Chapter 7

Stability and Bifurcation of a Prey-predator System with Additional Food and Two Discrete Delays [1]

7.1 Introduction

Living organisms on the surface of the earth adopt only the way that boosts their survival possibilities so that they can pass their genes to the next generation. There are several fundamental instincts in ecological communities and, predation is one of them that constitutes the building blocks for multispecies food webs. Initially, Lotka [132] and Volterra [217] studied the model for prey-predator interaction and observed the uniform fluctuations in the time series of the system. On later the fluctuations were removed from the system by taking logistic growth of prey population [133, 56]. Many researchers have widely studied prey-predator interactions for the last century [85, 38, 219, 61, 170, 117, 220, 119]. They have considered several essential concepts over time that play a vital role in the dynamics of the system like functional response, time delay, harvesting and conservation policies of species, stage structure, fear induced by predators, etc. The idea of functional response was proposed by Holling [85]. It is defined as the consumption rate of prey by predators. Holling considered it nonlinear function of prey species that saturates at a level. Further, it was considered a function of prey and predator both by several authors [38, 50, 117, 136].

In last few decades, many authors have studied the qualitative dynamics of prey-predator systems in the presence of additional food resources for predators [197, 44, 179, 110, 189, 68]. Additional food is an important component for predators like coccinellid which shapes the life history of many predator species [179]. Ghosh *et al.* [68] investigated the impact of additional food for predator on the dynamics of prey-predator model with prey refuge and they observed that predator extinction possibility in high prey refuge may be removed by providing additional food to predators. Again, to study the role of additional food in an eco-epidemiological system,

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a model was proposed and studied by Sahoo [180]. The author found that the system becomes disease free in presence of suitable additional food provided to predator. Recently, a prey-predator model with harvesting and additional food is analyzed by Rani and Gakkhar [175] and they have shown some local and global bifurcation with respect to different parameters. To incorporate the additional food into the model, they modified the Holling type II functional response.

Delayed models exhibit much more realistic dynamics than non delayed models [11, 56]. In prey-predator system, the impact of consumed prey individuals into predator population does not appear immediately after the predation, there is some time lag that is gestation delay [208]. We incorporate the effect of time delay into the model with delay differential equations. A delay differential equation demonstrates much more complex character than ordinary differential equation. On the other hand predators do not consume the additional food as soon as it is provided. They take some time to consume and digest the food. Delayed models are widely studied by researchers [71, 195, 173, 144, 27, 64, 92, 145, 130, 117, 136]. A delayed prey and predator density dependent system is investigated by Li *et al.* [117]. The authors analyzed stability, Hopf-bifurcation and its qualititative properties by using Poincare normal form and the formulae given in Hassard *et al.* [80]. Sahoo and Poria [181] examined prey-predator model with effects of supplying additional food to predators in a gestation delay induced prey-predator system and habitat complexity. They have pointed out that Hopf-bifurcation occurs in the system when delay crosses a threshold value that strongly depends on quality and quantity of supplied additional food. The effect of additional food along with fear induced by predators and gestation delay is discussed by Mondal *et al.* [148]. There are several studies carried out with multiple delays [120, 231, 229, 111]. Li and Wei [120] have done stability and Hopf-bifurcation analysis of a prey-predator model with two maturation delays. Gakkhar and Singh [64] explored the complex dynamics of a prey-predator system with multiple delays. They established the presence of periodic orbits via Hopf-bifurcation with respect to both delays. Recently, Kundu and Maitra [111] have discussed about the dynamics of two prey and one predator system with cooperation among preys against predators incorporating three discrete delays. The authors have found that all delays are capable to destabilize the system.

To the best of our knowledge, an ecological model including (i) effect of additional food supplies to predators, (ii) dependency factor of supplied additional food, (iii) Holling type II functional response, (iv) gestation delay in predator have not been considered. Inspired by this, we establish three dimensional non delayed and delayed models in section 7.2. We analyze the dynamics of non delayed model and validate it via some numerical simulations in section 7.3. In section 7.4, we analyze the dynamics of delayed model through Hopf-bifurcation. Direction and stability of Hopf-bifurcation are carried out in section 7.5. Section 7.6 is devoted to the numerical simulations for delayed model.

7.2 Proposed Mathematical Model

We consider a habitat where two biological populations, prey population and predator population are surviving and interacting with each other. It is assumed that prey population grows logistically and the interaction between prey and predator follows Holling type II functional response. We assume that the density of the additional food supplied to the predators is directly proportional to the density of predators present in the habitat. Keeping these in view, the dynamics of the system can be governed by the following system of differential equations:

$$\begin{aligned}
\frac{dx}{dt} &= rx\Big(1-\frac{x}{K}\Big) - \frac{\alpha(1-A_0)xy}{1+ax},\\
\frac{dA}{dt} &= \lambda A_0 y - \beta A - \phi A y,\\
\frac{dy}{dt} &= \frac{c_1\alpha(1-A_0)xy}{1+ax} + c_2\phi A y - dy - ey^2,\\
x(0) &\geq 0,\ A(0) \geq 0,\ y(0) \geq 0.
\end{aligned} \tag{7.1}$$

In the above model $x(t)$, $y(t)$ are number of prey and predator individuals at time t and A is quantity of additional food provided to predators. A_0 is dependency factor of predators on provided additional food resources. If $A_0 = 1$, then predators depend completely on additional food and prey population grows logistically. If $A_0 = 0$, then predators depend only on the prey population and in such a case additional food is not required. λ is maximum supply rate of additional food resources.

In real situations, each organism needs an amount of time to reproduce their progeny. Due to this fact the increment in predators does not appear immediately after consuming prey. It is assumed that a predator individual takes τ_1 time for gestation. Therefore, it seems reasonable to incorporate a gestation delay in the system. Thus, the delay τ_1 is considered in the numeric response only. Again, it is assumed that the additional food is provided to predators with another delay τ_2. The generalized model involving these two discrete delays takes the following form

$$\begin{aligned}
\frac{dx}{dt} &= rx\Big(1-\frac{x}{K}\Big) - \frac{\alpha(1-A_0)xy}{1+ax},\\
\frac{dA}{dt} &= \lambda A_0 y - \beta A - \phi A y,\\
\frac{dy}{dt} &= \frac{c_1\alpha(1-A_0)x(t-\tau_1)y(t-\tau_1)}{1+ax(t-\tau_1)} + c_2\phi A(t-\tau_2)y(t-\tau_2) - dy - ey^2,
\end{aligned} \tag{7.2}$$

subject to the non negative conditions $x(s) = \phi_1(s) \geq 0$, $A(s) = \phi_2(s) \geq 0$, $y(s) = \phi_3(s) \geq 0$, $s \in [-\tau, 0]$, where $\tau = max\{\tau_1, \tau_2\}$ and $\phi_i(s) \in C([-\tau, 0] \to R_+)$, $(i = 1, 2, 3)$.

The biological meaning of all parameters and variables in above models is provided in Table 7.1.

Table 7.1: Variables and parameters used in models (7.1) & (7.2).

Variables/ Parameters	Biological meaning	Unit
x	Density of prey population	Number per unit area (tons)
A	Quantity of additional food	Number per unit area (tons)
y	Density of predator population	Number per unit area (tons)
r	Intrinsic growth rate of prey	Per day
K	Carrying capacity of the prey population	Number per unit area (tons)
α	Attack rate of predator on prey	Per day
A_0	Dependency factor of predators on provided additional food	Constant & $0 \leq A_0 \leq 1$
a	Handling time	Days
λ	Maximum supply rate of additional food	Per day
β	Natural depletion rate of additional food	Per day
ϕ	Consumption rate of additional food by predators	Per day
c_1	Conversion efficiency of y on x	Constant & $0 \leq c_1 \leq 1$
c_2	Conversion efficiency of y on A	Constant & $0 \leq c_2 \leq 1$
d	Mortality rate of predators	Per day
e	Intra-specific interference among predators	Per day
τ_1	Gestation delay of predators	Days
τ_2	Delay in supply of the additional food	Days

7.3 Dynamics of Non-delayed Model

First of all, we examine the boundedness and persistence of the system (7.1).

7.3.1 Boundedness and persistence of the solution

Theorem 7.3.1. *The set*

$$\Omega = \left\{(x,A,y) : 0 \leq x \leq K,\ 0 \leq c_1 x + c_2 A + y \leq \frac{1}{\delta}\left[2c_1 rK + \frac{c_2^2\lambda^2 A_0^2}{4e}\right]\right\}$$

is a positive invariant set for all the solutions of model (7.1), initiating in the interior of the positive octant, where $\delta = min\{r, \beta, d\}$.

Proof. The model system (7.1) can be written in the matrix form

$$\dot{X} = G(X),$$

where $X = (x_1, x_2, x_3)^T = (x, A, y)^T \in R^3$, and G(X) is given by

$$G(X) = \begin{bmatrix} G_1(X) \\ G_2(X) \\ G_3(X) \end{bmatrix} = \begin{bmatrix} rx\left(1 - \frac{x}{K}\right) - \frac{\alpha(1-A_0)xy}{1+ax} \\ \lambda A_0 y - \beta A - \phi A y \\ \frac{c_1\alpha(1-A_0)xy}{1+ax} + c_2\phi A y - dy - ey^2 \end{bmatrix}.$$

Since $G : R^3 \rightarrow R^3_+$ is locally Lipschitz-continuous in Ω and $X(0) = X_0 \in R^3_+$, the fundamental theorem of ordinary differential equation guarantees the local existence and uniqueness of the solution. Since $[G_i(X)]_{x_i(t)=0,\ X \in R^3_+} \geq 0$, it follows that $X(t) \geq 0$ for all $t \geq 0$. In fact, from the first equation of model (7.1), it can easily be seen that $\dot{x}|_{x=0} \geq 0$, $\dot{y}|_{y=0} \geq 0$ and hence $x(t) \geq 0$, $y(t) \geq 0$ for all $t \geq 0$. Secondly, $\dot{A}|_{A=0} = \lambda A_0 y \geq 0$ for all $t \geq 0$ (as $y(t) \geq 0$ for all $t \geq 0$.) and hence $A(t) \geq 0$ for all $t \geq 0$.

From the first equation of model (7.1), we can write

$$\frac{dx}{dt} \leq rx\left(1 - \frac{x}{K}\right),$$

which yields

$$\limsup_{t \to \infty} x(t) \leq K.$$

Now, suppose

$$W(t) = c_1 x(t) + c_2 A(t) + y(t),$$

then we have

$$\frac{dW(t)}{dt} = c_1\frac{dx}{dt} + c_2\frac{dA}{dt} + \frac{dy}{dt} \leq 2c_1 rK - c_1 rx - dy + c_2\lambda A_0 y - c_2\beta A - ey^2$$

$$\leq 2c_1 rK + \frac{c_2^2\lambda^2 A_0^2}{4e} - \left(\sqrt{e}y - \frac{c_2\lambda A_0}{2\sqrt{e}}\right)^2 - \delta W,$$

where $\delta = min\{r, \beta, d\}$.
Hence, it follows that

$$\limsup_{t \to \infty} W(t) \leq \frac{1}{\delta}\left(2c_1 rK + \frac{c_2^2\lambda^2 A_0^2}{4e}\right) =: y_s.$$

We also note that if $x(t) \geq K$ and $W(t) \geq \frac{1}{\delta}\left(2c_1 rK + \frac{c_2^2\lambda^2 A_0^2}{4e}\right)$, then $\frac{dx(t)}{dt} \leq 0, \frac{dW(t)}{dt} \leq 0$. This shows that all solutions of system (7.1) are bounded and remains in Ω for all $t > 0$ if $(x(0), A(0), y(0)) \in \Omega$. □

Theorem 7.3.2. *Let the following inequalities are satisfied:*

$$r > \alpha(1-A_0)y_s, \quad \frac{c_1\alpha(1-A_0)x_a}{1+ax_a} > d.$$

Then model system (7.1) is uniformly persistence, where, x_a is defined in the proof.

Proof. System (7.1) is said to be permanence or uniform persistence if there are positive constants M_1 and M_2 such that each positive solution $X(t) = (x(t), A(t), y(t))$ of the system with positive initial conditions satisfies

$$M_1 \leq \liminf_{t\to\infty} X(t) \leq \limsup_{t\to\infty} X(t) \leq M_2.$$

Keeping the above in view, if we define

$$M_2 = max\left\{K, \frac{y_s}{c_2}, y_s\right\},$$

then from Theorem 7.3.1, it follows that

$$\limsup_{t\to\infty} X(t) \leq M_2.$$

This also shows that for any sufficiently small $\varepsilon > 0$, there exists a $T > 0$ such that for all $t \geq T$, the following holds:

$$x(t) < K + \varepsilon, \quad A(t) < \frac{y_s}{c_2} + \varepsilon, \quad y(t) < y_s + \varepsilon.$$

Now from the first equation of model system (7.1), for all $t \geq T$, we can write

$$\frac{dx}{dt} \geq rx\left(1 - \frac{x}{K}\right) - \alpha(1-A_0)(y_s+\varepsilon)x$$

$$= [r - \alpha(1-A_0)(y_s+\varepsilon)]x - \frac{rx^2}{K}.$$

Hence, it follows that

$$\liminf_{t\to\infty} x(t) \geq \frac{K}{r}[r - \alpha(1-A_0)(y_s+\varepsilon)],$$

which is true for every $\varepsilon > 0$, thus

$$\liminf_{t\to\infty} x(t) \geq \frac{K}{r}[r - \alpha(1-A_0)y_s] =: x_a,$$

where $r > \alpha(1-A_0)y_s$.

Now from the third equation of model system (7.1), we obtain

$$\frac{dy}{dt} \geq \frac{c_1\alpha(1-A_0)(x_a+\varepsilon)y}{1+a(x_a+\varepsilon)} - dy - ey^2,$$

which implies

$$\liminf_{t\to\infty} y(t) \geq \frac{1}{e}\Big[\frac{c_1\alpha(1-A_0)(x_a+\varepsilon)}{1+a(x_a+\varepsilon)} - d\Big],$$

which is true for every $\varepsilon > 0$, thus

$$\liminf_{t\to\infty} y(t) \geq \frac{1}{e}\Big[\frac{c_1\alpha(1-A_0)x_a}{1+ax_a} - d\Big] =: y_a,$$

for persistence, we must have $\frac{c_1\alpha(1-A_0)x_a}{1+ax_a} > d$.

Second equation of model system (7.1) yields

$$\frac{dA}{dt} \geq \lambda A_0(y_a+\varepsilon) - \beta A - \phi(y_s+\varepsilon)A.$$

Hence,

$$\liminf_{t\to\infty} A(t) \geq \frac{\lambda A_0 y_a}{\beta+\phi y_s} =: A_a.$$

Taking $M_1 = min\{x_a, A_a, y_a\}$, the theorem follows. □

Remark 7.3.1. *Theorem 7.3.2 shows that threshold values for the persistence of the system are dependent on the parameter A_0.*

7.3.2 Equilibrium points and their stability behavior

System (7.1) has four equilibrium points, trivial equilibrium $E_0(0,0,0)$, axial equilibrium $E_1(K,0,0)$, prey free equilibrium $E_2(0,\tilde{A},\tilde{y})$ and interior equilibrium $E^*(x^*,A^*,y^*)$. E_0 and E_1 always exist.

- **Existence of $E_2(0,\tilde{A},\tilde{y})$:** The prey free equilibrium E_2 is positive solution of the following system:

$$\begin{aligned} &\lambda A_0 y - \beta A - \phi A y = 0,\\ &c_2\phi A - d - ey = 0. \end{aligned} \tag{7.3}$$

From the second equation of above system, we have

$$A = \frac{d+ey}{c_2\phi}.$$

Putting the value of A in the first equation of system (7.3), we get

$$\phi e y^2 + (\phi d + \beta e - c_2\phi\lambda A_0)y + \beta d = 0. \tag{7.4}$$

Above equation has two positive roots if

$$c_2\phi\lambda A_0 > \phi d + \beta e, \ (c_2\phi\lambda A_0 - \phi d - \beta e)^2 > 4\phi d\beta e. \tag{7.5}$$

System (7.1) has two prey free equilibrium under conditions given in (7.5): $\tilde{E}_2(0,\tilde{A},\tilde{y})$ and $\hat{E}_2(0,\hat{A},\hat{y})$. Again, If $c_2\phi\lambda A_0 < \phi d + \beta e$, then equation (7.4) does not have any positive root. Therefore, E_2 does not exist in this case.

- **Existence of interior equilibrium** $E^*(x^*,A^*,y^*)$**:** It may be seen that x^*, A^* and y^* are the positive solution of the following system of algebraic equations:

$$\begin{aligned} & r\left(1-\frac{x}{K}\right) - \frac{\alpha(1-A_0)y}{1+ax} = 0, \\ & \lambda A_0 y - \beta A - \phi A y = 0, \\ & \frac{c_1\alpha(1-A_0)x}{1+ax} + c_2\phi A - d - ey = 0. \end{aligned} \tag{7.6}$$

From the second equation of system (7.6), we have

$$A = \frac{\lambda A_0 y}{\beta + \phi y}.$$

Putting this into the first and third equation of system (7.6), we obtain the following system:

$$y = \frac{r}{\alpha(1-A_0)}\left(1-\frac{x}{K}\right)(1+ax), \tag{7.7}$$

$$\frac{c_1\alpha(1-A_0)x}{1+ax} + \frac{c_2\phi\lambda A_0 y}{\beta+\phi y} - d - ey = 0 \tag{7.8}$$

We note the following points from equation (7.7):

1. When $y = 0$, then $x = -\frac{1}{a} < 0$ or $x = K > 0$.
2. When $x = 0$ then $y = \frac{r}{\alpha(1-A_0)} > 0$.

3.
$$\frac{dy}{dx} = \frac{r}{\alpha(1-A_0)}\left[a - \frac{1}{K} - \frac{2ax}{K}\right].$$

It also can be noted that $\frac{dy}{dx} > 0$ if $-\frac{1}{a} < x < \frac{1}{2}\left(K - \frac{1}{a}\right)$ and $\frac{dy}{dx} < 0$ if $x > \frac{1}{2}\left(K - \frac{1}{a}\right)$.

4. $y_{max} = \frac{1}{4}\left(1 + \frac{1}{ak}\right)(1+ak)$ at $x = \frac{1}{2}\left(K - \frac{1}{a}\right)$.

Similarly, from equation (7.8), we note the following:

1. When $y = 0$, then $x = \frac{d}{c_1\alpha(1-A_0)-ad}$.

2.
$$\frac{dy}{dx} = \frac{\frac{c_1\alpha(1-A_0)}{(1+ax)^2}}{e - \frac{\beta c_2\phi\lambda A_0}{(\beta+\phi y)^2}}.$$

It can be noted that $\frac{dy}{dx} > 0$ if

$$e(\beta + \phi y_a)^2 > \beta c_2\phi\lambda A_0. \tag{7.9}$$

From above analysis we can conclude that system (7.6) has a unique positive solution (x^*, A^*, y^*) if, in addition to condition (7.9), the following holds:

$$0 < \frac{d}{c_1\alpha(1-A_0)-ad} < K. \tag{7.10}$$

Hence, we can state the following theorem.

Theorem 7.3.3. *The system (7.1) has a unique positive equilibrium $E^*(x^*, A^*, y^*)$ if (7.9) and (7.10) hold.*

Remark 7.3.2. *The number of positive equilibrium for the system (7.1) depends on values of parameters, which we have chosen. Several possibilities are depicted in Figure 7.1.*

The local behavior of a system in the vicinity of any existing equilibrium is very close to the behavior of its Jacobian system. So, we compute the Jacobian matrix to see the local behavior of the system around its equilibrium and we observe that

- The trivial equilibrium $E_0(0,0,0)$ is always a saddle point having stable manifold along the A and y-axes and unstable manifold along the x-axis.

- The axial equilibrium $E_1(K,0,0)$ is locally asymptotically stable iff $\frac{c_1\alpha(1-A_0)K}{1+aK} < d$. If $\frac{c_1\alpha(1-A_0)K}{1+aK} > d$, then E_1 is a saddle point having stable manifold along the x and A-axes and unstable manifold along the y-axis.

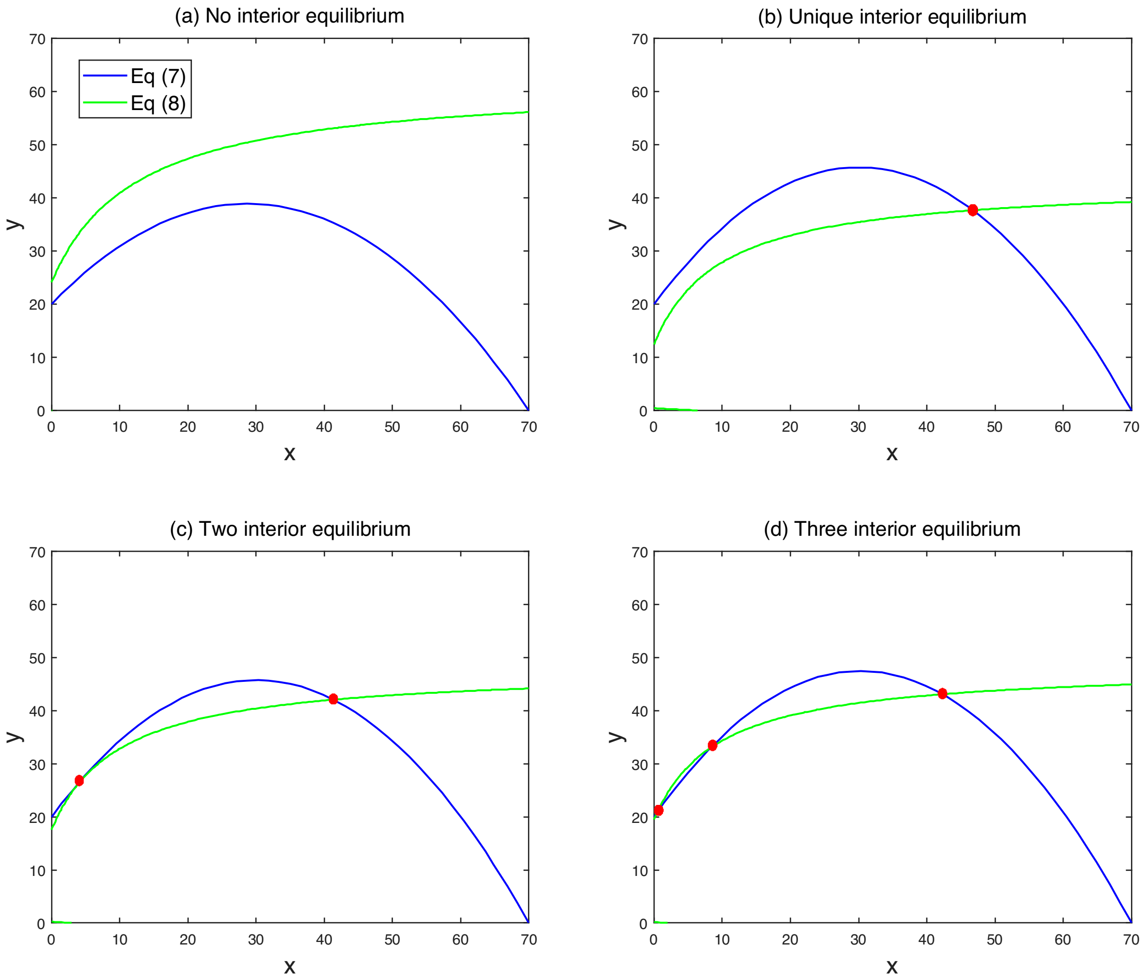

Fig. 7.1: Four possiblities of the prey and predator zero growth isoclines. (a) Interior equilibrium does not exist for the parametric values $a = 0.08, d = 0.01$, (b) Interior equilibrium exists uniquely for the values of parameters $a = 0.1, d = 0.235$, (c) Two interior equilibria for parameter values $a = 0.1, d = 0.137$, (d) Three interior equilibria for parameter values $a = 0.105, d = 0.1$. Rest of the parameters are same as that in (7.25).

- The Jacobian matrix evaluated at prey free equilibrium $E_2(0, \tilde{A}, \tilde{y})$ is given by

$$J|_{E_2} = \begin{bmatrix} r - \alpha(1 - A_0)\tilde{y} & 0 & 0 \\ 0 & -\frac{\lambda A_0 \tilde{y}}{\tilde{A}} & \lambda A_0 - \phi \tilde{A} \\ c_1 \alpha(1 - A_0)\tilde{y} & c_2 \phi \tilde{y} & -e\tilde{y} \end{bmatrix}.$$

Characteristic equation is given by

$$\left(\zeta - (r - \alpha(1 - A_0))\right)\left[\zeta^2 + (\lambda A_0 \tilde{y} + e\tilde{y})\zeta + (e\lambda A_0 \tilde{y}^2 - c_2 \phi \tilde{y}(\lambda A_0 - \phi \tilde{A}))\right] = 0. \quad (7.11)$$

The roots of equation (7.11) have negative real part if

$$r < \alpha(1-A_0)\tilde{y}, \;\; \lambda A_0 < \phi\tilde{A}. \tag{7.12}$$

Hence $\tilde{E_2}(0,\tilde{A},\tilde{y})$ is asymptotically stable under condition (7.12).
Equation (7.11) have at least one positive and one negative root if

$$e\lambda A_0\tilde{y} < c_2\phi(\lambda A_0 - \phi\tilde{A}) \tag{7.13}$$

Therefore, $\tilde{E_2}(0,\tilde{A},\tilde{y})$ is a saddle point under condition (7.13).

Remark 7.3.3. *By replacing $\tilde{A}$ by $\hat{A}$ and $\tilde{y}$ by $\hat{y}$, similar analysis holds good for the stability behavior of $\hat{E_2}(0,\hat{A},\hat{y})$.*

- In order to analyze the local stability of unique interior equilibrium $E^*(x^*,A^*,y^*)$, we evaluate the Jacobian matrix at E^* and it is given by

$$J|_{E^*} = \begin{bmatrix} -\frac{rx^*}{K} + \frac{\alpha(1-A_0)ax^*y^*}{(1+ax^*)^2} & 0 & -\frac{\alpha(1-A_0)x^*}{1+ax^*} \\ 0 & -\frac{\lambda A_0 y^*}{A^*} & \lambda A_0 - \phi A^* \\ \frac{c_1\alpha(1-A_0)y^*}{(1+ax^*)^2} & c_2\phi y^* & -ey^* \end{bmatrix}.$$

Characteristic equation corresponding to above matrix is given by

$$\Lambda^3 + A_1\Lambda^2 + A_2\Lambda + A_3 = 0, \tag{7.14}$$

where

$$A_1 = \frac{rx^*}{K} - \frac{\alpha(1-A_0)ax^*y^*}{(1+ax^*)^2} + \frac{\lambda A_0 y^*}{A^*} + ey^*,$$

$$A_2 = \frac{\lambda A_0 e y^{*2}}{A^*} - c_2\phi(\lambda A_0 - \phi A^*)y^* + \left(\frac{rx^*}{K} - \frac{\alpha(1-A_0)ax^*y^*}{(1+ax^*)^2}\right)(ey^*)$$
$$+ \frac{c_1\alpha^2(1-A_0)^2x^*y^*}{(1+ax^*)^3} + \left(\frac{rx^*}{K} - \frac{\alpha(1-A_0)ax^*y^*}{(1+ax^*)^2}\right)\left(\frac{\lambda A_0 y^*}{A^*}\right),$$

$$A_3 = -\left(-\frac{rx^*}{K} + \frac{\alpha(1-A_0)ax^*y^*}{(1+ax^*)^2}\right)\left[\frac{\lambda A_0 e y^{*2}}{A^*} - c_2\phi(\lambda A_0 - \phi A^*)y^*\right]$$
$$+ \frac{\alpha(1-A_0)x^*}{1+ax^*}\left[\frac{\lambda A_0 y^*}{A^*}\frac{c_1\alpha(1-A_0)y^*}{(1+ax^*)^2}\right].$$

Now using the Routh-Hurwitz criterion, all eigenvalues of $J|_{E^*}$ have negative real part iff

$$A_1 > 0, \;\; A_3 > 0, \;\; A_1A_2 > A_3. \tag{7.15}$$

Thus we can state the following theorem.

Theorem 7.3.4. *The system (7.1) is stable in the neighborhood of its positive equilibrium iff inequalities in (7.15) hold.*

It is also noted that inequalities in (7.15) hold if

$$\frac{r}{K} > \frac{\alpha(1-A_0)ay^*}{(1+ax^*)^2}, \quad \lambda A_0 < \phi A^*. \tag{7.16}$$

Infect, the above two conditions imply that $A_1 > 0$ and $A_3 > 0$. The third condition $A_1A_2 > A_3$ is also satisfied.

Remark 7.3.4. *The system (7.1) is stable around its positive equilibrium E^* if inequalities in (7.16) hold.*

In the following theorem we give a criterion for global asymptotic stability of interior equilibrium $E^*(x^*,A^*,y^*)$ of the system (7.1).

Theorem 7.3.5. *The interior equilibrium $E^*(x^*,A^*,y^*)$ of the system (7.1) is globally asymptotically stable under the following conditions:*

$$\frac{r}{K} > \frac{\alpha(1-A_0)ay^*}{1+ax^*}, \quad c_2\lambda^2A_0^2 < 4\beta eA^*. \tag{7.17}$$

Proof. We Choose a suitable Lyapunov function about E^* as

$$V(x,A,y) = \left(x - x^* - x^*\ln\frac{x}{x^*}\right) + \frac{\gamma_1}{2}(A-A^*)^2 + \gamma_2\left(y - y^* - y^*\ln\frac{y}{y^*}\right),$$

where γ_1 and γ_2 are positive constants, to be specified later. Now, differentiating V with respect to t along the solutions of system (7.1), we get

$$\frac{dV}{dt} = \left(\frac{x-x^*}{x}\right)\frac{dx}{dt} + \gamma_1(A-A^*)\frac{dA}{dt} + \gamma_2\left(\frac{y-y^*}{y}\right)\frac{dy}{dt}.$$

$$= -\left[\frac{r}{K} - \frac{\alpha(1-A_0)ay^*}{(1+ax)(1+ax^*)}\right](x-x^*)^2 - \gamma_1(\beta+\phi y)(A-A^*)^2 - \gamma_2 e(y-y^*)^2 + (\gamma_1\lambda A_0 - \gamma_1\phi A^* + \gamma_2 c_2\phi)(A-A^*)(y-y^*) + \left[-\frac{\alpha(1-A_0)}{1+ax} + \frac{\gamma_2 c_1\alpha(1-A_0)}{(1+ax)(1+ax^*)}\right](x-x^*)(y-y^*).$$

Choosing $\gamma_2 = \frac{1+ax^*}{c_1}$ and $\gamma_1 = \frac{\gamma_2 c_2}{A^*}$, we get

$$\frac{dV}{dt} = -\left[\frac{r}{K} - \frac{\alpha(1-A_0)ay^*}{(1+ax)(1+ax^*)}\right](x-x^*)^2 - \frac{(1+ax^*)c_2}{c_1 A^*}(\beta+\phi y)(A-A^*)^2$$
$$-\frac{1+ax^*}{c_1}e(y-y^*)^2 + \frac{(1+ax^*)c_2}{c_1 A^*}\lambda A_0(A-A^*)(y-y^*).$$

Applying Sylvester criterion, $\frac{dV}{dt}$ is negative definite if conditions in (7.17) hold. Hence E^* is globally stable under conditions in (7.17). □

7.3.3 Hopf-bifurcation and its properties

Hopf-bifurcation is a local phenomenon where a system's stability switches and a periodic solution arises around its equilibrium point by varying a parameter. In system (7.1), the parameter A_0 seems crucial, therefore we analyze the Hopf-bifurcation by taking A_0 as bifurcation parameter, then we have some $A_0 = A_0^*$. The necessary and sufficient conditions for occurrence Hopf-bifurcation at $A_0 = A_0^*$ are

(a) $A_1|_{A_0^*} > 0$, $A_3|_{A_0^*} > 0$,

(b) $f(A_0^*) \equiv (A_1A_2 - A_3)|_{A_0^*} = 0$,

(c) $Re\left[\frac{d\Lambda_i}{dA_0}\right]_{A_0=A_0^*}$ is either positive or negative, where Λ_i, $i = 1,2,3$ are roots of equation (7.14).

From $A_1A_2 - A_3 = 0$, we get an equation in A_0 and assume that it has at least one positive root A_0^*. Then for some $\varepsilon > 0$, there is an interval containing A_0^*, $(A_0^* - \varepsilon, A_0^* + \varepsilon)$ such that $A_0^* - \varepsilon > 0$ and $A_2 > 0$ for $A_0 \in (A_0^* - \varepsilon, A_0^* + \varepsilon)$. Thus, equation (7.14) cannot have any real positive root for $A_0 \in (A_0^* - \varepsilon, A_0^* + \varepsilon)$.

Therefore, at $A_0 = A_0^*$, equation (7.14) becomes

$$(\Lambda + A_1)(\Lambda^2 + A_2) = 0,$$

this gives us three roots

$$\Lambda_{1,2} = \pm i\rho, \ \Lambda_3 = \mu,$$

where $\rho = \sqrt{A_2}$ and $\mu = -A_1$.

For $A_0 \in (A_0^* - \varepsilon, A_0^* + \varepsilon)$, roots can be taken as

$$\Lambda_{1,2} = k_1(A_0) \pm ik_2(A_0), \ \Lambda_3 = -A_1(A_0).$$

Now, we have to verify the transversality condition. Differentiating equation (7.14) with respect to the bifurcation parameter A_0, we obtain

$$\left[\frac{d\Lambda}{dA_0}\right]_{A_0=A_0^*} = -\left[\frac{\dot{A_1}\Lambda^2+\dot{A_2}\Lambda+\dot{A_3}}{3\Lambda^2+2A_1\Lambda+A_2}\right]_{\Lambda=i\sqrt{A_2}}$$

$$= -\frac{\frac{dR}{dA_0}}{2(A_1^2+A_2)} + i\left[\frac{\sqrt{A_2}\dot{A_2}}{2A_2} - \frac{A_1\sqrt{A_2}\frac{dR}{dA_0}}{2A_2(A_1^2+A_2)}\right],$$

where $R = A_1A_2 - A_3$ and $\dot{A_i},\ i = 1,2,3$ denote the derivative of A_i with respect to time. Thus

$$Re\left[\frac{d\Lambda_i}{dA_0}\right]_{A_0=A_0^*} = -\frac{\frac{dR}{dA_0}}{2(A_1^2+A_2)}.$$

Thus, we can state the following theorem.

Theorem 7.3.6. *The system undergoes Hopf-bifurcation near interior equilibrium $E^*(x^*,A^*,y^*)$ under the necessary and sufficient conditions (a),(b) and (c). Critical value of bifurcation parameter A_0 is given by the equation $f(A_0^*) = 0$.*

In order to see the stability and direction of Hopf-bifurcation, we use center manifold theorem [80] and some concepts used in [146]. Now, consider the following transformation

$$x_1 = x - x^*,\ x_2 = A - A^*,\ x_3 = y - y^*.$$

Using this transformation, system (7.1) takes the following form

$$\dot{X} = M^*X + G(X), \tag{7.18}$$

where $X = (x_1,x_2,x_3)^T$,

$$M^* = \begin{bmatrix} -\frac{rx^*}{K} + \frac{\alpha(1-A_0)ax^*y^*}{(1+ax^*)^2} & 0 & -\frac{\alpha(1-A_0)x^*}{1+ax^*} \\ 0 & -\frac{\lambda A_0 y^*}{A^*} & \lambda A_0 - \phi A^* \\ \frac{c_1\alpha(1-A_0)y^*}{(1+ax^*)^2} & c_2\phi y^* & -ey^* \end{bmatrix},\ G = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} -\frac{rx_1^2}{K} - \frac{\alpha(1-A_0)x_1x_3}{1+ax_1} \\ -\phi x_2x_3 \\ \frac{c_1\alpha(1-A_0)x_1x_3}{1+ax_1} + c_2\phi x_2x_3 - ex_3^2 \end{bmatrix}.$$

Let v_1 and v_2 be the eigenvectors corresponding to eigenvalues $i\rho$ and μ of E^* at $A_0 = A_0^*$. Then v_1 and v_2 are given by

$$v_1 = \begin{bmatrix} -\rho^2 + i\left(\frac{\lambda A_0 y^*}{A^*} + ey^*\right)\rho + \frac{\lambda A_0 y^{*2} e}{A^*} - c_2\phi y^*(\lambda A_0 - \phi A^*) \\ \frac{c_1\alpha(1-A_0)y^*}{(1+ax^*)^2}(\lambda A_0 - \phi A^*) \\ \frac{c_1\alpha(1-A_0)y^*}{(1+ax^*)^2}\left(\frac{\lambda A_0 y^*}{A^*} - i\rho\right) \end{bmatrix},$$

and

$$v_2 = \begin{bmatrix} \mu^2 + i\left(\frac{\lambda A_0 y^*}{A^*} + ey^*\right)\mu + \frac{\lambda A_0 y^{*2} e}{A^*} - c_2\phi y^*(\lambda A_0 - \phi A^*) \\ \frac{c_1\alpha(1-A_0)y^*}{(1+ax^*)^2}(\lambda A_0 - \phi A^*) \\ \frac{c_1\alpha(1-A_0)y^*}{(1+ax^*)^2}\left(\frac{\lambda A_0 y^*}{A^*} - \mu\right) \end{bmatrix} = \begin{bmatrix} p_{13} \\ p_{23} \\ p_{33} \end{bmatrix} (say).$$

Let

$$H = (Im(v_1), Re(v_1), v_2) = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix},$$

where

$$p_{11} = \left(\frac{\lambda A_0 y^*}{A^*} + ey^*\right)\rho, \;\; p_{12} = -\rho^2 + \frac{\lambda A_0 y^{*2} e}{A^*} - c_2\phi y^*(\lambda A_0 - \phi A^*), \;\; p_{21} = 0,$$

$$p_{22} = \frac{c_1\alpha(1-A_0)y^*}{(1+ax^*)^2}(\lambda A_0 - \phi A^*), \;\; p_{31} = -\frac{c_1\alpha(1-A_0)y^*\rho}{(1+ax^*)^2}, \;\; p_{32} = \frac{c_1\alpha(1-A_0)y^*}{(1+ax^*)^2}\frac{\lambda A_0 y^*}{A^*}.$$

Then $H^{-1} = \frac{1}{\Delta}\begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}$, where

$$\Delta = p_{11}(p_{22}p_{33} - p_{23}p_{32}) + p_{12}(p_{23}p_{31} - p_{21}p_{33}) + p_{13}(p_{21}p_{32} - p_{22}p_{31}) \neq 0,$$

$$q_{11} = p_{22}p_{33} - p_{23}p_{32}, \;\; q_{12} = p_{21}p_{33} - p_{23}p_{31}, \;\; q_{13} = p_{21}p_{32} - p_{22}p_{31},$$

$$q_{21} = p_{12}p_{33} - p_{13}p_{32}, \;\; q_{22} = p_{11}p_{33} - p_{13}p_{31}, \;\; q_{23} = p_{11}p_{32} - p_{31}p_{12},$$

$$q_{31} = p_{12}p_{23} - p_{13}p_{22}, \;\; q_{32} = p_{11}p_{23} - p_{13}p_{21}, \;\; q_{33} = p_{11}p_{22} - p_{12}p_{21}.$$

Now let $X = HY$ or $Y = H^{-1}X$, where $Y = (y_2, y_2, y_3)^T$. Using this transformation, system (7.18) can be written as

$$\dot{Y} = (H^{-1}M^*H)Y + F(Y), \tag{7.19}$$

where

$$F(Y) = H^{-1}G(HY) = \begin{bmatrix} f^1 \\ f^2 \\ f^3 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} q_{11}m_1 + q_{12}m_2 + q_{13}m_3 \\ q_{21}m_1 + q_{22}m_2 + q_{23}m_3 \\ q_{31}m_1 + q_{32}m_2 + q_{33}m_3 \end{bmatrix}, \;\; H^{-1}M^*H = \begin{bmatrix} 0 & -\rho & 0 \\ \rho & 0 & 0 \\ 0 & 0 & \mu \end{bmatrix}.$$

So, we can write system (7.19) as

$$\begin{bmatrix} \dot{y_1} \\ \dot{y_2} \\ \dot{y_3} \end{bmatrix} = \begin{bmatrix} 0 & -\rho & 0 \\ \rho & 0 & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} f^1 \\ f^2 \\ f^3 \end{bmatrix}. \tag{7.20}$$

Thus, system (7.20) takes the following form

$$\begin{aligned} \dot{U} &= BU + f(U,V), \\ \dot{V} &= CU + g(U,V), \end{aligned} \tag{7.21}$$

where $U = (y_1, y_2)^T,\ V = (y_3),\ B = \begin{bmatrix} 0 & -\rho \\ \rho & 0 \end{bmatrix},\ C = (\mu),\ f = (f^1, f^2)$ and $g = (f^3)$. The eigenvalues of B and C may have zero real part and negative real parts, respectively. f, g vanish along with their first partial derivative at the origin.

Since the center manifold is tangent to $W^C(y = 0)$ we can represent it as a graph

$$W^C = \{(U,V) : V = h(U)\} : \quad h(0) = h'(0) = 0,$$

where $h : U \to R^2$ is defined on some vicinity $U \subset R^2$ of the origin [24, 113].

We consider the projection of vector field on $V = h(U)$ onto W^C :

$$\dot{U} = BU + f(U,V) = BU + f(U, h(U)). \tag{7.22}$$

Now we state the following theorem to approximate the center manifold.

Theorem 7.3.7. *Let Φ be a C^1 mapping of a neighborhood of the origin in R^2 into R with $\Phi(0) = 0$ and $\Phi'(0) = 0$. If for some $q > 1, (N\Phi)(U) = o(|U|^q)$ as $U \to 0$, then $h(U) = \Phi(U) + o(|U|^q)$ as $U \to 0$, where*

$$(N\Phi)(U) = \Phi'(U)[BU + f(U, \Phi(U))] - C\Phi(U) - g(U, \Phi(U)).$$

In order to approximate $h(U)$, we consider

$$y_3 = h(y_1, y_2) = \frac{1}{2}(b_{11}y_1^2 + 2b_{12}y_1y_2 + b_{22}y_2^2) + \text{h.o.t.}, \tag{7.23}$$

where h.o.t. stands for high order terms. Using (7.23), we get from (7.22)

$$\dot{y_3} = \frac{\partial h}{\partial y_1}\frac{dy_1}{dt} + \frac{\partial h}{\partial y_2}\frac{dy_2}{dt} = \mu y_3 + f^3.$$

After simplification, we get

$$\begin{aligned}\Big(\rho b_{12}-\frac{\mu}{2}b_{11}\Big)y_1^2+\Big(-\rho b_{12}-\frac{\mu}{2}b_{22}\Big)y_2^2+\{\rho(-b_{11}+b_{22})-\mu b_{12}\}y_1y_2 \\ =Q_1y_1^2+Q_2y_1y_2+Q_3y_2^2+\text{h.o.t.}\,,\end{aligned}\tag{7.24}$$

where

$$\begin{aligned}Q_1=\frac{1}{\Delta}\Big[&q_{31}\Big(-\frac{r}{K}p_{11}^2-\alpha(1-A_0)p_{11}p_{13}\Big)+q_{32}(-\phi p_{21}p_{31})+q_{33}(c_1\alpha(1-A_0)p_{11}p_{31}\\&+c_2\phi p_{21}p_{31}-ep_{31}^2)\Big],\end{aligned}$$

$$\begin{aligned}Q_2=\frac{1}{\Delta}\Big[&q_{31}\Big(-\frac{2r}{K}p_{11}p_{12}-\alpha(1-A_0)(p_{11}p_{32}+p_{12}p_{31}\Big)+q_{32}\big(-\phi(p_{21}p_{32}+p_{31}p_{22})\big)\\&+q_{33}(c_1\alpha(1-A_0)(p_{11}p_{32}+p_{12}p_{31})+c_2\phi(p_{21}p_{32}+p_{22}p_{31})-ep_{31}p_{32})\Big],\end{aligned}$$

$$\begin{aligned}Q_3=\frac{1}{\Delta}\Big[&q_{31}\Big(-\frac{r}{K}p_{12}^2-\alpha(1-A_0)p_{12}p_{32}\Big)+q_{32}(-\phi p_{32}p_{22})+q_{33}(c_1\alpha(1-A_0)p_{12}p_{32}\\&+c_2\phi p_{22}p_{32}-ep_{32}^2)\Big].\end{aligned}$$

Equating both the sides of equation (7.24), we get

$$\begin{aligned}&\rho b_{12}-\frac{\mu}{2}b_{11}=Q_1,\\&\rho(-b_{11}+b_{22})-\mu b_{12}=Q_2,\\&-\rho b_{12}-\frac{\mu}{2}b_{22}=Q_3.\end{aligned}$$

Using Crammer's rule,

$$b_{11}=-\frac{\rho^2(Q_1+Q_3)+\frac{\mu}{2}(\rho Q_2+\mu Q_1)}{\frac{\mu^3}{4}+\mu\rho^2},$$

$$b_{12}=-\frac{\frac{\mu^2}{4}Q_2+\frac{\rho\mu}{2}(Q_3-Q_1)}{\frac{\mu^3}{4}+\mu\rho^2},$$

$$b_{22}=-\frac{\frac{\mu^2}{2}Q_3-\frac{\mu}{2}\rho Q_2+\rho^2(Q_1+Q_3)}{\frac{\mu^3}{4}+\mu\rho^2}.$$

We can find the behavior of the solution of system (7.21) from the following theorem.

Theorem 7.3.8. *If the zero solution of (7.22) is stable (asymptotically stable/ unstable), then the zero solution of (7.21) is also stable (asymptotically stable/ unstable).*

Now from equation (7.22), we have

$$\begin{bmatrix} \dot{y_1} \\ \dot{y_2} \end{bmatrix} = \begin{bmatrix} 0 & -\rho \\ \rho & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} f^1 \\ f^2 \end{bmatrix},$$

where

$$f^1 = \frac{1}{\Delta}(q_{11}m_1 + q_{12}m_2 + q_{13}m_3), \;\; f^2 = \frac{1}{\Delta}(q_{21}m_1 + q_{22}m_2 + q_{23}m_3),$$

$$\begin{aligned} m_1 =& \Big[-\frac{r}{K}p_{11}^2 - \alpha(1-A_0)p_{11}p_{31}\Big]y_1^2 + \Big[-\frac{r}{K}p_{12}^2 - \alpha(1-A_0)p_{12}p_{31}\Big]y_2^2 \\ &+ \Big[-\frac{2r}{K}p_{11}p_{12} - \alpha(1-A_0)(p_{11}p_{32}+p_{12}p_{31})\Big]y_1y_2 + \text{h.o.t.}\,, \\ m_2 =& -\phi p_{21}p_{31}y_1^2 - \phi p_{22}p_{32}y_2^2 - \phi(p_{21}p_{32}+p_{22}p_{31})y_1y_2 + \text{h.o.t.}\,, \\ m_3 =& \big[c_1\alpha(1-A_0)p_{11}p_{31} + c_2\phi p_{21}p_{31} - ep_{31}^2\big]y_1^2 + \big[c_1\alpha(1-A_0)p_{12}p_{32} + c_2\phi p_{22}p_{32} - ep_{32}^2\big]y_2^2 \\ &+ \big[c_1\alpha(1-A_0)(p_{11}p_{32}+p_{12}p_{31}) + c_2\phi(p_{21}p_{32}+p_{22}p_{31}) - 2ep_{31}p_{32}\big]y_1y_2 + \text{h.o.t.} \end{aligned}$$

Let $f_{ij}^k = \left[\frac{\partial f^k}{\partial y_i \partial y_j}\right]_{(0,0)}$ and $f_{ijl}^k = \left[\frac{\partial f^k}{\partial y_i \partial y_j \partial y_l}\right]_{(0,0)}$. Therefore,

$$\begin{aligned} f_{11}^1 =& \frac{2}{\Delta}\Big[q_{11}\Big[-\frac{r}{K}p_{11}^2 - \alpha(1-A_0)p_{11}p_{31}\Big] + q_{12}[-\phi p_{21}p_{31}] + q_{13}\big[c_1\alpha(1-A_0)p_{11}p_{31} \\ &+ c_2\phi p_{21}p_{31} - ep_{31}^2\big]\Big], \end{aligned}$$

$$\begin{aligned} f_{11}^2 =& \frac{2}{\Delta}\Big[q_{21}\Big[-\frac{r}{K}p_{11}^2 - \alpha(1-A_0)p_{11}p_{31}\Big] + q_{22}[-\phi p_{21}p_{31}] + q_{23}\big[c_1\alpha(1-A_0)p_{11}p_{31} \\ &+ c_2\phi p_{21}p_{31} - ep_{31}^2\big]\Big], \end{aligned}$$

$$\begin{aligned} f_{22}^1 =& \frac{2}{\Delta}\Big[q_{11}\Big[-\frac{r}{K}p_{12}^2 - \alpha(1-A_0)p_{12}p_{31}\Big] + q_{12}[-\phi p_{22}p_{32}] + q_{13}\big[c_1\alpha(1-A_0)p_{12}p_{32} \\ &+ c_2\phi p_{22}p_{32} - ep_{32}^2\big]\Big], \end{aligned}$$

$$\begin{aligned} f_{22}^2 =& \frac{2}{\Delta}\Big[q_{21}\Big[-\frac{r}{K}p_{12}^2 - \alpha(1-A_0)p_{12}p_{31}\Big] + q_{22}[-\phi p_{22}p_{32}] + q_{23}\big[c_1\alpha(1-A_0)p_{12}p_{32} \\ &+ c_2\phi p_{22}p_{32} - ep_{32}^2\big]\Big], \end{aligned}$$

$$f^1_{12} = \frac{1}{\Delta}\bigg[q_{11}\Big[-\frac{2r}{K}p_{11}p_{12} - \alpha(1-A_0)(p_{11}p_{32}+p_{12}p_{31})\Big] + q_{12}[-\phi(p_{21}p_{32}+p_{22}p_{31})]$$
$$+q_{13}\big[c_1\alpha(1-A_0)(p_{11}p_{32}+p_{12}p_{31}) + c_2\phi(p_{21}p_{32}+p_{22}p_{31}) - 2ep_{31}p_{32}\big]\bigg],$$

$$f^2_{12} = \frac{1}{\Delta}\bigg[q_{21}\Big[-\frac{2r}{K}p_{11}p_{12} - \alpha(1-A_0)(p_{11}p_{32}+p_{12}p_{31})\Big] + q_{22}[-\phi(p_{21}p_{32}+p_{22}p_{31})]$$
$$+q_{23}\big[c_1\alpha(1-A_0)(p_{11}p_{32}+p_{12}p_{31}) + c_2\phi(p_{21}p_{32}+p_{22}p_{31}) - 2ep_{31}p_{32}\big]\bigg],$$

$$f^1_{111} = \frac{6}{\Delta}\bigg[q_{11}\Big(-\frac{r}{K}p_{11}p_{13}b_{11} - \alpha(1-A_0) + \Big(-ap_{11}^2p_{31} + \frac{p_{11}p_{33}b_{11}+p_{13}p_{31}b_{11}}{2}\Big)\Big)$$
$$+q_{12}\Big(-\frac{\phi}{2}\Big(p_{21}p_{33}b_{11}+p_{23}p_{31}b_{11}\Big)\Big) + q_{13}\Big(c_1\alpha(1-A_0)\Big(-ap_{11}^2p_{31}$$
$$+\frac{p_{11}p_{33}b_{11}+p_{31}p_{31}b_{11}}{2}\Big) + \frac{c_2\phi}{2}(p_{21}p_{33}b_{11}+p_{31}p_{23}b_{11} - ep_{31}p_{33}b_{11})\Big)\bigg],$$

$$f^2_{222} = \frac{6}{\Delta}\bigg[q_{21}\Big(-\frac{r}{K}p_{12}p_{13}b_{22} - \alpha(1-A_0) + \Big(-ap_{12}^2p_{32} + \frac{p_{12}p_{33}b_{22}+p_{13}p_{32}b_{22}}{2}\Big)\Big)$$
$$+q_{22}\Big(-\frac{\phi}{2}\Big(p_{22}p_{33}b_{22}+p_{23}p_{32}b_{22}\Big)\Big) + q_{23}\Big(c_1\alpha(1-A_0)\Big(-ap_{12}^2p_{32}$$
$$+\frac{p_{12}p_{33}b_{22}+p_{32}p_{13}b_{22}}{2}\Big) + \frac{c_2\phi}{2}(p_{22}p_{33}b_{22}+p_{32}p_{23}b_{22} - ep_{32}p_{33}b_{22})\Big)\bigg],$$

$$f^1_{122} = \frac{1}{\Delta}\bigg[q_{11}\Big(-\frac{r}{K}\big(2p_{11}p_{13}b_{22}+4p_{12}p_{13}b_{12}\big) - \alpha(1-A_0)\big(-4ap_{11}p_{32}b_{12} - 2ap_{12}^2p_{31}$$
$$+p_{11}p_{33}b_{22}+2p_{12}p_{33}b_{12}+p_{31}p_{13}b_{22}+2p_{32}p_{13}b_{12}\big)\Big) + q_{12}\Big(-\phi\big(p_{21}p_{33}b_{22}$$
$$+2p_{22}p_{33}b_{12}+p_{31}p_{23}b_{22}+2p_{32}p_{23}b_{12}\big)\Big) + q_{13}\Big(c_1\alpha(1-A_0)\big(-4ap_{11}p_{32}b_{12}$$
$$-2ap_{12}^2p_{31}+p_{11}p_{33}b_{22}+2p_{12}p_{33}b_{12}+p_{31}p_{13}b_{22}+2p_{32}p_{13}b_{12}\big) + c_2\phi\big(p_{21}p_{33}b_{22}$$
$$+2p_{22}p_{33}b_{12}+p_{31}p_{23}b_{22}+2p_{32}p_{23}b_{12}\big) - e\big(2p_{31}p_{33}b_{22}+4p_{32}p_{33}b_{12}\big)\Big)\bigg],$$

$$f^2_{112} = \frac{1}{\Delta}\bigg[q_{21}\Big(-\frac{r}{K}\big(2p_{12}p_{13}b_{11}+4p_{11}p_{13}b_{12}\big) - \alpha(1-A_0)\big(-4ap_{11}p_{31}b_{12} - 2ap_{11}^2p_{32}$$
$$+p_{12}p_{33}b_{11}+2p_{11}p_{33}b_{12}+p_{32}p_{13}b_{11}+2p_{31}p_{13}b_{12}\big)\Big) + q_{22}\Big(-\phi\big(p_{22}p_{33}b_{11}$$
$$+2p_{21}p_{33}b_{12}+p_{32}p_{23}b_{12}+2p_{31}p_{23}b_{12}\big)\Big) + q_{23}\Big(c_1\alpha(1-A_0)\big(-4ap_{11}p_{31}b_{12}$$
$$-2ap_{11}^2p_{32}+p_{11}p_{33}b_{12}+2p_{12}p_{33}b_{11}+p_{32}p_{13}b_{11}+2p_{31}p_{13}b_{12}\big) + c_2\phi\big(p_{22}p_{33}b_{11}$$
$$+2p_{21}p_{33}b_{12}+p_{32}p_{23}b_{11}+2p_{31}p_{23}b_{12}\big) - e\big(2p_{32}p_{33}b_{11}+4p_{31}p_{33}b_{12}\big)\Big)\bigg].$$

We determine the direction and stability of bifurcation periodic orbit of the system (7.21) by the following formula [125]

$$\nu\left[\frac{dK}{dA_0}\right]_{A_0=A_0^*}=\frac{1}{\rho}\left[f_{12}^1(f_{11}^1+f_{22}^1)-f_{12}^2(f_{11}^2+f_{22}^2)-f_{11}^1f_{11}^2+f_{22}^1f_{22}^2-(f_{111}^1+f_{112}^2+f_{122}^1+f_{222}^2)\right].$$

In above expression, if $\nu>0(<0)$, then the Hopf-bifurcation is supercritical (subcritical) and bifurcation periodic solution exists for $A_0=A_0^*$. The bifurcating periodic solution is stable (unstable) if

$$\nu\left[\frac{dK}{dA_0}\right]_{A_0=A_0^*}>0(<0).$$

The bifurcating direction of periodic solution of the system (7.1) is same as the system (7.21).

7.3.4 Numerical simulation

To validate our theoretical findings of model (7.1), we perform some numerical simulations using MATLAB R2018b. We have chosen the following dataset

$$\begin{aligned}&r=3,\ K=70,\ \alpha=0.3,\ a=0.07,\ A_0=0.5,\ \lambda=2,\\&\beta=0.32,\ \phi=0.7,\ c_1=0.4,\ c_2=0.5,\ d=0.3,\ e=0.02.\end{aligned}\tag{7.25}$$

For the above set of parameters, condition for existence of prey free equilibrium (7.5) and conditions for existence and uniqueness of interior equilibrium (7.9) and (7.10) are satisfied. Therefore, the system (7.1) has five equilibrium points (refer to Table 7.2).

Table 7.2: Existing equilibria and their stability nature.

Equilibrium Point	Eigenvalues	Stability nature
$E_0(0,0,0)$	3,-0.32,-0.3	Saddle point
$E_1(70,0,0)$	-3,-0.32,0.4113	Saddle point
$\tilde{E}_2(0,0.9019,0.7828)$	0.0898,-0.9734,2.8826	Saddle point
$\hat{E}_2(0,1.3577,8.7601)$	-0.1511,-6.4762,1.6860	Saddle point
$E^*(18.0971,1.4094,33.6153)$	$-0.0990\pm0.3860i,-23.7585$	Locally asymptotically stable

The eigenvalues of the Jacobian matrix at E_0 and E_1 are $(3,-0.32,-0.3)$ and $(-3,-0.32,0.4113)$, respectively. Therefore, E_0 and E_1 both are saddle points. Similarly $\tilde{E}_2$ and $\hat{E}_2$ are also saddle points. Again, all the inequalities in (7.15) are satisfied. So, according to Theorem 7.3.4, the interior equilibrium E^* is locally asymptotically stable. The stability of system in the vicinity of the positive equilibrium E^* is illustrated by Fig. 7.2. In Fig. 7.2(a), time evolution of species is shown and it is noted that they converge to their equilibrium levels after some oscillations. In Fig. 7.2(b), phase diagram is drawn in $xAy-$space which shows the asymptotic stability behavior of positive equilibrium E^*.

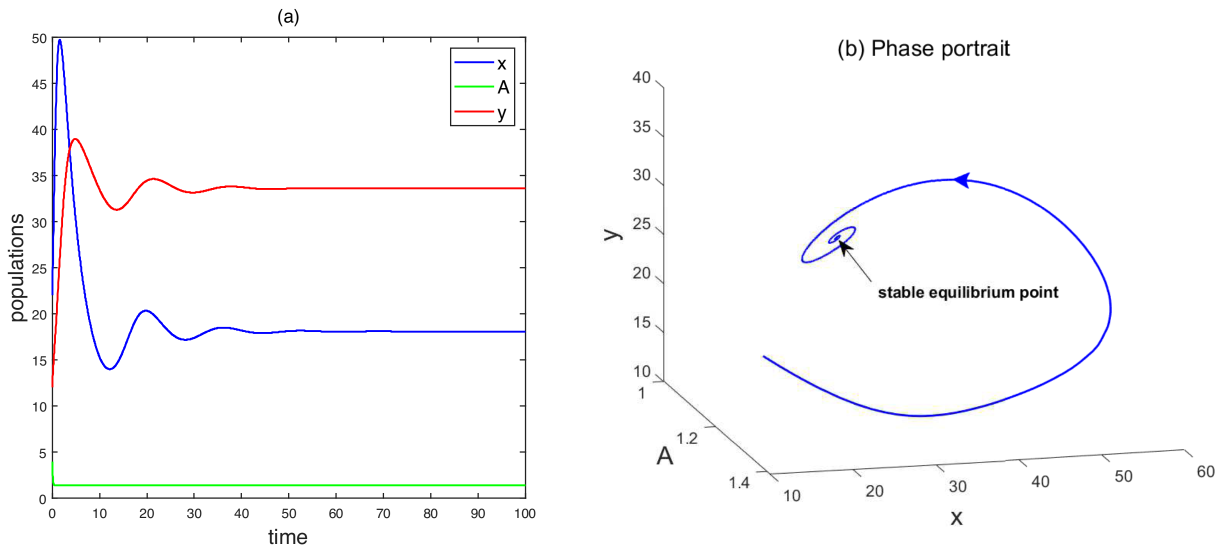

Fig. 7.2: Time series evolution (a) and phase portrait (b) of species for the set of parameters chosen in (7.25). Positive equilibrium E^* is locally asympeoeically stable.

In this chapter, we found that predators dependency factor A_0 on additional food plays an important role in the dynamics of the system. If it is less than a threshold value then it can be the cause of destabilizing the system. The threshold value can be calculated by solving $f(A_0^*) = 0$ (Theorem 7.3.6). By our computer simulation we obtain it as $A_0^* = 0.482$. All the conditions of Theorem 7.3.6 are satisfied, so the system undergoes a Hopf-bifurcation at $A_0^* = 0.482$. If we keep the value of parameter A_0 below its threshold value, then the system (7.1) always remains unstable. The instable behavior of solutions and presence of stable limit cycle at $A_0 = 0.45 < A_0^* = 0.482$ is shown in Fig. 7.3. In Fig. 7.4, we draw the bifurcation diagram with respect to parameter A_0 for both prey and predator species. From the figure, it is noted that the periodic solution present in the system when $A_0 \in [0, A_0^*]$ and oscillations can be removed from the system by increasing the parameter A_0 beyond A_0^*.

In the model (7.1), consumption rate of additional food ϕ is also a vital parameter. We have noted that if system is stable for parameter A_0 ($A_0 \in [0.482, 1]$) then it is stable for all range of parameter ϕ. But if $A_0 \in [0.2361, 0.482)$ then system undergoes a Hopf-bifurcation with respect to parameter ϕ. In Fig. 7.5, we have shown the bifurcation diagram when $A_0 = 0.4$ and other parameters are same as given in (7.25). The Hopf-bifurcation point is $\phi^* = 0.02847$.

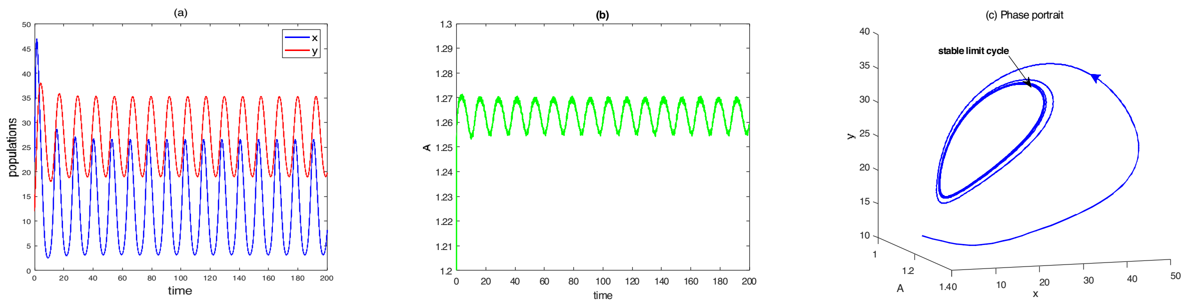

Fig. 7.3: Instable behavior of solutions and existence of stable limit cycle for $A_0 = 0.45(< A_0^*)$. Rest of the parameters are same as (7.25).

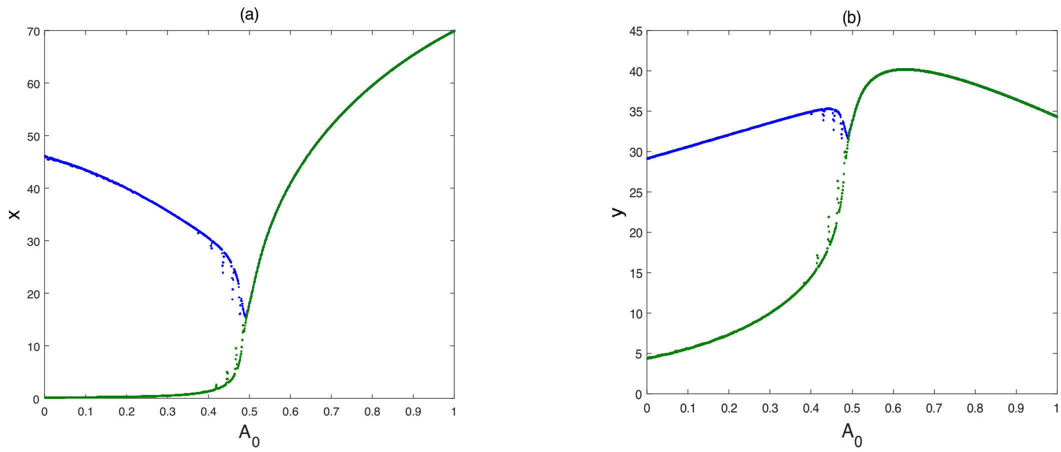

Fig. 7.4: Bifurcation diagram of the prey and predator population with respect to parameter A_0.

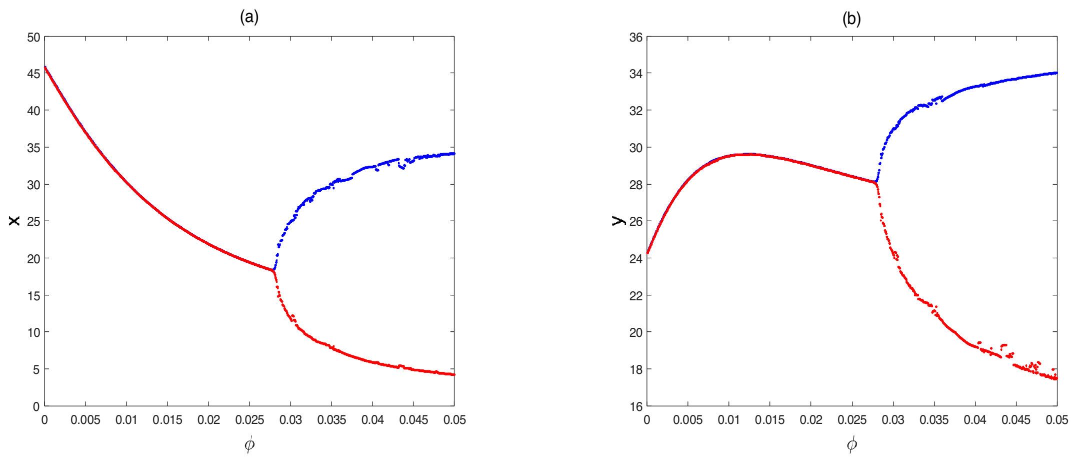

Fig. 7.5: Bifurcation diagram of the prey and predator population with respect to parameter ϕ.

As the system (7.1) shows Hopf-bifurcation with respect to parameters A_0 and ϕ, and direction of Hopf-bifurcation is opposite for both the parameters. Therefore, we can divide the $A_0\phi$-plane into two regions:

Region of stability (green) $S_1 = \{(A_0, \phi) :$ system (7.1) is locally asymptotically stable$\}$.

Region of instability (white) $S_2 = \{(A_0, \phi) :$ system (7.1) is unstable$\}$.

Both the regions are drawn in Fig. 7.6. The curve which separates both the regions is called Hopf-bifurcation curve.

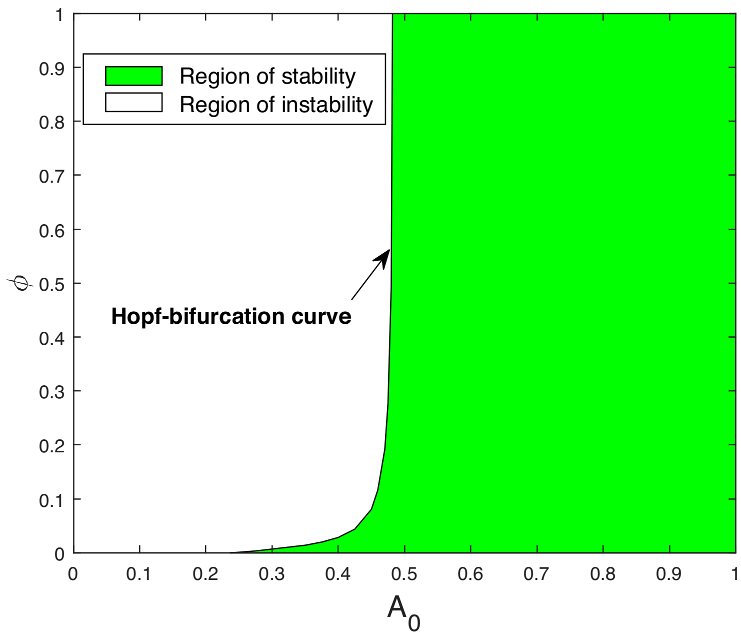

Fig. 7.6: Region of stability and instability for system (7.1) in $A_0\phi$-plane.

The number of interior equilibrium points depend on the values of parameters. In the Table 7.3, we have shown dependence of total number of interior equilibrium on parameters a and d and the nature of their stability. It is observed that when $a = 0.105$ and $d = 0.1$ (other parameters are as in (7.25)), then three interior equilibrium exist for the system (7.1), $E_1^*(0.5099, 1.398, 20.9174)$, $E_2^*(8.2355, 1.409, 32.9068)$ and $E_3^*(42.309, 1.4136, 43.0591)$. E_1^* and E_3^* are locally asymptotically stable and E_2^* is unstable. Since there are two locally asymptotically stable equilibrium in the system, so it shows bistability. Bistability is a phenomenon where a system converges to two different equilibrium points for the same parametric values based on the variation of the initial conditions. In Fig. 7.7, we initiated two trajectories from two nearby points and they converse to different interior equilibria. The black dotted curve is separatrix, which divides the $xy-$plane into two regions in such a way that if a solution is initiated from the left of the separatrix, it converses to E_1^* and if a solution is initiated from the right of the separatrix, it converses to E_3^*. In other words, left region is region of attraction for E_1^* and right region is region of attraction for E_3^*.

Remark 7.3.5. *For the best representation of bistability phenomenon and separatrix curve, the Fig. 7.7 is drawn in the xy−plane. But initial conditions and interior equilibrium points are written as they are.*

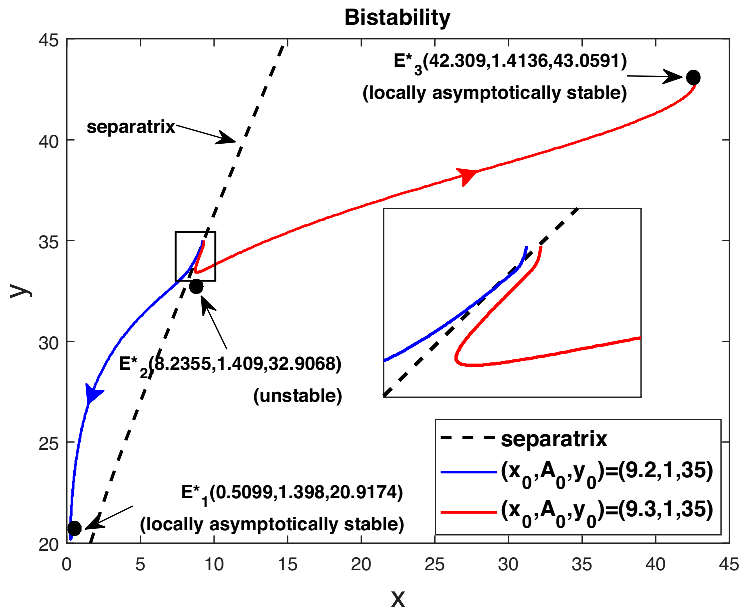

Fig. 7.7: Trajectories initiated from region of attraction of both the locally asymptotically stable equilibrium points, system (7.1) shows bistability.

Table 7.3: Dependence of total number of interior equilibria and their stability on parameters a and d. Rest of the parameters are same as in (7.25).

Parametric values	No. of interior equilibrium points	Equilibrium points	Nature of equilibrium points
$a = 0.08, d = 0.01$	0	-	-
$a = 0.1, d = 0.235$	1	(46.7827,1.4114,37.6669)	stable
		(4.4407,1.4048,27.0479)	unstable
$a = 0.1, d = 0.137$	2	(41.3432,1.4132,42.038)	stable

		(0.5099,1.398,20.9174)	stable
$a = 0.105, d = 0.1$	3	(8.2355,1.409,32.9068)	unstable
		(42.309,1.4136,43.0591)	stable

7.4 Analysis of Delayed Model

In this section, we discuss the local stability and Hopf-bifurcation phenomenon for the delayed system (7.2). The introduction of time delay does not affects the equilibria of the system. So, all the equilibria remain same as the non-delayed system (7.1). To see the effect of delay on the dynamical behavior of the interior equilibrium E^*, we rewrite the delayed system (7.2) as

$$\frac{dU(t)}{dt} = F(U(t), U(t-\tau_1), U(t-\tau_2)), \tag{7.26}$$

where

$$U(t) = [x(t), A(t), y(t)]^T, \;\; U(t-\tau_j) = [x(t-\tau_j), A(t-\tau_j), y(t-\tau_j)]^T, \;\; j = 1, 2.$$

Now we linearize the system (7.26) by using the following transformations:

$$x(t) = \bar{x}(t) + x^*, \;\; A(t) = \bar{A}(t) + A^*, \;\; y(t) = \bar{y}(t) + y^*,$$

where $\bar{x}, \bar{A}$ and $\bar{y}$ are small perturbations around x^*, A^* and y^*, respectively. Then the linearized system of (7.26) about the interior equilibrium E^* is given by

$$\frac{dZ}{dt} = PZ(t) + Q_1 Z(t-\tau_1) + Q_2 Z(t-\tau_2),$$

where

$$P = \left[\frac{\partial F}{\partial U(t)}\right]_{E^*}, \;\; Q_1 = \left[\frac{\partial F}{\partial U(t-\tau_1)}\right]_{E^*}, \;\; Q_2 = \left[\frac{\partial F}{\partial U(t-\tau_2)}\right]_{E^*}, \;\; Z = [\bar{x}(t), \bar{A}(t), \bar{y}(t)]^T.$$

Thus, the Jacobian matrix of the system (7.2) at E^* is given by

$$\begin{bmatrix} a_1 & 0 & -a_2 \\ 0 & a_3 & a_4 \\ c_1 a_5 e^{-\xi\tau_1} & c_2 \phi y^* e^{-\xi\tau_2} & a_6 + c_1 a_2 e^{-\xi\tau_1} + c_2 \phi A^* e^{-\xi\tau_2} \end{bmatrix},$$

where

$$a_1 = r - \frac{2rx^*}{K} - \frac{\alpha(1-A_0)y^*}{(1+ax^*)^2}, \; a_2 = \frac{\alpha(1-A_0)x^*}{1+ax^*}, \; a_3 = -\beta - \phi y^*,$$

$$a_4 = \lambda A_0 - \phi A^*,\ a_5 = \frac{\alpha(1-A_0)y^*}{(1+ax^*)^2},\ a_6 = -d - 2ey^*.$$

The characteristic equation corresponding to the above Jacobian matrix is

$$\xi^3 + b_1\xi^2 + b_2\xi + b_3 + (b_4\xi^2 + b_5\xi + b_6)e^{-\xi\tau_1} + (b_7\xi^2 + b_8\xi + b_9)e^{-\xi\tau_2} = 0, \tag{7.27}$$

where

$$b_1 = -(a_1 + a_3 + a_6),\ b_2 = a_3a_6 + a_1a_6 + a_1a_3,\ b_3 = -a_1a_3a_6,\ b_4 = -c_1a_2,$$

$$b_5 = c_1a_2(a_1 + a_3 + a_5),\ b_6 = -c_1a_2a_3(a_1 + a_2),\ b_7 = -c_2\phi A^*,$$

$$b_8 = c_2\phi(a_1A^* + a_3A^* - a_4y^*),\ b_9 = c_2a_1\phi(-a_3A^* + a_4y^*).$$

Remark 7.4.1. *When $\tau_1 = \tau_2 = 0$, then the characteristic equation (7.27) is same as the characteristic equation (7.14) for non-delayed system.*

Case (1): $\tau_1 > 0,\ \tau_2 = 0$. Then equation (7.27) becomes

$$\xi^3 + d_1\xi^2 + d_2\xi + d_3 + (b_4\xi^2 + b_5\xi + b_6)e^{-\xi\tau_1} = 0, \tag{7.28}$$

where

$$d_1 = b_1 + b_7,\ d_2 = b_2 + b_8,\ d_3 = b_3 + b_9.$$

For the delayed system (7.2), the positive equilibrium is locally asymptotically stable if and only if all the roots of the equation (7.28) have negative real parts. For switching of the stability, the root of the equation (7.28) must cross the imaginary axis. Therefore let $i\omega(\omega > 0)$ be a root of equation (7.28), then it follows that

$$\begin{aligned} (-b_4\omega^2 + b_6)\cos(\omega\tau_1) + b_5\omega\sin(\omega\tau_1) &= d_1\omega^2 - d_3, \\ b_5\omega\cos(\omega\tau_1) - (-b_4\omega^2 + b_6)\sin(\omega\tau_1) &= \omega^3 - d_2\omega. \end{aligned} \tag{7.29}$$

From the above set of equations, we can obtain

$$\omega^6 + h_1\omega^4 + h_2\omega^2 + h_3 = 0, \tag{7.30}$$

where

$$h_1 = d_1^2 - b_4^2 - 2d_2,\ h_2 = d_2^2 - b_5^2 - 2d_1d_3 + 2b_4b_6,\ h_3 = d_3^2 - b_6^2.$$

If we put $\omega^2 = z$, then equation (7.30) becomes

$$g(z) = z^3 + h_1z^2 + h_2z + h_3 = 0. \tag{7.31}$$

Theorem 7.4.1. *If equation (7.31) has no positive root, then there is no change in the stability behavior of E^* for all $\tau_1 \geq 0$.*

Corollary 7.4.1.1. *If inequalities in (7.15) hold and equation (7.31) has no positive root, then E^* is locally asymptotically stable for all $\tau_1 \geq 0$.*

Corollary 7.4.1.2. *If inequalities in (7.15) do not hold and equation (7.31) has no positive root, then E^* is unstable for all $\tau_1 \geq 0$.*

Now let inequalities in (7.15) hold and equation (7.31) has at least one positive root, say $z_1 = \omega_1^2$. Substituting ω_1 into equation (7.29), we obtain

$$\tau_{1_i} = \frac{1}{\omega_1}\cos^{-1}\left[\frac{(d_1\omega_1^2 - d_3)(-b_4\omega_1^2 + b_6) + (\omega_1^3 - d_2\omega_1)\omega_1 b_5}{(-b_4\omega_1^2 + b_6)^2 + \omega_1^2 {b_5}^2}\right] + \frac{2i\pi}{\omega_1},\ i = 0, 1, 2\ldots. \quad (7.32)$$

$(H_1):\ g'(\omega_1^2) > 0.$
Let $\xi(\tau_{1_i}) = \pm i\omega_1$ be the root of equation (7.28), a little calculation yields

$$Re\left[\frac{d\xi}{d\tau_1}\right]^{-1}_{\xi = i\omega_1,\ \tau_1 = \tau_{1_i}} = \frac{g'(\omega_1^2)}{(-b_4\omega_1^2 + b_6)^2 + \omega_1^2 {b_5}^2} > 0.$$

But sign of $\left[\frac{d}{d\tau_1}Re(\xi)\right]_{\xi = i\omega_1,\ \tau_1 = \tau_{1_i}}$ is same as the sign of $\left[Re\left(\frac{d\xi}{d\tau_1}\right)\right]_{\xi = i\omega_1,\ \tau_1 = \tau_{1_i}}$.
Hence, the transversality condition can be obtained under (H_1)

$$\left[\frac{d}{d\tau_1}(Re(\xi))\right]_{\tau_1 = \tau_{1_i}} > 0,$$

Thus, we can state the following theorem.

Theorem 7.4.2. *For system (7.2), with $\tau_2 = 0$ and assuming that (H_1) holds, there exists a positive number τ_{1_0} such that the equilibrium E^* is locally asymptotically stable when $\tau_1 < \tau_{1_0}$ and unstable when $\tau_1 > \tau_{1_0}$. Furthermore system (7.2) undergoes a Hopf-bifurcation at E^* when $\tau_1 = \tau_{1_0}$.*

Case (2): $\tau_1 = 0,\ \tau_2 > 0$. Then equation (7.27) becomes

$$\xi^3 + e_1\xi^2 + e_2\xi + e_3 + (b_7\xi^2 + b_8\xi + b_9)e^{-\xi\tau_2} = 0, \quad (7.33)$$

where

$$e_1 = b_1 + b_4,\ e_2 = b_2 + b_5,\ e_3 = b_3 + b_6.$$

Under an analysis similar to Case (1), one can easily deduce the following theorem.

Theorem 7.4.3. *For $\tau_1 = 0$, the interior equilibrium point is locally asymptotically stable for $\tau_2 < \tau_{2_0}$, unstable for $\tau_2 > \tau_{2_0}$, and it undergoes Hopf-bifurcation at $\tau_2 = \tau_{2_0}$ given by*

$$\tau_{2_0} = \frac{1}{\omega_2}\cos^{-1}\left[\frac{(e_1\omega_2^2 - e_3)(-b_7\omega_2^2 + b_9) + (\omega_2^3 - e_2\omega_2)\omega_2 b_8}{(-d_7\omega_2^2 + b_9)^2 + \omega_2^2 {b_8}^2}\right],$$

where $i\omega_2$ is root of characteristic equation (7.33).

Case (3): τ_1 is fixed in the interval $(0, \tau_{1_0})$ and assuming τ_2 as a variable parameter
We consider equation (7.27) with τ_1 as fixed in its stable interval $(0, \tau_{1_0})$ and τ_2 as a variable. Let $i\omega$ $(\omega > 0)$ be a root of characteristic equation (7.27). Then separating real and imaginary parts, we obtain

$$-b_1\omega^2 + b_3 + (-b_4\omega^2 + b_6)\cos(\omega\tau_1) + b_5\omega\sin(\omega\tau_1) = -(-b_7\omega^2 + b_9)\cos(\omega\tau_2) - b_8\omega\sin(\omega\tau_2), \tag{7.34}$$

$$-\omega^3 + b_2\omega - (-b_4\omega^2 + b_6)\sin(\omega\tau_1) + b_5\omega\cos(\omega\tau_1) = (-b_7\omega^2 + b_9)\sin(\omega\tau_2) - b_8\omega\cos(\omega\tau_2). \tag{7.35}$$

Squaring and then adding (7.34) and (7.35) to eliminate τ_2, we obtain

$$\begin{aligned}&(-b_1\omega^2 + b_3)^2 + (-\omega^3 + b_2\omega)^2 + (-b_4\omega^2 + b_6)^2 + b_5^2\omega^2 + 2[(-b_1\omega^2 + b_3)\\ &(-b_4\omega^2 + b_6) + (-\omega^3 + b_2\omega)b_5\omega]\cos(\omega\tau_1) + 2[-(-\omega^3 + b_2\omega)(-b_4\omega^2 + b_6)\\ &+ (-b_1\omega^2 + b_3)b_5\omega]\sin(\omega\tau_1) = (-b_7\omega^2 + b_9)^2 + b_8^2\omega^2.\end{aligned} \tag{7.36}$$

Equation (7.36) is a transcendental equation in complex form. So, it is not easy to predict the nature of roots. Without going detailed analysis with (7.36), it is assumed that there exist at least one positive root ω_0. Equations (7.34) and (7.35) can be re-written as

$$-(-b_7\omega_0^2 + b_9)\cos(\omega_0\tau_2) - b_8\omega_0\sin(\omega_0\tau_2) = D_1 \tag{7.37}$$

$$(-b_7\omega_0^2 + b_9)\sin(\omega_0\tau_2) - b_8\omega_0\cos(\omega_0\tau_2) = D_2 \tag{7.38}$$

where

$$D_1 = -b_1\omega_0^2 + b_3 + (-b_4\omega_0^2 + b_6)\cos(\omega_0\tau_1) + b_5\omega_0\sin(\omega_0\tau_1),$$

$$D_2 = -\omega_0^3 + b_2\omega_0 - (-b_4\omega_0^2 + b_6)\sin(\omega_0\tau_1) + b_5\omega_0\cos(\omega_0\tau_1).$$

Equations (7.37) and (7.38) lead to

$$\tau'_{2_n} = \frac{1}{\omega_0}\cos^{-1}\left[\frac{-(-b_7\omega_0^2 + b_9)D_1 - b_8\omega_0 D_2}{(-b_7\omega_0^2 + b_9)^2 + b_8^2\omega_0^2}\right] + \frac{2n\pi}{\omega_0},\ n = 0, 1, 2, \ldots. \tag{7.39}$$

Now, to verify the transversality condition of Hopf-bifurcation, differentiating equation (7.34) and (7.35) with respect to τ_2 and substitute $\tau_2 = \tau'_{2_0}$, we obtain

$$\begin{aligned} P\left[\frac{d(Re\ \xi)}{d\tau_2}\right]_{\tau_2=\tau_2^*} + Q\left[\frac{d\omega}{d\tau_2}\right]_{\tau_2=\tau_2^*} &= R, \\ -Q\left[\frac{d(Re\ \xi)}{d\tau_2}\right]_{\tau_2=\tau_2^*} + P\left[\frac{d\omega}{d\tau_2}\right]_{\tau_2=\tau_2^*} &= S, \end{aligned} \tag{7.40}$$

where

$$P = -3\omega_0^2 + b_2 + b_5\cos(\omega_0\tau_1) - (-b_4\omega_0^2 + b_6)\tau_1\cos(\omega_0\tau_1) + 2b_4\omega_0\sin(\omega_0\tau_1) - b_5\omega_0\tau_1 \sin(\omega_0\tau_1) - (-b_7\omega_0^2 + b_9)\tau_2\cos(\omega_0\tau_2) + b_8\cos(\omega_0\tau_2) - b_8\omega_0\tau_2\sin(\omega_0\tau_2) + 2b_7\omega_0\sin(\omega_0\tau_2),$$

$$\begin{aligned} Q = {} & -2b_1\omega_0 - 2b_4\omega_0\cos(\omega_0\tau_1) - (-b_4\omega_0^2 + b_6)\tau_1\sin(\omega_0\tau_1) + b_5\sin(\omega_0\tau_1) + b_5\omega_0\cos(\omega_0\tau_1) \\ & - 2b_7\omega_0\cos(\omega_0\tau_2) - (-b_7\omega_0^2 + b_9)\tau_2\sin(\omega_0\tau_2) - b_8\omega_0\tau_2\cos(\omega_0\tau_2) + b_8\sin(\omega_0\tau_2), \end{aligned}$$

$$R = (-b_7\omega_0^2 + b_9)\omega_0\sin(\omega_0\tau_2) - b_8\omega_0^2\cos(\omega_0\tau_2),$$

$$S = (-b_7\omega_0^2 + b_9)\omega_0\cos(\omega_0\tau_2) + b_8\omega_0^2\sin(\omega_0\tau_2).$$

Solving equation (7.40) for $\left[\frac{d(Re\ \xi)}{d\tau_2}\right]_{\tau_2=\tau'_{2_0}}$, it is obtained

$$\left[\frac{d(Re\ \xi)}{d\tau_2}\right]_{\tau_2=\tau'_{2_0},\ \xi=ß\omega_0} = \frac{PR - QS}{P^2 + Q^2}.$$

$(H_2) : PR - QS \neq 0.$

Theorem 7.4.4. *For system (7.2), with $\tau_1 \in (0, \tau_{1_0})$ and assuming that (H_2) holds, there exists a positive number τ'_{2_0} such that E^* is locally asymptotically stable when $\tau_2 < \tau'_{2_0}$ and unstable when $\tau_2 > \tau'_{2_0}$. Furthermore, system (7.2) undergoes a Hopf-bifurcation at E^* where $\tau_2 = \tau'_{2_0}$.*

Case (4): τ_2 is fixed in the interval $(0, \tau_{2_0})$ and assuming τ_1 as a variable parameter Under an analysis similar to Case (3), one can easily prove the following theorem.

Theorem 7.4.5. *For $\tau_2 \in (0, \tau_{2_0})$, the interior equilibrium point is locally asymptotically stable for $\tau_1 < \tau'_{1_0}$ and it undergoes Hopf-bifurcation at $\tau_1 = \tau'_{1_0}$, given by*

$$\tau'_{1_0} = \frac{1}{\omega_*}\cos^{-1}\left[\frac{-(-b_4\omega_*^2 + b_6)D_3 - b_5\omega_* D_4}{(-b_4\omega_*^2 + b_6)^2 + b_5^2\omega_*^2}\right],$$

where

$$D_3 = -b_1\omega_*^2 + b_3 + (-b_7\omega_*^2 + b_9)\cos(\omega_*\tau_2) + b_8\omega_*\sin(\omega_*\tau_2),$$

$$D_4 = -\omega_*^3 + b_2\omega_* - (-b_7\omega_*^2 + b_9)\sin(\omega_*\tau_2) + b_8\omega_*\cos(\omega_*\tau_2),$$

and $i\omega_*$ is characteristic root of equation (7.27).

7.5 Direction and Stability of Hopf-bifurcation

Now with the help of center manifold theory and normal form concept (see [80] for details), we shall study direction and stability of the bifurcated periodic solutions at $\tau_1 = \tau'_{1_0}$.

Without loss of generality, we assume that $\tau_2^* < \tau'_{1_0}$, where $\tau_2^* \in (0, \tau_{2_0})$. Let

$$x_1(t) = x(t) - x^*,\ A_1(t) = A(t) - A^*,\ y_1(t) = y(t) - y^*,$$

and still denote $x_1(t), A_1(t), y_1(t)$ by $x(t), A(t), y(t)$. Let $\tau_1 = \tau'_{1_0} + \mu,\ \mu \in R$ so that Hopf-bifurcation occurs at $\mu = 0$. We normalize the delay with scaling $t \mapsto (\frac{t}{\tau_1})$, then system (7.2) can be re-written as

$$\dot{U}(t) = \tau_1\big(PU(t) + Q_1U(t-1) + Q_2U\big(t - \frac{\tau_2^*}{\tau_1}\big) + f(x,A,y)\big), \tag{7.41}$$

where $U(t) = (x(t), A(t), y(t))^T$,

$$P = \begin{bmatrix} a_1 & 0 & -a_2 \\ 0 & a_3 & a_4 \\ 0 & 0 & a_6 \end{bmatrix},\ Q_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_1a_5 & 0 & c_1a_2 \end{bmatrix},\ Q_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_2\phi y^* & c_2\phi A^* \end{bmatrix},\ f(x,A,y) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}.$$

The nonlinear term f_1, f_2 and f_2 are given by

$$f_1 = \Big(-\frac{2r}{K} + \frac{2a\alpha(1-A_0)y^*}{(1+ax^*)^3}\Big)x^2(t) - \frac{\alpha(1-A_0)}{(1+ax^*)^2}x(t)y(t) + \text{h.o.t.}\,,$$

$$f_2 = -\phi A(t)y(t) + \text{h.o.t.}\,,$$

$$f_3 = -2ey^2(t) - \frac{2c_1\alpha(1-A_0)y^*}{(1+ax^*)^3}x^2(t-1) + \frac{c_1\alpha(1-A_0)}{(1+ax^*)^2}x(t-1)y(t-1)$$
$$+ c_2\phi A\Big(t - \frac{\tau_2^*}{\tau'_{1_0}}\Big)y\Big(t - \frac{\tau_2^*}{\tau'_{1_0}}\Big) + \text{h.o.t.}\,,$$

The linearization of equation (7.41) around the origin is given by

$$\dot{U}(t) = \tau_1\big(PU(t) + Q_1U(t-1)\big) + Q_2U\big(t - \frac{\tau_2^*}{\tau_1}\big).$$

For $\chi = (\chi_1, \chi_2, \chi_3)^T \in C([-1,0], R^3)$, define

$$L_\mu(\chi) = (\tau_1 + \mu)\Big(P\chi(0) + Q_1\chi(-1)\Big) + Q_2\chi\big(-\frac{\tau_2^*}{\tau_1}\big)\Big).$$

By the Riesz representation theorem, there exists a 3×3 matrix $\eta(\theta, \mu)$, $(-1 \le \theta \le 0)$ whose element are of bounded variation function such that

$$L_\mu(\chi) = \int_{-1}^{0} d\eta(\theta, \mu)\chi(\theta) \;\; for\; \chi \in C([-1,0], R^3). \tag{7.42}$$

In fact, we can obtain

$$\eta(\theta, \mu) = \begin{cases} (\tau'_{1_0} + \mu)(P + Q_1 + Q_2), & \text{if } \theta = 0 \\ (\tau'_{1_0} + \mu)(Q_1 + Q_2), & \text{if } \theta \in \big[-\frac{\tau_2^*}{\tau_1}, 0\big) \\ (\tau'_{1_0} + \mu)Q_2, & \text{if } \theta \in \big(-1, -\frac{\tau_2^*}{\tau_1}\big) \\ 0, & \text{if } \theta = -1. \end{cases}$$

Then equation (7.42) is satisfied.
For $\chi \in C^1([-1,0], R^3)$, define the operator $H(\mu)$ as

$$H(\mu)\chi(\theta) = \begin{cases} \frac{d\chi(\theta)}{d\theta}, & \text{if } \theta \in [-1, 0) \\ \int_{-1}^{0}[d\eta(\xi, \mu)]\chi(\xi), & \text{if } \theta = 0, \end{cases}$$

and

$$R(\mu)\chi(\theta) = \begin{cases} 0, & \text{if } \theta \in [-1, 0) \\ h(\mu, \chi), & \text{if } \theta = 0, \end{cases}$$

where

$$h(\mu, \chi) = (\tau'_{1_0} + \mu)\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}, \; \chi = (\chi_1, \chi_2, \chi_3)^T \in C([-1,0], R^3),$$

$$h_1 = \Big(-\frac{2r}{K} + \frac{2a\alpha(1 - A_0)y^*}{(1 + ax^*)^3}\Big)x^2(0) - \frac{\alpha(1 - A_0)}{(1 + ax^*)^2}x(0)y(0) + \text{h.o.t.}\,,$$

$$h_2 = -\phi A(0)y(0) + \text{h.o.t.}\,,$$

$$h_3 = -2ey^2(0) - \frac{2c_1\alpha(1 - A_0)y^*}{(1 + ax^*)^3}x^2(-1) + \frac{c_1\alpha(1 - A_0)}{(1 + ax^*)^2}x(-1)y(-1) + c_2\phi A\Big(-\frac{\tau_2^*}{\tau'_{1_0}}\Big)y\Big(-\frac{\tau_2^*}{\tau'_{1_0}}\Big) + \text{h.o.t.}\,,$$

Then system (7.2) is equivalent to the following operator equation

$$\dot{U}_t = H(\mu)U_t + R(\mu)U_t \ ,$$

where $U_t = U(t+\theta)$ for $\theta \in [-1,0]$.
For $\psi \in C^1([0,1],(R^3)^*)$, define

$$H^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & \text{if } s \in (0,1] \\ \int_{-1}^{0} \psi(-\xi)d\eta(\xi,0), & \text{if } s = 0, \end{cases}$$

and a bilinear form

$$\langle \psi(s), \chi(\theta) \rangle = \overline{\psi}(0)\chi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{\psi}(\xi-\theta)d\eta(\theta)\chi(\xi)d\xi \ ,$$

where $\eta(\theta) = \eta(\theta,0)$, $H = H(0)$ and H^* are adjoint operators. From the discussion in previous section, we know that $\pm i\omega_0\tau'_{1_0}$ are the eigenvalues of $H(0)$ and therefore they are also eigenvalues of H^*. It is not difficult to verify that the vectors $q(\theta) = (1,\alpha_1,\beta_1)^T e^{i\omega_0\tau'_{1_0}\theta}$ $(\theta \in [-1,0])$ and $q^*(s) = \frac{1}{D}(1,\alpha_1^*,\beta_1^*)e^{i\omega_0\tau'_{1_0}s}$ $(s \in [0,1])$ are the eigenvectors of $H(0)$ and H^* corresponding to the eigenvalue $i\omega_0\tau'_{1_0}$ and $-i\omega_0\tau'_{1_0}$ respectively, where

$$\langle q^*(s), q(\theta) \rangle = 1, \ \langle q^*(s), \overline{q}(\theta) \rangle = 1,$$

$$\beta_1 = \frac{c_1 a_5 e^{-i\omega_0\tau'_{1_0}}}{i\omega_0 - a_6 - c_1 a_2 e^{-i\omega_0\tau'_{1_0}} - c_2\phi A^* e^{-i\omega_0\frac{\tau_2^*}{\tau'_{1_0}}}}, \quad \alpha_1 = \frac{a_4\beta_1}{i\omega_0 - a_3},$$

$$\beta_1^* = -\frac{i\omega_0 + a_1}{c_1 a_5 e^{-i\omega_0\tau'_{1_0}}}, \quad \alpha_1^* = \frac{a_2 - \left[i\omega_0 + a_6 + c_1 a_2 e^{-i\omega_0\tau'_{1_0}} + c_2\phi A^* e^{-i\omega_0\frac{\tau_2^*}{\tau'_{1_0}}}\right]\beta_1^*}{a_4},$$

$$D = \Big[1 + \alpha_1\overline{\alpha_1^*} + \beta_1\overline{\beta_1^*} + \tau'_{1_0}\big(\overline{\beta_1^*}c_1 a_5 + \beta_1\overline{\beta_1^*}c_1 a_2\big)e^{-i\omega_0\tau'_{1_0}} + \tau_2^*\big(\alpha_1\overline{\beta_1^*}c_2\phi y^*$$
$$+ \beta_1\overline{\beta_1^*}c_2\phi A^*\big)e^{-i\omega_0\frac{\tau_2^*}{\tau'_{1_0}}}\Big].$$

Following the algorithms explained in Hassard *et al.* [80] and using a computation process similar to that in Song and Wei [195], which is used to obtain the properties of Hopf-bifurcation, we obtain

$$g_{20} = -\frac{2\tau'_{1_0}}{\overline{D}}\left[\frac{r}{K} + \beta_1\alpha(1-A_0) + \overline{\alpha_1^*}\phi\alpha_1\beta_1 + \overline{\beta_1^*}e\beta_1^2 - \overline{\beta_1^*}c_1\alpha(1-A_0)\beta_1 e^{-2i\omega_0\tau'_{1_0}} - \overline{\beta_1^*}c_2\phi\alpha_1\beta_1 e^{-2i\omega_0\tau_2^*}\right],$$

$$g_{11} = -\frac{\tau'_{1_0}}{\overline{D}}\left[2\frac{r}{K} + \alpha(1-A_0)(\beta_1+\overline{\beta_1}) + \overline{\alpha_1^*}\phi(\alpha_1\overline{\beta_1}+\overline{\alpha_1}\beta_1) + 2\overline{\beta_1^*}e\beta_1\overline{\beta_1} - \overline{\beta_1^*}c_1\alpha(1-A_0)(\beta_1+\overline{\beta_1}) + \overline{\beta_1^*}c_2\phi(\alpha_1\overline{\beta_1}+\overline{\alpha_1}\beta_1)\right],$$

$$g_{02} = -\frac{2\tau'_{1_0}}{\overline{D}}\left[\frac{r}{K} + \overline{\beta_1}\alpha(1-A_0) + \overline{\alpha_1^*}\phi\overline{\alpha_1}\overline{\beta_1} + \overline{\beta_1^*}e\overline{\beta_1}^2 - \overline{\beta_1^*}c_1\alpha(1-A_0)\overline{\beta_1}e^{2i\omega_0\tau'_{1_0}} - \overline{\beta_1^*}c_2\phi\overline{\alpha_1}\overline{\beta_1}e^{2i\omega_0\tau_2^*}\right],$$

$$\begin{aligned}g_{21} = -\frac{2\tau'_{1_0}}{\overline{D}}\Bigg[&\frac{r}{K}\left\{2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)\right\} + \alpha(1-A_0)\left\{W_{11}^{(3)}(0) + \frac{1}{2}W_{20}^{(3)}(0) + \frac{1}{2}W_{20}^{(1)}(0)\right.\\
&\left.+\beta_1 W_{11}^{(1)}(0) - a\overline{\beta_1} - 2a\beta_1\right\} + \phi\overline{\alpha_1^*}\left\{\alpha_1 W_{11}^{(3)}(0) + \frac{1}{2}\overline{\alpha_1}W_{20}^{(3)}(0) + \frac{1}{2}\overline{\beta_1}W_{20}^{(2)}(0) + \beta_1 W_{11}^{(2)}(0)\right\}\\
&+e\overline{\beta_1^*}\left\{2\beta_1 W_{11}^{(3)}(0) + \overline{\beta_1}W_{20}^{(3)}(0)\right\} - \overline{\beta_1^*}c_1\alpha(1-A_0)\left\{e^{-i\omega_0\tau'_{1_0}}W_{11}^{(3)}(-1) + \frac{1}{2}e^{i\omega_0\tau'_{1_0}}W_{20}^{(3)}(-1)\right.\\
&\left.+\frac{1}{2}\overline{\beta_1}e^{i\omega_0\tau'_{1_0}}W_{20}^{(1)}(-1) + \beta_1 e^{-i\omega_0\tau'_{1_0}}W_{11}^{(1)}(-1) - a\overline{\beta_1}e^{-i\omega_0\tau'_{1_0}} - 2a\beta_1 e^{-i\omega_0\tau'_{1_0}}\right\}\\
&-c_2\phi\overline{\beta_1^*}\left\{\alpha_1 e^{-i\omega_0\tau_2^*}W_{11}^{(3)}\left(-\frac{\tau_2^*}{\tau'_{1_0}}\right) + \frac{1}{2}\overline{\alpha_1}e^{i\omega_0\tau_2^*}W_{20}^{(3)}\left(-\frac{\tau_2^*}{\tau'_{1_0}}\right) + \frac{1}{2}\overline{\beta_1}e^{i\omega_0\tau_2^*}W_{20}^{(2)}\left(-\frac{\tau_2^*}{\tau'_{1_0}}\right)\right.\\
&\left.+\beta_1 e^{-i\omega_0\tau_2^*}W_{11}^{(2)}\left(-\frac{\tau_2^*}{\tau'_{1_0}}\right)\right\}\Bigg],\end{aligned}$$

where
$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau'_{1_0}}q(0)e^{i\omega_0\tau'_{1_0}\theta} + \frac{i\overline{g}_{02}}{3\omega_0\tau'_{1_0}}\overline{q}(0)e^{-i\omega_0\tau'_{1_0}\theta} + E_1 e^{2i\omega_0\tau'_{1_0}\theta}$,
$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau'_{1_0}}q(0)e^{i\omega_0\tau'_{1_0}\theta} + \frac{i\overline{g}_{11}}{\omega_0\tau'_{1_0}}\overline{q}(0)e^{-i\omega_0\tau'_{1_0}\theta} + E_2$,
$E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)})^T \in R^3$ and $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)})^T \in R^3$ are constant vectors, computed

as:

$$E_1 = 2\begin{bmatrix} 2i\omega_0 - a_1 & 0 & a_2 \\ 0 & 2i\omega_0 - a_3 & -a_4 \\ -c_1 a_5 e^{-2i\omega_0 \tau'_{1_0}} & -c_2\phi y^* e^{-2i\omega_0 \tau_2^*} & 2i\omega_0 - a_6 - c_1 a_2 e^{-2i\omega_0 \tau'_{1_0}} - c_2 \phi A^* e^{-2i\omega_0 \tau_2^*} \end{bmatrix}^{-1} \times \begin{bmatrix} \frac{r}{K} + \alpha(1-A_0)\beta_1 \\ \phi\alpha_1\beta_1 \\ e\beta_1^2 - c_1\alpha(1-A_0)\beta_1 e^{-2i\omega_0 \tau'_{1_0}} - c_2\phi\alpha_1\beta_1 e^{-2i\omega_0 \tau_2^*} \end{bmatrix},$$

$$E_2 = 2\begin{bmatrix} -a_1 & 0 & a_2 \\ 0 & -a_3 & -a_4 \\ -c_1 a_5 & -c_2\phi y^* & -a_6 - c_1 a_2 - c_2\phi A^* \end{bmatrix}^{-1} \begin{bmatrix} \frac{r}{K} + \frac{1}{2}\alpha(1-A_0)(\beta_1 + \overline{\beta_1}) \\ \frac{1}{2}\phi(\overline{\alpha_1}\beta_1 + \alpha_1\overline{\beta_1}) \\ e\beta_1\overline{\beta_1} - \frac{1}{2}c_1\alpha(1-A_0)(\beta_1 + \overline{\beta_1}) - \frac{1}{2}c_2\phi(\overline{\alpha_1}\beta_1 + \alpha_1\overline{\beta_1}) \end{bmatrix}.$$

Consequently, g_{ij} can be expressed by the parameters and delays τ'_{1_0} and τ_2^*. Thus, these standard results can be computed as:

$$c_1(0) = \frac{i}{2\omega_0\tau'_{1_0}}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},\ \mu_2 = -\frac{Re(c_1(0))}{Re(\lambda'(\tau'_{1_0}))},$$

$$\beta_2 = 2Re(c_1(0)),\ T_2 = -\frac{Im(c_1(0)) + \mu_2 Im(\lambda'(\tau'_{1_0}))}{\omega_0\tau'_{1_0}}.$$

These expressions give a description of the bifurcating periodic solution in the center manifold of system (7.2) at critical values $\tau_1 = \tau_{1_0}$ which can be stated in the form of following theorem:

Theorem 7.5.1.
- *μ_2 determines the direction of Hopf-bifurcation. If $\mu_2 > 0 (< 0)$ then the Hopf-bifurcation is supercritical (subcritical).*
- *β_2 determines the stability of bifurcated periodic solution. If $\beta_2 > 0 (< 0)$ then the bifurcated periodic solutions are unstable (stable).*
- *T_2 determines the period of bifurcating periodic solution. The period increases (decreases) if $T_2 > 0 (< 0)$.*

Remark 7.5.1. *When $\tau_1 > 0$ and $\tau_2 = 0$ or $\tau_1 = 0$ and $\tau_2 > 0$, then under an analysis similar to section 7.5, the corresponding values of μ_2, β_2 and T_2 can be computed. Depending upon the sign of μ_2, β_2 and T_2, the corresponding results can also be deduced.*

7.6 Numerical Simulation of Delayed Model

In order to validate our theoretical findings, obtained in previous sections, we perform some simulations by taking the same values of parameters in (7.25). We consider all four cases on delay parameters τ_1 and τ_2.

Case (I): When $\tau_2 = 0$ and $\tau_1 > 0$, then we see that condition (H_1) holds. Since the transversality condition is satisfied, therefore Hopf-bifurcation occurs in the system. To evaluate the critical value of delay parameter, taking $i = 0$ in equation (7.31) and (7.32), we obtain

$$\omega_1 = 0.3688, \quad \tau_{1_0} = 0.2889.$$

Thus, the positive equilibrium is locally asymptotically stable for $\tau_1 < \tau_{1_0} = 0.2889$, which is shown in Fig. 7.8. When $\tau_1 = \tau_{1_0}$, system undergoes a Hopf-bifurcation and periodic solution occurs around E^*. The time series analysis and periodic solution have been shown in Fig. 7.9. If we starts a trajectory from an initial point then it approaches to the periodic solution (Fig. 7.9). This shows that the periodic solution is stable. In Fig. 7.10, we made the bifurcation diagram for both the populations. The blue (red) curve represents the maximum (minimum) values of population at sufficiently large time. It is easy to see that Hopf-bifurcation occurs at $\tau_1 = \tau_{1_0} = 0.2889$.

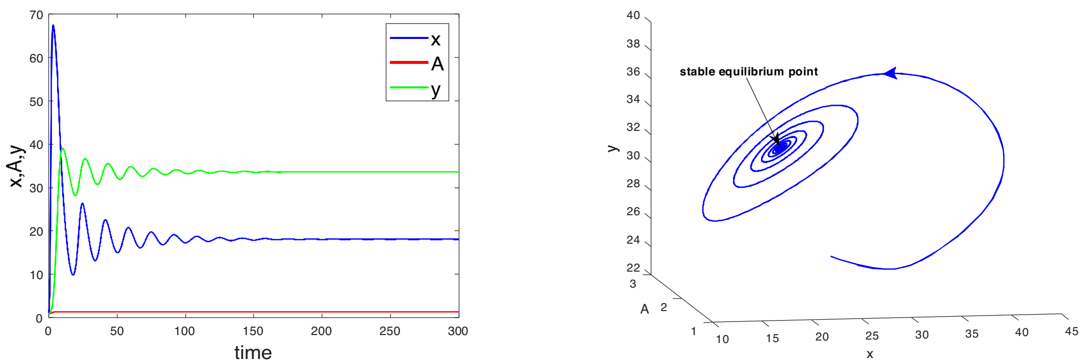

Fig. 7.8: Time series evolution and phase portrait of species for the set of parameters in (7.25) and $\tau_1 = 0.2 < \tau_{1_0} = 0.2889$ when $\tau_2 = 0$. System is locally asymptotically stable around the positive equilibrium E^*.

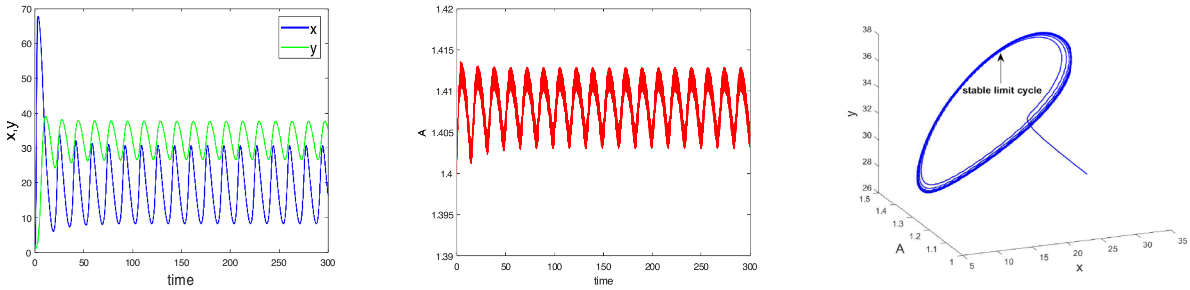

Fig. 7.9: System (7.2) is unstable when $\tau_1 = 0.35 > \tau_{1_0} = 0.2889$ and $\tau_2 = 0$. Hopf-bifurcation occurs and stable limit cycle arises in the system.

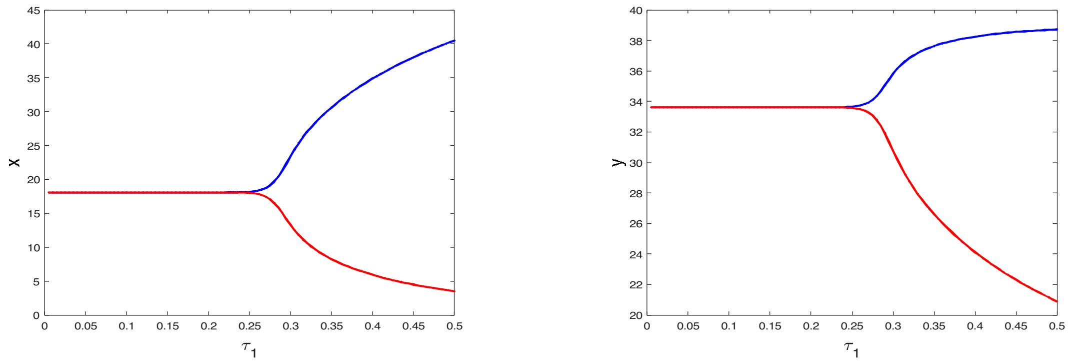

Fig. 7.10: Bifurcation diagram of the prey and predator population with respect to delay parameter τ_1 when $\tau_2 = 0$.

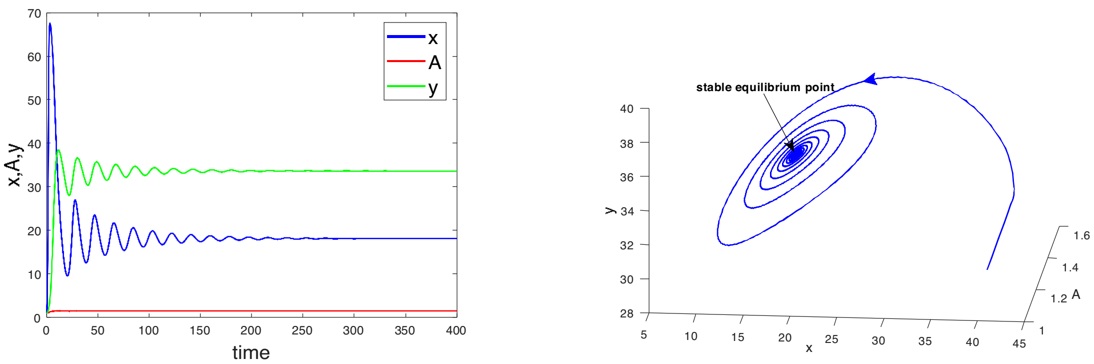

Fig. 7.11: Stable time series solutions and phase diagram of system (7.2) for $\tau_2 = 0.7 < \tau_{2_0} = 0.9618$ and $\tau_1 = 0$. Other parameters are same as in (7.25).

Case (II): When $\tau_1 = 0$ and $\tau_2 > 0$. In this case, the transversality condition is satisfied, so the system will show Hopf-bifurcation at a critical value of delay parameter τ_2. By some computation, we obtain

$$\omega_2 = 0.317, \quad \tau_{2_0} = 0.9618.$$

Therefore, according to our theoretical analysis, the system (7.2) is locally asymptotically stable for $\tau_2 < \tau_{2_0}$. In Fig. 7.11, we draw the time series of both the species for $\tau_2 = 0.7 < \tau_{2_0} = 0.9618$. From the figure, it can be seen that system is stable around the positive equilibrium E^*. At $\tau_2 = \tau_{2_0}$, the system goes through a Hopf-bifurcation and for $\tau_2 > \tau_{2_0}$, system becomes unstable and limit cycle produces. This behavior is depicted in Fig. 7.12. Again bifurcation diagram with respect to delay τ_2 for both the species is drawn in Fig. 7.13, which helps us to understand the Hopf-bifurcation phenomenon in the system.

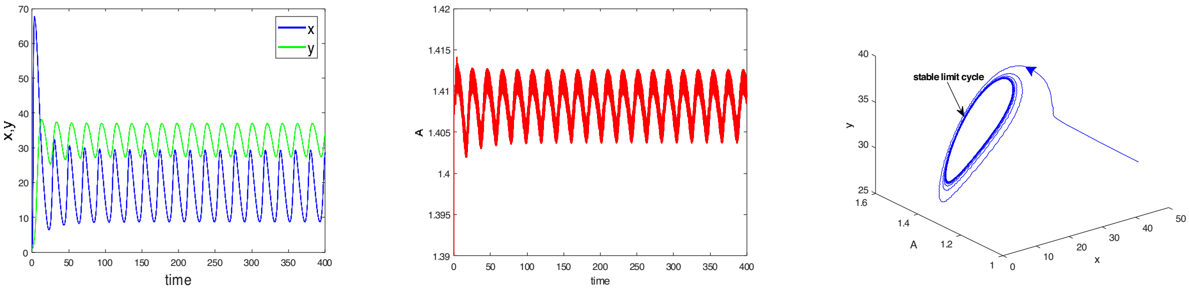

Fig. 7.12: Instable behavior and existence of periodic solutions of system (7.2) around the positive equilibrium E^* at $\tau_2 = 1.2 > \tau_{2_0} = 0.9618$ and $\tau_1 = 0$.

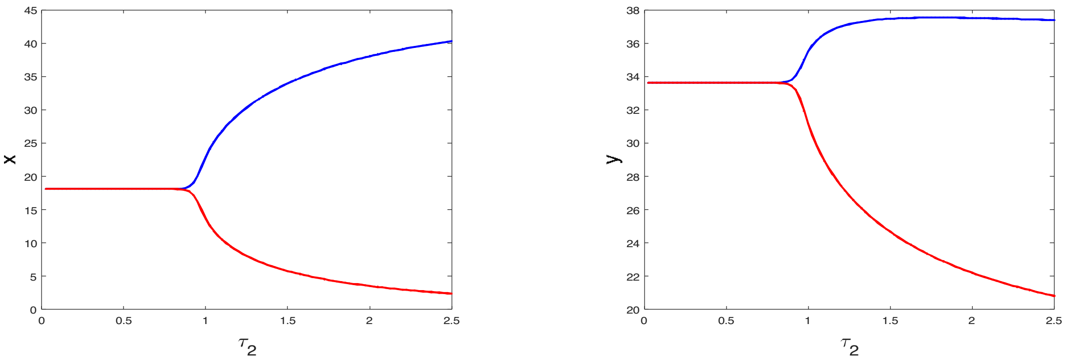

Fig. 7.13: Bifurcation diagram of the prey and predator population with respect to delay parameter τ_2 and $\tau_1 = 0$.

Case (III): When $\tau_1 = 0.12$ (fixed in the interval $(0, \tau_{1_0})$) and τ_2 as a parameter, then we observe that the condition (H_2) holds true. Therefore according to Theorem 7.4.4 system (7.2) undergoes a Hopf-bifurcation. Equations (7.36) and (7.39) give us the values of ω_0 and τ'_{2_0} as

$$\omega_0 = 0.445, \quad \tau'_{2_0} = 0.4731.$$

Thus the equilibrium point E^* is locally asymptotically stable for $\tau_2 < \tau'_{2_0} = 0.4731$ which is shown in Fig. 7.14 and unstable for $\tau_2 > \tau'_{2_0}$ (Fig. 7.15). When $\tau_2 = \tau'_{2_0}$, system undergoes a

Hopf-bifurcation around E^* and periodic solution arises in the system. Bifurcation diagram is also presented in Fig. 7.16 with respect to τ_2 for both the species when $\tau_1 = 0.12$ (fixed).

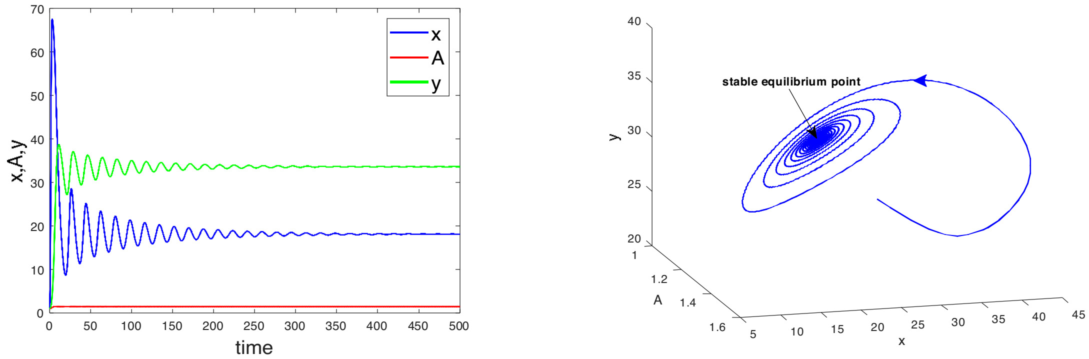

Fig. 7.14: E^* is locally asymptotically stable when $\tau_1 = 0.12$ is fixed in its stable range $(0, \tau_{1_0})$ and $\tau_2 = 0.4 < \tau'_{2_0} = 0.4731$.

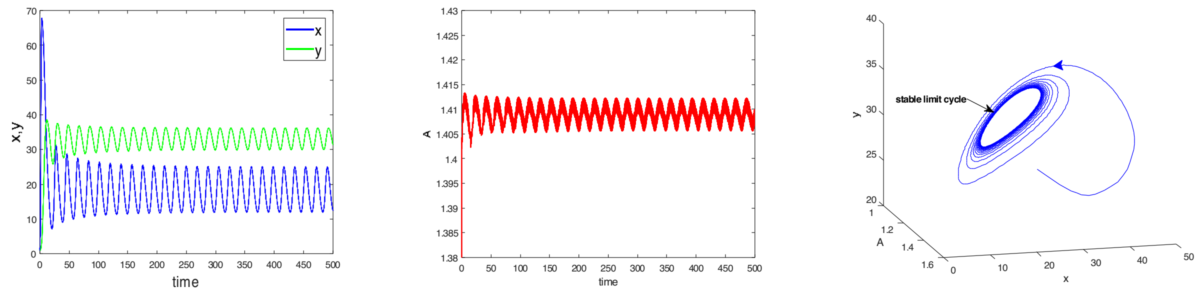

Fig. 7.15: E^* is unstable when $\tau_1 = 0.12$ is fixed in its range of stability $(0, \tau_{1_0})$ and $\tau_2 = 0.6 > \tau'_{2_0} = 0.4731$. Time series solution of species and existence limit cycle (right).

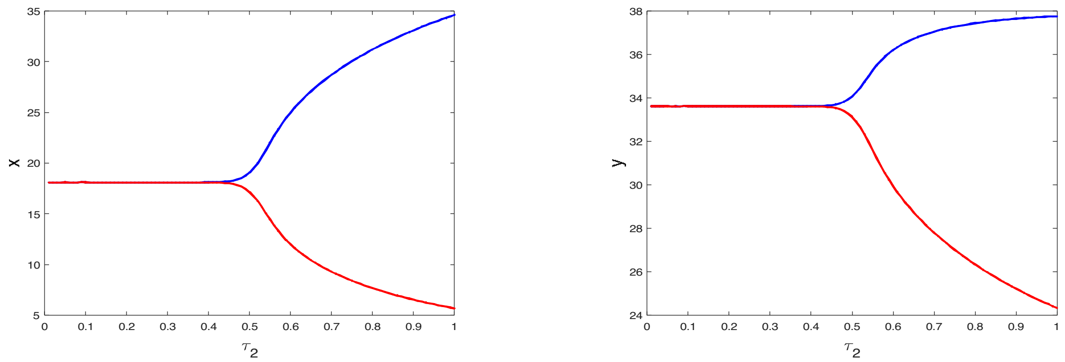

Fig. 7.16: Bifurcation diagram of the prey and predator species with respect to parameter τ_2 when $\tau_1 = 0.12$ is fixed in its range of stability $(0, \tau_{1_0})$.

Case (IV): When $\tau_2 = 0.42$ (fixed in the interval $(0, \tau_{2_0})$) and τ_1 as a parameter, then our computer simulation yields

$$\omega_* = 0.4491, \quad \tau'_{1_0} = 0.1336.$$

For $\tau_1 = 0.1 \in (0, \tau'_{1_0})$, the system is locally asymptotically stable (Fig. 7.17). But for $\tau_1 = 0.2 > \tau'_{1_0}$, the system becomes unstable (Fig. 7.18). Thus the model is stable for $\tau_1 < \tau'_{1_0}$. As τ_1 passes through τ'_{1_0}, it loses the stability and a Hopf-bifurcation occurs in the system. Fig 7.18 shows the existence of periodic solution (closed trajectory). The trajectory started from an initial point, approaches to the closed trajectory. This shows that the closed trajectory is stable. In Fig. 7.19, we present the bifurcation diagram of both the species with respect to τ_1 when $\tau_2 = 0.42$ (fixed).

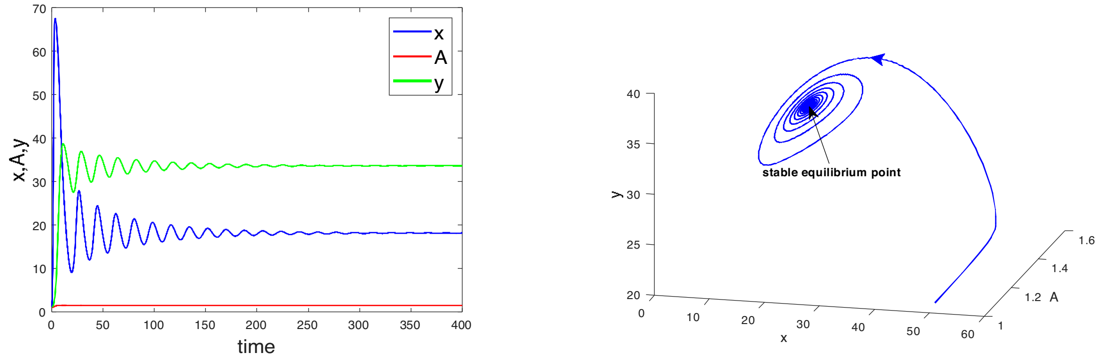

Fig. 7.17: E^* is locally asymptotically stable when $\tau_2 = 0.42$ is fixed in its stable range $(0, \tau'_{2_0})$ and $\tau_1 = 0.1 < \tau_{1_0} = 0.1336$.

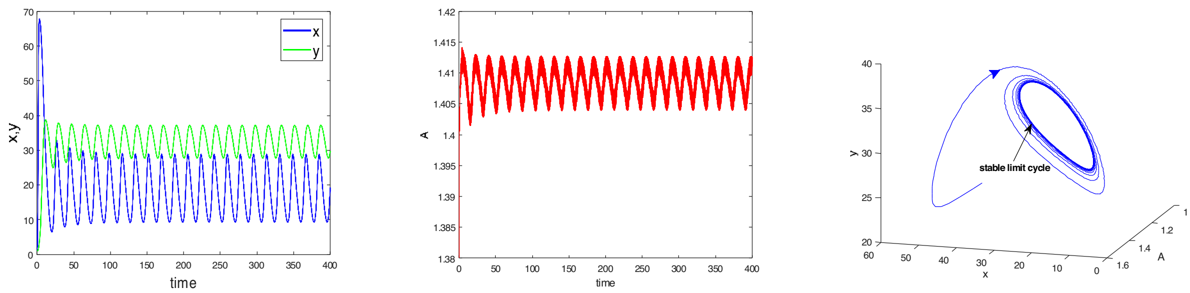

Fig. 7.18: E^* is unstable when $\tau_2 = 0.42$ is fixed in its stable range $(0, \tau_{2_0})$ and $\tau_1 = 0.2 > \tau'_{1_0} = 0.1336$. Time series solution of species and existence limit cycle (right).

As the system (7.2) shows Hopf-bifurcation with respect to both the delay parameters τ_1 and τ_2. Therefore, we can bisect the $\tau_1\tau_2-$plane into two regions, which are separated by Hopf-bifurcation curve.

Region of stability (sky blue) $S_3 = \{(\tau_1, \tau_2) :$ system (7.2) is locally asymptotically stable$\}$,

Region of instability (white) $S_4 = \{(\tau_1, \tau_2) :$ system (7.2) is unstable$\}$.

Both the regions are drawn in Fig. 7.20.

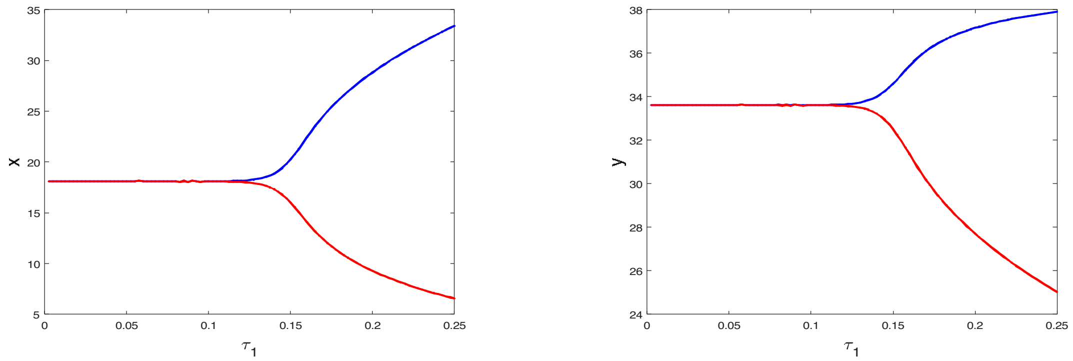

Fig. 7.19: Bifurcation diagram of the prey and predator species with respect to parameter τ_1 when $\tau_2 = 0.42$ is fixed in its stable range $(0, \tau_{2_0})$.

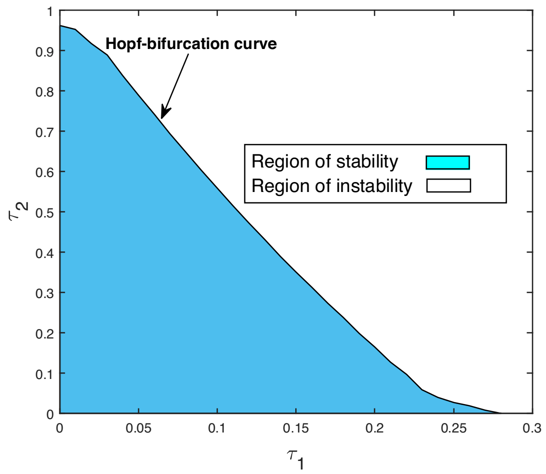

Fig. 7.20: Region of stability and instability for system (7.2) in $\tau_1\tau_2$-plane.

7.7 Conclusion

In this chapter, we have considered a habitat where two biological populations, prey population x and predator population y are surviving and interacting with each other. It is assumed that prey population follows logistic growth in the absence of predator and in the presence of predator, the interaction between them follows Holling type II functional response. We have shown the positivity, boundedness and persistence of the system, which implies that the proposed model is ecologically wellposed. We have defined a parameter A_0 $(0 \leq A_0 \leq 1)$ which denotes the dependency of predators on supplied additional food. Our system has four kinds of equilibria, trivial equilibrium $E_0(0,0,0)$, axial equilibrium $E_1(K,0,0)$, two prey free equilibria $\hat{E}_2$ and $\tilde{E}_2$ under condition (7.5) and unique positive equilibrium E^* under conditions (7.9) and (7.10). Local and global stability of the positive equilibrium are shown under several conditions which are dependent upon the parameter A_0. The parameter A_0 is crucial, so we have studied its effect via Hopf-bifurcation analysis which is also condescend by the numerical illustration. For a chosen set of parameters we calculated the threshold value of parameter A_0, that is $A_0 = 0.482$, where Hopf-bifurcation occurs and system stabilizes. It is also observed that after stabilization of system if predators are more dependent on additional food then prey population increase whereas predators remain in their range. We also have studied the Hopf-bifurcation with respect to consumption rate of additional food ϕ. Threshold value of ϕ is obtained as $\phi = 0.02847$. In table 7.3, we have shown the different number of positive equilibrium points by varying the parametric values, when $a = 0.105$ and $d = 0.1$ (other parameters are same as in (7.25)) then our system has two stable equilibrium together, therefore system shows the phenomenon of bistability, which is depicted in Fig 7.7.

Models with delay show comparatively more realistic dynamics than non delayed models. When a predator consumes a prey individual, then its effect does not come immediately, it takes some time i.e. time lag for gestation. Again, predators also take some time to consume and digest the supplied additional food to them. Therefore, to make our model ecologically more realistic, we incorporated two delays; one for gestation delay and other for consuming and digesting the supplied additional food.

For the delayed model, we have analyzed Hopf-bifurcation via local stability taking delay as a bifurcation parameter. We investigated the Hopf- bifurcation phenomenon for all combinations of both delays. We obtained the sufficient conditions for the stability of the positive equilibrium point and existence of Hopf-bifurcation for Case(1): $\tau_1 > 0,\ \tau_2 = 0$, Case(2): $\tau_1 = 0,\ \tau_2 > 0$, Case(3): τ_1 is fixed in the interval $(0, \tau_{1_0})$ and τ_2 as a variable parameter, Case(4): τ_2 is fixed in the interval $(0, \tau_{2_0})$ and τ_1 as a variable parameter. Our system undergoes Hopf-bifurcation in the vicinity of the interior equilibrium point with respect to both the delay parameters when they cross their critical values. The qualitative properties of Hopf-bifurcation are studied by using the Normal form theory and the formulae given in Hassard *et al.* [80].

We have performed some numerical simulations to illustrate our theoretical results. For a biologically feasible set of parameters, the system is stable initially, then we introduce delay and system remains stable till its critical value. If we increase the delay parameter over the critical value, then system goes through Hopf-bifurcation and becomes unstable. Bifurcation diagrams (Fig.7.10,7.13,7.16,7.19) with respect to different delays depict the dynamical behavior of the system.

Our study is important to conserve the prey population through providing additional food to predators and to establish their balance. Here, we have also shown the significance of delay parameters. We hope that this study will help to perceive the dynamics of an ecological system with additional food and two discrete delays.