Chapter 6

Conclusion and Future Research Work

6.1 Contribution of the Thesis

Throughout the thesis, we have focused on algebraic and graph-theoretic properties of cyclic graphs, enhanced power graphs and commuting graphs associated with groups and semigroups. The contribution of the thesis are highlighted below.

Since cyclic graph and enhanced power graph are equal for finite groups, in Chapter 2, we have studied the cyclic graph $\Gamma(S)$ of a semigroup S. First we described the structure of $\Gamma(S)$. Then various aspects of $\Gamma(S)$ have been studied, namely: completeness, bipartite, regularity etc. The independence number and clique number of $\Gamma(S)$, where S is an arbitrary semigroup are investigated. Finally, an upper bound of the chromatic number of $\Gamma(S)$ is supplied and it is shown that the chromatic number of $\Gamma(S)$, where S is an infinite semigroup, is countable.

In Chapter 3, we study enhanced power graph associated with groups and semigroups. First we investigate the graph-theoretic properties viz. minimum degree, independence number and matching number, vertex covering number and then determine them when G is a finite abelian p-group, the dihedral group D_{2n}, the semidihedral group SD_{8n}, the dicyclic group Q_{4n}, U_{6n} or V_{8n}. If G is any of these groups, we prove that $\mathcal{P}_e(G)$ is perfect and then obtain its strong metric dimension. Moreover, we obtain the independence number of $\mathcal{P}_e(G)$ for any finite abelian group G. These results along with certain known equalities yield the edge connectivity, vertex covering number and edge covering number of enhanced power graphs of the respective groups as well.

This thesis also initiated the study of the enhanced power graph of a semigroup. Necessary and sufficient conditions on S are provided such that $\mathcal{P}_e(S)$ is complete, bipartite, tree, regular, null graph and acyclic graph. Moreover, the results concerning the minimum degree and independence number of $\mathcal{P}_e(G)$ are extended to $\mathcal{P}_e(S)$. Also, planarity of $\mathcal{P}_e(S)$ is discussed for some cases. Then the question: Whether the chromatic number of $\mathcal{P}_e(S)$ is countable?, is answered negatively. In this connection, a semigroup has been constructed such that the chromatic number of $\mathcal{P}_e(S)$ is uncountable.

In Chapter 4, first we consider the commuting graph $\Delta(G)$ of an arbitrary group G. We determine the edge connectivity, minimum degree, matching number, clique number of $\Delta(G)$. It is shown that the minimum degree and edge connectivity both are equal. Also, we give necessary and sufficient condition on the group G such that the interior and center of $\Delta(G)$ are equal. Further, we investigate the commuting graph of the semidihedral group SD_{8n}. In this connection, we discuss various graph invariants of $\Delta(SD_{8n})$ including, vertex connectivity, independence number, matching number and detour properties, Laplacian spectrum, metric dimension and resolving polynomial of $\Delta(SD_{8n})$.

Then, we consider the commuting graph on an important class of inverse semigroups, namely: Brandt semigroup B_n. In this connection, we determine the minimum degree, girth, diameter, dominance number of $\Delta(B_n)$ and then investigated various properties including Eulerian, Hamiltonian, planarity, perfectness. The clique number, strong metric dimension, chromatic number of $\Delta(B_n)$ are obtained. A class of inverse semigroup is also identified such that its commuting graph is Hamiltonian. Further, it is shown that $\mathrm{Aut}(\Delta(B_n)) \cong S_n \times \mathbb{Z}_2$ for all $n \geq 2$ and $\mathrm{End}(\Delta(B_n)) = \mathrm{Aut}(\Delta(B_n))$ for all $n \geq 4$.

Finally, this thesis classifies finite semigroups such that the pair of graphs, viz. $\mathcal{P}(S), \Gamma(S), \mathcal{P}_e(S)$ and $\Delta(S)$, are equal. This extends the corresponding results of groups to semigroups.

The work embedded in the thesis has its own limitations. During the investigation of graphs associated with semigroups, we observe that the research problem "Classification of finite semigroups such that its associated graph satisfy certain property viz. metric dimension, Eulerian, planar etc." is not easy to handle. Similar problems are tackled in case of finite groups by using Lagrange's theorem. However, the Lagrange's theorem need not hold in case of finite semigroups. Moreover, the investigation of graphs associated with semigroups becomes limited because complete classification of finite semigroups of given cardinality is not known. In comparison to the graphs associated with groups, we observe that the inverse elements and idempotents of a semigroup acts differently in its associated graph. Therefore, the further investigation of such graphs (considered in this thesis) associated with semigroups is not easy.

6.2 Scope for Future Research

We conclude this thesis with some research questions which can be addressed in future.

- Compute the independence number and the chromatic number of $\Gamma(S)$?
- What are the necessary and sufficient condition on S such that the graphs $\Gamma(S)$ and $\mathcal{P}_e(S)$ are planar (or Hamiltonian)?

- Classification of S such that the graphs $\Gamma(S)$ and $\mathcal{P}_e(S)$ are perfect.
- What will be the automorphism group of $\Gamma(S)$ and $\mathcal{P}_e(S)$? Since semigroup of transformations play an important role in semigroup theory, one can begin this question by taking S to partial transformation semigroups or its various subsemigroups.
- Classify the semigroup S such that the chromatic number of $\mathcal{P}_e(S)$ is at most countable.
- Investigate the spectral properties, detour properties and resolving polynomial of the commuting graph of Brandt semigroups.
- Is $\kappa'(\Delta(S)) = \delta(\Delta(S))$ for an arbitrary semigroup S?