Chapter 1

Introduction

Ecology, a sub-discipline of biology, is the study of the distributions, abundance and relation of organisms and their interactions with the environment. It deals primarily with the descriptive study of organisms. Ecology includes the study of plant and animal communities and ecosystems. The fundamental goal of ecological research is to study all the factors which affect the interaction of individual organism with each other or environment and to get the results which help to sustainable development of ecosystem. Mathematical model, which describes dynamics of these organisms, has been playing an important role in understanding the dynamics of ecosystems. Predator feeds prey but there dynamics depends upon different situations. There are some biological factors such as hunting rate, handling time, search efficiency, prey escape ability, predator's interference, structure of prey habitat etc. In mathematical ecology, we study the impact of all these factors via some mathematical models, different analytical approaches and reach to some results which are helpful to keep intact the prey-predator relationship for a long time.

Nature provides us all the conveniences to survive on the surface of the world. Overall development of our system we live in, is based on available natural resources. Nowadays, more renewable resources like forestry and fishery are required in the consequences of the growing human population and modernization. In such a situation, exploitation of renewable resources is increasing and hence our society will face problems like shortage of resources and degradation of environment. Prey-predator is one of the dominant theme in ecology due to its universal existence and importance. Mathematical modelling has been recognized as an important tool to understanding and analyzing the prey-predator interactions. It has been an issue of attention among ecologists and naturalists since last three to four generations. First model on prey-predator organism was formulated and proposed by Lotka and Volterra, known as Lotka-Volterra system [132]. It contains a pair of first order, non-linear, differential equations frequently used to described the dynamics of biological system in which two species interact. After that a lot of work has been done to understand the prey-predator relationship [11, 15, 73, 100, 107, 120, 144, 167, 210]. In classical Lotka-Volterra systems, the functional response follows linear mass action law. But all organisms do not happen with same hunting or intake rate. So one vital factor of the prey-predator interaction is the intake rate of prey by a predator i.e. functional response. It helps to predict about a prey-predator dynamics with more accuracy. There are many types of functional response: Holling type I-IV, Ratio dependent, Beddington-DeAngelis, Crowley-Martin, Hassel-Verley. Holling type I-IV functional responses are prey dependent whereas Beddington-DeAngelis, Crowley-Martin, Hassel-Verley are predator dependent i.e. functional response is function of both the prey and predator's density.

The effect of intraspecific interference among predators has been investigated in prey-predator model with Holling type II functional response in [241, 104], with Holling type III functional response in [73] and with Beddington-DeAngelis type functional response in [51]. Crowley and Martin [38] assumed that predation will decrease when the predator density is high due to interference among predators. Some investigations have been conducted on prey predator model including Crowley-Martin functional response [50, 192, 214].

In recent years, ecological models with add-on of certain new factors such as prey refuge (creating safe zones for prey) [104, 216, 100], complex habitat [37, 11], strong and weak Allee effect [185, 4], providing additional food for predators [181, 170], harvesting of renewable resources [144], etc have also been investigated to reach more close to realistic situations. The effect of the degree of habitat complexity on the dynamics of prey-predator system is well studied by Bairagi and Jana [11]. They observed that the fluctuations in the population level can be controlled completely by increasing the degree of habitat complexity. Recently, Verma and Misra [216] have studied the impact of a constant prey refuge on the dynamics of a ratio-dependent predator-prey system with strong Allee effect in prey growth. They found that if prey refuge is less than the Allee threshold, the incorporation of prey refuge increases the threshold value of the predation rate and conversion efficiency at which unconditional extinction occurs. They also vindicated that the species can be protected by creating safe zones in accordance with the Allee threshold. Parsad *et al.* [170] investigated the characteristics of biological control agents and additional food permitting the eco-managers to enhance the success rate of biological control programs.

The rate of reproduction and survival largely depend upon age or the development stage and hence it is important to incorporate the effect of past life history during analysis of the system. A life cycle of species can be divided in two stages, viz., mature and immature. The analysis of stage structured prey-predator models has attracted good amount of attention recently, as a way to eliminate the shortcomings of classical Lotka-Volterra models [233, 219, 173, 39, 234, 99]. Bosch and Gabrial [23], and Kar [96] studied the stage and age structure of species without and with delay. A study of simultaneous effects of harvesting on the dynamics of Leslie-Gower prey-predator system has been done by Chakraborty *et al.* [26]. They suggested a policy of optimal harvesting for the system with stage structure.

Hopf-bifurcation is an important tool which helps to understand the behavior of system. It gives us such a critical value of a parameter that the stability behavior of system is contrasty in both the sides of the critical value [79, 45]. Patra *et al.* [167] proposed and analyzed three mathematical models to study the degradation and conservation of a wetland park. Models shows interaction among bird population, good biomass, bad biomass and applied effort to control the density of bad biomass. It has been shown that carrying capacity serves as a bifurcation parameter for the systems. If carrying capacity increases beyond a threshold value, then behavior of the system changes from stability to instability. Local Hopf-bifurcation in a delayed system has been studied by Song and Wei [194], Bairagi and Jana [11] and Deng *et al.* [46]. They have shown the existence of periodic solutions after a certain value of time delay.

Allee effect plays a major role in the structure of population. It creates the possibilities of extinction of species [185, 241] and has a huge impact in population dynamics [4]. The Allee effect can be classified into two types on the basis of per capita growth rate at low density. These are known as strong Allee effect and weak Allee effect. Strong Allee effect have negative per capita growth rate at low population level and implies the existence of a threshold level of population so that the species become extinct below this level. On the other hand in weak Allee effect, the per capita growth rate decreases but remains positive at low population level. Sexual selection [18, 147], reduced mating efficiency [47] and alleviated foraging efficiency [17] are some other reasons to give rise to Allee effect. Many researchers have found the natural evidence of weak Allee effect by experimental work on flour beetles of the genus *Tribolium* [4, 103, 211]. They have shown that the per capita growth of beetles reaches its maximum at a medium density and the rate is positive at low density.

Time delay occurs in a wide range of physical, chemical, economic, engineering and biological systems. The conversion of hunted prey population to predator population is not an instantaneous process, there is some time lag (gestation delay) [136]. Besides that, in stage structured systems maturation delay is the time lag for juvenile individuals to be mature [26]. Negative feedback delay [64, 61], hunting delay [235] are some other kind of time delay. Time delay makes the model more realistic. It is well known that delay differential equations exhibit much more complicated dynamics than ordinary differential equations as a time delay could cause a stable equilibrium to become unstable and cause the population to fluctuate. The characteristic equation corresponding to the linearized system of delay differential equation is no longer a polynomial, rather, it is exponential polynomial or quasi-polynomial as named in Bellman and Cooke [12]. We refer to the monographs of Cushing [42], Gopalsamy [70] and Kuang [107] for general delayed biological systems and to Song and Wei [195], Tripathi *et al.*. [210], Bairagi and Jana [11], Beretta and Kuang [15] and references cited therein for studies on delayed prey-predator models. Song and Wei [195] have done bifurcation analysis for Chen's system [28] with delayed feedback. They found that when the delay passes through certain critical values, chaotic oscillation is converted into a stable steady state or a stable periodic orbit. Feng and Hongwei [116] have analyzed a prey-predator model with gestation delay and observed that system changes its stability behavior beyond a certain critical value of the delay. Hopf bifurcation and it's stability and direction have been investigated using normal form theory and the center manifold theorem for functional differential equations by Zhang *et al.* [238] and Tripathi *et al.* [210] by taking maturation delay and gestation delay, respectively. Chakraborty *et al.* [26] have shown that the delay not only can cause a stable equilibrium to become unstable, it can cause a switching of stability sometimes.

Some investigations have been made with two or more delays involved in the system. In such cases, the complexity involved in obtaining the eigenvalues from the transcendental characteristic equation becomes difficult. Thus, the bifurcation problem will be complicated due to presence of multiple delays. Some systems with multiple delays have been well studied in [64, 120, 46, 122, 126, 128].

1.1 Objectives of the thesis

The objective of thesis is to deal with several problems of survival of biological populations subject to ecological stability and to study various characteristics that play significant role to stabilizing the ecosystem. The following objectives are fulfilled in this thesis using mathematical models:

(i) To study the dynamics of a fear induced prey-predator model with gestation delay.

(ii) To investigate global stability and Hopf-bifurcation in a prey-predator model including habitat complexity and prey refuge.

(iii) To study an optimal harvesting policy of a stage structured prey-predator model.

(iv) To analyze the dynamics of delay induced prey-predator system with stage structure in prey.

(v) To study the dynamics of prey-predator system with strong and weak Allee effect.

(vi) To examine the effect of additional food provided to predator in a ecosystem.

1.2 Basic population models

1.2.1 Malthusian model

This is the simplest case where the rate of change in population is proportional to existing population. The mathematical form is

$$\frac{dP}{dt} = rP,$$

where r is proportionality constant, called growth rate. This is known as exponential growth model as the population grows without any restrictions. Such growth may be followed for a short time, but not much applicable forever in real world.

1.2.2 Logistic model

Due to limitations of Malthusian model, logistic model came into the existence by introducing interference among individuals. The equation of growth is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right),$$

where K is carrying capacity, the number of individuals which is affordable by the environment. Initially population increases with exponential growth but as population size increases, interference also increases accordingly and keeps the population bounded.

1.2.3 Prey-predator model

The prey-predator interaction between two species occurs if one species consumes another as food resource. The first prey-predator model is proposed by Lotka [132] and Volterra [217].

$$\begin{aligned}
&\frac{dx}{dt} = ax - bxy, \\
&\frac{dy}{dt} = cxy - dy, \\
&a, b, c, d > 0 \;\text{ and }\; x(0) \geq 0, y(0) \geq 0,
\end{aligned}$$

where x and y are densities of prey and predator, a is growth rate of prey, b is attack rate of predator on prey, c is growth rate of predator due to predation and d is the death rate of predator.

1.3 Functional response

In the dynamics of prey-predator system, functional response is an important aspect, which represents the consumption rate of prey by predator. If the number of prey consumed by a predator per unit time is a linear function of prey then it is known Holling type-I functional response, which is represented as follows:

$$f_1(x) = \alpha x,$$

where α is capture or attack rate. Mathematically, it is represented by a straight line which passes through origin. This is conventional functional response which is used in the pioneer work of Lotka [132] and Volterra [217]. But this functional response has some limitations. If prey population is in abundance then predators will consume at maximum rate therefore further increment in prey population will not be able to increase the consumption rate. In this case the prey predator phenomenon is described by Holling type-II functional response. It can be written as follows:

$$f_2(x) = \frac{\alpha x}{1 + h\alpha x},$$

where α is attack rate and h is handling time required per prey. In the above expression, number of prey consumed per predator initially increases fast but as the density of prey increases it goes to its saturation level.

Holling type-III functional response is similar to Holling type-II functional response at high level of prey density. But at low prey density, the number of consumed prey and the density of the prey population is more linearly increasing function of prey consumed by predators. It can be represented by the following equation

$$f_3(x) = \frac{\alpha x^2}{1 + h\alpha x^2}.$$

This functional response is typical of generalist predators which readily switch from one food to another and concentrate their feeding in areas where certain resources are abundant.

Holling type-IV functional response is a non-monotonic function of prey density. Initially, it increases and reaches its maxima and then decreases for further values. Mathematically, it is represented by

$$f_4(x) = \frac{\alpha x}{x^2 + ax + b},$$

where a is the tolerance limit of predator and b is half-saturation constant. This functional response is used for such prey species that show group defence against predator.

Beddington and DeAngelis assumed that predator's feeding rate decreases at high predator density due to interference among them. It makes the functional response more realistic. The Beddington DeAngelis functional response is

$$g_1(x,y) = \frac{\alpha x}{1 + h\alpha x + by},$$

where b is magnitude of interference among predators. On later Crowley and Martin modified it by adding the interference between prey and predator. The Crowley Martin functional response is

$$g_2(x,y) = \frac{\alpha x}{(1 + h\alpha x)(1 + by)}.$$

1.4 Some useful definitions

We develop variety of ecological models to address our main objectives which are described by system of ordinary differential equations and delay differential equations. To investigate the stability of the deterministic models, many approaches are established. We have adopted the following techniques. We consider a nonlinear autonomous system

$$\frac{dx}{dt} = f(x), \tag{1.1}$$

where $x = (x_1, x_2, ..., x_n)$ is a vector and $f \in C[R^n, R^n]$ is a smooth enough function that ensure the existence and uniqueness of the solution of (1.1).

Definition 1.4.1. *(**Stability**) The solution of system (1.1) $x(t)$ with initial data $x(0) = x_0$, is called locally stable if for any $\varepsilon > 0$ there exist a $\delta(\varepsilon) > 0$ and t_0 such that for any solution $\bar{x}(t)$, the inequality $||\bar{x}_0 - x_0|| < \delta$ implies $||\bar{x}(t) - x(t)|| < \varepsilon$ for each $t > t_0$. In other words the solution $x(t)$ is locally stable if any solution initiated from $\delta-$neighborhood of x_0 always remains in $\varepsilon-$neighborhood of $x(t)$ after a considerable time.*

Definition 1.4.2. *(**Asymptotic stability**) The solution of system (1.1) $x(t)$ with initial data $x(0) = x_0$, is said to be locally asymptotically stable if there exist a $\delta > 0$ such that the inequality $||\bar{x}_0 - x_0|| < \delta$ implies $\lim_{t\to\infty} ||\bar{x}(t) - x(t)|| \to 0$. In other words the solution $x(t)$ is locally asymptotically stable if any solution initiated from $\delta-$neighborhood of x_0, converges to the solution $x(t)$.*

Definition 1.4.3. *(**Instability**) The solution of system (1.1) $x(t)$ is known as unstable if it is not locally stable.*

Definition 1.4.4. *(**Equilibrium point**) Equilibrium point is a solution of system (1.1) that does not change with time. It can be obtained as a solution of system of equation $f(x) = 0$.*

Definition 1.4.5. *(**Global stability**) An equilibrium point is called globally asymptotically stable if the solutions initiated from any point in a finite region, converge to the equilibrium point.*

Definition 1.4.6. *(**Basin of attraction**) Basin of attraction of an equilibrium point is collection of such points in phase space, that any solution trajectory started from them, approaches the equilibrium point.*

Definition 1.4.7. *(**Bistability**) In a system bistability occurs when there are two local stable equilibria (sometimes attractor). The curve that separates the basin of attraction of both the equilibria, known as **separatrix**. In the similar way, tristability and multistability can also be defined.*

Definition 1.4.8. *(**Invariant set**) A set $\Omega \in R^n$ is said to be invariant set if for every solution $x, x(t_0) \in \Omega$ implies $x(t) \in \Omega$ for all $t > t_0$. In other words if a solution is started from point in Ω, remains in Ω for all time.*

Definition 1.4.9. *(**Limit cycle**) A closed trajectory of a system is known as orbit and the motion along orbit is periodic. An orbit is said to be limit cycle if every trajectory that starts at a point close to the orbit, converges towards the orbit as $t \to \infty$. Any orbit that is not one of a family of concentric orbits, must be either a limit cycle or an originating cycle in the sense that all neighboring trajectories diverse from the orbit. Thus a limit cycle is a isolated periodic solution. The neighboring solutions may be attracted or repelled by the limit cycle. A limit cycle is stable if nearby trajectories are attracted by the limit cycle. A limit cycle is unstable if nearby trajectories are repelled by the limit cycle. A limit cycle is semistable if trajectories are attracted from one side and repelled from another side by the limit cycle.*

1.5 Mathematical tools used in the thesis

1. **Linearization of differential equations:** Suppose

$$\frac{dX(t)}{dt} = F(X(t)), \tag{1.2}$$

where $X(t) = [x_1(t), x_2(t), ..., x_n(t)]^T$ is a system of ODEs. Let $x_i(t) = x_i^*(t) + z_i(t),\ i = 1, 2, ...n$, then linearizing the system (1.2) about the equilibrium solution $E^* = (x_1^*, x_2^*, ..., x_n^*)$, we have

$$\frac{dZ(t)}{dt} = JZ(t)),$$

where J is the Jacobian matrix of the system, calculated at E^* and $Z(t) = [z_1(t), z_2(t), ..., z_n(t)]^T$.

2. **Eigenvalue Method for Local Stability:** Being a straight forward method, based purely on the sign of real parts of eigenvalues. We shall use Routh-Hurwitz criterion [1] to study the local stability of wide range of systems in homogeneous environment.
The Routh-Hurwitz Stability Criterion: According to this criterion, the necessary and sufficient condition for the negativity of the real parts of all the roots of the polynomial

$$\lambda^n + A_1\lambda^{n-1} + A_2\lambda^{n-2} + ... + A_n = 0$$

with real coefficients is the positivity of all the principal diagonals of the minors of the Hurwitz matrix

$$H_n = \begin{bmatrix} A_1 & 1 & 0 & ... & 0 \\ A_3 & A_2 & A_1 & ... & 0 \\ A_5 & A_4 & A_3 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & A_n \end{bmatrix}.$$

The alternate columns in this matrix consist of coefficients with only odd indices or with only even indices (including the coefficient $a_0 = 1$). Hence, the elements of the Hurwitz matrix $H_n = (b_{ik})$ are given by $b_ik = a_{2i-k}$, the missing coefficients (i.e., the coefficients with indices greater than n or less than zero being replaced by zeros.

$$D_1 = A_1 > 0, \quad D_2 = \begin{vmatrix} A_1 & 1 \\ A_3 & A_2 \end{vmatrix} > 0,$$

$$D_3 = \begin{vmatrix} A_1 & 1 & 0 \\ A_3 & A_2 & A_1 \\ A_5 & A_4 & A_3 \end{vmatrix} > 0,$$

$$D_n = det(H_n).$$

The Routh-Hurwitz criterion for particular values of $n = 2, 3$ and 4 are stated below:

$$n = 2; \quad A_1 > 0, \; A_2 > 0,$$

$$n = 3; \quad A_1 > 0, \; A_3 > 0, \; A_1A_2 > A_3,$$

$$n = 4; \quad A_1 > 0, \; A_2 > 0, \; A_3 > 0, \; A_4 > 0, \; A_1A_2A_3 > A_3^2 + A_1^2A_4.$$

The above method is useful to check the local stability of an equilibrium point. The local stability describes the qualitative behavior of the solution in a certain neighborhood. It does not give any information about the behavior of the solution out of that neighborhood. The Lyapunov's direct method [1] can be useful to study the stability behavior of nonlinear systems.

3. **Lyapunov's Direct Method:** The physical validity of this method is contained in the fact that stability of the system depend on the energy of the system which is a function of system variables. Lyapunov's direct methods consists in finding out such energy functions termed as Lyapunov functions which need not be unique. The major role in this process is played by positive or negative definite functions which can be obtained in general by trial of some particular functions of state variables, and in some cases with a planned procedure.

   We shall use the following important results for the stability analysis of our models. Consider the following system of autonomous differential equations:

$$\frac{dx}{dt} = f(x), \tag{1.3}$$

   where $f \in C[R^n, R^n]$ and $S_\rho = \{x \in R^n : ||x|| < \rho\}$.
   Assume that f is smooth enough to ensure the existence and uniqueness of the solution of (1.3). Let $f(0) = 0$ and $f(x) \neq 0$ for $x \neq 0$ in some neighborhood of the origin so that (1.3) admits the so-called zero solution $(x = 0)$ and the origin is an isolated critical point of (1.3).

   Theorem 1.5.1. *If there exists a positive definite scalar function $V(x)$ such that $\frac{dV}{dt} \leq 0$ on S_ρ, then zero solution of (1.3) is stable.*

   Theorem 1.5.2. *If there exists a positive definite scalar function $V(x)$ such that $\frac{dV}{dt} < 0$ on S_ρ, then zero solution of (1.3) is asymptotically stable.*

   Theorem 1.5.3. *If there exists a scalar function $V(x)$, $V(0) = 0$ such that $\frac{dV}{dt} > 0$ on S_ρ and if in every neighborhood N of the origin, $N \subset S_\rho$, there is a point x_0, where $V(x_0) > 0$ then the zero solution of (1.3) is unstable.*

4. **Bendixson-Dulac Theorem:** Consider a dynamical system

$$\frac{dx}{dt} = f(x,y), \quad \frac{dy}{dt} = g(x,y),$$

   where $f(x,y)$ and $g(x,y)$ are assumed to be smooth functions in a simple connected domain D. If there exist a smooth function $\phi(x,y)$ in domain D such that the expression

$$\nabla = \frac{\partial(\phi f)}{\partial x} + \frac{\partial(\phi g)}{\partial y}$$

does not change the sign in D. Then the above system has no limit cycle in the domain D.

5. **Center Manifold Theory:** We shall study the direction and stability of the periodic solution bifurcating from the steady state by using center manifold theory introduced by Hassard *et al.* [80].

Theorem 1.5.4 (Center Manifold Theorem). *Let $f : R^n \to R^n$ be a vector field in $C^{k+1}(k \geq 1)$, with $f(0) = 0$. Consider a matrix $A = Df(0)$, and let V^s, V^u, V^c be the corresponding stable, unstable and center sub-spaces. Then there exists $\delta > 0$ and a local center manifold M with the following properties.*

(i) There exist a C^k function $\phi : V^c \to R^n$ with $\pi_c \phi(x_c) = x_c$ such that

$$M = \{\phi(x_c) : x_c \in V^c, |x_c| < \delta\}.$$

(ii) The manifold M is locally invariant for the flow of (1.3), i.e. $x \in M$ implies $\tilde{x}(t,x) \in M$ for $|t|$ small.

(iii) M is tangent to V^c at the origin.

(iv) Every globally bounded orbit remaining in a suitably small neighborhood of the origin is entirely contained inside M.

(v) Given any trajectory such that $x(t) \to 0$ as $t \to +\infty$, there exists $\eta > 0$ and a trajectory $t \to y(t) \in M$ on the center manifold such that

$$e^{\eta t}|x(t) - y(t)| \to 0 \quad as \quad t \to +\infty.$$

5. **Numerical Simulation:** Numerical experiments have been carried out with the help of Mathematica / MATLAB to validate the theoretical findings.

1.6 Bifurcation

In dynamical systems, a bifurcation occurs when a small perturbation made to the parameter values of the system causes a sudden qualitative or topological change in its behavior. This qualitative change can happen in different ways, on their basis bifurcation can be categorized into several types.

(i) **Saddle-node:** In saddle-node bifurcation, two distinct equilibrium points (one is saddle point and another is node) come towards each other by varying a parameter and at a critical value of the parameter, both the equilibria collide and annihilate each other. A typical example of saddle-node bifurcation is

$$\frac{dx}{dt} = x^2 + \mu,$$

where μ is bifurcation parameter and the bifurcation value of μ is 0.

(ii) **Transcritical:** In transcritical bifurcation, two existing equilibrium points interchange their stability behavior as the bifurcation parameter is varied. It is characterized by an equilibrium having an eigenvalue whose real part passes through zero. A common example of this bifurcation is similar to the logistic growth equation without restriction on growth rate μ

$$\frac{dx}{dt} = \mu x(1-x),$$

where μ is bifurcation parameter and the bifurcation occurs at $\mu = 0$.

(iii) **Pitchfork:** In pitchfork bifurcation, the system transitions from one equilibrium point to three equilibrium points. The stability behavior of two newly born equilibria is alter to the existing equilibrium. A typical example of this bifurcation is

$$\frac{dx}{dt} = \mu x - x^3.$$

For $\mu < 0$, there is one stable equilibrium at $x = 0$. For $\mu > 0$ there is an unstable equilibrium at $x = 0$, and two stable equilibria at $x = \pm\sqrt{\mu}$.

(iv) **Hopf-bifurcation:** In this bifurcation, an equilibrium point of the system switches its stability and a periodic solution (limit cycle) arises. The eigenvalues of linearized system around the equilibrium point are complex in vicinity of the critical value of bifurcation parameter. They cross the imaginary axis as the bifurcation parameter is varied and at the critical value, we get a pair of pure imaginary eigenvalues. When a stable limit cycle surrounds an unstable equilibrium point, the bifurcation is called a *supercritical* Hopf bifurcation. If the limit cycle is unstable and surrounds a stable equilibrium point, then the bifurcation is called a *subcritical* Hopf bifurcation.

Let's consider a two dimensional system

$$\frac{dx}{dt} = f(x,y,\mu), \quad \frac{dy}{dt} = g(x,y,\mu)$$

where μ is a parameter. Suppose it has a fixed point, which without loss of generality we may assume to be located at $(x,y) = (0,0)$. Let the eigenvalues of the linearized system about the fixed point be given by $\lambda(\mu), \bar{\lambda}(\mu) = \alpha(\mu) + i\beta(\mu)$. Suppose further that for a certain value of μ (which we may assume to be μ_c) the following conditions are satisfied:

(a) non-hyperbolicity condition: conjugate pair of imaginary eigenvalues

$$\alpha(\mu_c) = 0, \ \ \beta(\mu_c) = \omega \neq 0,$$

(b) transversality condition: the eigenvalues cross the imaginary axis with non-zero speed

$$\left[\frac{d\alpha(\mu)}{d\mu}\right]_{\mu=\mu_c} = d \neq 0,$$

(c) genericity condition:

$$\sigma = \frac{1}{16}(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) + \frac{1}{16\omega}(f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}) \neq 0,$$

where $f_{xy} = \left[\frac{\partial^2 f}{\partial x \partial y}\right]_{\mu=\mu_c,(x,y)=(0,0)}$, etc.

Then a unique curve of periodic solutions bifurcates from the origin into the region $\mu > 0$ if $\sigma d < 0$ or $\mu < 0$ if $\sigma d > 0$. The origin is a stable fixed point for $\mu > 0$ (resp. $\mu < 0$) and an unstable fixed point for $\mu < 0$ (resp. $\mu > 0$) if $d < 0$ (resp. $d > 0$) whilst the periodic solutions are stable (resp. unstable) if the origin is unstable (resp. stable) on the side of $\mu = 0$ where the periodic solutions exist. The amplitude of the periodic orbits grows like $\sqrt{|\mu|}$ whilst their periods tend to $\frac{2\pi}{|\omega|}$ as $|\mu|$ tends to zero.

1.7 Delay differential equations

Delay differential equations are a type of differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. Many physical and biological systems include aftereffect phenomena in their inner dynamics. In such systems, its necessary to incorporate delay to approach reality.

A general delay differential equation have the following form for $x(t) \in R^n$

$$\frac{dx(t)}{dt} = f(t, x(t), x_\tau),$$

where $x_\tau = \{x(\tau) : 0 \leq \tau \leq t\}$ represents the solution trajectory in the past. There are three types of DDEs

(i) Discrete DDEs: The derivative of unknown function at a certain time depends on several specific values of the function in the past.

$$\frac{dx(t)}{dt} = f(t, x(t), x(t-\tau_1), x(t-\tau_2), ..., x(t-\tau_n)),$$

where $\tau_1, \tau_2, ..., \tau_n$ are positive and constants.

(ii) Continuous DDEs: The derivative of unknown function at a certain time depends on all the values of the function in the past.

$$\frac{dx(t)}{dt} = f\left(t, x(t), \int_{-\infty}^{0} x(t+\tau)d\mu(\tau)\right).$$

(iii) Scaled DDEs: The derivative of unknown function at a certain time depends on the values of the function at a multiple of that time.

$$\frac{dx(t)}{dt} = f(t, x(t), x(\rho\tau)), \quad 0 < \rho < 1.$$

Similar to ODEs, many properties of DDEs can be characterized and analyzed using the characteristic equation. The characteristic equation associated to linear system of DDEs with discrete delays

$$\frac{dx(t)}{dt} = A_0 x(t) + A_1 x(t-\tau_1) + A_2 x(t-\tau_2) + ... + A_n x(t-\tau_n)$$

can be given by

$$\det(-\lambda I + A_0 + A_1 e^{-\lambda\tau_1} + A_2 e^{-\lambda\tau_2} + ... + A_n e^{-\lambda\tau_n}) = 0.$$