APPENDIX A

A.1 Concavity of Π_1 in P^*_1 and α^* and Π_2 in P_2 and θ

Differentiating Eq. (3.2) wrt P_1 and α, yields,

$$\frac{\partial^2 \Pi_1}{\partial P_1^2} = -2e\,,\ \frac{\partial^2 \Pi_1}{\partial \alpha^2} = -\eta\,,\ \frac{\partial^2 \Pi_1}{\partial \alpha \partial P_1} = \frac{\partial^2 \Pi_1}{\partial P_1 \partial \alpha} = \gamma \tag{A.1}$$

For $\left(2e\eta - \gamma^2\right) > 0$,

$$\frac{\partial^2 \Pi_1}{\partial P_1^2} < 0\,,\ \frac{\partial^2 \Pi_1}{\partial \alpha^2} < 0\,, \text{ and } \frac{\partial^2 \Pi_1}{\partial P_1^2}\frac{\partial^2 \Pi_1}{\partial \alpha^2} - \left(\frac{\partial^2 \Pi_1}{\partial \alpha \partial P_1}\right)^2 = \left(2e\eta - \gamma^2\right) > 0 \tag{A.2}$$

Differentiating Eq. (3.9) wrt P_2 and θ, yields,

$$\frac{\partial^2 \Pi_2}{\partial P_2^2} = -\frac{2e^2\eta}{\left(2e\eta - \gamma^2\right)}\,,\ \frac{\partial^2 \Pi_2}{\partial \theta^2} = -\frac{2e\eta\lambda V_2 \nu}{\left(2e\eta - \gamma^2\right)} - \zeta\,,\ \frac{\partial^2 \Pi_2}{\partial \theta \partial P_2} = \frac{\partial^2 \Pi_2}{\partial P_2 \partial \theta} = \frac{e\eta}{\left(2e\eta - \gamma^2\right)}\left(\lambda + eV_2\nu\right) \tag{A.3}$$

For $\left(2e\eta - \gamma^2\right) > 0$ and $\left[2\zeta\left(2e\eta - \gamma^2\right) - \eta\left(\lambda - eV_2\nu\right)^2\right] > 0$,

$$\frac{\partial^2 \Pi_2}{\partial P_2^2} < 0\,,\ \frac{\partial^2 \Pi_2}{\partial \theta^2} < 0\,,\ \frac{\partial^2 \Pi_2}{\partial P_2^2}\frac{\partial^2 \Pi_2}{\partial \theta^2} - \left(\frac{\partial^2 \Pi_2}{\partial \theta \partial P_2}\right)^2 = \left[2\zeta\left(2e\eta - \gamma^2\right) - \eta\left(\lambda - eV_2\nu\right)^2\right] > 0 \tag{A.4}$$

A.2 Variation of Π^*_1, Π^*_2, P^*_1, α^*, P^*_2 and θ^* wrt η and ζ

Differentiating θ^*, α^*, P^*_2, P^*_1, Π^*_2, and Π^*_1 in Eqs. (3.14), (3.6), (3.12), (3.4), (3.15) and (3.5) wrt η and wrt ζ, yield,

$$\frac{\partial \theta^*}{\partial \eta} = -\frac{2\zeta\gamma^2\left(\lambda - eV_2\nu\right)\left[A - e\left(V_1 + V_2\right)\right]}{\left[2\zeta\left(2e\eta - \gamma^2\right) - \eta\left(\lambda - eV_2\nu\right)^2\right]^2} \tag{A.5}$$

$$\frac{\partial \alpha^*}{\partial \eta} = -\frac{\left\{4e\gamma\zeta\left[2\zeta\left(2e\eta - \gamma^2\right) - \eta\left(\lambda - eV_2\nu\right)^2\right] + \gamma^3\zeta\left(\lambda - eV_2\nu\right)^2\right\}\left[A - e\left(V_1 + V_2\right)\right]}{\left[2\zeta\left(2e\eta - \gamma^2\right) - \eta\left(\lambda - eV_2\nu\right)^2\right]^2} \tag{A.6}$$

$$\frac{\partial P_2^*}{\partial \eta} = -\frac{\zeta\gamma^2\left(\lambda - eV_2\nu\right)\left(\lambda + eV_2\nu\right)\left[A - e\left(V_1 + V_2\right)\right]}{e\left[2\zeta\left(2e\eta - \gamma^2\right) - \eta\left(\lambda - eV_2\nu\right)^2\right]^2} \tag{A.7}$$

$$\frac{\partial P_1^*}{\partial \eta} = -\frac{\zeta\left[2\gamma^2\lambda\left(1 + \gamma^2\right)\left(\lambda - eV_2\nu\right) + 4e\gamma^2\right]\left[A - e\left(V_1 + V_2\right)\right]}{2e\left[2\zeta\left(2e\eta - \gamma^2\right) - \eta\left(\lambda - eV_2\nu\right)^2\right]^2} \tag{A.8}$$

$$\frac{\partial \Pi_2}{\partial \eta} = -\frac{\zeta^2\gamma^2\left[A - e\left(V_1 + V_2\right)\right]^2}{\left[2\zeta\left(2e\eta - \gamma^2\right) - \eta\left(\lambda - eV_2\nu\right)^2\right]^2} \tag{A.9}$$

$$\frac{\partial \Pi_1}{\partial \eta} = -\frac{8e^2\zeta^2\gamma^2\left[A-e(V_1+V_2)\right]^2}{\left[2\zeta\left(2e\eta-\gamma^2\right)-\eta(\lambda-eV_2\nu)^2\right]^2} \tag{A.10}$$

$$\frac{\partial \theta^*}{\partial \zeta} = -\frac{2\eta\left(2e\eta-\gamma^2\right)(\lambda-eV_2\nu)\left[A-e(V_1+V_2)\right]}{\left[2\zeta\left(2e\eta-\gamma^2\right)-\eta(\lambda-eV_2\nu)^2\right]^2} \tag{A.11}$$

$$\frac{\partial \alpha^*}{\partial \zeta} = -\frac{2\eta\gamma\left[\lambda(e-1)+e^2V_2\nu\right](\lambda-eV_2\nu)\left[A-e(V_1+V_2)\right]}{\left[2\zeta\left(2e\eta-\gamma^2\right)-\eta(\lambda-eV_2\nu)^2\right]^2} \tag{A.12}$$

$$\frac{\partial P_2^*}{\partial \zeta} = -\frac{2\eta\left(2e\eta-\gamma^2\right)(\lambda+eV_2\nu)(\lambda-eV_2\nu)\left[A-e(V_1+V_2)\right]}{\left[2\zeta\left(2e\eta-\gamma^2\right)-\eta(\lambda-eV_2\nu)^2\right]^2} \tag{A.13}$$

$$\frac{\partial P_1^*}{\partial \zeta} = -\frac{2\eta(\lambda-eV_2\nu)\left[A-e(V_1+V_2)\right]\left[\eta\lambda+(\lambda+eV_2\nu)\left(e\eta-\gamma^2\right)\right]}{\left[2\zeta\left(2e\eta-\gamma^2\right)-\eta(\lambda-eV_2\nu)^2\right]^2} \tag{A.14}$$

$$\frac{\partial \Pi_2}{\partial \zeta} = -\frac{\eta^2(\lambda-eV_2\nu)^2\left[A-e(V_1+V_2)\right]^2}{2\left[2\zeta\left(2e\eta-\gamma^2\right)-\eta(\lambda-eV_2\nu)^2\right]^2} \tag{A.15}$$

$$\frac{\partial \Pi_1}{\partial \zeta} = -\frac{16\zeta e^2\eta^2\left(2e\eta-\gamma^2\right)(\lambda-eV_2\nu)^2\left[A-e(V_1+V_2)\right]^2}{\left[2\zeta\left(2e\eta-\gamma^2\right)-\eta(\lambda-eV_2\nu)^2\right]^3} \tag{A.16}$$

A.3 Concavity of Π_j in α^*, and θ^*

Differentiating Eq. (3.38) wrt α and θ, yield,

$$\frac{\partial^2\Pi_j}{\partial\alpha^2} = -\frac{\left(2e\eta-\gamma^2\right)}{2e},\ \frac{\partial^2\Pi_j}{\partial\theta^2} = -\frac{\left[2e\zeta-(\lambda-eV_2\nu)^2\right]}{2e},\ \frac{\partial^2\Pi_j}{\partial\theta\partial\alpha} = \frac{\partial^2\Pi_j}{\partial\alpha\partial\theta} = \frac{\gamma}{2e}(\lambda-eV_2\nu) \tag{A.17}$$

For $\left(2e\eta-\gamma^2\right)>0$, $\left[2e\zeta-(\lambda-eV_2\nu)^2\right]>0$, $(\lambda-eV_2\nu)>0$, and $\left[\zeta\left(2e\eta-\gamma^2\right)-\eta(\lambda-eV_2\nu)^2\right]>0$,

$$\frac{\partial^2\Pi_j}{\partial\alpha^2}<0,\ \frac{\partial^2\Pi_j}{\partial\theta^2}<0,\ \frac{\partial^2\Pi_j}{\partial\alpha^2}\frac{\partial^2\Pi_j}{\partial\theta^2}-\left(\frac{\partial^2\Pi_j}{\partial\theta\partial\alpha}\right)^2 = \left[\zeta\left(2e\eta-\gamma^2\right)-\eta(\lambda-eV_2\nu)^2\right]>0 \tag{A.18}$$

A.4 Variation of Π^*_j, P^*_1, α^*, P^*_2 and θ^* wrt η and ζ

Differentiating θ^*, α^*, P^*_1, and Π^*_j in Eqs. (3.43), (3.41), (3.39) and (3.44) wrt η and wrt ζ, yield,

$$\frac{\partial \theta^*}{\partial \eta} = -\frac{\zeta\gamma^2(\lambda - eV_2\nu)[A - e(V_1 + V_2)]}{\left[\zeta(2e\eta - \gamma^2) - \eta(\lambda - eV_2\nu)^2\right]^2} \tag{A.19}$$

$$\frac{\partial \alpha^*}{\partial \eta} = -\frac{\zeta\gamma(2e\eta - \gamma^2)[A - e(V_1 + V_2)]\left[2e\zeta - (\lambda - eV_2\nu)^2\right]}{\left[\zeta(2e\eta - \gamma^2) - \eta(\lambda - eV_2\nu)^2\right]^2} \tag{A.20}$$

$$\frac{\partial \theta^*}{\partial \zeta} = -\frac{\eta(2e\eta - \gamma^2)(\lambda - eV_2\nu)[A - e(V_1 + V_2)]}{\left[\zeta(2e\eta - \gamma^2) - \eta(\lambda - eV_2\nu)^2\right]^2} \tag{A.21}$$

$$\frac{\partial \alpha^*}{\partial \zeta} = \frac{\eta\gamma(\lambda - eV_2\nu)^2[A - e(V_1 + V_2)]}{\left[\zeta(2e\eta - \gamma^2) - \eta(\lambda - eV_2\nu)^2\right]^2} \tag{A.22}$$

$$\frac{\partial P_1^*}{\partial \eta} = -\frac{\zeta\gamma^2\left\{(2e\eta - \gamma^2)\left[2e\zeta - (\lambda - eV_2\nu)^2\right] + \left[\lambda^2 - (eV_2\nu)^2\right]\right\}[A - e(V_1 + V_2)]}{2e\left[2\zeta(2e\eta - \gamma^2) - \eta(\lambda - eV_2\nu)^2\right]^2} \tag{A.23}$$

$$\frac{\partial P_1^*}{\partial \zeta} = \frac{\eta(\lambda - eV_2\nu)\left[V_2\nu(e\eta - \gamma^2) + \eta\lambda\right][A - e(V_1 + V_2)]}{\left[\zeta(2e\eta - \gamma^2) - \eta(\lambda - eV_2\nu)^2\right]^2} \tag{A.24}$$

$$\frac{\partial \Pi_j}{\partial \eta} = -\frac{\zeta^2\gamma^2[A - e(V_1 + V_2)]^2}{2\left[\zeta(2e\eta - \gamma^2) - \eta(\lambda - eV_2\nu)^2\right]^2} \tag{A.25}$$

$$\frac{\partial \Pi_j}{\partial \zeta} = -\frac{\eta(\lambda - eV_2\nu)^2[A - e(V_1 + V_2)]^2}{2\left[\zeta(2e\eta - \gamma^2) - \eta(\lambda - eV_2\nu)^2\right]^2} \tag{A.26}$$

A.5 Proof of Lemma 3.1

$\theta^*{}_I > \theta^*{}_{II}$, on simplification results in;

$$\frac{\eta(\lambda - eV_2\nu)[A - e(V_1 + V_2)]}{\left[2\zeta(2e\eta - \gamma^2) - \eta(\lambda - eV_2\nu)^2\right]} > \frac{\eta(\lambda - eV_2\nu)\left[A - e\left(V_1 + V_2 + \frac{(S_2 + O_2)}{Q} + \frac{Qh_2}{2R}\right)\right]}{\left[2\zeta(2e\eta - \gamma^2) - \eta(\lambda - eV_2\nu)^2\right]} \tag{A.27}$$

which is true since $e\left(\frac{(S_2 + O_2)}{Q} + \frac{Qh_2}{2R}\right) > 0$

Similarly $P^*{}_{2,I} < P^*{}_{2,II}$ simplifies to;

$$\frac{A + \lambda\theta_I^*}{2e} - \frac{1}{2}\left[V_1 - V_2(1 + \nu\theta_I^*)\right] < \frac{A + \lambda\theta_{II}^*}{2e} - \frac{1}{2}\left(V_1 - V_2(1 + \nu\theta_{II}^*) - \frac{(S_2 + O_2)}{Q} - \frac{Qh_2}{2R}\right) \tag{A.28}$$

and is true if $\left[\zeta(2e\eta - \gamma^2) - \eta\lambda(\lambda - eV_2\nu)\right] > 0$.

Similarly $\alpha^*{}_I > \alpha^*{}_{II}$ simplifies to;

$$\frac{\gamma\left[A+\lambda\theta_I^* - e\left(V_1 + P_{2,I}^*\right)\right]}{2e\eta-\gamma^2} > \frac{\gamma\left[A+\lambda\theta_{II}^* - e\left(V_1 + P_{2,II}^*\right)\right]}{2e\eta-\gamma^2} \qquad \text{(A.29)}$$

and is true if $\frac{\left[2\zeta\left(2e\eta-\gamma^2\right)+\eta(\lambda-eV_2\nu)^2\right]}{\left[2\zeta\left(2e\eta-\gamma^2\right)-\eta(\lambda-eV_2\nu)^2\right]} > 0$.

Similarly $P^*{}_{1,I} > P^*{}_{1,II}$ simplifies to;

$$\frac{A+\gamma\alpha_I^*+\lambda\theta_I^*}{2e} + \frac{\left(V_1+P_{2,I}^*\right)}{2} > \frac{A+\gamma\alpha_{II}^*+\lambda\theta_{II}^*}{2e} + \frac{\left(V_1+P_{2,II}^*\right)}{2} \qquad \text{(A.30)}$$

and is true if $\left(2e\eta-\gamma^2\right)\left[\lambda\eta(\lambda-eV_2\nu)-\zeta\left(2e\eta-\gamma^2\right)\right] > 0$.

A.6 Proof of Lemma 3.2

$\theta^*{}_I > \theta^*{}_{III}$, on simplification results in;

$$\frac{\eta(\lambda-eV_2\nu)\left[A-e(V_1+V_2)\right]}{\left[2\zeta\left(2e\eta-\gamma^2\right)-\eta(\lambda-eV_2\nu)^2\right]} > \frac{\eta(\lambda-eV_2\nu)\left[A-e\left(V_1+V_2+\frac{O_1}{Q}\right)\right]}{\left[2\zeta\left(2e\eta-\gamma^2\right)-\eta(\lambda-eV_2\nu)^2\right]} \qquad \text{(A.31)}$$

which is true since $e\frac{O_1}{Q} > 0$.

Similarly $P^*{}_{2,I} > P^*{}_{2,III}$ simplifies to;

$$\frac{A+\lambda\theta_I^*}{2e} - \frac{1}{2}\left[V_1 - V_2\left(1+\nu\theta_I^*\right)\right] > \frac{A+\lambda\theta_{III}^*}{2e} - \frac{1}{2}\left(V_1 - V_2\left(1+\nu\theta_{III}^*\right) + \frac{O_1}{Q}\right) \qquad \text{(A.32)}$$

and is true if $\left[\zeta\left(2e\eta-\gamma^2\right)+\eta eV_2\nu(\lambda-eV_2\nu)\right] > 0$.

Similarly $\alpha^*{}_I > \alpha^*{}_{III}$ simplifies to;

$$\frac{\gamma\left[A+\lambda\theta_I^* - e\left(V_1 + P_{2,I}^*\right)\right]}{2e\eta-\gamma^2} > \frac{\gamma\left[A+\lambda\theta_{III}^* - e\left(V_1 + P_{2,III}^* + \frac{O_1}{Q}\right)\right]}{2e\eta-\gamma^2} \qquad \text{(A.33)}$$

and is true if $\frac{\left[3\zeta\left(2e\eta-\gamma^2\right)-\eta(\lambda-eV_2\nu)^2\right]}{\left[2\zeta\left(2e\eta-\gamma^2\right)-\eta(\lambda-eV_2\nu)^2\right]} > 0$

Similarly $P^*{}_{1,I} > P^*{}_{1,III}$ simplifies to;

$$\frac{A+\gamma\alpha_I^*+\lambda\theta_I^*}{2e} + \frac{\left(V_1+P_{2,I}^*\right)}{2} > \frac{A+\gamma\alpha_{III}^*+\lambda\theta_{III}^*}{2e} + \frac{1}{2}\left(V_1+P_{2,III}^*+\frac{O_1}{Q}\right) \qquad \text{(A.34)}$$

and is true if $\left[\eta(\lambda - eV_2\nu)(3\lambda - eV_2\nu) - 2\zeta\left(3e\eta - 2\gamma^2\right)\right] > 0$.

A.7 Proof of Lemma 3.3

$\theta^*{}_I > \theta^*{}_{IV}$, on simplification results in;

$$\frac{\eta(\lambda - eV_2\nu)\left[A - e(V_1 + V_2)\right]}{\left[2\zeta\left(2e\eta - \gamma^2\right) - \eta(\lambda - eV_2\nu)^2\right]} > \frac{\eta(\lambda - eV_2\nu)\left[A - e\left(V_1 + V_2 + \frac{O_1}{Q} + \frac{(S_2 + O_2)}{Q} + \frac{Qh_2}{2R}\right)\right]}{\left[2\zeta\left(2e\eta - \gamma^2\right) - \eta(\lambda - eV_2\nu)^2\right]}$$

(A.35)

which is true since $e\left(\frac{O_1}{Q} + \frac{(S_2 + O_2)}{Q} + \frac{Qh_2}{2R}\right) > 0$

Similarly $P^*{}_{2,I} > P^*{}_{2,IV}$ simplifies to;

$$\frac{A + \lambda\theta_I^*}{2e} - \frac{1}{2}\left[V_1 - V_2\left(1 + \nu\theta_I^*\right)\right] > \frac{A + \lambda\theta_{IV}^*}{2e} - \frac{1}{2}\left(V_1 - V_2\left(1 + \nu\theta_{IV}^*\right) + \frac{O_1}{Q} - \frac{(S_2 + O_2)}{Q} - \frac{Qh_2}{2R}\right)$$

(A.36)

and is true if

$$\left(\frac{O_1}{Q}\right)\frac{\left[\zeta\left(2e\eta - \gamma^2\right) + \eta eV_2\nu(\lambda - eV_2\nu)\right]}{\left[2\zeta\left(2e\eta - \gamma^2\right) - \eta(\lambda - eV_2\nu)^2\right]} + \left(\frac{(S_2 + O_2)}{Q} + \frac{Qh_2}{2R}\right)\frac{\left[\lambda\eta(\lambda - eV_2\nu) - \zeta\left(2e\eta - \gamma^2\right)\right]}{\left[2\zeta\left(2e\eta - \gamma^2\right) - \eta(\lambda - eV_2\nu)^2\right]} > 0$$

.

Similarly $\alpha^*{}_I > \alpha^*{}_{IV}$ simplifies to;

$$\frac{\gamma\left[A + \lambda\theta_I^* - e\left(V_1 + P_{2,I}^*\right)\right]}{2e\eta - \gamma^2} > \frac{\gamma\left[A + \lambda\theta_{IV}^* - e\left(V_1 + P_{2,IV}^* + \frac{O_1}{Q}\right)\right]}{2e\eta - \gamma^2} \quad \text{(A.37)}$$

and is true if $\frac{\left(2e\eta - \gamma^2\right)}{\left[2\zeta\left(2e\eta - \gamma^2\right) - \eta(\lambda - eV_2\nu)^2\right]} > 0$

Similarly $P^*{}_{1,I} > P^*{}_{1,III}$ simplifies to;

$$\frac{A + \gamma\alpha_I^* + \lambda\theta_I^*}{2e} + \frac{\left(V_1 + P_{2,I}^*\right)}{2} > \frac{A + \gamma\alpha_{III}^* + \lambda\theta_{III}^*}{2e} + \frac{1}{2}\left(V_1 + P_{2,III}^* + \frac{O_1}{Q}\right) \quad \text{(A.38)}$$

and is true if

$$\left(\frac{O_1}{Q}\right)\frac{\left[3\zeta\left(2e\eta - \gamma^2\right) + \eta eV_2\nu(\lambda - eV_2\nu)\right]}{\left[2\zeta\left(2e\eta - \gamma^2\right) - \eta(\lambda - eV_2\nu)^2\right]} + \left(\frac{(S_2 + O_2)}{Q} + \frac{Qh_2}{2R}\right)\frac{\left[2\lambda\eta(\lambda - eV_2\nu) - \zeta\left(2e\eta - \gamma^2\right)\right]}{\left[2\zeta\left(2e\eta - \gamma^2\right) - \eta(\lambda - eV_2\nu)^2\right]} > 0$$

A.8 Proof of Lemma 3.4

$\theta^*_{VI} > \theta^*_{VIII}$, on simplification results in;

$$\frac{\eta(\lambda - eV_2\nu)[A - e(V_1 + V_2)]}{\left[\zeta(2e\eta - \gamma^2) - \eta(\lambda - eV_2\nu)^2\right]} > \frac{\eta(\lambda - eV_2\nu)\left[A - e\left(V_1 + V_2 + \frac{(O_1 + O_2 + S_2)}{Q_j} + \frac{Q_j h_2}{2R}\right)\right]}{\left[\zeta(2e\eta - \gamma^2) - \eta(\lambda - eV_2\nu)^2\right]} \tag{A.39}$$

which is true since $e\left(\frac{(O_1 + S_2 + O_2)}{Q_j} + \frac{Q_j h_2}{2R}\right) > 0$

Similarly $\alpha^*_I > \alpha^*_{IV}$ simplifies to;

$$\frac{\gamma\left\{A + \lambda\theta^*_{VI} - e\left[V_1 + V_2(1 + \nu\theta^*_{VI})\right]\right\}}{2e\eta - \gamma^2} > \frac{\gamma}{2e\eta - \gamma^2}\left[A + \lambda\theta^*_{VIII} - e\left(V_1 + V_2(1 + \nu\theta^*_{VIII}) + \frac{(O_1 + O_2 + S_2)}{Q_j} + \frac{Q_j h_2}{2R}\right)\right] \tag{A.40}$$

and is true if $\frac{(2e\eta - \gamma^2)}{\left[\zeta(2e\eta - \gamma^2) - \eta(\lambda - eV_2\nu)^2\right]} > 0$

Similarly $P^*_{1,I} > P^*_{1,III}$ simplifies to;

$$\frac{A + \gamma\alpha^*_{VI} + \lambda\theta^*_{VI}}{2e} + \frac{\left[V_1 + V_2(1 + \nu\theta^*_{VI})\right]}{2} > \frac{A + \gamma\alpha^*_{VIII} + \lambda\theta^*_{VIII}}{2e} + \frac{1}{2}\left(V_1 + V_2(1 + \nu\theta^*_{VIII}) + \frac{(O_1 + O_2 + S_2)}{Q_j} + \frac{Q_j h_2}{2R}\right) \tag{A.41}$$

and is true if $\frac{\left[(\gamma - 1)\zeta(2e\eta - \gamma^2) + 2\lambda(\lambda - eV_2\nu)\right]}{\left[\zeta(2e\eta - \gamma^2) - \eta(\lambda - eV_2\nu)^2\right]} > 0$

APPENDIX B

B.1 Proof of Positive Definiteness of Hessain Matrix of *TRC*(*M,N,Q,θ*) in *Q* and *θ*

The determinant of Hessain matrix can be calculated as

$$\frac{\partial^2 TRC(M,N,Q,\theta)}{\partial Q^2}\frac{\partial^2 TRC(M,N,Q,\theta)}{\partial \theta^2}-\left(\frac{\partial^2 TRC(M,N,Q,\theta)}{\partial Q\partial\theta}\right)^2=\left(\frac{gND}{2}\right)^2\left[\frac{2Q\beta}{iq}-1\right]>0 \tag{B.1}$$

where

$$\frac{\partial^2 TRC(M,N,Q,\theta)}{\partial Q^2}=\frac{2\alpha}{Q^3}>0 \tag{B.2}$$

$$\frac{\partial^2 TRC(M,N,Q,\theta)}{\partial \theta^2}=\frac{iq}{\theta^2}>0 \tag{B.3}$$

$$\frac{\partial^2 TRC(M,N,Q,\theta)}{\partial Q\partial\theta}=\frac{gND}{2}>0 \tag{B.4}$$

The Hessain matrix is positive and $TRC(M,N,Q,\theta)$ is convex with respect to Q and θ.

B.2 Proof of Positive Definiteness of Hessain Matrix of *TRC*$_s$(*M,N,Q,θ*) in *Q* and *θ*

The determinant of Hessain matrix can be calculated as

$$\frac{\partial^2 TRC_s(M,N,Q,\theta)}{\partial Q^2}\frac{\partial^2 TRC_s(M,N,Q,\theta)}{\partial \theta^2}-\left(\frac{\partial^2 TRC_s(M,N,Q,\theta)}{\partial Q\partial\theta}\right)^2= \left(\frac{gND}{2}\right)^2\left[\frac{Qh_r}{iq}\frac{(S+A_m M)}{A_r N}-1\right] \tag{B.5}$$

where

$$\frac{\partial^2 TRC_s(M,N,Q,\theta)}{\partial Q^2}=\frac{2D(S+A_m M)}{NQ^3}>0 \tag{B.6}$$

$$\frac{\partial^2 TRC_s(M,N,Q,\theta)}{\partial \theta^2}=\frac{iq}{\theta^2}>0 \tag{B.7}$$

$$\frac{\partial^2 TRC_s(M,N,Q,\theta)}{\partial Q\partial\theta}=\frac{gND}{2}>0 \tag{B.8}$$

The Hessain matrix is positive when $\left(\frac{gND}{2}\right)^2\left[\frac{\sqrt{2DA_r h_r}}{iq}\frac{(S+A_m M)}{A_r N}-1\right]>0$ and then $TRC_s(M,N,Q,\theta)$ is convex with respect to Q and θ.

B.3 Proof of Positive Definiteness of Hessain Matrix of $TRC_b(Q,\theta)$ in Q and θ

The determinant of Hessain matrix can be calculated as

$$\frac{\partial^2 TRC_b(Q,\theta)}{\partial Q^2}\frac{\partial^2 TRC_b(Q,\theta)}{\partial \theta^2}-\left(\frac{\partial^2 TRC_b(Q,\theta)}{\partial Q\partial\theta}\right)^2=\left(\frac{gND}{2}\right)^2\left[\frac{\sqrt{2DA_r(h_r+gND\theta)}}{iq}-1\right] \tag{B.9}$$

where

$$\frac{\partial^2 TRC_b(Q,\theta)}{\partial Q^2}=\frac{2DA_r}{Q^3}>0 \tag{B.10}$$

$$\frac{\partial^2 TRC_b(Q,\theta)}{\partial \theta^2}=\frac{iq}{\theta^2}>0 \tag{B.11}$$

$$\frac{\partial^2 TRC_b(Q,\theta)}{\partial Q\partial\theta}=\frac{gND}{2}>0 \tag{B.12}$$

The Hessain matrix is positive when $\left(\frac{gND}{2}\right)^2\left[\frac{\sqrt{2DA_r(h_r+gND\theta)}}{iq}-1\right]>0$ and then $TRC_b(Q,\theta)$ is convex with respect to Q and θ.

B.4 Proof of Positive Definiteness of Hessain Matrix of $TRC_b(Q,\lambda,\theta)$ and $TRC_s(M,N,\lambda,\theta)$ in Q and θ

The determinant of Hessain matrix can be calculated as

$$\frac{\partial^2 TRC_b(Q,\lambda,\theta_b)}{\partial Q^2}\frac{\partial^2 TRC_b(Q,\lambda,\theta_b)}{\partial \theta_b^2}-\left(\frac{\partial^2 TRC_b(Q,\lambda,\theta_b)}{\partial Q\partial\theta_b}\right)^2= \left(\frac{\lambda gND}{2}\right)^2\left[\frac{\sqrt{2DA_r(h_r+\lambda gND\theta_b)}}{iq}-1\right] \tag{B.13}$$

where

$$\frac{\partial^2 TRC_b(Q,\lambda,\theta_b)}{\partial Q^2}=\frac{2DA_r}{Q^3}>0 \tag{B.14}$$

$$\frac{\partial^2 TRC_b(Q,\lambda,\theta_b)}{\partial \theta_b^2}=\frac{iq}{\theta_b^2}>0 \tag{B.15}$$

$$\frac{\partial^2 TRC_b(Q,\lambda,\theta_b)}{\partial Q\partial\theta_b}=\frac{\lambda gND}{2}>0 \tag{B.16}$$

The Hessain matrix is positive when $\left(\frac{\lambda gND}{2}\right)^2\left[\frac{\sqrt{2DA_r(h_r+\lambda gND\theta_b)}}{iq}-1\right]>0$ and then $TRC_b(Q,\theta_b)$ is convex with respect to Q and θ_b.

The determinant of Hessain matrix can be calculated as

$$\frac{\partial^2 TRC_s(M,N,\lambda,\theta)}{\partial Q^2}\frac{\partial^2 TRC_s(M,N,\lambda,\theta)}{\partial \theta^2}-\left(\frac{\partial^2 TRC_s(M,N,\lambda,\theta)}{\partial Q\partial\theta}\right)^2 = \left[\frac{(1-\lambda)gND}{2}\right]^2\left[\sqrt{\frac{2D(h_r+\lambda gND\theta_b)}{A_r}}\frac{(S+A_mM)}{iqN}-1\right] \tag{B.17}$$

where

$$\frac{\partial^2 TRC_s(M,N,\lambda,\theta)}{\partial Q^2}=\frac{2D(S+A_mM)}{NQ^3}>0 \tag{B.18}$$

$$\frac{\partial^2 TRC_s(M,N,\lambda,\theta)}{\partial \theta^2}=\frac{iq}{\theta^2}>0 \tag{B.19}$$

$$\frac{\partial^2 TRC_s(M,N,\lambda,\theta)}{\partial Q\partial\theta}=\frac{(1-\lambda)gND}{2}>0 \tag{B.20}$$

The Hessain matrix is positive when $\left[\frac{(1-\lambda)gND}{2}\right]^2\left[\sqrt{\frac{2D(h_r+\lambda gND\theta_b)}{A_r}}\frac{(S+A_mM)}{iqN}-1\right]>0$ and then $TRC_s(M,N,\lambda,\theta)$ is convex with respect to Q and θ.

B.5 Proof of Positive Definiteness of Hessain Matrix of *TRC*(*Q,m,θ,S,L*) in *Q, θ and S*.

For a given value of *L*, we first obtain the Hessain matrix as follows:

$$H=\begin{bmatrix}\frac{\partial^2 TRC(Q,L,m,\theta,S)}{\partial Q^2} & \frac{\partial^2 TRC(Q,L,m,\theta,S)}{\partial Q\partial\theta} & \frac{\partial^2 TRC(Q,L,m,\theta,S)}{\partial Q\partial S}\\ \frac{\partial^2 TRC(Q,L,m,\theta,S)}{\partial\theta\partial Q} & \frac{\partial^2 TRC(Q,L,m,\theta,S)}{\partial\theta^2} & \frac{\partial^2 TRC(Q,L,m,\theta,S)}{\partial\theta\partial S}\\ \frac{\partial^2 TRC(Q,L,m,\theta,S)}{\partial S\partial Q} & \frac{\partial^2 TRC(Q,L,m,\theta,S)}{\partial S\partial\theta} & \frac{\partial^2 TRC(Q,L,m,\theta,S)}{\partial S^2}\end{bmatrix} \tag{B.21}$$

For a fixed $L\in[L_i,L_{i-1}]$, *TRC(Q,m,θ,L,S)* is convex with respect to *Q, θ,* and *S*, since:

The first principal minor of H is

$$|H_{11}|=\frac{2D}{Q^3}\left[A+\frac{S}{m}+\left\{c_i(L_{i-1}-L)+\sum_{j=1}^{i-1}c_j(b_j-a_j)\right\}\right]>0 \tag{B.22}$$

The second principal minor of H is

$$\left|H_{22}\right| = \left(\frac{gmD}{2}\right)^2 \frac{1}{i_\theta q_\theta}\left[\left\{\left(m\left(1-\frac{D}{P}\right)-1+\frac{2D}{P}\right)rC_v + rC_p\right\}Q + i_\theta q_\theta\right] > 0 \qquad \text{(B.23)}$$

The third principal minor of H is

$$\left|H_{33}\right| = \frac{D}{Q^2 m}\left(\frac{D}{i_s q_s m}-1\right)\left|H_{22}\right| > 0 \text{ if } \left(\frac{D}{i_s q_s m}-1\right) > 0 \qquad \text{(B.24)}$$

APPENDIX C

C.1 Proof of Convergence of λ_2

For $i = 1,2,\ldots\ldots.,m$,

$$f(\lambda_2) = \sum_{i=1}^{m} s_i \sqrt{\frac{H_i + \frac{Q_i + R_i}{n_i}}{A_i + n_i v_i + \lambda_2 s_i}} - \kappa \quad \text{(C.1)}$$

The limiting values of $f(\lambda_2)$ and n_i are

$$\lim_{\lambda_2 \to 0} f(\lambda_2) = \sum_{i=1}^{m} s_i \sqrt{\frac{H_i + \frac{Q_i + R_i}{n_i}}{A_i + n_i v_i}} - \kappa \quad \text{(C.2)}$$

$$\lim_{\lambda_2 \to 0} n_i = \sqrt{\frac{A_i (Q_i + R_i)}{H_i v_i}} \quad \text{(C.3)}$$

Substituting Eq. (C.3) in Eq. (C.2) yields

$$\lim_{\lambda_2 \to 0} f(\lambda_2) = \sum_{i=1}^{m} s_i \sqrt{\frac{H_i}{A_i}} - \kappa > 0, \text{ if } \sum_{i=1}^{m} s_i \sqrt{\frac{H_i}{A_i}} - \kappa > 0 \quad \text{(C.4)}$$

$$\lim_{\lambda_2 \to \infty} f(\lambda_2) = -\kappa \quad \text{(C.5)}$$

$f'(\lambda_2) = -\frac{1}{2}\sum_{i=1}^{m} s_i^2 \frac{\sqrt{H_i + \frac{Q_i + R_i}{n_i}}}{(A_i + n_i v_i + \lambda_2 s_i)^{3/2}} < 0$ for all values of λ_2, which implies that $f(\lambda_2)$ is a strictly decreasing function of λ_2. Hence an unique optimal solution for λ_2 satisfying $f(\lambda_2)=0$ exists.

C.2 Proof of Convergence of $T(n_i)$

For $i = 1,2,\ldots\ldots.,m$,

$$f(T_i) = T_i g(T_i) = T_i^2 - \frac{A_i + n_i v_i + \lambda_2 s_i}{H_i + \frac{Q_i + R_i}{n_i}} \quad \text{(C.6)}$$

Substituting α_i in Eq. (5.58) in Q_i results in

$$Q_i = \frac{i_\alpha b_i n_i}{T_i} \quad \text{(C.7)}$$

Substituting A_i in Eq. (5.57) and Q_i in Eq. (C.7) in Eq. (C.6) yields

$$g(T_i) = T_i - \frac{n_i (i_a a_i T_i + n_i v_i + \lambda_2 s_i)}{(H_i n_i T_i + i_\alpha b_i n_i + T_i R_i)} \quad \text{(C.8)}$$

$$\lim_{T_i \to 0} g(T_i) = -\frac{n_i(n_i v_i + \lambda_2 s_i)}{i_\alpha b_i n_i} < 0 \tag{C.9}$$

$$\lim_{T_i \to \infty} g(T_i) = \infty \tag{C.10}$$

If $g'(T_i) = 1 + \frac{n_i(n_i v_i + \lambda_2 s_i)(H_i n_i + R_i) - i_a a_i i_\alpha b_i n_i^2}{(H_i n_i T_i + i_\alpha b_i n_i + T_i R_i)^2} > 0$ for all values of T_i, which implies that $g(T_i)$ is a strictly increasing function of T_i. Since T_i satisfying $g(T_i)=0$ always exists, the optimal solution for T_i satisfying $f(T_i)=0$ also always exists.

APPENDIX D

D.1 Derivation of Average Finished Goods Inventory

Fig. D.1 shows the inventory level at any time t, with regard to the quantity manufactured and the total inventory consumed.

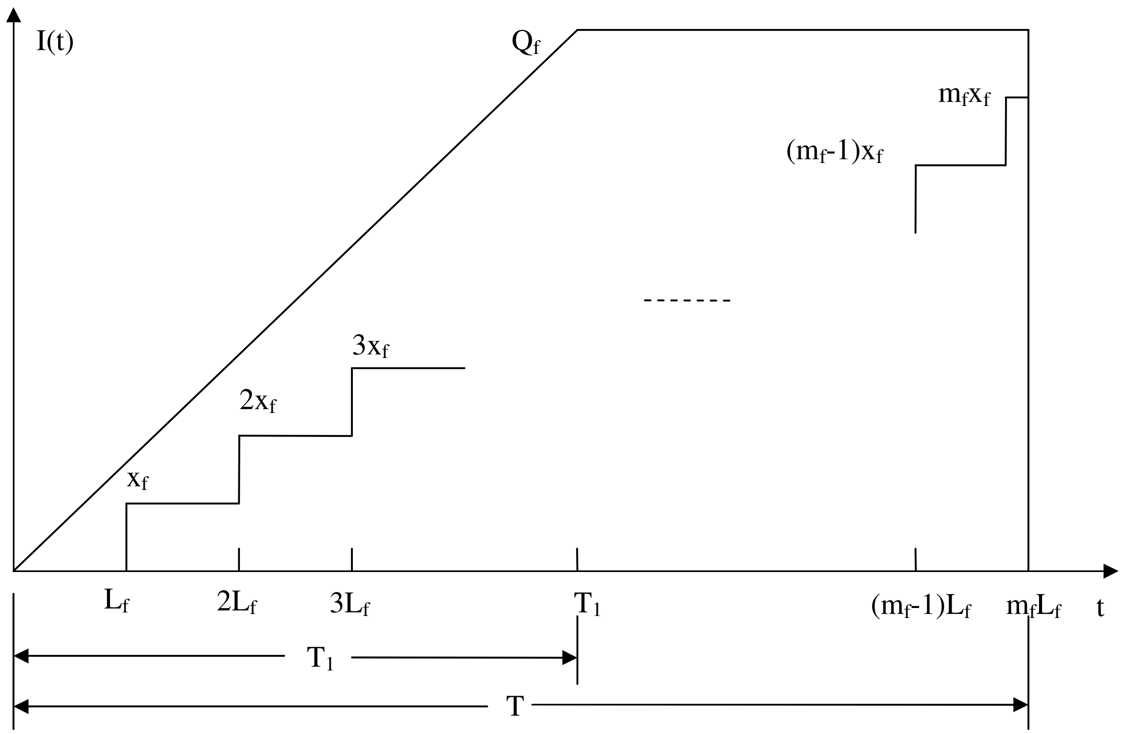

Fig. D.1 Inventory build-up and lumpy demand

The level of finished goods inventory, $Q_f(t)$ at any time t is the difference between the manufactured quantity, $Q_m(t)$ and shipping quantity, $Q_s(t)$. The average inventory per cycle, Q_{avg}, is expressed as (Sarker and Parija, 1996):

$$Q_{avg} = \frac{1}{T}\int_0^T Q_f(t)dt = \frac{1}{T}\left(\int_0^T Q_m(t)dt - \int_0^T Q_s(t)dt\right) \tag{D.1}$$

$$Q_m(t) = \begin{cases} P_f t, & 0 \le t \le T_1 \\ P_f T_{1f}, & T_1 \le t \le T \end{cases} \tag{D.2}$$

and

$$Q_s = jx_f \text{, where } jL_f \le t \le (j+1)L_f \text{, for } j = 0,1,2,......, m_f - 1 \tag{D.3}$$

$$\frac{1}{T}\int_0^T Q_m(t)dt = \frac{1}{T}\left[\int_0^{T_1} P_f t dt + \int_{T_1}^{T} P_f T_{1f}\, dt\right] = P_f T_1\left(1 - \frac{T_1}{2T}\right) \tag{D.4}$$

Substituting $P_f T_1 = TD_f$,

$$\frac{1}{T}\int_0^T Q_m(t)dt = TD_f\left(1 - \frac{D_f}{2P_f}\right) = Q_f\left(1 - \frac{D_f}{2P_f}\right) \tag{D.5}$$

Similarly (Sarker and Parija, 1996),

$$\frac{1}{T}\int_0^T Q_s(t)dt = \frac{1}{T}\int_0^{m_f L_f} Q_s(t)dt = \frac{1}{T}\sum_{j=0}^{m_f-1}\left(\int_{jL_f}^{(j+1)L_f} jx_f\, dt\right) = \frac{1}{T}\sum_{j=0}^{m_f-1}\left(jx_f L_f\right)$$
$$= \frac{m_f(m_f-1)x_f L_f}{2T} = \frac{(m_f-1)x_f}{2} = \frac{(m_f-1)Q_f}{2m_f} \tag{D.6}$$

since $m_f L_f = T$ and $m_f x_f = Q_f$

Substituting Eqs. (D.5) and (D.6) in Eq. (D.1), yield

$$Q_{avg} = Q_f\left(1 - \frac{D_f}{2P_f}\right) - \frac{(m_f-1)Q_f}{2m_f} \tag{D.7}$$

$$Q_{avg} = \frac{1}{2}\left\{m_f x_f\left(1 - \frac{D_f}{P_f}\right) + x_f\right\} \tag{D.8}$$

D.2 Derivation of Average Finished Goods Inventory with Rework

Fig. D.2 shows the inventory level at any time t, with regard to the quantity manufactured and the total inventory consumed.

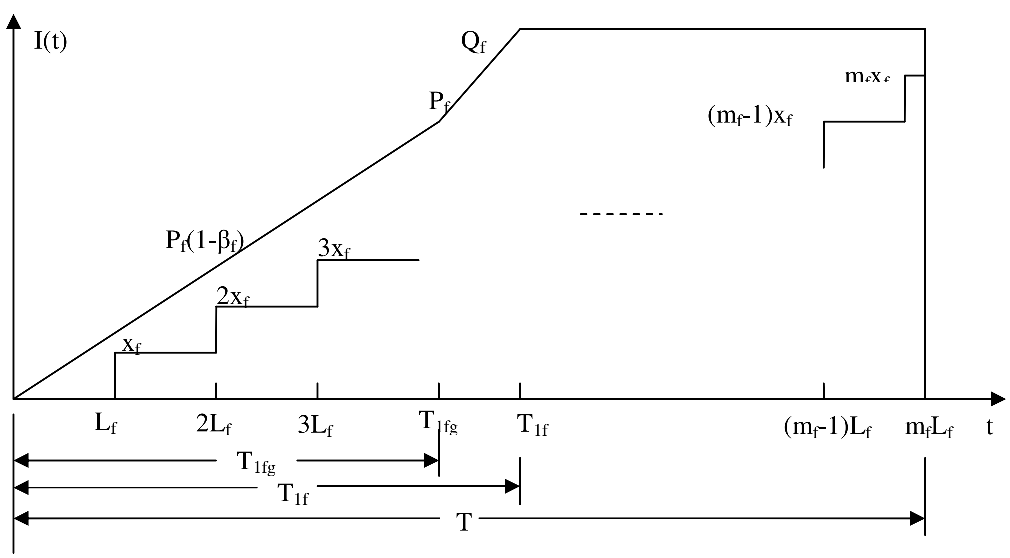

Fig. D.2 Inventory build-up and lumpy demand with rework

The level of finished goods inventory, $Q_f(t)$ at any time t is the difference between the manufactured quantity, $Q_m(t)$ and shipping quantity, $Q_s(t)$. The average inventory per cycle, Q_{avg}, is expressed as (Sarker and Parija, 1996):

$$Q_{avg} = \frac{1}{T}\int_0^T Q_f(t)dt = \frac{1}{T}\left(\int_0^T Q_m(t)dt - \int_0^T Q_s(t)dt\right) \quad \text{(D.9)}$$

where

$$Q_m(t) = \begin{cases} P_f(1-\beta_f)t, & 0 \le t \le T_{1fg} \\ \left[P_f(1-\beta_f)T_{1fg} + \left(P_f T_{1f} - P_f(1-\beta_f)T_{1fg}\right)\frac{(t-T_{1fg})}{(T_{1f}-T_{1fg})}\right], & T_{1fg} \le t \le T_{1f} \\ PT_{1f}, & T_{1f} \le t \le T \end{cases}$$

(D.10)

and

$Q_s = jx_f$, where $jL_f \le t \le (j+1)L_f$, for $j = 0,1,2,......,m_f - 1$ (D.11)

$$\frac{1}{T}\int_0^T Q_m(t)dt = \frac{1}{T}\left[\begin{array}{l} \int_0^{T_{1fg}} P_f(1-\beta_f)tdt \\ + \int_{T_{1fg}}^{T_{1f}}\left[P_f(1-\beta_f)T_{1fg} + \left(P_f T_{1f} - P_f(1-\beta_f)T_{1fg}\right)\frac{(t-T_{1fg})}{(T_{1f}-T_{1fg})}\right]dt \\ + \int_{T_{1f}}^{T} PT_{1f}dt \end{array}\right] \quad \text{(D.12)}$$

$$\frac{1}{T}\int_0^T Q_m(t)dt = \frac{1}{2T}P_f(1-\beta_f)T_{1fg}^2 + \frac{1}{T}P_f(1-\beta_f)T_{1fg}(T_{1f}-T_{1fg}) + \frac{1}{2T}P_f\left[T_{1f} - (1-\beta_f)T_{1fg}\right](T_{1f}-T_{1fg}) + \frac{1}{T}P_f T_{1f}(T - T_{1f}) \quad \text{(D.13)}$$

Substituting $P_f T_{1fg} = (1-\beta_f)TD_f = (1-\beta_f)Q_f$, $P_f(T_{1f}-T_{1fg}) = \beta_f TD_f = \beta_f Q_f$ and $P_f T_{1f} = TD_f = Q_f$

$$\frac{1}{T}\int_0^T Q_m(t)dt = (1-\beta_f)^3\frac{Q_f D_f}{2P_f} + (1-\beta_f)^2\frac{\beta_f Q_f D_f}{P_f} + \left[1-(1-\beta_f)^2\right]\frac{\beta_f Q_f D_f}{2P_f} + Q_f\left(1-\frac{D_f}{P_f}\right)$$

(D.14)

Similarly (Sarker and Parija, 1996),

$$\frac{1}{T}\int_0^T Q_s(t)dt = \frac{1}{T}\int_0^{m_f L_f} Q_s(t)dt = \frac{1}{T}\sum_{j=0}^{m_f-1}\left(\int_{jL_f}^{(j+1)L_f} jx_f\,dt\right) = \frac{1}{T}\sum_{j=0}^{m_f-1}\left(jx_f L_f\right)$$

$$= \frac{m_f(m_f-1)x_f L_f}{2T} = \frac{(m_f-1)x_f}{2} = \frac{(m_f-1)Q_f}{2m_f} \qquad \text{(D.15)}$$

since $m_f L_f = T$ and $m_f x_f = Q_f$

Substituting Eqs. (D.14) and (D.15) in Eq. (D.9), yield

$$Q_{avg} = (1-\beta_f)^3\frac{Q_f D_f}{2P_f} + (1-\beta_f)^2\frac{\beta_f Q_f D_f}{P_f} + \left[1-(1-\beta_f)^2\right]\frac{\beta_f Q_f D_f}{2P_f} + Q_f\left(1-\frac{D_f}{P_f}\right) - \frac{(m_f-1)Q_f}{2m_f} \qquad \text{(D.16)}$$

$$Q_{avg} = \frac{1}{2}\left\{m_f x_f\left(1-\frac{D_f\left(1+\beta_f-\beta_f^2\right)}{P_f}\right) + x_f\right\} \qquad \text{(D.17)}$$

$$TC(m_f, n_0) = \frac{D_f(n_0K_0+K_s)}{m_f x_f} + \frac{m_f x_f D_f H_0}{2fn_0P_f} + D_f\beta_f C_f + \frac{1}{2}\left\{m_f x_f\left(1-\frac{D_f\left(1+\beta_f-\beta_f^2\right)}{P_f}\right) + x_f\right\}H_f \qquad \text{(D.18)}$$

D.3 Proof for Global Optimality of Total Cost

$$H = \begin{bmatrix} \frac{\partial TC(m_f, n_o)}{\partial m_f^2} & \frac{\partial TC(m_f, n_0)}{\partial m_f \partial n_0} \\ \frac{\partial TC(m_f, n_0)}{\partial n_0 \partial m_f} & \frac{\partial TC(m_f, n_0)}{\partial n_o^2} \end{bmatrix} \qquad \text{(D.19)}$$

$$H_1 = \left|\frac{\partial TC(m_f, n_o)}{\partial m_f^2}\right| = \frac{2D_f(n_0K_0+K_s)}{m_f^3 x_f} > 0 \qquad \text{(D.20)}$$

$$H_2 = \left|\frac{\partial TC(m_f, n_o)}{\partial m_f^2}\frac{\partial TC(m_f, n_0)}{\partial n_o^2} - \frac{\partial TC(m_f, n_0)}{\partial m_f \partial n_0}\frac{\partial TC(m_f, n_0)}{\partial n_0 \partial m_f}\right| = \frac{2D_f^2 H_0(n_0K_0+K_s)}{m_f^2 n_o^3 fP_f} - \left(\frac{D_f K_0}{m_f^2 x_f} + \frac{x_f D_f H_0}{2fn_o^2 P_f}\right)^2 = \frac{2D_f^2 K_s H_0}{m_f^2 n_o^3 fP_f} > 0 \qquad \text{(D.21)}$$

In this case, optimal solution exists for total cost function and total cost function attains a minimum value for the unique values (m_f, n_o).