

Chapter 3

Global Stability and Hopf-bifurcation of Prey-predator System with Two Discrete Delays Including Habitat Complexity and Prey Refuge ¹

3.1 Introduction

The dynamics of prey-predator relationship is recognized as an important focus in an ecological system [16]. The important part of modelling in ecological system is to examine and validate whether the proposed mathematical model can demonstrate the proper behavior for the system under consideration. Predator's functional response, defined as the amount of prey catch per predator per unit of time, plays an important role to understand the complex dynamical behavior of the system. Stability and bifurcation dynamics of a system are also determined by a functional response. Different types of prey-predator models have been discussed by several researchers [43, 127, 29, 207].

On the other hand, to control the over exploitation of the biological resources, researchers have used many tools like reserve area/refuge, time delay, stage structure, harvesting etc. The hiding behavior of prey to escape from predation is common in nature. The effect of refuges on the dynamics of prey-predator interaction can be seen in two ways: Firstly, the effect is positive on the growth of prey and negative on the predator. It reduces the prey mortality due to decrease in the predation success. Secondly, the birth rate of prey population may reduce, because feeding and mating opportunities are also reduced in safe refuges. Thus, the idea of prey refuge is an important factor and it has received a great attention [29, 174, 203, 207, 184, 68, 119, 150]. Most of the researchers have shown that refugia have a stabilizing effects on predator-prey model.

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A Holling type II predator-prey model with a constant prey refuge has been presented by Tang *et al.* [203]. They have proved a global qualitative analysis to determine the global dynamics of the model. Tripathi *et al.* [207] discussed a prey-predator model with reserve area. They investigated the role of reserved region and degree of mutual interference among predators. Their results indicate that the predator species exist so long as prey reserve value does not cross a threshold value and after this value the predator species extinct. Ghosh *et al.* [68] investigated the impacts of additional food for predator on the dynamics of a prey-predator model with prey refuge. They observed that predator extinction possibility in high prey refuge ecological systems may be removed by supplying additional food to predator population. Some studies have been conducted on prey-predator model with two factors: refuge and harvesting [204, 177, 77]. The stage-structure prey-predator model with refuge region has also been studied [225, 224, 102]. A Leslie-Gower predator-prey model with disease in prey incorporating prey refuge has been discussed by Sharma and Samanta [190].

Habitat complexity plays a major role in the distribution and structure of population. It provides refuge from predation and that can reduce the probability of capturing prey by reducing the searching efficiency of predators. Thus by decreasing encounter rates between prey and predator, the structurally complex habitats may reduce predation rates. So complexity of the prey habitat can affect predator functional response [218, 67, 49]. By reducing predators foraging efficiency, the structural complexity of the habitat stabilizes the predator-prey interaction [11]. Ghorai and Poria [67] considered a predator-prey model with habitat complexity in the presence of cross diffusion. They observed that cross-diffusion, habitat complexity, birth rate of prey and mortality rate of predator play a significant role in the pattern formation.

Time delay occurs in every biological situation. It makes the model more realistic. Delay differential equation exhibits much more complicated behavior than ordinary differential equation. When delay crosses its critical value, we observe limit cycle through Hopf-bifurcation and then the system becomes unstable. Thus time delay makes a stable equilibrium to become unstable and the population to fluctuate. The conversion of prey population to predator population is not instantaneous, there is some time lag i.e. gestation delay. The prey-predator population model with different types of discrete gestation delay [71, 66, 144, 234, 27, 92, 130, 20, 140, 136] and feedback delay [61, 142, 229, 91, 46] has been studied. Some researchers have considered different types of delays together [120, 231, 64, 229, 51, 232, 91, 235, 157, 230, 122]. They studied local stability, Hopf-bifurcation, Stability and direction of Hopf-bifurcation with the help of Centre manifold theory and Normal form theory. Li *et al.* [117] investigated the stability and Hopf-bifurcation of a delayed density dependent predator-prey system with Beddington-DeAngelis functional response. The Beddington-DeAngelis functional response is similar to Holling type-II functional response and have an extra term describing mutual interference among predators. Several studies have been done involving this type of functional

response [118, 129, 75, 207].

The degree of habitat complexity and gestation delay on the stability of a predator prey model has been performed by Bairagi and Jana [11]. They observed that stability switches, and Hopf-bifurcation occurs when delay crosses its critical value. They proved that the fluctuations in the population levels can be controlled completely by increasing the degree of habitat complexity.

Tripathi *et al.* [208] described a predator-prey model incorporating prey refuge. They investigated the role of prey refuge and degree of mutual interference among predator. Then they analysed the dynamical behavior of the delayed system introducing discrete gestation delay of the predator. A delayed prey-predator model with Crowley-Martin type functional response including prey refuge has been considered by Maiti *et al.* [136]. They further modified the model incorporating gestation delay of the predator. They have also discussed the influence of prey refuge on densities of prey and predator species.

To the best of the knowledge of the authors, double delayed prey-predator model with habitat complexity and refuge with Holling type-II functional response has not been studied. Keeping all these in mind, we first modify Holling type-II functional response to incorporate the effect of habitat complexity and refuge. Then we consider a prey-predator model, where both prey and predator grow logistically. Predator partially depends on prey with Holling type-II functional response. Then we introduce two delays: one is negative feedback delay for prey population and another one is gestation delay for the predator population. The rest of this chapter is organized as follows. In the next section 3.2, we discuss formulation of the model. In section 3.3, we consider non-delayed system. In this section, we discuss positivity, boundedness, existence of equilibrium points and their local and global stability. Section 3.4 is devoted to delayed system. Here, we discuss Hopf-bifurcation through local stability of the positive equilibrium point, considering delay as a bifurcation parameter. In section 3.5, we analyze the stability and direction of Hopf-bifurcation using Normal form theory and Centre manifold theory. In section 3.6, we perform the global stability of delayed system. The influence of prey refuge is discussed in section 3.7. An extensive numerical simulation experiments have also been conducted to illustrate the theoretical results in section 3.8.

3.2 The Model Construction: Non-delayed Model

A general two dimensional model of interaction between prey and a generalist predator is represented by

$$\frac{du}{dt} = uf(u) - vg(u),$$

$$\frac{dv}{dt} = v(h(v) + \beta g(u)),$$

where u and v are the densities of prey and predator population respectively, $f(u)$ and $h(v)$ are per capita growth rate of prey and predator respectively. Here $g(u)$ and $\beta g(u)$ are functional and numerical responses of predator for a particular prey, where $\beta (0 < \beta < 1)$ is the conversion coefficient denoting the number of newly born predator for each captured prey.

If the number of prey consumed by a predator per unit time is a linear function of prey, then we have

$$g(u) = \alpha u,$$

which is known as linear mass-action law, $\alpha > 0$ is capture rate. This is traditional functional response. The feeding rate is proportional to the product of prey and predator populations. Mathematically, this represents a straight line which passes through the origin but this functional response has some drawbacks. In case of super abundant supply of food, predators will feed at maximum rate per individual predators and hence further increase in food supply will not be able to increase the feeding rate further. There are some biological factors (such as handling time, encounter rate, prey escape ability, searching efficiency, structure of the prey habitat) which affect the predator's functional response and hence the dynamics of the system also diverts. In linear mass-action law, it has been assumed that predator's handling time for prey is zero. It has been observed that predators often become saturated in nature. This phenomenon is described by Holling type II functional response in which the number of prey consumed per predator initially increases very fast as the density of the prey increases then it settle down at its steady state. In this case the feeding rate of predator increases with prey density and then attains its maximum saturated level. This type of response function is represented as follows:

$$g(u) = \frac{\alpha u}{1 + h\alpha u},$$

where α is attack or capture rate and h is the handling time required per prey. But in above expression, the effect of habitat complexity of the prey is not considered. In this chapter we will show the effect of complex habitats of prey, so above formula cannot be used directly, and thus a modified form is required. According to Winfield [226], the habitat complexity is more likely to affect the attack coefficient than the handling time for search, so the attack coefficient α has to be replaced by $\alpha(1 - c)$, $0 < c < 1$. Then according to Bairagi and Jana [11], the modified Holling type II functional response including the effect of complex habitat of prey can be written as follows:

$$g(u) = \frac{\alpha(1 - c)u}{1 + h\alpha(1 - c)u},$$

where $c(0 < c < 1)$ is a dimension less parameter that measures the strength of habitat complexity.

Now we consider another important factor from ecological point of view, If all amount of prey is available for predation then prey species might be at the brink of extinction. To avert this situation and maintain the balance between prey and predator, we divide the prey population into two regions: reserved region for prey where predator access is prohibited and unreserved or open access region in which predation is allowed. But the migration of the prey between these two regions is allowed. Suppose mu number of prey are reserved and $(1 - m)u$ number of prey are available for predation. The functional response in presence of complex habitat of prey and refuge factor will be represented by

$$g(u) = \frac{\alpha(1-c)(1-m)u}{1+h\alpha(1-c)(1-m)u},$$

where $m(0 < m < 1)$ is refuge parameter.

It is to be noted, when $c = 0$ and $m = 0$, i.e. when there is no complexity and no refuge for predator, we get back the original Holling type II functional response. Therefore, this modified functional would be suitable for prey-predator interaction with habitat complexity and refuge parameter.

Now consider an ecological system where we wish to model the interaction of a prey biomass with density $u(t)$ and a generalist predator biomass with density $v(t)$, both growing logistically in absence of each other. The dynamics of the system can be governed by the following differential equations:

$$\begin{aligned} \frac{du}{dt} &= ru\left(1 - \frac{u}{K}\right) - \frac{\alpha(1-c)(1-m)uv}{1+h\alpha(1-c)(1-m)u}, \\ \frac{dv}{dt} &= sv\left(1 - \frac{v}{L}\right) + \frac{\beta\alpha(1-c)(1-m)uv}{1+h\alpha(1-c)(1-m)u}, \\ u(0) &\geq 0, v(0) \geq 0. \end{aligned} \tag{3.1}$$

In the above model r, s, K and L are positive constants that stand for specific growth rates of prey and predator respectively and their respective carrying capacities. β ($0 < \beta < 1$) is the conversion coefficient denoting the number of newly born predator for each captured prey.

3.3 Dynamics of Non-delayed Model

3.3.1 Positivity and boundedness of the solution

Positivity refers that the population survives and boundedness represents as a natural restrictions to growth as a consequence of limited resources. As system (3.1) represents densities of populations, so it is important to show positivity and boundedness.

The model (3.1) can be written in the matrix form

$$\dot{X} = G(X),$$

$$X = (x_1, x_2)^T = (u, v)^T \in R^2$$

and $G(X)$ is given by

$$G(X) = \begin{bmatrix} G_1(X) \\ G_2(X) \end{bmatrix} = \begin{bmatrix} ru \left(1 - \frac{u}{K}\right) - \frac{\alpha(1-c)(1-m)uv}{1+h\alpha(1-c)(1-m)u} \\ sv \left(1 - \frac{v}{L}\right) + \frac{\beta\alpha(1-c)(1-m)uv}{1+h\alpha(1-c)(1-m)u} \end{bmatrix}.$$

Since $G(X)$ and $\frac{\partial G}{\partial X}$ is continuous in R_+^2 , $G : R_+^2 \rightarrow R^2$ is locally Lipschitz. By the standard theory of ODE system, it follows that model (3.1) has a unique solution for any initial condition $X(0) = X_0 = (u(0), v(0)) \in R_+^2$.

Further model (3.1) can be re-written as

$$\frac{du}{dt} = u\phi_1(u, v), \quad \frac{dv}{dt} = v\phi_2(u, v),$$

where

$$\phi_1(u, v) = r \left(1 - \frac{u}{K}\right) - \frac{\alpha(1-c)(1-m)v}{1+h\alpha(1-c)(1-m)u}, \quad \phi_2(u, v) = s \left(1 - \frac{v}{L}\right) + \frac{\beta\alpha(1-c)(1-m)u}{1+h\alpha(1-c)(1-m)u}.$$

It follows that

$$u(t) = u(0)e^{\int_0^t \phi_1(u(s), v(s)) ds} \geq 0,$$

$$v(t) = v(0)e^{\int_0^t \phi_2(u(s), v(s)) ds} \geq 0.$$

Thus, the solution $X(t) = (u(t), v(t))$ with initial condition $X(0) = X_0 = (u(0), v(0)) \in R_+^2$ remains positive throughout the region R_+^2 .

In the following lemma, we show that all solutions of model (3.1) are bounded which implies that the model is biologically well behaved.

Lemma 3.3.1. *The set $\Omega = \{(u, v) : 0 \leq u \leq K, 0 \leq v \leq v_m\}$ is a region of attraction for all solutions initiating in the positive quadrant, where*

$$v_m = \frac{L}{s} \left[s + \frac{\beta\alpha(1-c)(1-m)K}{1+h\alpha(1-c)(1-m)K} \right].$$

Proof. From the first equation of model (3.1), we have

$$\frac{du}{dt} \leq ru \left(1 - \frac{u}{K} \right),$$

which yields

$$\limsup_{t \rightarrow \infty} u(t) \leq K.$$

Now from the second equation of the model (3.1), we have

$$\frac{dv}{dt} \leq \left[s + \frac{\beta\alpha(1-c)(1-m)K}{1+h\alpha(1-c)(1-m)K} \right] v - \frac{sv^2}{L},$$

which implies

$$\limsup_{t \rightarrow \infty} v(t) \leq \frac{L}{s} \left[s + \frac{\beta\alpha(1-c)(1-m)K}{1+h\alpha(1-c)(1-m)K} \right] := v_m.$$

This proves the Lemma. □

3.3.2 Persistence

Ecologically, permanence or uniform persistence of a system implies that all species will be present in future and none of them will become extinct if they are initially present. System (3.1) is said to be uniformly persistence if there are positive constants M_1 and M_2 such that each positive solution $(u(t), v(t))$ of the system with positive initial conditions satisfies:

$$M_1 \leq \liminf_{t \rightarrow \infty} U(t) \leq \limsup_{t \rightarrow \infty} U(t) \leq M_2,$$

where $U(t) = (u(t), v(t))$.

The following theorem gives the criteria for the uniform persistence of the model (3.1).

Theorem 3.3.2. *Assume that Lemma 3.3.1 holds true. Then system (3.1) is uniform persistence if the following condition holds:*

$$r > \alpha(1-c)(1-m)v_m.$$

Proof. Keeping in view Lemma 3.3.1, we define

$$M_2 = \max\{K, v_m\},$$

then it follows that

$$\limsup_{t \rightarrow \infty} U(t) \leq M_2.$$

This also shows that for any sufficiently small $\varepsilon_i > 0$, there exist a $T_i > 0$, such that $u(t) < K + \varepsilon_1$ for all $t \geq T_1$, $v(t) < v_m + \varepsilon_2$ for all $t \geq T_2$. Taking $\varepsilon = \max \varepsilon_i$, $T = \max T_i, i = 1, 2$; one can say that for any $\varepsilon > 0$, there exists a $T > 0$ such that for all $t \geq T$, the following holds:

$$u(t) < K + \varepsilon, \quad v(t) < v_m + \varepsilon.$$

Now from the first equation of model (3.1), for all $t > T$, we can write

$$\frac{du}{dt} \geq ru - \frac{ru^2}{K} - \alpha(1-c)(1-m)(v_m + \varepsilon)u,$$

which implies

$$\liminf_{t \rightarrow \infty} u(t) \geq \frac{K}{r} [r - \alpha(1-c)(1-m)(v_m + \varepsilon)].$$

This is true for every $\varepsilon > 0$, thus

$$\liminf_{t \rightarrow \infty} u(t) \geq \frac{K}{r} [r - \alpha(1-c)(1-m)v_m] := u_a.$$

Clearly $u_a > 0$ under the condition mentioned in Theorem 3.3.1. From the second equation of model (3.1), for all $t > T$, we have

$$\frac{dv}{dt} \geq sv - \frac{sv^2}{L} + \frac{\beta\alpha(1-c)(1-m)u_a v}{1 + h\alpha(1-c)(1-m)u_a},$$

which gives

$$\liminf_{t \rightarrow \infty} v(t) \geq \frac{L}{s} \left[s + \frac{\beta\alpha(1-c)(1-m)u_a}{1 + h\alpha(1-c)(1-m)u_a} \right] := v_a.$$

Taking $M_1 = \min\{u_a, v_a\}$, the theorem follows. □

Remark 3.3.1. *The above theorem shows that the system is persistence if intrinsic growth rate of prey population greater than a threshold value. This threshold value depends on habitat complexity and refuge of prey also.*

3.3.3 Equilibrium points and their stability analysis

3.3.3.1 Existence of equilibrium points

Model (3.1) has four non-negative equilibria, namely, $E_0(0, 0)$, $E_1(K, 0)$, $E_2(0, L)$ and $E^*(u^*, v^*)$. E_0 , E_1 and E_2 always exist. From ecological and biological point of view, it is important to study the uniqueness of positive equilibrium point $E^*(u^*, v^*)$. It may be seen that u^* and v^* are the positive solution of the following algebraic equations:

$$r\left(1 - \frac{u^*}{K}\right) - \frac{\alpha(1-c)(1-m)v^*}{1+h\alpha(1-c)(1-m)u^*} = 0, \quad (3.2)$$

$$s\left(1 - \frac{v^*}{L}\right) + \frac{\beta\alpha(1-c)(1-m)u^*}{1+h\alpha(1-c)(1-m)u^*} = 0. \quad (3.3)$$

From equation (3.3), we have

$$v^* = \frac{L}{s} \left(s + \frac{\beta\alpha(1-c)(1-m)u^*}{1+h\alpha(1-c)(1-m)u^*} \right). \quad (3.4)$$

Substituting the above value of v^* in equation (3.2), we obtain

$$r\left(1 - \frac{u^*}{K}\right) - \frac{\alpha(1-c)(1-m)}{1+h\alpha(1-c)(1-m)u^*} \frac{L}{s} \left(s + \frac{\beta\alpha(1-c)(1-m)u^*}{1+h\alpha(1-c)(1-m)u^*} \right) = 0,$$

which implies

$$u^{*3} + P_1 u^{*2} + P_2 u^* + P_3 = 0, \quad (3.5)$$

where

$$P_1 = \frac{2}{\alpha(1-c)(1-m)h} - K, \quad P_2 = \frac{1-2h\alpha(1-c)(1-m)K}{\alpha^2(1-c)^2(1-m)^2h^2} + \frac{KL(\beta+sh)}{rsh^2}, \quad P_3 = \frac{K(L\alpha(1-c)(1-m)-r)}{r\alpha^2(1-c)^2(1-m)^2h^2}.$$

From Descarte's rule of sign, equation (3.5) has a unique solution $u = u^*$ in the interval $0 < u^* < K$ if any one of the following holds.

$$\begin{aligned} I: & P_1 > 0, P_2 > 0, P_3 < 0, \\ II: & P_1 > 0, P_2 < 0, P_3 < 0, \\ III: & P_1 < 0, P_2 < 0, P_3 < 0. \end{aligned} \quad (3.6)$$

Equation (3.5) has one or three positive solutions in u^* in the interval $(0, K)$ if

$$P_1 < 0, P_2 > 0, P_3 < 0, \quad (3.7)$$

holds true.

We also note that equation (3.5) has no solution or two positive solutions in u^* in the interval $(0, K)$ if any one of

$$\begin{aligned}
 I: & P_1 > 0, P_2 < 0, P_3 > 0, \\
 II: & P_1 < 0, P_2 > 0, P_3 > 0, \\
 III: & P_1 < 0, P_2 < 0, P_3 > 0,
 \end{aligned} \tag{3.8}$$

holds.

The value of v^* can be calculated easily from equation (3.4).

Thus, we can state the following theorem.

Theorem 3.3.3. (i) The model (3.1) has a unique positive equilibrium if any one of three conditions in (3.6) holds.

(ii) The model (3.1) has either one or three positive equilibria if condition (3.7) holds.

(iii) The model (3.1) has no positive equilibrium point or two positive equilibria if any one of three conditions in (3.8) holds.

3.3.3.2 Local stability analysis

To investigate the local stability of the equilibria whenever they exist, we compute the variational matrix for model (3.1). This matrix is further computed at each of equilibria. Then using Routh-Hurwitz criterion, we observe the following:

- i $E_0(0, 0)$ is unstable equilibrium point,
- ii $E_1(K, 0)$ is saddle point with stable manifold in u direction and with unstable manifold in v direction,
- iii (a) $E_2(0, L)$ is locally asymptotically stable in the uv plane if

$$r < \alpha(1 - c)(1 - m)L.$$

(b) $E_2(0, L)$ is saddle point with unstable manifold in u direction and with stable manifold in v direction if

$$r > \alpha(1 - c)(1 - m)L.$$

This shows that stability of equilibrium point $E_2(0, L)$ depends on habitat complexity, prey refuge and carrying capacity of the predator.

In order to study the stability behavior of E^* , let $M(E^*)$ denotes the variational matrix evaluated

at E^* . Then the characteristic equation of $M(E^*)$ is given by

$$\lambda^2 + A'_1 \lambda + A'_2 = 0, \quad (3.9)$$

where

$$A'_1 = -tr(M(E^*)) = \frac{ru^*}{K} + \frac{sv^*}{L} - \frac{h\alpha^2(1-c)^2(1-m)^2u^*v^*}{[1+h\alpha(1-c)(1-m)u^*]^2},$$

$$A'_2 = det(M(E^*)) = -\frac{sv^*}{L} \left[-\frac{ru^*}{K} + \frac{h\alpha^2(1-c)^2(1-m)^2u^*v^*}{[1+h\alpha(1-c)(1-m)u^*]^2} \right] + \frac{\beta\alpha^2(1-c)^2(1-m)^2u^*v^*}{[1+h\alpha(1-c)(1-m)u^*]^3}.$$

Using the Routh-Hurwitz criterion, all the eigenvalues of $M(E^*)$ have negative real part if and only if following conditions hold:

$$A'_1 > 0, \quad A'_2 > 0. \quad (3.10)$$

Thus, we can state the following theorem.

Theorem 3.3.4. *The positive equilibrium point E^* is locally asymptotically stable in the uv plane if and only if (3.10) holds true.*

We can easily note that (3.10) holds true if

$$r > \frac{h\alpha^2(1-c)^2(1-m)^2v^*K}{[1+h\alpha(1-c)(1-m)u^*]^2}, \quad (3.11)$$

i.e. intrinsic growth rate of the prey greater than a threshold value.

Thus, we will state the following theorem:

Theorem 3.3.5. *The positive equilibrium point E^* is locally asymptotically stable in uv plane under condition (3.11).*

In equation (3.9), if $A'_2 < 0$ then one eigenvalue is real and positive and other eigenvalue is real and negative thus, the following theorem follows.

Theorem 3.3.6. *If $A'_2 < 0$, then the positive equilibrium point E^* is a saddle point.*

Now assume that $A'_1 < 0$ and $A'_2 > 0$, then both eigenvalues are either positive or complex conjugate having positive real part. Hence the following theorem follows:

Theorem 3.3.7. *If $A'_1 < 0$ and $A'_2 > 0$, then the positive equilibrium point E^* is unstable.*

3.3.3.3 Global stability analysis

In this section, we will prove the global asymptotic stability of $E_2(0, L)$ and the unique positive equilibrium $E^*(u^*, v^*)$ by constructing a suitable Liapunov function.

Theorem 3.3.8. *The equilibrium point $E_2(0, L)$ is globally asymptotically stable if the following inequality holds:*

$$r < \frac{\alpha(1-c)(1-m)L}{1+h\alpha(1-c)(1-m)K}. \quad (3.12)$$

Proof. Consider a positive definite function

$$V_1 = u + \frac{1}{\beta} \left(v - L - L \ln \frac{v}{L} \right).$$

Differentiating V_1 with respect to time, we get

$$\frac{dV_1}{dt} = ru \left(1 - \frac{u}{K} \right) - \frac{s}{\beta L} (v - L)^2 - \frac{\alpha(1-c)(1-m)uL}{1+h\alpha(1-c)(1-m)u}.$$

It is easy to note that $\frac{dV_1}{dt}$ is negative definite under the condition (3.12), hence the theorem follows. \square

Theorem 3.3.9. *The positive equilibrium point E^* is globally asymptotically stable if*

$$r > \frac{\alpha^2(1-c)^2(1-m)^2hv^*K}{1+h\alpha(1-c)(1-m)u^*}. \quad (3.13)$$

holds.

Proof. Choosing a positive definite function

$$V_2 = u - u^* - u^* \ln \frac{u}{u^*} + \gamma \left(v - v^* - v^* \ln \frac{v}{v^*} \right),$$

where γ is a positive constant to be chosen suitably.

Differentiating V_2 with respect to time along the solution of system (3.1), we get

$$\frac{dV_2}{dt} = a_{11}(u - u^*)^2 + a_{12}(u - u^*)(v - v^*) + a_{22}(v - v^*)^2,$$

where $a_{11} = -\frac{r}{K} + \frac{\alpha^2(1-c)^2(1-m)^2hv^*}{(1+h\alpha(1-c)(1-m)u)(1+h\alpha(1-c)(1-m)u^*)}$,

$a_{12} = -\frac{\alpha(1-c)(1-m)}{(1+h\alpha(1-c)(1-m)u)} + \gamma \frac{\beta\alpha(1-c)(1-m)}{(1+h\alpha(1-c)(1-m)u)(1+h\alpha(1-c)(1-m)u^*)}$,

$a_{22} = -\gamma \frac{s}{L}$.

γ is arbitrary positive real number, so we choose

$$\gamma = \frac{(1+h\alpha(1-c)(1-m)u^*)}{\beta},$$

such that a_{12} becomes zero. Then $\frac{dV_2}{dt}$ is negative definite under condition (3.13). Hence E^* is globally asymptotically stable under condition (3.13). \square

Remark 3.3.2. As long as E^* exists, it is interesting to note here that condition (3.13) implies condition (3.11) because $1 + h\alpha(1-c)(1-m)u^* > 1$ holds.

3.4 The Delayed Model

Delayed models are much more realistic in nature. They occur in every biological moment. Delay differential equation exhibits more complicated behavior than ordinary differential equation. It makes a stable equilibrium to become unstable and then population starts regular fluctuation. In this section, we modify our model (3.1) in the presence of two types of discrete time delays. First we consider the effect of negative feedback, in which the system responds in an opposite direction to population. Initially, density of prey population decreases due to its predation. This causes reduction in the predator population, and thus less prey are consumed. This causes an increase in the prey population. Therefore, reduction in the prey population results in its increase. This mechanism is described as negative feedback. In order to study the effect of density dependent feedback mechanism, we consider a time delay τ_1 in the specific growth rate of the prey species. This feedback mechanism takes τ_1 units of time to respond to changes in the prey population. In such a case, the dynamics of prey species in the absence of predator is governed by

$$\frac{du}{dt} = ru(t) \left(1 - \frac{u(t - \tau_1)}{K} \right).$$

Again the reproduction of predator after consuming prey is not instantaneous. There is some time lag τ_2 required. This delay τ_2 can be regarded as gestation period of predator or reaction time of predation. Here it is assumed that change rate of predators depends on the number of prey and predators present at some previous time [107]. Thus, the second delay τ_2 is considered in the numeric response only. So for more realistic situations, we are showing the effect of both of the time delays in our model (3.1). Then the delayed prey predator model (3.1) takes the form

$$\begin{aligned} \frac{du}{dt} &= ru(t) \left(1 - \frac{u(t - \tau_1)}{K} \right) - \frac{\alpha(1-c)(1-m)u(t)v(t)}{1 + h\alpha(1-c)(1-m)u(t)}, \\ \frac{dv}{dt} &= sv(t) \left(1 - \frac{v(t)}{L} \right) + \frac{\beta\alpha(1-c)(1-m)u(t - \tau_2)v(t - \tau_2)}{1 + h\alpha(1-c)(1-m)u(t - \tau_2)}, \end{aligned} \quad (3.14)$$

subject to the non negative conditions $u(x) = \zeta_1(x) \geq 0$, $v(x) = \zeta_2(s) \geq 0$, $x \in [-\tau, 0]$, $\tau = \max(\tau_1, \tau_2)$, where $\zeta_i(x) \in C([-\tau, 0] \rightarrow R_+)$, ($i = 1, 2$), where τ_1 and τ_2 denote feedback time delay of the prey and gestation time delay of the predator respectively. For this model, we shall discuss local stability, Hopf-bifurcation and global stability. In the next section, we will analytically prove the existence of Hopf-bifurcation taking delay as a bifurcation parameter.

3.4.1 Local stability and Hopf-bifurcation

Theorem 3.4.1. *All the solutions of model (3.14) with initial conditions that initiate in R_+^2 are positive invariant and uniformly bounded.*

Proof. Solutions of model (3.14) which starts in R_+^2 are positive invariant as well as uniformly bounded. The proof is straight forward and can be omitted. \square

Rewriting model (3.14) as

$$\frac{dU(t)}{dt} = F(U(t), U(t - \tau_1), U(t - \tau_2)),$$

where $U(t) = [u(t), v(t)]^T$, $U(t - \tau_i) = [u(t - \tau_i), v(t - \tau_i)]^T$, $i = 1, 2$.

Let $u(t) = u^* + u'(t)$, $v(t) = v^* + v'(t)$. Then linearizing system (3.14) about the interior equilibrium solution $E^*(u^*, v^*)$, we have

$$\frac{dZ}{dt} = P'Z(t) + Q'Z(t - \tau_1) + R'Z(t - \tau_2),$$

where

$$P' = \left(\frac{\partial F}{\partial U(t)} \right)_{E^*}, \quad Q' = \left(\frac{\partial F}{\partial U(t - \tau_1)} \right)_{E^*}, \quad R' = \left(\frac{\partial F}{\partial U(t - \tau_2)} \right)_{E^*},$$

and $Z(t) = [u'(t), v'(t)]^T$.

Thus, the variational matrix of the system (3.14) at E^* is given by

$$J = P' + Q'e^{-\lambda\tau_1} + R'e^{-\lambda\tau_2}.$$

After a little calculation, we obtain

$$J = \begin{bmatrix} a_1 + d_1 e^{-\lambda\tau_1} & b \\ a_2 e^{-\lambda\tau_2} & d_2 - b_1 + b_1 e^{-\lambda\tau_2} \end{bmatrix},$$

where

$$a_1 = h\alpha(1-c)(1-m)au^*, \quad a = \frac{\alpha(1-c)(1-m)v^*}{(1+h\alpha(1-c)(1-m)u^*)^2}, \quad a_2 = \beta a, \\ b = \frac{\alpha(1-c)(1-m)u^*}{1+h\alpha(1-c)(1-m)u^*}, \quad b_1 = \beta b, \quad d_1 = -\frac{ru^*}{K}, \quad d_2 = -\frac{sv^*}{L},$$

and its corresponding characteristic equation is

$$\lambda^2 + A\lambda + B + (C\lambda + D)e^{-\lambda\tau_1} + (E\lambda + F)e^{-\lambda\tau_2} + Ge^{-\lambda(\tau_1+\tau_2)} = 0, \quad (3.15)$$

where

$$A = -(a_1 + d_2 - b_1), B = a_1(d_2 - b_1), C = -d_1, \\ D = d_1(d_2 - b_1), E = -b_1, F = a_1b_1 - ba_2, G = d_1b_1.$$

Case (1): $\tau_1 = \tau_2 = 0$. Then equation (3.15) becomes

$$\lambda^2 + (A + C + E)\lambda + (B + D + F + G) = 0. \quad (3.16)$$

Remark 3.4.1. *The characteristic equation (3.16) is same as the characteristic equation (3.9) of the non-delayed model (3.1) studied earlier.*

All the roots of equation (3.16) have negative real part if and only if

$$(H_1): A + C + E > 0, B + D + F + G > 0.$$

So, the interior equilibrium point (u^*, v^*) is locally asymptotically stable if and only if (H_1) holds.

Case (2): $\tau_1 = 0, \tau_2 > 0$. Then equation (3.15) becomes

$$\lambda^2 + (A + C)\lambda + (B + D) + (E\lambda + F + G)e^{-\lambda\tau_2} = 0. \quad (3.17)$$

Let $i\omega$ ($\omega > 0$) be a root of equation (3.17), then it follows that

$$E\omega \sin(\omega\tau_2) + (F + G) \cos(\omega\tau_2) = \omega^2 - (B + D), \\ (F + G) \sin(\omega\tau_2) - E\omega \cos(\omega\tau_2) = (A + C)\omega, \quad (3.18)$$

which leads to a quadratic equation in ω^2 as

$$\omega^4 + [(A + C)^2 - 2(B + D) - E^2]\omega^2 + (B + D)^2 - (F + G)^2 = 0. \quad (3.19)$$

If we put $\omega^2 = z$, then equation (3.19) becomes

$$z^2 + pz + q = 0,$$

where $p = (A + C)^2 - 2(B + D) - E^2$, $q = (B + D)^2 - (F + G)^2$.

Let $f(z) = z^2 + pz + q$,

$(H_2): p > 0, q > 0$.

Remark 3.4.2. 1. *If (H_2) holds, then equation (3.19) has no positive roots, therefore all the roots of (3.17) have negative real parts and hence $E^*(u^*, v^*)$ is asymptotically stable for all $\tau_2 \geq 0$ under (H_1) and (H_2) .*

2. If (H_1) fails and (H_2) holds true, then $E^*(u^*, v^*)$ is unstable for all $\tau_2 \geq 0$.

$(H_3) : q < 0$.

If (H_1) and (H_3) hold, then equation (3.19) have a unique positive root ω_1^2 . Substituting ω_1^2 into equation (3.18), we obtain

$$\tau_{2n} = \frac{1}{\omega_1} \cos^{-1} \left[\frac{(F+G)(\omega_1^2 - B - D) - (A+C)E\omega_1^2}{E^2\omega_1^2 + (F+G)^2} \right] + \frac{2n\pi}{\omega_1}, \quad n = 0, 1, 2, \dots$$

$(H_4) : p < 0, q > 0, p^2 > 4q$.

If (H_1) and (H_4) hold, then equation (3.19) have two positive roots ω_+^2 and ω_-^2 , substituting ω_{\pm}^2 into equation (3.18), we have

$$\tau_{2k}^{\pm} = \frac{1}{\omega_{\pm}} \cos^{-1} \left[\frac{(F+G)(\omega_{\pm}^2 - B - D) - (A+C)E\omega_{\pm}^2}{E^2\omega_{\pm}^2 + (F+G)^2} \right] + \frac{2k\pi}{\omega_{\pm}}, \quad k = 0, 1, 2, \dots$$

$(H_5) : f'(\omega_1^2) > 0$.

Let $\lambda(\tau_{2n}) = \pm i\omega_1$ be the root of equation (3.17), then the transversality condition can be obtained under (H_5)

$$\left[\frac{d(\operatorname{Re} \lambda)}{d\tau_2} \right]_{\tau_2 = \tau_{2n}} > 0,$$

and we can obtain

$$\left[\frac{d(\operatorname{Re} \lambda)}{d\tau_2} \right]_{\tau_2 = \tau_{2k}^+, \omega = \omega_+} > 0, \quad \left[\frac{d(\operatorname{Re} \lambda)}{d\tau_2} \right]_{\tau_2 = \tau_{2k}^-, \omega = \omega_-} > 0.$$

Thus, we can state the following theorem:

Theorem 3.4.2. For system (3.14), with $\tau_1 = 0$ and assume that (H_1) and (H_5) hold, there exists a positive number τ_{2_0} such that the equilibrium E^* is locally asymptotically stable when $\tau_2 < \tau_{2_0}$ and unstable when $\tau_2 > \tau_{2_0}$. Furthermore system (3.14) undergoes a Hopf-bifurcation at E^* when $\tau_2 = \tau_{2_0}$.

Case (3): $\tau_2 = 0, \tau_1 > 0$.

Theorem 3.4.3. For, $\tau_2 = 0$, the equilibrium point E^* is locally asymptotically stable when $\tau_1 < \tau_{1_0}$ and unstable when $\tau_1 > \tau_{1_0}$. Furthermore system (3.14) undergoes a

Hopf-bifurcation at E^* when $\tau_1 = \tau_{1_0}$, where

$$\tau_{1_0} = \frac{1}{\omega_2} \cos^{-1} \left[\frac{(D+G)(\omega_2^2 - B - F) - (A+E)C\omega_2^2}{C^2\omega_2^2 + (D+G)^2} \right].$$

Proof. Proof is similar as in Case (2). □

Case (4): $\tau_1 > 0$, $\tau_2 > 0$.

We consider the equation (3.15) with τ_2 as fixed in its stable interval $(0, \tau_{2_0})$ and τ_1 as a variable parameter. Let $i\omega$ ($\omega > 0$) be a root of equation (3.15). Separating real and imaginary parts, we have

$$\begin{aligned} -\omega^2 + B + E\omega \sin(\omega\tau_2) + F \cos(\omega\tau_2) &= \sin(\omega\tau_1)[G \sin(\omega\tau_2) - C\omega] \\ &\quad - \cos(\omega\tau_1)[G \cos(\omega\tau_2) + D], \end{aligned} \quad (3.20)$$

$$\begin{aligned} A\omega - F \sin(\omega\tau_2) + E\omega \cos(\omega\tau_2) &= \sin(\omega\tau_1)[G \cos(\omega\tau_2) + D] \\ &\quad + \cos(\omega\tau_1)[G \sin(\omega\tau_2) - C\omega]. \end{aligned} \quad (3.21)$$

On eliminating τ_1 , we have

$$\omega^4 + k_1\omega^3 + k_2\omega^2 + k_3\omega + k_4 = 0, \quad (3.22)$$

where

$$k_1 = -2E \sin(\omega\tau_2),$$

$$k_2 = A^2 + E^2 - C^2 - 2B + 2(AE - F) \cos(\omega\tau_2),$$

$$k_3 = 2(BE - AF + CG) \sin(\omega\tau_2),$$

$$k_4 = B^2 + F^2 - D^2 - G^2 + 2(BF - GD) \cos(\omega\tau_2).$$

Equation (3.22) is a transcendental equation in complicated form. It is not easy to predict about the nature of roots. Without going detailed analysis with (3.22) it is assumed that there exist at least one positive root ω_0 . Equation (3.20) and (3.21) can be re-written as

$$A_1 + B_1 \cos(\omega_0\tau_2) + C_1 \sin(\omega_0\tau_2) = -E_1 \cos(\omega_0\tau_1) + F_1 \sin(\omega_0\tau_1), \quad (3.23)$$

$$D_1 + B_1 \sin(\omega_0\tau_2) + C_1 \cos(\omega_0\tau_2) = E_1 \sin(\omega_0\tau_1) + F_1 \cos(\omega_0\tau_1), \quad (3.24)$$

where

$$A_1 = -\omega_0^2 + B, \quad B_1 = F, \quad C_1 = E\omega_0, \quad D_1 = A\omega_0,$$

$$E_1 = G \cos(\omega_0\tau_2) + D, \quad F_1 = G \cos(\omega_0\tau_2) - C\omega_0.$$

Equation (3.23) and (3.24) leads to

$$\tau'_{1_i} = \frac{1}{\omega^*} \cos^{-1} \left[\frac{C_2 F_1 - B_2 E_1}{E_1^2 + F_1^2} \right] + \frac{2i\pi}{\omega^*}, \quad i = 0, 1, 2, \dots,$$

where

$$B_2 = A_1 + B_1 \cos(\omega_0 \tau_2) + C_1 \sin(\omega_0 \tau_2),$$

$$C_2 = D_1 + B_1 \sin(\omega_0 \tau_2) + C_1 \cos(\omega_0 \tau_2),$$

and $\pm i\omega^*$ is purely imaginary root of equation (3.15) for $\tau_2 \in [0, \tau_{2_0}]$.

Now we will verify the transversality condition of Hopf-bifurcation. Differentiating equations (3.20) and (3.21) with respect to τ_1 and substituting $\tau_1 = \tau'_{1_0}$, we obtain

$$\begin{aligned} P \left[\frac{d(\operatorname{Re} \lambda)}{d\tau_1} \right]_{\tau_1=\tau'_{1_0}} + Q \left[\frac{d\omega^*}{d\tau_1} \right]_{\tau_1=\tau'_{1_0}} &= R, \\ -Q \left[\frac{d(\operatorname{Re} \lambda)}{d\tau_1} \right]_{\tau_1=\tau'_{1_0}} + P \left[\frac{d\omega^*}{d\tau_1} \right]_{\tau_1=\tau'_{1_0}} &= S, \end{aligned} \quad (3.25)$$

where

$$P = A + C \cos(\omega^* \tau'_{1_0}) - D \tau'_{1_0} \cos(\omega^* \tau'_{1_0}) - C \omega^* \tau'_{1_0} \sin(\omega^* \tau'_{1_0}) + E \cos(\omega^* \tau_2) -$$

$$F \tau_2 \cos(\omega^* \tau_2) - E \omega^* \tau_2 \sin(\omega^* \tau_2) - G(\tau'_{1_0} + \tau_2) \cos \omega^*(\tau'_{1_0} + \tau_2),$$

$$Q = -2\omega^* - D \tau'_{1_0} \sin(\omega^* \tau'_{1_0}) + C \sin(\omega^* \tau'_{1_0}) + C \omega^* \tau'_{1_0} \cos(\omega^* \tau'_{1_0}) - F \tau_2 \sin(\omega^* \tau_2) +$$

$$E \sin(\omega^* \tau_2) + E \omega^* \tau_2 \cos(\omega^* \tau_2) - G(\tau'_{1_0} + \tau_2) \sin \omega^*(\tau'_{1_0} + \tau_2),$$

$$R = D \omega^* \sin(\omega^* \tau'_{1_0}) - C \omega^{*2} \cos(\omega^* \tau'_{1_0}) + G \omega^* \sin \omega^*(\tau'_{1_0} + \tau_2),$$

$$S = D \omega^* \cos(\omega^* \tau'_{1_0}) + C \omega^{*2} \sin(\omega^* \tau'_{1_0}) + G \omega^* \cos \omega^*(\tau'_{1_0} + \tau_2).$$

Solving equation (3.25) for $\left[\frac{d(\operatorname{Re} \lambda)}{d\tau_1} \right]_{\tau_1=\tau'_{1_0}}$, it is obtained

$$\left[\frac{d(\operatorname{Re} \lambda)}{d\tau_1} \right]_{\tau_1=\tau'_{1_0}, \lambda=i\omega^*} = \frac{PR - QS}{P^2 + Q^2},$$

$$(H_6) : PR - QS \neq 0.$$

Theorem 3.4.4. For system (3.14), with $\tau_2 \in (0, \tau_{2_0})$ and assuming that (H_1) and (H_6) hold, there exists a positive number τ'_{1_0} such that E^* is locally asymptotically stable when $\tau_1 < \tau'_{1_0}$ and unstable when $\tau_1 > \tau'_{1_0}$. Furthermore, system (3.14) undergoes a Hopf-bifurcation at E^* where $\tau_1 = \tau'_{1_0}$.

Remark 3.4.3. • If we fix τ_2 beyond its stability interval, then system remains unstable for all $\tau_1 > 0$ and Hopf-bifurcation does not exist in the system. The region of stability and instability has been drawn in Fig.3.14.

i If $\tau_1 < \tau_{1_0}$ and $\tau_2 < \tau_{2_0}$, then system undergoes a Hopf-bifurcation.

ii If $\tau_1 > \tau_{1_0}$ or $\tau_2 > \tau_{1_0}$, then system is unstable and Hopf-bifurcation does not occur.

- The analysis of system (3.14) for the case, when τ_1 is fixed in its stable interval and τ_2 as a variable parameter, can be done by adopting the same mechanism.

3.5 Direction and Stability of Hopf-bifurcation

In previous section, we have obtained the condition under which the system undergoes Hopf-bifurcation for the different combination of the delay τ_1 and τ_2 . In this section we will determine the direction, stability and period of the bifurcated periodic solutions at $\tau_1 = \tau'_{1_0}$ using center manifold theory and normal form concept in Hassard *et al.* [80].

Without loss of generality, we assume that $\tau_2^* < \tau'_{1_0}$, where $\tau_2^* \in (0, \tau_{2_0})$. Let

$$u_1(t) = u(t) - u^*, \quad v_1(t) = v(t) - v^*,$$

and still denote $u_1(t), v_1(t)$ by $u(t), v(t)$. Let $\tau_1 = \tau'_{1_0} + \mu$, $\mu \in R$ so that Hopf-bifurcation occurs at $\mu = 0$. We normalize the delay with scaling $t \mapsto (\frac{t}{\tau_1})$, then system (3.14) can be re-written as

$$\dot{U}(t) = \tau_1 \left(A'U(t) + C'U \left(t - \frac{\tau_2^*}{\tau_1} \right) + B'U(t-1) + f(u, v) \right), \quad (3.26)$$

where $U(t) = (u(t), v(t))^T$

$$A' = \begin{bmatrix} a_1 & -b \\ 0 & d_3 \end{bmatrix}, \quad B' = \begin{bmatrix} d_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C' = \begin{bmatrix} 0 & 0 \\ a_2 & b_1 \end{bmatrix}, \quad d_3 = d_2 - b_1, \quad f(u, v) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

The nonlinear term f_1 and f_2 are given by

$$f_1 = g_1 u(t)u(t-1) + g_2 u^2(t) + g_3 u(t)v(t) + g_4 u^3(t) + g_5 u^2(t)v(t) + \dots,$$

$$f_2 = h_1 v^2(t) + h_2 u^2 \left(t - \frac{\tau_2^*}{\tau_1} \right) + h_3 u \left(t - \frac{\tau_2^*}{\tau_1} \right) v \left(t - \frac{\tau_2^*}{\tau_1} \right) + h_4 u^3 \left(t - \frac{\tau_2^*}{\tau_1} \right) + h_5 u^2 \left(t - \frac{\tau_2^*}{\tau_1} \right) v \left(t - \frac{\tau_2^*}{\tau_1} \right) + \dots,$$

where

$$g_1 = -\frac{r}{K}, \quad g_2 = -\frac{\alpha(1-c)(1-m)v^*h}{(1+h\alpha(1-c)(1-m)u^*)^3}, \quad g_3 = \frac{\alpha(1-c)(1-m)}{(1+h\alpha(1-c)(1-m)u^*)^2}, \quad g_4 = \frac{\alpha^3(1-c)^3(1-m)^3h^2v^*}{(1+h\alpha(1-c)(1-m)u^*)^4},$$

$$g_5 = -\frac{\alpha^2(1-c)^2(1-m)^2h}{(1+h\alpha(1-c)(1-m)u^*)^3}, \quad h_1 = -\frac{s}{L}, \quad h_i = \beta g_i, \quad i = 2, 3, 4, 5.$$

The linearization of equation (3.26) around the origin is given by

$$\dot{U}(t) = \tau_1 \left(A'U(t) + C'U \left(t - \frac{\tau_2^*}{\tau_1} \right) + B'U(t-1) \right).$$

For $\phi = (\phi_1, \phi_2)^T \in C([-1, 0], R^2)$, define

$$L_\mu(\phi) = (\tau_1 + \mu) \left(A'\phi(0) + C'\phi \left(-\frac{\tau_2^*}{\tau_1} \right) + B'\phi(-1) \right).$$

By the Riesz representation theorem, there exists a 2×2 matrix $\eta(\theta, \mu)$, ($-1 \leq \theta \leq 0$) whose element are of bounded variation function such that

$$L_\mu(\phi) = \int_{-1}^0 d\eta(\theta, \mu)\phi(\theta) \text{ for } \phi \in C([-1, 0], R^2). \quad (3.27)$$

In fact, we can obtain

$$\eta(\theta, \mu) = \begin{cases} (\tau'_{1_0} + \mu)(A' + C' + B'), & \text{if } \theta = 0 \\ (\tau'_{1_0} + \mu)(C' + B'), & \text{if } \theta \in \left[-\frac{\tau_2^*}{\tau_1}, 0\right) \\ (\tau'_{1_0} + \mu)A', & \text{if } \theta \in \left[-1, -\frac{\tau_2^*}{\tau_1}\right) \\ 0, & \text{if } \theta = -1. \end{cases}$$

Then equation (3.26) is satisfied.

For $\phi \in C^1([-1, 0], R^2)$, define the operator $A(\mu)$ as

$$A(\mu)\phi(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \text{if } \theta \in [-1, 0) \\ \int_{-1}^0 [d\eta(\xi, \mu)]\phi(\xi), & \text{if } \theta = 0, \end{cases}$$

and

$$R(\mu)\phi(\theta) = \begin{cases} 0, & \text{if } \theta \in [-1, 0) \\ h(\mu, \phi), & \text{if } \theta = 0, \end{cases}$$

where

$$h(\mu, \phi) = (\tau'_{1_0} + \mu) \begin{bmatrix} h'_1 \\ h'_2 \end{bmatrix}, \quad \phi = (\phi_1, \phi_2)^T \in C([-1, 0], R^2),$$

$$h'_1 = g_1u(0)u(-1) + g_2u^2(0) + g_3u(0)v(0) + g_4u^3(0) + g_5u^2(0)v(0) + \dots,$$

$$h'_2 = h_1v^2(0) + h_2u^2\left(-\frac{\tau_2^*}{\tau_1}\right) + h_3u\left(-\frac{\tau_2^*}{\tau_1}\right)v\left(-\frac{\tau_2^*}{\tau_1}\right) + h_4u^3\left(-\frac{\tau_2^*}{\tau_1}\right) + h_5u^2\left(-\frac{\tau_2^*}{\tau_1}\right)v\left(-\frac{\tau_2^*}{\tau_1}\right) + \dots$$

Then system (3.14) is equivalent to the following operator equation

$$\dot{U}_t = A(\mu)U_t + R(\mu)U_t,$$

where $U_t = U(t + \theta)$ for $\theta \in [-1, 0]$.

For $\psi \in C^1([0, 1], (R^2)^*)$, define

$$A^* \psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & \text{if } s \in (0, 1] \\ \int_{-1}^0 \psi(-\xi) d\eta(\xi, 0), & \text{if } s = 0, \end{cases}$$

and a bilinear form

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0) \phi(0) - \int_{-1}^0 \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta) d\eta(\theta) \phi(\xi) d\xi,$$

where $\eta(\theta) = \eta(\theta, 0)$, $A = A(0)$ and A^* are adjoint operators. From the discussion in previous section, we know that $\pm i\omega^* \tau'_{10}$ are the eigenvalues of $A(0)$ and therefore they are also eigenvalues of A^* . It is not difficult to verify that the vectors $q(\theta) = (1, \alpha_1)^T e^{i\omega^* \tau'_{10} \theta}$ ($\theta \in [-1, 0]$) and $q^*(s) = \frac{1}{D} (1, \alpha_1^*) e^{i\omega^* \tau'_{10} s}$ ($s \in [0, 1]$) are the eigenvectors of $A(0)$ and A^* corresponding to the eigenvalue $i\omega^* \tau'_{10}$ and $-i\omega^* \tau'_{10}$, respectively, where

$$\langle q^*(s), q(\theta) \rangle = 1, \quad \langle q^*(s), \bar{q}(\theta) \rangle = 1,$$

$$\alpha_1 = \frac{i\omega^* - a_1 - d_1 e^{-i\omega^* \tau'_{10}}}{-b}, \quad \alpha_1^* = -\frac{i\omega^* + a_1 + d_1 e^{i\omega^* \tau'_{10}}}{a_2 e^{i\omega^* \frac{\tau_2^*}{\tau'_{10}}}},$$

$$D = \left[1 + \bar{\alpha}_1 \alpha_1^* + \tau'_{10} \left(d_1 e^{-i\omega^* \tau'_{10}} + \alpha_1^* \frac{\tau_2^*}{\tau'_{10}} a_2 e^{-i\omega^* \frac{\tau_2^*}{\tau'_{10}}} + \bar{\alpha}_1 \alpha_1^* \frac{\tau_2^*}{\tau'_{10}} b_1 e^{-i\omega^* \frac{\tau_2^*}{\tau'_{10}}} \right) \right].$$

Following the algorithms explained in Hassard *et al.* [80] and using a computation process similar to that in Song and Wei [195], which is used to obtain the properties of Hopf-bifurcation, we obtain

$$g_{20} = -\frac{2\tau'_{10}}{D} \left[\frac{r}{K} e^{-i\omega^* \tau'_{10}} + \alpha(1-c)(1-m)\alpha_1 + \frac{s}{L} \alpha_1^2 \bar{\alpha}^*_1 - \bar{\alpha}^*_1 \alpha_1 \beta \alpha(1-c)(1-m) e^{-2i\omega^* \tau_2^*} \right],$$

$$g_{11} = -\frac{\tau'_{10}}{D} \left[\frac{r}{K} (e^{-i\omega^* \tau'_{10}} + e^{i\omega^* \tau'_{10}}) + 2\text{Re}(\alpha_1) \alpha(1-c)(1-m) + 2\frac{s}{L} \bar{\alpha}^*_1 |\alpha_1|^2 - 2\bar{\alpha}^*_1 \text{Re}(\alpha_1) \beta \alpha(1-c)(1-m) \right],$$

$$g_{02} = -\frac{2\tau'_{10}}{D} \left[\frac{r}{K} e^{-i\omega^* \tau'_{10}} + \alpha(1-c)(1-m) \bar{\alpha}_1 + \frac{s}{L} \bar{\alpha}^*_1 \bar{\alpha}_1^2 + \bar{\alpha}^*_1 \bar{\alpha}_1 \beta \alpha(1-c)(1-m) e^{2i\omega^* \tau_2^*} \right],$$

$$g_{21} = -\frac{2\tau'_{10}}{D} \left[\frac{r}{K} \left\{ W_{11}^{(1)}(-1) + \frac{1}{2} W_{20}^{(1)}(-1) + \frac{1}{2} W_{20}^{(1)}(0) e^{i\omega^* \tau'_{10}} + W_{11}^{(1)}(0) e^{-i\omega^* \tau'_{10}} \right\} \right. \\ \left. + \alpha(1-c)(1-m) \left\{ W_{11}^{(2)}(0) - h\alpha(1-c)(1-m)\bar{\alpha}_1 - 2h\alpha(1-c)(1-m)\alpha_1 + \frac{1}{2} W_{20}^{(2)}(0) \right. \right. \\ \left. + \frac{1}{2} W_{20}^{(1)}(0)\bar{\alpha}_1 + W_{11}^{(1)}(0)\alpha_1 \right\} + \frac{s}{L} \bar{\alpha}^*_1 \left\{ 2\alpha_1 W_{11}^{(2)}(0) + \bar{\alpha}_1 W_{20}^{(2)}(0) \right\} - \bar{\alpha}^*_1 \beta \alpha (1-c) \\ (1-m) \left\{ W_{11}^{(2)}\left(-\frac{\tau_2^*}{\tau'_{10}}\right) e^{-i\omega^* \tau'_{10}} - h\alpha(1-c)(1-m)\bar{\alpha}_1 e^{-i\omega^* \tau'_{10}} - 2h\alpha(1-c)(1-m)\alpha_1 \right. \\ \left. e^{-i\omega^* \tau'_{10}} + \frac{1}{2} W_{20}^{(2)}\left(-\frac{\tau_2^*}{\tau'_{10}}\right) e^{i\omega^* \tau'_{10}} + \frac{1}{2} \bar{\alpha}_1 W_{20}^{(1)}\left(-\frac{\tau_2^*}{\tau'_{10}}\right) e^{i\omega^* \tau'_{10}} + \alpha_1 W_{11}^{(1)}\left(-\frac{\tau_2^*}{\tau'_{10}}\right) e^{-i\omega^* \tau'_{10}} \right\} \Big],$$

where

$$W_{20}(\theta) = \frac{ig_{20}}{\omega^* \tau'_{10}} q(0) e^{i\omega^* \tau'_{10} \theta} + \frac{i\bar{g}_{02}}{3\omega^* \tau'_{10}} \bar{q}(0) e^{-i\omega^* \tau'_{10} \theta} + E_1 e^{2i\omega^* \tau'_{10} \theta},$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega^* \tau'_{10}} q(0) e^{i\omega^* \tau'_{10} \theta} + \frac{i\bar{g}_{11}}{\omega^* \tau'_{10}} \bar{q}(0) e^{-i\omega^* \tau'_{10} \theta} + E_2,$$

$E_1 = (E_1^{(1)}, E_1^{(2)})^T \in \mathbb{R}^2$ and $E_2 = (E_2^{(1)}, E_2^{(2)})^T \in \mathbb{R}^2$ are constant vectors, computed as:

$$E_1 = 2 \begin{bmatrix} 2i\omega^* - a_1 - d_1 e^{-2i\omega^* \tau'_{10}} & b \\ -a_2 e^{-2i\omega^* \tau_2^*} & 2i\omega^* - d_3 - b_1 e^{-2i\omega^* \tau_2^*} \end{bmatrix}^{-1} \begin{bmatrix} \frac{r}{K} e^{-i\omega^* \tau'_{10}} + \alpha(1-c)(1-m)\alpha_1 \\ \frac{s}{L} \alpha_1^2 - \alpha(1-c)(1-m)\beta\alpha_1 e^{-2i\omega^* \tau'_{10}} \end{bmatrix},$$

$$E_2 = 2 \begin{bmatrix} -a_1 - d_1 & b \\ -a_2 & -d_3 - b_1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{r}{2K} (e^{-i\omega^* \tau'_{10}} + e^{i\omega^* \tau'_{10}}) + \alpha(1-c)(1-m)Re(\alpha_1) \\ \frac{s}{L} |\alpha_1|^2 - \alpha(1-c)(1-m)\beta Re(\alpha_1) \end{bmatrix}.$$

Consequently, g_{ij} can be expressed by the parameters and delays τ'_{10} and τ_2^* . Thus, these standard results can be computed as:

$$c_1(0) = \frac{i}{2\omega^* \tau'_{10}} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2}, \quad \mu_2 = -\frac{Re(c_1(0))}{Re(\lambda'(\tau'_{10}))},$$

$$\beta_2 = 2Re(c_1(0)), \quad T_2 = -\frac{Im(c_1(0)) + \mu_2 Im(\lambda'(\tau'_{10}))}{\omega^* \tau'_{10}}.$$

These expressions give a description of the bifurcating periodic solution in the center manifold of system (3.14) at critical values $\tau_1 = \tau_{10}$ which can be stated in the form of following theorem:

Theorem 3.5.1. *1. μ_2 determines the direction of Hopf-bifurcation. If $\mu_2 > 0 (< 0)$ then the Hopf-bifurcation is supercritical (subcritical).*

2. β_2 determines the stability of bifurcated periodic solution. If $\beta_2 > 0 (< 0)$ then the bifurcated periodic solutions are unstable (stable).

3. T_2 determines the period of bifurcating periodic solution. The period increases (decreases) if $T_2 > 0 (< 0)$.

Remark 3.5.1. When $\tau_1 > 0$ and $\tau_2 = 0$ or $\tau_1 = 0$ and $\tau_2 > 0$, then under an analysis similar to section 3.5, the corresponding values of μ_2 , β_2 and T_2 can be computed. Depending upon the sign of μ_2 , β_2 and T_2 , the corresponding results can also be deduced.

3.6 Global Asymptotic Stability for the Delayed System

Before dealing with the global stability analysis for the delayed system (3.14), we present some inequalities [95] which will be used in the proofs. Consider the function $F(x) = x - 1 - \ln x$, $x > 0$. We note that $F(x) \geq 0$, $\forall x$ and $F(x) = 0$ if and only if $x = 1$. Let x_1, x_2, \dots, x_n be positive numbers then,

$$1 - x_i + \ln x_i = -F(x_i) \leq 0, \quad i = 1, 2, \dots, n.$$

Summing over $i = 1, 2, \dots, n$, we obtain

$$n - \sum_{i=1}^n x_i + \ln \prod_{i=1}^n x_i \leq 0.$$

Choosing $x_i = \frac{p_i}{q_i}$, where $p_i > 0$, $q_i > 0$ for $i = 1, 2, \dots, n$, it follows that

$$n - \sum_{i=1}^n \frac{p_i}{q_i} + \ln \prod_{i=1}^n \frac{p_i}{q_i} \leq 0.$$

3.6.1 Global stability of one delayed model (i.e. $\tau_1 > 0$, $\tau_2 = 0$)

Theorem 3.6.1. The positive equilibrium point E^* of the delayed system (3.14) for $\tau_1 > 0$ is globally asymptotic stable under condition (3.13).

Proof. We define a Liapunov function V_3 as follows:

$$V_3 = u - u^* - u^* \ln \frac{u}{u^*} + \gamma_1 \left(v - v^* - v^* \ln \frac{v}{v^*} \right),$$

where γ_1 is a positive constant to be chosen suitably.

The derivative of V_3 with respect to time t along the solution of system (3.14) is given by

$$\frac{dV_3}{dt} = \left(1 - \frac{u^*}{u} \right) \frac{du}{dt} + \gamma_1 \left(1 - \frac{v^*}{v} \right) \frac{dv}{dt}$$

$$\begin{aligned}
&= (u-u^*) \left[r \left(1 - \frac{u(t-\tau_1)}{K} \right) - \frac{\alpha(1-c)(1-m)v}{1+h\alpha(1-c)(1-m)u} \right] + \gamma_1 (v-v^*) \left[s \left(1 - \frac{v}{L} \right) \right. \\
&\quad \left. + \frac{\beta\alpha(1-c)(1-m)u}{1+h\alpha(1-c)(1-m)u} \right], \\
&= (u-u^*) \left[r \left(1 - \frac{u}{K} \right) - \frac{\alpha(1-c)(1-m)v}{1+h\alpha(1-c)(1-m)u} - r \left(1 - \frac{u^*}{K} \right) + \frac{\alpha(1-c)(1-m)v^*}{1+h\alpha(1-c)(1-m)u^*} \right] \\
&\quad + \gamma_1 (v-v^*) \left[s \left(1 - \frac{v}{L} \right) + \frac{\beta\alpha(1-c)(1-m)u}{1+h\alpha(1-c)(1-m)u} - s \left(1 - \frac{v^*}{L} \right) - \frac{\beta\alpha(1-c)(1-m)u^*}{1+h\alpha(1-c)(1-m)u^*} \right] \\
&\quad - \frac{r}{K} u(t-\tau_1)(u-u^*) + \frac{r}{K} u(u-u^*), \\
&= - \left[\frac{r}{K} - \frac{h\alpha^2(1-c)^2(1-m)^2v^*}{(1+h\alpha(1-c)(1-m)u)(1+h\alpha(1-c)(1-m)u^*)} \right] (u-u^*)^2 \\
&\quad - \left[\alpha(1-c)(1-m) + h\alpha^2(1-c)^2(1-m)^2u^* - \gamma_1\beta\alpha(1-c)(1-m) \right] \times \\
&\quad \frac{(u-u^*)(v-v^*)}{(1+h\alpha(1-c)(1-m)u)(1+h\alpha(1-c)(1-m)u^*)} - \gamma_1 \frac{s}{L} (v-v^*)^2 \\
&\quad - \frac{r}{K} u(t-\tau_1)(u-u^*) + \frac{r}{K} u(u-u^*).
\end{aligned}$$

Since γ_1 is a arbitrary positive real number, so we choose $\gamma_1 = \frac{(1+h\alpha(1-c)(1-m)u^*)}{\beta}$. Then we have

$$\begin{aligned}
\frac{dV_3}{dt} &= - \left[\frac{r}{K} - \frac{h\alpha^2(1-c)^2(1-m)^2v^*}{(1+h\alpha(1-c)(1-m)u)(1+h\alpha(1-c)(1-m)u^*)} \right] (u-u^*)^2 \\
&\quad - \frac{s}{L} \frac{(1+h\alpha(1-c)(1-m)u^*)}{\beta} (v-v^*)^2 - \frac{r}{K} u(t-\tau_1)(u-u^*) + \frac{r}{K} u(u-u^*).
\end{aligned}$$

Define

$$V'_3 = V_3 + \frac{r}{K} u^* \int_{t-\tau_1}^t \left[u(s) - u^* - u^* \ln \frac{u(s)}{u^*} \right] ds,$$

which yields

$$\begin{aligned}
\frac{dV'_3}{dt} &= - \left[\frac{r}{K} - \frac{h\alpha^2(1-c)^2(1-m)^2v^*}{(1+h\alpha(1-c)(1-m)u)(1+h\alpha(1-c)(1-m)u^*)} \right] (u-u^*)^2 \\
&\quad - \frac{s}{L} \frac{(1+h\alpha(1-c)(1-m)u^*)}{\beta} (v-v^*)^2 - \frac{r}{K} uu(t-\tau_1) + \frac{r}{K} u^* u(t-\tau_1) + \frac{r}{K} u^2 \\
&\quad - \frac{r}{K} uu^* + \frac{r}{K} u^* \left[u - u(t-\tau_1) - u^* \ln \left(\frac{u}{u(t-\tau_1)} \right) \right], \\
&= - \left[\frac{r}{K} - \frac{h\alpha^2(1-c)^2(1-m)^2v^*}{(1+h\alpha(1-c)(1-m)u)(1+h\alpha(1-c)(1-m)u^*)} \right] (u-u^*)^2 \\
&\quad - \frac{s}{L} \frac{(1+h\alpha(1-c)(1-m)u^*)}{\beta} (v-v^*)^2 + \frac{r}{K} uu(t-\tau_1) + \frac{r}{K} u^2 + \frac{r}{K} u^* \ln \left(\frac{u(t-\tau_1)}{u} \right).
\end{aligned}$$

Noting that $\frac{r}{K}u^2 = \frac{r}{K}u^{*2}$ at $E^*(u^*, v^*)$,

$$\begin{aligned} \frac{dV_3'}{dt} = & - \left[\frac{r}{K} - \frac{h\alpha^2(1-c)^2(1-m)^2v^*}{(1+h\alpha(1-c)(1-m)u)(1+h\alpha(1-c)(1-m)u^*)} \right] (u-u^*)^2 \\ & - \frac{s}{L} \frac{(1+h\alpha(1-c)(1-m)u^*)}{\beta} (v-v^*)^2 + \frac{r}{K}u^{*2} \left[1 - \frac{u}{u^*} \frac{u(t-\tau_1)}{u^*} + \ln \left(\frac{u(t-\tau_1)}{u} \right) \right]. \end{aligned}$$

Using $n=1$, it follows that

$$1 - \frac{u}{u^*} \frac{u(t-\tau_1)}{u^*} + \ln \left(\frac{u(t-\tau_1)}{u} \right) \leq 0.$$

Thus, $\frac{dV_3'}{dt} \leq 0$ and we have $\frac{dV_3'}{dt} = 0$ if and only if $u = u^*$, $v = v^*$. We now look for the invariant subset M within the set $M = \{(u, v) \mid u = u^*, v = v^*, \frac{u}{u^*} \frac{u(t-\tau_1)}{u^*} = 1\}$. Hence the only invariant set M is $M = \{(u^*, v^*)\}$. Using LaSalle invariance principle [139], E^* is globally asymptotic stable whenever condition (3.13) holds true. \square

3.6.2 Global stability of two delayed model

Theorem 3.6.2. *The positive equilibrium point E^* of the delayed system (3.14) is globally asymptotic stable under condition (3.13).*

Proof. We define a Liapunov function V_4 as follows:

$$V_4 = u - u^* - u^* \ln \frac{u}{u^*} + \gamma_2 \left(v - v^* - v^* \ln \frac{v}{v^*} \right),$$

where γ_2 is a positive constant to be chosen suitably.

The derivative of V_4 along the solution of system (3.14) is

$$\begin{aligned} \frac{dV_4}{dt} &= \left(1 - \frac{u^*}{u} \right) \frac{du}{dt} + \gamma_2 \left(1 - \frac{v^*}{v} \right) \frac{dv}{dt} \\ &= (u - u^*) \left[r \left(1 - \frac{u(t-\tau_1)}{K} \right) - \frac{\alpha(1-c)(1-m)v}{1+h\alpha(1-c)(1-m)u} \right] + \gamma_2 \left(1 - \frac{v^*}{v} \right) \left[sv \left(1 - \frac{v}{L} \right) \right. \\ &\quad \left. + \frac{\beta\alpha(1-c)(1-m)u(t-\tau_2)v(t-\tau_2)}{1+h\alpha(1-c)(1-m)u(t-\tau_2)} \right], \\ &= - \left[\frac{r}{K} - \frac{h\alpha^2(1-c)^2(1-m)^2v^*}{(1+h\alpha(1-c)(1-m)u)(1+h\alpha(1-c)(1-m)u^*)} \right] (u-u^*)^2 - \gamma_2 \frac{s}{L} (v-v^*)^2 \\ &\quad - \frac{r}{K} u(t-\tau_1)(u-u^*) + \frac{r}{K} u(u-u^*) + \gamma_2 \left(1 - \frac{v^*}{v} \right) \frac{\beta\alpha(1-c)(1-m)u(t-\tau_2)v(t-\tau_2)}{1+h\alpha(1-c)(1-m)u(t-\tau_2)} \\ &\quad - \gamma_2 \left(1 - \frac{v^*}{v} \right) \frac{\beta\alpha(1-c)(1-m)uv}{1+h\alpha(1-c)(1-m)u}. \end{aligned}$$

Define

$$V_4' = V_4 + \frac{r}{K} u^* \int_{t-\tau_1}^t \left[u(s) - u^* - u^* \ln \frac{u(s)}{u^*} \right] ds + \gamma_2 \beta \alpha (1-c)(1-m) \times \\ \int_{t-\tau_2}^t \left[\frac{u(s)v(s)}{1+h\alpha(1-c)(1-m)u(s)} - \frac{u^*v^*}{1+h\alpha(1-c)(1-m)u^*} \right. \\ \left. - \frac{u^*v^*}{1+h\alpha(1-c)(1-m)u^*} \ln \left(\frac{(1+h\alpha(1-c)(1-m)u^*)u(s)}{(1+h\alpha(1-c)(1-m)u(s))u^*} \right) \right] ds.$$

Then we have

$$\frac{dV_4'}{dt} = - \left[\frac{r}{K} - \frac{h\alpha^2(1-c)^2(1-m)^2v^*}{(1+h\alpha(1-c)(1-m)u)(1+h\alpha(1-c)(1-m)u^*)} \right] (u-u^*)^2 - \gamma_2 \frac{s}{L} (v-v^*)^2 \\ + \frac{r}{K} u^{*2} \left[1 - \frac{u}{u^*} \frac{u(t-\tau_1)}{u^*} + \ln \left(\frac{u(t-\tau_1)}{u} \right) \right] \\ - \gamma_2 \frac{v^* \beta \alpha (1-c)(1-m)u(t-\tau_2)v(t-\tau_2)}{v(1+h\alpha(1-c)(1-m)u(t-\tau_2))} + \gamma_2 \frac{\beta \alpha (1-c)(1-m)uv^*}{1+h\alpha(1-c)(1-m)u} \\ + \gamma_2 \frac{\beta \alpha (1-c)(1-m)u^*v^*}{1+h\alpha(1-c)(1-m)u^*} \ln \left(\frac{uv(1+h\alpha(1-c)(1-m)u(t-\tau_2))}{u(t-\tau_2)v(t-\tau_2)(1+h\alpha(1-c)(1-m)u)} \right).$$

Noting that $\frac{\beta \alpha (1-c)(1-m)uv^*}{1+h\alpha(1-c)(1-m)u} = \frac{\beta \alpha (1-c)(1-m)u^*v^*}{1+h\alpha(1-c)(1-m)u^*}$ at $E^*(u^*, v^*)$, then we have

$$\frac{dV_4'}{dt} = - \left[\frac{r}{K} - \frac{h\alpha^2(1-c)^2(1-m)^2v^*}{(1+h\alpha(1-c)(1-m)u)(1+h\alpha(1-c)(1-m)u^*)} \right] (u-u^*)^2 - \gamma_2 \frac{s}{L} (v-v^*)^2 \\ + \frac{r}{K} u^2 \left[1 - \frac{u(t-\tau_1)}{u} + \frac{u^{*2}}{u^2} \ln \left(\frac{u(t-\tau_1)}{u} \right) \right] \\ + \gamma_2 \frac{\beta \alpha (1-c)(1-m)u^*v^*}{1+h\alpha(1-c)(1-m)u^*} \left[1 - \frac{u(t-\tau_2)}{u^*} \frac{v(t-\tau_2)}{v^*} \frac{v^*}{v} \frac{1+h\alpha(1-c)(1-m)u^*}{1+h\alpha(1-c)(1-m)u(t-\tau_2)} \right. \\ \left. + \ln \left(\frac{u(t-\tau_2)}{u} \frac{v(t-\tau_2)}{v} \frac{1+h\alpha(1-c)(1-m)u}{1+h\alpha(1-c)(1-m)u(t-\tau_2)} \right) \right]$$

We note that

$$\left[1 - \frac{u(t-\tau_2)}{u^*} \frac{v(t-\tau_2)}{v^*} \frac{v^*}{v} \frac{1+h\alpha(1-c)(1-m)u^*}{1+h\alpha(1-c)(1-m)u(t-\tau_2)} \right. \\ \left. + \ln \left(\frac{u(t-\tau_2)}{u} \frac{v(t-\tau_2)}{v} \frac{1+h\alpha(1-c)(1-m)u}{1+h\alpha(1-c)(1-m)u(t-\tau_2)} \right) \right] \leq 0,$$

and $\left[1 - \frac{u}{u^*} \frac{u(t-\tau_1)}{u^*} + \ln \left(\frac{u(t-\tau_1)}{u} \right) \right] \leq 0$.

Thus $\frac{dV_4'}{dt} \leq 0$ and with equality iff $u = u^*, v = v^*$. We now look for the invariant subset M within the set $M = \{(u, v) \mid u = u^*, v = v^*\}$. Thus $M = \{E^*\}$. Using LaSalle invariance principle [139],

E^* is globally asymptotic stable whenever condition (3.13) is satisfied. \square

Remark 3.6.1. *If the system (3.1) is globally asymptotically stable, then it is also globally asymptotically stable for one delay and two delays.*

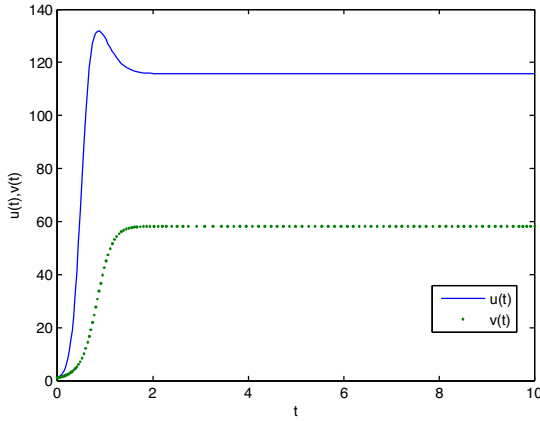


Fig. 3.1: Time series of u and v for the set of values of parameters given in (3.28).

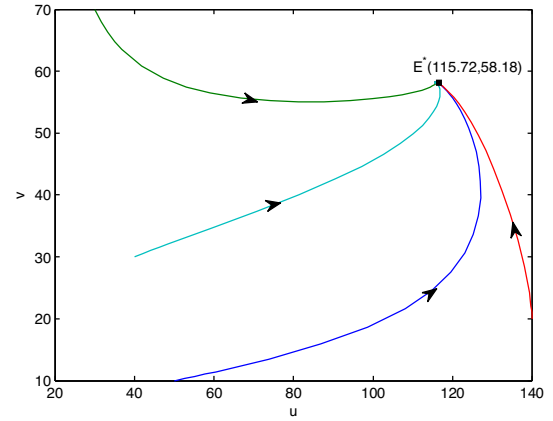


Fig. 3.2: Stable solution in uv -plane, starting from different initial conditions.

3.7 The influence of prey refuge

To investigate the influence of prey refuge, we calculate the derivative of the prey and predator along the positive equilibrium point with respect to refuge parameter m and we get

$$\frac{du^*}{dm} = \frac{p(u^*)}{q(u^*)},$$

$$\frac{dv^*}{dm} = \frac{L\alpha\beta(1-c)[(1-m)\frac{p(u^*)}{q(u^*)} - u^*]}{s[1+h\alpha(1-c)(1-m)u^*]},$$

where

$$p(u^*) = L\alpha(1-c)[1+h\alpha(1-c)(1-m)u^*] > 0,$$

$$q(u^*) = \frac{r}{K}\{1+h\alpha(1-c)(1-m)u^*\}^3 - Lh\alpha^2(1-c)^2(1-m)^2\{1+h\alpha(1-c)(1-m)u^*\} + \frac{L\beta\alpha^2(1-c)^2(1-m)^2}{s} - \frac{Lh\alpha^3(1-c)^3(1-m)^3u^*}{s}.$$

Now, we have $\frac{du^*}{dm} > 0$ if $q(u^*) > 0$ hence u^* is a strictly increasing function of m . Suppose $R_1 = \frac{q(u^*)}{p(u^*)}u^*$, then $\frac{dv^*}{dm} > 0$ if $0 < m < 1 - R_1$, which implies that v^* is a strictly increasing

function on $m \in (0, 1 - R_1)$; When $1 - R_1 < m < 1$, then $\frac{dv^*}{dm} < 0$, which yields that v^* is a strictly decreasing function on $m \in (1 - R_1, 1)$.

This shows that increment of the amount to refuge is in the favor to promote prey density. Predator density increases if amount of refuged prey is less than $1 - R_1$ and it decreases when quantity of refuged prey crosses the value $1 - R_1$. When $m = 1 - R_1$, predator species reaches its maximum.

Remark 3.7.1. *Refuge affects predator species, but not such efficient to extinct them because predators depend on other resources.*

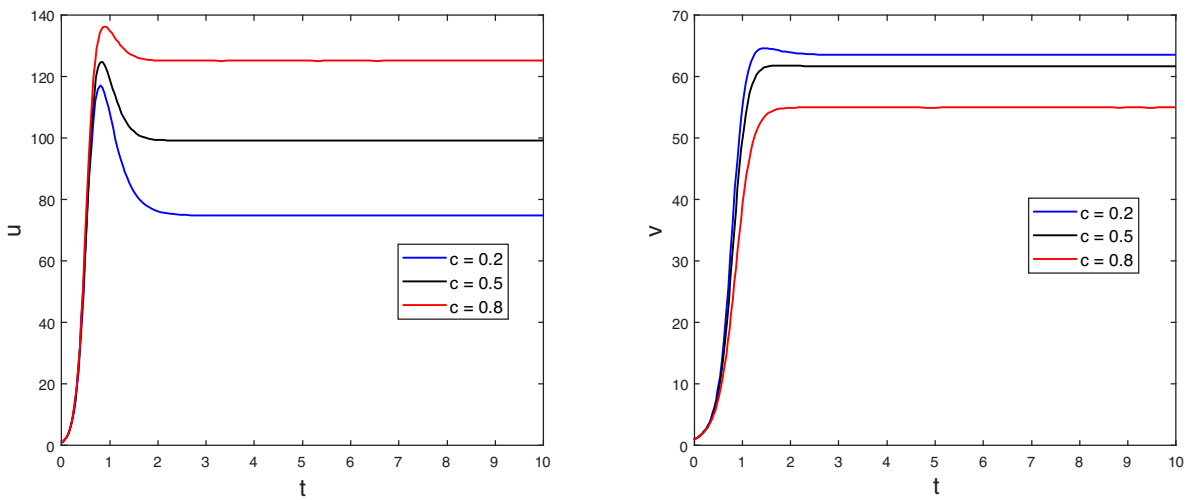


Fig. 3.3: Behavior of u and v with time t for different values of c and other values are same as in (3.28).

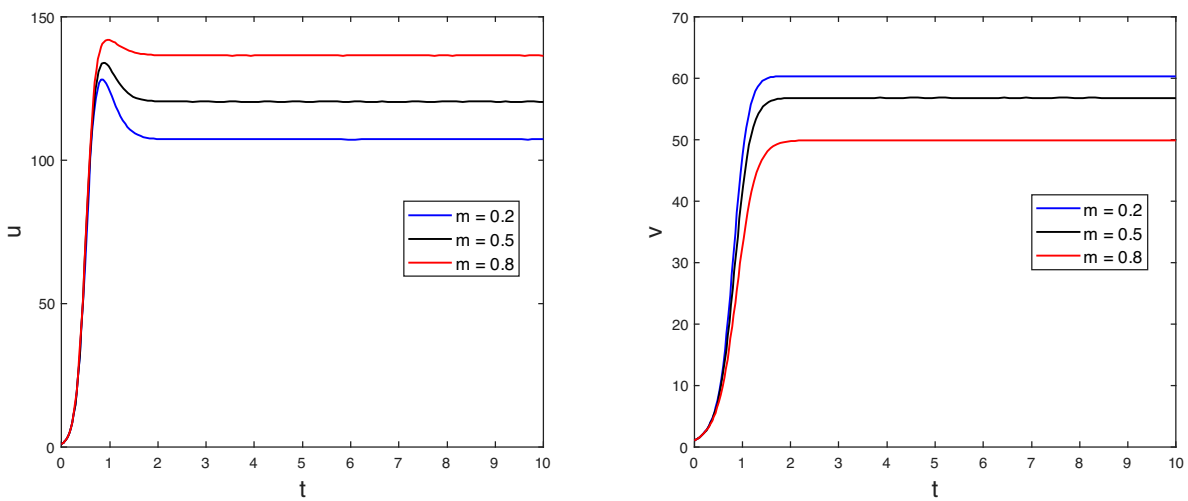


Fig. 3.4: Behavior of u and v with time t for different values of m and other values are same as in (3.28).

3.8 Numerical Simulation

In this section, we present numerical simulations to verify the analytical predictions obtained in previous section using MATLAB 2017a. As this is not a case study, real data are not available for this problem, so we consider some biologically feasible data.

3.8.1 Non-delayed system

For the model (3.1), parameters are chosen as follows:

$$\begin{aligned} r &= 10, K = 150, s = 4, L = 40, h = 0.1, \\ c &= 0.7, m = 0.4, \alpha = 0.4, \beta = 0.4, \end{aligned} \quad (3.28)$$

with initial conditions $u(0) = v(0) = 1$.

For the above set of values of parameters, second condition of (3.6) is satisfied. Thus, the positive equilibrium point $E^*(u^*, v^*)$ is given by

$$u^* = 115.72, v^* = 58.18 .$$

It is also to be noted that the condition of Theorem 3.3.9 is satisfied for the set of parameters chosen in (3.28). Hence, the equilibrium point $E^*(u^*, v^*)$ is globally asymptotically stable. The time series of u and v is presented in Fig. 3.1. This figure shows that the density of prey species and predator species both increase with time and finally settle down at their steady state level. But the predator population initially increases slowly and then goes to equilibrium level, whereas the prey population increases fast and then slightly decreases and goes to the steady state level. In Fig. 3.2, we plotted the trajectories of solution curves of u and v with different initial conditions. From this figure, we observe that all the trajectories starting from the various initial conditions converge to the equilibrium point $E^*(115.72, 58.18)$, which shows that E^* is globally asymptotically stable.

We also observe that habitat complexity c and refuge parameter m play an important role to understand the dynamics of the model (3.1). We have examined the dynamics of the system with respect to these parameters.

In Fig. 3.3, we show the behavior of u and v with time for three different values of the parameter c and rest of the parameters have the same values as that in (3.28). Here we note that as the value of the c increases, the density of u increases but the density of v decreases. u increases with respect to time very fast, then decreases and after that goes to the equilibrium level. When c is large ($c = 0.8$), u increases, then slightly decreases, and settles down at higher equilibrium level. Whereas v increases with respect to time and then goes to their respective

steady state level. The effect of m on the dynamics of u and v is similar to that of c , and it is shown in Fig. 3.4.

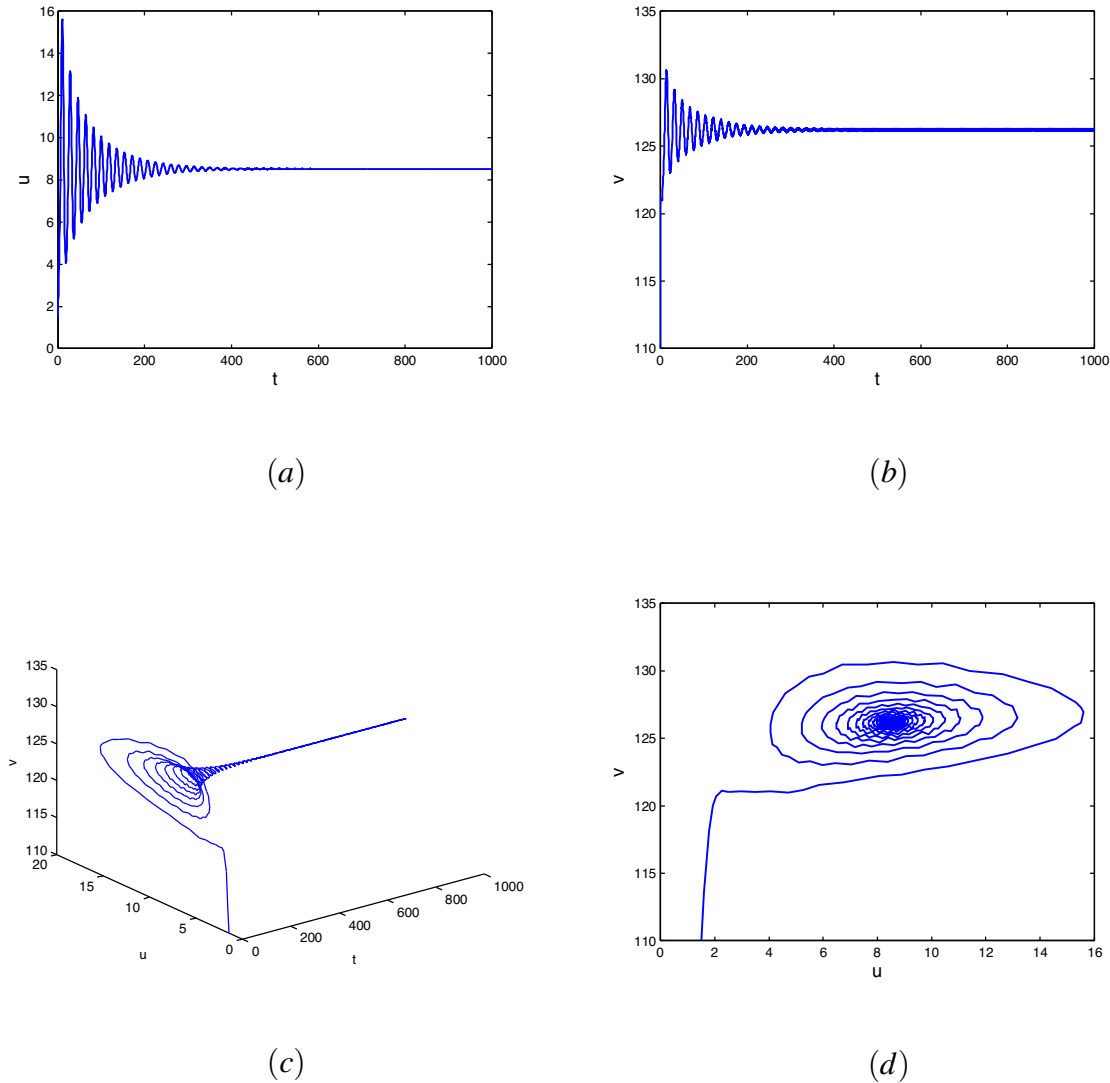


Fig. 3.5: E^* is asymptotically stable when $\tau_1 = 0$ and $\tau_2 = 3.6 < \tau_{2_0}$, (a-b) Time series evolution of species, (c) phase-portrait in tuv -space, (d) phase-portrait in uv plane.

3.8.2 Delayed system

In order to illustrate the theoretical results derived above in case of delayed system (3.14), we choose the parameters as follows:

$$\begin{aligned}
 r &= 8, K = 200, s = 5, L = 120, h = 0.1, \\
 c &= 0.6, m = 0.8, \alpha = 0.8, \beta = 0.5.
 \end{aligned}
 \tag{3.29}$$

Then system (3.14) has a unique positive equilibrium $E^*(u^*, v^*) = E^*(8.51, 126.1981)$. The complex dynamical behavior of the system has been observed with time delay.

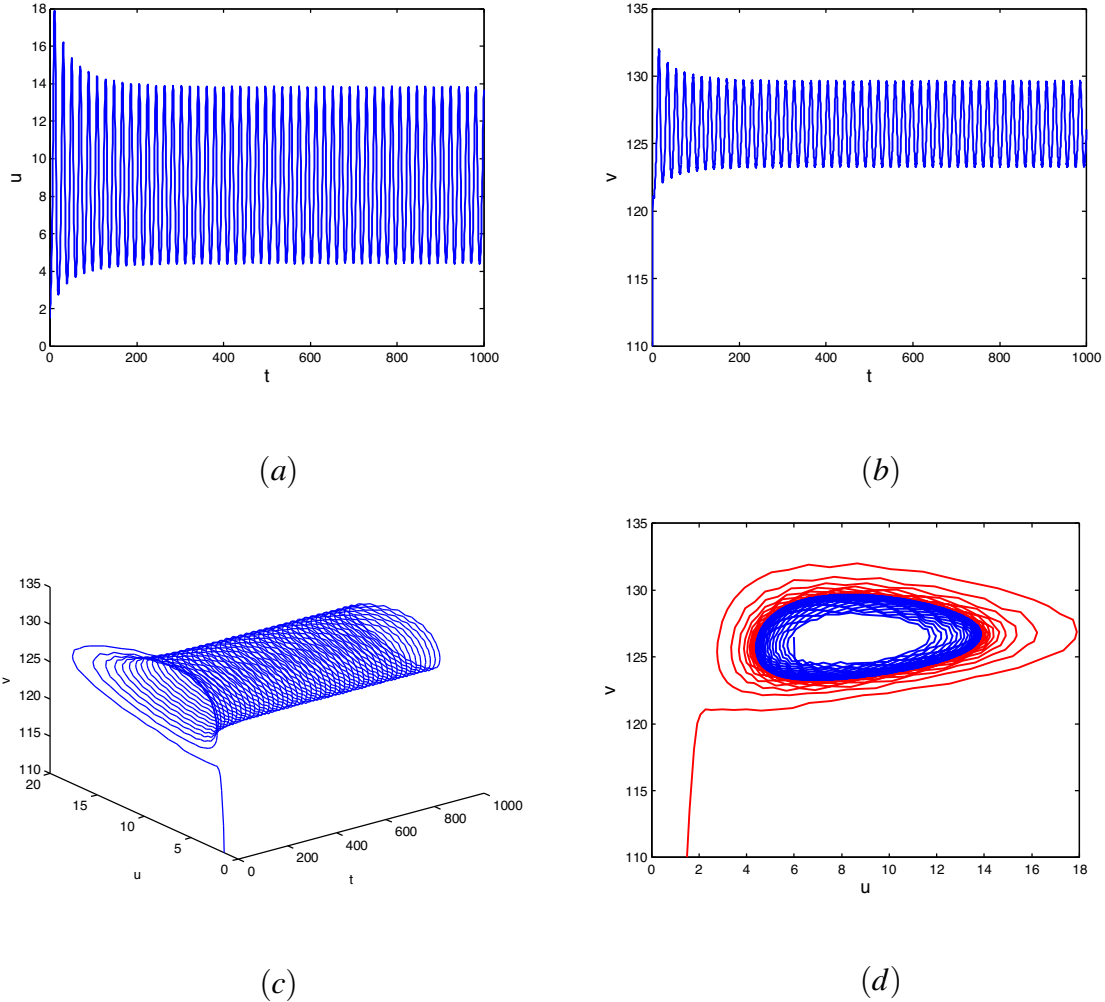


Fig. 3.6: E^* is unstable when $\tau_1 = 0$ and $\tau_2 = 4 > \tau_{2_0}$, (a-b) Time series evolution of species, (c) phase-portrait in uv -space, (d) existence of periodic solution.

Case (I): When $\tau_1 = 0$ and $\tau_2 > 0$, then we see that conditions (H_1) and (H_3) are satisfied. Taking $n = 0$, our computer simulation yields

$$\omega_1 = 0.3379, \tau_{2_0} = 3.8517,$$

and transversality condition is satisfied. Thus, the positive equilibrium E^* is asymptotically stable for $\tau_2 < \tau_{2_0} = 3.8517$ which are shown in Fig. 3.5 and unstable for $\tau_2 > \tau_{2_0}$ (Fig. 3.6). When $\tau_2 = \tau_{2_0}$, system undergoes a Hopf-bifurcation at the positive equilibrium E^* i.e. a small amplitude periodic solution occurs around E^* . The time series analysis has been shown in Fig. 3.6(a-b), phase portrait in the uv space in Fig. 3.6(c)

and phase-portrait in uv plane in Fig. 3.6(d). From Fig. 3.6(d), we note that a periodic solution exists. Any solution trajectory initiating from inside (blue) or outside (red) the closed trajectory, approaches towards the closed trajectory. This shows the existence of a stable periodic solution. By the algorithm derived in previous section, we can obtain

$$c_1(0) = -0.0111 - 0.0139i, \mu_2 > 0, \beta_2 = -0.0221, T_2 = 0.022.$$

This shows that the nature of Hopf-bifurcation is supercritical, bifurcated periodic solution is stable and its period increases

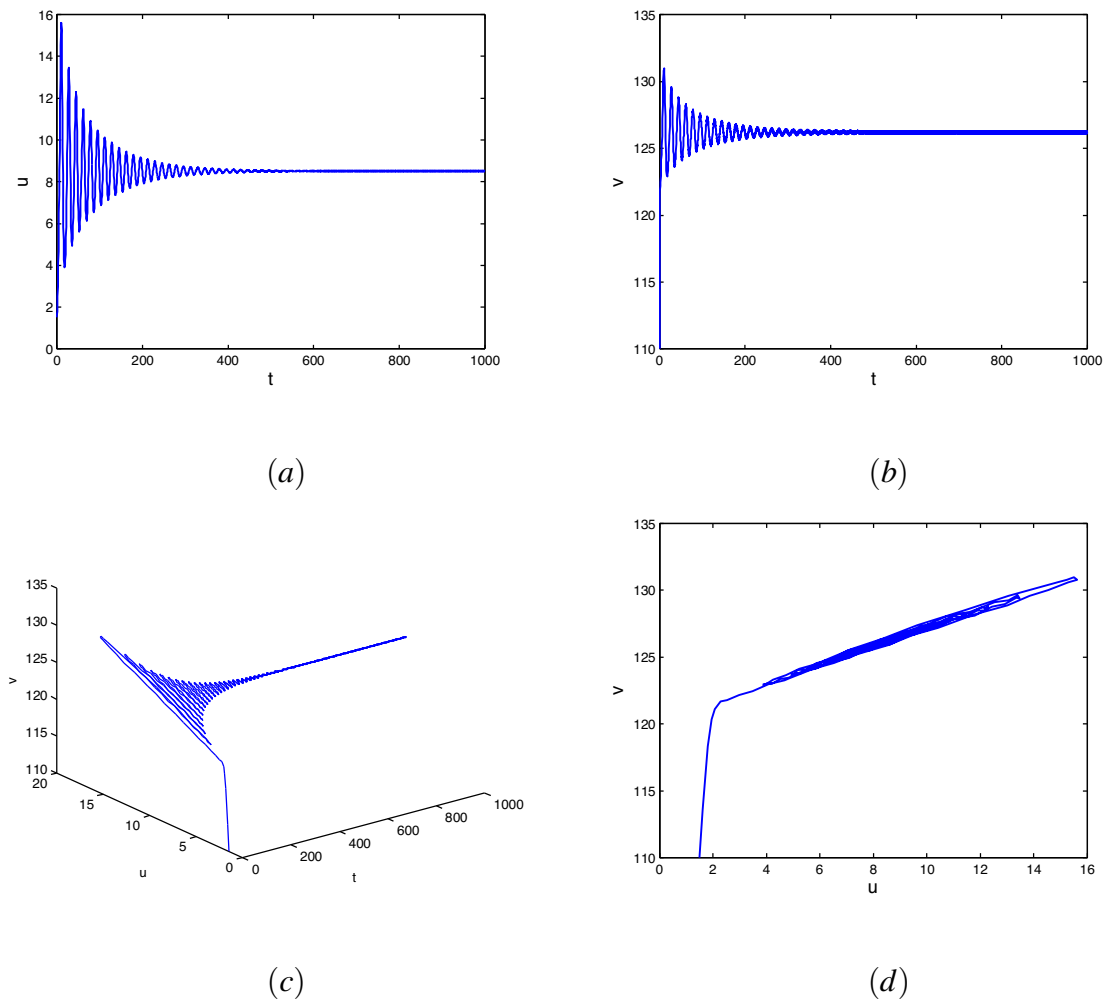


Fig. 3.7: E^* is asymptotically stable when $\tau_2 = 0$ and $\tau_1 = 3.8 < \tau_{1_0}$, (a-b) Time series evolution of species, (c) phase-portrait in tuv -space, (d) phase-portrait in uv plane.

Case (II): When $\tau_2 = 0$ and $\tau_1 > 0$, by some computation, we obtain

$$\omega_2 = 0.3625, \tau_{1_0} = 4.0038,$$

and transversality condition is satisfied. Thus, the equilibrium point E^* is asymptotically stable for $\tau_1 < \tau_{1_0} = 4.0038$, which is shown in Fig. 3.7 and unstable for $\tau_1 > \tau_{1_0}$ (Fig. 3.8). When $\tau_1 = \tau_{1_0}$, system undergoes a Hopf-bifurcation at positive equilibrium E^* and a periodic solution occurs around E^* . By the algorithm derived in section 3.5, we can obtain

$$c_1(0) = -0.1032 + 0.8913i, \mu_2 > 0, \beta_2 = -0.2064, T_2 = 0.1903.$$

Thus, the nature of Hopf-bifurcation remains same as in the previous case I.

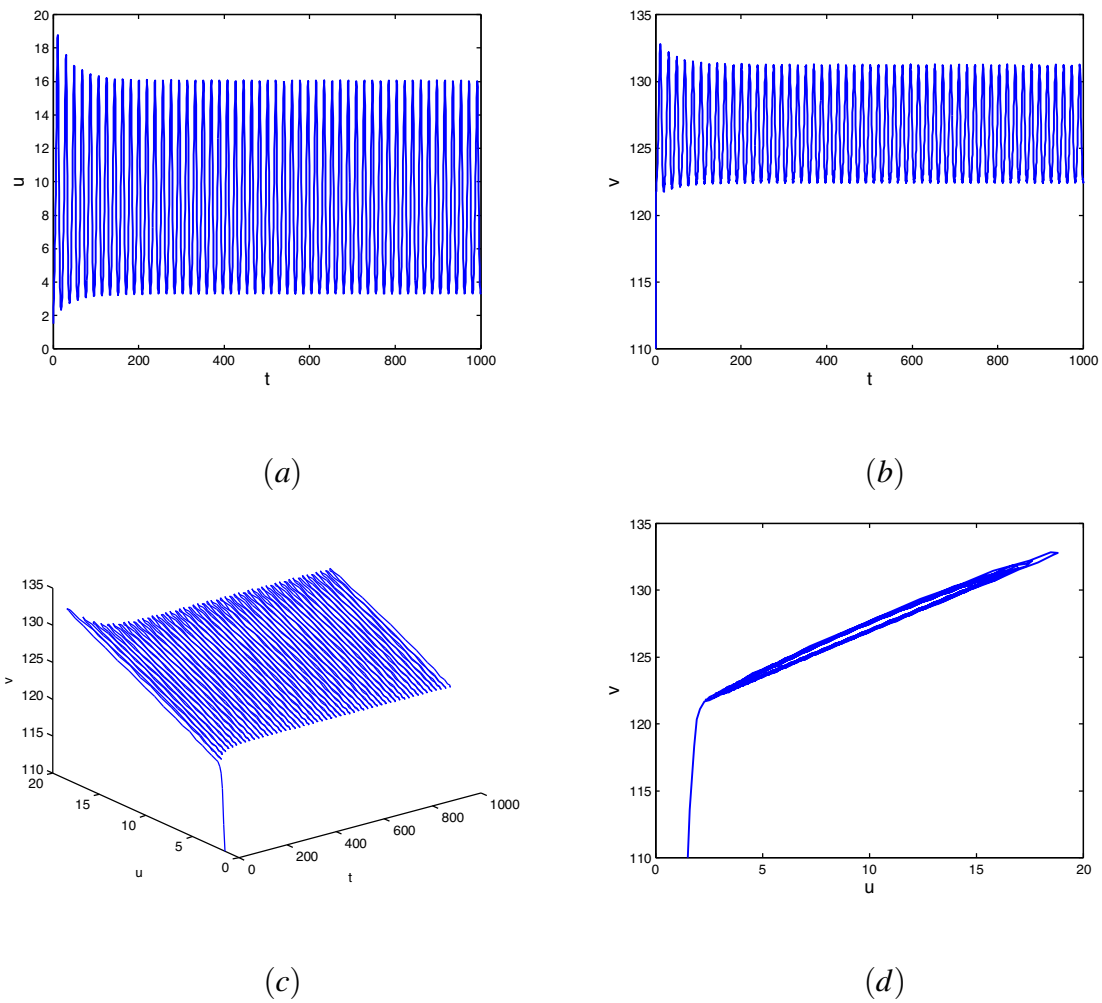


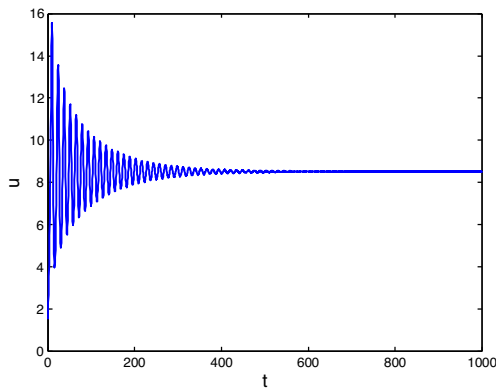
Fig. 3.8: E^* is unstable when $\tau_2 = 0$ and $\tau_1 = 4.2 > \tau_{1_0}$, (a-b) Time series evolution of species, (c) phase-portrait in tuv -space, (d) existence of periodic solution.

Case (III): $\tau_1 > 0$ and $\tau_2 > 0$. Let $\tau_2^* = 0.5 \in (0, 3.8517)$ and choose τ_1 as a parameter. We have $\tau'_{1_0} = 3.1302$ (critical value). For $\tau_1 \in (0, \tau'_{1_0})$, the system is asymptotically stable (Fig. 3.9). But for $\tau_1 = 3.3 > \tau'_{1_0}$, the system becomes unstable (Fig. 3.10). Thus the model is

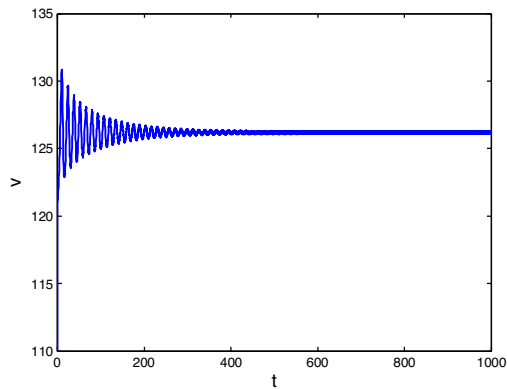
asymptotically stable for $\tau_1 < \tau'_{1_0}$. As τ_1 passes through τ'_{1_0} , the system loses its stability and a Hopf-bifurcation occurs into the system. Fig. 3.10(d) shows the existence of a periodic solution (closed trajectory). The blue trajectory represents any solution starting from inside the closed trajectory and the red trajectory denotes any solution starting from outside the closed trajectory. Both of the trajectories approach to the closed trajectory which represents a stable periodic solution. By the algorithm derived in previous section, we can obtain

$$\left[\frac{d}{d\tau_1} Re(\lambda) \right]_{\tau_1=\tau'_{1_0}} = 0.0715 > 0$$

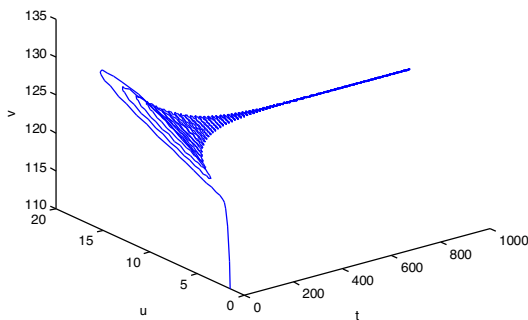
$$c_1(0) = -570.32 - 417.71i, \mu_2 = 7976.5, \beta_2 = -1140.6, T_2 = 842.3801.$$



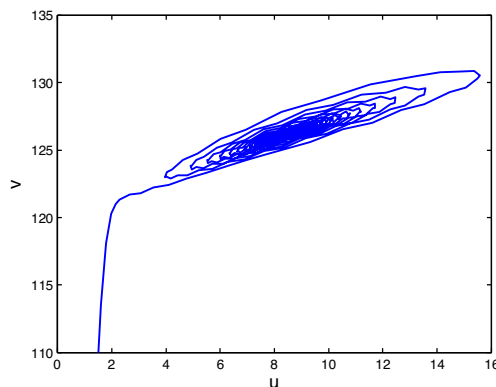
(a)



(b)



(c)



(d)

Fig. 3.9: E^* is asymptotically stable when $\tau_2 = 0.5$ and $\tau_1 = 3 < \tau'_{1_0}$ (a-b) Time series evolution of species, (c) phase-portrait in tuv -space, (d) phase-portrait in uv plane.

Since $\mu_2 > 0$, the Hopf-bifurcation is supercritical. $\beta_2 < 0$ implies that the bifurcated periodic solution is stable. $T_2 > 0$ shows that the period of bifurcated periodic solution increases.

The bifurcating diagrams have been shown in Fig.(3.11-3.13) by taking τ_1 as a bifurcation parameter and $\tau_2 = 0$ in Fig. 3.11, by taking τ_1 as a bifurcation parameter and $\tau_2 = 0.5$ in Fig. 3.12 and by taking τ_2 as a bifurcation parameter and $\tau_1 = 0$ in Fig. 3.13. These figures depict the dynamics of the system as the delay increases. From these figures, it is evident that for small values of the delay, the system is stable but as the value of delay crosses its critical value, the system loses its stability and undergoes Hopf-bifurcation. Thus the periodic solution exists for $\tau > \tau_{critical}$. In Fig.(3.14), the region of stability and instability has been drawn. It is to be noted that if $\tau_1 < \tau_{1_0}$ and $\tau_2 < \tau_{2_0}$, then system undergoes a Hopf-bifurcation and if $\tau_1 > \tau_{1_0}$ or $\tau_2 > \tau_{1_0}$, then system is unstable and Hopf-bifurcation does not exist.

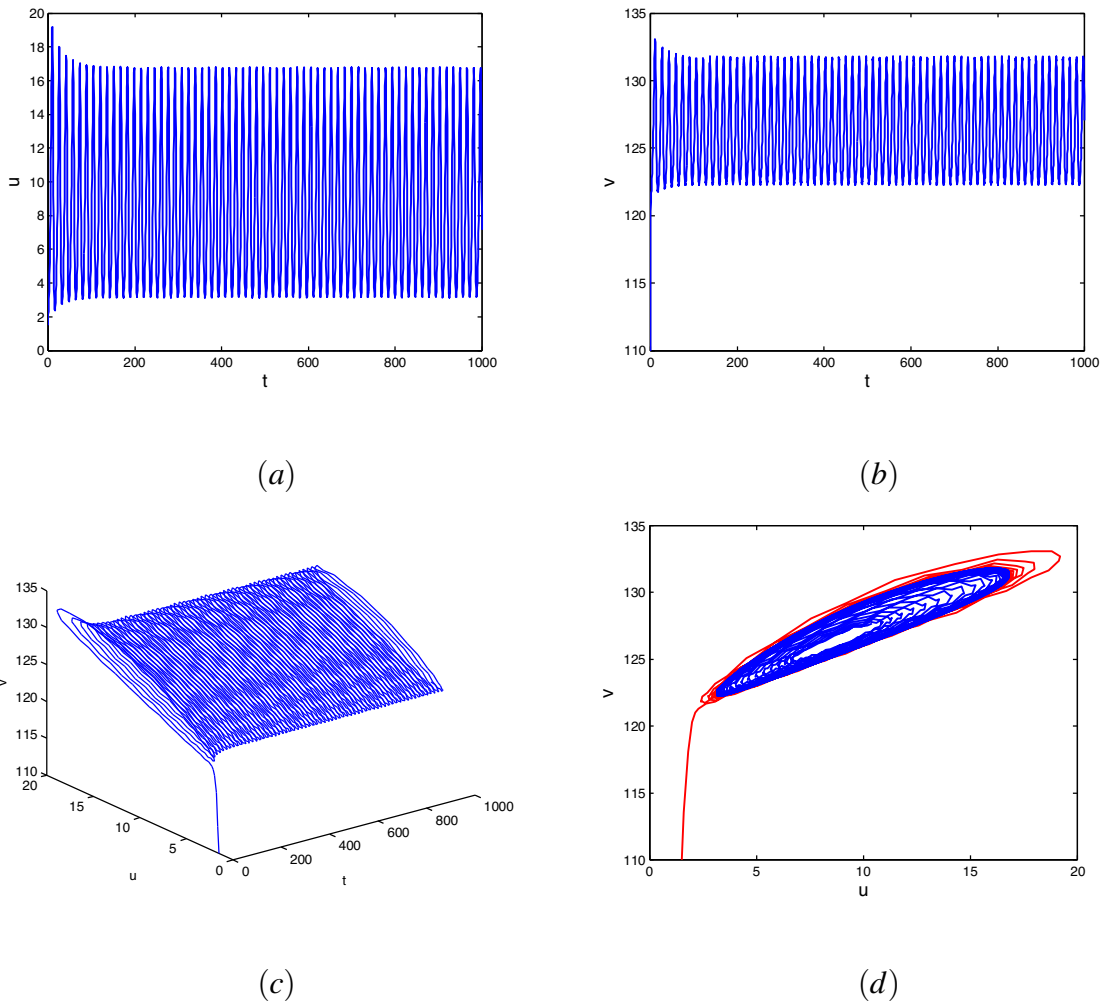


Fig. 3.10: E^* is unstable when $\tau_2 = 0.5$ and $\tau_1 = 3.3 > \tau'_{1_0}$, (a-b) Time series evolution of species, (c) phase-portrait in tuv -space, (d) existence of periodic solution.

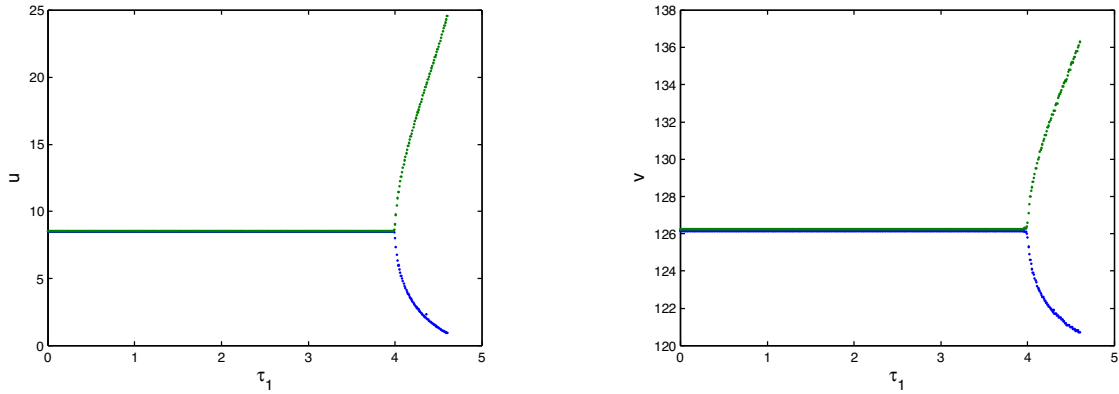


Fig. 3.11: Bifurcation diagram of the prey and predator population with respect to feedback time delay τ_1 when $\tau_2 = 0$.

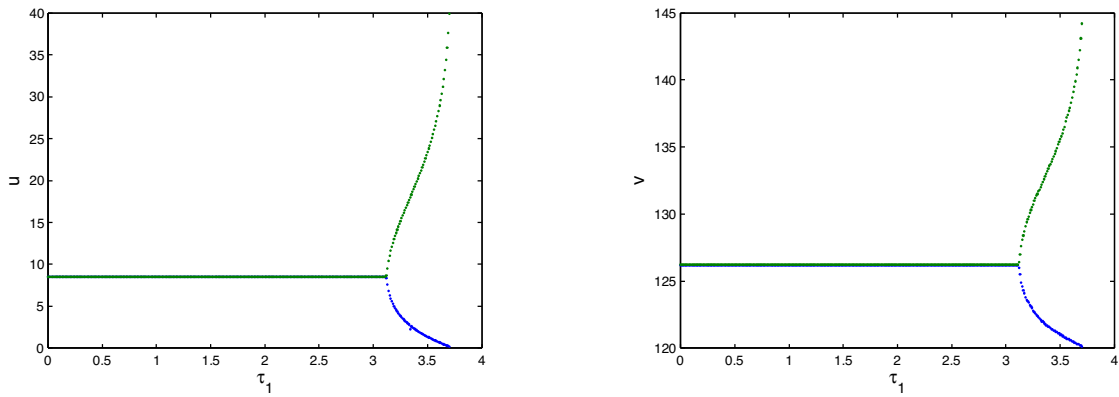


Fig. 3.12: Bifurcation diagram of the prey and predator population with respect to feedback time delay τ_1 when $\tau_2 = 0.5$.

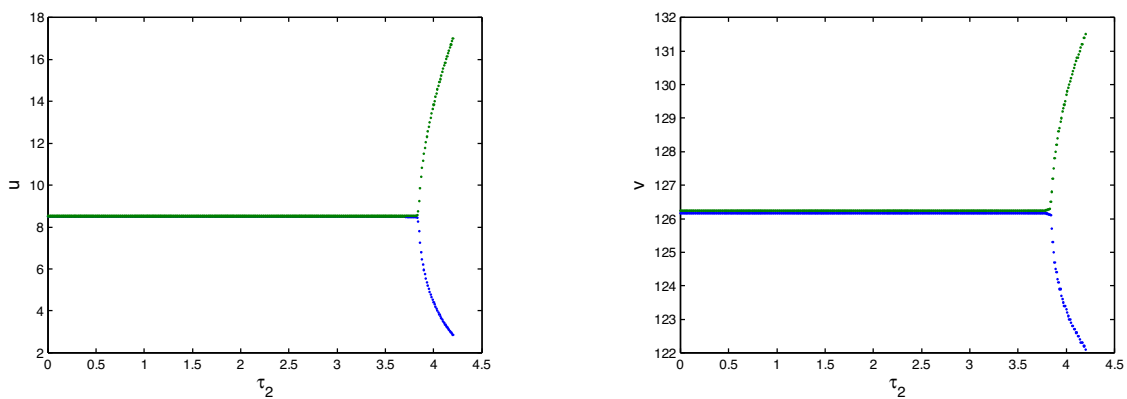


Fig. 3.13: Bifurcation diagram of the prey and predator population with respect to gestation time delay τ_2 when $\tau_1 = 0$.

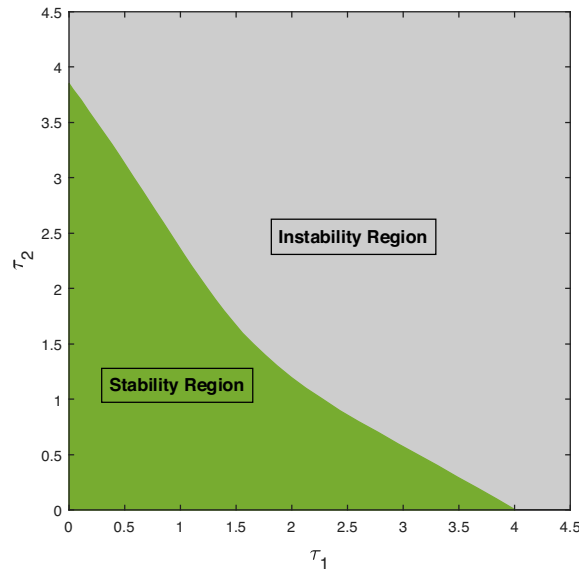


Fig. 3.14: Stability and instability regions of model (3.14) for the set of values of parameters in (3.29) with respect to τ_1 and τ_2 .

3.9 Conclusion

In this chapter, a non-linear mathematical model is proposed and analyzed to understand the dynamics of prey-predator relationship. Here prey and predator both grow logistically. The predator depends on prey for alternative food and that followed by Holling type-II functional response. The Holling type-I functional response is linear mass action and the predator's handling time for prey is assumed to be zero. It has been observed that predator often becomes saturated in nature. So, we considered Holling type-II functional response. In this response, the number of prey consumed per predator initially increases very fast as the density of prey increases, then it settle down at its steady state level. Again, the habitat complexity plays a major role in the distribution and structure of population. It can reduce the probability of capturing a prey by reducing the searching efficiency of predator. Predator's functional response is affected by the complexity of prey habitat [5, 8]. Refuge is also an important factor to maintain the balance between prey and predator. To control the over-exploitation of the biological resources, refuge is one of the tools suggested by ecologists as a control variable. Thus, we include these two factors in our model. Then we discussed the positivity, boundedness of the solution, persistence, existence of equilibrium points and their stability analysis. The model is confined within a compact set Ω in the non-negative quadrant, which indicates that none of the interacting species can grow exponentially or abruptly for a long period of time, which has a biological implications in the view of limited resources, species, the competition between species

and natural calamity. The model is persistence if the intrinsic growth rate of prey is greater than a threshold value. We also analyzed the local and global stability behavior of the model around the positive equilibrium point. We observed that, if intrinsic growth rate is greater than a threshold value, then the interior equilibrium point is globally stable.

Models with delay are much more realistic than ordinary differential equation. When predator consumes less prey, then prey population increases. Thus, the reduction in prey population also causes an increase in its growth rate. This phenomenon known as negative feedback delay, has been incorporated in our proposed model. Again, reproduction of predator after consuming prey takes some time i.e. time lag for gestation. Therefore, to make this prey-predator model biologically more realistic, we consider two delays: one for negative feedback delay of the prey and other for gestation delay for predator.

Now we considered modified delayed prey-predator model. Here we discussed boundedness, Hopf-bifurcation through local stability taking delay as a bifurcation parameter. We studied the existence of Hopf-bifurcation for possible combination of two delays. We discussed the sufficient conditions for the stability of interior equilibrium point and existence of Hopf-bifurcation when Case (1): $\tau_1 = 0, \tau_2 > 0$, Case (2): $\tau_1 > 0, \tau_2 = 0$, Case (3): $\tau_1 > 0, \tau_2 > 0$. As we know that the term 'delay' can destabilized the behavior of the system. The positive stable equilibrium undergoes Hopf-bifurcation with respect to one of the delays or both delays, when they cross their critical values. Thus, the quantitative level of abundance of system populations depends on the delay parameter. The direction and stability of Hopf-bifurcation are studied with the help of Centre manifold theorem and Normal form theory. The global stability behavior of the system with one delay and two delays have been discussed. We observe that if the intrinsic growth rate of the prey is greater than a threshold value, then the system with one delay and two delays are globally asymptotically stable.

Our numerical simulation results are based on some biologically feasible data to illustrate the analytical results. For a set of values of parameters, the system is stable, then introducing delay, the system remains stable if the delay is less than their critical value. If we increase delay, then the system becomes unstable (Hopf-bifurcation). We also proved that Hopf-bifurcated periodic solution is supercritical with stable periodic solution. Bifurcation diagram (Figs.3.11-3.13) with respect to delays τ_1 and τ_2 gives us information about the stable and unstable behavior of the system.

As we discussed earlier, habitat complexity and refuge are two important factors of this model. Fig. 3.3 and Fig. 3.4 show the behavior of the prey and predator with respect to c and m respectively. As c and m increase, the prey population increases but predator population decreases. The study of this chapter is important to control the over-exploitation of prey population from the predator population and to maintain their balance behavior. Here we have given importance on both the delays.