Chapter 4

Updation and Evaporation of Pheromone Trials

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4.1 Introduction

A fundamental fact in ant trial formation is the indirect interaction and communication of ants through the environment by deposited chemical pheromones (Dorigo and Blum, 2005). It enables ant colonies to make choices based purely on local knowledge. Besides the travel time experience (Badr and Fahmy, 2004), it leads individual ants to use the shortest among several paths to transport food from distant places into their nest. This stigmergy concept in general allows for a complex collective behavior of relatively simple agents (Dorigo and Stutzle, 2004) and inspired a large number of new algorithms and applications especially in the field of combinatorial optimization problems (Stutzle and Hoos, 2000).

4.2 Preliminaries

Before moving on to the main results, some description on the preliminary notions on the 'theory of pheromone' is necessary. It has been observed that a colony of ants is able to find the shortest path to a food source. As an ant moves and searches for food, it lays down a chemical substance called pheromone along its path. As the ant travels, it looks for pheromone trails on its path and prefers to follow trails with higher levels of pheromone deposits. If there are multiple paths to reach a food source, an ant will lay the same amount of pheromone at each step regardless of the path chosen. However, it will return to its starting point quicker when it takes the shorter path which contains more pheromone. It is then able to return to the food source to collect more food. Thus, in an equal amount of time, the ant would lay a higher concentration of pheromone over its path if it takes the shorter path, since it would complete more trips in the given time. The pheromone is then used by other ants to determine the shortest path to find food as described.

During the process, another factor affects on the amount of pheromone deposition namely, evaporation of pheromone, which can be seen as an exploration mechanism that delays faster convergence of all ants towards a suboptimal path. The decrease in pheromone intensity favors the exploration of different paths during the whole search process. In real ant colonies, pheromone trail also evaporate, but as we have seen, evaporation does not play an important role in real ant's shortest path finding. But on the contrary, the importance of pheromone evaporation in artificial ants is probably due to the fact that the optimization problems tackled by artificial ants are much more complex than those real ants can solve. A mechanism like evaporation helps in forgetting of errors or poor choices done in the past plays the important function of bounding the maximum value achievable by pheromone trails. In S-ACO (Gutjahr, 2000) the pheromone evaporation is interleaved with pheromone deposit of ants. After each ant has moved to a next node according to ant's search behavior, pheromone trails are evaporated by applying the following equation to all arcs;

$$\tau_{ij} \leftarrow (1 - \rho)\tau_{ij}, \ \ \forall i, j \in A$$

where $\rho \in (0, 1]$ is a parameter, A is the set of all nodes in the problem and τ_{ij} is the artificial pheromone trail associated with each $\text{arc}(i, j)$. The value of pheromone evaporation lies between 0 and 1.

4.3   Temporal Effects on Evaporation Mechanism

As pheromone evaporation plays some role in the efficiency of the algorithm, an effective formula for finding the rate at which the evaporation occurs is needed. A new formula is proposed, which improves all such attempts done by the researchers in the past. This section discuss in detail about the mathematical aspects and also more importantly, the convergence of the formula to authenticate its validity.

Theorem 4.3.1 *Let the pheromone evaporation at time t be ρ_t, where the value of ρ_t, lies in the closed interval $[0, 1]$. Now the recurrence relation for the evapora-*

tion of pheromone at time $t + 1$ is given by

$$\rho_{t+1} = \alpha \, \rho_t + \beta(1 - \rho_t) = k\rho_t + \beta \tag{4.1}$$

where α, β are two constants such that $0 \leq \alpha, \beta \leq 1$ and $k = \alpha - \beta$.

Proof. It is obvious that the formula given above is well defined at $t + 1$. By hypothesis, ρ_t at time t must lie in the interval $[0, 1]$. Thus if we show the value of ρ_{t+1} is always in $[0, 1]$, the proof is done.

We write $f(\rho_t) = \rho_{t+1}$. Then from formula (4.1) it follows that $f'(\rho_t) = \alpha - \beta$. But then $f'(\rho_t) = 0$ if $\alpha = \beta$ which leads us to conclude that the maximum value of ρ_{t+1} is just β, which is less than 1. It is quite obvious that the minimum value of ρ_{t+1} is achieved only at $\alpha = \beta = 0$, in which case $\rho_{t+1} = 0$. Thus we have $0 \leq \rho_{t+1} \leq 1$.

Now we establish a new expression for the pheromone evaporation through exponential generating function.

Theorem 4.3.2 *Let the pheromone evaporation at time t be ρ_t, where the value of ρ_t, lies in the closed interval $[0, 1]$ and the rate at which the evaporation occurs be given by the formula (4.1) with the additional condition $\alpha \geq \beta$. Then*

$$\rho_t = \frac{\beta(1 - k^t)}{(1 - k)} \quad if \quad k \neq 1 \tag{4.2}$$

Proof. The exponential generating function for ρ_t is given by,

$$f(x) = \sum_{t=0}^{\infty} \rho_t \frac{x^t}{t!}$$

$$= \rho_0 + \sum_{t=1}^{\infty} (k\rho_{t-1} + \beta) \frac{x^t}{t!}$$

$$= \rho_0 + k \sum_{t=1}^{\infty} \rho_{t-1} \frac{x^t}{t!} + \beta \sum_{t=0}^{\infty} \frac{x^t}{t!} - \beta$$

$$= \rho_0 + k \sum_{t=0}^{\infty} \rho_t \frac{x^{(t+1)}}{(t+1)!} + \beta e^x - \beta$$

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But then

$$f'(x) = k\sum_{t=0}^{\infty} \rho_t \frac{x^{(t)}}{t!} + \beta e^x = kf(x) + \beta e^x.$$

Thus we have a first order, first degree, linear differential equation in x as follows;

$$\frac{dy}{dx} - ky = \beta e^x. \tag{4.3}$$

where $y = f(x)$. One can easily find its general solution which is given by $y = \frac{\beta e^x}{1-k} + ce^{kx}$, where $k \neq 1$ and c being an arbitrary real constant. By taking $c = -\frac{\beta}{1-k}$, we get a particular solution of (4.3) as

$$y = \frac{\beta}{1-k}(e^x - e^{kx}).$$

Hence, by using the series expansions for e^x and e^{kx} we get,

$$f(x) = \sum_{t=0}^{\infty} \rho_t \frac{x^t}{t!} = \frac{\beta}{1-k}\sum_{t=0}^{\infty}(1-k^t)\frac{x^t}{t!}.$$

The comparison of corresponding co-efficients in the above two power series yields the desired formula(4.2).

Remark 1. The value of ρ_t must be taken as the initial pheromone evaporation value ρ_0, if $\alpha = 1$, $\beta = 0$ or in other way if $k = 1$. It should be observed that the above conclusion is not arbitrarily drawn and can be verified by substituting the values $\alpha = 1$, $\beta = 0$ in identity (4.1).

Remark 2. One can easily see that the theorem 4.3.2 and the remark 1 can be combined to get the expression for all values of k such that $0 \leq k \leq 1$ as follows:

$$\rho_t = \frac{\beta(1-k^t)}{(1-k)} \quad if \quad k \neq 1$$

$$= \rho_0 \quad\quad if \quad k = 1. \tag{4.4}$$

Theorem 4.3.3 *The value of ρ_t given by (4.4) converges.*

Proof. From the theorem 4.3.2, we have, $\rho_t = \frac{\beta(1-k^t)}{(1-k)}$ $\quad if \quad k \neq 1$. We are aware of the fact that $t \to \infty \Rightarrow k^t \to 0$, if $k \neq 1$. Therefore

$$\lim_{t\to\infty} \rho_t = \frac{\beta}{(1-k)} \quad if \quad k \neq 1.$$

If $k = 1$, it is obvious that the sequence $\{\rho_t\}$ converges to ρ_0.

Remark 3. By giving appropriate values for the parameters α, β, we can get the limit of convergence of ρ_t equal to zero or close to zero. Then the pheromone evaporation becomes almost nil which is expected during the implementation of ACO. Therefore the formulae given above shall enhance the performance quality of the ACO algorithms.

4.4   Runtime Analysis

We now do comparative runtime analysis between 1-ANT and a simple evolutionary algorithm called $(1 + 1)$ EA, which has been extensively studied with respect to its runtime distribution. Even though it is already done for a particular range of the value of the evaporation ρ we verify it for the new formula given in the paper (Prasanna and Raghavendra, 2011). The $(1 + 1)$ EA starts with a solution x_0 that is chosen uniformly at random and produces in each iteration a new solution x from a currently best solution x_0 by flipping each bit of x_1 with probability $1/n$. Hence, the probability of producing a certain solution x with Hamming distance $H(x, x_1)$ to x_0 is $(1/n)^{H(x,x_0)}(1 - 1/n)^{n-H(x,x_0)}$. In the following, we consider the 1-ANT with values of $\rho_t = \frac{\beta(1-k^t)}{1-k}$ if $k \neq 1$ in the theorem 4.3.2. Here one can see that the 1-ANT behaves as the (1+1) EA on each function. This also means that the 1-ANT has the same expected optimization time as the $(1 + 1)$ EA on each function. Before going to our prime theorem we state an important lemma due to Neumann and Witt, (2009).

Lemma 4.4.1 *For all $\rho \geq \frac{n-2}{3n-2}$, the 1-ANT has the same runtime distribution as the (1+1) EA on each function.*

For large values of n, lemma is true for $\rho \geq \frac{1}{3}$. With the help of this, we show that the new formula proposed in theorem 4.3.2 for ρ_t also makes 1-ANT algorithm to have the probability to produce a specific solution that has a Hamming distance as same as in the case of (1+1)EA.

Theorem 4.4.2 *Choosing the values of α, β in ρ_t such that $\alpha + 2\beta \geq 1$, the 1-ANT has the same runtime distribution as the (1+1) EA on each function.*

Proof. Note that $\rho_t \geq \frac{\beta}{(1-k)}$. In view of the lemma 4.4.1, it suffices to show that

$$\frac{\beta}{(1-k)} \geq \frac{1}{3}$$

We have $\alpha + 2\beta \geq 1 \Rightarrow 2\beta \geq 1 - \alpha \Rightarrow 3\beta \geq 1 - \alpha + \beta = (1-k)$. Simple rearrangement yields the desired inequality.

The below theorem is on the relative change of pheromone values before and after the pheromone updation. The part of this theorem has appeared in Neumann and Witt (2009) and we acknowledge the paper for motivating us to come up with this theorem.

Theorem 4.4.3 *Let e_1 and e_2 be two edges of connected graph of a combinatorial problem and let τ_1 and τ_2 respectively be their current pheromone values in the 1-ANT. Let τ_1' and τ_2' respectively be their updated pheromone values for the next accepted solution x. If e_1 and e_2 are in the path $P(x)$ of the accepted solution x then*

$$|\tau_1' - \tau_2'| = |\tau_1 - \tau_2|[1 - \frac{\beta}{1 - \alpha + 2n\beta}] \tag{4.5}$$

Proof. The pheromone values in 1-ANT are updated if edge (u, v) is contained in the path $P(x)$ of the accepted solution x. The pheromone value updation formulae are given by Neumann and Witt (2009) as follows;

$$\tau_1' = \frac{(1-\rho)\tau_1 + \rho}{1 - \rho + 2n\rho}$$

$$\tau_2' = \frac{(1-\rho)\tau_2 + \rho}{1 - \rho + 2n\rho}$$

But then,

$$\tau_1' - \tau_2' = \frac{(1-\rho)(\tau_1 - \tau_2)}{1 - \rho + 2n\rho}$$

Thus by taking $\rho = \frac{\beta}{(1-k)}$ and considering both the possibilities $\tau_1 \geq \tau_2$ and $\tau_2 \geq \tau_1$ we will arrive at the expression (4.5).

4.5   Concluding Remarks

In this chapter, an investigation on pheromone update mechanism is discussed. The modification we suggested for the pheromone evaporation, refines the existing algorithm. But in any case our results show that the efficiency of the ACO algorithm is vulnerable with respect to the choice of ρ_t.

The description on theoretical runtime analysis of ACO algorithms comprising of some formal techniques might be helpful for further investigations of this type with the available results. Moreover the results established in section 4.4 give some hints on the appropriate choice of rate of evaporation.