.1.5, note that a^2 is not adjacent to a^3 in $\Gamma(S)$. Clearly, $a^2 \sim a^3$ in $\mathcal{P}_e(S)$. Consequently, $\mathcal{P}_e(S) \neq \Gamma(S)$; again a contradiction. Hence, for each $a \in S$, $\langle a \rangle$ must be one of the form given in (i), (ii) and (iii).

Conversely, suppose that for each $a \in S$, $\langle a \rangle$ is one of the form given in (i), (ii) and (iii). Since $\Gamma(S)$ is a (spanning) subgraph of $\mathcal{P}_e(S)$ (cf. Lemma 1.2.16), it is sufficient to show that for any $x, y \in S$ such that $x \sim y$ in $\mathcal{P}_e(S)$, we have $x \sim y$ in $\Gamma(S)$. Let $x \sim y$ in $\mathcal{P}_e(S)$. Then there exists $z \in S$ such that $x, y \in \langle z \rangle$. By the hypothesis, $\langle z \rangle$ is one of the form given in (i), (ii) and (iii). By Theorem 2.1.5, $\Gamma(\langle z \rangle)$ is complete. Consequently, $\langle x, y \rangle$ is a monogenic subsemigroup of S. Hence, $x \sim y$ in $\Gamma(S)$. □

Corollary 5.0.3. *For a finite group G, we have $\Gamma(G) = \mathcal{P}_e(G)$.*

Example 5.0.4. Let $S = M(2,6) = \{a, a^2, a^3, a^4, a^5, a^6, a^7\}$ where $a^8 = a^2$. By Theorem 2.1.5, $\Gamma(S)$ is complete. Further, note that neither $a^2 \in \langle a^3 \rangle$ nor $a^3 \in \langle a^2 \rangle$. Thus, a^2 and a^3 is not adjacent in $\mathcal{P}(S)$. Hence, $\Gamma(S) \neq \mathcal{P}(S)$. See Figure 5.2.

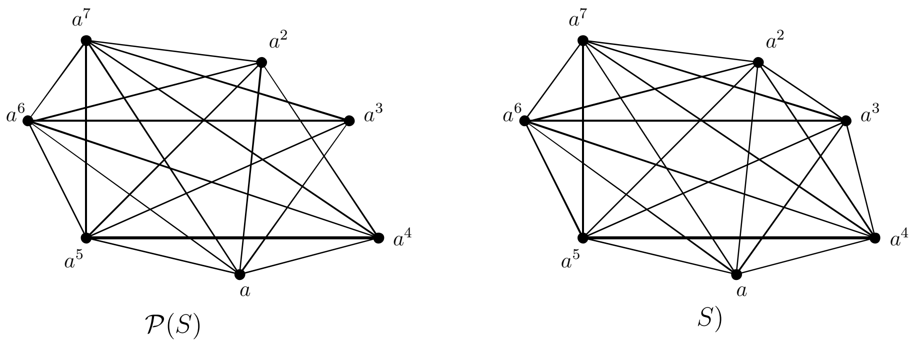

FIGURE 5.2: Power graph and cyclic graph of the semigroup $S = M(2,6)$

Lemma 5.0.5. *Let $S = M(m, r)$ be a monogenic semigroup. If $i < m$ and a^i is not adjacent to a^j in $\mathcal{P}(S)$, then a^i is not adjacent to a^j in $\Gamma(S)$.*

Proof. If possible, let $a^i \sim a^j$ in $\Gamma(S)$. Then $\langle a^i, a^j \rangle = \langle a^k \rangle$ for some $k \in \mathbb{N}$. Note that $k \neq i, j$. Otherwise, we have $a^i \sim a^j$ in $\mathcal{P}(S)$; a contradiction. Now we have the following cases on i, j, k:

Case 1: $i, j < k$. Since $a^i \in \langle a^k \rangle$, we have $a^i = a^{tk}$ for some $t \in \mathbb{N}$. It follows that $m \leq i$; a contradiction .

Case 2: $k < i, j$. Since $a^k \in \langle a^i, a^j \rangle$, we get $m \leq k < i$; a contradiction.

Case 3: $i < k < j$. Since $a^i \in \langle a^k \rangle$, we have $a^i = a^{tk}$ for some $t \in \mathbb{N}$. Consequently, $m \leq i$; again a contradiction.

Case 4: $j < k < i$. Since $a^j \in \langle a^k \rangle$, we get $a^j = a^{tk}$ for some $t \in \mathbb{N}$. Consequently, $m \leq j < k < i$; again a contradiction. □

The following theorem will be useful in the sequel.

Theorem 5.0.6 ([Chakrabarty, Ghosh and Sen, 2009, Theorem 2.12])**.** *Let G be a finite group. Then $\mathcal{P}(G)$ is complete if and only if G is a cyclic group of order 1 or p^m, for some prime p and $m \in \mathbb{N}$.*

Theorem 5.0.7. *For a semigroup S, the following are equivalent:*

(i) *The cyclic graph $\Gamma(S)$ is equal to $\mathcal{P}(S)$*

(ii) *Every cyclic subgroup of S has a prime power order*

(iii) *For each $a \in S$, we have $\langle a \rangle = M(m, p^n)$ for some prime p and $m, n \in \mathbb{N}_0$.*

Proof. (i) $\iff$ (iii). First assume that for each $a \in S$, we have $\langle a \rangle = M(m, p^n)$ for some prime p and $m, n \in \mathbb{N}_0$. In view of Lemma 1.2.16, it is sufficient to show that $\Gamma(S)$ is a subgraph of $\mathcal{P}(S)$. Let $a \sim b$ in $\Gamma(S)$. Then $\langle a, b \rangle = \langle c \rangle$ for some $c \in S$. Thus $\langle c \rangle = M(m, p^n)$ for some prime p and $m, n \in \mathbb{N}_0$. By Theorem 5.0.6, $\mathcal{P}(K_c)$ is complete. If $a, b \in K_c$, then $a \sim b$ in $\mathcal{P}(K_c)$ so that one of a, b is power of other. Thus, $a \sim b$ in $\mathcal{P}(S)$. Without loss of generality, assume that $a \notin K_c$. Since $a, b \in \langle c \rangle$, we have $a = c^i$ and $b = c^j$ such that $i < m$. By Lemma 5.0.5, we have $a \sim b$ in $\mathcal{P}(S)$.

Conversely, suppose that $\mathcal{P}(S) = \Gamma(S)$. For $a \in S$, clearly $\langle a \rangle = M(m, r)$ for some $m, r \in \mathbb{N}$. Then it is routine to verify $\mathcal{P}(K_a) = \Gamma(K_a)$. Since $\Gamma(K_a)$ is complete (cf. Corollary 2.1.5) so is $\mathcal{P}(K_a)$. By Theorem 5.0.6, we have $|K_a| = p^n$ for some prime p and $n \in \mathbb{N}_0$. Thus $r = p^n$ for some prime p and $n \in \mathbb{N}_0$.

(ii) $\iff$ (iii). Suppose every cyclic subgroup of S has prime power order. For $a \in S$, we have $\langle a \rangle = M(m, r)$. Since K_a is a cyclic subgroup of S of order r, we have $r = p^n$ for some prime p and $n \in \mathbb{N}_0$. Thus, $\langle a \rangle = M(m, p^n)$ for some prime p and $n \in \mathbb{N}_0$. Conversely, let H be a cyclic subgroup of S so that $H = \langle a \rangle$ for some $a \in S$. Clearly, $H = M(1, r)$. By the hypothesis, we have $r = p^n$ for some prime p and $n \in \mathbb{N}_0$. Thus, the order of H is a prime power. □

In view of the Corollary 5.0.3, we have the following corollary of the above theorem.

Corollary 5.0.8 ([Aalipour et al., 2017, Theorem 28])**.** *For a finite group G, $\mathcal{P}(G)$ is equal to $\mathcal{P}_e(G)$ if and only if every cyclic subgroup of G has prime power order.*

Theorem 5.0.9. *The enhanced power graph $\mathcal{P}_e(S)$ is equal to $\mathcal{P}(S)$ if and only if for each $a \in S$, we have either $\langle a \rangle = M(m, p^n)$ where $m \in \{1, 2\}$ or $\langle a \rangle = M(3, p^n)$ such that p is an odd prime.*

Proof. Suppose that $\mathcal{P}(S) = \mathcal{P}_e(S)$. Since $\mathcal{P}(S) \preceq \Gamma(S) \preceq \mathcal{P}_e(S)$ (cf. Lemma 1.2.16), we have $\Gamma(S) = \mathcal{P}_e(S)$ and $\mathcal{P}(S) = \Gamma(S)$. By Theorems 5.0.2 and 5.0.7, the result holds. □

In general, the cyclic graph and the commuting graph of S are not equal (see Example 5.0.10). Now we present a necessary and sufficient condition on S such that these two graphs are equal.

Example 5.0.10. Let $S = \mathbb{Z}_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$ be a semigroup with respect to multiplication modulo 4. Being a commutative semigroup, clearly $\Delta(S)$ is complete but $\Gamma(S)$ is not complete as $\langle\, \overline{0}, \overline{1}\,\rangle$ is not a monogenic semigroup. Hence, $\Delta(S) \neq \Gamma(S)$. See Figure 5.3.

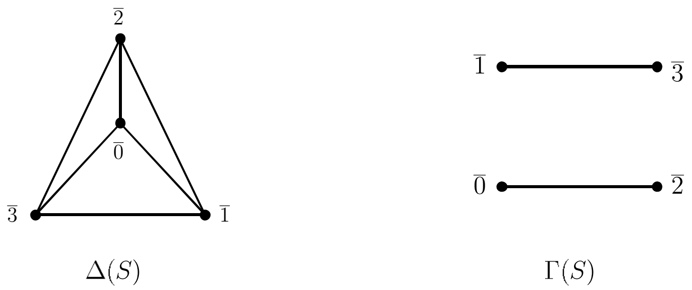

FIGURE 5.3: Commuting graph and cyclic graph of the semigroup $(\mathbb{Z}_4, \odot_4)$

Proposition 5.0.11. *If the cyclic graph $\Gamma(S)$ is equal to $\Delta(S)$, then for $f, f' \in E(S)$ such that $ff' = f'f$, we have $f = f'$.*

Proof. If possible, let $f \neq f'$. Since $ff' = f'f$, we have $f \sim f'$ in $\Delta(S)$. Consequently, by the hypothesis, we have $f \sim f'$ in $\Gamma(S)$ which is a contradiction of the fact that every connected component of $\Gamma(S)$ contains exactly one idempotent (cf. [Afkhami et al., 2014, Theorem 2.3]). Hence, $f = f'$. □

Proposition 5.0.12. *If the cyclic graph $\Gamma(S)$ is equal to $\Delta(S)$, then for each $a \in S$, we have either $\langle a \rangle = M(m, r)$ with $m \in \{1, 2\}$ or $\langle a \rangle = M(3, r)$ with r is odd.*

Proof. Suppose that $\Gamma(S) = \Delta(S)$. If possible, for some $a \in S$, let $\langle a \rangle$ is not of the given form. Then either $\langle a \rangle = M(3, r)$ with r is even or $\langle a \rangle = M(m, r)$ with $m \geq 4$. Then by the similar argument used in the proof of Theorem 2.1.5, in each of the case, we have a^2 is not adjacent to a^3. Clearly, $a^2 \sim a^3$ in $\Delta(S)$. Thus $\Gamma(S) \neq \Delta(S)$; a contradiction. Hence, the result holds. □

Theorem 5.0.13. *The cyclic graph $\Gamma(S)$ is equal to $\Delta(S)$ if and only if every commutative subsemigroup of S is monogenic.*

Proof. Let $\Gamma(S) = \Delta(S)$ and H be an arbitrary commutative subsemigroup of S. First, we prove that $\Gamma(H) = \Delta(H)$. For that, let $x \sim y$ in $\Delta(H)$, thus $xy = yx$. Consequently, we have $x \sim y$ in $\Delta(S) = \Gamma(S)$. Thus, $\langle x, y \rangle = \langle z \rangle$ for some $z \in S$. Because of $z \in \langle x, y \rangle$, we get $z \in H$. Therefore, $x \sim y$ in $\Gamma(H)$ so that $\Delta(H)$ is a subgraph of $\Gamma(H)$. As a result, $\Gamma(H) = \Delta(H)$ (cf. Lemma 1.2.16). Since H is commutative, we have $\Delta(H)$ is complete and so is $\Gamma(H)$. By Theorem 2.1.5, H is monogenic.

Conversely, suppose that every commutative subsemigroup of S is monogenic. In order to prove $\Gamma(S) = \Delta(S)$, it is sufficient to show $\Delta(S) \preceq \Gamma(S)$ (cf. Lemma 1.2.16). Let $a, b \in S$ such that $a \sim b$ in $\Delta(S)$, we have $ab = ba$. Consequently, $\langle a, b \rangle$ is a commutative subsemigroup of S. By the hypothesis, $\langle a, b \rangle$ is a monogenic subsemigroup of S. Thus, $a \sim b$ in $\Gamma(S)$. Hence, we have the result. □

Example 5.0.14. Let $S = \{-1, 0, 1\}$ be a semigroup with respect to usual multiplication. Note that $0 \sim 1$ in $\Delta(S)$ but there is no edge between 0 and 1 in $\mathcal{P}_e(S)$. Thus, $\Delta(S) \neq \mathcal{P}_e(S)$. See Figure 5.4.

Remark 5.0.15. Let S be a commutative semigroup. Then $\mathcal{P}_e(S) = \Delta(S)$ if and only if S is monogenic.

Theorem 5.0.16. *The enhanced power graph $\mathcal{P}_e(S)$ is equal to $\Delta(S)$ if and only if the following holds:*

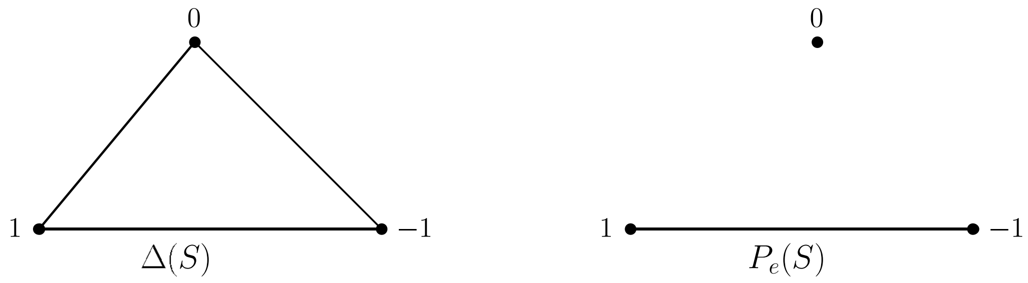

FIGURE 5.4: $\Delta(S)$ and $\mathcal{P}_e(S)$ of the semigroup $S = (\{-1, 0, 1\}, \cdot)$

(i) *For $f, f' \in E(S)$ such that $ff' = f'f$, we have $f = f'$*

(ii) *S has no subgroup $C_p \times C_p$ for prime p*

(iii) *For $x, y \in S$ with $xy = yx$ and at least one of m_x, m_y is greater than 1, we have $x, y \in \langle z \rangle$ for some $z \in S$.*

Proof. First, suppose that $\mathcal{P}_e(S) = \Delta(S)$. In order to prove (i), let $f, f' \in E(S)$, $ff' = f'f$ so that $f \sim f'$ in $\Delta(S)$. Since $\mathcal{P}_e(S) = \Delta(S)$, we have $f \sim f'$ in $\mathcal{P}_e(S)$. Thus $f, f' \in \langle z \rangle$ for some $z \in S$. Consequently, $f = f'$ (cf. Lemma 1.1.4). Next, we shall show that S has no subgroup of the form $C_p \times C_p$ for some prime p. On contrary, we assume that S has a subgroup $C_p \times C_p$ for some prime p. For each $x \in C_p \times C_p$, we have $o(x) = 1,\ p,\ p^2$. Since $C_p \times C_p$ is a non-cyclic subgroup of S, we get $o(x) = p$ for all $x \in (C_p \times C_p) \setminus \{\mathfrak{e}\}$, where $\mathfrak{e}$ is the identity element of the group $C_p \times C_p$. For $x \in (C_p \times C_p) \setminus \{\mathfrak{e}\}$, we get $\langle x \rangle \subsetneq C_p \times C_p$. Thus, there exists $y \in C_p \times C_p$ such that $y \notin \langle x \rangle$. Note that $\langle x \rangle \cap \langle y \rangle = \{\mathfrak{e}\}$. Otherwise, if there exists a nonidentity element $z \in \langle x \rangle \cap \langle y \rangle$, then we have $\langle z \rangle \subseteq \langle x \rangle$ and $\langle z \rangle \subseteq \langle y \rangle$. Since $o(x) = o(y) = o(z) = p$, we get $\langle x \rangle = \langle z \rangle = \langle y \rangle$. Consequently, $y \in \langle x \rangle$; a contradiction. Further, note that $|\langle x, y \rangle| = |\langle x \rangle| \cdot |\langle y \rangle| = p^2 = |C_p \times C_p|$ and $\langle x, y \rangle \subseteq C_p \times C_p$, we get $\langle x, y \rangle = C_p \times C_p$. Thus, $xy = yx$ so that $x \sim y$ in $\Delta(S)$. Since $\mathcal{P}_e(S) = \Delta(S)$, we get $x, y \in \langle z \rangle$ for some $z \in S$. Also, we have $x, y \in C_p \times C_p$ so that $m_x = m_y = 1$. It follows that $x, y \in \mathcal{K}_z$ which is a cyclic subgroup of S. Thus $\langle x, y \rangle = C_p \times C_p$ is a cyclic subgroup of S; a contradiction. Thus, (ii) holds. To prove (iii), let $x, y \in S$, $xy = yx$ and at least one of m_x, m_y is greater than 1. Thus, $x \sim y$ in $\Delta(S)$. Since $\mathcal{P}_e(S) = \Delta(S)$, we have $x \sim y$ in $\mathcal{P}_e(S)$. Hence $x, y \in \langle z \rangle$ for some $z \in S$.

Conversely, suppose S satisfies (i), (ii), and (iii). Since $\mathcal{P}_e(S) \preceq \Delta(S)$ (cf. Lemma 1.2.16), we need to show that $\Delta(S)$ is a subgraph of $\mathcal{P}_e(S)$. Let $x \sim y$ in $\Delta(S)$ so that $xy = yx$. If at least one of m_x, m_y is greater than 1. Then by (iii), there exists $z \in S$ such that $x, y \in \langle z \rangle$ so that $x \sim y$ in $\mathcal{P}_e(S)$. Hence, $\mathcal{P}_e(S) = \Delta(S)$. If $m_x = m_y = 1$, then $\langle x \rangle$ and $\langle y \rangle$ are the cyclic subgroup of S. Since, $x, y \in S$, by Remark 1.1.26, $x \in S_f,\ y \in S_{f'}$ for some $f, f' \in E(S)$. Then there exist $m, n \in \mathbb{N}$ such that $x^m = f,\ y^n = f'$. Note that $(xy)^{mn} = x^{mn}y^{mn} = (x^m)^n(y^n)^m = ff'$ and $(yx)^{mn} = f'f$. Since $xy = yx$, we have $ff' = f'f$. By (i), we get $f = f'$. Since $f \in E(S)$ and $f \in \langle x \rangle \cap \langle y \rangle$, thus f is the identity element of the subgroups $\langle x \rangle$ and $\langle y \rangle$. Consequently, f becomes the identity element of $\langle x, y \rangle$. Because of $xy = yx$, note that

$$(x^iy^j)^{mn} = (x^i)^{mn}(y^j)^{mn} = (x^m)^{in}(y^n)^{jm} = f^{in}f^{jm} = f$$

so that $x^iy^j \in S_f$. As a result, we have $\langle x, y \rangle \subseteq S_f$. Thus $\langle x, y \rangle$ contains exactly one idempotent f. Since $\langle x, y \rangle$ is a finite monoid containing exactly one idempotent so that $\langle x, y \rangle$ becomes a subgroup of S, and hence is the direct product of two cyclic groups, say $C_r \times C_s$ for some $r, s \in \mathbb{N}$. Let $\gcd(r, s) = d$. If $d = 1$, then $C_r \times C_s$ is a cyclic subgroup which makes $\langle x, y \rangle$ to a cyclic subgroup of S. Consequently, $x, y \in \langle z \rangle$ for some $z \in S$. Thus, $x \sim y$ in $\mathcal{P}_e(S)$.

If $d > 1$, then there exists a prime p such that p divides r and s. By Cauchy's theorem, there exist $x \in C_r$ and $y \in C_s$ such that $o(x) = o(y) = p$. Consequently, we get (x, e_2) and (e_1, y) in $C_r \times C_s$ such that $o(x, e_2) = o(e_1, y) = p$, where e_1, e_2 are the identity elements of C_r and C_s, respectively. Note that (x, e_2) and (e_1, y) commute with each other and $\langle (x, e_2) \rangle \cap \langle (e_1, y) \rangle = \{(e_1, e_2)\}$. It follows that $|\langle (x, e_2), (e_1, y) \rangle| = p^2$. Now $\big((x, e_2)^i(e_1, y)^j\big)^p = (e_1, e_2)$ so that there does not exist an element of order p^2 in the group $\langle (x, e_2), (e_1, y) \rangle$. Thus, $\langle (x, e_2), (e_1, y) \rangle$ is non-cyclic group of order p^2. Consequently, $\langle (x, e_2), (e_1, y) \rangle$ is of the form $C_p \times C_p$;

a contradiction of (ii). □

Now, we have the following corollary of the above theorem.

Corollary 5.0.17 ([Aalipour et al., 2017, Theorem 30]). *Let G be a finite group. Then the enhanced power graph $\mathcal{P}_e(G)$ is equal to $\Delta(G)$ if and only if G has no subgroup $C_p \times C_p$ for prime p.*

Example 5.0.18. For $S = B_2$, note that $0 \sim (1,1)$ in $\Delta(S)$ whereas there is no edge between 0 and $(1,1)$ in $\mathcal{P}(S)$. Hence, $\Delta(S) \neq \mathcal{P}(S)$. See Figure 5.5.

FIGURE 5.5: Commuting graph and power graph of the Brandt semigroup B_2

Theorem 5.0.19. *The power graph $\mathcal{P}(S)$ is equal to the commuting graph $\Delta(S)$ if and only if*

(i) the order of every cyclic subgroup of S is of prime power.

(ii) every commutative subsemigroup of S is monogenic.

Proof. In view of Lemma 1.2.16, the proof is strightforward by Theorems 5.0.7 and 5.0.13. □

Lemma 5.0.20. *Let G be a nontrivial group and its every cyclic subgroup has prime power order. Then every commutative subgroup of G is cyclic if and only if G has no subgroup of the form $C_p \times C_p$, for some prime p.*

Proof. Suppose every commutative subgroup of G is cyclic. On contrary, let G has a subgroup of the form $C_p \times C_p$, where p is a prime. Then by the similar argument used in the proof of Theorem 5.0.16, there exist $x, y \in C_p \times C_p$ such that $\langle x, y\rangle = C_p \times C_p$. As a result, we have a commutative subgroup $\langle x, y\rangle$ of G which is non-cyclic; a contradiction. Thus, G has no subgroup of the form $C_p \times C_p$, for some prime p.

Conversely, suppose G has no subgroup of the form $C_p \times C_p$. Let H be an arbitrary commutative subgroup of G. To prove H is cyclic i.e. $H = \langle x\rangle$, for some $x \in H$, we choose an element $x \in H$ such that $o(x)$ is maximum. First, we shall show that for an arbitrary $y \in H$, we have either $x \in \langle y\rangle$ or $y \in \langle x\rangle$. For $y \in H$, we get $\langle x, y\rangle$ is a commutative subgroup of H. Consequently, $\langle x, y\rangle$ is a cyclic subgroup of H (see proof of Theorem 5.0.16). By the hypothesis, $|\langle x, y\rangle| = q^n$, where q is a prime and $n \in \mathbb{N}$. Then by Theorem 5.0.6, $\mathcal{P}(\langle x, y\rangle)$ is complete so that either $x \in \langle y\rangle$ or $y \in \langle x\rangle$. Now we claim that $H = \langle x\rangle$. If possible, let $\langle x\rangle \subsetneq H$. Then there exists $y \in H$ such that $y \notin \langle x\rangle$. We must have $x \in \langle y\rangle$. Because of $o(x)$ is maximum, we have $\langle x\rangle = \langle y\rangle$; a contradiction of $y \notin \langle x\rangle$. Hence, the subgroup H is cyclic. □

In view of Lemma 5.0.20, we have the following corollary of the Theorem 5.0.19.

Corollary 5.0.21 ([Aalipour et al., 2017, Theorem 22]). *The power graph $\mathcal{P}(G)$ of a group G is equal to $\Delta(G)$ if and only if*

(i) *every cyclic subgroup of G has prime power order.*

(ii) *G has no subgroup of the form $C_p \times C_p$ for some prime p.*