Chapter 2

Invariants of families of flat connections

In this chapter[1] we consider the problem of associating $\mathbb{R}/\mathbb{Z}$ cohomology elements to families of flat connections on a fixed vector bundle. This chapter is organised as follows. In section 2.1, we recollect the results concerning geometric chains and fiber integration of differential characters. Then in section 2.2, we describe our construction, and in section 2.3 we discuss the comparison of our results with other constructions in the literature.

2.1 Preliminaries

In this section we briefly state the definitions and results that are used later in the paper. Subsection 2.1.1 describes a geometric chain model of differential characters on smooth spaces, and subsection 2.1.2 discusses the construction of fiber integration of differential characters and its properties.

We assume that the reader is familiar with smooth spaces and stratifolds. The required notions and results are summarised in section 2 of [1], and the book [35] contains a detailed description.

[1]This chapter is based on the author's article 'Invariants of families of flat connections using fiber integration of differential characters', *Lett. Math. Phys.* 110, 639–657 (2020). https://doi.org/10.1007/s11005-019-01234-3

The discussion in this section is based on the framework developed in [1]. Nothing contained in this section is original.

2.1.1 Geometric chain model of differential characters

Definition 4. *Let $\mathcal{C}_k(X)$ be the set of equivalence classes of smooth maps $\zeta : M \to X$ where M is an oriented $k-$stratifold such that ∂M is an oriented $(k-1)-$stratifold under the following equivalence relation:*
$(\zeta : M \to X) \sim (\zeta' : M' \to X)$ iff $\exists$ an orientation preserving diffeomorphism $\psi : M \to M'$ such that $\zeta' \circ \psi = \zeta$. The operation of disjoint union makes $\mathcal{C}_k(X)$ an abelian semi-group. Elements of $\mathcal{C}_k(X)$ are called geometric chains.

Definition 5. *The boundary operator $\partial : \mathcal{C}_k(X) \to \mathcal{C}_{k-1}(X)$is defined as $\partial[\zeta : M \to X] = [\zeta|_{\partial M} : \partial M \to X]$.*

By $[\bar{\zeta} : \bar{M} \to X]$, we denote the element in $\mathcal{C}_k$ given by the same map $[\zeta : M \to X]$ but with the orientation of M reversed.

Definition 6. *The set $ker\partial := \mathcal{Z}_k(X)$ is called the group of geometric $k-$ cycles. The set $im\partial := \mathcal{B}_{k-1}(X)$ is called the group of geometric boundaries. The quotient $\mathcal{H}_k(X) := \frac{\mathcal{Z}_k(X)}{\mathcal{B}_k(X)}$ is the homology of the complex defined by ∂.*

The homology $\mathcal{H}_k(X)$ is an abelian group, where the inverse of $[\zeta : M \to X]$ is given by $[\bar{\zeta} : \bar{M} \to X]$.

Let $C_n(X;\mathbb{Z})$ denote the group of smooth singular $n-$chains on X. A chain $c \in C_n(X;\mathbb{Z})$ is called thin if $\forall \omega \in \Omega^n(X), \int_c \omega = 0$. We denote the group of thin $n-$chains by $S_n(X,\mathbb{Z})$.

We now define maps $\psi : \mathcal{C}_n(X) \to C_n(X,\mathbb{Z})/S_n(X,\mathbb{Z})$ by $[\zeta : M \to X] \mapsto [\zeta_*(c)]_{S_n}$ where c denotes a fundamental cycle of $H_n(M,\partial M,\mathbb{Z})$ (or $H_n(M,\mathbb{Z})$ if $\partial M = \phi$). This map takes boundaries to boundaries, and chains to chains thereby giving us the commutative diagram:

$$
\begin{array}{ccccccccc}
\longrightarrow & \mathcal{C}_{n+1}(X) & \longrightarrow & \mathcal{B}_n(X) & \longrightarrow & \mathcal{Z}_n(X) & \longrightarrow & \mathcal{C}_n(X) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & \frac{C_{n+1}(X,\mathbb{Z})}{S_{n+1}(X,\mathbb{Z})} & \longrightarrow & \frac{B_n(X,\mathbb{Z})}{\partial S_{n+1}(X,\mathbb{Z})} & \longrightarrow & \frac{Z_n(X,\mathbb{Z})}{\partial S_{n+1}(X,\mathbb{Z})} & \longrightarrow & \frac{C_n(X,\mathbb{Z})}{S_n(X,\mathbb{Z})} & \longrightarrow
\end{array}
$$

This chain map induces a map on homology of the two chain complexes: $\mathcal{H}_n(X) := \frac{\mathcal{Z}_n(X)}{\mathcal{B}_n(X)} \to H_n(X,\mathbb{Z})$. This map is an isomorphism (see Theorem 20.1 in [35]). Further it is shown there that the product $\mathcal{H}_m(X) \times \mathcal{H}_n(Y) \to \mathcal{H}_{m+n}(X \times Y)$ given by $[\zeta : M \to X] \times [\eta : N \to Y] \mapsto [\zeta \times \eta : M \times N \to X \times Y]$ is compatible with the isomorphism above and the usual multiplication in smooth singular homology.

Bär and Becker show (lemma 7 of [1]) that

Proposition 7. *There exist homomorphisms $\zeta : C_{n+1}(X,\mathbb{Z}) \to \mathcal{C}_{n+1}(X)$, $a : C_n(X,\mathbb{Z}) \to C_{n+1}(X,\mathbb{Z})$, and $y : C_{n+1}(X,\mathbb{Z}) \to Z_{n+1}(X,\mathbb{Z})$ such that the following hold:*

$$\partial\zeta(c) = \zeta\partial(c) \quad \forall c \in C_{n+1}(X,\mathbb{Z}), \tag{2.1}$$

$$[\zeta(c)]_{S_{n+1}} = [c - a(\partial c) - \partial a(c + y(c))]_{S_{n+1}} \qquad \forall c \in C_{n+1}(X,\mathbb{Z}), \tag{2.2}$$

and

$$[\zeta(z)]_{\partial S_{n+1}} = [z - \partial a(z)]_{\partial S_{n+1}} \qquad \forall z \in Z_{n+1}(X,\mathbb{Z}). \tag{2.3}$$

2.1.2 Fiber integration of differential characters

Before discussing fiber integration, let us fix our conventions regarding orientation. We use the same convention as in [1] which is: if M is a manifold with boundary ∂M, a tangent vector pointing outward at a boundary point $p \in \partial M$ followed by an oriented basis for $T_p(\partial M)$ gives us an oriented basis for the manifold M at p. For a fiber bundle with oriented fiber and oriented base, the orientation on the total space is chosen to be given by an oriented basis of the base followed by an oriented basis of the fiber. We now discuss fiber bundles over smooth spaces. As remarked in [1] (sec 7.1), there are multiple non-equivalent generalisations of the concept of fiber bundles over smooth spaces. We use the same definition as by the above authors, which is stated below:

Definition 8. *A smooth surjective map $p : E \to B$ is a fiber bundle with fiber F, if for any smooth map $f : M \to B$ from a finite dimensional stratifold M, the pull back $f^*E \to M$ is locally trivial with fiber F.*

Let $F \hookrightarrow E \to B$ be a fiber bundle where M, F, and E are smooth manifolds of finite dimension and F is compact oriented. Then we have a fiber integration map of differential forms (see, for example, [6])
$\fint_F : \Omega^n(E, \mathbb{R}) \to \Omega^{n-r}(B)$ for $n \geq r$ where $dimF = r$. This map satisfies the Stokes theorem:

$$\fint_F d\omega = d\fint_F \omega + (-1)^{deg\omega + dimF} \fint_{\partial F} \omega \tag{2.4}$$

Similarly there is a push-forward map (see [5, 15]) for singular cohomology with coefficients in an arbitrary group $\pi_F : H^n(E, G) \to H^{n-r}(B, H^r(F, G))$. Generally F is a connected, closed and oriented manifold of dimension r, and the above map becomes $\pi_F : H^n(E, G) \to H^{n-r}(B, G)$. Fiber integration maps have been studied for various models of differential cohomology, see for example [19, 28, 37]. For our purpose, the construction given by Bär and Becker [1] is suitable. We briefly describe this construction below. For this purpose, they use the transfer maps at the level of chains $\lambda : C_{k-r}(B, \mathbb{Z}) \to C_k(E, \mathbb{Z})$ satisfying $[\lambda(z)]_{\partial S_{k+1}} = [PB_E(\zeta(z))]_{\partial S_{k+1}} \quad \forall z \in Z_{k-r}(B, \mathbb{Z})$ where PB_E denotes the pull-back along the map $E \to B$.

Definition 9. *Let $E \to B$ be a fiber bundle with oriented closed fibers F, and let $dimF = r$. We then have fiber integration $\hat{\pi}_{!F} : \hat{H}^k(E) \to \hat{H}^{k-r}(B)$ for $k > r$ given by $h \mapsto \hat{\pi}_{!F}(h)$, where*

$$(\hat{\pi}_{!F}(h))(z) = h(\lambda(z)) \times exp(2\pi i \int_{a(z)} \fint_F curv(h)) \tag{2.5}$$

The authors of [1] show that this construction does not depend on the choice of functions λ,ζ, and a.

Remark 10. *The above construction in [1] is done for the case where M, and E are smooth spaces, but F is assumed to be a finite dimensional closed manifold. However, their construction as well as the properties cited below hold for the case when F is a compact boundary-less finite dimensional stratifold. This is because all that is required in their construction and proofs is that (a) the fiber integration of differential forms is defined, and that (b) the integration of forms satisfies the Stokes theorem. These hold good when F is a compact oriented stratifold by the virtue of results proved in [21].*

When the fiber has a boundary, the fiber integration (along the boundary) of the restriction of a differential character, finds an expression in terms of the integral of the curvature of the differential character. This is a very useful identity, and in fact several of our results are a direct consequence of this formula. The precise statement (Proposition 54 of [1]) is as follows:

Proposition 11. *Let $F \hookrightarrow E \to B$ be a fiber bundle where F is a compact manifold with boundary ∂F, such that $\partial E \to B$ is a fiber bundle with fiber ∂F. If $h \in \hat{H}^k(E, \mathbb{Z})$, then*

$$\hat{\pi}_{!\partial F}(h|_{\partial E}) = \iota((-1)^{k-dimF} \fint_F curv(h)) \tag{2.6}$$

As observed in example 56 of [1], this is a generalisation of the famous Cheeger–Simons homotopy formula (2.7).

They further show that fiber integration is compatible with the exact sequences i.e. the diagrams:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^{k-1}(E)/\Omega_0^{k-1}(E) & \longrightarrow & \hat{H}^k(E) & \longrightarrow & H^k(E,\mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \fint_F} & & \downarrow{\scriptstyle \hat{\pi}_{!F}} & & \downarrow{\scriptstyle \pi_{!F}} & & \\ 0 & \longrightarrow & \Omega^{k-r-1}(B)/\Omega_0^{k-r-1}(B) & \longrightarrow & \hat{H}^{k-r}(B) & \longrightarrow & H^{k-r}(B,\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{k-1}(E,\mathbb{R}/\mathbb{Z})) & \longrightarrow & \hat{H}^k(E) & \longrightarrow & \Omega_0^k(E) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_{!F}} & & \downarrow{\scriptstyle \hat{\pi}_{!F}} & & \downarrow{\scriptstyle \fint_F} & & \\ 0 & \longrightarrow & H^{k-r-1}(B,\mathbb{R}/\mathbb{Z}) & \longrightarrow & \hat{H}^{k-r}(B) & \longrightarrow & \Omega_0^{k-r}(B) & \longrightarrow & 0 \end{array}$$

commute.

2.2 Cohomological invariants of the space of flat connections using fiber integration of differential characters

In the seminal paper [14] Cheeger and Simons showed how to associate a differential character to a bundle equipped with a connection. We denote this differential character by $cs_{P,u}(E \to B, \theta) \in \hat{H}^{2p}(B)$. Here $P \in I^p_{\mathbb{Z}}(GL(n,\mathbb{R}))$, and $u \in H^{2p}(BGL(n,\mathbb{R}),\mathbb{Z})$ are assumed to be compatible with each other. Often when there is no possibility of confusion, we simply write these as $cs(E,\theta)$. When the connection is flat, this characters lies in the image of the inclusion $H^{2p-1}(B,\mathbb{R}/\mathbb{Z}) \to \hat{H}^{2p}(B)$. They further prove that for $p \geq 2$, $cs(E,\theta_1) = cs(E,\theta_0)$ if the connections θ_0 and θ_1 are connected by a smooth family of flat connections and are therefore rigid invariants of the space of flat connections. This is a consequence of their 'homotopy formula':

$$cs_{(P,u)}(\nabla_1) - cs_{(P,u)}(\nabla_0) = p\int_I P(\frac{d}{dt}\nabla_t, \nabla_t^2, \nabla_t^2, \cdots, \nabla_t^2). \tag{2.7}$$

In this article, we consider the problem of attaching invariants to a family of flat connections. More precisely, we formulate the problem as done in [34]: Let $\mathcal{D}(E)$ be the simplicial set, whose r-simplices are $(r+1)-$ tuples[2] $(D^0, D^1, \cdots, D^r)$ of relatively flat connections on the vector bundle $E \to B$ of rank k. (The connections $D^0, D^1, \cdots, D^r$ are called relatively flat if the linear combination $t_0D^0 + t_1D^1 + \cdots + t_rD^r$ is a flat connection for each choice of $t_0, \cdots, t_r$ such that $\sum t_i = 1$.) If $\phi : [r] \to [s]$ is an increasing function (where $[r] := \{0, 1, \cdots, r\}$), we then define the corresponding map $\phi^* : \mathcal{D}(E)_s \to \mathcal{D}(E)_r$ by $\phi^*(D^0, D^1, \cdots, D^s) =$

[2] To avoid duplicity, we shall choose any ordering on the set of all connections and require that $D^0 \leq D^1 \leq \ldots \leq D^r$

$(D^{\phi(0)}, D^{\phi(1)}, \cdots, D^{\phi(r)})$. In this section we construct maps:

$$\tilde{\psi}_{p,r} : \mathbb{H}_r(\mathcal{D}(E)) \to H^{2p-r-1}(B, \mathbb{R}/\mathbb{Z}) \quad for\ p \neq r, r+1 \tag{2.8}$$

using fiber integration of differential characters. In the next section we show that in the $p > r+1$ case they agree with the maps constructed in [34], and vanish in the $p < r$ case.

Let $\Sigma \in \mathcal{Z}_r(\mathcal{D}(E))$ be a cycle representing the homology class $[\Sigma] \in \mathbb{H}_r(\mathcal{D}(E))$. Then Σ decomposes as $\Sigma = \Sigma^1 + \cdots + \Sigma^m$ where $\Sigma^i = \pm(D^{0,i}, D^{1,i}, \cdots, D^{r,i})$. For simplicity, we use the same symbol Σ^i to denote the geometric realization Δ^r of the chain Σ^i. The geometric realisation of the cycle Σ (again denoted by the same symbol Σ) is obtained by first taking the disjoint union $\bigsqcup_i \Sigma^i$ and then identifying the suitable faces in the boundaries $\partial\Sigma^i$ i.e. l^{th}-face of Σ^i is identified with m^{th}-face of Σ^j if $\Sigma^i_{(l)} = \pm\Sigma^j_{(m)}$.

On the bundle $E \times \Sigma \to B \times \Sigma$, we define a connection as follows. Δ^r is conveniently parameterised by tuples $(t_0, \cdots, t_r)$ such that $t_0 + t_1 + \cdots + t_r = 1$. Let $D^i = \sum_j t_j D^{j,i}$ be a connection on the bundle $E \times \Sigma^i \to B \times \Sigma^i$. Schematically speaking, we could patch the connections D^i on $E \times \Sigma^i \to B \times \Sigma^i$, to get a connection D on $E \times \Sigma \to B \times \Sigma$ and then apply Cheeger–Simons theory to obtain a character $h_{B\times\Sigma} \in \hat{H}^{2p}(B \times \Sigma)$. We could then integrate along fibers of the bundle $B \times \Sigma \to B$ to get a character in $\hat{H}^{2p-r}(B)$, and show that it lies in the image of the inclusion $H^{2p-r-1}(B; \mathbb{R}/\mathbb{Z}) \to \hat{H}^{2p-r}B$. Broadly, this is indeed the idea used in this paper. However carrying it out rigorously requires some care since Σ is a stratifold, and not a manifold in general. In order to apply Cheeger–Simons theory to the bundle $E \times \Sigma \to B \times \Sigma$, we need to ensure that (i) the connection we endow it with is indeed smooth (in the sense described below), and (ii) that this connection is the pull back, under some smooth classifying map, of the universal connection on a classifying bundle (in the sense of Narasimhan–Ramanan [39]). To ensure smoothness, we need to modify each connection D^i so that it commutes with a small collar near the boundary $\partial\Sigma^i$. We do this in a precise manner below.

First we describe the notion of smoothness of a connection ω on the bundle $E\times\Sigma\to B\times\Sigma$. Choose a trivializing open cover $\mathcal{U}$ of B . For each $U\in\mathcal{U}$, let $\{\phi^{\mu}:U\to E|_U\}$ for $\mu\in\{1,\ldots,n\}$ be a frame of smooth sections of this vector bundle. Then $\{\phi^{\mu}\times id_{\Sigma}:U\times\Sigma\to E|_U\times\Sigma\}$ form a frame for the bundle $E\times\Sigma\to B\times\Sigma$ restricted to $U\times\Sigma$. Let $\{\omega^{\mu}_{\nu}\}$ be the 1-forms corresponding to the connection ω with respect to this frame. We say that ω is a smooth connection if for each U, ω^{μ}_{ν} are smooth[3] 1-forms on the stratifold $U\times\Sigma$. This definition depends neither on the choice of the cover, nor on the frame.

Now let us describe the stratifold structure on the geometric realisation of Σ. Note that geometric realisation of Σ^i is an $r-$simplex Δ^r for each i. Geometric realisations of simplicial complexes, and hence simplices Σ^i in particular, can be given a stratifold structure by an inductive process as in section 2.1.2 of [38] (described below). Thereafter, the c-stratifolds ("collared-stratifolds") Σ^i are pasted together[4] to obtain the required stratifold structure on Σ so that a real valued continuous function on Σ is smooth iff for each i, its restriction to Σ^i is smooth with respect to the c-stratifold structure on Σ^i.

Let us now briefly recall the construction of a c-stratifold structure [38] on $(\Delta^r,\partial\Delta^r)$. The construction proceeds by induction on r. $(\Delta^1,\partial\Delta^1)$ is simply a line segment with two end points, and already carries the structure of a $c-$stratifold. Assume that we have equipped $(\Delta^{i-1},\partial\Delta^{i-1})$ with a $c-$stratifold structure. To complete the inductive step, we shall construct a stratifold structure on $(\Delta^i,\partial\Delta^i)$. The boundary $\partial\Delta^i$ has $i+1$ faces, each of which is a geometric realisation of Δ^{i-1}. The stratifold structure on the boundary is the stratifold structure obtained by pasting together the faces (which already have the stratifolds structure by the inductive hypothesis). Thereafter, we choose a bijective map $\eta_i:\mathbb{D}^i\to\Delta^i$ which is a diffeomorphism on the interiors and takes the boundary $\partial\mathbb{D}^i=S^{i-1}$ smoothly to $\partial\Delta^i$.

[3] For background on the notions of smoothness of forms on stratifolds, we refer the reader to the book [35], or to C.-O. Ewald's works [21, 20].

[4] Collared stratifolds can be pasted along their boundary, or part thereof, to obtain another stratifold. For the details of this construction, see Propositions 3.1 (p.37) and Proposition A1(p.194) of [35]

There is a collar $\gamma : S^{i-1} \times [0,\epsilon) \to \mathbb{D}^i$ which in the polar coordinates $(r, \theta_1, ..., \theta_{i-1})$ is given by $(r = 1, \theta_1, ..., \theta_{i-1}) \times t \mapsto ((1-t), \theta_1, ..., \theta_{i-1})$. Using the bijective map η_i, this translates into a collar (denoted by the same symbol γ for convenience) $\gamma : \partial\Delta^i \times [0,\epsilon) \to \Delta^i$. A map $\zeta : \Delta^i \to \mathbb{R}$ is declared to be smooth if (i) ζ restricts to a smooth map on $\partial\Delta^i$, (ii) $\zeta \circ \eta_i : \mathbb{D}^i \to \mathbb{R}$ restricts to a smooth map on the interior of $\mathbb{D}^i$ and (iii) if it is compatible with the germ of γ i.e. if there exists $\epsilon' > 0$ such that $\zeta(\gamma(p,t)) = \zeta(p) \forall p \in \partial\Delta^i$ and $t \in [0, \epsilon')$. In this manner $(\Delta^i, \partial\Delta^i)$ becomes a c-stratifold with boundary thereby completing the induction step.

We now proceed to define a smooth connection on $E \times \Sigma^i \to B \times \Sigma^i$ for each i. Without loss of generality, assume that $D^{0,i}, \ldots, D^{r,i}$ are distinct. For each i, the bundle $E \times \Sigma^i \to B \times \Sigma^i$ is an isomorphic copy of $E \times \Delta^r \to B \times \Delta^r$. On $E \times \Delta^r \to B \times \Delta^r$, the smooth connection is defined inductively starting from the $0-$skeleton of Δ^r. The $0-$skeleton of Δ^r is the disjoint union of $r+1$ points $\{0, 1, \ldots, r\}$. If $D^i = (D^{0,i}, D^{1,i}, \ldots, D^{r,i})$, then the connection on $E \times \{l\} \to B \times \{l\}$ is defined to be $D^{l,i}$. To carry out the inductive step, assume that a smooth connection has been constructed on $E \times S_{j-1} \to B \times S_{j-1}$ where S_{j-1} is the $(j-1)-$skeleton of Δ^r. Now $\partial S_j \subset S_{j-1}$ consists of several faces (each of which is diffeomorphic as a stratifold to Δ^{j-1}) which combine to yield copies of $\partial\Delta^j$. S_j is obtained by attaching $j-$cells Δ^j to S_{j-1}. In this manner, the problem reduces to extending a smooth connection on the bundle $E \times \partial\Delta^j \to B \times \partial\Delta^j$ to a smooth connection on the bundle $E \times \Delta^j \to B \times \Delta^j$. We do this below.

There is a projection map $pr_1 : \partial\Delta^j \times [0,\epsilon) \to \partial\Delta^j$. The pullback under the bundle map $(id_E \times pr_1, id_B \times pr_1)$ of the given smooth connection on the bundle $E \times \partial\Delta^j \to B \times \partial\Delta^j$ yields a connection D_{collar} on $E \times (\partial\Delta^j \times [0,\epsilon)) \to B \times (\partial\Delta^j \times [0,\epsilon))$. Via the collar map, we consider $\partial\Delta^j \times [0,\epsilon)$ as an open subset of Δ^j.

Let $\nabla^0, \nabla^1, \ldots, \nabla^j$ be the $(j+1)$ connections on $E \to B$ corresponding to the corners of Δ^j. Let $D^j = t_0\nabla^0 + \ldots + t_j\nabla^j$ be a smooth connection on $E \times \mathring{\Delta}^j \to B \times \mathring{\Delta}^j$ where $\mathring{\Delta}$ denotes the interior of Δ and $(t_0, \ldots, t_j)$ are the standard simplex coordinates on $\mathring{\Delta}^j$. Choose a smooth function $f : [0,1] \to [0,1]$ such that $f(t) = 1 on [0, \frac{\epsilon}{2}]$

and $supp(f) \subset [0, \epsilon)$. Let $D^j_f := f(1-r)D_{collar} + [1 - f(1-r)]D^j$. Here r denotes the radial coordinate on the disk $\mathbb{D}^j$. Since we have identified Δ^j with $\mathbb{D}^j$, we can use the same coordinate on Δ^j.

In this manner, given a smooth connection on $E \times S_{j-1} \to B \times S_{j-1}$, we extend it to a connection on $E \times S_j \to B \times S_j$. Continuing the inductive procedure, we obtain a smooth connection on $E \times \Delta^r \to B \times \Delta^r$ (which is $E \times \Sigma^i \to B \times \Sigma^i$). By construction, these connections $\{D^i_f\}$ are compatible with the respective collars on Σ^i, and hence combine to yield a smooth connection D_f on $E \times \Sigma \to B \times \Sigma$.

We now apply the Cheeger–Simons theory to the bundle $E \times \Sigma \to B \times \Sigma$ of rank k with connection D_f to obtain a differential character $h^f_{B\times\Sigma} \in \hat{H}^{2p}(B \times \Sigma)$. Notice that the original Cheeger–Simons construction [14] was done for the case when the base is a manifold. In our case, $B \times \Sigma$ is a stratifold. This is not a problem however, because all that goes into the proof of Cheeger–Simons theorem is that the given connection is the pull back under a map from the bundle to the tautological bundle on a Grassmannian manifold $G(N, k)$ with its canonical connection for some sufficiently large N (for simplicity of notation, we shall denote the Grassmannian $G(N, k)$ by A_N, and the tautalogical bundle by $E_N \to A_N$). While this may or may not be true for a general connection when the base is a general stratifold, it suffices for our purposes to show this for the connection D_f on the bundle $E \times \Sigma \to B \times \Sigma$. We do this below.

We shall again use an inductive procedure to find a desired bundle map $(G : E \times \Sigma \to E_N, g : B \times \Sigma \to A_N)$ such that the pullback of the universal connection on $E_N \to A_N$ under (G, g) is the connection D_f on $E \times \Sigma \to B \times \Sigma$. Let C_k denote the $k-$skeleton of Σ. C_0 is a collection of points, and hence (by using the result of Narasimhan-Ramanan [39])there is a smooth bundle map $(G_0 : E \times C_0 \to E_N, g_0 : B \times C_0 \to A_N)$, pullback of the universal connection under which is the connection D_f restricted to $E \times C_0 \to B \times C_0$. Suppose that there is a smooth bundle map $(G_{k-1} : E \times C_{k-1} \to E_{N_1}, g_{k-1} : B \times C_{k-1} \to A_{N_1})$ such that the universal connection θ on $E_{N_1} \to A_{N_1}$ pulls-back under g_{k-1} to the restriction of D_f to $E \times C_{k-1} \to$

$B \times C_{k-1}$. Now, C_k is obtained by attaching (possibly several) copies of Δ^k to C_{k-1}. For each copy of Δ^k, let $g_{k-1}^{collar} : B \times \partial\Delta^k \times [0, \epsilon) \to A_{N_1}$ be given by $g_{k-1}^{collar} = g_{k-1} \circ pr_{12}$ where $pr_{12} : B \times \partial\Delta^k \times [0, \epsilon) \to B \times \partial\Delta^k$. Similarly, let $G_{k-1}^{collar} : E \times \partial\Delta^k \times [0, \epsilon) \to E_{N_1}$ be given by $G_{k-1}^{collar} = G_{k-1} \circ pr_{12}$. We consider $\partial\Delta^k \times [0, \epsilon)$ to be a subset of Δ^k via the collar map. Let U denote the complement of $\partial\Delta^k \times [0, \frac{\epsilon}{2}]$ in Δ^k. This is an open subset of the interior of Δ^k, and hence is a manifold. By the Narasimhan-Ramanan result, there is a bundle map $(G_U : E \times U \to E_{N_2}, g_U : B \times U \to A_{N_2})$ which induces the connection D_f on $E \times U \to B \times U$. By using a partition of unity argument (as in [39]), we get an extension $(G_k : E \times C_k \to E_{N_3}, g_k : B \times C_k \to A_{N_3})$, for a sufficiently large N_3, thereby completing the induction step. Since Σ is a cycle, its geometric realization is a compact stratifold without boundary. Therefore, by remark 10, we can apply fiber integration to the bundle $B \times \Sigma \to B$, to obtain a differential character $\hat{\pi}_{!\Sigma}(h^f_{B\times\Sigma})$ on B. (Notice that, as remarked in [34], geometric realization of Σ^i (or Δ^r)is not boundary-less and hence fiber integration of differential characters can not be applied to the bundle $B \times \Sigma^i \to B$. However, since Σ is a cycle, the geometric realization of Σ has no boundary. This is what makes it possible to apply fiber integration to the bundle $B \times \Sigma \to B$). We have thus obtained a map:

$$\psi^f_{(P,u,r)} : \mathcal{Z}_r(\mathcal{D}(E)) \to \hat{H}^{2p-r}(B) \tag{2.9}$$

given by

$$\Sigma \mapsto \hat{\pi}_{!\Sigma}(h^f_{B\times\Sigma}) \tag{2.10}$$

We often drop one or more of the subscripts P, u and r so as to avoid cluttered notation.

Below we show that this map vanishes on the boundaries and hence descends to a map on the homology $H_r(\mathcal{D}(E))$. We further show that this map is independent of the choice of the function f or ϵ.

Let us now compute the curvature of the differential character $\psi^f(\Sigma)$. To do this, we note that by Proposition 46, and equation (62) of [1], we have a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{k-1}(B\times\Sigma,\mathbb{R}/\mathbb{Z}) & \longrightarrow & \hat{H}^k(B\times\Sigma) & \xrightarrow{curv} & \Omega^k(B\times\Sigma) & \longrightarrow & 0\\
 & & \Big\downarrow{\scriptstyle \pi_{!\Sigma}} & & \Big\downarrow{\scriptstyle \hat{\pi}_{!\Sigma}} & & \Big\downarrow{\scriptstyle \fint_\Sigma} & & \\
0 & \longrightarrow & H^{k-r-1}(B,\mathbb{R}/\mathbb{Z}) & \longrightarrow & \hat{H}^{k-r}(B) & \xrightarrow{curv} & \Omega^{k-r}(B) & \longrightarrow & 0
\end{array}$$

Thus we have

$$curv(\psi^f(\Sigma)) = \fint_\Sigma curv(h^f_{B\times\Sigma}) = \fint_\Sigma P(\Omega^f) \tag{2.11}$$

where Ω^f is the curvature of the connection D_f i.e. $\Omega^f = dD_f + D_f \wedge D_f$. (Strictly speaking, we should write$(\Omega^f)^\mu_\nu = d(D_f)^\mu_\nu + (D_f)^\mu_\lambda \wedge (D_f)^\lambda_\nu$ where $\{(D_f)^\mu_\nu\}$ are connection 1-forms on $U \times \Sigma$ with respect to a frame $\{\phi^\mu\}$ over a trivializing subset $U \subseteq B$. However, in order to avoid clutter of notation, we use the same symbol D_f to denote the connection, and the connection 1-form w.r.t. a frame $\{\phi^\mu\}$ and omit the indices μ, and ν.)

Now we use the fact that for each $t \in \Sigma$, the restriction of the connection D_f to the bundle $E \times \{t\} \to B \times \{t\}$ is flat, together with a standard argument in the literature (see for example, [13, 30, 34]) to prove below that when $p > r+1$ or when $p < r$, the curvature of $\psi^f(\Sigma)$ is zero.

First, note that forms on a product manifold can be decomposed as $\Omega^k(M \times N) = \bigoplus_{0\le l\le k} \Omega^{l,k-l}(M \times N)$. On $E \times \Sigma^i \to B \times \Sigma^i$, the connection D_f restricts to $D^i_f = \sum c_j(t_0,\ldots,t_r)D^{j,i}$. Now if Ω^i_f is the curvature of D^i_f (or equivalently the restriction of Ω^f to Σ^i), then we have $\Omega^i_f = d_{\Sigma^i\times B}D^i_f + D^i_f \wedge D^i_f$.

We now use local coordinates $(x_\alpha)_{1\le\alpha\le m}$on on open set $U \subset B$. This gives us a chart for $U \times \mathring{\Sigma}^i$ with coordinates $(x_1,\ldots,x_m,t_0,\ldots,t_{r-1})$. In these coordinates, we have

$$d_{\Sigma^i\times B}D^i_f = \sum_{j,k} \frac{\partial c_j}{\partial t_k} dt_k \wedge D^{j,i} + d_B D^i_f \tag{2.12}$$

Thus $\Omega^i_f = \sum_{j,k} \frac{\partial c_j}{\partial t_k} dt_k \wedge D^{j,i} + d_B D^i_f + D^i_f \wedge D^i_f = \sum_{j,k} \frac{\partial c_j}{\partial t_k} dt_k \wedge D^{j,i}$.

The last equality follows since the restriction of D^i_f to any slice $E \times \{t\} \to B \times \{t\}$ gives a flat connection, thereby making the sum of the last two terms vanish.

Therefore $\Omega_f \in \Omega^{1,1}(B \times \Sigma, \mathfrak{gl}(n, \mathbb{R}))$. Since P is a homogeneous polynomial of degree p, we have $P(\Omega^f) \in \Omega^{p,p}(B \times \Sigma)$. Thus

$$P(\Omega^f) = 0 \quad for \quad p > r \tag{2.13}$$

In the $p < r$ case, even though $P(\Omega^f)$ need not vanish, the integral $\fint_\Sigma P(\Omega^f)$ still vanishes. We have thus obtained

Proposition 12. *When $p \neq r$, the curvature of $\psi^f(\Sigma)$ vanishes, and hence ψ^f takes values in $H^{2p-r-1}(B, \mathbb{R}/\mathbb{Z})$.*

When $p = r$, the above map gives a differential character in $\hat{H}^r(B)$ rather than an $\mathbb{R}/\mathbb{Z}$ cohomology class. We now proceed to show that ψ evaluated on boundaries vanishes, and hence it gives rise to a map $\tilde{\psi}^f : \mathbb{H}_r(\mathcal{D}(E)) \to H^{2p-r-1}(B, \mathbb{R}/\mathbb{Z})$. To see this, let $\Sigma = \partial K$ where $K \in \mathcal{Z}_r(\mathcal{D}(E))$. The geometric realization of Σ is the boundary of the geometric realization of K, and hence $h^f_{B\times\Sigma}$ is the restriction of $h^f_{B\times K}$ along its boundary. Therefore, we apply Proposition 11 to the bundle $E \times K \to B \times K$ to obtain:

$$\psi^f(\Sigma) = \hat{\pi}_{!\Sigma}(h^f_{B\times\Sigma}) \tag{2.14}$$

$$= \iota((-1)^{2p-dimK} \fint_K curv(h^f_{B\times K})) \tag{2.15}$$

$$= (-1)^{2p-dimK} \iota(\fint_K P(\Omega_{B\times K})) \tag{2.16}$$

Now $P(\Omega_{B\times K}) \in \Omega^{p,p}(B \times K)$, and $dimK = r + 1$. Hence if $p \neq r + 1$, $\psi^f_{p,r}(\Sigma) = 0 \ \forall \Sigma \in \mathcal{B}_r(\mathcal{D}(E))$. We have thus obtained

Proposition 13. *$\psi^f_{p,r}(\Sigma) = 0$ if Σ is an r-boundary and $p \neq r + 1$. Thus, for $p \neq r, r+1$ the map $\psi^f_{p,r}$ induces a map $\tilde{\psi}^f_{p,r} : \mathbb{H}_r(\mathcal{D}(E)) \to H^{2p-r-1}(B, \mathbb{R}/\mathbb{Z})$.*

We are now ready to see that the induced map on homology $H_r(\mathcal{D}(E))$ does not depend upon the choice of f or ϵ. Let f_0 and f_1 be two distinct choices of maps for ϵ_0, and ϵ_1 respectively. Choose a small $\delta > 0$, and a smooth monotonic function

$g : I = [0,1] \to [0,1]$ such that $g(t) = 0\ \forall t \in [0, \delta/2]$, $g(t) = t\ \forall t \in [\delta, 1-\delta]$, and $g(t) = 1\ \forall t \in [1-\delta/2, 1]$. Now consider the bundle $E \times \Sigma \times I \to B \times \Sigma \times I$. On this, define a smooth connection F such that F is trivial in the directions tangent to $\Sigma \times I$, and (ii) restriction of F to $E \times \Sigma \times \{s\} \to B \times \Sigma \times \{s\}$ is given by $g(s)D_{f_0} + (1-g(s))D_{f_1}$. Then, by Proposition (11), we have

$$\psi^{f_1}(\Sigma) - \psi^{f_0}(\Sigma) = (-1)^{r+1}\iota(\fint_{\Sigma\times I} curv(h^F_{B\times\Sigma\times I})) \tag{2.17}$$

Since $curv(h^F_{B\times\Sigma}) \in \Omega^{p,p}(B \times (\Sigma \times I))$, and since $p \neq r+1$, the R.H.S. vanishes, thereby proving the result:

Proposition 14. *When $p \neq r+1$, $\psi^f_{p,r}(\Sigma) \in H^{2p-r-1}(B;\mathbb{R}/\mathbb{Z})$ does not depend upon the choice of the function f.*

We therefore omit the superscript f, and simply write ψ or $\tilde{\psi}$.

Remark 15. *Let $\mathcal{G}$ be the group of bundle isomorphisms $\xi : E \to E$ which cover the identity map on B. Then $\mathcal{G}$ acts on the simplicial set $\mathcal{D}(E)$ by $\xi \cdot (D^0, \ldots, D^r) = ((\xi^{-1})^*D^0, \ldots, (\xi^{-1})^*D^r)$. Let $\xi \in \mathcal{G}$ be connected to the identity map via a smooth path in $\mathcal{G}$. An argument similar to the proof of Proposition 14 shows that $\psi(\xi\cdot\Sigma) = \psi(\Sigma)$ for any cycle $\Sigma \in \mathcal{Z}_r(\mathcal{D}(E))$ and $p \neq r+1$.*

Remark 16. ***The** $p = r$ **case.** In view of the Proposition 14 , for $\psi^f_{p,r}(\Sigma)$ to be independent of the choice of f, it suffices to have $p \neq r+1$. Thus $\psi^f_{r,r}(\Sigma)$ is a well defined differential character in $\hat{H}^r(B)$ which depends only on the homology class of Σ and not on f. We therefore get a well defined map $\tilde{\psi}_{r,r} : H_r(\mathcal{D}(E)) \to \hat{H}^r(B)$. Note, however, that since Proposition 12 does not guarantee vanishing of $curv(\psi^f_{r,r}(\Sigma))$, we may not conclude that $\tilde{\psi}_{r,r}$ takes values in the image of the inclusion map $H^{r-1}(B,\mathbb{R}/\mathbb{Z}) \xrightarrow{i_1} \hat{H}^r(B)$ (see the exact sequence 1.5).*

Remark 17. ***The** $p = r+1$ **case.** In this case, the Proposition 14 does not imply the independence of $\psi^f_{r+1,r}(\Sigma) \in H^{r+1}(B,\mathbb{R}/\mathbb{Z})$ from the choice of f. Further Proposition 13 does not apply, and thus we may not conclude that the maps $\psi^f_{r+1,r}$:*

$\mathcal{Z}_r(\mathcal{D}(E)) \to H^{r+1}(B, \mathbb{R}/\mathbb{Z})$ descend to maps $\tilde{\psi}^f_{r+1,r} : H_r(\mathcal{D}(E)) \to H^{r+1}(B, \mathbb{R}/\mathbb{Z})$. Also compare remark 7 in [13].

2.3 Comparison with other constructions in the literature

In this section, we first derive an explicit computable formula for $\psi(\Sigma)$. We find that this matches with the formula given by [34] thereby showing that the maps ρ' in [34] are equal to the maps $\tilde{\psi}$ here for the $p > r + 1$ case, and are zero for the $p < r$ case. We then show that as a consequence of Proposition 11, our invariants match with the ones constructed in [4] using entirely different methods. We further show that the results here are compatible with the ones obtained in [13].

Fix any flat connection D_0 on $E \to B$, and let $\bar{D}_0$ be the connection on $E \times \Sigma \to B \times \Sigma$ obtained by pulling back the connection D_0 on $E \to B$ under the pull back diagram

$$\begin{array}{ccc} E \times \Sigma & \longrightarrow & E \\ \downarrow & & \downarrow \\ B \times \Sigma & \longrightarrow & B \end{array}$$

Now since the space of connections is convex, there exists a connection $\tilde{D}$ on $E \times \Sigma \times I \to B \times \Sigma \times I$ such that its restriction to $E \times \Sigma \times \{0\} \to B \times \Sigma \times \{0\}$ is $\bar{D}$, and restriction to the slice $E \times \Sigma \times \{1\} \to B \times \Sigma \times \{1\}$ is D. Note that the restriction of $\tilde{D}$ to the slice $E \times \{p\} \times \{t\} \to B \times \{p\} \times \{t\}$ for $0 < t < 1, p \in \Sigma$ need not be flat.

By Proposition 11, we have

$$h^D_{B\times\Sigma\times\{1\}} - h^D_{B\times\Sigma\times\{0\}} = (-1)\iota(\fint_I curv(h^{\tilde{D}}_{B\times\Sigma\times I})) \in \hat{H}^{2p-r}(B \times \Sigma) \tag{2.18}$$

Applying further integration over the fiber Σ in the bundle $B \times \Sigma \to B$, we get

$$\psi(\Sigma) = (-1)^{r+1}\iota(\int_{\Sigma\times I} P(\Omega^{\tilde{D}})) \tag{2.19}$$

This is precisely[5] the construction given in [34] (since the R.H.S. in 2.19 does not depend upon ϵ, we can take $\epsilon \to 0+$). Therefore, we obtain:

Proposition 18. *The maps $\tilde{\psi} : \mathbb{H}_r(\mathcal{D}(E)) \to H^{2p-r-1}(B, \mathbb{R}/\mathbb{Z})$ are equal to the maps ρ' constructed in [34] for $p \geq r+2$.*

Notice that the equation 2.19 shows that $\psi_{p,u}$ does not depend on u, but only on P. Further it shows that that a different choice of the path of simplices connecting the trivial simplex to D changes the integral in the R.H.S. of equation 2.19 at most by a closed form with integral periods. In [34] R.H.S. is taken as the definition of the invariant associated to the family of flat connections.
Note that $\tilde{\psi}([\Sigma])$ considered as an element of $\hat{H}^{2p-r}(B)$ lies in the image of both the inclusion maps $H^{2p-r-1}(B, \mathbb{R}/\mathbb{Z}) \to \hat{H}^{2p-r}(B)$, and $\frac{\Omega^{2p-r-1}}{\Omega_0^{2p-r-1}} \xrightarrow{\iota} \hat{H}^{2p-r}(B)$. Thus not only does the curvature of this differential character vanish, but also its characteristic class. The vanishing of the characteristic class can also be understood in the following way.

Proposition 19. *The characteristic class of the differential character $\tilde{\psi}([\Sigma])$ vanishes.*

Proof. By compatibility of fiber integration of differential characters, and singular cohomology we have $ch(\hat{\pi}_{!\Sigma}(h_{B\times\Sigma})) = \pi_{!\Sigma} ch(h_{B\times\Sigma})$. Now we have an explicit description of the fiber integration map in singular cohomology (see the discussion in remark 4.5 of [1]). If $\mu \in H^{2p}(B \times \Sigma, \mathbb{Z})$, then $\pi_{!\Sigma}(\mu) \in H^{2p-r}(B, \mathbb{Z})$ is given by $\pi_{!\Sigma}(\mu)(c) = \mu(EZ(c \otimes \Sigma))$ where Σ denotes the fundamental class of the top homology $H_r(\Sigma, \mathbb{Z}) \simeq \mathbb{Z}$, and EZ is the Eilenberg-Zilber map. Now observe that if $u \in H^{2p}(G(k, \infty))$ is an element of cohomology of the base of the classifying bundle,

[5] In [34], the results are stated for the case $p > r$. However, to the best of our understanding, this is an error, and the results there hold for $p > r+1$. See remark 17 .

the characteristic class μ of the bundle $E \times \Sigma \to B \times \Sigma$ is given by $\mu(c') = u(f_*(c'))$ where $f : B \times \Sigma \to G(k, \infty)$ is any classifying map. In our case, if $g : B \to G(k, \infty)$ is a classifying map for the bundle $E \to B$, we can choose $f = g \circ pr_B$ where $pr_B : B \times \Sigma \to B$ is the projection map. Thus $f_*(EZ(c \otimes \Sigma)) = EZ(g_*(c) \otimes 0) = 0$. Hence we have $\pi_{!\Sigma}(ch(h_{B\times\Sigma})) = 0$. □

Since $ch \circ i_1$ is (up to a sign) the connecting homomorphism in the long exact sequence in cohomology corresponding to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$, we have the:

Corollary 20. *For any* $[\Sigma] \in H_r(\mathcal{D}(E))$, $\tilde{\psi}([\Sigma])$ *lies in the image of the map* $H^{2p-r-1}(B, \mathbb{R}) \to H^{2p-r-1}(B, \mathbb{R}/\mathbb{Z})$.

We shall now prove that the invariants vanish in case $p < r$.

Proposition 21. *For* $p < r$, $\psi_{p,r}(\Sigma) = 0 \;\; \forall \; \Sigma \in \mathcal{Z}_r(\mathcal{D}(E))$

Note that the differential character $h^f_{B\times\Sigma} \in \hat{H}^{2p}(B\times\Sigma)$ itself may not necessarily be zero. The above result therefore is a vanishing theorem about the the push-forward (fiber integral) $\hat{\pi}_{!\Sigma}(h^f_{B\times\Sigma})$.

Proof. $\Omega^{\tilde{D}} = \Omega^{\tilde{D}\ 2,0} + \Omega^{\tilde{D}\ 1,1} + \Omega^{\tilde{D}\ 0,2} \in \Omega^2(B \times (\Sigma \times I))$. However, as has been argued above, $\Omega^{\tilde{D}\ 0,2} = 0$ since $\tilde{D}$ is trivial in the directions of $\Sigma \times I$. Thus

$$P(\Omega^{\tilde{D}}) \in \Omega^{p,p}(B\times(\Sigma\times I))\oplus\Omega^{p+1,p-1}(B\times(\Sigma\times I))\oplus\cdots\oplus\Omega^{2p,0}(B\times(\Sigma\times I)) \quad (2.20)$$

Since $dim(\Sigma \times I) = r + 1$, the R.H.S. in equation 2.19 vanishes. □

Though this result regarding vanishing of invariant for the $p < r$ case has not been explicitly stated in [34], it is a consequence of Jaya Iyer's formula i.e. equation 2.19 in this paper.

There is an alternate way of formulating the study of the topology of the space of flat connections on a bundle $E \to B$. Rather than constructing a simplicial set $\mathcal{D}(E)$ of relatively flat connections, one considers the set of all connections $\mathcal{A}$ and gives it the structure of an infinite-dimensional Fréchet manifold. The space of flat connections $\mathcal{F}$ is a closed subspace of this space. One then associates cohomological invariants to elements of smooth singular homology groups [13] or to smooth maps $f : S \to \mathcal{F}$ from a suitable manifold S [4]. Our approach of employing fiber integration works in this formulation of the problem and yields the same answers (up to a sign, which arises due to differences in orientation conventions) as in [4, 13], as shall be shown below. Note that though the discussion in [4, 13] is formulated in terms of principal bundles, it can be written in terms of vector bundles as well.

In [4] the authors consider the set of smooth maps $Maps(S, \mathcal{F})$ where S is a null cobordant manifold and $\mathcal{F}$ denotes the space of all flat connections. Employing the framework of Atiyah bundle, and the bundle of connections, they construct certain $(r+1)$-differential forms $\beta^p_{r+1} \in \Omega^{r+1}(\mathcal{A}, \Omega^{2p-r-1}(B))$ on $\mathcal{A}$ taking values in $\Omega^*(B)$. They then define maps $\Lambda^p_{r+1} : Maps(S, \mathcal{F}) \to H^{2p-r-1}(B, \mathbb{R})$ given by $f \mapsto \Lambda^p_{r+1}(f) = \int_T \bar{f}^* \beta^p_{r+1}$ where $\bar{f} : T \to \mathcal{A}$ is any extension of the given map $f : S \to \mathcal{F}$ to a manifold T whose boundary is S. They show that these maps are well defined i.e the R.H.S. does not depend (considered as an element of cohomology group $H^{2p-r-1}(B, \mathbb{R})$) upon the choice of T or $\bar{f}$. As a consequence of certain identities in their paper, they further show that if $f_0, f_1 \in Maps(S, \mathcal{F})$ are homologous, then $\Lambda^p_{r+1}(f_0) = \Lambda^p_{r+1}(f_1)$. Also they prove that

$$\Lambda^p_{r+1}(f) \equiv \int_T \bar{f}^* \beta^p_{r+1} = \fint_T P(\Omega) \tag{2.21}$$

where Ω is the curvature of the connection induced on the bundle $T \times E \to T \times B$. In this formulation of the problem our approach works as follows. For $f \in Maps(S, \mathcal{F})$, consider the induced connection ∇^f on the bundle $E \times S \to B \times S$. This is the unique connection which restricts to the connection $f(s)$ on the slice $E \times \{s\} \to B \times \{s\} \forall s \in S$, and vanishes in the tangent directions along S. Let $h \in \hat{H}^{2p}(B \times S)$ denote the differential character corresponding to this connection. Then $f \mapsto \hat{\pi}_{!S}(h_{B\times S})$ gives us a map $\tilde{\psi}_{p,r} : Maps(S, \mathcal{F}) \to H^{2p-r-1}(B, \mathbb{R}/\mathbb{Z})$ for $p > r+1$. We have the following proposition:

Proposition 22. *For all $f \in Maps(S, \mathcal{F})$*

$$\tilde{\psi}_{p,r}(f) = (-1)^{2p-r-1}\Lambda^p_{r+1}(f)\ mod\ \mathbb{Z} \quad p \neq r, r+1 \tag{2.22}$$

This is a direct consequence of Proposition 11 and the identity 2.21 proved in [4].The fact that the form $\Lambda^p_{r+1}(f)$ is closed is automatic since the exterior derivative of this form is the curvature of the differential character $(-1)^{2p-r-1}\hat{\pi}_{!S}(h_{S\times B})$ by virtue of equation 2.11 which vanishes for $p \neq r$ (see eqn. 2.13). Note that Lopez and Biswas obtain maps into $H^{2p-r-1}(B, \mathbb{R})$ instead of $H^{2p-r-1}(B, \mathbb{R}/\mathbb{Z})$. From our standpoint, this is because the differential character has trivial characteristic class (see corollary 20). Invariance under the homology relation (Proposition 3.8 of [4]) is a consequence of Proposition 11 by an argument similar to the one used in proving Proposition 14. Their invariants for $p < r$ vanish because the forms β^p_r themselves vanish. From our point of view, even though $P(\Omega_{B\times T})$ does not vanish, $\oint_T P(\Omega_{B\times T})$ vanishes. Also note that their construction does not require a choice of $u \in H^{2p}(BGL(n, \mathbb{R}), \mathbb{Z})$ in the first place, while ours turns out to be independent of this choice (see the discussion after Proposition 18). Also compare this discussion with remark 3.4 in [4] where the authors show that Λ^p_{r+1} can be expressed as fiber integrals of certain transgression forms which can be used to prove some of their results.

As a special case, they consider the case where $S = S^r$, and obtain the maps $\Lambda^p_{r+1} : \pi_r(\mathcal{F}) \to H^{2p-r-1}(B, \mathbb{R})$ from homotopy groups of $\mathcal{F}$ to $H^{2p-r-1}(B, \mathbb{R})$. In contrast to homotopy groups, general homology classes can not be represented by maps from smooth manifolds, which is why we have used stratifolds in this paper.

In [13], the authors use the theory of differential characters and equivariant characteristic classes to obtain maps $H_{2p-r-1}(B, \mathbb{Z}) \times H_r(\mathcal{F}/\mathcal{G}, \mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ for $p > r+1$ where $\mathcal{G} \subset Gau(P)$ is a subgroup of the group of global gauge transformations which acts freely on $\mathcal{A}$. Their approach is to use the canonical connection $\mathbb{A}$ on the bundle $E \times \mathcal{A} \to B \times \mathcal{A}$ which is trivial in the directions of $\mathcal{A}$, and whose restriction to $E \times \{\theta\} \to B \times \{\theta\}$ is θ. They then choose a connection $\mathcal{U}$ on the principal bundle $\mathcal{A} \to \mathcal{A}/\mathcal{G}$. They show that this data determines a connection $\underline{\mathcal{U}}$ on the bundle $(E \times \mathcal{A})/\mathcal{G} \to B \times \mathcal{A}/\mathcal{G}$. Cheeger–Simons theory on this bundle yields a character $\chi_{\underline{\mathcal{U}}} \in \hat{H}^{2p}(B \times \mathcal{A}/\mathcal{G})$. Their maps are the composition $Z_{2p-r-1}(B,\mathbb{Z}) \times Z_r(\mathcal{F}/\mathcal{G},\mathbb{Z}) \to Z_{2p-1}(B \times \mathcal{F}/\mathcal{G},\mathbb{Z}) \xrightarrow{\chi_{\mathcal{U}}} \mathbb{R}/\mathbb{Z}$. The first map here is the standard multiplication map. They then show that these maps vanish when either factor is a boundary, and hence descend to homology groups $\chi_{p,r} : H_{2p-r-1}(B,\mathbb{Z}) \times H_r(\mathcal{F}/\mathcal{G}) \to \mathbb{R}/\mathbb{Z}$. They further show that these maps do not depend on the choice of the connection $\underline{\mathcal{U}}$.

Our method is a variant of their approach, and has the disadvantage that it does not prove the invariance of the invariants under the action of the Gauge group (or more precisely, a subgroup $\mathcal{G}$ of $Gau(E)$ which acts freely on $\mathcal{A}$). Thus we have to restrict our attention to obtaining maps $H_r(\mathcal{F}) \to H^{2p-r-1}(B,\mathbb{R}/\mathbb{Z})$ rather than $H_r(\mathcal{F}/\mathcal{G}) \to H^{2p-r-1}(B,\mathbb{R}/\mathbb{Z})$. In this sense our results are a special case of the results of [13] for the choice $\mathcal{G} = \{e\}$. (Notice that nothing in our approach actually uses the fact that the connection on the bundle corresponding to a point on the geometric realization of the cycle is a linear combination of the connection corresponding to the vertices. All that is required is that the connection corresponding to any point in the parameter space is flat.) However our approach does not assume that the Cheeger–Simons construction is applicable in the case when the base is an infinite dimensional Fréchet manifold. The Cheeger–Simons construction of a differential character given a bundle with a connection uses the fact (due to Narasimhan-Ramanan [39]) that any smooth connection on the bundle $E \to B$ (with the base B a finite dimensional manifold) can be obtained from a universal connection on a Stiefel bundle by pullback under a smooth classifying map. However to the best of knowledge of the present author, an analogous result when the base is an infinite dimensional Fréchet manifold has not yet been proved in the literature (also see the discussion in section 2.1.1, and footnote 3, p.9 in [2]).

Notice also that our result that the character $\tilde{\psi}(\Sigma)$ does not depend on the choice of u, proves that the maps $\chi_{P,u} : H_r(\mathcal{F}, \mathbb{Z}) \times H_{2p-r-1}(B, \mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ constructed in [13] do not depend upon the choice of u. In this formulation, we construct maps $\Psi_{p,r} : H_r(\mathcal{F}) \to H^{2p-r-1}(B, \mathbb{R}/\mathbb{Z})$ as follows. Let $g : \Sigma \to \mathcal{F}$ represent an element $[\Sigma]$ of the smooth singular homology group $H_r(\mathcal{F})$. Let ∇ be the induced connection on $E \times \Sigma \to B \times \Sigma$, and let $h_{B\times\Sigma} \in \hat{H}^{2p}(B \times \Sigma)$ be the differential character corresponding to ∇. Then $[\Sigma] \mapsto \hat{\pi}_{!\Sigma}(h_{B\times\Sigma})$ defines a map $\Psi_{p,r} : H_r(\mathcal{F}) \to H^{2p-r-1}(B, \mathbb{R}/\mathbb{Z})$. We have already seen that the maps constructed in this article are compatible with the maps in [4]. In Proposition 10 of [13], it is shown that the maps constructed there are compatible with the ones in [4]. Also see the discussion in section 8 of [13] where they consider the relation of their invariants with those in [34]. Below we show that the the maps χ^r (of [13]) are compatible with the maps $\Psi_{p,r}$ defined above. For this purpose we need the following lemma which follows directly from the definition of fiber integration.

Lemma 23. *Let $h \in \hat{H}^k(E)$, and let $f : \Sigma \to E$ represent a $(k-1)$-cycle z. Then*

$$h(z) = \hat{\pi}_{!\Sigma}(f^* h) \tag{2.23}$$

where the fiber integration is along the fiber Σ of the fiber bundle $\Sigma \to *$.

Let $\chi_{\mathbb{A}} \in \hat{H}^{2p}(B \times \mathcal{A})$ be the character corresponding to the connection $\mathbb{A}$ on $E \times \mathcal{A} \to B \times \mathcal{A}$. Let $f : \Sigma \to \mathcal{F} \to \mathcal{A}$ be a cycle representing a homology class in $H_r(\mathcal{F})$. Then the connection that we used in our construction is the pull back of the connection $\mathbb{A}$ in the diagram

$$\begin{array}{ccc} E \times \Sigma & \longrightarrow & E \times \mathcal{A} \\ \downarrow & & \downarrow \\ B \times \Sigma & \longrightarrow & B \times \mathcal{A}. \end{array}$$

Hence $h_{B\times\Sigma} \in \hat{H}^{2p}(B \times \Sigma)$ is given by $h_{B\times\Sigma} = (id_B \times f)^* \chi_{\mathbb{A}}$. Let $g : K \to B$ is a

$(2p-r-1)$-cycle in B. Then consider $g \times f : K \times \Sigma \to B \times \mathcal{F}$ We then have

$$\begin{aligned}\Psi(\Sigma)(K) &= \hat{\pi}_{!\Sigma}(h_{B\times\Sigma})(K) \\ &= \hat{\pi}_{!K}(g^*(\hat{\pi}_{!\Sigma}(h_{B\times\Sigma}))) \\ &= \hat{\pi}_{!\Sigma\times K}((g\times f)^*\chi_{\mathbb{A}}) \\ &= \chi_{\mathbb{A}}(\Sigma\times K).\end{aligned}$$

The right hand side of the last equality is precisely the definition of the maps $\chi^r : H_{2p-r-1}(B) \times H_r(\mathcal{F}) \to \mathbb{R}/\mathbb{Z}$ for the special case $\mathcal{G} = \{e\}$ constructed in [13], see equations 8 and 14 in their paper. As they mention in section 8 of their article, the maps χ^r above induce maps (denoted by the same symbol) $\chi^r : H_r(\mathcal{F}) \to H^{2p-r-1}(B, \mathbb{R}/\mathbb{Z})$. We thus have the proposition

Proposition 24. *The maps $\Psi_{p,r}$ are equal to the maps χ^r constructed in [13].*

We would like to remark that the maps on $H_r(\mathcal{D}(E))$ constructed in section 2.2 factor through the maps on $H_r(\mathcal{F})$ constructed in this section. A choice of smoothening function f as in the construction in section 2.2 gives a map $\mathcal{Z}_r(\mathcal{D}(E)) \to \mathcal{Z}_r(\mathcal{F})$ as follows. Given a cycle $\Sigma \in \mathcal{Z}_r(\mathcal{D}(E))$, one constructs a smooth connection D_f on $E \times \Sigma \to B \times \Sigma$ (for the construction of this connection, see section 2.2). But this connection can be thought of as induced by a smooth map (also denoted D_f) $D_f : \Sigma \to \mathcal{F}$ on the stratifold Σ. This geometric chain (see section 2.1.1) gives a cycle in $\mathcal{Z}_r(\mathcal{F})$. The map $\mathcal{Z}_r(\mathcal{D}(E)) \to \mathcal{Z}_r(\mathcal{F})$ takes boundaries in the simplicial set $\mathcal{D}(E)$ to boundaries $\mathcal{B}_r(\mathcal{F})$. Further a different choice f' of the smoothening function changes the smooth singular cycle by at most a boundary. Consequently we have an induced map $\lambda : H_r(\mathcal{D}(E)) \to H_r(\mathcal{F})$ which does not depend on the choice of the function f. The map $\tilde{\psi}_{p,r} : H_r(\mathcal{D}(E)) \to H^{2p-r-1}(B, \mathbb{R}/\mathbb{Z}) forp > r+1$ (of section 2.2 or equivalently the map $\rho'_{p,r}$ of [34]) can now be seen, by construction, to be the composition of the maps $\lambda : H_r(\mathcal{D}(E)) \to H_r(\mathcal{F})$ and the maps $\Psi_{p,r} = \chi^r : H_r(\mathcal{F}) \to H^{2p-r-1}(B, \mathbb{R}/\mathbb{Z})$. We thus have the following

Proposition 25. *The following diagram commutes for $p > r+1$:*

$$\begin{array}{ccc} H_r(\mathcal{D}(E)) & & \\ \Big\downarrow \lambda & \overset{\tilde{\psi}_{p,r}=\rho'_{p,r}}{\searrow} & \\ H_r(\mathcal{F},\mathbb{Z}) & \xrightarrow{\Psi_{p,r}=\chi^r} & H^{2p-r-1}(B,\mathbb{R}/\mathbb{Z}). \end{array}$$