Chapter 1

Introduction

Differential equations describe the relation between a function and its one or more derivatives. The solutions of the differential equations are the functions and not just numbers unlike algebraic equations. The most natural expression of many general laws of nature in chemistry, physics, economics, biology, astronomy and engineering lies in the language of the differential equation. The discovery of a new planet (Neptune) is an example of successful study of a mathematical model and observation of some analyzing results that was carried out long before the arrival of modern computers.

A differential equation where the unknown and/or its derivative(s) at the current time depend on the solution and possibly its derivatives at previous times (known as delay/lag) is known as delay differential equation (DDE). The solution of the DDEs depends not only on the solution at a present stage but also on the solution at some past stages. Mostly the delay represents gestation times, transport delays, incubation periods, or simply it leads to complicated biological processes together, accounting only for the time required for the process to occur. The mathematical modeling of physical models becomes more interesting when traditional modeling assumptions are replaced by some more realistic assumptions, for example, in predator-prey model, the birth rate of predators is not only effected by current levels but also by prior levels of predator or prey. Thus, the more realistic model includes some of the past and future states of the system and hence this type of real system should be modeled by a differential equation with delay and/or advance term.

1.1 Singularly Perturbed Boundary Value Problems (SP-BVPs)

"While studying SPPs always try the easiest case first it may be harder than you expect!! "

- MARTIN STYNES,

Definition 1.1.1. In the field of differential equation an initial value problem is a differential equation with a specified value which is called an initial condition of the unknown function at a given point in the domain of the solution.

Definition 1.1.2. In mathematics, in the field of differential equations, a boundary value problem is a differential equation with a set of additional constraints, called the boundary conditions. The solution of a boundary value problem also satisfies the boundary conditions.

For example, if the independent variable is time over the domain $[0,2]$, a boundary value problem would specify values for $u(t)$ at both $t = 0$ and $t = 2$, whereas an initial value problem would specify a value of $u(t)$ and $u'(t)$ at time $t = 0$.

A differential equation with a small positive parameter multiplying the highest derivative term subject to the boundary conditions belongs to a class of problems known as singular perturbation problems. The solutions of these problems have singularities related to boundary layers. The justification for the name "Singular Perturbation" is that the nature of the differential equations changes completely in the limit case when singular perturbation parameter is equal to zero. For example, the conservation of momenta and the conservation of energy equations change from being nonlinear parabolic equations to nonlinear hyperbolic equations. These problems have steep gradients in the narrow region(s) of the domain which are called boundary and/or interior layer(s). Mathematically, let us consider a boundary value problem; say P_ε depending on a small parameter $\varepsilon \in (0,1]$, known as a perturbation parameter. We assume that, for each ε, the problem P_ε has a unique smooth solution $u_\varepsilon(x)$. Our goal is to construct approximations of $u_\varepsilon(x)$ for small values of ε. The differential equation and boundary conditions constituting P_ε approach limiting forms and define a “reduced boundary value problem”, say P_0. Some of the original boundary conditions are no longer necessary for P_0. The problem P_ε is said to be a perturbed problem (perturbed model), while the problem P_0 is called an unperturbed problem (reduced model). The solution $u_\varepsilon(x)$ of P_ε depends on ε and boundary conditions as well. The problem P_ε is called regularly perturbed if there exists a solution $u_0(x)$ of the problem P_0 such that $u_\varepsilon(x) \to u_0(x)$ uniformly as $\varepsilon \to 0$, otherwise, P_ε is said to be singularly perturbed. Moreover, in the case of regular perturbation, $u_\varepsilon(x)$ can be constructed by the usual perturbation calculations as a power series in ε. On the other hand, in the case of singular perturbation one cannot represent the solution $u_\varepsilon(x)$ as an asymptotic expansion in powers of ε.

Prandtl’s boundary layer theory is the prototype of singularly perturbed problem. The term "boundary layer" is introduced by Prandtl [1] at the third international congress of mathematicians in 1904 in Heidelberg and proved that the flow about a body can be treated in two regions: a very thin layer (which is called boundary layer) where frictional effects are prominent, and the remaining outer region (which is called regular part). However, the term "singular perturbation" was first used by Friedrichs and Wasow in their paper [2]. After the introduction by Prandtl, this work got much more generality in the substantial work of Wasow [3]. There are many practical models which contains a small parameter multiplied by highest order derivative like in the modeling of steady and unsteady viscous flows problems with large Reynolds number, combustion, fluid dynamics etc. One of the most striking example of singular perturbation is the Navier-Stokes equation of fluid dynamics

$$\frac{\partial(u^2+p)}{\partial x}+\frac{\partial(uv)}{\partial y}=\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \tag{1.1}$$

with some suitable boundary and initial conditions. Here, u and v represents the velocity components in x and y directions respectively and p is the pressure. The parameter Re is the ’Reynolds number’ which is proportional to the velocity scale, length scale and inversely proportional to the kinematic viscosity of the fluid. The equation (1.1) will be transformed into singularly perturbed differential equation for sufficiently large value of $Re(\gg 1)$. There are many physical phenomena like water quality problems in river networks [4], simulation of oil extraction from underground reservoirs [5] etc. in which these type of problems arise. There are some books in which a comprehensive overview of singularly perturbed problems and their theoretical and numerical treatment can be found like Kaplun [6], Hemker [7], Doolen *et al.*

[8], O'Malley [9], Farrell *et al.* [10], Bender and Orszag [11], Ardema [12], Meyer and Parter [13], Wasow [14] and the citations therein.

The numerical treatment of these type of problems are found to be interesting not only because of their applications but also because their solution exhibit multi-scale phenomena and have small regions (boundary and/or interior layer(s)) with steep gradients which creates serious difficulties for numerical techniques.

1.1.1 Singularly perturbed BVPs for differential-difference equations

A singularly perturbed differential equation in which the dependent variable or/and its derivative(s) has some delay and/or advance term is called a singularly perturbed differential-difference equation. These problems have numerous applications in science and engineering, especially in biological, chemical, electronic and transportation systems. Some of these include simulation of oil extraction from underground reservoirs, chemical kinetics [15–17], chemical processes [18], the drift-diffusion model of semiconductor devices [19, 20] etc. Several other phenomena especially appearing in many branches of biological modeling like the study of drug therapy [21], circadian rhythms [22], industrial processes [23], neural networks [24], primary infection [25], fluid flows, water quality problems in river networks [26], mechanical systems [27] can also be modeled by using DDEs. For further development of the theory and the applications of DDEs the readers may refer to [28–30].

A nonlinear example from population dynamics is the Hutchinson's equation

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} + \lambda u(1 - u(x, t - \tau)),$$

which arises as a rough model for the evolution of a population in mathematical ecology with density u. The parameter λ represents the growth rate of the population. It is observed that in many practical applications, the parameter ε is usually small.

1.1.2 Some singularly perturbed differential-difference models

1. **Optical and physiological models**

   The singularly perturbed differential-difference equation [31–35]

$$\varepsilon x'(t) = -x(t) + f(x(t-1)),$$

describes the mathematical models in optics, physiology, biology etc. Here ε is inversely proportional to the product of the time delay in the physical system and its rate of decay. This delay differential equation becomes singularly perturbed when this time lag is large relative to the reciprocal of the decay rate.

2. **Processing of metal sheets**

   Equation modeling a furnace used to process metal sheets [36] is given by

$$\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} = v(g(u(x,t-\tau)))\frac{\partial u}{\partial x} + c[f(u(x,t-\tau)) - u(x,t)].$$

   Here, u is the temperature distribution in a metal sheet, moving at an instantaneous material strip velocity v and heated by distributed temperature source specified by the function f; both v and f are dynamically adapted by a controlling device monitoring the current temperature distribution. The finite speed of the controller, however, introduces a fixed delay of length τ. When $\tau = 0$, this problem becomes a thermal problem without time delay.

3. **Motion of the Sunflower**

   Consider a mathematical model proposed by Israelson and Johnson [37] to explain the helical movements of the tip of the growing plants. In Sunflower, the upper part of the stem performs rotating movement. The mathematical model describing the motion of the Sun flower is the equation

$$\varepsilon x''(t) + ax'(t) + b\sin(x(t-\varepsilon)) = 0,$$

$$x(t) = \phi(t), \ -\varepsilon \le t \le 0, \quad x'(0) = y_0.$$

   Here x is the angle of the plant with the vertical and ε is the time lag; a and b are positive parameters which can be obtained experimentally.

4. **Van der Pol equation**

   Van der Pol equation is a model of electronic circuit which appeared in very early radios. This circuit appeared back in the days of vacuum tubes. When the current is high, the tube acts like a normal resistor, but if the current is low, the tube acts like a negative resistor. So this circuit drags down large oscillations but pumps up small oscillations and this behavior is recognized as a relaxation oscillation . In 1927, Van der Pol derived the following equation

$$y''(t) + \varepsilon(y^2 - 1)y'(t) + y = 0,$$

where ε is a constant which effects that how non linear the system is. For $\varepsilon = 0$, the system is a linear oscillator. The non linearity of the system becomes impossible to ignore as ε grows. At initial stage, he did this purely as a matter of mathematical interest, but very soon it appeared that the relaxation-oscillations described by this equation were also of great practical importance. Van der Pol derived several properties of the oscillation directly from the mathematical expression [38] without solving the above equation analytically. For more detail readers can refer [39].

Oliveira [40] considered a Van der Pol model with a shift parameter τ (delayed time)

$$y''(t) - \varepsilon y'(t) + \varepsilon y^2(t-\tau)y'(t-\tau) + y(t) = 0,$$

where $0 < \varepsilon \ll 1$ and $0 \leq \tau < \frac{\pi}{2}$ are the real parameters and proved the existence and stability of a periodic solution.

1.1.3 Singularly perturbed turning point problems

Singularly perturbed turning point boundary value problems (SPTPBVPs) have always been attractive in both applied and pure mathematics, because the solution of such type of problems exhibits an interesting behavior like boundary /interior layer(s)/resonance phenomena [41–46]. These problems are more difficult to handle as compare to the non turning point problems. Boundary turning point problem on the other hand, arise for instance, in laminar flow [47], heat flow [48] transport phenomenon [49] etc. When the coefficient of convection term have a simple zeros, the turning point is said to be simple and for the multiple zeros of the convection coefficient it becomes multiple turning point. For the applications of the multiple boundary turning points problems the readers are referred to [47].

A linear turning point problem in one dimension is given by

$$\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x),\ x \in (x_1, x_2),\ x_1 < 0,\ x_2 > 0,$$

where $\varepsilon \in (0,1)$, $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth functions. These type of problems become more interesting for mathematicians and physicist due to the different solution behavior as discussed above. When $a(x)$ does not change sign in the interval $[x_1, x_2]$ then the solution have a boundary layer near one endpoint as $\varepsilon \to 0$. Any point $x_0 \in [x_1, x_2]$ say $x_0 = 0$ is called turning point, when $a(x)$ has a simple zero at x_0 and the problem is classified as turning point problem. In this case, the solution behavior depend upon the properties of the coefficient $a'(x)$ and $b(x)$ at the turning point $x_0 = 0$. Indeed, let, for some constants α, β, $a(x) \approx \alpha x$ and $b(x) \approx \beta$ as $x \to 0$ then following cases arise:

i. If $\alpha > 0$, $\beta/\alpha \neq 1, 2, 3, \ldots$, an internal layer occurs near the turning point $x_0 = 0$. This type of turning point sometimes called converging flow turning point.

ii. If $\alpha < 0$, $\beta/\alpha \neq 0, -1, -2, \ldots$, two boundary layers occurs at the both end points of the interval. This type of turning point sometimes called diverging flow turning point.

iii. If $\alpha < 0$, $\beta/\alpha = 0, -1, -2, \ldots$, or if $\alpha > 0, \beta/\alpha = 1, 2, 3, \ldots$, the solution exhibits a very interesting phenomenon named Ackerberg-O'Malley's resonance phenomenon [50].

1.1.4 Some singularly perturbed turning point models

1. **Fokker-Plank equation**

   Fokker-Plank equation [51] is given by

$$\varepsilon^2 \frac{d^2\phi}{dx^2} + b(x)\frac{d\phi}{dx} = 0, 0 < \varepsilon \ll 1, x \in (0,1),$$
$$\phi(0) = A, \phi(1) = B,$$

   where $b(x)$ is the gradient field. Under the assumption that $b'(x)$ is strictly negative throughout the interval $[0,1]$ and $b(\gamma) = 0$ for some $0 < \gamma < 1$, the solution of the problem exhibits resonant turning point behavior.

2. **Homogenization process of a passive tracer**

The Homogenization process of a passive tracer in a flow is described by a time-dependent singularly perturbed system of partial differential equations [52]

$$
\begin{aligned}
u_t &= \varepsilon u_{xx} - v, \quad (x,t) \in (0,1) \times (0,\infty),\\
u(x,0,\varepsilon) &= \phi_1(x,\varepsilon), \quad x \in [0,1],\\
u(0,t,\varepsilon) &= u(1,t,\varepsilon) = 0, \quad t \in [0,\infty),\\
v_t + f(u,u_x)v_x + g(x,u,u_x)v &= \varepsilon v_{xx}, \quad (x,t) \in (0,1) \times (0,\infty),\\
v(x,0,\varepsilon) &= \phi_2(x,\varepsilon), \quad x \in [0,1],\\
v(0,t,\varepsilon) &= v_0(t),\, v(1,t,\varepsilon) = v_1(t), \quad t \in [0,\infty),
\end{aligned}
$$

where $f(u,u_x) = h(u)$ or $h(u_x)$ and $g(x,u,u_x) > 0$.

3. **Black-Scholes model**

   Black-Scholes model [53] which are modeled for some financial data is given by

$$
\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0, \quad (S,t) \in R^+ \times [0,T),
$$

   with the following boundary conditions at $S = 0$ and $S \to \infty$,

$$
C(0,t) = 0,\; C(S,t) \to S \text{ when } S \to \infty,\; t \in [0,T),
$$

   and the final condition

$$
C(S,T) = \max(S - E, 0), \quad S \in R^+.
$$

   Here $C = C(S,t)$ is an European call option, S and t are the current values of the underlying asset and time, E, σ, r and T denote exercise price, volatility, the risk free interest rate, and expiry time respectively. The last condition gives the value of option at the time when the option matures. The solution of *PDE* gives the value of the option at any earlier time.

4. **Quantum mechanics model**

Consider the model which describes a quantum mechanical particle in a potential $V(x)$, see [11],

$$\left(-\varepsilon^2 \frac{d^2}{dx^2} + V(x) - E\right) y(x) = 0,$$

where $V(x)$ and E are the potential energy and the total energy of the particle. Here $Q(x) = V(x) - E$, which vanishes at points where $V(x) = E$ and these points are called turning points. The classical orbit of a particle in the potential $V(x)$ is confined to regions where $V(x) < E$. The particle moves until it reaches a point where $V = E$ and then it stops, turns around and moves off in opposite direction.

1.1.5 Two parameter singularly perturbed BVPs

Singularly perturbed differential equations in which there are two small parameters multiplying the highest order derivative and convective term are called two parameter problems. The two-parameter SPBVPs have applications in lubrication theory, electrical networks, control theory [9, 54, 55] etc. To the best of our knowledge, O'Malley was the first who studied these problems asymptotically (see for the reference [56–60]). The solution to these problems depends not only on the parameters ε_1 and ε_2 (highest order derivative and convective term coefficients respectively) but it is significantly different depending upon the ratios $\varepsilon_1/\varepsilon_2^2 \to 0$ as $\varepsilon_2 \to 0$, and $\varepsilon_2^2/\varepsilon_1 \to 0$ as $\varepsilon_1 \to 0$. For $\varepsilon_2 = 1$, an exponential boundary layer exhibits of width $O(\varepsilon_1)$ in the neighbourhood of the left lateral boundary whereas when $\varepsilon_2 = 0$, parabolic boundary layers of width $O(\sqrt{\varepsilon_1})$ appear at both the left and right lateral boundary as $\varepsilon_1 \to 0$. In this thesis, we will consider two Cases: I. $\varepsilon_1/\varepsilon_2^2 \to 0$ as $\varepsilon_2 \to 0$, and II. $\varepsilon_2^2/\varepsilon_1 \to 0$ as $\varepsilon_1 \to 0$. For Case I, two boundary layers of widths $O(\varepsilon_1/\varepsilon_2)$ and $O(\varepsilon_2)$ occurs at left and right lateral boundary respectively, while in Case II, both twin layers are of $O(\sqrt{\varepsilon_1})$.

1.2 Numerical Approximations of SPPs

Firstly, we will explain the Landau's order symbol O (big-oh) and o(little-oh) which are used throughout the thesis to define the order of convergence of the numerical method.

Definition 1.2.1. The expression $f(\varepsilon) = O(g(\varepsilon))$ as $\varepsilon \to 0$, means there exist some positive constants C and ω where $\varepsilon \in (0, \omega]$ such that

$$|f(\varepsilon)| \leqslant C|g(\varepsilon)|, \quad \varepsilon \to 0.$$

Definition 1.2.2. The expression $f(\varepsilon) = o(g(\varepsilon))$ as $\varepsilon \to 0$, means

$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{g(\varepsilon)} = 0.$$

For more details of these two definitions one can refer to [61].

Definition 1.2.3. A matrix $A = (a_{ij}) \in R^{k \times k}$ is said to be M-matrix if A is non singular, $A^{-1} \geq 0, a_{ij} \leq 0$ for all $i \neq j$, $1 \leq i, j \leq k$.

Definition 1.2.4. A square matrix A is said to be diagonally dominant if, for every row of the matrix A, the magnitude of the diagonal entry in a row is greater than or equal to the sum of the magnitudes of all the other non-diagonal entries in that row *i.e.,*

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|.$$

1.2.1 Numerical instability

Definition 1.2.5. ε-uniform numerical method: Let u and u^N be the exact and numerical solutions respectively of the given problem, where N is the number of mesh subdivisions used, $\|u\|_{\Omega^N} = \max_{x_i \in \Omega^N} |u(x_i)|, 0 \leq i \leq N$ be the maximum point-wise norm. A numerical method is said to be parameter uniform of order p if there exist a positive integer N_0, such that

$$\sup_{\varepsilon} \|u - u^N\|_{\Omega^N} \leq CN^{-p}, \quad p > 0, \quad \text{for all} \quad N \geq N_0,$$

where Ω^N is discretization of Ω, the constant C is ε-uniform error constant which is independent of ε and N.

The boundary layer behavior of the SPBVPs leads to the failure of the classical/standard numerical methods unless an unacceptable large number of mesh elements are used, which is practically very tedious. Therefore, the classical numerical methods are not appropriate when the value of the perturbation parameter ε is close to zero. Let us take a simple example for the better understanding of the behavior of approximated solution by using a central difference scheme on a uniform mesh with spacing $h = \frac{1}{N}$.

$$\varepsilon u''(x) + u'(x) - u(x) = 0, \tag{1.2}$$

$$u(0) = 1, u(1) = 1. \tag{1.3}$$

After applying numerical scheme, we obtain

$$\varepsilon u''(x) + u'(x) - u(x) \equiv U_{i-1}\left(\frac{\varepsilon}{h^2} - \frac{1}{2h}\right) - U_i\left(\frac{2\varepsilon}{h^2} + 1\right) + U_{i+1}\left(\frac{\varepsilon}{h^2} + \frac{1}{2h}\right),$$
$$U_0 = 1, \quad U_N = 1.$$

After solving this system of equations, we will get the numerical solution which is shown in the following figures.

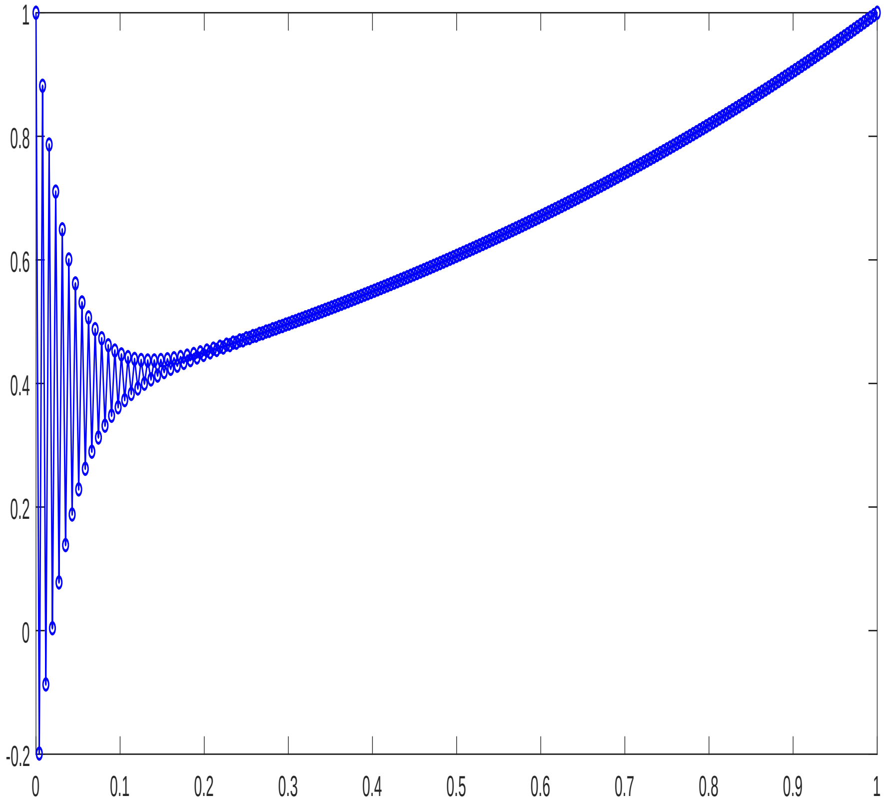

Figure 1.1: Numerical solution of (1.2) using uniform mesh with $\varepsilon = 10^{-4}$ and $N = 256$.

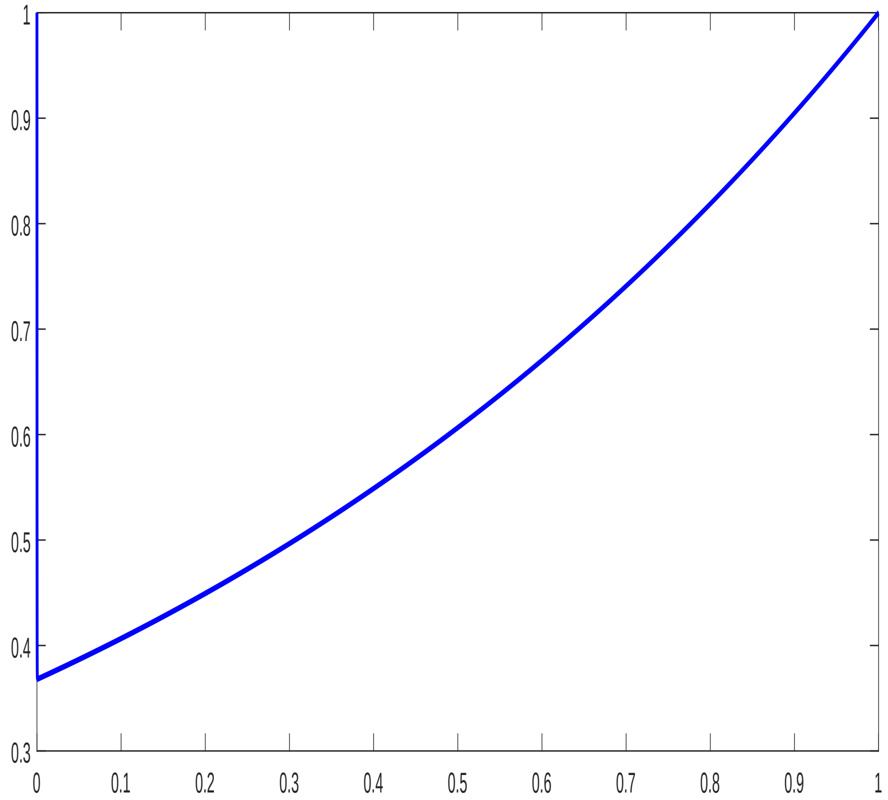

Figure 1.2: Numerical solution of (1.2) using uniform mesh with $\varepsilon = 10^{-4}$ and $N = 2^{13}$.

The explanation of the failure of classical methods is clearly visible from the Figure 1.1 because one can see the non physical oscillations in the numerical solution when $\varepsilon = 10^{-4}$ and $N = 256$ and these oscillations get smeared when $N = 2^{13}$ (see Figure 1.2). After a deep study of the matrices which are obtained after the approximation of differential equation, we observed that they are not M-matrices when the perturbation parameter ε is small relative to the mesh size. Therefore, the inverse can be expected to have both positive and negative entries. As a result, the approximated solution will have non physical oscillations unless the mesh size is small relative to ε. The conclusion is that for sufficiently small mesh size, the central difference method is stable for solving SPPs. Briefly, not just central difference scheme, but any classical numerical scheme on uniform mesh gives satisfactory result only if one uses an unacceptable large number of points. In this sense, the classical methods fails.

1.2.2 Fitted mesh

The solutions of the singularly perturbed boundary value problems have a complicated behavior in narrow part/s of the domain. The introduction of special adapted meshes overcomes these difficulties. To illustrate it, let us consider the one dimensional problem on the interval (0,1) and choose a point σ in the domain such that the layer region contains the sufficient number of points so that we can have better resolution of the solution in the layer region. The point σ is called the transition parameter. Due to the variation in the width of the layer with respect to the small perturbation parameter(s), several difficulties are experienced in solving the singular perturbation problems using standard numerical methods with uniform mesh. Then the mesh needs to be refined substantially to grasp the solution within the boundary layers. Layer adapted meshes have been first proposed by Bakhvalov for reaction diffusion equations and later on by Gartland [62] and others for convection diffusion equations. In early 90s, special piecewise uniform meshes have been introduced by Shishkin [63]. Because of simple structure of Shishkin meshes they have attracted much attention and are used in the whole thesis work for numerically approximation of singularly perturbed problems.

Shishkin Mesh:

(a) When boundary layer is at the left end: A Shishkin mesh is a piecewise-uniform mesh. The difference between a Shishkin mesh and any other piecewise-uniform mesh is the choice of the so-called transition parameter(s), which are the point(s) at which the mesh size changes abruptly. On the interval $\Omega = (0,1)$, choose a point σ satisfying $0 < \sigma \leq 1/2$ and assume $N = 2^r$, for some $r \geq 2$, the number of points which ensures that there is at least one point in each region. We define the mesh transition parameter as

$$\sigma = \min\{0.5, \sigma_0 \varepsilon \ln N\},$$

where σ_0 depends upon the convective coefficient. Then

- divide the interval $[0,1]$ into two sub-intervals $[0,\sigma)$ and $[\sigma,1]$.
- Divide the interval $[0,\sigma)$ into $N/2$ equal subdivisions each of width h and $[\sigma,1)$ into $N/2$ equal subdivisions each of width H.

- Hence, the Shishkin mesh is given by $\Omega^N = x_i, i = 0, 1, \ldots, N$ with $x_0 = 0, x_N = 1$ and the mesh width $h_i = x_i - x_{i-1}$ satisfies $h_i = h$ for $i = 1, 2, \ldots, N/2$ and $h_i = H$ for $i = N/2 + 1, \ldots, N$.
- The piecewise-uniform mesh is entirely determined by the two chosen parameters N and ε.

(b) When boundary layer is at the right end: In this case, the mesh is just the image of the mesh obtained in the left boundary layer case by using the transformation $x \to 1 - x$.

(c) When boundary layer is at the both ends: On the interval $\Omega = (0, 1)$, choose a point σ satisfying $0 < \sigma \leq 1/2$ and assume that $N = 2^r$ for some $r \geq 3$, the number of points which ensures that there is at least one point in each region. In this case, define a mesh transition parameter as

$$\sigma = \min\{0.25, \sigma_0 \sqrt{\varepsilon} \ln N\},$$

where σ_0 depends upon the convective coefficient.

- Divide the interval $[0, 1]$ into three sub-intervals $[0, \sigma)$, $[\sigma, 1 - \sigma)$ and $(1 - \sigma, 1]$.
- Divide the intervals $[0, \sigma)$ and $(1 - \sigma, 1]$ into $N/4$ equal subdivisions each of width h and the interval $[\sigma, 1 - \sigma)$ into $N/2$ equal subdivisions each of width H.
- Hence, the Shishkin mesh is given by $\Omega^N = \{x_i\}$, $i = 0(1)N$ with $x_0 = 0$,$x_N = 1$ and the mesh width $h_i = x_i - x_{i-1}$ satisfies $h_i = h$ for $i = 1, 2, \ldots, N/2$ and $h_i = H$ for $i = N/2 + 1, \ldots, 3N/2$, and $h_i = h$ for $i = 3N/2 + 1, \ldots, N$.
- Thus the piecewise-uniform mesh is entirely determined by the two chosen parameters N and ε.

Now, we will give the brief relevant literature review for the numerical solutions of singularly perturbed boundary value problems.

1.2.3 One parameter singularly perturbed boundary value problems

Qiu and Sloan [64] performed an analysis of upwind finite difference scheme on adaptive mesh for the singularly perturbed model problem and shown that adaptivity may be used to generate a mesh for which ε-uniform convergence is readily achieved. The mesh is produced by equidistributing a monitor function that is based on the exact solution. The mesh is an approximation to that which is produced by a fully adaptive scheme based on the equidistribution of a computed approximation to the monitor function. For the monitor function that was selected in this work, the equidistribution process gives rise to an exponentially graded mesh. Clavero *et al.* [65] constructed and analyzed a finite difference scheme used to solve a class of $2D$ time-dependent convection–diffusion problems, for which they supposed that the convection term is positive in both spatial directions. Authors used the Peaceman and Rachford method to discretize in time and high-order differences via an identity expansion finite difference scheme, defined on a piecewise uniform Shishkin mesh, to discretize in space. They have shown that the method is uniformly convergent with respect to the diffusion parameter, reaching almost order two in space. Motivated by the asymptotic behavior of the SPPs in 2009, Kadalbajoo and Kumar [66] presented an approximate method (Initial value technique) for the numerical solution of quasi-linear SPBVPs having a boundary layer at one end (left or right) point. Mohapatra and Natesan [67] presented the analysis of an upwind scheme for obtaining the solution of the convection-diffusion two-point boundary value problem with Robin boundary conditions. The solution is obtained on a suitable non-uniform mesh which is formed by equidistributing the arc-length monitor function. It is shown that the discrete solution obtained by the upwind scheme converges uniformly with respect to the perturbation parameter.

In 2017, Kumar and Kumar [68] considered a discrete Monotone Iterative Domain Decomposition (MIDD) method based on Schwarz alternating algorithm for solving singularly perturbed parabolic partial differential equations. A discrete iterative algorithm was proposed which combines the monotone approach and the iterative non-overlapping Domain Decomposition Method based on the Schwarz alternating procedure using three-step Taylor Galerkin Finite Element approximation for solving singularly perturbed parabolic partial differential equations. The convergence of the MIDD method have been established.

Shiralashetti *et al.* [69] presented a comparative study of Haar wavelet collocation method (HWCM) and Finite Element Method (FEM) for the numerical solution of parabolic type PDEs such as $1D$ singularly perturbed convection-dominated diffusion equation and $2D$ transient heat conduction problems validated against exact solution. The distinguishing feature of HWCM is that it provides a fast converging series of easily computable components. Compared with FEM, this approach needs substantially shorter computational time with better accuracy. It is found that higher accuracy can be attained by increasing the level of Haar wavelets. As consequences, it avoids more computational costs, minimize errors and speeds up the convergence. Xiong *et al.* [70] deduced the error estimates of the finite element method on Shishkin mesh for the solution of semi-linear SPBVPs.

1.2.4 Singularly perturbed differential- difference equations

Due to the presence of the small arbitrary singular perturbation parameter in its highest spatial derivative term, there exist narrow regions, in the neighborhood of outflow boundary (the lateral sides of the rectangle in our case), where the solution has the steep gradient. Due to the presence of the boundary layer(s), these problems are difficult to solve by using classical numerical methods on a uniform mesh. In fact, the classical methods for solving singular perturbation problems are unstable and fail to give accurate results and unexpected oscillations occur when the perturbation parameter is small. Therefore, in connection with the stiff behavior, it is of interest to develop suitable numerical methods, whose accuracies do not depend on the perturbation parameter *i.e.,* the methods which are uniformly convergent with respect to the perturbation parameter. The analysis of DDEs is not as straightforward as that of ODEs. There have been extensive analytical and numerical methods developed for the solutions of singularly perturbed PDEs and many kinds of DDEs in the context of ODEs thoroughly over the last two decades (see [10, 71–74] and the references therein).

The study of singularly perturbed delay partial differential equations (SPDPDEs) has provided challenging analytic and numerical problems for over forty years, and space precludes an exhaustive discussion of the many issues. Most widely used numerical methods for solving ODEs/PDEs are based on a stepwise approach, that is, given values at the time they construct new values by stepping to the new time for a positive step. In contrast, the solution of a DDEs requires some type of interpolation for evaluating the delayed arguments at points well-removed from the local step. A stepwise algorithm must save enough information about early solution steps to construct the current step. Thus in a computational method, an interpolation must be used for previous steps. Only a few researchers focused on the numerical studies of SPDPDEs. Some of them are listed here for reference.

In 2006, Wang [75] constructed a new technique for a class of singularly perturbed delay parabolic PDEs in which the outer and inner solutions of the linear problem can be expressed in the form of a series and the outer and inner approximate solutions of the nonlinear problem were given by the iterative formula. Ansari *et al.* [76] proposed a robust finite difference method on a rectangular piecewise-uniform fitted mesh condensing in the boundary layers. They have shown that the method is first-order accurate in time and almost second-order accurate in space. Bashier and Patidar [77] designed a robust fitted operator finite difference method for the numerical solution of the singularly perturbed delay parabolic partial differential equation. The method was shown unconditionally stable and parameter-uniform convergent of order one in the temporal direction and of order two in the spatial direction. In 2015, Das and Natesan [78] proposed a numerical study for the solution of singularly perturbed delay parabolic convection-diffusion initial-boundary value problems. To obtain the ε-uniform convergence the time derivative was discretized on the uniform mesh by the implicit-Euler scheme and the spatial derivatives were discretized on a piecewise-uniform mesh by the hybrid scheme. They have shown that the proposed scheme is ε-uniformly convergent of first-order in time and second-order up to a logarithmic factor in space.

Aziz and Amin [79] applied Haar wavelet collocation method on a linear and nonlinear DDEs as well as on the systems involving these DDEs. The method was also extended to the SPDPDEs with the delay in time. For the solution of a class of time-dependent parabolic SPPDEs with general shift arguments in the reaction term, Bansal *et al.* [80] designed two different finite difference schemes. To tackle the shift terms they used a special type of mesh and an interpolation. But in the contest of singular perturbations both of these methods have a major drawback as these are not parameter-uniformly convergent. For the numerical solution of singularly perturbed DPDEs a modified version of the barycentric interpolation collocation method was discussed in [81] by Wang *et al.*

Bansal and Sharma [82] presented a finite difference scheme on an appropriate piecewise-uniform mesh for the time-dependent singularly perturbed reaction–diffusion problem with large delay. They have shown that the method is ε-uniformly convergent and first-order accurate in time and second order accurate (up to the logarithm factor) in space. In 2018, Kumar [83] has given an implicit scheme for a class of time-dependent singularly perturbed parabolic convection-diffusion initial-boundary value problems having the delay as well as advance terms on a rectangular domain. He [84] has also developed a collocation method for SPDDEs with turning point exhibiting boundary/interior layers.

1.2.5 Singularly perturbed turning point BVPs

These problems are more difficult to handle as compare to the non turning point problems. The numerical study of the single boundary turning point problems for ordinary differential equations has been considered by many researchers. Few of these includes [85–88]. Liseikin [86] constructed a first-order parameter-uniform scheme in the discrete maximum norm on a uniform mesh by using some transformation. Vulanović [87] used Bakhvalov-type mesh to find the solution of mildly non-linear SPBVP with a turning point. An extensive analysis of the continuous problem has been considered. He has shown the parameter-uniform convergence in a discrete ℓ_1-norm. Later in 1992, Vulanović and Lin [88] extended the result of [87] for the singularly perturbed quasi-linear BVP with the attractive turning point. To solve SPBVP with a multiple boundary turning point, Vulanović and Farrell [45] constructed an exponentially fitted scheme. They have suggested a modified scheme on a special discretization mesh to improve the order of convergence from first-order to second-order.

Many authors have used finite difference schemes for solving such type of problems. Clavero [89] constructed a uniformly convergent finite difference method for turning point problems, whereas Sun and Stynes [90] constructed various piecewise-uniform meshes and used Galerkin finite element method on these meshes. They also proved the uniform convergence of the method in weighted energy norm and in usual L^2 norm. Surla and Uzelac [91] have taken a linear combination of the two spline difference schemes to solve these problems. Natesan and Ramanujam [92] combined the exponentially fitted scheme with the classical difference scheme for the solution of singularly perturbed turning point boundary value problems(SPTPBVPs) whose solution exhibit twin boundary layers. In the same year, for the solution of the same problem they have also developed an initial-value technique [93]. Linß and Vulanović [94] developed the first and second-order finite difference schemes for the solution of convection-diffusion semi linear problems having turning points.

Later in 2003, to establish an ε-uniform convergence for the solution of same problem, Natesan *et al.* [95] used classical finite difference scheme on an appropriate piecewise-uniform mesh. Based on the cubic spline approximation on a piecewise-uniform mesh, Kadalbajoo and Gupta [96] developed a second-order parameter-uniform numerical method for the solution of SPTPBVPs whose solution exhibit twin boundary layers.

However, in the literature, there are only a few articles on the numerical solution of singularly perturbed partial differential equations including a boundary turning point of multiplicity ≥ 1. For instance, based on the finite differences and using the classical grid approximations [97], Shishkin [49] constructed parameter-uniform convergent schemes for the solution of singularly perturbed partial differential equations including a boundary turning point of multiplicity ≥ 1. In 2003, Dunne *et al.* [98] developed a parameter-uniform first-order upwind finite difference scheme on a fitted-mesh for the solution of a time-dependent convection-diffusion SPBVPs for PDEs with a boundary turning point whose solution exhibits a parabolic boundary layer.

1.2.6 Two parameter singularly perturbed BVPs

Several numerical methods have been constructed for two-parameter SPBVPs but in the context of ODEs for instance, on the basis of fitted finite difference operators on uniform meshes, Shishkin and Titov [99] constructed a parameter-uniform method for two-parameter SPBVPs. Vulanović [100] used higher-order finite difference scheme on both Shishkin and Bakhavalov type piecewise-uniform meshes for solving quasi-linear boundary value problem with two small parameters. He has shown that for the higher order schemes the Shishkin meshes are more suitable as compared to the Bakhvalov meshes. Roos and Uzelac [101] constructed a second-order parameters uniform streamline-diffusion parameter-uniform finite element method for the solution of two-parameter SPBVPs. Surla *et al.* [102] suggested a quadratic spline collocation method for the solution of two-parameter SPBVPs. Based on the exponential spline functions on a Shishkin mesh, Zahra and El Mhlawy [103] constructed almost second-order scheme for the two-parameter semi-linear SPBVP. Brdar and Zarin [104] analyzed a uniformly convergent Galerkin finite element method in the energy norm on a Bakhvalov-type mesh for two-parameter SPBVPs.

However, not much work has been done for the numerical solution of two-parameter SPBVPs in the context of PDEs(the problem considered in the present study). Some numerical methods developed for the solution to these problems for PDEs are mentioned here, for instance, Riordan *et al.* [105] constructed a parameter-uniform upwind scheme on a piecewise-uniform mesh for the solution of two-parameter SPBVPs. In 2012, Kadalbajoo and Yadaw [106] suggested a first order implicit scheme comprising a finite difference scheme on a uniform mesh in the temporal direction and a finite element scheme on a piecewise-uniform mesh in the spatial direction for the solution of the two parameter SPBVPs. Munyakazi [107] constructed a fitted operator finite difference method for the solution of a class of two-parameter time-dependent singularly perturbed problems. He has shown that the method is robust with respect to the perturbation parameters.

Hence, several numerical methods have been constructed in literature for two-parameter SPBVPs with smooth data but the numerical study is limited in the case of non smooth data and the development of the parameters-uniform numerical methods for two-parameter SPBVPs with non-smooth data is at the initial stage. Few of these as a reference is given here. An almost first-order accurate robust numerical method is constructed by Shanthi *et al.* [108] for two-parameter SPBVPs for ODEs with a discontinuous source term. A classical upwind scheme is used on an appropriate piecewise-uniform mesh. Clavero *et al.* [109] constructed almost first-order uniformly convergent scheme for the solution of two-parameter parabolic SPBVP having a discontinuity in the source term. They used an implicit Euler method on a uniform mesh in the time direction and the classical upwind scheme on Shishkin mesh in the space direction. Prabha *et al.* [110] developed almost second-order parameters-uniform convergent method for two-parameter SPBVPs for ODEs with a discontinuous source term. The method comprises a five-point second-order scheme at the interior layer and the central, mid-point, and upwind difference schemes for other regions.

Chandru *et al.* [111] constructed an almost second-order hybrid monotone difference scheme for two-parameter SPBVPs for second-order ODEs with a discontinuous source term. Almost first-order uniformly convergent scheme comprising implicit Euler scheme in time and upwind scheme in space is suggested by Chandru *et al.* [112] for two-parameter parabolic SPBVP with a jump discontinuity in the convection-coefficient and source terms. The method of averaging is used at the point of discontinuity. Based on the standard upwind finite difference scheme on a piecewise uniform mesh, recently, Prabha *et al.* [113] developed a first-order parameters uniform numerical scheme for two-parameter SPBVP with a discontinuity in the convection-coefficient and source terms. Recently, by using an adaptive mesh, Chandru *et al.* [114] developed almost first-order parameters uniform numerical scheme for two-parameter singularly perturbed parabolic PDE with discontinuous convection-coefficient and source terms.

1.3 Organization of the Thesis

This thesis contains eight chapters. Chapter 1 reveals the basic introduction of singular perturbation and applications, governing equations, a brief literature regarding the applications and numerical solution of singularly perturbed boundary value problems. This chapter also contains a brief introduction about delay, turning point and two parameter singularly perturbed differential equations with their applications and a brief of the relevant literature review. The historical works with initiation has also been done in this chapter which gave an idea to develop the numerical schemes which has not been covered earlier. Chapters 2-7 deal the main contribution of research work and the last Chapter 8 presents the summary of whole contributions with their future scopes. Chapter 2 deals with a numerical scheme for a class of singularly perturbed parabolic partial differential equation with the time delay on a rectangular domain in the *x*-*t* plane. In Chapter 3, we have taken the large delay in space which is more difficult to handle because the solution of these type of problems, in general, exhibits twin boundary layers (due to the presence of the perturbation parameter) and an interior layer (due to the presence of the large delay parameter in the reaction term).

In Chapter 4, we are going to construct a numerical scheme for turning point problems having twin boundary layers at both ends of the domain. To resolve the boundary layer a fitted-mesh is constructed and the cubic *B*-spline basis functions on this mesh are used to discretize the given equation. A numerical scheme for singularly perturbed parabolic boundary value problems (SPBVPs) including a multiple boundary turning point at left end point of the spatial direction is developed in Chapter 5. The solution of these problems exhibits a boundary layer of parabolic type near the left lateral surface of the domain of consideration. In Chapter 6, a parameter-uniform implicit scheme for two-parameter singularly perturbed boundary value problems has been developed in the present work. Through rigorous analysis, the theoretical results for two different cases: Case I. $\varepsilon_1/\varepsilon_2^2 \to 0$ as $\varepsilon_2 \to 0$, and Case II. $\varepsilon_2^2/\varepsilon_1 \to 0$ as $\varepsilon_1 \to 0$ which show that the method is convergent irrespective to the size of the parameters $\varepsilon_1, \varepsilon_2$ are provided.

Chapter 7 deals the development of a parameters uniform numerical method for two parameter parabolic convection-diffusion-reaction SPBVPs where the convection-coefficient and source term have a discontinuity inside the domain of consideration, which in addition to the twin boundary layers leads to an interior layer near the point of discontinuity.

Last Chapter 8 contains the summary of whole contributions of the study throughout the work followed by the scope of future work in this area.