Chapter 3

A parameter uniform numerical method for the singularly perturbed initial-interval boundary value problems with unit space delay

3.1 Introduction

There are some articles that start to fill the numerics void have appeared in the literature (see Chapter 1), in which the finite differences on piecewise-uniform meshes were used for the solution of SPDDEs. However, the main drawback of all these articles is the restriction of the study to the small shift arguments and so these articles are based on Taylor series expansion approach. In fact, these methods work only when the delay term is small and is of $o(\varepsilon)$. Motivated by these works, in this chapter, we initiate a numerical approach for the study of singularly perturbed time-dependent differential equations which works for unit space delay and thus Taylor series expansion is not required. One can see Chapter 1 for more real life applications of these type of problems.

A parameter-uniform implicit scheme is constructed for a class of singularly perturbed parabolic reaction-diffusion initial-boundary value problems with unit space delay. In general, the solution of these problems exhibit twin boundary layers and an interior layer (due to the presence of the delay in the reaction term). Crank-Nicolson difference formula (on a uniform mesh) is used in time to semi-discretize the given PDE, and then the standard finite difference scheme (on a piecewise-uniform mesh) is used for the system of ordinary differential equations obtained in the semi-discretization. The convergence analysis shows that the method is ε-uniformly convergent with order two in the temporal direction and almost first-order in the spatial direction. Two test examples are encountered to show the efficiency of the method, validation of the computational results, and to confirm the predicted theory.

The chapter is designed as follows. Some preliminary and *a priori* estimates are presented in Section 3.2 followed by the temporal semi-discretization in Section 3.3. The full discretized scheme on a predefined Shishkin mesh is constructed in Section 3.4. The parameter-uniform convergence analysis of the proposed scheme is given in Section 3.5. The applicability of the proposed method is demonstrated by considering two test examples in Section 3.6. The last Section 3.7 is devoted to the concluding remarks and the future scope.

3.2 The Continuous Problem: Preliminary and a Priori Estimates

Let T be a fixed positive real number. We consider the following SP-DDEs for PDEs on $\mathscr{D} = \Omega \times \Lambda = (0,2) \times (0,T]$

$$Ly \equiv y_t(x,t) - \varepsilon y_{xx}(x,t) + r(x)y(x,t) + s(x)y(x-1,t) = \omega(x,t), \tag{3.1a}$$

with the initial condition

$$y(x,0) = y_0(x), \quad x \in \overline{\Omega}, \tag{3.1b}$$

and the interval-boundary conditions

$$y(x,t) = \varphi(x,t),\ (x,t) \in \mathscr{D}_L, \tag{3.1c}$$

$$y(2,t) = \psi(t),\ (x,t) \in \mathscr{D}_R, \tag{3.1d}$$

where $0 < \varepsilon \ll 1$, $\mathscr{D}_L = \{(x,t) : -1 \leq x \leq 0;\ t \in \overline{\Lambda}\}$ and $\mathscr{D}_R = \{(2,t) : t \in \overline{\Lambda}\}$. It is assumed that the functions involved in (3.1) are smooth enough. Furthermore, it is also assumed that for some positive constant α; $r(x)$ and $s(x)$ satisfy

$$r(x) + s(x) \geq 2\alpha > 0, \text{ and } s(x) < 0,\ \forall x \in \overline{\Omega}. \tag{3.2}$$

Since y is given on the interval $[-1,0]$, so the equation (3.1a) is equivalent to the following two equations

$$y_t(x,t) - \varepsilon y_{xx}(x,t) + r(x)y(x,t) = \omega(x,t) - s(x)\varphi(x-1,t),\ (x,t) \in (0,1] \times \Lambda,$$
$$y_t(x,t) - \varepsilon y_{xx}(x,t) + r(x)y(x,t) + s(x)y(x-1,t) = \omega(x,t),\ (x,t) \in (1,2) \times \Lambda.$$

If we let $\Omega_1 = (0,1]$, $\Omega_2 = (1,2)$, $\mathscr{D}_1 = \Omega_1 \times \Lambda$, $\mathscr{D}_2 = \Omega_2 \times \Lambda$ and denote by y_1 (respectively y_2) the restriction of y to $\mathscr{D}_1$ (respectively $\mathscr{D}_2$), then the problem (3.1a)-(3.1d) is equivalent to the following problem

$$(y_1)_t(x,t) - \varepsilon(y_1)_{xx}(x,t) + r(x)y_1(x,t) = \omega(x,t) - s(x)\varphi(x-1,t),\ (x,t) \in \Omega_1 \times \Lambda,$$
$$(y_2)_t(x,t) - \varepsilon(y_2)_{xx}(x,t) + r(x)y_2(x,t) + s(x)y_2(x-1,t) = \omega(x,t),\ (x,t) \in \Omega_2 \times \Lambda,$$
$$y_1(x,0) = y_0(x),\ x \in \overline{\Omega}_1,\ y_2(x,0) = y_0(x),\ x \in \overline{\Omega}_2,$$
$$y_1(0,t) = \varphi(0,t),\ y_2(2,t) = \psi(t),$$
$$y_1(1,t) = y_2(1,t),\ (y_1)_x(1,t) = (y_2)_x(1,t).$$

The limiting problem associated with above problem, as $\varepsilon \to 0$, is

$$(y_1^0)_t(x,t) + r(x)(y_1^0)(x,t) = \omega(x,t) - s(x)\varphi(x-1,t),\ (x,t) \in \Omega_1 \times \Lambda,$$
$$(y_2^0)_t(x,t) + r(x)(y_2^0)(x,t) + s(x)(y_2^0)(x-1,t) = \omega(x,t),\ (x,t) \in \Omega_2 \times \Lambda.$$

Since y_1^0, y_2^0 need not necessarily satisfy the following conditions

$$y_1^0(0,t) = \varphi(0,t),\ y_2^0(2,t) = \psi(t),\ y_1^0(1,t) = y_2^0(1,t),\ (y_1^0)_x(1,t) = (y_2^0)_x(1,t),$$

so, in general, the solution y exhibits twin boundary layers (one at each end of the interval) and an interior layer at $x = 1$. It is assumed that the initial function $y_0(x)$ satisfies the compatibility conditions which together with the Hölder continuity of the data confirm the existence and uniqueness of the solution of (3.1a)-(3.1d). Moreover, the compatibility conditions ensure the existence of the following estimates

$$|y(x,t) - y(x,0)| \leq Ct,$$

whose proof can be seen in [26]. Let $\partial\mathscr{D}$ be the boundary of $\mathscr{D}$ consisting the left, right and the bottom sides of $\mathscr{D}$.

Lemma 3.2.1. *Let* $\Phi \in C^{2,1}(\overline{\mathscr{D}})$ *be such that* $\Phi(x,t) \geq 0$ *on* $\partial\mathscr{D}$. *Then* $L\Phi(x,t) \geq 0$ *in* $\mathscr{D}$ *implies* $\Phi(x,t) \geq 0$ *in* $\overline{\mathscr{D}}$.

Proof. Suppose $\Phi(\hat{x},\hat{t}) = \min_{(x,t)\in\mathscr{D}} \Phi(x,t) < 0$ for some $(\hat{x},\hat{t}) \in \mathscr{D}$. Then $\Phi_{xx}(\hat{x},\hat{t}) \geq 0$ and $\Phi_t(\hat{x},\hat{t}) = 0$. Clearly by the given condition $(\hat{x},\hat{t})$ does not lie on the boundary $\partial\mathscr{D}$. Then, we consider the following two cases.

Case I. $0 \leq x \leq 1$.

$$\begin{aligned} L\Phi(\hat{x},\hat{t}) &= \Phi_t(\hat{x},\hat{t}) - \varepsilon\Phi_{xx}(\hat{x},\hat{t}) + r(\hat{x})\Phi(\hat{x},\hat{t}) \\ &= -\varepsilon\Phi_{xx}(\hat{x},\hat{t}) + r(\hat{x})\Phi(\hat{x},\hat{t}) < 0. \end{aligned}$$

Case II. $1 < x \leq 2$.

$$\begin{aligned} L\Phi(\hat{x},\hat{t}) &= \Phi_t(\hat{x},\hat{t}) - \varepsilon\Phi_{xx}(\hat{x},\hat{t}) + r(\hat{x})\Phi(\hat{x},\hat{t}) + s(\hat{x})\Phi(\hat{x}-1,\hat{t}) \\ &\leq -\varepsilon\Phi_{xx}(\hat{x},\hat{t}) + (r(\hat{x}) + s(\hat{x}))\Phi(\hat{x},\hat{t}) < 0. \end{aligned}$$

Thus, in both cases, we obtain a contradiction. Hence, the proof is completed. □

The following result is an application of Lemma 3.2.1.

Lemma 3.2.2. *The solution y of (3.1) satisfies the following stability estimate*

$$\|y\|_{\overline{\mathscr{D}}} \leq \|y\|_{\partial\mathscr{D}} + \frac{\|\omega\|_{\overline{\mathscr{D}}}}{2\alpha}.$$

Proof. The functions $\Pi^{\pm}(x,t) = \|y\|_{\partial\mathscr{D}} + \frac{\|\omega\|_{\overline{\mathscr{D}}}}{2\alpha} \pm y(x,t)$ satisfy $\Pi^{\pm}(x,0) \geq 0$, $\Pi^{\pm}(0,t) \geq 0$, and $\Pi^{\pm}(2,t) \geq 0$. Also, for all $(x,t) \in \mathscr{D}$, we have

$$\begin{aligned} L\Pi^{\pm}(x,t) &= (r(x) + s(x))\left[\|y\|_{\partial\mathscr{D}} + \frac{\|\omega\|_{\overline{\mathscr{D}}}}{2\alpha}\right] \pm Ly(x,t) \\ &\geq 2\alpha\left[\|y\|_{\partial\mathscr{D}} + \frac{\|\omega\|_{\overline{\mathscr{D}}}}{2\alpha}\right] \pm \omega(x,t) \\ &= 2\alpha\|y\|_{\partial\mathscr{D}} + \|\omega\|_{\overline{\mathscr{D}}} \pm \omega(x,t) \\ &\geq 0. \end{aligned}$$

Thus the result is obtained as a consequence of Lemma 3.2.1. □

Under some assumptions, the solution of (3.1) and its first two time derivatives are bounded and given by the following lemma.

Lemma 3.2.3. *Under the assumption* $y_0(x) = \varphi(x,t) = \psi(t) = 0$*, the time derivatives of the solution y of (3.1) are bounded i.e.,*

$$\left|\frac{\partial^i y}{\partial t^i}\right| \leq C, \quad i = 0, 1, 2.$$

Proof. For the proof for $i = 0$ the readers are referred to [83]. The proof for $i = 1$ is as follows. The assumption $\varphi(x,t) = \psi(t) = 0$ gives $y = 0$ along the left and right sides of $\mathscr{D}$ and so $y_t = 0$. Also, $y_0(x) = 0$ gives $y = 0$ along the bottom line of $\mathscr{D}$ and so $y_x = y_{xx} = 0$. Using (3.1a) which gives

$$y_t(x,0) - \varepsilon y_{xx}(x,0) + r(x)y(x,0) + s(x)y(x-1,0) = \omega(x,0),$$

and so

$$y_t(x,0) = \omega(x,0).$$

Thus,

$$|y_t| \leq C \text{ on } \partial\mathscr{D}.$$

Now on differentiating (3.1a), we obtain

$$Ly_t(x,t) = \omega_t(x,t),$$

which gives

$$|Ly_t(x,t)| \leq C \text{ on } \mathscr{D}.$$

Finally, on applying Lemma 3.2.1, it yields

$$|y_t(x,t)| \leq C \text{ on } \overline{\mathscr{D}}.$$

Now for $i = 2$ along the left and right sides of $\mathscr{D}$, we have $y_{tt} = 0$ (as $y_t = 0$ there). Also, along the bottom side of $\mathscr{D}$, we have $y = y_x = y_{xx} = 0$ (as $y = 0$ there). Now on differentiating (3.1a), we get

$$y_{tt}(x,t) - \varepsilon y_{xxt}(x,t) + r(x)y_t(x,t) + s(x)y_t(x-1,t) = \omega_t(x,t). \tag{3.4}$$

Since $y_t(x,0) = \omega(x,0)$ (from $i = 1$ case), so $y_{xxt}(x,0) = \omega_{xx}(x,0)$. Also, by definition

$$y_t(x-1,0) = \lim_{\Delta t \to 0} \frac{y(x-1,\Delta t) - y(x-1,0)}{\Delta t} = 0.$$

Thus along $t = 0$, we get from (3.4)

$$y_{tt}(x,0) = \varepsilon \omega_{xx}(x,0) - r(x)\omega(x,0) + \omega_t(x,0),$$

which gives $|y_{tt}| \le C$ along $t = 0$. Therefore, $|y_{tt}| \le C$ on $\partial\mathscr{D}$. Now differentiating (3.1a) twice with respect to t to obtain

$$Ly_{tt}(x,t) = \omega_{tt}(x,t),$$

which yields

$$|Ly_{tt}(x,t)| \le C \text{ on } \mathscr{D}.$$

Hence, the proof is completed by using Lemma 3.2.1. □

3.3 Semi-discretization for time variable

Let M be a positive integer. The uniform mesh Ω_t^M which is used in the time semi-discretization is defined as

$$\Omega_t^M = \{t_j = j\Delta t,\ j = 1,2,\ldots,M,\ \Delta t = T/M\}.$$

Using the Crank-Nicolson scheme on Ω_t^M the discretized problem in the temporal direction associated with the continuous problem (3.1) is given by

$$\mathscr{L}u^{j+1}(x) = g(x,t_{j+1}), \quad x \in \Omega,\ j = 0,1,\ldots,M-1, \tag{3.5a}$$

with the initial and boundary conditions

$$u^0(x) = y_0(x), \quad x \in \overline{\Omega}, \tag{3.5b}$$

$$u^{j+1}(x) = \varphi(x, t_{j+1}),\ -1 \le x \le 0,\ j = 0, 1, \ldots, M-1, \tag{3.5c}$$

$$u^{j+1}(2) = \psi(t_{j+1}),\ j = 0, 1, \ldots, M-1, \tag{3.5d}$$

where the discrete operator $\mathscr{L}$ is defined as

$$\mathscr{L}u^{j+1}(x) :\equiv -\frac{\varepsilon}{2}(u^{j+1}(x))_{xx} + \frac{p(x)}{2}u^{j+1}(x) + \frac{s(x)}{2}u^{j+1}(x-1).$$

The functions $p(x)$ and $g(x, t_{j+1})$ are given by

$$\begin{aligned} p(x) &= \frac{r(x)\Delta t + 2}{\Delta t}, \\ g(x, t_{j+1}) &= \frac{\varepsilon}{2}(u^j(x))_{xx} + \frac{q(x)}{2}u^j(x) - \frac{s(x)}{2}u^j(x-1) + \frac{1}{2}(\omega(x, t_{j+1}) + \omega(x, t_j)), \end{aligned}$$

where $q(x) = \frac{2 - r(x)\Delta t}{\Delta t}$. Equation (3.5a) can be written as

$$\mathscr{L}_k u^{j+1}(x) = g_k(x, t_{j+1}), \quad k = 1, 2,\ x \in \Omega,\ j = 0, 1, \ldots, M-1,$$

where

$$\begin{aligned} \mathscr{L}_1 u^{j+1}(x) &:\equiv -\frac{\varepsilon}{2}(u^{j+1}(x))_{xx} + \frac{p(x)}{2}u^{j+1}(x),\ x \in \Omega_1, \\ \mathscr{L}_2 u^{j+1}(x) &:\equiv -\frac{\varepsilon}{2}(u^{j+1}(x))_{xx} + \frac{p(x)}{2}u^{j+1}(x) + \frac{s(x)}{2}u^{j+1}(x-1),\ x \in \Omega_2, \end{aligned}$$

and

$$\begin{aligned} g_1(x, t_{j+1}) &= \frac{\varepsilon}{2}(u^j(x))_{xx} + \frac{q(x)}{2}u^j(x) - \frac{s(x)}{2}u^j(x-1) + \frac{1}{2}(\omega(x, t_{j+1}) + \omega(x, t_j)) \\ &\quad - \frac{s(x)}{2}\varphi(x-1, t_{j+1}),\ x \in \Omega_1, \\ g_2(x, t_{j+1}) &= \frac{\varepsilon}{2}(u^j(x))_{xx} + \frac{q(x)}{2}u^j(x) - \frac{s(x)}{2}u^j(x-1) + \frac{1}{2}(\omega(x, t_{j+1}) + \omega(x, t_j)), \\ &\quad x \in \Omega_2. \end{aligned}$$

The operator $\mathscr{L}$ satisfies the following lemma which can be proved by splitting the domain Ω in two parts Ω_1 and Ω_2.

Lemma 3.3.1. *For* $j = 0,1,\dots,M-1$ *assume* $\Psi^{j+1}(0) \geq 0$ *and* $\Psi^{j+1}(2) \geq 0$. *Then* $\mathscr{L}\Psi^{j+1}(x) \geq 0 \forall x \in \Omega$ *implies* $\Psi^{j+1}(x) \geq 0 \forall x \in \overline{\Omega}$.

Proof. For contrary assume $\Psi^{j+1}(\eta) = \min_{x\in\Omega} \Psi^{j+1}(x) < 0$ for some $\eta \in \Omega$. Then $(\Psi^{j+1})_{xx}(\eta) \geq 0$ and $(\Psi^{j+1})_x(\eta) = 0$. Now, consider the following two cases.

Case I. $\eta \in \Omega_1$.

$$\mathscr{L}\Psi^{j+1}(\eta) = \mathscr{L}_1\Psi^{j+1}(\eta) = -\frac{\varepsilon}{2}(\Psi^{j+1})_{xx}(\eta) + \frac{p(\eta)}{2}(\Psi^{j+1})(\eta) < 0.$$

Case II. $\eta \in \Omega_2$.

$$\begin{aligned}
\mathscr{L}\Psi^{j+1}(\eta) &= \mathscr{L}_2\Psi^{j+1}(\eta) \\
&= -\frac{\varepsilon}{2}(\Psi^{j+1})_{xx}(\eta) + \frac{p(\eta)}{2}\Psi^{j+1}(\eta) + \frac{s(\eta)}{2}\Psi^{j+1}(\eta-1) \\
&\leq -\frac{\varepsilon}{2}(\Psi^{j+1})_{xx}(\eta) + \frac{p(\eta)}{2}\Psi^{j+1}(\eta) + \frac{s(\eta)}{2}\Psi^{j+1}(\eta) \\
&= -\frac{\varepsilon}{2}(\Psi^{j+1})_{xx}(\eta) + \left(\frac{r(\eta)}{2} + \frac{1}{\Delta t}\right)\Psi^{j+1}(\eta) + \frac{s(\eta)}{2}\Psi^{j+1}(\eta) \\
&= -\frac{\varepsilon}{2}(\Psi^{j+1})_{xx}(\eta) + \left(\frac{r(\eta)+s(\eta)}{2}\right)\Psi^{j+1}(\eta) + \frac{1}{\Delta t}\Psi^{j+1}(\eta) \\
&\leq -\frac{\varepsilon}{2}(\Psi^{j+1})_{xx}(\eta) + \alpha\Psi^{j+1}(\eta) + \frac{1}{\Delta t}\Psi^{j+1}(\eta) \\
&< 0.
\end{aligned}$$

It follows the required result. □

An application of the above lemma is the following uniform stability estimate.

Lemma 3.3.2. *The solution* $u^{j+1}(x)$ *of (3.5) satisfies*

$$\|u^{j+1}\|_{\overline{\Omega}} \leq \max\left\{|u^{j+1}(0)|, |u^{j+1}(2)|, \frac{\Delta t}{\alpha\Delta t+1}\|g\|_{\overline{\Omega}}\right\}.$$

Proof. The proof can be done by considering Ω_1 and Ω_2 separately.

Case I. $x \in \Omega_1$. Define the comparison functions

$$\zeta^{\pm}(x,t_{j+1}) = \max\left\{|u^{j+1}(0)|, \frac{\Delta t}{\alpha\Delta t+1}\|g_1\|_{\overline{\Omega}_1}\right\} \pm u^{j+1}(x).$$

Then it is easy to check $\zeta^{\pm}(0,t_{j+1}) \geq 0$. Also,

$$\begin{aligned}
\mathscr{L}\zeta^{\pm}(x,t_{j+1}) &= \mathscr{L}_1\zeta^{\pm}(x,t_{j+1}) \\
&= \frac{p(x)}{2}\max\left\{|u^{j+1}(0)|, \frac{\Delta t}{\alpha\Delta t+1}\|g_1\|_{\overline{\Omega}_1}\right\} \pm \mathscr{L}_1 u^{j+1}(x) \\
&= \frac{p(x)}{2}\max\left\{|u^{j+1}(0)|, \frac{\Delta t}{\alpha\Delta t+1}\|g_1\|_{\overline{\Omega}_1}\right\} \pm g_1 \\
&\geq \frac{p(x)\Delta t}{2(\alpha\Delta t+1)}\|g_1\|_{\overline{\Omega}_1} \pm g_1 \\
&= \frac{r\Delta t+2}{\Delta t}.\frac{\Delta t}{2(\alpha\Delta t+1)}\|g_1\|_{\overline{\Omega}_1} \pm g_1 \\
&\geq \frac{(r(x)+s(x))\Delta t+2}{2(\alpha\Delta t+1)}\|g_1\|_{\overline{\Omega}_1} \pm g_1 \\
&\geq \frac{2\alpha\Delta t+2}{2(\alpha\Delta t+1)}\|g_1\|_{\overline{\Omega}_1} \pm g_1 \\
&= \|g_1\|_{\overline{\Omega}_1} \pm g_1 \\
&\geq 0.
\end{aligned}$$

Case II. $x \in \Omega_2$**.** Define the comparison functions

$$\zeta^{\pm}(x,t_{j+1}) = \max\left\{|u^{j+1}(2)|, \frac{\Delta t}{\alpha\Delta t+1}\|g_2\|_{\overline{\Omega}_2}\right\} \pm u^{j+1}(x).$$

Then it is easy to check $\zeta^{\pm}(2,t_{j+1}) \geq 0$. Also,

$$\begin{aligned}
\mathscr{L}\zeta^{\pm}(x,t_{j+1}) &= \mathscr{L}_2\zeta^{\pm}(x,t_{j+1}) \\
&= \frac{p(x)+s(x)}{2}\max\left\{|u^{j+1}(2)|, \frac{\Delta t}{\alpha\Delta t+1}\|g_2\|_{\overline{\Omega}_2}\right\} \pm \mathscr{L}_2 u^{j+1}(x) \\
&= \left(\frac{1}{\Delta t} + \frac{r(x)+s(x)}{2}\right)\max\left\{|u^{j+1}(2)|, \frac{\Delta t}{\alpha\Delta t+1}\|g_2\|_{\overline{\Omega}_2}\right\} \pm g_2 \\
&\geq \left(\frac{1}{\Delta t} + \alpha\right)\max\left\{|u^{j+1}(2)|, \frac{\Delta t}{\alpha\Delta t+1}\|g_2\|_{\overline{\Omega}_2}\right\} \pm g_2 \\
&\geq \|g_2\|_{\overline{\Omega}_2} \pm g_2 \\
&\geq 0.
\end{aligned}$$

Therefore, the result follows by the maximum principle Lemma 3.3.1. □

Let $\tilde{u}(x)$ be the computed solution of (3.5). Then the local truncation error (LTE)

$e_{j+1} = u^{j+1}(x) - \tilde{u}(x)$ for the temporal semi-discretization is estimated by the following lemma.

Lemma 3.3.3. *The LTE in the temporal direction at $(j+1)$-th time level is estimated as*

$$\|e_{j+1}\| \leq C(\Delta t)^3.$$

Proof. For the proof, the readers are referred to [119]. □

The global truncation error $E_j = y(x,t_j) - u^j(x)$ of the time semi-discretization is the contribution of the local truncation error at each time step. The following estimate for the global error can be easily deduced.

Lemma 3.3.4. *The bound on E_j is estimated as*

$$\|E_j\| \leq C(\Delta t)^2, \quad j \leq T/\Delta t.$$

The following theorem estimates the bounds on the derivatives of the solution $u^{j+1}(x)$ of (3.5).

Theorem 3.3.1. *The solution $u^{j+1}(x)$ and its first three derivatives satisfy the following estimates*

$$\begin{aligned} \left|\frac{d^k u^{j+1}(x)}{dx^k}\right| &\leq C\left(1+\varepsilon^{-k/2}e_1(x,\alpha)\right),\ x \in \overline{\Omega}_1,\ k=0,1,2,3, \\ \left|\frac{d^k u^{j+1}(x)}{dx^k}\right| &\leq C\left(1+\varepsilon^{-k/2}e_2(x,\alpha)\right),\ x \in \overline{\Omega}_2,\ k=0,1,2,3, \end{aligned}$$

where

$$\begin{aligned} e_1(x,\alpha) &= \exp\left(-x\sqrt{\alpha/\varepsilon}\right) + \exp\left(-(1-x)\sqrt{\alpha/\varepsilon}\right), \\ e_2(x,\alpha) &= \exp\left(-(x-1)\sqrt{\alpha/\varepsilon}\right) + \exp\left(-(2-x)\sqrt{\alpha/\varepsilon}\right). \end{aligned}$$

Proof. The proof can be done by following the approach given in [120]. □

As in the continuous case $u^{j+1}(x)$ can be written in its decomposition form as

$$u^{j+1}(x) = v^{j+1}(x) + w^{j+1}(x),$$

where $v^{j+1}(x)$ is the solution of the following differential equation in $(0,1]$

$$
\begin{aligned}
&-\frac{\varepsilon}{2}\frac{d^2v^{j+1}(x)}{dx^2}+\frac{p(x)}{2}v^{j+1}(x)=g_1(x,t_{j+1})-\frac{s(x)}{2}\varphi(x-1,t_{j+1}),\\
&\qquad v^{j+1}(0)=v_0^{j+1}(0),\\
&\qquad v^{j+1}(1)=2p(1)^{-1}\left(g_1(1,t_{j+1})-\frac{s(1)}{2}\varphi(0,t_{j+1})\right),
\end{aligned}
$$

while it is the solution of the following differential equation in $(1,2)$

$$
\begin{aligned}
&-\frac{\varepsilon}{2}\frac{d^2v^{j+1}(x)}{dx^2}+\frac{p(x)}{2}v^{j+1}(x)+\frac{s(x)}{2}v^{j+1}(x-1)=g_2(x,t_{j+1}),\\
&\qquad v^{j+1}(1)=2p(1)^{-1}\left(g_2(1,t_{j+1})-\frac{s(1)}{2}v_0^{j+1}(0)\right),\\
&\qquad v^{j+1}(2)=v_0^{j+1}(2),
\end{aligned}
$$

where $v_0^{j+1}(x)$ is the solution of associated reduced problem. On the other hand $w^{j+1}(x)$ is the solution of

$$
\begin{aligned}
&-\frac{\varepsilon}{2}\frac{d^2w^{j+1}(x)}{dx^2}+\frac{p(x)}{2}w^{j+1}(x)=0,\ x\in(0,1],\\
&-\frac{\varepsilon}{2}\frac{d^2w^{j+1}(x)}{dx^2}+\frac{p(x)}{2}w^{j+1}(x)+\frac{s(x)}{2}w^{j+1}(x-1)=0,\ x\in(1,2),\\
&\qquad w^{j+1}(0)=u^{j+1}(0)-v^{j+1}(0),\\
&\qquad w^{j+1}(2)=u^{j+1}(2)-v^{j+1}(2).
\end{aligned}
$$

Lemma 3.3.5. *The first three derivatives of $v^{j+1}(x)$ and $w^{j+1}(x)$ satisfy the following estimates*

$$
\left|\frac{d^kv^{j+1}(x)}{dx^k}\right|\le C\begin{cases}\left(1+\varepsilon^{-(k-2)/2}e_1(x,\alpha)\right), & x\in\overline{\Omega}_1,\ k=0,1,2,3,\\ \left(1+\varepsilon^{-(k-2)/2}e_2(x,\alpha)\right), & x\in\overline{\Omega}_2,\ k=0,1,2,3,\end{cases}
$$

$$
\left|\frac{d^kw^{j+1}(x)}{dx^k}\right|\le C\begin{cases}\varepsilon^{-k/2}e_1(x,\alpha), & x\in\overline{\Omega}_1,\ k=0,1,2,3,\\ \varepsilon^{-k/2}e_2(x,\alpha), & x\in\overline{\Omega}_2,\ k=0,1,2,3.\end{cases}
$$

Proof. For the proof of the lemma, the readers are referred to [120–122]. □

3.4 The Spatial Discretization

The piecewise-uniform mesh having $N\,(\geq 8)$ mesh elements on $[0,2]$ is generated by dividing the first half interval $[0,1]$ into three subintervals as

$$[0,1] = [0,\tau] \cup (\tau, 1-\tau] \cup (1-\tau, 1],$$

where the transition parameter τ separates the non-uniform mesh into uniform meshes and is given by

$$\tau = \min\left\{1/4, \sqrt{\varepsilon/\alpha}\,\ln(N/2)\right\}.$$

To obtain a piecewise-uniform mesh, we place $N/4$ mesh elements in $(\tau, 1-\tau]$ and $N/8$ mesh elements in each of the subintervals $[0,\tau]$ and $(1-\tau, 1]$. Hence, the piecewise-uniform mesh is given by

$$x_i = \begin{cases} 0, & i = 0 \\ x_{i-1} + h_i, & i = 1, \ldots, N/2, \end{cases}$$

where h_i's are given by

$$h_i = x_i - x_{i-1} = \begin{cases} 8\tau/N, & i = 1, 2, \ldots, N/8, \\ 4(1-2\tau)/N, & i = N/8+1, \ldots, 3N/8, \\ 8\tau/N, & i = 3N/8+1, \ldots, N/2. \end{cases}$$

Similarly, the second half interval $(1,2]$ is also divided into three subintervals as

$$(1,2] = (1, 1+\tau] \cup (1+\tau, 2-\tau] \cup (2-\tau, 2].$$

The nodal points are then given by

$$x_i = x_{i-1} + h_i, \quad i = N/2+1, \ldots, N,$$

where h_i's are given by

$$h_i = x_i - x_{i-1} = \begin{cases} 8\tau/N, & i = N/2+1,\dots,5N/8, \\ 4(1-2\tau)/N, & i = 5N/8+1,\dots,7N/8, \\ 8\tau/N, & i = 7N/8+1,\dots,N. \end{cases}$$

Thus, we obtain a piecewise-uniform mesh $\Omega^N = \{x_i\}_{i=0}^{N}$. We define $\Omega_1^N = \{x_i\}_{i=0}^{N/2}$ and $\Omega_2^N = \{x_i\}_{i=N/2+1}^{N}$ so that $\Omega^N = \Omega_1^N \cup \Omega_2^N$. Following difference formulae are used to discretize the problem (3.5)

$$\begin{aligned} D_x^+ u^{j+1}(x) &= \frac{u_{i+1}^{j+1} - u_i^{j+1}}{h_{i+1}}, \\ D_x^- u^{j+1}(x) &= \frac{u_i^{j+1} - u_{i-1}^{j+1}}{h_i}, \\ D_x^+ D_x^- u^{j+1}(x) &= 2\frac{(D_x^+ - D_x^-)u_i^{j+1}}{h_i + h_{i+1}}. \end{aligned}$$

Then, the problem (3.5) is discretized as

$$\mathscr{L}^N u_i^{j+1} = g(x_i, t_{j+1}), \quad i = 1,2,\dots,N-1, \tag{3.6a}$$

with the initial condition

$$u_i^0 = y_0(x_i), \quad i = 0,1,\dots,N, \tag{3.6b}$$

and the interval-boundary conditions

$$u_i^{j+1} = \varphi(x_i, t_{j+1}),\ u_N^{j+1} = \psi(t_{j+1}), \quad i = -N/2, -N/2+1,\dots,0,\ j = 0,1,\dots,M-1. \tag{3.6c}$$

The discrete operator $\mathscr{L}^N$ is defined as

$$\mathscr{L}^N u_i^{j+1} = \begin{cases} \mathscr{L}_1^N u_i^{j+1} :\equiv -\frac{\varepsilon}{2} D_x^+ D_x^- u_i^{j+1} + \frac{p_i}{2} u_i^{j+1},\ i = 1,2\dots,N/2, \\ \\ \mathscr{L}_2^N u_i^{j+1} :\equiv -\frac{\varepsilon}{2} D_x^+ D_x^- u_i^{j+1} + \frac{p_i}{2} u_i^{j+1} + \frac{s_i}{2} u_{i-N/2}^{j+1},\ i = N/2+1,\dots,N-1, \end{cases} \tag{3.7}$$

and $p_i = p(x_i)$, $s_i = s(x_i)$, etc. The operator $\mathscr{L}^N$ satisfies the following discrete maximum principle.

Lemma 3.4.1. *Assume* $\Psi_0^{j+1} \geq 0$ *and* $\Psi_N^{j+1} \geq 0$, $j = 0, 1, \ldots, M-1$. *Then* $\mathscr{L}^N \Psi_i^{j+1} \geq 0, \forall i = 1, 2, \ldots, N-1$ *implies* $\Psi_i^{j+1} \geq 0, \forall i = 0, 1, \ldots, N$.

Proof. For contrary suppose $\Psi_k^{j+1} = \min\limits_{i=0,1,\ldots,N} \Psi_i^{j+1} < 0$ for some index k. We consider the following two cases:

Case I. $i = 1, 2, \ldots, N/2$**.**

$$\begin{aligned}
\mathscr{L}^N \Psi_k^{j+1} &= \mathscr{L}_1^N \Psi_k^{j+1} \\
&= -\frac{\varepsilon}{2} D_x^+ D_x^- \Psi_k^{j+1} + \frac{p_k}{2} \Psi_k^{j+1} \\
&= -\varepsilon \frac{(D_x^+ - D_x^-)\Psi_k^{j+1}}{h_k + h_{k+1}} + \frac{p_k}{2} \Psi_k^{j+1} \\
&= -\varepsilon \frac{\frac{\Psi_{k+1}^{j+1} - \Psi_k^{j+1}}{h_{k+1}} - \frac{\Psi_k^{j+1} - \Psi_{k-1}^{j+1}}{h_k}}{h_k + h_{k+1}} + \frac{p_k}{2} \Psi_k^{j+1} \\
&< 0.
\end{aligned}$$

Case II. $i = N/2 + 1, N/2 + 2, \ldots, N-1$

$$\begin{aligned}
\mathscr{L}^N \Psi_k^{j+1} &= \mathscr{L}_2^N \Psi_k^{j+1} \\
&= -\frac{\varepsilon}{2} D_x^+ D_x^- \Psi_k^{j+1} + \frac{p_k}{2} \Psi_k^{j+1} + \frac{s_k}{2} \Psi_{k-N/2}^{j+1} \\
&\leq -\frac{\varepsilon}{2} D_x^+ D_x^- \Psi_k^{j+1} + \frac{p_k}{2} \Psi_k^{j+1} + \frac{s_k}{2} \Psi_k^{j+1} \\
&= -\frac{\varepsilon}{2} D_x^+ D_x^- \Psi_k^{j+1} + \left(\frac{r_k}{2} + \frac{1}{\Delta t} \right) \Psi_k^{j+1} + \frac{s_k}{2} \Psi_k^{j+1} \\
&= -\frac{\varepsilon}{2} D_x^+ D_x^- \Psi_k^{j+1} + \left(\frac{r_k + s_k}{2} \right) \Psi_k^{j+1} + \frac{1}{\Delta t} \Psi_k^{j+1} \\
&\leq -\varepsilon \frac{(D_x^+ - D_x^-)\Psi_k^{j+1}}{h_k + h_{k+1}} + \alpha \Psi_k^{j+1} + \frac{1}{\Delta t} \Psi_k^{j+1} \\
&< 0.
\end{aligned}$$

Hence the proof is completed. □

The following stability estimate is an immediate consequence of the above maximum principle.

Lemma 3.4.2. *Let* Ψ_i^{j+1}*,* $i = 0, 1, \ldots, N$ *be any mesh function. Then*

$$|\Psi_i^{j+1}| \leq \max\left\{|\Psi_0^{j+1}|,\ |\Psi_N^{j+1}|,\ \frac{\Delta t}{\alpha\Delta t + 1}\|\mathscr{L}^N\Psi_i^{j+1}\|\right\}.$$

Proof. Consider two different cases.

Case I. $i = 1, \ldots, N/2$**.** Consider the barrier functions

$$(\xi_i^{j+1})^{\pm} = \max\left\{|\Psi_0^{j+1}|, \frac{\Delta t}{\alpha\Delta t + 1}\|\mathscr{L}_1^N\Psi_i^{j+1}\|\right\} \pm \Psi_i^{j+1}.$$

Then, clearly $(\xi_0^{j+1})^{\pm}$ is non-negative. Also,

$$\begin{aligned}
\mathscr{L}_1^N(\xi_i^{j+1})^{\pm} &= \frac{p_i}{2}\max\left\{|\Psi_0^{j+1}|, \frac{\Delta t}{\alpha\Delta t + 1}\|\mathscr{L}_1^N\Psi_i^{j+1}\|\right\} \pm \mathscr{L}_1^N\Psi_i^{j+1} \\
&= \frac{p_i}{2}\max\left\{|\Psi_0^{j+1}|, \frac{\Delta t}{\alpha\Delta t + 1}\|g_1\|\right\} \pm g_1 \\
&\geq \frac{p_i\Delta t}{2(\alpha\Delta t + 1)}\|g_1\| \pm g_1 \\
&= \frac{r_i\Delta t + 2}{\Delta t}.\frac{\Delta t}{2(\alpha\Delta t + 1)}\|g_1\| \pm g_1 \\
&\geq \frac{(r_i + s_i)\Delta t + 2}{2(\alpha\Delta t + 1)}\|g_1\| \pm g_1 \\
&\geq \frac{2\alpha\Delta t + 2}{2(\alpha\Delta t + 1)}\|g_1\| \pm g_1 \\
&= \|g_1\| \pm g_1 \\
&\geq 0.
\end{aligned}$$

Case II. $i = N/2 + 1, N/2 + 2, \ldots, N - 1$**.** Consider the barrier functions

$$(\xi_i^{j+1})^{\pm} = \max\left\{|\Psi_N^{j+1}|, \frac{\Delta t}{\alpha\Delta t + 1}\|\mathscr{L}_2^N\Psi_i^{j+1}\|\right\} \pm \Psi_i^{j+1}.$$

Then, clearly $(\xi_N^{j+1})^{\pm}$ is non-negative. Also,

$$\begin{aligned}
\mathscr{L}_2^N(\xi_i^{j+1})^{\pm} &= \frac{p_i+s_i}{2}\max\left\{|\Psi_N^{j+1}|, \frac{\Delta t}{\alpha\Delta t+1}\|g_2\|\right\} \pm \mathscr{L}_2^N\Psi_i^{j+1} \\
&= \left(\frac{1}{\Delta t}+\frac{r_i+s_i}{2}\right)\max\left\{|\Psi_N^{j+1}|, \frac{\Delta t}{\alpha\Delta t+1}\|g_2\|\right\} \pm g_2 \\
&\geq \left(\frac{1}{\Delta t}+\alpha\right)\max\left\{|\Psi_N^{j+1}|, \frac{\Delta t}{\alpha\Delta t+1}\|g_2\|\right\} \pm g_2 \\
&\geq \|g_2\| \pm g_2 \\
&\geq 0.
\end{aligned}$$

Thus the result is obtained by applying the discrete maximum principle. □

3.5 Parameter Uniform Convergence Analysis

In this section, the parameter-uniform error will be estimated by decomposing the solution u_i^{j+1} into the smooth and singular components as

$$u_i^{j+1} = v_i^{j+1} + w_i^{j+1},$$

where v_i^{j+1} is the solution of

$$\begin{aligned}
\mathscr{L}^N v_i^{j+1} &= g_1,\ i = 1,2,\dots,N/2-1, \\
v_0^{j+1} &= v^{j+1}(0),\ v_{N/2-1}^{j+1} = v^{j+1}(1^-), \\
\mathscr{L}^N v_i^{j+1} &= g_2,\ i = N/2+1,\dots,N-1, \\
v_{N/2+1}^{j+1} &= v^{j+1}(1^+),\ v_N^{j+1} = v^{j+1}(2),
\end{aligned}$$

and w_i^{j+1} is the solution of

$$\begin{aligned}
\mathscr{L}^N w_i^{j+1} &= 0,\ i = 1,2,\dots,N-1, \\
w_0^{j+1} &= w^{j+1}(0),\ w_N^{j+1} = w^{j+1}(2).
\end{aligned}$$

The error estimate between the solution of the semi-discretized problem and the solution of full discretized problem (at $(j+1)^{\text{th}}$ time level) is given by the following theorem.

Theorem 3.5.1. *The maximum absolute error between $u^{j+1}(x)$, the solution of the problem (3.5), and u_i^{j+1} the solution of the problem (3.6) is given by*

$$|u^{j+1}(x_i) - u_i^{j+1}| \leq CN^{-1}\ln(N/2),\ i = 0,1,\ldots,N.$$

Proof. The result is proved by dividing the interval into two parts. For the proof in the interval $(0,1]$, the readers are referred to [122]. For the interval $(1,2)$ the proof is as follows. The nodal error given by

$$v_{i,j+1} = u^{j+1}(x_i) - u_i^{j+1} = \left(v^{j+1}(x_i) - v_i^{j+1}\right) + \left(w^{j+1}(x_i) - w_i^{j+1}\right),$$

will be estimated by estimating the errors in the smooth and singular components separately. For $i = \frac{N}{2}+1,\ldots,N-1$, using the classical argument we have

$$\mathscr{L}^N(v_i^{j+1} - v^{j+1}(x_i)) = -\frac{\varepsilon}{2}\left(D_x^+D_x^- - \frac{d^2}{dx^2}\right)v^{j+1}(x_i).$$

By using the result given in [123] it gives

$$|\mathscr{L}^N(v_i^{j+1} - v^{j+1}(x_i))| \leq C\varepsilon(x_{i+1} - x_{i-1})|v^{j+1}(x_i)|_3, \quad i = \frac{N}{2}+1,\ldots,N-1.$$

Since $x_{i+1} - x_{i-1} \leq 4N^{-1}$ for all i, so by using Lemma 3.3.5 to estimate the value of $|v^{j+1}(x_i)|_3$, we obtain

$$|\mathscr{L}^N(v_i^{j+1} - v^{j+1}(x_i))| \leq CN^{-1}, \quad i = \frac{N}{2}+1,\ldots,N-1.$$

An application of the maximum principle to the mesh function $v_i^{j+1} - v^{j+1}(x_i)$ gives the following estimate

$$|v_i^{j+1} - v^{j+1}(x_i)| \leq CN^{-1}, \quad i = \frac{N}{2}+1,\ldots,N-1. \tag{3.8}$$

The error in the singular component of the solution will be estimated for $\tau = 1/4$ and $\tau = \sqrt{\varepsilon/\alpha}\,\ln(N/2)$ separately. In the former case we have $\sqrt{\varepsilon/\alpha}\,\ln(N/2) \geq \frac{1}{4}$ (as the mesh is uniform) and so the classical argument yields

$$|\mathscr{L}^N(w_i^{j+1} - w^{j+1}(x_i))| \leq C\varepsilon(x_{i+1} - x_{i-1})|w^{j+1}(x_i)|_3, \quad i = \frac{N}{2}+1,\ldots,N-1.$$

Again, the estimate for $|w^{j+1}(x_i)|_3$ obtained in Lemma 3.3.5 gives

$$|\mathscr{L}^N(w_i^{j+1} - w^{j+1}(x_i))| \le CN^{-1}\varepsilon^{-1/2}, \quad i = \frac{N}{2}+1,\ldots,N-1.$$

Also, since $\varepsilon^{-1/2} \le \frac{4}{\sqrt{\alpha}}\ln(N/2)$, so

$$|\mathscr{L}^N(w_i^{j+1} - w^{j+1}(x_i))| \le CN^{-1}\ln(N/2), \quad i = \frac{N}{2}+1,\ldots,N-1.$$

Thus it gives

$$|w_i^{j+1} - w^{j+1}(x_i)| \le CN^{-1}\ln(N/2), \quad i = \frac{N}{2}+1,\ldots,N-1. \tag{3.9}$$

In the latter case, in the subinterval $[1+\tau, 2-\tau]$ the mesh spacing is $\frac{4(1-2\tau)}{N}$ while in $(1, 1+\tau]$ and $[2-\tau, 2)$ it is $\frac{8\tau}{N}$. Depending on the mesh spacing, a different argument is used for the bound of $|w_i^{j+1} - w^{j+1}(x_i)|$. For x_i lying in the subintervals $(1, 1+\tau]$ and $[2-\tau, 2)$, the classical argument gives

$$|\mathscr{L}^N(w_i^{j+1} - w^{j+1}(x_i))| \le C\varepsilon(x_{i+1} - x_{i-1})|w^{j+1}(x_i)|_3.$$

Since $|w^{j+1}(x_i)|_3$ is bounded by $C\varepsilon^{-3/2}$, therefore

$$|\mathscr{L}^N(w_i^{j+1} - w^{j+1}(x_i))| \le CN^{-1}\tau/\sqrt{\varepsilon}.$$

Replacing the value of τ, it yields

$$|\mathscr{L}^N(w_i^{j+1} - w^{j+1}(x_i))| \le CN^{-1}\ln(N/2). \tag{3.10}$$

On the other hand, for $x_i \in [1+\tau, 2-\tau]$, the local truncation error for the singular component is estimated as follows.

$$|\mathscr{L}^N(w_i^{j+1} - w^{j+1}(x_i))| \le C\varepsilon\left|\left(D_x^+D_x^- - \frac{d^2}{dx^2}\right)w^{j+1}(x_i)\right|.$$

But $|D_x^+D_x^- w^{j+1}(x_i)| \le \max_{x\in[x_{i-1},x_{i+1}]}|w^{j+1}(x_i)|_2$, and so

$$|\mathscr{L}^N(w_i^{j+1} - w^{j+1}(x_i))| \le C\varepsilon \max_{x\in[x_{i-1},x_{i+1}]}|w^{j+1}(x_i)|_2. \tag{3.11}$$

Now we have

$$|w^{j+1}(x_i)|_2 \le C\varepsilon^{-1}\begin{cases} e^{-\sqrt{\alpha}(x_{i-1}-1)/\sqrt{\varepsilon}}, & \text{if} \quad x_i \le \frac{3}{2}, \\ e^{-\sqrt{\alpha}(2-x_i)/\sqrt{\varepsilon}}, & \text{if} \quad x_i \ge \frac{3}{2}. \end{cases}$$

In the case $x_i \le 3/2$ if $x_i > 1+\tau$ then $e^{-\sqrt{\alpha}(x_{i-1}-1)/\sqrt{\varepsilon}} \le e^{-\sqrt{\alpha}\tau/\sqrt{\varepsilon}} = (N/2)^{-1}$, and if $x_i = 1+\tau$, then $e^{-\sqrt{\alpha}(x_{i-1}-1)/\sqrt{\varepsilon}} = e^{-\sqrt{\alpha}\left(\tau-\frac{8\tau}{N}\right)/\sqrt{\varepsilon}} = e^{-\ln(N/2)}e^{8N^{-1}\ln(N/2)} = (N/2)^{-1}\left((N/2)^{1/N}\right)^{8} \le CN^{-1}$. Thus, it follows from (3.11)

$$|\mathscr{L}^N(w_i^{j+1} - w^{j+1}(x_i))| \le CN^{-1}. \tag{3.12}$$

The same result is obtained for the case of $x_i \ge 3/2$. Combining (3.10) and (3.12) gives

$$|\mathscr{L}^N(w_i^{j+1} - w^{j+1}(x_i))| \le CN^{-1}\ln(N/2),\ x_i \in (1,2).$$

Thus we obtained the following error estimate to the singular component

$$|w_i^{j+1} - w^{j+1}(x_i)| \le CN^{-1}\ln(N/2), \quad i = \frac{N}{2}+1,\ldots,N-1. \tag{3.13}$$

The proof is thus completed by combining (3.8) and (3.13) and using triangle inequality. □

The error estimate for the solution of the continuous problem and the solution of the fully discretized scheme is given by the following theorem.

Theorem 3.5.2. *Let $y(x,t)$ and u_i^{j+1} are the solutions of the problem (3.1a)-(3.1d) and (3.6a)-(3.6c), respectively. Then the error estimate is given by*

$$\sup_{0<\varepsilon\ll 1}\ \max_{i=0,1,\ldots,N}\ \max_{j=0,1,\ldots,M} |y(x_i,t_{j+1}) - u_i^{j+1}| \le C((\Delta t)^2 + N^{-1}\ln N/2).$$

Proof. The proof follows from Lemma 3.3.4 and Theorem 3.5.1. □

3.6 Numerical Illustrations

Two test problems whose solutions exhibit interior and boundary layers are encountered to verify the theoretical results estimated in the previous section. Since the analytical/exact solutions of the problems considered are unknown, for a fixed ε, the maximum pointwise absolute error is calculated as

$$e^{\varepsilon}_{N,\Delta t} = \max_{1\leq i\leq N-1}\left(\max_{1\leq j\leq M-1}\left|U_i^j - \widetilde{U}_{2i}^{2j}\right|\right),$$

where $\widetilde{U}_{2i}^{2j}$ is the solution obtained by keeping the transition parameter fixed (same as in the original mesh obtained by taking N points in the spatial direction and the spacing Δt in the temporal direction) and doubling the points in the spatial direction and half spacing in the temporal direction. The following formula estimates the order of convergence of the scheme

$$\rho^{\varepsilon}_{N,\Delta t} = \ln_2(e^{\varepsilon}_{N,\Delta t}/e^{\varepsilon}_{2N,\Delta t/2}).$$

The ε-uniform absolute error $e_{N,\Delta t}$ is calculated by taking the maximum over the specific range of ε as

$$e_{N,\Delta t} = \max_{\varepsilon=2^{-4},2^{-8},\ldots,2^{-32}} e^{\varepsilon}_{N,\Delta t},$$

and the ε-uniform order of convergence is determined as

$$\rho_{N,\Delta t} = \ln_2(e_{N,\Delta t}/e_{2N,\Delta t/2}).$$

Example 3.6.1. $r(x) = 3$, $s(x) = -1$, $\omega(x,t) = 1$, $y_0(x) = 0$, $\varphi(x,t) = 0$, $\psi(t) = 0$.

Example 3.6.2. $r(x) = x+6$, $s(x) = -(x^2+1)$, $\omega(x,t) = 3$, $y_0(x) = 0$, $\varphi(x,t) = 0$, $\psi(t) = 0$.

Table 3.1: $e^{\varepsilon}_{N,\Delta t}$, $e_{N,\Delta t}$, $\rho^{\varepsilon}_{N,\Delta t}$ and $\rho_{N,\Delta t}$ for Example 3.6.1.

	N				
ε	18	36	72	144	288
2^{-4}	3.56e-03	2.09e-03	1.36e-03	7.68e-04	4.06e-04
	0.7684	0.6199	0.8244	0.9196	
2^{-8}	1.12e-02	7.01e-03	2.97e-03	9.94e-04	2.52e-04
	0.6760	1.2390	1.5791	1.9798	
2^{-12}	1.12e-02	6.98e-03	2.97e-03	1.14e-03	3.90e-04
	0.6822	1.2328	1.3814	1.5475	
2^{-16}	1.12e-02	6.98e-03	2.96e-03	1.14e-03	3.89e-04
	0.6822	1.2376	1.3766	1.5512	
2^{-20}	1.12e-02	6.98e-03	2.96e-03	1.14e-03	3.89e-04
	0.6822	1.2376	1.3766	1.5512	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-32}	1.12e-02	6.98e-03	2.96e-03	1.14e-03	3.89e-04
	0.6822	1.2376	1.3766	1.5512	
$e_{N,\Delta t}$	1.12e-02	7.01e-03	2.97e-03	1.14e-03	4.06e-04
$\rho_{N,\Delta t}$	0.6760	1.239	1.3814	1.4895	

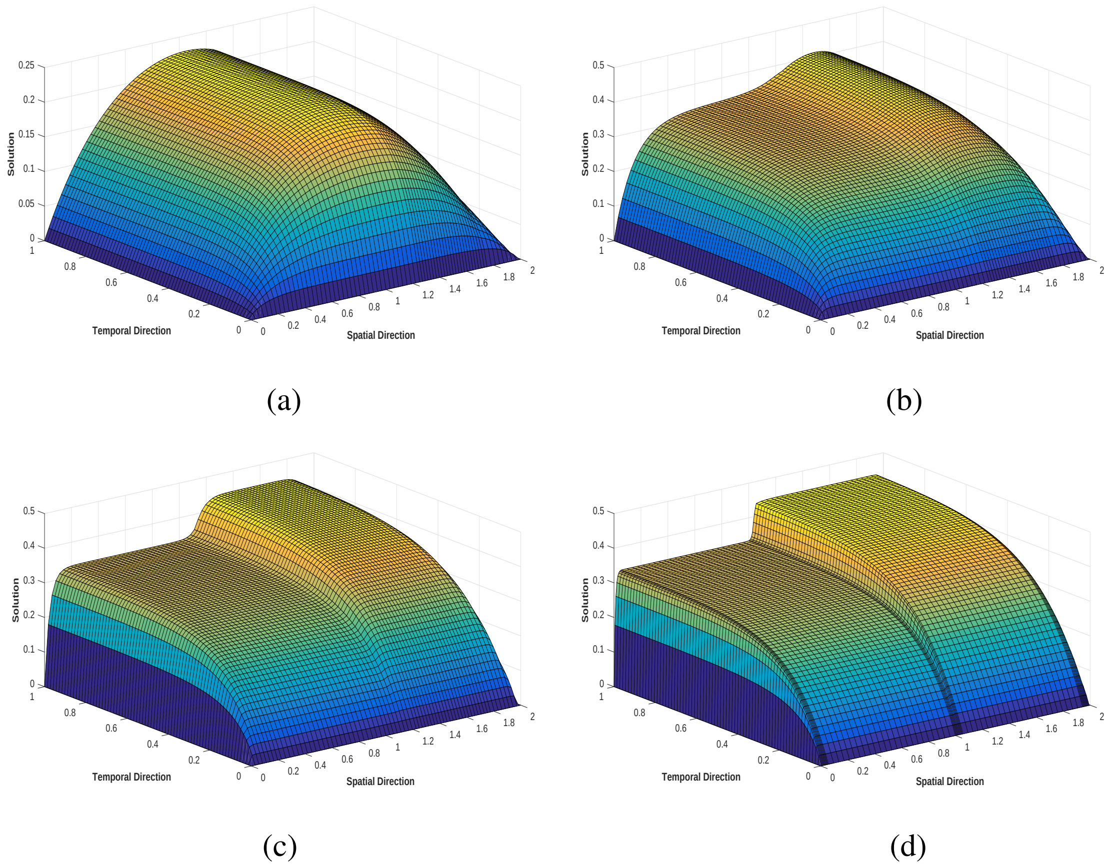
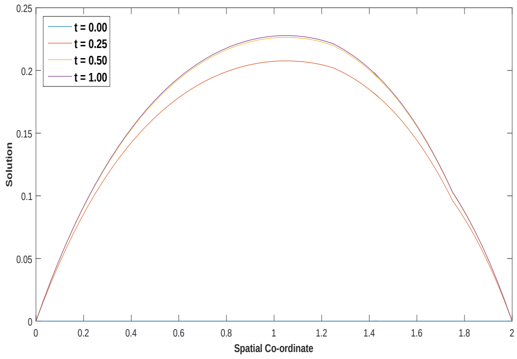
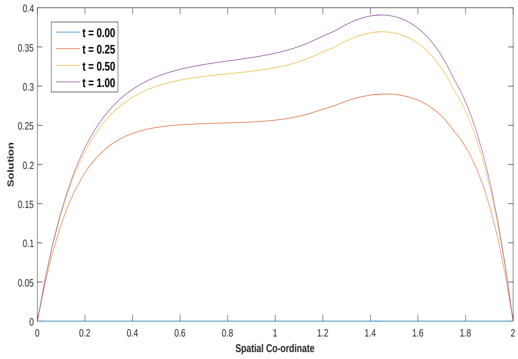


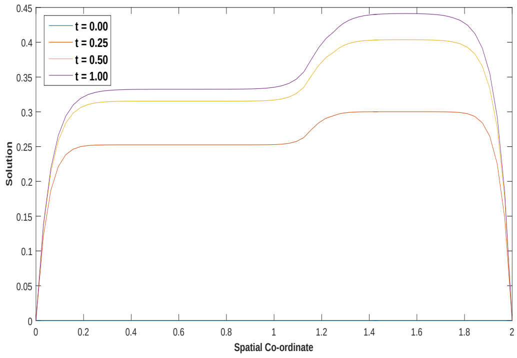
Figure 3.1: Surface plots for Example 3.6.1 for (a) $\varepsilon = 1$ (b) $\varepsilon = 2^{-4}$ (c) $\varepsilon = 2^{-8}$ and (d) $\varepsilon = 2^{-12}$.



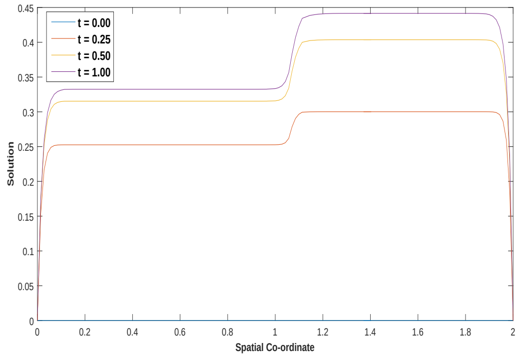
(a)



(b)



(c)



(d)

Figure 3.2: Numerical solution for Example 3.6.1 at different time levels for (a) $\varepsilon = 1$ (b) $\varepsilon = 0.1$ (c) $\varepsilon = 0.01$ and (d) $\varepsilon = 0.001$.

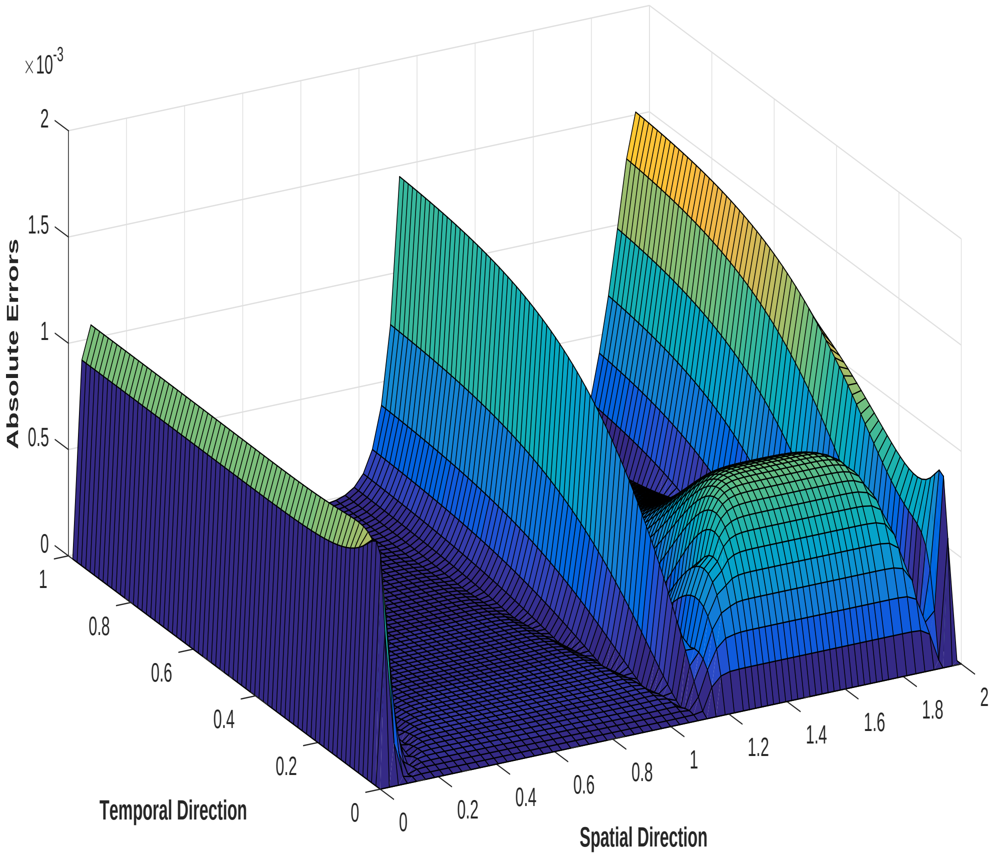


Figure 3.3: Absolute errors profile for Example 3.6.1 for $\varepsilon = 0.01$.

Table 3.2: $e^{\varepsilon}_{N,\Delta t}$, $e_{N,\Delta t}$, $\rho^{\varepsilon}_{N,\Delta t}$ and $\rho_{N,\Delta t}$ for Example 3.6.2.

	N				
ε	18	36	72	144	288
2^{-4}	1.28e-02	4.79e-03	3.50e-03	2.28e-03	1.24e-03
	1.4180	0.4527	0.6183	0.8787	
2^{-8}	1.38e-02	1.09e-02	5.56e-03	2.10e-03	7.26e-04
	0.3403	0.9712	1.4047	1.5323	
2^{-12}	1.36e-02	1.07e-02	5.47e-03	2.05e-03	7.28e-04
	0.3460	0.9680	1.4159	1.4936	
2^{-16}	1.36e-02	1.07e-02	5.44e-03	2.04e-03	7.24e-04
	0.3460	0.9759	1.4150	1.4945	
2^{-20}	1.36e-02	1.07e-02	5.44e-03	2.04e-03	7.22e-04
	0.3460	0.9759	1.4150	1.4985	
⋮	⋮	⋮	⋮	⋮	⋮
2^{-32}	1.36e-02	1.07e-02	5.43e-03	2.04e-03	7.22e-04
	0.3460	0.9786	1.4124	1.4985	
$e_{N,\Delta t}$	1.36e-02	1.09e-02	5.56e-03	2.28e-03	1.24e-03
$\rho_{N,\Delta t}$	0.3193	0.9712	1.2861	0.8787	

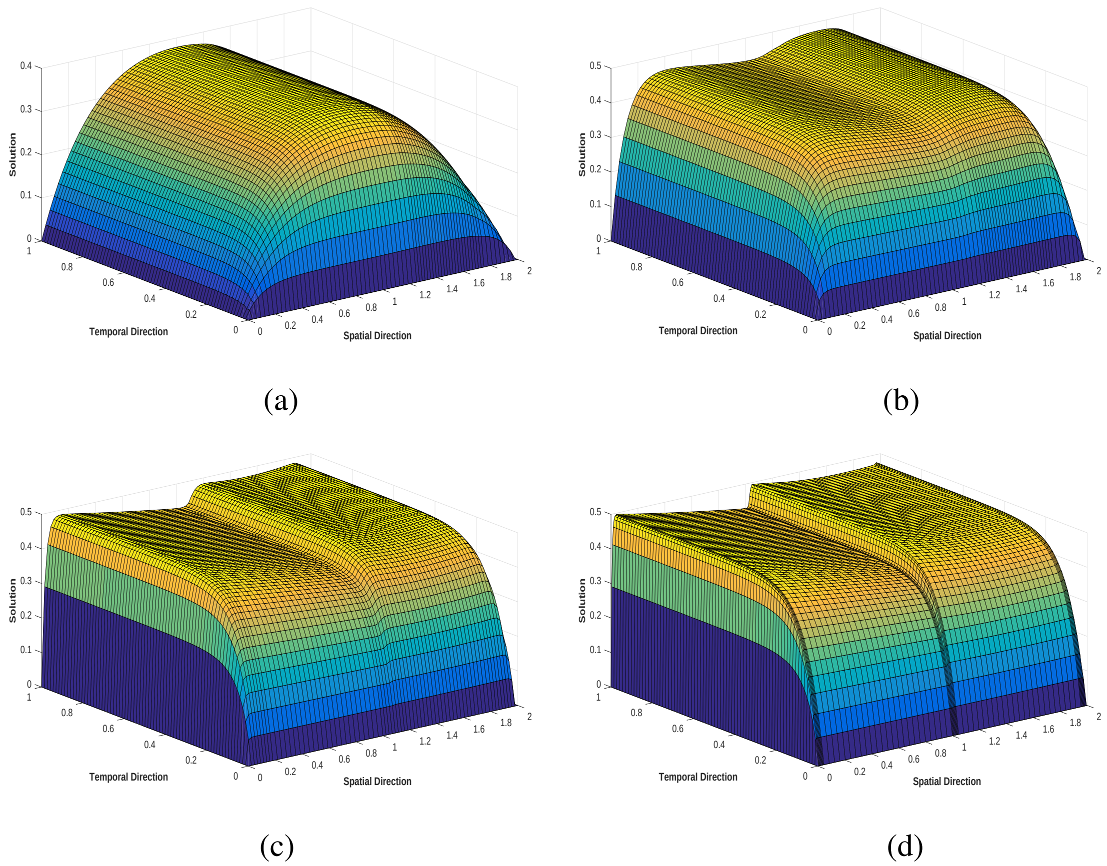


Figure 3.4: Surface plots for Example 3.6.2 for (a) $\varepsilon = 1$ (b) $\varepsilon = 2^{-4}$ (c) $\varepsilon = 2^{-8}$ (d) $\varepsilon = 2^{-12}$.

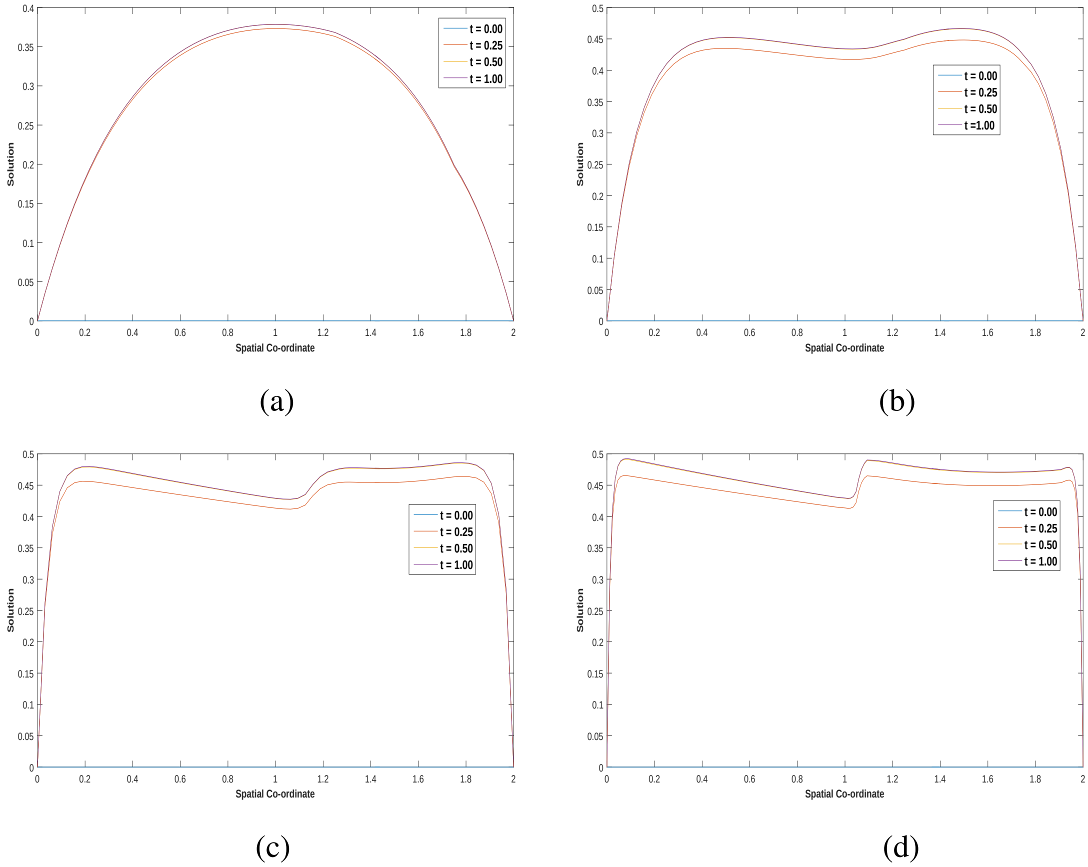


Figure 3.5: Numerical solution for Example 3.6.2 at different time levels for (a) $\varepsilon = 1$ (b) $\varepsilon = 0.1$ (c) $\varepsilon = 0.01$ and (d) $\varepsilon = 0.001$.

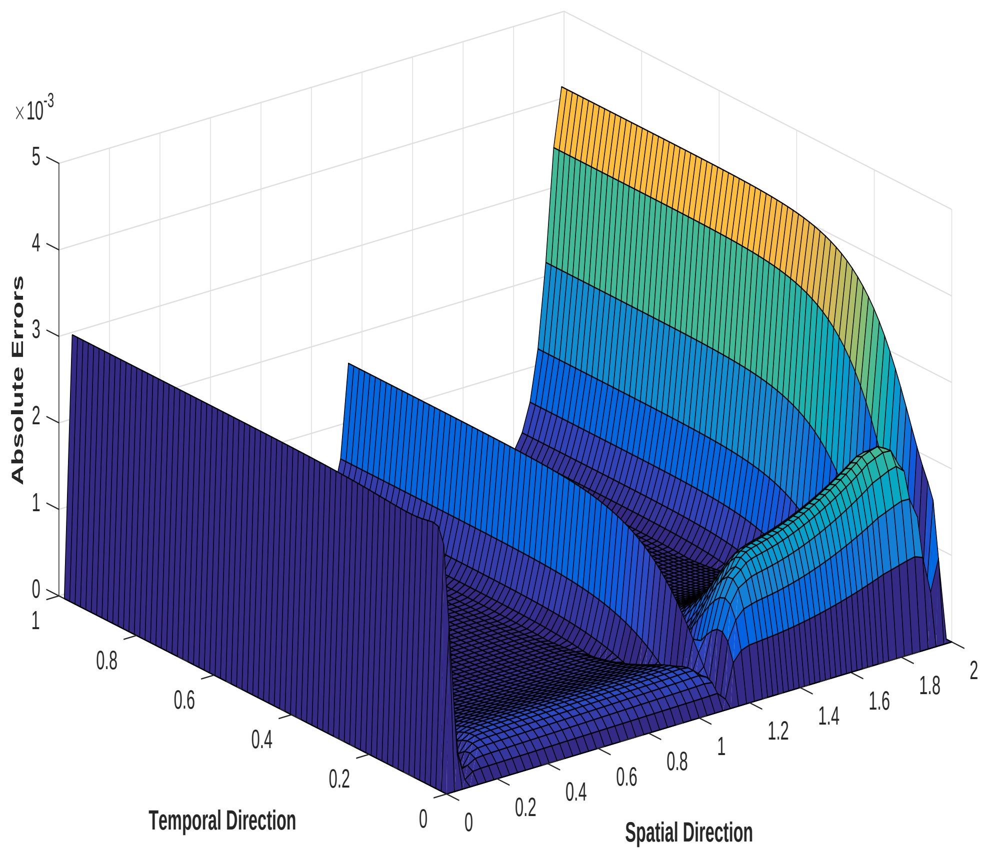


Figure 3.6: Absolute errors profile for Example 3.6.2 for $\varepsilon = 0.01$.

From Tables 3.1 and 3.2, it can be observed that after a certain value of ε ($= 2^{-20}$ for both problems) the maximum pointwise errors stabilized, which clearly indicates the parameter-uniform convergence of the method. Also, it can be noted that these results are in good agreement with the theoretical bounds given in Theorem 3.5.2. These results are obtained by using $\Delta t = 1/N$ and taking $\alpha = 1$ and 1.5, respectively. To plot all graphs (Figures 3.1-3.6) we have used $N = 72$ and $\Delta t = 1/N$. To observe the change in the boundary layer width with respect to ε, and to show the physical phenomenon of the solution the surface plots (Figures 3.1 and 3.4) have been plotted. From these figures for small ε close to zero, twin boundary layers and an interior layer can be seen in the solution. Also, it can be observed that the boundary/interior layers width decrease as the parameter ε decreases. The solution behavior for different time levels is also plotted (refer to Figures 3.2 and 3.5). The plots of estimated pointwise absolute errors in the solutions are also plotted (Figures 3.3 and 3.6). Due to the abrupt change in the mesh width, from these figures, it can be seen that the absolute pointwise error is large near the transition points as compared to the other part of the given domain.

3.7 Conclusion

A robust unconditionally stable first-order implicit numerical scheme for the time-dependent parabolic SP-PDEs whose solution exhibits an interior layer and twin boundary layers is developed. Since the width of the boundary/interior layers depends on ε, the transition parameter τ should be defined carefully. The physical behavior of the solutions to the given problems is plotted in Figures 3.1 and 3.4. It can be seen from the graphs that the boundary layer width continuously depends on ε and it decreases as ε decreases. Also, the numerical results presented in the tables are in good agreement with the theoretical results and the proposed method is convergent irrespective of ε and the convergence is as per the Theorem 3.5.2. The proposed difference scheme can be implemented for the solution of higher dimensional delay problems and the system of equations involving the delay parameter. In particular, for future consideration, the authors shall consider the problems in two-dimensions.