Chapter 6

Conclusion and Future Scope

6.1 Conclusion

This thesis is devoted to study different periodically driven quantum systems. These types of systems are studied under Floquet theory formalism. We have also done the same and studied some quantum systems extensively under the Floquet theory. A major goal of the Floquet theory is to derive the effective time independent Hamiltonian of the time-periodic Hamiltonian. Very recently, numerous papers have been published on this subject. The major goal of these studies is to design a periodic driving protocol and apply that to a static system and generate a desired effective Hamiltonian, which may not be possible to obtain by any static means.

In general, theoretically, it is not possible to derive the effective Hamiltonian exactly by any analytical means. Some perturbation theories are developed to derive the effective Hamiltonian approximately at a suitable limit. The high frequency driving is one such suitable limit where one can derive the effective Hamiltonian perturbatively using the inverse of the driving frequency as the perturbation parameter. However, there is a class of driven systems, particularly the quantum systems with two energy bands driven by the Dirac delta kind of periodic kicks, which can be studied analytically and possible to derive its effective Hamiltonian analytically exact way. We have studied extensively one such prototype system, i.e., periodically driven hard core bosons on a one-dimensional lattice with periodic boundary condition.

Most of the Floquet theory-based analysis of the periodically driven nonintegrable systems have obtained only the effective Hamiltonians, which are integrable. Here we have studied a couple kicked top system which is a nonintegrable system and shows chaos; its effective Hamiltonian, obtained at the high frequency limit, also shows chaos, and interestingly its energy spectrum follows a non-standard statistics with respect to a more recently introduced tenfold classification of the quantum spectrum.

We now summarize the thesis chapter wise. This thesis has total *six* chapters, which include an introductory chapter, a chapter on the mathematical formalism, and the present concluding chapter. Chapter 1 was the introduction chapter. In this chapter, we have introduced the problems addressed in this thesis and presented these problems in their proper perspectives. The next chapter, Chapter 2, is the backbone of this thesis. In this chapter, we have presented the mathematical formalism which we have used throughout this thesis. This chapter gives an overall picture of the Floquet theory and its exploitation in deriving the effective Hamiltonian at the high frequency limit using Van Vleck and Brillouin-Wigner based perturbation theories. In chapters 3 to 5, we have studied three physical systems from the Floquet theory perspectives.

In Chapter 3, we have studied the double kicked top system, where the systems experience two δ-kicks of opposite polarities within a single time-period. This system is interesting because its underlying classical dynamics is chaotic, but its quantum spectrum does not follow the standard statistics predicted in quantum chaos literature. Rather, the quantum spectrum of this system shows interesting butterfly like fractal spectrum. This property remains preserved in the spectrum of its effective Hamiltonian. We have presented a detailed study of the energy eigenvalues and the eigenvectors of the effective Hamiltonian. Our particular finding is that the spectrum is multifractal, and thus it is described by multiple scales. Instead of studying only the local properties, we have considered the whole butterfly spectrum as a single object and investigated its (multi)fractal properties extensively. We have found that this butterfly spectrum shares similar number theoretical property with the celebrated Hofstadter butterfly spectrum.

Chapter 4 has presented our results of the hard core bosonic (HCB) system under the δ-kicked onsite staggered potential. We have considered two driving protocols: single kicked driving and double kicked driving, where two kicks are acting on the system at a different instant and opposite polarity. This system is a two-bands model, and therefore, its Hamiltonian in the momentum space can be expressed as a linear combination of the Pauli matrices. Exploiting the δ-kicked driving and the algebraic properties of the Pauli matrices, we have derived the effective Hamiltonian analytically exact for both the driving schemes. Here we have mainly studied two observables: particle current flowing through the system and the work done on the system by the driving to maintain the current flow. In the case of the single kicked driving, we have observed decay of current flow through the system, which is a manifestation of the dynamical localization in the system. The same was reported in an earlier publication using slightly different mathematical formulation. In the case of the double kicked system, the second kick controls the effect of the first kick, and thus tunes the current flow through the system. This tuning was achieved by varying the driving strength and the time interval between the two kicks. We could manage to avoid any drop in the current by a particular tuning. In the other cases, we have observed partial decay in the current, which is the signature of the partial dynamical localization. A general behavior of this system is the anticorrelation between the current flow and the work done on the system. We have also studied the dynamics of the density of particles and observed a light-cone like expansion of the particles on the lattice. Finally, we have presented an overall behavior of the system by a phase diagram in a parameter space. We have found that the superfluidity is the most common feature of the dynamics of this HCB system, whereas the Mott insulator was observed in a very small parameter regime. The intermediate of these two phases was also observed in a significant area of the parameter space.

In the penultimate chapter, which is in Chapter 5, we have studied the couple kicked top (CKT) system under the Floquet theory. This system is interesting because its effective time independent Hamiltonian at a very large frequency limit shows chaos. A special case of the obtained effective Hamiltonian was studied extensively earlier, and it is known as the Feingold-Peres (FP) model in the literature. The FP model is interesting because its energy spectrum follows one of the nonstandard tenfold statistics. Besides the FP limit, the effective Hamiltonian of the CKT also follows the nonstandard statistics at some other limit also.

6.2 Future Scope

In this thesis, we have mainly studied two classes of systems: a nonintegrable system, i.e., the kicked top (double kicked and coupled single kicked) and an integrable system, i.e., the hard core bosons. In the case of the double kicked top (DKT), the observed butterfly spectrum is different than the Hofstadter butterfly. Even though these two butterflies share certain number theoretical property, but their intricate features are different. In the case of the Hofstadter butterfly, its wings are clean due to the absence of any energy eigenvalues. This indicates the presence of gaps in the spectrum, even for the irrational flux strength. On the other hand, the wings of the DKT butterfly is not clean, which suggests that the spectrum is gapless. This feature was also revealed in the density of states of the spectrum. An extensive analysis of the presence of those energy levels at the wings is one immediate extension of the present studies. In *prima facie*, we expect that the reason behind these energy levels' appearance is due to the effect of the presence of edge states in the spectrum.

In the case of the hard core bosonic system, we can think of following important extensions:

1. Role of the interaction between the HCBs in the system. Even though two bosons cannot be at the same site due to the strong repulsive interaction, but we can think of the interaction between two bosons placed at two neighboring sites.

2. We can extend the driving from the onsite to the hopping part. In the case of the δ-kicked driving, this exchange of driving will not affect the dynamics. However, any other kind of driving protocol, including a sinusoidal driving, the driving induced hopping will generate different dynamics and also spectral properties. This particular type of driving is known as kinetic driving. Therefore, an extensive study of the kinetically driven HCB system can also be an important extension of this thesis.

In case of CKT, we can calculate the entanglement between the two coupled kicked tops. Entanglement is a unique property of coupled systems if we know the exact state of the system, it is in general not possible to assign any pure state to the subsystems.

Appendix A

Perturbation methods for calculation of H_{eff}

A.1 Van Vleck expansion based method

We use the following identities in the expression of $H_{\text{eff}} = \sum\limits_{0\leq n<\infty} \frac{1}{\omega^n} H^{(n)}$,

$$e^{iG} H e^{-iG} \; = \; H \; + \; i[G, H] \; - \; \frac{1}{2}\big[G, [G, H]\big] \; - \; \frac{i}{6}\,\big[G, \big[G, [G, H]\big]\big] \; + ... \tag{A.1}$$

and

$$\left(\frac{\partial}{\partial t} e^{iG}\right) e^{-iG} \; = \; i\left[\frac{\partial G}{\partial t}\right] - \frac{1}{2}\left[G, \frac{\partial G}{\partial t}\right] - \frac{i}{6}\left[G, \left[G, \frac{\partial G}{\partial t}\right]\right], \tag{A.2}$$

and consider terms upto $\mathcal{O}\big(\frac{1}{\omega^2}\big)$ which gives

$$\begin{aligned} H_{\text{eff}} = H_0 + V(t) + i\left[\frac{G^{(1)}}{\omega}, H\right] + i\left[\frac{G^{(2)}}{\omega}, H\right] - \frac{1}{2}\left[\frac{G^{(1)}}{\omega}\left[\frac{G^{(1)}}{\omega}, H\right]\right] - \frac{1}{\omega}\frac{\partial G^{(1)}}{\partial t} \\ - \frac{1}{\omega^2}\frac{\partial G^{(2)}}{\partial t} - \frac{1}{\omega^3}\frac{\partial G^{(3)}}{\partial t} - \frac{i}{2}\left[\frac{G^{(1)}}{\omega} + \frac{G^{(2)}}{\omega^2}, \frac{1}{\omega}\frac{\partial G^{(1)}}{\partial t} + \frac{1}{\omega^2}\frac{\partial G^{(2)}}{\partial t}\right] \\ + \frac{1}{6}\left[\frac{G^{(1)}}{\omega}, \left[\frac{G^{(1)}}{\omega}, \frac{1}{\omega}\frac{\partial G^{(1)}}{\partial t}\right]\right]. \end{aligned}$$

Here, $G^{(n)}$ is periodic $G^{(n)}(t+T) = G^{(n)}(t)$, and have zero mean over a time period T. Therefore $\langle G^{(n)} \rangle = 0$. At each order of perturbation, the time independent average is retained in H_{eff} and $\hat{G}$ is designed to nullify the time dependent part. For order ω^0

$$\begin{aligned} & H_0 + V(t) - \frac{1}{\omega}\frac{\partial G^{(1)}}{\partial t} \\ H_{\text{eff}}^{(0)} &= \left\langle H_0 + V(t) - \frac{1}{\omega}\frac{\partial G^{(1)}}{\partial t} \right\rangle \\ &= \langle H_0 \rangle + \langle V(t) \rangle - \left\langle \frac{1}{\omega}\frac{\partial G^{(1)}}{\partial t} \right\rangle \\ &= \frac{H_0}{T}\int_0^T dt + \frac{1}{T}\int_0^T V(t)dt - \frac{1}{\omega T}\int_0^T \frac{\partial G^{(1)}}{\partial t} dt. \end{aligned} \tag{A.3}$$

Potential $V(t)$ is periodic; $V(t+T) = V(t)$ and may be expanded in a Fourier series as

$$V(t) = V_0 + \sum_{1 \le n < \infty} V_n e^{in\omega t} + \sum_{1 \le < \infty} V_{-n} e^{-in\omega t}.$$

Similarly $G^1(t)$ can be expanded as Fourier series and has zero mean therefore $\frac{\partial G^{(1)}}{\partial t}$ also has zero mean. Therefore $H_{\text{eff}}^0 = H_0 + V_0$, where $V_0 = \frac{1}{T}\int_0^T V(t)dt$. Time dependent part can be written as

$$\begin{aligned} H_0 + V(t) - \frac{1}{\omega}\frac{\partial G^{(1)}}{\partial t} - \left\langle H_0 + V(t) - \frac{1}{\omega}\frac{\partial G^{(1)}}{\partial t} \right\rangle \\ = V(t) - V_0 - \frac{1}{\omega}\frac{\partial G^{(1)}}{\partial t}. \end{aligned} \tag{A.4}$$

By equating this time dependent part with zero, we have $V(t) = V_0 + \frac{1}{\omega}\frac{\partial G^{(1)}}{\partial t}$. Here $V(t) = V_0 + \sum_{1 \le n < \infty} V_n e^{in\omega t} + \sum_{1 \le n < \infty} V_{-n} e^{-in\omega t}$, therefore $\sum_{1 \le n < \infty} V_n e^{in\omega t} + \sum_{1 \le n < \infty} V_{-n} e^{-in\omega t} = \frac{1}{\omega}\frac{\partial G^{(1)}}{\partial t}$. As a result at order ω_0

$$\begin{aligned} H_{\text{eff}}^0 &= H_0 + V_0 \\ G^{(1)} &= \frac{1}{i}\sum_n \frac{1}{n}\left(V_n e^{in\omega t} - V_{-n} e^{-in\omega t}\right). \end{aligned} \tag{A.5}$$

Similarly, for order $\mathcal{O}(\omega^{-1})$

$$H^1_{\text{eff}} = \sum_n \frac{1}{\omega n}[V_n, V_{-n}]$$

and

$$\begin{aligned} G^{(2)} = &\frac{1}{i}\sum_n \frac{1}{n^2}\Big(\big[V_n, H_0 + V_0\big]e^{in\omega t} - \text{h.c.}\Big) \\ &+ \frac{1}{2i}\sum_{n,m=1}^{\infty}\frac{1}{n(n+m)}\Big(\big[V_n, V_m\big]e^{i(n+m)\omega t} + \text{h.c.}\Big) \\ &+ \frac{1}{2i}\sum_{n\neq m=1}^{\infty}\frac{1}{n(n+m)}\Big(\big[V_n, V_{-m}\big]e^{i(n-m)\omega t} + \text{h.c.}\Big). \end{aligned} \tag{A.6}$$

Following the same procedure, finally we obtain

$$\begin{aligned} H_{\text{eff}} = H_0 + V_0 + \frac{1}{\omega}\sum_{n=1}^{\infty}\frac{1}{n}[V_n, V_{-n}] + \frac{1}{2\omega^2}\sum_{n=1}^{\infty}\Big(\Big[\big[V_n, H_0\big], V_{-n}\Big] + \text{h.c.}\Big) \\ + \frac{1}{3\omega^2}\sum_{n,m=1}^{\infty}\frac{1}{nm}\Big(\Big[V_n, \big[V_m, V_{-(n+m)}\big]\Big] - 2\Big[V_n, \big[V_{-n}, V_{(n-m)}\big]\Big] + \text{h.c.}\Big)...., \end{aligned} \tag{A.7}$$

and

$$\begin{aligned} G(t) = &\frac{1}{i\omega}\sum_{n=1}^{\infty}\frac{1}{n}\Big(V_n e^{in\omega t} - V_{-n}e^{-in\omega t}\Big) + \frac{1}{i\omega^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\Big([V_n, H_0 + V_0]e^{in\omega t} - \text{h.c.}\Big) \\ &+ \frac{1}{2i\omega^2}\sum_{n,m=1}^{\infty}\frac{1}{n(n+m)}\Big([V_n, V_m]e^{i(n+m)\omega t} - \text{h.c.}\Big) \\ &+ \frac{1}{2i\omega^2}\sum_{n\neq m=1}^{\infty}\frac{1}{n(n-m)}\Big([V_n, V_{-m}]e^{i(n-m)\omega t} - \text{h.c.}\Big)...... \end{aligned} \tag{A.8}$$

Now, we consider a system with periodic Dirac δ driving and calculate the effective time independent Hamiltonian for the same. The Hamiltonian for the corresponding periodically driven system can be written as: $H(t) = H_0 + V_0\sum_n \delta(t - nT)$.

A.1.1 General expression of effective Hamiltonian for a periodically driven system with Dirac δ kick

A time dependent Hamiltonian for a system with periodic δ-function kick potential is written as

$$H(t) = H_0 + V(t), \tag{A.9}$$

where $V(t) = V \sum_n \delta(t - nT)$ can be expressed as Fourier series in terms of Fourier coefficients as $V(t) = V_0 + \sum_n (V_n\, e^{in\omega t} + V_{-n}\, e^{-in\omega t})$, where V_0, V_n and V_{-n} can be obtained as

$$\begin{aligned} V_0 &= \frac{1}{T}\int_0^T V(t)\, dt = \frac{V}{T}\int_0^T \delta(t - nT)\, dt = \frac{V}{T}, \\ V_n &= \frac{1}{T}\int_0^T e^{-in\omega t} V \delta(t - nT)\, dt = \frac{V}{T} e^{-in\omega T} = \frac{V}{T}, \\ V_{-n} &= \frac{1}{T}\int_0^T e^{in\omega t} V \delta(t - nT)\, dt = \frac{V}{T} e^{in\omega T} = \frac{V}{T}. \end{aligned} \tag{A.10}$$

Following the expression of H_{eff} given in Eq. (A.7), and also from the above equation $[V_n, V_{-n}] = 0$ and $[[V_n, H_0], V_{-n}] = \frac{1}{T^2}[[V, H_0], V]$, the effective Hamiltonian for the δ-kicked system becomes

$$H_{\text{eff}} = H_0 + \frac{V}{T} + \frac{1}{\omega^2 T^2}[[V, H_0], V] \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right) + \mathcal{O}\left(\frac{1}{\omega^3}\right), \tag{A.11}$$

where $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Finally we obtain the general expression of effective Hamiltonian H_{eff} as

$$H_{\text{eff}} = H_0 + \frac{V}{T} + \frac{1}{24}[[V, H_0], V] + \mathcal{O}\left(\frac{1}{\omega^3}\right). \tag{A.12}$$

A.2 Brillouin-Wigner Method

According to Brillouin-Wigner theory effective Hamiltonian can be obtained by $H_{\text{BW}} = \mathcal{P}\overline{H}\Omega_{\text{BW}}\mathcal{P}$, where $\mathcal{P}$ is the Projection operator and the wave operator Ω_{BW} is obtained by substituting the series of $1/\omega$ in Ω_{BW} as

$$\Omega_{\text{BW}} = \sum_{n=0}^{\infty} \Omega_{\text{BW}}^{(n)}, \tag{A.13}$$

where $\Omega_{\text{BW}}^{(n)}$ corresponds to $(1/\omega^n)$ coefficient in the iterative solution to Ω_{BW}, in the recursion relation of Ω_{BW}. Similarly, effective Hamiltonian can also be expanded in a series of $(1/\omega)$ as

$$H_{\text{BW}} = \sum_{n=0}^{\infty} H_{\text{BW}}^{(n)}, \tag{A.14}$$

and

$$H_{\text{BW}}^{(n)} = \mathcal{P}\overline{H}\Omega_{\text{BW}}^{(n)}\mathcal{P}. \tag{A.15}$$

Here $\overline{H} = (\mathcal{H} - \mathcal{M})$ plays the role of effective Hamiltonian in model space. Now, apply this method to the wave operator as

$$\sum_{n=0}^{\infty} \Omega_{\text{BW}}^{(n)} = \mathcal{P} + \sum_{n=0}^{\infty} \frac{\mathcal{Q}}{\mathcal{M}\omega}\overline{H}\Omega_{\text{BW}}^{(n)} + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\mathcal{Q}}{\mathcal{M}\omega}\Omega_{\text{BW}}^{(n)}\mathcal{P}\overline{H}\Omega_{\text{BW}}^{(n)}. \tag{A.16}$$

Now, by comparing the terms of same order of $(1/\omega)$, we obtain

$$\begin{aligned} \Omega_{\text{BW}}^{(0)} &= \mathcal{P} \\ \Omega_{\text{BW}}^{(1)} &= \frac{\mathcal{Q}}{\mathcal{M}\omega}\overline{H}\mathcal{P} \\ \Omega_{\text{BW}}^{(n+1)} &= \frac{\mathcal{Q}}{\mathcal{M}\omega}\overline{H}\Omega_{\text{BW}}^{(n)} - \sum_{m=0}^{\infty} \frac{\mathcal{Q}}{\mathcal{M}\omega}\Omega_{\text{BW}}^{(n)}\mathcal{P}\overline{H}\Omega_{\text{BW}}^{(n-m)} \end{aligned} \tag{A.17}$$