Chapter 7

Reliability and Vacation: The Critical Issue

7.1 Introduction

In today's highly technological environment, there is a requirement of the specific machining system for almost every task. Our dependence on machines is not only observed, but it is also increasing with the needs of our day-to-day life. But over time, all machines with their components will experience breakdowns, *i.e.* machines are unreliable. The random failure leads to unexpected failures, expensive repairs, costly replacement of their components, and even caused the significant interruption in the business of any industry. Also, the increasing complexity of fault-tolerant machining systems, as well as the cumulative cost incurred due to the failure of the operation, have brought the issue of reliability to the forefront. Hence, it creates a challenge for system analysts and engineers for the better reliability of machining systems, and as a result, researchers need to pay special attention to modeling and analysis. Besides this, time variability and random failure events also lead to interrupt the working of the redundant machining system. Over the last few decades, the reliability of machining systems has become a more challenging problem because the highly technological industries include more complex engineering systems with increasing levels of sophistication. Mathematically, reliability is the probability that the system will work properly without interruption over the time interval $[0,t)$ under certain operating conditions. Reliability is the much narrower concept than dependability and has the mathematical formulation in contrast to availability.

Under the preventive-maintenance policy, to avoid any loss of production, hindrance in functioning, data losses, monetary losses, the system designer always keeps

[0]The partial content of this chapter has been accepted in
C. Shekhar, **S. Varshney**, and A. Kumar, "Reliability and Vacation: The Critical Issue", *Advances in Reliability Analysis and its Applications*, Springer Series in Reliability Engineering, Springer Nature (Switzerland AG), (`https://www.springer.com/gp/book/9783030313746`).

some standby machines as the safety measure so that the standby machine can immediately act as the substitute when the operating machine fails. In many critical applications of computer, communication, and manufacturing systems, the provision of standby redundancy besides active redundancy has been the essential architectural attribute for achieving high reliability and grade of service (GoS). The standby machines can be classified into three types, hot-standby, warm-standby, and cold-standby. If the failure rate of the standby machine is similar to the failure rate of the operating machine, then it is known as hot-standby. The warm-standby is the one whose failure rate is less than that of the operating machine and greater than zero. However, the failure rate of the cold-standby is zero. The standbys to be considered in the present investigation are warm-standby and can be extended for other types just by setting the failure rate as per need. Queueing modeling is being used tremendously and effectively in congestion problems encountered due to the failure of machines in day-to-day life as well as industrial scenario, including manufacturing systems, computer systems, web services, and communication networks. In this chapter, we consider the fundamental machine repair problem where the group of the finite number of identical and unreliable operating machines functions in the parallel, is under the supervision of the repairman in the repair facility as corrective maintenance measure and provision of the finite number of standby machines as the preventive maintenance measure.

Under the strategic-maintenance policy, in many practical machining/service systems, the repairman may become unavailable deliberately for some time due to the variety of reasons like minimizing the idle period, utilizing the efficiency of repairman for secondary services, etc. The vacation is defined as the random period when the maintenance specialist is unavailable for the random period in the repair facility. The duration of the vacation of the repairman is a critical issue in the redundant repairable machining system. Longer vacation period increases the loss of production/revenue and cost incurred due to the failed machines in the queue, while the shorter vacation period increases idleness and cost of the system. Optimal vacation policies for the repairman balance among the loss of production, idleness of repairman, loss of revenue, and extra cost. Vacation policies like N-policy, single vacation, multiple vacations, Bernoulli vacation, and working vacation et cetera are some important strategies to leave the repair facility for the vacation of random length. Each vacation policy differs from other policies either on starting or terminating criterion. Besides the expected total cost, all vacation policies directly affect the reliability characteristics also.

In the forthcoming sections, we analyze the effect of different vacation policies on the reliability characteristics of the fault-tolerant machining system via the queueing-theoretic approach. Based on the rule of the resumption of service after the vacation, the machine repair models have been analyzed in different categories, which are given below. For that purpose, we first introduce the underlying machine repair problem, and then we implement the machine repair model with several types of vacation policies.

7.2 Machine Repair Problem (MRP)

Machine repair problem (MRP) is the typical example of the real-time finite population queueing model, where the machines represent the population of prospective customers, the arrival corresponds to the failure of the machine in the system, and the caretaker who provides the repair to the failed machines is known as server/repairman. Now, for the analysis, we have considered the finite population machine repair problem consisting of M identical operating machines and S warm standby machines under the care of the single repairman. When the operating machine breakdowns, it is immediately replaced perfectly by the available standby machine with the negligible switch-over time and has the same failure characteristics as of the operating machine. In the normal mode, the system requires M operating machines, but it continues to function properly until there is at least $m\,(1 \leq m < M)$ machines remain in the system, *i.e.* the system also works in short mode with the increased likelihood of the failure. Therefore, for the well functioning of the system, only $K = M + S - m + 1$ machines are allowed to fail. For the modeling purpose, we assume that the time-to-failure of the operating machine as well as standby machine follow the exponential distribution with failure rates λ & $\nu\,(0 < \nu < \lambda)$ and the time-to-repair of the failed machine is identically and independently exponentially distributed random variate with rate μ. Hence, the overall state-dependent failure rate of machines is represented as

$$\lambda_n = \begin{cases} M\lambda + (S-n)\nu; & 0 \leq n < S \\ (M+S-n)\lambda; & S \leq n < K \end{cases}$$

Some good reference books for machine repair problem via queueing-theoretic approach have been authored by following learners (*cf.* [55], [29], [53], [192], [87], [193], [86]). During last few decades, the MRP with several essential and/or optional queueing terminologies has been investigated by many researchers (*cf.* [26], [265], [245], [167], [97], [59], [127], [187], [104], [293], [217], [201], [72], [286], [158], [128], [207]).

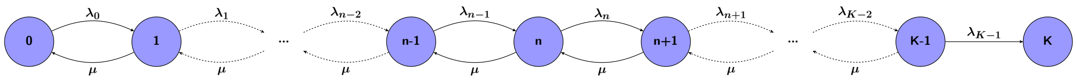

Figure 7.1: State transition diagram of the basic machine repair problem

For the reliability analysis at time instant t, we define the state probabilities of the system at time t as $P_n(t) = \Pr[N(t) = n; n = 0, 1, \cdots, K]$, which represent that there are n failed machines in the system at time t. For the better understanding of the governing model, the state transition diagram for continuous time Markov chain (CTMC) involved in machine repair model is depicted in Fig. 7.1.

Using the concept of quasi-birth and death (QBD) process and balancing the input and output flow in Fig. 7.1, the governing time-dependent Chapman-Kolmogorov differential-difference equations are constructed as follows.

$$\frac{dP_0(t)}{dt} = -\lambda_0 P_0(t) + \mu P_1(t) \tag{7.1}$$

$$\frac{dP_n(t)}{dt} = -(\lambda_n + \mu)P_n(t) + \lambda n - 1P_{n-1}(t) + \mu P_{n+1}(t); \quad 1 \le n \le K-2 \tag{7.2}$$

$$\frac{dP_{K-1}(t)}{dt} = -(\lambda_{K-1} + \mu)P_{K-1}(t) + \lambda_{K-2}P_{K-2}(t) \tag{7.3}$$

$$\frac{dP_K(t)}{dt} = \lambda P_{K-1}(t) \tag{7.4}$$

Now, one can easily transform the above mentioned system of differential-difference equations into the matrix form as follows

$$\mathbf{DX} = \mathbf{Q}_1\mathbf{X} \tag{7.5}$$

where $\mathbf{X}$ is the column vector of all the time-dependent state probabilities and $\mathbf{DX}$ is the derivative of the probability vector $\mathbf{X}$ while $\mathbf{Q}_1$ be the block-square matrix which is obtained by using the tri-diagonal characteristics of the above system of differential-difference equations. The structure of the block-square matrix $\mathbf{Q}_1$ is represented as follows.

$$\mathbf{Q}_1 = \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{C}_1 & \mathbf{D}_1 \end{bmatrix}$$

where $\mathbf{A}_1$ is the scalar matrix, $\mathbf{B}_1$ & $\mathbf{C}_1$ are the vectors of order $(1 \times K)$ & $(K \times 1)$ respectively. and $\mathbf{D}_1$ is the tri-diagonal square matrix of order $(K \times K)$. The structures of these block sub-matrices are given as follow.

$$\mathbf{A}_1 = [-\lambda_0]\,;\ \mathbf{B}_1 = [\lambda_0, 0, 0, \cdots, 0]\,;\ \mathbf{C}_1 = [\mu, 0, 0, \cdots, 0]^T$$

and

$$\mathbf{D}_1 = \begin{bmatrix} u_1^1 & v_1^1 & 0 & \cdots & 0 & 0 \\ w_2^1 & u_2^1 & v_2^1 & \cdots & 0 & 0 \\ 0 & w_3^1 & u_3^1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & u_{K-1}^1 & v_{K-1}^1 \\ 0 & 0 & 0 & \cdots & w_K^1 & u_K^1 \end{bmatrix}$$

where

$$u_n^1 = \begin{cases} -(\lambda_n + \mu); & 1 \le n \le K-1 \\ 0; & \text{otherwise} \end{cases}$$

$$v_n^1 = \begin{cases} \lambda_n; & 1 \le n \le K-1 \\ 0; & \text{otherwise} \end{cases}$$

$$w_n^1 = \begin{cases} \mu; & 2 \le n \le K-1 \\ 0; & \text{otherwise} \end{cases}$$

The system performance measures are scaled to calculate the efficiency of the system. These performance measures may either be quantitative or qualitative and help the system designers to rank the complex machining/service systems. Following are some basic queueing/reliability measures which are required to analyze the machine repair problem.

- Expected number of failed machines in the system at time t

$$E_N(t) = \sum_{n=0}^{K} nP_n(t) \tag{7.6}$$

- Throughput of the system at time t

$$\tau_p(t) = \sum_{n=1}^{K-1} \mu P_n(t) \tag{7.7}$$

- Probability that the repairman is on vacation/idle

$$P_V(t) = P_0(t) \tag{7.8}$$

- Reliability of the system

$$R_Y(t) = 1 - P_K(t) \tag{7.9}$$

- Mean time-to-failure of the system

$$MTTF = \int_0^\infty R_Y(t)dt \tag{7.10}$$

- Failure frequency of the system at time t

$$FF(t) = \lambda_{K-1} P_{K-1}(t) \tag{7.11}$$

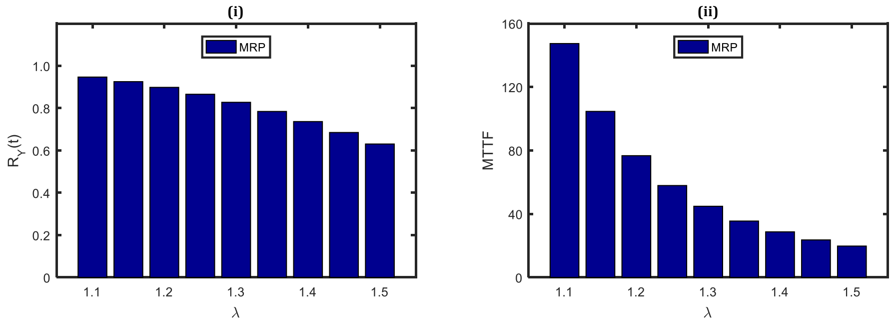

Figure 7.2: Effect of failure rate λ on reliability and $MTTF$ of the system respectively for MRP

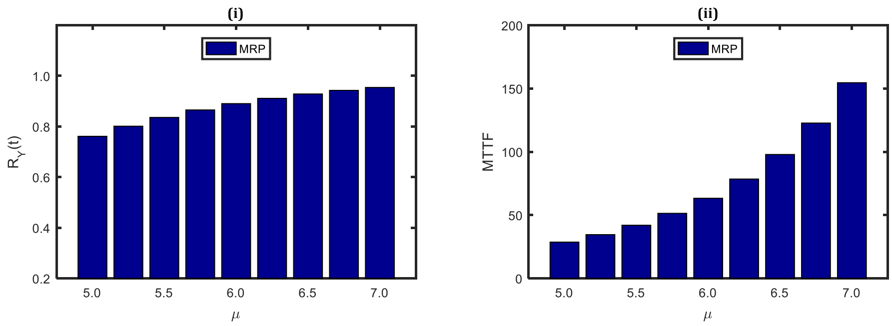

Figure 7.3: Effect of repair rate μ on reliability and $MTTF$ of the system respectively for MRP

For the comparative analysis purpose, first we fix the default values of several system parameters as $M = 10$, $S = 5$, $m = 2$, $\nu = 0.2$ and vary the values of λ from 1.1 to 1.5 with $\mu = 10$ and μ from 5 to 7 with $\lambda = 0.8$ for all the test instances. We consider same default values for other MRP with different vacation policies in forthcoming sections also. In practice, the incremental changes in failure rate and service rate show the decreasing and increasing effects on the reliability characteristics of the machining systems. From Fig. 7.2, the obvious results are observed that on increasing the value of λ, the reliability of the system and $MTTF$ decreases. Similarly, in accordance with Fig. 7.3, the reliability and $MTTF$ of the system increase correspondingly with incremental change in service rate μ.

In the following sections, we develop the queueing models for MRP with different vacation policies, and comparative analysis of reliability characteristics is done. For that purpose, we consider the pre-define assumptions, notations, performance measures, and default parameters value for forthcoming MRP models also.

7.3 MRP with N-Policy

In the redundant repairable machining systems, if the repairman becomes unavailable at the end of the busy period and resumes repairing of the failed machines instantly when the queue length of failed machines reaches the critical number N, then it is known as N-policy vacation of the repairman. In the past, many of the researches have been done using the concept of N-policy in the queueing literature. For the quick glance, refer [103], [266], [93], [146], [267], [316], [117], [283], [236], [116], [41], [20] and references therein.

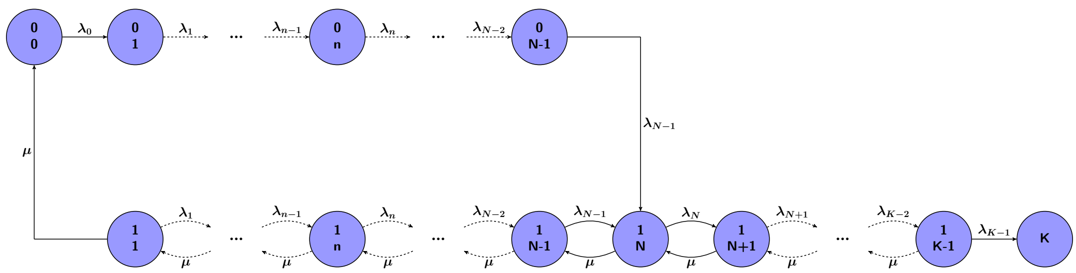

Figure 7.4: State transition diagram of the machine repair model with N-policy

Define the state of the repairman and state probabilities as follows

$$I(t) = \begin{cases} 0; & \text{Repairman is in vacation state} \\ 1; & \text{Repairman is in working state} \end{cases}$$

and

$$P_{0,n}(t) = \Pr[I(t) = 0 \,\text{and}\, N(t) = n;\, n = 0, 1, \cdots, N-1]$$
$$P_{1,n}(t) = \Pr[I(t) = 1 \,\text{and}\, N(t) = n;\, n = 0, 1, \cdots, K-1]$$
$$P_K(t) = \Pr[\text{The system is in failed state}]$$

Using the assumptions and notations of MRP model in previous section 7.2 and state transition diagram in Fig. 7.4, following governing differential-difference equations in terms of transient-state probabilities are developed.

$$\frac{dP_{0,0}(t)}{dt} = -\lambda_0 P_{0,0}(t) + \mu P_{1,1}(t) \tag{7.12}$$
$$\frac{dP_{0,n}(t)}{dt} = -\lambda_n P_{0,n}(t) + \lambda_{n-1} P_{0,n-1}(t); \quad 1 \le n \le N-1 \tag{7.13}$$
$$\frac{dP_{1,1}(t)}{dt} = -(\lambda_1 + \mu) P_{1,1}(t) + \mu P_{1,2}(t) \tag{7.14}$$

$$\frac{dP_{1,n}(t)}{dt} = -(\lambda_n+\mu)P_{1,n}(t)+\lambda_{n-1}P_{n-1}(t)+\mu P_{1,n+1}(t); \quad 2\le n\le N-1 \quad (7.15)$$

$$\frac{dP_{1,N}(t)}{dt} = -(\lambda_N+\mu)P_{1,N}(t)+\lambda_{N-1}P_{1,N-1}(t)+\lambda_{N-1}P_{0,N-1}(t)+\mu P_{1,N+1}(t) \quad (7.16)$$

$$\frac{dP_{1,n}(t)}{dt} = -(\lambda_n+\mu)P_{1,n}(t)+\lambda_{n-1}P_{1,n-1}(t)+\mu P_{1,n+1}(t); \quad N+1\le n\le K-2 \quad (7.17)$$

$$\frac{dP_{1,K-1}(t)}{dt} = -(\lambda_{K-1}+\mu)P_{K-1}(t)+\lambda_{K-2}P_{1,K-2}(t) \quad (7.18)$$

$$\frac{dP_K(t)}{dt} = \lambda_{K-1}P_{K-1}(t) \quad (7.19)$$

The generator matrix $\mathbf{Q}_2$ of the quasi-birth-death process for MRP with N-policy is as follows

$$\mathbf{Q}_2 = \begin{bmatrix} \mathbf{A}_2 & \mathbf{B}_2 \\ \mathbf{C}_2 & \mathbf{D}_2 \end{bmatrix}$$

where $\mathbf{A}_2$, $\mathbf{B}_2$, $\mathbf{C}_2$, and $\mathbf{D}_2$ are the block matrices of order $(N\times N)$, $(N\times K)$, $(K\times N)$, and $(K\times K)$ respectively and have the following matrix form.

$$\mathbf{A}_2 = \begin{bmatrix} x_1^2 & y_1^2 & 0 & \cdots & 0 & 0 \\ 0 & x_2^2 & y_2^2 & \cdots & 0 & 0 \\ 0 & 0 & x_3^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x_{N-1}^2 & y_{N-1}^2 \\ 0 & 0 & 0 & \cdots & 0 & x_N^2 \end{bmatrix}$$

$$x_n^2 = \begin{cases} -\lambda_{n-1}; & 1\le n\le N \\ 0; & \text{otherwise} \end{cases}$$

$$y_n^2 = \begin{cases} \lambda_{n-1}; & 1\le n\le N-1 \\ 0; & \text{otherwise} \end{cases}$$

and

$$\mathbf{B}_2[N,N] = \lambda_{N-1};\ \mathbf{C}_2[1,1] = \mu$$

$$\mathbf{D}_2 = \begin{bmatrix} u_1^2 & v_1^2 & 0 & \cdots & 0 & 0 \\ w_2^2 & u_2^2 & v_2^2 & \cdots & 0 & 0 \\ 0 & w_3^2 & u_3^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & u_{K-1}^2 & v_{K-1}^2 \\ 0 & 0 & 0 & \cdots & w_K^2 & u_K^2 \end{bmatrix}$$

$$u_n^2 = \begin{cases} -(\lambda_n + \mu); & 1 \le n \le K-1 \\ 0; & \text{otherwise} \end{cases}$$

$$v_n^2 = \begin{cases} \lambda_n; & 1 \le n \le K-1 \\ 0; & \text{otherwise} \end{cases}$$

$$w_n^2 = \begin{cases} \mu; & 2 \le n \le K-1 \\ 0; & \text{otherwise} \end{cases}$$

Using the transient-state probabilities obtained by above differential-differential equations on employing the Runge-Kutta method of the fourth order, performance measures for MRP with N-policy corresponds to MRP model are as follows.

$$E_N(t) = \sum_{n=0}^{N-1} nP_{0,n}(t) + \sum_{n=1}^{K-1} nP_{1,n}(t) \tag{7.20}$$

$$\tau_p(t) = \sum_{n=1}^{K-1} \mu P_{1,n}(t) \tag{7.21}$$

$$P_V(t) = \sum_{n=0}^{N-1} P_{0,n}(t) \tag{7.22}$$

$$R_Y(t) = 1 - P_K(t) \tag{7.23}$$

$$MTTF = \int_0^\infty R_Y(t)dt \tag{7.24}$$

$$FF(t) = \lambda_{K-1} P_{1,K-1}(t) \tag{7.25}$$

For the graphical analysis, we fix $N = 4$ besides default value of governing parameters in previous section. From Fig. 7.5, it is observed that on increasing the value of λ, the value of $R_Y(t)$ and $MTTF$ decreases but the observed values of these indices in the case of N-policy vacation queue is lesser than the basic machine repair model. Similarly, if we increase the value of μ, the corresponding increasing values of $R_Y(t)$ and $MTTF$ are observed which validates the mathematical formulation.

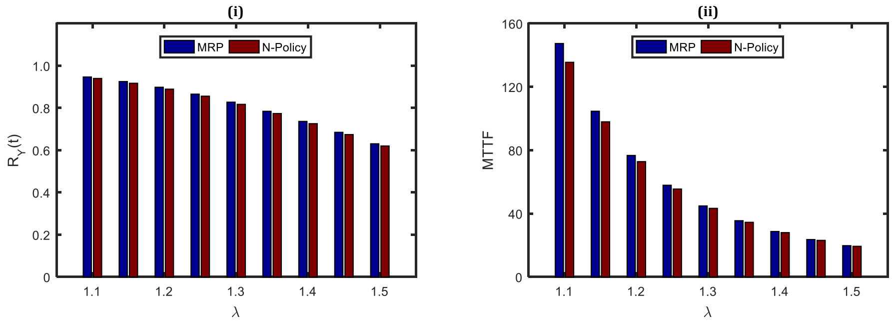

Figure 7.5: Effect of failure rate λ on reliability and $MTTF$ respectively wrt (i) MRP, and (ii) N-policy.

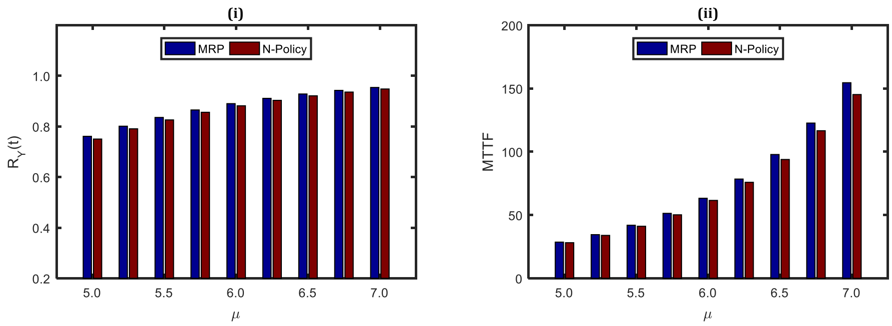

Figure 7.6: Effect of repair rate μ on reliability and $MTTF$ respectively wrt (i) MRP, and (ii) N-policy.

7.4 MRP with Bernoulli Vacation Policy (BV)

The repairing system in which the repairman decides whether to leave for the vacation of random duration with some certain probability p or continue to repair the next failed machine, if any, with the complementary probability $1-p$ after inspecting the system state is known as Bernoulli scheduled vacation of the repairman. The duration-of-vacation follows the exponential distribution with mean time $1/\theta$. To deal MRP and queueing problem with Bernoulli vacation, many studies have been done in the past also (*cf.* [31], [47], [48], [46], [183], [235], [213], [238], [42]).

For the Markovian analysis, we consider identical assumptions and notations for arrival process and service process as in previous sections. By using the appropriate transitions and balancing the input-output flow in Fig. 7.7, the governing system of

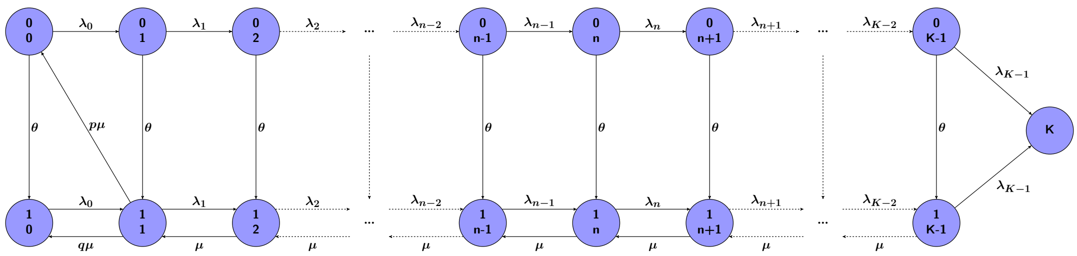

Figure 7.7: State transition diagram of the machine repair model with Bernoulli vacation policy

differential-difference equations in the transient state are constructed as follows.

$$\frac{dP_{0,0}(t)}{dt} = -(\lambda_0+\theta)P_{0,0}(t) + p\mu P_{1,1}(t) \tag{7.26}$$

$$\frac{dP_{0,n}(t)}{dt} = -(\lambda_n+\theta)P_{0,n}(t) + \lambda_{n-1}P_{0,n-1}(t); \quad 1 \le n \le K-1 \tag{7.27}$$

$$\frac{dP_{1,0}(t)}{dt} = -\lambda_0 P_{1,0}(t) + q\mu P_{1,1}(t) + \theta P_{0,0}(t) \tag{7.28}$$

$$\frac{dP_{1,n}(t)}{dt} = -(\lambda_n+\mu)P_{1,n}(t) + \lambda_{n-1}P_{1,n-1}(t) + \mu P_{1,n+1}(t) + \theta P_{0,n}(t); \quad 1 \le n \le K-2 \tag{7.29}$$

$$\frac{dP_{1,K-1}(t)}{dt} = -(\lambda_{K-1}+\mu)P_{1,K-1}(t) + \lambda_{K-2}P_{1,K-2}(t) + \theta P_{0,K-1}(t) \tag{7.30}$$

$$\frac{dP_K(t)}{dt} = \lambda_{K-1}P_{0,K-1}(t) + \lambda_{K-1}P_{1,K-1}(t) \tag{7.31}$$

To calculate the transient probabilities from the flow balance equations, we construct the generator matrix $\mathbf{Q}_3$ in the following form

$$\mathbf{Q}_3 = \begin{bmatrix} \mathbf{A}_3 & \mathbf{B}_3 & \mathbf{E}_3 \\ \mathbf{C}_3 & \mathbf{D}_3 & \mathbf{F}_3 \\ \mathbf{G}_3 & \mathbf{H}_3 & \mathbf{I}_3 \end{bmatrix}$$

where $\mathbf{A}_3$, $\mathbf{B}_3$, $\mathbf{C}_3$, and $\mathbf{D}_3$ are the square matrices of order $(K \times K)$, both $\mathbf{E}_3$ and $\mathbf{F}_3$ are the column vectors of order $(K \times 1)$ whereas the vectors $\mathbf{G}_3$, $\mathbf{H}_3$, $\mathbf{I}_3$ are the null vectors respectively.

$$\mathbf{A}_3 = \begin{bmatrix} x_1^3 & y_1^3 & 0 & \cdots & 0 & 0 \\ 0 & x_2^3 & y_2^3 & \cdots & 0 & 0 \\ 0 & 0 & x_3^3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x_{K-1}^3 & y_{K-1}^3 \\ 0 & 0 & 0 & \cdots & 0 & x_K^3 \end{bmatrix}$$

$$x_n^3 = \begin{cases} -(\lambda_{n-1}+\theta); & 1 \le n \le K \\ 0; & \text{otherwise} \end{cases}$$

$$y_n^3 = \begin{cases} \lambda_{n-1}; & 1 \le n \le K-1 \\ 0; & \text{otherwise} \end{cases}$$

and

$$\mathbf{B}_3 = diag[\theta,\theta,\cdots,\theta];\ \mathbf{C}_3[2,1] = p\mu$$

$$\mathbf{D}_3 = \begin{bmatrix} u_1^3 & v_1^3 & 0 & \cdots & 0 & 0 \\ w_2^3 & u_2^3 & v_2^3 & \cdots & 0 & 0 \\ 0 & w_3^3 & u_3^3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & u_{K-1}^3 & v_{K-1}^3 \\ 0 & 0 & 0 & \cdots & w_K^3 & u_K^3 \end{bmatrix}$$

$$u_n^3 = \begin{cases} -\lambda_0; & n=1 \\ -(\lambda_{n-1}+\mu); & 2 \le n \le K \\ 0; & \text{otherwise} \end{cases}$$

$$v_n^3 = \begin{cases} \lambda_{n-1}; & 1 \le n \le K-1 \\ 0; & \text{otherwise} \end{cases}$$

$$w_n^3 = \begin{cases} q\mu; & n=2 \\ \mu; & 3 \le n \le K \\ 0; & \text{otherwise} \end{cases}$$

$$\mathbf{E}_3 = \mathbf{F}_3 = [0,0,\cdots,0,\lambda_{K-1}]^T;\ \mathbf{G}_3 = \mathbf{H}_3 = [0,0,\cdots,0]^T;\ \mathbf{I}_3 = [0]$$

On the basis of computed transient-state probabilities, the expressions for system performance indices, $E_N(t)$, $P_V(t)$, $\tau_p(t)$, $FF(t)$, $R_Y(t)$ and $MTTF$, are delineated as follows.

$$E_N(t) = \sum_{j=0}^{1}\sum_{n=0}^{K-1} nP_{j,n}(t) + KP_K(t) \tag{7.32}$$

$$\tau_p(t) = \sum_{n=1}^{K-1} \mu P_{1,n}(t) \tag{7.33}$$

$$P_V(t) = \sum_{n=0}^{K-1} P_{0,n}(t) \tag{7.34}$$

$$R_Y(t) = 1 - P_K(t) \tag{7.35}$$

$$MTTF = \int_0^{\infty} R_Y(t)dt \tag{7.36}$$

$$FF(t) = \lambda_{K-1}P_{0,K-1}(t) + \lambda_{K-1}P_{1,K-1}(t) \tag{7.37}$$

For observing the comparative reliability characteristics for different vacation policies, we fix $p = 0.5$ & $\theta = 0.5$ and other default parameters as same as in the previous section and depict the results in Figs. 7.8 and 7.9 by varying the values of failure rate λ and service rate μ.

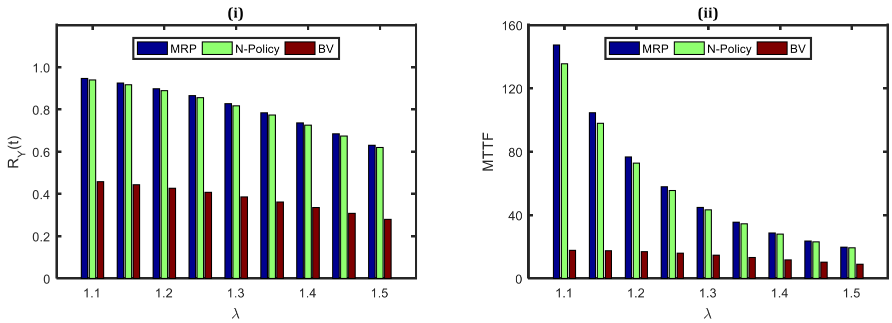

Figure 7.8: Effect of failure rate λ on reliability and $MTTF$ respectively wrt (i) MRP, (ii) N-policy, and (iii) BV.

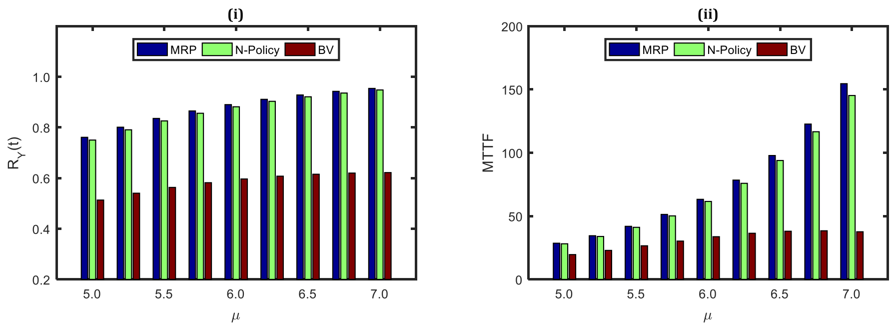

Figure 7.9: Effect of repair rate μ on reliability and $MTTF$ respectively wrt (i) MRP, (ii) N-policy, and (iii) BV.

For the varied values of λ & μ range from 1.1 to 1.5 & 5.0 to 7.0, respectively, we depict that the observed values of $R_Y(t)$ and $MTTF$ follow the obvious trend. For Bernoulli vacation policy, these reliability characteristics are quite lesser than the N-policy vacation.

7.5 MRP with Multiple Vacation Policy (MV)

In this section, we study the MRP with multiple vacation policy. According to this policy, the repairman leaves for the vacation of random duration when there is no failed machine for repair in the repair station. On return from the vacation, if the repairman finds no waiting failed machine in the system, he takes another vacation and continues this process until he finds at least one failed machine awaiting for repair. Numerous problems on machine repair model with multiple vacation policy have been proposed by many researchers in the queueing literature (*cf.* [259], [326], [129], [117], [130], [315], [180], [243], [20]).

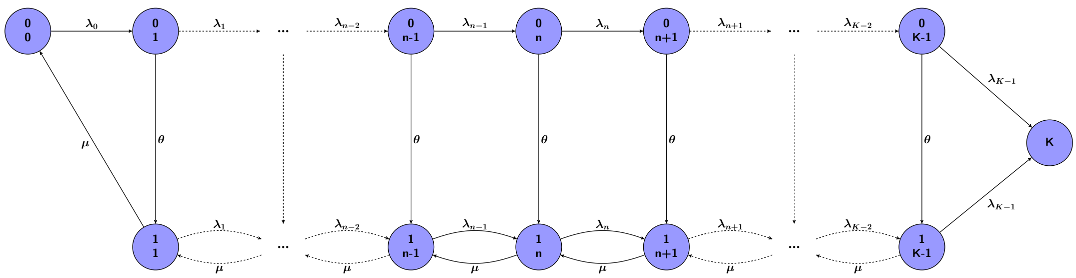

Figure 7.10: State transition diagram of the machine repair model with multiple vacation policy

The basic assumptions and notations are identical as for previous sections. Referring to the state transition diagram for the MRP with the multiple vacation policy shown in Fig. 7.10, we have following time-dependent Chapman Kolmogrove equations in terms of state probabilities.

$$\frac{dP_{0,0}(t)}{dt} = -\lambda_0 P_{0,0}(t) + \mu P_{1,1}(t) \tag{7.38}$$

$$\frac{dP_{0,n}(t)}{dt} = -(\lambda_n + \theta)P_{0,n}(t) + \lambda_{n-1}P_{0,n-1}(t); \quad 1 \leq n \leq K-1 \tag{7.39}$$

$$\frac{dP_{1,1}(t)}{dt} = -(\lambda_1 + \mu)P_{1,1}(t) + \mu P_{1,2}(t) + \theta P_{0,1}(t) \tag{7.40}$$

$$\frac{dP_{1,n}(t)}{dt} = -(\lambda_n + \mu)P_{1,n}(t) + \lambda_{n-1}P_{1,n-1}(t) + \mu P_{1,n+1}(t) + \theta P_{0,n}(t); \quad 2 \leq n \leq K-2 \tag{7.41}$$

$$\frac{dP_{1,K-1}(t)}{dt} = -(\lambda_{K-1} + \mu)P_{1,K-1}(t) + \lambda_{K-2}P_{1,K-2}(t) + \theta P_{0,K-1}(t) \tag{7.42}$$

$$\frac{dP_K(t)}{dt} = \lambda_{K-1}P_{0,K-1}(t) + \lambda_{K-1}P_{1,K-1}(t) \tag{7.43}$$

For employing the Runge-Kutta numerical method to compute the state probabilities, the corresponding matrix representation is provided for the set of differential equations for the MRP with multiple vacation policy

$$\mathbf{Q}_4 = \begin{bmatrix} \mathbf{A}_4 & \mathbf{B}_4 & \mathbf{E}_4 \\ \mathbf{C}_4 & \mathbf{D}_4 & \mathbf{F}_4 \\ \mathbf{G}_4 & \mathbf{H}_4 & \mathbf{I}_4 \end{bmatrix}$$

where $\mathbf{A}_4, \mathbf{B}_4, \mathbf{C}_4$, and $\mathbf{D}_4$ are the matrices of order $(K \times K)$, $(K \times K-1)$, $(K-1 \times K)$, and $(K-1 \times K-1)$ respectively, $\mathbf{E}_4$ and $\mathbf{F}_4$ are the column vectors of order $(K \times 1)$ and $(K-1 \times 1)$ respectively while $\mathbf{G}_4$, $\mathbf{H}_4$, and $\mathbf{I}_4$ are the null vectors. The structures of the block matrices are represented as

$$\mathbf{A}_4 = \begin{bmatrix} x_1^4 & y_1^4 & 0 & \cdots & 0 & 0 \\ 0 & x_2^4 & y_2^4 & \cdots & 0 & 0 \\ 0 & 0 & x_3^4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x_{K-1}^4 & y_{K-1}^4 \\ 0 & 0 & 0 & \cdots & 0 & x_K^4 \end{bmatrix}$$

$$x_n^4 = \begin{cases} -\lambda_0; & n = 1 \\ -(\lambda_{n-1} + \theta); & 2 \le n \le K \\ 0; & \text{otherwise} \end{cases}$$

$$y_n^4 = \begin{cases} \lambda_{n-1}; & 1 \le n \le K-1 \\ 0; & \text{otherwise} \end{cases}$$

and

$$\mathbf{B}_4[l, l-1] = \theta; \quad l = 2, 3, \cdots, K; \ \mathbf{C}_4[1,1] = \mu$$

$$\mathbf{D}_4 = \begin{bmatrix} u_1^4 & v_1^4 & 0 & \cdots & 0 & 0 \\ w_2^4 & u_2^4 & v_2^4 & \cdots & 0 & 0 \\ 0 & w_3^4 & u_3^4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & u_{K-2}^4 & v_{K-2}^4 \\ 0 & 0 & 0 & \cdots & w_{K-1}^4 & u_{K-1}^4 \end{bmatrix}$$

$$u_n^4 = \begin{cases} -(\lambda_n + \mu); & 1 \le n \le K-1 \\ 0; & \text{otherwise} \end{cases}$$

$$v_n^4 = \begin{cases} \lambda_n; & 1 \le n \le K-2 \\ 0; & \text{otherwise} \end{cases}$$

$$w_n^4 = \begin{cases} \mu; & 2 \le n \le K-1 \\ 0; & \text{otherwise} \end{cases}$$

$$\mathbf{E}_4 = [0,0,\cdots,0,\lambda_{K-1}]^T;\ \mathbf{F}_4 = [0,0,\cdots,0,\lambda_{K-1}]^T$$
$$\mathbf{G}_4 = [0,0,\cdots,0]^T;\ \mathbf{H}_4 = [0,0,\cdots,0]^T;\ \mathbf{I}_4 = [0]$$

Using transient-state probabilities derived from above generator matrix Q_4 on employing Runge-Kutta method of fourth order, the explicit expression of different system performance measures in closed form are as follow.

$$E_N(t) = \sum_{j=0}^{1}\sum_{n=j}^{K-1} nP_{j,n}(t) + KP_K(t) \tag{7.44}$$

$$\tau_p(t) = \sum_{n=1}^{K-1} \mu P_{1,n}(t) \tag{7.45}$$

$$P_V(t) = \sum_{n=0}^{K-1} P_{0,n}(t) \tag{7.46}$$

$$R_Y(t) = 1 - P_K(t) \tag{7.47}$$

$$MTTF = \int_0^\infty R_Y(t)dt \tag{7.48}$$

$$FF(t) = \lambda_{K-1}P_{0,K-1}(t) + \lambda_{K-1}P_{1,K-1}(t) \tag{7.49}$$

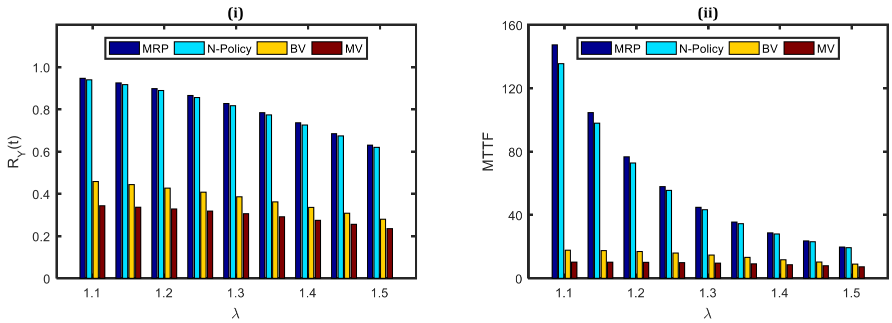

Figure 7.11: Effect of failure rate λ on reliability and $MTTF$ respectively wrt (i) MRP, (ii) N-policy, (iii) BV, and (iv) MV.

The effects of λ and μ on $R_Y(t)$ and $MTTF$ are shown in Figs. 7.11–7.12. For that purpose, we fix $\theta = 0.5$ and other parameters as in previous section as default value. These figures reveal that the values of reliability and $MTTF$ of the system are almost constant for increasing values of λ and μ for the multiple vacation case. Also, for the fixed value of λ and μ, we see that the values of $R_Y(t)$ and $MTTF$ are lesser as compared to other vacation policies as expected.

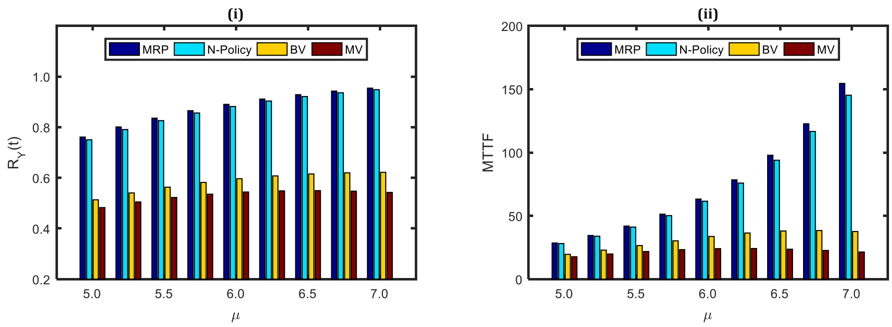

Figure 7.12: Effect of repair rate μ on reliability and $MTTF$ respectively wrt (i) MRP, (ii) N-policy, (iii) BV, and (iv) MV.

7.6 MRP with Single Vacation Policy (SV)

In the single vacation policy, the repairman takes exactly one vacation $i.e.$ at the end of the vacation of random duration, if the repairman finds no failed machine in the repair station for repair, he then waits idly in the system unless the prospect failed machine arrives in the repair station for the seek of repair. Over the last few decades, vacation queues have been emerged as the significant area of research in queueing literature. A number of researchers including [162], [191], [296], [277], [294], [291], [324] and many more studied the vacation queues with single vacation policy.

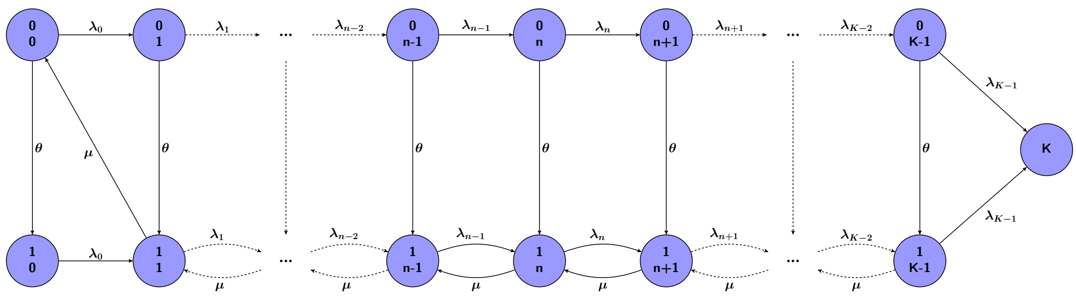

Figure 7.13: State transition diagram of the machine repair model with single vacation policy

Now, connecting the states of the system according to the transitions in Fig. 7.13, we obtain the following set of the transient system of differential equations.

$$\frac{dP_{0,0}(t)}{dt} = -(\lambda_0+\theta)P_{0,0}(t)+\mu P_{1,1}(t) \tag{7.50}$$

$$\frac{dP_{0,n}(t)}{dt} = -(\lambda_n+\theta)P_{0,n}(t)+\lambda_{n-1}P_{0,n-1}(t); \quad 1 \le n \le K-1 \tag{7.51}$$

$$\frac{dP_{1,0}(t)}{dt} = -\lambda_0 P_{1,0}(t) + \theta P_{0,0}(t) \tag{7.52}$$

$$\frac{dP_{1,n}(t)}{dt} = -(\lambda_n + \mu)P_{1,n}(t) + \lambda_{n-1}P_{1,n-1}(t) + \mu P_{1,n+1}(t) + \theta P_{0,n}(t); \quad 1 \le n \le K-2 \tag{7.53}$$

$$\frac{dP_{1,K-1}(t)}{dt} = -(\lambda_{K-1} + \mu)P_{1,K-1}(t) + \lambda_{K-2}P_{1,K-2}(t) + \theta P_{0,K-1}(t) \tag{7.54}$$

$$\frac{dP_K(t)}{dt} = \lambda_{K-1}P_{0,K-1}(t) + \lambda_{K-1}P_{1,K-1}(t) \tag{7.55}$$

The matrix-form is presented to compute the state probabilities through Runge-Kutta method from the resulting system of differential equations mentioned earlier. Using the results given by [200], the generator matrix $\mathbf{Q}_5$ of the Markov chain in MRP with single vacation is partitioned in the following form

$$\mathbf{Q}_5 = \begin{bmatrix} \mathbf{A}_5 & \mathbf{B}_5 & \mathbf{E}_5 \\ \mathbf{C}_5 & \mathbf{D}_5 & \mathbf{F}_5 \\ \mathbf{G}_5 & \mathbf{H}_5 & \mathbf{I}_5 \end{bmatrix}$$

where $\mathbf{A}_5$, $\mathbf{B}_5$, $\mathbf{C}_5$, and $\mathbf{D}_5$ are the square matrices of order $(K \times K)$, $\mathbf{E}_5$ and $\mathbf{F}_5$ are the column vectors of order $(K \times 1)$, and $\mathbf{G}_5$, $\mathbf{H}_5$ & $\mathbf{I}_5$ are the zero vectors.

$$\mathbf{A}_5 = \begin{bmatrix} x_1^5 & y_1^5 & 0 & \cdots & 0 & 0 \\ 0 & x_2^5 & y_2^5 & \cdots & 0 & 0 \\ 0 & 0 & x_3^5 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x_{K-1}^5 & y_{K-1}^5 \\ 0 & 0 & 0 & \cdots & 0 & x_K^5 \end{bmatrix}$$

$$x_n^5 = \begin{cases} -(\lambda_{n-1} + \theta); & 1 \le n \le K \\ 0; & \text{otherwise} \end{cases}$$

$$y_n^5 = \begin{cases} \lambda_{n-1}; & 1 \le n \le K-1 \\ 0; & \text{otherwise} \end{cases}$$

and

$$\mathbf{B}_5 = diag[\theta, \theta, \cdots, \theta]; \ \mathbf{C}_5[2,1] = \mu$$

$$\mathbf{D}_5 = \begin{bmatrix} u_1^5 & v_1^5 & 0 & \cdots & 0 & 0 \\ w_2^5 & u_2^5 & v_2^5 & \cdots & 0 & 0 \\ 0 & w_3^5 & u_3^5 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & u_{K-1}^5 & v_{K-1}^5 \\ 0 & 0 & 0 & \cdots & w_K^5 & u_K^5 \end{bmatrix}$$

$$u_n^5 = \begin{cases} -\lambda_0; & n = 1 \\ -(\lambda_{n-1} + \mu); & 2 \le n \le K \\ 0; & \text{otherwise} \end{cases}$$

$$v_n^5 = \begin{cases} \lambda_{n-1}; & 1 \le n \le K-1 \\ 0; & \text{otherwise} \end{cases}$$

$$w_n^5 = \begin{cases} \mu; & 3 \le n \le K \\ 0; & \text{otherwise} \end{cases}$$

$$\mathbf{E}_5 = \mathbf{F}_5 = [0, 0, \cdots, 0, \lambda_{K-1}]^T;\ \mathbf{G}_5 = \mathbf{H}_5 = [0, 0, \cdots, 0]^T;\ \mathbf{I}_5 = [0]$$

Hence, the explicit expressions in the closed form of system performance measures are derived in terms of state probabilities as follow.

$$E_N(t) = \sum_{j=0}^{1} \sum_{n=0}^{K-1} nP_{j,n}(t) + KP_K(t) \tag{7.56}$$

$$\tau_p(t) = \sum_{n=1}^{K-1} \mu P_{1,n}(t) \tag{7.57}$$

$$P_V(t) = \sum_{n=0}^{K-1} P_{0,n}(t) \tag{7.58}$$

$$R_Y(t) = 1 - P_K(t) \tag{7.59}$$

$$MTTF = \int_0^\infty R_Y(t)dt \tag{7.60}$$

$$FF(t) = \lambda_{K-1} P_{0,K-1}(t) + \lambda_{K-1} P_{1,K-1}(t) \tag{7.61}$$

For the study purpose, we set the parameters as same as in previous sections. Fig. 7.14 and 7.15 display the variation of reliability characteristics with respect to failure rate and repair rate respectively. It is observed that the single vacation is better than multiple vacation for enhancing the reliability measures.

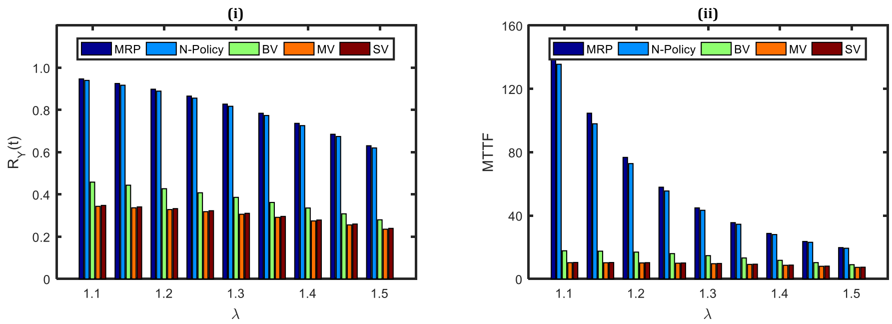

Figure 7.14: Effect of failure rate λ on reliability and $MTTF$ respectively wrt (i) MRP, (ii) N-policy, (iii) BV, (iv) MV, and (v) SV.

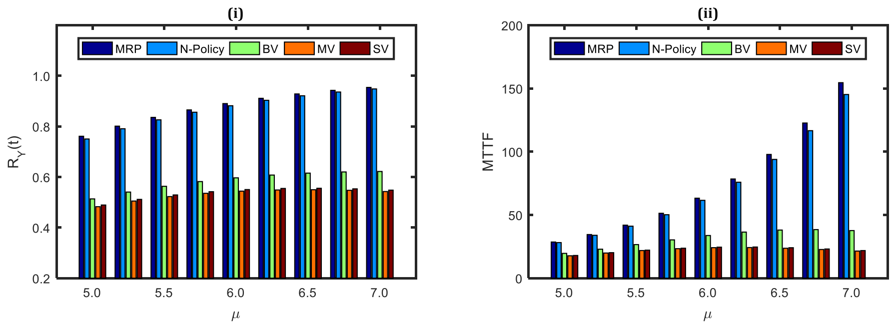

Figure 7.15: Effect of repair rate μ on reliability and $MTTF$ respectively wrt (i) MRP, (ii) N-policy, (iii) BV, (iv) MV, and (v) SV.

7.7 MRP with Multiple Working Vacation Policy (MWV)

The working vacation policy was firstly introduced by [222], in which the repairman works at a slower rate rather than completely terminating the repair in the vacation period. The repairman continues the repair at a slower rate until his random vacation period ends. There are mainly two special cases of the working vacation policy depending on vacation terminating criterion; the first one is multiple working vacation policy and the second is a single working vacation policy. In the multiple working vacation policy, at the end of the vacation period, if the repairman does not find any failed machine in the system for repair then he repeatedly takes the working vacation of random duration otherwise provides the repair to the failed machine if any present in the system with normal mean rate. Several researchers investigated the queueing problems with the multiple working vacation policy in the past (*cf.* [22], [262], [221], [89], [212]).

The basic assumptions and notations are simillar as in previous section. The time-to-repair in busy mode and vacation mode follow the exponential distribution with

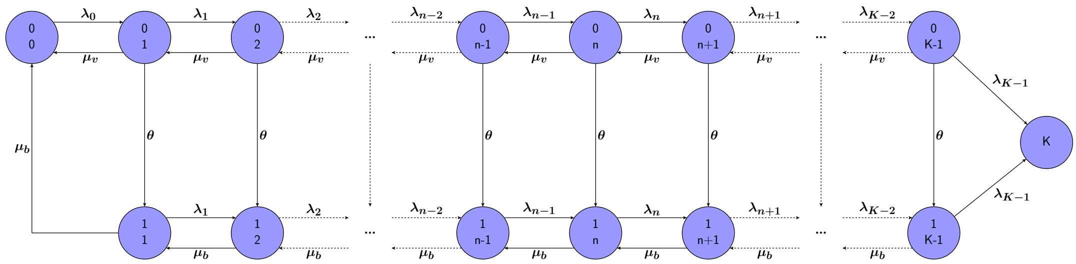

Figure 7.16: State transition diagram of the machine repair model with multiple working vacation policy

mean rate μ_b and μ_v respectively. The following transient-state differential equations are obtained by balancing the transitions at time instant t in Fig. 7.16.

$$\frac{dP_{0,0}(t)}{dt} = -\lambda_0 P_{0,0}(t) + \mu_v P0,1(t) + \mu_b P_{1,1}(t) \tag{7.62}$$

$$\frac{dP_{0,n}(t)}{dt} = -(\lambda_n + \mu_v + \theta)P_{0,n}(t) + \lambda_{n-1}P_{0,n-1}(t) + \mu_v P_{0,n+1}(t); \quad 1 \le n \le K-2 \tag{7.63}$$

$$\frac{dP_{0,K-1}(t)}{dt} = -(\lambda_{K-1} + \mu_v + \theta)P_{0,K-1}(t) + \lambda_{K-2}P_{0,K-2}(t) \tag{7.64}$$

$$\frac{dP_{1,1}(t)}{dt} = -(\lambda_1 + \mu_b)P_{1,1}(t) + \mu_b P_{1,2}(t) + \theta P_{0,1}(t) \tag{7.65}$$

$$\frac{dP_{1,n}(t)}{dt} = -(\lambda_n + \mu_b)P_{1,n}(t) + \lambda_{n-1}P_{1,n-1}(t) + \mu_b P_{1,n+1}(t) + \theta P_{0,n}(t); \quad 2 \le n \le K-2 \tag{7.66}$$

$$\frac{dP_{1,K-1}(t)}{dt} = -(\lambda_{K-1} + \mu_b)P_{1,K-1}(t) + \lambda_{K-2}P_{1,K-2}(t) + \theta P_{0,K-1}(t) \tag{7.67}$$

$$\frac{dP_K(t)}{dt} = \lambda_{K-1}P_{0,K-1}(t) + \lambda_{K-1}P_{1,K-1}(t) \tag{7.68}$$

Suppose that $\mathbf{Q}_6$ denotes the corresponding transition rate matrix, then $\mathbf{Q}_6$ can be expressed in block structures as

$$\mathbf{Q}_6 = \begin{bmatrix} \mathbf{A}_6 & \mathbf{B}_6 & \mathbf{E}_6 \\ \mathbf{C}_6 & \mathbf{D}_6 & \mathbf{F}_6 \\ \mathbf{G}_6 & \mathbf{H}_6 & \mathbf{I}_6 \end{bmatrix}$$

where the block-matrices $\mathbf{A}_6$, $\mathbf{B}_6$, $\mathbf{C}_6$, and $\mathbf{D}_6$ are of order $(K \times K)$, $(K \times K-1)$, $(K-1 \times K)$ and $(K-1 \times K-1)$ respectively, $\mathbf{E}_6$ and $\mathbf{F}_6$ are the column vectors of order $(K \times 1)$ and $(K-1 \times 1)$ respectively, and $\mathbf{G}_6$, $\mathbf{H}_6$, $\mathbf{I}_6$ are the zero vectors of appropriate order. These sub-matrices are given as follow.

$$\mathbf{A}_6 = \begin{bmatrix} x_1^6 & y_1^6 & 0 & \cdots & 0 & 0 \\ z_2^6 & x_2^6 & y_2^6 & \cdots & 0 & 0 \\ 0 & z_3^6 & x_3^6 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x_{K-1}^6 & y_{K-1}^6 \\ 0 & 0 & 0 & \cdots & z_K^6 & x_K^6 \end{bmatrix}$$

$$x_n^6 = \begin{cases} -\lambda_0; & n = 1 \\ -(\lambda_{n-1} + \mu_v + \theta); & 2 \le n \le K \\ \ \ 0; & \text{otherwise} \end{cases}$$

$$y_n^6 = \begin{cases} \lambda_{n-1}; & 1 \le n \le K-1 \\ 0; & \text{otherwise} \end{cases}$$

$$z_n^6 = \begin{cases} \mu_v; & 2 \le n \le K \\ 0; & \text{otherwise} \end{cases}$$

and

$$\mathbf{B}_6[l, l-1] = \theta; \quad l = 2, 3, \cdots, K; \ \mathbf{C}_6[1,1] = \mu_b$$

$$\mathbf{D}_6 = \begin{bmatrix} u_1^6 & v_1^6 & 0 & \cdots & 0 & 0 \\ w_2^6 & u_2^6 & v_2^6 & \cdots & 0 & 0 \\ 0 & w_3^6 & u_3^6 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & u_{K-2}^6 & v_{K-2}^6 \\ 0 & 0 & 0 & \cdots & w_{K-1}^6 & u_{K-1}^6 \end{bmatrix}$$

$$u_n^6 = \begin{cases} -(\lambda_n + \mu_b); & 1 \le n \le K-1 \\ \ \ 0; & \text{otherwise} \end{cases}$$

$$v_n^6 = \begin{cases} \lambda_n; & 1 \le n \le K-2 \\ 0; & \text{otherwise} \end{cases}$$

$$w_n^6 = \begin{cases} \mu_b; & 2 \le n \le K-1 \\ 0; & \text{otherwise} \end{cases}$$

$$\mathbf{E}_6 = [0,0,\cdots,0,\lambda_{K-1}]^T;\ \mathbf{F}_6 = [0,0,\cdots,0,\lambda_{K-1}]^T$$
$$\mathbf{G}_6 = [0,0,\cdots,0]^T;\ \mathbf{H}_6 = [0,0,\cdots,0]^T;\ \mathbf{I}_6 = [0]$$

The system performance measures of the machine repair model with multiple working vacation policy are given by

$$E_N(t) = \sum_{j=0}^{1}\sum_{n=j}^{K-1} nP_{j,n}(t) + KP_K(t) \tag{7.69}$$

$$\tau_p(t) = \sum_{n=1}^{K-1} \mu_v P_{0,n}(t) + \sum_{n=1}^{K-1} \mu_b P_{1,n}(t) \tag{7.70}$$

$$P_{WV}(t) = \sum_{n=0}^{K-1} P_{0,n}(t) \tag{7.71}$$

$$R_Y(t) = 1 - P_K(t) \tag{7.72}$$

$$MTTF = \int_0^\infty R_Y(t)dt \tag{7.73}$$

$$FF(t) = \lambda_{K-1}P_{0,K-1}(t) + \lambda_{K-1}P_{1,K-1}(t) \tag{7.74}$$

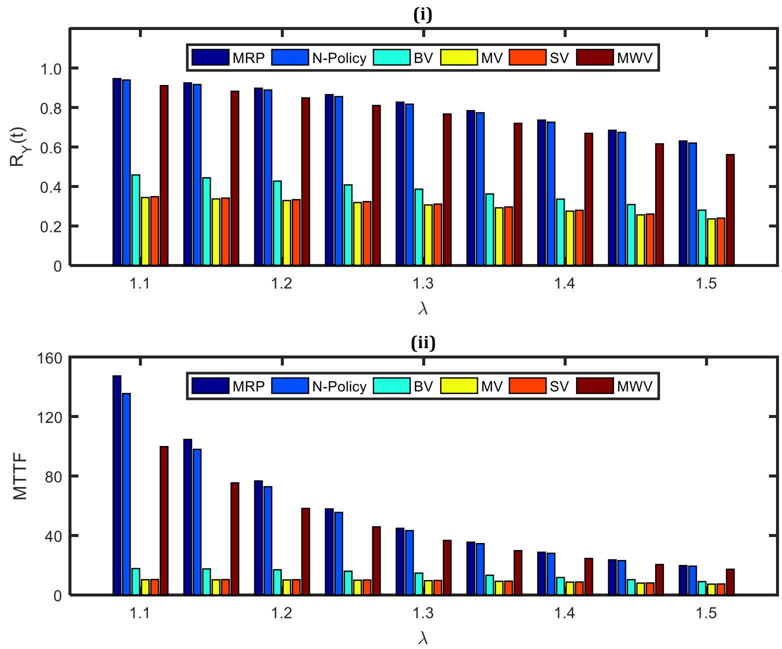

Figure 7.17: Effect of failure rate λ on reliability and $MTTF$ respectively wrt (i) MRP, (ii) N-policy, (iii) BV, (iv) MV, (v) SV, and (vi) MWV.

For the graphical comparative purpose, we fix the value of system parameters as in the last section along with $\mu_b = \mu = 10$ & $\mu_v = 8$ for the Fig. 7.17 wrt failure rate (λ) and $\mu_b = \mu = 6$ & $\mu_v = 4$ for Fig. 7.18 wrt service rate (μ or μ_b) respectively.

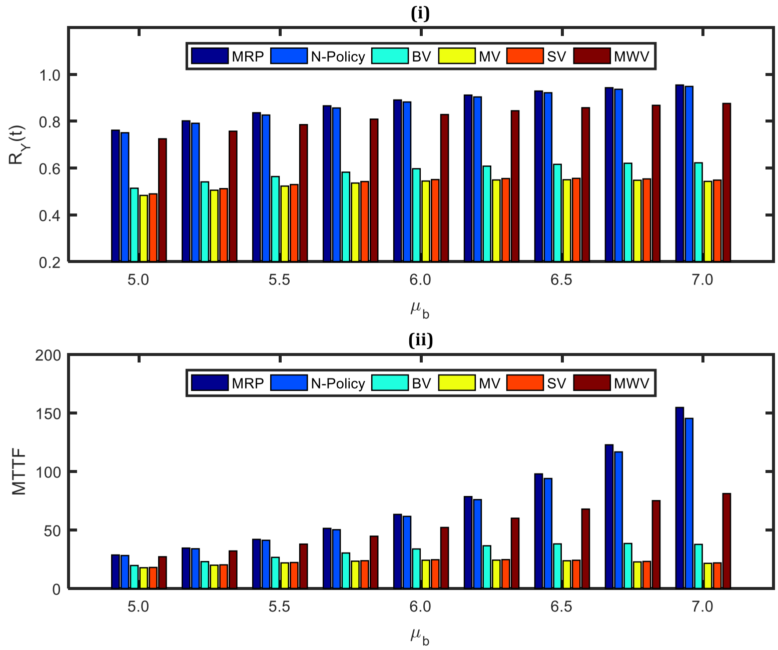

Figure 7.18: Effect of repair rate μ_b on reliability and $MTTF$ respectively wrt (i) MRP, (ii) N-policy, (iii) BV, (iv) MV, (v) SV, and (vi) MWV.

Fig. 7.17 and 7.18 portray the changing trend of reliability measures with respect to breakdown rate and repair rate. The inferences from these figures are obvious that reliability and mean time-to-failure are decreasing on the increased value of breakdown rate. To achieve the high reliability of the fault-tolerant machining system, high repair rate should be maintained. It also observed that multiple working vacation is better than the multiple vacation for the seek of the high level of reliability measures.

7.8 MRP with Single Working Vacation Policy (SWV)

In the single working vacation policy, the working vacation starts when there is no failed machine in the system at service completion instant. When the working vacation ends, if there is no failed machine for repair in the service station, the repairman stays in system idle and is ready for serving the newly arriving failed machines. Otherwise, after returning from the vacation if he finds the non-empty system, he immediately switches to the normal service rate. The number of research papers have appeared in the queueing literature in which the MRP and several Makov/Non-Markov queueing models with single working vacation are considered (*cf.* [177], [301], [171], [88], [89], [137]).

Fig. 7.19 represents the state transition diagram of the MRP with single working vacation policy. For the modeling purpose, we take assumptions and notations similar

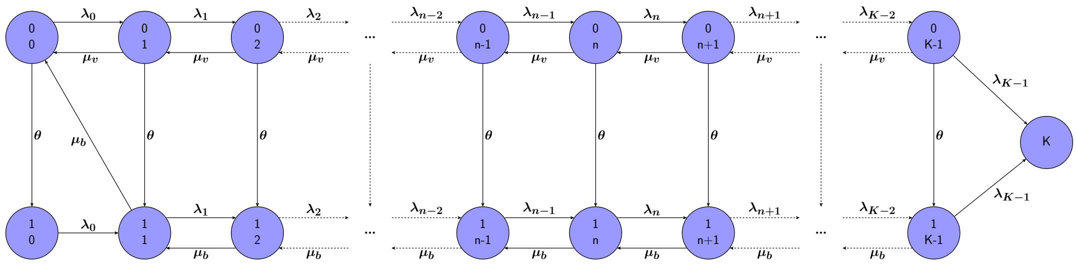

Figure 7.19: State transition diagram of the machine repair model with single working vacation policy

as in the previous section. The governing set of differential-difference equations is given as follows.

$$\frac{dP_{0,0}(t)}{dt} = -(\lambda_0+\theta)P_{0,0}(t)+\mu_v P_{0,1}(t)+\mu_b P_{1,1}(t) \tag{7.75}$$

$$\frac{dP_{0,n}(t)}{dt} = -(\lambda_n+\mu_v+\theta)P_{0,n}(t)+\lambda_{n-1}P_{0,n-1}(t)+\mu_v P_{0,n+1}(t); \quad 1\le n\le K-2 \tag{7.76}$$

$$\frac{dP_{0,K-1}(t)}{dt} = -(\lambda_{K-1}+\mu_v+\theta)P_{0,K-1}(t)+\lambda_{K-2}P_{0,K-2}(t) \tag{7.77}$$

$$\frac{dP_{1,0}(t)}{dt} = -\lambda_0 P_{1,0}(t)+\theta P_{0,0}(t) \tag{7.78}$$

$$\frac{dP_{1,n}(t)}{dt} = -(\lambda_n+\mu_b)P_{1,n}(t)+\lambda_{n-1}P_{1,n-1}(t)+\mu_b P_{1,n+1}(t)+\theta P_{0,n}(t); \quad 1\le n\le K-2 \tag{7.79}$$

$$\frac{dP_{1,K-1}(t)}{dt} = -(\lambda_{K-1}+\mu_b)P_{1,K-1}(t)+\lambda_{K-2}P_{1,K-2}(t)+\theta P_{0,K-1}(t) \tag{7.80}$$

$$\frac{dP_K(t)}{dt} = \lambda_{K-1}P_{0,K-1}(t)+\lambda_{K-1}P_{1,K-1}(t) \tag{7.81}$$

Using the lexicographical sequence of the states of the system, the generator matrix $\mathbf{Q}_7$ in the terms of block-matrices is represented as

$$\mathbf{Q}_7 = \begin{bmatrix} \mathbf{A}_7 & \mathbf{B}_7 & \mathbf{E}_7 \\ \mathbf{C}_7 & \mathbf{D}_7 & \mathbf{F}_7 \\ \mathbf{G}_7 & \mathbf{H}_7 & \mathbf{I}_7 \end{bmatrix}$$

where the matrices $\mathbf{A}_7$, $\mathbf{B}_7$, $\mathbf{C}_7$, and $\mathbf{D}_7$ are the block matrices of order $(K\times K)$, $\mathbf{E}_7$ and $\mathbf{F}_7$ are the column vectors of order $(K\times 1)$, and $\mathbf{G}_7$, $\mathbf{H}_7$, $\mathbf{I}_7$ are the null vectors.

$$\mathbf{A}_7 = \begin{bmatrix} x_1^7 & y_1^7 & 0 & \cdots & 0 & 0 \\ z_2^7 & x_2^7 & y_2^7 & \cdots & 0 & 0 \\ 0 & z_3^7 & x_3^7 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x_{K-1}^7 & y_{K-1}^7 \\ 0 & 0 & 0 & \cdots & z_K^7 & x_K^7 \end{bmatrix}$$

$$x_n^7 = \begin{cases} -(\lambda_0+\theta); & n=1 \\ -(\lambda_{n-1}+\mu_v+\theta); & 2\leq n\leq K \\ 0; & \text{otherwise} \end{cases}$$

$$y_n^7 = \begin{cases} \lambda_{n-1}; & 1\leq n\leq K-1 \\ 0; & \text{otherwise} \end{cases}$$

$$z_n^7 = \begin{cases} \mu_v; & 2\leq n\leq K \\ 0; & \text{otherwise} \end{cases}$$

and

$$\mathbf{B}_7 = diag[\theta,\theta,\cdots,\theta];\ \mathbf{C}_7[2,1]=\mu_b$$

$$\mathbf{D}_7 = \begin{bmatrix} u_1^7 & v_1^7 & 0 & \cdots & 0 & 0 \\ w_2^7 & u_2^7 & v_2^7 & \cdots & 0 & 0 \\ 0 & w_3^7 & u_3^7 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & u_{K-1}^7 & v_{K-1}^7 \\ 0 & 0 & 0 & \cdots & w_K^7 & u_K^7 \end{bmatrix}$$

$$u_n^7 = \begin{cases} -\lambda_0; & n=1 \\ -(\lambda_{n-1}+\mu_b); & 2\leq n\leq K \\ 0; & \text{otherwise} \end{cases}$$

$$v_n^7 = \begin{cases} \lambda_{n-1}; & 1\leq n\leq K-1 \\ 0; & \text{otherwise} \end{cases}$$

$$w_n^7 = \begin{cases} \mu_b; & 3\leq n\leq K \\ 0; & \text{otherwise} \end{cases}$$

$$\mathbf{E}_7=\mathbf{F}_7=[0,0,\cdots,0,\lambda_{K-1}]^T;\ \mathbf{G}_7=\mathbf{H}_7=[0,0,\cdots,0]^T;\ \mathbf{I}_7=[0]$$

Following are some system performance measures on which the overall mathematical analysis of the machine repair model with single working vacation is depended.

$$E_N(t) = \sum_{j=0}^{1} \sum_{n=0}^{K-1} nP_{j,n} + KP_K(t) \tag{7.82}$$

$$\tau_p(t) = \sum_{n=1}^{K-1} \mu_v P_{0,n}(t) + \sum_{n=1}^{K-1} \mu_b P_{1,n}(t) \tag{7.83}$$

$$P_{WV}(t) = \sum_{n=0}^{K-1} P_{0,n}(t) \tag{7.84}$$

$$R_Y(t) = 1 - P_K(t) \tag{7.85}$$

$$MTTF = \int_0^{\infty} R_Y(t)dt \tag{7.86}$$

$$FF(t) = \lambda_{K-1} P_{0,K-1}(t) + \lambda_{K-1} P_{1,K-1}(t) \tag{7.87}$$

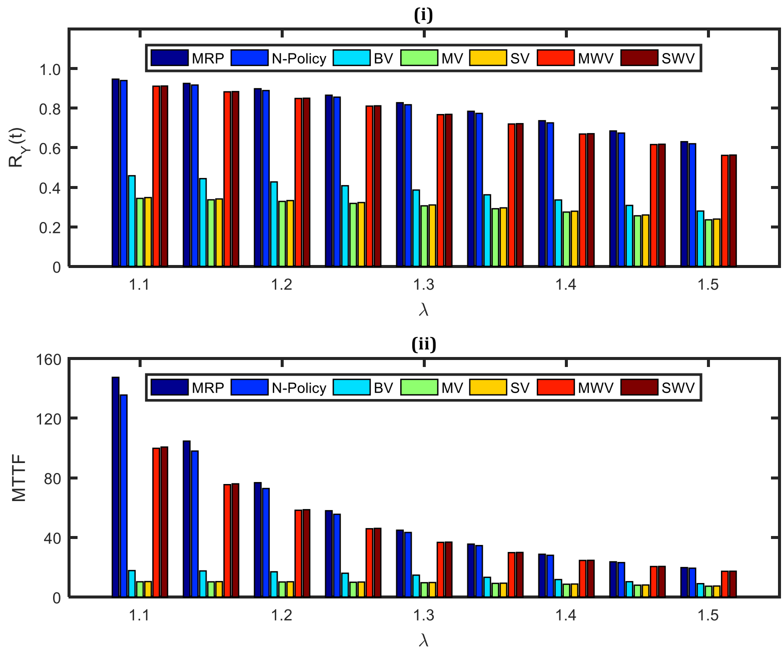

Figure 7.20: Effect of failure rate λ on reliability and $MTTF$ respectively wrt (i) MRP, (ii) N-policy, (iii) BV, (iv) MV, (v) SV, (vi) MWV, and (vii) SWV.

On fixing the default parameters as same as for the previous section, we observe from Figs. 7.20 and 7.21 that single working vacation is superior to multiple working vacation and single vacation as well for the high demand of better reliability.

7.9 MRP with Vacation Interruption Policy (VI)

In the working vacation policy, it is assumed that the server provides the service with the slower service rate in the vacation period but in the real-time scenario, this

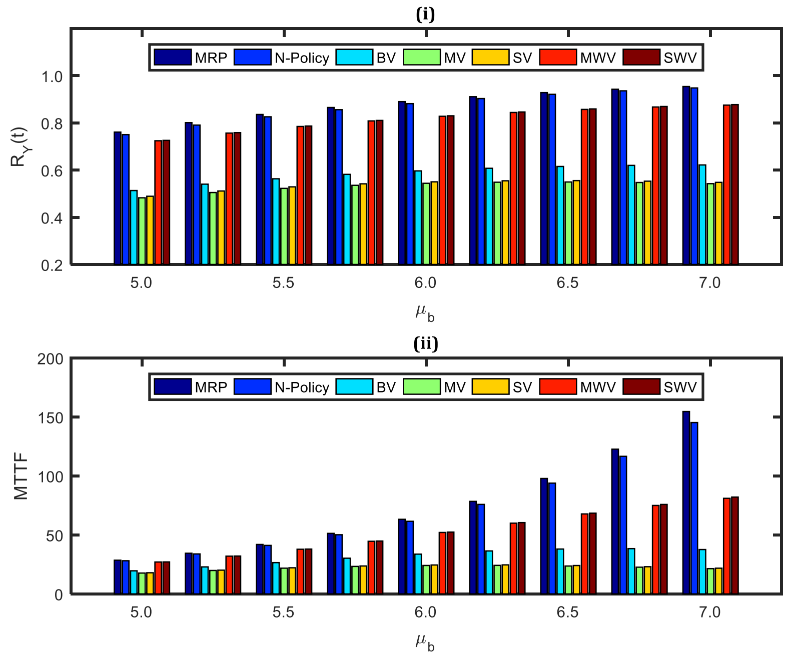

Figure 7.21: Effect of repair rate μ_b on reliability and $MTTF$ respectively wrt (i) MRP, (ii) N-policy, (iii) BV, (iv) MV, (v) SV, (vi) MWV, and (vii) SWV.

assumption seems more restrictive. Therefore, to overcome this limitation, [170] proposed the vacation interruption policy for the $M/M/1$ queue. According to this policy, if there are more than the certain pre-specified number of failed machines in the system (T) waiting for repair after the service completion instant during the vacation period, the repairman ends his vacation and resumes the normal working attribute. Machining systems with vacation interruption and other features in different contexts have been investigated by many researchers (*cf.* [254], [78], [256], [273], [214]).

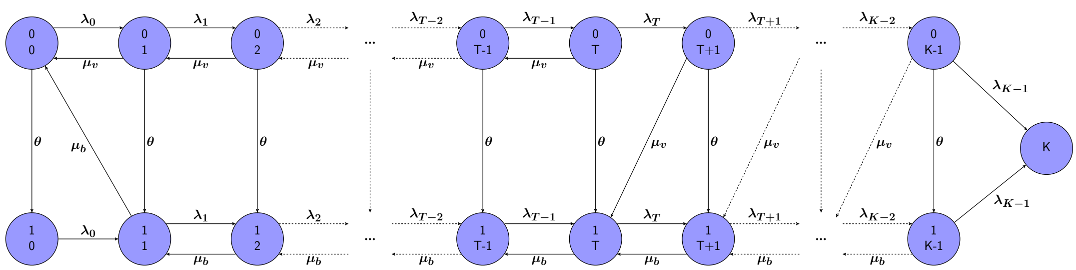

Figure 7.22: State transition diagram of the machine repair model with vacation interruption policy

The assumptions and notions are identical to respective assumptions/notations in the previous section. Using the state transition diagram depicted in Fig. 7.22 of the machine repair model with vacation interruption policy, the following differential-difference equations governing the system are developed.

$$\frac{dP_{0,0}(t)}{dt} = -(\lambda_0+\theta)P_{0,0}(t)+\mu_v P_{0,1}(t)+\mu_b P_{1,1}(t) \tag{7.88}$$

$$\frac{dP_{0,n}(t)}{dt} = -(\lambda_n+\mu_v+\theta)P_{0,n}(t)+\lambda_{n-1}P_{0,n-1}(t)+\mu_v P_{0,n+1}(t); \quad 1\le n\le T-1 \tag{7.89}$$

$$\frac{dP_{0,n}(t)}{dt} = -(\lambda_n+\mu_v+\theta)P_{0,n}(t)+\lambda_{n-1}P_{0,n-1}(t); \quad T\le n\le K-1 \tag{7.90}$$

$$\frac{dP_{1,0}(t)}{dt} = -\lambda_0 P_{1,0}(t)+\theta P_{0,0}(t) \tag{7.91}$$

$$\frac{dP_{1,1}(t)}{dt} = -(\lambda_1+\mu_b)P_{1,1}(t)+\lambda_0 P_{1,0}(t)+\mu_b P_{1,2}(t)+\theta P_{0,1}(t) \tag{7.92}$$

$$\frac{dP_{1,n}(t)}{dt} = -(\lambda_n+\mu_b)P_{1,n}(t)+\lambda_{n-1}P_{1,n-1}(t)+\mu_b P_{1,n+1}(t)+\theta P_{0,n}(t); \quad 2\le n\le T-1 \tag{7.93}$$

$$\frac{dP_{1,n}(t)}{dt} = -(\lambda_n+\mu_b)P_{1,n}(t)+\lambda_{n-1}P_{1,n-1}(t)+\mu_b P_{1,n+1}(t)+\theta P_{0,n}(t) + \mu_v P_{0,n+1}(t); T\le n\le K-2 \tag{7.94}$$

$$\frac{dP_{1,K-1}(t)}{dt} = -(\lambda_{K-1}+\mu_b)P_{1,K-1}(t)+\lambda_{K-2}P_{1,K-2}(t)+\theta P_{0,K-1}(t) \tag{7.95}$$

$$\frac{dP_{K}(t)}{dt} = \lambda_{K-1}P_{0,K-1}(t)+\lambda_{K-1}P_{1,K-1}(t) \tag{7.96}$$

The generator matrix, denoted by $\mathbf{Q}_8$, is composed by the block matrices obtained by corresponding transitions between states of the system. The structure of the generator matrix is expressed as follows

$$\mathbf{Q}_8 = \begin{bmatrix} \mathbf{A}_8 & \mathbf{B}_8 & \mathbf{E}_8 \\ \mathbf{C}_8 & \mathbf{D}_8 & \mathbf{F}_8 \\ \mathbf{G}_8 & \mathbf{H}_8 & \mathbf{I}_8 \end{bmatrix}$$

where, the block matrices $\mathbf{A}_8$, $\mathbf{B}_8$, $\mathbf{C}_8$, and $\mathbf{D}_8$ are of order $(K\times K)$, $\mathbf{E}_8$ and $\mathbf{F}_8$ are the column vectors of order $(K\times 1)$, and $\mathbf{G}_8$, $\mathbf{H}_8$, $\mathbf{I}_8$ are the zero vectors.

$$\mathbf{A}_8 = \begin{bmatrix} x_1^8 & y_1^8 & 0 & \cdots & 0 & 0 \\ z_2^8 & x_2^8 & y_2^8 & \cdots & 0 & 0 \\ 0 & z_3^8 & x_3^8 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x_{K-1}^8 & y_{K-1}^8 \\ 0 & 0 & 0 & \cdots & z_K^8 & x_K^8 \end{bmatrix}$$

$$x_n^8 = \begin{cases} -(\lambda_0 + \theta); & n = 1 \\ -(\lambda_{n-1} + \mu_v + \theta); & 2 \le n \le K \\ 0; & \text{otherwise} \end{cases}$$

$$y_n^8 = \begin{cases} \lambda_{n-1}; & 1 \le n \le K-1 \\ 0; & \text{otherwise} \end{cases}$$

$$z_n^8 = \begin{cases} \mu_v; & 2 \le n \le T+1 \\ 0; & \text{otherwise} \end{cases}$$

and

$$\mathbf{B}_8 = \begin{bmatrix} k_1^8 & 0 & 0 & \cdots & 0 & 0 \\ m_2^8 & k_2^8 & 0 & \cdots & 0 & 0 \\ 0 & m_3^8 & k_3^8 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & k_{K-1}^8 & 0 \\ 0 & 0 & 0 & \cdots & m_K^8 & k_K^8 \end{bmatrix}$$

$$k_n^8 = \begin{cases} \theta; & 1 \le n \le K \\ 0; & \text{otherwise} \end{cases}$$

$$m_n^8 = \begin{cases} \mu_v; & T+2 \le n \le K \\ 0; & \text{otherwise} \end{cases}$$

$$\mathbf{C}_8[2,1] = \mu_b$$

$$\mathbf{D}_8 = \begin{bmatrix} u_1^8 & v_1^8 & 0 & \cdots & 0 & 0 \\ w_2^8 & u_2^8 & v_2^8 & \cdots & 0 & 0 \\ 0 & w_3^8 & u_3^8 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & u_{K-1}^8 & v_{K-1}^8 \\ 0 & 0 & 0 & \cdots & w_K^8 & u_K^8 \end{bmatrix}$$

$$u_n^8 = \begin{cases} -\lambda_0; & n = 1 \\ -(\lambda_{n-1} + \mu_b); & 2 \le n \le K \\ 0; & \text{otherwise} \end{cases}$$

$$v_n^8 = \begin{cases} \lambda_{n-1}; & 1 \le n \le K-1 \\ 0; & \text{otherwise} \end{cases}$$

$$w_n^8 = \begin{cases} \mu_b; & 3 \le n \le K \\ 0; & \text{otherwise} \end{cases}$$

$$\mathbf{E}_8 = \mathbf{F}_8 = [0,0,\cdots,0,\lambda_{K-1}]^T;\ \mathbf{G}_8 = \mathbf{H}_8 = [0,0,\cdots,0]^T;\ \mathbf{I}_8 = [0]$$

The closed-form expressions for the expected number of failed machines in the system $E_N(t)$, throughput of the system $\tau_p(t)$, probability that the repairman is on working vacation $P_{WV}(t)$, probability that the vacation of the repairman is interrupted $P_{VI}(t)$, reliability of the system $R_Y(t)$, mean time-to-failure of the system $MTTF$, and failure frequency $FF(t)$ of the system are obtained respectively as follows.

$$E_N(t) = \sum_{j=0}^{1}\sum_{n=0}^{K-1} nP_{j,n}(t) + KP_K(t) \tag{7.97}$$

$$\tau_p(t) = \sum_{n=1}^{K-1} \mu_v P_{0,n}(t) + \sum_{n=1}^{K-1} \mu_b P_{1,n}(t) \tag{7.98}$$

$$P_{WV}(t) = \sum_{n=0}^{K-1} P_{0,n}(t) \tag{7.99}$$

$$P_{VI}(t) = \sum_{T+1}^{K-1} P_{0,n}(t) \tag{7.100}$$

$$R_Y(t) = 1 - P_K(t) \tag{7.101}$$

$$MTTF = \int_0^\infty R_Y(t)dt \tag{7.102}$$

$$FF(t) = \lambda_{K-1}P_{0,K-1}(t) + \lambda_{K-1}P_{1,K-1}(t) \tag{7.103}$$

For the numerical illustration, we consider the values of governing threshold as $T = 8$ and other system parameters as same as in the previous section. The variation of reliability measures is depicted in Figs. 7.23 and 7.24 for the varied value of failure rate λ and service rate μ respectively. It is clearly observed that vacation interruption always be the better solution for getting the better reliability of the machining system.

7.10 Discussion

In this chapter, the comparative analysis of different vacation policies on the reliability characteristics of the fault-tolerant machining system is presented. For that

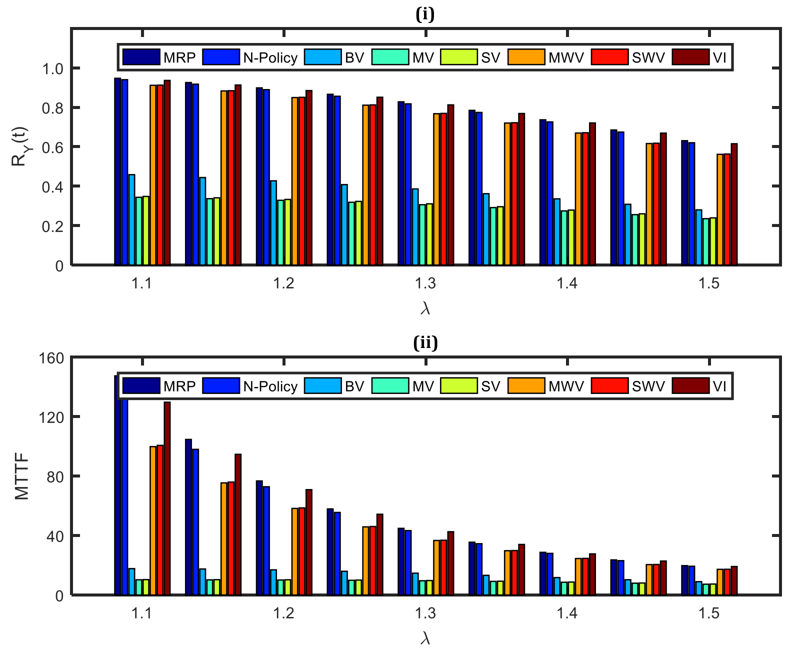

Figure 7.23: Effect of failure rate λ on reliability and $MTTF$ respectively wrt (i) MRP, (ii) N-policy, (iii) BV, (iv) MV, (v) SV, (vi) MWV, (vii) SWV, and (viii) VI.

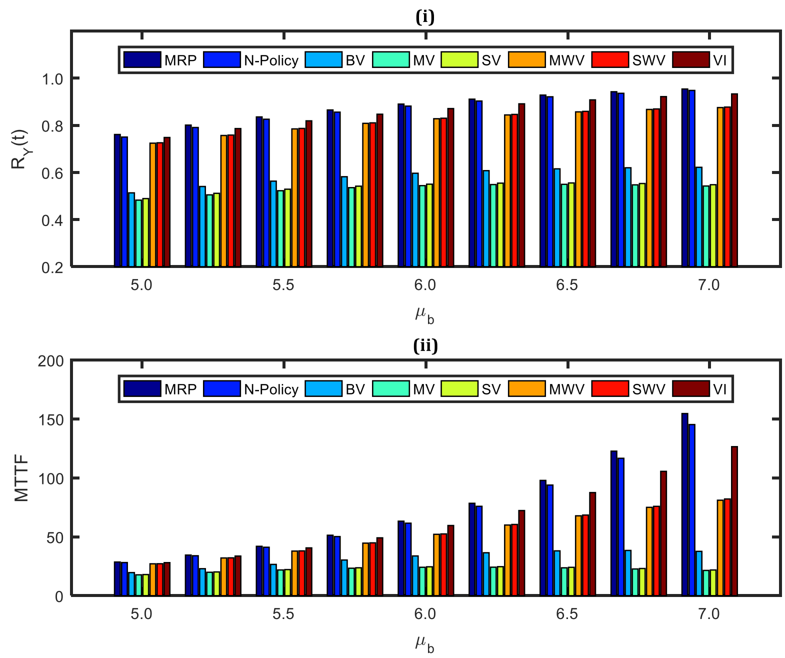

Figure 7.24: Effect of repair rate μ_b on reliability and $MTTF$ respectively wrt (i) MRP, (ii) N-policy, (iii) BV, (iv) MV, (v) SV, (vi) MWV, (vii) SWV, and (viii) VI.

purpose, the queueing-theoretic approach has been used, and Markovian models are developed. The governing system of Chapman Kolmogrove differential-difference equations has been derived with some assumptions. We employ the numerical scheme, namely the Runge-Kutta method of the fourth order, and program the code in MATLAB (2018b) software for computing the transient-state probabilities and henceforth system performance measures. For illustrative purposes, we present the numerical experiment for some hypothetical default parameters, and results are depicted with bar graphs for comparison at a glance.

The vacation of the repairman terminates when either number of failed machines exceeds the pre-specified threshold (e.g. N-policy), random duration ends (e.g. Bernoulli vacation, multiple/single vacation, multiple/single working vacation), or due to interruption (e.g. vacation interruption). In general, the vacation of the repairman starts when there is no failed machine in the system to be repaired. But, in Bernoulli's vacation, the repairman may opt for the vacation in the presence of failed machines to be repaired probabilistically. In a single vacation, the repairman may remain idle even when there is no failed machine to be repaired in the system after the completion of the vacation period. All vacation policies directly affect the reliability of the system. Under all vacations, reliability of the system and $MTTF$ reduce.

Next, for the comparative analysis purpose, we range the values of failure rate (λ) and service rate ($\mu = \mu_b$) from 1.1 to 1.5 and 5.0 to 7.0, respectively. Figs. 7.23 and 7.24 depict the combined illustration of all the studied vacation policies on the reliability ($R_Y(t)$) and $MTTF$ of the system. Fig. 7.23 reveals that as the value of λ increases, correspondingly, the reliability and $MTTF$ of the system decreases, which is quite obvious. Fig. 7.24 exhibits the reverse trend of variation for the increasing rate of service. For any test instance, if we fix the value of λ and μ, it is easily observed that the value of reliability and $MTTF$ is higher for working vacation and vacation interruption policies in comparison of fully vacation policies. Therefore, the selection of the working vacation policies is the much better choice for the system analyst and engineers instead of opting for the full vacation policies. So, as the conclusive remark, the system analyst and engineers may opt for the working vacation and vacation interruption policies to achieve the maximal reliability of the system.