REINFORCED CONCRETE

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PREFACE

THIS book has been written for students and engineers who wish to learn the theory and practice of reinforced concrete. It has been assumed that the reader is familiar with the elementary theory of structures, but the more advanced analysis of indeterminate structures has been included. The fundamental theory of the stresses in reinforced concrete based on stress-strain relations as commonly accepted, with suitable margins of safety, has been presented and also extended to simplified forms so that the student may not only learn to base calculations on established theory but become accustomed to developing methods of design which enable him to evolve and select the best solution of a practical problem. The undergraduate will be more concerned with fundamental principles, and the post-graduate with the application of theory, but both should aim at attaining a balance between the academic and practical viewpoints.

It is seldom that the first edition of a book is produced without the inclusion of some errors, and the writer would appreciate being informed of any that readers may find so that they may be corrected in future editions.

Grateful acknowledgment is made to all who have assisted by their comments and interest, to Mr. J. H. Creed who prepared the drawings, and to Miss B. M. Gandy and Miss M. J. Trewin who typed the manuscript. The late Dr. G. S. Coleman and Mr. F. E. Drury, O.B.E., M.Sc. (formerly of the Manchester College of Technology), Dr. R. Gartner (of South Africa), and Colonel Frank Clarke, M.A. (former headmaster of Queen Elizabeth's School, Crediton, who is quoted in the last chapter), are especially remembered as having contributed, although indirectly, through their educational zeal.

A. L. L. B.

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CONTENTS

CHAPTER I

GENERAL PRINCIPLES OF DESIGN

[Certain points in this chapter will only be fully appreciated after reading the remainder of this book. It is therefore recommended that it be read again at that stage.]

Factor of Safety, Working Loads, and Stresses.

A CIVIL engineer must be not only a reliable calculator of stresses but a man of courage, resource, and sound judgment. In one sense the pioneering of the nineteenth century has passed. Some structural materials have been improved and more may be known about their strength and durability, but on the other hand adverse conditions on sites of engineering works have still to be overcome, lower factors of safety may have to be adopted, and the replacement of structures with pin-joints by those with rigid joints leads to more complex calculations. Thus the task of the engineer is no easier.

Professor Sir Charles Inglis, O.B.E., M.A., in an address given to the members of the Institution of Civil Engineers in 1948 on the subject of "Mathematics in Relation to Engineering," said: "In relation to engineering the qualities required in a mathematical process are utility, generality, and simplicity, and of these the greatest is simplicity," and his answer to the question "How can engineers get the best out of mathematics?" was as follows: "By using the simplest and most straightforward methods and avoiding those which obscure the physical aspects of the problem. The Greek philosopher Aristotle spoke words of profound wisdom when he said, 'Let your reasoning be that of an intelligent man but let your meaning be expressed in the language of the common people.' That precept should be the guiding star directing our mathematical explorations. We should strive to discern the natural connection between cause and effect, and in all cases where it seems possible to do so the mathematical reasoning involved is crystal clear, beauteous in its simplicity, and most frequently the illumination is of a geometrical character which is convincing to the eye."

The engineer must ensure the safety of a structure at all stages of construction and when it is in use and subjected to loads applied under normal conditions, and often he must allow for these loads being exceeded. He has to decide what loads the structure may receive, how it is to be supported, and the stresses on which his calculations should be based. He may have to assess the force of wind and tide or the compression of the ground, with little data to guide him. In the case of a simple structure such as the floor of a building the local by-laws generally specify the superimposed loads and the safe working stresses which must be included in the calculations. Thus the factor of safety which must be adopted is clearly defined. Frequently the determination of the factor of safety will be a much more complex matter; it may vary according to circumstances and for different parts of a structure.

The term factor of safety is sometimes understood to be the ratio of the ultimate load to the assumed working load as in the case of the stability of a retaining wall, and sometimes, when considering the strength of a structure, as the ratio of the stress at failure to the working stress. The latter definition is only satisfactory so long as the stress in a structure increases in proportion to the load. When this does not happen the term "load factor" is sometimes used, and is defined as the ratio of the ultimate load to the working load. To avoid confusion it is proposed to adopt in all cases the definition that factor of safety is the ultimate load divided by the working load. The ultimate load is the load which causes failure of the structure, or an appreciable reduction in strength due to the yielding of a member or deformation so that the structure is of no further use. The working load is the load or the combination of loads which may occur and which causes the maximum stress or change of stress in a part of the structure.

It is convenient to make most calculations in terms of working loads and working stresses but, in order to ensure that an adequate factor of safety is provided, possible altered conditions of stress distribution or increased eccentricity under ultimate load may have to be considered. Conversely if, as is advisable, for example, in prestressed concrete, calculations are made in terms of ultimate loads and stresses, the distribution and magnitude of the stresses under the working load should also be considered to ensure that there is no overstressing.

It is sufficient to adopt a factor of safety of two since it is defined as the ultimate load divided by the working load provided that the following assumptions in the calculations are within safe limits: (1) Stress and distribution of stress; (2) Working loads; (3) Strength and yielding of supports; (4) Stresses due to changes in temperature, and to creep and shrinking of the concrete; (5) Reduction of strength due to fatigue, corrosion, or wear and tear; (6) Redistribution of stress or bending moments due to plasticity or creep.

The assessment of reasonably safe assumptions can sometimes be made after tests and are often, as a result of experience and research, specified in by-laws and codes of practice as maximum permissible stresses, minimum superimposed loads, and other conditions of design. In many cases, however, as will be seen from the following example, an engineer must use his own judgment and experience in making these assumptions and in assessing safe numerical values.

The value of two for the factor of safety may be reduced in some special cases which will be mentioned, but there is no need to use higher values, as is often suggested, provided that safe limits of loads and stresses have been assumed. A safe limit for a particular factor is sufficiently outside the range of all known values to ensure that the worst possible conditions will not cause failure. It is most important to ensure this, and the assumption of safe limits is a much sounder way of producing a safe design than the use of a higher factor of safety combined with smaller loads. For example, in the case of the force of the wind which increases as the square of its velocity, the assumption of the pressure due to an extreme case of a gale of 100 miles per hour in conjunction with a factor of safety of two might be safe for the worst condition of loading, whereas the assumption of the pressure of a gale of 70 miles per hour and a factor of safety of four would not. If in the case of a high building there is the remotest possibility of a hurricane of 100 miles per hour occurring it should be allowed for. Therefore, in order to keep the issue clear, stresses and loads which are, if necessary, quite extreme must be assumed in order to be absolutely safe. A factor of safety of two is then sufficient.

Safe Limits of Assumptions in Design.

STRESS AND DISTRIBUTION OF STRESS.—In the case of steel the yield-point stress of a test specimen is assumed to be the same as the yield stress of similar steel used as reinforcement for concrete. The yield-point stress of a test specimen with a slight reduction for possible corrosion divided by two is therefore the extreme limiting value of the working stress in cases in which the stress varies in proportion to the load. Steel that exhibits no definite yield point under test cannot be summarily considered in this manner. For example, in prestressed reinforced concrete beams the total strain of the wires at the ultimate load must not be so great that the compressive stress in the concrete is excessive, and the calculated stress at working load must not exceed 50 per cent. to 75 per cent. of the arbitrary yield stress.

In the case of concrete certain effects, depending on the type of structural member, must be taken into account. The strength of concrete in compression in a structural member depends on the amount of lateral support it receives. Thus, using the same material, a cube crushes at a higher stress than a cylinder the height of which is twice the diameter. In a beam, before failure takes place, the distribution of the compressive stress alters from a triangular distribution to a form having about 30 per cent. greater total compression, but the design of beams at the working load is based on a triangular distribution of stress, which is a safe assumption. In a column subjected to a load for a long period, the stress undergoes a redistribution due to creep of the concrete, and the load is partly transferred from the concrete to the steel. For columns, therefore, the permissible working stress is lower than for beams.

The ratio of the moduli of elasticity m of steel and concrete is sometimes assumed in calculations to be 40,000 divided by three times the permissible bending stress for concrete. This is based on the approximate value of the modulus for concrete a few days after casting, and for high-quality concrete this may be doubled in twelve months on account of creep, or quadrupled in the case of low-quality concrete. Because of the uncertainty of the value of m, an arbitrary value of 15 is frequently used. It is seen in Chapter III that the effect of variations in the value of the modular ratio are almost negligible.

The factors that may cause the ultimate stress in concrete in a structural member to be less than that in a test cube made from the same concrete include less lateral support, less compaction, loss of cement grout through joints in the shuttering, delay between mixing and placing and tamping in position, trapping of air in the concrete, damage by rain, frost, or injurious substances, and careless curing. In addition, even with the most efficient control, weak concrete may occur in an important part of a member on account of the use of defective cement, weak aggregates, or the intrusion of dirt. It may be that no test cubes have been taken from such a batch and, even if they were so taken, the results might be received too late. It is customary (D.S.I.R. code) to use permissible working stresses of $\frac{x}{3 \cdot 75}$ for concrete in columns subjected to direct stress and $\frac{x}{3}$ for concrete subjected to bending, where x is the crushing strength of cubes of similar concrete made on the site. It can then be assumed that a factor of safety of at least two is provided. In most reinforced concrete structures the factor of safety in the concrete is probably higher than two, but this is necessary in order to guard against excessively weak concrete occurring in an important part of the structure.

Since an advantageous redistribution of the stress in the concrete takes place in a rectangular beam before failure, the factor of safety for the concrete is probably again increased. In a tee-beam with a low neutral axis, however, the redistribution of stress before failure has less influence on the strength since the average stress is not greatly altered. For this reason, when further research has been carried out on stress distribution, the view that calculations based on conditions at failure result in a sounder structure may be confirmed.

Failures rarely occur due to weak concrete, but occasionally due to wrongly-placed reinforcement. It is very easy, for example, for the top reinforcement of a slab or cantilever to be trodden down during concreting and thereby to increase the stress in the concrete and the steel. This type of failure can only be guarded against by a considerable reduction in the working stress. The fault lies in the method of fixing the reinforcement and inadequate supervision. It is equally important to ensure that the stress in the reinforcement is not excessive, as the risk of failure due to overstressing the steel is probably greater than in the case of the concrete, in which the working stress in bending is one-third of the cube-strength on the assumption that the factor of safety is not less than two after the redistribution of stress and defects of workmanship have been taken into account.

Working Loads.—Minimum working loads covering most of the extreme cases likely to occur are now generally specified in regulations and codes of practice. These include superimposed loads on floors of buildings, live loads on bridges, and wind pressures. If a factor of safety of only two is used care must be taken to ensure that in no circumstances can these loads be exceeded. As a general rule, therefore, an exceptional extreme working load, such as wind pressure from a hurricane, and a low factor of safety should be used, rather than a normal maximum working load and a higher factor of safety, in order to appreciate clearly the margin of safety provided.

In some cases it is practical to design only for maximum loads under normal conditions, it being quite impractical to design for abnormal conditions such as when a jetty receives a blow from a vessel completely out of control. In such a case the engineer must assume the maximum speed at collision which may occur under normal conditions in the case of the largest vessel likely to approach the jetty. He must then design all parts of the structure to withstand the forces resulting from such a collision with a factor of safety of two, which factor should also apply to the general stability of the whole structure. In the design of fendering or other shock-absorbing devices a factor of safety of 1·5 only should be applied to the known strength so that the shock-absorber fails under excessive load before the main structure, thereby absorbing considerable kinetic energy from the colliding vessel or other moving body and reducing the amount of energy to be absorbed by the structure.

STRENGTH AND YIELDING OF SUPPORTS.—For a footing, raft, pile, an abutment receiving thrust from an arch, or a pile resisting uplift, the supporting power of the ground should be great enough to ensure that negligible yielding takes place when the structure is subjected to the ultimate load, otherwise the stresses in the structure are increased by the yielding of the supports and may reach the ultimate values under loads less than the working load multiplied by the factor of safety.

If the yielding is not negligible the structure must be designed to fail under the stresses due to twice the working load plus the additional stresses due to the yielding of the supports caused by twice the working load. If the stresses in the structure increase in proportion to the load, it may be sufficient to consider the working-load conditions only but, as in the case of long columns, the stresses may increase more rapidly than the load and the yielding of the supports may do the same. In the case of a flexible arch it may be necessary to take into account the increase in eccentricity of the thrust due to deflection under ultimate load.

TEMPERATURE AND SHRINKAGE AND THE REDISTRIBUTION OF STRESS DUE TO CREEP.—Temperature and shrinkage stresses may sometimes be added algebraically to other stresses under either working load or ultimate load. In some cases, however, change in temperature or shrinking may cause cracking in an otherwise uncracked member, and as a result the combination of stresses is more complex than an algebraic summation.

The transfer of stress from the concrete to the steel due to creep is a function of time and of the average compressive stress in the concrete during the period under consideration. It appears that after twelve months creep has practically reached its limit, and the resulting transfer of stress may be taken into account by assuming a higher value for the modular ratio. This effect of creep is ignored in ordinary reinforced concrete, but is taken partly into account in the design of columns. In long columns in some cases the effect of creep in increasing lateral deflection and eccentricity is not negligible. Creep is also considered in the design of prestressed members.

DETERIORATION OF STRENGTH DUE TO FATIGUE, CORROSION, OR WEAR AND TEAR.—When these factors are considerable the dimensions of members should be increased in order to ensure that they do not decrease below what is required, under proper maintenance, for whatever period is required.

REDISTRIBUTION OF BENDING MOMENTS.—Redistribution of bending moments may take place in a continuous frame, at the stage when stiff members yield, before failure of the whole structure, and this possibility is sometimes used to justify designing on the assumption that bending moments are slightly redistributed to advantage under working-load conditions. For example, in a continuous beam, partly on account of creep and partly on account of the redistribution of bending moments, bending moments at the supports calculated by the ordinary elastic theory may be reduced up to 15 per cent., and the bending moments at mid-span increased by the same amount. Such a redistribution would not take place until the reinforcement at the supports commenced to yield, or sufficient creep of concrete had taken place, but the ultimate load would be the same as for a beam designed without the adjustment. Such redistribution of moments in design are permissible so long as the stresses in the reinforcement under working load are not high enough to cause serious cracking of the concrete, and so long as the ultimate strength is not affected. It is generally considered that a tensile stress of 27,000 lb. per square inch in the reinforcement is the greatest that will not cause objectionable cracking.

The result of failure considerably influences the safe working load adopted, and may occasionally justify a slight reduction in the factor of safety. If failure means that life may be endangered, working loads which can in no circumstances be exceeded must obviously be used. If failure means the cracking of a pile during driving or of a precast member during erection, which can easily, if necessary, be replaced, then the minimum working load and a factor of safety of 1·5 may be sufficient. Other cases are when a member is cracked because of the settlement of a footing, or because of excessive stress in the reinforcement in a continuous frame due to local stiffness being ignored in the calculations; the adverse effects disappear after cracking, so that a factor of safety of at least two is still provided against complete failure. In practice a great variety of such problems exists, and in each case the general principles which have been outlined must be applied. The following are typical examples.

In the case of long columns it is customary to reduce the working stresses below the ordinary value as the slenderness ratio increases, in order to take into account the bending moment which develops due to eccentric loading as the column deflects laterally.

In designing beams for high shearing stresses the reinforcement is assumed to develop the full working stress, regardless of the stress in the surrounding concrete. Tests show that this is a satisfactory method of design, and the factor of safety is based on ultimate-load conditions.

In the case of swimming pools, basements, or underground storage tanks there is sometimes a risk of the floor lifting under external hydrostatic pressure after a wet season has raised the water-saturation level of the ground. If such a possibility is remote, and if it is not possible to ensure disposal of the water by draining or to release the pressure through vent-pipes, a factor of safety of 1·25 might be used.

In the case of a retaining wall liable to overturning or sliding, a factor of safety of 1·5 can be used if the ground pressure can be accurately determined.

On some sites there is a risk of gypsum, salts, or other chemicals causing deterioration of concrete foundations. In such cases high-alumina cement concrete should be used, or the site drained so that the risk is removed.

Where there is a risk of bad workmanship or of poor-quality materials being supplied the working stresses generally assumed for the materials should be reduced.

In marine structures, fendering should in many cases be designed for normal conditions only, so that if failure takes place under abnormal conditions replacement or repair is simple. Where high speeds of berthing are unavoidable, shock-absorbing fenders should be used.

In flexible raft foundations, high and low safe limiting values for the coefficient of settlement of the ground should be assumed in order to cover the worst possible conditions, and the deflection of the raft limited to an amount which will not cause cracking of the building it supports.

In piled structures the piles should be spaced so that their total resistance to load is not less than the sum of their individual resistances. In marine structures loss of resistance to uprooting in clay due to lubrication by water in some cases must be allowed for; in other cases the resistance can be assumed to increase as time passes.

In the case of bridges allowance must sometimes be made for the effect of impact or "hammer" from locomotives.

In determining the safe supporting power of the ground the sub-stratum should be explored by test borings. Settlements obtained by tests should be multiplied by three or four, as when a complete site is loaded the settlement is generally much greater than when only a small area is loaded.

In an arch liable to yielding of the abutments, and therefore to much deflection, the factor of safety is the ultimate load divided by the working load, the value of the ultimate load being based on calculations which take into account the yield of the ground under the ultimate load and the bending moment in the arch due to thrust, eccentricity, and deflection due to the ultimate load.

In prestressed beams the permissible working stresses of ordinary reinforced concrete beams may be exceeded provided a suitable factor of safety is used, since the stresses do not increase at the same rate as the load.

As an example which includes a variety of considerations, consider a concrete jetty with dolphins. The structure is exposed to gales of 80 miles per hour in the open sea and it can be approached only by a narrow channel in which the direction of the current is liable to vary. The sea-bed is clay and for this reason, and in the interests of economy and speed of construction, it was necessary to provide a light, open, piled structure to support the crane and to provide an approach jetty. The most important loads governing the design of the jetty were therefore the large impact forces expected from vessels in berthing or leaving the berth, and possible pulls from mooring ropes attached to the bollards on the top of the dolphins. Other forces were the vertical loads and uplift from the crane, vertical live load from wagons travelling on the deck, and the horizontal force due to wave action (particularly at high tide) on the parapet walls of the approach jetty.

In the case of the impact forces from ships, it was necessary to assume the maximum speed of berthing expected under the worst possible conditions of wind, tide, and current. Such an assumption could only be made by study of vessels berthing under various conditions and estimating the degree of control it might be possible to maintain in this case. Such an estimate was bound to be very approximate and the decision a difficult one to make, as the force of the impact on a jetty increases as the square of the velocity of the ship. It was decided to adopt a maximum safe working speed of 1·25 ft. per second for a glancing blow from a 15,000-tons vessel and to use a factor of safety of two. These values were assumed partly because 1·25 ft. per second for a collision represents a high berthing speed, and partly because, even for these assumptions, a structure is required which is fairly elaborate and which is comparable in strength with other marine structures which have withstood normal blows from large vessels and which would therefore only fail under abnormal accidental conditions. The dolphins are a shock-absorbing type, and the deck between the dolphins and the supporting sub-structure are also protected by shock-absorbing fenders. In designing the dolphins the safe working load was assumed to be the force of impact from a 15,000-tons ship, or a static horizontal pull of 270 tons on the bollard at the top of the dolphin. These forces were considered to be applied in a direction normal to the face of the jetty, that is in the direction from which the worst blows were expected. In a longitudinal direction the horizontal working load was assumed to be reduced to 200 tons. For the intermediate shock-absorbing fenders a lower speed of collision was assumed ; three fenders were therefore provided at the centre but only two at intermediate points, as the manner of berthing, the shape of the ships, and the relative position of the faces of the fenders determined that blows would probably be received in this order of magnitude.

The working stresses used in the design of the jetty were those recommended in the D.S.I.R. Code of Practice for Reinforced Concrete in Buildings, namely, 18,000 lb. per square inch in the reinforcement and 750 lb. per square inch in bending for 1 : 2 : 4 concrete. In the parts of the structure liable to tensile stresses and submerged by sea-water a stress of 12,000 lb. per square inch was used in the reinforcement. Thus an effective factor of safety of about two is provided at the weakest critical sections in regard to the concrete and steel. A factor of safety of two was used against the dolphins failing by overturning due to the piles being uprooted, test piles having shown an anchorage resistance of 20 tons, which was assumed to increase to 30 tons after a short time ; the maximum uplift on any pile was 15 tons. A slightly higher factor of safety would have been used if there had been sufficient space to increase the number of piles in the group, but the value adopted was satisfactory as the working loads were high. It would have been ideal if the factor of safety throughout could have been arranged so that uprooting of the piles and the fracture of every member in bending, direct stress, bearing, or shearing took place simultaneously so that uniformity of strength was provided throughout except in the fenders on the dolphins. If the fenders are pressed to their extreme position a considerable amount of the remaining kinetic energy from the ship can be absorbed by partly crushing them and partly indenting the ship's plates without uprooting the piles, thereby involving only minor damage instead of a major collapse. Similarly in the design of the fenders, the suspension links have a higher factor of safety than the remainder of the structure as wear and tear was involved and their failure would lead to much more serious results than, say, the buckling of one of the struts in the supporting frame which, after failure, could continue to absorb kinetic energy from the oncoming vessel and thus perhaps avert the collapse of the deck.

The foregoing example illustrates that in a complicated marine structure the assumptions of safe working loads and stresses can only be made by judgment, rather than by calculation, after considering the many factors which vary with every site. The example demonstrates how important it is that a civil engineer should always have in mind the fundamental principles of design. He must also take into account experience as recorded in the papers of engineering societies and in technical journals, as published as codes of practice, or established as by-laws with which he may be compelled to comply. By-laws are generally definite with regard to the working loads and stresses that must be used in buildings, and are based on the results of experience, tests of materials, and perhaps tests on full-size beams, columns, and slabs. Over-cautious or out-of-date by-laws sometimes seriously restrict the development and scope of reinforced concrete, and engineers should continually press for their revision as progress is made. Some most enterprising and successful reinforced concrete construction has been executed in cities where by-laws are not restrictive and engineers are fully aware of their greater responsibility.

It may be advisable to carry out experiments to guide the design of an important or unusual structure, as was done in the case of the pierheads used in the Mulberry Harbours in connection with which collision tests with model landing craft were made in a tank to determine the forces of impact when berthing at the abnormally high speed of four knots. Sometimes it is advisable to conduct wind-tunnel tests for structures such as large hangars or light buildings on exposed sites.

Two further principles may be stated. In designing a structure, for economy and appearance a uniform factor of safety should be provided throughout so that failure due to all possible causes takes place simultaneously, except where it is desirable that parts of the structure, such as fenders, should fail before the remainder of the structure and by so doing prevent further damage.

The degree of accuracy employed in calculations should be intelligently related to the degree of accuracy with which the working loads and stresses can be assessed. It is important to know how to calculate theoretical stresses exactly from first principles, using approximate but safe assumptions, and to be able to produce factors and formulæ for design.

When these fundamental principles have been firmly grasped an engineer should make as much use as possible of quick approximate methods, provided he is certain that they produce results within the required safe limits. He will then not only save much time but be able to think more clearly about the design of the structure as a whole.

Influence on Design of Cost and other Factors.

The design of most structures can best be determined by a systematic consideration of the structural requirements, the function of the structure, æsthetic considerations, cost, site conditions, plant available for construction, supply of materials and labour, probable future requirements, the life of the structure, the possibility of extensions and alterations, and the machinery to be installed, the equipment required for heating and ventilating, the provision of sound and heat insulation, and the degree of fire resistance required.

First it must be assured that reinforced concrete is the most suitable material for the structure and that there is no advantage in constructing the structure or parts of it in other materials. Then the best structural system for supporting the loads must be considered. The supporting power of the ground may be an important factor, and different types of foundation, such as rafts or piles, may have to be considered. The purpose of a building, æsthetic considerations, cost, ease of extension, and the accommodation of machinery may determine whether flat-slabs, load-bearing walls, or beam-and-slab and framed construction is the best. The scarcity of timber may cause precast concrete to be advantageous for parts of the structure. Considerations of cost and the size of beams and columns may determine whether concrete of high quality should be used in, for example, the columns and beams of residential flats. In such a structure, reduction of weight and saving in timber and transport may be more economical than using a cheaper concrete, and it may be cheaper to precast or prestress some of the members. Having decided on the general structural arrangement the most economical dimensions of the members must be determined, bearing in mind the relative costs of concrete, reinforcement, and shuttering (see Chapter III).

In any structure only a complete estimate of cost for each possible design, taking into account the cost of plant, labour, and materials, can determine the most economical design. The cost of maintenance should also be considered. In preparing each design considerable guidance can be obtained from approximate estimates of the cost of different sizes and forms of structural members which are repeated many times.

The influence of the purpose of the structure and of æsthetic considerations on the design in any particular case will be fairly obvious. Functional and structural requirements should be combined as far as possible. For example, the deck slab and parapet walls of a jetty can form a beam of channel shape to distribute horizontal forces over a large number of piles. (Æsthetic considerations in relation to cost are discussed later.) Site conditions determine such structural forms as light structures of high-quality concrete; portal-frame bridges bearing on bad ground; hinged arches or arches of high ratio of rise: span on medium ground, and flat arches and continuous beams on firm ground; whether or not cantilever and floating pile-frames or screwing plant are available may determine the best form of a jetty; and so on. Whether reinforced concrete of high strength can be made, or whether precasting is an advantage, or whether steel shuttering is used in preference to timber, may depend upon the supply of materials. The possibility of future extensions sometimes determines whether beam-and-slab, load-bearing walls, or flat-slab construction is most suitable in a building. The equipment to be installed, for example, cranes, fenders, or machinery, often considerably influences the form and dimensions of the structural members by reason of their suitability to support concentrated loads or to resist vibration.

Organisation and Essential Experience.

The preparation of the design of an important structure and the placing of the contract are matters which require much care. Various procedures are used. Consulting engineers are often employed by Government departments and industrial concerns to produce designs in detail, and contractors are invited to tender for the construction. Often the design is prepared in outline only, since the preparation of the details is a long operation which can be to a certain extent carried on at the same time as the construction. It is, moreover, often better to postpone some decisions on the final design until the contractor has been appointed, as the method of construction he proposes to adopt, the plant that he proposes to use, or special conditions only revealed after trial borings have been made, may affect the final design.

When a contractor is invited to tender on a preliminary design it is important that he be given sufficient details of the structure on which to base a firm price, including the assumed quantities of concrete, shuttering, and steel. He may also be asked to give rates by which the price can be adjusted due to variations between the estimated and the final amounts of work. Sometimes a contractor is asked to submit an alternative design or to include the cost of preparing the working drawings. In this case the working loads and factors of safety must be clearly specified to ensure that all contractors tender on the same basis, and so that any advantages in price some contractors may offer are not due to increases in the working stresses or reductions in the loads.

Industrial organisations frequently ask for tenders for the design and construction of a complete structure, and in describing the structure they merely define the purpose of the structure thereby leaving the contractor wide scope with regard to the design and the stresses which he adopts. This type of competitive design is not always satisfactory as it may enable a less cautious contractor to obtain a contract simply because he is prepared to take a greater risk. This practice has led to working stresses higher than those used for ordinary designs being used for what are known as competitive designs. An engineer accustomed to prepare competitive designs often has an outlook with regard to working stresses and factors of safety different from that of an engineer accustomed to work as a consultant. For a particular structure there should be only one design from the point of view of the strength required, and a civil engineer should try to retain an objective view unbiassed by the outlook prevailing in the organisation in which he is working.

It is sometimes difficult for a young engineer working with reinforced concrete specialists to acquire the ability to view large civil engineering works as a whole, and therefore to visualise the parts constructed in reinforced concrete in true perspective. His employer will naturally have in mind the monetary aspect of the contract, and it is important that the young engineer should understand his employer's viewpoint and effect all the economies he can in his designs; at the same time he should not lose sight of the fact that in large civil engineering works large savings can frequently be made by careful planning of the project as a whole. Often the savings made by working to the standards of competitive design in the reinforced concrete parts of the work are insignificant compared with savings which can be made, say, by the bulk purchase of materials, or by avoiding extensive draining or pumping in the excavations if a study of the natural characteristics of the site during the planning of the works shows that this can be avoided.

When preparing a competitive design an engineer should inform the owners that for a little extra cost, which may be insignificant in comparison with the total cost of the works, a better structure could be built, if he is of opinion that this is possible. The owner may be glad to receive this advice, and in any case he will be encouraged to compare carefully the competitive designs he receives so as to ensure that the cheapest is based on an adequate factor of safety.

In contrast to competitive design, it is equally unsatisfactory for a young engineer to become entirely accustomed to the rather extravagant standards of design produced by some authorities and engineers when there is no competition. The Codes of Practice define standards which are now fairly generally adopted, and which help to dispose of the disparity existing between competitive and other designs.

A serious defect in civil engineering is the absence of a recognised procedure for a young engineer to follow in order to become fully experienced. His education as an engineer is only beginning when he completes his technical training as a student. Ideally he should begin with work on outside construction, and, as soon as possible, take full responsibility on small works, planning the programme and method of construction, dealing through a foreman with labour problems, ordering materials, requisitioning plant, and generally seeing the work through from beginning to end with all its difficulties at every phase. Visits to sites are not enough. Only by working in close contact with the men carrying out the work can the extent of the gap between the drawing office and the completed structure be appreciated. The most brilliant computor of stresses is liable to fail if in preparing his drawings he does not continually keep in mind, as a result of outside experience, the practical and human problems of construction, with the foreman struggling to elucidate inadequate blueprints after exhausting days coping with mud, rain, a truculent gang, and complaints about delays from his employer.

After a variety of site experience the young engineer should work in a design office, preferably where he is responsible for producing tenders and competitive designs for different reinforced concrete structures and subsequently preparing the working drawings. Later he may obtain a post in a Government department, with a local authority, or with an industrial concern. In this capacity he will learn to employ contractors and consultants and to organise design and construction on a broader basis. His previous experience will enable him to understand the point of view of the contractors whom he employs, and to find little difficulty in embodying in the final design of a structure not only his own ideas as expressed in his preliminary drawings but also improvements suggested by the contractor to facilitate construction or to improve the design. If he obtains a good position with a firm of consulting engineers he will be well advised to employ specialists, or to form a specialist department in his own organisation, to assist him in producing the working drawings. However experienced and brilliant an engineer may be, he will always be wise to incorporate in his designs the best ideas gathered from as wide an experience as possible; in fact, it is probably true to say that no engineering project of any importance has ever been carried out successfully in recent years except as a result of the collaboration of a number of specialists led by a competent engineer with understanding and imagination.

There may be a tendency in some specialist firms to design everything in reinforced concrete whereas sometimes it would be better to use other materials. Engineers should choose the best materials from every point of view. For example, in a reinforced concrete wharf it may be better to use sheet-steel piles than concrete, and it is sometimes better to use brick walls with a reinforced concrete frame, welded sheet-steel to line underground oil tanks, structural steel for bridges of large span, and alloys for aeroplane hangars.

Æsthetic Principles.

The designer should not only produce a structure which provides sufficient strength at minimum cost and which efficiently serves its purpose; the structure should also be attractive in appearance. In some cases the engineer will work under the direction of an architect, with whom he will collaborate in reconciling the ideas of the architect with economical construction. In other cases the engineer will be entirely responsible but may think it advisable to consult an architect in connection with the elevation.

Reinforced concrete is still a new material of construction, and architects are by no means agreed on the æsthetic principles which should govern its design. Some architects stress the importance of efficiency of purpose and boldly reveal in their designs the structural system in order to emphasise the fact that concrete is the principal material in the structure; this may be so not only in the case of industrial structures but also in commercial, monumental, and domestic buildings. More conservative architects preserve a semblance of traditional style and employ reinforced concrete as a structural frame behind a façade of masonry or brickwork.

The architecture of reinforced concrete is still in an early stage of development, and widely different views on the æsthetics of design will continue to be held until more experience has been gained and improvements made in regard to such matters as surface finishes and joints between external precast wall slabs.

Salisbury Cathedral (*Fig.* 1) has been selected as the first illustration in this book to show the grandeur that can be achieved by designing according to tradition, and building with a pride in craftsmanship. It is suggested that structures quite unlike mediæval cathedrals, but perhaps equally inspiring, may be built in reinforced concrete if its use is guided by technical knowledge and a pride in good workmanship. A reinforced concrete church in the United States is shown in *Fig.* 2.

Æsthetic quality in a structure depends on many factors, such as the form and structural system, the efficiency with which it fulfils its purpose, the pattern of light and shade formed by openings and projections, the position of the joints, the texture and hue of the surface, and the skill and artistry with which special features or ornaments are used. The degree of emphasis to apply to any of these factors in any particular case is a matter of taste. An attractive result obtained by the arrangement of joints and by contrasting features in precast concrete construction is shown in *Fig.* 3, which illustrates part of Kelly's College, Tavistock. There is seldom general agreement about the æsthetic quality of any structure, but few people disagree about the ugliness of most structures which are produced for purely utilitarian purposes. It is therefore important that the engineer, who is mainly concerned with utility, should be aware of his responsibility in this matter and should have a pride in the appearance of his work. He should seek not only to embody efficiency in his design but also refinement and, generally, simplicity.

The engineer has an important part to play in the architecture of reinforced concrete, and even if he has no artistic talent his skill in design can be greatly increased by studying structures of æsthetic quality and attempting to determine the essential factors upon which that quality depends. This should be done not only in regard to concrete but also other materials such as brickwork and masonry in traditional architecture, particularly the simpler styles of the eighteenth century. By a study of this kind certain principles of design can be discovered which can be applied to concrete, for example, suitable shapes, sizes, and arrangements of window openings. There should, of course, be no attempt to reproduce in concrete details only suitable in masonry or brickwork. Concrete details should suit the method of construction, but a study of the features, lines, proportions, and details of good traditional architecture can suggest adaptations and developments for concrete, and contribute much towards the success of concrete as a complete material of construction for buildings, and not just for the hidden structural frame. Architectural considerations apply to all structures, including factories, grandstands, bridges, silos, and other industrial or utilitarian structures, and not only to churches, colleges, and monumental buildings built in the past in traditional styles. The reinforced concrete grain silos shown in *Fig.* 4, for example, might have been a mere battery of bins had not thought been given to their form by emphasising the height of the bins by vertical lines and arranging the tower as a pleasing feature of the front elevation.

It is impossible to give definite rules which consistently apply in the production of good æsthetic design. In some structures, such as bridges, the structural system is the dominating factor; in others, such as blocks of residential flats, the windows and the finish of the external walls are the most prominent features. The emphasis in some structures is therefore on the purpose of the structure, in others on the structural system, and in others on the façade which may obscure the structural system but have the important object of providing protection against the effects of weather.

The purpose of a structure determines very largely the degree to which it is possible to produce a design of æsthetic appeal. A cathedral, or a large community building such as a college, is an ideal subject, but in most industrial and engineering structures the degree of efficiency in satisfying functional requirements very largely determines such æsthetic quality as can be achieved. This is exemplified in the Boulder Dam in the United States (*Fig.* 5). The reverse is, of course, obviously true. Few structures are more depressingly ugly than a badly planned group of houses, factory, or railway station. Attempts to beautify by superimposing architectural style or features quite unsuited to the function of the structure are equally, if not more, objectionable. Concrete structures should be designed to allow repeated use of the shuttering. Many examples of simple orderly architecture have resulted from the application of this principle.

In industrial and engineering structures such as factories, bunkers, tanks, and gantries, all that the engineer can do from an æsthetic standpoint is to design the structure so that it can fulfil its purpose in the most efficient manner, using a structural system which is simple and economical to construct and not wasteful in material. In the "daylight" factory at Stockholm shown in *Fig.* 6 the main staircase is used as a feature to break the regularity of the long rows of windows. Another reinforced concrete factory is shown in *Fig.* 7. Simplicity and neatness are the outstanding qualities of the reinforced concrete factory at Welwyn Garden City (*Fig.* 8), and the simple design and the continuity and regularity of the main structural members in the office building at The Hague (*Fig.* 9) are well suited to construction in reinforced concrete. Well-consolidated concrete of good quality should be used and placed so as to avoid ugly joints and honeycombing. The shuttering should be well made and liquid-tight, and constructed so that the joints form a tidy pattern on the concrete surface. The exposed faces of the concrete should not be subject to any treatment liable to peeling or cracking. In factories emphasis on lighting, regularity of window openings and columns, the use of thin vaulted roofs and flat slabs where suitable can contribute to æsthetic quality. Occasionally the provision of a purely decorative treatment may be justified, for example fluting on a tall chimney. Some structures designed with cheapness as the main consideration could have been much improved in appearance by the provision of a little more material.

Concrete as a material for industrial purposes has special scope. It can be used for many of the structures for which structural steel is suitable, but it can be shaped more readily than steel into the structural form required to support the load, for example the tapered cantilever. The use of reinforced concrete avoids the unsightliness of the standard connections of structural steel. The reinforcement is protected by a fire- and corrosion-resisting material.

Concrete panel walls retaining solids or liquids can be conveniently combined with the main structural frame, and this gives scope for design in bunkers, retaining walls, tanks, and water towers. Due to its high resistance to bending reinforced concrete can approach steel in slenderness, and masonry in finish and durability.

In the case of a bridge some architects and engineers prefer a structure which provides the best solution from a purely structural standpoint, while others prefer decorative balustrading and piers or, particularly where there is a risk of discoloration, a facing of stone on the spandrels. In some cases attractive finishes of piers and spandrels have been obtained by leaving untouched a regular board-mark pattern from the shuttering, in others bush-hammering or scrubbing has been employed to expose a selected aggregate. *Fig.* 11 shows a bridge with exposed aggregate spandrels, fluted abutments, and deep-jointed wing walls. Waterloo bridge, London (*Fig.* 10), recently rebuilt in reinforced concrete, is a very fine structure having as main members continuous cantilevered beams deepened over the supports and with artificially widened piers; the concrete is faced with stone. The long span of Fürstenland bridge, Switzerland (*Fig.* 14), depends for its slenderness on the use of high-quality concrete, and its form is determined almost entirely by structural requirements and the conditions at the site. The same remarks apply to Sandö bridge, Stockholm (*Fig.* 12), which has a clear span of 866 ft. and for which concrete with a minimum crushing strength of 7000 lb. per square inch was specified; the tall and slender unbraced columns supporting the approach of this bridge (*Fig.* 13) are ideal structurally as supports for a viaduct and are a testimony to the high quality of the workmanship. Another Swedish bridge is shown in *Fig.* 15. The bridge shown in *Fig.* 16 is a simple functional design in which the arch follows strictly a polygonal line of thrust, and the marks of the shutter boards provide a pattern on the concrete without extra cost. *Fig.* 17 shows a prestressed concrete bridge in France. It has a span of 180 ft. and a rise of only 9 ft. 6 in.

Residential flats, office buildings, and houses built in reinforced concrete provide a more difficult problem because the elevations need not be related to the form of the structural members. A study of traditional architecture shows that plain façades, which are most suitable for concrete construction, depend for their æsthetic quality on their silhouette, on the pattern formed by the openings and by the window frames, on features such as occasional bay windows, on the pattern formed by the joints in the brickwork or masonry, and on the hue and texture of the bricks or stones. Modern reinforced concrete flats are shown in *Fig.* 18, and indicate the type of structure that is beginning to replace the slums built in the last century.

In concrete it is generally easy to provide a pleasing silhouette and pattern of openings, together with a special feature, say, at a main entrance. In towns where brickwork and masonry soon become begrimed it is necessary to provide a surface which will weather satisfactorily by becoming fairly uniformly dirty. Surfaces which are very smooth or not uniformly slightly rough tend to show dirt in streaks and patches. It is therefore advisable to use precast slabs with a fine aggregate which is slightly exposed by brushing the outside face while it is still green, thereby obtaining a uniformly slightly rough surface. Large aggregate tends to give the appearance of pebble-dash. A variety of detailed pattern and texture can be provided by the use of aggregates and cement of various hues ; in any particular case samples should be prepared from which a selection can be made. The shape and size of the slab and the detail of the joints and pointing are important factors to consider. In brickwork, small rustic bricks and special pointing often make an attractive wall whereas the same wall would be less pleasing if built in common brick. The detailed pattern of a concrete wall is equally important. In cast-in-situ work horizontal grooves or projections are often used to form a pattern and to obscure construction joints. Bad finishes showing ugly joint marks, patched-up honeycombing, and dirty streaks and patches due to non-uniformity of surface texture, have been the cause of some of the chief objections to the use of concrete in buildings. Photographs of structures often show attractive patterns in black and white, which in reality are dilapidated and dirty in appearance. However excellent the general pattern of the elevation may be it does not excuse or make up for a bad surface finish. It is not impossible to provide a concrete surface which remains clean in rural or seaside districts and which becomes, like masonry, fairly uniformly begrimed in a smoky atmosphere. It has been demonstrated that cracking and discoloration of rendering can be eliminated by the use of selected fine aggregate, careful grading, low water-cement ratio, and low cement content.

Suitability of style in relation to environment is an important consideration. A " slick " style may be a welcome addition at a popular seaside resort but an offensive intrusion in a cathedral city or a city predominatingly built of stone in the eighteenth century. In a big project the use of large precast concrete slabs or other elements having a good finish may produce an appearance comparable with the best traditional styles. Regularity, skilfully interrupted here and there, is one of the characteristics of good architecture. There is no reason why large buildings or groups of houses should not be constructed with large precast units which, with suitable finishes, joints, and modelling will not look out of place among existing traditional buildings, and may give a new residential or business centre an architectural character of its own. *Fig.* 19 is a view of

Fig. 1.—Salisbury Cathedral.

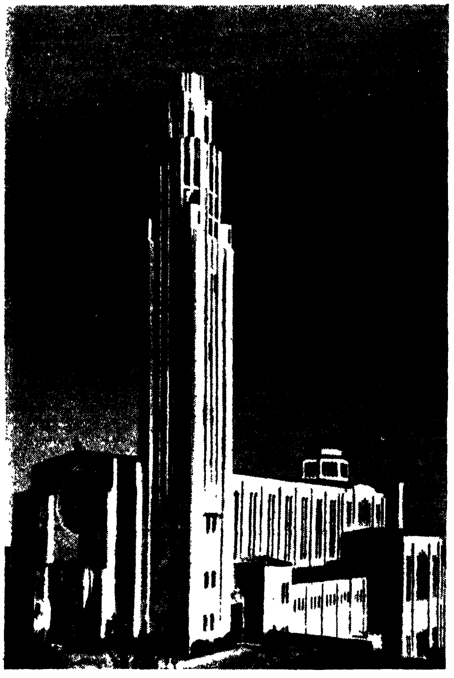


FIG. 2.—ST. JOSEPH'S CHURCH, SEATTLE, WASHINGTON, U.S.A.

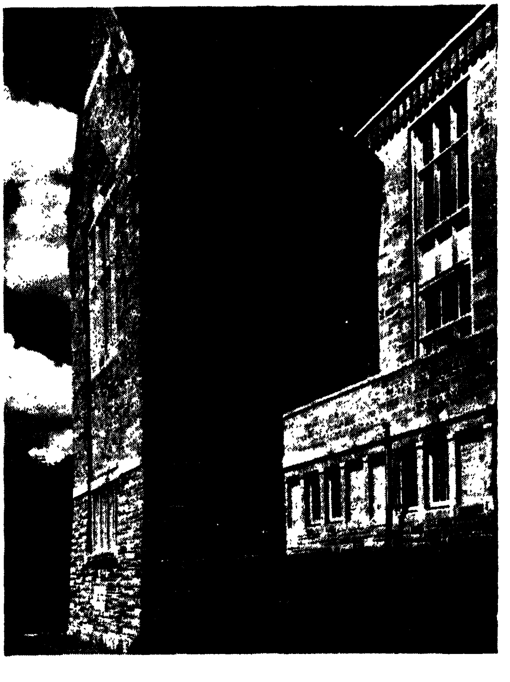


FIG. 3.—KELLY'S COLLEGE, TAVISTOCK.

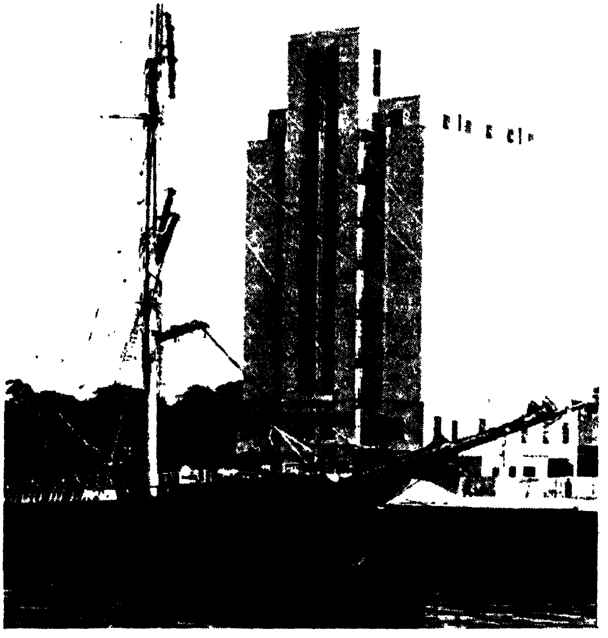


FIG. 4.—GRAIN SILOS AT CORK, EIRE.

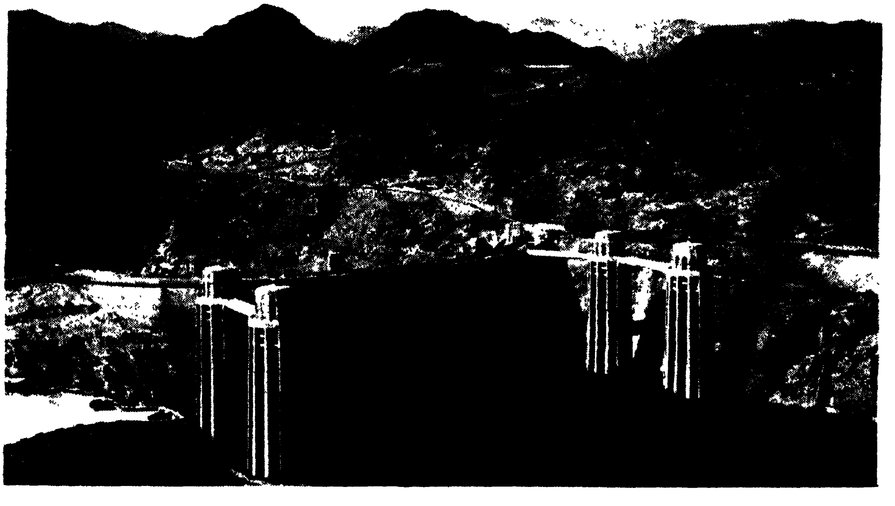


FIG. 5.—BOULDER DAM, UNITED STATES.

FIG. 6.—FACTORY AT STOCKHOLM, SWEDEN.

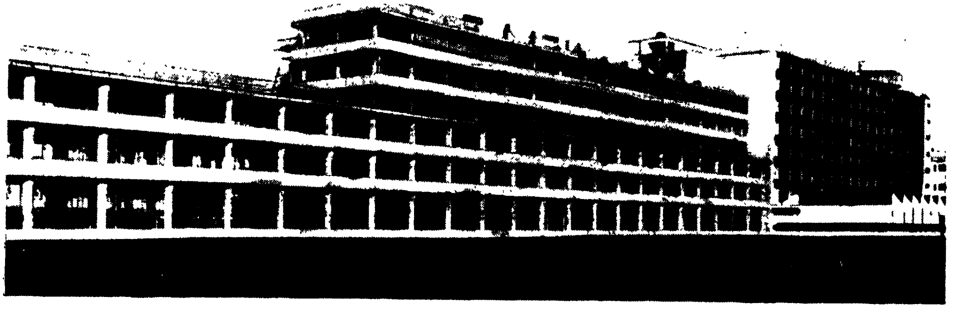


FIG. 7.—FACTORY AT ROTTERDAM, HOLLAND.

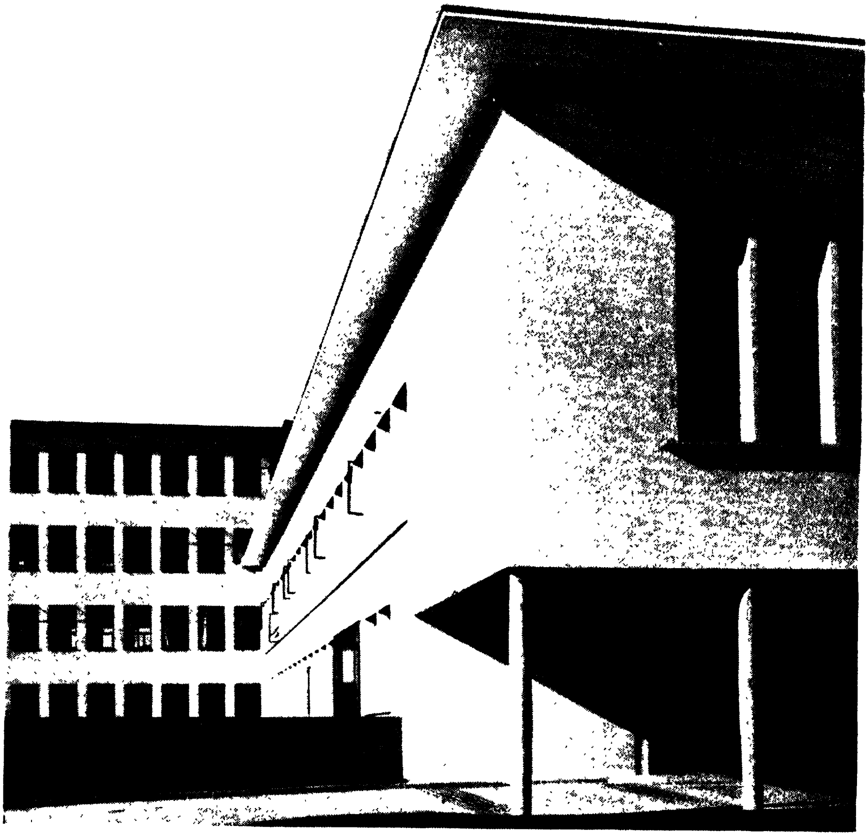


FIG. 8.—FACTORY AT WELWYN GARDEN CITY.

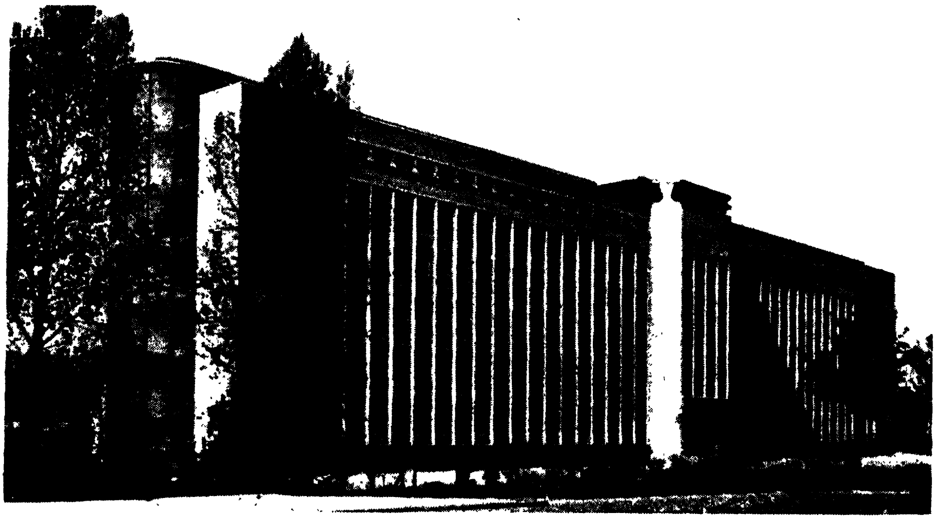


FIG. 9.—OFFICE BUILDING, THE HAGUE, HOLLAND.

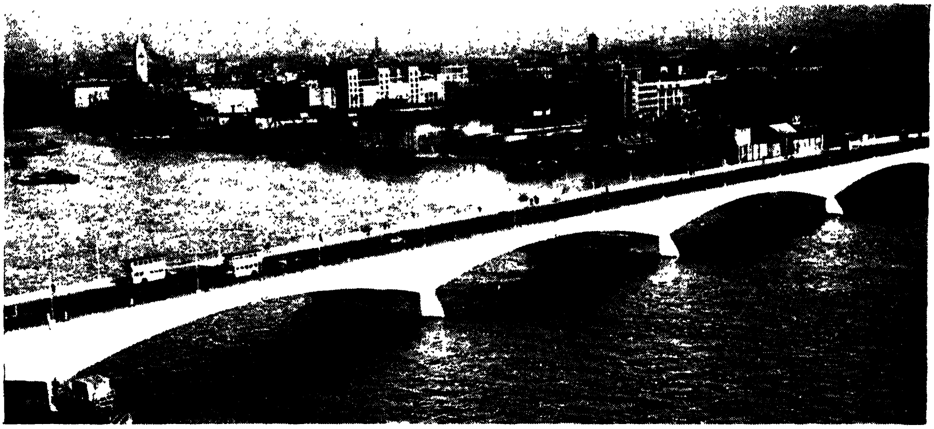


Fig. 10.—Waterloo Bridge, London.

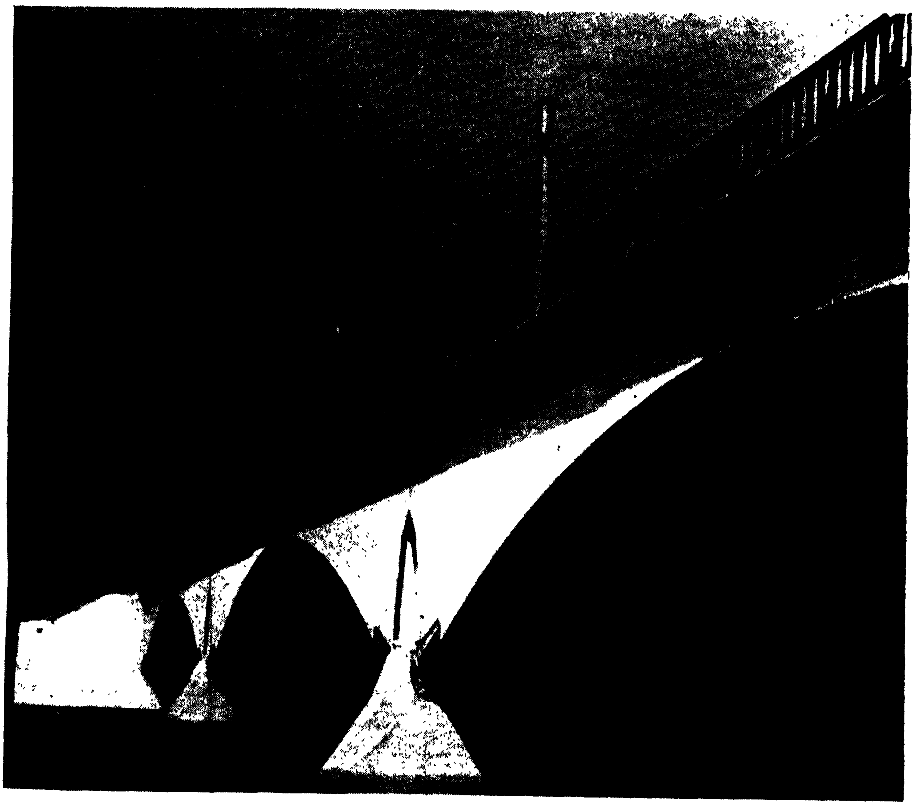


Fig. 11.—Twickenham Bridge, London.

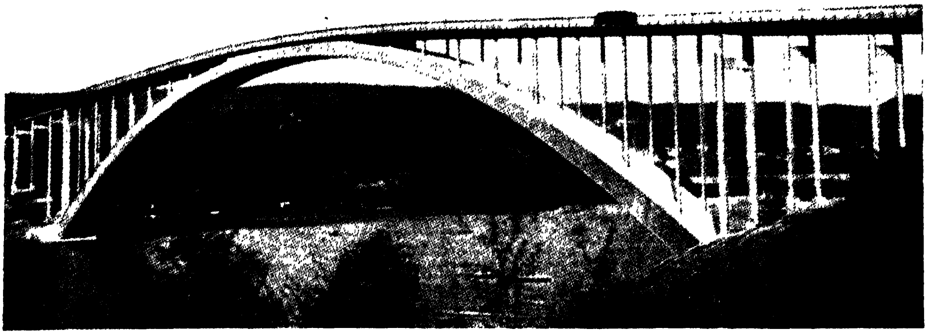


FIG. 12.—SANDÖ BRIDGE, SWEDEN.

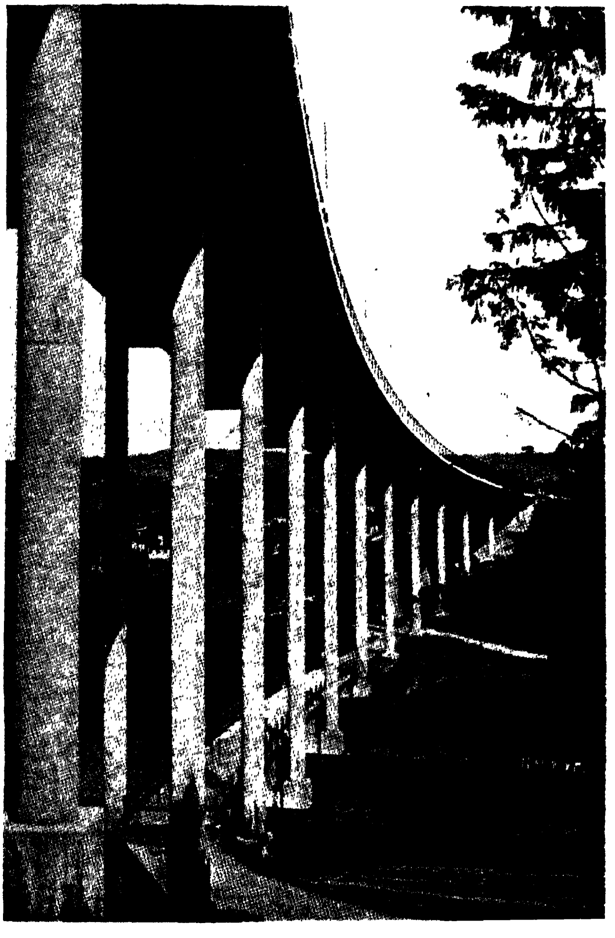


FIG. 13.—AN APPROACH TO SANDÖ BRIDGE.

FIG. 14.—FÜRSTENLAND BRIDGE, SWITZERLAND.

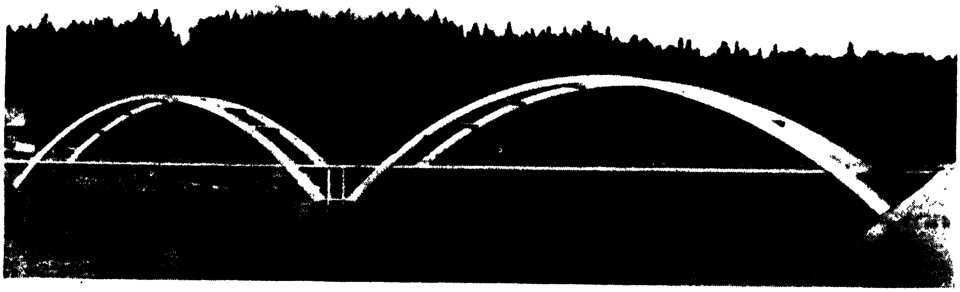


FIG. 15.—A BRIDGE IN SWEDEN.

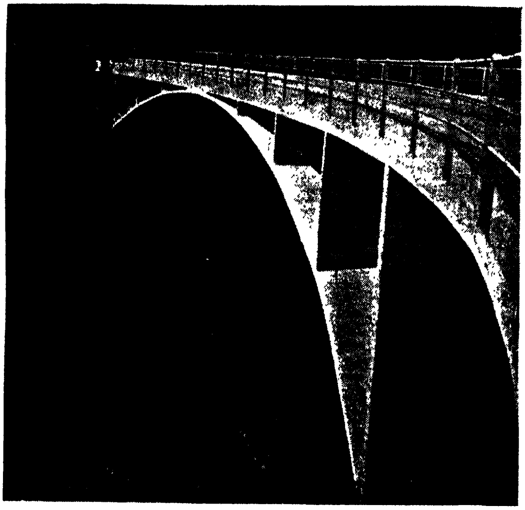


FIG. 16.—A BRIDGE IN SWITZERLAND.

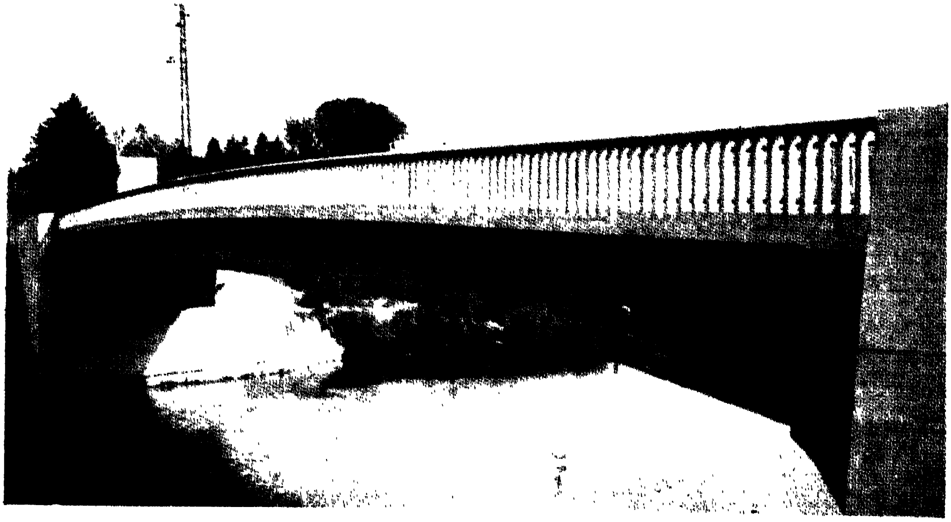


FIG. 17.—PRESTRESSED CONCRETE BRIDGE OVER THE RIVER MARNE AT LUZANCY.

FIG. 18.—FLATS AT WALTHAMSTOW, LONDON.

Fig. 19.—A Terrace at Bath.

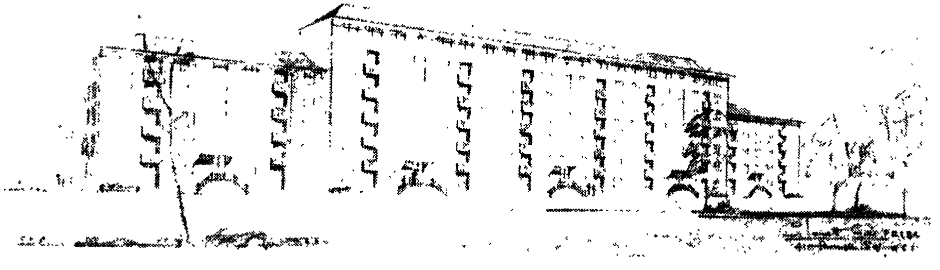


Fig. 20.—Design for Block of Flats to be Built with Large Precast Members.

FIG. 21.—HOUSES AT WELWYN GARDEN CI

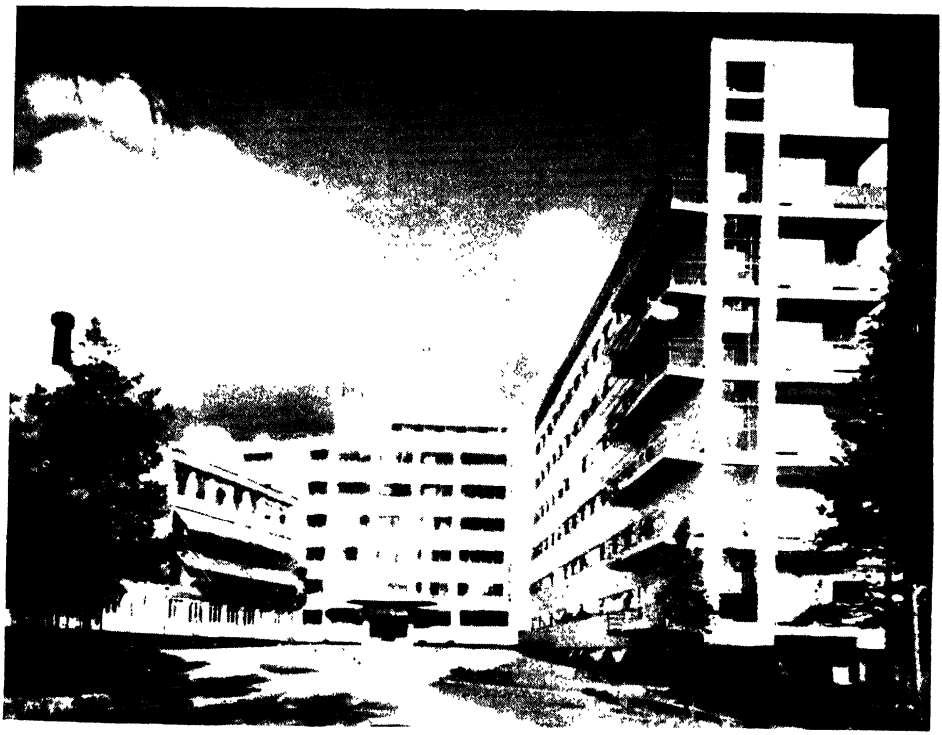


FIG. 22.—SANATORIUM IN FINLAND.

FIG. 23.—A PRESTRESSED CONCRETE WATER TANK IN THE UNITED STATES.

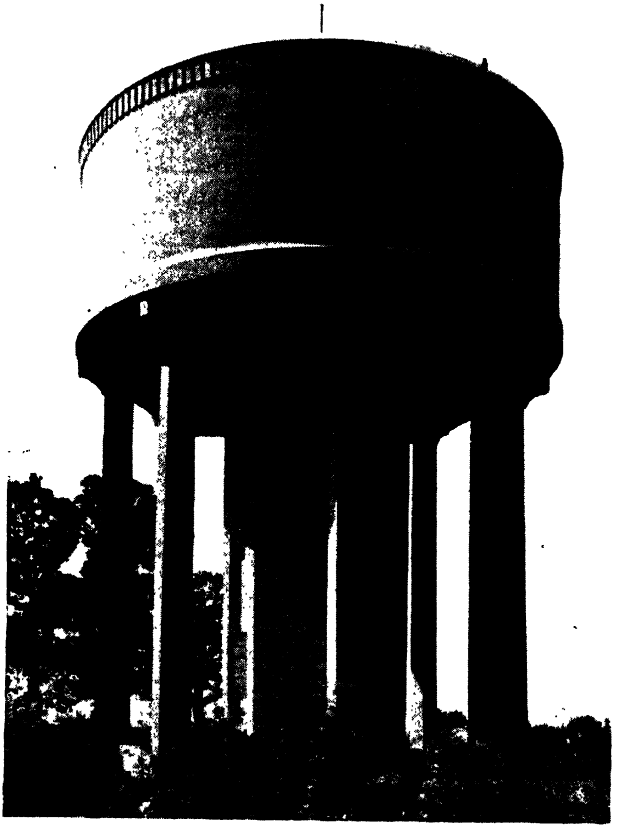


FIG. 24.—WATER TOWER IN ESSEX.

an eighteenth-century terrace at Bath, in which the design comprises a repetition of large units; the same principle could be followed with large precast concrete units erected by a crane. *Fig.* 20 is a design for a block of flats to be built of large precast concrete wall and floor slabs, while *Fig.* 21 shows a group of houses with cast-in-situ concrete walls. To preserve or create local character is important. Much late nineteenth-century and early twentieth-century building is lacking in this respect. The use of local aggregates should ensure some harmony with environment.

A reinforced concrete sanatorium in Finland is shown in *Fig.* 22. In modern architecture multi-story buildings tend to have similar elevations, regardless of their particular function. This simplicity results from the economical use of materials, and by allocating more of the cost to the provision of good amenities and less to external ornament.

The early designs in concrete domestic architecture were a healthy, but too extreme, reaction from the speculative builders' attempts to provide a house with a showy imitative exterior at a low cost. Few people really enjoy possessing as a home an efficiently designed concrete box just because it is what is known as "modern functional architecture" and photographs well. Efficiency is very appropriately emphasised in the design of an office building or a factory, but a home is built to contain the human jumble of family life. It is therefore appropriate for the exterior to symbolise a friendly atmosphere in which adults can relax and children play; it should accord with our conception of a home. The popular type of house has aimed at this by providing an assortment of gables, vertical tiling, rustic brick, and half-timber construction. Efficient houses which also possess charm can be produced in concrete.

Departure from tradition is generally not popular, and sometimes popular resentment is justified, but if the departure is based on sound architectural principles new styles are gradually accepted and liked. The buildings in a new colonial town often at first appear to a visitor to be ugly, but after a long stay they can appear more attractive, and by contrast some traditional styles are found to be irritating.

To summarise, it is impossible to reduce the production of good æsthetic designs to a systematic procedure. Wadsworth, referring to the work of the engineer, says "Beauty is a refinement on function which induces visual comfort or stimulates visual delight or visual excitement." It is not always realised that long experience has evolved "refinements on function" in attractive brick façades. The main pattern of the façades of a concrete domestic building should be primarily based on function and structural requirements but, as in the case of brickwork, the introduction of refinements in the detailed pattern may change a very ugly wall into a most attractive one. It should not be forgotten, however, that simplicity may be attractive in itself. There is much truth in Bossiney's outburst to Soames in Galsworthy's "Forsyte Saga," which might well be applied to concrete structures: "In architecture as in life you will get no self-respect without regularity. Fellows tell you that it's old fashioned. It appears to be peculiar anyway, it never occurs to us to embody the main principle of life in our buildings. We load our houses with decoration, gim cracks, corners, anything to distract the eye. On the contrary the eye should rest; get your effects with a few strong lines. The whole thing is regularity—there is no self-respect without it."

To the importance of regularity in architecture must be added the use of a structural system which is rational and economical, and which satisfies the requirements of the structure. Employ simple continuous members; emphasise the tension of slender members and the compression of massive supports; let openings in shadow, the joints, and the pattern of the façade be simple and pleasing. Sketch many possibilities and gradually evolve the design that is the best. Provide a finish which weathers well. See that the structure suits the environment or, if it dominates the landscape, that it is worthy of its site. In small engineering structures have the members in good proportion and approach perfection of purpose as nearly as possible. Collaborate with an architect, who should be both an artist and a structural engineer in his approach to the problem.

The most attractive designs are sometimes the cheapest. When the form of a structure is arranged with the help of calculations of stress to reduce waste of materials, its æsthetic quality is often very nearly assured. In a purely utilitarian structure there is, however, often something lacking in its appeal. Architects speak of "the feel" of a structure, and if possible an engineer should obtain the assistance of an architect who understands reinforced concrete and who, by some slight alterations to the form of the structure, may change it from a mediocre object into one of character and charm. For example, an artificial device such as a rise in the soffit of the girders of a bridge will improve its appearance; an odd number of spans in a bridge is generally more attractive than an even number; the outside columns of a warehouse may be used to obtain an effect of vertical lines; and better proportions and arrangements of openings will often improve the appearance of a building without extra cost.

The finish of the surface is important. Due to neglect of this matter, concrete as a building material has quite unnecessarily acquired a bad name. In an industrial structure the employment of shuttering marks to form a pleasing pattern is often all that is needed. In the case of a bridge or a building it is sometimes better to employ a bush-hammered face or provide precast slabs which may mellow with time.

Water towers can spoil a landscape, and care should be taken to ensure that a light and high tower is elegant in appearance, or that a low tower of large capacity has supports that are well shaped and proportioned. The prestressed concrete tank shown in *Fig.* 23 is a simple functional design and is a pleasing and utilitarian structure. The water tower illustrated in *Fig.* 24 has an attractive appearance obtained at little extra cost by care in the proportions of the tank and the columns and by a little decoration.

The importance of æsthetic considerations cannot be over-emphasised. Structures and buildings reflect the spirit of the times. The industrial revolution left an ugly mark, not so much because materials of construction were limited or that planning could not keep pace with expansion, but because the attitude of mind of the industrialists was wrong. Even to-day the old saying "Where there's muck there's money" is still quoted with a curious shameless admiration. The æsthetes of those days deplored the products of machines and sought to revert to handicrafts, instead of striving for the better planning and design of industrial expansion which mechanical power made possible. Thus factories just spread and became a chaos of untidy sheds mixed indiscriminately with rows of drab terraces in which the workers lived.

In contrast to industrial structures, architects were given considerable scope with regard to public buildings, but again the mental outlook was wrong. Consequently the outstanding feature of their works was solidity at the expense of amenity. Such structures were built at enormous cost and remain to-day as monuments of their period, sepulchral within and heavily ornate without. In mediæval times great cathedrals were built "to the glory of God," a purpose which called for the highest skill from architect and craftsman in order to satisfy the requirements of the ecclesiastical owner. To state the problem in present-day language, the purpose of these great structures was to house large assemblies, to inspire and symbolise the highest principles, and to endure as long as the spiritual needs of mankind. Stone was used as the material of construction, with arches and vaulting supported by buttresses and piers, as structural systems for large spans on account of the weakness of mortar joints in tension. The craftsman fashioned the bare structural system with all kinds of tooled stone finishes, joints, and groining, thereby producing the fine architecture. To-day mechanical power makes it possible to extend structural permanency to most common uses. There is, however, a danger that this may result in producing ugly buildings such as were typical of the period of the "industrial revolution," unless the development is controlled by town-planners, architects, and engineers with a zeal for good design and workmanship.

CHAPTER II

THEORY OF STATICALLY-INDETERMINATE STRUCTURES

I.—FUNDAMENTAL PRINCIPLES OF ELASTIC DEFORMATION

It is important for a young engineer to learn a form of fundamental theory which he finds the simplest to understand and remember. When he starts to design he will then be able to pay more attention to the arrangement of the structure as a whole, the general problem of construction, and other factors affecting the design. Some engineers later in their careers, to the disadvantage of engineering progress, seem to lose their grasp of the theory of design, their mental energy becoming completely absorbed by administrative and commercial tasks. They may then be unwilling to approve of the construction of unfamiliar types of structures partly because they are dependent on calculations, prepared by subordinates, which they do not fully understand. Most engineers find that methods of calculation which have been impressed on them with the help of diagrams and visual conceptions remain clear in their minds long after the meaning of the corresponding mathematical expressions have been forgotten. For example, it may be easier to remember that the slope at the support of a member of a framework is the reaction at the support due to a load on the member equal to the area of the bending-moment diagram divided by EI than as the evaluation of $\frac{1}{l}\int\frac{M(l-x)dx}{EI}$. The visual conception also keeps in mind a picture of the relation between the distribution of the bending moments and moments of inertia and deformation, which is helpful in the preparation of a design. In cases in which it is advantageous to use the mathematical expression of integration, because the moment of inertia I varies as a function of x or for other reasons, it is easy to use the calculus, but most of an engineer's work deals with cases to which the familiar terms of the simple theory of bending can be applied, thereby saving much mental energy. In some cases it is necessary to integrate by measuring an area. The graphical conception of integration is therefore really fundamental, and the mathematical form a convenient means of expression which it is sometimes an advantage to use.

Definitions.

In the following some terms frequently used are defined. The notation, which in most cases is shown on the accompanying diagrams, is as follows:

E : Modulus of elasticity of the material of which the structural member is made.
I (or I_x, I_1, I_2, etc.) : Moment of inertia of the member.
M (or M_x, M_A, M_B, etc.) : Bending moment.
g_A, g'_A : Distance of centroids of $\frac{M_x}{EI_x}$ and M_x diagrams from support A.
l (or l_1, l_2, l_r, etc.) : Span of member.
K_r (or K_1, K_2, etc.) : Coefficient of restraint.
u (or u_1, u_2, etc.) : Coefficient of stiffness.

ELASTIC WEIGHT.—Referring to *Fig.* 25, the elastic weight is a convenient term denoting the area of the diagram obtained by dividing each ordinate of the bending-moment diagram by the corresponding value of EI. The total

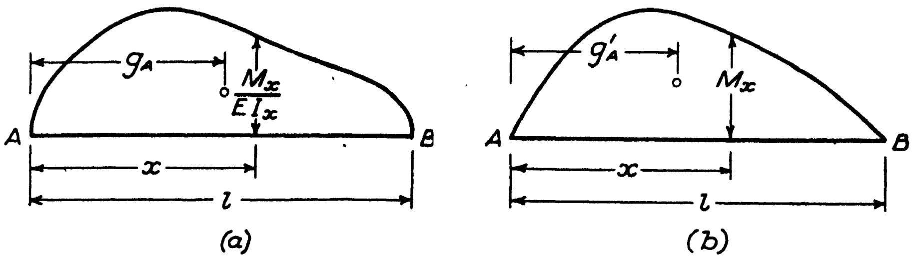


FIG. 25.

elastic weight acting on a structural member is $\int_0^l \frac{Mdx}{EI}$. If I is constant the total elastic weight is $\frac{1}{EI}\int_0^l Mdx$, that is the area of the bending-moment diagram divided by EI. The reaction at the support A due to the elastic weight is the total elastic weight $\times (l - g_A) \div l$. When I is constant the reaction at A, *Fig.* 25 (*b*), is the area of the bending-moment diagram multiplied by $\frac{l - g'_A}{EI.l}$.

It is shown later that the slope of a member at A is the support reaction of the elastic weight at A, and the deflection at any point of a member is the bending moment at the point due to the elastic weight acting as a distributed load. The deflection is a maximum at the point where there is no shearing force when the member is assumed to be loaded with the elastic weight.

COEFFICIENT OF STIFFNESS.—Referring to *Fig.* 26, the coefficient of stiffness

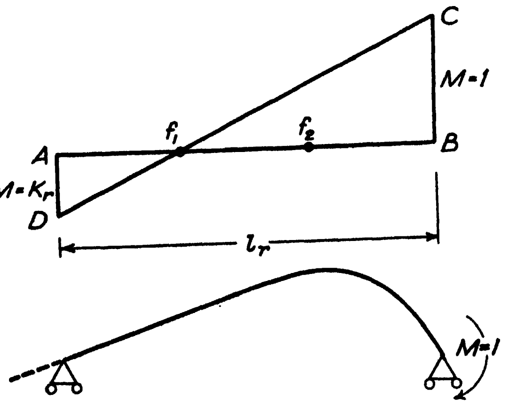


FIG. 26.

u of a member acted on by unit bending moment at a free support B is the reciprocal of the change of slope which takes place at B. If K_r is the coefficient of restraint, as explained later, of support A in relation to support B, the change of slope at B, as is also seen later, for unit bending moment acting at B is $\frac{1}{u}$, which equals $\frac{l_r}{6EI}\cdot\frac{2M_B - M_A}{M_B}$. Substituting $M_A = K_rM_B$, $\frac{1}{u} = \frac{l_r}{6EI}(2 - K_r)$. Therefore the coefficient of stiffness is

$$u = \frac{6EI}{l_r(2 - K_r)}.$$

When support A is fixed, $K_r = 0{\cdot}5$; therefore $u = \frac{4EI}{l_r}$.

When support A is free, $K_r = 0$; therefore $u = \frac{3EI}{l_r}$.

Calculations are usually only concerned with the relative stiffness of members. Thus

$$\frac{u_1}{u_2} = \frac{I_1l_2(2 - K_2)}{I_2l_1(2 - K_1)}.$$

When the end conditions are the same,

$$\frac{u_1}{u_2} = \frac{I_1l_2}{I_2l_1}.$$

COEFFICIENT OF RESTRAINT.—The coefficient of restraint at support A (*Fig.* 26) with regard to any unloaded span (AB) of a beam continuous over several supports is K_r when the bending moment at support A is K_rM and at B is M; the bending moment M is applied at B either externally, if B is an end support, or otherwise from a span to the right of B. If support A is free, $K_r = 0$. If support A is fixed, $K_r = 0{\cdot}5$. Values of K_r are always between 0 and 0·5.

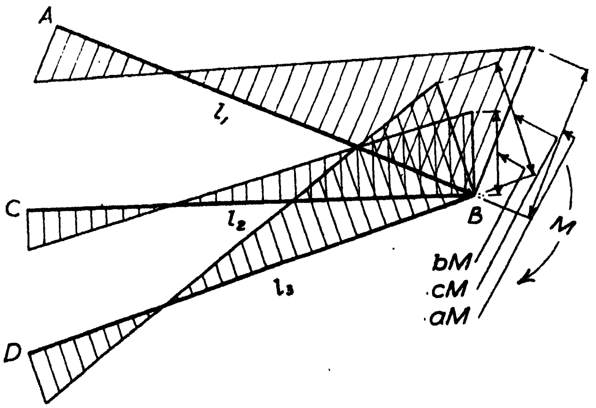


FIG. 27.

COEFFICIENT OF MOMENT DISTRIBUTION.—Referring to *Fig.* 27, if a moment M is applied at B to three rigidly-connected members BA, BC, and BD of a frame, the total moment M is divided between the three members as follows: $m_1 = aM$ acting on BA; $m_2 = bM$ acting on BC; $m_3 = cM$ acting on BD;

and $m_1 + m_2 + m_3 = M$. The terms a, b, and c are moment-distribution factors:

$$a = \frac{u_1}{u_1 + u_2 + u_3}; \quad b = \frac{u_2}{u_1 + u_2 + u_3}; \quad \text{and } c = \frac{u_3}{u_1 + u_2 + u_3},$$

where u_1, u_2, and u_3 are the coefficients of stiffness of BA, BC, and BD respectively. Thus M is divided among the three members in proportion to their stiffness. When A, C, and D are either all fixed supports or all free supports and I is constant in each member, u_1, u_2, and u_3 can be replaced by $\frac{I_1}{l_1}$, $\frac{I_2}{l_2}$, and $\frac{I_3}{l_3}$ respectively.

RESIDUAL MOMENTS.—Referring to *Fig.* 28, AB and BC are any two consecutive spans of a continuous beam or framework supporting loads. All supports are assumed to be fixed, the bending moments at the supports being M_A,

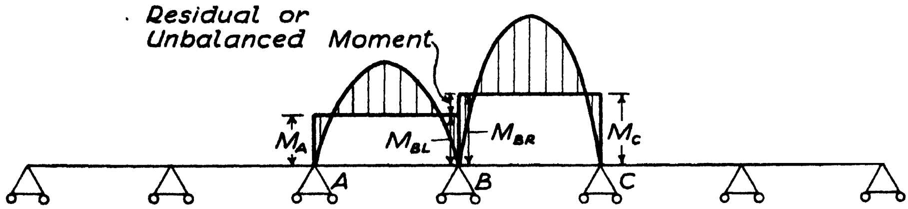


FIG. 28.

M_{BL}, M_{BR}, and M_C. Then M_A, $M_{BR} - M_{BL}$, and M_C are termed the residual moments which act on the adjacent spans in proportion to their stiffness if supports A, B, and C are assumed to be unfixed. The distribution and dispersion of residual moments are explained later.

CHARACTERISTIC POINTS.—The position of the characteristic points, which are defined later, in a bending-moment diagram depends upon the slope of the member at the supports due to the free bending moment and the variation of the moment of inertia. Bending moments at supports can be quickly determined by trial and error when the positions of the characteristic points have been plotted.

FIXED POINTS.—Referring to *Fig.* 33 (*a*), given later in this chapter, if AB is one of a series of unloaded spans of a continuous beam to which a moment is applied to the right of support B, the moment at support A always equals K_{r-1} times the moment at support B. The line DE therefore always passes through a point F_{r-1} (left) in the left-hand part of AB; this point is called a fixed point because its position is independent of the magnitude of the support moments. Similarly in the case of a moment applied to the left of A [*Fig.* 33 (*b*)] there is a fixed point F_{r-1} (right) in the right-hand part of AB at which there is no bending moment provided that the spans to the right of A are unloaded.

CONVENTION OF SIGNS.—Support bending moments are often plotted on the tensile side of a member of a frame and free bending moments on the same side in order to obtain a diagram of the resultant bending moments. If the line connecting the ordinates of the bending moments at two consecutive supports is then regarded as a base line, support bending moments are then below the base line and may be termed negative. If, as is usual in continuous beams, the resultant bending moments at mid-span are above the base line, they are positive. The important point is that opposite signs be given to bending moments represented by ordinates on opposite sides of the base line.

Elastic Deformation of a Reinforced Concrete Member.

Fig. 29 shows the elevation of the neutral-axis plane OPQ of a reinforced concrete member having no slope at $x = l$. The segment PQ of the member has an infinitely short length ds. Other symbols are: r, the radius of the curva-

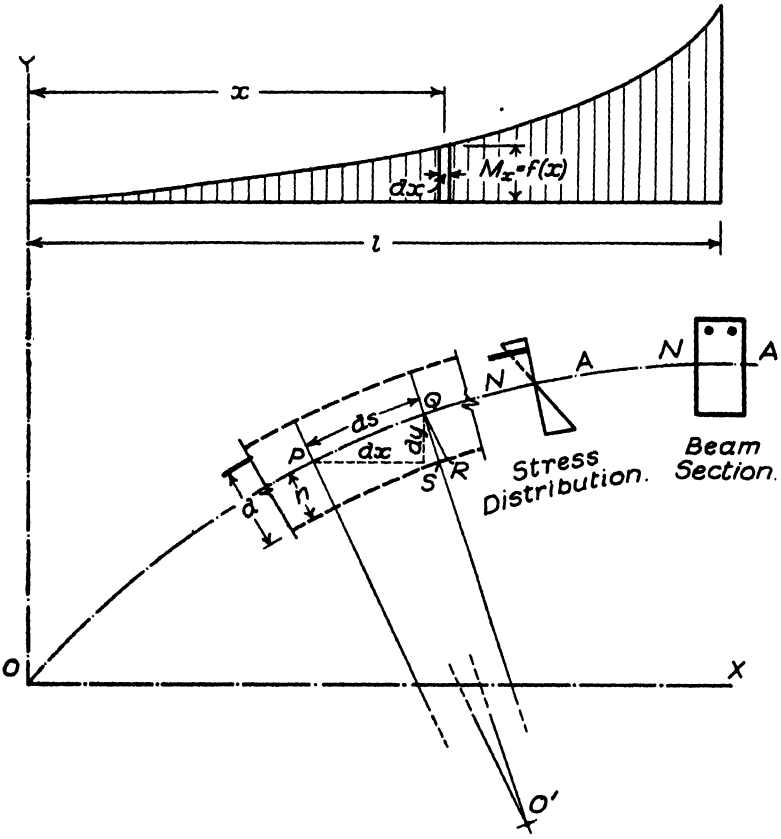


FIG. 29.

ture of the member at P; n, the depth of the neutral axis; E_c, the modulus of elasticity of concrete; c, the maximum stress in the concrete in segment PQ; and M, the applied bending moment at P (M is a function of x, hence M and the curvature increase in the direction of x).

The slope of the member at P is $\frac{dy}{dx}$. So long as the slope of the member is small dx equals ds very nearly. The rate of change of slope at P is $\frac{d^2y}{dx^2}$, which equals the degree of curvature $\frac{1}{r}$. Therefore the slope is $\frac{dy}{dx} = \int \frac{1}{r}dx$. The deflection is $y = \iint \frac{1}{r}dx.dx$.

In *Fig.* 29, QR is drawn parallel to PO′. Therefore SR is the maximum contraction of the concrete in the length dx, and is $\frac{c\,dx}{E_c}$. Since QR is parallel to PO′, $\frac{PQ}{PO'} = \frac{SR}{SQ}$. Since $\frac{PQ}{PO'} = \frac{ds}{r} = \frac{dx}{r}$ nearly, and $SQ = n$, $\frac{dx}{r} = \frac{c\,dx}{E_c n}$. Therefore $\frac{1}{r} = \frac{c}{E_c n}$. The slope at P is therefore $\int \frac{c}{E_c n}dx$. If M_r is the known moment of resistance for a particular compressive stress c', $c = \frac{Mc'}{M_r}$. Thus the slope at P is

$$\int \frac{M}{E_c}.\frac{c'}{M_r n}.dx = \int \frac{M}{E_c I_c}dx \quad . \quad . \quad . \quad . \quad . \quad (1)$$

since by definition $I_c = \frac{M_r n}{c'}$, which is a convenient term to apply to a section of a reinforced member, since the value of M_r for a particular concrete stress c' and the value of n are usually available from tables or curves.

The expression $\int \frac{M}{E_c I_c}dx$ is, of course, the same as that for a homogeneous elastic member. The value of I_c for any section of a member depends only on the dimensions of the section and the value of n which varies slightly as the amount of reinforcement varies.

In the case of prestressed beams n may be greater than the total depth D for combined stresses but is about equal to $0{\cdot}5D$ for bending stresses only. In the case of columns subject to bending n may be greater than D for combined stresses, and care must be taken in a given case to ascertain whether the value of n applies to the change of stresses due to bending only or to the resultant stresses due to bending and direct stress. An alternative approximate method of evaluating I_c is to calculate the second moment of the concrete section about its centroid. The deflection at P is $\iint \frac{M}{E_c I_c}dx.dx$, and, if I_c is constant, is

$$\int \frac{Mx\,dx}{E_c I_c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Slope and Deflection expressed in Terms of Elastic Weights.

Fig. 30 shows a beam of span l simply supported at A and B and having a constant moment of inertia I. If A_m is the area of the bending-moment diagram for the beam and g_A is the horizontal distance of the centroid from A, then $\frac{A_m}{EI}$ is the total elastic weight acting on the beam. If I varies, a diagram must be substituted in which the ordinate at a point x from B is $\frac{M_x}{I_x}$ where M_x and I_x are the values of M and I at x.

Referring to *Fig.* 30, AG is drawn tangential to the deformed beam at A. Provided that the slope is small BG is approximately the deflection of point B relative to A referred to a base line AG, the slope at A being zero relative to AG; the actual slope at A is $\frac{BG}{l}$. The deflection BG is $\int_0^l \frac{Mxdx}{EI}$ with B as origin and a line parallel to AG as the x-axis as shown in *Fig.* 30.

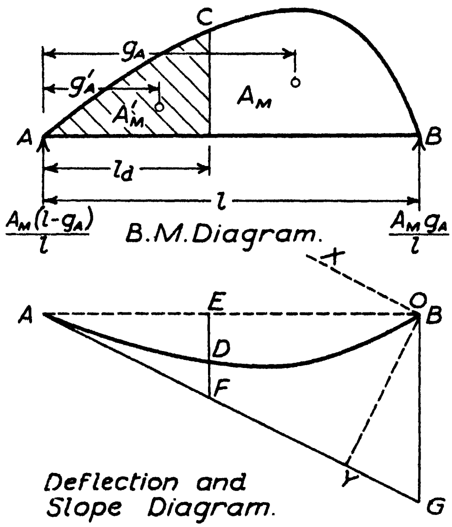


FIG. 30.

Thus BG is $\frac{A_m(l - g_A)}{EI}$, and

the actual slope at A is $$\frac{A_m(l - g_A)}{EIl} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Similarly, the slope at B is $\frac{A_m g_A}{EIl}$.

Therefore the slope at A or B can be expressed as the reaction at A or B due to the beam loaded with the elastic weight $\frac{A_m}{EI}$.

Let A'_m be the area of the bending-moment diagram between A and any point D, l_d from A and let g'_A be the horizontal distance of its centroid from A. The deflection at D is DE = FE − DF, where DF is the deflection of a beam having no slope at A and subjected to the bending moments A_m. Again with D as origin and the x-axis parallel to GA,

$$DF = \int_0^{l_d} \frac{Mxdx}{EI} = \frac{A'_m(l_d - g'_A)}{EI}.$$

By proportion $\quad FE = BG \times \frac{l_d}{l} = \frac{A_m(l - g_A)}{EI} \cdot \frac{l_d}{l}$.

The deflection at D is

$$FE - DF = \frac{A_m(l - g_A)l_d}{EIl} - \frac{A'_m(l_d - g'_A)}{EI} \quad . \quad . \quad . \quad (4)$$

Thus the deflection at D can be expressed as the bending moment at D due to the beam loaded with the elastic weight.

The relation between slope and bending moment is therefore seen to be the same as the relation between the reactions at the supports and a distributed load carried by a simply-supported beam, and the relation between deflection and bending moment to be the same as the relation between the bending moment and the distributed load. The theory of bending of simply-supported beams therefore applies. Maximum deflection and no slope occur at the point where there is no shearing force when a simply-supported beam is loaded with the elastic weight and the following simple rule, Mohr's Rule, always applies:

The slope at a support of a simply-supported beam is equal to the reaction at the support when the beam is loaded with the elastic weight, and the deflection at any point is equal to the bending moment due to the same load.

If I is constant, the total elastic weight is the area of the bending-moment diagram divided by EI. If I varies, the total elastic weight is the area of the $\frac{M_x}{EI_x}$ diagram.

Application of Mohr's Rule.

Mohr's Rule for calculating slopes and deflections forms the basis of most of the theory of statically-indeterminate structures given in the following. The notation is

- α : Slope of beam at support A.
- β : Slope of beam at support B.
- A_m : Area of bending moment diagram.
- g_A : Horizontal distance of centroid of bending-moment diagram from A.
- g_B : Horizontal distance of centroid of bending-moment diagram from B.
- l : Span of beam AB.
- I : Moment of inertia of beam (generally assumed to be constant).
- E : Modulus of elasticity.

Slopes at Supports.—Expressions for the slopes at the supports of a beam loaded in various ways are in *Fig.* 31. For any symmetrical load, $\alpha = \beta = \frac{A_m}{2EI}$ from which the expressions for a uniformly-distributed load and for a central concentrated load are obtained. If bending moments M_A and M_B are applied at supports A and B respectively, and the moments act in contrary directions, α is derived from $\left(\frac{M_A l}{2} \cdot \frac{2l}{3} + \frac{M_B l}{2} \cdot \frac{l}{3}\right)\frac{1}{lEI}$; if the moments act in the same direction, the plus sign in this expression should be minus, and in this case if $M_B = K_r M_A$, $\alpha = \frac{lM_A}{6EI}(2 - K_r)$ and $\beta = \frac{lM_A}{6EI}(1 - 2K_r)$.

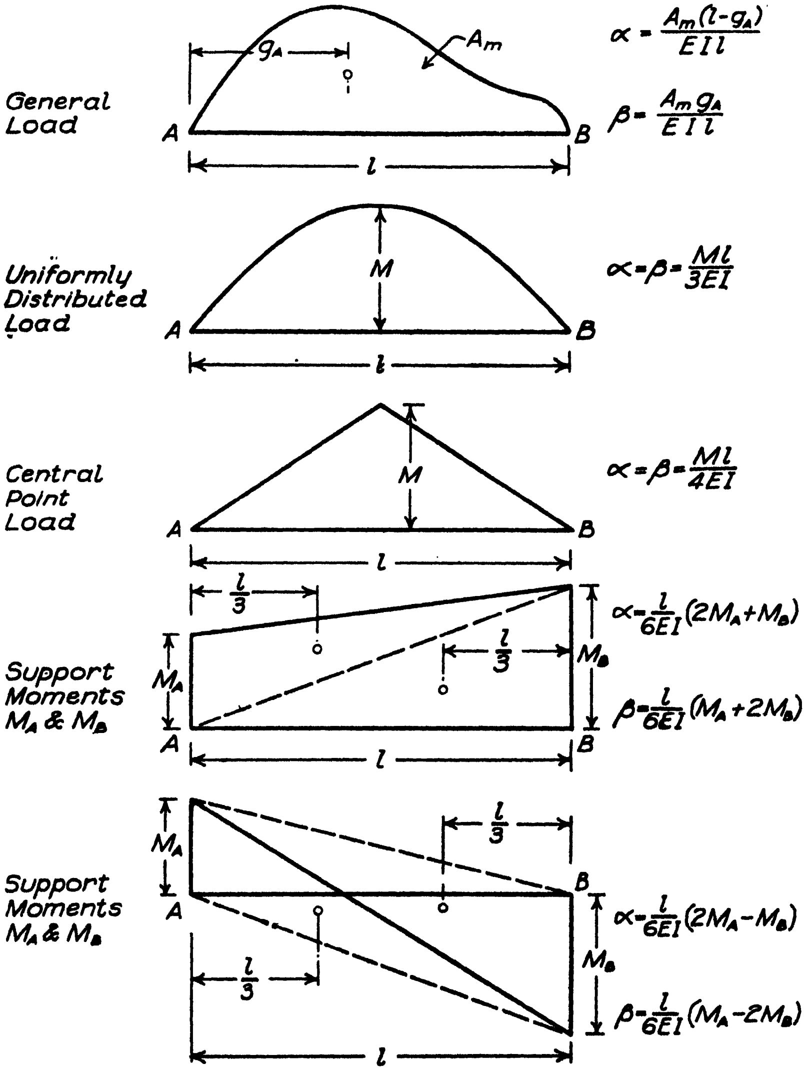


FIG. 31.—BENDING MOMENTS AND SLOPES.

Bending Moments at Fixed Supports.

If the beam AB (*Fig.* 32) is fixed at supports A and B in a direction parallel to the axis of the beam, $\alpha = \beta = 0$. Therefore the slope at A or B due to the free moments represented by the diagram A_m is equal to the slope at A or B due to support moments M_A and M_B.

(*a*) For any load, therefore, considering the slope at A,

$$A_m\frac{(l - g_A)}{EIl} = \frac{l}{6EI}(2M_A + M_B).$$

Similarly, considering the slope at B, $\frac{A_m g_A}{lEI} = \frac{l}{6EI}(M_A + 2M_B)$.

Therefore $\frac{A_m}{l}(l - 3g_A) = \frac{l}{6}(-3M_B)$, and $M_B = \frac{2A_m}{l^2}(3g_A - l)$ . . (5)

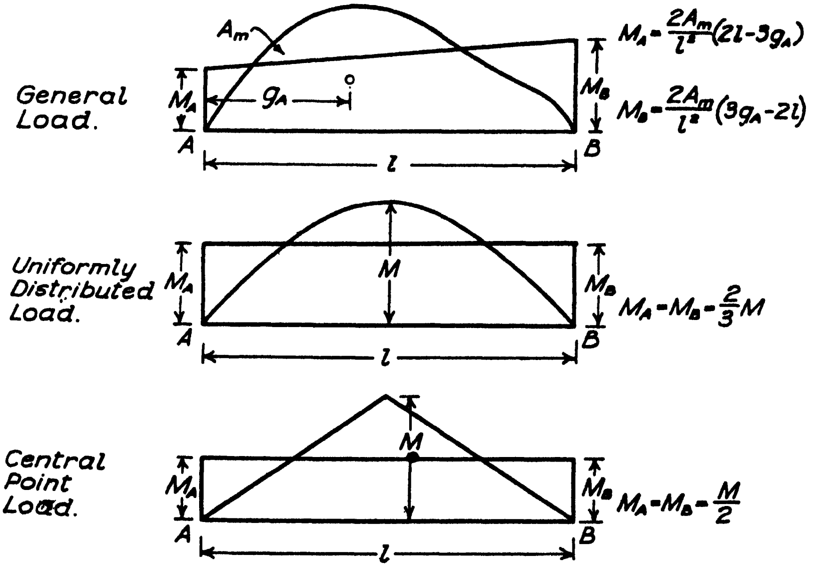


FIG. 32.—FIXED-SUPPORT MOMENTS.

Also, $\frac{A_m}{l}(2l - 3g_A) = \frac{l}{6}(3M_A)$, and $M_A = \frac{2A_m}{l^2}(2l - 3g_A)$ . . (6)

(*b*) For a uniformly-distributed load, by substituting in (5) or (6),

$$M_A = M_B = \frac{2 \times 2Ml}{3 \times l^2}\cdot\left(\frac{3l}{2} - l\right) = \frac{2M}{3}.$$

(*c*) For a central concentrated load, by substituting in (5) or (6),

$$M_A = M_B = \frac{2 \times \frac{1}{2}Ml}{l^2}\cdot\left(\frac{3l}{2} - l\right) = \frac{M}{2}.$$

II.—CONTINUOUS BEAMS.

Bending Moments at the Supports of Continuous Beams Determined by the Methods of the Coefficient of Restraint and Fixed Points.

Fig. 33 shows a continuous beam of any number of spans. At (*a*) the bending moments due to a moment applied to a support to the right of C, if the spans are not loaded, is shown, and at (*b*) the bending moments due to a moment applied to a support to the left of A, if the spans are not loaded. The diagram (*c*) shows the distribution of the bending moments in spans AB and

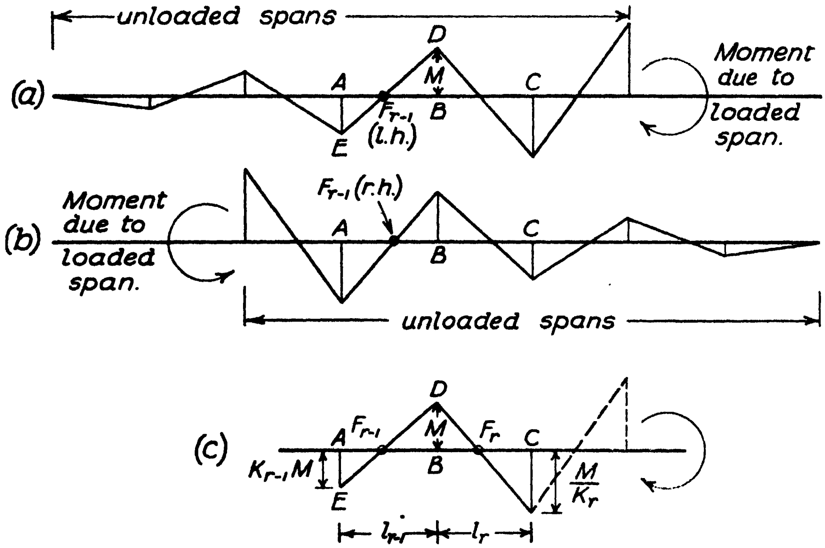


FIG. 33.—CONTINUOUS BEAM.

BC due to a moment applied to a support to the right of C, as at (*a*). Let M be the bending moment at B, $\frac{M}{K_r}$ be the bending moment at C, and $K_{r-1}M$ be the bending moment at A. K_r is therefore the coefficient of restraint of support B with respect to span BC and a moment at support C. K_{r-1} is the coefficient of restraint of support A with respect to span AB and a moment at support B. Let span AB be l_{r-1}, and span BC be l_r. Assume that I is constant throughout each beam and is I_{r-1} for span AB and I_r for span BC. Since the slopes at B due to the moments in spans AB and BC are the same, the reactions at B of the elastic weights on spans AB and BC must be equal. Therefore

$$\frac{l_{r-1}}{6EI_{r-1}}(K_{r-1}M - 2M) = -\frac{l_r}{6EI_r}\left(-2M + \frac{M}{K_r}\right),$$

$$\frac{l_{r-1}}{I_{r-1}}.K_{r-1} - \frac{2l_{r-1}}{I_{r-1}} = \frac{2l_r}{I_r} - \frac{l_r}{I_r K},$$

and $$\frac{1}{K_r} = \frac{I_r}{l_r}\left(\frac{2l_r}{I_r} + \frac{2l_{r-1}}{I_{r-1}} - K_{r-1}.\frac{l_{r-1}}{I_{r-1}}\right).$$

Therefore $$K_r = \frac{1}{2 + \frac{I_r l_{r-1}}{l_r I_{r-1}}(2 - K_{r-1})} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

or $$K_r = \frac{1}{2 + \frac{I_r}{l_r}.\frac{6E}{u_{r-1}}},$$

where u_{r-1} is the coefficient of stiffness of the $(r-1)$th span for a moment at B.

When $I_r = I_{r-1}$, $$K_r = \frac{1}{2 + \frac{l_{r-1}}{l_r}(2 - K_{r-1})}.$$

It should be observed that, in the foregoing, K_r and K_{r-1} are considered as purely quantitative terms and positive throughout, and that the slope of AB at B has the same sign as the slope of BC at B. The expressions for K_r therefore give positive values, and positive values of K_{r-1} must be substituted in these expressions. K_r then increases as K_{r-1} is increased, which is reasonable. Confusion of signs is avoided if slopes are thought of in quantitative terms only and as reactions of elastic weights. If signs are used in the strictly mathematical sense, K_{r-1} would be of opposite sign throughout and the final expression for K_r would be of opposite sign, because a slope considered from opposite sides of a support has opposite signs. A negative value of K_{r-1} would then be substituted in the expressions for K_r, and K_r would have a negative value.

The value of K_r depends only on the value of K_{r-1} and the ratio $\frac{I_r l_{r-1}}{I_{r-1} l_r}$, and is independent of the value of M. Therefore starting with the left-hand end support, where K_1 is nothing for a free support and $\frac{1}{2}$ for a fixed support, values of K can be obtained for each support in turn. Similarly, starting from the extreme right-hand support, another series of values of K can be obtained for the case shown in *Fig.* 33 (*b*), when the moment is applied to the left of the spans under consideration without load. Curves such as those in *Fig.* 34 relating K_r to $\frac{I_r l_{r-1}}{I_{r-1} l_r}$ for various values of K_{r-1} can be used to facilitate calculations. It should be noted that the influence of the value of K_{r-1} on K_r is almost negligible.

To calculate the support moments of a continuous beam such as that shown in *Fig.* 35, the procedure is as follows:

(1) Determine the free moments of each span as shown in *Fig.* 35 (*a*).

(2) Determine the residual support moments such as M'_A and M'_B for each span; that is first assume that all the supports are fixed and determine by subtraction the moment exerted by the restraint of each support.

(3) Determine the coefficient of restraint for each support for moments applied to the right and left of the unloaded spans.

(4) Distribute the residual moments, such as M_A', as shown in *Fig.* 35 (*b*),

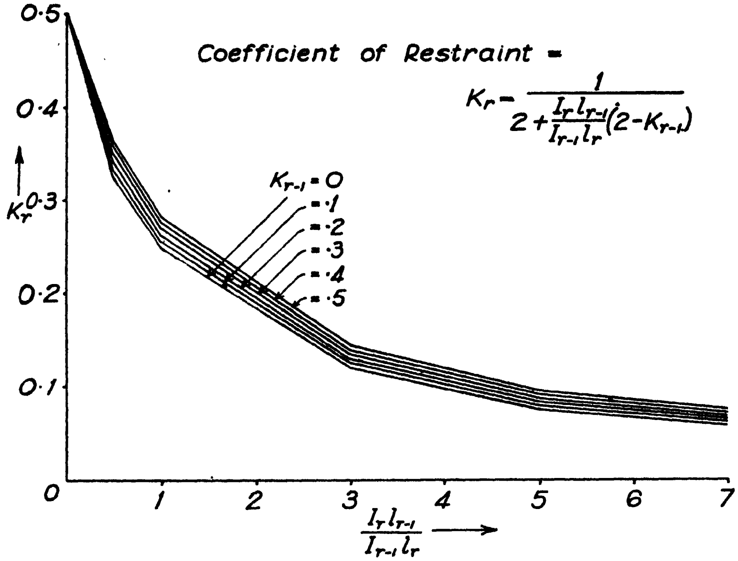


FIG. 34.

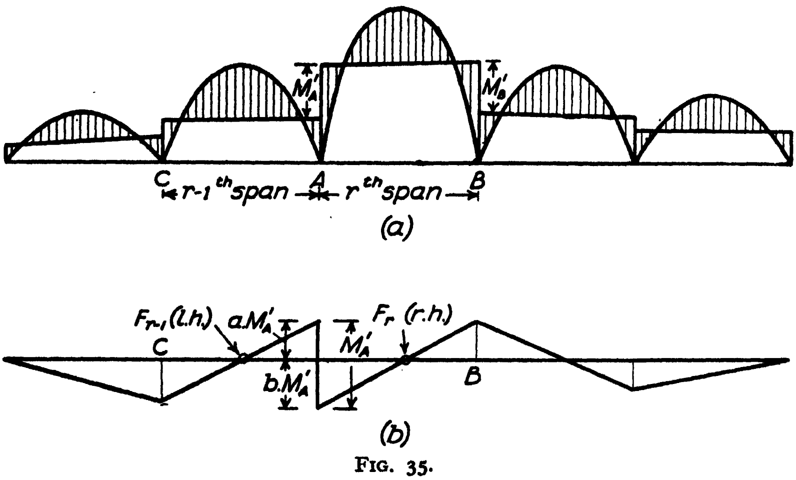


FIG. 35.

to the adjacent spans in proportion to the stiffness of the adjacent spans and disperse the moments to the ends of the series of spans, using the coefficients of restraint calculated in stage (3). Coefficients of moment-distribution are $a = \frac{u_{r-1}}{u_{r-1} + u_r}$ and $b = \frac{u_r}{u_{r-1} + u_r}$, where u_r and u_{r-1} are the coefficients of stiffness of the $(r - 1)$th and rth spans with respect to moments applied at A.

$$\frac{u_{r-1}}{u_{r-1} + u_r} = \frac{1}{1 + \frac{u_r}{u_{r-1}}} = \frac{1}{1 + \frac{I_r}{l_r(2 - K_r)} \cdot \frac{l_{r-1}(2 - K_{r-1})}{I_{r-1}}}.$$

It should be noted that K_{r-1} is the coefficient of restraint with respect to moments applied to the right of the unloaded span and K_r with respect to moments applied to the left of the unloaded spans.

It is clear that when a fixed support is unfixed a residual moment will be distributed to adjacent spans as shown in *Fig.* 35 (*b*) if the procedure of unfixing is considered as the application of an external moment to the beam at the support equal to the residual moment, and at the same time an equal and opposite moment to the support in order to relieve the restraint, thus adding no moments to the system. The dispersal of residual moments to the ends of a beam can be considered separately, and the results added algebraically, by assuming all supports to be unfixed but only one residual moment to act at a time.

(5) Having distributed and dispersed all residual moments such as M'_A, the

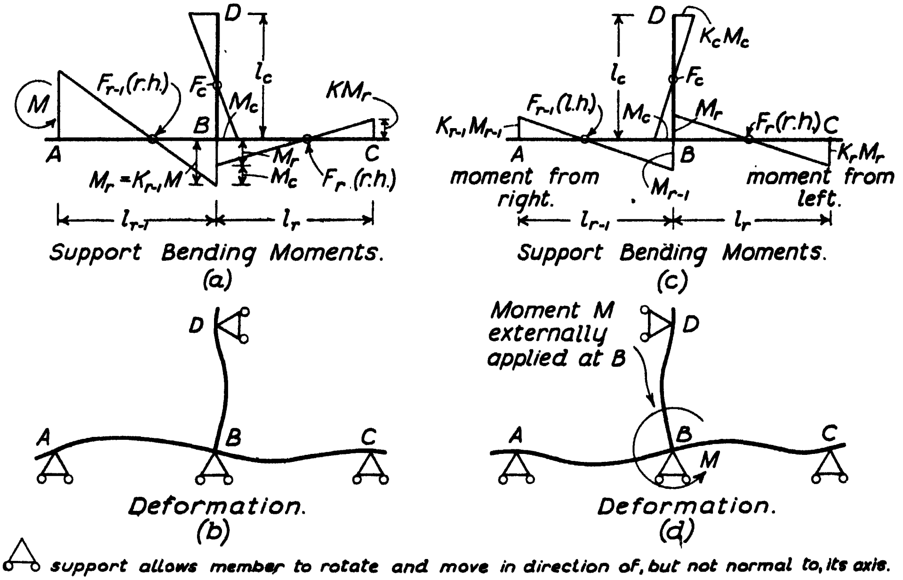


FIG. 36.

final step is therefore to obtain the resultant bending moment at each support by algebraically adding the original fixed moment and all distributed and dispersed moments acting at the support.

DISTRIBUTION AND DISPERSAL OF RESIDUAL MOMENTS.—The distribution and dispersal of residual moments is an important procedure, and the following additional explanation may help to establish the general principles of the method.

Figs. 36 (*a*) and (*b*) show the bending moments and deformation of two spans AB and BC of a continuous beam and an adjoining column BD. A bending moment M is applied to AB at A either from an adjacent span or by any other external means. The joint B is free to rotate and the intersecting members are all rigidly connected. Therefore each member develops the same slope at B. The moment developed at B in the member BA is $K_{r-1}M$, where K_{r-1} is the coefficient of restraint of support B with respect to span BA. The stiffness of the column BD obviously influences the value of K_{r-1}, which will be greater than for a continuous beam without monolithic branch members. The moment $K_{r-1}M$ is balanced by the equal and opposite moment $M_c + M_r$ exerted by the spans BD and BC.

Figs. 36 (*c*) and (*d*) show the bending moments and deformation of the three members when an external moment M is applied to the ends of the members at the rigid joint B. The slopes of each of the members at B are equal but $M_c + M_r + M_{r-1}$ is equal and opposite to the externally-applied moment M, which is distributed among the members in proportion to their stiffnesses. It is important not to confuse the distribution of moments at B in *Fig.* 36 (*a*) with those at (*c*). When a support such as A [*Fig.* 36 (*a*)] is unfixed the result is the same as applying an external moment M'_A to the ends of the members intersecting at A and at the same time applying an equal and opposite moment M'_A to the support in order to relieve the support of all restraint, thereby unfixing the support without adding any bending moment to the system. The residual moment M'_A is therefore distributed to members adjoining A in the same way as the moment M is distributed in *Fig.* 36 (*c*). Successive resulting moments (dispersed residual moments) at the supports beyond B and C [see *Fig.* 35 (*b*)] are then related by the coefficient of restraint of each support. For spans to the right of A successive coefficients of restraint can be found by commencing the evaluation with the extreme right-hand support, and for spans to the left of A by commencing the evaluation with the extreme left-hand support.

Since the distribution and dispersal of residual moments depend only on the coefficient of restraint, it is possible to assume that one support is unfixed at a time and the residual moment distributed and dispersed to the end of the system, although dispersal is not complete until all the supports have been unfixed. The process of unfixing each support in turn adds no moments to the system but relieves the restraint at the supports and adds the distributed and dispersed residual moments to the continuous beam or frame. The resultant moment at a support such as A (*Fig.* 35) is the original fixed moment plus the residual moment M'_A [distributed as in *Fig.* 35 (*b*)] plus the dispersed residual moments from all other unfixed supports.

Since the values of the coefficients of restraint are independent of the support moments, the support-moment line must always pass through the same points such as F_{r-1} and F_r in each span [*Fig.* 35 (*b*)]. These points are the fixed points as previously defined. Their positions are easily determined from the coefficients of restraint, and they are convenient for determining graphically the dispersal of residual moments.

Practical Coefficient-of-Restraint Method for the Distribution and Dispersal of Residual Moments.

In equation (7), let $\dfrac{I_r l_{r-1}}{I_{r-1} l_r} = x$. The limits of K_{r-1} are 0 and 0·5. Also let y be the difference between the values of K_r when $K_{r-1} = 0$ and 0·5; that is

$$y = \frac{1}{2 + x(2 - 0{\cdot}5)} - \frac{1}{2 + 2x} = \frac{0{\cdot}5x}{4 + 7x + 3x^2}.$$

The maximum value of y occurs when $\frac{dy}{dx} = 0$, that is when

$$(4 + 7x + 3x^2)0{\cdot}5 - 0{\cdot}5x(7 + 6x) = 0,$$

that is when $x = 1{\cdot}15$. When $K_{r-1} = 0$ and $x = 1{\cdot}15$, equation (7) becomes

$$K_r = \frac{1}{2 + (1{\cdot}15 \times 2)} = 0{\cdot}23.$$

When $\quad K_{r-1} = 0{\cdot}5$ and $x = 1{\cdot}15$, $K_r = \frac{1}{2 + (1{\cdot}15 \times 1{\cdot}5)} = 0{\cdot}27.$

When $\quad K_{r-1} = 0{\cdot}25$ and $x = 1{\cdot}15$, $K_r = \frac{1}{2 + (1{\cdot}15 \times 1{\cdot}75)} = 0{\cdot}25.$

Therefore if K_{r-1} is assumed to be 0·25 and x is given its actual value of $\frac{I_r l_{r-1}}{I_{r-1} l_r}$, the calculated value of K_r must be within 0·02 of its correct value. Thus, when dispersing a residual moment M by assuming that $K_{r-1} = 0{\cdot}25$, the error in calculating the moment at the adjacent support cannot exceed $\frac{M}{50}$ approximately. It is therefore possible to distribute and disperse residual moments with sufficient accuracy for practical purposes assuming any coefficient of restraint $K_r = \cfrac{1}{2 + 1{\cdot}75\frac{I_r l_{r-1}}{I_{r-1} l_r}}$, which avoids the laborious procedure of calculating K_r commencing with K_0, K_1, etc., at the extreme supports of the continuous member under consideration.

Coefficient-of-Restraint Method : General Case.

The coefficient-of-restraint or fixed-point method of calculation can be extended to deal with continuous beams monolithic with columns or other branch members. Consider any two adjacent unloaded spans AB and BC (*Fig.* 37) continuous with two branch members or columns. A residual moment $\frac{M}{K_r}$ to the right of C is to be dispersed to the spans to the left of C. The span BC applies a moment M to the ends B of the members BA, BD, and BE; M is distributed between these members in proportion to their stiffnesses.

Let u_1, u_2, and u_B be the coefficients of stiffness of BD, BE, and BA respectively, and m_1 and m_2 the moments acting on BD and BE at B.

Then $\quad m_1 = \frac{M_{u1}}{u_B + u_1 + u_2}$, and $m_2 = \frac{M_{u2}}{u_B + u_1 + u_2}$.

$$u_1 = \frac{6EI_1}{l_1(2 - K_1)}, \qquad u_2 = \frac{6EI_2}{l_2(2 - K_2)},$$

and $\quad u_B = \frac{6EI_{r-1}}{l_{r-1}(2 - K_{r-1})}$, where l_1, I_1, and K_1 relate to member BD and l_2, I_2, and K_2 to member BE.

In the case of the beam, starting at the extreme left-hand support, K_{r-1} is known, and is between 0 and 0·5 ; K_r can be related to K_{r-1} and the corresponding coefficient of restraint for the column. Similarly K_{r+1} can be related to K_r and the corresponding coefficient of restraint of the column, and so on. Values of K_1 and K_2 can be determined by treating the columns, starting from their supports remote from B, as unloaded continuous beams allowing for branch members if necessary, or by using the approximate method and assuming the coefficient of restraint of the next but one supports to D and E to be 0·25.

Considering then the two adjacent spans AB and BC (*Fig.* 37), K_r is related

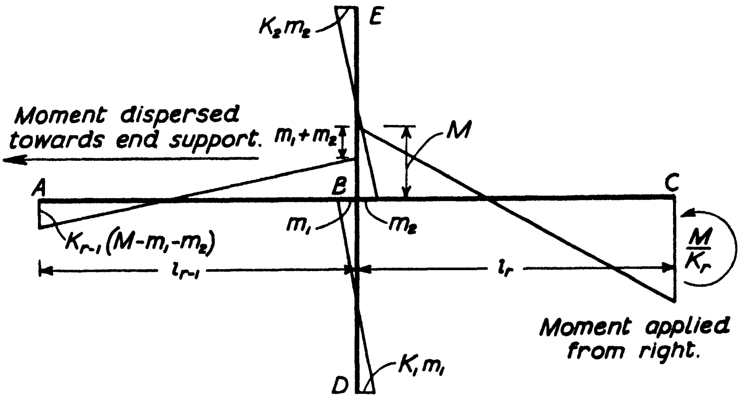


FIG. 37.

to K_{r-1} by equating the slopes, or reactions of the elastic weights, of spans AB and BC at B. Expressing the slope of member BA in terms of the end-moment and coefficient of stiffness,

$$-\frac{(M - m_1 - m_2)}{u_B} = \frac{Ml_r}{6EI_r}\left(2 - \frac{1}{K_r}\right)$$

Therefore

$$-\frac{M\left[1 - \dfrac{u_1}{u_B + u_1 + u_2} - \dfrac{u_2}{u_B + u_1 + u_2}\right]}{u_B} = \frac{Ml_r}{6EI_r}\left(2 - \frac{1}{K_r}\right),$$

$$-\frac{1}{u_B + u_1 + u_2} = \frac{l_r}{6EI_r}\left(2 - \frac{1}{K_r}\right),$$

$$-\frac{6EI_r}{l_r(u_B + u_1 + u_2)} = 2 - \frac{1}{K_r},$$

and therefore

$$K_r = \frac{1}{2 + \dfrac{6EI_r}{l_r(u_B + u_1 + u_2)}} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Equation (8) *indicates, as might be expected, that when one end of the rth span of a continuous system is restrained by more than one adjoining member the coefficient of restraint of the support is the same as for a single continuous adjoining member except that the coefficient of stiffness of the single member is replaced by the sum of the coefficients of stiffness of the several adjoining members.*

In most cases K_1 and K_2 can be considered to lie between the safe limiting values of 0 and 0·5 for free and fixed supports respectively, or the coefficient of restraint of the support next but one to the end support is assumed to be 0·25.

The foregoing procedure can be repeated for each span of the beam in turn starting with a known value of K_r at the extreme support, the moments at each end of each span always being related by the restraint coefficient, and the moments on each side of the support always differing by the sum of the moments transmitted to the columns.

The support-moment line of the beam always passes through the fixed points since equation (8) for K_r is independent of M, m_1, and m_2. The positions of the fixed points are not, of course, the same as those for a continuous beam discontinuous with columns or branch members.

The Method of Moment-Distribution.

The coefficient-of-restraint and fixed-point methods are very laborious, particularly when continuity with supporting columns is included. Unless definite expressions for the bending moments are required in order to determine, say, an ideal ratio for the stiffnesses of the members of a continuous system, the method of characteristic points, the approximate coefficient-of-restraint method, or the method of moment distribution is preferable.

The basis of the method of moment distribution is as follows. In *Fig.* 38, A is any support of a continuous beam 1, 2, 3, 4, etc., M_A is the residual moment which occurs at A if support A is unfixed, and a_1, b_1, c_1, d_1, and a_2, b_2, c_2, d_2, etc., are the coefficients of moment distribution of each of the members with respect to moments applied at supports 1, 2, etc.

In the method of Professor Hardy Cross all supports are assumed to be fixed. Each support or joint in turn is then assumed to be unfixed, and the residual moment distributed and dispersed on the assumption that all other joints remain fixed. The joint is assumed to be refixed after distribution.

If u_1, u_2, etc., are the coefficients of stiffness of the members meeting at 2,

$$a_2 = \frac{u_1}{u_1 + u_2 + u_3 + u_4} \text{ and } b_2 = \frac{u_2}{u_1 + u_2 + u_3 + u_4}, \text{ etc.}$$

The coefficient of stiffness of any member is $u_n = \dfrac{6EI_n}{l_n(2 - K_n)}$, but since the joints of the members are assumed to be fixed, except for the support which is assumed temporarily to be unfixed, K_n is always 0·5. Therefore

$$u_n \text{ is } \frac{4EI_n}{l_n} \text{ and } a_1 = \frac{\dfrac{I_1}{l_1}}{\dfrac{I_1}{l_1} + \dfrac{I_2}{l_2} + \dfrac{I_3}{l_3} + \dfrac{I_4}{l_4}}$$

the term $4E$ being common to all factors.

Fig. 38 shows at (*a*) a continuous system of members with all supports fixed. The fixed-end moments at each support are calculated from the formulæ already established. If each support is released in turn, the residual moment M_A

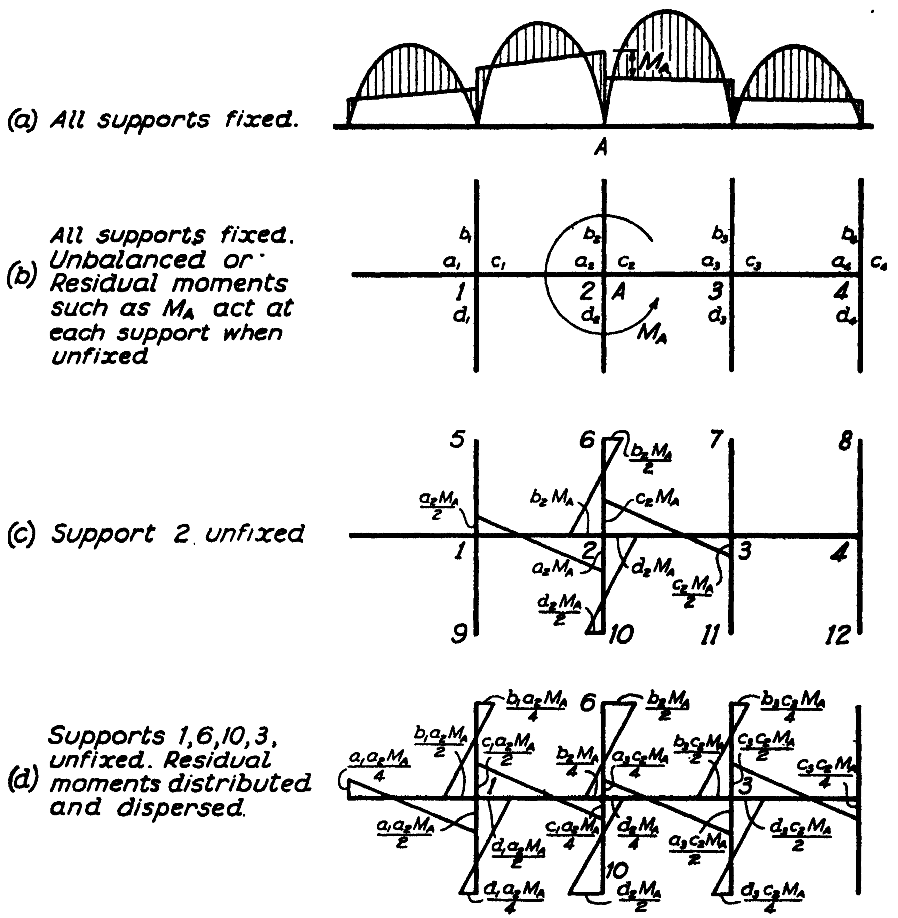


FIG. 38.—MOMENT-DISTRIBUTION METHOD.

acts on the members meeting at joint 2. Consider therefore the distribution and dispersal of the residual moment M_A.

The first stage is: (*a*) Unfix joint 2 [see *Fig.* 38 (*b*)]. (*b*) Distribute the residual moment M_A among the members meeting at 2 so that a_2M_A, b_2M_A, etc., are the moments acting at the ends of the members. (*c*) Disperse moments a_2M_A, b_2M_A, etc., to the opposite fixed-end of each member so that $\frac{a_2M_A}{2}$ acts at 1, $\frac{b_2M_A}{2}$ acts at 6, etc. (*d*) Refix joint 2.

The second stage is to unfix and refix in turn the joints 1, 3, 6, and 10. Each has a residual moment such as $\frac{a_2M_A}{2}$, $\frac{b_2M_A}{2}$, etc. At each joint therefore (*a*) Distribute the residual moment to the adjoining members $\frac{a_1a_2M_A}{2}$, $\frac{b_1a_2M_A}{2}$, etc., acting at 1. (*b*) Disperse the residual moment to the opposite fixed-end of each adjoining member giving, say, from joint 1 $\frac{c_1 a_2 M_A}{4}$ acting at joint 2, etc. [see *Fig.* 38 (*d*)].

Other residual moments can be treated in the same way. In the case of free end supports there is no need to fix the support before a residual moment is dispersed provided that the end condition is considered when determining the relative stiffness of adjoining members. A fixed end support can remain fixed.

The process can be repeated indefinitely. In practice a schedule of moments is made for the end of each member of the system and divided into stages defined by unfixing and refixing all supports once, twice, etc. All residual moments such as M_A can be distributed and dispersed simultaneously at each stage. At the second or third stage the residual moments are generally small enough to be negligible and support moments sufficiently accurate for practical purposes can be determined. The resultant moment acting at each support is the algebraic sum of the original fixed-support moments and all the distributed residual moments received by dispersal from other supports.

Approximate Coefficient-of-Restraint Method.

The foregoing problem can be solved by the approximate coefficient-of-restraint method which in some cases may be simpler. Referring to *Fig.* 39, the procedure is first to assume that all joints are free to rotate. At joint 2, M_A is distributed among the four branch members in proportion to their stiffnesses, giving $a_2 M_A$, $b_2 M_A$, $c_2 M_A$, and $d_2 M_A$. As before, $a_2 = \frac{u_1}{u_1 + u_2 + u_3 + u_4}$, where u_1, u_2, etc. are the coefficients of stiffness of the members 1–2, 2–6, etc.

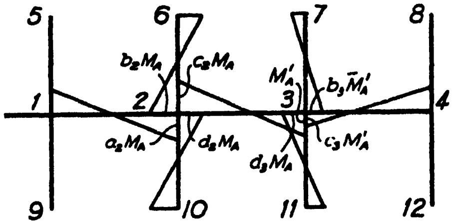


FIG. 39.—APPROXIMATE COEFFICIENT-OF-RESTRAINT METHOD.

Also $u_1 = \frac{6EI_1}{l_1(2 - K_1)}$, where I_1 and l_1 relate to the member 1–2, and K_1 is the coefficient of restraint of joint 1 with respect to joint 2. I_1 and l_1 are known, and K_1 is found from equation (8) by substituting values of u_B, u_1, and u_2 for the members branching from joint 1 and restraining member 1–2, and assuming when evaluating each coefficient of stiffness that $K_{r-1} = 0{\cdot}25$. Similarly values of u for the other members meeting at joint 2 can be found.

The second stage is to disperse to joint 3 the moment $c_2 M_A$ acting on member 2–3. The moment at joint 3 is $M'_A = K_2 c_2 M_A$, where K_2 is the coefficient of restraint of joint 3 with respect to joint 2. The value of K_2 can be found

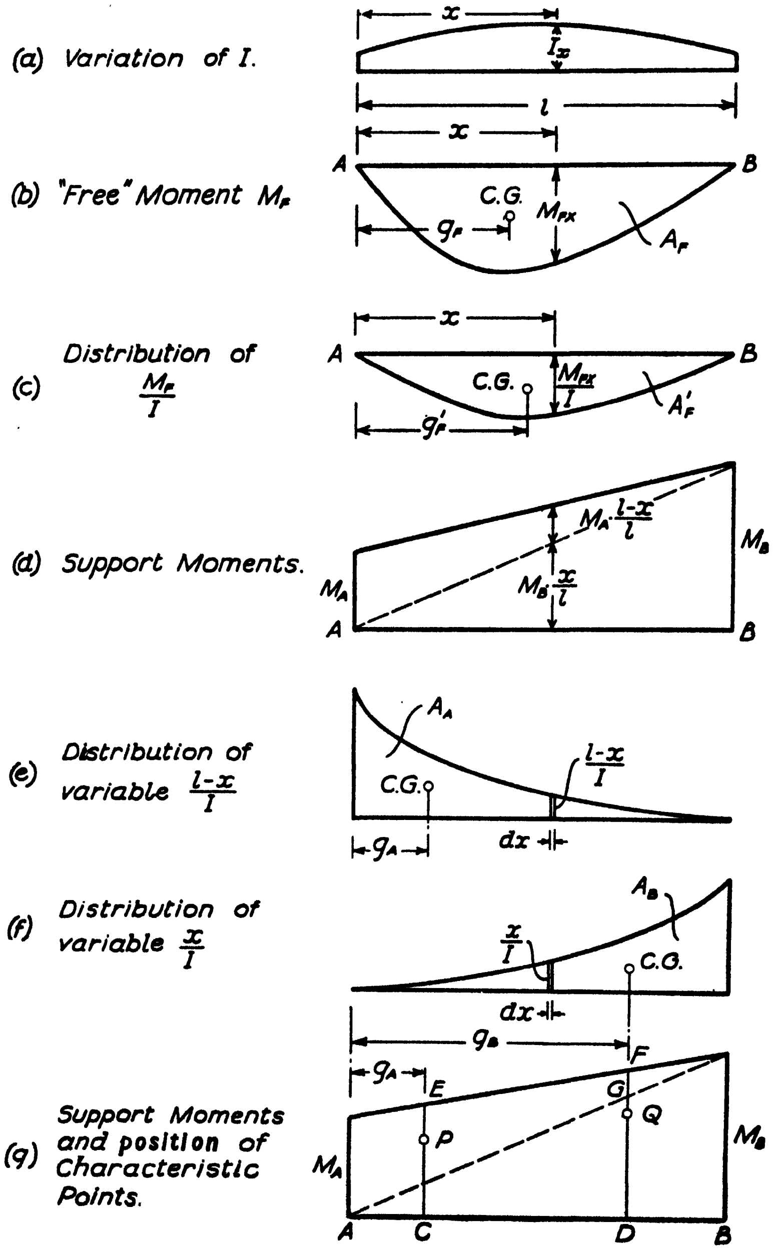

(a) Variation of I.
(b) "Free" Moment M_F
(c) Distribution of $\frac{M_F}{I}$
(d) Support Moments.
(e) Distribution of variable $\frac{l-x}{I}$
(f) Distribution of variable $\frac{x}{I}$
(g) Support Moments and position of Characteristic Points.
x
I_x
l
A
B
C.G.
M_{FX}
g_F
A_F
$\frac{M_{FX}}{I}$
g'_F
A'_F
$M_A \cdot \frac{l-x}{l}$
$M_B \cdot \frac{x}{l}$
M_A
M_B
A_A
$\frac{l-x}{I}$
g_A
dx
A_B
$\frac{x}{I}$
g_B
E
F
G
P
Q
C
D

FIG. 40.

by applying equation (8) and assuming $K_{r-1} = 0{\cdot}25$ when evaluating the coefficient of stiffness of each branch member to the right of joint 3.

The next stage is to distribute the moment M'_A among the members 3–7, 3–4, 3–11 in proportion to their stiffnesses, the moments acting on each being $b_3M'_A$, $c_3M'_A$, and $d_3M'_A$. The coefficient of stiffness of each member is calculated in a similar manner to the coefficient of stiffness of member 1–2 when acted upon by a moment at joint 2.

The fourth stage is to disperse the moments $b_3M'_A$, $c_3M'_A$, etc., to joints 7, 4, and 11, using coefficients of restraint calculated in a similar manner to K_1 in the foregoing. Distribution and dispersion should be continued until the residual moments become negligible. Finally all support moments are added together algebraically.

Characteristic Points.

The following is a consideration of the general case, the notation (*Fig.* 40) being:

l: the span of the beam or structural member supported at A and B.
I_x: the moment of inertia of a section x from A.
M_{Fx}: the free bending moment at x from A.
g_F: the distance of the centroid of the free-moment diagram from A.
g'_F: the distance of the centroid of the $\frac{M_{Fx}}{I_x}$ diagram from A.
A_F: the total area of the M_{Fx} diagram.
A'_F: the total area of the $\frac{M_{Fx}}{I_x}$ diagram.
M_A: the bending moment at support A.
M_B: the bending moment at support B.
g_A: the distance of the centroid of the $\frac{l-x}{I_x}$ diagram from A.
A_A: the total area of the $\frac{l-x}{I_x}$ diagram.
g_B: the distance of the centroid of the $\frac{x}{I}$ diagram from A.
A_B: the total area of the $\frac{x}{I}$ diagram.

Fig. 40 shows the following for any span AB of a continuous beam or for a structural member which, at the supports, is free to rotate and to move in the direction of its axis, but not to move normal to its axis.

(*a*) A diagram showing the variation of I_x throughout the span AB.
(*b*) The bending-moment diagram for the span AB if freely supported at A and B, that is assuming that there is no continuity with adjacent spans.
(*c*) A diagram showing the variation of the free bending moment divided by I_x.
(*d*) The diagram of the bending moments at the supports.
(*e*) A diagram showing the variation of $\frac{l-x}{I_x}$.
(*f*) A diagram showing the variation of $\frac{x}{I_x}$.
(*g*) The diagram of the bending moments at the supports showing the position of the characteristic points.

Slope of Beam at A and B Expressed Graphically by Characteristic Points.—The slope of the beam at B due to the free bending moment only is the reaction at B of the beam loaded with the elastic weights $\frac{M_F}{EI}$, and is $\frac{A'_F g'_F}{El}$ (E is constant). The slope of the beam at B due to the support moments is the reaction at B of the beam loaded with the elastic weights derived from the support-moment diagram. This diagram can be divided, as in *Fig.* 40 (*d*), into two diagrams the ordinates of which are $M_A\left(\frac{l-x}{l}\right)$ and $M_B\frac{x}{l}$. The terms $\frac{M_A}{El}$ and $\frac{M_B}{El}$ are constant throughout the beam, and $\frac{l-x}{I_x}$ and $\frac{x}{I_x}$ are variables as shown in *Fig.* 40 (*e*) and (*f*) respectively. The reaction at B of the elastic weights due to the support moments only is

$$\int_0^l \frac{M_A}{EI_x}\left(\frac{l-x}{l}\right).\frac{x}{l}dx + \int_0^l \frac{M_B}{EI_x}\left(\frac{x}{l}\right)\left(\frac{x}{l}\right)dx.$$

According to the notation already given, $\int_0^l \left(\frac{l-x}{I_x}\right)x.dx$ is $A_A g_A$, the moment about A of the area of the $\frac{l-x}{I_x}$ diagram, and $\int_0^l \left(\frac{x}{I_x}\right)x.dx$ is $A_B g_B$, the moment about A of the area of the $\frac{l}{I_x}$ diagram. Thus the reaction at B is

$$\frac{1}{El^2}(M_A A_A g_A + M_B A_B g_B) \quad . \quad . \quad . \quad . \quad (9)$$

The moment about A of any strip of infinitely small width dx of the diagram in *Fig.* 40 (*e*) is $\frac{l-x}{I_x}.x.dx$. The moment about B of similar strips of the diagram in *Fig.* 40 (*f*) is $\frac{x}{I_x}(l-x)dx$. Thus the sums of all such moments are equal; that is $A_A g_A = A_B l - g_b$. Substituting in (9), the slope of the beam at B due to the support moments only is

$$\frac{1}{El^2}[M_A A_B(l-g_B) + M_B A_B g_B] \quad . \quad . \quad . \quad . \quad (10)$$

Now, in *Fig.* 40 (*g*), FD = FG + GD. Thus FD $= M_A\frac{l-g_B}{l} + \frac{M_B g_B}{l}$, which from equation (10) is the slope of the beam at B due to the support moments only multiplied by $\frac{El}{A_B}$. If DQ is $\frac{A'_F g'_F}{A_B}$, which is the slope at B due to the free moment only multiplied by $\frac{El}{A_B}$, then QF = FD — DQ is the resultant slope of the beam at B multiplied by $\frac{El}{A_B}$.

Similarly if PC is $\frac{A_F'(l-g'_F)}{A_A}$, PE is the resultant slope of the beam at A multiplied by $\frac{El}{A_A}$.

The points P and Q are the characteristic points for the span AB and their position is determined by the distribution of the free bending moment and the variation of I.

MOMENTS AT THE SUPPORTS OF A CONTINUOUS BEAM OR A MEMBER OF A FRAME. (SUPPORTS FREE TO ROTATE.)—Referring to *Fig.* 41, AB and BC are

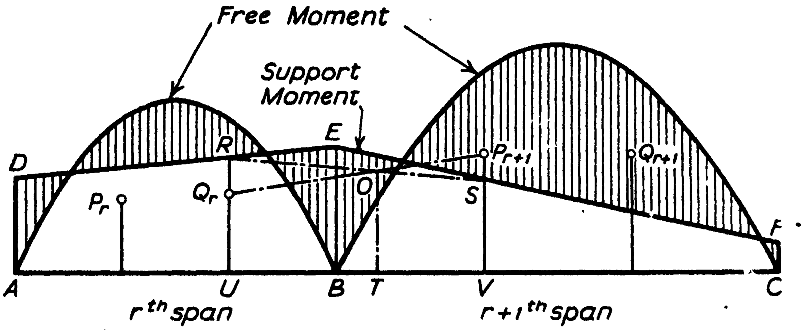


FIG. 41.

the rth and $(r + 1)$th spans of a continuous beam or frame on supports fixed in position but free to rotate. The thick lines indicate the distribution of the bending moments at the supports and the free bending moments; P_r, Q_r, P_{r+1}, and Q_{r+1} are the characteristic points for the rth and $(r + 1)$th spans, the positions of which for each span are found by setting up $UQ_r = \frac{A'_F g'_F}{A_B}$, calculated as already explained for the rth span, and $VP_{r+1} = \frac{A'_F(l - g'_F)}{A_A}$ for the $(r + 1)$th span. Also, $UA = g_B$ for the rth span, and $BV = g_A$ for the $(r + 1)$th span.

Now Q_rR is the slope of the beam at B due to the free and support bending moments of the rth span multiplied by $\frac{El_r}{A_{Br}}$, where l_r is the rth span and A_{Br} is the area of the $\frac{x}{I_x}$ diagram for the rth span. Also, SP_{r+1} is the slope of the beam at B due to the free and support bending moments of the $(r + 1)$th span multiplied by $\frac{El_{r+1}}{A_{A(r+1)}}$, where $A_{A(r+1)}$ is the area of the $\frac{l - x}{I_x}$ diagram for the $(r + 1)$th span. The slope of the beam at B due to bending moments of the rth span is equal to, but of opposite sign to, the slope of the beam at B due to bending moments of the $(r + 1)$th span.

Therefore
$$\frac{Q_rR}{\frac{El_r}{A_{Br}}} = -\frac{SP_{r+1}}{\frac{El_{r+1}}{A_{A(r+1)}}}.$$

The terms Q_rR and SP_{r+1} refer to the lengths between points Q_r and R, and S and P_{r+1}, respectively in *Fig.* 41.

Thus
$$\frac{Q_rR}{SP_{r+1}} = -\frac{l_rA_{A(+1)}}{l_{r+1}A_{Br}} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In a given case, therefore, at support B the bending moment represented by

BE must be such that the lines joining E to D and F (where AD and CF represent the bending moments at supports A and C respectively) pass above and below the characteristic points Q_r and P_{r+1} at vertical distances Q_rR and SP_{r+1} respectively such that equation (11) is satisfied.

If T is set out so that $\frac{UT}{TV}$ equals the right-hand side of equation (11) and O is a point vertically above T on the line joining the characteristic points Q_r and P_{r+1}, then R and S, the points of intersection of the vertical lines through the characteristic points with the support-moment lines, must lie on a straight line through O since, by similar triangles, $\frac{Q_rR}{P_{r+1}S} = \frac{Q_rO}{OP_{r+1}}$, and $\frac{RO}{OS} = \frac{UT}{TV}$ because RU, OT, and SV are parallel. The height of E vertically above B, which repre-

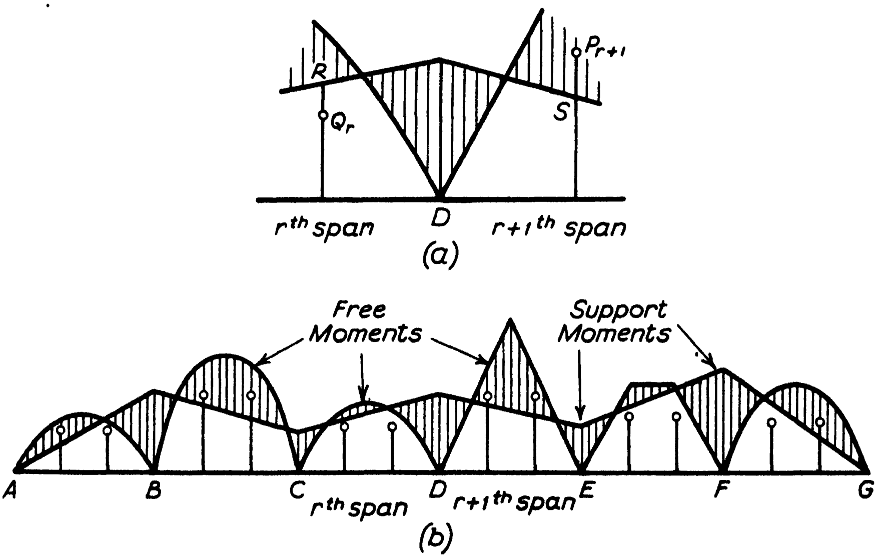


FIG. 42.

sents the support moment at B, must therefore be such that when lines ED and EF are drawn they intersect the verticals through the characteristic points at points on a straight line passing through O.

Spans such as AB and BC (*Fig.* 41), will generally be two of a series of spans such as are shown in *Fig.* 42. To determine the support-moment line for such a case, the characteristic points for each span must be plotted and the support-moment line drawn so that it passes above and below the characteristic points adjacent to each support at vertical distances related in accordance with equation (11). The line satisfying this condition can only be drawn by trial and error assisted by setting out the points corresponding to O as described. With a little experience a support-moment line can be drawn very quickly giving bending moments sufficiently accurate for practical purposes. Frequently, in the early stages of a design, only an approximate diagram is required, and the support-moment line can be immediately drawn with sufficient accuracy on approximate free-moment diagrams, sketched on squared paper, with the bending moments at critical sections and with precise safe limiting values for the heights of the characteristic points expressed in terms of wl^2. This procedure applies particularly to the usual case in which I is constant. The method of characteristic points is useful for determining quickly the effect of the variation of I on the distribution of the bending moments, particularly when the depth at the supports is greater than that at mid-span.

The following should be noted when setting out the support-moment lines:

(i) Since the height of the support-moment line above or below the characteristic point is a function of the slope of the adjacent support, the slope due to the free moment is greater than the slope due to the bending moment at the support if the line passes below a characteristic point. The beam therefore slopes downwards from the support towards the middle of the span concerned and upwards towards the span on the other side of the support; thus in the span on the other side of the support the support-moment line passes above the characteristic point.

(ii) If the beam is horizontal at a support, as in the case of a fixed-end or of the middle support of a symmetrical series of spans, the support-moment line must pass through the characteristic points on either side of the support since there is no slope at the support.

(iii) If the beam is freely supported at an end support, the support-moment line must obviously pass through the support. The vertical distance between the support-moment line and the characteristic point, which is generally above the line, is a function of the slope at the end support.

SIMPLIFICATION WHEN I IS CONSTANT.—Often the moment of inertia is constant throughout each span of a continuous beam. Referring to *Fig.* 40, for this case the following values are obtained: $g_A = \frac{l}{3}$; $g_B = \frac{2l}{3}$; $A_A = \frac{l}{I} \times \frac{l}{2} = \frac{l^2}{2I}$; $A_B = \frac{l^2}{2I}$; $A'_F = \frac{A_F}{I}$, where A_F is the area of the free-moment diagram [see *Fig.* 40 (*b*)], and $g'_F = g_F$. Therefore the characteristic points are situated on verticals through the third-points of each span.

The height of a characteristic point Q_r is $\frac{A'_F g'_F}{A_B} = \frac{A_F g_F}{I \times \frac{l^2}{2I}} = \frac{2A_F g_F}{l^2}$, A_F and g_F having suitable values for the rth span. Although I is therefore omitted from the expression for the height of Q_r, it must be included when scaling from the diagram a definite value of the slope of the beam. Similarly, the height of characteristic point P_r is $\frac{2A_F(l - g_F)}{l^2}$.

From equation (11),

$$\frac{Q_rR_r}{SP_{r+1}} = -\frac{l_r}{l_{r+1}} \cdot \frac{\frac{l^2_{r+1}}{2I_{r+1}}}{\frac{l^2_r}{2I_r}} = -\frac{l_{r+1}}{l_r} \cdot \frac{I_r}{I_{r+1}}$$

$$= -\frac{l_{r+1}}{l_r} \text{ if } I_r = I_{r+1}.$$

The bending-moment diagram for a continuous beam of constant moment of inertia can therefore be plotted by the following simple procedure (*Fig.* 42):

(i) Draw the free-moment diagram for each span. Generally it is only necessary to sketch this approximately on squared paper with only the maximum bending moment accurately set out, as this is at or near a critical section of the beam.

(ii) Plot the characteristic points at vertical distances of $\frac{2A_F(l - g_F)}{l^2}$ and $\frac{2A_F g_F}{l^2}$ above the first and second third-points respectively of each span; the values of A_F, g_F, and l are those applicable to each span. If the free-moment diagram is for a uniformly-distributed load, $A_F = \frac{2M_F l}{3}$, where M_F is the maximum value of the free bending moment. The height of both characteristic points is then the same and is $2 \times \frac{2M_F l}{3l^2} \cdot \frac{l}{2} = \frac{2M_F}{3}$, because $g_F = \frac{l}{2}$. If the free-moment diagram is for a central concentrated load, the height of both characteristic points is $\frac{2 \times \frac{M_F l}{2}}{l^2} \cdot \frac{l}{2} = \frac{M_F}{2}$.

(iii) Draw the support-moment lines so that at each pair of characteristic points, such as Q_r and P_{r+1} [*Fig.* 42 (*a*)], $\frac{RQ_r}{SP_{r+1}} = \frac{l_{r+1}}{l_r}$. The beam generally slopes downwards from the support towards the span in which the free bending moment is greatest. The support-moment line then passes below the characteristic points in that span. Thus in *Fig.* 42 (*a*) the beam slopes from support D downwards towards the right. The support-moment line can generally be sketched sufficiently accurately by eye when I is constant since the ratio $\frac{l_{r+1}}{l_r}$ is simple, and the construction involving the point O as in *Fig.* 41 is not required.

Most cases of irregular load occurring in practice can be dealt with by a free-moment diagram which is either that for a central concentrated load or for a uniformly-distributed load, and for which the bending moment at any section of the beam exceeds that for the worst distribution of the working load. As I is generally constant, the procedure is very simple and only entails remembering the following two points.

(i) The vertical distances of the characteristic points above the first and second third-points of each span are $\frac{2M_F}{3}$ for a span carrying a uniformly-distributed load and $\frac{M_F}{2}$ for a span carrying a central concentrated load, where M_F is the maximum free bending moment.

(ii) The support-line must pass above and below the characteristic points adjacent to the support between the rth and $(r + 1)$th spans at distances in the ratio of $\frac{l_{r+1}}{l_r}$.

The method of characteristic points can be applied to any continuous member bearing on supports which cannot yield in a direction normal to the axis of the member but are free to rotate, because it is merely a device for finding the bending moments developed at a support in order that the slope of the beam at the support is the same when calculated from the distribution of the bending moment in adjacent spans. For example, the method can be applied to find the bending moments in the walls of a rectangular tank if the walls span horizontally, or in a portal frame if the joints do not sway sideways due to horizontal forces or unsymmetrical loads.

CONTINUOUS BEAMS MONOLITHIC WITH COLUMNS.—Most continuous beams are sufficiently stiff relative to the supporting columns to make bending moments in the columns negligible except at the end supports. A check can be made by

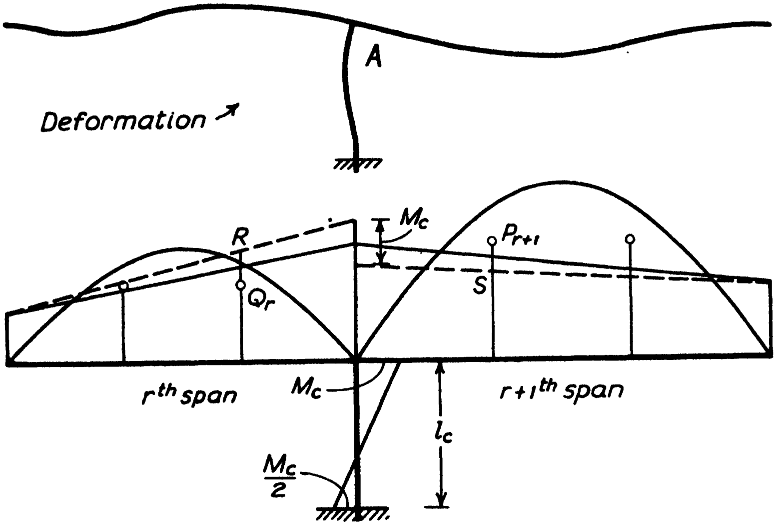


FIG. 43.

the following adjustment of the support-moment line obtained by characteristic points.

Referring to *Fig.* 43, assume that I_c is the moment of inertia of a column at an intermediate support A of a continuous beam, l_c is the height of the column, and M_c the bending moment at the head of the column. The most adverse condition is when the column is fixed at the base. The slope of the column at the head equals the reaction of the elastic weights, that is

$$\frac{l_c}{6EI_c}\left(2M_c - \frac{M_c}{2}\right) = \frac{M_c l_c}{4EI_c}.$$

The support-moment line must pass the characteristic points Q_r and P_{r+1} so that $\dfrac{Q_rR}{P_{r+1}S} = \dfrac{l_{r+1}I_r}{l_rI_{r+1}}$, provided that I is constant in each span of the beam.

Also the slope of the beam at A is $Q_rR.\dfrac{l_r}{2EI_r} = P_{r+1}S.\dfrac{l_{r+1}}{2EI_{r-1}}$, which must equal

$\frac{M_c l_c}{4EI_c}$. If necessary, the support-moment line must be adjusted until these conditions are satisfied. Generally the adjustment required is negligible, and therefore the value of M_c is negligible.

III.—FRAMED STRUCTURES AND ARCHES.

It must be emphasised that the methods of characteristic points, fixed points, coefficient of restraint, and moment distribution are based on the assumption that the supports are not displaced. The supports of building frames, portal frames, and similar structures are, with the exception of the supports on the ground, liable to horizontal displacements due to unsymmetrical vertical loads or to unsymmetrical arrangement of the members of the frame, or to horizontal forces such as those due to the wind. In buildings, the effect of horizontal displacement due to unsymmetrical load or arrangement of members is often ignored. In cases in which sway must be taken into account it is possible to assume first that no horizontal displacements occur and to calculate the external horizontal forces required for equilibrium; the bending moments due to these forces are algebraically added to those obtained with the supports assumed not displaced. In such cases, however, it is preferable to use other methods which are described later.

The method of moment distribution is popular, but engineers familiar with the method of characteristic points may find that, by sketching on squared paper the bending moments at the critical sections of a beam, they can obtain sufficiently accurate results more quickly and with less risk of error as mistakes are more noticeable in a graphical method. The plotting of the free-moment diagrams and the characteristic points can be done with absolute certainty in most cases in a few minutes, and with little experience the support-moment line can be established after very few trials. If the bending moment at a support is slightly under-estimated the bending moment in the adjacent span is automatically slightly over-estimated, so that under ultimate load the strength of the beam is not affected. The method of characteristic points is also easier when dealing with beams having varying moments of inertia and for determining quickly the influence of deep haunches. The approximate method of coefficient of restraint is also as quick as the moment-distribution method and is sufficiently accurate for practical purposes. An engineer should, however, adopt the method of calculation which he finds to be the easiest to understand, thereby ensuring the reliability of the results. Some other methods are referred to in the following.

The Solution of Statically-Indeterminate Frames by the δ_{ik}-Method.

The stresses in vertical frames subjected to horizontal forces and displacement of the supports, portal frames, frames of foundation rafts, and arches can also be analysed by using the δ_{ik} table [*Fig.* 46 (v)], by which standard solutions of the general elastic equations are evaluated.

In *Fig.* 44 (*a*) P_1 and P_2 are known external forces or moments acting on any frame. If the frame is n times statically indeterminate, n unknowns X_1, X_2 . . . X_n are assumed to act at suitable points of the frame, so that the frame becomes statically determinate.

The continuity at any point in the frame may be broken by inserting a hinge and replacing the internal moment of resistance by two equal and opposite external unknown moments X_1 acting on the member on either side of the hinge

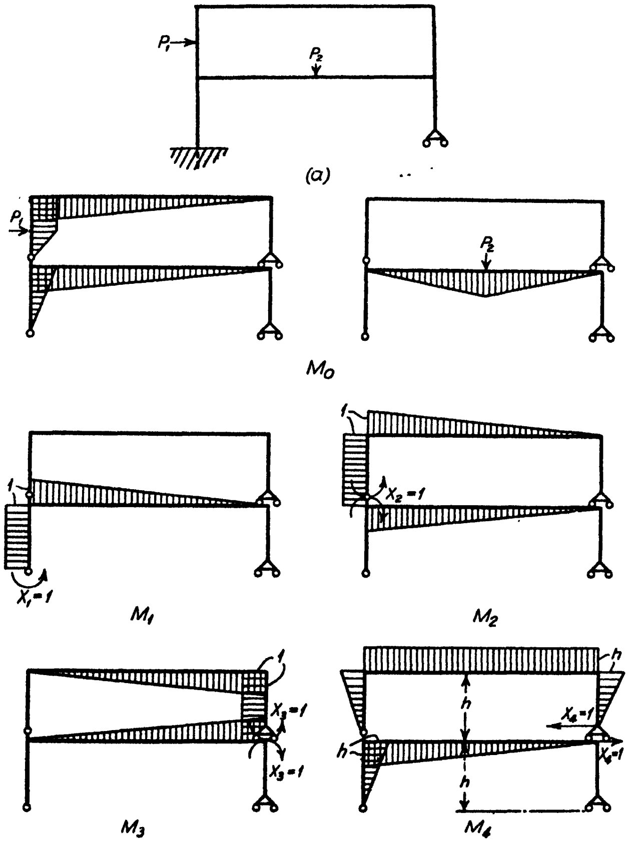


FIG. 44.

[*Fig.* 45 (*a*)] ; or by making a cut and replacing the internal moment of resistance, shearing force, and tension or compression by equal and opposite unknown moments X_2, equal and opposite unknown forces X_3 acting normal to the member, and equal and opposite forces X_4 acting in the direction of the member on either side of the cut [*Fig.* 45 (*b*)].

A fixed bearing may be replaced by a hinged bearing and an unknown external moment ; or by a roller bearing and an unknown moment, and an unknown external force X_5 acting on the end of the member in the direction in which the bearing is free to roll ; or by a free unsupported end and two unknown forces acting in two directions at right-angles, and an unknown bending moment.

A hinged bearing may be replaced by a roller bearing and an unknown force acting in the direction in which the roller is free to roll; or by a free unsupported end and two unknown forces acting at right-angles on the end of the member.

In a frame which is n times statically indeterminate, cuts, hinges, roller bearings, or unsupported ends must be provided and unknown forces or moments assumed to act in place of all internal and supporting forces and moments removed, until n unknowns have been assumed. The positions of the cuts, hinges, and the like must be chosen so that when the n unknowns are applied the frame is statically determinate in all parts and is stable.

In a particular case, having assumed n unknowns, the forces acting on the frame can be split up into $n + 1$ statically-determinate systems. *Fig.* 44 shows a frame which is four times statically indeterminate. The division of the forces

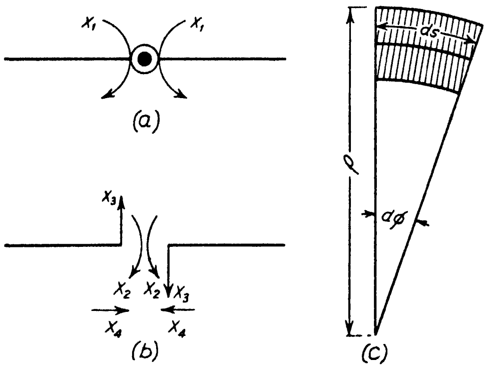


FIG. 45.

acting on the frame into $n + 1$ statically-determinate systems may be regarded as typical. The systems are:

System (0) (marked M_0): P_1 and P_2, the known forces or moments acting on the frame and their corresponding reactions at the supports.

System (1) (marked M_1): X_1 (always assumed at first to be unity to simplify the calculations) and its corresponding reactions at the supports. The forces and moments at any point in the frame due to X_1 having a value other than unity are X_1 times the forces and moments due to $X_1 = 1$.

System (2) (marked M_2): $X_2 = 1$ and its corresponding reactions at the supports.

System (3) (marked M_3): $X_3 = 1$ and its reactions.

System (4) (marked M_4): $X_4 = 1$ and its reactions.

The bending moment, shearing force, and tension or compression at any point in the frame is the sum of the moments, shearing forces, and direct forces at the point due to each of the $n + 1$ systems. For systems (2), (3), and (4) the moment or shearing force due to $X_1 = 1$, $X_2 = 1$, $X_3 = 1$, and $X_4 = 1$ must be multiplied by the value of X_1, X_2, etc., when found.

Let δ_{ik} indicate the movement in the direction of X_i taken up by any force or moment $X_i = 1$ already acting on the frame, when any other force, say, $X_k = 1$, and its reactions, are applied to the frame, which has been made statically determinate by the insertion of suitable hinges, cuts, and the like. δ_{ik} is a deflection when X_i is a force and an angle of rotation when X_i is a moment.

Let M_i and M_k indicate the moments at any point in the frame caused by the system of forces (*i*) and (*k*) respectively acting on the frame, that is by $X_i = 1$ and $X_k = 1$, and their reactions, acting alone.

The bending-moment diagrams for M_i and M_k are plotted on the tensile side of the members with ordinates normal to the members (*Fig.* 44). The moments are easily obtained from the forces $X_i = 1$ and $X_k = 1$, since the frame has been made statically determinate. Let M_0 indicate the moments caused by P_1 and P_2 only acting [system (0)], and let δ_{i0} and δ_{k0} indicate the movement of X_i and X_k in the direction X_i or X_k, when the moments M_0 only act on the frame. In *Fig.* 44 the moments M_0 caused by P_1 and P_2 are separated for clearness.

Since δ_{ik} is the movement of $X_i = 1$ in the direction X_i due to $X_k = 1$ only acting, $1 \times \delta_{ik}$ is the virtual work done by $X_i = 1$ in being moved δ_{ik} when $X_k = 1$ acts. When X_k having a value other than unity acts, the virtual work done by $X_i = 1$ is equal to $X_k\delta_{ik}$. The force $X_i = 1$ causes bending moments M_i throughout the frame, that is the force $X_i = 1$ is equilibrated by internal moments M_i acting throughout the frame and by external reactions which do not move when other forces or moments are applied to the frame. Therefore, if the force or moment $X_k = 1$ is applied to the frame when already acted on by $X_i = 1$, each internal resisting couple M_i turns through an angle $d\phi = \frac{M_k}{EI}ds$, since $\frac{M}{I} = \frac{E}{\rho}$ and $d\phi = \frac{ds}{\rho}$ [*Fig.* 45 (*c*), where ds is an infinitely short length of the member and $d\phi$ is the angle subtended by ds at the centre of curvature].

The total virtual work done by all couples M_i throughout the frame rotating through an angle $d\phi$ is $\int M_i d\phi = \int \frac{M_i M_k ds}{EI}$. Since the variable ds is always measured along each member and the moments M_i and M_k are always plotted normal to the members, this expression is the integration of the products of the ordinates of the M_i-diagram and the M_k-diagram throughout the frame, with ds as the horizontal variable measured along the members. Hence δ_{ik}, the virtual work done by $X_i = 1$ when $X_k = 1$ is applied to the frame, is $\int \frac{M_i M_k ds}{EI}$, and the virtual work done by $X_i = 1$ when X_k having any other value than unity is applied to the frame is $X_k \int \frac{M_i M_k ds}{EI}$. Similarly, the virtual work done by $X_i = 1$ when P_1 and P_2 are applied to the frame is $\delta_{i0} = \int \frac{M_i M_0 ds}{EI}$.

The total virtual work done by $X_i = 1$ when the forces P_1 and P_2 and X_1 to X_4 of the systems 0 to 4, or 0 to *n* in the general case, are applied to the frame is $\delta_{i0} + X_1\delta_{i1} + X_2\delta_{i2} + \ldots X_n\delta_{in}$

$$= \int \frac{M_i M_0 ds}{EI} + X_1 \int \frac{M_i M_1 ds}{EI} + X_2 \int \frac{M_i M_2 ds}{EI} \ldots + X_n \int \frac{M_i M_n ds}{EI} \quad . \quad (1)$$

When the point of application of X_i does not move due to sinking of a support or yielding of a fixed end, or when movements due to shearing force, temperature, and direct stress are neglected, the forces P_1 to P_n and X_1 to X_n and their reactions form a system in equilibrium, and the total virtual work done by $X_i = 1$ when all these forces act is nil. Therefore expression (1) equals 0.

Similarly, the total virtual work done by $X_1 = 1$, $X_2 = 1$, . . . $X_n = 1$ when all forces P_1 to P_n and X_1 to X_n act on the frame is nil.

Thus there can be derived n equations of the general form (1), from which the n unknowns X_1 to X_n can be found. It is useful to derive general solutions with three unknowns, thus covering many practical cases, as the expressions can be used to solve any frame that is three or fewer times statically indeterminate by substituting the values of δ_{10}, δ_{11}, etc., for the particular case.

The general equations for a frame which is three times statically indeterminate are

$$\begin{aligned} \delta_{10} + X_1\delta_{11} + X_2\delta_{12} + X_3\delta_{13} &= 0, \\ \delta_{20} + X_1\delta_{21} + X_2\delta_{22} + X_3\delta_{23} &= 0, \\ \delta_{30} + X_1\delta_{31} + X_2\delta_{32} + X_3\delta_{33} &= 0. \end{aligned}$$

Using determinants, the solutions of these equations are

$$X_1 = \frac{\begin{vmatrix} -\delta_{10} & \delta_{12} & \delta_{13} \\ -\delta_{20} & \delta_{22} & \delta_{23} \\ -\delta_{30} & \delta_{32} & \delta_{33} \end{vmatrix}}{D}; \quad X_2 = \frac{\begin{vmatrix} \delta_{11} & -\delta_{10} & \delta_{13} \\ \delta_{21} & -\delta_{20} & \delta_{23} \\ \delta_{31} & -\delta_{30} & \delta_{33} \end{vmatrix}}{D}; \quad X_3 = \frac{\begin{vmatrix} \delta_{11} & \delta_{12} & -\delta_{10} \\ \delta_{21} & \delta_{22} & -\delta_{20} \\ \delta_{31} & \delta_{32} & -\delta_{30} \end{vmatrix}}{D},$$

where

$$D = \begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{vmatrix}.$$

By expanding the determinants

$$X_1 = \frac{-\delta_{10}(\delta_{22}\delta_{33} - \delta_{32}\delta_{23}) + \delta_{20}(\delta_{12}\delta_{33} - \delta_{32}\delta_{13}) - \delta_{30}(\delta_{12}\delta_{23} - \delta_{22}\delta_{13})}{D};$$

$$X_2 = \frac{\delta_{10}(\delta_{21}\delta_{33} - \delta_{31}\delta_{23}) - \delta_{20}(\delta_{11}\delta_{33} - \delta_{31}\delta_{13}) + \delta_{30}(\delta_{11}\delta_{23} - \delta_{21}\delta_{13})}{D};$$

$$X_3 = \frac{-\delta_{10}(\delta_{21}\delta_{32} - \delta_{33}\delta_{22}) + \delta_{20}(\delta_{11}\delta_{32} - \delta_{33}\delta_{12}) - \delta_{30}(\delta_{11}\delta_{22} - \delta_{21}\delta_{12})}{D};$$

$$D = \delta_{11}(\delta_{22}\delta_{33} - \delta_{32}\delta_{23}) - \delta_{21}(\delta_{12}\delta_{33} - \delta_{32}\delta_{13}) + \delta_{31}(\delta_{12}\delta_{23} - \delta_{22}\delta_{13}).$$

It is convenient to expand the determinants with δ_{10}, δ_{20}, and δ_{30} outside the brackets if more than one case of loading is to be investigated, since δ_{10}, δ_{20}, and δ_{30} only are affected by changes in the known forces, δ_{11}, δ_{12}, and δ_{13} being the same for all cases of loading.

Since $\int \frac{M_i M_k ds}{EI} = \int \frac{M_k M_i ds}{EI}$, δ_{ik} is always equal to δ_{ki}, a relation which considerably reduces the work of calculating δ_{12}, δ_{22}, etc., from the moment diagrams.

Evaluation of δ_{ik} from the Bending-Moment Diagrams.—The virtual work δ_{ik} is $\int \frac{M_i M_k ds}{EI}$. In actual cases either M_i or M_k can always be split up into segments along each member in which either M_i or M_k varies between the ends of the segment according to a straight-line law, making it possible to derive a simple rule for evaluating $\int \frac{M_i M_k ds}{EI}$ for each segment. Suppose that

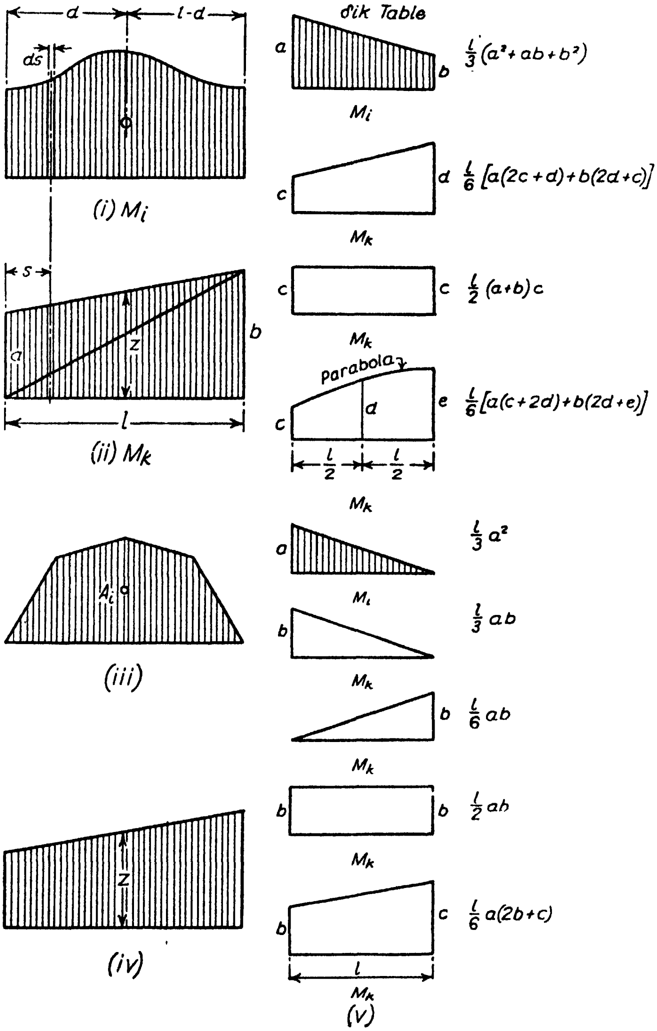


FIG. 46.

the M_i-diagram over a segment of length l [*Fig.* 46 (i) and (ii)] has a general form, and the M_k-diagram varies according to a straight-line law from a at one end to b at the other end. Let A_i be the area of the M_i-diagram and let the distance of its centre of gravity from the left-hand end be d. Let the value of M_k at a distance d from the left-hand end be Z. The value of M_k at a point distant S from the left-hand end is $a\frac{(l-S)}{l} + b\frac{S}{l}$. Therefore

$$\int M_i M_k ds = \int M_i\left[a\frac{(l-S)}{l}\right]ds + \int M_i\left[\frac{bS}{l}\right]ds$$

$$= A_i\left[\frac{a}{l}(l-d) + \frac{b}{l}d\right] = A_i\left(a - \frac{ad}{l} + \frac{bd}{l}\right);$$

Thus
$$\delta_{ik} = \frac{A_i Z}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since in practice it is possible to divide the M_i- and M_k-diagrams into segments, as in *Fig.* 46 (i) and (ii), over various lengths of the members, it is possible to evaluate $E\delta_{ik}$ for any case by applying expression (2) to the various segments of the moment diagrams, adding them together, and dividing the total by E. If the moment of inertia is constant throughout the frame, A_iZ should be evaluated for each part and the total divided by EI. If the moment diagrams are plotted on the tensile side of each member, the integral $\int M_i M_k ds$ is positive when M_i and M_k are on the same side of a member and negative when on opposite sides.

Fig. 46 (v), known as the δ_{ik}-table, shows shaded a trapezoidal and triangular distribution of M_i over a straight length l. The unshaded areas represent various forms of M_k, and opposite each is shown the value of $\int M_i M_k ds$. Sometimes it is quicker to use these values instead of A_iZ. When the M_i-diagram of a member varies according to a curve or straight lines of more than one gradient, but is symmetrical, $\int M_i M_k ds$ can be quickly obtained from A_iZ if the M_k-diagram of the same member varies according to a straight-line law, since the centroid of the M_i-diagram is at the centre [*Fig.* 46 (iii) and (iv)]. To find the values of δ_{10}, δ_{11}, etc. the moment diagrams must be plotted for each of the systems 0 to n. The values of δ_{10}, δ_{11}, . . . δ_{ik} are then obtained by adding the values of $\int \frac{M_0 M_1 ds}{EI}$, $\int \frac{M_1 M_2 ds}{EI}$, . . . $\int \frac{M_i M_k ds}{EI}$ for each segment of each member to which the δ_{ik}-table or the expression A_iZ is applied. The careful choice of the positions of cuts, hinges, and the like simplifies the calculations. If possible, they should be chosen so that each member extends over a short length only, thus keeping the values of δ_{ik}, etc., as small as possible thereby avoiding determining the differences of large quantities. Having calculated the values of X_1, X_2, etc., the bending moment at any point is determined by adding together the moments at the point due to the forces P_1, P_2, etc., and X_1, X_2, etc., which are quickly obtained from the moment diagrams for M_0, M_1, M_2, etc., already plotted. Similarly, the shearing force at any section can be obtained by summing the shearing forces due to each case of loading. In frames symmetrically loaded the unknowns at two or more cuts or hinges may be identical. If, for example, X_1 occurs at four cuts, the virtual work done by $X_1 = 1$ when all other forces act on the frame is, if there are 3 unknowns,

$$4\delta_{10} + 4X_1\,\delta_{11} + 4X_2\,\delta_{12} + 4X_3\,\delta_{13} = 0\,;$$

as 4 cancels and the usual equation is obtained.

Application of the δ_{ik}-Method to a Vertical Vierendeel Cantilever (Building) Frame.—The calculation of the bending moments due to horizontal forces on a vertical Vierendeel cantilever (*Fig.* 47) is important since the wind stresses in a high framed building can usually be determined by dividing the building into a series of single-span frames. The internal columns are assumed to be halved. The moments of inertia of the vertical members of the single-span frames are assumed to be equal to obtain symmetry and small enough to ensure that the actual complete frame is at least as strong as the combined strength of the assumed single-span frames in resisting bending due to horizontal loads. The footings of the columns are shown hinged in *Fig.* 47 but the effect of fixed supports should also be investigated in an actual case; this involves only one additional unknown. All joints are assumed to be fully continuous, a condition which applies to reinforced concrete structures but not always to steel structures.

Let l be the horizontal span of the frame, $h_0, h_1, h_2, \ldots h_r$ heights between horizontal members and $W_0, W_1, W_2, \ldots W_r$ total shearing forces due to wind acting at each horizontal member. Due to symmetry there is never a bending moment at the middle of any horizontal member; it can therefore be assumed that all horizontal members are hinged at mid-span.

Assume a hinge to be inserted in each vertical member just above each horizontal member and each hinge to be replaced by unknown moments $X_1, X_2, X_3, \ldots X_r \ldots X_n$ on the left-hand side, and $X'_1, X'_2, X'_3, \ldots X'_r \ldots X'_n$ on the right-hand side. Referring to *Fig.* 47, case 1′ is opposite hand to case 1, case 2′ is opposite hand to case 2, etc. As usual, X_1, X_2, etc., are at first assumed to be unity.

Let δ'_{11}, δ'_{12}, etc., be the deformations of $X_1 = 1$ when $X'_1 = 1$, $X'_2 = 1$, etc., are acting. Then $\delta'_{11} = \int \frac{M_1 M'_1 . ds}{EI}$, where M'_1 is the moment due to $X'_1 = 1$.

The equations of elasticity obtained by equating the total virtual work done by $X_1 = 1$, $X_2 = 1$, etc., to zero, when all other forces are acting, are:

$$X_1 \delta_{11} + X'_1 \delta'_{11} + X_2 \delta_{12} + X'_2 \delta'_{12} = -\delta_{10}$$
$$X_1 \delta_{21} + X'_1 \delta'_{21} + X_2 \delta_{22} + X'_2 \delta'_{22} + X_3 \delta_{23} + X'_3 \delta'_{23} = -\delta_{20}$$

and so on.

Referring again to *Fig.* 47, it is seen that $\delta_{13} = 0$, $\delta_{24} = 0$, etc., and since $X_1 = -X'_1$, the elastic equations may be written as (3), (4), and (5), etc., in the following. In a case having n unknowns there are n equations, and in each equation there are only three unknowns.

$$X_1(\delta_{11} - \delta'_{11}) + X_2(\delta_{12} - \delta'_{12}) = -\delta_{10} \quad (3)$$
$$X_1(\delta_{21} - \delta'_{21}) + X_2(\delta_{22} - \delta'_{22}) + X_3(\delta_{23} - \delta'_{23}) = -\delta_{20} \quad (4)$$
$$X_2(\delta_{32} - \delta'_{32}) + X_3(\delta_{33} - \delta'_{33}) + X_4(\delta_{34} - \delta'_{34}) = -\delta_{30} \quad (5)$$

and so on. These equations up to any number, say, n if applying to a building having $n + 1$ stories, can be solved by assuming that the deformations δ_{10}, δ_{20}, etc., due to the applied load occur in separate stages.

The general principle of solving three-member equations which can be applied

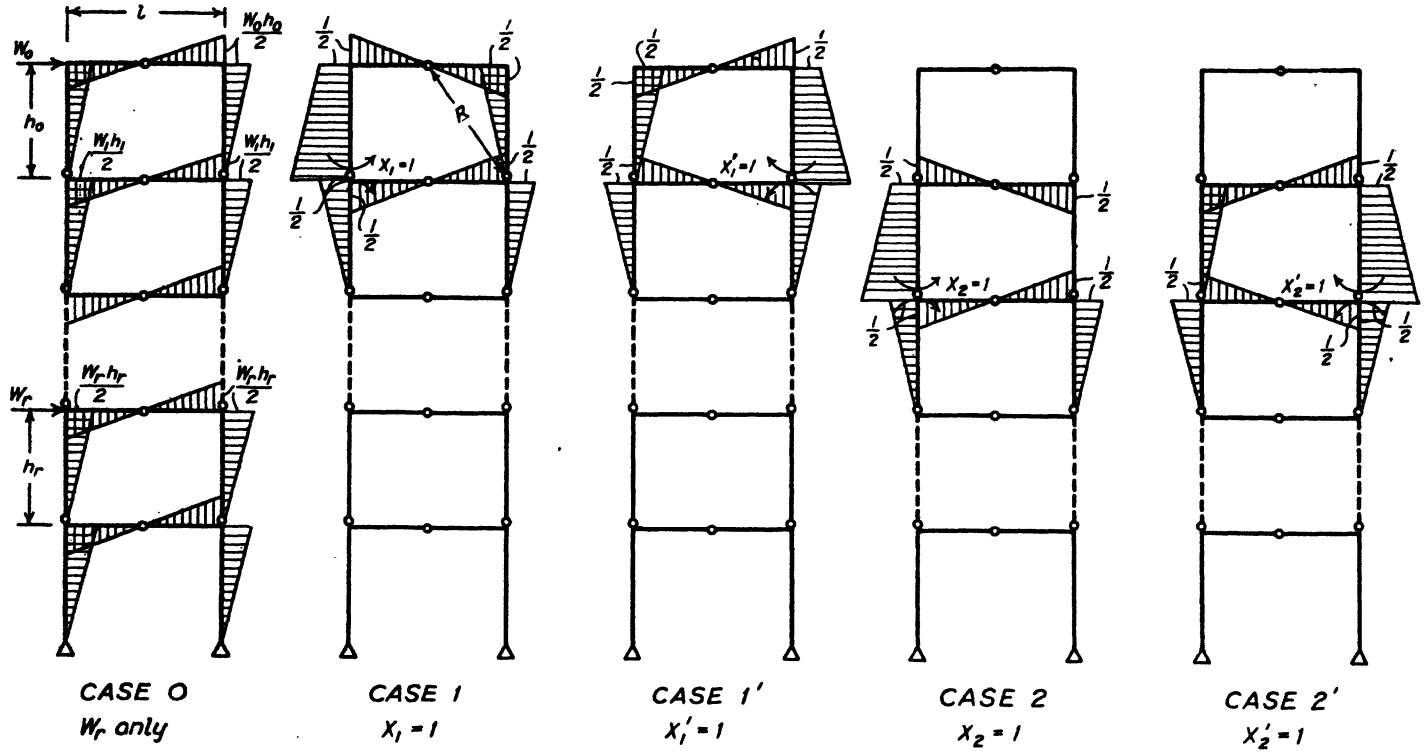

l
W_0
$\frac{W_0 h_0}{2}$
h_0
$\frac{W_1 h_1}{2}$
W_r
$\frac{W_r h_r}{2}$
h_r
$\frac{1}{2}$
$X_1 = 1$
B
$X_1' = 1$
$X_2 = 1$
$X_2' = 1$
CASE 0
W_r only
CASE 1
$X_1 = 1$
CASE 1′
$X_1' = 1$
CASE 2
$X_2 = 1$
CASE 2′
$X_2' = 1$

FIG. 47.

to many similar systems, can be established by considering the case shown in *Fig.* 48, in which a beam continuous over any number of spans has been

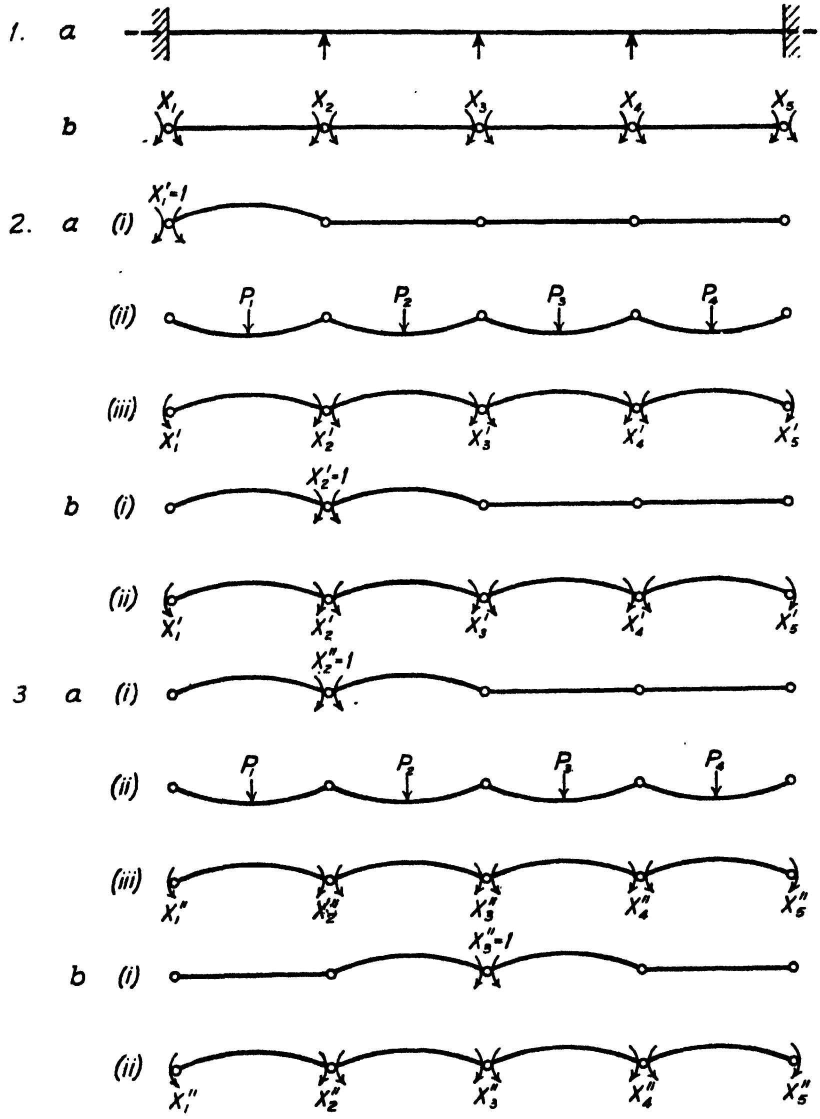


FIG. 48.

chosen for diagrammatic simplicity. The indeterminate system shown at 1(a) is first made statically determinate by the insertion of hinges, the unknown moments of resistance at each hinge being X_1, X_2, X_3, etc., as at 1(b). If there are n unknowns, n cases of loading such as 2(a), 2(b), etc., and n cases such as 3(a), 3(b), etc., must be considered. The various cases of loading and the corresponding equations are:

Case 2(a).—In (i), $X'_1 = 1$ is assumed to act as an internal moment of resistance opposed to (ii), the externally applied loads P_1, P_2, P_3, etc., and in (iii) the unknowns X'_1, etc., are regarded as applied loads. The elastic equation for this condition is $X'_1\,\delta_{11} + X'_2\,\delta_{12} + \ldots = -\,\delta_{10}$.

Case 2(b).—In (i), $X'_2 = 1$ is the internal moment of resistance, and in (ii) X'_1, X'_2, etc., are considered as applied loads. The equation is

$$X'_1\,\delta_{21} + X'_2\,\delta_{22} + \ldots = 0.$$

The moments at the hinges, X_3, X_4, etc., are considered similarly, giving n equations and n unknowns X'_1, X'_2, etc.

Case 3(a).—In (i), $X''_2 = 1$ is considered as the internal moment of resistance; the applied loads are shown in (ii), and in (iii) X''_1, X''_2, etc., are considered as applied loads. The equation is $X''_1\,\delta_{21} + X''_1\,\delta_{22} + \ldots = -\,\delta_{20}$.

Case 3(b).—In (i), consider $X''_3 = 1$ as the internal moment of resistance only, and in (ii) consider X''_1, X''_2, etc., as loads. The equation is

$$X''_1\,\delta_{11} + X''_2\,\delta_{12} + \ldots = 0.$$

The moments at the hinges, X''_3, X''_4, etc., are treated similarly and give n equations.

If conditions 1, 2, 3, etc. (*Fig.* 48), are assumed to be applied simultaneously, the equations for each hinge can be added giving a series of n equations of the form

$$(X'_1 + X''_1 + X'''_1 + \ldots)\delta_{11} + (X'_2 + X''_2 + \ldots)\delta_{12} + \ldots = -\,\delta_{10},$$
$$(X'_1 + X''_1 + X'''_1 + \ldots)\delta_{21} + (X'_2 + X''_2 + \ldots)\delta_{22} + \ldots = -\,\delta_{20},$$

and so on. It is seen that these equations have the same form as the general elastic equations for X_1, X_2, etc. Therefore

$$X_1 = X'_1 + X''_1 + X'''_1 + \ldots,\quad X_2 = X'_2 + X''_2 + X'''_2 + \ldots,$$

and so on. Referring to equations (3), (4), (5), etc., if δ_{10}, δ_{20}, δ_{30}, etc., equal zero, from equation (3),

$$\frac{X_1}{X_2} = -\,\frac{\delta_{12} - \delta'_{12}}{\delta_{11} - \delta'_{11}} =, \text{ say, } K_1.$$

From equation (4), since $X_1 = K_1X_2$,

$$K_1X_2(\delta_{21} - \delta'_{21}) + X_2(\delta_{22} - \delta'_{22}) + X_3(\delta_{23} - \delta'_{23}) = 0.$$

Therefore $$\frac{X_2}{X_3} = -\,\frac{\delta_{23} - \delta'_{23}}{K_1(\delta_{21} - \delta'_{21}) + (\delta_{22} - \delta'_{22})} =, \text{ say, } K_2.$$

Generally, $$K_r = \frac{X_r}{X_{r+1}} = -\,\frac{\delta_{r(r+1)} - \delta'_{r(r+1)}}{K_{r-1}(\delta_{r(r-1)} - \delta'_{r(r-1)}) + (\delta_{rr} - \delta'_{rr})}.$$

Similarly, starting with the nth equation and assuming

$$\delta'_{n0} = \delta_{n-10} = \delta_{n-20} = \ldots = 0,$$

$$K'_r = \frac{X_r}{X_{r-1}} = -\,\frac{\delta_{r(r-1)} - \delta'_{r(r-1)}}{K_{r+1}(\delta_{r(r+1)} - \delta'_{r(r+1)}) + (\delta_{rr} - \delta'_{rr})}.$$

Values of K and K' can be obtained for each horizontal support of a vertical Vierendeel frame, as in the coefficient-of-restraint method for continuous beams, by applying the two foregoing general expressions successively, starting from the top in the case of K_r and from the bottom in the case of K'_r (*Fig.* 49). Referring to *Fig.* 47, if h and I are constant, $\delta_{r(r+1)}$, δ_{rr}, δ_{r-1}, etc., are also constant. This

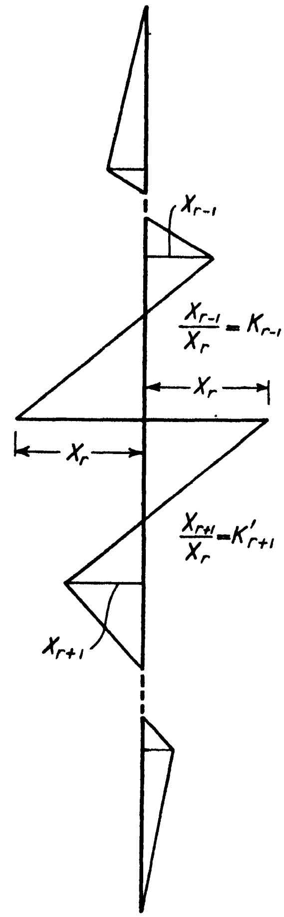


FIG. 49.

is generally the case for several stories and the calculation of K_1, K_2, etc., and K'_1, K'_2, etc., is not therefore too laborious. The expression

$$\delta_{ik} = \int \frac{M_i M_k}{EI} ds = A_i Z,$$

as is seen from *Fig.* 47, gives a general expression in terms of h, l, and I, which can be applied throughout the frame. Since the frame is symmetrical the moment M_0 at the top of each vertical member is $\frac{W_r h_r}{2}$.

In the case of moments M_1, M_2, etc., the reactions at the hinges must all be in the direction of R [*Fig.* 47, Case (1)]. The reactions and moments can then be evaluated as shown. If the values of each δ_0 term, except δ_{r0}, is zero,

$$X_{r-1}(\delta_{r(r-1)} - \delta'_{r(r-1)}) + X_r(\delta_{rr} - \delta'_{rr}) + X_{r+1}(\delta_{r(r+1)} - \delta'_{r(r+1)}) = -\delta_{r0},$$

and $X_{r-1} = K_{r-1}X_r$, and $X_{r+1} = K'_{r+1}X_r$. Therefore

$$K_{r-1}X_r(\delta_{r(r-1)} - \delta'_{r(r-1)}) + X_r(\delta_{rr} - \delta'_{rr}) + K'_{r+1}X_r(\delta_{r(r+1)} - \delta'_{r(r+1)}) = -\delta_{r0}.$$

Thus $$X^r_r = -\frac{\delta_{r0}}{K_{r-1}(\delta_{r(r-1)} - \delta'_{r(r-1)}) + (\delta_{rr} - \delta'_{rr}) + K'_{r+1}(\delta_{r(r+1)} - \delta'_{r(r+1)})}.$$

As already stated, X^r_r is the value of X_r when the value of each δ_0 term, except δ_{0r}, is zero. By applying values of K_r, K_{r-1}, etc., and K'_r, K'_{r-1}, etc., values of X^r_{r-1}, X^r_{r-2}, X^r_{r+1}, X^r_{r+2}, etc., can be determined. Similarly, it can be assumed that the values of each δ_0 term, except δ_{10}, δ_{20}, etc., in turn, are zero, and the corresponding values X'_{10}, X''_{10}, etc., can be determined. Finally, $X_1 = X'_1 + X''_1 + X'''_1 + \ldots$, and $X_2 = X'_2 + X''_2 + X'''_2 + \ldots$ The resultant moment at any point in the frame is the algebraic sum of the moments due to X_1, X_2, X_3, etc.

Calculations can be conveniently tabulated as shown in *Fig.* 50, each column of which is used as described in the following.

Column (1).—The values of δ_{rr} and $\delta_{r(r+1)}$ are functions of the frame and external load and repeat for the same values of h and I.

Column (2).—The values of K_r are determined from δ_{rr} and $\delta_{r(r+1)}$ commencing from the top of the frame and proceeding with successive ratios.

Column (3).—The values of K'_r are determined in the same way as those of K_r but starting from the bottom of the frame and working upwards.

Column (4).—The values of δ_{r0} are determined by applying $\delta_{0k} = A_iZ$ to the diagrams for the load M_1, M_2, etc. If h is constant and W_r is expressed as a simple function of h a general expression for δ_{0k} can be derived.

Columns (5), (6), (7), (8), and (9).—The values of X^r_r are determined in each of these columns first for the condition $\delta_{r0} \neq 0$. Other values in each column are obtained from the corresponding value of $\delta_{r0} \neq 0$ by applying successive values of K_r working upwards and of K'_r working downwards.

Most of the values of δ_{rr}, etc., obtained by substituting values of h and l in the expression $\delta'_{ik} = \int \frac{M_iM_k}{EI}ds = A_iZ$ can be reduced to simple terms.

For example, if I is constant,

$$\delta'_{rr} = -\left\{\frac{h_r}{6}.\frac{1}{2}\left[\left(2 \times \frac{1}{2}\right) + 1\right]2\right\} - \left[4.\frac{l}{6} \times \left(\frac{1}{2}\right)^2\right] + \left[2.\frac{h_{r+1}}{3}\left(\frac{1}{2}\right)^2\right]$$

$$= -\frac{h_r}{3} - \frac{l}{6} + \frac{h_{r+1}}{6}.$$

Some numerical examples of the foregoing method are given by Dr. R. Gartner in "Statically-Indeterminate Structures."

r	$\delta_{rr} - A_i Z$ $\delta_{rr+1} - A_i Z$	$K_r = -\frac{\delta_{rr+1} - \delta'_{rr+1}}{K_{r-1}(\delta_{rr-1} - \delta'_{rr-1}) + (\delta_{rr} - \delta'_{rr})}$	$K'_r = -\frac{\delta_{rr-1} - \delta'_{rr-1}}{K_{r+1}(\delta_{rr+1} - \delta'_{rr+1}) + (\delta_{rr} - \delta'_{rr})}$	δ_{r0}	X_1		X_2		X_3		X_4		X_5	
1	$\delta_{11} - \delta'_{11}$ $\delta_{12} - \delta'_{12}$	$K_1 =$	$K'_1 =$		X'_1	$\delta_{10} \neq 0$	X'_2		X'_3		X'_4		X'_5	
2	$\delta_{22} - \delta'_{22}$ $\delta_{23} - \delta'_{23}$	$K_2 =$	$K'_2 =$		X''_1		X''_2	$\delta_{20} \neq 0$	X''_3		X''_4		X''_5	
3	$\delta_{33} - \delta'_{33}$ $\delta_{34} - \delta'_{34}$	$K_3 =$	$K'_3 =$		X'''_1		X'''_2		X'''_3	$\delta_{30} \neq 0$	X'''_4		X'''_5	
4	$\delta_{44} - \delta'_{44}$ $\delta_{45} - \delta'_{45}$	$K_4 =$	$K'_4 =$		X''''_1		X''''_2		X''''_3		X''''_4	$\delta_{40} \neq 0$	X''''_5	
5	$\delta_{55} - \delta'_{55}$	$K_5 =$	$K'_5 =$		X'''''_1		X'''''_2		X'''''_3		X'''''_4		X'''''_5	$\delta_{50} \neq 0$
	1	2	3	4	5		6		7		8		9	

FIG. 50.

Müller-Breslau's Solution of Symmetrical Arches and Portal Frames.

The application, originally due to Professor Müller-Breslau, of the general elastic equations to symmetrical arches and portal frames is a simple method of calculation when a frame subjected to forces in any direction has fixed supports even if the moment of inertia varies.

A frame of single span having fixed supports at A and B (*Fig.* 51) can be made statically determinate by: (i) Replacing the fixed support A by a roller bearing free to rotate and move horizontally; (ii) replacing the fixed support B by a hinge free to rotate but restrained against horizontal and vertical movements; and (iii) replacing the restraint moments and horizontal thrusts, removed by the preceding steps, by equal and opposite unknowns X_1, X_2, and X_3 (see *Fig.* 51) each of which is at first assumed to be unity acting at the point O at the ends of two perfectly rigid arms continuous with the frame at A and B. X_1 is a moment; X_2 and X_3 are forces. It will be seen that the position of O can be chosen so that only one unknown occurs in each of the three elastic equations. Since the arms are assumed to be rigid, the moments of inertia of the arms are infinite, and for the arms $\frac{M}{EI} = 0$.

The analysis proceeds as follows. Draw the x and y axes through the point O. Calculate the moments about the centre-lines of members of the frame, as seen in elevation, referred to the axes by the ordinates x and y. Plot the ordinates of the moments normal to the centre-lines of the members of the frame. Divide one-half of the frame into equal segments each of length ds. The forces and moments are divided into the following cases as shown in the diagrams in *Fig.* 51:

Case (0) [*Fig.* 51 (*a*)]. The ordinates M_0 of the bending-moment diagram are the bending moments due to the applied load acting on the frame which is now statically determinate. When the loads act vertically the moments M_0 are the same as for a simply-supported beam.

Case (1) [*Fig.* 51 (*b*)]. The ordinates M_1 are the bending moments due to $X_1 = 1$, and each ordinate is therefore unity.

Case (2) [*Fig.* 51 (*c*)]. The ordinates M_2 are the bending moments due to $X_2 = 1$, and each ordinate therefore equals y.

Case (3) [*Fig.* 51 (*d*)]. The ordinates M_3 are the bending moments due to $X_3 = 1$, and each ordinate therefore equals x.

The resultant bending moment and forces acting at any section of the frame are the algebraic sums of moments and forces respectively acting at the section due to each of the cases (0) to (3).

The position of the point O is chosen so that $\int \frac{x}{I} ds = 0$ and $\int \frac{y}{I} ds = 0$, that is, O is at the centre of gravity of the elastic weights for unit bending moments acting throughout the frame, the elastic weights being concentrated at the centre-line of the elevation of the members of the frame. Then

$$\delta_{12} = \delta_{21} = \int \frac{M_1 M_2}{EI} ds = \int \frac{y}{EI} ds = 0,$$

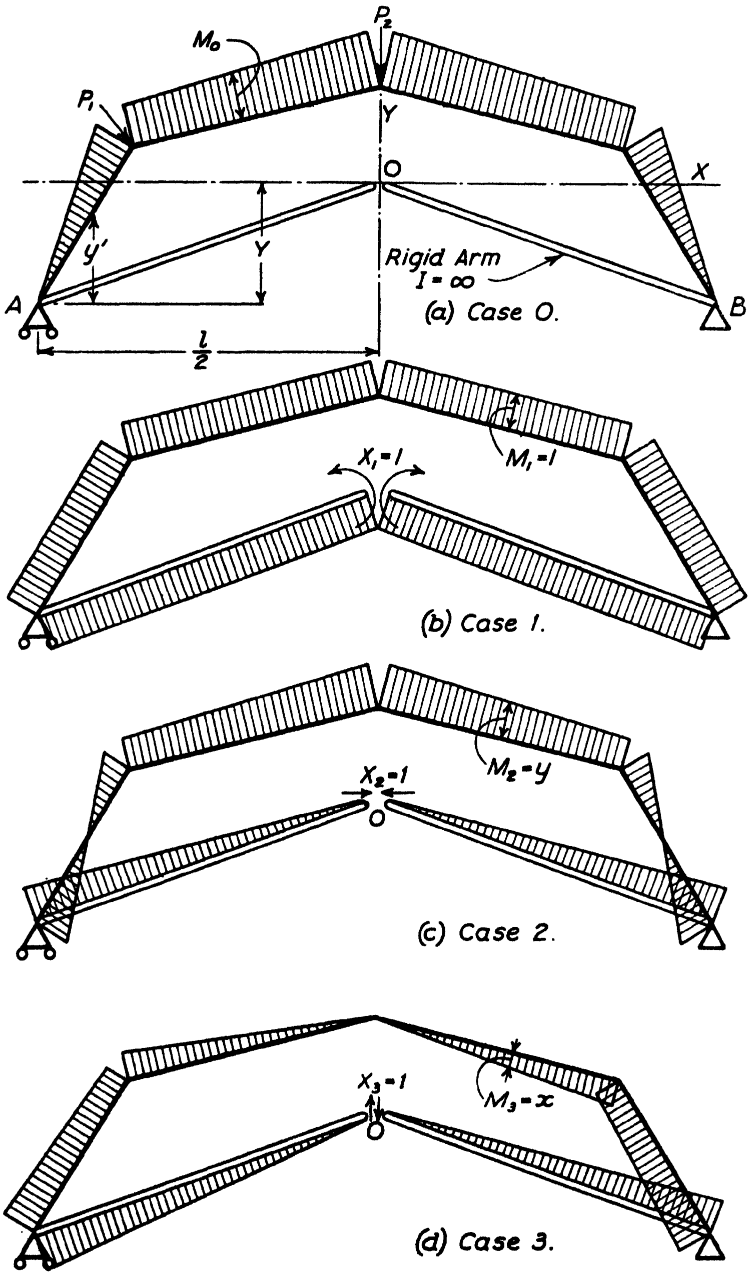

P2
M0
P1
Y
O
X
y'
Y
Rigid Arm
I = ∞
A
B
(a) Case 0.
l/2
X1 = 1
M1 = 1
(b) Case 1.
X2 = 1
M2 = y
O
(c) Case 2.
X3 = 1
M3 = x
O
(d) Case 3.

FIG. 51.

$$\delta_{13} = \delta_{31} = \int \frac{M_1 M_3}{EI} ds = \int \frac{x}{EI} ds = 0,$$

and
$$\delta_{23} = \delta_{32} = \int \frac{M_2 M_3}{EI} ds = \int \frac{xy}{EI} ds = 0,$$

since the frame is symmetrical and each x-ordinate for case (3) is of opposite sign to the corresponding ordinate on the opposite side of the axis of symmetry.

Since the point O is the centre of gravity of the elastic weights, by symmetry O must be on the central vertical axis of the frame and, if y' represents the height of a small segment ds of the frame above the supports and Y is the height of O above the supports, $Y = \dfrac{\int \frac{y'ds}{I}}{\int \frac{1}{I} ds}$. If y' and I are functions of s, this expression can be exactly integrated. Otherwise the expression $Y = \dfrac{\sum \frac{y'ds}{I}}{\sum \frac{ds}{I}}$, which can always be evaluated, is sufficiently accurate for practical purposes. The evaluation is made by dividing the members of the frame into short lengths and applying Simpson's rule.

Having chosen the point O so that $\delta_{12} = \delta_{21} = \delta_{13} = \delta_{31} = \delta_{23} = \delta_{32} = 0$, the general elastic equations for three unknowns are

$$(1)\ \delta_{10} + X_1 \delta_{11} + X_2 \delta_{12} + X_3 \delta_{13} = 0,$$
$$(2)\ \delta_{20} + X_1 \delta_{21} + X_2 \delta_{22} + X_3 \delta_{23} = 0,$$
and
$$(3)\ \delta_{30} + X_1 \delta_{31} + X_2 \delta_{32} + X_3 \delta_{33} = 0.$$

These reduce to

$$(1)\ \delta_{10} + X_1 \delta_{11} = 0, \quad \text{or} \quad X_1 = -\frac{\delta_{10}}{\delta_{11}}.$$

$$(2)\ \delta_{20} + X_2 \delta_{22} = 0, \quad \text{or} \quad X_2 = -\frac{\delta_{20}}{\delta_{22}}.$$

$$(3)\ \delta_{30} + X_3 \delta_{33} = 0, \quad \text{or} \quad X_3 = -\frac{\delta_{30}}{\delta_{33}}.$$

Since $\delta_{ik} = \int \frac{M_i M_k}{EI} . ds$,

$$X_1 = -\frac{\int \frac{M_1 M_0}{EI} ds}{\int \frac{M_1^2}{EI} ds}$$ and since $M_1 = 1$ throughout the frame,

$$X_1 = -\frac{\int \frac{M_0}{I} ds}{\int \frac{1}{I} ds} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$X_2 = -\frac{\int \frac{M_2 M_0}{EI} ds}{\int \frac{M_2^2}{EI} ds} \text{ and since } M_2 = y,$$

$$X_2 = -\frac{\int \frac{M_0 y}{I} ds}{\int \frac{y^2}{I} ds} \quad \ldots \ldots \quad (7)$$

$$X_3 = -\frac{\int \frac{M_3 M_0}{EI} ds}{\int \frac{M_3^2}{EI}} \text{ and since } M_3 = x,$$

$$X_3 = -\frac{\int \frac{M_0 x}{I} ds}{\int \frac{x^2}{I} ds} \quad \ldots \ldots \quad (8)$$

These integrals can be evaluated in any given case by either (i) dividing the frame into straight lengths throughout each of which I is constant and applying the expression $\int \frac{M_i M_k}{EI} ds = A_i Z$, or (ii) expressing M and I as functions of s and integrating mathematically, or (iii) integrating approximately by summation, which is sufficiently accurate for practical purposes.

In method (iii) one-half of the frame is divided into equal segments and, using the average values of M and I for each segment, the integrals are evaluated by direct summation, using Simpson's rule if desired for slightly greater accuracy. Therefore

$$X_1 = -\frac{\sum \frac{M_0}{I} ds}{\sum \frac{1}{I} ds} \quad \ldots \ldots \quad (9)$$

$$X_2 = -\frac{\sum \frac{M_0 y}{I} ds}{\sum \frac{y^2}{I} ds} \quad \ldots \ldots \quad (10)$$

$$X_3 = -\frac{\sum \frac{M_0 x}{I} ds}{\sum \frac{x^2}{I} ds} \quad \ldots \ldots \quad (11)$$

It is best to tabulate the calculations as shown in *Fig.* 52. *ds* can be omitted

DETERMINATION OF Y					
	Segment	I	$\frac{1}{I}$	y'	$\frac{y'}{I}$
Left-hand side of frame only	1				
	2				
	3				
	etc				
		$\sum\frac{1}{I}=$		$\sum\frac{y'}{I}=$	
$Y=\sum\frac{y'}{I}\Big/\sum\frac{1}{I}$					

DETERMINATION OF X_1, X_2 & X_3												
	Segment	$\frac{1}{I}$	M_0	$\frac{M_0}{I}$	y	$\frac{M_0y}{I}$	y^2	$\frac{y^2}{I}$	x	$\frac{M_0x}{I}$	x^2	$\frac{x^2}{I}$
Left-hand side of frame	1											
	2											
	etc											
Right-hand side of frame	1											
	2											
	etc.											
	$\sum\frac{1}{I}=$		$\sum\frac{M_0}{I}=$		$\sum\frac{M_0y}{I}=$		$\sum\frac{y^2}{I}=$		$\sum\frac{M_0x}{I}=$		$\sum\frac{x^2}{I}=$	
	$X_1=\dfrac{\sum\frac{M_0}{I}}{\sum\frac{1}{I}}$				$X_2=\dfrac{\sum\frac{M_0y}{I}}{\sum\frac{y^2}{I}}$				$X_3=\dfrac{\sum\frac{M_0x}{I}}{\sum\frac{x^2}{I}}$			

FIG. 52.

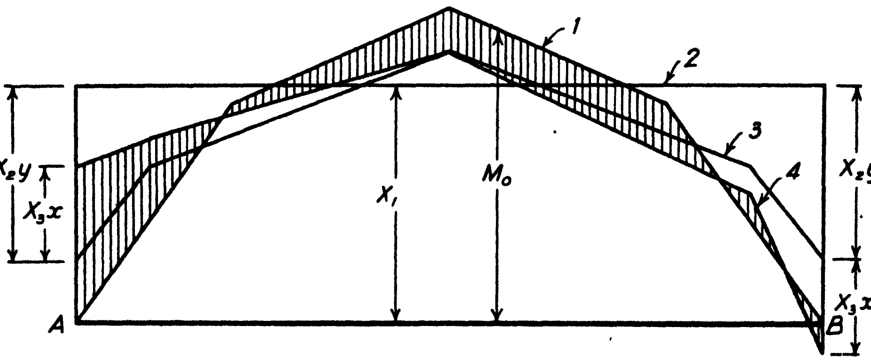


FIG. 53.

throughout since all segments are equal in length. The values of $\frac{1}{I}$ for the rigid arms are omitted since I is infinity.

Having determined X_1, X_2, and X_3 it is easy to tabulate the bending moments due to X_1, X_2, X_3, and M_0 for each section of the frame or to draw a diagram thereof. Thus on the base AB (*Fig.* 53), representing the span of the frame, the ordinates of line (1) are the values of M_0, and the ordinates of line (2) are values of $X_1M_1 = X_1$, since $M_1 = 1$. Line (3) is obtained by deducting the values of X_2y from the ordinates of line (2), and line (4) is obtained by deducting the values of X_3x from the ordinates of line (3). The sign changes on the left-hand side of the central vertical axis. The ordinates between lines (1) and (4) are then $M_0 + X_1 + X_2y + X_3x$; X_1 has the opposite sign to M_0, and X_2y and X_3x are opposed to X_1 with y negative and x positive, and vice versa.

Alternatively it may be more convenient to assume the rigid arms to be removed and replaced by the internal forces and moments at the supports acting as external forces and moments; the factors to be tabulated, or plotted, and added algebraically are: (i) the free moments M_0; (ii) the moments due to the horizontal thrust X_2y' at the level of the supports, and (iii) the moments due to the restraint moments at the supports being distributed trapezoidally along the length of the frame and at A being $X_1 - X_2y + X_3\frac{l}{2}$, and at B being $X_1 - X_2y - X_3\frac{l}{2}$.

The moment due to the thrust is always opposite in sign to the free moment, which for a vertical load is the same as for a simply-supported beam. The moments at the supports always provide a restraint that reduces the difference between the free moment and the moment due to the thrust. Having determined the moments at the supports and the horizontal thrust, the vertical reaction at each support can be calculated, and hence the direction, amount, and eccentricity of the resultant thrust. The line of thrust for the frame can then be plotted.

Since the moment due to the thrust is X_2y', it is seen that if the outline of the frame is identical with the form of the free bending-moment diagram, plotted to a scale in which its height at mid-span equals the height of the frame at mid-span, the free moment and moment due to the thrust are equal and opposite, and the frame is not subjected to bending moment but to direct stresses only.

Profile of an Ideal Arch.

For a single member carrying loads to span between two points without bending moment developing, the profile of the member, if in tension, must be the same as that of the link polygon connecting the load-forces and the two points, that is the profile assumed by a weightless cable supporting the loads and suspended between the two points. If the member is to be in compression only, the profile must be the same as that of the cable but must be concave downwards. Referring to the upper diagram in *Fig.* 54, P_1, P_2, etc., are any loads acting on a single member having the form of a link polygon. The thrusts developed in each length of the member are, as shown in the lower diagram, T_1, T_3, etc., the right-hand reaction is R_R, and the left-hand horizontal and vertical reactions are H and R_L respectively. Both supports are hinged. Consider the bending moment at any point X. The load P_2 can be replaced by the thrusts T_3 and T_4; the thrust T_4 passes through X and therefore causes no moment at X. T_3 can be combined with P_1 to give T_1. T_1 can be combined with R_L to give H. Therefore the bending moment at X due to R_L, P_1, and P_2 is $-Hy$, and due to horizontal thrust H is Hy. This again shows that the outline of the member is the same as that of the link polygon, there is no bending moment at X or at other points, only direct thrusts being developed.

Also the vertical ordinates of the link polygon for a member of any form having a hinge at one support and a roller bearing at the other, if drawn to a suitable scale on a base connecting the supports, represents the bending moments in the member. Also, a single member spanning between two unyielding sup-

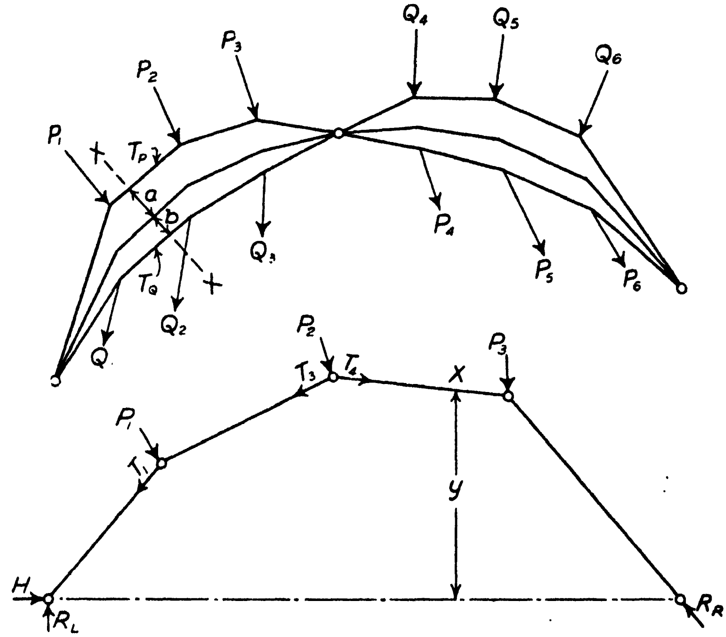


FIG. 54.

ports in the direction of the member is not subjected to bending moment provided the gravity axis of the member coincides with the bending-moment diagram, plotted to any vertical scale, for the loads acting on the member and with the supports replaced by one hinged support and one roller-bearing support. The thrust developed in the member depends on the rise of the member above the line joining the supports.

To determine accurately the ideal profile of an arch it is often better to calculate the ordinates of the free bending-moment diagram instead of using the graphical construction for the link polygon. A suitable vertical scale must be used to give the arch the required rise and thrust on the abutments. By considering either the form of the link polygon or of the bending-moment diagram, it is seen that the shape of the ideal arch is a parabola for uniformly-distributed vertical load, a circle for uniformly-distributed radial load, a catenary for a load uniformly distributed along the length of the arch, and a regular link polygon for a series of equally-spaced concentrated loads.

If an arch is subjected to unsymmetrical live loads, for example the forces from high winds on one side of a hangar, link polygons between assumed hinged supports and passing through a common crown should be constructed for the two extreme conditions of load as in *Fig.* 54. The ideal profile of the arch lies between the two link polygons, so that at the point of maximum bending moment, say, at section X – X, $T_P a = T_Q b$, where T_P and T_Q are the thrusts for the two extreme cases of loading. In the case of heavy concentrated rolling loads on bridges, influence lines, which can be plotted by finding X_1, X_2, and X_3 for unit load at suitable intervals along the span, can be used to obtain a bending-moment envelope from which, combined with a consideration of any particularly influential concentrated loads, the ideal profile of the arch can be determined.

The profile is therefore determined by the loads. The rise may depend on functional requirements or æsthetic considerations, or a high rise may be required to reduce the horizontal thrust on the supports. The moment of inertia is often increased near the abutments for the sake of appearance, or to concentrate weight near the supports, or to increase the bending moments at the supports where extra depth is easily provided. Sometimes, as in the case of a hangar on yielding foundations, it may be advisable to reduce the thickness at the crown and at the supports in order to reduce the bending moments at these points and thus to reduce the increase of stress due to possible yielding of the supports.

If a support yields an amount, say, δ_1, δ_2, or δ_3 in directions X_1, X_2, or X_3 respectively, or if a support rotates, the total virtual work done in the direction of $X_1 = 1$, $X_2 = 1$, or $X_3 = 1$ is δ; thus the elastic equations become

$$\delta_{10} + X_1\delta_{11} = -\delta_1 \quad (12)$$

$$\delta_{20} + X_2\delta_{22} = -\delta_2 \quad (13)$$

and

$$\delta_{30} + X_3\delta_{33} = -\delta_3 \quad (14)$$

Generally equation (13) is the only one requiring consideration and this condition often occurs in a flat arch having a large horizontal thrust. A safe limiting value for δ should be used. To ensure that a sufficiently large factor of safety F is provided, yielding of the supports under the ultimate load should be considered, because the yield might then be more than $F \times$ (yield under working load). In some cases it may be necessary to consider the effect of either the vertical or lateral deflection of the arch under ultimate load increasing the eccentricity of the thrust by an appreciable amount. Deflection due to bending can be calculated as the bending moment with the elastic weights as the load at a section assuming the full length of the arch to be flattened and the arch to span as a simply-supported beam. A fall in temperature in a frame or arch, or shortening due to shrinking, creep, or thrust, has the same effect as a yield of the support and a safe value of δ should be assumed. In some cases the shape of the arch should be such that the ideal outline develops when deformation due to the ultimate load, maximum creep, and change of temperature occurs.

In a tied arch or bowstring girder, where the stretch of the tie is a function of the tension, δ is a function of X_2; thus the second elastic equation becomes

$$\delta_{20} + X_2\delta_{22} = -X_2\frac{l}{EA}$$

where l is the length, E the modulus of elasticity, and A the cross-sectional area of the tie.

Application of the General Elastic Equations for Fixed Arches.

PORTAL-FRAME PIPE BRIDGE.—A portal-frame is an arch of rectangular profile and can be designed as such. *Fig.* 55 shows the outline of an arch of this kind of 45-ft. span which was designed as a bridge to carry pipes. The method of calculation is to divide the semi-arch into twelve parts of equal length. The free bending-moment diagram from which values of M_0 are obtained is

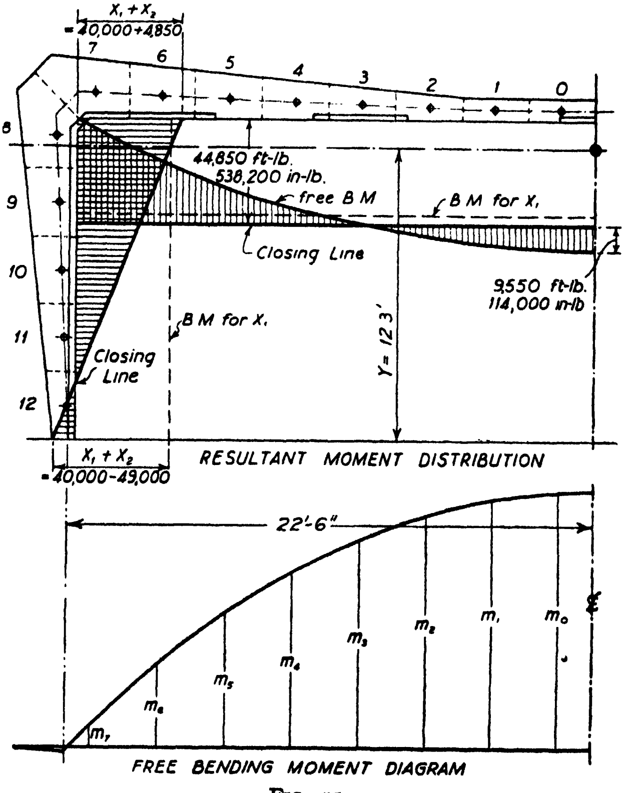


FIG. 55.

shown in the lower part of *Fig.* 55. The values of y' and M_0 are tabulated in *Fig.* 56 with the extensions necessary for obtaining the values of Y, X_1, and X_2. The distribution of bending moments M_0 and of moments due to X_1 and X_2 is shown in *Fig.* 57. The resultant distribution of bending moment is shown in the upper part of *Fig.* 55. The closing line of this diagram is obtained from the sum of the moments due to X_1 and X_2, and represents bending moments opposite in sign to the free bending moments. The diagram of the resultant bending moments for the horizontal part of the frame is therefore the shaded

DETERMINATION OF Y						DETERMINATION OF X_1 & X_2				
Divn	y'	I (ft)	$\frac{1}{I}$	$\frac{y'}{I}$	$\frac{Y}{I}$	free B.M. (ft-lb.)	y	$\frac{M_0}{I}$	$\frac{y^2}{I}$	$\frac{M_0 y}{I}$
0	14·0	·056	18·0	250	30·4	54,000	1·6	970,000	46	1,550,000
1	14·0	·056	18·0	250	30·4	52,000	1·6	935,000	46	1,490,000
2	14·0	·096	10·4	145	17·6	49,000	1·7	510,000	30	870,000
3	14·2	·282	3·5	50	6·7	44,000	1·8	155,000	12	280,000
4	14·3	·274	3·6	52	7·3	37,000	1·9	133,000	13	252,000
5	14·4	·38	2·6	38	5·5	28,000	2·0	73,000	10	146,000
6	14·5	1·01	1·0	14	2·2	18,000	2·2	18,000	5	40,000
7	14·7	1·43	0·7	10	1·7	5,000	2·3	3,500	3·7	8,000
8	12·8	1·51	0·66	8·5	·3	− 400	0·5	− 270	0·2	− 135
9	10·0	·92	1·1	11	−2·2	0	−2·4	0	6·3	0
10	7·1	·49	2·2	14	−11·4	0	−5·2	0	60·0	0
11	4·3	·33	3·3	13	−24·3	0	−8·0	0	211·0	0
12	1·4	·17	5·9	8	−64·0	0	−10·9	0	703·0	0
TOTALS.			69·96	863	+0·2			2,797,200	1146·2	4,636,000

$$Y = \frac{\sum \frac{y'}{I}}{\sum \frac{1}{I}} = \frac{863}{69 \cdot 96} = 12 \cdot 3 \text{ ft}$$

$$X_1 = \frac{\sum \frac{M_0}{I}}{\sum \frac{1}{I}} = 40{,}000 \text{ ft-lb.} = 480{,}000 \text{ in-lb.}$$

$$X_2 = \frac{\sum \frac{M_0 y}{I}}{\sum \frac{y^2}{I}} = 4{,}030 \text{ ft-lb.} = 48{,}400 \text{ in-lb.}$$

FIG. 56.

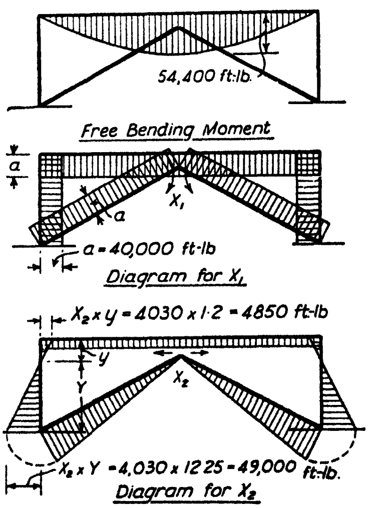
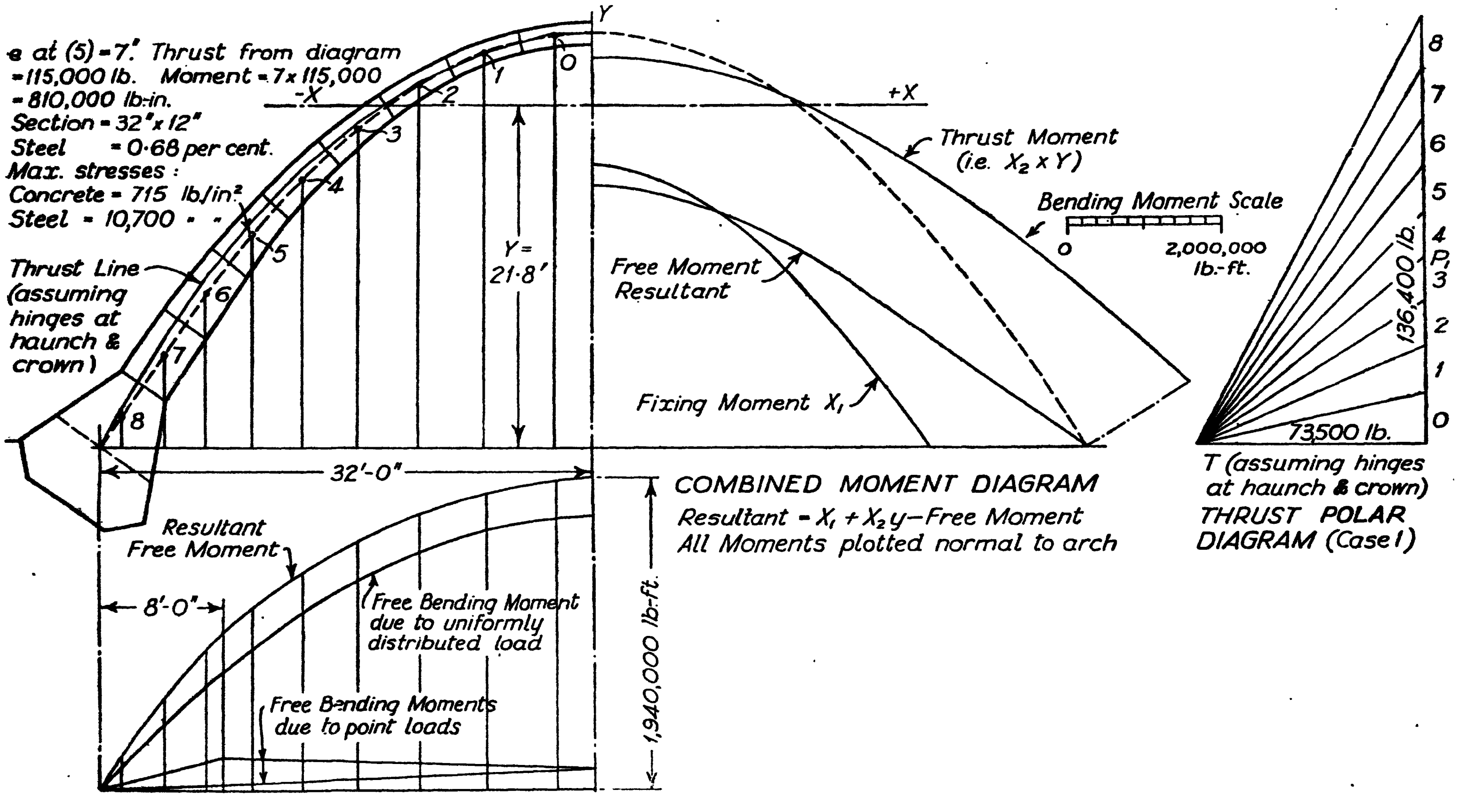
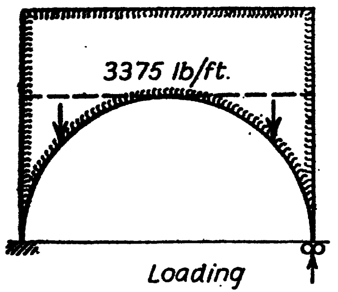


FIG. 57.

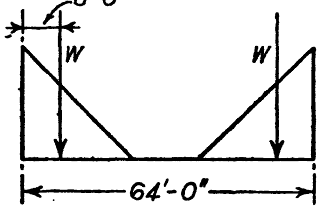


e at (5) = 7." Thrust from diagram = 115,000 lb. Moment = 7 x 115,000 = 810,000 lb-in.
Section = 32" x 12"
Steel = 0·68 per cent.
Max. stresses:
Concrete = 715 lb/in²
Steel = 10,700 " "
Thrust Line (assuming hinges at haunch & crown)
Y
-X
+X
0
1
2
3
4
5
6
7
8
Y = 21·8'
32'-0"
Thrust Moment (i.e. X₂ x Y)
Bending Moment Scale
0
2,000,000 lb.-ft.
Free Moment Resultant
Fixing Moment X₁
COMBINED MOMENT DIAGRAM
Resultant = X₁ + X₂y − Free Moment
All Moments plotted normal to arch
Resultant Free Moment
8'-0"
Free Bending Moment due to uniformly distributed load
Free Bending Moments due to point loads
1,940,000 lb-ft.
136,400 lb.
P₁
73500 lb.
T (assuming hinges at haunch & crown)
THRUST POLAR DIAGRAM (Case 1)

Fig. 58.



CASE 1.



Uniform Load Free B.M

Loading $= 25 \times 120 = 3000$

O.W. 2'-6" $= 375$

$3{,}375$ lb./ft.^{2}

$$\text{B.M.} = \frac{3375 \times 64^2}{8} = 1{,}720{,}000 \text{ lb-ft.}$$

$$\text{Point Load} = 28{,}800 \times 8 \times \frac{56}{64} = 201{,}000 \text{ lb-ft.}$$

(1 only)

DETERMINATION OF Y								DETERMINATION OF X_1 & X_2				
Division	d	I	y_1	$\frac{1}{I}$	$\frac{y_1}{I}$	$Y-y_1$	$\frac{Y-y_1}{I}$	Free m (Kips)	y	$\frac{m}{I}$	$\frac{y^2}{I}$	$\frac{my}{I}$
0	1·6	·34	26·1	2·94	76·8	4·3	12·6	1,940	4·4	5,700	57·0	25,000
1	1·7	·41	25·0	2·43	61·0	3·2	7·8	1,860	3·3	4,540	26·5	15,000
2	1·8	·49	23·0	2·04	47·0	1·2	2·45	1,730	1·2	3,530	2·9	4,240
3	2·0	·67	20·1	1·49	30·0	-1·7	-2·5	1,550	-1·6	2,720	3·8	-4,350
4	2·3	1·01	17·0	·99	16·8	-4·8	-4·7	1,360	-4·7	1,350	22·0	-6,340
5	2·7	1·64	13·5	·61	8·2	-7·3	-4·4	1,140	-8·3	694	42·0	-5,750
6	3·2	2·73	9·7	·37	3·6	-12·1	-4·4	870	-12·0	318	53·0	-3,810
7	3·5	3·56	5·9	·28	1·6	-16·9	-4·75	570	-15·9	160	71·0	-2,540
8	5·4	13·20	2·0	·076	·15	-19·8	-1·5	200	-19·9	15	30·0	- 292
Totals				11·226	245·15		+0·6			19,027	308·2	+21,448

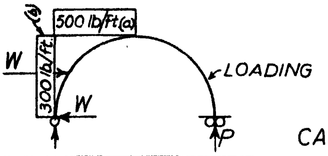
$$Y = \frac{\sum \frac{y_1}{I}}{\sum \frac{1}{I}} = 21{\cdot}8 \text{ ft} \qquad X_1 = \frac{\sum \frac{m}{I}}{\sum \frac{1}{I}} = \frac{19{,}027 \times 1{,}000}{11{\cdot}226} = 1{,}690{,}000 \text{ lb-ft.}$$

$$X_2 = \frac{\sum \frac{my}{I}}{\sum \frac{y^2}{I}} = \frac{21{,}448{,}000}{308{\cdot}2} = 69{,}600 \text{ lb.}$$

$Y \times X_2 = 1{,}520{,}000$ lb-ft.

(crown) $y_0 \times X_2 = 314{,}000$ lb-ft.

Fig. 59.



$W = 26 \times 300 = 7{,}800$ lb.

$P = \frac{7800 \times 13}{64} = 1{,}580$ lb.

CASE 2

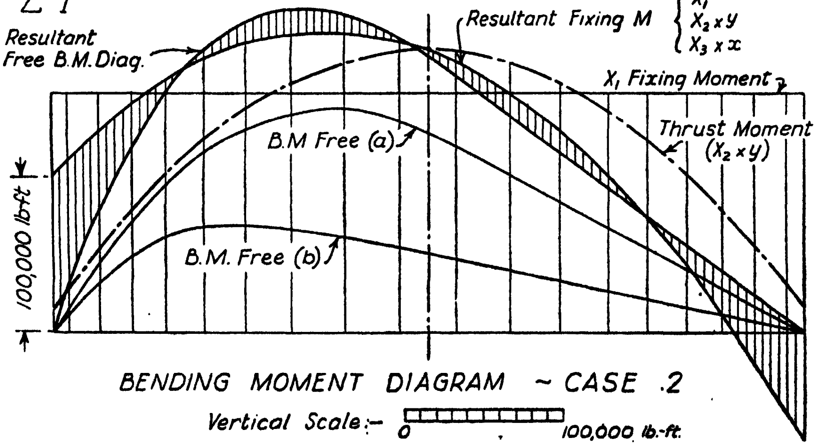
DETERMINATION OF X_1 & X_2						Thrust M	DETERMINATION OF X_3			
	Free M (Kips)	y	$\frac{m}{I}$	$\frac{y^2}{I}$	$\frac{my}{I}$	$X_2 \times y$	x	$\frac{m_0 x}{I}$	$\frac{x^2}{I}$	$x \times X_3$
LEFT 8	31	−19·9	2·3	30·0	+ 46	−124,000	30·6	− 70	72	82700
7	78	−15·9	22	71·0	+350	−99500	27·9	−615	219	74200
6	120	−12·0	44	53·0	+528	−74500	25·2	−1110	233	68000
5	154	− 8·3	94	42·0	+790	−52000	22·2	−2090	320	60000
4	181	− 4·7	179	22·0	+840	−29400	19·0	−3410	357	51400
3	199	− 1·6	297	3·8	+475	−10000	15·4	−4580	353	41600
2	208	1·2	425	2·9	−510	7000	11·4	−4860	265	30800
1	204	3·3	497	26·5	−1640	20600	7·1	−3520	123	19200
0	189	4·4	556	57·0	−2450	27500	2·4	−1340	17	6470
RIGHT 0	165	4·4	484	57·0	−2130	27500	2·4	+1160	17	−6470
1	140	3·3	341	26·5	−1130	20600	7·1	+2420	123	−19200
2	115	1·2	235	2·8	−282	7000	11·4	+2670	265	−30800
3	92	− 1·6	137	3·8	+220	−10000	15·4	+2110	353	−41600
4	73	− 4·7	72	22·0	+348	−29400	19·0	+1370	357	−51400
5	55	− 8·3	34	42·0	+282	−52000	22·2	+750	320	−60000
6	38	−12·0	14	53·0	+168	−74500	25·2	+353	233	−68000
7	23	−15·9	6	71·0	+ 95	−99500	27·9	+167	219	−74200
8	8	−19·9	0·7	30·0	+ 14	−124000	30·6	+21	72	−82700
Total			3440·0	616·4	−3986			+10574	3918	

$$X_1 = \frac{-\sum \frac{m_0}{I}}{\sum \frac{1}{I}} = \frac{3{,}440{,}000}{22{\cdot}452} = +153{,}000 \text{ lb-ft} \qquad X_3 = \frac{-\sum \frac{m_0 x}{I}}{\sum \frac{x^2}{I}} = \frac{10{,}574 \times 1{,}000}{3918} = +2{,}700 \text{ lb.}$$

$$X_2 = \frac{-\sum \frac{m_0 y}{I}}{\sum \frac{y^2}{I}} = \frac{3{,}986{,}000}{616{\cdot}4} = +6{,}250 \text{ lb.}$$

$Y \times X_2 = 137{,}000$ lb-ft.

$y \times X_2 = 28{,}200$ lb-ft.



BENDING MOMENT DIAGRAM ~ CASE 2

Vertical Scale:— 0 100,000 lb.-ft.

Notes. (1) x is positive for the right-hand side and negative for the left-hand side. (2) $y \times X_2 = 28{,}200$ lb.-ft. is at the crown.

Fig. 60.

area. The free bending moment on the vertical members is negligible, so that the bending moments represented by the closing line are the sum of the moments due to X_1 and X_2 only.

FIXED ARCH OF 64-FT. SPAN.—*Figs.* 58 to 62 show stages in the design of an earth-covered fixed parabolic arch roof of 64-ft. span and rise of 26 ft. 6 in.; the ratio of rise to span is 0·414, which is high compared with common arch bridges. The symmetrical dead load is shown as Case 1, *Fig.* 59, and the unsymmetrical live load as Case 2, *Fig.* 60. The diagram of the free bending moments due to the dead load is given in *Fig.* 58. The values of X_1 and X_2 are calculated in the tabulation in *Fig.* 59 for which purpose the semi-arch is divided into eight equal segments. Bending moments due to X_1 and X_2 and the free bending moments are plotted along the arch axis in *Fig.* 58. Values of X_1, X_2, and X_3 for the live load are obtained by the tabulation in *Fig.* 60, the diagram of the free bending moments and the bending moments due to X_1, X_2, and X_3 is given in the lower part of *Fig.* 60, where the shaded area is the diagram of the resultant bending moments.

From the resultant bending moments for the dead and live loads and the

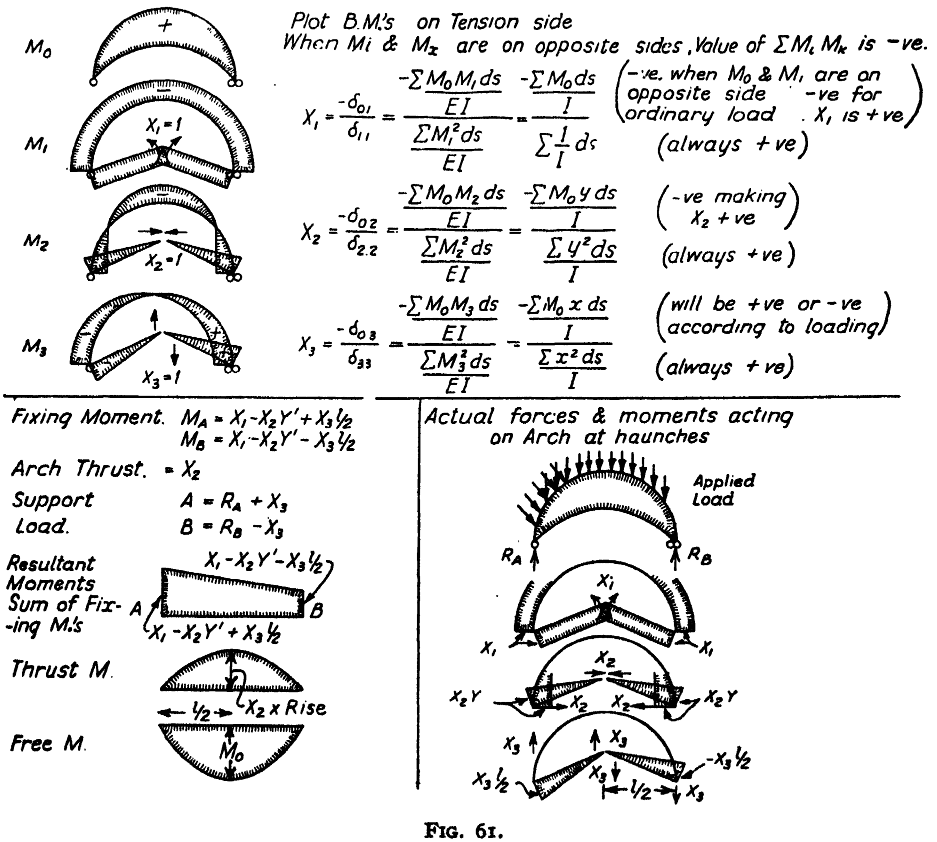


FIG. 61.

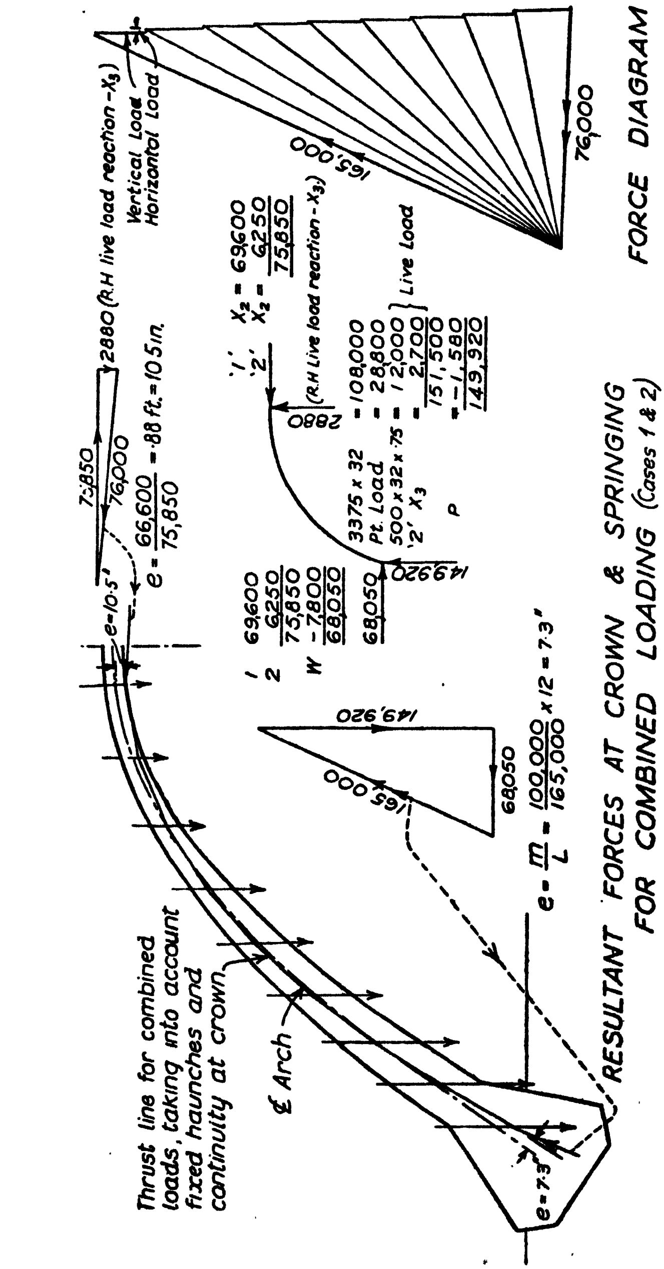

RESULTANT FORCES AT CROWN & SPRINGING FOR COMBINED LOADING (Cases 1 & 2)
FORCE DIAGRAM
Thrust line for combined loads, taking into account fixed haunches and continuity at crown
℄ Arch
e = m/L = 100,000/165,000 x 12 = 7.3"
e = 66,600/75,850 = ·88 ft. = 10·5 in.
e=10·5"
e=7·3"
2880 (R.H live load reaction - X3)
Vertical Load
Horizontal Load
165,000
76,000
75,850
149,920
68,050
'1' X2 = 69,600
'2' X2 = 6250
75,850
(R.H Live load reaction - X3)
3375 x 32 = 108,000
Pt. Load = 28,800
500 x 32 x ·75 = 12,000
'2' X3 = 2,700
151,500
P = -1,580
149,920
Live Load
1 69600
2 6250
75,850
W -7,800
68,050

FIG. 62.

resultant forces acting at the crown and the springing, the direction and eccentricity of the line of thrust at the crown and the springing can be obtained as in the subsidiary diagrams in *Fig.* 62. The force diagram for the semi-arch can then be drawn as in *Fig.* 62 and then the line of thrust throughout the semi-arch. The resultant bending moment at any section as obtained from the diagrams of the bending moments should agree with the value obtained by multiplying the thrust at the section by the eccentricity of the line of thrust at that section.

The formulæ and procedure are given in *Fig.* 61, and the following is a summary of the results. The net bending moments (in lb.-ft.) at the crown are:

Case 1 (*Fig.* 59).	$y_0 \times X_2$	$= -$	314,000
	X_1	$= -$	1,690,000
Case 2 (*Fig.* 60).	Resultant (by scale)	$= -$	2,000
	Free bending momen (*Fig.* 58)	$= +$	1,940,000
Net bending moment		$-$	66,000 lb.-ft.

The corresponding thrust is $69{,}600 + 6250 = 75{,}850$ lb. (*Fig.* 62) and the eccentricity is $\dfrac{66{,}000 \times 12}{75{,}850} = 10 \cdot 5$ in.

The bending moments (in lb.-ft.) at the left-hand haunch are:

Case 1 (*Fig.* 59).	X_1	$= +$	1,690,000
	$21 \cdot 8X_2$	$= -$	1,520,000
Case 2 (*Fig.* 60).	Resultant	$= -$	100,000
Net bending moment		$-$	70,000, say, $-$ 100,000 lb.-ft.

The corresponding thrust is 165,000 lb. (*Fig.* 62) and the eccentricity is

$$\frac{100{,}000 \times 12}{165{,}000} = 7 \cdot 3 \text{ in.}$$

Since the net bending moments are the differences of large quantities, possible errors in the calculation of X_1 and X_2 should be assumed in order to obtain safe limiting values of the eccentricities. In a flat arch it would be necessary to investigate the possible increase in eccentricity of the thrust due to temperature stresses, creep of the concrete in the arch, and yield of the abutments for the ultimate load (since creep and yield occur increasingly rapidly at high stresses). It is not difficult, however, to assume safe limiting effective values of E for this purpose. It is interesting to note, in *Fig.* 58, that if it is assumed that there are hinges at the springings and at the crown, the eccentricity of the thrust is within safe limits at the first quarter-point. The stresses in the concrete and reinforcement, assuming full fixity at the haunches and continuity at the crown, are not excessive. If excessive stresses were induced at those sections plasticity would produce the effect of a partial hinge thereby reducing the eccentricity of the thrust at the haunches and crown, and the line of thrust would then approach the position shown in *Fig.* 58.

A Simple Method, within Safe Limits, of Calculating Wind Moments on Building Frames.

The foregoing exact method of using three-member elastic equations is generally too long for preliminary designs. In many buildings the stresses due to wind are calculated on the assumption that the points of contraflexure occur at mid-span of the beams and at mid-height of the columns, thus making the whole structure statically determinate. The total shearing force due to wind at any level is divided among the columns in proportion to their stiffnesses, and the vertical reactions on the columns due to wind are assumed to be proportional to their distances from the centre of the building. These assumptions, which are true only if the beams are infinitely stiff relative to the columns, are

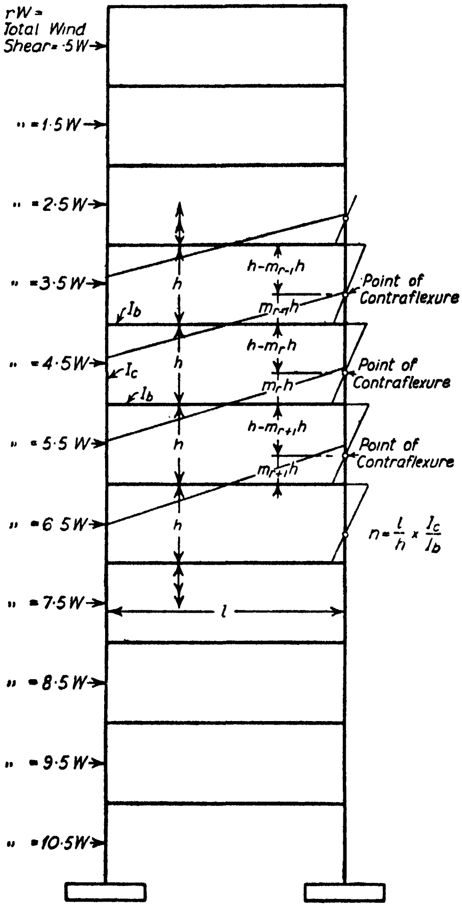


FIG. 63.

nearly correct in most cases and provide an adequate factor of safety, particularly when the panels of the frame in the same vertical plane as the direction of the wind are filled solid with brickwork or other material. Also, it must be remembered that the ratio of stiffnesses of the beams to the columns does not greatly affect the value of the ultimate load, since often excessive stiffness in the columns is removed by plastic yield before failure. If only light partitions are provided the rule given in the following may be used, unless the column bases are almost fixed and the columns immediately above the bases are very stiff relative to the beams when full investigation using three-member equations should be made.

Referring to Fig. 63 *the rule is that it can be assumed that the points of contra-flexure in the columns are within the middle third of the height between any two floors provided that the column in the* r*th story from the top is not greater than* r *times as stiff as the main beams of the floors and that the coefficient of stiffness of the ground is equal to the coefficient of stiffness of the beams and that the heights of each story and the sizes of the beams are constant.*

The stiffness of the ground is discussed later. In a building frame in which the limiting ratio of the stiffness of columns to that of the beams is maintained throughout, the maximum possible wind moment in any column is calculated simply as the shearing force on the stanchion due to the wind multiplied by

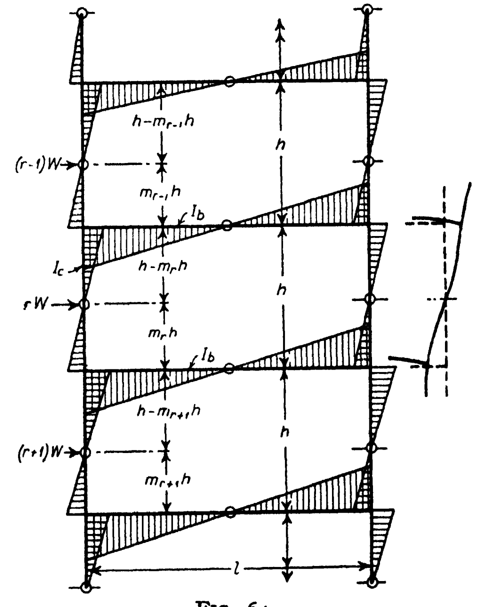
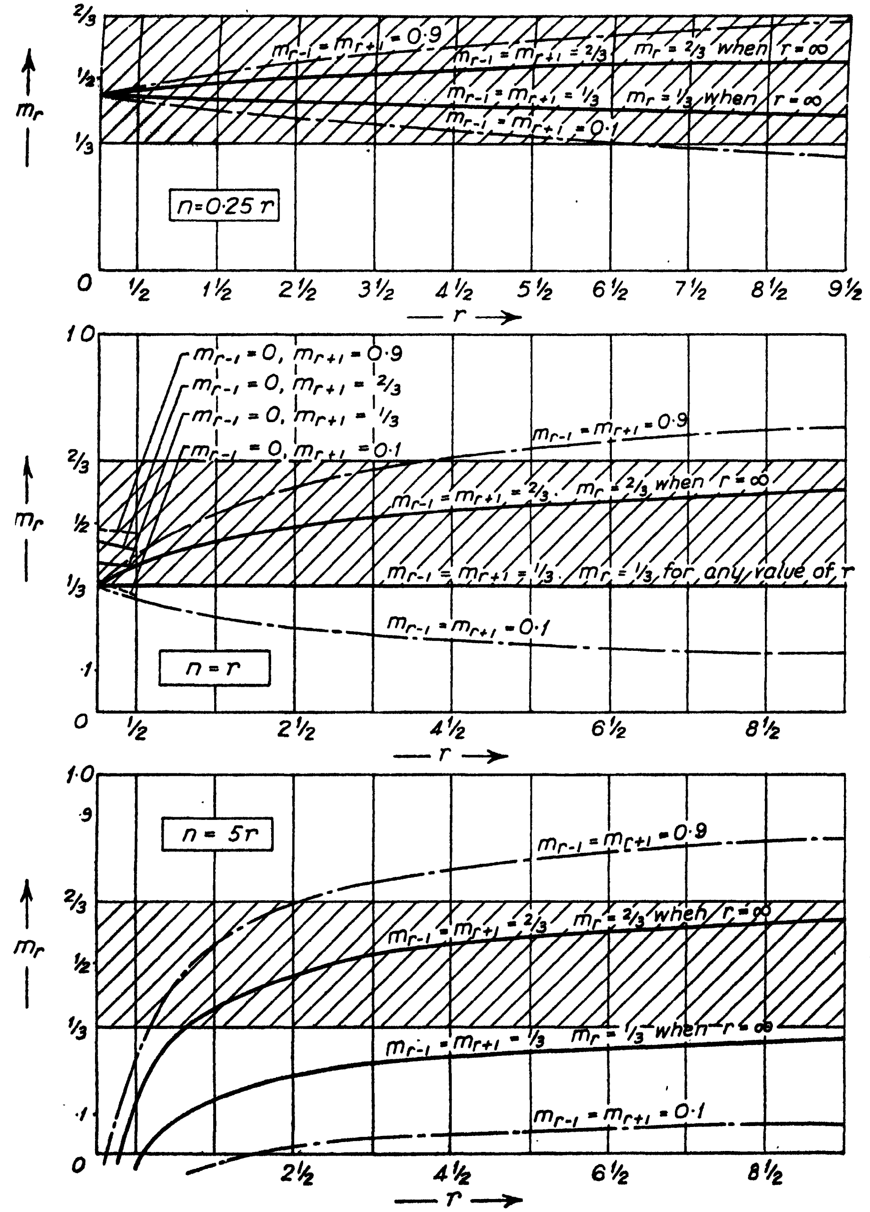


FIG. 64.

two-thirds of the distance between the floors. The shearing force in any column is obtained by dividing the total shearing force among the columns in proportion to their stiffnesses.

The correctness of the rule can be proved by considering any bay of a typical building frame (*Fig.* 64). Assume that the frame is symmetrical. In determining the resistance to wind of a building frame, it is always possible to divide the frame into a number of symmetrical frames which are weaker than the actual frame. Let the shearing force due to the wind on each column be rW; proceeding from the top story downwards, $r = \frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, etc. The shearing force on the column in the story above is $(r - 1)W$, and in the story below, $(r + 1)W$. The story-height h is constant throughout; small variations in height can be neglected. It is also assumed that the value of I_c is constant between any two floors, and that the value of I_b is constant throughout the



Curves relating to m_{r-1} for values of m_{r-1} and m_{r+1} such as ⅓ and ⅔, 0·1 and 0·9, 0·9 and 0·1, etc., have not been plotted as they are within the Curves for ⅓ and ⅔, 0·1 and 0·9, etc.

FIG. 65.

frame. If l is the span of the beam, $\frac{l}{I_b} = \frac{nh}{I_c}$; that is the column is n times as stiff as the beam.

By equating the slope of the rth column above and below the point of contraflexure, the relationship between the points of contraflexure in the $(r - 1)$th, rth, and $(r + 1)$th columns can be established from

$$m_r = \frac{nr(m_{r-1} + m_{r+1}) - n(m_{r-1} - m_{r+1}) - n + 3r}{r(2n + 6)} \qquad . \qquad . \quad (15)$$

The derivation of this expression is given later.

In *Fig.* 65, m_r is plotted for various values of m_{r-1}, m_{r+1}, and n and r, and it is seen that, provided that n is not greater than r, for values of m_{r-1} and m_{r+1} within the middle third of the height of the column, the corresponding values of m_r also fall within the middle third. By substituting in (15) $m_{r-1} = m_r = m_{r+1} = \frac{1}{3}$, $n = r$. Further, by substituting in (15) all possible extreme values of m_{r-1} and m_{r+1} within the middle third of the height of the column, it is shown later that the corresponding value of m_r must be within the middle third of the height of the column provided that $n = r$. As is also shown later, and in *Fig.* 65, in the top story where $r = \frac{1}{2}$ and $m_{r-1} = 0$, if m_{r+1} is within the middle third and n is not greater than r, then m_r is within the middle third. The value of m_r for the second story will also be within the middle third provided that the point of contraflexure of the third story is within the middle third, and so on until the footings are reached. Provided that the depth of the footing is suitable and the stiffness of the ground supporting the footing is equivalent to the stiffness of the beam, the point of contraflexure in the bottom story is within the middle third. Therefore, provided that the depth of the footing and the stiffness of the supporting ground are suitable, and that n is not greater than r throughout the frame, all points of contraflexure will be within the middle third of the heights of the columns. If $n = 0$, that is if all the beams are rigid, then all points of contraflexure are at the mid-point in the height of the columns, shown later. It is seen from the curves in *Fig.* 65 that if n is not less than r throughout the frame, errors can occur if it is assumed that the points of contraflexure of the columns are midway between the floors. In a particular case it may be desirable to depart slightly from the condition that n does not exceed r or to vary the height of a story. By following a similar procedure to that described in the foregoing, suitable safe limiting positions of the points of contraflexure for the particular case can be established.

If, for the purpose of analysis, a building frame is divided into a number of vertical Vierendeel frames, the internal columns should be considered as having a moment of inertia of one-half the actual value. If the frame is not symmetrical the values of n assumed for a symmetrical frame should allow for the worst characteristics of the frame under consideration.

The bending moments and shearing forces due to the wind calculated in this way are added in accordance with the law of superimposition the moments and forces in the frame due to the vertical loading or settlement of the foundations. The columns must be designed to withstand the worst possible combinations of moments and forces due to each cause. An extension of the foregoing theory by Mr. E. Horne in a paper presented to the Institution of Structural

Engineers, June 1948, " Wind Stresses in Multi-Story Buildings " considers the effect of variation in the story-heights and the influence of stiff footings.

POINT OF CONTRAFLEXURE OF THE COLUMN IMMEDIATELY ABOVE THE FOOTING (FIG. 66).—The slope at the point of contraflexure of the column immediately above the footing is the same as the slope at the point of contraflexure of the rth bay of a vertical Vierendeel frame, which continues indefinitely beyond the rth bay provided that the stiffness factor of the supporting ground is equal to that of the corresponding horizontal beam of the Vierendeel frame. This is true since the slope at the point of contraflexure is the sum of the angle of the slope due to bending of the column over a length $m_r h$ and the angle through which the footing rotates. The reciprocal of the coefficient of stiffness for the beam of the frame is $\frac{nh}{6EI_c}$ (see the next subsection). The bending moment

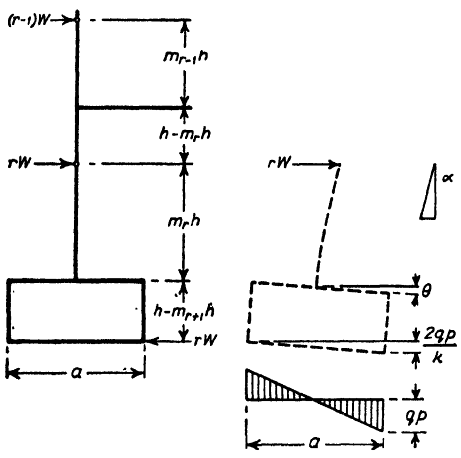


FIG. 66.

acting on the beam is $rW(m_r h) + (r+1)W(h - m_{r+1}h)$, which can be considered to be equal to $rW(m_r h) + rW(h - m_{r+1}h)$, which is the bending moment acting on the footing; a negligible error on the safe side results from this assumption.

The reciprocal of the coefficient of stiffness of a footing of width a and length b can be obtained as follows. If the average unit pressure on the ground is p, the variation in unit pressure at the edges of the footing due to the bending moment is qp, and the coefficient of elasticity or settlement of the ground is k, the bending moment acting on the footing due to the variation in the pressure on the ground is $\frac{qpba^2}{6}$. The settlement of the footing at the edge due to the extra pressure qp is $\frac{qp}{k}$. Therefore the angle of rotation of the footing is $\theta = \frac{2qp}{ak}$. The bending moment multiplied by the reciprocal of the coefficient of stiffness is θ; thus $\frac{1}{\text{coefficient of stiffness}} = \frac{2qp}{ak} \cdot \frac{6}{qpba^2} = \frac{12}{ba^3k}$. Therefore in a given case, provided that $\frac{12}{ba^3k} = \frac{nh}{6EI_c}$, all relationships established between the $(r-1)$th, rth, and $(r+1)$th stories of a vertical Vierendeel frame still apply when the equivalent of the $(r+1)$th story is a footing.

Generally, for the footing m_{r+1} is between one-third and, say, three-quarters due to the resultant horizontal reaction of the ground to the wind force acting near the base of the footing. If m_{r-1} is between one-third and two-thirds, by substitution in the expression for m_r it is found that m_r is between one-third and two-thirds when $n = r$ for all ordinary values of r. Since for a footing $h - hm_{r+1}$ will generally be about $\frac{h}{3}$, it is seen from *Fig.* 65 that in many cases n could be smaller or greater than r without the middle-third rule breaking down.

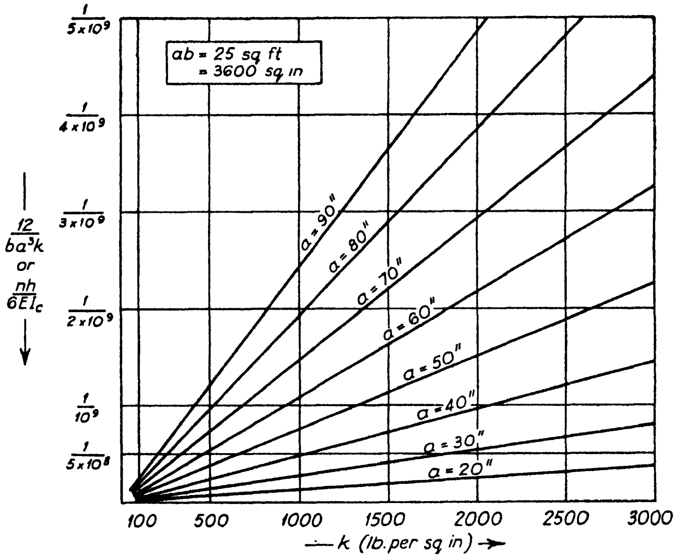


FIG. 67.

In a particular case, if it were desirable to use a footing with n less or more than r, a special investigation could easily be made. It should be noted that the stiffness factor of a particular footing can be varied considerably in a given case by altering the relationship of a to b (*Fig.* 67).

To show that, for Various Values of m_{r+1} and m_{r-1} within the Middle Third, m_r is within the Middle Third if n does not exceed r.

(i) If $n = r$.

(*a*) $m_{r+1} = \frac{1}{3}$ and $m_{r-1} = \frac{1}{3}$. By substitution in formula (15),

$$m_r = \frac{r^2(\frac{1}{3}+\frac{1}{3}) - r(\frac{1}{3}-\frac{1}{3}) + 2r}{2r^2+6r} = \frac{\frac{2}{3}r+2}{2r+6} = \frac{1}{3}.$$

(*b*) $m_{r+1} = \frac{2}{3}$ and $m_{r-1} = \frac{2}{3}$.

$$m_r = \frac{r^2(\frac{2}{3} + \frac{2}{3}) - r(\frac{2}{3} - \frac{2}{3}) + 2r}{2r^2 + 6r} = \frac{1\frac{1}{3}r + 2}{2r + 6}.$$

If $r = 1$, $m_r = 0{\cdot}416$; if $r = \infty$, $m_r = 0{\cdot}667$.

(*c*) $m_{r+1} = \frac{1}{3}$ and $m_{r-1} = \frac{2}{3}$.

$$m_r = \frac{r^2(\frac{2}{3} + \frac{1}{3}) - r(\frac{2}{3} - \frac{1}{3}) + 2r}{2r^2 + 6r} = \frac{r + 1\frac{2}{3}}{2r + 6}.$$

If $r = 1$, $m_r = 0{\cdot}333$; if $r = \infty$, $m_r = 0{\cdot}5$.

(*d*) $m_{r+1} = \frac{2}{3}$ and $m_{r-1} = \frac{1}{3}$.

$$m_r = \frac{r^2(\frac{1}{3} + \frac{2}{3}) - r(\frac{1}{3} - \frac{2}{3}) + 2r}{2r^2 + 6r} = \frac{r + 2\frac{1}{3}}{2r + 6}.$$

If $r = 1$, $m_r = 0{\cdot}416$; if $r = \infty$, $m_r = 0{\cdot}5$.

(ii) If $n = xr$.

(*a*) $m_{r+1} = m_{r-1} = \frac{2}{3}$.

$$m_r = \frac{xr^2(\frac{4}{3}) - rx + 3r}{2xr^2 + 6r} = \frac{4xr - 3x + 9}{6xr + 18}.$$

For any value of x, m_r does not exceed $\frac{2}{3}$.

(*b*) $m_{r+1} = m_{r-1} = \frac{1}{3}$.

$$m_r = \frac{xr^2(\frac{2}{3}) - rx + 3r}{2xr^2 + 6r} = \frac{2xr - 3x + 9}{6xr + 18}.$$

If x exceeds unity, m_r is less than $\frac{1}{3}$; that is n must not exceed r.

To show that for a Top Story ($r = \frac{1}{2}$) m_r is within the Middle Third if $m_{r-1} = 0$, $m_{r+1} = \frac{1}{3}$ or $\frac{2}{3}$, and n does not exceed r.—Substituting in formula (15), $n = r = \frac{1}{2}$, $m_{r-1} = 0$, and $m_{r+1} = \frac{1}{3}$,

$$m_r = \frac{r^2(\frac{1}{3}) + \frac{1}{3}r - r + 3r}{2r^2 + 6r} = \frac{\frac{1}{3}r + 2\frac{1}{3}}{2r + 6}.$$

Therefore $m_r = \dfrac{2\frac{1}{2}}{7} = 0{\cdot}357$.

Also substituting in formula (15) $n = r = \frac{1}{2}$, $m_{r-1} = 0$, and $m_{r+1} = \frac{2}{3}$,

$$m_r = \frac{r^2(\frac{2}{3}) + \frac{2}{3}r - r + 3r}{2r^2 + 6r} = \frac{\frac{2}{3}r + 2\frac{2}{3}}{2r + 6}.$$

Therefore $m_r = \dfrac{3}{7} = 0{\cdot}429$.

To show that all Points of Contraflexure are at $\dfrac{h}{2}$ if all Beams are Rigid.—For a rigid beam $n = 0$, and by substituting in (15)

$$m_{r-1} = m_{r+1} = 0{\cdot}5, \text{ and } n = 0,$$

$$m_r = \frac{0 \times r(0{\cdot}5 + 0{\cdot}5) - 0(0{\cdot}5 - 0{\cdot}5) - 0 + 3r}{7(0 + 6)} = 0{\cdot}5.$$

Derivation of Formula (15).—Referring to *Fig.* 68, the slope at A, from the bending-moment diagram, is $\frac{P_1h_1^2}{2EI_{c1}} + \frac{2}{3}(P_1h_1 + P_2h_2)\frac{l}{4EI_b}$

$$= \frac{P_1h_1^2}{2EI_{c1}} + \frac{l}{6EI_b}(P_1h_1 + P_2h_2) \quad . \quad . \quad . \quad . \quad (a)$$

Similarly the slope at B is

$$\frac{P_2h_2^2}{2EI_{c2}} + \frac{l}{6EI_b}(P_1h_1 + P_2h_2) \quad . \quad . \quad . \quad . \quad (b)$$

Equating the slope of the column above the point of contraflexure in bay r to the slope below by substituting in expressions (a) and (b)

$$\frac{rW(h - m_rh)^2}{2EI_c} + \frac{l}{6EI_b}[(r - 1)Wm_{r+1}h + rW(h - m_rh)]$$

$$= \frac{rW(m_rh)^2}{2EI_c} + \frac{l}{6EI_b}[rWm_rh + (r + 1)W(h - m_{r+1}h)].$$

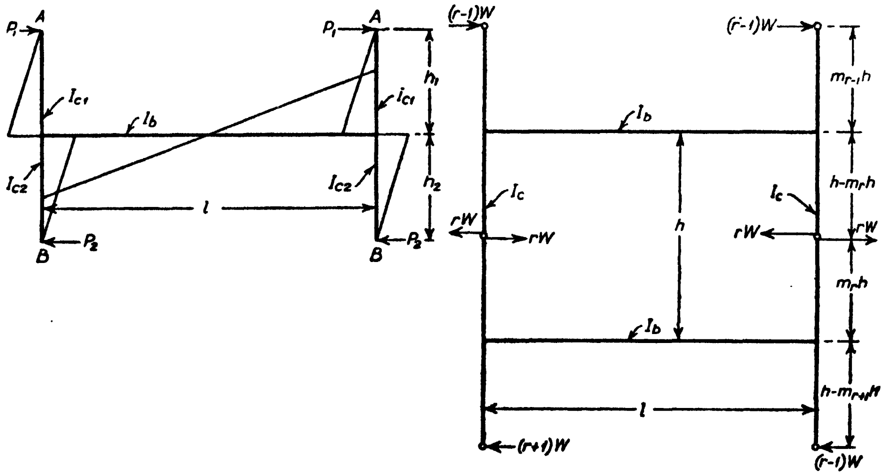


Fig. 68.

Substituting $\frac{l}{I_b} = \frac{nh}{I_c}$, that is assuming that the column is n times as stiff as the beam,

$$\frac{rW(h - m_rh)^2}{2EI_c} + \frac{nh}{6EI_c}[(r - 1)Wm_{r-1}h + rW(h - m_rh)]$$

$$= \frac{rW(m_rh)^2}{2EI_c} + \frac{nh}{6EI_c}[rWm_rh + (r + 1)W(h - m_{r+1}h)].$$

The terms W, h, E, and I_c cancel out; note that the values of I_c for the $(r - 1)$th story and $(r + 1)$th story do not affect the slope in the rth story. Therefore

$$m_r = \frac{nr(m_{r-1} + m_{r+1}) - n(m_{r-1} - m_{r+1}) - n + 3r}{r(2n + 6)} \quad . \quad . \quad (15)$$

IV.—DISTRIBUTION OF LOAD ON A GROUP OF PILES.

In this analysis of the distribution of the load on a group of piles the assumptions are that the pile-cap is rigid relative to the piles, that the piles are sufficiently flexible relative to the pile-cap for the piles to offer no restraint to bending, that any raking piles are sufficiently inclined so that horizontal forces are resisted by direct forces and not by bending, that the longitudinal compression or extension of a pile is elastic, that the dimensions of the cross sections are the same for all piles, that the group is symmetrical on plan about one axis, that external forces act within the plane of symmetry, that no settlement of the support of the piles occurs, and that the points of support of the piles are at the same level. The notation used is as follows.

h, the height from the point of support to the head of the pile.
θ, the angle of inclination of the pile to the vertical.
V, the vertical component of the resultant external force acting on the pile-cap.
H, the horizontal component of the resultant external force acting on the pile-cap.
e, the eccentricity of V; a, the eccentricity of H.
U_v, the coefficient of vertical stiffness of the head of the pile.
U_h, the coefficient of horizontal stiffness of the head of the pile.
v_1, v_2, etc., the vertical components of the forces in the piles.
P_1, P_2, etc., the forces in the piles.
r_1, r_2, etc., the vertical components of the forces in the piles due to external bending moment only.
h_1, h_2, etc., the horizontal components of the forces in the piles.
n_1, n_2, etc., the horizontal distances from the heads of the piles to the neutral axis of the group.

Analysis.

Referring to *Fig.* 69 (*a*), AB is the unloaded position of any pile in a symmetrical group such as in *Fig.* 69 (*b*). If the head of the pile is subjected to a horizontal force H and a vertical force V, CB represents the vertical movement due to V and CE the horizontal movement due to H; $\text{BC} = \frac{V}{U_v}$ and $\text{CE} = \frac{H}{U_h}$, U_v and U_h being the coefficients of stiffness of the head of the pile as expressed in formulæ (1) and (4).

Since the displacements due to load are very small compared with the length of the pile, and AB is inclined at θ to the vertical, $\text{AC} = \frac{h}{\cos\theta}$, which is the length of the pile. The total compression of the pile is $\text{CF} = \frac{H}{U_h}\sin\theta$. Then the force in the pile is $P = \alpha\frac{H\sin\theta}{U_h} \times \frac{\cos\theta}{h}$, where α is an elastic constant (elastic modulus × area) applying to all the piles.

Also, from the triangle of forces at the head of the pile, $P = \frac{H}{\sin\theta}$.

Therefore $$\frac{H}{\sin\theta} = \alpha\frac{H\sin\theta}{U_h} \times \frac{\cos\theta}{h},$$

and $$U_h = \alpha.\frac{\sin^2\theta\cos\theta}{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If a horizontal force H acts on the heads of several piles inclined at θ_1, θ_2, etc., each having the same dimensions, elastic constant, and height h, then H is opposed by the horizontal reaction from the head of each pile proportional to the stiffness of the head of the pile [*Fig.* 69 (*c*)]. If there is no rotation of the pile-cap and the group is symmetrical in elevation,

$$h_1 = H.\frac{U_{h_1}}{U_{h_1} + U_{h_2} + U_{h_3} + \ldots} = \frac{\frac{\alpha}{h}}{\frac{\alpha}{h}}.\frac{(\sin^2\theta_1\cos\theta_1).H}{(\sin^2\theta_1\cos\theta_1 + \sin^2\theta_2\cos\theta_2 + \ldots)}$$

$$= H.\frac{\sin^2\theta_1\cos\theta_1}{\sin^2\theta_1\cos\theta_1 + \sin^2\theta_2\cos\theta_2 + \ldots} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Similarly $$h_2 = H.\frac{\sin^2\theta_2\cos\theta_2}{\sin^2\theta_1\cos\theta_1 + \sin^2\theta_2\cos\theta_2 + \ldots} \quad . \quad . \quad . \quad . \quad (3)$$

Referring to *Fig.* 69 (*a*), by an analysis similar to that for U_h,

$$P = \alpha.\frac{V}{U_v}\cos\theta \times \frac{\cos\theta}{h} = \frac{V}{\cos\theta}.$$

Thus $$U_v = \frac{\alpha\cos^3\theta}{h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Consider a vertical force V [*Fig.* 69 (*b*)] acting on a group of piles and assume there is no rotation or horizontal movement of the pile-cap, that is each pile is compressed vertically by the same amount, which is the case in a concentrically-loaded symmetrical or unsymmetrical group having a horizontal support. The vertical components of the forces in each pile are

$$v_1 = V.\frac{\cos^3\theta_1}{\cos^3\theta_1 + \cos^3\theta_2 + \ldots},\ v_2 = V.\frac{\cos^3\theta_2}{\cos^3\theta_1 + \cos^3\theta_2 + \ldots},\ \text{etc.}$$

Referring to *Fig.* 69 (*b*), (*c*), and (*d*), the point or horizontal axis through which the lines of action of the horizontal and vertical forces must pass in order that no rotation of the pile-cap takes place may be termed the " neutral axis " of the system. The problem of finding the ideal arrangement of the piles in plan is comparable to finding the ideal cross section of a stanchion subject to direct load, bending moment, and shearing force. In a symmetrical group of piles the " neutral axis " is on the axis of symmetry. If the group is unsymmetrical the piles must be divided into a left-hand group and right-hand group. The left-hand group develops compression for a counter-clockwise moment and the right-hand group develops tension. When a vertical load acts and causes no rotation, the " neutral axis " is on the vertical axis through the point of intersection of the resultants of the forces in the piles in the left-hand and right-hand groups (including the vertical piles). Values of v_1, v_2, etc., in the left-hand group are each proportional to U_{v1}, U_{v2}, etc., and similarly in the right-hand group. The

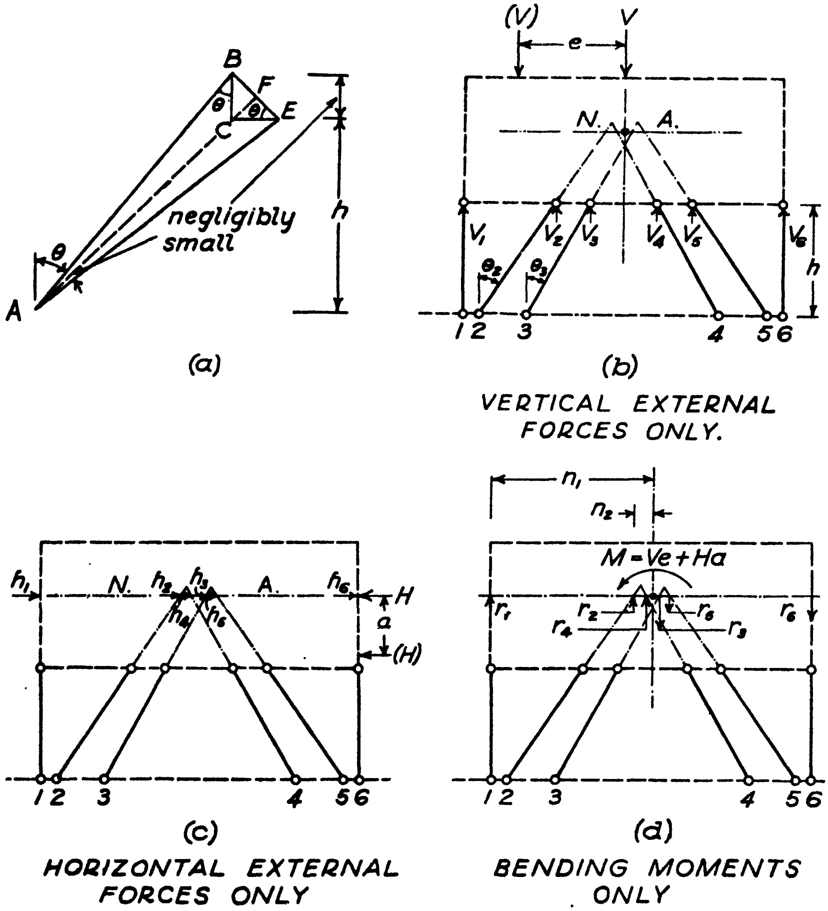


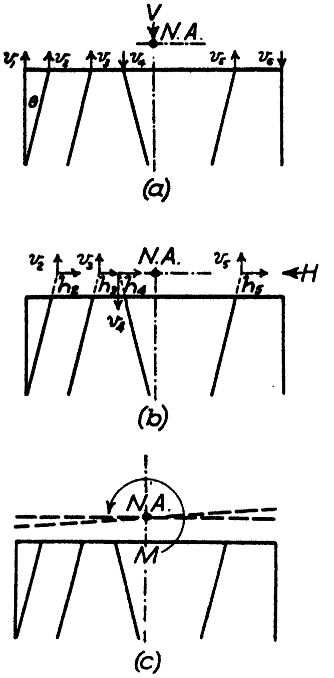
FIG. 69.

sum of the horizontal components of the forces in the piles, that is $\Sigma v \tan \theta$, in each group, must be equal and opposite when only a vertical load acts [*Fig.* 70 (*a*)]. Alternatively, when dealing with vertical loads only, it may be sufficient to determine simply the position of the vertical axis passing through the point of action of the resultant or centre of gravity of v_1, v_2, etc. If the pile-cap is restrained from horizontal movement, as in a wharf with raking piles, v_1, v_2, etc., as a whole group are proportional to U_{v1}, U_{v2}, etc., and the condition that the horizontal components of the left-hand and right-hand groups must balance does not apply.

If horizontal forces are to be considered it is necessary to determine the position on the vertical concentric axis of the "neutral axis" by finding the point of intersection of the resultant of the forces in the left-hand and right-hand groups of raking piles. Alternatively, the level of an imaginary pile-cap can be determined, the level being that at which, with the piles extended, $\Sigma h_n \cot \theta$ and Σ [moment of $(h_n \cot \theta)$ about any point] are both equal to zero. This condition must be satisfied since there are no external vertical forces. Forces in vertical piles can be ignored since their stiffness factor U_h is zero [*Fig.* 70 (*b*)].

In a symmetrical group, the analysis is simple because the horizontal components of the left-hand and right-hand groups balance with equal forces in the corresponding left-hand and right-hand piles.

If the lines of action of the vertical and horizontal forces V and H are



Free to move vertically and horizontally. No rotation. $\Sigma v \tan \theta$ is equal and opposite in both groups. The total vertical force V acts on the line of the resultant of v_1, v_2, etc., or through the point of intersection of the resultant of both groups.

Free to move vertically and horizontally. No rotation. H acts at a level at which Σv and Σ (moments v about any point) = 0, or through the point of intersection of the resultants of the left-hand and right-hand forces in the piles. $v = h_n \cot \theta$, where h_n is the horizontal component of the force in the pile.

The pile-cap rotates about a horizontal axis at the intersection of the vertical and horizontal axes of (*a*) and (*b*), since there can be no horizontal or vertical movement of the " neutral axis " due to rotation about any other point without internal vertical and horizontal reactions developing as at (*a*) and (*b*).

FIG. 70.

distant e and a respectively from the " neutral axis," then the group is subjected to a bending moment equal to $Ve + Ha$. When a bending moment only is applied the pile-cap must rotate about the " neutral axis," in order that the sums of the resulting vertical components of the forces in the piles and of the corresponding horizontal components be both zero. The vertical components of the reactions at the heads of the piles resisting an external moment are proportional to the product of the vertical coefficients of stiffness of each pile and the vertical movement, which is proportional to the horizontal distance of each pile from the " neutral axis."

Any system of forces acting on the pile-cap can be reduced to

(i) A vertical force V acting through the neutral axis [*Fig.* 69 (*b*)].

(ii) A horizontal force H acting through the neutral axis [*Fig.* 69 (*c*)].

(iii) A moment $M = Ve + Ha$ [*Fig.* 69 (*d*)] causing rotation about the neutral axis.

The reactions at the heads of the piles due to each of the foregoing can be found as follows. Referring to *Figs.* 69 and 70,

(a) $v_1 = V.\dfrac{\cos^3\theta_1}{\cos^3\theta_1 + \cos^3\theta_2 + \ldots}$, if no horizontal movement occurs.

(b) $h_1 = H.\dfrac{\sin^2\theta_1\cos\theta_1}{\sin^2\theta_1\cos\theta_1 + \sin^2\theta_2\cos\theta_2 + \ldots}$, if the raking piles are symmetrical in elevation.

(c) Since no external vertical or horizontal forces are applied, no horizontal or vertical movement of the "neutral axis" takes place. The vertical components of the forces in the piles r_1, r_2, etc., are proportional to $U_{v1}n_1$, $U_{v2}n_2$, etc. Assume $r_1 = kU_{v1}n_1$, $r_2 = kU_{v2}n_2$, etc. Then

$$kU_{v1}n^2_1 + kU_{v2}n^2_2 + \ldots = M.$$

Therefore

$$k = \frac{M}{U_{v1}n^2_1 + U_{v2}n^2_2 + \ldots},$$

$$r_1 = \frac{U_{v1}n_1}{U_{v1}n^2_1 + U_{v2}n^2_2 + \ldots}.M,$$

and

$$r_2 = \frac{U_{v2}n_2}{U_{v1}n_1^2 + U_{v2}n^2_2 + \ldots}.M.$$

The resultant force in a pile is the sum of the forces of which the three foregoing vertical and horizontal forces are the components. Thus in pile No. 1 the resultant force is

$$\left[\frac{V\cos^3\theta_1}{\cos^3\theta_1 + \cos^3\theta_2 + \ldots} \times \frac{1}{\cos\theta_1}\right] + \left[\frac{H\sin^2\theta_1\cos\theta_1}{\sin^2\theta_1\cos\theta_1 + \sin^2\theta_2\cos\theta_2 + \ldots} \times \frac{1}{\sin\theta_1}\right] + \left[\frac{U_{v1}n_1}{U_{v1}n_1^2 + U_{v2}n_2^2 + \ldots} \times \frac{M}{\cos\theta_1}\right]$$

$$= \frac{V\cos^2\theta_1}{\cos^3\theta_1 + \cos^3\theta_2 + \ldots} + \frac{H\sin\theta_1\cos\theta_1}{\sin^2\theta_1\cos\theta_1 + \sin^2\theta_2\cos\theta_2 + \ldots} + \frac{M\cos^2\theta_1 n_1}{\cos^3\theta_1 n_1^2 + \cos^3\theta_2 n_2^2 + \ldots} \quad . \quad . \quad (5)$$

This expression only applies to a vertical load on a group of piles symmetrical in elevation, or a vertical load on an unsymmetrical group having a horizontal support thereby allowing vertical movement but not horizontal movement, and for a horizontal load on a symmetrical group.

For an unsymmetrical group with a vertical load acting through the "neutral axis," V must be divided into the resultant vertical components of the left-hand and right-hand groups, because horizontal movement takes place, so that $\Sigma v \tan\theta$ for the left-hand group equals $\Sigma v \tan\theta$ for the right-hand group. The respective resultants of the left-hand and right-hand groups can then be divided among the piles in proportion to their stiffnesses.

For a horizontal load acting through the neutral axis of an unsymmetrical group, H must be divided into the resultant horizontal components of the left-hand and right-hand groups so that $\Sigma h_n \cot\theta = 0$. The respective resultants of the left-hand and right-hand groups can then be divided among the piles in proportion to their stiffnesses.

Example.

Apply formula (5) to the symmetrical group shown in *Fig.* 71(*a*), in which $\theta_1 = \theta_4 = 0$ and $\theta_2 = \theta_3 = 30$ deg. $\left(\cos\theta = \frac{\sqrt{3}}{2}\right)$. V acts centrally; therefore $e = 0$. The resultant force in each of the vertical piles due to V only is

$$\frac{V}{2 + 2\cos^3 30\text{ deg.}} = \frac{V}{3{\cdot}298}.$$

The corresponding force in each of the raking piles is

$$\frac{V\cos^2 30\text{ deg.}}{2 + 2\cos^3 30\text{ deg.}} = \frac{V}{4{\cdot}43}.$$

As a check, the sum of the vertical components is

$$\frac{2 \times V}{2 + 2\cos^3 30\text{ deg.}} + \frac{2 \times V\cos^2 30\text{ deg.} \times \cos 30\text{ deg.}}{2 + 2\cos^3 30\text{ deg.}} = V;$$

that is,
$$\frac{2V}{3{\cdot}30} + \frac{2V}{4{\cdot}4}.\cos 30\text{ deg.} = V.$$

There is no force in the vertical piles due to H only, if the raking piles meet at the line of action of H. The resultant force in each of the raking piles due to H only is

$$\pm\frac{H\sin 30\text{ deg.}\cos 30\text{ deg.}}{2\sin^2 30\text{ deg.}\cos 30\text{ deg.}} = \pm\frac{H}{2\sin 30\text{ deg.}} = \pm H.$$

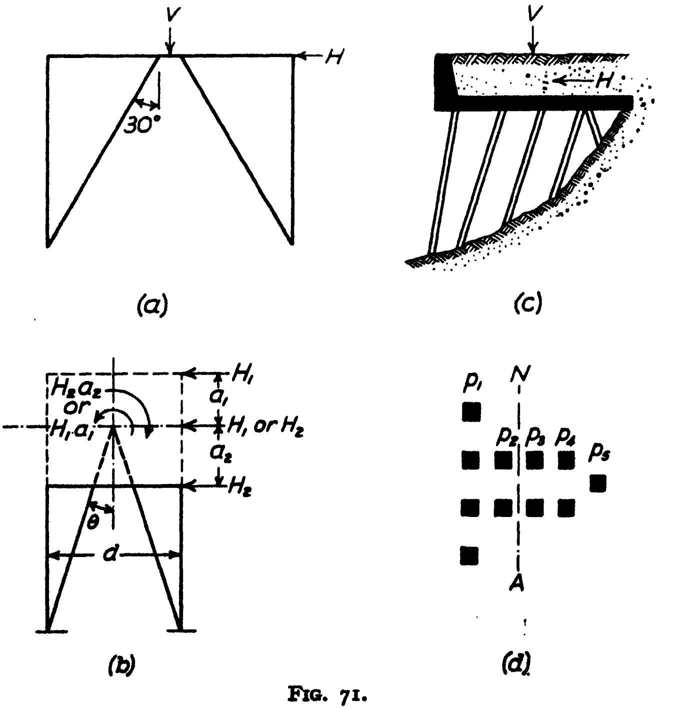


FIG. 71.

The forces in the piles for this simple system due to H are therefore the components of H obtained from a triangle of forces for the junction of the raking piles only, since the vertical piles have no load due to H.

Arrangement of Piles in Jetties.

The arrangement of piles shown in *Fig.* 71 (*b*) frequently occurs in jetties. The principal forces acting on the structure are H_1 at the level of the deck and H_2 at the level of the pile-cap, H_1 being greater than H_2 because, due to the action of the wind, greater forces are imparted to the structure by ships colliding therewith at high tide. The least forces are developed in the raking piles when θ has the greatest value within practical limits. If the raking piles are so inclined that the point of intersection of the projected centre-lines of the piles is a_2 above the level of the pile-cap, and $H_1a_1 = H_2a_2$, then it is seen that all the forces due to the bending moment are resisted by the vertical piles only, because the heads of the raking piles are at the neutral axis, and the magnitudes of the forces in the vertical piles due to external horizontal forces are therefore minima, since any other position of the point of intersection of the raking piles increases either H_1a_1 or H_2a_2. The forces in the raking piles are $\frac{H_1}{2 \sin \theta}$ when H_1 acts, and $\frac{H_2}{2 \sin \theta}$ when H_2 acts. These forces cannot be reduced by varying the position of the raking piles, but only by increasing θ. The level of the intersection does not affect the distribution of a vertical load applied at the middle of the jetty. The vertical load must be great enough when combined with the uplift due to the horizontal forces to prevent the piles being uprooted, and not too great when combined with the downward load due to the horizontal forces to over-load the piles or the ground. The forces in the vertical piles are obtained by substitution in (5); hence, $\frac{H_1a_1\frac{d}{2}}{2\left(\frac{d}{2}\right)^2} = \frac{H_1a_1}{d}$ or $\frac{H_2a_2}{d}$ and, as expected for a symmetrical system subjected to a horizontal force only, are therefore equal to the external moment about the intersection of the raking piles divided by the distance between the vertical piles.

Such an arrangement can therefore be analysed by reducing the external forces to a resultant vertical force acting at the middle, a resultant horizontal force acting at the intersection of the raking piles, and a resultant moment about the point of intersection of the raking piles due to the horizontal forces. It is not difficult then to determine the arrangement of piles which develops the least forces in the piles and downward and upward reactions within the safe resistance of the ground and anchorage. If necessary additional permanent vertical load is provided by ballast.

In a wharf that acts as a retaining wall and is supported on raking piles, the principal horizontal force acts in one direction only, and the most economical arrangement of piles includes raking piles inclined in one direction and spread as in *Fig.* 71 (*c*).

The General Case of a Group of Piles.

In a group of piles in which the number of piles in each row varies, as in *Fig.* 71 (*d*), the expressions already obtained for the forces in the piles can be adapted by treating each row of piles as a unit subjected to the components pv, ph, and pr, where p is the number of piles in a row. In a given case the ideal arrangement of the piles depends on the following factors: (i) The vertical load, (ii) The eccentricity of the actual loads, (iii) The horizontal load, (iv) The spacing of the piles to obtain full support from the ground, (v) The spacing of the piles to reduce bending stresses in the pile-cap to a minimum, and (vi) The practical limitations of handling and driving the piles. These factors influence the arrangement of the piles in the following manner.

(i) A vertical load induces the smallest forces in vertical piles.

(ii) The eccentricity of the actual loads should be reduced to a minimum and, where unavoidable as in a jetty subjected to horizontal forces acting above the level of the pile-cap, vertical piles should be concentrated at the edges of the group to produce a large equivalent moment of inertia.

(iii) Raking piles resist horizontal forces and the smallest forces are induced in piles having the greatest inclination. The inclination is normally limited to 1 in 3 or 1 in 4, as it is difficult to drive a pile at a greater inclination, and pile-drivers suitable for the purpose are not made. In a jetty the lower parts of the piles must be reasonably clear of ships in the berth, and the horizontal forces are usually the greatest forces that act on the groups of piles; the smallest forces are induced in the vertical piles when the extensions of the raking piles intersect at a point between the levels of the deck and pile-cap.

(iv) The nature of the ground determines the closest spacing of the piles that ensures each pile receiving full support from the ground when all the piles are loaded. It also determines the penetration required to develop safe resistances to vertical loads acting downward or upward.

(v) Small distances between the heads of piles reduce the stresses in the cap.

(vi) The length and weight of the pile are limited by the means available for handling and driving. Long piles tend to crack because of shrinking stresses and are difficult to lift without causing large bending moments. The points of lifting must be arranged to reduce the stresses due to handling to safe values. It is particularly difficult at sea to handle and pitch piles weighing from 5 to 10 tons; for such piles a hammer weighing from 4 to 6 tons is required.

Groups of piles are often designed by using a vector diagram to obtain the forces on the piles, regardless of the varying stiffnesses of the pile heads. When θ is small the factor of safety of such groups cannot be greatly affected, since overloaded piles would yield before failure and transfer the load to other piles. In some cases varying yielding of the ground greatly affects the distribution of the working load, but this effect becomes negligible under conditions of ultimate load. Raking piles having only a small inclination must be investigated to ensure that bending stresses are negligible.

V.—SLABS SPANNING IN TWO DIRECTIONS.

The common method of designing reinforced concrete slabs supported on all sides is a good example of the use of a quick approximate method giving results within safe limits in preference to a more lengthy but strictly mathematical method. The factors used have some theoretical basis which has been confirmed by tests and experience, and at ultimate load give results which differ by a negligible amount from those obtained by more exact but long calculations.

Slabs supported by Beams.

A complete investigation, which in some special cases may be necessary, of the bending moments in slabs supported by beams along all edges involves calculating the bending moments in each direction in each panel, taking into consideration the effect of continuity with adjacent panels and the most adverse arrangement of the live loads. Such a calculation, which is a useful exercise and leads to an appreciation of the fundamental principles of continuous slabs, is unnecessary in practice and may prove to be only a hindrance to clear thinking in determining the best arrangement of the beams and the panels of slabs.

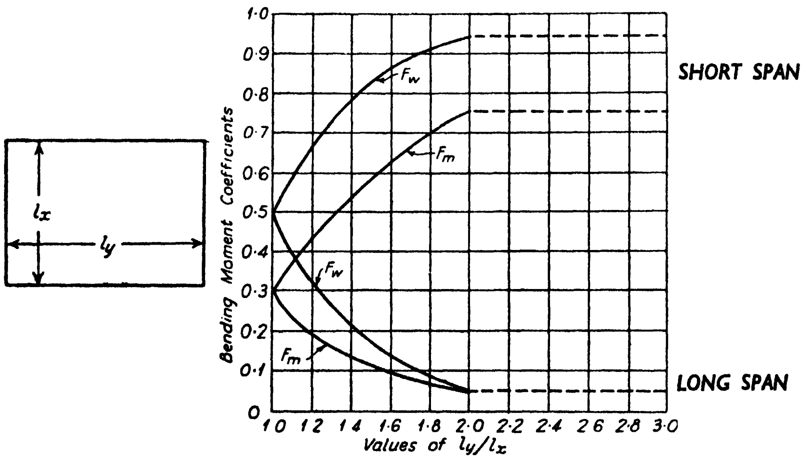


FIG. 72.

First it is necessary to establish factors which enable the bending moments in a simply-supported slab spanning in two directions to be calculated within safe limits. The slab in *Fig. 72* is freely supported on four sides, and l_x is the shorter span, l_y the longer span, F_w the load-distribution factor, F_m the moment-distribution factor, and w the intensity of the uniformly-distributed load. In the middle of the slab the load is transferred to the supports by the slab acting very approximately as a series of independent strips, the load being divided between the short and long strips roughly in the proportions of $\frac{l_y^4}{l_x^4 + l_y^4}$ and

$\frac{l_x^4}{l_x^4 + l_y^4}$, since deflections are proportional to wl^4. Near the supports this relationship does not apply. The slab spans across the corners, and near its mid-span each beam, either by contortion or continuity of the slab, supports almost the whole of the load of the adjacent strip of slab. The partly-empirical distribution shown by the curves in *Fig.* 72 therefore relates F_w and F_m to $\frac{l_y}{l_x}$ by a fourth-degree curve up to $\frac{l_y}{l_x} = 2$; up to, say, 3 we may assume F_w and F_m to be constant. The moment-distribution factor F_m is about 40 per cent. less than the load-distribution factor F_w, and may be used if at the corners of the slab one-half of the reinforcement necessary in the bottom of the slab at mid-span is continued to the supports and if the remaining one-half is turned up into the top of the slab and continued to the supports (or beyond if the slab is continuous) in order to provide "corner restraint." It can be shown that, if 10 per cent. of the total load on the slab is assumed to be supported at each corner by a short strip spanning across the corner normal to the diagonal at the corner and supporting a strip spanning from the corner along the diagonal, the reinforcement required for corner restraint is less than one-half of that required at the middle of the longer span. Although the reinforcement is at an angle of 45 deg. to the direction of the bending, the tensile forces in the two bands of reinforcement mutually at right-angles, when added vectorally, act approximately in the direction of the bending. Thus the moment-distribution factors for the short span can, on account of corner restraint, be assumed to be 40 per cent. less than the load-distribution factors, and in any given case a slab spanning in two directions and simply supported on four sides should be designed for bending moments per unit width of slab of the values:

$$\text{Short span: } \frac{F_m w l_x^2}{8}. \quad \text{Long span: } \frac{F_m w l_y^2}{8}.$$

The value of F_m is that for the particular ratio $\frac{l_y}{l_x}$ and, except for a square panel, has different values for the short and long spans.

If the slab is continuous, before obtaining the moment-distribution and load-distribution factors, the values of l_x and l_y should be adjusted by multiplying by 0·8 if continuous at one support only, and by 0·67 if continuous at both supports. The bending moments are then $\frac{F_m w l_x^2}{10}$ and $\frac{F_m w l_y^2}{10}$ if continuous at one end only, and $\frac{F_m w l_x^2}{12}$ and $\frac{F_m w l_y^2}{12}$ if continuous at both ends.

In cases in which adjacent slabs have spans of very different lengths, the method of characteristic points or other suitable method should be used to determine the bending-moment factors for continuous strips of slab subjected to the most adverse incidence of live load. A slab of short span between two slabs of longer spans must generally be provided with reinforcement in the top throughout.

Although the foregoing method of calculation is only approximate, the results obtained are sufficiently accurate for most practical purposes. It should be remembered that the whole of the load on the slab is calculated as being transferred to the beams and a reduction made only to the bending moments. The assumed reduction of about 40 per cent. for the short span is on the safe side, and any errors in the distribution of the moments are negligible since, within the plastic range at the stage of yielding, a redistribution of moments to utilise the maximum strength of the slab takes place.

By-laws often allow the superimposed load on the slab to be reduced by, say, 10 lb. per square foot, when designing the beams. This reduction has nothing to do with the way in which the slab behaves in supporting the load, but is allowed because it is unlikely that the slabs supported by one beam will be simultaneously subjected to the full load.

The load on a beam supporting a slab spanning in two directions has approximately a trapezoidal distribution but, when calculated by the appropriate load-distribution factor, the load may be assumed to be uniformly distributed, thereby facilitating the calculations for the beam which will usually support another slab and perhaps a partition, each of which may contribute an equivalent uniformly-distributed load towards the total load on which the design of the beam is based.

The ordinary procedure for the design of slabs can be summarised as follows:

(1) Reduce all partition or concentrated loads to equivalent uniformly-distributed loads and add to the uniformly-distributed live load required by the by-laws or suitable for the structure.

(2) If the slab is continuous, adjust the ratio of $\frac{l_y}{l_x}$ by multiplying each span by a suitable continuity-factor, that is, 0·8 for continuity at one support, or 0·67 for continuity at both supports.

(3) For the adjusted ratio of $\frac{l_y}{l_x}$ determine from the curves in *Fig.* 72 the appropriate load-distribution and moment-distribution factors F_w and F_m.

(4) Determine the reactions from the slab on each beam as follows:

$$\text{Long beam: } \frac{F_w w l_x}{2}. \quad \text{Short beam: } \frac{F_w w l_y}{2}.$$

Assume these reactions to be distributed uniformly along each beam, and make any reduction in the live load allowed by by-laws (such as using 40 lb. per square foot of slab instead of 50 lb. per square foot as required for the design of the slab).

(5) Determine the bending moments for a slab of unit width in each direction using the formulæ

$$\text{Long span: } \frac{F_m w l_y^2}{N}. \quad \text{Short span: } \frac{F_m w l_x^2}{N}.$$

The numerator N has the following values:

Mid-span: Both ends simply supported, 8; continuous at one end only, 10; continuous at both ends, 12.

Support: Support next to end support, 10; all other supports except end supports, 12.

These values of N allow the worst possible arrangements of live load provided that consecutive spans are about equal in length; suitable adjustments must be made if consecutive spans are unequal, using one of the ordinary methods of dealing with continuous beams. The bending moments at the supports are actually greater than those at mid-span within the elastic range, but due to creep, and in the event of excessive load at the yield stage, a distribution of the moments takes place to utilise the maximum strength of the slab. The factor of safety is therefore reduced by a negligible amount by the assumption of equal bending moments at support and mid-span, and practical design is greatly convenienced by this assumption since the amount of reinforcement and the thickness of the slab can be the same at the supports and at mid-span without waste. The D.S.I.R. Code (1934) and some by-laws do not permit the bending moment at mid-span to be assumed to be equal to that at the supports. The B.S. code gives coefficients for several cases (see Appendix I).

Tables have been published that give the load-distribution and moment-distribution factors for various ratios of live to dead load and consider more accurately the effect of continuity on the distribution of the load. In special cases, such as when the slab is subjected to a heavy live load, the refinements in such tables are a great help, but generally there is no advantage in departing from the simple procedure explained, as slightly inaccurate distribution of the bending moments assumed in the design is always adjusted under excessive load. Experience has shown also that corner restraint and the torsional resistance of the beam assist the slabs to a greater extent than is assumed, and even when the permissible reductions in bending moment are made the slabs in a monolithic reinforced concrete floor have a higher factor of safety than the beams.

In calculating areas of the reinforcement in slabs spanning in two directions, the effect on the effective depth of two layers of bars must be considered.

Flat-Slab Beamless Construction.

The distribution of bending moments and stresses in flat slabs cannot satisfactorily be determined by the elastic theory. Formulæ in general are based on the results of full-scale tests. Those recommended in some of the latest codes and by-laws in Britain assume higher bending moments at some sections than do some American regulations. Many buildings in the United States, South Africa, and elsewhere have, however, been designed in accordance with the American regulations with satisfactory results. It can therefore be assumed that the recommendations of the B.S. Code (1948) and the by-laws of the London County Council (1938) are safe.

The load on the slab is assumed to be transferred to the columns by the "middle bands" spanning between the "column bands" (*Fig.* 73) which span between the columns. In addition to the overlapping of bands mutually at right-angles, spanning also takes place in a diagonal direction between the columns. This effect, combined with the restraining effect of the columns when alternate panels are loaded, reduces the bending moments obtained from the first assumption. Reinforcement in two directions only is the most common practice; four-way systems have few advantages and are difficult to fix. The recommendations of the B.S. Code are given in the following.

BENDING MOMENTS IN SLABS.—The total bending moment over the whole width of the band, if fully continuous, should be:

		Bending moment at mid-span	Bending moment at support
Panels without drops	Column band Middle band	$0{\cdot}022wL(L - \frac{2}{3}D)^2$ $0{\cdot}018wL(L - \frac{2}{3}D)^2$	$0{\cdot}042wL(L - \frac{2}{3}D)^2$ $0{\cdot}018wL(L - \frac{2}{3}D)^2$
Panels with drops	Column band Middle band	$0{\cdot}022wL(L - \frac{2}{3}D)^2$ $0{\cdot}016wL(L - \frac{2}{3}D)^2$	$0{\cdot}046wL(L - \frac{2}{3}D)^2$ $0{\cdot}016wL(L - \frac{2}{3}D)^2$

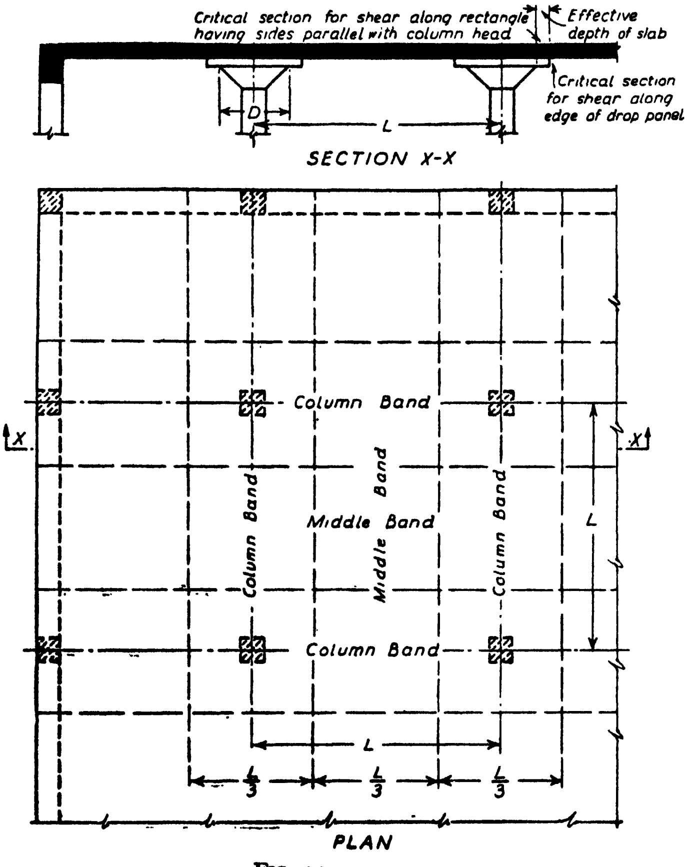


FIG. 73.

In the foregoing, L is the distance between the centres of the columns, D is the width of the column head, and w is the total load per unit area.

For panels that are not square, the dimension L within the brackets is the distance between the centres of the columns in the direction of the span being considered, and the dimension L outside the brackets is the distance between the centres of the columns in the direction at right angles to the direction of the span.

END PANELS.—Assume bending moments at mid-span 25 per cent. greater than for fully continuous panels. The area of the reinforcement at the end support should be 90 per cent. of that at fully continuous supports for column bands and 60 per cent. for the middle bands. The edges should be reinforced and an edge-beam provided so as to give some restraint which is transmitted by torsion to the columns. The area of the reinforcement in the semi-column band parallel to the edge beam should be one-half of the area required in a column strip of full width.

SHEAR.—The critical sections are assumed to be around the perimeter of the dropped panel, and at a distance equal to the effective depth of the slab from the side of the column head (*Fig.* 73). The shearing stresses permitted for beams are allowed in these sections.

BENDING MOMENTS ON COLUMNS.—The bending moments in internal and external columns are assumed to be 50 per cent. and 90 per cent. respectively of the support bending moment in the column band, and are divided between the columns above and below the slab in proportion to the stiffness of the columns.

DIMENSIONS OF PANELS.—At least three panels must be provided in either direction. The length of the longer side of a panel that is not square must not exceed 1⅓ times the length of the shorter side. In a series of panels the length of any one panel must not differ by more than 10 per cent. from the greatest length. End spans may be shorter than but not longer than interior spans.

DIMENSIONS OF COLUMNS.—The heads of columns are splayed at an angle of 45 deg. to form enlarged heads having a width at least equal to $0{\cdot}2L$ and not more than $0{\cdot}25L$.

For complete details of the limiting dimensions, the arrangement of the reinforcement, and other matters the reader should refer to the British Standard Code (1948) or the London by-laws. The design of flat slabs is simple and it is only necessary carefully to follow the requirements of the recommendations or regulations. It is advisable to concentrate the reinforcement in the column bands near the centre line of the panels. The bars in the top of the slab should be supported on very strong stools which cannot be trodden down during concreting, thereby preventing displacement of bars. The work of the men fixing the reinforcement is greatly assisted if the order of fixing the bars is clearly indicated on the drawings; otherwise it is easy for more than two layers of bars to occur in the top or bottom of the slab. In the calculations of the effective depth allowance must be made for the two layers of bars.

VI.—THIN VAULTS.

A slab supporting a load acts as a shell or membrane when it is sufficiently thin to offer negligible resistance to bending, has a suitable shape, and is supported along the edges so that the load is distributed by direct stresses only acting in the plane of the membrane. Common examples are domes and semi-cylindrical, partly cylindrical, and semi-ellipsoidal reinforced concrete roofs. The fundamental theory, upon which the following is largely based, was first proposed by Dischinger in "Handbuch für Eisenbetonbau," volume VI.

The Membrane Theory.

Fig. 74 shows at (*a*) a cross section and at (*b*) a plan of a semi-ellipsoidal or semi-cylindrical thin vault. Consider the equilibrium of the small element ABCD shown at (*c*) and having a length dx in the longitudinal direction of x, a width dy measured along the curve, a slope ψ, and a radius of curvature R, and subtending an angle $d\psi$ at the point of intersection of the normal to the tangent at B with the vertical axis of symmetry. When it is infinitely small, the element is subject to the following stresses: (i) A direct stress T_1 in the longitudinal direction x; (ii) A direct stress T_2 in the direction of y tangentially to the curve; and (iii) A shearing stress S. These three stresses can be combined, as explained later, into two principal stresses acting mutually at right angles. The values of the stresses depend on the position of the element in the longitudinal and transverse spans, the curvature, and the load. If the whole of the transverse thrust is transmitted to the supports as a shearing force, the membrane as a whole spans longitudinally as a hollow beam with the lower edges acting as flanges and resisting tensile forces, the upper segment of the vault resisting longitudinal compression, and the remainder of the vault acting as a curved sloping web subjected mainly to shearing stress.

Let the components of unit load applied to the element be: (i) In the longitudinal direction of x, X per unit area of the membrane; (ii) In the tangential transverse direction, Y per unit area of the membrane; and (iii) Normal to the element, Z per unit area of the membrane.

The area of the element is $R.d\psi.dx$. Assuming the membrane to have unit thickness, the cross-sectional area longitudinally is dx. Transversely, the cross-sectional area is $R.d\psi$.

The resultant internal forces acting on the element are the rates of increase in stress multiplied by the dimensions of the element. In the longitudinal direction of x, the force X acting on an area $R.d\psi.dx$ is balanced by the increment of stress $S.dx$ acting along the transverse width $R.d\psi$ and the increment of stress $T_1.R.d\psi$ acting along dx. Therefore

$$\frac{\partial S}{R\partial\psi}.dx.R.d\psi + \frac{\partial T_1}{\partial x}.R.d\psi.dx + X.R.d\psi.dx = 0,$$

or

$$\frac{\partial T_1}{\partial x} = -\frac{\partial S}{R\partial\psi} - X \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In the tangentially transverse direction y, the component Y of the load acts on the area $R.d\psi.dx$ and is resisted by the increment of stress $T_2.dx$ over the width $R.d\psi$ and by the increment of $SR.d\psi$ along dx. Therefore,

$$Y.R.d\psi.dx + \frac{\partial T_2}{R\partial\psi}R.d\psi.dx + \frac{\partial S}{\partial x}.R.d\psi.dx = 0,$$

or

$$\frac{\partial S}{\partial x} = -\frac{\partial T_2}{R\partial\psi} - Y \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the direction normal to the element, the load Z acting on an area $R.d\psi.dx$ is resisted by the component of $T_2.dx$ acting in the direction normal

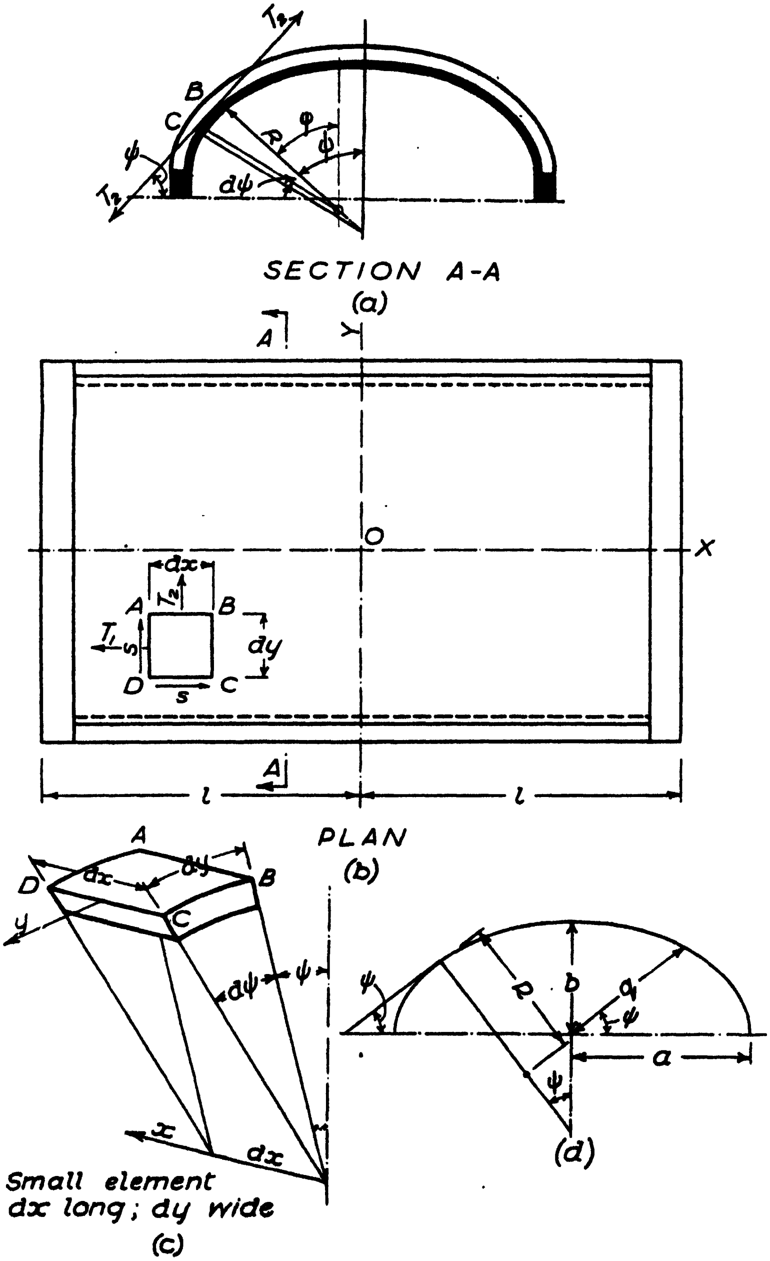


FIG. 74.

to the tangent to the element. Therefore $Z.R.d\psi.dx + T_2.dx.d\psi = 0$, from which

$$T_2 = -RZ \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Thus the circumferential tension is the load normal to the element multiplied by the radius of curvature.

In equations (1), (2), and (3), R is a function of ψ, and the general solutions are, by integrating with respect to x,

$$T_1 = -\int\frac{\partial S}{R\partial\psi}dx - \int X.dx + f_2(\psi) \quad . \quad . \quad . \quad (4)$$

$$S = -\int\frac{\partial T_2}{R\partial\psi}dx - \int Y.dx + f_1(\psi) \quad . \quad . \quad . \quad (5)$$

and

$$T_2 = -RZ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The functions $f_1(\psi)$ and $f_2(\psi)$ depend on the edge-conditions and the system of co-ordinates selected. For curves in which R, X, Y, and Z can be expressed in terms of ψ the solution of the equations is straightforward. Consider the case of a vault of uniform thickness supporting only its own weight g per unit area, that is, $X = 0$, $Y = g \sin\psi$, and $Z = g\cos\psi$. Substituting in (3),

$$T_2 = -gR\cos\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$S = -x\left(g\sin\psi + \frac{\partial T_2}{R\partial\psi}\right) + f_1(\psi).$$

Substituting, $$S_0 = g\sin\psi + \frac{\partial T_2}{R\partial\psi},$$

$$S = -xS_0 + f_1(\psi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Substituting S from (7) in (4), with $X = 0$,

$$T_1 = -\int\frac{\partial[-xS_0 + f_1(\psi)]\,dx}{R\partial\psi} + f_2(\psi)$$

$$T_1 = \frac{x^2}{2}\frac{\partial S_0}{R\partial\psi} - \frac{x\partial[f_1(\psi)]}{R\partial\psi} + f_2(\psi) \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In general, $R = R_0\cos^n\psi$ where R_0 is the radius of curvature at the crown of the vault and n is -3 if the curve is a parabola, -2 if a catenary, 0 if a circle, and 1 if a cycloid.

Therefore, for a catenary, substituting $R = R_0\cos^{-2}\psi$ in (6) and (7),

$$T_2 = -gR_0\cos^{-1}\psi,$$

and

$$S = -x\left[g\sin\psi + \frac{\partial(-gR_0\cos^{-1}\psi)}{R_0\cos^{-2}\psi.\partial\psi}\right] + f_1(\psi)$$

$$= -x[g\sin\psi - g\sin\psi] + f_1(\psi) = f_1(\psi).$$

Therefore there is no shearing force S if $f_1(\psi) = 0$, that is when the shearing force is not opposed by a longitudinal direct stress, in which case $T_1 = 0$. This might be expected since the form of the line of thrust for a vertical load uniformly distributed along the axis of an arch is an inverted catenary. The thrust T_2 is

$-gR_0\cos^{-1}\psi$ and the horizontal component is $H = -T_2\cos\psi = -gR_0$, which is constant and which must be resisted at the springing by a tie or abutment.

If the radius of curvature increases with ψ, $H(=T_2\cos\psi)$ decreases as ψ increases, being gradually taken up by shear tangential to the section, the shear in turn being taken up by longitudinal tension. If the tangent to the curve is vertical at the springing, as in a semicircle or semi-ellipse, from the equation $T_2 = -gR\cos\psi$ it is seen that at the springing $T_2 = 0$; therefore no abutment or transverse tie is required, and the vault spans as a hollow beam of large moment of inertia. The total shearing force on the beam at any section is the sum of the vertical components of the shearing force S, and the moment of resistance is the moment of the internal longitudinal stresses T_1 and edge-tension Z' (see later). The vertical distribution of shearing and bending stresses at any section is not the same as in a beam, but depends on the distribution of the load.

DETERMINATION OF $f_1(\psi)$ AND $f_2(\psi)$ WHEN ψ IS 90 DEG. AT THE SPRINGING AND THE VAULT SPANS AS A HOLLOW BEAM.—It is best to assume a co-ordinate system in which at the point where there is no shearing stress $x = 0$ and $f_1(\psi)$ and the bending moment both equal nothing when $x = l$. Therefore $f_2(\psi)$ is obtained from equation (8) with $T_1 = 0$ when $x = l$, remembering that l is the semi-span.

Equation (6) remains as before.

Equation (7) becomes

$$S = -S_0 x \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where $S_0 = g\sin\psi + \dfrac{\partial T_2}{R\partial\psi}$.

Equation (8) becomes $T_1 = \dfrac{x^2}{2}\dfrac{\partial S_0}{R\partial\psi} - \dfrac{l^2}{2}\dfrac{\partial S_0}{R\partial\psi}$

$$= -\frac{(l^2 - x^2)}{2}\frac{(\partial S_0)}{R\partial\psi} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$T_2 = 0$ when $\psi = 90$ deg. S is maximum when $\sin\psi = 1$. Substituting for T_2 and $\sin\psi = 1$ in (9),

$$S = -x\left[g\sin\psi + \frac{\partial(gR\cos\psi)}{R\partial\psi}\right] = -x(g+g)$$

Thus $\quad S_{max.} = -2gx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$

This greatest shearing force occurs at the springing of the vault and transmits local longitudinal tension to the edge, which must be reinforced as the tension flange of the hollow beam.

The greatest tension in the edge flange occurs at mid-span, $x = 0$, and is obtained by integrating the shear $S_{max.}$ over the half-span l. The edge-tension Z' at any point x from mid-span is

$$\int_l^x S_{max.}\,dx = \int_l^x -2gx\,.dx = g(l^2 - x^2) \quad . \quad . \quad . \quad (12)$$

and is greatest when $x = 0$; that is

$$Z'_{max.} = gl^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The edge-tension Z' is therefore parabolically distributed and at any section is proportional to the bending moment and, with the force T_1 at the section, provides the resistance-moment couple. The sum of the tension Z' in each edge equals the sum of the forces T_1 acting across any section. Equation (11) shows that the shearing force at the edges of vaults which are vertical at the springing is independent of the transverse span and of the form of the cross section, and that it is a function of the longitudinal span only. For large longitudinal spans the edge reinforcement and surrounding concrete add considerable dead load to the roof at the edges. This load is spread over the roof by bending in the plane of the roof, which therefore cannot act as a perfect membrane. It must be ensured that local bending stresses are not excessive. Because the symbols are similar, the edge-tension Z' should not be confused with the normal component Z of the load.

Semi-ellipsoidal Vault.

A semi-ellipsoidal vault is vertical at the springings and is a suitable shape for a reinforced concrete roof. For an ellipse, $R = \dfrac{a^2b^2}{(a^2 \sin^2 \psi + b^2 \cos^2 \psi)^{\frac{3}{2}}}$, where $2a$ is the transverse span and b is the rise [*Figs.* 74 (*d*) and 75 (*a*)].

If the conjugate radius is $q = \dfrac{ab}{(a^2 \sin^2 \psi + b^2 \cos^2 \psi)^{\frac{1}{2}}}$, $R = \dfrac{q^3}{ab}$.

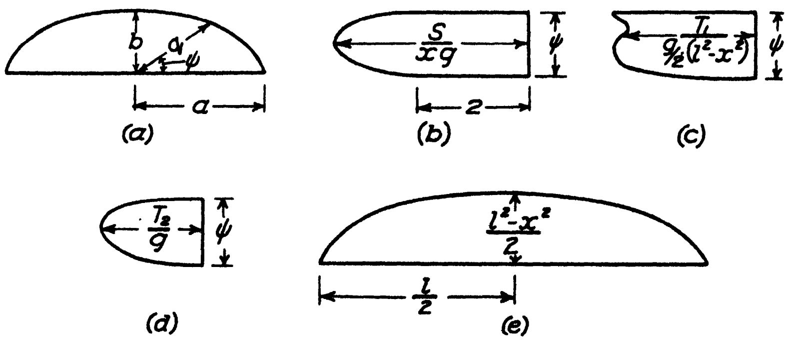


FIG. 75.—STRESSES IN ELLIPSOIDAL VAULT.

DEAD WEIGHT.—Substituting for R in (6),

$$T_2 = -g \cos \psi . \frac{a^2b^2}{(a^2 \sin^2 \psi + b^2 \cos^2 \psi)^{\frac{3}{2}}}$$

$$\frac{\partial T_2}{R\partial\psi} = -g\frac{a^2b^2}{R}[-\sin\psi(a^2\sin^2\psi + b^2\cos^2\psi)^{-\frac{3}{2}}$$

$$-\cos\psi\,\tfrac{3}{2}(a^2\sin^2\psi + b^2\cos^2\psi)^{-\frac{5}{2}}\,.2\sin\psi\cos\psi(a^2 - b^2)].$$

Let $a^2 - b^2 = e^2$. Substituting for R,

$$\frac{\partial T_2}{R\partial\psi} = g \sin \psi[1 + 3e^2 \cos^2 \psi(a^2 \sin^2 \psi + b^2 \cos^2 \psi)^{-1}]$$

$$S_0 = g \sin \psi + \frac{\partial T_2}{R\partial\psi} = g \sin \psi[2 + 3e^2 \cos^2 \psi(a^2 \sin^2 \psi + b^2 \cos^2 \psi)^{-1}].$$

$$\frac{\partial S_0}{R\partial\psi} = \frac{1}{R}g \cos \psi[2 + 3e^2 \cos^2 \psi(a^2 \sin^2 \psi + b^2 \cos^2 \psi)^{-1}]$$

$$+ \frac{1}{R}g \sin \psi\left[- 3e^2(a^2 \tan^2 \psi + b^2)^{-2}\, 2a^2 \tan \psi \frac{1}{\cos^2 \psi}\right]$$

$$= \frac{1}{R}g \cos \psi\left[2 + \frac{3e^2 \cos^2 \psi}{a^2 \sin^2 \psi + b^2 \cos^2 \psi} - \frac{6e^2a^2 \sin^2 \psi}{(a^2 \sin^2 \psi + b^2 \cos^2 \psi)^2}\right].$$

Substituting this value of $\frac{\partial S_0}{R\partial\psi}$ in (6), (9), and (10) with

$$R = \frac{q^3}{ab} \text{ and } q = \frac{ab}{(a^2 \sin^2 \psi + b^2 \cos^2 \psi)^{\frac{1}{2}}}$$

$$T_2 = - gR \cos \psi = - g \cos \psi \frac{q^3}{ab} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$S = - S_0 x = - xg \sin \psi\left[2 + \frac{3e^2q^2 \cos^2 \psi}{a^2b^2}\right]. \quad . \quad . \quad . \quad (15)$$

$$T_1 = - g.\frac{l^2 - x^2}{2} \cos \psi\left[\frac{2ab}{q^3} + \frac{3e^2}{abq} - \left(\cos^2 \psi - \frac{2q^2}{b^2} \sin^2 \psi\right)\right] \quad . \quad (16)$$

At the crown of the semi-ellipse, where $\psi = 0$ and $q = a$, $S = 0$,

$$T_2 = - g.\frac{a^2}{b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14a)$$

$$T_1 = - g.\frac{l^2 - x^2}{2}.\frac{3a^2 - b^2}{a^2b} \quad . \quad . \quad . \quad . \quad (16a)$$

At the springing of the semi-ellipse, where $\psi = 90$ deg. and $q = b$, $T_1 = T_2 = 0$ and $S = - 2gx$, as in (11).

The tension in the edge-member which resists the shearing force from the roof is obtained from (12), and is $Z' = g(l^2 - x^2)$.

Similarly a transverse edge-member must be provided to resist the shearing force at the sections where $x = l$. Generally gable walls are required at the ends of the vault, and these provide the resistance required. Considering the horizontal forces across a longitudinal section, the force across the end panel at the crown is $\int_0^l T_2.dx$, since the external forces are vertical and there is no shearing force S.

$\int_0^l T_2\, dx = \int_0^l Rg \cos \psi.dx = Rg \cos \psi l$, which, when $\psi = 0$, equals Rgl.

Thus the tension across the crown of the end panel is Rgl.

The determination of the distribution of the stresses throughout an elliptical vault such as a roof is greatly facilitated by plotting diagrams as in *Fig.* 75. By choosing a simple ratio of $\frac{b}{a}$, say, $\frac{1}{2}$, $\frac{S}{xg} = 2 \sin \psi + 9\frac{q^2}{a^2} \cos^2 \psi \sin \psi$. Values of q can be obtained for various values of ψ from *Fig.* 75 (*a*). Values of $\frac{S}{xg}$ are plotted at (*b*) for various values of ψ, values of $\frac{T_1}{\frac{g}{2}(l^2 - x^2)}$ at (*c*), values of $\frac{T_2}{g}$ at (*d*), and values of $\frac{l^2 - x^2}{2}$ for various values of x at (*e*). Values of T_2 for any particular value of ψ are constant throughout the length of the vault. Values of S are readily obtained from (*b*) for any value of x and ψ, and values of T_1 from (*c*) in conjunction with (*e*). The value of Z' (the tension in the longitudinal edge member at any position) can be readily obtained from (*e*).

The direct stresses in a longitudinal and transverse direction and the shearing force can be obtained at any point of the vault, and thus the thickness of the vault and the reinforcement required in each direction can be determined. If required, the principal stresses at each point can be obtained by combining the direct stress and shearing stress by the formulæ given later, and the reinforcement can then be arranged along the lines of the tensile principal stresses. This arrangement is not essential, and it may be easier to provide transverse and longitudinal reinforcement only, the reinforcement at any point being calculated to resist transverse and longitudinal components of the tensile principal stress.

SNOW LOAD.—Assuming a load p_0 is uniformly distributed over the horizontally projected area of the shell, the vertical load (p) is $p_0 \cos \psi$, the components of which are $X = 0$, $Y = p_0 \sin \psi \cos \psi$, and $Z = p_0 \cos^2 \psi$. Substituting these values in (6), (9), and (10),

$$T_2 = -p_0\frac{q^3}{ab} \cos^2 \psi \quad . \quad . \quad . \quad (17)$$

$$S = -1{\cdot}5p_0 \sin 2\psi\left[1 + \frac{e^2q^2}{a^2b^2} \cos^2 \psi\right]x \quad . \quad . \quad . \quad (18)$$

and
$$T_1 = -\frac{l^2 - x^2}{2}3p_0\left[\left(\frac{ab}{q^3} + \frac{e^2 \cos^2 \psi}{abq}\right) \cos 2\psi + \frac{qe^2}{2ab^3} \sin^2 2\psi\right] \quad . \quad (19)$$

At the crown, $\psi = 0$ and $q = a$; therefore $S = 0$,

$$T_2 = -p_0\frac{a^2}{b} \quad . \quad . \quad . \quad (17a)$$

and
$$T_1 = -\frac{l^2 - x^2}{2}.\frac{3p_0}{b} \quad . \quad . \quad . \quad (19a)$$

At the springing, $\psi = 90$ deg., $q = b$, $T_2 = 0$, therefore $S = 0$,

and
$$T_1 = \frac{l^2 - x^2}{2}3p_0\frac{a}{b^2} \quad . \quad . \quad . \quad (19b)$$

Since the shearing force at the springing is zero the tension in the edge member is zero. If snow accumulates in the valleys between a series of vaults then the concentrated load must be distributed by bending over the whole cross section and the local bending stresses investigated as the roof is no longer acting purely as a membrane, but very approximately so.

WIND LOAD.—Wind load is seldom important and the snow load is generally considered sufficient to include the effects of wind. A horizontal wind pressure w_0 on the edge member causes a thrust $T_2 = w_0 R_k$, where R_k is the radius of curvature at the springing. Where high winds are expected, the distribution of pressure should be determined from wind-tunnel tests. The thrust T_2 must be distributed by bending throughout the cross section of the roof as is the weight of the edge member. Thus the roof again acts as a slab and not as a pure membrane, and local bending stresses should be investigated.

If the radius of curvature of a vault roof is large at the crown, where the compressive stresses are high, the effects of errors in construction, and of concentration instead of distribution of loads leading to possible local buckling, should be investigated, and if necessary stiffening ribs should be provided.

Circular-Arc Roofs.

A common form of thin roof is that in which the vault is a part of the surface of a cylinder, that is the cross section of the vault is an arc of a circle. If the cross section is semi-circular, the tangents at the springings being vertical, the formulæ for an elliptical vault apply with $a = b$. There is no transverse thrust from the vault at the springings.

In most cases, however, the circular-arc vault is less than a semi-cylinder, and edge beams are provided. These beams are often designed as members separate from the vault and resist the transverse thrust from the latter. Alternatively the vault can be designed to resist transverse and longitudinal bending moments and torsion. The edge of the vault at the middle of the longitudinal span then tends to act in bending as a horizontal cantilever. Mathematical analyses of this case have been made, but they are necessarily complex. Further research may lead to a simpler theory based on the distribution of the stresses at ultimate load, when plasticity and the commencement of local buckling probably cause a fairly uniformly-distributed longitudinal compressive stress in the top of the vault. This stress, combined with the tension in the beams at the edges, forms the internal couple resisting longitudinal bending. The compressive zone extends downwards towards the supports so that the vault may possibly act like two arches, arranged diagonally on plan and tied longitudinally at the edges of the vault and transversely at the ends. Due to the vault spanning between these arches and the ties, bending and torsional stresses are developed in the roof and the longitudinal ties sag and act slightly in suspension.

Principal Stresses.

When the stresses T_1, T_2, and S have been determined for any point in the vault, the principal stresses at the point are given by the common expression

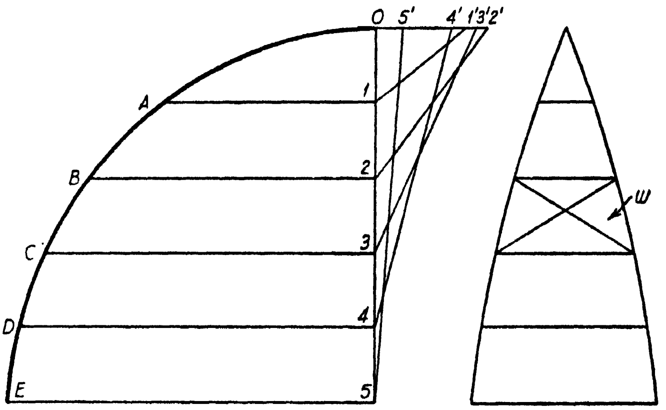
$$\begin{matrix} p_1 \\ p_2 \end{matrix} = \frac{T_1 + T_2}{2} \pm \sqrt{\frac{(T_1 - T_2)^2}{4} + S^2}.$$

The direction in which p_1 acts is normal to the plane that makes an angle θ

with the direction of the stress T_1 such that $\tan 2\theta = \frac{2S}{T_1 - T_2}$. The stress p_2 acts at right-angles to p_1, both principal stresses being in the plane of the slab.

Graphical Analysis of a Dome.

The stresses in a spherical thin dome can be obtained very simply by graphical analysis. Referring to *Fig.* 76, 05 is the axis of rotation of a dome and is divided into equal parts 01, 12, etc. The horizontal planes through 1, 2, 3, etc., divide the surface of the dome into rings each projecting an equal area on the surface of a cylinder having the same axis of rotation and radius R as the dome. Let 01, 12, etc., represent the weight w of a segment of a ring projecting unit cir-



CROSS SECTION

Elevation: Segments of Rings projecting unit length on the circumference of the circumscribing cylinder.

FIG. 76.

cumferential length on the cylinder. If W is the total weight of the dome, in the case illustrated, $2\pi Rw \times 5 = W$, and $w = \frac{W}{10\pi R}$. Draw 11^1, 22^1, 33^1, etc., parallel to the tangents at A, B, C, etc. Then 11^1, 22^1, etc., represent the tangential thrusts exerted by each segment on each vertically adjacent segment, and 01^1, 1^12^1, 2^13^1, etc., represent the radial thrusts or pulls exerted on each ring by each segment. The total circumferential thrust or tension at the bottom of any ring is therefore equal to the lengths 01^1, 1^12^1, 2^13^1, etc., multiplied by the horizontal radius of the dome at the bottom of the ring. The component 1^12^1 causes thrust in the ring but 2^13^1 causes ring-tension.

If the dome is supported at, say, the level of 3 or 4, a ring beam must be provided to resist the thrusts 33^1 or 44^1, and the ring tension is 03^1 or 04^1 multiplied by the horizontal radius of the dome at levels 3^1 and 4^1 respectively.

CHAPTER III

BEAMS AND SLABS

I.—THEORY OF REINFORCED CONCRETE IN BENDING.

THE distribution of the stresses in a reinforced concrete beam is too complicated for exact mathematical analysis as is possible in the case of a beam of homogeneous elastic material forming part of a pin-jointed structure. The theory of the bending and shearing stresses generally accepted to-day is based on two different, but justified, series of assumptions. Bending stresses are, of course, quite inseparable from shearing stresses. That the assumptions provide an adequate factor of safety is confirmed by tests on actual structures. With these assumptions as a basis, formulæ are derived by strictly mathematical procedures and are therefore safe within the limits of the assumptions.

In calculating the bending stresses normal to a cross section of a member, it is assumed that the member acts as a solid beam of homogeneous material except that on the tension side of the neutral axis the concrete, being weak in tension, is assumed to be cracked and the whole of the tension is resisted by steel reinforcement embedded in and gripped by the surrounding concrete. The variation of strain on the tension side of the neutral axis is assumed to be identical with that on the compression side, except that it is of opposite sign; thus plane sections before bending are assumed to remain plane after bending and the strain at any point is assumed to be proportional to the distance of the point from the neutral axis.

Due to the grip of the concrete, which must be sufficient to ensure that no slip takes place between the concrete and the reinforcement, the same strain is developed in the reinforcement as in the surrounding concrete and therefore the stress in the reinforcement is equal to the imaginary tensile stress in the surrounding concrete multiplied by the ratio of the moduli of elasticity of the steel and the concrete.

Shearing stresses are generally calculated on the assumption that a reinforced concrete beam acts as a combination of several pin-jointed frames [*Fig.* 77 (*a*) to (*e*)] which can be superimposed regardless of their relative stiffnesses. The diagonal and vertical steel tensile members are assumed to develop their safe working stress regardless of the stress, or imaginary stress after cracking, in the surrounding concrete. Beams are designed so that the stress in the imaginary diagonal concrete compressive members is low, the value being based on tests; this limits the width of cracks in the concrete. These assumptions differ from those made for the bending stresses since, unless the stress in the vertical and inclined steel tensile members is very low, the strain of the steel will be greater than can be developed in the surrounding concrete without plane sections no longer remaining plane after bending, as is assumed for bending stresses [*Fig.* 77 (*g*) and (*h*)]. The tendency of the imaginary shear members to act as those of a pin-jointed frame, however, causes a more uniform distribution of stress in the upper compression zone. The assumptions made for shearing and bending stresses can be reconciled to a certain extent by considering the redistribution of stresses which must take place due to diagonal cracking and plasticity before failure.

The ideal distribution of stress is one which causes the imaginary compression boom to act before failure as if it were perfectly articulated with the diagonal

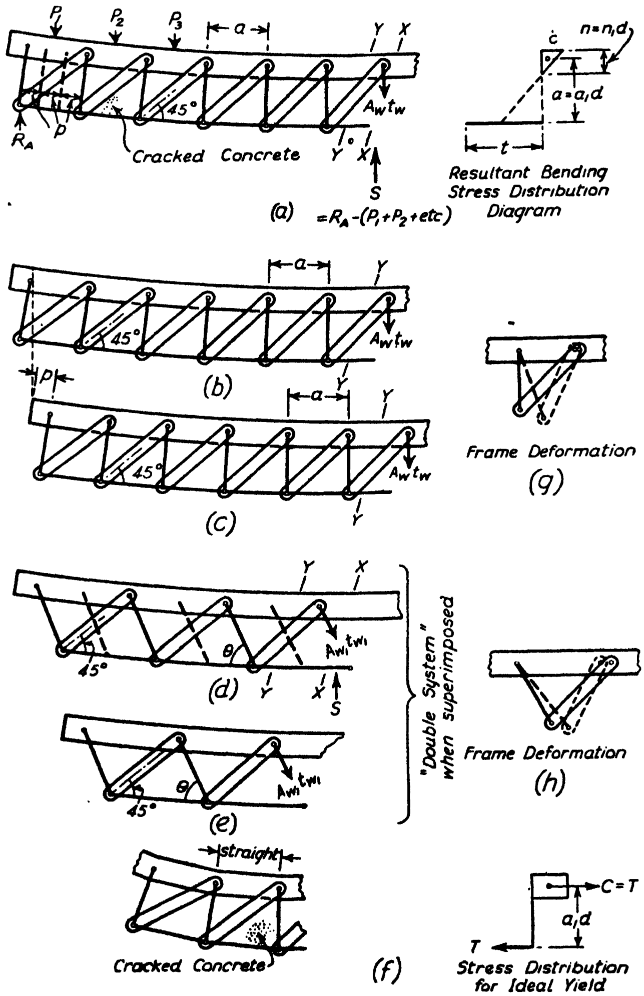


FIG. 77.

members at the imaginary pin-joints [*Fig.* 77 (*f*)]. The compressive stress in the concrete in the boom then becomes uniformly distributed, and before failure the beam acts as a pin-jointed framed girder in which, regardless of the imaginary tensile stress in the concrete, all the reinforcement is stressed at its yield stress. The depth n' of the imaginary compression boom would be sufficient to provide a total compression equal to the total tension available at the yield stress in the main tensile reinforcement. Such a distribution of stress is shown in *Figs.* 77 (*f*) and 78. The moment of resistance of the beam when the reinforcement yields is Ta, where T is $A_t t_y$, A_t is the cross-sectional area of the reinforcement, and t_y is the yield stress of the steel. The total compression in the concrete, C, is equal to T. If c_y is the "yield" stress of the concrete and b is the breadth of the beam, $n'bc_y = C = T$. Therefore $n' = \frac{T}{bc_y}$. The lever arm is $a' = d - \frac{n'}{2}$. The moment of resistance based on the resistance of the concrete is therefore nearly twice as great as if a triangular distribution of stress is assumed. Moreover, even without perfect articulation, if x is the maximum strain of a thin diagonal

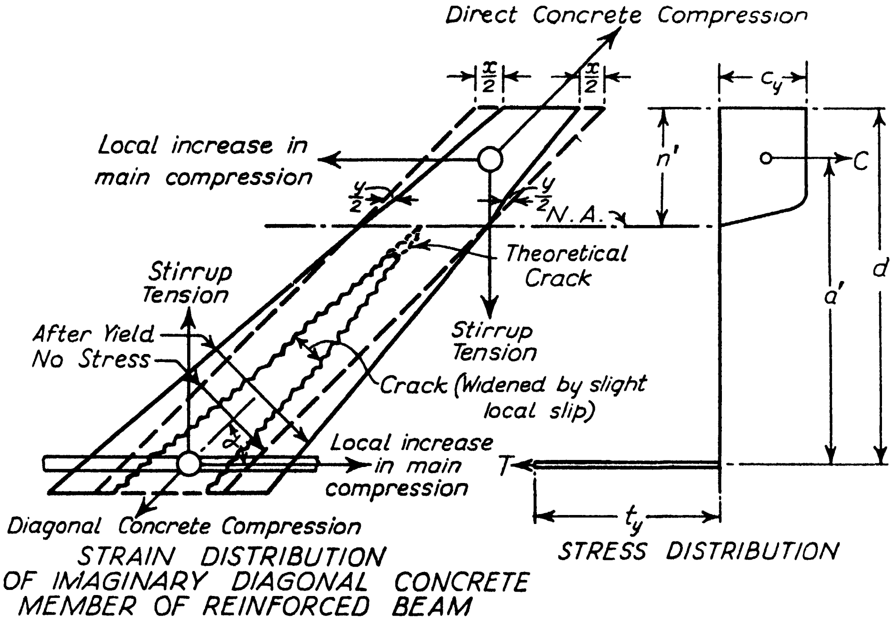
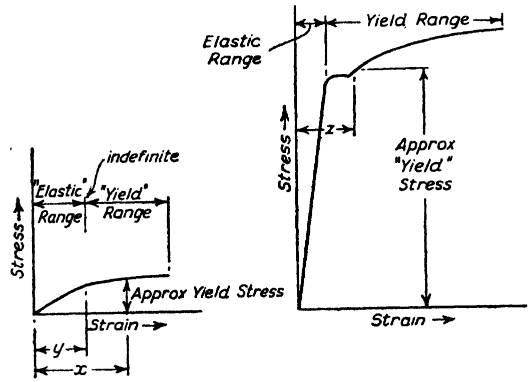


FIG. 78.

strip of the beam (*Fig.* 78) and y is the strain at the point where the compressive stress has just reached the "yield" stress of the concrete, the rectangular distribution of stress shown is possible provided that when the "yield-point" is reached the stress-strain curve is horizontal and x is less than the strain at failure (*Fig.* 79). Also, if the total strain z of the reinforcement is within the yield-range (*Fig.* 79), the reinforcement can develop its full yield stress before failure due to a suitable reduction of the depth to the neutral axis. If the reinforcement does not yield or slip locally the value of n' will be greater and the distribution of the concrete stress nearer to triangular. In actual beams the distribution of the stress before failure will be intermediate between the ideal condition just described and conditions based on the common assumptions made for calculating bending and shearing stresses. This doubtless accounts for the fact that the results of tests on beams appear to indicate that the strength of concrete in compression is greater than that given by crushing tests on prisms.

Although the assumptions made in calculating shearing and bending stresses are generally safe, it is probably unwise to depart from them until further research has been made. Eventually a very simple theory may be devised in which design for bending and shearing stresses is based on the distribution of stresses at the yield stage. Such a theory would completely disregard the relation of the stress in the reinforcement to the imaginary tensile stress in the surrounding concrete, and the moment of resistance of a beam would be the total tension multiplied by a lever arm calculated as the distance between the centre of the tensile reinforcement and the centre of an imaginary compressive flange in which the stress in the concrete is uniformly distributed. The result would be divided by a factor of safety which would take into account the fact that before failure



CONCRETE STEEL
IN A REINFORCED CONCRETE BEAM

FIG. 79.

the distribution of stress in the imaginary compressive flange would be intermediate between a triangular and a rectangular distribution. Only a considerable number of tests could establish a value of the factor of safety which could be applied to a large range of beams having various proportions of tensile reinforcement. The use of higher concrete stresses in bending than in direct compression is a step in this direction (see Appendix III).

The assumptions made in calculating shearing and bending stresses are therefore not only justified by tests but are reconciled by considering conditions at the yield stage, when the relative stiffnesses of the various structural systems of which a beam is imagined to be composed become a convenient variable factor, which is adjusted so that the stresses are redistributed to give the greatest strength before failure.

Grip.

The strength of reinforced concrete members depends on stress being induced in the reinforcement by means of the grip of the surrounding concrete. The safe working grip-stress s_b has been determined by pulling embedded bars out of blocks of concrete. The total resistance to extraction of a bar, or the total grip of the concrete, is only approximately proportional to the area of contact between the bar and the concrete. Therefore, since the safe working tensile or compressive force in a bar is proportional to its cross-sectional area, the safe grip-length L_0 of a round bar is very approximately Fd, where d is the diameter of the bar and F is a factor the value of which depends on the value of s_b for the concrete and on the stress t in the steel. The value of F is 45 for ordinary 1 : 2 : 4 concrete and mild steel if s_b is assumed to be 100 lb. per square inch and t is 18,000 lb. per square inch. Values of F for other qualities of concrete and steel are obtained by direct proportion using the values of s_b and t given, say, in the D.S.I.R. or British Standard codes of practice. If the bar is in tension an anchorage equivalent to an additional length of $14d$ should be provided. If a semi-circular hook as in *Fig.* 80 is formed on the end of the bar the grip-length L_0 should be Fd, but if the bar is straight and is in tension the length must be $L_1 = L_0 + 14d = (F + 14)d$. The grip-length required for a bar in compression is also L_0.

In a reinforced concrete member the bars must extend beyond any section where the greatest stress is developed for a distance at least equal to the safe

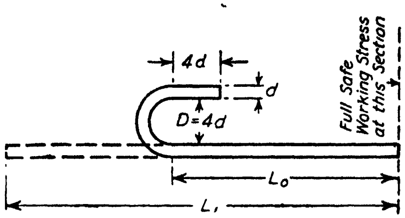


FIG. 80.

grip-length, and generally for a considerably greater distance depending on the rate at which the bending moment is reduced. Where the bending moment is sufficiently reduced, the number of effective bars is often reduced either by bending up some of the bars or by curtailing some of them. To determine the point at which a bar can be curtailed, a good method is to extend every bar by at least the safe grip-length (plus an end anchorage) beyond the point at which the stress in the bar is greatest. If it is possible to reduce the number of bars in the beam as the bending moment decreases, each bar may be initially curtailed at the point at which the remaining bars provide sufficient resistance to the bending moment at that point ; the bar is then extended beyond the point by a distance (plus an end anchorage) at least equal to

$$\frac{\text{actual stress in bar}}{\text{permissible stress in bar}} \times \text{safe working grip-length.}$$

This method needs to be carried out carefully at supports of continuous beams where the negative bending moment decreases very rapidly, and it is necessary to extend the bars for a distance equal to nearly the whole of their greatest grip-length beyond the point at which they are no longer required to provide resistance to bending. Hooks should be avoided if possible in places where the concrete is in tension and an additional straight length provided as an end-anchorage, or the bars should be hooked and turned up (or down) into the compressive zone beyond the neutral axis.

Shear.

Experience and tests on structures show that the shearing resistance of a reinforced concrete beam can be based on the assumption that the total shearing resistance of the beam at any section is the sum of the shearing resistances provided by a number of imaginary pin-jointed framed girders, as shown in *Fig.* 77 (*a*) to (*e*), in which the vertical tensile members are the stirrups and the inclined tensile members are the inclined reinforcement bars, generally being the main bars bent up, the concrete being assumed to resist no tensile forces. The compressive members are imaginary concrete struts acting in the direction of the bisector of the angle made by the inclined bars, and are often assumed to be inclined at 45 deg. The top compressive flange is an imaginary concrete member of depth n_1d, the stress distribution being triangular (or trapezoidal in a tee-beam), and the centre of compression being a_1d above the centroid of the main tensile reinforcement which forms the bottom flange of the girder; the calculations for n_1 and a_1 are given later. The stress in the vertical and inclined reinforcement is assumed to be independent of the stress in the surrounding concrete, and the stress in the inclined concrete compressive members is assumed to be uniform. In any beam the separate frames can be superimposed regardless of their relative stiffnesses; tests show that this assumption is justified and that, provided that the shearing stresses in the concrete are low and the bars have sufficient end-anchorage, an adequate factor of safety is provided. Resistance to shear is therefore really based on the conditions beyond the elastic limit when plastic yield allows the assumed superimposition to be effective before failure.

NOTATION [*Fig.* 77 (*a*) to (*e*)].—

S, total shearing force at any section X–X $= R_A - (P_1 + P_2,$ etc.)
R_A, reaction at left-hand support of beam.
P_1, P_2, etc., loads between any section X–X and the left-hand support of the beam.
a, lever arm.
b_r, breadth of a rectangular beam or breadth of a rib of a tee-beam.
n, depth of compression flange $= n_1d$.
p, pitch of stirrups.
θ, slope of inclined reinforcement.
A_w, total cross-sectional area of vertical stirrups.
A_{wi}, total cross-sectional area of inclined reinforcement in each frame.
s_c, diagonal compressive stress in concrete due to shear.
t_{wi}, tensile stress in inclined reinforcement.
t_w, tensile stress in stirrups.
c_t, tensile stress in concrete due to shear.

ANALYSIS OF SHEARING RESISTANCE.—There are two types of imaginary frames to be considered: (1) Vertical tension-member frames [*Fig.* 77 (*a*) to (*c*)]; the number of such frames is $\frac{a}{p}$; and (2) Two inclined tension-member frames [*Fig.* 77 (*d*) and (*e*)].

The imaginary compressive concrete members may be assumed to act at a slope of 45 deg. The vertical stirrups are sub-divided into $\frac{a}{p}$ imaginary systems

[*Fig.* 77 (*b*) and (*c*)] in which the space between the vertical members of each frame is a; when the several frames are superimposed the pitch of the stirrups is p. If a cut is made at any section X–X the algebraic sum of the external vertical forces to the left of the cut equals the algebraic sum of the vertical components of the forces in the members of the frames which have been cut. Therefore

$$R_A - (P_1 + P_2 + P_3, \text{ etc.}) = \frac{a}{p} A_w t_w + 2A_{wi} t_{wi} \sin\theta,$$

as there are $\frac{a}{p}$ vertical systems and two inclined systems. Since

$$R_A - (P_1 + P_2 + P_3, \text{ etc.})$$

is the total shearing force S at section X–X,

$$S = \frac{aA_w t_w}{p} + 2A_{wi} t_{wi} \sin\theta.$$

The expression for the shearing resistance of the inclined reinforcement, $2A_{wi}t_{wi}\sin\theta$, becomes $A_{wi}t_{wi}\sqrt{2}$ if θ is 45 deg., and $A_{wi}t_{wi}$ if θ is 30 deg. It is not common for θ to be less than 30 deg.

COMPRESSIVE STRESS DUE TO SHEAR.—Make an imaginary cut Y-Y [*Fig.* 77 (*a*) to (*d*)] through the concrete compressive members. Equating the total external vertical forces to the left of Y–Y to the total vertical components of the inclined compressive members, $S = a\sqrt{2}b_r s_c \sin 45$ deg., since the sum of the widths of the members is $a\sqrt{2}$ and their thickness is b_r. Hence $S = ab_r s_c$, from which $s_c = \frac{S}{ab_r}$.

TENSILE STRESS DUE TO SHEAR.—When there is no inclined reinforcement $S = \frac{aA_w t_w}{p}$, and therefore $t_w = \frac{pS}{aA_w}$. Assume that the vertical reinforcement is replaced by concrete. The cross-sectional area of each concrete member is pb_r and therefore $c_t = \frac{pS}{apb_r} = \frac{S}{ab_r}$.

Alternatively, if reinforcement inclined at an angle of 45 deg. is replaced by concrete, the cross-sectional area of each concrete member being $ab_r\sqrt{2}$, then $S = ab_r\sqrt{2}c_t \sin 45$ deg., and $c_t = \frac{S}{ab_r}$.

Therefore, if the imaginary shear members in an uncracked beam are at an angle between 45 deg. and 90 deg. to the horizontal and that the tensile stress due to shear is uniformly distributed, it is safe to assume that $c_t = \frac{S}{ab_r}$. The actual distribution of shearing stress can be considered under three conditions.

(1) Small shearing stresses may be combined with small bending stresses, as near the supports of a lightly loaded beam, or near the point of contraflexure. If cracks do not occur the bending and shearing stresses are distributed as in a beam of homogeneous elastic material, that is, in any horizontal plane the compressive and tensile stresses due to bending are proportional to the distance of the plane from the neutral axis, which passes through the centroid of the equivalent concrete section, and the shearing stresses are distributed parabolically

from no stress at the top and bottom fibres to a maximum at the neutral axis. The greatest shearing stress in a rectangular beam is $\frac{1\cdot5S}{Db_r}$ (D being the total depth of the beam) and is $\frac{S}{ab_r}$ approximately.

(2) Small shearing stresses combine with large bending stresses near the middle of the span of the beam. The imaginary concrete tensile members resisting shear are nearly vertical, and are usually assisted by stirrups.

(3) Large shearing stresses combine with small bending stresses near the supports of a freely-supported beam. The concrete cracks due to the excessive diagonal tension, and the shearing tension is resisted by the reinforcement mainly.

The distribution of shearing stress in an uncracked beam is assumed to be parabolic as in a homogeneous beam. If cracks occur the concrete between the cracks transmits alone the increase in tension to the main reinforcement if the shearing stress is small, and in conjunction with the shear reinforcement if the stress is great.

Tests and experience show that provided c_t is not greater than $0\cdot1c$, where c is the safe working compressive stress in the concrete, the whole of the shearing force may be resisted by the concrete. The ability of concrete to resist tensile stresses is therefore made use of in this case, but the conditions for resistance to shear are quite different from those for resistance to bending. In bending, a crack due to shrinking or other cause would result in complete failure of a beam with no tensile reinforcement even though it were designed for a small tensile stress in the concrete.

When c_t is greater than $0\cdot1c$, reinforcement should be provided to resist the whole of the shearing force, and the value of c_t should not exceed $0\cdot4c$ otherwise excessive cracks occur in the concrete. In many members it is advisable to restrict the shearing stress to $0\cdot3c$ or even to $0\cdot2c$, although the whole shearing force is resisted by reinforcement.

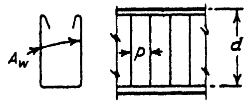
The foregoing analysis is only one way of considering the action of shearing forces on a reinforced concrete beam and the reader should study, for comparative purposes, the works of other writers.

DESIGN.—The calculation of the amount of reinforcement required to resist shear is greatly facilitated by tables such as those in *Fig.* 81. The table at (*a*) gives the resistance of stirrups with two legs and is based on $S = \frac{aA_wt_w}{p}$; $\frac{a}{d}$ is assumed to be 0·86, and t_w to be 18,000 lb. per square inch. The values given in the table multiplied by the effective depth of the beam d give the shearing resistance provided by the stirrups.

The table at (*b*) in *Fig.* 81 gives the shearing resistance of inclined reinforcement for a "double system" [*Fig.* 77 (*d*) and (*e*)] from the formula $S = 2A_{wi}t_{wi} \sin\theta$, assuming t_{wi} is 18,000 lb. per square inch and θ 30 deg. or 45 deg. For a "single system" the resistances are half of those tabulated. For intermediate angles the resistances can be interpolated from the data in the table. If the permissible stress is other than 18,000 lb. per square inch, the resistance of stirrups and inclined bars can be determined by direct proportion to the stress.

The procedure for determining the shearing reinforcement for any beam can therefore be summarised as follows: (1) Determine the length of the middle part of the span where c_t is less than 0·1c. Theoretically, reinforcement for shear is unnecessary over this part of the beam but it is usual to provide some stirrups to hold the main bars in position. (2) Determine c_t at the supports where the shear is greatest, and if it exceeds 0·4c the depth or width of the beam or rib must be increased. (3) In those parts of the beam where c_t exceeds 0·1c the whole of the shear must be resisted by reinforcement.

The sizes of the main bars at the middle of the span, and probably at the supports, should be selected so that at points where the bars are no longer required to resist bending stresses they can be bent up to act as reinforcement to resist shear. A " double system " of bars at 45 deg. is the best arrangement. In some cases it may not be possible to provide the inclined reinforcement from



Stirrups

$$S = \frac{a_1 d A_w t_w}{p} = Qd$$

Values of Q, assuming $a_1 = ·86$, $t_w = 18{,}000$ lb/in²
Stirrups have 2 legs

Diam of Stirrup leg \ p	3"	6"	9"	12"
1/4"	500	250	167	125
5/16"	800	400	266	200
3/8"	1,140	570	380	285
1/2"	2,030	1,015	677	507

(a)

"Double System"

$$S = \sqrt{2}\, A_{w_1} t_{w_1} : \theta = 45^\circ \quad ; \quad = A_{w_1} t_{w_1} . \ \theta = 30^\circ \qquad t_{w_1} = 18{,}000 \text{ lb./in}^2$$

Diam	1/2"	5/8"	3/4"	7/8"	1"	1 1/8"
θ = 45°	4,970	7,800	11,250	15,300	19,980	24,980
θ = 30°	3,530	5,510	7,960	10,860	14,150	17,900

(b)

FIG. 81.—REINFORCEMENT FOR RESISTANCE TO SHEAR.

the main bars, and additional bars are then used, or sufficient resistance may be obtained by bending-up at 30 deg. or by using a " single " or " intermediate " system. Having arranged the available bars to the best advantage, the table in *Fig.* 81 (*a*) should be used to determine the size and pitch of the stirrups required at each part of the beam in order to provide sufficient total resistance to shear. It is difficult to state any rule for determining the pitch and size of stirrups in relation to the size of the beam. The reinforcement should be an arrangement of inclined bars and vertical stirrups decreasing in spacing towards the supports at a rate similar to the increase in shearing force. In large deep beams four-legged stirrups of ½-in. bars closely pitched may sometimes be necessary. Small beams should have ¼-in. or $\frac{5}{16}$-in. stirrups, which are easily made and provide sufficient grip-length in the restricted depth of the beam. The pitch of stirrups should be not less than 3 in. The addition of inclined bars for the sole purpose of resisting shear should if possible be avoided, as it is necessary to bend them up into the compression zone of the beam to provide a satisfactory anchorage (*Fig.* 82).

It is difficult to visualise how the stresses in a reinforced concrete beam are distributed below the neutral axis after cracks due to shear have occurred. *Fig.* 78 shows an imaginary concrete inclined compressive strut as it might crack at a section of large shearing stress and still continue to act satisfactorily in transmitting the force by grip to the main bars and stirrups. In the foregoing derivation of formulæ for reinforcement for shear, the inclination of the imaginary concrete compressive strut is assumed to be 45 deg., but for beams in which the inclined bars are spaced at not more than the distance a (the lever arm) apart, any angle between 45 deg. and 90 deg. could have been assumed without increasing s_c. Generally the inclination of cracks or planes of potential cracking due to excessive shear is between about 45 deg. at sections near the supports where the shearing stresses are large and bending stresses small, and approximately vertical near the middle of the span where shearing stresses are small and bending stresses large. In these cases the spacing of the reinforcement for shear has some influence on the width and number of cracks due to

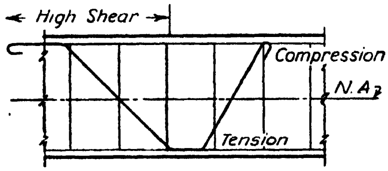


FIG. 82.

shear, as the more closely the tensile members are together the smaller will be the local stresses developed by the imaginary concrete struts in transmitting their stress to the tensile members.

When c_t is less than 0·1c no cracks due to shear should occur and the concrete acts as a web of homogeneous material resisting inclined compression and tension, unless cracks due to bending occur, when the concrete between the cracks resists the shear.

As a precaution against excessive local stresses developing when inclined bars are used, the working stress in the inclined part should be reduced so that the stress in the horizontal part of the bar is not excessive when calculated on the assumption that the beam acts as a pin-jointed frame. For bars inclined at 45 deg. the inclination of the imaginary concrete compressive member should not be less than 67½ deg. if a reduced working stress in the inclined bar is to be avoided. The triangle of forces for the pin-joint is then an isosceles triangle.

The total safe grip on the main tensile steel in unit length of a beam, based on double the permissible grip-stress, should be not less than the variation in the total tension in the bars in unit length of the beam. The variation in tension in a short length dl is $\frac{dM}{a.dl}$, where dM is the variation in the bending moment in the length dl. Since $\frac{dM}{dl}$ is the total shearing force at the section considered, the variation in total tension in unit length of the beam is $\frac{S}{a}$, which must not exceed the total grip on unit length based on double the grip-stress ordinarily permissible. If S is in pounds and a is in inches, the total grip would be calculated in pounds for a length of 1 in. This requirement may restrict the size of the bars used near the supports of continuous beams where bending moments and shearing stresses are large.

Bending.

The resistance of a reinforced concrete member to bending, direct force, or a combination of both, can be determined from equations based on the fundamental principles that at any cross section the algebraic sum of the internal forces acting normal to and on one side of the section equals the algebraic sum of the external forces, and that the algebraic sum of the moments, about any point, of the internal forces acting normal to and on one side of the section equals the algebraic sum of the bending moments at the cross section. For a member subjected to bending only the algebraic sum of the internal forces is zero.

General Assumptions.—The assumptions which are made regarding the physical properties of concrete and steel, and which have been proved by tests to be reasonable and safe, are:

(1) The tensile strength of concrete is only about one-tenth of its compressive strength and is neglected entirely (except sometimes in considering resistance to shear and in some liquid-containing structures).

(2) Steel and concrete are elastic materials for which the stress-strain curves are straight lines within the elastic range. Actually concrete develops a certain amount of plastic yield and creep, but both are neglected except in some methods of calculating the stress in compressive reinforcement and in prestressed concrete.

(3) Plane sections before bending remain plane after bending and the strain at any point is directly proportional to the distance of the point from the neutral axis. (This assumption differs from that made in calculating shearing stresses, but is safe.)

(4) If the reinforcement is extended sufficiently far beyond the section to develop the grip strength, no slip takes place between the concrete and the steel. The same strain is therefore developed in the reinforcement and the surrounding concrete.

(5) The coefficients of thermal expansion for steel and concrete are for practical purposes equal. The stresses developed due to the difference in the expansion and contraction of the concrete and reinforcement are negligible, and there is practically no stress due to variations in temperature which are uniform throughout the member provided that the member is free to expand and contract.

(6) Stresses due to shrinking of the concrete during setting and hardening are generally neglected, but are not neglected in the design of arches and prestressed concrete members, and in some other structures, such as long walls and roads, measures are taken to restrict the stresses due to this cause.

A general formula for bending combined with a direct force for a reinforced concrete member of any cross-sectional shape is too complex to be of use in practice. Expressions are therefore derived in the following for cases which occur frequently in practice. As a tee-beam without compressive reinforcement is the most common type of reinforced concrete beam, this is considered first. The general expressions so obtained are then adapted to the special cases of a rectangular beam, subjected to bending only, with and without compressive reinforcement, and a slab, subject to bending only, with no compressive reinforcement.

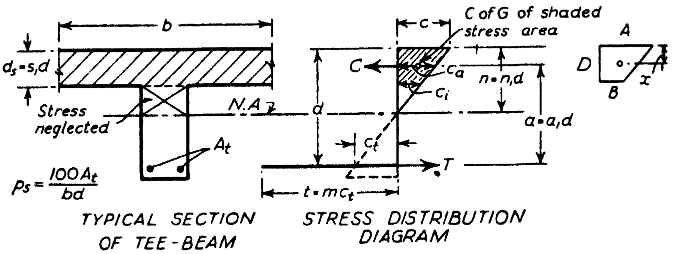


FIG. 83.—TEE-BEAM.

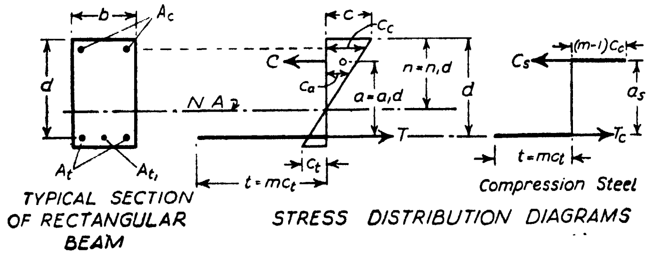


FIG. 84.—RECTANGULAR BEAM.

NOTATION.—Refer to *Figs.* 83 and 84:

b, breadth of flange of beam.
d, effective depth of beam, that is, the depth to the centre of the tensile reinforcement.
$d_s = s_1 d$, thickness of flange of beam.
A_t, area of tensile reinforcement.
p_s, percentage of tensile reinforcement $= \frac{100A_t}{bd}$.
A_{t_1}, area of tensile reinforcement additional to the "economic percentage."
A_c, area of compressive reinforcement.
$n = n_1 d$, depth of neutral axis.
$a = a_1 d$, lever arm of resistance couple (concrete).
a_s, lever arm of resistance couple (compressive reinforcement).
c, maximum compressive stress in concrete.
c_a, average compressive stress in concrete.
c_i, compressive stress in concrete at bottom of flange.
c_t, imaginary tensile stress in concrete around tensile reinforcement.
c_c, compressive stress in concrete around compressive reinforcement.
t, tensile stress in reinforcement.
m, $\frac{\text{modulus of elasticity of steel}}{\text{modulus of elasticity of concrete}}$.

C, total or resultant compressive force in concrete.
C_s, total compressive force in reinforcement.
T_c total tensile force (with no compressive reinforcement).
$T_{,,}$ total extra tensile force forming couple with C_s.
M_r, moment of resistance of concrete.
M, applied bending moment.

Tee-Beam (General Case for Bending).

When analysing the distribution of the bending stresses in a tee-beam the expressions are generally simplified by neglecting the stress in the rib between the neutral axis and the underside of the flange (*Fig.* 83). Experience and tests show that the breadth of the flange b can be safely assumed to be equal to the least of the following: (*i*) the breadth of the rib plus 12 times the thickness of the flange; (*ii*) the distance between the centre-lines of adjacent beams; (*iii*) the actual width of the flange of an isolated tee-beam or ell-beam; or (*iv*) one-third of the effective span of the tee-beam.

The moment of resistance is the total force in the concrete multiplied by the lever arm, that is,

$$M_r = bd_s c_a . a_1 d \qquad (1)$$

For the purpose of plotting curves from which values of M_r can be quickly obtained, the width of flange (omitting the rib) can be assumed to be $12d_s$. Since the width of the rib is often at least $2d_s$, an assumed value for b is $12d_s + 2d_s = 14d_s$, and (1) becomes

$$M_r = 14d_s^2 c_a a_1 d \qquad (1a)$$

Before expressions for a and c_a can be derived, n_1 must be determined in order that the distribution of the stresses may be known.

DERIVATION OF EXPRESSION FOR n_1.—Since the strain of the reinforcement equals the strain of the surrounding concrete, $t = mc_t$. The stress is proportional to the distance from the neutral axis, that is $c_t = c\frac{(d - n_1 d)}{nd}$. Therefore

$$t = \frac{mc(d - n_1 d)}{nd} = \frac{mc}{n_1}(1 - n_1),$$

and

$$t = mc\left(\frac{1}{n_1} - 1\right). \qquad (1b)$$

The total tensile force equals the total compressive force, that is $A_t t = s_1 dbc_a$. Substituting $A_t = \frac{p_s}{100}bd$, and t from (1b), and c_a from (4a) (as derived later),

$$\frac{p_s bd}{100}mc\left(\frac{1}{n_1} - 1\right) = s_1 dbc\left(1 - \frac{s_1}{2n_1}\right).$$

Since b, d, and c cancel, $p_s m - p_s m n_1 = 100 s_1 n_1 - 50 s_1^2$, and

$$n_1 = \frac{50 s_1^2 + p_s m}{100 s_1 + p_s m} \qquad (2)$$

Thus n_1 depends only on s_1, p_s, and m, and the distribution of the stress in any given case likewise depends only on these factors and is independent of b, d, and c.

Derivation of Expression for a_1.— Since stress is proportional to the distance from the neutral axis, $c_i = \dfrac{c(n_1 d - s_1 d)}{n_1 d} = \dfrac{c}{n_1}(n_1 - s_1)$. The resultant compressive force C acts through the centroid of the trapezium representing the distribution of the stresses (*Fig.* 83). Referring to this diagram, the centroid of a trapezium is distant x from the long side A and $x = \dfrac{D}{3} \cdot \dfrac{A + 2B}{A + B}$. In the stress diagram, $A = c$, $D = s_1 d$, $B = c_i$, and $x = d - a_1 d$. Therefore

$$d - a_1 d = \frac{s_1 d}{3} \cdot \frac{c + 2c_i}{c + c_i}.$$

Cancelling d and substituting for c_i,

$$1 - a_1 = \frac{s_1}{3}\left[\frac{c + \dfrac{2c}{n_1}(n_1 - s_1)}{c + \dfrac{c}{n_1}(n_1 - s_1)}\right],$$

and

$$a_1 = 1 - \frac{s_1}{3}\left[\frac{n_1 + 2n_1 - 2s_1}{n_1 + n_1 - s_1}\right].$$

Therefore

$$a_1 = 1 - \frac{s_1}{3}\left(\frac{3n_1 - 2s_1}{2n_1 - s_1}\right) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Thus a_1 depends only on n_1, which in turn depends only on s_1, p_s, and m.

Derivation of Expression for c_a.—Since stress varies as the distance from the neutral axis (*Fig.* 83),

$$c_a = c\frac{n_1 d - \dfrac{s_1 d}{2}}{n_1 d} = c\left(1 - \frac{s_1}{2n_1}\right). \quad . \quad . \quad . \quad . \quad (4a)$$

Substituting for n_1 from (2),

$$c_a = c\left[1 - \frac{s_1}{2\dfrac{(50s_1^2 + p_s m)}{100s_1 + p_s m}}\right]$$

which reduces to

$$c_a = \frac{\left(1 - \dfrac{s_1}{2}\right)c}{1 + \dfrac{50s_1^2}{p_s m}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4b)$$

Rectangular Beams and Slabs.

For a tee-beam, n_1 is given by formula (2). Since the stress in the concrete below the neutral axis is neglected the distribution of stress in a rectangular beam is the same as in a tee-beam in which $n_1 = s_1$. Therefore $n_1 = \dfrac{50n_1^2 + p_s m}{100n_1 + p_s m}$, from which

$$p_s = \frac{50n_1^2}{m(1 - n_1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In a rectangular beam, therefore, n_1 and the distribution of stress depend only on m and p_t and are independent of b, d, and c.

The trapezoidal distribution diagram of a tee-beam becomes triangular in a rectangular beam (*Fig.* 84). The centroid of the triangle is $\frac{n_1 d}{3}$ from the top of the beam. The lever arm $a_1 d$ is therefore $d - \frac{n_1 d}{3}$, from which

$$a_1 = 1 - \frac{n_1}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

DERIVATION OF EXPRESSION FOR M_r.—In a rectangular beam the distribution of stress (*Fig.* 84) is triangular, and $c_a = 0{\cdot}5c$. As before, the moment of resistance is the total compression in the concrete multiplied by the lever arm, that is $M_r = bn_1\frac{c}{2}d\left(1 - \frac{n_1}{3}\right)d$. Therefore

$$M_r = 0{\cdot}5\left(1 - \frac{n_1}{3}\right)n_1 cbd^2 \quad . \quad . \quad . \quad . \quad . \quad (7)$$

or $M_r = Rbd^2$, where

$$R = 0{\cdot}5n_1.\left(1 - \frac{n_1}{3}\right)c \quad . \quad . \quad . \quad . \quad . \quad (8)$$

CONDITIONS FOR MAXIMUM STRESSES IN CONCRETE AND REINFORCEMENT BEING AT THE SAFE WORKING VALUES SIMULTANEOUSLY.—It has been shown that the distribution of stress depends on the value of n_1, which in a tee-beam depends on the values of s_1, p_t, and m, and in a rectangular beam on p_t and m only. As already shown, the maximum stresses in the concrete and the reinforcement are related by the formula (1*b*), which can be rearranged to give

$$n_1 = \frac{1}{1 + \frac{t}{mc}}.$$

The curves in *Fig.* 85 show n_1 plotted against c for the various values of m and t, and from these the value of n_1 which produces simultaneously the safe working values of t and c (for a given value of m) can be obtained. In a beam without compressive reinforcement it is therefore necessary to provide a value for p_t suitable for producing the required value of n_1, and hence the safe working values of t and c under working load. This value of p_t is often called the "economic percentage." In practice so many other factors govern the dimensions of a beam, such as shearing resistance, thickness of flange required by the design of the slab only, fire resistance, and the provision of uniform sizes of concrete, that the "economic percentage." is seldom a factor that governs the dimensions of the beam. Often the maximum concrete stress under the safe working load is less than the safe working stress. The area of the tensile reinforcement A_t required to develop the full safe working stress in the steel can nearly always be provided because the size of the bars can be adjusted without greatly affecting other considerations.

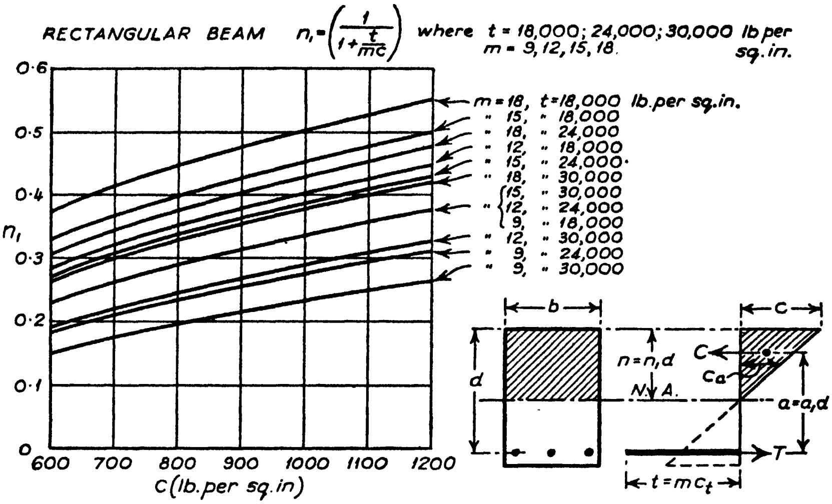


Fig. 85.

Compressive Reinforcement in Beams.—Where continuous rectangular beams and tee-beams are subjected to negative bending moments over the supports, it is frequently necessary to supplement the moment of resistance of the concrete M_r by the provision of compressive reinforcement. To calculate the area of the compressive steel A_c the total moment of resistance should be separated into M_r, the moment of resistance of the concrete when p_s equals the "economic percentage" for the particular values adopted for m, c, and t, and a couple, $M - M_r$, where M is the applied bending moment. The value of c is the safe working value, and the value of n_1 is that corresponding to the "economic percentage." The compressive stress c_c in the concrete surrounding the compressive reinforcement can then be calculated, since the stress is proportional to the distance from the neutral axis. The stress in the compressive reinforcement is mc_c. The lever arm for moment of resistance $M - M_r$ is then a_s (*Fig.* 84) and

$$A_c = \frac{M - M_r}{a_s(m - 1)c_c}.$$

The area of tensile reinforcement A_{t1} required in addition to the area corresponding to the "economic percentage" is $\frac{M - M_r}{ta_s}$. By dividing the total moment of resistance in this way it is seen that the distribution of stress for the "economic percentage" of steel is in no way altered provided that compressive and tensile reinforcement are added so that $C_s = T_c$. In practice an approximate, but sufficiently accurate, and quicker procedure is often adopted. The average lever arm a_a is assumed to be $\frac{a_s + a_1 d}{2}$, and $A_t + A_{t1} = \frac{M}{ta_a}$. The stress in the compressive reinforcement is assumed to be 7000 lb. to 10,000 lb.

per square inch depending on the value of c but regardless of the true value of n_1; A_c is calculated from $\dfrac{M - M_r}{(7000 \text{ to } 10{,}000)a_s}$.

If the calculated amount of compressive reinforcement equals or exceeds the total amount of tensile reinforcement, it is sometimes assumed that the compressive reinforcement can be stressed to the same value as the tensile reinforcement, regardless of the stress in the surrounding concrete, provided that the cross-sectional area of the concrete is such that c would not exceed, say, 1200 lb. per square inch if there were no compressive reinforcement. Apparently this method is justified by experience and tests, and theoretically it appears to be reasonable since before failure the overstressed concrete must yield and allow the compressive steel to develop its yield stress before the beam fails.

BEAM WITH NARROW COMPRESSION FLANGE.—The design of a beam with a narrow compression flange, which may be liable to buckle laterally, is best governed by the requirements of the B.S. Code and some regulations which stipulate that the unsupported lateral length of beam must not exceed

$$20b\left(3 - 2 \times \frac{\text{calculated compressive stress}}{\text{permissible compressive stress for beams}}\right).$$

Elastic Deformation of Reinforced Concrete Members.

Fig. 86 shows the elevation of a segment of a reinforced concrete beam subjected to bending. The slope of the beam is $\dfrac{dy}{dx}$, and if the slope is small

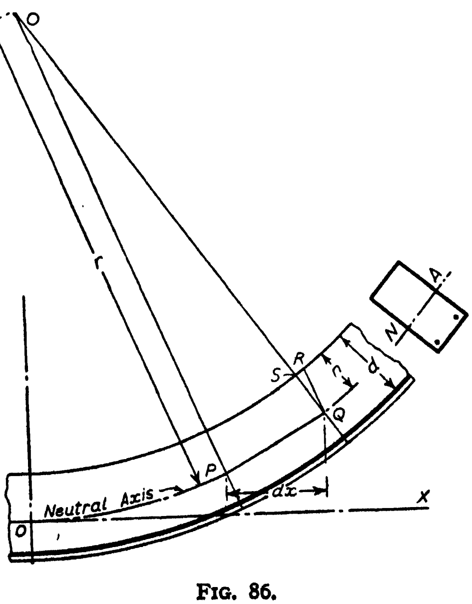


FIG. 86.

dx is very approximately equal to PQ. The rate of change of slope is $\frac{d^2y}{dx^2}$, which is equal to the curvature $\frac{1}{r}$. Thus the slope is

$$\frac{dy}{dx} = \int \frac{1}{r} dx \text{ and the deflection is } y = \iint \frac{1}{r} dx . dx.$$

Referring to *Fig.* 86, SR is the maximum strain on the length dx and equals $\frac{fdx}{E_c}$, where f is the maximum stress and E_c is the modulus of elasticity of concrete. Since RQ is parallel to OP, $\hat{POQ} = \hat{SQR}$.

Therefore $\frac{dx}{r} = \frac{SR}{RQ} = \frac{fdx}{E_c n}$, and $\frac{1}{r} = \frac{f}{E_c n}$. Thus the slope, or angular deformation of a member, is

$$\int \frac{f}{E_c n} . dx.$$

If M is the applied bending moment at any section of the beam and M_r is the moment of resistance of the section in terms of a definite value of f, that is c, $f = \frac{Mc}{M_r}$. The slope is

$$\int \frac{Mc}{E_c M_r n} dx = \int \frac{M}{E_c I} dx.$$

The term $\frac{M_r n}{c}$ for a homogeneous elastic beam is conveniently termed the second moment of the section, or moment of inertia I, and M_r for such a beam can be expressed in terms of I. In a reinforced concrete beam it is more convenient to reverse this procedure and calculate I from $\frac{M_r n}{c}$, since M_r and n for various sections of beam can be readily obtained from the curves and tables usually prepared for the purpose of design. When values of I are required for evaluating stiffness factors or values of $\frac{M}{EI}$, as in moment-distribution methods, the amount of reinforcement is not generally known. It is sufficiently accurate, however, in view of the indeterminate value of E_c, to assume A_t and n corresponding to the " economic percentage " of p_s. The value of A_t generally varies throughout the length of a beam; therefore I also varies, but this variation is generally ignored.

The value of I for a rectangular beam with no compressive reinforcement is

$$\frac{M_r n}{c} = \frac{bnc}{2} . \frac{n}{c}\left(d - \frac{n}{3}\right) = \frac{bn^2}{2}\left(d - \frac{n}{3}\right).$$

The values of M_r for a tee-beam should be obtained from curves from which a value of n is selected corresponding to the " economic percentage " for substitution in $I = \frac{M_r n}{c}$; if a more exact value is required the two safe limiting values of n, corresponding to the maximum and minimum values of A_t likely to be used, may be combined with the maximum and minimum values of E_c.

Sometimes I is calculated as the second moment of the whole sectional area of the concrete about an axis through the centroid. This method is approximately correct, and if it is followed consistently in the analysis of a frame the results will be sufficiently accurate, because before failure creep and yield redistribute the bending moments to give increased resistance; small errors in the assumed distribution of moments do not seriously reduce the factor of safety. In calculations in which the numerical values, and not only relative values, of the deflection or stiffness after creep has occurred are required, the assumed value of E_c should be between 2×10^6 and 3×10^6, and the slope and deflection are then between corresponding limits. Reference should also be made to *Figs.* 109, 110, and 112. Calculations can be only approximate.

For a beam in which the compressive reinforcement has been calculated by the ultimate-load theory, the value of M_r used for calculating I should be based on the stress in the compressive reinforcement being $(m - 1)$ times the stress in the surrounding concrete, that is on the distribution of stresses occurring within the elastic range. For columns and struts subjected to bending, the direct stresses cause no lateral deformation of the member; but the bending stresses cause lateral deformation and must be separated from the direct stresses. For such members I must then be based on the deformation of the concrete on the compression side of the neutral axis with respect to the same neutral axis. The deflection of prestressed concrete beams is discussed in Chapter VI.

II.—PRACTICAL METHODS OF DESIGNING BEAMS.

It is not practical to use the foregoing expressions for M_r, n_1, a_1, etc., to calculate directly the dimensions of reinforced concrete beams to resist specified bending moments, because the distribution of stresses depends on the value of n_1, which cannot be calculated until p_s is known. The value of p_s depends on the value of A_t, which in turn depends on n_1. In a tee-beam, n_1 also depends on s_1. The direct calculation of n_1 involves the solution of a quadratic equation, which is too long for repeated trial calculations. The value of d for a tee-beam, and often of d and b for a rectangular beam, is generally governed by shearing resistance. The relative cost of concrete, reinforcement, and shuttering must also be taken into account. The most economical beam is not usually that in which the " economic percentage " of reinforcement is provided. Generally only in the case of a slab, and occasionally in a rectangular beam, is it possible to provide a percentage of reinforcement which at the same time develops the safe working stresses in the concrete and reinforcement at the working load. The depth of a beam is often restricted to obtain a certain clearance below, and for economy of shuttering and for appearance it is often desirable to adopt a constant depth for a continuous beam although the bending moment varies considerably.

In practice it is nearly always necessary to assume first the dimensions of the beam and then to ascertain that the safe working stress in the concrete is not exceeded; the amount of reinforcement is then calculated. In a beam of constant depth it is easy to vary the area of reinforcement and thereby provide the correct amount of reinforcement at the safe working stress. For the reasons stated, however, it is not always possible to provide beams of such dimensions that the permitted working stress in the concrete is developed.

Diagrams for Design.

The diagrams in *Figs.* 85, 87 to 91, and 93 to 98, and the table in *Fig.* 92 save labour and time in determining the several factors required in the design of beams and slabs.

FIGS. 85 AND 87 TO 91.—The curves in *Figs.* 85 and 87 relate n_1, corresponding to the "economic percentage," to c for various values of t, the general equation being $n_1 = \dfrac{1}{1 + \dfrac{t}{mc}}$. *Fig.* 85 applies when $m = 9$, 12, 15, and 18, and *Fig.* 87 when $m = 40$. The value of n_1 increases with the increase in the values

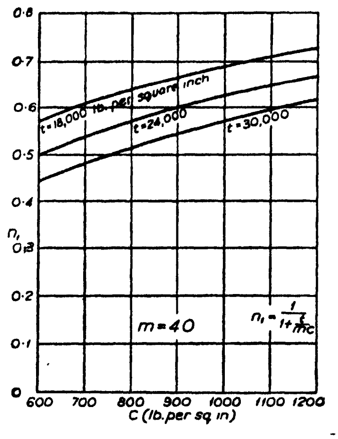
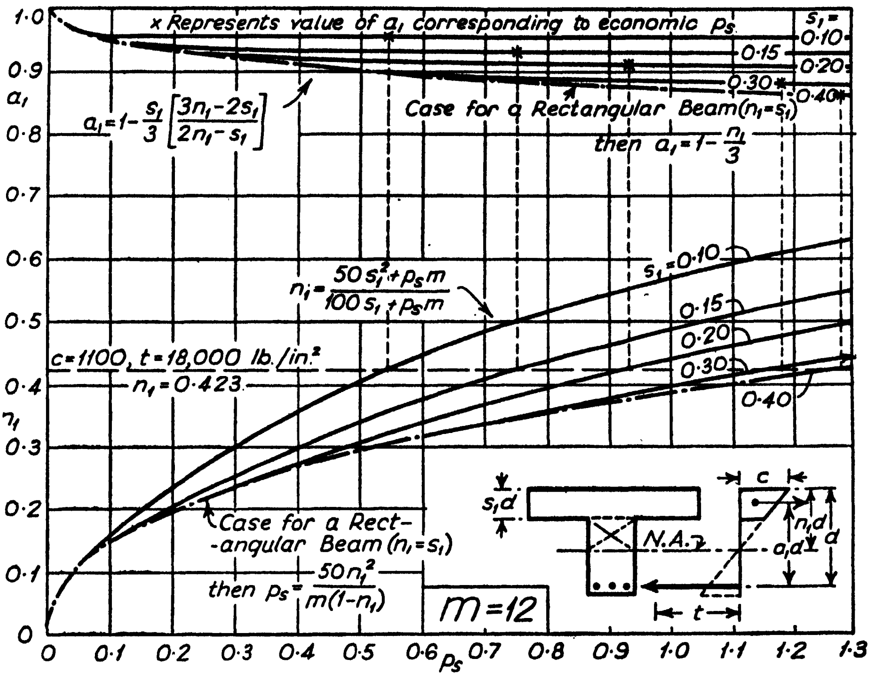
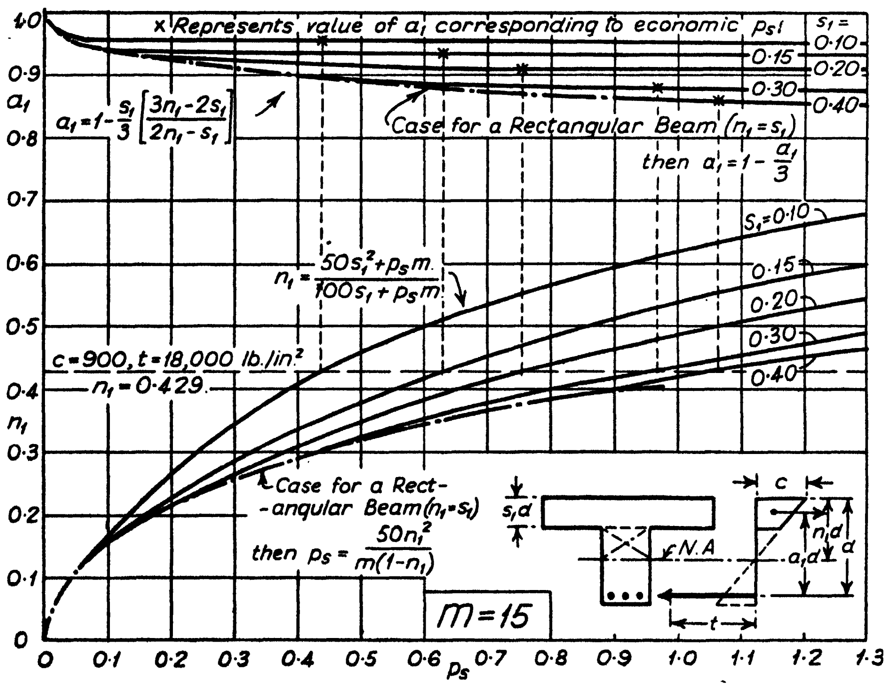


FIG. 87.

of m and c, and decreases as t increases. It is important to be able to obtain readily the value of n_1 for various combinations of stresses so that the "economic percentage" of reinforcement can be determined for finding the corresponding values of a_1 for tee-beams and rectangular beams from *Figs.* 88 to 91, and also the corresponding values of R. *Figs.* 88 to 91 give curves relating n_1 and a_1 to p_s for tee-beams for various values of s_1, and are based on the formulæ given on each of the diagrams which are drawn for different values of m from 12 to 40. The points on the curves at which s_1 is equal to n_1 are joined by other curves to the left of which the neutral axis is within the flange and the calculations for the tee-beam are as for a rectangular beam. The curves joining the points at which s_1 is equal to n_1 relate n_1 and a_1 to p_s for rectangular beams and slabs, and the formulæ are as already established.


x Represents value of a_1 corresponding to economic p_s
s_1 =
0·10
0·15
0·20
0·30
0·40
$a_1 = 1 - \frac{s_1}{3}\left[\frac{3n_1 - 2s_1}{2n_1 - s_1}\right]$
Case for a Rectangular Beam ($n_1 = s_1$) then $a_1 = 1 - \frac{n_1}{3}$
$n_1 = \frac{50 s_1^2 + p_s m}{100 s_1 + p_s m}$
s_1 = 0·10
0·15
0·20
0·30
0·40
c = 1100, t = 18,000 lb./in.²
n_1 = 0·423
Case for a Rect-angular Beam ($n_1 = s_1$) then $p_s = \frac{50 n_1^2}{m(1 - n_1)}$
m = 12
$s_1 d$
c
$n_1 d$
d
N.A.
$a_1 d$
t
a_1
n_1
p_s
1·0 0·9 0·8 0·7 0·6 0·5 0·4 0·3 0·2 0·1 0
0 0·1 0·2 0·3 0·4 0·5 0·6 0·7 0·8 0·9 1·0 1·1 1·2 1·3
FIG. 88.


x Represents value of a_1 corresponding to economic p_s!
s_1 =
0·10
0·15
0·20
0·30
0·40
$a_1 = 1 - \frac{s_1}{3}\left[\frac{3n_1 - 2s_1}{2n_1 - s_1}\right]$
Case for a Rectangular Beam ($n_1 = s_1$) then $a_1 = 1 - \frac{a_1}{3}$
$n_1 = \frac{50 s_1^2 + p_s m}{100 s_1 + p_s m}$
S_1 = 0·10
0·15
0·20
0·30
0·40
c = 900, t = 18,000 lb./in.²
n_1 = 0·429
Case for a Rect-angular Beam ($n_1 = s_1$) then $p_s = \frac{50 n_1^2}{m(1 - n_1)}$
m = 15
$s_1 d$
c
$n_1 d$
d
N.A
$a_1 d$
t
a_1
n_1
p_s
1·0 0·9 0·8 0·7 0·6 0·5 0·4 0·3 0·2 0·1 0
0 0·1 0·2 0·3 0·4 0·5 0·6 0·7 0·8 0·9 1·0 1·1 1·2 1·3
FIG. 89.

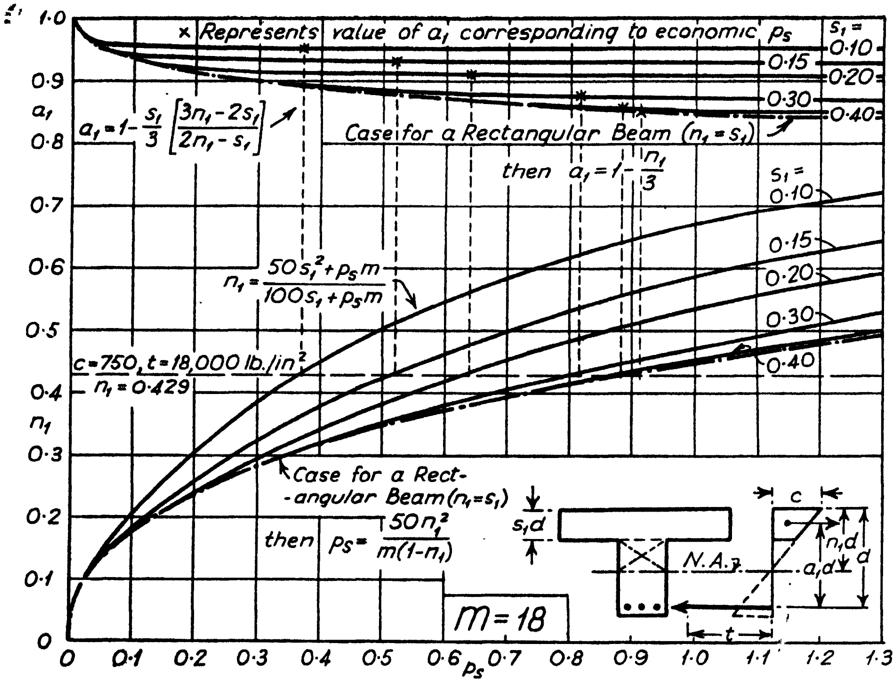

× Represents value of a_1 corresponding to economic p_s
$a_1 = 1 - \frac{s_1}{3}\left[\frac{3n_1 - 2s_1}{2n_1 - s_1}\right]$
Case for a Rectangular Beam ($n_1 = s_1$)
then $a_1 = 1 - \frac{n_1}{3}$
$n_1 = \frac{50s_1^2 + p_s m}{100s_1 + p_s m}$
c = 750, t = 18,000 lb./in.²
n_1 = 0·429
Case for a Rectangular Beam ($n_1 = s_1$)
then $p_s = \frac{50n_1^2}{m(1-n_1)}$
m = 18
N.A.
$s_1 =$ 0·10 0·15 0·20 0·30 0·40

FIG. 90.

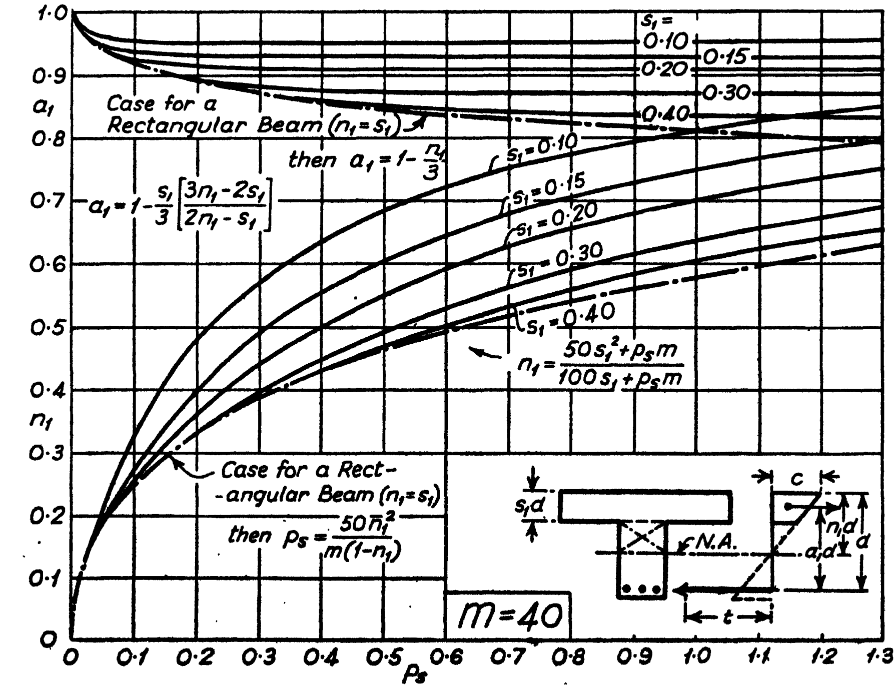

Case for a Rectangular Beam ($n_1 = s_1$)
then $a_1 = 1 - \frac{n_1}{3}$
$a_1 = 1 - \frac{s_1}{3}\left[\frac{3n_1 - 2s_1}{2n_1 - s_1}\right]$
$n_1 = \frac{50s_1^2 + p_s m}{100s_1 + p_s m}$
Case for a Rectangular Beam ($n_1 = s_1$)
then $p_s = \frac{50n_1^2}{m(1-n_1)}$
m = 40
N.A.
$s_1 =$ 0·10 0·15 0·20 0·30 0·40

FIG. 91.

It is seen that n_1 increases as p_s and s_1 increase, and varies considerably in magnitude, so that for a tee-beam it is not economical to use a safe limiting value $n_1 = s_1$ to calculate c_a. The value of a_1 decreases as p_s and s_1 increase. As s_1 increases, the level of the underside of the flange of the beam approaches the neutral axis. In a tee-beam and a rectangular beam, therefore, the value of a_1 never exceeds the value corresponding to the "economic percentage" of reinforcement, since if more reinforcement is used on the tensile side compressive reinforcement is necessary and a_1 is thereby increased.

Therefore, for a tee-beam in which s_1 is not less than n_1 and for a rectangular beam the safe limiting value of a_1 is the value corresponding to the "economic percentage" of reinforcement, and for a tee-beam in which s_1 is less than n_1 this same value (or approximately $d - 0{\cdot}5d_s$) is the safe limiting value of a_1.

The safe limiting values of a_1 for stresses and values of m most commonly used are shown in *Fig.* 92. It is seen that a_1 increases only slightly for small values of s_1 and p_s. The use of a safe limiting value to facilitate calculations is therefore not wasteful, and may offset in part the effect of inaccuracy in placing the reinforcement or disturbance during concreting. It is wise therefore to err

Concrete	Reinforcement Stress-t	Concrete Bending Stress-c	Modular Ratio m	'Economic' value of p_s	Corresponding value of n_1	Safe limiting values of a_1	R
1:2:4	18,000	750	17·78	0·887	0·425	0·858	130
1:1·2:2·4	18,000	925	14·41	1·090	0·425	0·858	160
1:2:4	25,000	750	17·78	0·522	0·347	0·884	115
1:1·2:2·4	25,000	925	14·41	0·644	0·347	0·884	142

Fig. 92.

on the low side when computing a_1 for determining A_t. Other comments on *Fig.* 92 are made later.

Fig. 93.—These curves relate c_a to s_1 for various values of p_s and m, assuming that c is 600 lb., 750 lb., or 900 lb. per square inch. If c is 600 lb. per square inch, for example, c_a cannot be less than 300 lb. per square inch since this is the stress at which a tee-beam becomes equivalent to a rectangular beam. It is seen that the value of c_a increases very rapidly as s_1 decreases, and increases as p_s increases. The only safe limiting value is 300 lb. per square inch, or $0{\cdot}5c$, and often an assumed value of 300 lb. per square inch, or $0{\cdot}5c$, combined with the safe limiting value of a_1, is sufficient to show that a tee-beam has an adequate moment of resistance. However, if s_1 is small and the moment of resistance calculated in this way is insufficient, a more accurate value for the moment of resistance is calculated by determining c_a for values of p_s and s_1 applicable to the beam being analysed. The more accurate value of c_a can exceed $0{\cdot}5c$ considerably. It is interesting to note that in some theories of reinforced concrete in which the distribution of the stresses at the yield stage is the basis of calculation, the calculations are simplified as s_1 and p_s are omitted as they do not directly influence the values of a_1 and c_a.

Figs. 94 to 98.—These curves relate M_r for the concrete to d for various values of d_s, p_s, and m when c is 600 lb. per square inch. Since values of A_t

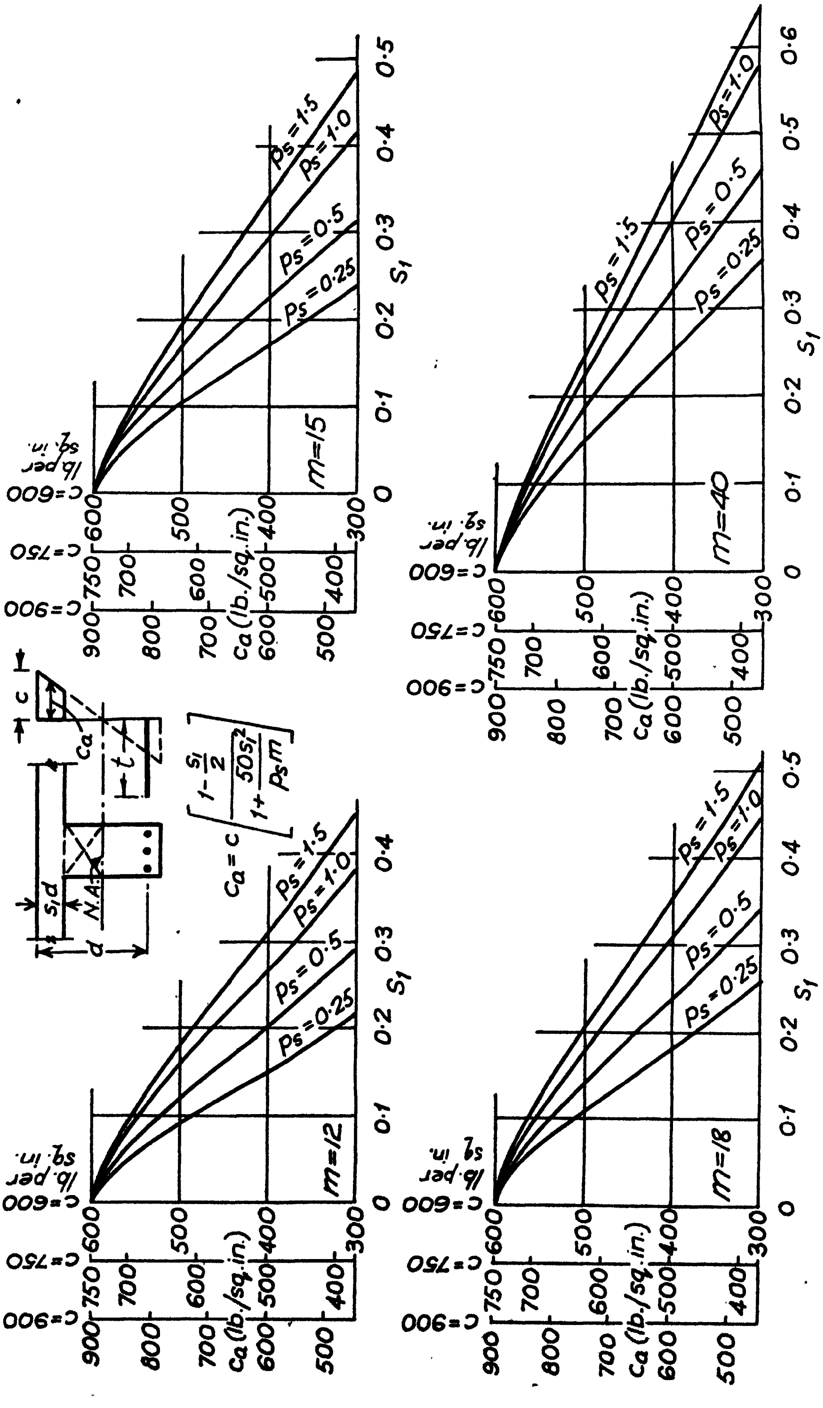

m=15
m=40
m=12
m=18
Ps=1.5
Ps=1.0
Ps=0.5
Ps=0.25
s_1
Ca (lb./sq.in.)
c=600 lb. per sq. in.
c=750
c=900
$C_a = c\left[\frac{1-\frac{s_1}{2}}{1+\frac{50s_1^2}{p_s m}}\right]$
N.A.
s_1d
d
t
c
Ca

FIG. 93.

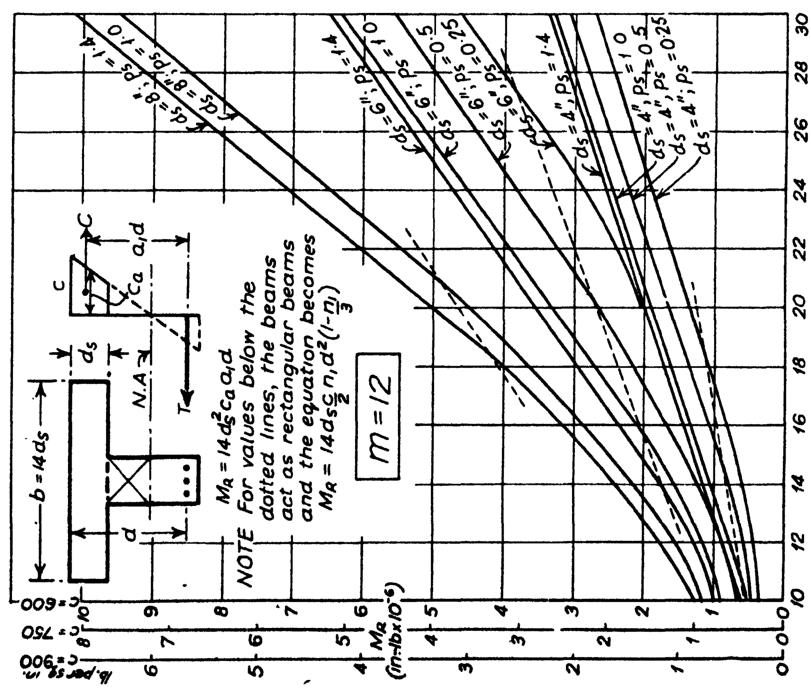


FIG. 94.

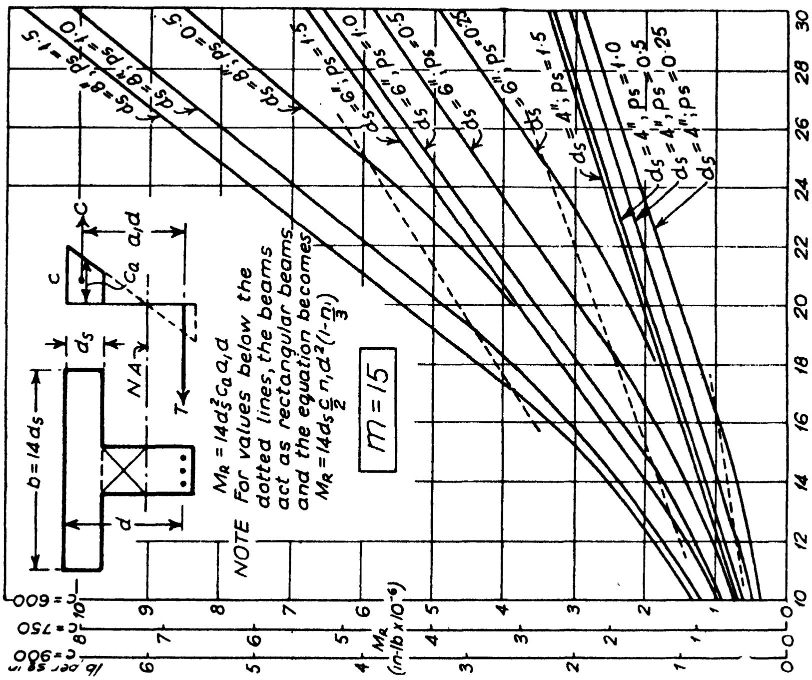


FIG. 95.

These diagrams are not to be used for determining A_t.

These diagrams are not to be used for determining A_t.

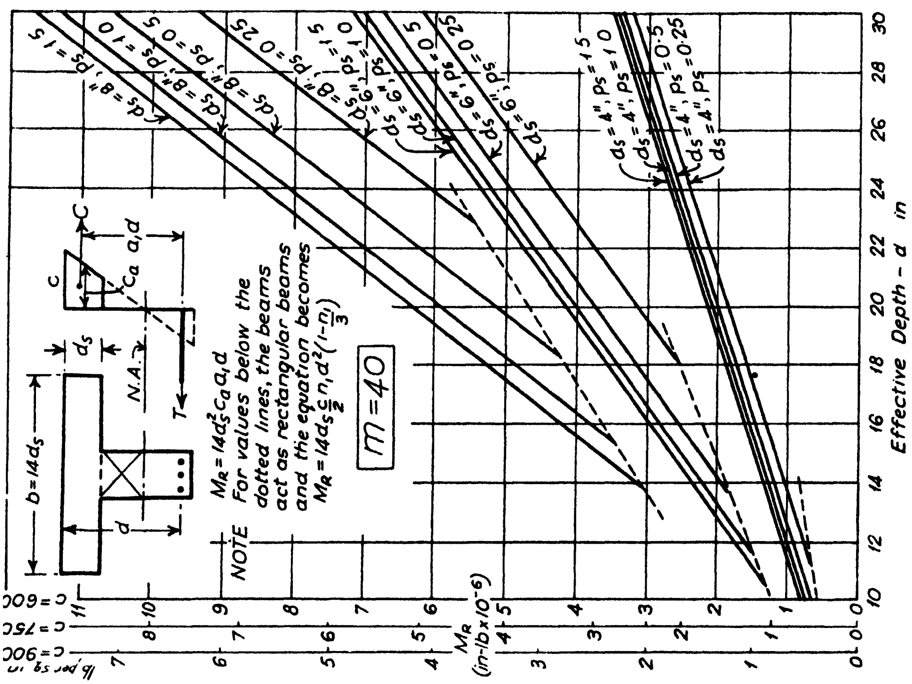


FIG. 97.

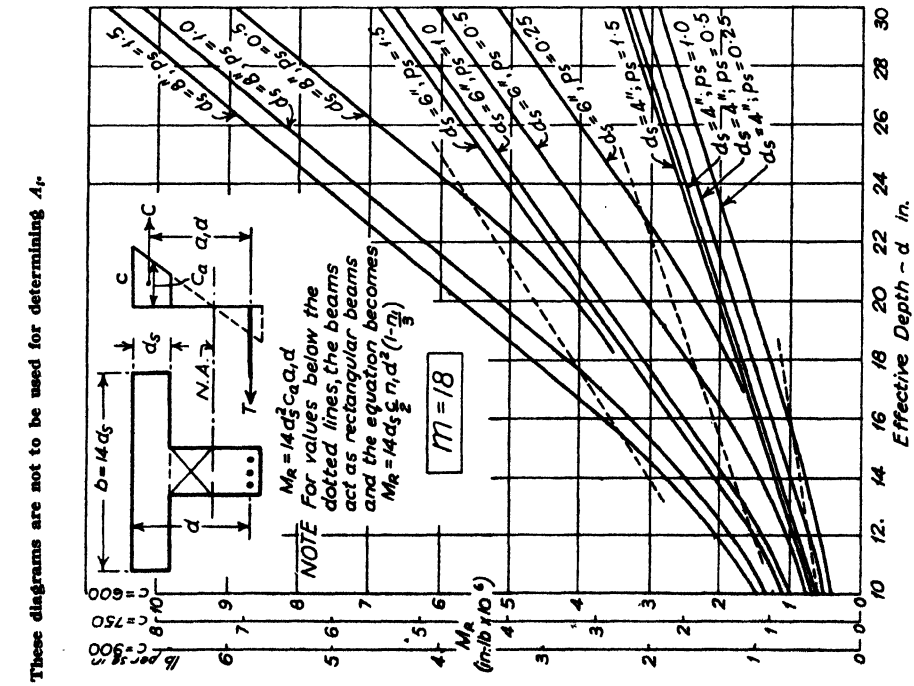


FIG. 96.

are determined by calculation the curves are not curtailed at small values of p_s at which the permissible stress in the reinforcement might be exceeded. Proportional scales are added to the M_r-scale so that M_r for $c = 750$ lb. and 900 lb. per square inch, or intermediate stresses, may readily be obtained. At the point on each curve where s_1 is equal to n_1 the neutral axis coincides with the bottom of the flange of the tee-beam. To the left of these points c_a is constant and is $0{\cdot}5c$, and $a_1 = 1 - \frac{n_1}{3}$ as for a rectangular beam. Values of M for tee-beams in which the neutral axis is within the flange are readily obtained from the extensions of the curves. The formulæ for M_r for tee-beams and for rectangular beams are on the diagrams, which must not be used for determining A_t.

Since the accurate calculation of the moment of resistance of a tee-beam is long, and since it is only required to know the moment of resistance approximately in order to ensure that the greatest stress in the concrete is not excessive, the curves are useful and sufficiently accurate for most cases likely to occur. If, however, the moment of resistance as determined from the curves is not sufficient a more accurate calculation based on correct values of n_1 and c_a can be made, although in border-line cases the " more accurate " value may be of little more use as the width of the flange assumed for a tee-beam is arbitrary. In cases where the actual or permissible width of the flange b is less than $14d_s$, such as in an ell-beam, M_r can be calculated from the value given by the curves since M_r is directly proportional to b, which in the curves is assumed to be $14d_s$.

FIGS. 92 AND 98.—The diagram in *Fig.* 98 relates R to p_s for $c = 600$ lb. per square inch and for various values of m. Proportional scales enable the values of R for $c = 750$ lb. and 900 lb. per square inch or intermediate stresses to be obtained, the value of R being directly proportional to c. It is seen that R increases only very slowly for values of p_s above the " economic percentage," indicating that to increase the moment of resistance of a beam above that corresponding to the " economic percentage " of reinforcement by increasing A_t is not economical; in such cases compressive reinforcement should be used, or the depth of the beam should be increased, or stronger concrete should be used.

In order to check or determine the dimensions of a rectangular beam required to have a specified moment of resistance, the value of R (see the table in *Fig.* 92) corresponding to the " economic percentage " of reinforcement for the particular stresses being used is a safe value to employ. This is true since, if p_s is less than the " economic percentage," the actual moment of resistance of the concrete is sufficient because the value of Rbd^2 decreases less rapidly as p_s decreases than does the moment of resistance when calculated from $A_t a_1 t$. Thus if the dimensions of a beam are such that the value of $A_t = \frac{M}{a_1 dt}$ is less than that corresponding to the " economic percentage " it is not necessary to determine the value of Rbd^2 for the actual value of p_s provided. The value of m is often assumed to be 15 for all qualities of concrete, but is assumed to be $\frac{40{,}000}{3c}$ in the D.S.I.R. code. Due to creep under constant compression the latter value may be trebled after twelve months, but the " no-creep " value of $\frac{40{,}000}{3c}$ is safe since the effect of creep is to reduce the stress in the concrete and to increase very slightly the stress in the reinforcement. Creep cannot reduce the factor of safety seriously, since *Figs.* 89 and 91 show that for $m = 40$ the value of a_1 cannot decrease more than about 5 per cent. compared with the value for $m = 15$. Substituting $m = \frac{40{,}000}{3c}$ in $n_1 = \frac{1}{1 + \frac{t}{mc}}$ gives $n_1 = \frac{1}{1 + \frac{t}{13{,}333}}$. It is seen therefore that the value of n_1 corresponding to the "economic percentage" depends only on t; for $t = 18{,}000$ lb. per square inch $n_1 = 0{\cdot}425$, and for $t = 25{,}000$ lb. per square inch $n_1 = 0{\cdot}347$, as seen in the table in *Fig.* 92.

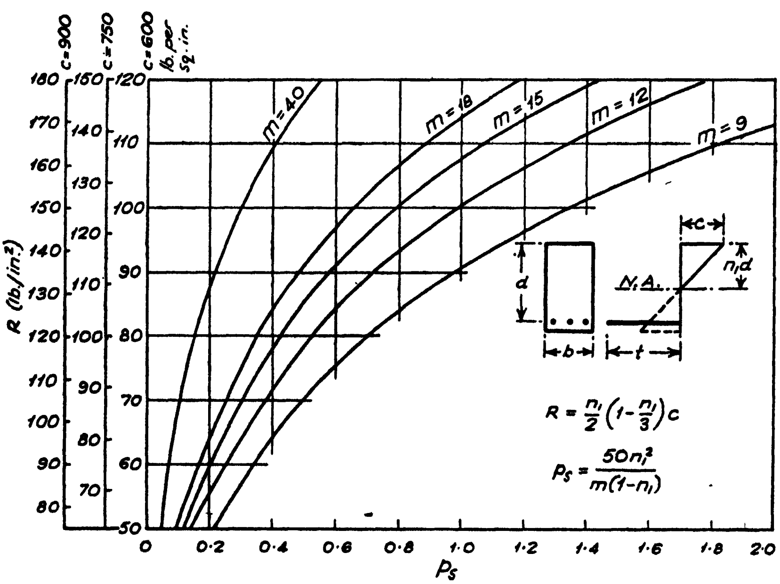


FIG. 98.

The safe limiting value of a_1 for tee-beams and rectangular beams has been shown to be the value of a_1 corresponding to the "economic percentage" for a rectangular beam, that is $1 - \frac{n_1}{3}$. The safe limiting value of a_1 therefore also depends only on the value of t. For $t = 18{,}000$ lb. per square inch, $n_1 = 0{\cdot}425$ and $a_1 = 0{\cdot}86$; for $t = 25{,}000$ lb. per square inch, $n_1 = 0{\cdot}35$ and $a_1 = 0{\cdot}88$. Therefore for all qualities of concrete the safe limiting value of a_1 for tee-beams and rectangular beams is 0·86 provided that t is not less than 18,000 lb. per square inch (see *Fig.* 92).

By substituting $n_1 = 0{\cdot}425$, $c = 750$ lb. per square inch, and $t = 18{,}000$ lb. per square inch in the expression for R, the "economic" value of R is 130, and the corresponding values of R for concrete of other strengths is independent of

m and directly proportional to c provided that t is unaltered. The values of R are also given in *Fig.* 92.

Substituting $m = \frac{40,000}{3c}$ and $n_1 = 0{\cdot}425$ in the expression for p_e, the "economic percentage" is $0{\cdot}00118c$ if t is 18,000 lb. per square inch; if c is 750 lb. per square inch the "economic percentage" is 0·887. Therefore, for a specified value of t, the "economic percentage," some values of which are given in *Fig.* 92, is directly proportional to c.

In the curves in *Figs.* 85, 87 to 91, and 93 to 98 for n_1, a_1, M_r, R, and other factors, the relation of m to c expressed by $m = \frac{40,000}{3c}$ might have been taken into account, but it is desirable to be able to study the effect of the value of m on the values of n_1, a_1, etc. When other factors are constant, for an increase in m from 12 to 18 there is only a slight increase in the values of n_1, c_a, M_r, and R and a slight decrease in a_1. The value of m does not therefore greatly affect the calculations. On the other hand, the values of c and t have a considerable influence. Variations in the value of t are easily taken into account by using a suitable safe limiting value of a_1 in the expression $A_t = \frac{M}{a_1 dt}$. The factors R and M_r are directly proportional to the values of c.

Effect of Creep.

The simplest way to ascertain the effect of creep on the strength of a beam is to substitute suitable safe limiting values of m in the various expressions for a_1, R, M_r, etc. *Fig.* 110 (Chapter V) shows that $m = 40$ is safe for concrete having a crushing strength of 3500 lb. per square inch. *Figs.* 95 and 97 show that when $m = 40$, compared with $m = 15$, M_r is increased from 5 per cent. to 25 per cent. depending on the position of the neutral axis, and a_1 is reduced by about 5 per cent. The greatest increase of M_r is for a rectangular beam or for a tee-beam in which the neutral axis is within the slab. It appears that the permissible stresses in the concrete for a design based on conditions of greatest creep, or on a consideration of the distribution of stress for conditions of ultimate load, should therefore be reduced by about 20 per cent. for a rectangular beam, but should remain unaltered for a tee-beam having a small value of s_1 and a high value of n_1. In some cases the stress in the reinforcement may be increased due to creep by not more than 5 per cent., which is negligible. The present basis of design using a small value of m appears to be safe (see Appendix III).

Determination of the Economical Dimensions of a Beam.

The dimensions of a beam having the lowest cost can only be calculated approximately and the simplest method is to plot curves, as in *Fig.* 99, relating the depth of the beam to the costs of the concrete, reinforcement, and shuttering for 1 ft. of the beam. The depth of beam for which the sum of these costs is the lowest may be the lowest cost, but the depth required for resistance to

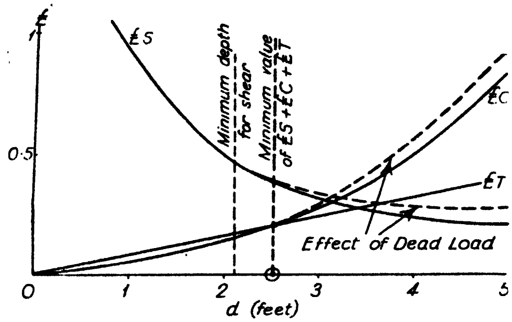


FIG. 99.

shear must be considered; some adjustment is possible by altering the value of b. The unit costs on which the curves are based should be current costs which are not always easily determined for a particular type of structure. The cost of 1 ft. of the rib of a tee-beam can be approximately estimated as follows. If the price of supplying, bending, and fixing the reinforcement is £C_T per ton, the cost of the steel in 1 ft. of beam, including the main bars, bent-up bars, overlaps, hooks, and stirrups, is £$\frac{C_T A_t}{2240}$ multiplied by, say, 5, = £T, where A_t is the cross-sectional area of the steel in the bottom at mid-span. If the price of a cubic yard of concrete placed in the ribs of tee-beams is £C_c, the cost of the concrete in 1 ft. of beam (£C) is £$\frac{C_c B_r D_r}{27}$ if the breadth of the rib, B_r, and the depth of the rib, D_r, are in feet.

The cost of shuttering is more difficult to estimate. If the cost of the material for 1 sq. yd. of shuttering is £C_M and the cost of the labour for each use is £C_L, the total cost of the shuttering for 1 ft. of beam, assuming three uses, is roughly £$\left(\frac{C_M}{3} + C_L\right)\left(\frac{2D_r + B_r}{9}\right)$ = £S. Many factors influence the cost, such as standard sizes of timber, the cost of hiring steel shuttering, the amount of plant to be used, overhead charges, the number of times the shuttering can be re-used, and the size of the reinforcement bars. The estimated costs of separate members of different proportions are thus generally unreliable, but more reliance can often be placed on comparative estimates of two or more designs for a whole structure for the purpose of determining the cheapest design.

Methods of Designing Beams and Slabs.

There are three methods of designing tee-beams, rectangular beams, and slabs, namely, the exact method, the safe-limit method, and the M_r-curve method. There should be no difficulty in deciding which of these methods should be followed in a particular case, although generally it is best to use the M_r-curve method to determine the dimensions of the concrete and the safe-limit method as a check when the stress in the concrete is nearly the safe working stress. The three methods are described in the following, first for a tee-beam, and then, as special cases of the tee-beam, for a rectangular beam with and without compressive reinforcement and for a slab.

Tee-Beams.

Exact Method.—The exact method has only academic value. The procedure is to assume m to suit the concrete to be used, and d, b_r, and d_s to suit the shearing resistance, the design of the slab, and other relevant conditions. Calculate n_1 from

$$n_1 = \frac{50s_1^2 + p_s m}{100s_1 + p_s m} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or read n_1 from the curves (*Figs.* 88 to 91), assuming the value of p_s. Calculate a_1 from

$$a_1 = 1 - \frac{s_1}{3}\left(\frac{3n_1 - 2s_1}{2n_1 - s_1}\right) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since the applied bending moment, M, equals $14d_s^2c_aa_1d$, the average stress in the concrete can be calculated from (1) transposed,

$$c_a = \frac{M}{14d_s^2a_1d} \text{ or } = \frac{M}{bs_1a_1d^2} \quad . \quad . \quad . \quad (9)$$

The maximum stress in the concrete is then calculated from

$$c = \frac{c_a\left(1 + \frac{50s_1^2}{p_s m}\right)}{1 - 0{\cdot}5s_1} \text{ or } = \frac{n_1c_a}{n_1 - 0{\cdot}5s_1}. \quad . \quad . \quad (10)$$

Adjustments of d and b_r should now be made, if necessary, within the limits of the other requirements mentioned until c is equal to or less than the safe working compressive stress.

Calculate A_t from $$A_t = \frac{M}{a_1dt} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Calculate $p_s\left(= \frac{100A_t}{bd}\right)$; the value obtained will probably differ from that assumed in (*a*) and (*d*), and for an exact calculation the preceding steps should be repeated with other assumed values of p_s until, by trial and error, the calculated value of A_t corresponds to the assumed value of p_s. It is seen, therefore, that the " exact " method is involved and may be inaccurate if the assumptions are not valid.

Safe-Limit Method.—Assume, b_r, d_s, d, and m to suit the requirements for resistance to shear and other conditions. With a safe limiting value of a_1 applicable to the " economic percentage " for the value of t used, calculate c_a from (9). With a safe assumed value of $p_s = 0{\cdot}25$, calculate c from (10). Adjust the assumed dimensions insofar as other requirements allow, and repeat the foregoing steps until c is less than, and as nearly as possible equal to, the safe working stress in the concrete. Calculate A_t from (11). The assumed value of p_s should be checked and the procedure repeated with other values if necessary. By using safe limiting values of n_1 and a_1 a considerable amount of work is saved and the results obtained differ very little from those obtained by the exact method.

The depth of the rib of a tee-beam required to provide sufficient shearing resistance nearly always prevents the use of an "economic percentage" of reinforcement, and so prevents the concrete developing the safe working stress under the safe working load. If the neutral axis falls within the flange, the method of designing a tee-beam is the same as for a rectangular beam.

M_r-CURVE METHOD.—Since the effective width of the flange of a tee-beam is generally obtained by an arbitrary safe assumption, the curves in *Figs.* 94 to 97 are sufficiently accurate for obtaining the moment of resistance of the concrete, particularly as in most cases the moment of resistance is much greater than that required to resist the applied bending moment because of the requirements for the design of the slab and for the resistance to shear. The curves are extended so as to be applicable to beams in which the neutral axis is within the flange; there is therefore no need for a different method in such a case. If the width of the flange b, as in an ell-beam, is less than $14d_s$ it is easy to adjust the value of M_r read from the curves, since M_r is directly proportional to b; that is the moment of resistance is $\frac{M_r b}{14d_s}$ where M_r is the value read from the curve.

Although the use of curves for checking the stress in the concrete is satisfactory, the area of main tensile reinforcement should always be calculated as described for the safe-limit method. If the safe limiting value of a_1 is slightly small, the excess reinforcement is negligible. In fact the use of a small value can be regarded as a wise precaution since the reinforcement, especially in the top of a beam or slab, although fixed correctly in the first instance, may be moved slightly during concreting by men standing on it, by a barrow slipping off its runway, or other accidental causes. Precautions should be taken to support the reinforcement so that displacement is unlikely to occur, since this is one of the most common causes of constructional defects. The accuracy of the value of a_1 is not so important as regards the concrete, because the safe working stresses are assumed to be about one-third of the strength of test cubes, and before failure the distribution of stress tends to become rectangular.

Rectangular Beams.

EXACT METHOD.—Beams designed as rectangular beams occur most frequently as the ribs of tee-beams resisting negative bending moments at the supports, and as strips of solid slabs. The dimensions are therefore generally determined by considerations other than the provision economically of resistance to the bending moment. If the depth and the width of the rib can be varied without being detrimental to other requirements, values of d and b should be adopted so that the reinforcement and concrete are simultaneously subjected to the safe working stresses at the working load. The method of calculation, which is not cumbersome, is first to select values of m and c to suit the concrete to be used. Calculate the value of R in the expression $M_r = Rbd^2$ from $R = \frac{n_1}{2}\left(1 - \frac{n_1}{3}\right)c$, using the value of n_1 corresponding to the "economic percentage" (see the table in *Fig.* 92). Assume suitable values of b and d so that $Rbd^2 = M$. Calculate A_t from (11), using the value of a_1 corresponding to the "economic percentage." If the values of b and d necessary for resistance to shear or other requirements are such that the value of Rbd^2 exceeds M, the exact method is as follows:

Assume a value for p_s and calculate n_1 from

$$n_1 = \sqrt{\left(\frac{mp_s}{100}\right)^2 + \frac{mp}{50}} - \frac{mp_s}{100},$$

which is the expression $p_s = \frac{50n_1^2}{m(1 - n_1)}$ transformed. Calculate a_1 from $a_1 = 1 - \frac{n_1}{3}$, and A_t from (11). Calculate p_s from $p_s = \frac{100A_t}{bd}$, and if this value differs from that assumed the method should be repeated until the assumed and calculated values are the same. The value of c is less than the safe working stress, since Rbd^2 exceeds M. Obviously this method is too long to be of practical use, and the safe-limit method should be used.

SAFE-LIMIT METHOD.—Calculate the value of R for the permissible stresses, or obtain R from the table in *Fig.* 92. Select values of b and d to suit the requirements for shear and other considerations so that Rbd^2 equals or exceeds the applied bending moment M. Calculate A_t from (11), using the safe limiting value of a_1 corresponding to the "economic percentage."

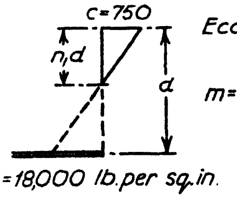
BEAM WITH COMPRESSIVE REINFORCEMENT.—If Rbd^2 is less than M, compressive reinforcement is required, and the method of design is as follows. Let M_c be the moment of the couple formed by the compressive reinforcement and that area of the tensile reinforcement additional to the "economic percentage." If M is the applied bending moment, $M_c = M - Rbd^2$, the value of R being calculated or obtained from the table in *Fig.* 92 for the required stresses. Let c_c' be the compressive stress in the concrete surrounding the compressive reinforcement. With the value of n_1 corresponding to the "economic percentage," calculate c_c by proportion to the distance of the compressive reinforcement from the neutral axis. Then A_c (the area of the compressive reinforcement) is $\frac{M_c}{a_s(m - 1)c_c}$, where a_s is the distance between the compressive and tensile reinforcement. The total tensile reinforcement to be provided is the area corresponding to the "economic percentage" plus $\frac{M_c}{a_s t}$. To save time $(m - 1)c_c$ can sometimes be assumed to be 7000 lb. per square inch. Based on an ultimate-load theory, the value of mc_c is sometimes assumed to be equal to t regardless of the stress in the surrounding concrete. This is commonly known as the "steel-beam" theory.

Slabs.

In calculating M_r for a slab it is common to consider a strip 12 in. wide. As the width of the strip is therefore fixed, the only variable factors are d and p_s, since m and c are determined by the class of concrete. It is therefore simple to calculate directly a thickness of slab in which at the working load the safe working stresses in the reinforcement and concrete are simultaneously developed. The procedure is to calculate d from $M = M_r = Rbd^2$, where b is 12 in. and the value of R is that corresponding to the "economic percentage" for the permissible stress. The area of the tensile reinforcement can be found from the "economic percentage." The table in *Fig.* 100 can be used to select a slab to resist a specified bending moment when the permissible stresses are 750 lb. and 18,000 lb. per square inch and m is 18.

If for any reason the thickness of the slab is greater than required by the foregoing calculation, A_t should be calculated from (11) using the value of a_1 corresponding to the "economic percentage" as a safe limiting value. If a

Slab Depth (in.)	Approx. effective depth (in)	Rbd^2 (in.-lb.)	Economic area of Reinforcement (sq. in)
4	3·25	17,500	0·350
5	4·25	·30,000	0·457
6	5·25	45,600	0·564
7	6·25	64,700	0·671
8	7·10	83,500	0·762
9	8·10	108,600	0·870



Economic $p_s = \dfrac{50\times 0{\cdot}429^2}{18(1-0{\cdot}429)} = 0{\cdot}895$

$m=18$ $\quad n_1 = \dfrac{18\times 750}{18{,}000+(18\times 750)} = 0{\cdot}429$

$R = \dfrac{0{\cdot}429}{2}\left(1-\dfrac{0{\cdot}429}{3}\right)750 = 138$

FIG. 100.—MOMENT OF RESISTANCE OF SLABS WITH "ECONOMIC PERCENTAGE" OF REINFORCEMENT.

more accurate design is required, the larger value of a_1 corresponding to the actual value of p_s provided can be found by trial and error; the saving in reinforcement is generally so slight, however, that this method is rarely worth while. It is useful when making slide-rule calculations to remember the value of a_1t for the most commonly used stress in the reinforcement. Thus, if t is 18,000 lb. per square inch and the safe limiting value of a_1 is 0·86, a_1t is 15,500, and A_t is $M \div 15{,}500d$.

Summary of Methods of Designing Beams.

The curves in *Figs.* 85, 87 to 91, and 93 to 98 should not be used until one is familiar with their derivation and satisfied with their accuracy by checking them. After a little practice the dimensions of a reinforced concrete beam can be calculated with sufficient accuracy in a few moments with a slide rule and a set of curves. The complicated procedure for the calculation of the stresses due to shear and bending can be reduced to the following.

TEE-BEAMS.—Assume values of d and b_r so that $s = \dfrac{S}{a_1db_r}$ does not exceed 0·4c. Check the value of M_r from the curves, adjusting the assumed values of d and b_r if necessary. Calculate A_t from $\dfrac{M}{a_1dt}$, the safe limiting value of a_1 being that corresponding to the "economic percentage." (For example, for $c = 750$ lb. per square inch and $t = 18{,}000$ lb. per square inch, the safe limiting value of a_1t is 15,500, as calculated previously.)

Calculate the reinforcement required to resist the shearing force at each section where s exceeds $0{\cdot}1c$. Select inclined bars and stirrups for each section from the tables in *Fig.* 81 (*a*) and (*b*); $S = \frac{a_1 A_w t_w}{p} + 2A_{w1}t_{w1} \sin\theta$.

Extend all bars that are in tension for a distance of not less than 59 times the diameter of the bar (for $t = 18{,}000$ lb. per square inch and $c = 750$ lb. per square inch) beyond the point of greatest stress; bars in compression should be extended for a distance of not less than 45 times the diameter of the bar.

Rectangular Beams.—Assume values of d and b so that s does not exceed $0{\cdot}4c$. Check that the value of M_r ($= Rbd^2$) is not less than M. It is often, but not always, economical for M to equal Rbd^2. (The value of R corresponding to the "economic percentage" is 130 for $c = 750$ lb. per square inch and $t = 18{,}000$ lb. per square inch and $m = 18$.)

At sections where s is less than $0{\cdot}1c$ nominal, stirrups spaced at not more than a_1d only are required. At sections where s exceeds $0{\cdot}1c$, reinforcement to resist the whole shearing force should be provided as described for tee-beams. The main bars should be extended to provide sufficient grip, also as described for tee-beams.

If M exceeds Rbd^2 the compressive reinforcement required is given by $A_c = \frac{M - Rbd^2}{a_s(m - 1)c_c}$. If c, neglecting the compressive reinforcement, does not exceed twice the permissible compressive stress in the concrete, the stress in the compressive reinforcement may be assumed to be 18,000 lb. per square inch and the resistance of the concrete can be neglected.

Slabs.—Select from the table in *Fig.* 100 a slab of a thickness such that M_r is not less than M. Calculate A_t from $\frac{M}{a_1dt}$; the safe limiting value of a_1t is 15,500 for $c = 750$ lb. per square inch and $t = 18{,}000$ lb. per square inch. For other stresses, the procedure is as for rectangular beams with $b = 12$ in.

Examples of calculations for beams and slabs are in Appendix II.

Factor of Safety.

It is important to keep in mind the basic principles of design in order to know what degree of accuracy is necessary in the calculations. *Figs.* 101 and 102 may help to do this, as the curves provide a visual conception of the meaning of the factor of safety. The curves in *Fig.* 101 show values of M_r for various depths of a tee-beam assuming that $m = 18$, $d_s = 4$ in., and $p_s = 1$ per cent.

Concrete in compression on one side of a beam is in the form of a prism bounded by stirrups and longitudinal bars. On the other side, cracked concrete is subjected to a diagonal compression due to shear. Longitudinal stress is greatest near the outsides of the beam and reduces to nothing at the neutral axis. A prism of double the length of a standard test cube has about 67 per cent. of the compressive strength of a cube. The compressive strength of concrete, calculated from a test on a beam assuming a triangular distribution of stress, is about 30 per cent. higher than that obtained from a test on a prism of the same concrete ; this difference is due mainly to plastic yield before failure causing the stress-distribution diagram to approach a rectangle. The D.S.I.R. code recommends that the permissible strength of concrete in bending should be one-third of the crushing strength of cubes made on the site ; this rule is presumably based on tests on rectangular beams which show that this assumption is safe. The weakness of the prismoidal form of the concrete in the beam compared with the cubical form in the test is compensated by the distribution of stress at the ultimate load being more effective than the triangular distribution assumed in the calculations. In a tee-beam with a low neutral axis the increased strength due to plasticity would probably be found to be negligible because of the trapezoidal distribution assumed in the flange for elastic conditions. *Fig.* 158

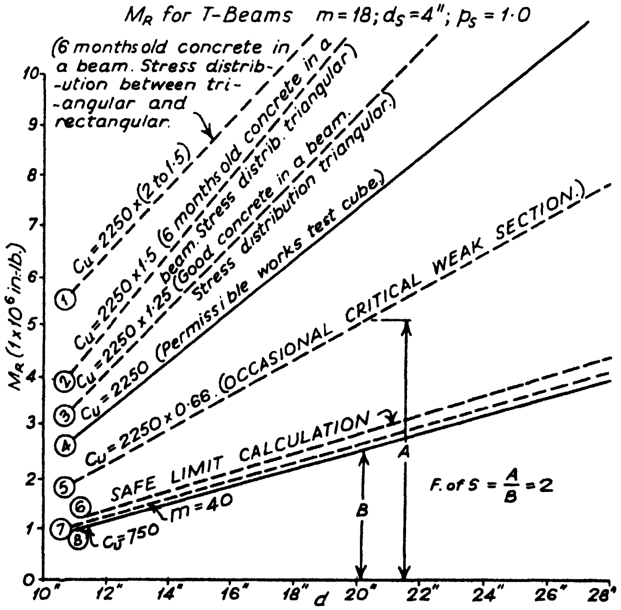


FIG. 101.—VARIATION OF RATIO $\frac{\text{ULTIMATE LOAD}}{\text{WORKING LOAD}}$ (CONCRETE).

shows that after 28 days the crushing strength of ordinary concrete, having a low water-cement ratio, in a beam may be 4800 lb. per square inch. Apart from the effects of plastic yield, therefore, it is reasonable to adopt a crushing strength c_u of 2250 lb. per square inch in calculations for beams made of ordinary concrete under average works conditions. The values of M_r in the curves in *Fig.* 101 are therefore about 30 per cent. higher than the values based on the D.S.I.R. code, but they are reasonable values to assume in assessing the real factor of safety in view of the test results shown in *Fig.* 158. Referring to *Fig.* 101, curve (4) is for $c_u = 2250$ lb. per square inch, which is a reasonable crushing strength for ordinary 1 : 2 : 4 concrete in a beam. The strength of test cubes would need to be 30 per cent. to 50 per cent. in excess of this stress which, as already stated, is not unreasonable according to the curves shown in *Fig.* 158. The effect of plasticity is not included. Curve (3) is for $c_u = 2250 \times 1{\cdot}25 = 2813$ lb. per square inch, which is a possible strength for well-graded and properly compacted 1 : 2 : 4 concrete, having a low water-cement ratio, in a beam, the concrete being well bound by stirrups; the effect of plasticity is not included. Curve (2) is for $c_u = 2250 \times 1{\cdot}5 = 3375$ lb. per square inch, which is for the same concrete as in curve (3) assuming that the concrete has matured for six months. Curve (1) is for $c_u = 2250 \times 2$, which is a suitable value of c_u for concrete similar to that in curve (2) but taking into account the redistribution of stress from a triangular form towards a rectangular form due to plasticity under ultimate load. For small values of d the increase in M_r for this reason is high, but for large values of d the increase is almost negligible, since the neutral axis is well below the flange and the average stress in the flange approaches the greatest stress because the stress distribution is trapezoidal, the effects of plasticity being neglected. Curve (5) is for $c_u = 2250 \times 0{\cdot}66 = 1500$ lb. per square inch, a possible effective value of c_u (that is, taking into account plasticity at a critical section of a beam) for local patches of weak concrete due to leaking of grout, poor compaction, weak cement or aggregate, the presence of foreign matter, insufficient cement, delay in placing after mixing, the action of frost, or any of the other occasional defects that are not always avoided even by careful supervision on the site. Curve (8) is for $c_u = 750$ lb. per square inch, the permissible stress (D.S.I.R. code) for ordinary 1 : 2 : 4 concrete. The higher value of M_r for curve (7), compared with curve (8), is due to creep increasing the value of m to 40 after a long period. The increased value of M_r for curve (6) compared with curve (8) occurs at many sections of a beam due to the calculations being based on safe limiting values of a_1.

The conclusions drawn from *Fig.* 101 are:

(1) Until the control and testing of completed concrete structures is sufficiently developed to justify using larger values of c, the design must be based on a small value of c, say, 750 lb. per square inch if ordinary concrete is used since, in spite of the greater effective strength of concrete in beams than in cubes, experience shows that it is possible for the concrete to be occasionally so weak at a critical position that the strength of the beam corresponds to an effective value of c_u of only two-thirds of 2250 lb. per square inch. Advantage cannot be taken of the greater strength after, say, six months due to greater age and the effects of creep unless the load is restricted for that period.

(2) The value of $c_u = 2250 \times 0{\cdot}66$ may occur only occasionally; values of $c_u = 2250 \times (1 \text{ to } 2)$ are shown by tests to occur in most parts of a beam. It is therefore evident that saving in the cost of rectangular beams and slabs may be made (particularly if the dead load forms a high proportion of the total load), provided that further research confirms that the higher values of c_u assumed at ultimate load are reasonable, and if occasional weak places can be avoided by better supervision or detected by tests and strengthened. For this reason greater working stresses in precast and prestressed concrete beams, which can be tested and matured before use, are often permitted. The thickness of the flange of cast-in-situ tee-beams is generally determined by the requirements of the slab, and the dimensions of the rib by the requirements for resistance to shear; therefore the possible saving in a tee-beam is not so great.

(3) Curve (6) shows that if the calculations are based on safe limiting values

of a_1, the factor of safety is affected very little, and that there is no reason for making laborious and more exact calculations.

(4) Curve (7) shows that the effect of creep, which may increase the effective value of m from 18 to 40, increases by only a small amount the factor of safety as regards the concrete stresses.

(5) When designs are based on the ordinary assumption of the distribution of stress and $c = 750$ lb. per square inch, the factor of safety at 28 days is about 2, this being the ratio of the ultimate load (causing bending stresses at a critical section) to the working load based on "exact" calculations, the value of m being that at 28 days, that is, before creep due to the dead load and the average live load has fully taken place.

The curves in *Fig.* 102 relate the moment of resistance of beams to the stress in the reinforcement and show the following. Curve (1) is based on "exact"

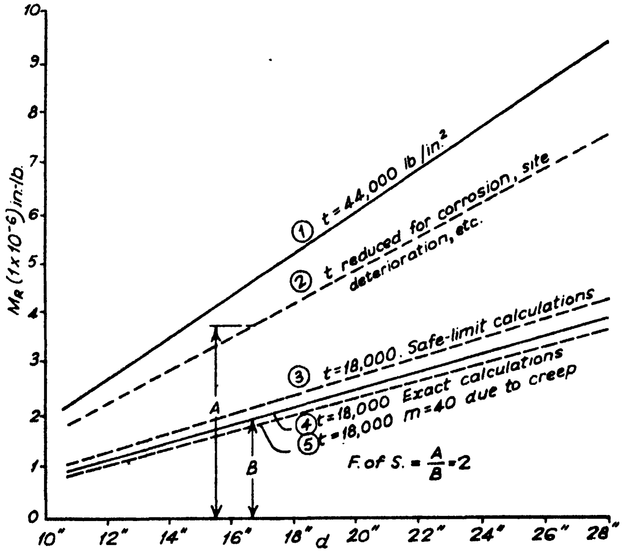


FIG. 102.—VARIATION OF RATIO $\frac{\text{ULTIMATE LOAD}}{\text{WORKING LOAD}}$ (REINFORCEMENT).

calculations with $t = 44{,}000$ lb. per square inch. Curve (2) is based on a reduction of the tensile strength by 17·5 per cent. due to corrosion or deterioration of the steel or errors in placing the bars. Curve (3) is based on safe limiting values of a_1 which, when $s_1 = 0{\cdot}1$, may be 10 per cent. greater than would be obtained by exact calculations. Curve (4) is based on exact calculations with $t = 18{,}000$ lb. per square inch, the area of steel being assumed so that the values of M_r equal those based on $c = 750$ lb. per square inch as in the case of the curves in *Fig.* 101. Curve (5) is for the same condition as (4) after creep has taken place, thereby increasing m from 18 to 40; the values of M_r are reduced by nearly 5 per cent.

The yield stress of reinforcement in a concrete beam is generally assumed to be the same as the yield stress of specimens of the steel unless deterioration has taken place due to bending or corrosion. The yield-point stress of mild steel may be less than 40,000 lb. per square inch, but for medium-tensile steel or mild steel with a guaranteed yield-point a stress of 44,000 lb. per square inch is a safe upper limit to assume. If precautions are taken in fixing the reinforcement and for maintaining it in position during concreting, a reduction of 17·5 per cent. is a reasonable allowance for the risk of corrosion and bad workmanship. Curves (1) and (2), *Fig.* 102, show how easily the factor of safety can be dangerously reduced by displacement of the bars. Curves (3) and (5) show that creep reduces the factor of safety by a negligible amount, and that the use of a safe limiting value of a_1 provides a little additional strength to provide for this reduction. Therefore it is not generally worth while to calculate exact values of a_1 in order to save reinforcement.

In addition to the factors mentioned in the foregoing as varying the factor of safety of a beam, the bending moments in framed structures and continuous beams tend to be redistributed under increasing load before failure takes place, thereby increasing the strength of the structure. Advantage is taken of this fact in beams, support sections of which are often designed for a bending moment 15 per cent. less than that given by elastic calculations, and the bending moments at mid-span are correspondingly increased. This reduction should not be applied beyond the stage at which, under working load, the stress in the reinforcement at the section where the bending moment is reduced exceeds 27,000 lb. per square inch, otherwise cracking and corrosion may occur. (See Appendix III.)

CHAPTER IV

COLUMNS AND STRUTS

In a reinforced concrete member subjected to compression the strain of the reinforcement is the same as the strain of the surrounding concrete, if the bond between the concrete and the steel is not destroyed. If conditions remained constant the stresses in the two materials would be proportional to their elastic moduli, that is if the stress in the concrete is c the stress in the reinforcement would be cm. This is the basis of some methods, such as that in the London Building By-laws of 1938, of the design of columns and other members subjected to concentric compressive load. The safe load on a short column in accordance with this method is the effective cross-sectional area of the column multiplied by the permissible compressive stress in the concrete; the effective cross-sectional area is the gross area of the concrete plus $(m - 1)$ times the area of the longitudinal reinforcement. Owing to the effect of creep, the effective modulus of elasticity of concrete under load decreases with time and, unless a suitable value of m is chosen, this method for the design of columns is not theoretically sound and is now largely replaced by a method, described in later paragraphs, in which the stress in the reinforcement is assumed to be independent of the stress in the surrounding concrete.

To enable the long thin reinforcement bars to act vertically as columns within the concrete, it is necessary to provide binders or lateral ties to prevent them buckling. The strength of a reinforced concrete column is increased if binding is provided in excess of that required to prevent the bars buckling, and particularly if the binders are in the form of a helix. The effect of helical binding is to reduce the lateral deformation of the column under compression and thereby to reduce the longitudinal deformation; in effect the binding increases the resistance to "bursting" of the concrete. The consequent increase in strength of a helically-bound column is discussed in a later paragraph.

In spite of the fact that it is now common to design columns under direct load on the basis that the stress in the reinforcement is independent of that in the surrounding concrete, it is still mainly the practice to design reinforced concrete columns and other members subjected simultaneously to direct compression (or tension) and bending on the assumption that the steel and concrete conform to Hooke's law, and that the strain of the steel is the same as that of the surrounding concrete, as in the design of beams. The permissible stress in the concrete is, however, greater than that for columns subjected to direct load alone.

Short Columns under Axial Load.

Tests show that under continual loading the modular ratio in a reinforced concrete column may increase from 20 to 60 in a year if the compressive strength of the concrete is about 2000 lb. per square inch, or from 10 to 20 if the strength of the concrete is 6000 lb. per square inch. Also, at failure load the concrete develops only about two-thirds of the strength shown by cubes, while the stress in the reinforcement is the yield stress of the steel regardless of the stress in the surrounding concrete. In the D.S.I.R. code the safe working load on a short axially-loaded column is based on the assumptions that the permissible working stress in mild steel is 13,500 lb. per square inch, which for the purpose of design is assumed to be developed under working load regardless of the stress in the surrounding concrete, and that the permissible working stress in concrete is as given in the following table.

Nominal volumetric proportions	Direct stress c in the concrete (lb. per square inch)		
	Ordinary-grade	High-grade	Special-grade
1 : 1 : 2	780	1000	1250
1 : 1·2 : 2·4	740	960	1200
1 : 1½ : 3	680	880	1100
1 : 2 : 4	600	760	950

The strength of works cubes must be 3·75 times the tabulated stresses, which are 80 per cent. of the permissible stresses for concrete compressed in bending (see Chapter III and Appendix I).

In accordance with the British Standard code the permissible compressive stresses in columns are 18,000 lb. per square inch in mild steel, 1140 lb. in 1 : 1 : 2 concrete, 950 lb. in 1 : 1½ : 3 concrete, and 760 lb. in 1 : 2 : 4 concrete. Under certain conditions the stresses in the concrete may be increased, and under other conditions must be decreased; reference should be made to the code for these conditions. The stresses are also given in Appendix I.

If P is the safe axial load on a short column, c the permissible working stress in the concrete as given in the foregoing table, t_s the permissible compressive stress in the longitudinal reinforcement, A_c the cross-sectional area of the concrete, and A_s the cross-sectional area of the longitudinal reinforcement,

$$P = cA_c + t_sA_s.$$

When it is first loaded excessive stress is developed in the concrete but, due to the effect of creep or of a slight overload causing plastic yield, some of the compressive force is transferred to the reinforcement. Since the required load before failure is assured, this method of design provides a factor of safety that is high enough and is probably sounder than ignoring creep and assuming the stress in the reinforcement to be, say, 15 times the stress in the surrounding concrete.

To bind the concrete adequately and prevent the main reinforcement from buckling outwards, ties are provided. The volume of the ties must be at least 0·4 per cent. of the gross volume of the column, and the pitch of the ties should not exceed the smallest width of the column or twelve times the diameter of the bars forming the longitudinal reinforcement.

The safe axial load on a column with helical binding is calculated from the formula, based on tests,

$$P = cA_k + t_sA_s + 2t_bA_b,$$

where A_k is the cross-sectional area of concrete in the core of the column, t_b the permissible tensile stress in the helical binding (13,500 lb. per square inch for mild steel), and A_b the equivalent area of helical binding, that is the volume of helical binding in unit length of the column. The safe load P must also not exceed $cA_c + t_sA_s$, and $cA_k + 2t_bA_b$ must not exceed $0{\cdot}5uA_c$, where u is the crushing strength of works cubes of concrete as in Appendix I.

Examples of calculations for axially-loaded columns are given in Appendix II.

Long Columns under Axial Load.

The permissible axial working load on a long column is the same as that on a short column of the same size reduced by an amount depending on the ratio of the effective length of the column, l, to the least radius of gyration k. For columns of rectangular cross section the least width d is used as the basis of this ratio, and the reduction coefficient C_r, the value of which is based on tests, is obtained from $C_r = 1{\cdot}5 - \frac{l}{30d}$. For a column with helical binding d is the diameter of the core. The value of C_r is unity when $\frac{l}{d}$ is 15, and zero when $\frac{l}{d}$ is 45. Intermediate values can be interpolated linearly. The corresponding formula based on the radius of gyration is

$$C_r = 1{\cdot}5 - \frac{l}{100k}.$$

The effective length of a column is its actual length when its ends are restrained in position but not direction, or three-quarters of its actual length when its ends are restrained in position and direction.

Professor Ross has shown that, to ensure that the factor of safety (or load

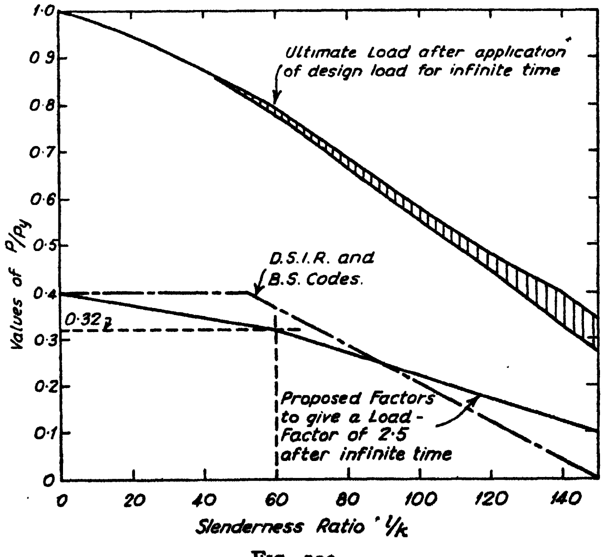


FIG. 103.

factor) of 2·5 is obtained when full creep has taken place, a slight modification of the factors should be made as in *Fig.* 103, where $\frac{P}{P_y}$ is the ratio of the working load to the buckling load.

Columns Subjected to Compression and Bending.

It is not practical to design directly a column subjected to compression combined with bending. A section must be assumed and the maximum stresses or amounts of reinforcement determined either by calculation or from curves. It is customary to adopt the stresses permissible for members subjected to pure bending and to base the calculation of the distribution of the stress on the assumption that the effects of creep, which are taken into account in the case of an axially-loaded column, are ignored. The steel and concrete are assumed to be elastic, and the stresses are assumed to be directly proportional to the distance from the neutral axis. The strength of concrete in tension is ignored. In a column supporting only a slightly eccentric load conditions obviously

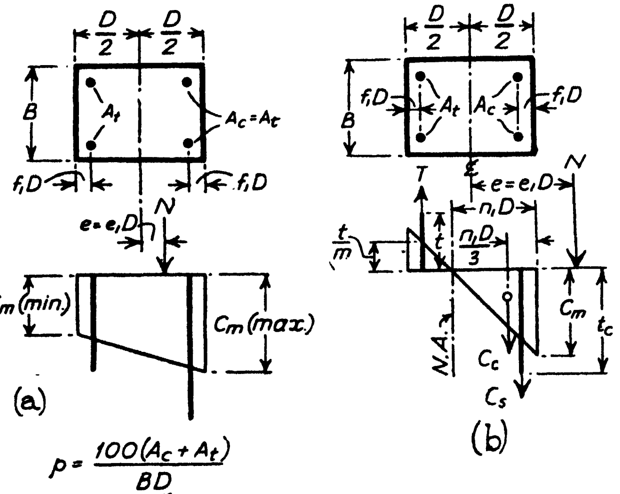


FIG. 104.

approach those of an axially-loaded column, and the area of reinforcement may be a little too great and the area of concrete a little too small; the effect on the factor of safety is, however, negligible.

CASE I. NO TENSILE STRESS DEVELOPED.—Referring to *Fig.* 104, the notation is:

D, the overall depth of the cross section of column.
B, the breadth of the cross section.
f_1D, the distance from the centre of the reinforcement to the edge of the concrete.
A_c, the area of reinforcement on the side of greatest compression.
A_t, the area of reinforcement on the side of least compression.

$$p = \frac{100(A_c + A_t)}{BD}.$$

c_m, the maximum stress in the concrete.

N, the externally applied force acting normal to the cross section.

e, the eccentricity of N; $e_1 = \frac{e}{D}$.

Sometimes the actual line of action of N is known, for example if a load is carried on a bracket on the side of a column, or when the position of the line of thrust is known in an arch. In such cases e must be measured from the line of action of N to the centroid of the equivalent area of the cross section (the equivalent area being the area in which the reinforcement is replaced by an equivalent area of concrete). The centroid of a symmetrical cross section is on the centre-line. If the bending moments M and N are known, the eccentricity, measured about the centroid, is $e = \frac{M}{N}$. The remarks given later should be noted.

In a beam of homogeneous material, if A is the cross-sectional area, I the moment of inertia about the centroid, and y the distance from the centroid to the point of maximum stress, $c_m = \frac{N}{A} + \frac{Ny}{I}$. This well-known expression can be adapted to a reinforced concrete column by considering the area of the reinforcement as equivalent to an area of concrete m times as great. The most common cross section is a rectangle in which $A_c = A_t$, and only this case will be considered.

Referring to *Fig.* 104 (*a*),

$A = BD + (m - 1)(A_c + A_t)$, or $= BD[1 + 0{\cdot}01p(m - 1)]$.

$M = Ne = Ne_1D$. $y = 0{\cdot}5D$.

$$I = \frac{BD^3}{12} + (m - 1)\frac{pBD}{100}\left(\frac{D}{2} - f_1D\right)^2, \text{ or } = BD^3[\tfrac{1}{12} + 0{\cdot}01p(m - 1)(0{\cdot}5 - f_1)^2].$$

Therefore

$$c_m = \frac{N}{BD}\left\{\frac{1}{1 + 0{\cdot}01p(m - 1)} + \frac{e_1}{2[\frac{1}{12} + 0{\cdot}01p(m - 1)(0{\cdot}5 - f_1)^2]}\right\}.$$

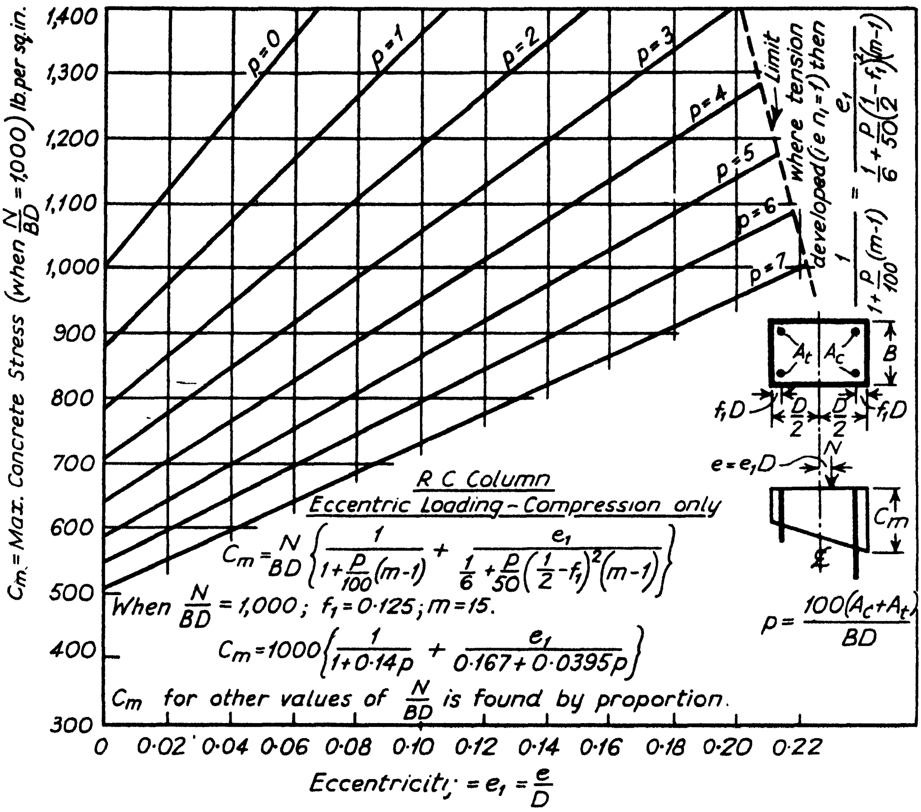
If $m = 15$, and $f_1 = 0{\cdot}125$, $c_m = \frac{N}{BD}\left(\frac{1}{1 + 0{\cdot}14p} + \frac{e_1}{0{\cdot}167 + 0{\cdot}039p}\right)$.

Curves relating p, e_1, and c_m can be plotted if a value, say, 1000 lb. per square inch, is assumed for $\frac{N}{BD}$; such curves are given in *Fig.* 105. For a given value of $\frac{N}{BD}$, c_m can be derived by multiplying the value read from the curves by $\frac{N}{1000BD}$, because c_m is directly proportional to $\frac{N}{BD}$.

Since the minimum compressive stress in the concrete is $\frac{N}{A} - \frac{My}{I}$, the curves end where $\frac{1}{1 + 0{\cdot}14p}$ equals $\frac{e_1}{0{\cdot}167 + 0{\cdot}039p}$. Beyond this point tensile stress is developed; this condition affects the magnitude and distribution of the stresses since it is assumed that concrete cannot resist tension.

CASE II. TENSILE STRESS DEVELOPED.—The notation is the same as for Case I with additional symbols [*Fig.* 104 (*b*)] as follows:

n_1D, the depth of the neutral axis.
C_c, the total compressive force in the concrete.
t_c, the stress in the compressive reinforcement.
t, the tensile stress in the tensile reinforcement.
T, the total tensile force in the tensile reinforcement.
C_s, the total compressive force in the compressive reinforcement.
$c_s = mc_c$.



$$C_m = \frac{N}{BD}\left\{\frac{1}{1+\frac{p}{100}(m-1)} + \frac{e_1}{\frac{1}{6}+\frac{p}{50}\left(\frac{1}{2}-f_1\right)^2(m-1)}\right\}$$

$$C_m = 1000\left\{\frac{1}{1+0\cdot14p} + \frac{e_1}{0\cdot167+0\cdot0395p}\right\}$$

$$e_1 = \frac{1}{1+\frac{p}{100}(m-1)} = \frac{1}{\frac{1}{6}+\frac{p}{50}\left(\frac{1}{2}-f_1\right)^2(m-1)}$$

FIG. 105.

Since stress is proportional to the distance from the neutral axis,

$$t = \frac{D - n_1D - f_1D}{n_1D}c_m m = \frac{1 - n_1 - f_1}{n_1}c_m m,$$

and

$$t_c = \frac{n_1D - f_1D}{n_1D}c_m m = \frac{n_1 - f_1}{n_1}c_m m.$$

Since the algebraic sum of the internal forces equals the external forces, $N = C_c + C_s - T$. The total compressive force in the concrete, C_c, is $\frac{Bn_1Dc_m}{2}$

if the concrete displaced by the compressive reinforcement is included. The total forces in the compressive and tensile reinforcement respectively are

$$C_s = \frac{pBD}{200} \cdot \frac{(n_1 - f_1)}{n_1} c_m m, \text{ and } T = \frac{pBD}{200} \cdot \frac{(1 - n_1 - f_1)}{n_1} c_m m.$$

The expression for C_s ignores the fact that the concrete displaced by compressive reinforcement has already been included in calculating C_c; a more accurate expression is obtained if $(m - 1)$ is substituted for m.

By substituting for C_c, C_s, and T in the expression for N and transposing,

$$c_m = \frac{N}{BD}\left[\frac{200n_1}{100n_1^2 + p(n_1 - f_1)m - p(1 - n_1 - f_1)m}\right] \quad . \quad . \quad (1)$$

Since the algebraic sum of the moments of the internal and external forces about, say, the centre-line of the cross section must equal nothing,

$$Ne_1D = C_c\left(\frac{D}{2} - \frac{n_1D}{3}\right) + C_s\left(\frac{D}{2} - f_1D\right) + T\left(\frac{D}{2} - f_1D\right).$$

Substituting for C_c, C_s, and T, and transposing,

$$c_m = \frac{Ne_1}{BD}\left[\frac{600n_1}{100n_1^2(1{\cdot}5 - n_1) + 3pm(n_1 - f_1)(0{\cdot}5 - f_1) + 3pm(1 - n_1 - f_1)(0{\cdot}5 - f_1)}\right] . \quad (2)$$

From (1), assuming a constant value of $\frac{N}{BD}$, curves can be drawn, relating c_m to n_1 for various values of p, from which values of c_m for particular values of n_1 can be obtained for substitution in (2) to calculate values of e_1 for the corresponding values of c_m and p. The procedure is to assume values of n_1 and p and calculate the corresponding values of c_m. For these values of c_m and the corresponding values of n_1 and p, e_1 can be calculated from (2). Curves can then be drawn relating c_m to e_1. If $\frac{N}{BD}$ is assumed to be 1000 lb. per square inch, values of c_m can be readily obtained by proportion, and the curves used to determine a suitable section in which the safe working stresses are not exceeded. The curves should not be used beyond points where $n_1 = (1 - f_1)n_e$, where n_e is the value of n_1 that is obtained when the permissible stresses in the concrete and reinforcement are simultaneously produced, otherwise the safe working stress in the reinforcement will be exceeded. Neither should they be used beyond the points where $n_1 = 1$, which is the point where tensile stresses cease to occur.

The curves in *Figs.* 106 and 107 are plotted for the common case of $m = 15$, $f_1 = 0{\cdot}125$ (a safe limiting value), and $\frac{N}{BD} = 1000$ lb. per square inch (a convenient value for obtaining c_m by proportion). Substituting these values in (1),

$$c_m = \frac{200{,}000n_1}{100n_1^2 + 30pn_1 - 15p}.$$

Substituting the same values in (2),

$$c_m = \frac{600{,}000n_1e_1}{150n_1^2 - 100n_1^3 + 12{\cdot}7p}.$$

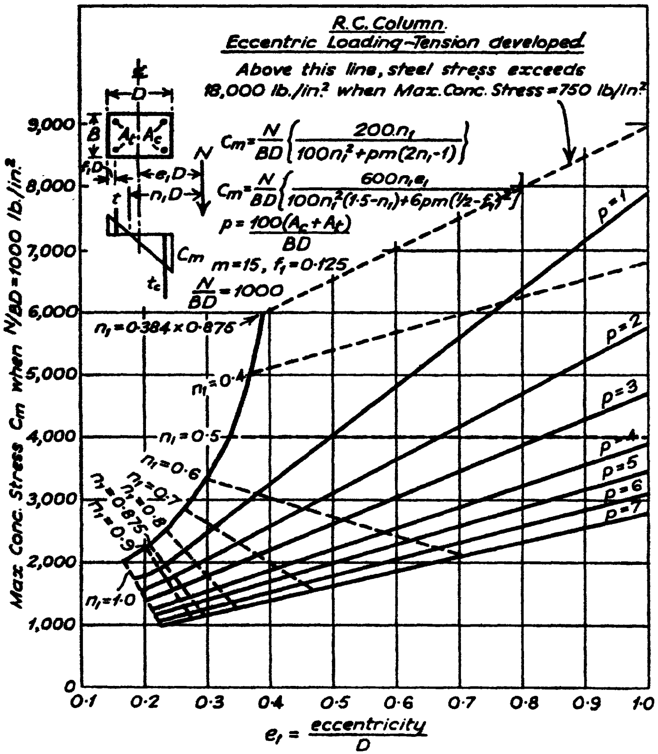


Fig. 106.

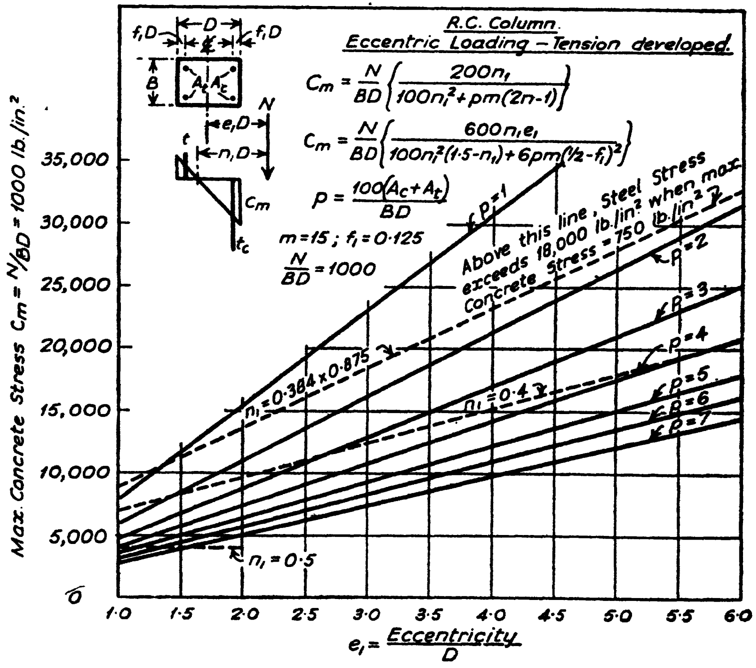


Fig. 107.

The line of action of N has been assumed to be distant $e = M \div N$ from the centre-line of the cross section as is commonly done when bending and direct stresses are developed. It is more accurate theoretically, however, to measure e from the centroid of the stressed area, when the bending moment is given and not the position of the line of action of the force, but the common method of measuring e from the centre-line of the section is sufficiently accurate if e is large compared with D, say, if e_1 exceeds 1·5; but if e_1 is less than unity the theoretical inaccuracy may be considerable and the error is not on the safe side. It is recommended that, as a useful exercise, the student should derive the more accurate formulæ involving $(m - 1)$ and the true value of e, and draw curves, similar to those in *Figs.* 106 and 107, for other stresses, say, those specified in the British Standard code. It should be noted that when tensile and compressive stresses are developed the stressed area is only the area over which the compressive stresses are induced and the area of the tensile reinforcement; the areas of the tensile and compressive reinforcement are increased to m and $(m - 1)$ times their actual values respectively, as described earlier, to give their equivalent values when determining the position of the centroid. In deriving the accurate formulæ it is still most convenient to take moments about the centre-line.

BEAMS SUBJECTED TO END COMPRESSION OR TENSION.—Suitable cross sections are best found by assuming the dimensions and the stresses in the concrete and reinforcement, and calculating the area of reinforcement required to balance all forces acting on these cross sections. The notation is as for the analysis of bending stresses in beams with compressive steel, as given in Chapter III and in *Fig.* 108; it should be noted that d is the effective depth and not the overall depth of the beam.

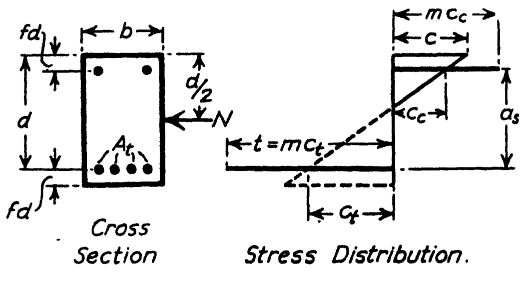


FIG. 108.

Let the resultant bending moment at the cross section due to loads acting normal and parallel to the axis of the beam be M, and let the resultant thrust parallel to the axis of the beam be N acting axially (or more accurately, at the centroid of the stressed area), since the moment due to the eccentricity of N is included in M. The moment of the compressive resistance of the concrete about the tensile reinforcement is Rbd^2, the value of R being obtained by assuming values for the maximum stress in the concrete c and the tensile stress in the reinforcement t from the expression

$$R = \frac{cn_1}{2}\left(1 - \frac{n_1}{3}\right), \text{ where } n_1 = \frac{mc}{mc + t}.$$

In an example, if the thrust is small, assume c to be the permissible stress in the concrete and t the permissible stress in the reinforcement. If the thrust is great, assume c to be the permissible stress in the concrete and t to be equal to mc.

The moment of the force in the compressive reinforcement about the tensile reinforcement is $A_c c_c(d - fd)m$, where $c_c = \frac{c(n_1 - f)}{n_1}$. The moment of N about the bottom reinforcement is $N\left(\frac{d}{2} - \frac{fd}{2}\right)$. Since the sum of the external moments must equal the sum of the moments of the internal forces, the moments being taken in this instance about the tensile reinforcement,

$$N\left(\frac{d}{2} - \frac{fd}{2}\right) + M = Rbd^2 + A_c mc_c(d - fd).$$

$$\text{Therefore, } A_c = \frac{\frac{Nd}{2}(1 - f_1) + M - Rbd^2}{mc_c d(1 - f_1)}.$$

The terms in this expression are either known or are obtained from the assumptions.

Since the algebraic sum of the internal and external forces must be nothing, the resultant internal tensile force, $T = A_t t$, is $\frac{Rbd^2}{a} + A_c mc_c - N$.

$$\text{Therefore, } A_t = \frac{1}{t}\left(\frac{Rbd^2}{a} + A_c mc_c - N\right).$$

If the values of A_c and A_t determined by these expressions are unsuitable, the assumed dimensions and the assumed stress in the tensile reinforcement should be adjusted and the calculation repeated. If A_c is negative the compressive reinforcement can be omitted, and the actual compressive stress in the concrete will be less than that assumed. For a tee-beam or beam of other special cross section a similar procedure can be followed using a suitable value of R for the assumed stresses and cross section. The assumed stresses will be theoretically developed with the applied loads provided that the stresses and external forces form a balanced system obtained by using a suitable area of compressive and tensile reinforcement. In the case of beams subjected to a direct tensile force the same procedure can be followed if the sign of N is reversed.

Long Columns Eccentrically Loaded.

To calculate exactly the required cross section of a long column, the deflection δ of the column due to bending under the failure load must be assumed and added to the original eccentricity. The deflection δ and the assumed total eccentricity for an assumed cross section must then be calculated by separating bending stresses from direct stresses in a similar way to that used for prestressed beams in Chapter VI. If the assumed and calculated deflections agree and the stresses at failure load do not exceed the strengths of the materials, the assumed cross section is correct. If not, the calculation must be repeated with other assumed dimensions until agreement is reached or the strength of the column is greater than is required.

The following is an approximate method. From the curves derived for columns subjected to bending, the eccentricity, which produces in a short column an increase of stress, indicated by the reduction of the permissible load for a long column, can be determined. The sum of this eccentricity, the eccentricity of the load, and the calculated deflection due to bending, caused by the original eccentricity, are determined. It is also advisable to add a further eccentricity of, say, 1 in. for every 20 ft. of height, for possible errors in construction. The column should then be designed for direct compression combined with a bending moment equal to the direct load multiplied by the total eccentricity.

CHAPTER V

SECONDARY EFFECTS

Creep.

CREEP of concrete is the yield which takes place over a period of time under compressive stress, and is greater for weak concrete than for strong concrete. It probably reaches a limiting value in one to two years. The influence of creep on the distribution of the stresses in ordinary structural members of reinforced concrete is commonly ignored. When the full effect of creep has developed, the effective value of the elastic modulus E_c for concrete may be halved in the case of strong concrete or divided by 4 in the case of weak concrete. The variation

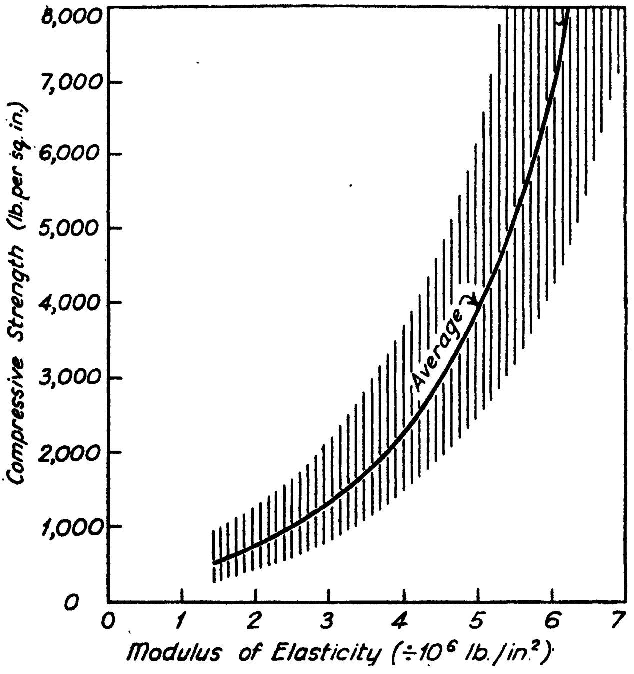


FIG. 109.

of E_c for concretes of different strengths is shown in *Fig.* 109, the curve in which is based on investigations in France. The value of the modular ratio m may therefore be doubled or quadrupled due to creep under dead load in course of time; this is shown by the curves in *Fig.* 110 which are based on similar curves given in " Explanatory Handbook on the Code of Practice (D.S.I.R.) for Reinforced Concrete " by W. L. Scott and W. H. Glanville. The curves for R in *Fig.* 98, Chapter III, show that the moment of resistance of a beam is

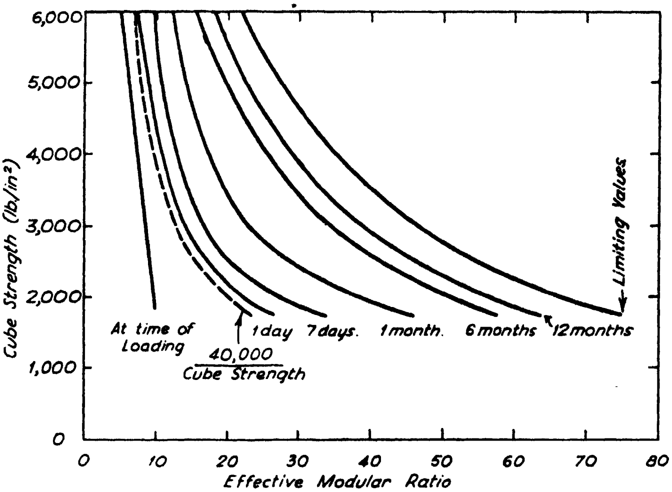


FIG. 110.

increased by increasing the value of *m*. Creep therefore causes no reduction in the factor of safety of the resistance of the concrete in a beam, but a slight reduction may occur in the resistance of the steel due to the smaller lever arm, as discussed under "Factor of Safety" in Chapter III.

The effect of the creep of concrete in a short column is to transfer load from the concrete to the reinforcement. This is partly taken into account by assuming a stress of 13,500 lb. or 18,000 lb. per square inch in the reinforcement regardless of the stress in the surrounding concrete. The factor of safety of a short column is not reduced by creep since, under the ultimate load, the load is so distributed, due to the yield of the steel, that the steel and concrete reach their yield stresses simultaneously before failure takes place. In a long column the effect of creep is to cause increased lateral deflection and consequently an increase in the eccentricity of the load. Professor Ross has shown ("Concrete and Constructional Engineering," September, 1948) that the amount of the reduction of load or stress in long columns as recommended in the D.S.I.R. code (1934), and subsequently in the British Standard code (1948), should be slightly increased for this reason (see *Fig.* 103 in Chapter IV). An increase of eccentricity also occurs in the compression flange of a long narrow beam. In both cases it is advisable to assume an initial deflection of about $\frac{1}{2}$ in. for every 20 ft. of length to allow for possible errors in the shuttering. In prestressed concrete, the reduction, due to creep, in the prestress is important and must be taken into account as is described in Chapter VI.

Shrinkage.

Shrinkage of concrete is caused by the reduction in volume of the cement paste due to the chemical combination of the water and the cement, by the evaporation of the water during the hardening of the concrete, and by the drawing together of the particles of aggregate by the internal tension of the cement paste. The aggregates are only very slightly compressed. Concrete develops about half of its total shrinkage during the early stages of setting and hardening. A concrete slab, which is not generally kept wet as in a reservoir, has a coefficient of linear shrinkage of about 0·00025 after 28 days, 0·00035 after 3 months, and 0·0005 after 12 months, when it becomes almost stable. If alternate bays of a long slab harden before the intermediate bays are concreted, the total effective shrinkage may be reduced by about 25 per cent.

If the concrete is free to contract no excessive internal tension develops, but generally friction with the ground or anchorage to a rigid structure restrains contraction, and unless contraction joints, or sufficient reinforcement, are provided cracking takes place. In a slab requiring no reinforcement for structural purposes in the direction in which shrinkage tends to occur, it is usual to provide reinforcement in this direction near each face, the area of such reinforcement being about 0·1 per cent. of the area of the concrete or 20 per cent. of that of the main reinforcement. Shrinkage joints are often provided by dividing the slab into lengths of 15 ft. to 20 ft. The joints may be joggled to resist shearing forces and packed with a plastic material to make them watertight. In ducts and tunnels the joints may be butt joints across which no reinforcement passes and which form a straight tidy gap when shrinkage occurs. If it is necessary to avoid closely-spaced joints, either the concrete member must be supported so that it can contract freely (for example on flexible piles), or reinforcement must be provided to distribute the shrinkage cracks uniformly throughout the slab so that the width of each crack will be negligible. A reliable method of calculating the amount of reinforcement required has not yet been established, but some idea of the amount can be obtained from the following.

The maximum tensile force which can develop at any section is $A_c c_t$, where A_c is the cross-sectional area of the concrete and c_t the tensile strength of the concrete. If c_t is 400 lb. per square inch, and the permissible tensile stress in

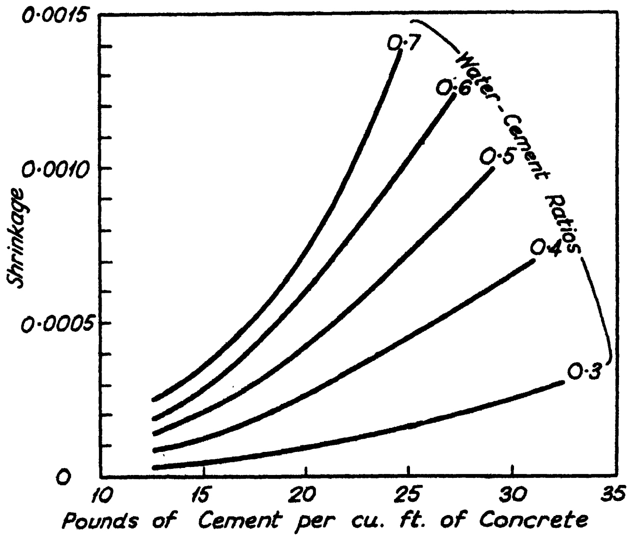


Fig. 111.

the reinforcement is 30,000 lb. per square inch (a high stress is suggested since cracking is not a serious failure, and the steel can only fail if the concrete is stronger), and if the area of the reinforcement is A_t, then for the force in the concrete to be equal to that in the reinforcement $A_t \times 30{,}000 = A_c \times 400$, and the amount of reinforcement is $\frac{100A_t}{A_c} = \frac{100 \times 400}{30{,}000} = 1{\cdot}33$ per cent. If this amount of reinforcement is provided wide cracks cannot occur.

In a beam or column subjected to constant stress, shrinkage and creep take place at the same time. Some authorities think that creep is merely the process of shrinkage assisted by externally applied compression. The amount of shrinkage which develops in a given time increases with an increase in the amount of cement and water in the concrete and the dryness of the atmosphere; this is seen by the curves in *Fig.* 111, which are based on tests made in America.

Plastic Yield.

Concrete has no definite yield-point like that of mild steel, and from repeated loading tests on a cube a series of stress-strain curves similar to those in *Fig.* 112 is obtained. A certain amount of plastic yield therefore takes place under each compression, since recovery is not complete. When the cube is loaded to

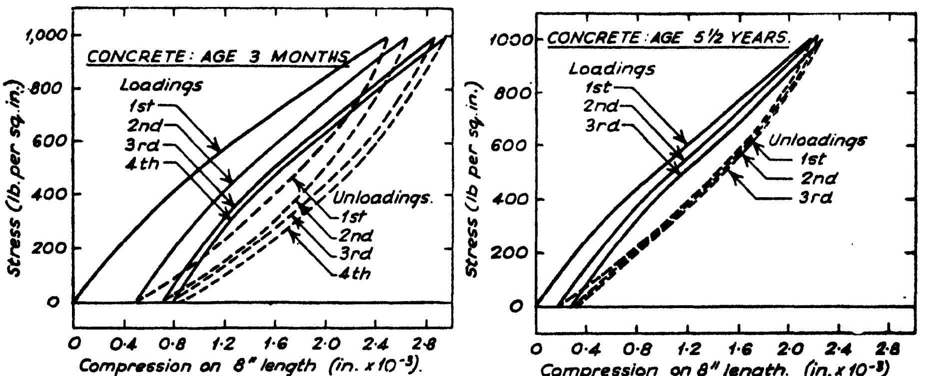


FIG. 112.—TYPICAL STRESS-STRAIN CURVES FOR CONCRETE.

destruction, the stress-strain curve, while having no definite yield-point, becomes almost flat before failure takes place. With the design methods described in Chapters III and IV the stresses due to the working loads in some cases may be excessive at first, but after one to two years the concrete becomes compact because of unrestrained shrinkage and creep, or due to plastic yield caused by repeated applications of the live load in a shorter period. The stresses are reduced to such values that the factor of safety of the structure is adequate even though at first the permissible stresses are exceeded.

Torsion.

The resistance of concrete to torsion is low on account of the high diagonal tensile stresses produced by twisting actions. Where possible torsional strains should be avoided. Twisting, however, can occur in beams supporting cantilevered hoods (*Fig.* 113) if the hoods are not continuous with the floor slabs; twisting also occurs in bow-girders, that is beams that are curved in plan. In a rectangular section the maximum intensity of shearing stress s_t due to twisting occurs at the middle of the long sides a and, since on one side of the beam the shearing stress acts in the same direction as the ordinary shearing stress due to the load, the stresses due to both causes must be added together. If b is the short side of the section, the moment of resistance to twisting is $T = \frac{a^2 b^2 s_t}{3a + 1{\cdot}8b}$, as is shown in text books on the strength of materials.

In a tee-beam the resistance to twisting contributed by the slab is negligible.

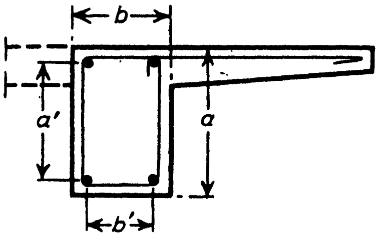


FIG. 113.

Stirrups must be closed, continuous with the reinforcement in the slab causing torsion (*Fig.* 113), and sloped at 45 deg. in both directions to provide diagonal reinforcement in the direction of tensile stresses on each side of the beam. Additional longitudinal reinforcement should be distributed equally in the four corners of the beam. In the accompanying table the amount of additional reinforcement (expressed as a percentage) is the sum of the area of the longitudinal steel expressed as a percentage of the cross-sectional area of the member, and the volume of the diagonal closed stirrups expressed as a percentage of the volume of the member. The additional reinforcement is extra to that required

Safe working stress s_t	Quality of concrete	Additional reinforcement to resist torsion
60	Ordinary	Nominal amount of longitudinal reinforcement; closed stirrups.
75	High-grade	Do.
100	Ordinary High-grade	2 per cent. 1 per cent.
120	Ordinary High-grade	2 per cent. 1 per cent. } Helical binding (instead of stirrups) at 45 deg.

for resistance to bending or direct compression. The safe working stress must not be exceeded when the bending and torsional shearing stresses are added together. The longitudinal bars should be stiff enough to provide a rigid frame of reinforcement and they should constitute about one-half of the total torsional reinforcement required, the remaining amount of torsional reinforcement being provided by stirrups or helical binding. A formula sometimes used on the Continent to calculate the area A_h of helical binding to resist torsion is (referring to *Fig.* 113)

$$A_h = \frac{T\sqrt{2}(a' + b')p}{2a'b's_t},$$

where p is the pitch of the helical binding and s_t is 180 lb. per square inch for high-grade concrete. The amount of longitudinal reinforcement provided is 0·2 per cent.

Temperature Stresses.

In structures such as chimneys, the difference T between the temperatures of the inside and outside faces causes tensile stresses on the low-temperature face if the member is unable to take up freely the deformation due to expansion on the other side. Reinforcement must be provided to resist the tension, the area of such reinforcement being calculated on the assumption that the distribution of stresses in the concrete is as shown in *Fig.* 114. This distribution of stresses corresponds to the variation of strain shown in the same diagram, due to expansion $\frac{\mathcal{E}T}{2}$ being prevented from taking place, where $\mathcal{E}$ is the coefficient of thermal expansion of concrete (about 1×10^{-5} per degree C.). If E_c is the

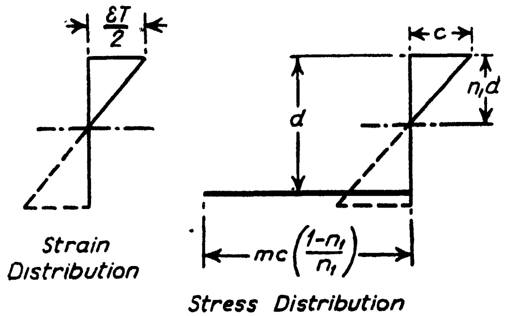


FIG. 114.

modulus of elasticity of concrete, c the stress in the concrete at the outside of the beam or slab, and t the permissible stress in the reinforcement, and since stress is "strain × E," $c = \frac{\mathcal{E}T}{2}E_c$. Since $M = \frac{n_1}{2}\left(1 - \frac{n_1}{3}\right)cbd^2$ and the area of tensile reinforcement is $A_T = \frac{M}{ta_1d}$, by substitution for M and c and $a_1 = 1 - \frac{n_1}{3}$

$$A_T = \frac{n_1\mathcal{E}TE_cbd}{4t}.$$

The stress c must be added to the stresses due to the load. The value of n_1 depends on the percentage of temperature reinforcement provided in the same way as for ordinary bending. It is seen that A_t decreases as d decreases.

CHAPTER VI

PRESTRESSED CONCRETE

METHODS of inducing compression in a concrete member before it is loaded, known as prestressing, are used to great advantage for precast concrete beams of long span in which the dead load forms a high proportion of the total load. The use of high-strength concrete and high-tensile steel is the most effective way of reducing the dead load, and consequently the bending moment, of a beam, but high-tensile steel cannot be used at a high working stress without excessive cracking of the concrete and deflection of the beam unless the concrete is prestressed. When a load is applied to a prestressed beam a greater increase in the compressive stress in the concrete can take place before failure, so that when failure is due to crushing of the concrete a prestressed beam can resist a higher bending moment than an ordinary beam of the same size and concrete strength. The effect of prestressing may be regarded as increasing the ultimate moment of resistance of the concrete by lowering the neutral axis for the stresses produced by bending after cracking has occurred. Prestressing also increases the resistance to shearing forces, and in some cases reduces the amount of reinforcement required to resist shear.

Prestressing is more reliably carried out and high-strength concrete is best made in factories. Prestressed concrete is therefore becoming increasingly used for precast members which might otherwise develop serious cracks during handling or under load, especially if they are designed for high stresses in order to reduce the weight. Prestressing is being successfully used in the manufacture of reinforced concrete sleepers. Ordinary reinforced concrete sleepers quickly fail in service due to the rapid opening and closing of cracks when loads are applied and removed in rapid succession. It has been found that some prestressed sleepers do not crack, and disintegration due to the opening and closing of cracks is therefore avoided. Hollow beams for bridges and concrete piles constructed of precast blocks are other examples of the use of prestressed members. It is not often possible to predict the length required for a pile, and it is a great convenience to be able to extend a pile during driving by adding blocks; the handling of long piles is avoided and, provided the lateral support of the ground is adequate, the prestressing cables can be removed upon completion of driving and used again. Other examples of the application of prestressed concrete are dams, caissons, and runways. Prestressing is also used in the construction of tanks, the elimination of ring tension and shrinkage stress in the wall of a cylindrical tank preventing cracks and ensuring liquid-tightness. Prestressed concrete thin-slab vaulted roofs have also been constructed.

Another advantage of prestressed concrete is that the concrete and the steel are severely tested during the prestressing operation, and a lower factor of safety is justified. The permissible working stress in the concrete is generally one-third of the compressive strength, thus allowing a margin to cover the risk of poor concrete occurring at a critical section. The risk is reduced by prestressing, because the stress induced in the concrete during the prestressing operation may be 50 per cent. to 75 per cent. of its compressive strength.

Methods of Prestressing Concrete.

There are three principal ways in which concrete is prestressed, namely, by stretched wires fixed at their ends, by stretched wires gripped by the concrete, and by applying an external load ; experiments are also being made in France with expanding cement concrete.

End-anchored wires are used in long-span concrete beams. The reinforcement is separated from the surrounding concrete by a sheath or other wrapping which allows the wires to move freely during stretching. The wires are anchored at the ends of the beam either by fixing to an anchor plate, or by other devices such as wedging with concrete cones. A disadvantage of this method of prestressing is that the wires must be protected from corrosion by forcing cement grout into the sheath. The grout provides protection against corrosion and also provides a bond between the wires and the sheath and thus with the concrete ; it also supplements the resistance of the wires to slip, without which the security of the wires would depend entirely on the permanence of the end anchorages. With proper care these disadvantages do not involve serious risk. The object of allowing the wires to move while the prestress is being established is to make it possible to prestress the beam after most of the shrinkage of the concrete has taken place, and so reduce the loss of prestress due to shrinkage and to eliminate loss of prestress due to elastic contraction of the concrete. In some cases it is possible to increase the prestress when the beam begins to carry its own weight, the prestress thus relieving the member of a considerable proportion of compressive stresses due to its weight.

In the concrete-gripped type of prestressed concrete, the wires are stretched before concreting. When it has hardened the concrete grips the wires as in ordinary reinforced concrete, except that the grip may be increased slightly when the wires are released from the stretching device on account of a slight shortening and swelling of the wires that occur as the concrete member shortens under compression. The shortening of the concrete member is also a slight disadvantage of this method of prestressing, since the consequent shortening of the stretched wires is accompanied by a reduction in the prestress, in addition to that due to shrinkage of the concrete. On the other hand it is not necessary to provide anti-corrosive treatment, the wires are gripped throughout their length, and the security of the beam is not dependent upon anchorage of the wires at the ends. Prestressing may increase the ultimate moment of resistance of the concrete in a beam ; failure generally occurs by the yielding of the wires. If, therefore, sufficient prestress can be induced by this method to eliminate cracking under a small overload, it is probably better than the end-anchored method, since there are no anchorages to fail. The application of the end-anchored method when prestressing by compressing together precast blocks is often simpler to carry out on the site than the concrete-gripped method. It is important to use a strong concrete with the concrete-gripped method since the wires have been known to slip in concrete of average strength.

An example of a concrete member being prestressed by the application of an external load is the jacking of arches during construction. Otherwise there are few opportunities of adopting this method which entails prestressing an arch slab or arch rib spanning between two unyielding abutments. A loaded column can be regarded as prestressed against stresses due to bending moments, as also can an arch if it is uniformly compressed by a large dead load.

The use of expanding-cement concrete for inducing a prestress was introduced by M. Lossier and is still in the experimental stage. The prestress is caused by the concrete swelling instead of shrinking during setting and hardening. If this process can be developed so that the initial prestresses are sufficiently high to prevent cracking occurring under load, it may become a useful method of construction.

Various means of inducing the prestress and anchoring the wires are in use. Hydraulic jacks, the pressure gauges of which simplify the measurement of the force in the wires, are most commonly used. Turnbuckles or a travelling winding and tensioning machine are used for stretching the circumferential reinforcement of cylindrical tanks; steam-heating has also been used. Obviously there is a means of prestressing which is most suitable for each particular case. In the manufacture of sleepers and similar mass-produced small products prestressed by the concrete-gripped method, an efficient process is to stretch the high-tensile wires in long lengths through several moulds placed end to end. When the concrete has sufficiently matured the wires between the moulds are cut by an oxy-acetylene flame.

Principles of Prestressed Concrete.

In determining the distribution of stresses in a prestressed concrete beam it is essential to divide the stresses into the following categories: (1) Initial stresses due to prestressing; (2) Losses of prestress (*a*) in transferring the prestressing force to the concrete, (*b*) due to the concrete contracting under compression, (*c*) due to shrinkage of the concrete, (*d*) due to creep of the concrete and steel; and (3) Stresses due to dead load and live load. The loss of prestress in (*b*), and to some extent that in (*c*), can be reduced by using the end-anchored method.

It is necessary to ensure that at each stage of manufacturing, transporting, erecting, and loading, the stresses in the beam are not excessive. The resultant stresses at any stage are calculated by superimposing the various stress-distribution diagrams, that is by adding algebraically the stresses in those of categories (1), (2), and (3) which apply at each stage. It will be seen later that it is an advantage when establishing the prestress to subject the steel and the concrete to stresses considerably in excess of the working stresses permissible in ordinary reinforced concrete. The steel and the concrete are then severely tested before the beam is loaded, and this, as stated before, justifies using a lower factor of safety. The maximum compressive stress in the concrete under full working load may exceed the ordinary permissible compressive stress provided it is accepted that the factor of safety in regard to the concrete can be reduced to, say, 2·5 on account of the conditions of manufacture. The stress in the wires is, however, only slightly increased when the load is applied to the beam. Generally it is necessary to apply at least the ultimate load (which is, say, 2½ times the working load) to the beam in order to cause the steel to yield. A reasonable factor of safety is therefore provided even though under ordinary working conditions the stress in the wires exceeds considerably that commonly permissible.

In analysing the distribution of stress the basic principles and assumptions used in the case of reinforced concrete beams apply and suffice to determine the position of the neutral axis, the maximum stresses, etc., in terms of the bending moment, the dimensions of the beam, and the percentage of steel; in addition shrinkage and creep must be taken into account. The basic principles are as follows.

(1) At any cross section of a beam subjected to simple bending the algebraic sum of the internal forces acting normal to and on one side of the section is zero. In addition, in a prestressed beam the prestressing force in the stretched wires acts like an external force on the cross section, causing equal and opposite internal forces in the concrete.

(2) At any cross section the algebraic sum of the moments, about any point, of the internal forces acting normal to and on one side of the section equals the algebraic sum of the moments, about the same point, of the external loads and forces acting on the beam. In addition, in a beam the prestressing force acts as an external load inducing an additional moment and thrust, or only a thrust, on the section.

The following assumptions of the properties of concrete and steel are made. (1) The tensile strength of concrete is about one-tenth of its compressive strength. (2) Steel and concrete are elastic, and the stress-strain curves for these materials are straight lines within the elastic range. (3) Plane sections before bending remain plane after bending has taken place, and the strain at any point is directly proportional to the distance of the point from the neutral axis. (4) In concrete-gripped wires no slip takes place between the concrete and the wires provided that the reinforcement is extended sufficiently beyond the cross section under consideration to develop the full grip resistance. In the case of end-anchored wires, the wires are free to slip relative to the concrete between the anchorages, but no movement of the anchorage relative to the adjacent concrete takes place. (5) The coefficients of thermal expansion for steel and concrete are the same. No internal stresses, therefore, develop due to variations in temperature, which are uniform throughout the beam. (6) Loss of prestress due to local yielding of the anchorage at the time of releasing the stretching devices must be taken into account. (7) Loss of prestress due to shrinkage of the concrete must be taken into account. In the case of concrete-gripped wires the shrinkage may be considerable. In the case of end-anchored wires most of the shrinkage of the concrete will have taken place before the prestressing force is transferred to the beam. Any shrinkage that occurs after this stage must be taken into account. (8) Loss of prestress due to creep over a long period must be taken into account. (9) The redistribution of bending or shearing stresses which takes place when cracking occurs must be taken into account. The object of prestressing is to eliminate cracks under the working load or under a slight over-load; but generally cracks occur under increasing load before the ultimate load is applied and cause a sudden increase in the stress in the wires, which increase must be taken into account to ensure that the steel does not yield until the ultimate resistance of the concrete is reached.

When designing a prestressed beam it is important to consider the magnitude and distribution of stresses under the ultimate load as well as under the working load. It is possible to design a beam which under working load develops a stress of, say, 750 lb. per square inch in the concrete, but which under double the working load develops a stress of 3000 lb. per square inch. The reason is explained later. It should be emphasised that, although an ordinary reinforced concrete beam designed for the working stresses is known for certain to have an ample factor of safety, in a prestressed beam the stresses in the concrete increase at a greater rate than the applied load. It is therefore necessary to base designs upon the ultimate load in addition to ensuring that at various stages in the construction excessive stresses in the concrete do not occur. For the stretched wires the reverse applies. Stresses in excess of the permissible working stresses can be induced with no risk of the yield stress being exceeded before the ultimate load is applied.

By comparing the expression for the moment of resistance of a prestressed beam with that for an ordinary reinforced concrete beam of the same dimensions, it is seen that an increase of about 30 per cent. can be obtained in the ultimate moment of resistance of the concrete under favourable conditions. This increase is valuable in the case of long-span beams in which the ratio of the live to the dead load is small. By far the greatest saving in weight, and therefore cost, is made by using concrete and steel of high strength, but it is necessary to induce a prestress to avoid excessive cracking. It is not easy to determine under what circumstances it is best to use prestressed concrete, but in a particular case it is not difficult to estimate the saving in cost made by using a smaller quantity of high-grade materials at a higher price compared with ordinary materials at a lower price. From this saving must be deducted the cost of the jacks, anchorages, and other equipment, and of carrying out the prestressing process. When a large number of products is produced under factory conditions considerable saving can be made in the cost of materials and equipment and also in the cost of transport, handling, and stacking.

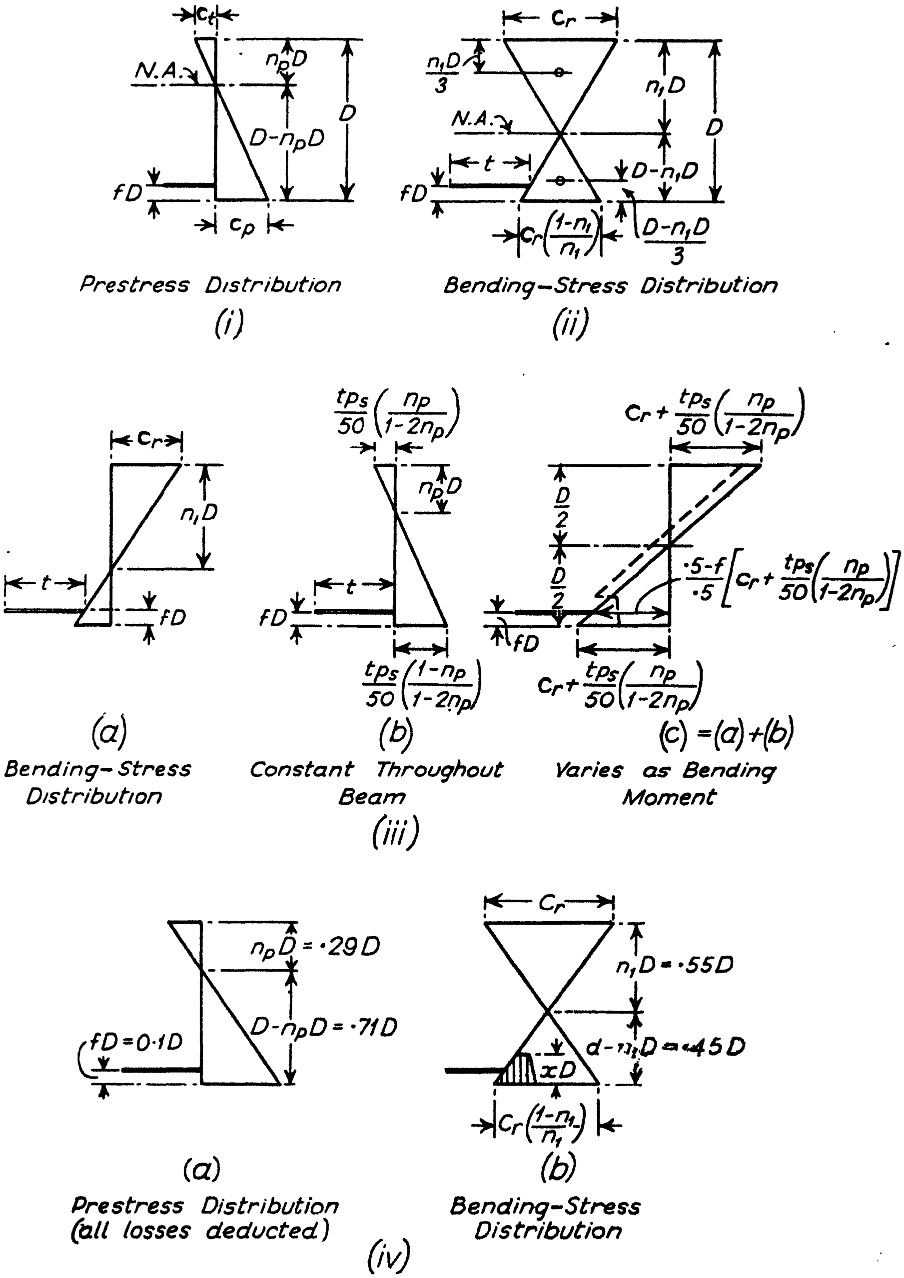
The fire-resistant properties of prestressed concrete are not yet established.

Theory of Rectangular Prestressed Concrete Beams.

In the following the stresses in a rectangular concrete beam due to prestressing, loss of prestress upon release of the stretching device and transfer to the anchorages, shrinkage, creep, and applied bending moment are considered. The distribution of the stresses at any cross section of the beam is obtained by superimposing the stresses due to these causes. If the stress due to the applied bending moment added algebraically to the stresses due to the remaining causes exceeds the tensile strength of the concrete, the distribution of stress must be adjusted to take into account the failure of the concrete in tension over the cracked part of the beam. Formulæ for the various stresses [*Fig.* 115 (i) and (ii)] are derived in the following, and apply particularly to beams prestressed by an end-anchored method. The modifications necessary for the concrete-gripped method are described later.

NOTATION.—

A_t, area of the stretched wires or reinforcement.
b, width of the beam.
c_p, maximum compressive stress in the concrete due to prestress.



Prestress Distribution
(i)
Bending–Stress Distribution
(ii)
(a)
Bending–Stress Distribution
(b)
Constant Throughout Beam
(c) =(a)+(b)
Varies as Bending Moment
(iii)
(a)
Prestress Distribution (all losses deducted)
(b)
Bending–Stress Distribution
(iv)

FIG. 115.

c_r, increase in stress in the concrete on compressive side of the beam due to an externally-applied bending moment.
c_{r1}, compressive stress in the concrete adjacent to the steel due to full prestress and dead load.
c_{r2}, loss of compressive stress in the concrete adjacent to the steel due to creep.
c_t, maximum tensile stress in the concrete due to prestress.
c_{td}, maximum available tensile prestress in the concrete due to the dead load relieving the compressive stresses.
c_u, compressive strength of the concrete.
D, overall depth of the beam.
E_c, E_s, modulii of elasticity of concrete and steel respectively; $m = \dfrac{E_s}{E_c}$.
fD, distance of centroid of the wires or of the reinforcement from the outside of the beam.
F_m, ratio of the average bending moment throughout the length of a beam to the maximum bending moment (= 0·67 for a uniformly-distributed load and 0·5 for a concentrated central load).
M_r, moment of resistance of the cross section.
n_1D, depth of the neutral axis for stresses due to the externally-applied bending moment only.
n_pD, depth of the neutral axis for prestress only.
p_s, percentage of wires or reinforcement $= \dfrac{100A_t}{bD}$.
t, increase in tensile stress in the wires or reinforcement due to externally-applied bending moment.
t_c, loss of tensile stress in the steel due to creep.
t_m, reduction of initial tensile stresses in the stretched wires due to elastic compression of the concrete.
t_p, initial tensile stress in the stretched wires.

Distribution of Prestresses.—By proportion [*Fig.* 115 (i)],

$$c_t = c_p\frac{n_p}{1 - n_p} \qquad . \quad . \quad . \quad . \quad . \quad (1)$$

The conditions for equilibrium are, (*a*) the moment of the total compression in the concrete minus the moment of the total tension in the concrete about the bottom edge of the cross section equals the moment of the total tension in the wires about the same edge, and (*b*) the total tension in the wires equals the total compression in the concrete minus the total tension in the concrete. Therefore,

$$\frac{c_p(D - n_pD)b}{2}\cdot\frac{(D - n_pD)}{3} - \frac{c_pn_p}{1 - n_p}\cdot\frac{n_pD\left(1 - \dfrac{n_p}{3}\right)Db}{2}$$
$$= \left[\frac{c_p(D - n_pD)b}{2} - \frac{c_pn_p.n_pDb}{2(1 - n_p)}\right]fD,$$

from which c_p, b, and D can be cancelled, giving

$$\frac{(1 - n_p)^2}{6} - \frac{n_p^2\left(1 - \dfrac{n_p}{3}\right)}{2(1 - n_p)} = f\left[\frac{1 - n_p}{2} - \frac{n_p^2}{2(1 - n_p)}\right],$$

from which $$n_p = \frac{1 - 3f}{3 - 6f} \qquad . \quad . \quad . \quad . \quad . \quad (2)$$

Therefore n_p depends only on f. When f is 0·1, n_p is 0·29, and, from (1), $c_t = 0{\cdot}41\ c_p$.

It should be noted that by placing the wires as closely as possible to the bottom of the beam the greatest ultimate strength is provided in the event of loss of prestress through slip, and also, for a given safe limiting precompression, the greatest pre-tension in the concrete is provided, which allows a maximum addition of compressive stress due to the externally-applied bending moment. When stretched wires are also in the top of the beam or elsewhere f applies to the position of the line of action of the resultant stretching forces.

The stress in the stretched wires can be derived for the condition (*b*), from which the total tensile force in the wires is

$$c_p b\left[\frac{D - n_p D}{2} - \frac{n_p . n_p D}{2(1 - n_p)}\right].$$

Since A_t is $\dfrac{p_s bD}{100}$, the stress t_p in the wires is $\dfrac{100c_p}{p_s}\left[\dfrac{(1 - n_p)}{2} - \dfrac{n_p^{\,2}}{2(1 - n_p)}\right]$.

That is

$$t_p = \frac{50c_p}{p_s}\left(\frac{1 - 2n_p}{1 - n_p}\right) \qquad (3)$$

When f is 0·1, n_p is 0·29, and t_p is $\dfrac{50c_p}{p_s}.\dfrac{0{\cdot}42}{0{\cdot}71} = 29{\cdot}5\dfrac{c_p}{p_s}$.

MOMENT OF RESISTANCE WHEN SUBJECTED TO AN INCREASE OF STRESS c_r IN THE CONCRETE AT THE TOP OF THE BEAM.—It is assumed that the tensile strength of the concrete is not exceeded, that is, the beam is not cracked; the depth of the neutral axis, $n_1 D$ [*Fig.* 115 (ii)] depends only on p_s for given values of m, f, and bending moment. The increase in the total compression is

$$\frac{c_r b n_1 D}{2} \qquad (4)$$

The reduction in the total compression below the neutral axis is

$$\frac{c_r(1 - n_1)}{n_1}.\frac{(1 - n_1)}{2}bD = \frac{c_r bD(1 - n_1)^2}{2n} \qquad (5)$$

The increase of the total tension in the wires is given by (4) minus (5), that is,

$$c_r bD\left(1 - \frac{1}{2n_1}\right) \qquad (6)$$

The moment of resistance of the section is the sum of moments of (5) and (6) about the centroid of (4) [see *Fig.* 115 (ii)], which is

$$\frac{c_r bD}{2n_1}(1 - n_1)^2\left[\left(D - \frac{n_1 D}{3}\right) - \frac{(D - n_1 D)}{3}\right] + c_r bD\left(1 - \frac{1}{2n_1}\right)\left(D - \frac{n_1 D}{3} - fD\right)$$

which reduces to

$$M_r = \frac{c_r bD^2}{6n_1}(3f - 1 + 3n_1 - 6n_1 f) \qquad (7)$$

When f is 0·1, $M_r = c_r bD^2\left(0{\cdot}4 - \dfrac{0{\cdot}11}{n_1}\right)$.

RELATION BETWEEN n_1 AND p_s, t, c_r, AND p_s.—*Fig.* 115 (iii *a*) shows the distribution of stresses due to applied bending moment alone (that is excluding the prestress). These stresses vary as the bending moment varies throughout the length of the beam.

The stresses in *Fig.* 115 (iii *b*) are those due to the stress t (caused by the externally-applied bending moment) considered as comparable with a pre-tension in the wires; these stresses are constant throughout the beam since the stress in any part of the steel in an anchored-end system is constant. The resultant stresses, which form a couple equal and opposite to the externally-applied bending moment, are shown in *Fig.* 115 (iii *c*) and result from an algebraic summation of (*a*) and (*b*). Although the stresses at (*b*) are constant throughout the length of the beam, since those at (*a*) vary, stresses at (*c*) vary throughout the beam.

As it is assumed that the tensile strength of the concrete is not exceeded, no cracking occurs to affect the stress distribution. From (3) the increase in compressive stress in *Fig.* 115 (iii *b*) is $\frac{tp_s}{50}\left(\frac{1 - n_p}{1 - 2n_p}\right)$. The increase in tensile stress in the concrete is $\frac{tp_s n_p}{50(1 - 2n_p)}$. The increase in compressive stress in the concrete adjacent to the wires is $\frac{tp_s}{50}.\frac{1 - n_p - f}{1 - 2n_p}$. By subtraction, the maximum increase, and decrease, of the stresses in the concrete, as shown in *Fig.* 115 (iii *c*), is $c_r + \frac{tp_s n_p}{50(1 - 2n_p)}$, the two stresses being equal since the stress in the wires is excluded and the internal stresses are balanced by an external moment.

The decrease in compressive stress in the concrete adjacent to the wires in *Fig.* 115 (iii *c*) is $\frac{0{\cdot}5 - f}{0{\cdot}5}\left[c_r + \frac{tp_s n_p}{50(1 - 2n_p)}\right]$.

Since the wires are free to move relative to the concrete (except at the ends of the beam) the relation between the change of stress in the wires and the change of stress in the adjacent concrete can be established by equating the total strain of the wires to the total strain of the concrete between the anchorages. The stress in the wires is constant throughout the beam, but the stress in the concrete varies with the bending moment so that the average stress in the concrete adjacent to the steel is F_m multiplied by the stress in the concrete at the point of maximum bending moment. With a uniformly-distributed load, since the bending-moment diagram is a parabola, F_m is 0·67. For a central concentrated load $F_m = 0{\cdot}5$. The total average reduction of the stress in the concrete adjacent to the wires is

$$\frac{0{\cdot}5 - f}{0{\cdot}5}\left[c_r + \frac{tp_s n_p}{50(1 - 2n_p)}\right]F_m - \frac{tp_s}{50}\left(\frac{1 - n_p - f}{1 - 2n_p}\right).$$

Since the total strain of the wires equals the total strain of the adjacent concrete, $t = m \times$ (average change of stress in the concrete adjacent to the steel), that is,

$$t = \frac{m(0{\cdot}5 - f)}{0{\cdot}5}\left(c_r + \frac{tp_s n_p}{50(1 - 2n_p)}\right)F_m - \frac{mtp_s}{50}\left(\frac{1 - n_p - f}{1 - 2n_p}\right),$$

which reduces to

$$t = \frac{-c_r m F_m\left(\dfrac{0{\cdot}5 - f}{0{\cdot}5}\right)}{\dfrac{p_s m}{50(1-2n_p)}\left[F_m n_p\left(\dfrac{0{\cdot}5 - f}{0{\cdot}5}\right) - (1 - n_p - f)\right] - 1} \quad . \quad . \quad (8)$$

Substituting $F_m = 0{\cdot}67$, $m = 15$, $f = 0{\cdot}1$, and $n_p = 0{\cdot}29$, in (8),

$$t = \frac{8c_r}{1 + 0{\cdot}32p_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

From (3) the tensile stress in the stretched wires is $29{\cdot}5\frac{c_p}{p_s} = 72{\cdot}1\frac{c_t}{p_s}$. Therefore, for a beam subjected to a prestress c_t and a bending moment causing an increase of compressive stress c_r, the resultant stress in the wires (excluding losses due to creep and other causes) is:

$$72{\cdot}1\frac{c_t}{p_s} + \frac{8c_r}{1 + 0{\cdot}32p_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Since the stresses shown in *Fig.* 115 (iii *b*) are in equilibrium, the moment of the stresses in *Fig.* 115 (iii *c*) equals M_r [as given by (7)], that is,

$$\frac{c_r bD^2}{6n_1}(3f - 1 + 3n_1 - 6n_1 f) = \frac{bD^2}{6}\left[c_r + \frac{tp_s n_p}{50(1-2n_p)}\right],$$

which reduces to

$$n_1 = \frac{3c_r f - c_r}{c_r + \dfrac{tp_s n_p}{50(1-2n_p)} - 3c_r(1-2f)} \quad . \quad . \quad . \quad . \quad (11)$$

If expression (8) is substituted for t, c_r is eliminated and n_1 is expressed in terms of p_s, m, F_m, and f, if n_p is expressed in terms of f by expression (2). The resulting expression is too cumbersome, and it is better to substitute values of F_m, f, and m in (8), thus obtaining (9), and then substitute the resulting value of t in (11) in which the assumed value of f and n_p are also substituted.

Substituting $F_m = 0{\cdot}67$, $f = 0{\cdot}1$, and $n_p = 0{\cdot}29$ in (11),

$$n_1 = \frac{0{\cdot}7c_r}{1{\cdot}4c_r - 0{\cdot}0138tp_s} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Substituting expression (9) for t when $f = 0{\cdot}1$, $m = 15$, and $F_m = 0{\cdot}67$,

$$n_1 = \frac{0{\cdot}22p_s + 0{\cdot}7}{0{\cdot}34p_s + 1{\cdot}4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

For given values of f, m, and F_m, therefore, the value of n_1 depends only on the value of p_s.

Increase of Tensile Prestress at Points where there is No Bending Moment due to an Externally-applied Load.—Referring to *Fig.* 115 (iii *b*), when f is 0·1 and n_p is 0·29 the tensile stress c_t is $\frac{tp_s}{72}$ and is caused by increase in stress t in the wires, which is constant throughout the length of the beam. At points where there is no bending moment, therefore, the tensile prestress in the beam is increased by $\frac{tp_s}{72}$. Substituting $t = \frac{8c_r}{1 + 0{\cdot}32p_s}$, the increase in the tensile prestress where there is no bending moment is $\frac{8c_r}{1 + 0{\cdot}32p_s} \times \frac{p_s}{72}$, which is

$$\frac{c_r}{2{\cdot}88 + \frac{9}{p_s}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

To avoid excessive tensile stresses in the concrete where there is no bending moment under full load, the stretched wires can be inclined upwards towards the ends of the beam. This increases the value of f where there is no bending moment and causes a compressive stress in the direction of the tensile stress due to shear. Alternatively reinforcement can be provided in the top of the beam to prevent excessive cracking due to tensile stresses. In a beam in which c_r is 10,000 lb. per square inch and p_s is 2, the increase in the tensile prestress in the concrete where there is no bending moment is $\frac{10{,}000}{2{\cdot}88 + \frac{9}{2}} = 1360$ lb. per square inch; this stress would cause serious cracking unless extra reinforcement were provided. The foregoing example is an extreme case.

INCREASE IN TENSILE STRESS IN THE WIRES DUE TO CRACKING.—*Fig.* 115 (iv) shows at (*a*) the distribution of the prestress in a beam after deducting all possible losses of prestress, and at (*b*) the distribution of the stresses due to bending only. When the value of the bending stress at any point above the bottom of the beam exceeds the value of the prestress at that point by an amount equal to the tensile strength of the concrete, cracks occur below the point and there is consequently no stress at and below that point. Since the algebraic sum of the stresses in the cracked zone at (*b*) is nothing, the stresses represented by the shaded trapezoidal area must be resisted by the wires. The increase in the total force in the wires because of cracking will therefore be very approximately [since the lever arm of the forces due to the stresses at (*b*) is only slightly altered] equal to the total force represented by the trapezium. The length of the base of the trapezium is the difference between the imaginary maximum tensile stress due to bending and the maximum precompressive stress, so that when the stresses at (*a*) and (*b*) are superimposed there is no resultant stress in the concrete in the cracked zone. The length of the top of the trapezium is the tensile strength of the concrete.

As an example, assume that c_r is 15,000 lb. per square inch, f is 0·1, m is 15, and n_p is 0·29. The maximum precompressive stress in the concrete is 8000 lb. per square inch. The tensile strength of the concrete is 1000 lb. per square inch. A suitable assumed value of n_1 will be determined. The tensile side of the beam will commence to crack just as the ultimate load is applied if $c_r\frac{1 - n_1}{n_1}$ is 9000 lb. per square inch, that is when $\frac{1 - n_1}{n_1}$ is $\frac{9000}{15{,}000} = 0{\cdot}6$. Therefore $n_1 = \frac{1}{1 \cdot 6} = 0 \cdot 63$. If n_1 is assumed to be 0·55, cracking will occur before the ultimate load is applied. Thus $0 \cdot 55 = \frac{0 \cdot 22p_s + 0 \cdot 7}{0 \cdot 34p_s + 1 \cdot 4}$, very approximately, since the points of application of the resultant total stress are not greatly altered by cracking. Therefore $p_s = 2 \cdot 1$ per cent.

The prestress at a point xD [*Fig.* 115 (iv *b*)] above the bottom of the beam is $8000\left(1 - \frac{x}{0 \cdot 71}\right)$. The bending stress at a point xD above the bottom of the beam is $15{,}000\frac{0 \cdot 45 - x}{0 \cdot 55}$. Therefore

$$\frac{15{,}000(0 \cdot 45 - x)}{0 \cdot 55} - 8000\left(1 - \frac{x}{0 \cdot 71}\right) = 1000 \text{ lb. per square inch,}$$

from which $x = 0 \cdot 21$. The length of the base of the shaded trapezium is $15{,}000 \times \frac{0 \cdot 45}{0 \cdot 55} - 8000 = 4280$ lb. per square inch. The length of the top of the trapezium is 1000 lb. per square inch. The total stress represented by the trapezium is $\frac{(4280 + 1000)0 \cdot 21bD}{2} = 554bD$. The area of the wires is

$$\frac{p_s bD}{100} = \frac{2 \cdot 1bD}{100},$$

and therefore the increase in stress in the wires is $\frac{554bD \times 100}{2 \cdot 1bD} = 26{,}400$ lb. per square inch.

The foregoing method is sufficiently accurate for practical purposes to determine to what extent a beam will crack on the tensile side before the ultimate load is applied, and to determine whether there is a risk of the wires yielding at loads smaller than the ultimate load. To obtain an exact result it is necessary to follow the procedure already given for determining the increase in tensile force in the wires due to bending, but with the stress distribution in *Fig.* 115 (iii *c*), modified as shown by the broken line to allow for the effect of cracking. The value of F_m is also affected, but the calculation is very complicated and hardly worth while as it is seen that the position of the neutral axis and value of the lever arm are only altered very slightly when cracking occurs. Alternatively, and particularly if cracking develops at an early stage, the simple method of determining the ultimate moment of resistance described later is applicable.

Loss of Prestress due to Creep.—Tests on columns show that the modular ratio, if the concrete has a crushing strength of about 5000 lb. per square inch, increases from about 10 to about 25 in twelve months. The corresponding increase is from 20 to 60 if the concrete has a crushing strength of about 2000 lb. per square inch. The rate of increase of m decreases in course of time, and it is probable that after twelve months m approaches its greatest value. If m increases due to creep from 10 to 25, then, before creep takes place, the strain of the concrete adjacent to the wires is $\frac{c_{r1}}{\frac{E_s}{10}}$. After creep has occurred the strain is $\dfrac{c_{r1} - c_{r2}}{\dfrac{E_s}{25}}$. Therefore the increase in the strain of the concrete due to creep is $\dfrac{c_{r1} - c_{r2}}{\dfrac{E_s}{25}} - \dfrac{c_{r1}}{\dfrac{E_s}{10}}$. Referring to *Fig.* 116, A and B are the points on the assumed equivalent straight stress-strain diagram for concrete at which full elastic strain and full strain due to creep develop. The decrease of strain in the reinforcement or wires due to creep of the concrete is $\dfrac{t_c}{E_s}$. Therefore

$$\frac{t_c}{E_s} = \frac{25(c_{r1} - c_{r2})}{E_s} - \frac{10c_{r1}}{E_s}, \text{ and } t_c = 15c_{r1} - 25c_{r2}.$$

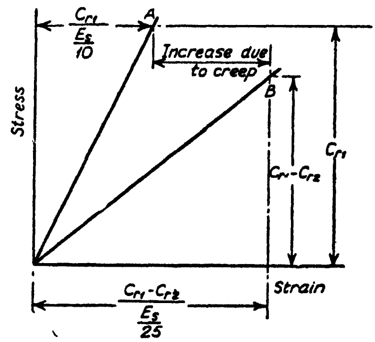


FIG. 116.

It has already been shown that for an increase of tensile stress t in the wires the corresponding increase of the compressive stress in the adjacent concrete is $\dfrac{tp_s(1 - n_p - f)}{50(1 - 2n_p)}$, which equals $0{\cdot}029tp_s$ if $f = 0{\cdot}1$. Therefore, for a decrease of tensile stress of t_c, the decrease of the compressive stress in the adjacent concrete is $0{\cdot}029t_cp_s$. Thus $c_{r2} = 0{\cdot}029t_cp_s$, and $t_c = 15c_{r1} - 0{\cdot}73t_cp_s$, from which $t_c = \dfrac{15c_{r1}}{1 + 0{\cdot}73p_s}$. For high-grade concrete c_{r1} might be 5000 lb. per square inch, and, if p_s is 1·5 per cent., $t_c = \dfrac{15 \times 5000}{1 + (0{\cdot}73 \times 1{\cdot}5)} = 35{,}500$ lb. per square inch. Thus c_{r2} is $0{\cdot}029 \times 35{,}500 \times 1{\cdot}5 = 1540$ lb. per square inch, which still leaves a prestress of $5000 - 1540 = 3460$ lb. per square inch.

If low-grade concrete is used m may increase from 20 to 60. Adjustment of the foregoing expressions gives

$$t_c = 40c_{r1} - 60c_{r2} = 40c_{r1} - 1{\cdot}74t_cp_s, \text{ from which } t_c = \frac{40c_{r1}}{1 + 1{\cdot}74p_s}.$$

If c_{r1} is 1500 lb. per square inch and p_s is 1·5 per cent., t_c is $\dfrac{40 \times 1500}{3{\cdot}62} = 16{,}600$ lb.

per square inch. Also c_{r2} is $0{\cdot}029 \times 16{,}600 \times 1{\cdot}5 = 730$ lb. per square inch, which represents about 50 per cent. loss of prestress in the concrete.

These losses of prestress due to creep are based on a high prestress in the concrete, which probably accounts for the large loss of prestress in the wires of 35,000 lb. per square inch compared with losses of 5000 lb. to 10,000 lb. per square inch referred to in reports of tests. For design purposes it is wise to base the calculations on the higher values which are derived from long-period tests for creep of short columns. The foregoing example shows that it is necessary to use high-grade concrete in order to avoid a large increase in m due to creep, which causes large losses of prestress. It is necessary to use high-strength steel in order that losses of prestress due to creep are a small proportion of the initial prestress.

Loss of Prestress on Release of Jacks.—Experiments show that when the jacks used to stretch the steel are released a reduction of stress occurs in the steel due to a slight yield at the anchorages. Allowance for this loss must be made in calculating the resultant stresses in a beam, and may be based on a yield of 0·05 in. to 0·08 in. irrespective of the length of the stretched wires. It is therefore impractical to stretch short wires sufficiently.

Summary of Formulæ.—

		When $f = 0{\cdot}1$, $F_m = \frac{2}{3}$, and $m = 15$.
Prestress in concrete (tensile):	$c = \dfrac{c_p n_p}{1 - n_p}.$	—
Depth of neutral axis:	$n_p = \dfrac{1 - 3f}{3 - 6f}$	0·29
Prestress in wires:	$\dfrac{50c_p(1 - 2n_p)}{p_s(1 - n_p)}$	$\dfrac{29{\cdot}5c_p}{p_s}$
Moment of resistance:	$M_r = \dfrac{c_r bD^2}{6n_1}(3f - 1 + 3n_1 - 6n_1 f)$	$c_r bD^2\left(0{\cdot}4 - \dfrac{0{\cdot}11}{n_1}\right)$

Increase of stress in wires due to bending (no cracking):

$$t = \frac{-2c_r m(0{\cdot}5 - f)F_m}{\dfrac{p_s m}{50(1 - 2np)}[2F_m n_p(0{\cdot}5 - f) - (1 - n_p - f)] - 1} \cdot \frac{8c_r}{1 + 0{\cdot}32p_s}.$$

Depth of neutral axis for bending stresses (with $F_m = 0{\cdot}67$):

$$n_1 = \frac{0{\cdot}7 + 0{\cdot}22p_s}{1{\cdot}4 + 0{\cdot}34p_s}.$$

Increase in prestress in wires where there is no bending moment:

$$\frac{c_r}{2{\cdot}88 + \dfrac{9}{p_s}}.$$

Comparison of Ordinary Reinforced Concrete and Prestressed Concrete Beams.

The most satisfactory way of comparing the strengths in bending of an ordinary reinforced concrete beam and a prestressed concrete beam of the same dimensions is by comparing for each the values of R' in the expression $M_r = R'bD^2$. I-beams are generally best for long-span prestressed beams, but for the purpose of comparison rectangular beams will be considered. The comparative strengths of I-beams having the same values of b and D are practically the same.

To examine the possible advantages of prestressing, the following advantageous conditions are assumed:

Compressive strength of the concrete, 10,000 lb. per square inch.

Tensile strength of the concrete, 1000 lb. per square inch.

Permissible maximum compressive prestress in the concrete after full creep and shrinkage have occurred, 8000 lb. per square inch.

Permissible maximum tensile stress in the concrete after full creep and shrinkage have occurred, 800 lb. per square inch.

Sufficient wires are provided to ensure that the steel does not yield before the concrete fails.

$f = 0{\cdot}1$; therefore $n_p = 0{\cdot}29$ and $c_p = 2{\cdot}44c_t$.

It is assumed that any cracking that occurs has a negligible effect on n_1.

Referring to *Fig.* 117, if c_{td} is the increase of the maximum compressive

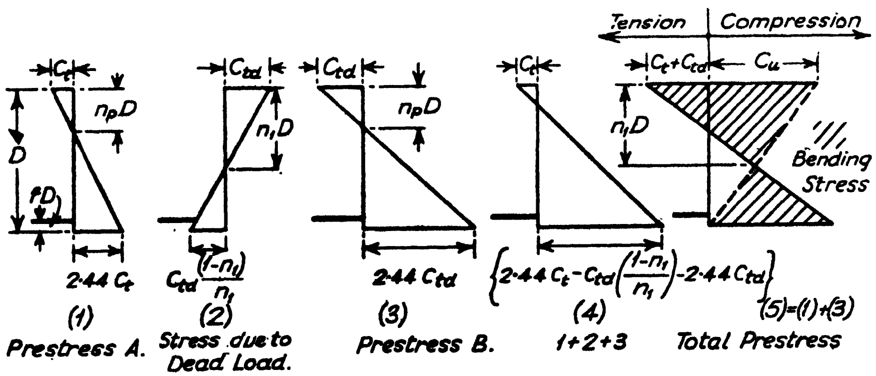


FIG. 117.

stress due to the dead load only, the corresponding decrease of compressive stress on the less compressed side of the beam is $\frac{c_{td}(1 - n_1)}{n_1}$. The maximum possible additional tensile prestress in the concrete after applying the dead load is equal to c_{td} and the corresponding increase in compressive prestress is $2{\cdot}44c_{td}$.

The resultant maximum stresses in the concrete after applying (1) the prestress A only, (2) the stress due to dead load, and (3) the prestress B are shown at (4) in *Fig.* 117 and are the maximum tensile prestress c_t, and the maximum compressive prestress $c = 2{\cdot}44c_t - c_{td}\frac{1 - n_1}{n_1} + 2{\cdot}44c_{td}$. After all shrinkage and creep have occurred the value of c_t must not exceed 800 lb. per square inch, and c_p must not exceed 8000 lb. per square inch. These stresses are slightly exceeded at the time when the beam carries the full dead load at which stage the prestress B is applied.

Substituting $c_t = 800$ lb. and $c_p = 8000$ lb. per square inch,

$$2{\cdot}44(800 + c_{td}) - \frac{(1 - n_1)c_{td}}{n_1} = 8000.$$

Therefore
$$c_{td} = \frac{6050}{3{\cdot}44 - \dfrac{1}{n_1}}.$$

It is seen that the maximum possible value of c_{td} varies with n_1, and that the permissible value of the prestress B decreases as n_1 increases.

The ultimate value of M_r for any value of n_1 can be obtained by substituting the maximum permissible values of $c_r = c_t + c_{td} + c_u$ in formula (7). Similarly, by substituting $c_r = c_t + c_{td} + c_u$, values of M_r based on a permissible safe working compressive stress c for concrete can be obtained. *Table* I gives values of R' for prestressed and ordinary beams at the ultimate and safe working stresses in the concrete for various values of n_1. The formula on which c_{td} in *Table* I is based is $c_{td} = \dfrac{6050}{3{\cdot}44 - \dfrac{1}{n_1}}$.

For prestressed beams the ultimate value of $R' = \left(0{\cdot}4 - \dfrac{0{\cdot}11}{n_1}\right)c_r$ in which $c_r = c_t + c_{td} + c_u$, or, if there is no relief of the stresses due to the dead load, in the formula for the ultimate value of R for the prestressed beam $c_r = c_t + c_u$. For ordinary reinforced concrete beams the ultimate value of R' is

$$\frac{n_1}{2}\left(1 - \frac{n_1}{3}\right)c_u(0{\cdot}9)^2$$

in which the factor $(0{\cdot}9)^2$ converts the overall depth to the effective depth.

TABLE I.—COMPARISON OF VALUES OF $R' = \dfrac{M_r}{bD^2}$.

$c_u = 10{,}000$ lb. per square inch.

n_1	c_{td} (lb. per square inch)	Ultimate values of R'					Values of R' at working stresses		
		Prestressed beam				Ordinary reinforced concrete beam R'	Prestressed beam		Ordinary reinforced concrete beam R'
		c_r (lb. per square inch)	R'	No relief of stresses due to dead load: c_r (lb. per square inch)	No relief of stresses due to dead load: R'		c_r (lb. per square inch)	R'	
0·5	4200	15,000	2500	10,800	1940	1710	7500	1250	423
0·6	3400	14,260	3130	10,800	2380	1944	6700	1480	486
0·7	3000	13,800	3300	10,800	2600	2285	6300	1520	558
0·8	2760	13,560	3500	10,800	2820	2358	6060	1580	594

For the comparison of the values of R' at working stresses, for the prestressed beam $c_r = c + c_{td} + \frac{c_u}{4}$ and $R' = \frac{M_r}{bD^2} = c_r\left(0{\cdot}4 - \frac{0{\cdot}11}{n_1}\right)$, and for the ordinary reinforced concrete beam $R' = \frac{n_1}{2}\left(1 - \frac{n_1}{3}\right)\cdot\frac{c_u}{4}(0{\cdot}9)^2$. Curves relating R' to n_1 are given in *Fig.* 118 (*a*).

It is seen that for the same value of n_1 under ultimate load a prestressed beam, in which dead-load stresses have been relieved by further prestressing, has an ultimate moment of resistance about 30 per cent. greater than an ordinary reinforced concrete beam. If the dead-load stresses are not relieved a slight advantage only is gained. At working stresses the value of R' for a prestressed

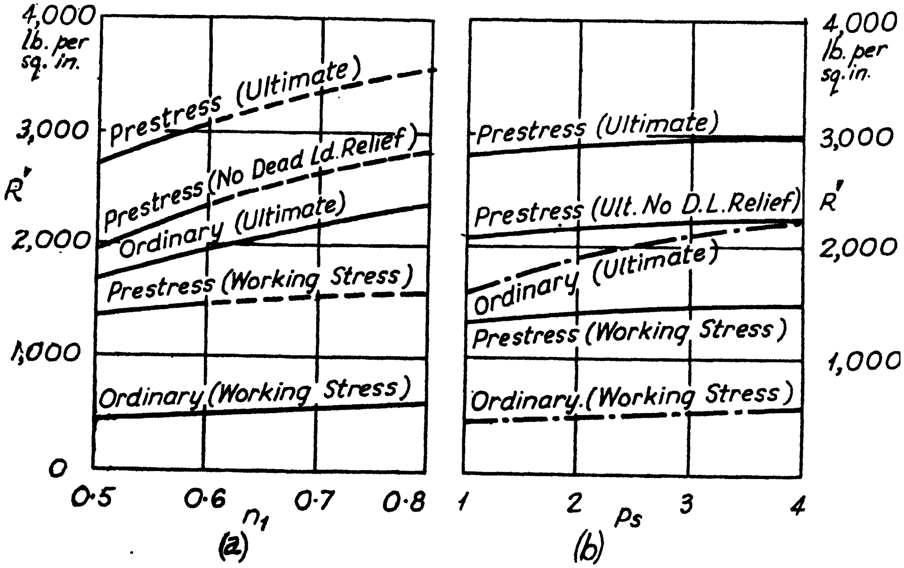


FIG. 118.

beam is about 150 per cent. greater than for a reinforced concrete beam, which shows the danger of designing a prestressed beam for the working load only. *Fig.* 119 shows the relation between n_1 for prestressed and reinforced concrete beams for various values of p_s. It is seen that only when p_s is as high as 2 per cent. does n_1 for an ordinary beam exceed the value for a prestressed beam by more than about 25 per cent. The curves in *Fig.* 118 (*b*) relate R' to p_s. A reinforced concrete beam at ultimate stress is then seen to have a higher value of R' for a large value of p_s because n_1 increases compared with a prestressed beam. In *Fig.* 118 (*a*) the curves for prestressed beams are seen to be imaginary if n_1 exceeds 0·6. In comparing a prestressed beam with an ordinary reinforced concrete beam, the lower value of n_1 obtained when using high-tensile steel, and consequently a smaller value of p_s, ought to be taken into account as it is disadvantageous to the prestressed beam; this is discussed later.

For values of n_1 and c_r for which $c_r\frac{1 - n_1}{n_1}$ is greater than the total compressive prestress plus the tensile strength of the concrete a prestressed beam will crack before the ultimate load is applied. This condition generally occurs.

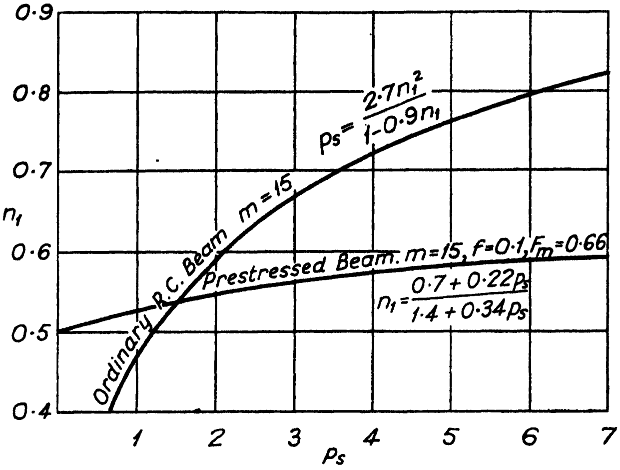


FIG. 119.

Assuming that sufficient wires are provided to prevent yielding of the steel, the effect on the ultimate value of M_r is very small but, as already explained, the tension in the wires is increased considerably. The alteration in n_1 due to the transfer of stress from the concrete to the wires is small.

Determination of the Wires Required.

As an example (*Fig.* 120), assume $f = 0{\cdot}1$ and $m = 15$; the stress in the wires is then $\frac{29{\cdot}5c_p}{p_s}$. Let F_m be 0·67 (that is a uniformly-distributed load), and

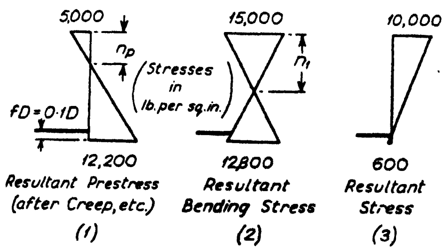


FIG. 120.

assume that c_t is 800 lb. and $c_t + c_{dt} + c_u$ is 15,000 lb. per square inch. If c_{td} is 4200 lb. per square inch, which is possible if $n_1 = 0{\cdot}5$ (see *Table* I), the resultant tensile prestress in the concrete is 5000 lb. per square inch. The resultant compressive prestress (since $f = 0{\cdot}1$) is $2{\cdot}44 \times 5000 = 12{,}200$ lb. per square inch; this stress, of course, is never developed since stresses due to dead load are applied before the final prestress. If the yield stress of the steel is assumed to be 200,000 lb. per square inch, the permissible tensile stress in the wires, after all losses due to creep, shrinkage, and transfer of anchorage have occurred, is $0{\cdot}75 \times 200{,}000 = 150{,}000$ lb. per square inch. Assume that the compressive strength of the concrete is 10,000 lb. per square inch, and the tensile strength is 1000 lb. per square inch. The stress in the wires due to establishing the prestress, and when losses due to creep, etc., have occurred, is

$$\frac{29{\cdot}5c_p}{p_s} = \frac{29{\cdot}5 \times 12{,}200}{p_s},$$

which must equal 150,000 lb. per square inch. Therefore

$$p_s = \frac{29{\cdot}5 \times 12{,}200}{150{,}000} = 2{\cdot}4 \text{ per cent.}$$

It has already been shown that the bending stress t is $\dfrac{8c_r}{1 + 0{\cdot}32p_s}$ when $f = 0{\cdot}1$, $m = 15$, and $F_m = 0{\cdot}67$. Therefore $t = \dfrac{8 \times 15{,}000}{1 + (0{\cdot}32 \times 2{\cdot}4)} = 68{,}000$ lb. per square inch.

The increase in tension due to cracking of the concrete is determined in the following manner. From *Fig.* 119, when $p_s = 2{\cdot}4$, $n_1 = 0{\cdot}54$, which is 0·04 greater than the assumed value of n_1 for the prestresses. The reduction in the compressive prestress due to bending is $15{,}000 \times \dfrac{0{\cdot}46}{0{\cdot}54} = 12{,}800$ lb. per square inch. Since c_p is 12,200 lb. per square inch, under ultimate load a tensile stress of 600 lb. per square inch is developed in the concrete, which may not crack. In the event of cracking, which in practice generally occurs because smaller values of p_s are used, an addition must be made to the pre-tension and the bending tension as already shown. Therefore the resultant tensile stress in the wires, after losses due to creep have occurred, is the pre-tension plus the tensile stress due to bending, that is, $150{,}000 + 68{,}000 = 218{,}000$ lb. per square inch.

The area of steel must therefore be increased slightly. Under working load, before creep has taken place, the stress in the wires is 20,000 lb. per square inch higher than the sum of the permissible pre-tension and the bending stress. Also, the stress in the wires at the time of prestressing and before transferring the anchorage is $150{,}000 + 20{,}000 = 170{,}000$ lb. per square inch.

It is seen from *Table* I that, within the specified concrete stresses, c_t cannot be quite as high as 5000 lb. per square inch for $n_1 = 0{\cdot}54$. The strength of the beam as regards failure of the concrete is therefore slightly less than for $n_1 = 0{\cdot}5$. By trial calculations aided by curves relating M_r to p_s and n_1 to p_s, it is not difficult to determine a value of p_s which would avoid excessive stress in the wires and also result in a value of n_1 which allows sufficient relief of prestress by the stress due to dead load for the exact value required for $c_t + c_{td} + c_u$. The size of the beam would, of course, be increased if c_{td} were limited by a high value of n_1 due to a high value of p_s. *Fig.* 119 shows that values of n_1 vary only slightly in relation to p_s.

The results in the foregoing example would be sufficiently accurate for practical purposes. Generally cracking occurs, and the simplified method explained later gives results more directly.

The Advantages of Prestressing and Using High-Grade Concrete and Steel.

As an example to show the advantages of prestressing and of using concrete and steel of high strength, consider a simply-supported reinforced concrete beam having a dead load of 100 lb. per foot and carrying a live load of 25 lb. per foot; the total load is 125 lb. per foot. The moment of resistance required is $\frac{125l^2}{8}$ ft.-lb. By using high-grade concrete and high-tensile steel the ultimate stresses might be trebled, and by prestressing cracks could be avoided and the moment of resistance for a beam of the same size quadrupled. Therefore, by reducing the width of the flange and the web, within the limits of shearing resistance, the dead weight could be reduced to about a quarter, that is to 25 lb. per foot, and the effective area of the flange to, say, one-sixth. Adding the live load of 25 lb. per foot results in the moment of resistance required being $\frac{50l^2}{8}$. The moment of resistance of the prestressed beam is approximately $\frac{125l^2}{8} \times \frac{2}{3} = \frac{82l^2}{8}$ which is still higher than is required. This example shows that when the dead load forms a large proportion of the total load the quantities of concrete and steel can be reduced to one-quarter or less by using high-strength materials and by prestressing.

Prestressing with Concrete-gripped Wires.

In this system the concrete is cast around the stretched wires or reinforcement and the temporary external anchorages are released as soon as the concrete has matured sufficiently. Thereafter the concrete grips the reinforcement in the same way as in ordinary reinforced concrete, except that the grip is claimed to be better on account of the small increase in the diameter of the wires which occurs when the wires are subjected to a slight reduction in pre-tension on account of the elastic contraction of the concrete under compression and the shrinkage of the concrete.

Loss of Prestress due to Shrinkage.—Shrinkage causes a reduction in the length of a concrete beam of about 0·0005 times the original length (see *Fig.* 111 for shrinkage related to the concrete mixture and the water-cement ratio.) The strain of the reinforcement is therefore reduced by 0·0005, corresponding to a reduction in stress of $0{\cdot}0005E_s = 0{\cdot}0005 \times 30{,}000{,}000 = 15{,}000$ lb. per square inch.

Experimental data regarding shrinkage are rather indefinite. Tests indicate that a loss of prestress of about 20 per cent. can occur for creep and shrinkage in 18 months, using high-strength concrete and a high pre-tension in the reinforcement. It is not necessary for this value to be known very accurately, since the ultimate strength of a beam is not greatly affected by considerable variations so long as cracking is taken into account.

Loss of Prestress due to Compressive Strain in the Concrete.—In beams in which concrete is cast around the stretched reinforcement and grips the latter throughout its length, loss of prestress occurs when the concrete is compressed by the stretched wires. Using the symbols already given, the tensile stress in the reinforcement after transfer is $t_p - t_m$. The compressive stress $c_{p'}$ in the concrete surrounding the wires is $\frac{t_p - t_m}{50} \cdot \frac{(1 - n_p - f)p_s}{1 - 2n_p}$, and the strain in the concrete is $c_{p'}E_c = \frac{c_{p'}m}{E_s}$. The reduction in the strain in the steel is $\frac{t_m}{E_s}$ and equals the strain in the concrete, that is

$$\frac{t_m}{E_s} = \frac{t_p - t_m}{50} \cdot \frac{p_s m(1 - n_p - f)}{E_s(1 - 2n_p)}.$$

Therefore, if $f = 0{\cdot}1$ and $n_p = 0{\cdot}29$, $t_m = 0{\cdot}029 p_s m(t_p - t_m)$ and

$$t_m = \frac{t_p}{1 + \frac{1}{0{\cdot}029 p_s m}}.$$

If $m = 15$ and $p_s = 1$ per cent., $t_m = \frac{t_p}{3{\cdot}3}$, that is, when t_p is 100,000 lb. per square inch t_m is 30,000 lb. per square inch.

This example demonstrates the advantage of the end-anchorage system as the loss of prestress due to compression of the concrete in the concrete-gripped system is high. There is generally no advantage in inducing a greater prestress than is required to prevent cracking under an overload of 25 per cent. The cost and difficulty of providing end anchorage and preventing corrosion and grip by the concrete are, however, avoided if the concrete-gripped system is used, and the security of anchorage does not depend on the durability of a single point of anchorage or on pressure grouting after prestressing.

Relation between M_r, c_r, t, and n_1.—*Fig.* 115 (ii) shows the distribution of stress due to bending only for an end-anchored prestressed beam. The distribution for a concrete-gripped prestressed beam is the same except for a small variation due to the tensile stress in the reinforcement varying in accordance with the bending moment throughout the length of the beam, whereas in the case of an end-anchored beam this stress is constant. The intensity of stress is proportional to the distance from the neutral axis. Therefore, provided that the precompression is sufficient to prevent cracking, the maximum decrease of the compressive prestress in the concrete is $c_r\frac{1 - n_1}{n_1}$, and the decrease of the compressive prestress in the concrete adjacent to the reinforcement is $c_r\frac{(1 - n_1 - f)}{n_1}$. The increase in the tensile stress in the reinforcement due to bending only is

$$t = mc_r\frac{1 - n_1 - f}{n_1} \qquad . \quad . \quad . \quad . \quad . \quad (15)$$

since the steel and adjacent concrete are subjected to equal strain. Because the beam is subjected to bending moment only, the algebraic sum of the internal forces is nothing, or the total force in the reinforcement is the difference between the compressive and tensile forces in the concrete. For a rectangular beam

$$\frac{tp_s bD}{100} = \frac{c_r n_1 bD}{2} - c_r bD\frac{1 - n_1}{n_1}.\frac{1 - n_1}{2}, \text{ that is}$$

$$t = \frac{50c_r}{p_s}\left(2 - \frac{1}{n_1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Equating (15) and (16) and reducing,

$$n_1 = \frac{50 + p_s m - p_s mf}{100 + p_s m} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

If $f_1 = 0{\cdot}1$ and $m = 15$, $\quad n_1 = \dfrac{50 + 13{\cdot}5p_s}{100 + 15p_s}$.

Taking moments about the centre of the reinforcement, the moment of resistance of the section is

$$M_r = \frac{c_r n_1}{2}\left(1 - \frac{n_1}{3} - f\right)bD^2 - c_r\frac{1 - n_1}{n_1}.\frac{1 - n_1}{2}\left(\frac{1 - n_1}{3} - f\right)bD^2$$

which reduces to

$$M_r = \frac{c_r bD^2}{6}\left(3 - 6f - \frac{1 - 3f}{n_1}\right) \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Formulæ (15) to (18) are comparable with the corresponding formulæ for ordinary reinforced concrete beams, and it is seen that n_1 depends on p_s, m, and f only, that t depends on c_r, p_s, and n_1, and that M_r depends on c_r, b, D, f, and n_1. Curves similar to those for ordinary reinforced concrete beams can be plotted from which values of b, D, and p_s can be obtained by trial for a given case, provided that cracking does not occur. If, as may be, cracking does occur, t must be increased as already described for end-anchored beams, or the simplified method for calculating ultimate strength used as explained later. Since, as will be seen, a safe limiting value of n_1 equal to 0·5 gives results very close to the exact values it is probably not worth while drawing curves.

SAFE-LIMIT METHOD OF CALCULATION WHEN CRACKING DOES NOT OCCUR.—When the compressive prestress is great enough, or the load is small enough, to avoid cracking the method of calculation is as follows. If it is assumed that m is 15 and f_1 is 0·1,

$$n_1 = \frac{50 + 15p_s - 1{\cdot}5p_s}{100 + 15p_s} = \frac{50 + 13{\cdot}5p_s}{100 + 15p_s}.$$

If $p_s = 5$, $n_1 = 0{\cdot}67$, and if $p_s = 0$, $n_1 = 0{\cdot}5$. Therefore, for values of p_s not exceeding 5 per cent., n_1 is between 0·5 and 0·67.

If f is 0·1, $M_r = \dfrac{c_r bD^2}{6}\left(2{\cdot}4 - \dfrac{0{\cdot}7}{n_1}\right)$; thus if n_1 is 0·5, $M_r = \dfrac{c_r bD^2}{6}$, and if $n_1 = 0{\cdot}67, M_r = 1{\cdot}35\dfrac{c_r bD^2}{6}$. Therefore, for preliminary calculations of the bending strength or for determining an approximate safe dimension, a concrete-gripped prestressed beam can be regarded as a beam of homogeneous elastic material having a safe working stress of c_r. The expression $M_r = \frac{c_r bD^2}{6}$ is the common expression for a rectangular beam.

By substituting n_1 from formula (17) in formula (16),

$$t = \frac{50c_r}{p_s}\left(2 - \frac{100 + p_s m}{50 + p_s m - p_s m f}\right).$$

If f is 0·1 and m is 15, $t = \frac{12c_r}{1 + 0{\cdot}27p_s}$.

If p_r is 1 per cent., $t = 9{\cdot}4c_r$, which is a greater increase in tensile stress in the reinforcement than occurs in an end-anchored beam.

Simplified Method of Calculating the Ultimate Moment of Resistance of Cracked Prestressed Beams.

In the description that follows of a simplified method of calculating the ultimate moment of resistance of end-anchored or concrete-gripped prestressed beams after cracking has occurred, the notation is the same as for reinforced concrete in Chapter III. The following assumptions are made: (1) The concrete cracks on the tensile side of the beam before the tensile reinforcement commences to yield. (2) The concrete on the tensile side provides no resistance to tensile forces. (3) Strain is directly proportional to the distance from the neutral axis (tests show that this is approximately true). (4) The resultant tension in the tensile reinforcement at the load that produces no stress in the concrete adjacent to the reinforcement is $\beta \times$ yield stress of the steel. The yield stress of high-tensile steel can be considered as slightly less than the tensile strength. In any given case the value of the stress in the reinforcement at this stage will be slightly greater than the tensile stress required to establish the prestress after creep and, in the case of concrete-gripped beams, elastic contraction have occurred, and can be calculated as already shown. β does not generally exceed 0·75. (5) The distribution of compressive stress due to plasticity and creep of the concrete is such that in a rectangular beam the total concrete compressive force is $\alpha n_1 dbc$; α is 1 for rectangular distribution and 0·5 for triangular. This compressive force acts at a depth of $\gamma n_1 d$; γ cannot exceed 0·5, and for triangular distribution of stress it is 0·33. (Note that n_1 is related to d and not to D as before.)

CONCRETE-GRIPPED BEAMS.—Consider first a loaded beam without prestress. The depth of the neutral axis is the same as for an ordinary reinforced concrete beam. It is now assumed that the reinforcement can be shortened relative to the concrete to produce a condition which corresponds to prestressing plus loading and in which the steel just commences to yield. The distribution of stress is now the same as for an ordinary reinforced concrete beam in which E_s is increased to $\frac{E_s}{1 - \beta}$, that is m is increased to $\frac{m}{1 - \beta}$. The tensile stress in the reinforcement is $\frac{c(1 - n_1)m}{n_1(1 - \beta)}$.

The value of n_1 for any value of p_s, the lever arm, and the moment of resistance when the steel commences to yield, can therefore be obtained in the same way as for an ordinary reinforced concrete beam, but with an effective modular ratio $\frac{m}{1-\beta}$, which, if β is 0·75, is $4m$; thus for concrete for which a ratio of 10 ordinarily applies curves relating n_1 to p_s for ordinary reinforced concrete with $m = 40$ can be used. In determining the value of m for a beam that is not prestressed, the effective value of E_s for high-tensile steel, say, the value at 80 per cent. of the breaking stress, must be used. Thus E_s might be 20×10^6, which with $E_c = 2 \times 10^6$ gives $m = 10$.

End-anchored Beams.—In an end-anchored prestressed beam, since the stress in the wires is constant between the anchors, equating the total strain of the wires to the total strain of the adjacent concrete results in the stress in the wires being $F'_m m\frac{(1-n_1)c}{n_1(1-\beta)}$, where F'_m is a factor depending on the distribution of the bending moment and the extent of the beam that is cracked; since n^1 must be greater for an uncracked than for a cracked beam, F'_m may be less than the value of F_m given in preceding paragraphs, the extent of the cracking determining how much less. This deficiency does not occur if the wires are effectively grouted.

Alternative Derivation.—The foregoing result may be derived as follows. Consider the loading of a prestressed beam, up to the time when the steel begins to yield, in two parts, namely (1) up to the stage at which there is no stress in the concrete adjacent to the wires, and (2) from stage (1) to the time when the wires commence to yield. If t_y is the yield stress of the steel, the increase in the stress in the wires from (1) to (2) is $(1-\beta)t_y$. Since the resultant stress in the wires at stage (2) is t_y, the distribution of stress at (2) is the same as for an ordinary reinforced concrete beam of identical cross section but having a modulus of elasticity of steel equal to $E_s \times \frac{t_y}{(1-\beta)t_y}$. In a concrete-gripped prestressed beam, therefore, at the yield stage the stress in the reinforcement is $\frac{cm(1-n_1)}{n_1(1-\beta)}$. In an end-anchored beam, since the total change of strain of the wires is equal to that of the adjacent concrete for the full length of the beam, the stress in the wires is $F'_m cm\frac{1-n_1}{n_1(1-\beta)}$, and F'_m can be expressed as the ratio of the average strain to the maximum strain in the concrete adjacent to the wires. For a beam cracked throughout the ratio is about 0·67 for a uniformly-distributed load and 0·5 for a central concentrated load. Since only a part of the beam is generally cracked the average change of stress in the concrete may be increased, and F'_m is therefore approximately equal to, but may be less than 0·67 or 0·5, and does not exceed unity for any load.

The Effects of Plasticity and Creep.—Referring to *Fig.* 121, since strain is proportional to the distance from the neutral axis the stress in the wires is $\frac{cm(1-n_1)}{n_1(1-\beta)}$, and the total tensile force is $\frac{p_s bdcm(1-n_1)}{100n_1(1-\beta)}$. The expressions apply to concrete-gripped beams, and should be multiplied by F'_m for end-anchored beams. The total compressive force in the concrete is $\alpha n_1 bdc$. Equating the total tensile force to the total compressive force (since the internal forces must form a couple opposed to the applied bending moment),

$$p_s = \frac{2\alpha(1-\beta)}{m} \cdot \frac{50n_1^2}{1-n_1} \quad . \quad . \quad . \quad . \quad . \quad (19)$$

In the foregoing m is the effective modular ratio $E'_s \div E'_c$ where E'_s is (stress at failure of beam $\div$ corresponding total strain) and E'_c is (stress in concrete at outside of beam at failure $\div$ corresponding total strain).

From (19) it is seen that the ultimate moment of resistance can be conveniently calculated by a procedure (using the formulæ, curves, and tables) similar to that described in Chapter III for ordinary reinforced concrete but with an "effective" modular ratio of Zm, where Z is $\frac{F'_m}{2\alpha(1-\beta)}$. F'_m is 1 for concrete-gripped beams, and for end-anchored beams supporting uniformly-distributed load is approximately equal to 0·67. Generally α does not exceed unity nor is less than 0·5, and, without the risk of excessive stress when the prestress is being established, β cannot exceed 0·75. Therefore Z cannot exceed 4. Generally Z is about 4, because α is about 0·5 since the steel commences to yield before plasticity develops in the concrete unless the beam is designed for very high stresses in the concrete. A high-grade concrete is generally used for which

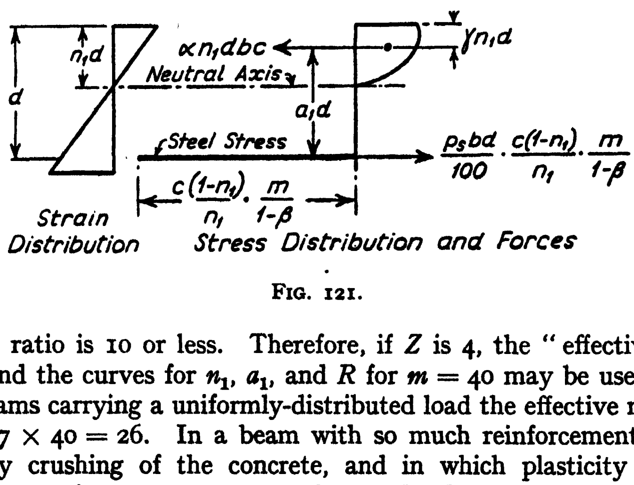


FIG. 121.

the modular ratio is 10 or less. Therefore, if Z is 4, the "effective" modular ratio is 40 and the curves for n_1, a_1, and R for $m = 40$ may be used. For end-anchored beams carrying a uniformly-distributed load the effective modular ratio is about $0{\cdot}67 \times 40 = 26$. In a beam with so much reinforcement that failure would be by crushing of the concrete, and in which plasticity occurs (say, $\alpha = 0{\cdot}75$), m may increase to, say, 20 due to the decrease in E_c. The value of Zm for a concrete-gripped beam is $20 \div (2 \times 0{\cdot}75 \times 0{\cdot}25) = 53$. Insufficient prestressing and ineffective grouting may result in low values of Zm which reduce n_1 and therefore R'.

A similar analysis can be made for tee-beams and I-beams, but the effect of plasticity in increasing the moment of resistance of the concrete is less than for a rectangular beam. Since α cannot exceed 1 and γ must lie between 0·33 and 0·5 it is easy to determine limits within which the moments of resistance of the concrete and reinforcement (or wires) must lie.

Comparison of the Moments of Resistance of Reinforced and Prestressed Concrete Beams.

If $\frac{2\alpha(1-\beta)}{m} = K$, formula (19) is $p_s = \frac{50Kn_1^2}{1-n_1}$, and therefore

$$n_1 = \frac{-p_s + \sqrt{p_s^2 + 200Kp_s}}{100K} \quad . \quad . \quad . \quad . \quad (20)$$

CASE I. ORDINARY REINFORCED CONCRETE BEAM: NO PLASTICITY OR CREEP.—For the purpose of comparison, assume p_s is 1 per cent. and m is 15. If α is 0·5, K is $\frac{1}{15}$, and n_1 is 0·41. In the expression $M_r = Rbd^2$, $R = \alpha(1 - \gamma n_1)cn_1$. If γ is 0·33, $R = 0·5 \times 0·862 \times 0·41c = 0·18c$.

CASE II. PRESTRESSED BEAM: NO PLASTICITY OR CREEP.—If α is 0·5 and β is 0·75, K is $\frac{1}{60}$ for $m = 15$. Since high-tensile steel is used, p_s is only, say, 0·25 per cent.

Substituting in (20) gives $n_1 = 0·42$, which is about the same value as in case I. Therefore the moment of resistance is also about the same. Thus prestressing and the use of high-tensile steel have not increased the strength of the beam.

CASE III. ORDINARY REINFORCED CONCRETE BEAM: GREATEST POSSIBLE PLASTICITY.—If α is 1 and β is nothing, and the effective modular ratio due to plasticity at the top of the beam is assumed to be 60, K is $\frac{2}{60}$, and n_1 is 0·53. If γ is 0·5, $R = n_1\left(1 - \frac{n_1}{2}\right)c = 0·4c$, which is an increase of 120 per cent. compared with case I.

CASE IV. PRESTRESSED CONCRETE BEAM: GREATEST POSSIBLE PLASTICITY.—Assume p_s to be 0·25 per cent. If α is 1 and β is 0·75, K is $\frac{2}{60 \times 4}$, and n_1 is 0·52. If γ is 0·5, $R = 0·52\left(1 - \frac{0·52}{2}\right)c = 0·4c$, which is the same increase as case III.

From the foregoing it is seen that when 1 per cent. of mild steel or 0·25 per cent. of high-tensile steel is used the moment of resistance of the concrete at the yield stage of the steel in a rectangular beam is about the same for an ordinary reinforced concrete beam as for a prestressed beam. If the greatest amount of plasticity develops in the compressive zone the moment of resistance of the concrete of an ordinary beam and a prestressed beam are both increased by about 120 per cent. In practice, in a beam designed for the concrete and steel to fail simultaneously the distribution of stress due to plasticity, would be intermediate between a triangle and a rectangle with α probably 0·75, thus producing a smaller increase in the moment of resistance of the concrete. However, due to the working stresses in the concrete generally adopted, failure takes place by yielding of the steel, the stress in the concrete being considerably below the crushing strength. It is likely, therefore, that the concrete develops only a little plastic yield, but the possibility of plastic yield developing before failure must be considered in deciding the permissible working stresses if triangular distribution of stress is expected. Therefore, the criteria of safe design are generally the strength of the main tensile reinforcement, that the yield stress occurs for approximately the same load whether prestressing is adopted or not, and that, in calculating the area of reinforcement, safe limiting values similar to those used for ordinary reinforced concrete can be used with m increased to take into account the effect of prestressing.

Generally there is, therefore, no great advantage in prestressing to a greater degree than is necessary to prevent cracking under a small over-load, because the strength of the concrete is only increased by the increased effective modular ratio lowering the neutral axis, and because failure is generally due to the yield of the steel.

If the values of α, β, and γ are known for concrete-gripped prestressed rectangular beams, in order to determine the "economic" dimensions for a specified moment of resistance the values of p_s and n_1 for which the concrete and reinforcement develop ultimate stress c_u and yield stress t_y simultaneously must be determined. For this condition $t_y = \frac{c_u(1 - n_1)m}{n_1(1 - \beta)}$, and if t_y is 200,000 lb. per square inch, c_u is 10,000 lb. per square inch, m is 10 (a low effective modular ratio for high stress), and β is 0·75, substitution gives the "economic" value of n_1 as 0·67. If α is 0·75, the "economic" value of p_s as found by substituting in formula (19) is 2·5 per cent. It is unlikely that such a high percentage of high-tensile steel would ever be used. Failure is therefore always likely to result from yielding of the reinforcement.

If γ is 0·4, the moment of resistance of the concrete is

$$0{\cdot}75\left[1 - \left(0{\cdot}4 \times \frac{2}{3}\right)\right] \times 10{,}000 \times \frac{2}{3}bd^2 = 3700bd^2.$$

The moment of resistance of the reinforcement is

$$\frac{2{\cdot}5bd}{100} \times 200{,}000\left[1 - \left(0{\cdot}4 \times \frac{2}{3}\right)\right]d = 3700bd^2.$$

Prestressed Beams of Any Cross Section.

The derivation of expressions for M_r, t, n_1, etc., for I-beams, which are commonly used, are not given as the expressions are too cumbersome. Approximate values of M_r can be determined by assuming a rectangular section and deducting the moment of resistance about the mid-axis of the omitted area [shown shaded in *Fig.* 122 (*a*)] adopting the distribution of stress of the enveloping rectangle. For an exact analysis of the distribution of stress the same procedure as for rectangular beams can be followed, but in evaluating the internal forces the variation of breadth must be taken into account. Each internal force is calculated by a suitable integration, and is the volume of a "wedge of stress" having no thickness at the neutral axis and a width that varies in accordance with the width of the beam. Curves for M_r, n_1, t, etc., can be drawn for I-beams in the same way as has been done for reinforced concrete tee-beams and should be prepared by those intending to specialise in prestressed concrete. Otherwise the procedure described for rectangular beams can be followed and adapted for a beam of any shape by assuming the dimensions and stresses, thereby simplifying the calculation of the volumes of the "wedges of stress" and positions of the centres of gravity of the wedges. Alternatively, moments of resistance can be expressed in terms of moment of inertia provided that cracking does not occur.

Shearing Stresses in Prestressed Concrete Beams.

If the tensile stress in the concrete due to shear on a prestressed beam does not exceed the tensile strength of the concrete until the ultimate load is applied, shear reinforcement is unnecessary as the shearing stresses at the working load can be safely resisted by the concrete. If the tensile stress due to the shear is excessive, however, the whole of the shear must be resisted by reinforcement as in ordinary reinforced concrete since, until cracking occurs, the reinforcement can only take up a small portion of the tension due to shear. Therefore the provision in a prestressed beam of unprestressed shear reinforcement does

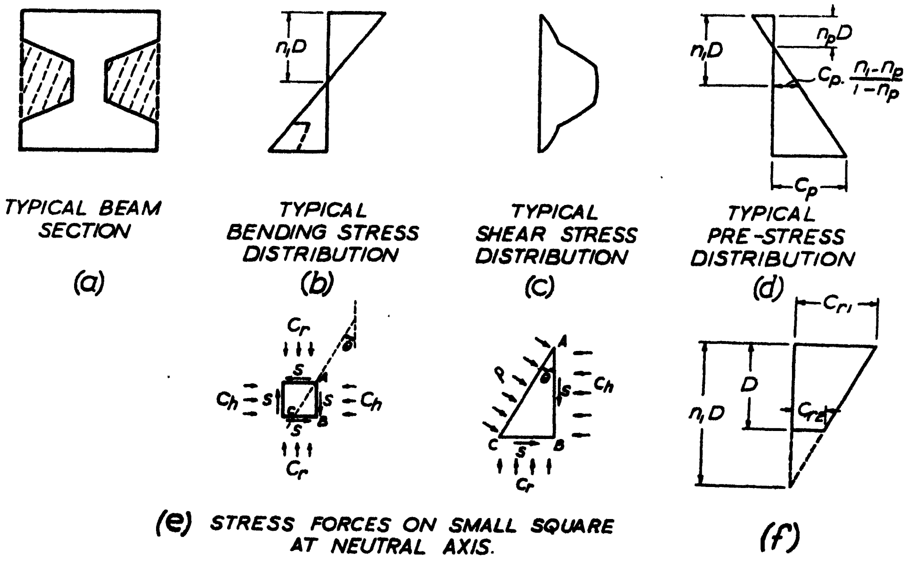


FIG. 122.

not prevent cracking, and before the ultimate load is applied the whole of the tension due to shear is resisted by the steel.

The greatest tensile stress due to shear occurs at the neutral axis of the section subjected to the maximum shear force. *Fig.* 122 shows (*a*) a typical cross section of a prestressed beam; (*b*) a typical distribution of the stresses due to bending; if cracking occurs the trapezoidal area shown dotted is deducted but, since the maximum bending stresses in a prestressed beam do not usually occur at the section subjected to the maximum shearing stresses, this deduction does not normally apply; (*c*) the distribution of shearing stress neglecting the increase in the stress in the steel, which is the same as for a beam of homogeneous elastic material if cracking does not occur; (*d*) a typical distribution of prestress (acting alone); and (*e*) the stresses acting on an elementary square at the neutral axis.

If S is the total shearing force acting on a section, A the area of the section above the neutral axis, y the distance of the centroid of A above the neutral axis, b the breadth of the section at the neutral axis, and I the moment of inertia of the section, the vertical and horizontal shearing stresses s at the neutral axis are $\frac{SAy}{Ib}$. For a rectangular section of depth D, $s = \frac{3S}{2bD}$.

If the horizontal compressive stress at the neutral axis due to the prestress is c_h, and if c_p is the maximum horizontal compressive stress due to the prestress, by proportion [*Fig.* 122 (*d*)], $c_h = c_p\frac{n_1 - n_p}{1 - n_p}$. Also, in *Fig.* 122 (*e*), c_r is the vertical compressive stress due to the prestress. The system of forces acting on the elementary square can be reduced to two principal stresses mutually at right-angles. If it is assumed that one of the principal stresses p acts normal to the plane AC, which makes an angle of θ with the vertical, and if a cut is made along AC, the known forces can be balanced by a single stress acting normal to AC.

Consider the equilibrium of the triangle ABC. Resolving the forces horizontally gives $p.\mathrm{AC}.\cos\theta + s.\mathrm{CB} = c_h.\mathrm{AB}$. Thus $p.\mathrm{AC}.\cos\theta + s.\mathrm{AC}.\sin\theta = c_h.\mathrm{AC}.\cos\theta$. By reducing, $p = c_h - s\tan\theta$.

Resolving the forces vertically gives $p.\mathrm{AC}.\sin\theta + s.\mathrm{AC}.\cos\theta = c_r.\mathrm{AC}.\sin\theta$.

Thus $p\tan\theta + s = c_r\tan\theta$, and $\tan\theta = \frac{s}{c_r - p}$.

By substitution,
$$p = c_h - \frac{s^2}{c_r - p},$$
and the solution of the resulting quadratic equation is

$$p = \frac{c_r + c_h}{2} \pm 0{\cdot}5\sqrt{(c_r - c_h)^2 + 4s^2}.$$

By substituting in the expression for $\tan\theta$, the two values of θ differing by 90 deg. are obtained, thus giving the direction of the two principal planes of stress. The larger of the two values of p is the compressive stress due to shear and the smaller value, if negative, is the tensile stress due to shear. The value of the compressive stress under the ultimate load must not exceed the compressive strength of the concrete. To avoid tensile stress due to shear

$$\tfrac{1}{2}(c_r + c_h) - \tfrac{1}{2}\sqrt{(c_r - c_h)^2 + 4s^2} = 0, \text{ from which } c_r = \frac{s^2}{c_h},$$

c_r in the foregoing and *Fig.* 122 must not be confused with c_r used elsewhere.

It is not necessary entirely to eliminate the tension due to shear, but to ensure that under ultimate load its value does not exceed the tensile strength of the concrete. The horizontal precompression is often sufficient to do this, but if not it is necessary to provide vertical precompression by inclining the prestressed reinforcement at the ends of the beam; c_r is then the vertical component of the inclined precompression. The inclination of the reinforcement also prevents excessive tensile stresses in the top of the beam near the supports. The ideal beam is that in which the pre-tensioned steel conforms to the line of the principal tensile stress, but this is difficult to follow exactly in practice.

Deflection of Prestressed Concrete Beams.

Referring to *Fig.* 122 (*f*), c_{r1} and c_{r2} are respectively the maximum and minimum compressive stresses in the concrete of a prestressed beam under the working load after taking into account the loss of prestress due to shrinkage and creep at the particular time for which the deflection is to be calculated. The depth of the imaginary neutral axis is given by n_1D and can be calculated by proportion from c_{r1} and c_{r2}.

The local deformation of a prestressed beam is the same as that of an ordinary reinforced concrete beam having a maximum compressive stress of c_{r1} and neutral-axis depth of n_1D. Thus in the general expression for deflection, $\Delta = \iint \frac{M}{EI} . dxdx$, $\frac{c_{r1}}{E_c n_1 D}$ may be substituted for $\frac{M}{EI}$, since $\frac{M}{I} = \frac{f}{y} = \frac{c_{r1}}{n_1 D}$. In a given case c_{r1} can be expressed as a function of x.

In beams where the stretched reinforcement is bonded in the concrete, that is in concrete-gripped prestressed beams, c_{r1} must be divided into the compression which is constant throughout the beam, and the compression due to bending, which varies in direct proportion to the bending moment. In beams where the reinforcement is not bonded in the concrete, that is in end-anchored beams, c_{r1} must also be divided into two parts—one being the initial prestress in the concrete, which is constant throughout the beam, plus the effective increase in the prestress corresponding to the increase in the tension in the reinforcement due to bending, which is also constant throughout the beam, and the other being the compressive stress which is produced by the bending moment due to the load and which varies in direct proportion to the bending moment; the deflection is the difference between the integrals for the varying and constant stress. The appropriate values of n must be used in each of the integrations.

Thus for a concrete-gripped beam, the deflection is

$$\iint \frac{c_r}{E_c n_1 D} dx . dx - \iint \frac{c_t}{E_c n_p D} dx . dx,$$

where c is the maximum compressive stress in the concrete due to bending (and is a function of x), and c_t (a constant stress) is the tensile stress in the concrete due to the prestress. For an end-anchored beam (*Fig.* 115), for c_r substitute $c_r + \frac{tp_s}{50}\left(\frac{n_p}{1 - 2n_p}\right)$ and for n_1D substitute $\frac{D}{2}$; t is constant and c_r is a function of x. For c_t substitute $c_t + \frac{tp_s}{50}\left(\frac{n_p}{1 - 2n_p}\right)$ which is constant throughout the length of the beam.

Summary of Procedure for the Design of Prestressed Concrete Beams.

It is not practical to calculate the required cross section directly. A cross section must be assumed and by trial and error or by the use of the curves in

Chapter III, relating a_1 and R to p_s and m, and a cross section selected which satisfies the following requirements:

(1) The ultimate load for the beam should be at least 2·5 times the working load. The ultimate moment of resistance of the assumed beam can be determined in the same way as for an ordinary reinforced concrete beam but with safe limiting values of n_1 and a_1 corresponding to an effective modular ratio of 40, and assuming a triangular distribution of stress; or the stresses due to the prestress in the assumed beam can be calculated after the losses have occurred, and the stresses due to the load, assuming no cracking, can then be calculated, the increase in the stress in the wires due to cracking being added separately. The expressions for rectangular beams must be modified for I-beams. In calculating the area of the steel the safe limiting value of a_1 (for $m = 40$) may be used except that it is an advantage to use the higher value of a_1 obtained with the generally low value of p_s.

(2) The tensile stress in the concrete under 1·25 times the working load should not exceed the tensile strength of the concrete for no cracks to occur. For this reason sufficient wires should be provided, and the stress in the stretched wires should be about 75 per cent. of the yield stress. Very approximately n_1 is 0·5 in a prestressed beam before cracking takes place; therefore M_r is $\frac{c_r bD^2}{6}$ very approximately for a rectangular beam and the reduction of compressive prestress under the load is c_r very approximately.

(3) Bend up the stretched wires so that (i) the tensile strength of the concrete is not exceeded due to shearing stresses (otherwise the whole shearing force should be resisted by reinforcement), and (ii) under the worst conditions of load the concrete in the top of the beam near the supports does not crack or is reinforced to make the cracks harmless.

Uses of Prestressed Concrete.

Consideration of the use of prestressed concrete in place of ordinary reinforced concrete may be justified for the following structures:

(1) Long-span precast beams supporting a small live load, since the use of materials of great strength reduces the dead load. Prestressing eliminates the excessive cracking and deflection which occur when reinforcement is highly stressed, provides a greater range of concrete compressive stress, and justifies designing with a slightly lower factor of safety for the concrete strength, since prestressing thoroughly tests the concrete of each beam. High-grade concrete is less liable to creep than low-grade concrete, and high-tensile steel at a high stress suffers a smaller loss of stress due to shrinkage and creep of concrete than does mild steel at lower stresses.

(2) Structures containing liquids, as seepage through cracks due to shrinkage or direct tension is prevented.

(3) Structures such as bridges and piles which can be conveniently constructed of hollow precast high-grade concrete blocks held together by stretched wires.

(4) Those parts of thin-slab vaulted roofs in which cracks can be avoided by stretched wire reinforcement.

(5) Railway sleepers in which resilience and resistance to impact are required.

(6) Aeroplane runway slabs where flexibility and freedom from cracks are required.

(7) Any concrete structure or precast product in which economies can be made by using high-tensile wire instead of mild steel bars.

Recommended Practice.

In view of the foregoing and of developments in prestressed concrete the writer recommends compliance with the following when designing members of prestressed concrete.

The lowest tensile strength of the wires should be 200,000 lb. per square inch. The crushing strength of 6-in. works cubes of concrete at 28 days should be not less than 6000 lb. per square inch and the tensile strength should be assumed to be one-tenth of the crushing strength.

At seven days after casting the effective modular ratio should be assumed to be 10, and 25 at twelve months after casting.

The stress in the stretched wires should not exceed three-quarters of the tensile strength, and the prestress in the concrete should not exceed half the strength of works cubes of the same age.

The factor of safety against cracking, that is the ratio of the cracking load to the working load, should be 1·25. The factor of safety against failure, that is the ratio of the failure load to the working load, should be 2·5. The failure load is considered to be the load that causes the steel to yield or that causes crushing of the concrete. Yield of the steel is assumed to occur at a stress equal to 85 per cent. of the tensile strength. (See Appendix III.)

CHAPTER VII

DESIGN PRACTICE

In this chapter the principles of the design of buildings, bridges, tanks, retaining walls, jetties, wharves, and foundations are discussed. Some observations on the arrangement of reinforcement in various structural members, on the design of joints, and on drawing-office practice are also included.

I.—PRINCIPLES OF DESIGN.

Buildings.

A building is usually planned by an architect who employs a structural engineer to design the frame, floors, and foundations. The engineer should be consulted at an early stage so that the building may serve its purpose and satisfy æsthetic and structural requirements to the greatest degree.

In a multiple-story building short spans are cheaper than long ones, but the span is generally limited by the size of the compartments into which columns must not intrude. Deep beams restrict head room or increase the height of the building. Flat slabs are advantageous in this respect, but are not economical unless the superimposed load is large, concentrated loads are light, and the spacing of the columns can be about equal in both directions. Precast hollow floor beams and hollow-tile floor slabs are often economical, but may not have the structural advantages of ordinary monolithic solid slabs and tee-beams. In slab-and-beam construction the arrangement of the main beams is dictated by the positions of the columns. The positions of the secondary beams may be determined by the positions of partitions and the span of the slabs. The shape and size of panels of slabs spanning either in one or in two directions should be considered from the points of view of reducing the dead weight as much as possible, of satisfying fire regulations, of enabling shuttering to be used repeatedly, and of avoiding an unsatisfactory appearance of secondary beams where they are exposed.

When the arrangement of the beams and columns has been decided, the area supported by each beam and column is marked on a plan of each floor in order to estimate, with sufficient accuracy for the preliminary calculations, the load carried by each member. In the final calculations the reactions of the slabs on the secondary beams, of the secondary beams on the main beams, and of the main beams on the columns must be computed, continuity being taken into account.

Approximate rules, which have been used in the design of many reinforced concrete framed buildings up to 22 stories in height, are that the reaction of a beam on an outside column is half the load on the beam ; on a penultimate column, 0·6 of the load on the end span and half the load on the next span ; and on other interior columns, half the load on each adjacent span. Some codes, regulations, and books give more accurate factors for such calculations, and also permissible reductions of the superimposed load for the design of beams, columns, and foundations. The improbability of all panels of slabs of every floor being fully loaded simultaneously justifies a reduction in the superimposed loads, and such reductions must be considered in the design of continuous footings or raft foundations, as large bending moments may be developed if the superimposed load supported by the middle part of the raft is considerably less than the load assumed in the calculations for the raft. Slabs are designed for the total superimposed load, and the bending-moment factors given in Chapter II should be used. The bending moments caused by the worst arrangement of superimposed load on continuous beams of nearly equal spans may then be calculated as $\frac{Wl}{10}$ at the middle of an end span and at a penultimate support, and as $\frac{Wl}{12}$ at the middle of an interior span and at other interior supports. Concentrated loads can be converted to equivalent distributed loads. Alternatively the theoretical bending moments may be calculated by one of the methods described in Chapter II. The British Standard Code recommends that the maximum negative and positive bending moments can be reduced by 15 per cent. ($7\frac{1}{2}$ per cent. in the case of end spans) if similar increases are made to adjacent bending moments. This rule, the application of which often results in relieving congestion of reinforcement, is justified because of the advantageous adjustment by creep of the bending moments.

It is usual to ignore the restraint moments due to monolithic construction of beams and internal columns except when adjacent spans of the beams are unequal, or the load is unsymmetrical, or in other cases where it is obvious that the restraining effect on the beams of the columns should not be ignored. The British Standard Code gives formulæ by which this effect on interior columns can be estimated. Outside columns and beams framing into them should always be designed for the restraint bending moments, and the expressions giving these bending moments approximately are, for frames of one bay, $\frac{K_u M_e}{K_l + K_u + 0{\cdot}5K_b}$ at the foot of the upper column at the intersection with the beam and $\frac{K_l M_e}{K_l + K_u + 0{\cdot}5K_b}$ at the top of the lower column. For frames of two or more bays the corresponding expressions are $\frac{K_u M_e}{K_l + K_u + K_b}$ and $\frac{K_l M_e}{K_l + K_u + K_b}$. The negative bending moment at the end of the beam framing into the column is the sum of the bending moments at the foot of the upper column and at the head of the lower column. In the foregoing M_e is the bending moment at the end of the beam framing into the external columns assuming that both ends of the beam are fixed, K_b is the stiffness of the beam, and K_l and K_u are the stiffnesses of the lower and upper columns respectively.

Columns should be designed as described in Chapter IV. The use of high-grade concrete is generally economical, and for small columns helical binding should be used. The ends of columns are partially restrained by the intersecting beams, and are held in position, unless the heads of a group of columns are free to sway in one direction. The bending moments due to the mutual restraining action of beams should be combined with the direct load. Columns should be designed for bending moments induced by wind when the height of a building is more than twice the width at the base. The wind forces should be calculated in accordance with the requirements of the British Standard Code of Practice for Loading (see Appendix I). It is common to calculate bending moments on columns due to wind on the assumption that a point of contraflexure occurs at mid-height between each floor. The maximum bending moments are then assumed to be the greater of those obtained by the two following methods: (1) Assume that the inside columns are twice as stiff as the outside columns and divide the total shearing force due to wind at each mid-height between the columns in proportion to their stiffnesses. The maximum bending moment is then obtained by regarding each half-height of column above and below the point of contraflexure as a cantilever free at the point of contraflexure and fixed at the level of the floor, and acted upon by a horizontal load equal to the shearing force at the point of contraflexure. (2) Assume that the points of contraflexure of the beams occur at mid-span. Calculate the shearing force acting on each column at the point of contraflexure (mid-way between floors) on the assumption that the vertical forces in the columns resisting the overturning moment are proportional to their distances from an axis through the centroid of a plan of the columns. These assumptions are safe in most buildings because the tee-beams of the floors are generally sufficiently stiff relative to the columns to cause the points of contraflexure under working load to occur very approximately at mid-height or mid-span. Under over-load the points of contraflexure will occur at these points very soon after yielding in any excessively stiff columns begins to take place. If the columns are very stiff relative to the beams or the heights between floors are irregular, a special investigation should be made as explained in Chapter II. The design of flat-slab floors and the supporting columns is also explained in Chapter II.

Concentric footings should be provided to support columns where possible. If the footings of outside columns are eccentric the moment due to the eccentricity should be resisted by a strap to the nearest column as described later. The design of continuous footings and of rafts supporting columns are also explained later in this chapter. If piles are provided the pile-caps must be designed for the bending moments and large shearing forces developed in transferring the load from the column to the piles. The dimensions of isolated footings and pile-caps should be large enough to obviate the use of reinforcement to resist shear and thereby avoid the risk of cracks and possible corrosion of the reinforcement by ground water. If there is a risk of one column settling relative to an adjacent column the bending moments in the floor beams should be increased by 5 per cent. or 10 per cent. A safe value for this increase can be estimated by assuming a safe limiting value for the relative settlement from a consideration of the ground, and calculating the additional bending moments caused in the beams by equating the relative settlement to the deflection between the two supports A and C (*Fig.* 123) due to a point load acting at B. The deflection can be obtained from the "elastic weights" on the beam (see Chapter II), and the bending moment can be distributed to the adjacent spans as shown in *Fig.* 123.

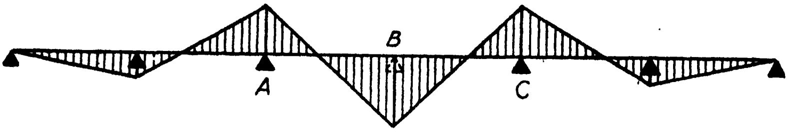


FIG. 123.

Beams in staircases are often cranked. The span in this case is the horizontal distance between the supports, and the calculations are made in the same manner as for horizontal beams. To prevent the bars at the corners of the crank pulling out, separate bars should be provided in each part of the member and should cross and extend sufficiently far beyond the point of intersection to provide an adequate length of grip as in *Fig.* 138.

The design of load-bearing reinforced concrete walls, which are sometimes provided instead of columns, should be based on the formulæ for columns, or on regulations based on tests, taking into account the restraint to lateral deflection provided by spanning horizontally between other walls or piers.

Breaks should be made in long buildings at points where cracks might occur due to unequal settlement, shrinkage, or thermal contraction.

A reinforced concrete building transmits sound readily, and floors and party walls should be insulated so that noise is not transmitted to the concrete. Outside walls and roofs must be insulated to prevent extremes of temperature within the building. Roofs must be made watertight, asphalt being frequently used for this purpose. Outside walls should be protected from staining by the use of projecting sills with drips and throatings to break the flow of water over the surface of the wall.

PRECAST BUILDINGS.—Many single-story buildings are constructed with precast frames, and in the construction of large blocks of flats or office buildings the use of large precast slabs may be economical. An application of the latter method is shown in the design for residential flats illustrated in *Fig.* 20. The advantages of using large panels are the elimination of much work on the site, a considerable saving in shuttering, the use of high-grade concrete cured under factory conditions, and the use of reconstructed stone or other special finishes for external walls regardless of weather conditions which interfere seriously with in-situ construction. Large panels enable satisfactory structural systems to be designed, both as regards the support of vertical loads and the stability of the structure under wind forces. Joints with plastic packing enable shrinkage or unequal settlement to take place without straining the walls. The savings in labour on the site are partly offset by the cost of the moulds and the plant required for erection and of transport, but these costs, if they can be recovered on one or more large buildings, will be small. Jigs for setting out cored holes are necessary. The dimensions of the moulds must be accurate, and the panels must have accurately placed seatings, in order to ensure precision in erection.

Bridges.

The structural system of a bridge is determined by the load, the span, and the rigidity of the supports. Slab-and-beam bridges are used up to about 200 ft. span. If beams are continuous over more than two supports there must be no risk of the supports yielding or settling unequally unless the additional bending moments due to the greatest probable relative settlement are taken into account. Where excessive relative settlement is expected an articulated structure, as shown in *Fig.* 124, is required. If the bases of the intermediate supports are sufficiently restrained against horizontal movement the vertical members may be continuous with the beams, and the structure considered as two or more portal frames with cantilevered extensions supporting freely-suspended beams in alternate spans. To improve the elevation, and to transfer some dead load from the middle of the spans to positions nearer the supports, the beams are often deepened near the supports and the vertical members widened at the top as shown in *Fig.* 124. Due to the consequent variation of moment of inertia, the bending moments are increased at the deep parts near the supports, and are reduced at the thinner parts at mid-span and at the bottom of the vertical members, where a horizontal thrust is transmitted to the ground. If the ground is weak it is necessary to

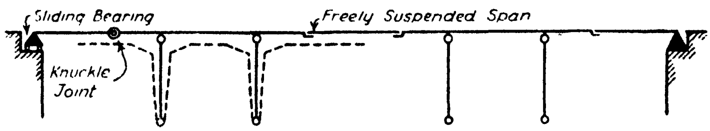


FIG. 124.

provide a tie between adjacent foundations to resist this thrust. The vertical members are subjected to vertical load, bending moment, and horizontal shearing force, and should not be widened at the bottom to look like arch abutments, although in other respects the structure looks like a flat arch bridge if the soffits of the beams are curved.

To calculate the maximum bending moments and shearing forces on slab-and-beam bridges, influence lines for the critical sections are first drawn from bending-moment diagrams for unit point loads as described in Chapter II. The influence lines, which for many common cases can be obtained from books, are then used to obtain the envelopes of the diagrams of bending moment and shearing force for each of the main structural members. For spans of less than 100 ft. it is necessary to decide from rough estimates of cost whether or not it is advantageous to provide continuity with the vertical members or to vary the moment of inertia of the beams. If firm ground is available an arched slab may be suitable for small spans. An arch can be used where weak ground overlies firmer ground by providing abutments on a group of piles, vertical and raking (see Chapter II), which must resist the horizontal thrust without excessive yield.

For spans exceeding 200 ft. arch bridges are general and may be either open-spandrel or solid-spandrel structures. In an open-spandrel bridge, arched ribs are provided as the main structural members (*Fig.* 125). The load from the deck is transmitted to the arches through either vertical columns [*Fig.* 125 (*b*)] or hangers [*Fig.* 125 (*a*)] or both [*Fig.* 125 (*c*)] according to the height of the deck relative to the supports of the arch. The shape of the arch is determined as explained in Chapter II. For an arch supporting concentrated loads at intervals a polygonal profile is the best structurally, but a continuous curve approximating to the ideal profile is generally preferred. If the thrust from the arch is not resisted by abutments on rock or by piled abutments, a horizontal tie combined with the deck, as in *Fig.* 125 (*a*), is provided and forms a bowstring girder. Details of a bowstring girder are given later (*Fig.* 147).

The rise of an arch may be determined by the clearance required below the deck and the levels of the deck and the springings. If the resistance of the abutments to horizontal thrust is small and the provision of a tie is not practicable, it may be possible to increase the rise of the arch in order to reduce the horizontal thrust. Before an arch can be designed the sizes of the rib must be assumed, and approximate dimensions can be obtained by considering the arch as the top boom of a framed girder of depth equal to the rise of the arch. The stress assumed should be small, however, to allow for the effects of bending and

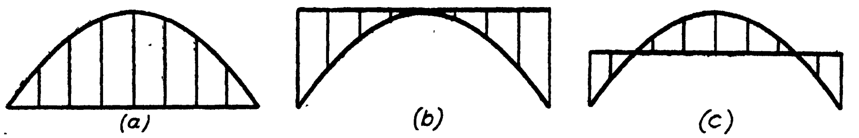


FIG. 125.—TYPES OF OPEN-SPANDREL ARCH BRIDGES.

direct thrust. Influence lines for the bending moments, thrusts, and shearing forces at the crown, the springing, and, say, the first quarter-point, should then be drawn, and diagrams of the maximum bending moments, thrusts, and shearing forces prepared as if the load were concentrated at a series of points. The construction of influence lines for unit point loads on arches is explained in Chapter II. Arch ribs must be braced laterally at suitable intervals to prevent lateral buckling. It may be an advantage to use the deck to assist the arch to resist the bending moments due to the live load, and to incline the hangers, if the deck is suspended, so that they act more closely in line with the direction of the deflection of the arch.

In a solid-spandrel bridge the deck is generally carried on filling on an arched slab. The worst combination of vertical load and horizontal pressure of the filling should be assumed to act on the arched slab.

If the supports of an arched rib or slab are liable to horizontal or vertical movement, hinges are provided at the springings and crown; the arch is then a three-pin arch and is statically determinate, and the bending moments and thrusts are independent of small movements of the abutments. The stresses in three-hinged arches are also practically independent of variation of temperature, creep, and shrinkage, the effects of which must be taken into account in arches with fixed ends or with hinges only at the springings (see Chapter II). In large bridges the transverse forces due to wind must be taken into account.

The best arrangement of main beams and secondary beams and slabs forming the deck of a bridge should be determined after making rough designs and approximate estimates. Dead weight may sometimes be reduced by providing a thin slab with beams more closely spaced. The slab and transverse secondary beams sometimes need to be strong enough to transmit heavy concentrated loads to two or more of the main longitudinal beams, and a calculation to determine the best relative stiffness using general elastic equations is necessary. Footpaths are best supported on cantilevers so that the outside longitudinal beams carry about the same load as the inside beams. Rail tracks are generally best supported by main beams immediately beneath them.

The abutments of a bridge may serve one or more of the following purposes: (1) to transmit vertical loads to the ground; (2) to retain an approach embankment; (3) to transmit horizontal thrusts to the ground; and (4) to provide stability to the bridge when the structure is subjected to longitudinal forces, due to braking or other causes, and transverse forces due to wind. For resisting vertical loads only, columns or walls suffice. If the horizontal thrusts can be transmitted directly to rock a solid concrete haunched abutment is generally provided. An abutment wall must act also as a retaining wall, and must be sufficiently wide at the base to prevent excessive settlement, sliding, or overturning. If a wall, which may be stiffened by vertical ribs, supports an arch, it may be necessary to provide a foundation of piles, including rakers, to resist vertical and horizontal forces. A massive pile-cap is generally required to withstand the shearing forces and bending moments developed in transmitting the forces from the members of the bridge to the piles. The stability of an abutment subjected to the reactions from the bridge due to dead load only and to dead and live load combined must be investigated, with or without earth pressures acting. Sliding bearings are necessary for long beam bridges to allow for movement due to changes of temperature and shrinkage, and movement due to settlement if the supports are liable to yield unequally.

Tanks and Bins.

The principal load on the walls and bottoms of containers is a fluid pressure or the pressure from contained materials. A horizontal pressure p is best expressed in pounds per square foot per foot of depth, and is 62·5 lb. for water, and, say, 30 lb. for dry sand or ordinary earth. The horizontal pressure is related to the angle of repose of the material and to the shape of the container. Walls subjected to horizontal pressures in rectangular containers [*Fig.* 126 (*a*)] may span horizontally or vertically or both, and be supported by ribs or adjacent

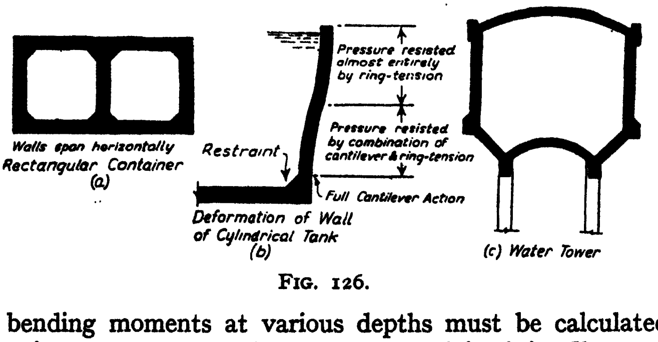


FIG. 126.

walls. The bending moments at various depths must be calculated and continuity taken into account by the methods explained in Chapter II. Slabs should not be less than 4 in. thick. Full advantage should be taken of continuity around corners, either vertically or horizontally, splays being provided in the corners to give increased depth and rigidity and to facilitate fixing the reinforcement. In deep containers or bins, called silos, the support of non-fluid contents by friction on the walls is taken into account. In the design of the walls near the bottom of a deep bin, the possibility of large horizontal thrusts due to arching of the stored material must be considered.

In cylindrical bins or tanks the walls resist the horizontal pressures by ring tension, which is *pr* per foot of depth where *r* is the radius of the container. It is common to provide reinforcement to resist the whole of the ring tension and to place the reinforcement in the middle of the wall, or to place the reinforcement in two equal layers, one near each face of the wall. In a cylindrical tank to contain liquid the reinforcement should not be stressed to more than 12,000 lb. per square inch. To prevent cracks, the thickness of the wall should be such that the tensile stress in the concrete does not exceed 200 lb. per square inch when it is assumed that the whole of the ring tension is resisted by the concrete. The concrete should be as dense as possible and should comply with the specification for liquid-tight concrete given in, say, the Code for Liquid-Containing Structures published by the Institution of Civil Engineers. Near the bottom of the wall reinforcement must be provided to resist the vertical bending moments induced by continuity with the base and the consequent restraint on free circumferential stretching under ring tension. An exact analysis of this condition is difficult, but it is not wasteful to provide reinforcement to resist full cantilever action of the wall up to the height at which most of the pressure is resisted by ring tension [*Fig.* 126 (*b*)]; this method simplifies the calculations. Alternatively, the method of Dr. Reissner, which is given in some books on tanks, can be used to determine the bending moments and direct tension.

The slabs in the bottoms of elevated tanks can be reduced in thickness by dishing in various ways [*Fig.* 126 (*c*)] but, except in large tanks, the cost of the shuttering more than cancels any saving effected by the reduction in the quantities of concrete and reinforcement.

To ensure watertightness, sudden changes of thickness should be avoided and concreting should be carried out so that shrinkage stresses are reduced as much as possible and construction joints, if unavoidable, made as tight as possible by providing an effective key between the new and old concrete. A water-bar, formed by a strip of copper or other non-corrodible metal, is often embedded in the concrete across the joint. It is difficult to ensure absolute watertightness in a concrete tank. A well-graded dense concrete thoroughly compacted is the first requirement. Vibration helps compaction, but must not be overdone. The additions of fatty materials or the use of special cements may help to produce a dense concrete, and waterproof renderings are sometimes applied to the surface. Where absolute watertightness is essential, the application of two or three layers of asphalt is probably the only certain method.

In large cylindrical tanks the walls are often considered to cantilever from the base or span between the roof and the base; the roof or a ring-beam at the top resists the accumulated ring-tension. Large cylindrical water and oil tanks have been prestressed, the circumferential bars or wires in the walls being stretched by turnbuckles, jacks, or travelling winding machines which maintain a constant tension in a continuous helix of wire; shrinkage stresses in the concrete are eliminated, and under the full internal pressure the concrete walls remain in compression. The roof of a large cylindrical tank is usually domical, and in some cases the dome is maintained in compression throughout by a prestressed ring beam.

Retaining Walls and Dams.

Retaining walls and dams generally act as cantilevers restrained by suitable footings at the base, although some dams are designed as horizontal arches if abutments on rock can be provided. A cantilevered retaining wall may be either a slab that itself cantilevers vertically [*Fig.* 127 (*a*)], or the slab may span

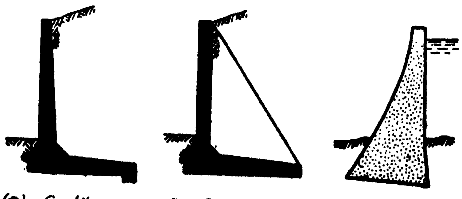


FIG. 127.

horizontally between vertical cantilevers, or counterforts, which are restrained at the base [*Fig.* 127 (*b*)]. The best design of a particular wall can only be determined by comparing approximate estimates of cost. The base of the wall is continuous with the stem of the wall and is designed to resist the bending moments and shearing forces caused by stabilising the vertical and horizontal loads. The whole structure must be stable with a factor of safety of 1·5 to 2 against overturning and against sliding. It must be remembered that overturning takes place about the point of action of the resultant upward earth pressure; this point may be at the edge of the base if on rock; if on yielding ground it is at the centre of gravity of the earth pressures. The width of the footing must be sufficient to reduce the ground pressure to a safe amount under the working load, and if necessary the edge must be deepened or provided with additional resistance to sliding.

Dams must be watertight and should comply with the requirements for tanks. The cross section of a typical plain mass concrete gravity dam is shown in *Fig.* 127 (*c*). The pressure due to the weight of concrete on each horizontal plane must be sufficient to eliminate, with a suitable factor of safety, all tensile stresses due to bending and shearing forces. Where anchorage into rock is available, vertical prestressing has been employed to induce a compressive stress and it is then possible to reduce the thickness of the dam; this method depends for its security on the resistance to corrosion of the vertical ties and is not satisfactory for high dams. When rock abutments are available at the sides of a dam, the structure may be designed to span horizontally as an arch; if the arch is circular in plan, the line of thrust of the arch is central since the external pressure is radial. Provision must be made for the greatest possible vertical restraining moments. The design and construction of a large arch dam require careful attention to the problems of reducing the stresses due to shrinkage, to variation of the ambient temperature, and to the higher temperature of the large mass of concrete when setting.

Underground Structures.

Underground structures such as tanks, pump-wells, air-raid shelters, culverts, ducts, and tunnels must be designed to resist the external pressures of the soil, which may be dry or waterlogged or in any intermediate state. The external pressures may affect in part the internal pressures on the walls of underground tanks, but before considering any reduction in the internal pressure on this account it must be certain that the external pressure assumed will act permanently. If there is any doubt, the external pressure should be ignored when designing for the condition of " tank full." Sound concrete structures in permanently dry ground may be sufficiently watertight but may not be damp-proof. Damp-proofness and, if the ground is waterlogged, watertightness, can generally be assured only if two or three layers of asphalt are applied to the outside of the structure.

Jetties and Wharves.

The design of a jetty or wharf is determined by the weight, draught, and type of vessels to be berthed; the depth of water at low tide, the rise of the tide, and the navigational conditions of approach; the direction and force of the prevailing wind; the direction and speed of the prevailing current; the nature of the ground forming the bed of the sea or river; the type and volume of cargo to be handled; the method of access to the shore; and the plant available for construction.

The length of the berth is determined by the accommodation required for vessels. Generally fenders must be provided throughout the length of the berth and must be spaced sufficiently closely to ensure that the vessels bear against the fenders without over-stressing the vessel. Vessels at the berth must be free to move up and down with the rise and fall of the tide. Since a vessel is rarely exactly parallel to the face of the wharf when contact with the fenders occurs, there is no need to provide fenders throughout if large vessels only are to use the berth. Fenders capable of withstanding heavy blows need only be provided at intervals and should be supported by strong-points, or dolphins, connected to the jetty by a light deck.

The direction and position of a jetty must suit the approach of vessels and provide sufficient depth of water at low spring-tides. When they are berthed, vessels should lie in the direction of the currents, and if possible in the direction of the prevailing wind and waves. If the speed of the current is low, say, about half a knot, the wind is often the governing factor. The vertical extent of the fenders will depend on the rise of the tide and free-board of the vessels, the shape of which should be studied in relation to the fenders at all states of the tide and loading to ensure that the position and spacing of the fenders are suitable, allowance being made for possible listing of the vessel and glancing blows.

The level of the deck of a jetty is determined by the water level at high tide, and for an exposed jetty should be 8 ft. to 10 ft. above the level of the highest known tide. The deck should be above the level of the crests of waves to avoid upward pressures under the deck. The width of the deck depends not only on the operational requirements but on structural requirements when the deck forms the cap of groups of piles and acts as a horizontal beam.

The substructure should present the least obstruction to waves and flow of water, and must withstand the vertical loads due to the weights of the deck, cargo stacked thereon, cranes, and vehicles; the horizontal forces transmitted by bollards, capstans, and fenders; the horizontal pressures of waves and wind; and the longitudinal forces transmitted by friction or otherwise from vessels or transmitted by the bollards. The force from a bollard will not exceed the force necessary to rupture or pull out the holding-down bolts, nor will it exceed the breaking load of the strongest mooring rope likely to be used. In extreme cases the breaking load of the ropes may be from 100 tons to 200 tons, but mostly bollards may fail at loads from 20 tons to 50 tons.

The force of impact of vessels entering or leaving the berth depends on the size and speed of the vessel and the resilience of the fenders and the jetty. Head-on blows do not occur unless a vessel is completely out of control, and the greatest blows are glancing blows delivered from near the bow or stern, in which case only 40 per cent. of the kinetic energy of the vessel need be absorbed by the fenders and structure. Since the mass of the part of a jetty moved by impact is generally small compared with the weight of the vessel, the greatest impact may be determined by equating a proportion of the kinetic energy of the vessel to the work done in deforming the structural members of the jetty plus the energy absorbed by the fenders. The work done in deforming the jetty can be expressed as the sum of the internal work done by the resulting internal forces in each member of the jetty. The internal work done in one member is $\frac{F^2 l}{2AE_c}$, where F is the axial force in pounds (or tons) in the member, l and A are the length in inches and effective cross-sectional area of the member expressed in square inches of concrete, and E_c is the modulus of elasticity of concrete in pounds (or tons) per square inch.

For an exposed jetty, or where it is difficult to manœuvre vessels into position on account of currents or an awkward approach, a collision speed of not less than 1 ft. per second should be assumed for vessels of 15,000 tons to 20,000 tons. In such cases the kinetic energy to be absorbed is very high and the work done in deforming the members of the jetty may become almost negligible unless the structure comprises closely-spaced piled trestles. It is therefore economical to provide shock-absorbing fenders designed to absorb the whole of the kinetic energy of the vessel. Shock-absorbing fenders are also used if it is desired to reduce the impact in order to reduce loads on the foundation or to enable a light structure to be provided. Fenders are best placed in groups so that they act together, and where severe glancing blows are expected provision for rotary and longitudinal movement should be made.

The horizontal pressure of wave crests and wind on the sides of deck beams may be assumed to be 3 cwt. per square foot if the deck is above the crests of the waves under the worst possible conditions. The pressure of waves on rigid breakwaters, however, may exceed 2 tons per square foot. The longitudinal horizontal forces due to the pressure of waves or wind and from mooring ropes or the friction of vessels can be assumed to be 50 per cent. to 75 per cent. of the corresponding transverse horizontal forces.

Many factors which vary considerably with the conditions of the site influence the design of the structure. Generally a slab-and-beam deck supported on piled trestles is the most suitable. Piling avoids working under water and reduces the risk of scour. Well-driven piles can resist upward or downward forces. The resistance of a pile to withdrawal is best determined by testing one or two piles, and it can generally be assumed that the anchorage of other piles is proportional to their penetration in the same stratum, and that it may increase in time. The resistance of piles to downward loads can be determined from the set of the pile during driving. For this purpose Mr. Hiley's formula is recommended and, for a factor of safety of 3, is $L = \frac{WH\eta}{3(s + 0{\cdot}5c)}$, where L is the safe load in tons, W is the weight of the monkey in tons, H is the free fall of the monkey in inches, c is the temporary elastic compression (in inches) of the pile, helmet, dolly, and ground, and s is the set in inches under the final blow. The factor η is the efficiency of the blow and, when W is not less than Pe, $\eta = \frac{W + Pe^2}{W + P}$; when W is less than Pe, $\eta = \frac{W + Pe^2}{W + P} - \frac{W - Pe^2}{W + P} = \frac{2Pe^2}{W + P}$. The term e, for a cast-iron monkey driving a reinforced concrete pile without a helmet but with the head protected by a mat 1 in. to 1½ in. thick, is equal to 0·4, and to 0·25 with a helmet with 3 in. to 4 in. of highly-compressed packing. For small sets it is only possible to determine L approximately.

The piles may be arranged in groups or trestles placed 15 ft. to 20 ft. apart, each group being designed to withstand the horizontal and vertical forces and connected to the deck by suitable extensions and bracing. Another arrangement is to provide a number of strong-points at 100-ft. to 200-ft. intervals, each comprising vertical and raking piles to which the horizontal forces are transmitted by the deck acting as a horizontal beam. The fenders on the strong-points project beyond the face of the jetty so that large impact forces are transmitted directly to the strong-points. Vertical loads between the strong-points are supported by vertical piles. When vessels of different sizes are to use the berth, a jetty incorporating a number of trestles is generally most suitable. The strong-point method is suitable when only large and very small vessels will use the berth.

Many other factors influence the arrangement of the piles. The nature of the sea-bed may require piles to be distributed. The pile-driving plant available may be more suitable for driving rakers in a few groups than as single piles at regular intervals. Approximate estimates of costs of a number of preliminary designs should be compared, the cost of the plant and craft necessary to carry out the construction being taken into account.

The deck comprises slabs and beams, but in some cases it may be economical to provide a thick deck slab without beams in order to provide weight to prevent overturning. The deck is generally supported on braced extensions of the piles which form a rigid vertical frame transmitting horizontal and vertical loads to the piles. *Fig.* 91 in Chapter II shows a typical piled trestle. The frame between the pile heads and the deck is considered to be a rigid structure relative to the piles and the forces in the frame are calculated on the assumption that it is pin-jointed. The forces in the piles can be calculated as explained in Chapter II. The heads of driven piles may be several inches out of their true position, and the pile-caps must be large enough to allow for this. It is sometimes an advantage to provide horizontal braces between the piled trestles at the level of the pile-cap in order to reduce the unsupported length of the piles and to distribute horizontal forces at that level to adjacent frames. Braces and extensions of the piles below high-water level cost several times as much as similar work above high-water level. If the distance from the bearing stratum to the deck is not excessive it is better for the piles to extend to the level of the deck, and any braces required below high-water level should be precast.

Although load-bearing piles are generally the most suitable support for a deep-sea jetty, in some cases strong-points can be economically constructed by retaining ballast within a cofferdam of steel or reinforced concrete sheet-piles. In other cases where the vertical load is great, reinforced concrete or steel cylinders driven by screwing or by pressure on a cutting-edge are sometimes

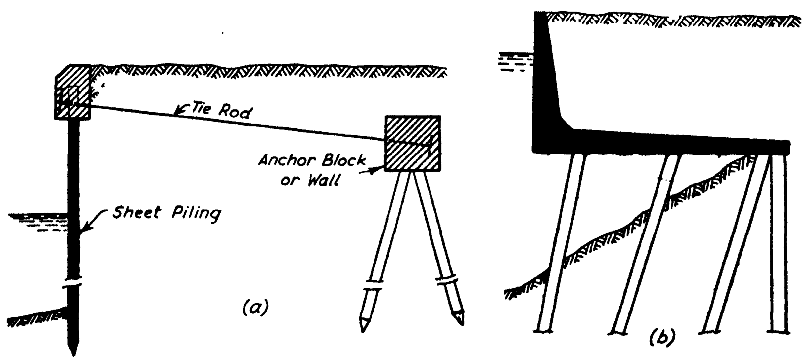


FIG. 128.

used, horizontal resistance being provided by the strength of the cylinders in bending; the bending moments can be reduced by shock-absorbing fenders and bracing above low-water level.

Wharves are sometimes constructed similarly to deep-sea jetties, but often it is cheaper to take advantage of the support provided by the adjacent land. Reinforced concrete or steel sheet-piles retain the earth [*Fig.* 128 (*a*)] and span vertically, the reactions at the tops of the piles being transmitted horizontally by a capping beam to ties carried back to anchorages formed by concrete blocks or raking piles. Another design suitable for deep-water wharves is shown in *Fig.* 128 (*b*). The retaining wall is cantilevered from the pile-cap and the supporting piles are set out, as explained in Chapter II, to give the pile-cap sufficient horizontal resistance to earth pressure with the smallest number of piles.

Foundations.

ISOLATED FOOTINGS.—The most common foundation for a column is an isolated footing that is generally square but sometimes rectangular. It is safe to assume the footing to be divided into four cantilevered slabs each similar to ABED, *Fig.* 129 (*b*). The greatest bending moment occurs on FG, but if the column is monolithic with the footing the critical section is at DE. The moment of resistance of the concrete is based on the width FG, and the reinforcement resisting bending tension may be distributed over the width FG but with bars slightly more closely spaced over DE. If the load on the column is W the upward pressure assumed to be uniformly distributed is $\frac{W}{AC}$. The total bending moment over FG is the moment about FG of the upward pressure on the area CAB minus the moment of the part of the downward-acting load of the column on DEC. If the footing and the column are square and the width of column is B and the width of the base is A, the bending moment over FG is

$$\frac{W}{4}\cdot\frac{A}{3} - \frac{W}{4}\cdot\frac{B}{3} = \frac{W(A - B)}{12}.$$

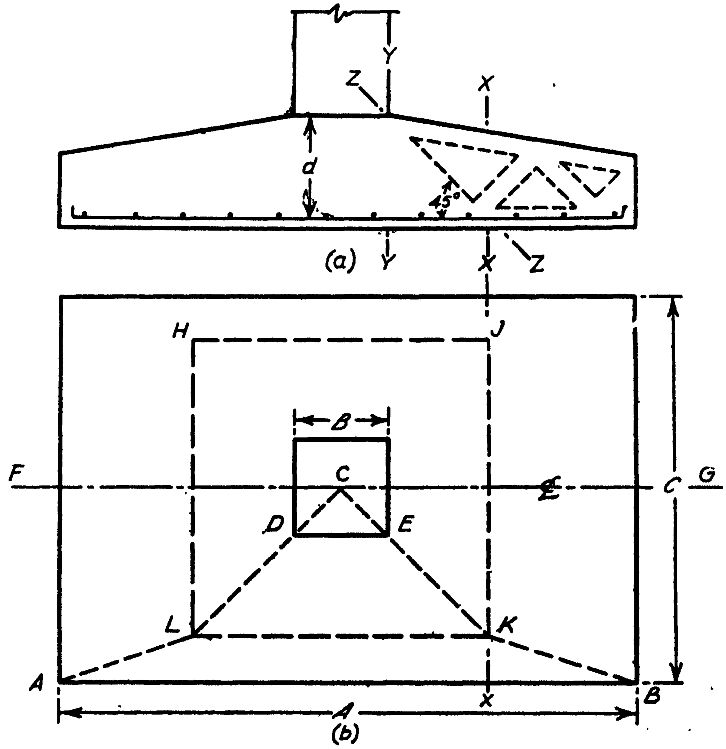


FIG. 129.

If W is in pounds and A and B are in feet, this bending moment is $W(A - B)$ in.-lb., which is a convenient form for use in practice. The depth of a footing is determined by the shearing stress. Tests show that shearing failure occurs along a plane such as Z–Z [*Fig.* 129 (*a*)], and it is therefore justifiable to assume that the critical plane for shear is X–X, that is a plane distant d from the face of the column where d is the effective depth of the footing. If P is the perimeter of HJKL, and a is the lever arm of the moment of resistance at section X–X, the maximum shearing stress s can be assumed to be $\frac{S}{aP}$ where S is the total shearing force on HJKL, which is the total upward pressure on the part of the footing outside HJKL. The value of *s* must not exceed 0·1 times the permissible compressive stress in the concrete due to bending. The justification for this method of calculating the maximum shearing stress is that before failure the parts of the footing within HJKL must act as a frustrum of a pyramid subjected mainly to direct compressive stresses which act vertically at the centre and diagonally at the sides, the reinforcement acting as ties in the base, thus reducing the shearing force at Y–Y. The remainder of the footing acts as a series of superimposed framed members [*Fig.* 129 (*a*)] which before failure combine and adjust their relative stiffnesses so as to give the greatest strength.

Concrete in footings is cheap and, as it is advisable to avoid cracked reinforced concrete because of the risk of corrosion, it is usual to provide sufficient concrete to avoid having to provide reinforcement to resist shearing forces. If, however, it is necessary to use reinforcement to resist shearing forces, the method of calculation used for beams applies and the whole of the shearing force is assumed to be resisted by the reinforcement.

It is common to provide at least 1½ in. of concrete under the reinforcement in the bottom of the footing and to protect the reinforcement and concrete from the earth by depositing a mat of plain concrete at least 2 in. thick on the bottom of the excavation. The sides of the footing should be shuttered if there is a risk of cement grout leaking out of the concrete or of earth being mixed with the concrete. If the top of the footing is sloped at about 1 in 3, top shuttering is unnecessary. Splice bars for the column should be fixed in position before the footing is cast.

COMBINED FOOTINGS.—If columns or other loads are closely spaced, or if the areas of the footings required for the columns overlap, a single footing may support two or more loads.

The eccentricity of the resultant downward force R on an irregular footing is $e = \frac{\Sigma Cx}{\Sigma C}$, where C_1, C_2, etc., are the column loads, and x_1, x_2, etc., are the distances of the points of action of the loads from an axis passing through the centroid CG of the footing (*Fig.* 130). The ground has therefore to resist a con-

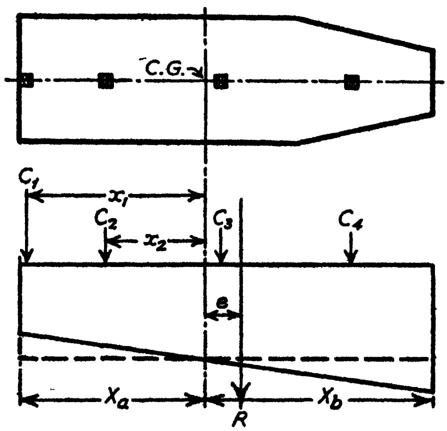


FIG. 130.

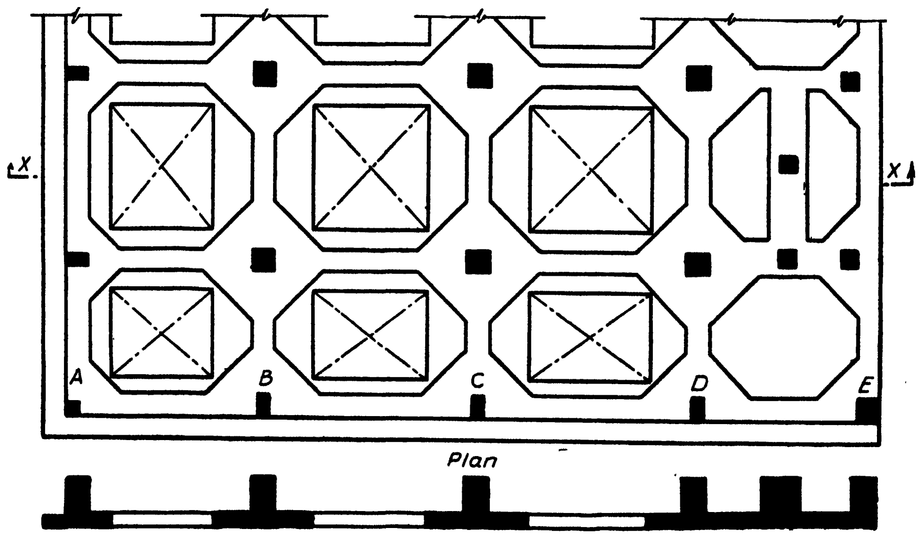
centric vertical load R and a couple Re. If I is the moment of inertia of the area on plan of the base about an axis through its centroid, and if the footing and the loads are symmetrical about a longitudinal axis through the centroid, the increase above the average ground pressure at the end nearer to R is $\frac{X_b Re}{I}$ and the reduction at the other end is $\frac{X_a Re}{I}$ where X_b and X_a are the distances of the respective ends from the centroid. It is assumed that the ground pressure varies uniformly. Therefore the bending moment at any section of the footing can be found by adding together the moments about the section of all forces to the right, or left, of the section. The moment of the ground pressure can be calculated by regarding the total pressure on the right, or left, of the section as the volume of a wedge and a prism, the resultants acting at the centres of gravity of the wedge and prism.

On firm ground it is sometimes assumed that less pressure occurs under lightly-loaded columns, thus greatly reducing the assumed bending moments in the footing. This method is risky unless the yielding of the soil is small compared with the flexibility of the raft. Another method is to regard a combined footing or raft as a reversed floor with the soil pressure as the superimposed load. This method is only safe when the ground is sufficiently unyielding, and the raft sufficiently flexible, to allow local increases or decreases in pressure proportional to local heavy or light loads to take place by the raft undergoing deflections for which the corresponding bending moments are negligible. Whether this condition can be relied on or not can only be guessed unless the flexibility of the footing or raft and the elasticity of the soil are in some way taken into account in the calculations. The bending moments are, of course, the same as those in a continuous beam if the column loads have the same relation to the ground pressure as the reactions at the supports of a continuous beam supporting a load equivalent to the assumed distribution of ground pressure. It is often a great advantage to vary the width of the footing, by widening the footing under the greater loads, so that this condition is obtained.

Raft Foundations.—Assuming that piling is not suitable, a raft is generally more economical than isolated footings if more than three-quarters of the site or part of the site would be covered by footings and if a number of footings overlap. A method of designing raft foundations is described in the writer's book "Raft Foundations: the Soil-line Method of Design." Only the essentials of the method are given in this chapter.

A slab-and-beam raft is generally the most economical, as conditions favouring a flat slab rarely occur, and the form of such a raft is shown in *Fig.* 131. If large positive bending moments occur between the columns a cellular raft is suitable, as the top slab will provide a compression flange. The arrangement of the beams in *Fig.* 131 is found in the following manner. Having decided on a suitable ground pressure, the area of foundation required for each column is marked around it in the form of a square with sides parallel to the diagonals of the column. Beams are provided from each corner of each square towards the corners of the squares under adjacent columns. By a process of adjustment of the areas, an arrangement is eventually obtained whose total area multiplied by the safe ground pressure (less the weight of unit area of the raft) is equal to the sum of the column loads. The following conditions must also be satisfied: (1) On clay the centre of gravity of the loads must coincide approximately with the centroid of the raft. On more unyielding soil the agreement need not be exact. The raft should be capable of division into small groups of columns carried by a corresponding small portion of the raft to which the foregoing condition applies. (2) As far as possible each column should be carried by a piece of raft which, if broken away from the remainder, would act as an isolated footing and continue to support the column.

If these conditions are fulfilled, an arrangement is obtained in which the bending moments are the least, and which will not be overstressed if soft patches of ground occur. The slab is at the bottom, so that the surface of the ground receiving the load is level. The slab also transfers the upward pressure directly to the beams and provides resistance to compression between the columns where



Section through Raft at X-X.

FIG. 131.

high negative bending moments occur. The beams are made wide to resist large shearing forces and to provide flexibility, and are splayed at the ends to resist the shearing force and the compression due to large positive bending moments under the columns. It is an advantage, where possible, to cantilever the raft beyond the outside columns, as the raft then tends to deflect downwards in the middle.

The beams of a raft form a horizontal frame kept in equilibrium by the column loads and the reaction from the ground. Calculations are simplified if the forces keeping the raft in equilibrium are divided into three balanced systems. System F_A.—The average ground pressure acting upwards uniformly over the whole of the raft, and the reactions acting downwards at each load point, the values of the reactions being obtained by treating the raft as a reversed floor carrying a uniformly distributed load equal in value to the average ground pressure, with the load points remaining at the same level. System F_B.—A system of forces each equal to the difference between the actual column loads and the reactions in system F_A; these forces act downwards under the greater loads and upwards under the lighter loads, and their sum is zero if the raft is not eccentrically loaded; if the raft is eccentrically loaded a couple is introduced which must be balanced by an assumed variation in the ground pressure. System F_C.—A system of forces comprising the difference between the ground pressure under the raft at any point and the average ground pressure, acting downwards in some parts of the raft and upwards in others; the upward and downward pressures must be distributed throughout the raft in such a way that a balanced system is produced. These differential pressures can be divided into (*a*) the variation in pressure due to the form of distribution of ground pressure; (*b*) the variation in pressure due to deflection of the raft; and (*c*) the variation in pressure due to variation in the properties of the ground.

The resultant system of forces acting on the raft is $F_A + F_B + F_C$. The moments and deflections due to F_A, F_B, and F_C are referred to as M_A, M_B, M_C, and Y_A, Y_B, Y_C; at any point in the raft the resultant bending moment or deflection is the sum of the bending moments or deflections due to each of the three balanced systems. The bending moment and deflection due to (F_A) can be calculated in the same way as those for the beams of a floor. To calculate those for (F_B) is more difficult, as the points of application of forces in this system do not remain at one level, and the calculation of the reactions of the beams on each other requires the solution of an indeterminate frame. When these reactions have been determined the calculation of the forces, bending moments, and deflections for each beam needs only statics, and the design of the beam is similar to that of a continuous footing as described in the following.

CONTINUOUS-BEAM FOOTINGS.—The notation is given in *Fig.* 132 and in the following, the three systems being those described in the previous section.

P_1, P_2, etc., are the column loads for the worst arrangement of superimposed load on the superstructure.

w_e is the average ground pressure per unit area of the footing.

W is the average earth pressure per unit length of the beam.

$2qW$ is the difference between the upward distributed pressure at the middle and ends of the beam; $W + pqW$ is the distributed pressure at the ends of the beam; $W - mqW$ is the distributed pressure at the middle of the beam; $p + m = 2$.

L is the length of the beam.

b is the breadth of the beam.

I is the moment of inertia of the beam.

E is Young's modulus for the beam.

F_A is the system (1) of forces shown in *Fig.* 132 (*b*), that is W per unit length of beam upwards and R_1, R_2, etc., the equilibrating support reactions when the points of support remain at the same level. F_B is the system (2) of forces shown in *Fig.* 132 (*d*), that is $P_1 - R_1$, $P_2 - R_2$, etc. F_C is the system (3) of the forces shown in *Fig.* 132 (*f*), that is the difference between the average distributed upward pressure W and the actual distributed upward pressure.

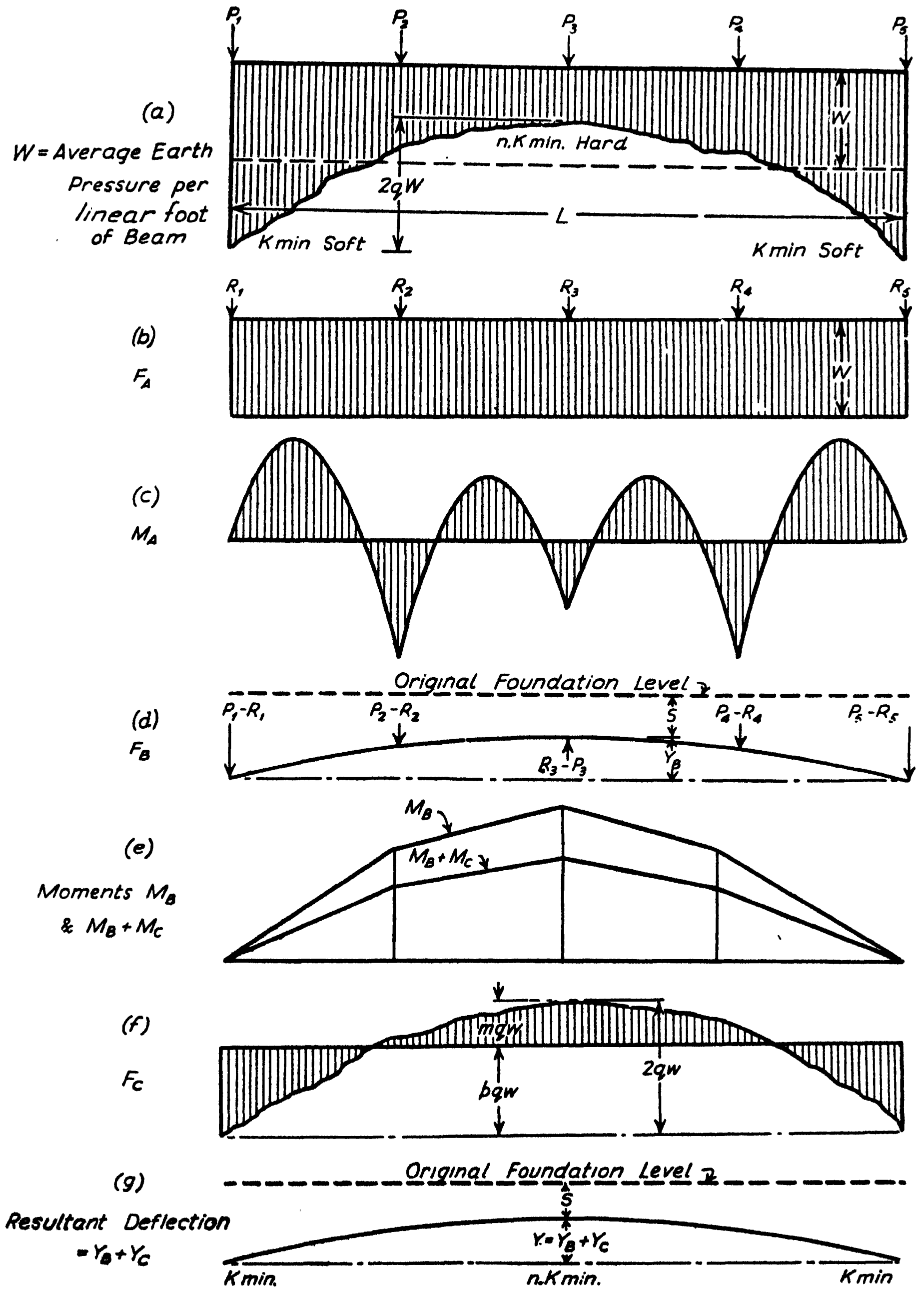
M_A, M_B, and M_C are the bending moments due to the forces F_A, F_B, and F_C respectively.

Y_A, Y_B, and Y_C are the deflections due to the forces F_A, F_B, and F_C respectively.

S is the settlement of the beam at mid-span.

K is the modulus of elasticity of the soil or the coefficient of settlement. $K_{min.}$ and $nK_{min.}$ are the minimum and maximum safe limiting values of K for the site.

$C_{min.}$ is the minimum safe limiting value of C in the expression $Y_C = \dfrac{CqwL^4}{EI}$.



P1 P2 P3 P4 P5
(a)
W = Average Earth Pressure per linear foot of Beam
W
n.K min. Hard
2qW
L
K min Soft
K min Soft
R1 R2 R3 R4 R5
(b)
FA
W
(c)
MA
Original Foundation Level
(d)
FB
P1-R1
P2-R2
S
P4-R4
P5-R5
R3-P3
YB
MB
(e)
MB+MC
Moments MB & MB+MC
(f)
FC
mqw
bqw
2qw
Original Foundation Level
(g)
Resultant Deflection = YB+YC
S
Y=YB+YC
K min.
n.K min.
K min

FIG. 132.

The resultant forces acting on the beam are $F_A + F_B + F_C$, the resultant bending moments are $M_A + M_B + M_C$, and the resultant deflections of the beam are $Y_A + Y_B + Y_C = Y$. When F_B deflects the ends of the beam downwards, the worst case of soil support occurs when K is $K_{min.}$ at the ends and $nK_{min.}$ at the middle of the beam. The settlement of the beam at mid-span is $-S = \frac{W - mqW}{bnK_{min.}}$, and at the ends $-S - Y = \frac{W + pqW}{bK_{min.}}$. Therefore

$$Y = -\frac{qW}{bK_{min.}}\left(p + \frac{m}{n}\right) - \frac{W}{bK_{min.}}\left(1 - \frac{1}{n}\right).$$

Since $p + \frac{m}{n}$ can never exceed 2, this can be taken as a safe limiting value; therefore within safe limits

$$Y = -\frac{2qw_e}{K_{min.}} - \frac{w_e}{K_{min.}}\left(1 - \frac{1}{n}\right) \quad . \quad . \quad . \quad . \quad \text{(A)}$$

There is no deflection Y_A at points of support and it is usually negligible at intermediate points. Y_B can be calculated in a given case from known values of F_B and M_B. Y_C can be expressed as $\frac{C_{min.}qWL^4}{EI}$. Thus $Y = Y_A + Y_B + Y_C$ within safe limits is

$$Y = Y_B + C_{min.}q\frac{WL^4}{EI} \quad . \quad . \quad . \quad . \quad \text{(B)}$$

Combining equations (A) and (B) to eliminate q, within safe limits,

$$Y = \frac{\frac{2w_eY_B}{K_{min.}} - \frac{w_e}{K_{min.}}\left(1 - \frac{1}{n}\right)C_{min.}\frac{WL^4}{EI}}{C_{min.}\frac{WL^4}{EI} + \frac{2w_e}{K_{min.}}}.$$

At mid-span, and very approximately at intermediate points, $\frac{Y_B}{Y_C} = \frac{M_B}{M_C}$. Therefore $M_B + M_C = M_B\frac{Y_B + Y_C}{Y_B} = \frac{M_BY}{Y_B}$. The resultant bending moment acting on the beam is $M_A + M_B + M_C$, that is, within safe limits, $M_A + \frac{M_BY}{Y_B}$.

In a particular case it is best to plot equations (A) and (B) as in *Fig.* 133 when determining safe limiting values. The straight line for equation (A) is conveniently termed the "soil line" for the bearing area, and the line for equation (B) the "beam line." For a case such as is shown in *Fig.* 132, $p + \frac{m}{n}$ should be as great as possible and $K_{min.}$ and $C_{min.}$ as small as possible in order to make $M_A + M_B \cdot \frac{Y}{Y_B}$ as great as possible. The value of $K_{min.}$ is determined from pressure tests, boreholes, and records of settlements on similar sites. It can be shown that 0·0035 is a safe value to adopt for $C_{min.}$ and, even if $\frac{p}{m}$ or $\frac{m}{p}$ is as much as 4, little variation in the value of C is obtained.

If the load on the beam is not symmetrical it is simple to design for an assumed symmetrical loading which covers the case, or, if the greatest value of Y_B is substituted in equation (B), the error will be negligible unless the point of greatest deflection is, say, outside the middle-third of the beam.

In the foundation of a framed building the resistance of the frame to bending of the footing beams is often considerable, and a method of taking this into account is described in the book previously mentioned, where also are given several numerical examples of the design of rafts.

The bending moments in raft beams are least in beams of greatest flexibility, but the flexibility must be limited so that the deflection does not exceed, say, the length ÷ 1350 if the raft supports a building with panels liable to crack.

ECCENTRIC FOOTINGS.—The design of eccentric footings (*Fig.* 134) is generally based on the assumption that the earth pressure is uniformly distributed under

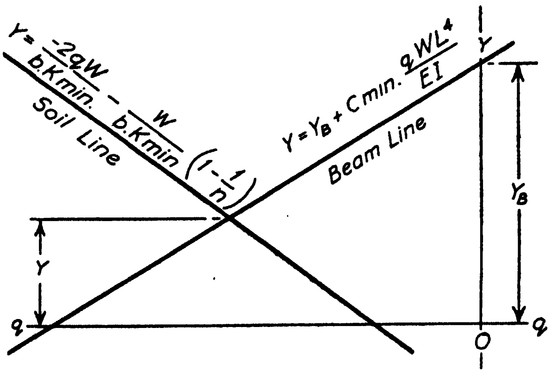


FIG. 133.

the eccentric base and that there is no pressure below the strap. If it is further assumed that the eccentric base is infinitely stiff, and that one base does not settle relatively to another, the upward reaction to be resisted by the inner column due to the load P_1 on the eccentric column is $P_2 = \frac{P_1 e}{b + 0{\cdot}5a}$. The total load to be resisted by the eccentric base is $P_1 + P_2$, and the average ground pressure is $W = \frac{P_1 + P_2}{ad}$ [*Fig.* 134 (*b*)]. The bending moment in the strap can be calculated by simple statics and its greatest value, which is attained at XX, is $P_2 b$. The shearing force in the strap is P_2.

If an eccentric footing bears on hard ground it may be worth while investigating the reduction in bending moment in the strap that occurs if a variable earth pressure is taken into account. A method of so doing is given in the book previously mentioned.

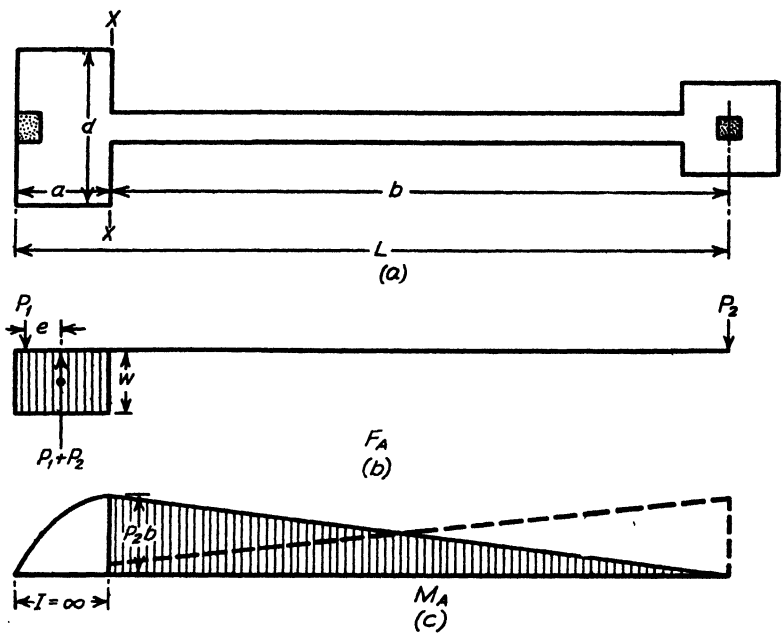


Fig. 134.

Joints.

Joints in reinforced concrete structures are either temporary construction joints or are permanent joints provided to allow freedom of movement due to shrinkage, changes of temperature, or unequal settlement.

Construction Joints.—A construction joint is a joint at a halt in the process of concreting and, to maintain the monolithic nature of a structure, it should be formed in such a way that so far as possible the member is as strong at the joint as elsewhere. Horizontal construction joints, unless in a plane subjected to a great horizontal shearing force, are formed by screeding the concrete to a horizontal surface, which is cleaned and brushed with cement grout before continuing concreting. Vertical construction joints should only be made where the shearing force is least. The concrete first deposited should be retained at the joint by a vertical board. A joggle, which will resist any small shearing forces that may occur, should be provided as in *Fig.* 135 (*a*). Such a joint should be used in beams and slabs.

Contraction Joints.—Contraction joints are sometimes formed in roads, retaining walls, ducts, and the linings of tunnels at intervals of 15 ft. to 60 ft. to avoid serious cracking due to shrinkage. Such joints may not be necessary if there is sufficient reinforcement or uniformly-distributed external friction to resist the movement due to shrinkage (see Chapter V). The theoretical amount by which a joint can open due to unrestrained shrinkage can be calculated by assuming a coefficient of linear shrinkage of 0·0005. If alternate panels of slabs or walls are cast and allowed to harden before the intermediate panels are cast, the amount of the shrinkage is about half, since the coefficient of linear shrinkage after, say, 28 days is about 0·00025. A similar procedure adopted in concreting an arch also reduces the total shrinkage. *Fig.* 135 (*b*) shows a suitable shrinkage joint in a wall. The joint in *Fig.* 135 (*c*) is a shrinkage or contraction joint suitable for a road, and forms a weak plane which determines the position of the crack. Bituminous filling is generally provided to seal the joints. Some reinforced concrete roads, notably in the United States, have been constructed without contraction joints and have been satisfactory; fine well-distributed cracks occur, but there is rarely any arching due to thermal expansion.

EXPANSION JOINTS.—In the decks of jetties and bridges in which there is sufficient longitudinal reinforcement and sufficient flexibility in the piles or other

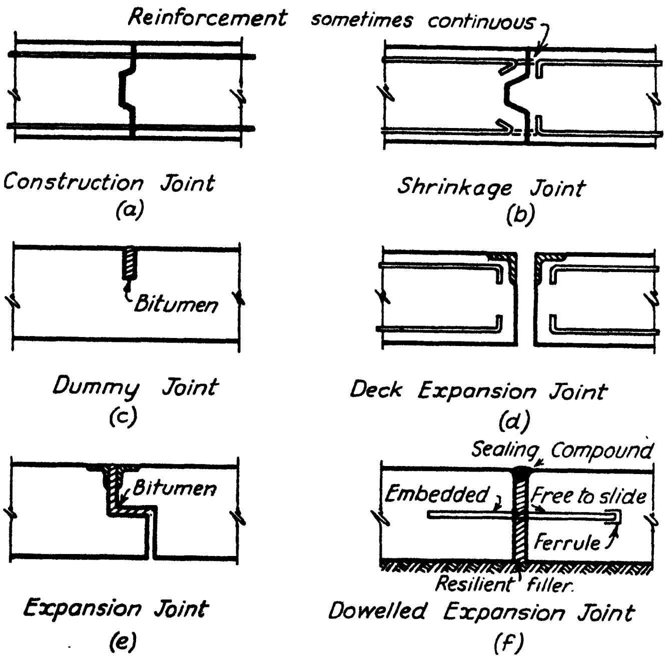


FIG. 135.

supports to reduce shrinkage stresses to a negligible amount over a long length of slab, joints as in *Fig.* 135 (*d*) may be used. The gap permits expansion due to changes of temperature to take place. The spacing of the joints and the greatest probable width of the gap can be calculated approximately for a known rise in temperature above that at the time of casting. The coefficient of thermal expansion of concrete is about 0·0000035 per deg. Fahr. If a closed joint is desired, a joint as in *Fig.* 135 (*e*) may be used; this joint can transmit vertical shearing force, if necessary, and is therefore suitable for suspended slabs in which an unequal settlement of the supports is liable to occur. Expansion joints in roads should also be capable of transferring load, and this is effected by a dowelled-joint as in *Fig.* 135 (*f*).

II.—DETAILS OF REINFORCEMENT AND DRAWING-OFFICE PRACTICE.

Principles of Design.

For the purpose of detailed design the members of a reinforced concrete structure are generally either slabs, beams (including continuous frames and arches), or columns.

Slabs can be classified as: (*a*) Spanning in one direction and supported on beams or walls; (*b*) Spanning in two directions and supported on beams or walls; (*c*) Spanning in two directions and supported only on columns (mushroom or flat-slab construction); (*d*) Arching between abutments or tied supports; (*e*) Spanning as a vault between beams, walls, or columns; (*f*) Acting in ring tension; and (*g*) Spanning in one or in two directions as a foundation to distribute the loads from columns on to the ground.

Beams, continuous frames, and arches generally support loaded slabs and can be classified as: (*a*) Beams (rectangular, ell-beams, or tee-beams) unrestrained at their ends; (*b*) Beams (rectangular, ell-beams, or tee-beams) restrained at their ends by continuity with beams, columns, or other members of a frame; and (*c*) Single-span or multiple-span frames and arches subjected to bending combined with thrusts resisted at the supports by ties or rigid abutments.

Columns can be classified as: (*a*) Short columns subjected to axial loads but no bending moment; (*b*) Long columns subjected to axial loads and bending only due to buckling; (*c*) Columns subject to bending due to eccentric loads; and (*d*) Vertical or sloping members of a frame subjected to direct load, bending moment, and shearing force.

The best structural system for supporting the loads is selected by consideration of the purpose of the structure, the nature of the supports, æsthetic considerations, cost, and the facilities available for construction. The greatest bending moments, shearing forces, and thrusts or pulls are then determined for every critical section of each member. Suitable dimensions of the members are selected so that at each critical section the compressive and shearing stresses in the concrete do not exceed the permissible working stresses. Adjustments are made to the sizes of the members so that the shuttering can be re-used the greatest number of times without alteration. At the same time the sizes of the members and the quality of the concrete are considered in order that the total cost of shuttering, reinforcement, and concrete is as low as possible (see Chapter III). Haunches, splays, changes of size, awkward intersections of members, and all such factors causing difficulty in fixing and removing the shuttering and placing the concrete are so far as possible eliminated. Having finally determined the dimensions of all members, the area of reinforcement required at each critical section of every member is then determined with respect to tension, compression due to bending, the shearing resistance of stirrups and inclined bars, direct stresses due to tension or compression, and the stresses due to tension or compression combined with bending.

Reinforcement.—The reinforcement provided at every section must be carried sufficiently far beyond the section to enable the required grip to be developed (see Chapter III). In continuous framed structures the bending moments close to the supports and at the supports chiefly determine the tensile and compressive reinforcement required. Bars must be lapped or extended far enough beyond a section near a support to enable the full grip to be developed, and where possible the reinforcement should follow the main lines of tensile stress in the intersecting members.

Hooks should not be provided in tensile zones of concrete as they may cause cracking. In such a case the bars should be bent to project into a compressive zone, or made longer to compensate for the omission of the hook.

In a cranked beam (as in a staircase) or a slab continuous around a corner (as in a tank) the radius r of the bend in a bar to resist a tensile force T [*Fig.* 136 (*a*)] must be great enough to reduce the outward radial pressure to that which can be safely resisted by the concrete in tension and shear [*Fig.* 136 (*c*)]. If contact with the main bars is assured, stirrups as in *Fig.* 136 (*b*) can be provided to resist the outward component of the force in the bar. The best design,

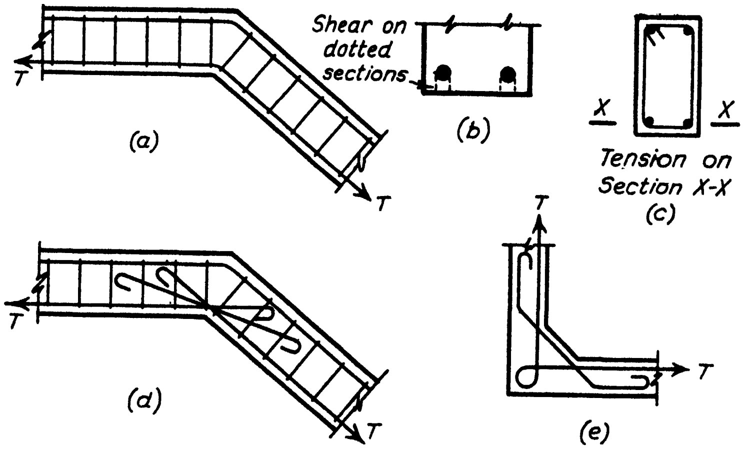


FIG. 136.

however, is for the bars in a beam or a member of a frame to extend beyond the intersection for a straight length equal to that required to attain sufficient grip as in *Fig.* 136 (*d*), or, in the case of a corner formed by two slabs, the bars should be looped as in *Fig.* 136 (*e*).

In addition to the reinforcement for which definite calculations are made, bars must also be provided for other purposes. Compressive reinforcement in beams and columns is liable to burst out of the concrete by buckling, and binders must be provided to counteract this tendency. Stirrups must be provided throughout the length of a beam to bind the concrete together, whether or not such stirrups are required for resistance to shearing force or other purposes. Bars must be provided in each corner throughout the length of a beam or column to hold the stirrups in position and generally to make the reinforcement a stiff frame which can be kept in position during concreting. Bars are provided at right angles to the main bars in a slab spanning in one direction to prevent shrinkage cracks and to prevent displacement of the main bars during concreting;

the amount of these secondary bars required is indeterminable but may be from 0·1 per cent. to 0·2 per cent. (by volume of the concrete). Bars in the form of a mesh should be placed near the surface of the concrete in any part where shrinkage cracks may develop and where no reinforcement is provided for structural purposes.

Horizontal spacing bars are provided between the layers of the main longitudinal bars in beams when there are two or more layers. Stools or supports must be provided to ensure that the reinforcement in the top of a slab is not trodden down during concreting. All points of intersection of bars must be tightly wired together so that the reinforcement is securely tied together into a mesh or cage and prevented from being displaced during concreting. Bars

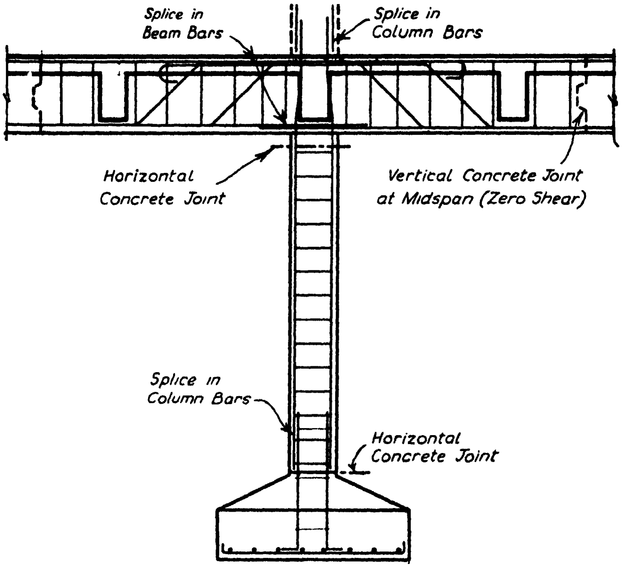


FIG. 137.

should be packed clear of shuttering by pieces of precast concrete of such a size that the correct cover is assured.

CONSTRUCTIONAL PROCEDURE.—In all structures the position of construction joints must be specified by the engineer. Horizontal joints in columns are generally made above and below the planes of intersection of horizontal beams and slabs. Vertical joints in beams and slabs should be made only where there is no shearing force, which is generally at the middle of the span. Splices or overlapping of reinforcement bars must be provided so that, as concreting and fixing of shuttering proceed, construction joints in the concrete can be made at permissible positions, and so that the projecting reinforcement is long enough to provide sufficient grip length to form a splice with the reinforcement in the succeeding section but not so long that it is difficult to keep the bars in position before the shuttering for the next section is erected.

In the construction of ordinary slabs, beams, and columns, which is typical of much reinforced concrete construction, it is usual to proceed in the following stages (*Fig.* 137) : (1) Support the bars in the footing on rough concrete covering the bottom of the excavation ; (2) Erect the splice bars (held together by stirrups) for the columns ; (3) Place the concrete for the footing ; (4) Erect the shuttering for the columns ; (5) Fix the reinforcement for the columns, the bars extending for a distance equal to the grip length above the level of the first floor ; (6) Place the concrete for the columns ; (7) Erect the shuttering for the beams and slabs ; (8) Fix the reinforcement for the beams in the first floor, most of the reinforcement for each beam being wired together to form a stiff cage ; (9) Fix any additional continuity-bars over the supports of the beams ; (10) Fix the main reinforcement in the slab ; fix the secondary reinforcement and tie it with wire to the main bars ; (11) Place the concrete for the beams and slabs ; (12) Repeat (4) to (11) for the upper stories.

Types of Reinforcement Bars.

Only a few different shapes of bars and stirrups are necessary for the more common reinforced concrete members, and, as shown in *Fig.* 138, these can be classified as bars suitable for slabs, beams, and columns (including members subjected to compression and bending), and footings.

BARS IN SLABS.—The shapes of bars (a) to (g) are typical of bars in slabs and walls. Bar (a) is for an end span of a slab and is bent up into the top at the supports of a slab, and bar (b) remains in the bottom of the slab throughout an end span. Bar (c) is for an interior span of a slab and is bent so that it provides reinforcement in the bottom of the slab at the middle of the span and in the top over the supports. Bar (d) is a stool, made from a piece of reinforcement bar, for supporting a mesh of bars in the top of a slab. Bar (e) is a loose bar in the top over the support of the slab and is most likely to be used in flat slabs. Corner bars (f) and (g) are used, as shown in *Fig.* 136 (*e*), at the junction of two slabs or walls.

Square bends are often formed at the ends of bars in slabs if the diameter is not greater than ½ in. ; otherwise a semi-circular hook is provided. The bend in a bar of type (a) occurs at about one-fifth of the span from the inner support and about 6 in. to 12 in. from the end support, and the bar extends to within 2 in. of the outer face of the support. It is common for half of the bars to be of type (a) and half of type (b), bars (a) and (b) alternating. By reversing alternate bars of type (c) in interior spans of equal length, the same amount of reinforcement is provided over a support as there is at the middle of the span. Although the outer support of an end span is considered to be free, there is always a certain amount of continuity due to the weight of the wall or torsional resistance of the supporting beam. In a flat slab the torsional moment is large, and the stirrups in the beam should be extended to splice with the bars in the top of the slab. Stools such as (d) should be shown in the schedule of reinforcement, as should also the straight lengths of secondary or shrinkage reinforcement ; this ensures that these bars and stools are not overlooked by the steel-fixer, that all the reinforcement is in the form of a mesh before concreting commences, and that no bar can be displaced or trodden down during concreting. Ordinary slabs spanning in two directions and flat slabs are reinforced with bars of types (a) to (c). In flat slabs additional short bars as type (e),

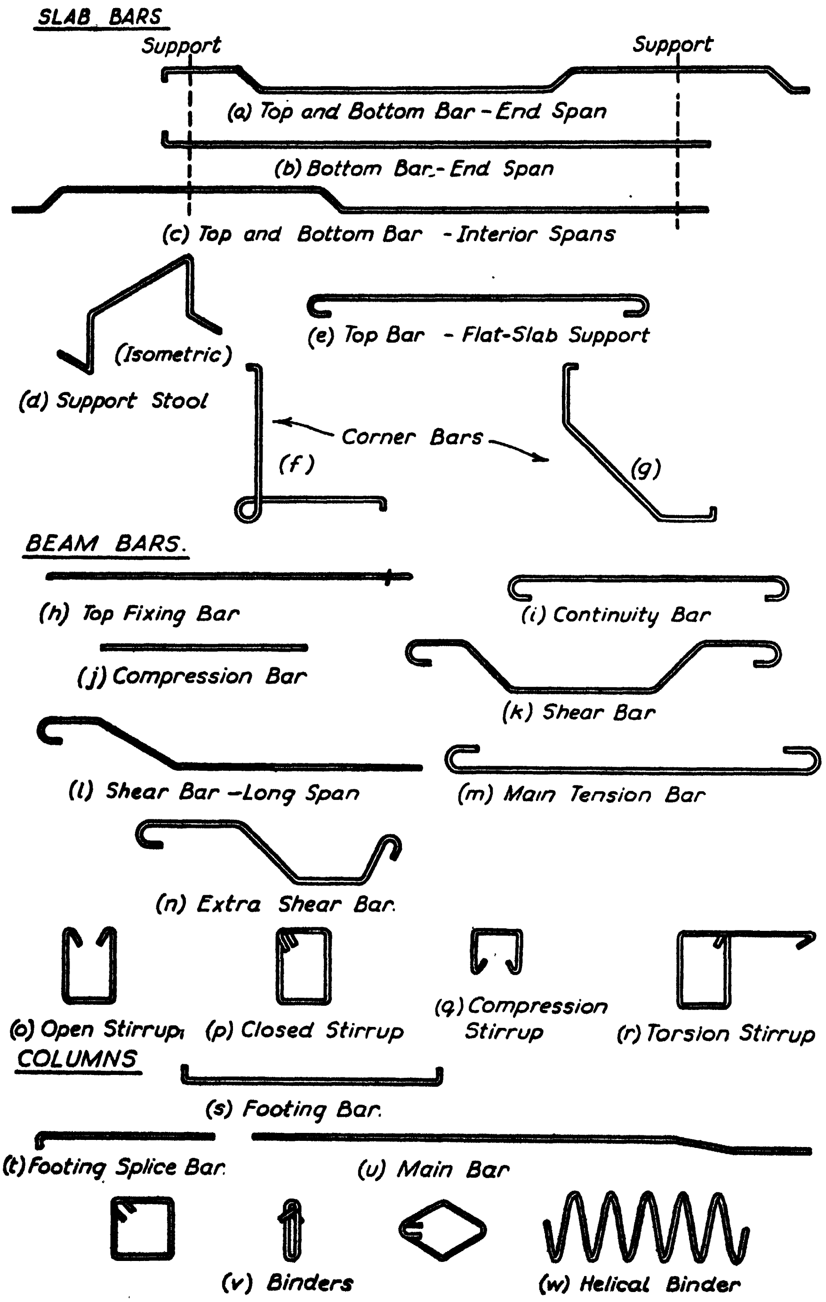

SLAB BARS
Support
Support
(a) Top and Bottom Bar - End Span
(b) Bottom Bar - End Span
(c) Top and Bottom Bar - Interior Spans
(Isometric)
(d) Support Stool
(e) Top Bar - Flat-Slab Support
Corner Bars
(f)
(g)
BEAM BARS.
(h) Top Fixing Bar
(i) Continuity Bar
(j) Compression Bar
(k) Shear Bar
(l) Shear Bar - Long Span
(m) Main Tension Bar
(n) Extra Shear Bar.
(o) Open Stirrup
(p) Closed Stirrup
(q) Compression Stirrup
(r) Torsion Stirrup
COLUMNS
(s) Footing Bar.
(t) Footing Splice Bar.
(u) Main Bar
(v) Binders
(w) Helical Binder

FIG. 138.

wired together to form a mesh, are generally used at points where the negative bending moment is greatest. Bars suitable for the corners of slabs are shown at (g) and (h), and help to maintain continuity around a corner as at the junction between the floor and wall of a rectangular tank.

BARS IN BEAMS.—Bars (h) to (r) are typical bars for beams. If it is not provided with compressive reinforcement throughout, a beam must have at least two fixing bars in the top ; these bars may be type (h), that is, straight. There are generally at least two main bars in the bottom, which, with the top bars and stirrups, are wired together to form a cage which fits into the shuttering with an amount of clearance to give the required thickness (or cover) of concrete over the reinforcement. It is advisable to allow a tolerance of about ½ in. to allow for the cage of reinforcement not being perfectly straight. In a beam continuous over two or more spans the bottom bars are generally like type (k), which provides resistance to shear as well as providing reinforcement for tension in the bottom of the beam at midspan and in the top over supports. Additional resistance to shear is provided by bars such as types (l) and (n). If they are not bent up, the main bars in the bottom are as type (m), which should extend over the supports sufficiently to lap with the bars in adjacent spans and to prevent shrinkage cracks and to provide sufficient grip to resist the compressive force in the bar. Additional bars, of type (i), may be required over the supports and these, and the bent-up bars, should extend far enough to provide the grip necessary to resist the tensile force in the bar. Extra compressive reinforcement is usually provided by straight bars of type (j).

Open stirrups, type (o), are used for tee-beams (the bars in the slab close the open side of the cage), as fixing the bars is easier than with closed stirrups. Closed stirrups, type (p), are used for ell-beams, rectangular beams, and where compressive reinforcement is provided in the top of a beam. Stirrups of type (q) used in conjunction with open stirrups, type (o), facilitate fixing the main bars which can be dropped easily into the lower open stirrups before the top stirrups are fixed. Torsional stirrups, type (r), are used for beams subjected to torsional moments such as those that restrain the edge of a flat slab.

BARS IN COLUMNS AND FOOTINGS.—The main bars in a column are generally as type (u), the set in which must fit easily into the reinforcement in the column above. The length beyond the set must be enough to provide the overlap required with the reinforcement in the upper column. Rectangular binders or links, types (v), or helical binding, type (w), must be provided throughout the length of the column, the pitch being determined as described in Chapter IV. Bars in isolated footings of columns and vertical bars projecting from a footing and splicing with the main bars in a column are types (s) and (t) respectively.

Detailing Reinforcement.

BENDING SCHEDULES.—In the bending schedule the bars must be carefully dimensioned so that when assembled they fit into the shuttering with sufficient cover of concrete. It is best to give overall dimensions and not dimensions from centre to centre of the bar, particularly for the set in a bent-up bar in a slab or beam. For the binders, the internal dimension should be given so that they determine accurately the dimensions of the cage of assembled reinforcement. A method of measuring and showing the bending dimensions of various types of reinforcement bars is given in a British Standard.

Suitable dimensions of hooks, the minimum radius to which a bar should be bent, the amount of concrete cover to the reinforcement, and other factors governing the detailed dimensions of the reinforcement, are specified in codes and regulations. Some notes on the cover, or thickness of concrete over the reinforcement, are given later.

For a binder or stirrup it is necessary to add about 3 in. to the straight dimensions to allow for each hook. For a shear bar that extends from the top row to the bottom row of bars in a beam, the set is generally the overall depth of the beam less 3 in. The set in a bar in a slab is generally the thickness of the slab less 1 in. for the bottom reinforcement; the deduction is greater for bars in layers other than the top or bottom and for bars with more than ½ in. of cover. It is most important for the dimensions of the set to be correct if serious difficulties in fixing are to be avoided. The internal dimension of a binder or stirrup is the overall dimension of the beam less 2 in. in each direction; this gives theoretically a cover of 1 in. over the main bars, but if the deduction is 2½ in. there is a margin of ½ in. for fitting the reinforcement into the shuttering; care should be taken to ensure that this tolerance does not reduce the space between adjacent bars below the amount permissible. In beams the horizontal distance between two bars should be not less than the diameter of the bar and not less than ¾ in.; when more than one layer of bars is provided there should be at least ½ in. between each layer.

The lengths of bars should, so far as possible, be such that the bars can be cut from stock lengths. It is useful to prepare a list of lengths which are convenient subdivisions of stock lengths. Lengths of bars should be multiples of 3 in. The number of different sizes of bars should be as small as possible.

Detailed Drawings.—Precautions to be observed in detailing reinforcement include ensuring that the concrete can flow reasonably freely among the bars to all parts of the shuttering and that there is no congestion of top reinforcement that is likely to make concreting difficult. It is necessary that bars projecting from a column can pass between the bars in adjacent beams, and that bars in secondary beams can pass between the bars in main beams. Splices in the reinforcement should be arranged to suit the positions of the construction joints in the concrete and to conform to the stages in the erection of the shuttering.

Main bars should terminate in compressive zones where possible. Hooks in concrete tensile zones should be avoided, and elsewhere they should be distributed by being staggered.

In drawings showing the details of the reinforcement the outline of the concrete is best shown in thick lines, and the reinforcement in thin lines conforming to the position of the centre-lines of the bars. Parts in section should be shaded. Cross sections showing all the dimensions should be drawn to a large scale at each critical part of a member, and the reinforcement should be shown on the cross section correct to scale so that it is easy to see whether the reinforcement is congested. The spacing and arrangement in slabs are best shown on a plan of the slab, only one bar being drawn in full to show the shape. Sections through the slab should be drawn to show clearly the position of sets and hooks, and whether the bars are in the top or bottom of the slab. Either the bars in the bottom or those in the top should be shown by broken lines.

In a slab one half of the bars at mid-span are generally extended to the supports. In a beam about one-third are so extended, the remainder being bent up or curtailed in so far as the reduction in the bending moment and requirements for length of grip permit. Bars in slabs are generally bent-up about the first fifth-point of the span. Distribution steel or shrinkage reinforcement should be not less than ¼ in. diameter at 18-in. centres; 0·2 per cent. of the area of the concrete is preferable. Where the shrinkage of long thin slabs may be restrained and joints are not provided, sufficient reinforcement should be provided in the direction of shrinkage so that the tensile strength of the reinforcement is greater than the tensile strength of the concrete (see Chapter V). In some structures, by concreting alternate parts and allowing shrinkage to occur before completion of the remaining parts, cracks may be reduced to a width which is not serious unless watertightness is required.

Additional bars to resist tension over the supports of continuous beams should be avoided where possible and the greatest use made of extensions to shear bars for this purpose. The reinforcement over the supports should generally be extended each way to the first quarter-point of the adjacent spans to provide the length of grip required, but each case should be checked. As a rule the outside bars in the bottom of the beam should be extended as far as the bottom of the bent-up part of the nearest shear bars in the adjacent span to provide compressive reinforcement and to aid in preventing shrinkage cracks. Additional binders should be provided when the load is applied close to the bottom of the beam.

Bars in columns should be set in sufficiently to fit inside the reinforcement of the column above, and should extend for a length of grip at least above the level of the floor. Binders in columns should be ¼-in. or $\frac{5}{16}$-in. bars at 9-in. centres, or as recommended in the D.S.I.R. Code, or the British Standard Code, or as specified in regulations. In the topmost 12 in. to 18 in., and in the bottom 3 ft., of a pile binders should be spaced at about 2 in. to 3 in. The ends of the main bars should all be at the same level at the top, and at the bottom should be in contact with the pile-shoe.

Cover of Concrete.—The least thickness (or cover) of concrete over a bar in a slab should be ½ in. In a beam the cover over the main bars should be 1 in., and not less than the diameter of the bar, and over the stirrups not less than ½ in., but an extra ¼ in. should be allowed on each side for inaccurate bending and for the main bars not being straight; the cover over the ends of bars should be 2 in. The cover over main bars in columns should be 1 in. (or 1½ in. for columns greater than 12 in. square), and ½ in. over the binders.

The foregoing amounts should be increased by ½ in. in external or other exposed members, and in marine structures, where contact with chemicals is possible, and foundations the cover should be at least 1½ in. to 2½ in.

III. PROCEDURE IN THE DRAWING OFFICE AND ON THE SITE.

To ensure economy and a high standard of workmanship it is essential to provide the carpenters and steel-fixers with clear drawings and bending schedules, and to simplify the work of cutting, bending, and fixing the reinforcement. The method described in the following is suitable where labour is unskilled. It is interesting to observe that this method has been used in South Africa, with native labour, particularly as in some parts of Africa reinforced concrete construction has frequently been avoided because local labour has been considered to be insufficiently skilled or reliable, although many reinforced concrete framed buildings have been constructed by native labour at the rate of a floor a week.

The work is divided into that done in the drawing office, in the bar-bending yard, and at the site.

Work in the Drawing Office.

The drawings and schedules of bars required include a drawing showing the arrangement of the beams and columns (*Fig.* 139), schedules of the reinforcement in the beams (*Fig.* 140) and columns (*Fig.* 141), and a table giving the sizes of the columns (*Fig.* 142). In addition to the schedules of bars, a cutting

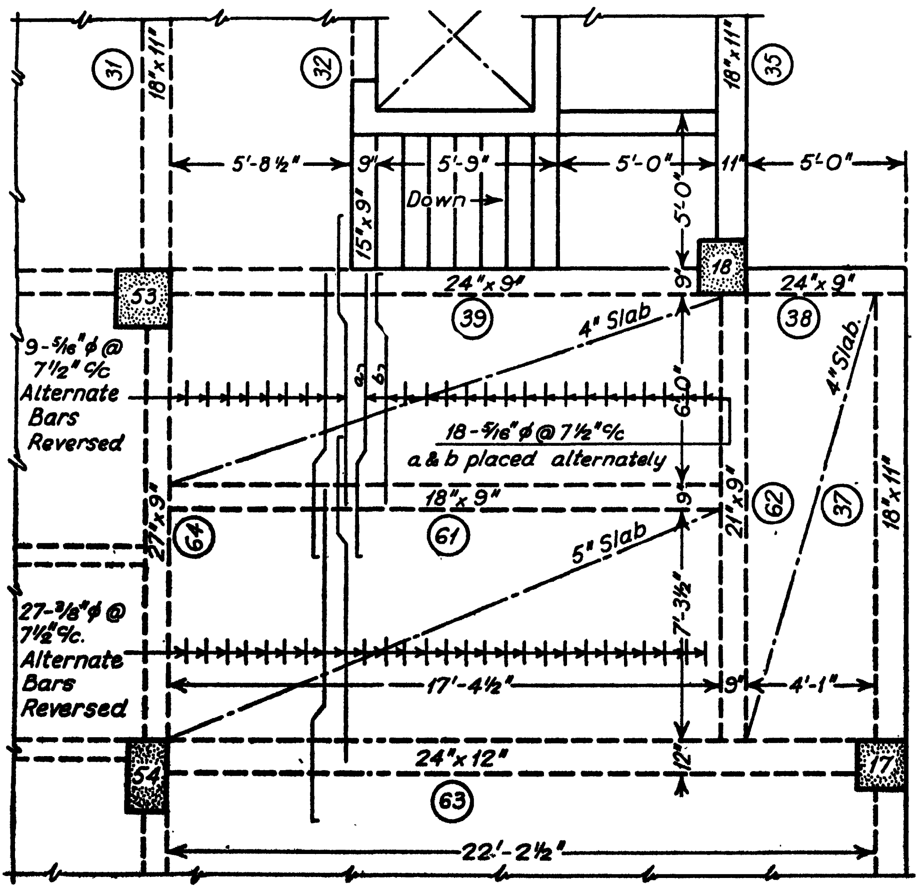


FIG. 139.

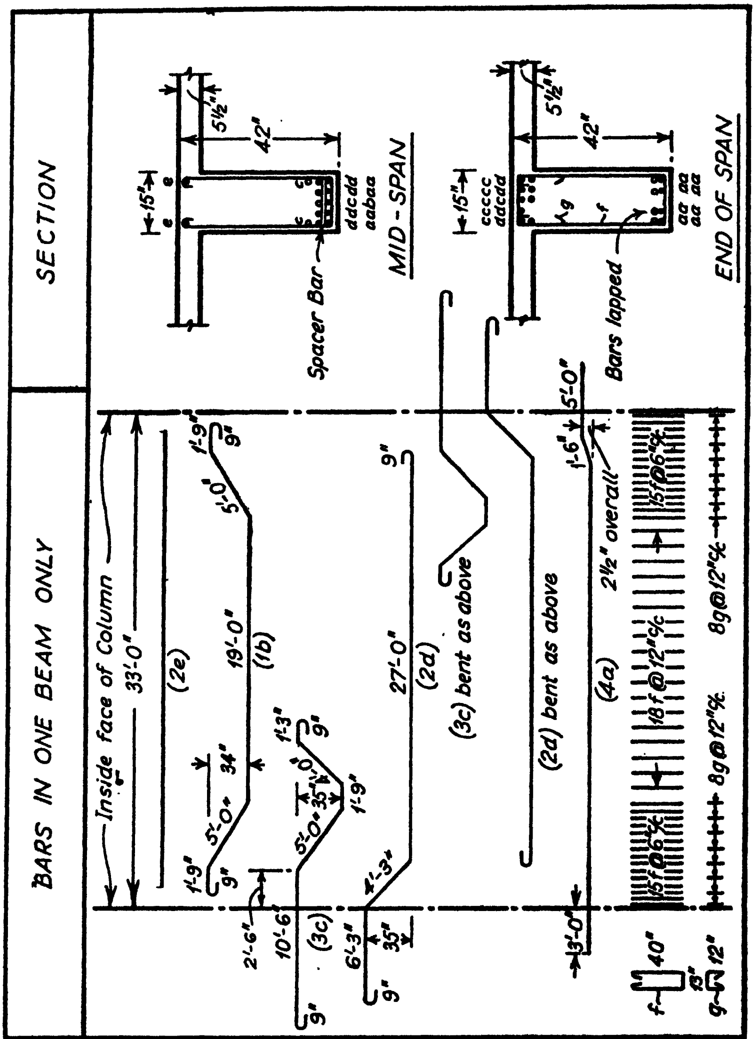
list of every bar required is prepared for the bending yard, the bars being grouped under diameters and lengths.

SLABS AND BEAMS.—The drawing (*Fig.* 139) of the arrangement of the beams is prepared on tracing cloth placed over the architect's plan and shows the setting-out of every beam, the reference numbers of every beam and column, the thickness of each floor slab, and the size and spacing of the reinforcement in the slabs.

The drawing showing the reinforcement in the beams combines the bending schedule, cross sections, and details of the beams, which in other methods are usually given on drawings of larger scale. It is easier for a draughtsman to draw the reinforcement for a beam when the bars are set out one below the other as shown in the schedule in *Fig.* 140, and the bending dimensions can be more easily calculated and more clearly shown. Also, by having the schedule of bars in each beam immediately opposite the cross sections and elevations, the lengths, numbers, and dimensions of the bars are readily summarised and checked. In most slab-and-beam structures, many beams are repeated, and the method shown in *Fig.* 140 saves much work in the drawing office as it is only necessary to enter the number of each type of beam in the appropriate column. *Fig.* 140 shows the schedule and detail of one span of a typical continuous beam. The purpose of each bar is as follows: Bars (e) are two small fixing bars in the top; bar (b) is a shear bar which also resists tension due to bending at mid-span. Bars (d) are two long shear bars which also resist tension due to bending at mid-span and over the supports. Bars (c) are two sets of three extra shear bars which also resist tension due to bending over the supports and are bent up so as to be anchored in the compressive zone. Bars (a) are four main bars resisting tension at mid-span and extended to resist compression at the supports; because of their size these bars are cranked to lap with similar bars from the next span. Bars similar to (d) and (c) project from the adjacent spans, and would be given in the schedule of bars in the next span, and supplement bars (d) and (c) in resisting tension at the support. Stirrups (f) are open at the top to facilitate fixing the bars in the beam and are closely spaced near the supports to resist shear; stirrups (g) are closing stirrups to support the top bars over the supports; similar stirrups could be used to support any top bar required for compressive reinforcement at mid-span, such as might be required if an opening in the slab reduced the width of the flange of the tee-beam. The cross sections in *Fig.* 140 show the arrangement of the bars and that the bars are not too congested, although placing concreting at the supports may be a little difficult.

COLUMNS.—The schedules of bars in the columns (*Fig.* 141) are prepared similarly to those for beams, but since the reinforcement in most columns is more standardised the schedule can be a printed form, the dimensions and quantities being filled in by a draughtsman. A table of columns in *Fig.* 142 gives the sizes of the columns and the sizes of main bars, and enables the details to be determined systematically from the calculations of the loads, and indicates to the carpenters the reduction in size that occurs at each floor level and which sides of the column are set back.

Calculations for typical slabs, beams, and columns similar to those in *Figs.* 139 to 142 are given in Appendix II.

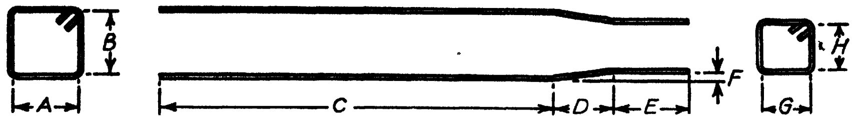


Beam Mk.	Beam No.	Ref.	No.	Dia.	Length
21	6	e	12	½"	30'-0"
		b	6	1¼"	34'-0"
		c	36	1¼"	24'-0"
		d	24	1¼"	39'-0"
		a	24	1¼"	39'-0"
		f	288	⅜"	8'-3"
		g	96	⅜"	3'-6"

FIG. 140.

COLUMN DETAILS. Job .... Sheet No ....

Supporting 2nd Floor .... Date ....



Column		Material			Binders		Column Bars				Top Binder		Spacing of
Mark	No. off	No.	Size	Length	A	B	C	D	E	F	G	H	Binders
2		36	1¼"	15'-6"			11'-0"	1'-6"	3'-0"	3¼"			
6	12	24	1¼"	13'-0"			11'-0"	1'-6"	6"	3¼"			
14		36	1¼"	15'-6"			11'-0"	1'-6"	3'-0"	6¼"			
22		144	5/16"	8'-0"	1'-9½"	1'-9½"					11½"	15"	12"
		36	5/16"	5'-0"									

Fig. 141.

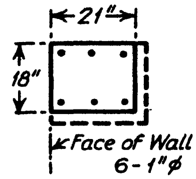
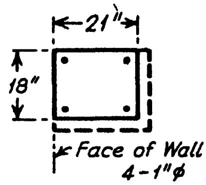
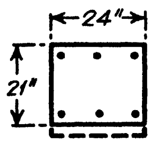
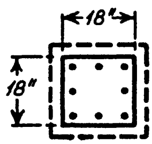
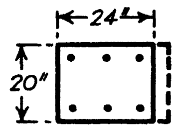
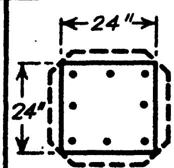
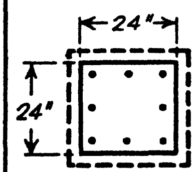
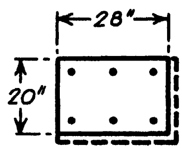
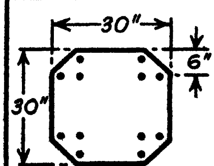
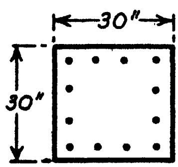
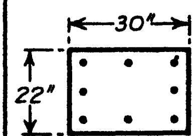
Supporting Floor	COLUMN No 2, 6, 14, 22	3, 7, 11, 5	4, 8, 10
4th	 6-1"ϕ	—	 4-1"ϕ
3rd	 6-1⅛"ϕ	 8-⅞"ϕ	 6-1"ϕ
2nd	 8-1¼"ϕ	 8-1"ϕ	 6-1⅛"ϕ
1st	 12-1¼"ϕ	 12-1"ϕ	 8-1⅛"ϕ

Fig. 142.

Work in the Bending Yard and on the Site.

The schedules of bars in the beams and columns and the cutting lists are sent to the bending yard, where the lengths are marked out by a competent man and a specimen of each type of bar is bent. The remaining bars are copied by less skilled men who have, however, been trained to know the importance of a bar having its correct shape and correct length beyond each bend. A labelling machine stamps a metal disk with the reference number of the beam or column to which a bundle of bent bars of one type applies. The disks are attached by wire to the bundle of bars. This system of labelling is useful for identification of the bars when they are fixed at the site by unskilled labour under the supervision of a competent foreman.

Reinforcement for a complete floor or part of a floor is despatched from the bending yard to the site and is delivered on to the shuttering for fixing in one operation. The foreman relates the beam references to those shown on the general drawing, and by referring to the bending schedules tells unskilled labourers which bars are required for each beam and in which order they should be placed. The labourers then have no difficulty in finding the bars, identified by the disks. The separate elevations of each bar indicate where some bars should turn up or how far some bars should extend beyond supports. The schedule of reinforcement in the slabs is given at the end of the schedules for the beams, and the position of the bars in the slabs is shown on the general drawing. The bars are tied in position by short pieces of wire twisted with large pliers, the efficiency of tying depending on the use of the right size and type of pliers.

Checking the reinforcement is quick and simple, as it is only necessary to confirm that all the bars in the schedule are in the beam and are fixed in the positions shown in the cross sections and elevations. By the method described in the foregoing, the reinforcement for the slabs and beams of a floor 50 ft. wide and 100 ft. long can be fixed in two or three days.

Checking Calculations and Drawings.

The most reliable designer is liable to make mistakes in computations, and a designer who knows his limitations with regard to accuracy, but who has a clear knowledge of structural principles and employs a sound system of checking, is far less liable to produce unsafe structures than one who is always confident in the accuracy of his calculations. It is always advisable to check designs by making approximate independent calculations. The chart in *Fig.* 143 is useful for this purpose. The loads on a beam are reduced to equivalent distributed loads. The bending moments are obtained from the vertical axis, adjusted for end-conditions by using the proportional scales marked FF and FS, and the area of the main tensile reinforcement is obtained from the left-hand horizontal axis. Values of $b_r a_1 d$ can be added to the chart for beams of common sizes, and with one operation of a slide-rule the shearing stress is determined. Similarly the " economic " area of reinforcement and the moment of resistance of the concrete can be determined, and the need, or otherwise, for compressive reinforcement is immediately ascertained. The reinforcement for shear is checked by using the tables in Chapter III. The thicknesses of slabs and areas of reinforcement generally become so familiar that an error can often be detected without reference to a chart.

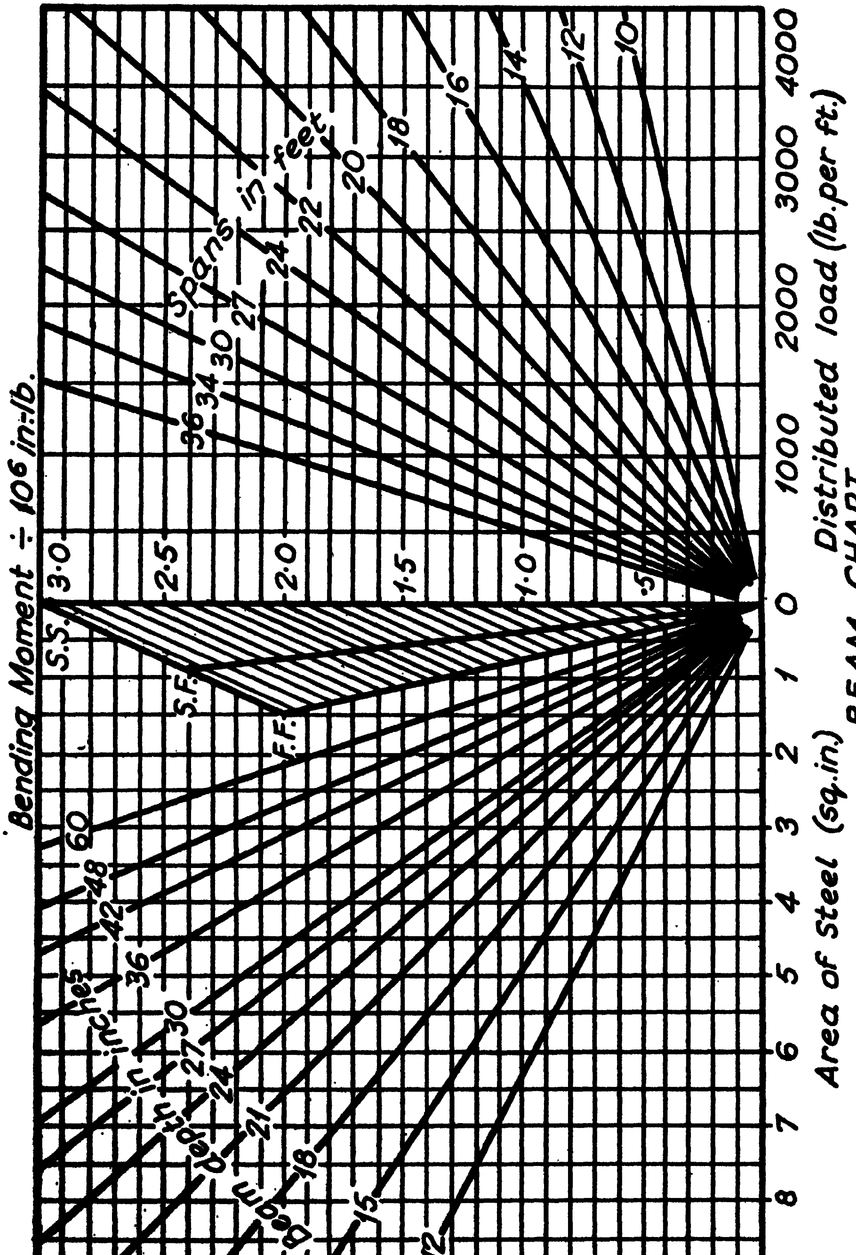

Bending Moment ÷ 10⁶ in-lb.
Distributed load (lb. per ft.)
Area of Steel (sq. in.)
BEAM CHART
Spans in feet
Beam depth in inches
S.S.
S.F.
F.F.

FIG. 143.

In checking designs, particularly those for buildings, it is advisable to apply a system such as considering each item in a list containing every factor which may affect the design. An example of such a list is as follows.

SLABS.—

(1) Superimposed distributed load, dead load, and any concentrated loads.
(2) Thickness of the slab and the area of reinforcement in each direction.

BEAMS.—

(1) Distributed superimposed and dead loads, and any concentrated loads.
(2) Check the area of the tensile reinforcement from the chart (*Fig.* 143), taking into account the end conditions.
(3) Determine whether compressive reinforcement is required at the supports or where openings in the slab occur.
(4) Shearing stress and reinforcement for shear, particularly if heavy loads are concentrated near the supports.
(5) Length of grip, particularly where the slope of the bending-moment diagram is steep or if large bars are used.
(6) Check that bending moments due to cantilevers and short spans are correctly taking into account in calculating the bending moments in continuous beams.

COLUMNS.—

(1) Dead loads and proportion of superimposed loads assumed.
(2) Size and amount of reinforcement.
(3) Slenderness ratio and end conditions, and consequent effect on permissible stresses.
(4) Working stresses applicable to the quality of the concrete.
(5) Bending moments due to wind forces or restrained beams.

REINFORCEMENT DETAILS.—

(1) Splices suit construction joints.
(2) Bars not too congested, particularly in the tops of beams.
(3) Shrinkage reinforcement, stools, and bars necessary to hold the reinforcement in position during concreting.
(4) Dimensions of sets in shear bars, dimensions of stirrups and binders, and lengths of bars.
(5) Hooks not in tensile zones, or corner bars not liable to pull out.
(6) Bars in columns and secondary beams can pass bars in main beams.
(7) Flanges of tee-beams suitably reinforced.

Examples of Drawings for Reinforced Concrete.

Additional examples to illustrate the details of the reinforcement in various structural members are given in *Figs.* 144 to 150.

BEAM IN A JETTY (*Fig.* 144).—This large transverse beam in the deck of a jetty cantilevers a short distance at one end, where it is supported by a pile; at the other end it is supported on a deep longitudinal beam. The shearing force is very great at the haunched end and is resisted by a combination of long closely-spaced double stirrups and inclined bars, the latter being extended at the top to assist in resisting tension over the support. The haunch bars and the bottom reinforcement are necessary to resist compression at the bottom of the beam at the support. At the other end the shearing force is less, but the top reinforcement is continued down the end of the beam to develop sufficient grip to balance the tensile stress in the top of the short cantilever over the support. The slab is sufficient to resist the compression in the top of the beam at

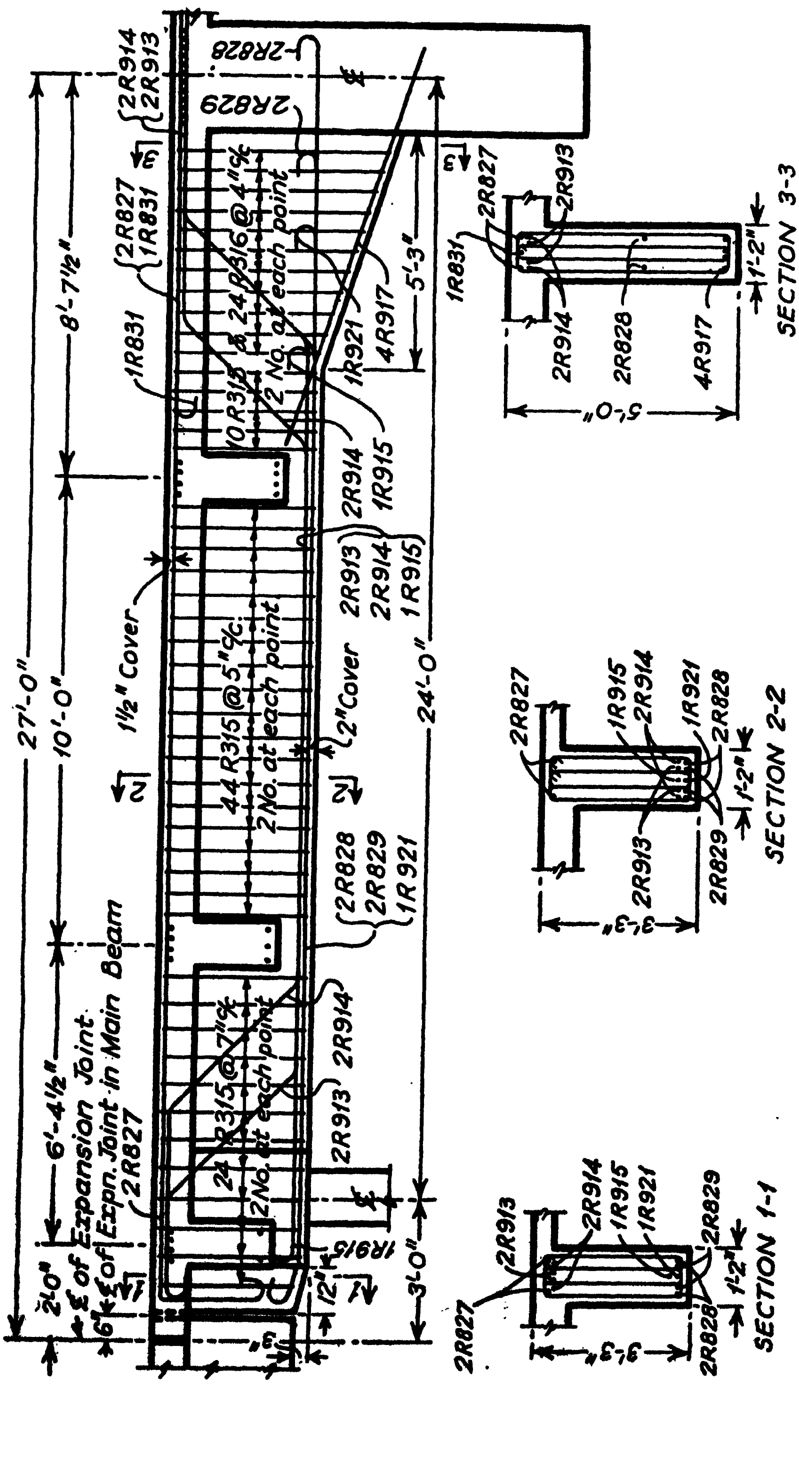

27'-0"
10'-0"
8'-7½"
6'-4½"
2'-0"
6"
℄ of Expansion Joint
℄ of Expn. Joint in Main Beam
1½" Cover
2" Cover
24'-0"
5'-3"
3'-0"
12"
44 R315 @ 5" c/c.
2 No. at each point
24 R315 @ 7" c/c
2 No. at each point
10 R315 & 24 R316 @ 4" c/c
2 No. at each point
2R827
1R831
2R828
2R829
2R913
2R914
1R915
1R921
4R917
SECTION 1-1
SECTION 2-2
SECTION 3-3
3'-3"
5'-0"
1'-2"

FIG. 144.

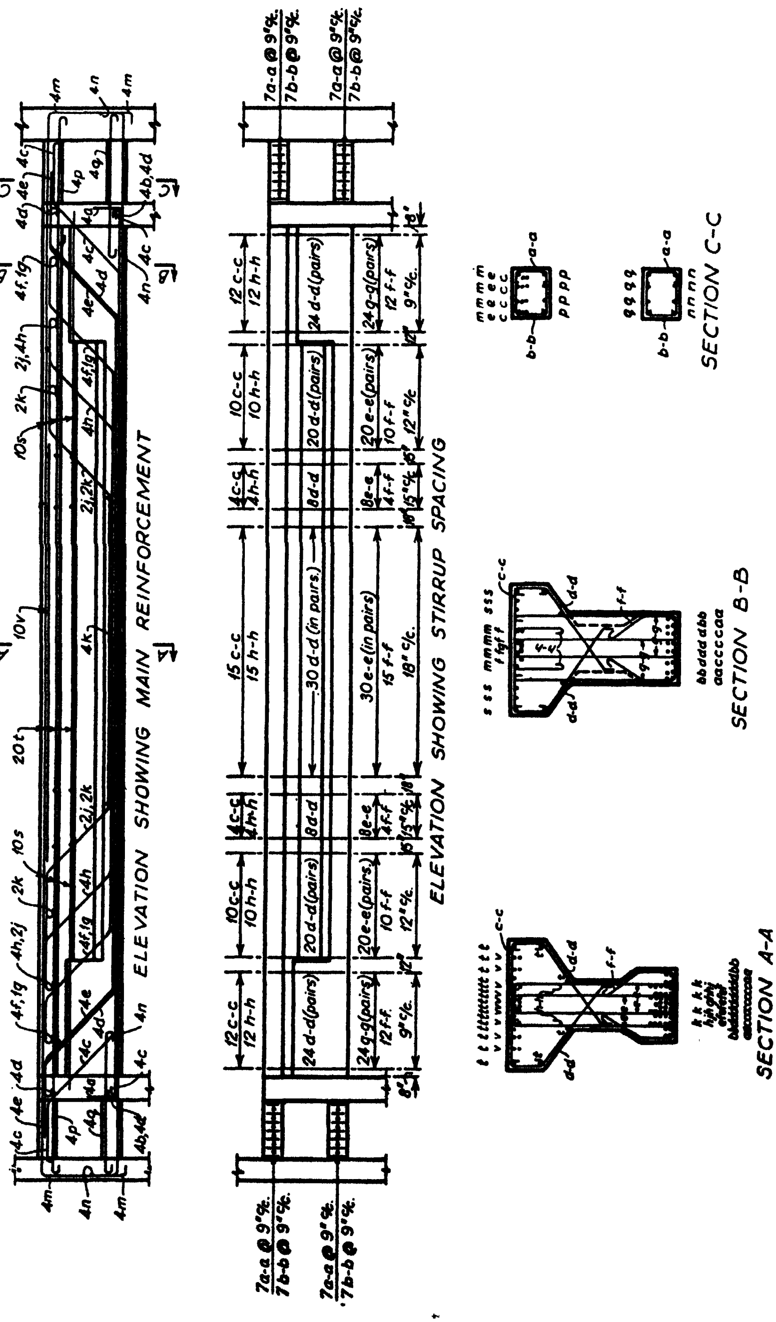
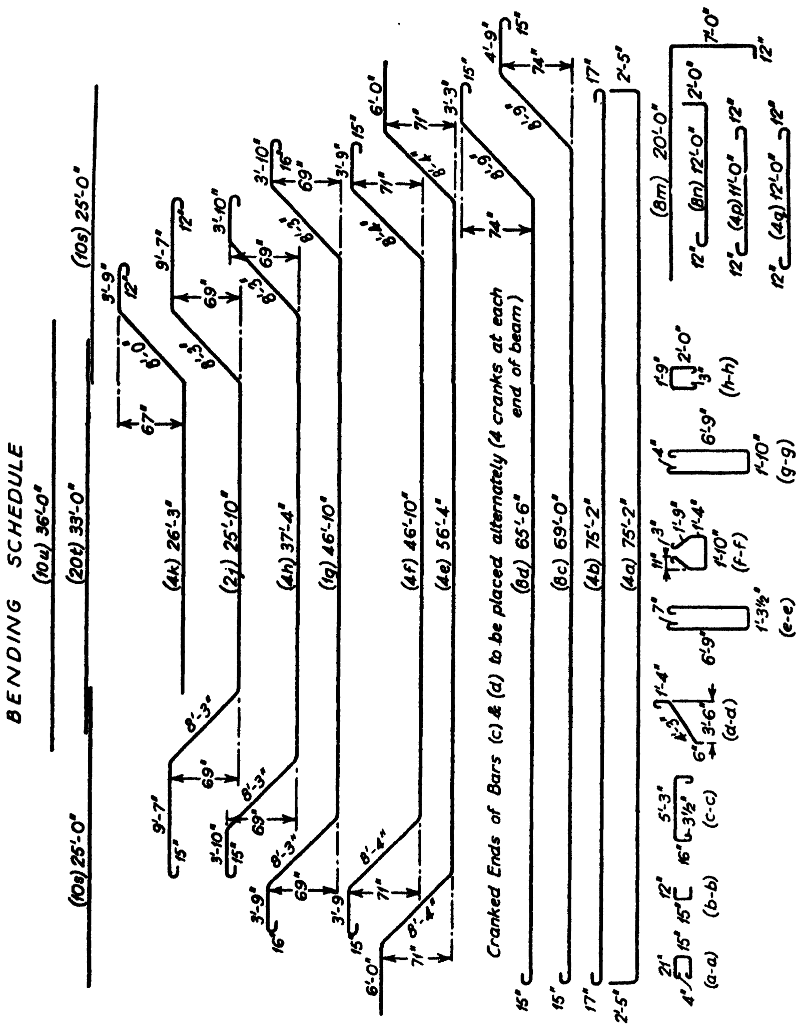

ELEVATION SHOWING MAIN REINFORCEMENT
ELEVATION SHOWING STIRRUP SPACING
SECTION A-A
SECTION B-B
SECTION C-C
7a-a @ 9" c/c.
7b-b @ 9" c/c.
30 d-d (in pairs)
30 e-e (in pairs)
15 c-c
15 h-h
15 f-f
18" c/c.

FIG. 145A.

mid-span. The bars in the secondary beams easily pass between the bars in the main beams. The bar references are those used by some designers and are interpreted thus: 3 *R* 833 means three (3) round bars (*R*) 8 × ⅛ in. = 1 in. diameter, No. 33 in the schedule.

Gallery Beam [*Figs.* 145 (*a*) and (*b*)].—In this single-span heavy beam



Total Steel			
Ref.	No.	dia.	Length
u	10	1¼"	36'-0"
s	20	1⅛"	25'-0"
t	20	1¼"	33'-0"
k	4	1⅛"	39'-0"
j	2	1½"	64'-0"
h	4	1½"	64'-0"
g	1	1½"	73'-6"
f	4	1½"	73'-6"
e	4	1½"	85'-0"
d	8	1½"	80'-0"
c	8	1½"	85'-0"
b	4	1½"	78'-0"
a	4	1½"	80'-0"
m	8	1¼"	28'-0"
n	8	1¼"	15'-0"
p	8	1¼"	13'-0"
q	8	1⅛"	14'-0"
a-a	28	3/8"	5'-6"
b-b	28	3/8"	3'-6"
c-c	67	1/2"	8'-6"
d-d	134	1/2"	6'-6"
e-e	86	1/2"	16'-0"
f-f	67	3/8"	9'-0"
g-g	48	1/2"	16'-0"
h-h	67	3/8"	6'-3"

Fig. 145B.

supporting a theatre gallery the main reinforcement comprises long bars that were specially rolled (or could have been made by welding shorter lengths). Each bar in the bottom is bent up immediately it is no longer required to resist the bending moment, and as an inclined bar resists the shearing force in combination with large closely-spaced stirrups. The beam supports secondary beams and cantilevers but, as there is no slab at the top, a large amount of compressive reinforcement is provided and is tied to the web of the beam by a considerable number of binders which are long enough to develop sufficient grip. The binders and stirrups are framed together by small longitudinal bars.

RECTANGULAR FRAME (*Fig.* 146).—This is one of a series of rectangular frames, of varying moment of inertia, which support the wall, roof, and floor of

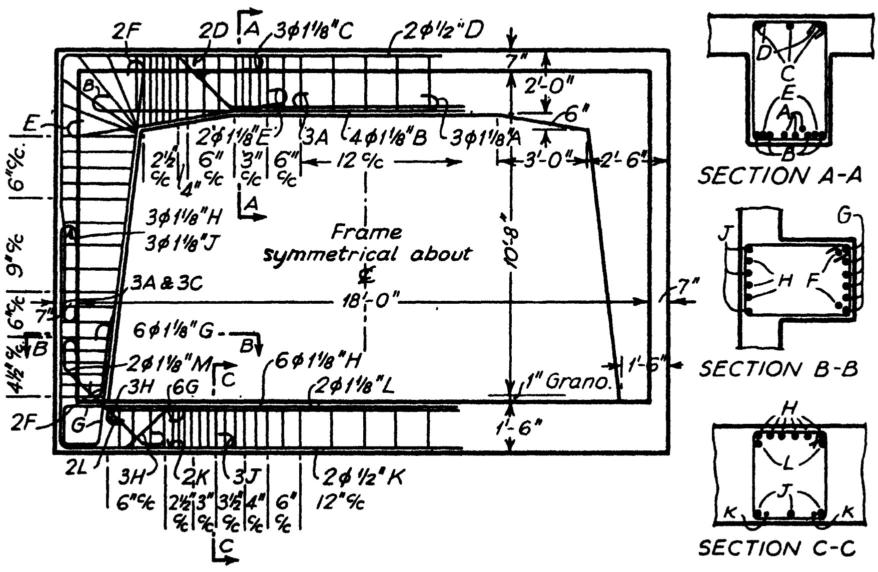


FIG. 146.

a deep underground pump-house subjected to large and varying external pressures. The shearing force in the roof and in the floor beams near the supports is great and is resisted by inclined bars and closely-spaced stirrups. The large bending moments at the corners, producing a tension on the outside and compression on the inside, are resisted by heavy reinforcement passing around the outside and across the inside of the corners, the bars being extended without bending to develop grip. The bars on the inside of the corners, the loop bars G and corner bars M, are also effective in the case of tension being developed on the inside by external pressures. The vertical rib is subjected to a direct thrust in addition to a bending moment. The stirrups therefore act as do binders in a column and as stirrups resisting shearing force. Closed stirrups are provided throughout to support the heavy reinforcement near both faces of the ribs. Splices in the reinforcement are arranged to suit a construction joint at the level of the floor, but the vertical rib is too short and the bars too large to justify an additional splice at the level of the soffit of the roof beam. The fixing of bar C before concreting the vertical rib would be a little difficult. The hooks being in the tensile zone in the bottom of the roof beam is not the best arrangement. It would also be better if the pitch of the stirrups in the horizontal rib were either constant or decreased uniformly towards the supports.

Bowstring Girder (*Fig.* 147).—This drawing, which is reproduced by the permission of the Institution of Civil Engineers, shows the reinforcement in the

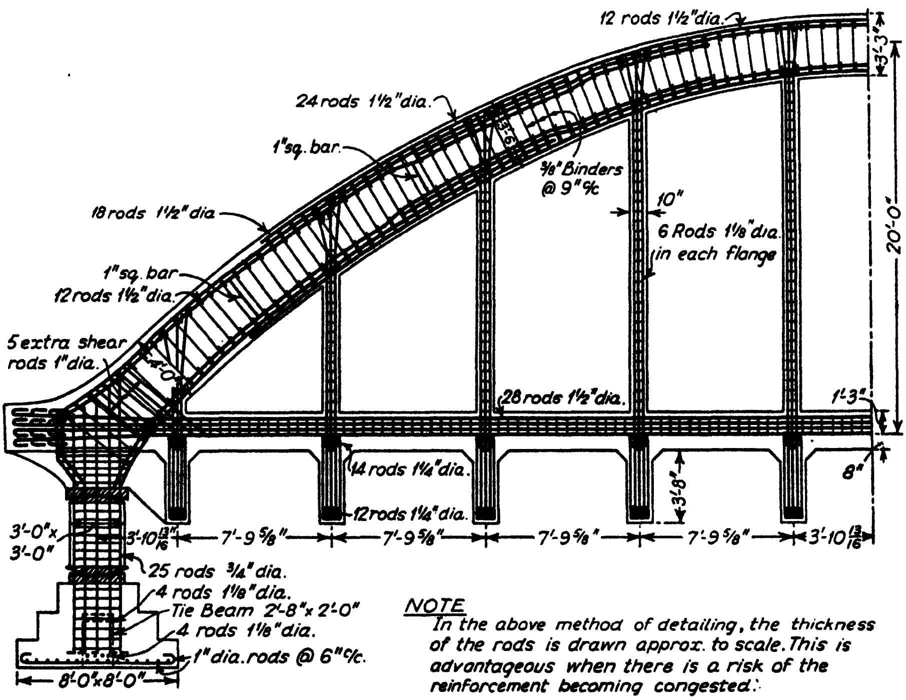
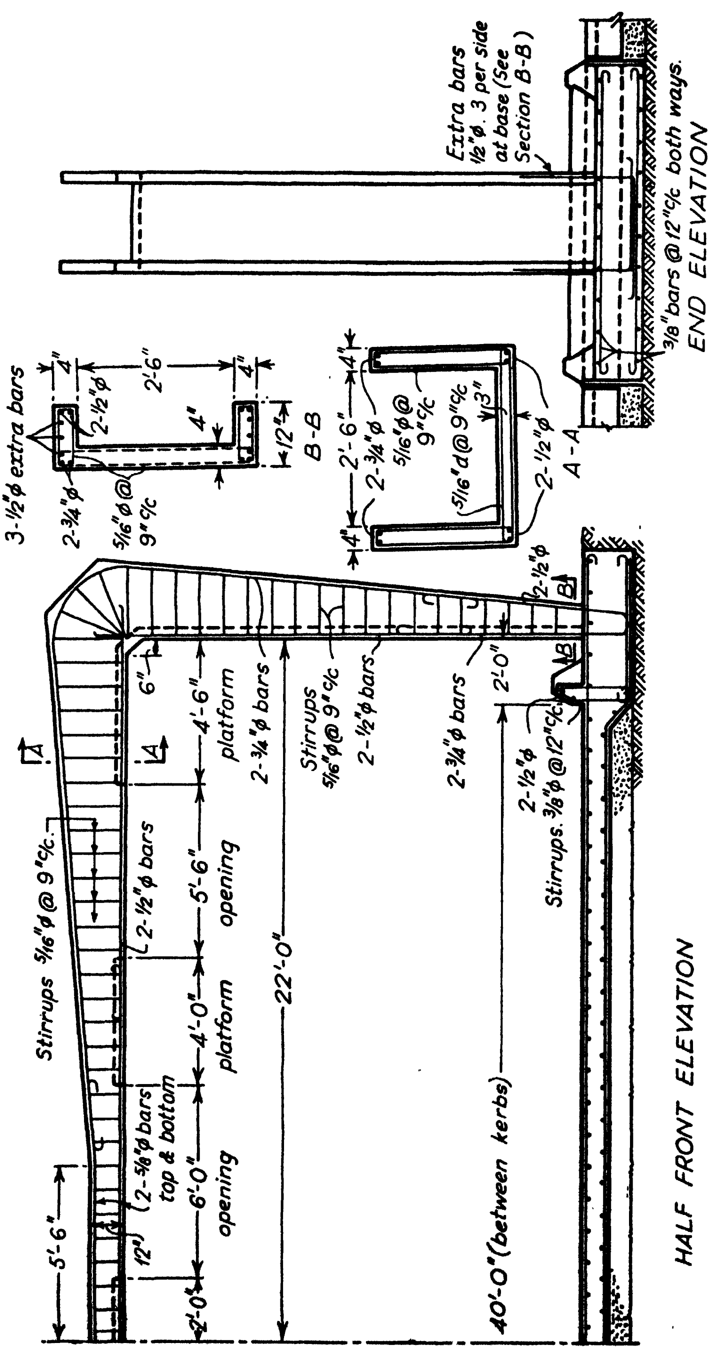


Fig. 147.

ties, hangers, and arch ribs of a bridge. The arch ribs are heavily reinforced members which resist compression and bending moment and therefore require strong closely-spaced binders. At the intersection of the tie and the rib at the springing the main reinforcement is extended well beyond the point of intersection to develop the full grip strength. A similar extension is provided at the intersection of the hangers and the rib and cross beams. The rocker bearing, providing free longitudinal movement due to expansion and contraction, is also shown.

Pipe Bridge (*Fig.* 148).—The reinforcement in the portal-frame pipe-bridge, for which calculations are given in Chapter II, is shown on this drawing. The shearing force on the beam is small, and inclined bars are not required. The main bars around the corner at the top of the vertical rib resist tension due to bending and have a large radius of curvature in order to restrict the radial pressure on the concrete. The compressive reinforcement on the inside of the corner



Extra bars 1/2"ϕ. 3 per side at base (See Section B-B)
3/8" bars @12" c/c both ways.
END ELEVATION
3-1/2"ϕ extra bars
2-3/4"ϕ
2-1/2"ϕ
5/16"ϕ@ 9" c/c
B-B
A-A
5/16"d@9" c/c
Stirrups 5/16"ϕ@9" c/c.
2-1/2"ϕ bars
2-5/8"ϕ bars top & bottom
2-3/4"ϕ bars
Stirrups 3/8"ϕ@12" c/c
40'-0" (between kerbs)
22'-0"
HALF FRONT ELEVATION

FIG. 148.

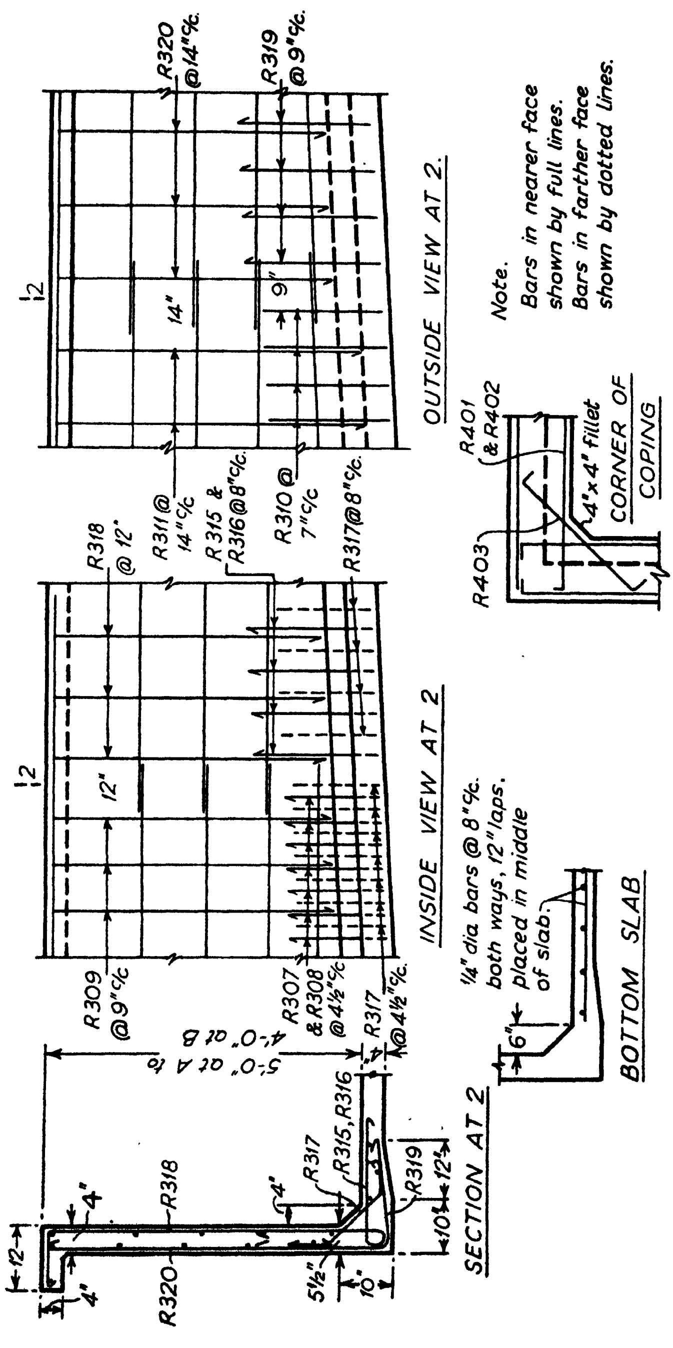

OUTSIDE VIEW AT 2.
INSIDE VIEW AT 2
SECTION AT 2
BOTTOM SLAB
CORNER OF COPING
Note.
Bars in nearer face shown by full lines.
Bars in farther face shown by dotted lines.
R320 @14"c/c
R319 @9"c/c
R311 @ 14"c/c
R315 & R316 @8"c/c
R310 @ 7"c/c
R317 @8"c/c
R318 @ 12"
R309 @9"c/c
R307 & R308 @4½"c/c
R317 @4½"c/c
R401 & R402
R403
4"x4" fillet
¼" dia bars @ 8"c/c both ways, 12" laps placed in middle of slab.
5'-0" at A to 4'-0" at B
R318
R320
R317
R315, R316
R319
14"
9"
12"
12
4"
5½
10"
6"

FIG. 149.

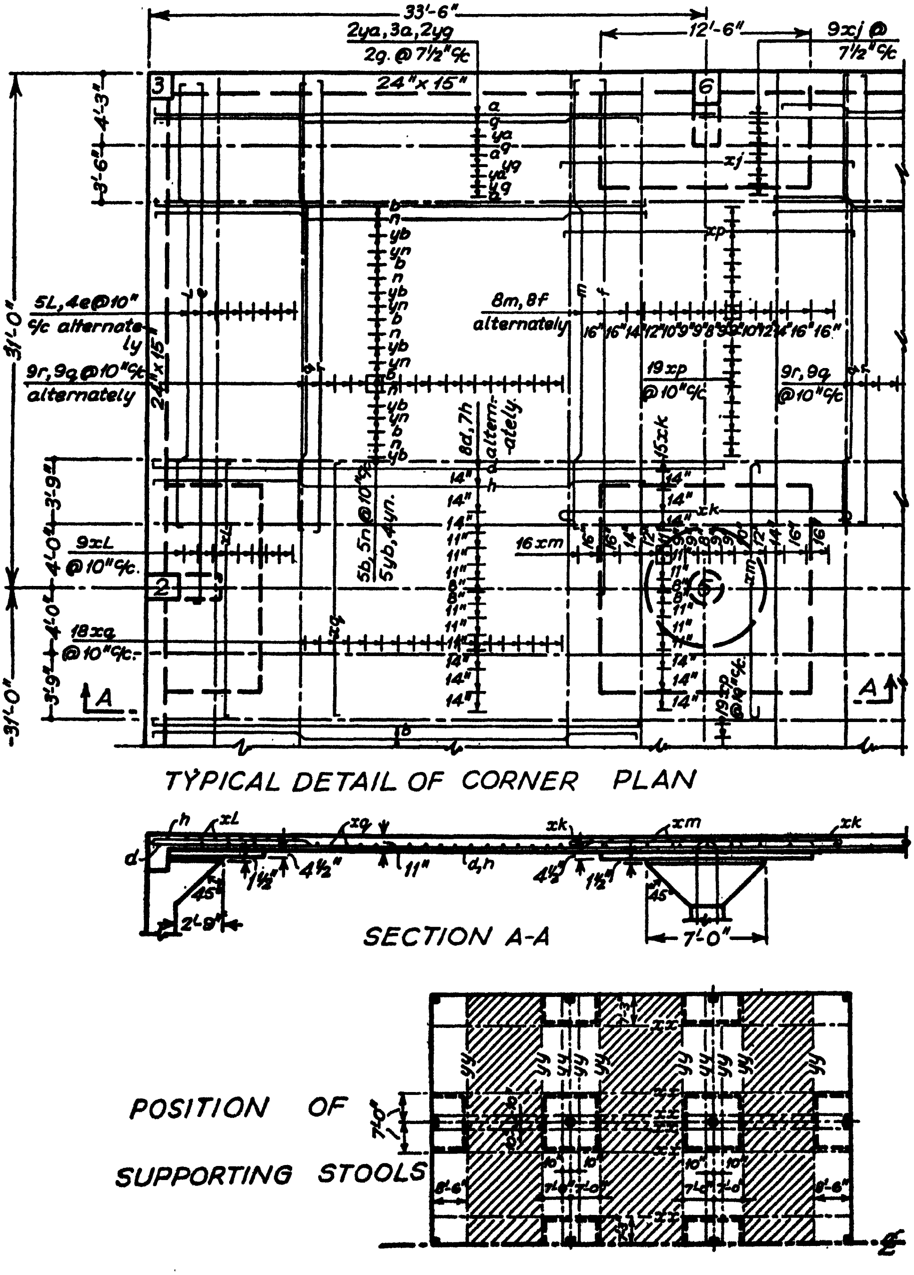

33'-6"
12'-6"
2ya, 3a, 2yg
2g. @ 7½" c/c
24" x 15"
9xj @
7½" c/c
5L, 4e @ 10"
c/c alternately
9r, 9q @ 10" c/c
alternately
8m, 8f
alternately
19 xp
@ 10" c/c
9r, 9q
@ 10" c/c
8d, 7h
alternately
15 xk
9xL
@ 10" c/c.
16 xm
5b, 5n @ 10" c/c
5yb, 4yn.
18 xq
@ 10" c/c.
31'-0"
TYPICAL DETAIL OF CORNER PLAN
SECTION A-A
7'-0"
2'-8"
45°
POSITION OF
SUPPORTING STOOLS
8'-6"

FIG. 150.

is extended to develop sufficient grip. The vertical rib resists direct compression and bending moment, and closed binders as in a column are therefore provided. The reinforcement in the road slab extends into the footing sufficiently far to develop full grip and to act as a tie.

SWIMMING POOL (*Fig.* 149).—The wall acts as a cantilever in resisting earth pressure on one face and water pressure on the other, and is restrained by the floor slab. The arrangement of the corner bars avoids subjecting the concrete to an outward pressure under the action of bending moments tending to open the corner. The vertical bars on both faces of the wall are curtailed as the bending moments reduce upwards. The horizontal distribution or shrinkage bars are placed on the inside of the main bars so that the latter can have the greatest effective depth. The coping stiffens the top edge of the wall and is tied in plan across the corners of the pool, where cracks would otherwise occur because of shrinkage and because of the tension which develops due to the walls tending to deflect outwards at the top in two directions at right angles. The central position of the mesh of bars in the floor slab resists shrinkage stresses and tension due to bending, either upwards or downwards, due to non-uniform settlement.

FLAT SLABS (*Fig.* 150).—The plan shows the arrangement of reinforcement of a typical flat-slab floor. The reinforcement in the bottom of the slab at mid-span comprises bars cranked at a distance of about one-fifth of the span from the centre-line of the columns and extending in the top beyond the centre-line. There are also straight bars extending to the centre-line in the bottom. In the "column bands" additional straight bars are provided in the top to supplement the extended cranked bars, as some of the latter are too short to extend beyond the centre-line of the columns unless very long bars are provided. Torsional stirrups in the edge beams lap with the bars in the top of the slab. Stools, fixed in the positions shown on the separate plan in *Fig.* 150, are advisable in flat slabs to prevent the bars being trodden down during concreting and reducing the effective depth.

The order in which the bars must be fixed should be given on the drawing as this assists the steel-fixer in placing the bars in the correct layers, and avoids mistakes, resulting in a decrease of the effective depth of the slab, because of the bars being in more than two layers in the top or bottom of the slab. The bars and stools should be fixed in the following order: (1) Bars a, ya, b, yb, c, and d; (2) Bars e and f; (3) Stools xx, except those within the shaded area (*Fig.* 150); (4) Stools yy; (5) Bars g, yg, h, j, xj, k, and xk; (6) Bars L, xL, m, xm; (7) Bars n, yn, p, xp; (8) Bars r; (9) Remaining stools xx; (10) Bars q and xq. Lines indicating the position of the first fifth-points of the spans, at which the bars are to be cranked, are shown and assist the draughtsman who prepares the bending schedule and the steel-fixer. The straight bars over the columns have semi-circular hooks to ensure good anchorage in the top of the slab (otherwise the ends would not be deeply embedded), and because they are, in this example, $\frac{7}{8}$ in. diameter. The reinforcement in the columns is not shown but, if the column does not continue upwards, the bars would extend in an easy bend to lap with the main tensile reinforcement in the top of the slab. Thus the columns are continuous with the slab as is assumed in the calculations.

CHAPTER VIII

ESTIMATING, COSTING, AND PROGRESS CHARTS

METHODS of estimating and costing are generally developed by each contractor to suit the type of work in which he specialises. In this chapter are briefly described various forms and charts which have been found useful in preparing estimates for reinforced concrete work, analysing costs, and controlling construction.

Fig. 151 shows a convenient form for a quantity estimate. In column (1) are entered short descriptions of the various members of the structure, which should be given in the order of construction, that is, say, footings, columns, beams, slabs, etc. The items should also be grouped so that the sum of the quantities of those having the same unit price can readily be obtained. Column (2) gives the number of identical members. In column (3), the lengths of columns, beams, and similar members, and the lengths and breadths of slabs

Particulars of Members					Unit Quantities			Total Quantities		
Description	No.	Dimensions	Total	Unit	Concrete (cu. ft.)	Reinforcement (lb.)	Shuttering (sq. ft.)	Concrete (cu. ft.)	Reinforcement (lb.)	Shuttering (sq. ft.)
Footing (4 ft. square)	*10*	—	*10*	*No.*	*16*	*48*	*16*	*160*	*480*	*160*
Column (14 in. square)	*4*	*15 ft.*	*60*	*lin. ft.*	*1·36*	*12*	*4·67*	*82*	*720*	*280*
Beam (20 in. × 8 in.)	*1*	*25 ft.*	*25*	*lin. ft.*	*1·11*	*15*	*4*	*28*	*375*	*100*
Slab (4½ in.) .	*12*	*10 ft.* by *25 ft.*	*250*	*sq. ft.*	*0·375*	*1·25*	*1*	*94*	*313*	*250*
(1)	(2)	(3)	(4)		(5)	(6)	(7)	(8)	(9)	(10)

FIG. 151.—FORM FOR CALCULATING QUANTITIES.

and footings are entered, and in column (4) the product of the figures in columns (2) and (3). Columns (5), (6), and (7) give the quantities of concrete, reinforcement, and shuttering for 1 ft. length of column, beam, or similar member, and for 1 sq. ft. of slab, wall, footing, etc. The total quantities of concrete, reinforcement, and shuttering obtained by multiplying the entries in columns (4) and (5), (4) and (6), and (4) and (7) are entered in columns (8), (9), and (10) respectively. The quantities for like members are abstracted from these forms, and the totals are sometimes converted into cubic yards of concrete, tons of reinforcement, and square yards of shuttering.

Fig. 152 is a convenient form for cost estimating. The items, separated into groups comprising like work in each part of the contract and with the items and groups listed in the order of construction, are given in column (1). In each group excavation, concrete, reinforcement, and shuttering are given first, and if

Item No.	Description of Work	Quantity	Unit	Unit Rates (shillings)		Costs in Pounds (£)				
						Material		Labour		Total
				Materials	Labour	Each Item	Sub-Total	Each Item	Sub-Total	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
	Part A of Contract.									
	Excavation :									
	Class (1)									
	,, (2) etc.									
	Concrete :									
	Class (1)									
	,, (2) etc.									
	Shuttering :									
	Class (1)									
	,, (2) etc.									
	Reinforcement :									
	Class (1)									
	,, (2) etc.									
	Sundries :									
	Special finishes									
	Asphalt									
	Openings, etc.									
	Site Overheads.									
	Foreman									
	Plant : Mixers									
	Hoists									
	Small tools etc.									
	Access road									
	Temporary drains									
	Pumping									
	Transport									
	Canteen									
	Insurance, etc.									
		Head office overheads . . £								
		Risk and profit . . . £								
		Total . . . . £								

FIG. 152.—FORM FOR CALCULATION OF COST ESTIMATE.

necessary each item is classified to take into account variations of unit price due to factors such as different qualities of concrete ; different costs of concreting due to the size of the member, the height the concrete has to be hoisted, and the distance it has to be transported ; differences in the size of the reinforcement bars ; and differences in the type of shuttering (for example simple or complicated, plane or curved, the degree of repetition, the amount of propping and strutting, etc.). Greater accuracy is not gained by using a large number of headings. Analyses of costs of a few completed contracts soon reveal the relative influences of various factors on unit prices and the total cost compared with the effects of unpredictable difficulties on the site. It is most important to include the total cost of materials, and to base the prices on definite quotations from suppliers. Generally it is simpler and more accurate to include items such as splayed corners, forming openings, etc., in the rates for shuttering. Large numbers of difficult openings should be given separately, as they may result in considerable extra cost. Items such as special finishes, asphalt, and granolithic, which require special workmen and considerable additional work, should be given separately. A " sundries " item, related to the materials cost, should be included to allow for the cost of handling and storing materials after delivery.

Columns (2) to (10) of *Fig.* 152 are arranged so that the costs of materials and labour and intermediate and final totals are kept separate. The costs of materials are definite and predictable, but labour costs are uncertain and must

Description of Work	1	2	3	4	5	6	7	8	9	10	Weeks
Preliminary Work	10	10									
Excavation		10	10	5							
Raft Construction				15	15						
Columns : Ground to 1st Floor						10					
Walls Ditto						10	10				
1st Floor: Slab and Beams etc.								20	20	20	
Total No. of Men	20	30	40	45	50	55	60	60	60	60	
Plant: 21/14 Mixer No 1											
Ditto No 2											
Hoist											
Delivery of Materials (Weekly Supply)											
Cement (Tons)	20	30	30	40	etc						
Sand (Cu. yd)	40	60	60	80	etc.						
Gravel (Cu. yd.)	80	120	120	160	etc						

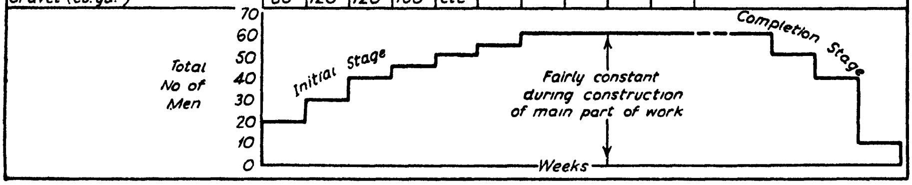


FIG. 153.

be estimated after consideration of common rates for the class of work. Small tools, workmen's sheds and canteen, and insurances are some of the costs on the site which are directly related to labour costs. The cost of mixers and other plant, fuel and water, cement sheds, and transport are some of the items that are directly related to materials costs. *Fig.* 152 is intended only to indicate the general lines on which estimates are made, and experience only will ensure that every item of construction and other costs such as access roads, pumping, special timbering, shoring, scaffolding, concreting in cold weather, tidal work, etc, are included.

A chart of anticipated progress (*Fig.* 153) should be prepared in conjunction with the estimate in order to decide the economical degree of concentration of men and plant on the site and the duration of the contract. The items of work should correspond to those in the estimate and be in the order of construction.

The anticipated duration of each part of the work is shown by horizontal lines across the "time" columns, each of which represents a week. The estimated number of men occupied each week on each item is also given. The estimated total number of men employed each week is shown at the bottom of each weekly column, and is also plotted as a curve which rises gradually to a maximum during the early period while the work is being got under way, remains more or less constant during the progress of the main part of the work, and falls away towards the end of the contract. The duration of each item of work must be planned so that the labour required is compatible with the total labour available as shown by the curve, thereby avoiding violent fluctuations of the total number of men required. The items of work must follow one another in accordance with a practical programme, and the total labour costs in the estimate must conform to the number of men shown in the progress chart.

The costs of the quantities of materials shown on the progress chart must also agree with the costs in the estimate. Delivery of materials must be sufficient for the work planned for each week, the latter being obtained from the estimated quantities in conjunction with the progress chart. Reference at par-

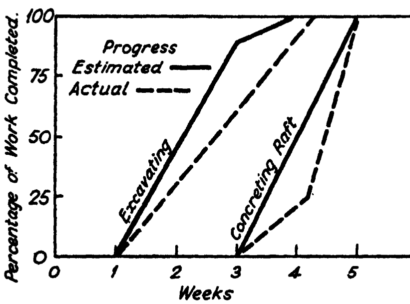


FIG. 154.

ticular stages of the work should be made to all materials, fittings, etc., to be supplied and fixed, so that prompt delivery is ensured.

Plant costs in the estimate must correspond with the period of hire or use shown on the progress chart, and the capacity of the plant must correspond with the work being done each week. Delivery dates and the capacity of the plant should have ample margins in case of delays and breakdowns. In this connection, the method of indicating the capacity of concrete mixers must be understood. A 21/14 mixer can take about 21 cu. ft. of dry unmixed materials and will give about 14 cu. ft. of wet concrete. The cost of special plant can often be more accurately estimated from the anticipated cost and the probable value on completion of the contract; this is one of the risks which a contractor cannot avoid or insure against. Contractors generally tender for work for which the plant in their yards is suited, and the estimators use hire rates which do not include a departmental profit. When, as often happens, plant can be hired more cheaply from outside, the contractor is evidently retaining uneconomical plant.

Fig. 154 shows the type of progress chart which should be kept by the resident engineer. The percentage of each item of work to be completed each week is plotted in full lines. Actual progress is plotted each week in dotted lines, and it is easy to see at any time the relationship between work done and the contract programme, and the acceleration required to catch up when any of the work is delayed. Only the main items of construction should appear on this chart, as too much detail is confusing.

Fig. 155 is a suitable form for the analysis of costs. Each week the foreman enters in column (1) items of work corresponding to those in the estimate (of which he should have a copy giving only those details with which he is concerned) which have made progress. Each day, with the assistance of the timekeeper, the total man-hours worked by each gang is allocated to the work done. At the end of the week columns (3), (4), and (5) are completed, and at the

	(1)	(2)						(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Item No.	Description of Work	Time (man-hours)							Rate per hour	Cost for week (£)	Work completed (per cent.)			Quantity of work completed this week	Unit cost
		F.	S.	M.	T.	W.	Th.	Total			Total to date	Last week	This week		
	(Items to correspond to estimate)														
				Sundries (by subtraction)											
				....											
				....											
				....											
								Total = £			= Total wages paid				

FIG. 155.—FORM FOR COST ANALYSIS.

bottom of column (5) a sum is added for sundry items (a list of which should be given) so that the total labour cost for the week is equal to the total wages paid. It is necessary to allow for sundries in this manner as it is unreasonable to expect a busy foreman to allocate man-hours other than for the main items of work. Each week the foreman enters in column (6) the percentage of the total amount of the work completed for each item. The essential information for costing is then on record and, if they are required at any time, details can be worked out for any week to find the cause of excessive expenditure, or on completion of the contract the average costs over a long period can be found for guidance in preparing future estimates. The quantity of work done, expressed as a percentage of the whole, during the week is obtained by subtracting the amounts in column (7) from those in column (6) and is entered in column (8), and, from the quantities given in the estimate, is converted into quantities and units and entered in column (9). The unit costs in column (10) for the week's work can only be approximate, but they are a great help in controlling labour costs and detecting and avoiding waste and inefficiency. Excessive time spent on unproductive work is quickly noticed by reference to the cost of sundries. Over a long period average costs obtained in this way are a reliable basis for future estimates. It is important to obtain the measurement of work done from totals to date, as weekly totals are apt to be unreliable. Costing systems in which each man reports daily the task he has accomplished are unnecessarily complicated, expensive to operate, and often unreliable.

CHAPTER IX

CONCRETE CONSTRUCTION

Aggregates.

IN this chapter, some important aspects of making concrete and constructing reinforced concrete structures are considered.

Most structures have their own problems with regard to the kind of concrete to be used. If the engineer has a knowledge of the aggregates available in the locality, as he should, he will know whether concrete having a certain strength can be obtained at a reasonable cost. The type of aggregate available locally governs the problem most often, as good cement is generally obtainable and the cost of transporting the cement to a site is much less than the cost of transporting the aggregates. The contractor's problem is to produce from local aggregates, at not more than the estimated cost, concrete which will satisfy the resident engineer. The aggregates must be clean and chemically inert, and of high crushing strength.

The choice and grading of aggregates, partly due to the arrangement of the particles and partly to the amount of water required to produce workability, has a considerable influence on the strength of concrete. If it were possible for the stones to be true cubes which fitted perfectly together, the compressive strength of the concrete would be the same as the strength of the stone, the density (which is related to strength) would be a maximum, and the cement paste required would be a minimum. If the stones were spherical and certain sizes could be selected, the strongest and most dense concrete would be obtained by the arrangement shown in *Fig.* 156. If the large spherical stones are arranged in a layer as shown in section 1–1, and similar layers of identical stones were arranged above and below as shown, then the vertical distance between the centres of two successive horizontal layers is the length of the side AB of the triangle ABC in section 2–2. If R is the radius of the large spheres, AB is $R\sqrt{2}$. Smaller spherical stones can be fitted between the large stones and will be in contact with six of the large stones. If the radius of the small stone is r, from the triangle ABC shown on the plan, $R + r = R\sqrt{2}$, that is $r = 0{\cdot}41R$. It is seen that the small spheres fit between the large spheres in both plan and elevation.

A cubical space measuring $2nR$ in each direction horizontally and $nR\sqrt{2}$ vertically comprises n^3 large spheres of radius R and n^3 small spheres of radius $0{\cdot}41R$. Neglecting the necessity to halve the outer smaller spheres, the total volume of the space is $4n^3R^3\sqrt{2} = 5{\cdot}7n^3R^3$. The total volume of the large spheres is $\frac{4}{3}\pi n^3R^3 = 4{\cdot}2n^3R^3$. The total volume of small spheres is $\frac{4}{3}\pi(0{\cdot}41)^3R^3n^3 = 0{\cdot}28n^3R^3$. Therefore the large spheres occupy $\frac{4{\cdot}2}{5{\cdot}7} \times 100 = 75$ per cent. of the total space and the small spheres occupy $\frac{0{\cdot}28}{5{\cdot}7} \times 100 = 5$ per cent. of the total space. The remaining space is $(5{\cdot}7 - 4{\cdot}2 - 0{\cdot}28)n^3R^3 = 1{\cdot}22n^3R^3$, which is 20 per cent. of the whole space, and can only be filled with very small particles. It is important to analyse the arrangement and fitting together of the spheres and particles of aggregate as a three-dimensional problem. Two-dimensional cross sections are deceptive with respect to the space between large particles.

To ensure adhesion, each particle of aggregate must be coated with cement-and-sand mortar, and therefore from the volume of the large spheres in the case

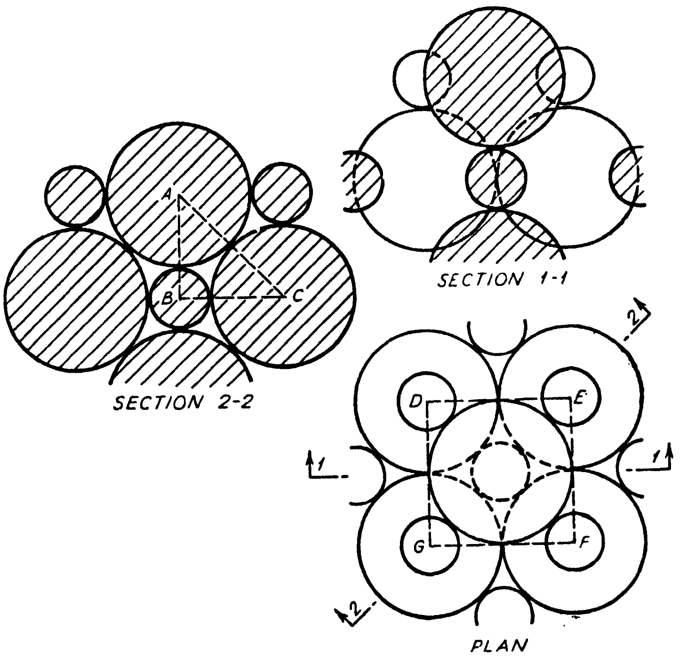


FIG. 156.

of 1-in. diameter stones must be deducted, say, 25 per cent. If the spheres could be perfectly packed the percentage of each size to give the greatest density would be about 50 per cent. of large spheres, 5 per cent. of small spheres, and 45 per cent. of fine particles. These proportions are not greatly different from the proportions used in practice for a concrete of greatest strength using rounded gravel as aggregate. The difference is presumably accounted for by the non-uniformity of size of the large stones, shapes differing from a sphere, and the arrangement of the relative positions of the particles due to mixing, placing, and consolidating the concrete. Particles of natural gravel and crushed stone vary considerably in shape. While cubical particles would give maximum strength if they could be packed together, particles that are nearly spherical are preferable since cubes cannot be fitted together by shaking or be poured into a tightly-packed mass, while rounded stone when "lubricated" by sufficient fine particles, water, and cement can form a semi-fluid substance and make a concrete which is termed "workable," that is one which can be easily consolidated around the reinforcement and into the corners of the shutters with the assistance of suitable tools. Flat or ellipsoidal stones do not produce a good workable concrete and require more cement paste to cover their relatively larger surface areas, and produce a weak concrete because they are more brittle.

From a consideration of price, shape, quality, and grading, the aggregates should be selected which will produce the required strength and quality of concrete, if necessary mixing together large and small aggregate of more than one type in order to obtain a grading of particles which gives a concrete of the required strength, workability, and density; surface finish or watertightness

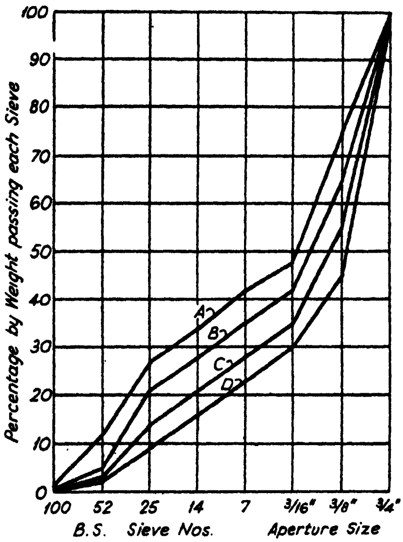


FIG. 157.

may also be important factors. Most quarries have washing and screening plant, so that cleanliness, size, and grading are to a certain extent controlled. Dry samples should be analysed by sieving, and the best sources of supply determined for obtaining a combined aggregate having a grading corresponding as closely as possible to one which has been proved by laboratory tests to give concrete of suitable workability combined with almost maximum density and strength. Typical curves showing such gradings for ordinary ¾-in. aggregate are given in *Fig.* 157. For a given water-cement ratio concrete made with aggregate having the coarsest grading *D* would require a slightly larger proportion of cement than concrete made with aggregates having either of the intermediate gradings *B* and *C*. It is often found, however, that the greatest proportion of cement is required if aggregate having the finest grading *A* is used because of the large superficial area of the smaller and more numerous particles. It is important to avoid using an aggregate containing a large proportion of fine particles passing B.S. sieve No. 100, as their presence requires additional water to give the same workability. For a concrete of medium workability, as is required in ordinary reinforced concrete, with a water-cement ratio (by weight) of 0·5, and using uncrushed ¾-in. gravel aggregate, the ratio of cement to aggregate is 1 : 4·5, 1 : 4·8, 1 : 4·8, and 1 : 4·6 for gradings *A*, *B*, *C*, and *D*, respectively. The corresponding proportions for ¾-in. crushed rock, with a slightly higher water-cement ratio, say, 0·55, are 1 : 4·3, 1 : 4·5, 1 : 4·7, and 1 : 4·7. (Many data on this matter are given in "Design of Concrete Mixes," published by H.M. Stationery Office in 1947, and in books on this subject. It is important that in this respect British students should consult books published in Great Britain because, although American books are useful for the purpose of understanding the principles of concrete mixtures, the difference between the British and the United States standard gallon, and the different bases used for the density of cement in the two countries, make American data of little practical use in this country.)

It is seen from *Fig.* 157 that a fairly uniform grading of the fine particles is desirable but, as has been shown, it is possible to have a very dense mixture with particles in close contact in all directions, so providing increased strength, with the large stones uniform in size and forming about 50 per cent. of the total and the next size less than half the diameter and forming only about 5 per cent. of the total. This explains why excessive intermediate sizes can cause bad workability, and such a grading tends to cause separation.

If the mixture determined from the sieve analysis, with the grading curves as a guide, is too fluid or insufficiently workable, an adjustment should be made to the proportion of fine particles. An increase of fine particles by 5 per cent. to 10 per cent. can be made without reducing the strength appreciably. The shape of the mould affects to a certain extent the proportion of fine particles required. For example, a very thin slab requires a higher proportion of fine particles than a bulky mass of concrete. For such work trial mixtures should be made with increasing proportions of fine material until a smooth surface can be obtained on the resulting concrete with a steel or wooden float. Crushed stone, on account of its angular shape, requires more fine material and also more cement and water than rounded gravel for the same workability. In all cases the sieve analysis and laboratory-established grading should be used as a preliminary guide, and the final mixture determined by trial, crushing tests and workability being the final criteria of the most suitable mixture to use.

The space between the reinforcement and the thickness of the cover to the reinforcement generally limits the maximum size of aggregate which can be used. A general rule is that the size of the largest particles should not exceed three-quarters of the thickness of the minimum cover of concrete or the smallest distance between reinforcement bars. If large aggregates of suitable shape and size are available, since less water is required for workability, greater strength and density can be obtained provided that the grading ensures the maximum possible contact between the large particles and that the voids are completely filled. Again a grading table or curve for the maximum size of aggregate being considered should be used as a guide to determine the best grading obtainable from the available supplies, and this will not be the same as for a smaller aggregate which requires a larger proportion of fine material for workability and more water because of the larger ratio of surface area to volume.

Workability and Water Content of Concrete.

In adjusting unworkable mixtures under site conditions by the addition of fine material, the aim should be to produce a mixture which is workable with the smallest quantity of water, and which allows the mortar-coated heavy particles to be tamped or shaken down into a compact mass containing just sufficient cement paste, with fine sand in suspension, to fill the voids, cover the aggregate with a thin film, and rise to the surface to provide suction for a screed or float. Excessive vibration must be avoided, as it causes the concrete to separate with the mortar at the top and the coarse particles at the bottom.

The strength of concrete increases as the water-cement ratio decreases (*Fig.* 158), provided that the mixture is sufficiently workable to allow proper compaction. The strength also increases as the proportion of cement increases, mainly on account of the reduction in the water-cement ratio made possible by the increased workability of richer mixtures. Excessive cement causes greater

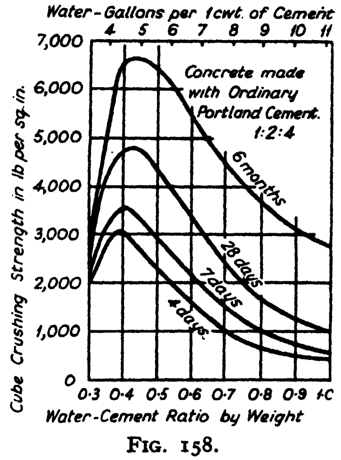


FIG. 158.

shrinkage. Too little water, particularly in highly-reinforced members of small cross section, reduces the workability so that it is not possible properly to consolidate the concrete. Vibratory consolidation enables concrete with small water content to be properly compacted, and thereby produces stronger concrete because workability is obtained with a lower water-cement ratio. The use of rise-and-fall tables, which shake the concrete in a downward direction only, produces very strong and dense concrete in precast members.

When mixing concrete it is necessary to increase the quantity of water required to give the desired water-cement ratio to allow for absorption by the aggregate, or to reduce the quantity if the aggregate is wet. The driest and densest concrete which can be compacted is the strongest for a given ratio of cement to aggregate.

The slump test is sometimes used as a check on the consistency of the concrete. An open-ended mould, 12 in. high by 8 in. diameter at the bottom and 4 in. diameter at the top, is filled with fresh concrete. When the mould is lifted, the wet concrete should not settle more than 1 in. to 6 in. The settlement, or slump, varies with the grading and shape of the aggregate, with the proportion of cement to aggregate, and with the water content. The amount of slump which will give the desired workability should be determined for the mixture and water-cement ratio specified, and thereafter should be kept as nearly as possible constant throughout the work in which this class of concrete is used.

Density and Strength of Concrete.

The use of excessive sand or water can be detected by the density of samples of fresh concrete ; the density will be low if the concrete is defective. Generally the density should be about 146 lb. per cubic foot, but the density required should be determined at the beginning of the work and maintained. The principle check on the quality of the concrete should be the results of compression tests on standard cubes.

For a given ratio of cement to total aggregate the strongest concrete is produced by the densest mixture that can be properly placed and compacted in the shutters. High density is produced by the use of the largest size aggregate which can be worked in among the reinforcement and between the reinforcement and the shuttering. The water-cement ratio must be kept as low as possible compatible with workability. The water content may sometimes be slightly reduced, and the degree of workability maintained, by increasing slightly the amount of sand.

The strength of concrete increases with (*a*) age (see *Fig.* 158), (*b*) reduction of the water-cement ratio (also see *Fig.* 158) provided that satisfactory workability is retained, (*c*) the proportion of cement (mainly because of the increased workability of richer mixtures), (*d*) the density of the mixture, which is increased by providing satisfactory workability by suitable grading and shape of the particles, and (*e*) thorough compaction by ramming or vibrating.

The control of the strength of concrete is therefore seen to be a fairly complicated problem. It is important for the engineer to have in mind the difficulties of site conditions and, in specifying water-cement ratios, to ensure that there is no risk of having insufficient cover to the reinforcement due to the use of too little water resulting in an unworkable mixture. The factor of safety of the concrete in a structure is generally higher than that of the reinforcement (see Chapter III). This, together with the increase in the strength of concrete which may take place after a period of six to twelve months, indicates that it is unwise to incur the very serious risk of the corrosion of the reinforcement by being too fastidious about the water-cement ratio and sand content in order to obtain only slightly stronger concrete. Maximum density of the concrete in the structure must be the objective, with compression and slump tests as useful guides.

Concrete Mixtures.

In defining a concrete mixture it is best to specify the proportion of total aggregate to cement but, since aggregates are generally available in two definite classes, namely gravel or crushed stone (that is coarse aggregate) and sand (that is fine aggregate), it is more common to specify the proportions of each class of material to one part of cement. The amount of cement in unit volume of the compacted concrete is an important factor in the cost. With ordinary aggregates and workability, about 6½ cu. ft. of dry combined aggregate to 1 cwt. of cement gives the same amount of cement in 1 cu. yd. of concrete as a mixture of 1 part of cement, 2 parts of fine aggregate, and 4 parts of coarse aggregate (that is 1 : 2 : 4 by volume). For other mixtures the corresponding equivalents are about 5 cu. ft. for a 1 : 1½ : 3 mixture, and 9½ cu. ft. for a 1 : 3 : 6 mixture. As it is important that the cement at least should be measured by weight; a 1 : 2 : 4 nominal volumetric mixture is given by 1 cwt. of cement, 2½ cu. ft. of fine aggregate, and 5 cu. ft. of coarse aggregate.

The total volume of combined aggregates is less than the sum of the volumes of the fine and coarse aggregates, and the volume of mixed and consolidated concrete is less than the total volume of the unmixed cement and aggregates. Approximate quantities of materials in 1 cu. yd. of 1 : 2 : 4 concrete are 4¾ cwt. of cement, 11¾ cu. ft. of fine aggregate, and 23½ cu. ft. of coarse aggregate. The corresponding quantities in 1 cu. yd. of 1 : 1½ : 3 concrete are 6 cwt., 11¼ cu. ft., and 22½ cu. ft. Many tables have been published giving similar data to the foregoing, but they must be treated with reserve. The only way in which to determine accurately the amount of concrete produced by mixing given materials in specified proportions is to make trial mixtures with the actual materials and of the same workability.

The measurement of the materials should be by weight if possible. A 1-cwt. bag of cement is a convenient unit on which to base the weights of water and aggregates. Generally, however, cement is measured by weight and the aggregates by volume. In measuring sand by volume account should be taken of bulking, which may be as much as 40 per cent. with fine sand containing up to 10 per cent. of moisture. Fully saturated sand has the same volume as dry sand. This property is a useful guide in determining the amount of bulking in order to adjust the proportions.

Making and Placing Concrete.

The water used for making concrete must be clean and free from chemicals in solution, especially sulphates.

The mixing of concrete must be sufficient to coat all the particles with cement paste and uniformly distribute the mortar within the mass. After it has been mixed the concrete must be placed before the initial setting occurs, that is within about twenty minutes. Concreting must not be done if the temperature of the concrete is liable to be reduced below 40 deg. Fahr. during casting or below 32 deg. Fahr. during hardening. The temperature of concrete in cold weather can sometimes be raised by heating the aggregates or by using hot mixing water, and can be maintained during curing by a covering of straw. Rapid-hardening Portland cement generates more internal heat during setting and reduces the period of hardening during which the concrete can be damaged by frost. Calcium chloride hastens the setting but should not be used in reinforced concrete in case it causes corrosion of the steel. If the action of frost is not excessive, the hardening of the concrete will be retarded by a period at least equal to the duration of the frost, but it may cause the surace to flake off.

Concrete must be thoroughly compacted and entrapped air released by ramming and tamping or by vibration. The space between the reinforcement and the shutters of an exposed face must be thoroughly worked with a tamping tool to ensure that air is released and the voids between large particles of aggregate filled with mortar. Over-vibration, causing the separation of the small from the large particles, must be avoided.

CURING AND PROTECTION.—After it has been placed the concrete should be kept damp at a temperature above 32 deg. Fahr. until it has hardened, and should be prevented from drying out for at least seven days if ordinary Portland cement is used or four days in the case of rapid-hardening Portland cement. In concrete products factories hardening can be accelerated and the output per mould considerably increased by curing the concrete in steam-heated chambers. Curing by electrical heating has been tried, and in the manufacture of pipes high pressure combined with heat and rapid-hardening cements have produced high-strength concretes in a few hours.

Concrete must also be protected after placing from injury from exposure to the sun, wind, or rain, and from abrasion and excessive loads.

AIR-ENTRAINED CONCRETE.—Air-entrained concrete, which must not be confused with aerated concrete used for lightness or thermal insulation, is being increasingly used in the United States. In order to obtain workability with a low water-cement ratio the sand content is slightly reduced and a chemical added which causes from 3 per cent. to 6 per cent. by volume of air to be uniformly distributed throughout the concrete. Examples of the chemicals used are resinous substances and a triethanolamine salt of sulphonated hydrocarbon. Some manufacturers supply air-entraining cement. The inclusion of the air reduces the compressive strength of the concrete by about 15 per cent. with the same water-cement ratio for rich mixtures and considerably less for lean mixtures. The workability, however, is considerably increased, segregation and " bleeding " are reduced, and, under conditions where excessive water is liable to be used to produce workability, air-entraining can be used with advantage. By avoiding the use of excessive water, air-entraining considerably increases the resistance of concrete to attack by sulphates and chlorides and to disintegration by frost. It is thought that the presence of air bubbles reduces capillarity in surface cracks and allows in the cracks freedom for expansion. In the United States air-entrained concrete is often used for roads and runways, and also to a small extent in general structural work. American engineers appear to have concluded that air-entraining ensures, under practical conditions of work, a more durable concrete than can be obtained by ordinary " dry " mixtures. The process of air-entraining has been developed as a result of considerable research, and further details are given in American literature, for example the Journal of the American Concrete Institute, 1943 (14), pages 1–19; and 1944 (15), pages 477–507.

Surface Finishes.

For durability, the concrete immediately behind the surface should have high density in order to prevent the intrusion of water, which accelerates disintegration. The requirements for a durable surface finish are therefore the same as for strength, except that additional mortar at the surface may be necessary to produce a close texture. While a smooth surface weathers more slowly, a rough surface darkens more uniformly and is generally sufficiently resistant to deterioration. Smooth surfaces have a hard appearance and tend to discolour in patches and streaks. If it results in too rough a finish a mixture prepared

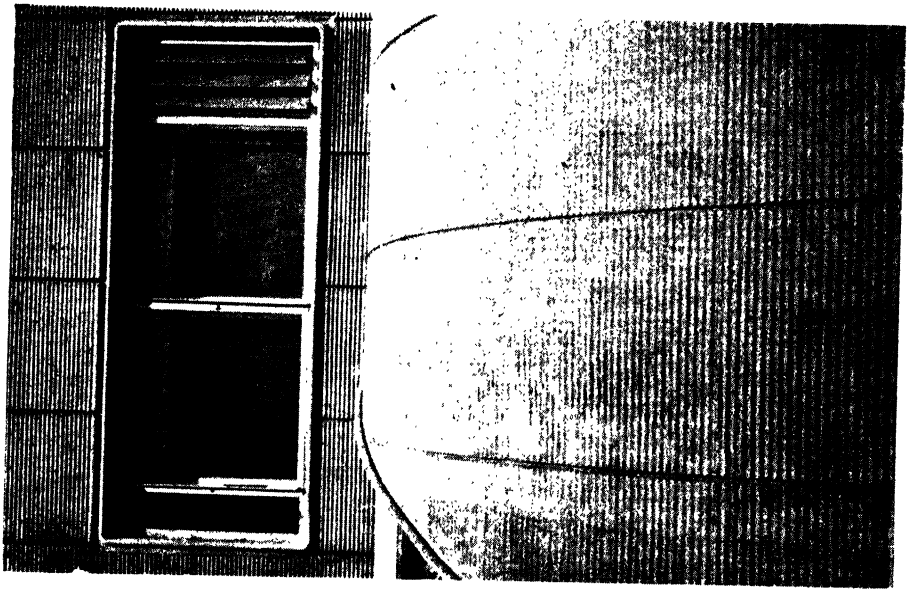


FIG. 159.—RIBBED CONCRETE SURFACE ON AN OFFICE BUILDING AT COPENHAGEN.

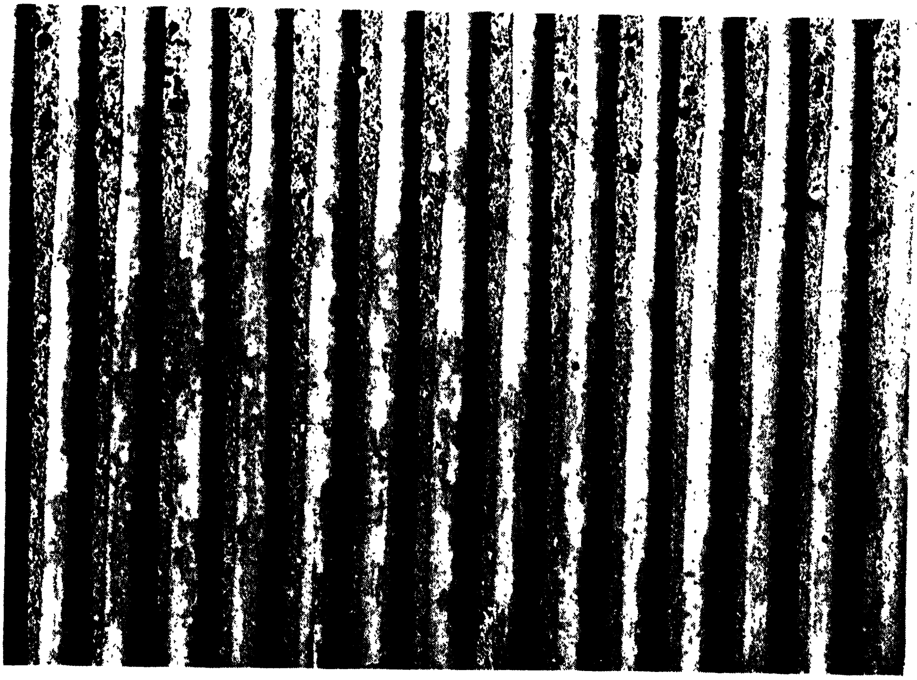


FIG. 160.—DETAIL OF RIBBED CONCRETE SURFACE ON THE ABUTMENTS OF TWICKENHAM BRIDGE, LONDON.

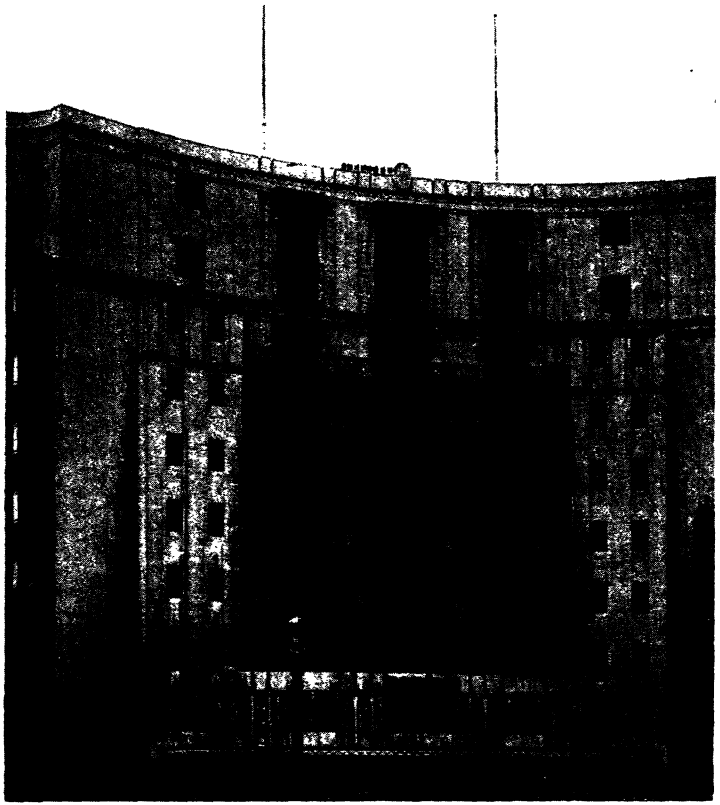


Fig. 161.—Polished Concrete Slabs, Dorchester Hotel, London: Showing Staining due to Rain, and Local Darkening due to absence of Sills.

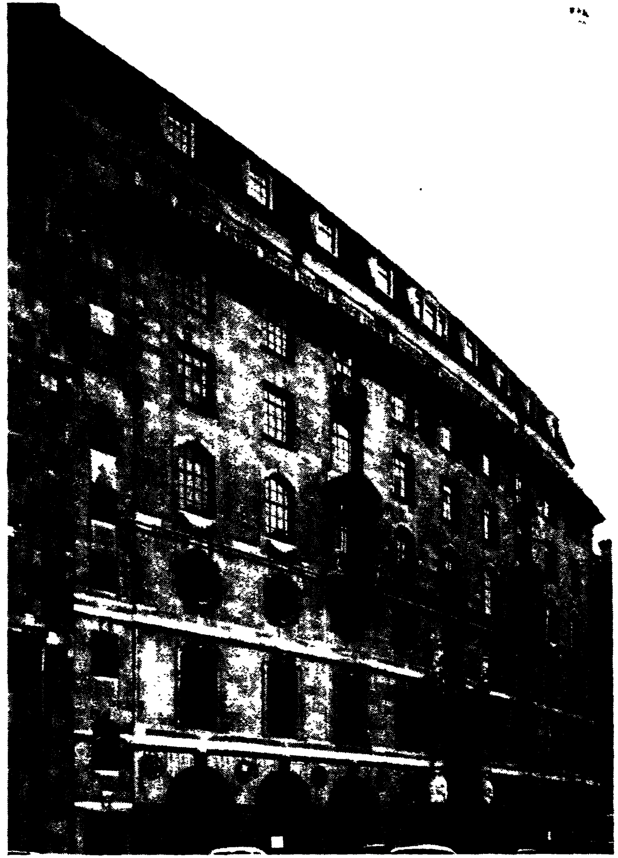


Fig. 162.—Reconstructed Portland Stone, India House, London: Uniform Darkening due to slightly Rough Surface and the Presence of Sills.

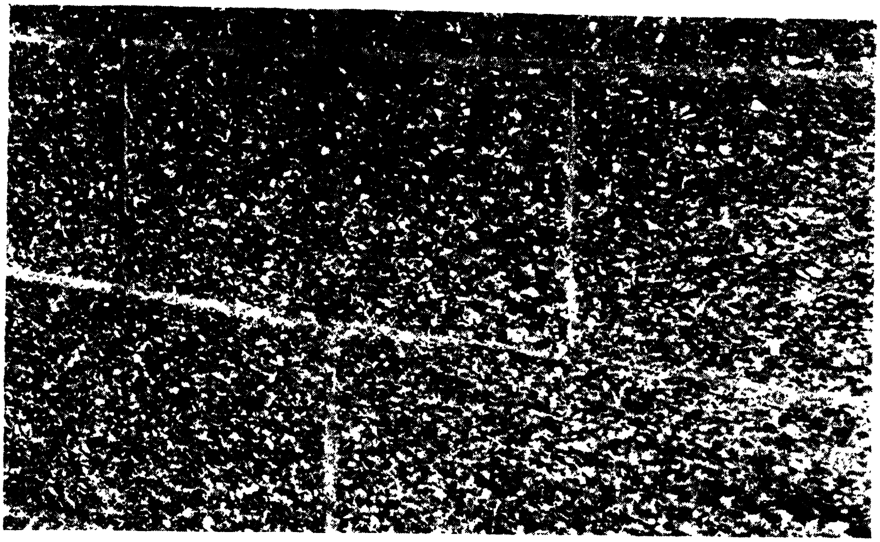


Fig. 163.—Precast Blocks with Bush-hammered Surface: Darkening is uniform due to the Rough Surface.

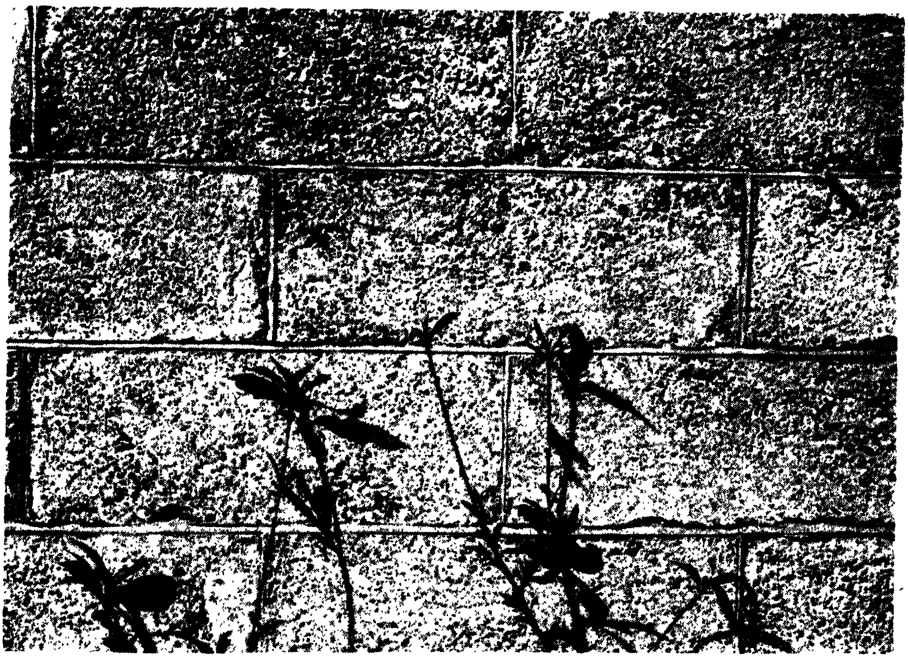


Fig. 164.—Detail of a House in Devonshire Thirty Years old: The Scrubbed Surface has Weathered Uniformly.

FIG. 165.—SMOOTH WHITE CONCRETE: CRACKS AND STAINS DUE TO RICH CONCRETE, SMOOTH SURFACE, AND ABSENCE OF SILLS AND DRIPS.

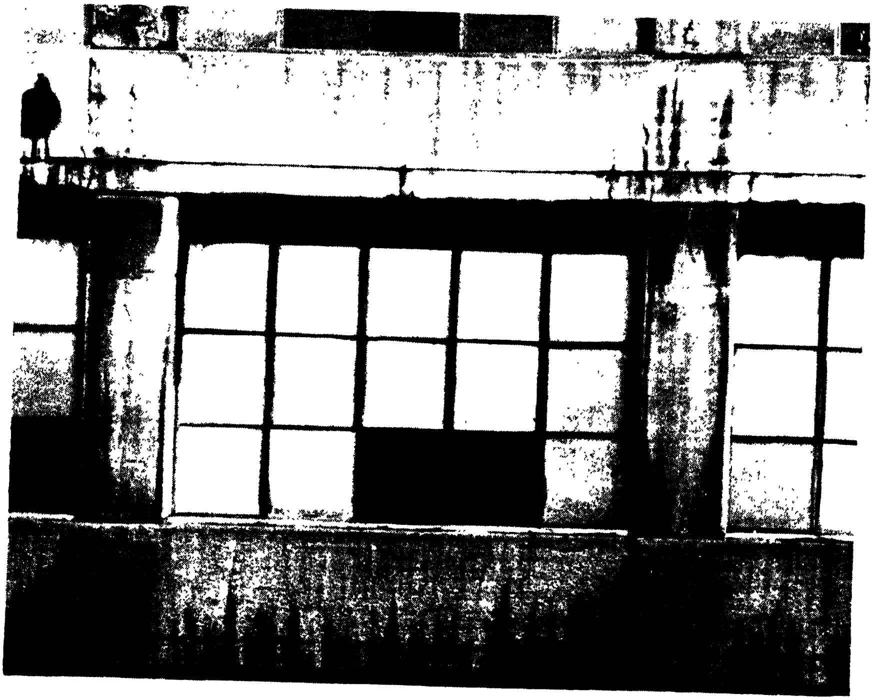


FIG. 166.—A SMOOTH CONCRETE SURFACE: CRACKS AND UNEVEN DARKENING DUE TO SAME CAUSES AS THE EXAMPLE IN FIG. 165.

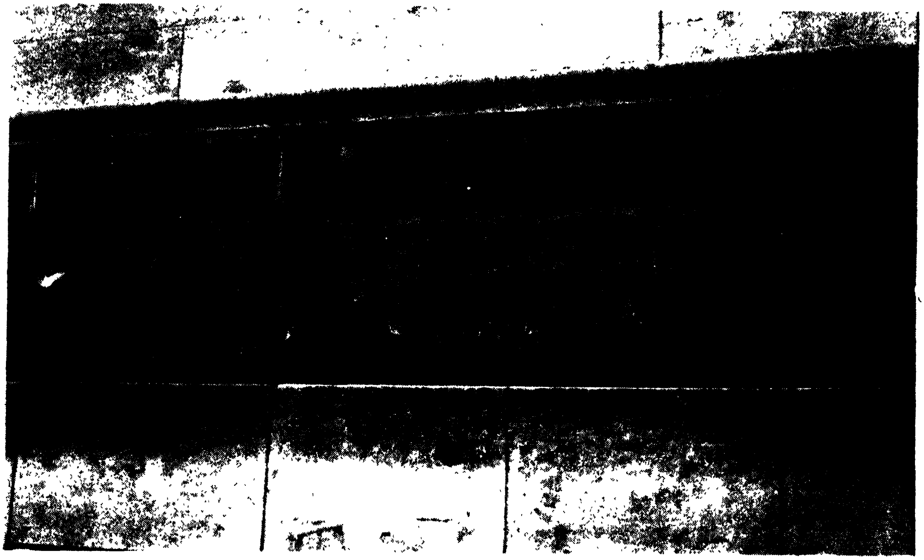


Fig. 167.—A Weathered Limestone Surface.

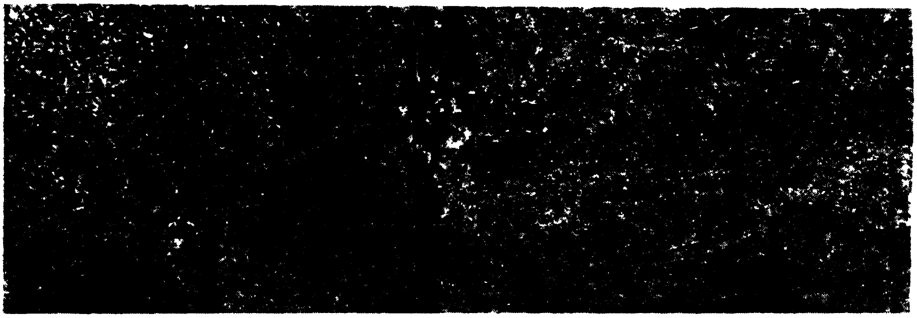


Fig. 168.—A Weathered Limestone Surface and Joint.

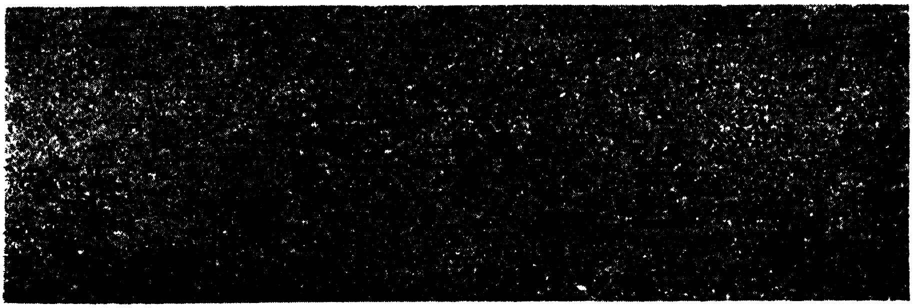


Fig. 169.—A Concrete Surface Ten years old: Slightly Rough Scrubbed Surface.

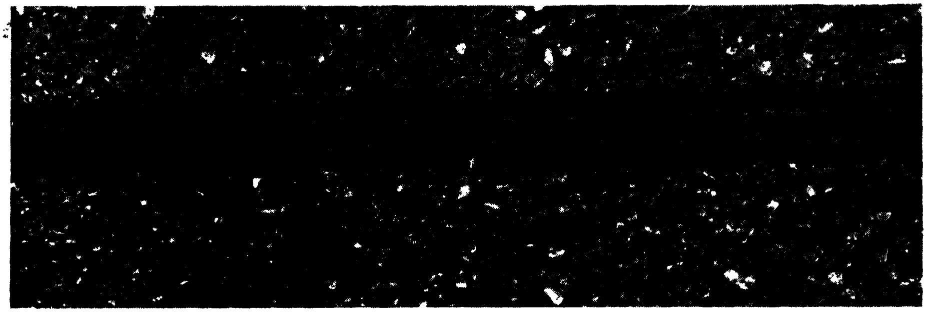


FIG. 170.—TWICKENHAM BRIDGE, LONDON: HEAVILY BUSH-HAMMERED CONCRETE WITH TILE-COURSE AT JOINT.

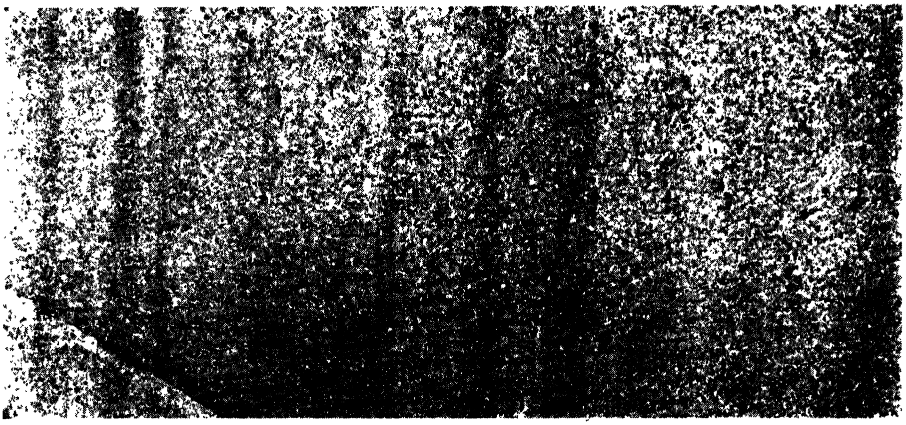


FIG. 171.—TWICKENHAM BRIDGE, LONDON: LIGHTLY BUSH-HAMMERED CONCRETE SURFACE.

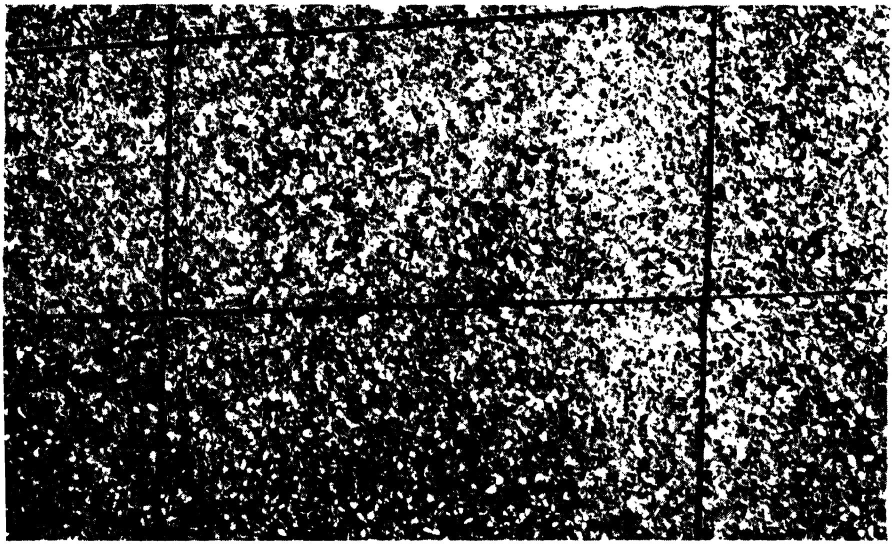


FIG. 172.—TWICKENHAM BRIDGE, LONDON: PRECAST CONCRETE SLABS WITH SCRUBBED SURFACE.

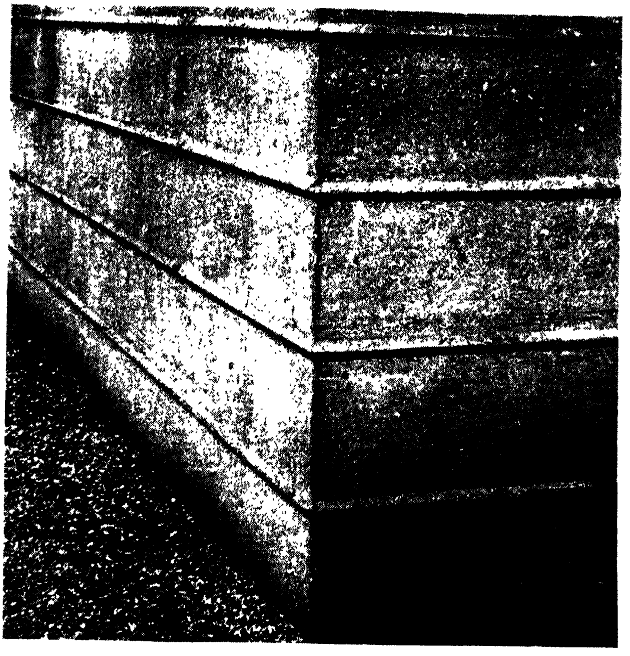


FIG. 173.—MARLBOROUGH COLLEGE: CONCRETE SHOWING MARKS OF WOODEN SHUTTER BOARDS AND CONSTRUCTION JOINTS HIDDEN BY VEE-SHAPED PROJECTIONS.

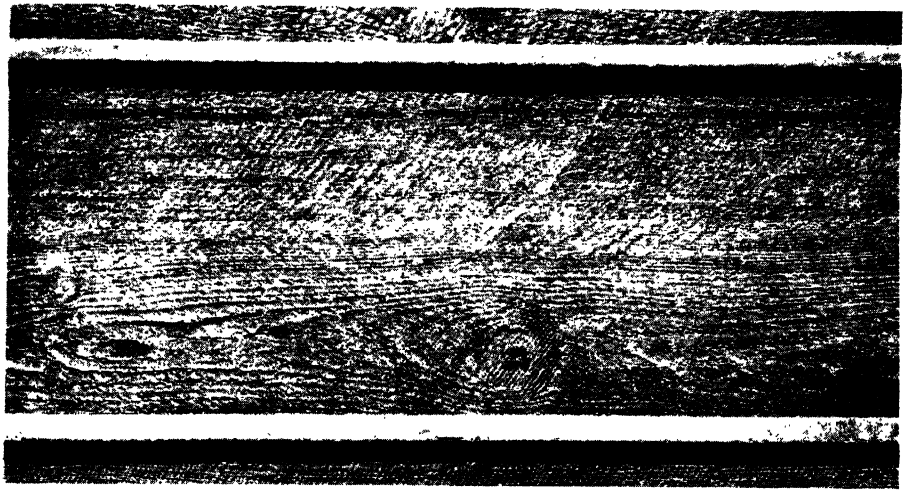


FIG. 174.—RIBBED CONCRETE SURFACE SHOWING THE GRAIN OF WOODEN SHUTTER BOARDS AND RESEMBLING PETRIFIED TIMBER.

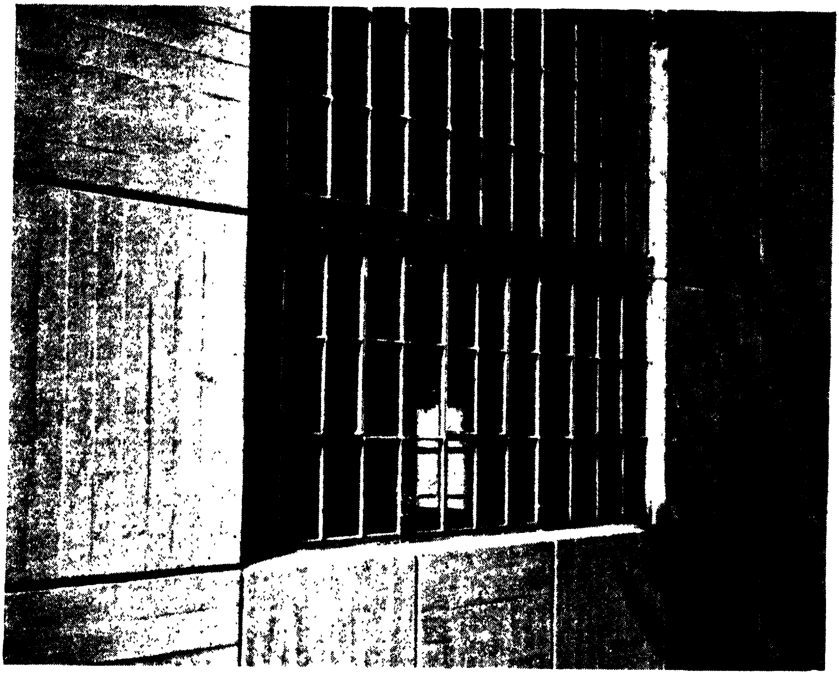


FIG. 175.—CONCRETE SURFACE SHOWING SHUTTER-BOARD MARKS AND WITH GROOVES TO HIDE CONSTRUCTION JOINTS AND TO FORM A PATTERN.

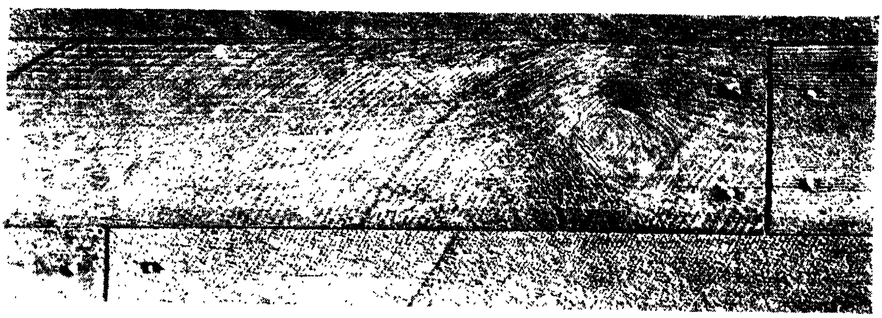


FIG. 176.—CONCRETE SURFACE SHOWING GRAIN OF SHUTTER BOARDS AND IMPRESSIONS OF NAIL HEADS.

with the use of grading curves should if necessary be altered by the addition of more sand until the surface of the concrete can be brought to a smooth finish when worked with a steel or wooden float. The surface of the concrete, if it is properly worked with tamping tools to release entrapped air, will also be smooth when the shutters are stripped. Industrial structures are best finished in this way without any surface treatment after the shutters are removed. Cement washes and renderings peel off in course of time, and rubbing over with cement bags, as is sometimes done, does not improve the finish except to fill pin-holes due to insufficient tamping. Rubbing should be lightly done, and a cement slurry used having a cement-to-sand ratio equal to the cement-aggregate ratio used in the concrete. For this reason the shutters must be liquid-tight, and if tamping or vibration of the shutters is properly done the surface will be free from honeycombing and pin-holes. The pattern of the shutter joints will be reproduced on the concrete, and the shutters should be designed so that the surface is at least tidy. On the other hand, at little extra cost an attractive pattern can be obtained. Horizontal construction joints can be obscured by forming horizontal grooves or projecting fillets on the concrete face at the joints, and these can be arranged to enhance the appearance of the wall. A surface left directly from the shuttering is more durable than any other.

In buildings and bridges in which architectural refinements are justified, exposed aggregate finishes can be very attractive. The size of the aggregate exposed should be related to the character of the structure. In domestic buildings $\frac{3}{8}$-in. aggregate, slightly exposed, is attractive. Excessively-exposed gravel resembles pebbledash and weakens the resistance of the surface to the action of the weather. The colour of the coarse aggregate should if possible contrast with the colour of the mortar. A black or dark aggregate contrasts well with white cement or ordinary cement used with silver sand. Coloured cements and coloured aggregates can be used to provide a large variety of patterns or textures, or they may be used to match natural stone. Exposed-aggregate finishes are best obtained by brushing with water the top surfaces of slabs cast horizontally as soon as the concrete has set sufficiently to prevent the stones from being dislodged by the brushing. In the case of walls cast in situ, in order to expose the aggregate by brushing, either the shutters must be removed a little earlier than usual (which is safe for low-stressed concrete) or the setting action of the surface delayed by a cement-retarding agent painted on the shuttering. Alternatively, dilute hydrochloric acid can be used to remove the surface cement, or the surface may be roughened by bush hammering. Horizontal surfaces are sometimes fluted or reeded by means of a roller, and cast-in-situ walls can be treated in the same way as stone to give any type of tooled masonry finish. Of these possibilities the simplest and most attractive finish, which is also cheap and durable, is obtained by slightly exposing fine dark stone aggregate used in conjunction with a white cement or silver sand. Smooth surfaces tend to become streaky with weathering. Cast-in-situ surfaces, cast as a veneer of special concrete at the same time as the ordinary concrete with the help of a vertical steel separating plate, particularly if they are too rich in cement, tend to crack and craze in course of time. A slightly rough exposed-aggregate surface can weather and darken uniformly like stone without developing a drab or dirty appearance. Pleasing elevations can be obtained by the use of grooves at construction and expansion joints, or at horizontal or vertical intervals, to form a pattern designed as in masonry structures as a feature of the façade.

The photographs in *Figs.* 159 to 176 illustrate many of the points made in the foregoing. The illustrations show that in a smoky atmosphere a reeded or slightly rough surface is necessary to ensure that darkening takes place uniformly and that fine cracks are invisible. Surfaces that are very light in hue make cracks conspicuous. Rich and wet concretes increase the width of cracks due to their greater shrinkage. Sills with drips and oversailing courses prevent local surface washing by rain. Fine, dark, slightly-exposed aggregates are suitable for residential flats and office buildings. Coarse and deeply-exposed aggregates suit bridge parapets and massive structures. Horizontal construction joints may be obscured by special constructional features such as raised or rebated joints or tile courses.

The office at Copenhagen (*Fig.* 159) is built in cast-in-situ concrete with a reeded finish formed by the use of a lining in the shutters; a detail of this type of finish is shown in *Fig.* 160. The Dorchester Hotel, London (*Fig.* 161), is faced with precast polished concrete slabs with marble aggregate on the face, and rain washing over the smooth surface has caused slight streakiness in places. India House, London (*Fig.* 162), is faced with reconstructed Portland stone and the slightly rough surface has caused fairly uniform darkening, the sills and oversailing courses having prevented local surface washing. The Admiralty " citadel " in The Mall, London (a detail of which is shown in *Fig.* 163), is faced with precast blocks with coarse and deeply-exposed shingle aggregate and buff-coloured cement. Darkening is either generally uniform or uniform throughout each block, the rough surface and dark colour having prevented visible cracks or streaky surface-washing by rain. The house known as Four Winds, Clyst St. George, Devon, was built about the year 1918 of concrete mixed in the proportions of 1 : 1½ : 3, using Chesil beach pea gravel aggregate. As is seen in *Fig.* 164 the surface has weathered perfectly in a rural atmosphere and is generally thought to be limestone. The fine dark aggregate was slightly exposed by brushing the concrete before it had hardened.

A typical example of unsatisfactory weathering due to the omission of sills and drips, and the formation of visible cracks probably due to the lightness of the colour, the smoothness of the surface, and the richness of the concrete, is shown in *Fig.* 165. Cast-in-situ concrete about ten years old, on which fine cracks are visible on account of the smoothness, the light colour, and probably the richness and wetness of the concrete required for placing as a veneer, is shown in *Fig.* 166. A typical example of the weathering of limestone (*Fig.* 167) shows that the dirt to a certain extent conforms to a pattern and accentuates the shadows, but there are patches and streaks. *Fig.* 168 shows the weathering of limestone at a joint; when masonry is examined critically many defects of finish can nearly always be found. A precast concrete wall slab about ten years old, with a slightly rough surface, fine dark aggregate slightly exposed, and of medium light colour is shown in *Fig.* 169. There are no visible cracks, and darkening is fairly uniform. This has proved a satisfactory finish for residential flats and office buildings in a smoky atmosphere. *Figs.* 165 to 169 are examples of the weathering of concrete and natural stone in London on buildings not more than a quarter of a mile apart.

The four photographs in *Figs.* 170 to 173 are of cast-in-situ concrete with the aggregate exposed by bush hammering; darkening has taken place uniformly, and there are no visible cracks. The abutments of Twickenham Bridge (*Fig.* 170) were bush-hammered and the tile courses were inserted to obscure the horizontal construction joints. The arch spandrel (*Fig.* 171), and the precast face slabs on the parapet walls (*Fig.* 172) were also bush-hammered; there are no visible cracks, darkening is uniform, and the texture is attractive. At Marlborough College (*Fig.* 173) the laboratory is built of cast-in-situ concrete, the joints being obscured by vee-shaped projections formed in the shuttering. Textures formed by the shutter boards on concrete surfaces are also shown in *Figs.* 174 to 176.

Site Planning and Inspection.

During the course of the work the contractor's engineer on the site and the buying department should use estimate and progress charts to facilitate the timely ordering of materials, delivery of plant, employment of labour, etc. The contractor's engineer should be given full details of the estimate so that he can arrange the work to the best advantage. He should make a plan showing the positions, at each stage of the work, in which he proposes to store materials, erect sheds, offices, plant, etc. On many works the greatest losses are made through lack of planning in this respect. Money spent on good access roads that allow materials to be transported by lorry to the point at which they are to be used is nearly always amply repaid. From the quantities in the estimate and from his knowledge of the availability, capacity, and hire charges of plant, the contractor's engineer must decide, for every part of the work, the type of mixer, hoist, batching plant, concrete pump, crane, excavator, or other plant which it is best to use. In some cases there are many possibilities, such as the use of skips, wheelbarrows, concrete pump, or hoist and chute in concreting. In the case of shuttering, sometimes it is best to use steel, sometimes sliding shuttering, and in other cases ordinary timber shutters. Experience, helped by cost analyses, must decide which method is to be used. The contractor's buyer generally purchases all materials, but the contractor's engineer should have the right to decide on the type and quality to be supplied, and the approval of the resident engineer (representing the consulting engineer and owners) must also be obtained.

The contractor's engineer should demand from his foreman a high standard of workmanship which will satisfy the resident engineer. The resident engineer (or clerk of works) should frequently inspect the work and ensure that the contractor maintains the required standard. Common faults to watch for are poor quality and badly-graded aggregates and unsatisfactory measuring of materials, especially the cement. Centering and shuttering and supporting props must be checked for their strength in carrying the load and in resisting the pressure of wet concrete. Bad joints in the shutters, which allow leakage of grout and cause unsightly surface honeycombing and weakening of the concrete, may be a much more serious fault than the use of too much water in the concrete. It must be ensured that tamping and punning are not neglected, and if vibration is employed care must be taken to avoid separation of the concrete or bulging of the shutters. The treading down of the top reinforcement in slabs, and the disturbance of reinforcement generally during concreting, are common faults. All reinforcement should be inspected before concreting and, in addition to ensuring that all the reinforcement is correctly placed, proper supports, packing, and fastening together by wire should be insisted upon so that during concreting there is no risk of the reinforcement being displaced. The proper curing and protection of the concrete, as described previously, must be insisted upon. The removal of centering or shuttering or shutter props too soon is a very serious mistake which must be prevented.

Frequent strength test cubes should be made of the concrete. If the concrete in the structure is sound, it should harden in the required time, and sound and feel really hard when it is struck with a hammer. If there is any doubt about the strength of any part of the concrete, cores for testing can be cut out or a test load may be applied.

Some foremen, being very much pre-occupied with securing as much profit as possible for the contractor, and possibly a bonus for themselves, may be tempted to adopt devices for evading the full requirements of the specification. The only way of ensuring that the right standard of concrete is provided is by frequent reliable inspection. Defective work may be covered early in the morning or at a time when it is known that the resident engineer is elsewhere. In pile driving a false set may sometimes be obtained by bending the straightedge used to mark the pile or by cushioning the fall of a steam hammer. When a concrete pump or chute is used a wet mixture flows more easily than a stiffer one, and before a slump test is taken the water supply may be temporarily reduced. Most foremen know many such tricks and, while a suspicious attitude is undesirable, it is as well for a young engineer to know that such practices are possible.

Some contractors sub-let parts of the work to the foremen. This procedure may result in unsatisfactory work, as a foreman's idea of sound concrete, when he is working for his own profit, may be far from satisfactory, particularly in the case of structures having long spans and liable to carry the full design load.

Specification.

The following is a typical specification for concrete, reinforcement, and shuttering, and provides a brief description of the standard of workmanship required and common faults to be avoided.

CEMENT.—The cement shall comply with the current British Standard specification for the class of cement used.

AGGREGATE.—The coarse aggregate shall consist of clean, hard, strong, durable, non-absorbent, natural ballast or broken stone, free from alkali, salt, silt, clay, dirt, stone dust, or organic matter. It shall be mechanically washed if so directed. No flat or flaky material or crushed sandstone shall be used. The coarse aggregate shall be uniformly graded from the maximum sizes specified and shall all be retained on a sieve with meshes measuring $\frac{3}{16}$ in. square.

The fine aggregate shall be coarse, hard, well-graded pit or freshwater sand, free from alkali, salt, silt, clay, dirt, or organic matter, and shall be mechanically washed if so directed. The material shall all pass through a sieve with meshes measuring $\frac{3}{16}$ in. square, and for reinforced concrete not more than 3 per cent. by weight shall pass a No. 100 B.S. sieve.

WATER.—Only clean fresh water, free from impurities, shall be used. Where the water is not obtained from a public supply the contractor shall submit an analysis from an approved analyst.

RATIO OF FINE TO COARSE AGGREGATE.—The contractor shall conduct at his own cost such preliminary tests as may be required with the dry aggregate to ascertain the ratio of fine to coarse aggregate and the water-cement ratio required for the various qualities of concrete specified. Where mixed fine and coarse aggregate is specified the quantity of sand shall be not less than that required to fill the voids in the coarse aggregate, and this quantity may be increased by an amount not exceeding 25 per cent. according to the requirements of workability and surface finish. For low-quality concrete the coarse and fine aggregate may be brought on the site already mixed.

The consistency of the concrete shall be measured by the slump test in accordance with the D.S.I.R. Code of Practice for Reinforced Concrete in Buildings (1934), Appendix VI, and the amount of slump shall be recorded. This test shall be repeated whenever the materials are changed.

QUALITIES OF CONCRETE.—*Plain Concrete*: One volume of cement to eight volumes of mixed fine and coarse aggregate of 2-in. maximum size. *Ordinary Concrete*: 1 cwt. of cement to 6¼ cu. ft. of mixed fine and coarse aggregate of ¾-in. maximum size; the minimum compressive strength at 28 days shall be 3000 lb. per square inch.

CONSISTENCY OF CONCRETE.—Ordinary concrete shall be of the consistency necessary to give complete consolidation in the shutters and around the reinforcement, shall have no excess water on the surface after compaction by hand ramming or approved vibratory methods, and shall have a smooth surface with no trace of voids, separation, or honeycombing when the shutters are removed.

CONCRETE MIXING.—The proportions of fine and coarse aggregates and cement shall be as determined by the preliminary tests. The aggregates shall be measured by volume and the cement by weight. The amount of water to be added shall depend on the moisture content of the aggregate, but shall be such as to give a slump of not more than 1 in. more or less than that recorded in the preliminary tests. The dry materials shall be mixed for at least three revolutions of a mechanical batch mixer of approved design, after which the required amount of water shall be added gradually while the drum rotates and the concrete mixed for at least two minutes or for a longer period if necessary until uniform colour and consistency are obtained. On the cessation of work, including short stops, the mixer and all handling plant shall be washed out with clean water.

PLACING CONCRETE.—Shuttering and reinforcement shall be approved by the resident engineer before concrete is placed. The concrete may be conveyed in any suitable manner from the mixer provided that there is no separation or loss of any ingredients and provided that it is placed in its final position before initial setting of the cement takes place and within 30 minutes of the addition of water in the mixer.

Plain Concrete.—Unless otherwise specified, plain unreinforced concrete shall be deposited in layers from 9 in. to 12 in. thick which shall follow one another as quickly as possible to prevent any distinct joint between successive layers. The concrete shall be thoroughly consolidated by working it with shovels and ramming it with wooden beaters. Care shall be taken that the concrete is rammed closely against the face of any excavation in order that there shall be no subsequent settlement.

Reinforced Concrete.—Horizontal slabs shall be laid the full thickness in one operation and beams shall be completed in one operation. The concrete shall be carefully rammed into position, and in the case of beams may be assisted into position by tapping or hammering the shutters next to the freshly-deposited concrete so that the concrete is thoroughly worked around the reinforcement and embedded fixtures and into the corners of the shutters. The surfaces of floors shall be tamped with heavy wooden screeds to a true and even surface.

TESTS OF CONCRETE.—Works cube tests of ordinary concrete shall be made at least twice weekly, and at least once for each part of a structure, not less than four cubes being cast at a time. The method of manufacture and curing of the cubes shall be as described in Appendix VIII of the D.S.I.R. Code of Practice. The contractor shall supply and deliver the cubes to an approved tester and pay all fees in connection therewith. Reports of tests shall be submitted in duplicate.

The works cubes shall have the strength specified at 28 days, and if the tests are not satisfactory all concreting shall be stopped and not proceed without the permission of the resident engineer. The contractor may be required completely to cut out and replace at his own expense any work represented by the cubes failing to comply with the foregoing. In the event of strengths consistently higher than those specified being obtained, a reduction in the number of tests may be authorised.

Slump tests in accordance with Appendix VI of the D.S.I.R. Code of Practice shall be made daily and when directed by the resident engineer.

CONSTRUCTION JOINTS.—The concrete shall be placed so that the number of construction joints shall be as few as possible. Construction joints shall be made on horizontal and vertical planes only, unless otherwise shown on the drawings, and only at places approved by the resident engineer. Where concreting is stopped on a vertical plane, an approved stop-board shall be provided and provision made to allow the reinforcement to pass through the joint without being temporarily bent or otherwise displaced. In the case of slabs and walls a 2-in. by 1-in. fillet slightly splayed to permit easy removal shall be nailed on to the stop-board to form a joggle running throughout the length of the joint. Any concrete flowing past the joint shall be removed as soon as initial set occurs. When concreting against a hardened vertical surface is resumed, the surface shall be well roughened, swept clean, and thoroughly wetted. When concrete is stopped on a horizontal plane the surface shall be well roughened and thoroughly cleaned of all scum as soon as the concrete has set. At horizontal joints in walls on all external surfaces a 2-in. by 1-in. slightly-splayed fillet shall be nailed to the shutters throughout the length of the wall and projecting 1 in. above and below the top of the concrete, the fillet being removed before placing the next lift of concrete. When work is resumed the joint shall be well washed.

CONCRETING IN COLD WEATHER.—When depositing concrete at a temperature below 35 deg. F. on a rising thermometer, precautions shall be taken, to the satisfaction of the resident engineer, to ensure that the concrete shall have a temperature of at least 40 deg. F. at the time it is placed, and that the temperature of the concrete shall be maintained at not less than 33 deg. F. until it has thoroughly hardened. When necessary, concrete materials shall be warmed before they are mixed. Salt or other materials shall not be used for the prevention of freezing, and no frozen materials or materials containing ice or snow shall be used. The contractor shall cut out and replace at his own expense any concrete damaged by frost.

CURING CONCRETE.—The concrete after it is placed and during the early stages of hardening shall be protected from the harmful effects of sunshine, drying winds, cold, rain, running water, and shock. Concrete shall be prevented from drying out for at least seven days.

CONCRETE FINISH.—Concrete when placed shall be adequately tamped with approved tools, and the shuttering hammered, to eliminate air pockets and produce a compact mass without voids and with smooth surfaces requiring no further treatment on sides retained by shuttering; top surfaces shall be given a smooth finish by screeding or floating. Unprotected slabs of concrete shall not be concreted during heavy rain.

REINFORCEMENT.—Steel reinforcement shall comply with the requirements of the current British Standard for the class of reinforcement concerned. Manufacturers' test reports shall be supplied when required.

All reinforcement shall be free from dirt, oil, paint, loose rust, or scale when in position for concreting. The bars shall be accurately bent to the shapes shown on the drawings and schedules, and the bending shall be completed before the steel is fixed in position. Straight parts of bars shall be true, and bends shall be true to shape and dimensions shown on the bending schedules supplied to the contractor. The internal radii of bends shall be not less than twice the diameter of the bar, except that stirrups, column binders, and wall shear bars shall be bent to fit closely around the main bars. The internal radii of hooks shall be not less than four times the diameter of the bar. Stirrups and binders shall be bent to the sizes shown within a tolerance of $\pm \frac{1}{8}$ in. Bars shall in general be bent cold, but bending hot at cherry-red heat (not exceeding 1500 deg. F.) may be allowed except for cold-worked steel.

Bars shall not be cooled by quenching in water or other liquid. Bars shall be welded or rebent only when approved.

The contractor shall provide supports of approved type for maintaining the top reinforcement of slabs in position during concreting.

SHUTTERING.—Shuttering shall be erected true to line and to the shapes required in the work, and shall carry without deformation the loads due to wet concrete and any incidental load, and where the concrete is vibrated the shuttering shall withstand the effects of vibration without appreciable deflection, bulging, distortion, or loosening of its component parts. The contractor shall be responsible for the sufficiency of all shuttering, centering, and moulds.

Wire or similar ties shall not be left in concrete having a face exposed to the weather. Bolts shall be permitted provided that they are greased to allow for easy withdrawal and the holes subsequently made good.

The shuttering shall be designed so that the soffits of slabs and the sides of beams, columns, and walls may be removed first, leaving the shuttering to the soffits of beams and their supports in position. Wedges or other suitable means shall be provided to allow accurate adjustment of the shuttering and to allow it to be removed gradually without jarring the concrete.

All shuttering shall be thoroughly cleaned of old concrete, and immediately before concreting it shall be thoroughly washed out with water, holes being provided in the shuttering to permit the escape with the water of sawdust, shavings, or other rubbish.

The contractor shall be responsible for the safe removal of the shuttering, but the resident engineer may delay the time of removal if he thinks that this is necessary. Any work showing sign of damage through premature removal of shuttering or loading shall be entirely reconstructed.

All joints shall be sufficiently tight to prevent the leakage of grout, and if necessary the boards shall be planed and thicknessed in order that the surface against the concrete shall not be broken at joints between the boards.

All shuttering shall be coated with an approved type of oil before it is fixed in position.

CHAPTER X

COMMERCIAL PRACTICE

WHILE it is only possible by experience to acquire much of the knowledge required for supervising and organising construction, the beginner should know something about general practice in the civil engineering industry. Although the planning and construction may be carried out in a variety of ways, according to whether the work is for a Government department, a local authority, or a private owner, or whether the engineer is a consultant or is an employee of the owner, the general relationship between owner, engineer, and contractor is the same. In the first place the owner's engineer prepares preliminary plans and estimates for the approval of his employer. He then prepares specifications, conditions of contract and, in some cases, detailed quantities and drawings with sufficient information to enable contractors who are asked to tender clearly to understand the amount of work entailed and legally to bind the selected contractor.

It is important for the engineer to define clearly the standard of workmanship required in all parts of the work, the services to be provided by the contractor during construction, the risks which are to be the contractor's responsibility and those which are to be insured, the duration of the work, rates to be quoted, and so on. If the contractor is to carry out a certain amount of design, such as the provision of detailed working drawings, it is important to define clearly the basis of the calculations so that all the tenders provide for the same factor of safety.

CONDITIONS OF CONTRACT.—The organising of the work is greatly facilitated by writing conditions of contract which are equitable, and specifications requiring standards of workmanship which are really practicable. Many specifications demand a standard of workmanship which may be quite impossible of achievement on account of weather conditions or the difficulties of the site. On the other hand the specification must make it very clear that poor workmanship will not be accepted. Generally-accepted engineering terms should be used, and obscuration by legal phraseology avoided; in order to avoid disputes, however, there must be no ambiguity in defining the responsibilities of the contractor, and it is necessary to use caution and precision to safeguard the owner from unreasonable claims. It is customary to place on the contractor all responsibility in regard to site matters; this is reasonable, since his organisation is most suited to deal with them. The engineer on his part, however, should be prepared to meet all reasonable claims in respect of variations of the contract arising out of alterations or faulty or impractical design, unless, of course, the detailed working drawings were made by the contractor. It is usual to make the general contractor responsible for all services such as pumping, shoring and timbering, watching, lighting, fencing, huts and canteens, plant and equipment, access roads, temporary works, notices, instruments, setting out, trial holes for foundations, and so on. He should also be responsible for any interference with traffic and claims arising from damage to property, and it should be his duty to pay any rents or fees due in connection with the site. Permissible arrangements for the employment of sub-contractors, and the division of work and responsibility if other specialist contractors are employed directly by the owner, should be defined. The arrangements for regular measurement of and payment for completed work, the possible variations to the works due to site conditions or which may be made by the owner, the method to be used for the adjustment of payments, and the arrangement for arbitration in the event of disputes, should also be defined. Such conditions of contract have now become standardised and recommended for general use by the Association of Consulting Engineers and other bodies. Finally it should be emphasised that the most complete and carefully worded documents will not produce satisfactory work from a guileful or incompetent contractor, whereas a brief and simple specification and a clear outline drawing are all that is necessary in negotiating with firms of honest intention and good repute.

TENDERS.—Provided that the contractors asked to tender are of the right type and that the work is clearly defined, the lowest tender is usually that of the contractor most able at the time to carry out the work. Generally such a contractor, in anticipation of having suitable plant and men available, in order to secure the work not only allows for a small profit but uses ingenuity to think out cheap methods of doing the work. He finds the cheapest sources of materials and, before submitting his tender, ensures that he can purchase all the materials at the rates allowed in his tender, that the plant he intends to use will be available in his store or can be hired at economical rates, and that he will have all the resources necessary to carry through the work in the required time.

From the time of asking for tenders the engineer and his staff are continually watching the interests of the owner, while the contractor and his staff endeavour first to secure the contract and then to ensure that a reasonable, and in most cases maximum, profit is made. The owner's engineer and the contractor's engineer become central figures in a complicated commercial business in which, on the whole, the rules of fair play are observed; but under pressure from either side, or when special site difficulties are encountered, these rules are sometimes, it is to be regretted, ignored. One of the unexpected difficulties which are inevitable in civil engineering work may alarm the accountants on either side. "Realistic" attitudes may be adopted. Friendliness may disappear, and fair play and creditable enterprise may even degenerate into attempts of one side to get the better of the other. In such circumstances an engineer of sound judgment who can visualise the work as a whole, and is unbiased by immediate financial matters, can often avert the complete failure of the enterprise. Clearly defined and imaginative contract documents are a great help in avoiding disputes in these circumstances.

Some contractors before tendering search in the contract for every loop-hole which may be used later as the basis of an extra claim, and take this factor into account in deciding on their tender price. Up to a point this is legitimate, but it does not make for the smooth running of the contract. A contractor, when he submits his tender, should state his intentions fully and openly, and the experienced engineer can discriminate between this type of tender and one which is based on a number of intentionally-made omissions which, as the work proceeds, gradually come to light in the form of unexpected extra claims. The essence of all satisfactory business is complete honesty of intention and the elimination of all forms of deception. Consultants and owner's engineers may also have undesirable practices. The engineer who is never prepared to make a concession to a contractor whatever difficulties may arise, or who at the tender stage discloses to one contractor the price submitted by another in order to get a still lower price, provokes contractors into adopting undesirable means of securing orders and of making a profit.

Tenders should be very carefully compared, and it should be ensured that all items have been included. Full details should be obtained from the contractor of the plant he proposes to use, in order to make certain that the plant is available and that it is sufficient to complete the work in the required time. Full details should be obtained of the aggregate and cement it is proposed to use, and the names of the contractor's staff who are to supervise the work should be ascertained, so that the owner's engineer is assured that the contractor is able to carry out the work satisfactorily. The engineer should also investigate any special methods of construction the contractor proposes to adopt, such as the handling and placing of the concrete, the shuttering, and whether any part of the work is to be precast. The contractor on his part can greatly improve his chances of obtaining the contract if he works out in detail the procedure of construction he proposes to employ and the most economical way of handling materials while at the same time ensuring a good standard of workmanship.

RELATIONSHIP BETWEEN ENGINEER AND CONTRACTOR.—When a tender has been accepted the relationship between the contractor and the engineer enters a new phase. The successful contractor may have tendered for other work at the same time and so obtain two or more new contracts almost simultaneously with the result that he may be short of plant and staff. The engineer then may find it difficult to ensure that the contractor fulfils his undertaking with regard to the date of commencement of the work and the rate of progress.

Excavation for the foundations may reveal unforeseen difficulties leading to variations of the original contract, and the contractor may then ask for concessions and make extra claims on account of these variations. In preparing the conditions of contract and specifications it is important to anticipate such possibilities so far as possible by asking contractors to tender for clearly defined and assumed conditions and quantities and to give rates for all variations and possibilities which may arise. Provisions of this sort cannot be complete. A contractor may claim that variations or alterations do not fall into the category for which he has quoted a variation rate but cause additional difficulties not mentioned or defined, and he may submit an extra claim based on higher rates. During the course of most work the process of bargaining over extras, and of the engineer demanding the specified requirements of time of completion and quality of work, continues. Many of the extra claims can be dealt with as the work proceeds by the resident engineer agreeing with the contractor's engineer on the variations from the original contract. The rates quoted in the tender may be agreed in the same way, and the variation or extra claims settled month by month together with payment for the amount of the originally specified work satisfactorily completed. The settlement of doubtful claims is generally postponed until suitable stages of the work are reached. Towards the end of a contract the owner's engineer can review such claims in the light of his knowledge of whether the work has been done in a sound manner or whether the contractor has shown a tendency to evade his commitments. The validity of some claims may not be clearly defined by the contract, and over these the owner's engineer must adjudicate in a fair manner. After discussion and a certain amount of forbearance on either side most contractors will accept his decision, and the construction work, the tidying of the site, and the settlement of claims will be brought to an amicable completion. In some cases, however, the contractor may feel that the decisions of the engineer have not been fair, and he may decide to sue the owners or submit the case for arbitration.

It is important at the beginning of a contract for the engineer to ensure that the contractor includes in his tender and puts into effect insurance against such risks as loss of plant, damage to construction due to storms (particularly in marine work), collapse of shoring, earth slips and consequential damage, and so on, because unless the contractor is insured against these risks a partially-completed contract may be carried on by a contractor in a state of potential bankruptcy. The main object of the engineer is to get the work satisfactorily completed within the required time, and in his negotiations with the contractor he should continually have this in mind. A situation in which a disgruntled contractor is losing money and seeking every opportunity of evading his commitments in order to avoid further losses hinders progress and encourages bad workmanship. The engineer should facilitate in every way the execution of the work and do all in his power to allow the contractor to make a reasonable profit, expecting in return a good standard of workmanship. In interpreting the clauses of the contract, which are more often written with a successful law suit rather than an engineering achievement in mind, it is better to err on the liberal side. It is the spirit rather than the letter of the contract which matters. The exacting engineer who employs every artifice of buying to obtain low tenders, and who adopts a mean and uncompromising attitude with regard to contractors' claims, creates bad conditions for good collaboration, and he certainly does not obtain the best results. On the other hand, the contractor who seeks by every means to evade his commitments and responsibilities, whose only concern is to make the greatest possible profit, and whose work is only superficially satisfactory, ultimately gains a bad reputation and is not invited to tender for other work.

The Engineer in Industry.

Two of the most important mental qualities an engineer should possess are intellectual honesty and a completely objective viewpoint in regard to any problem he may encounter, whether purely scientific or concerned with the ordering of human affairs. The gaining of experience of actual working conditions, and of the possibilities and limitations of human beings and human organisation, are also important, so that his plans may be practical. This does not mean that upon entering commercial life he must abandon as quite unpractical the ideal standards which may have been presented to him at school or the University. On the contrary, a noble philosophy in regard to human affairs and the giving of his best in all his work are the only sound standards to aim at in life. But the use of expedients to overcome difficulties in civil engineering work should be accepted as inevitable, and as an immediate objective the best standards attainable in the prevailing circumstances should be adopted, provided, of course, that the circumstances are beyond the control of the engineer. A sound commercial concern continually strives to overcome such difficulties; others simply take advantage of them and perhaps even trade on them. If he is unfortunately employed by the latter type of organisation an engineer should do his best to overcome the prejudices and narrow sectional interests which prevail around him. It may cost him his post, or mean less chance of promotion, but preservation of his reputation and self-respect are more important. He can also be sure that if he has natural ability he can, with hard work, attain success. Against weakness in such circumstances a supporting philosophy, or definite purpose underlying his work, is the only real protection. He should remember that much of the present way of life owes its existence to the engineer's creative mind. He has, therefore, a great moral responsibility, although he may claim that he is but a hired servant and has no control over policy. But this is only partly true. In industry, those responsible for sales and finance, and who are generally only concerned with profits, certainly wield most of the power, but an engineer can have considerable influence on their policy. He must not be so absorbed in his work, or so complacent, as to serve industrial enterprise regardless of its character and purpose. The story of Mitchell, the designer of the Spitfire, shows the influence that a brilliant and persistent engineer, who was concerned about national defence, was able to bring to bear. Engineers of vision may be able to guide policy far better than those concerned with administration or sales, and they should be well represented on boards of management since ability in design can initiate new developments and so create new markets.

The present may be the beginning of a new phase in industrial history which will offer great opportunities to the engineer. There appears to be a strong desire to control the development of industry, and to remove the ugly shadow of the industrial revolution. This is a great and long-needed step forward, but slums will only give way to garden cities or suburbs, chaos to efficiency, unsightliness to decency, and homes and factories will only become pleasant places in which to live or work, if the right spirit prevails. To examine this qualification further, and construct thereby a sound philosophy for the future, it is pertinent to review appropriate events of history.

Engineers, with their essentially practical outlook, are accustomed to rely upon a machine or a structure, provided that it is satisfactory under test and in use. Ability to analyse fully the internal stresses is an enormous help but not an essential. A similar approach may be followed in attempting to understand the forces at work in human affairs. Some economists believe that history follows a purely economic law. Professor T. H. Huxley as a scientist, it is interesting to observe, recognised other forces at work: "Whoso calls to mind what I may venture to term the bright side of Christianity—that ideal of manhood, with its strength and its patience, its justice and its pity for human frailty, its helpfulness to the extremity of self-sacrifice, its ethical purity and nobility, which apostles have pictured, in which armies of martyrs have placed their unshakable faith, and whence obscure men and women, like Catherine of Sienna and John Knox, have derived courage to rebuke popes and kings—is not likely to underrate the importance of the Christian faith as a factor in human history." The effect of this factor can be studied in the works of the great historians, from its revolutionary power in the days of the early Christians to its confusion with politics in the Middle Ages and to its less obviously effective but kindly influence in modern times. Where this has been manifest in the family business, and small industries have thrived in the country towns of England, life has been pleasant, and had a quality which the industrial revolution largely destroyed but which we must strive to regain. The late Earl Baldwin, referring to his own factory, has well described this kind of works: "It was a place where I knew, and had known from childhood, every man on the ground; a place where I was able to talk with the men not only about the troubles in the works, but troubles at home and their wives. It was a place where strikes and lock-outs were unknown. It was a place where the fathers and grandfathers of the men then working there had worked, and where their sons went automatically into the business. It was also a place where nobody ever got the sack and where we had a natural sympathy for those who were less concerned with efficiency than is this generation, and where a large number of old gentlemen used to spend their days sitting on the handles of wheelbarrows, smoking their pipes. Oddly enough it was not an inefficient community. It was the last survival of that type of works which ultimately became swallowed up in one of those great combinations towards which the industries of to-day are tending."

In our social history since the beginning of the industrial revolution there have been some creditable achievements due to men and women imbued with the spirit of true religion, such as the Factory Acts passed mainly through the enthusiasm of Lord Shaftesbury, and the industrial planning of the pioneers of garden cities and model factories. The earliest of these experiments are still amongst the finest examples of successful industrial planning, and stand in striking contrast to the character of other light industries in the adjoining "Black Country" and elsewhere, where similar experiments might equally well have been tried, resulting probably in a lower cost of production. These great human achievements must be attributed mainly to the enterprise and indomitable Christian spirit of the particular factory owners, and to the persistence of people who dedicated their lives to the service of others. So it has been throughout history. Wherever this spirit governs human activity a new factor operates for the benefit of humanity. Experience of working for a variety of employers in different parts of the world confirms this truth, on which must be built our philosophy for the future.

No man can give of his best unless he is convinced that he is working for the benefit of others as well as himself, that he and his work take their place in a world which, in spite of recurring set-backs, has always moved towards a higher conception of life. Contentment and self-respect result from a mental outlook which considers others as well as oneself. The engineer responsible for industrial structures is often in an exceptional position to put into practice those principles which he inwardly knows are for the good of mankind, or to persuade others to do so, and an outlook which seeks beyond immediate profit will lead to happiness and self-respect. The removal of unnecessary hardships, consideration for the comfort and happiness of the workers, is an important part of the duty of an engineer, for production depends upon the workers and no effort or expense on other means of securing efficiency are of avail if the workers are unhappy and discontented; even in the stage we have now reached towards the "machine age" the output of the machine depends upon the human being.

The engineer needs to beware of the danger of over-specialised technical training which tends to develop a purely mechanistic conception of production and efficiency. While early in his career he will have little authority and must be content to do his share of routine work, he should train for and aim at higher responsibility later, and learn to envisage an industrial organisation not as a machine but as a community, a sensitive living organism, requiring for efficiency the stimulation of growth and development, and the maintenance of goodwill among the workers and between the departments. The direct contact between master and man of the old guild system and the small family industry must have its counterpart in the combines and corporations. Organisation, costing, and progress charts should be used as guides rather than as infallible indicators. General interest can be aroused in the purpose of the work by the deliberate divulgence of full information. The sphere of responsibility of all grades of staff must be defined to avoid delays in decision. The young engineer must in fact become familiar with and sympathetic towards what is known as industrial psychology and so systematically apply, under the conditions of modern large-scale industry, the precepts of the wise Bishop Butler: "Perception of distress in others is a natural excitement, passively to pity and actively to relieve it: but let a man set himself to attend to, inquire out, and relieve distressed persons, and he cannot but grow less and less sensibly affected by the various miseries of life with which he must become acquainted; when yet, at the same time, benevolence considered not as a passion but as a practical principle of action, will strengthen and, whilst he passively compassionates the distressed less, he will acquire a greater aptitude actively to assist and befriend them." But invaluable as industrial psychology is as an aid to good management, it must only be regarded as a means of facilitating the work. Science and technology too, truth revealing and mentally satisfying as they are, must not be allowed, in the very full programme of modern life, to replace or exclude the wisdom of the humanities and acceptance of true religion with its indisputable practical implications and indispensable impelling spiritual power. Science and religion should be complementary, in fact they have mainly clashed through lack of imagination in interpreting the poetic language of the ancient mystics who were concerned with spiritual values rather than historical or scientific accuracy. Thomas á Kempis in the fourteenth century seemed to have had the right mental attitude towards science: "The more a man is undistracted and becomes inwardly simple, so much the more will he be able to enter easily into profound subjects, because his mind will be enlightened from above." Lord Avebury, writing on "The Use of Life" in the nineteenth century, points out the great value but inadequacy of Greek and Roman philosophy: "There are noble sentiments in Plato, Aristotle and Epictetus, in Seneca and Marcus Aurelius, but there is no such Gospel of Love as that in the New Testament." Commerce and industry, on account of special difficulties, are too often regarded as particular departments of life, exempt from the necessity of applying these sentiments.

A well-known headmaster and mathematician recently said, "True education consists of the enrichment of all the powers of the human personality. It must not be confused with the acquisition of knowledge, which is merely a tool used in the process. Schools and colleges must prepare their pupils for their future careers, for the right use of leisure, and for keen participation in the life of the community. Activities must train the body as well as the mind. The main aims of education, however, are spiritual: the development of intelligence, imagination, a habit of enquiry, a spirit of adventure, an appreciation of the beautiful things of life, a love of truth, justice, and fair dealing, a sense of duty and responsibility, and a scorn of any distortion of the views of others from whom we might differ."

New problems in industry will continually arise. For example, the civil engineer on public works, until quite recently, if he was at all conscientious, did his best to cope with the hardships of casual labour, but these unhappy conditions have largely gone. Problems in the future may be quite different, and concerned perhaps, in contrast, with the hardships of management, and of obtaining adequate production with the fear of unemployment removed. For this reason the engineer must learn to be adaptable, and if he is wise he will take advantage of his special opportunities by training, not only to be technically efficient in his special subject, but also to be a wise and sound leader for the better direction of industry and the happiness of humanity.

APPENDIX I

DESIGN DATA

THE tables in this appendix give some data relating to the working stresses and loads assumed in the design of reinforced concrete structures. The data are taken mainly from codes of practice published by the Department of Scientific and Industrial Research and the British Standards Institution.

TABLE 1.—WORKING STRESSES IN CONCRETE: D.S.I.R. CODE (1934).

Concrete	Reference	Nominal	Proportions. Cub. ft. of Aggregate per 112 lb. of cement		Minimum Cube Strength Requirements at 28 Days. Lb. per sq. in.		Modular Ratio	Permissible Concrete Stresses. Lb. per sq. in.			
			Fine	Coarse	Preliminary Tests	Works Tests		Bending	Direct	Shear	Bond
					$4{\cdot}5x$	$3x$	$\frac{40{,}000}{3x}$	x	$0{\cdot}8x$	$0{\cdot}1x$	$0{\cdot}1x + 25$
Ordinary Grade	I	1 : 1 : 2	1¼	2½	4,388	2,925	14	975	780	98	123
	II	1 : 1·2 : 2·4	1½	3	4,163	2,775	14	925	740	93	118
	III	1 : 1·5 : 3	1⅞	3¾	3,825	2,550	16	850	680	85	110
	IV	1 : 2 : 4	2½	5	3,375	2,250	18	750	600	75	100
High Grade	I	1 : 1 : 2	1¼	2½	5,625	3,750	11	1,250	1,000	125	150
	II	1 : 1·2 : 2·4	1½	3	5,400	3,600	11	1,200	960	120	145
	III	1 : 1·5 : 3	1⅞	3¾	4,950	3,300	12	1,100	880	110	135
	IV	1 : 2 : 4	2½	5	4,275	2,850	14	950	760	95	120
Special Grade (Limiting Values)	I	1 : 1 : 2	1¼	2½	Strength requirements, modular ratios and permissible stresses to be calculated from above factors and to be based on preliminary test results. The values for the permissible stresses shall not exceed those for similar mixes of high grade concrete (Table 2) by more than 25 per cent. Shear and bond stresses shall also not exceed 150 lb. per sq. in. Modular ratios may be taken to the nearest whole number.						
	II	1 : 1·2 : 2·4	1½	3							
	III	1 : 1·5 : 3	1⅞	3¾							
	IV	1 : 2 : 4	2½	5							

Where other proportions of fine to coarse aggregate are used the requirements shall be based on the ratio of the sum of the volumes of the fine and coarse aggregates, each measured separately, to the quantity of cement, and shall be obtained by proportion from the two nearest defined mixes.

The tabulated values of the modular ratios are given to the nearest whole number; nevertheless, more exact values, calculated from the formula $40{,}000/3x$, may be used.

TABLE 2.—WORKING STRESSES IN CONCRETE: BRITISH STANDARD CODE CP. 114 (1948).

Proportions of Concrete		Quantity of Aggregate (1) per 112 lb. of cement (cu. ft.)		Crushing strength at 28 (3) days (lb. per sq. in.)		Permissible stresses (lb. per square inch)				
						Compression		Shear	Bond	
		Fine	Coarse	Preliminary test	Works test	Direct	Bending		Average	Local
P.C.	1 : 1 : 2	1¼	2½	5250	4500	1140	1500	130	150	220
	1 : 1½ : 3	1⅞	3¾	4375	3750	950	1250	115	135	200
	1 : 2 : 4	2½	5	3500	3000	760	1000	100	120	180
A.C.	1 : 2 : 4	2½	5	6000 (2)	5000 (2)	1140	1500	130	150	220

P.C. = Portland cement concrete and Portland-blastfurnace cement concrete.
A.C. = High-alumina cement concrete.
(1).—Aggregate complying with B.S. No. 882.
(2).—Strengths at two days.
(3).—Alternative: Crushing strength of works cubes at 7 days must be not less than two-thirds of the specified strength at 28 days.

TABLE 3.—WORKING STRESSES IN REINFORCEMENT: BRITISH STANDARD CODE CP. 114 (1948).

Stress	Reinforcement	Permissible stresses (lb. per square inch)				
		Mild steel (1)	Medium-tensile steel (2) and mild steel with guaranteed yield-point stress	High-tensile steel (2)	Cold-twisted steel (3)	
					Single bars	Twin bars
Tensile	In reinforcement other than helical binding in columns and shear reinforcement	18,000	Half guaranteed yield-point stress but not more than 27,000			
	In helical binding in columns	13,500	15,000	18,000	18,000	18,000
	In shear reinforcement	18,000	Half guaranteed yield-point stress but not more than 20,000			
Compressive(1)	In columns	18,000	Ditto			—
	In beams and slabs: (a) Compressive resistance of concrete taken into account.	Stress in surrounding concrete multiplied by the modular ratio.				—
	(b) Compressive resistance of concrete neglected.	18,000	Half guaranteed yield-point but not more than 20,000			—

(1).—Not applicable to twin-twisted or other bent bars.
(2).—Complying with B.S. No. 785 (1938).
(3).—Complying with B.S. No. 1144 (1943).

TABLE 4.—FACTORS FOR RESISTANCE TO BENDING: D.S.I.R. CODE (1934).

GRADE OF CONCRETE	NOMINAL CONCRETE MIX	PERMISSIBLE STRESS IN BENDING (x) LB. PER SQ. IN.	MODULAR RATIO m $\left(\frac{40,000}{3x}\right)$	18,000 LB. PER SQ. IN. STEEL STRESS ($n = 0{\cdot}42553\ d$, $a = 0{\cdot}85816\ d$) ECONOMIC PERCENTAGE ($0{\cdot}0011820\ x$)	18,000 LB. PER SQ. IN. STEEL STRESS $R\left(=\frac{M}{bd^2}\right)$ ($0{\cdot}18258\ x$)	20,000 LB. PER SQ. IN. STEEL STRESS ($n = 0{\cdot}40000\ d$, $a = 0{\cdot}86667\ d$) ECONOMIC PERCENTAGE ($0{\cdot}0010000\ x$)	20,000 LB. PER SQ. IN. STEEL STRESS $R\left(=\frac{M}{bd^2}\right)$ ($0{\cdot}17333\ x$)	25,000 LB. PER SQ. IN. STEEL STRESS ($n = 0{\cdot}34782\ d$, $a = 0{\cdot}88406\ d$) ECONOMIC PERCENTAGE ($0{\cdot}00069564\ x$)	25,000 LB. PER SQ. IN. STEEL STRESS $R\left(=\frac{M}{bd^2}\right)$ ($0{\cdot}153747\ x$)
	1 : 1 : 2	975	13·68	1·153	178·0	0·975	169·0	0·678	149·9
ORDINARY	1 : 1·2 : 2·4	925	14·41	1·093	168·9	0·925	160·3	0·644	142·2
GRADE	1 : 1½ : 3	850	15·69	1·005	155·2	0·850	147·3	0·591	130·7
	1 : 2 : 4	750	17·78	0·887	136·9	0·750	130·0	0·522	115·3
	1 : 1 : 2	1250	10·67	1·478	228·2	1·250	216·7	0·870	192·2
HIGH	1 : 1·2 : 2·4	1200	11·11	1·418	219·1	1·200	208·0	0·835	184·5
GRADE	1 : 1½ : 3	1100	12·12	1·300	200·8	1·100	190·7	0·765	169·1
	1 : 2 : 4	950	14·03	1·123	173·5	0·950	164·7	0·661	146·1
	1 : 1 : 2	1563	8·53	1·848	285·4	1·563	270·9	1·087	240·3
SPECIAL	1 : 1·2 : 2·4	1500	8·89	1·773	273·9	1·500	260·0	1·044	230·6
GRADE	1 : 1½ : 3	1375	9·70	1·625	251·1	1·375	238·3	0·957	211·4
	1 : 2 : 4	1188	11·22	1·404	216·9	1·188	205·9	0·826	182·7

d = Effective depth.

TABLE 5.—REINFORCEMENT: AREAS OF ROUND BARS (SQUARE INCHES).

Diam. of bars in inches.	Number of bars.													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1/4	0·049	·098	·147	·196	·245	·294	·343	·392	·441	·491	·540	·589	·638	·687
5/16	0·076	·153	·230	·306	·383	·460	·536	·613	·690	·767	·843	·920	·997	1·073
3/8	0·110	·220	·331	·441	·552	·662	·772	·883	·993	1·104	1·214	1·324	1·435	1·545
7/16	0·150	·300	·450	·601	·751	·901	1·052	1·202	1·352	1·503	1·653	1·803	1·953	2·104
1/2	0·196	·392	·588	·785	·981	1·177	1·374	1·570	1·766	1·963	2·159	2·355	2·551	2·748
9/16	0·248	·497	·745	·994	1·242	1·491	1·739	1·988	2·236	2·485	2·733	2·982	3·230	3·479
5/8	0·306	·613	·920	1·227	1·534	1·840	2·147	2·454	2·761	3·068	3·374	3·681	3·988	4·295
11/16	0·371	·742	1·113	1·484	1·856	2·227	2·598	2·969	3·340	3·712	4·083	4·454	4·825	5·196
3/4	0·441	·883	1·325	1·767	2·209	2·650	3·092	3·534	3·976	4·418	4·859	5·301	5·743	6·185
13/16	0·518	1·037	1·555	2·074	2·592	3·111	3·629	4·148	4·665	5·185	5·703	6·222	6·740	7·259
7/8	0·601	1·202	1·803	2·405	3·006	3·607	4·209	4·810	5·411	6·013	6·614	7·215	7·816	8·418
15/16	0·690	1·380	2·070	2·761	3·451	4·141	4·832	5·522	6·212	6·903	7·593	8·283	8·973	9·664
1	0·785	1·570	2·356	3·141	3·927	4·712	5·497	6·285	7·068	7·854	8·639	9·424	10·210	10·995
1 1/16	0·886	1·772	2·659	3·545	4·432	5·318	6·204	7·091	7·977	8·864	9·750	10·636	11·523	12·409
1 1/8	0·994	1·988	2·982	3·976	4·970	5·964	6·958	7·952	8·946	9·940	10·934	11·928	12·922	13·916
1 3/16	1·107	2·215	3·322	4·430	5·537	6·645	7·752	8·860	9·967	11·075	12·182	13·290	14·397	15·505
1 1/4	1·227	2·454	3·681	4·908	6·136	7·363	8·590	9·817	11·044	12·272	13·499	14·726	15·953	17·180
1 5/16	1·353	2·706	4·059	5·412	6·765	8·118	9·471	10·824	12·177	13·530	14·883	16·236	17·589	18·942
1 3/8	1·484	2·969	4·454	5·939	7·424	8·909	10·394	11·879	13·364	14·849	16·333	17·818	19·303	20·788
1 7/16	1·623	3·246	4·869	6·492	8·115	9·738	11·361	12·984	14·607	16·230	17·853	19·476	21·099	22·722
1 1/2	1·767	3·534	5·301	7·068	8·835	10·602	12·369	14·136	15·903	17·671	19·438	21·205	22·972	24·739
1 9/16	1·917	3·836	5·752	7·670	9·587	11·505	13·422	15·340	17·257	19·175	21·092	23·010	24·927	26·845
1 5/8	2·073	4·147	6·221	8·295	10·369	12·443	14·517	16·591	18·665	20·739	22·812	24·886	26·960	29·034
1 11/16	2·236	4·473	6·709	8·946	11·182	13·419	15·655	17·892	20·128	22·365	24·601	26·838	29·074	31·311
1 3/4	2·405	4·810	7·215	9·621	12·026	14·431	16·837	19·242	21·647	24·053	26·458	28·863	31·268	33·674
1 13/16	2·580	5·160	7·740	10·320	12·901	15·481	18·061	20·641	23·221	25·802	28·382	30·962	33·542	36·122
1 7/8	2·761	5·522	8·283	11·044	13·086	16·567	19·328	22·089	24·850	27·612	30·375	33·134	35·895	38·656
1 15/16	2·948	5·896	8·844	11·793	14·741	17·689	20·638	23·586	26·534	29·483	32·431	35·379	38·327	41·276
2	3·141	6·283	9·424	12·566	15·708	18·849	21·991	25·132	28·274	31·416	34·557	37·699	40·840	43·982

TABLE 6.—SUPERIMPOSED LOADS FOR BUILDINGS: BRITISH STANDARD CODE CP. 3, CHAPTER V (1944).

Loading class	Floor space occupancy.	Superimposed floor load per sq. ft.	Minimum* loads on slabs or floorboards per ft. width, uniformly distributed	Minimum* loads on beams uniformly distributed
		lb.	lb.	lb.
I	Private dwellings of not more than two stories	30	240	1920
II	Rooms in private dwellings of more than two stories, including flats, hospital rooms and wards, bedrooms and private sitting rooms in hotels and tenement houses, and similar occupancies	40	320	2560
III	Rooms used as offices	50	400	3200
IV	Classrooms in schools and colleges; minimum for light workshops	60	480	3840
V	Banking halls and offices where the public may congregate	70	560	4480
VI	Retail shops; places of assembly with fixed seating†; churches and chapels; restaurants; garages for vehicles not exceeding 2½ tons gross weight (private cars, light vans, etc.); circulation space in machinery halls, power stations, pumping stations, etc., where not occupied by plant or equipment	80	640	5120
VII	Places of assembly without fixed seating (public rooms in hotels, dance halls, etc.); minimum for filing or record rooms in offices; light workshops generally, including light machinery	100	800	6400
VIII	Garages to take all types of vehicles	100	Worst combination of actual wheel loads or, if the slab is capable of effective lateral distribution of load, 1·5 × maximum wheel load, but not less than 2000 lb., considered to be distributed over a floor area 2 ft. 6 in. square	
IX	Light storage space in commercial and industrial buildings; medium workshops	150	—	—
X	Minimum for warehouses and general storage space in commercial and industrial buildings; heavy workshops. (The loads imposed by heavy plant and machinery should be determined and allowed for.)	200	—	—

* Minimum load for slabs becomes operative at spans of less than 8 ft. Minimum load for beams becomes operative on areas less than 64 sq. ft. Beams, ribs and joists spaced at not more than 3-ft. centres may be calculated for slab loadings.

† Fixed seating implies that the removal of the seating and the use of the space for other purposes are improbable.

TABLE 7.—VELOCITY AND PRESSURE OF WIND: BRITISH STANDARD CODE CP. 3, CHAPTER V (1944).

Assumed Velocity of Wind (v) (miles per hour).

District	Where natural protection is afforded	Open country inland	Conditions of maximum exposure
West Scotland N.W. Ireland N.W. England Wales S.W. England	60	70	85
East Scotland N.E. Ireland N.E. England East England South and S.E. England	55	65	75
Central England (including Severn Estuary)	55	60	70

Pressure of Wind (lb. per square foot).

Effective height (h—s) in feet	Wind velocity (v) m.p.h.						
	55	60	65	70	75	80	85
0	7·0	7·0	7·0	8·4	9·6	10·6	12·4
5	7·0	7·0	8·0	9·3	10·7	12·1	13·7
10	7·0	7·6	8·9	10·3	11·9	13·5	15·2
15	7·0	8·2	9·7	11·2	12·9	14·7	16·6
20	7·5	8·9	10·4	12·1	13·9	15·8	17·9
25	8·0	9·5	11·2	12·9	14·9	16·9	19·0
30	8·4	10·0	11·8	13·7	15·7	17·9	20·2
35	8·9	10·6	12·4	14·4	16·5	18·8	21·2
40	9·3	11·1	13·0	15·0	17·2	19·6	22·2
45	9·7	11·6	13·5	15·7	18·1	20·5	23·2
50	10·1	12·0	14·1	16·3	18·7	21·3	24·1
60	10·8	12·9	15·1	17·3	20·1	22·9	25·8
70	11·5	13·7	16·1	18·4	21·4	24·3	27·5
80	12·1	14·4	16·9	19·6	22·5	25·6	29·0
90	12·7	15·1	17·7	20·6	23·6	26·9	30·5
100	13·3	15·8	18·6	21·6	24·8	28·2	31·8
120	14·4	17·1	20·1	23·4	26·8	30·5	34·5
140	15·4	18·4	21·6	25·0	28·7	32·7	36·9
160	16·4	19·6	22·9	26·6	31·5	34·8	39·2
180	17·3	20·6	24·2	28·0	32·1	36·6	41·4
200 and over	18·2	21·6	25·4	29·4	33·8	38·4	43·5

TABLE 8.—BEARING PRESSURES ON GROUND: BRITISH STANDARD CODE CP. 101 (1948).

Type of soil	Permissible bearing pressure (tons per sq. ft.)
Coarse-grained, non-cohesive	
Compact gravel or sand-gravel mixtures	4 – 6
Loose gravel or sand-gravel mixtures	3 – 4
Compact well-graded sand	2 – 3
Loose well-graded sand	1½– 2
Compact poorly-graded or uniform sand	1½– 2
Loose poorly-graded or uniform sand	1 – 1½
Fine-grained, cohesive	
Very stiff boulder clays: shale-like clays	4 – 6
Stiff clay and sandy clay	2 – 4
Firm clay and sandy clay	1 – 2
Soft clay and silt	½– 1
Very soft clay and silt	½– less
'Made' ground	To be determined after thorough investigation
Peat	

NOTES.

1.—The pressures in *Table* 8 relate mainly to the foundations of buildings of not more than two stories, but the range of pressures is approximately correct for heavier structures except on clays.

2.—*Sand and gravel:* The bearing pressures in the table assume that the width of the foundation is not less than 3 ft. For narrower foundations on sand or gravel the permissible bearing pressure decreases as the width decreases, and should be one-third the value given in the table multiplied by the width of the foundation in feet. In such soils the permissible bearing pressure can be increased by one-eighth of a ton per square foot for each foot of depth of the loaded area below the lowest ground surface immediately adjacent. If the ground water level in sand or gravel is likely to approach foundation level the permissible bearing pressure should be reduced to about one-half those given in the table.

3.—*Clay soils:* For light buildings the width and depth of the foundation do not have an appreciable influence on the permissible bearing pressure on clay soils. If a clay is examined under dry summer conditions the probable deterioration in winter should be borne in mind.

4.—*Mixed soils:* Soils intermediate between the main types given in the table need to be assessed by examination. For example, if a soil comprises a mixture of gravel, sand, and clay, of sand and clay, or silt and clay, the permissible bearing pressure would depend largely on the proportion of clay and on the strength of the clay.

TABLE 9.—SLABS SPANNING IN TWO DIRECTIONS: BRITISH STANDARD CODE CP. 114 (1948).

Type of panel and moments considered	Short span Z'_x								Long span Z'_y for all values of $\frac{l_y}{l_x}$
	Values of $\frac{l_y}{l_x}$								
	1·0	1·1	1·2	1·3	1·4	1·5	1·75	2·0 or more	
Case 1. Interior panels. Negative moment at continuous edge.	0·033	0·040	0·045	0·050	0·054	0·059	0·071	0·083	0·033
Positive moment at mid-span.	0·025	0·030	0·034	0·038	0·041	0·045	0·053	0·062	0·025
Case 2. One edge discontinuous. Negative moment at continuous edge.	0·041	0·047	0·053	0·057	0·061	0·065	0·075	0·085	0·041
Positive moment at mid-span.	0·031	0·035	0·040	0·043	0·046	0·049	0·056	0·064	0·031
Case 3. Two adjacent edges discontinuous. Negative moment at continuous edge.	0·049	0·056	0·062	0·066	0·070	0·073	0·082	0·090	0·049
Positive moment at mid-span.	0·037	0·042	0·047	0·050	0·053	0·055	0·062	0·068	0·037
Case 4. Two short edges discontinuous. Negative moment.	0·056	0·061	0·065	0·069	0·071	0·073	0·077	0·080	
Positive moment.	0·044	0·046	0·049	0·051	0·053	0·055	0·058	0·060	0·035
Case 5. Two long edges discontinuous. Negative moment.									0·056
Positive moment.	0·035	0·053	0·060	0·065	0·068	0·071	0·077	0·080	0·044
Case 6. Three edges discontinuous. Negative moment at continuous edge.	0·058	0·065	0·071	0·077	0·081	0·085	0·092	0·098	0·058
Positive moment at mid-span.	0·044	0·049	0·054	0·058	0·061	0·064	0·069	0·074	0·044
Case 7. Four edges discontinuous. Positive moment at mid-span.	0·050	0·057	0·062	0·067	0·071	0·075	0·081	0·083	0·050

The bending-moment coefficients given in the table are for rectangular panels carrying a uniformly-distributed load and supported along four edges with provision for resisting torsion at the corners.

APPENDIX II

CALCULATIONS FOR REINFORCED CONCRETE SLABS, BEAMS, AND COLUMNS

THE calculations necessary for the design of slabs, beams, and columns, similar to those given in *Figs.* 139 to 142, are as follows. The purpose of the examples is to show that, although the theoretical analyses of reinforced concrete members may be complex, the arithmetical work in practical design is straightforward.

The notation and formulæ are those given in Chapters III and IV. The designs are in accordance with the British Standard Code, 1948 (see Appendix I), upon which the following data are based:

For beams and slabs.—1 : 2 : 4 concrete, $c = 1000$ lb. per square inch, $t = 18{,}000$ lb. per square inch, $s = 100$ lb. per square inch without shear reinforcement, $s = 400$ lb. per square inch with shear reinforcement, $m = 15$; for these stresses $n_1 = 0{\cdot}455$ (*Fig.* 85), $a_1 = 1 - \frac{0{\cdot}455}{3} = 0{\cdot}85$, and

$$Rbd^2 = 0{\cdot}455 \times 0{\cdot}85 \times 0{\cdot}5 \times 1000bd^2 = 195bd^2.$$

For columns.—1 : $1\frac{1}{2}$: 3 concrete, $c = 950$ lb. per square inch, $t_s = 18{,}000$ lb. per square inch.

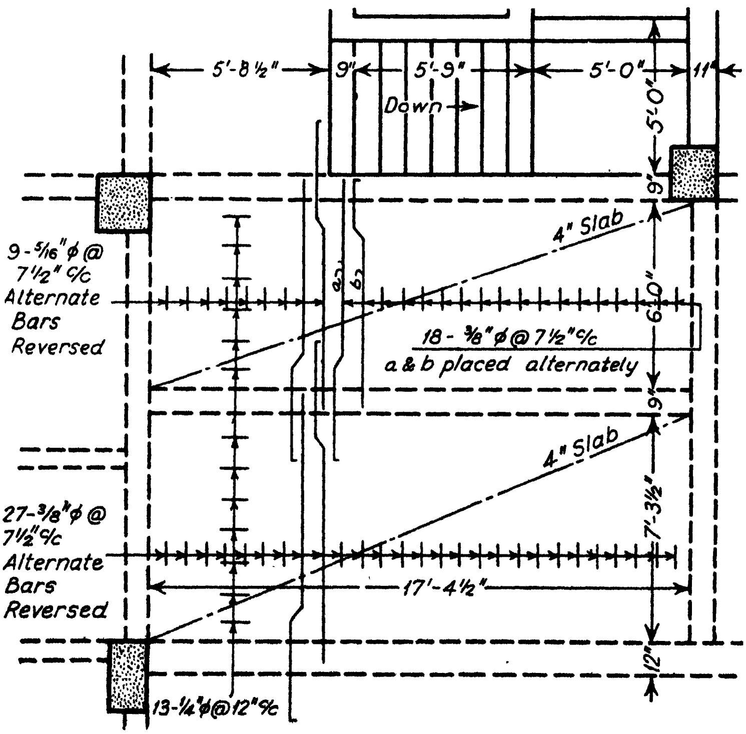


FIG. 177.

Slabs.

The two panels of slab shown in *Fig.* 177 are similar to those in *Fig.* 139, but are designed for different loads and working stresses. The safe-limit value of the effective depth of a 4-in. slab with ½ in. of cover is 3·25 in. Assume that the floor is in Class VI (*Table* No. 6, Appendix I).

INTERIOR PANEL.—Clear span, 7 ft. 3½ in.; effective span, 7·6 ft. Since the span is less than 8 ft., the minimum superimposed load of 640 lb. per foot width applies.

Dead load: Weight of slab,		48 lb. per square foot	
,, ,, finishes, say,		15 ,, ,, ,, ,,	
Total dead load	= 63 × 7·6 =	478 lb. per foot width.	
Superimposed load,		640 ,, ,, ,, ,,	
	Total	1118 ,, ,, ,, ,,	

Bending moment, $\frac{1118 \times 7{\cdot}6 \times 12}{12} = 9000$ in.-lb.

$d \nless \sqrt{\frac{9000}{195 \times 12}} = 1{\cdot}95$ in. The effective depth provided (3·25 in.) is ample.

It is often more economical to provide a slab thicker than that theoretically required; the thicker slab may also be required for resistance to fire. The consequent reduction in reinforcement may offset in cost the additional concrete. For slabs of large area it is advisable to estimate the approximate cost of two or more designs with different thicknesses of slab, and to select the cheapest.

$A_t = \frac{9000}{18{,}000 \times 0{\cdot}85 \times 3{\cdot}25} = 0{\cdot}18$ sq. in.; ⅜-in. bars at 7½ in. centres provide 0·177 sq. in.

Distribution bars: 0·1 per cent. of 12 × 4 = 0·048 sq. in.; ¼-in. bars at 12 in. centres.

END PANEL.—Clear span, 6 ft.; effective span, 6·3 ft.

Total load, (63 × 6·3) + 640 = 1037 lb. per foot width.

Bending moment, $\frac{1037 \times 6{\cdot}3 \times 12}{10} = 7850$ in.-lb. Therefore the same thickness of slab and amount of reinforcement as in the interior panel are satisfactory.

Beams.

The details of one span of the main beams to which the bending schedule and sections in *Fig.* 140 apply are given in *Fig.* 178. If it is assumed that the distance between adjacent lines of main beams (that is the span of the secondary beams) is 25 ft. and the superimposed load is 80 lb. per square foot of floor, the load on one span of a main beam is calculated as follows.

Load from each secondary beam:

Slab,	65 lb. per square foot	
Finishes,	15 ,, ,, ,, ,,	
Superimposed load,	80 ,, ,, ,, ,,	
	160 × 23·75 × 8·625 =	32,600 lb.
Rib of secondary beam,	say,	4,750 ,,
	Total,	37,350 ,,

Load from three secondary beams,	3 × 37,350 =	112,050 lb.
Rib of main beam,	550 lb. per foot	
Slab, finishes, and superimposed load above rib (160 × 1·25),	200 ,, ,, ,,	
	750 × 33 =	24,700 ,,
Total load on main beam,	say,	137,000 ,,

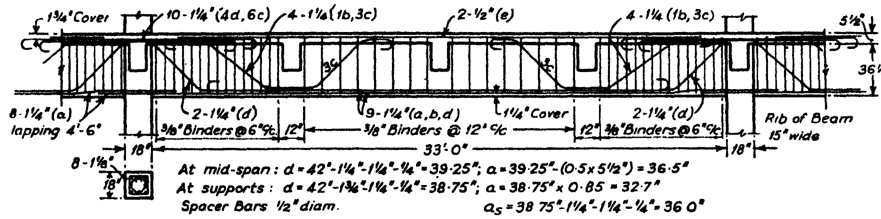


FIG. 178.

Allowing for the ratio of live to dead load (about 1 : 2) and for the fact that some of the load is uniformly distributed and that some is transferred to the main beam as loads concentrated at the quarter points, it is reasonable to assume in this example that the maximum bending moments at the middle of the span and over the supports are $\frac{WL}{10}$ for a beam continuous over both supports.

Bending moment, $\frac{137{,}000 \times 34{\cdot}5 \times 12}{10} = 5{,}670{,}000$ in.-lb.

At mid-span, $A_t = \frac{5{,}670{,}000}{36{\cdot}5 \times 18{,}000} = 8{\cdot}6$ sq. in. Nine $1\frac{1}{4}$-in. bars $= 11{\cdot}04$ sq. in.

$$b = (12 \times 5{\cdot}5) + 15 = 80 \text{ in.} \quad s_1 = \frac{5{\cdot}5}{39{\cdot}25} = 0{\cdot}14.$$

$$p_t = \frac{11{\cdot}04 \times 100}{39{\cdot}25 \times 80} = 0{\cdot}35 \text{ per cent.}$$

From *Fig.* 89, $n_1 = 0{\cdot}325$, $a_1 = 0{\cdot}94$.

$$c_a = \frac{5{,}670{,}000}{80 \times 5{\cdot}5 \times 0{\cdot}94 \times 39{\cdot}25} = 350 \text{ lb. per square inch.}$$

$$c = \frac{0{\cdot}325 \times 350}{0{\cdot}325 - 0{\cdot}07} = 446 \text{ lb. per square inch.}$$

At supports, $Rbd^2 = 195 \times 15 \times 38{\cdot}75^2 = 4{,}350{,}000$ in.-lb.

$$M_c = 5{,}670{,}000 - 4{,}350{,}000 = 1{,}320{,}000 \text{ in.-lb.}$$

$$n_1 d = 0{\cdot}455 \times 38{\cdot}75 = 17{\cdot}6 \text{ in.}$$

$$c_c = \frac{1000(17{\cdot}6 - 2{\cdot}75)}{17{\cdot}6} = 845 \text{ lb. per square inch.}$$

$A_c = \frac{1{,}320{,}000}{36 \times 14 \times 845} = 3{\cdot}07$ sq. in. Eight $1\frac{1}{4}$-in. bars in compression overlapping 4 ft. 6 in. are equivalent only to about four bars (4·91 sq. in.), which are sufficient.

$A_t = \frac{4{,}350{,}000}{32{\cdot}7 \times 18{,}000} + \frac{1{,}320{,}000}{36 \times 18{,}000} = 7{\cdot}4 + 2{\cdot}03 = 9{\cdot}43$ sq. in. At the centre of the support (where the bending moment is greatest) the area of tensile reinforcement is 12·27 sq. in. and just beyond the face of the column it decreases to 9·82 sq. in., which are sufficient.

Near the supports: $S = \frac{137{,}000}{2} = 68{,}500$ lb.

$s = \frac{68{,}500}{32{\cdot}7 \times 15} = 140$ lb. per square inch. Therefore shear reinforcement is required. By proportion, or directly from *Fig.* 81,

Two $1\frac{1}{4}$-in. bars at 45 deg. double-system resist 62,400 lb.
$\frac{3}{8}$-in. binders at 6 in. centres resist $570 \times 38{\cdot}75 = 22{,}000$,,

84,400 ,,

Between the support and the first secondary beam:

Four $1\frac{1}{4}$-in. bars at 30 deg. resist 88,000 lb. This is more resistance than is required, but it is automatically provided by the bars that are bent-up from the bottom of the beam to provide sufficient tensile resistance in the top towards the supports.

Between the first secondary beam and the middle of the span:

$$S = 68{,}500 - (8 \times 750) - 37{,}350 = 25{,}150 \text{ lb.}$$

$$s = \frac{25{,}150}{36{\cdot}5 \times 15} = 46 \text{ lb. per square inch;}$$ no shear reinforcement is required.

Columns.

Assume that the 18-in. square columns supporting the beam in *Fig.* 178 support two similar floors and a roof. The load on one column from one fully-loaded floor is

Reactions from main beams, 2 × 68,500 =	137,000 lb.
Direct reactions from secondary beams,	37,350 ,,
	174,350 ,,

The greatest load on the column from two floors and the roof is:

From roof, say,	110,000 lb.
,, top floor,	174,350 ,,
,, floor below top floor (0·9 × 174,350),	157,000 ,,
Weight of column, say, 40 ft. high and 12 in. square in top story,	8,650 ,,
	449,000 ,,
Resistance of concrete (18 × 18 × 950), including the area occupied by the main bars,	308,000 ,,
	141,000 ,,

Reinforcement required to resist $A_s = \frac{141{,}000}{18{,}000 - 950} = 8{\cdot}27$ sq. in. Eight $1\frac{1}{8}$-in. bars give 7·95 sq. in., which is sufficiently close.

The details of the columns and the bending schedules would be similar to those in *Figs.* 141 and 142. If the story height is about 10 ft. the column is not treated as a "long" column, and as the beams it supports are symmetrically disposed the load is assumed to act axially.

APPENDIX III

PLASTIC THEORY

RECENT research provides much data from which can be developed a general theory of the ultimate strength in bending of rectangular reinforced concrete beams, the plastic deformation, and the consequent redistribution of bending moments. The assumptions are :—

(1) Strain is directly proportional to the distance from the neutral axis ; tests show that this is roughly true even after plasticity has developed [Fig. 179(*a*)].

(2) Concrete has no tensile strength.

(3) A beam is loaded to its greatest capacity when (*a*) the concrete begins to crush without the tensile reinforcement yielding ; or (*b*) the concrete begins to crush due to contraction of the compressive zone upon yield, or excessive extension of the tensile reinforcement ; or (*c*) the stress in the tensile reinforcement is such that a small increase of load or a small number of repetitions of load causes fracture of the steel.

The plastic behaviour of concrete is important when considering prestressed concrete beams, but to obtain the data required and to understand the effect of the factors involved it is necessary first to analyse the plastic theories relating to ordinary reinforced concrete beams.

Reinforced Concrete Beams.

The notation used in the following and as shown in *Fig.* 179 is :

b, the breadth of the beam.
d, the effective depth of the beam.
n_1d, the depth to the neutral axis.
$\gamma n_1 d$, the depth to the centre of compression ; γ is the centre-of-compression factor.
p_t, the percentage of tensile reinforcement.
c', the compressive stress in the concrete at the top edge of the beam.
c_c and c_u, the unit crushing strengths of concrete cylinders and cubes respectively.
t_y, the tensile stress in the reinforcement at failure.
$\alpha n_1 dc'$, the total compression on unit width of beam (that is the area of the concrete stress diagram) ; α is the shape-factor.

E'_s, the modulus of plasticity of steel, that is $\frac{\text{AB}}{\text{OB}}$ in *Fig.* 180(*a*).

E'_c, the modulus of plasticity of the concrete at the top edge of the beam, that is $\frac{\text{CD}}{\text{OD}}$ in *Fig.* 180(*c*).

$m' = \frac{E'_s}{E'_c}$.

M', the ultimate moment of resistance.

$z = \frac{1}{2\alpha}$ for reinforced concrete beams.

Most plastic theories and experimental evidence establish that (1) the neutral axis is at about the mid-depth of a rectangular beam in a balanced design (that is when the tensile stress in the reinforcement reaches the yield point just as the concrete begins to fail in compression) ; (2) the distribution of stress is as in *Fig.* 179(*b*) just before failure, the maximum compressive stress being about equal to the cylinder strength ; (3) the ultimate moment of resistance of a rectangular reinforced concrete beam is

$$M' = 0{\cdot}33bd^2c_c \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The ultimate moment of resistance of the concrete can also be expressed as

$$M' = \alpha b n_1 d^2 c'(1 - \gamma n_1) = R'bd^2 \quad . \quad . \quad . \quad . \quad (2)$$

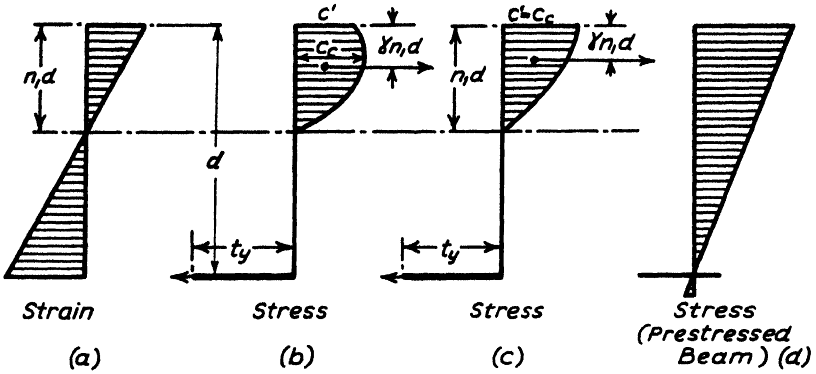


FIG. 179.

The stress c' at the top of the beam may be equal to or less than c_c (value of $0{\cdot}85c_c$ is assumed as a reasonable lower value in the following). It is reasonable to assume that the value of the shape-factor α is about unity when c' is $0{\cdot}85c_c$, because any value less than unity (with $c' = 0{\cdot}85c_c$) means that the neutral axis is lower than is shown by tests. As c' approaches equality with c_c, α may decrease to, say, 0·8 [see *Fig.* 179(*c*)]. The centre-of-compression factor γ is probably about 0·5 when $\alpha = 1$, but may be less, say 0·4 to 0·45, when $c' = c_c$ and $\alpha = 0{\cdot}8$. Therefore the following limiting conditions are reasonable.

Case 1 [*Fig.* 181(*b*)].—$c' = 0{\cdot}85c_c$; $\alpha = 1$; $\gamma = 0{\cdot}5$.

Case 2 [*Fig.* 181(*c*)].—$c' = c_c$; $\alpha = 0{\cdot}8$; $\gamma = 0{\cdot}45$.

The purpose is now to establish, for both cases, a relation between E'_c and c_c. Equating (1) to (2) and transposing,

$$n_1 = \frac{1 - \sqrt{1 - \dfrac{4\gamma c_c}{3\alpha c'}}}{2\gamma} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

By substitution, in case 1 $n_1 = 0{\cdot}537$, and in case 2 $n_1 = 0{\cdot}556$. Both values of n_1 are seen to be about one-half which, as already stated, agrees with tests.

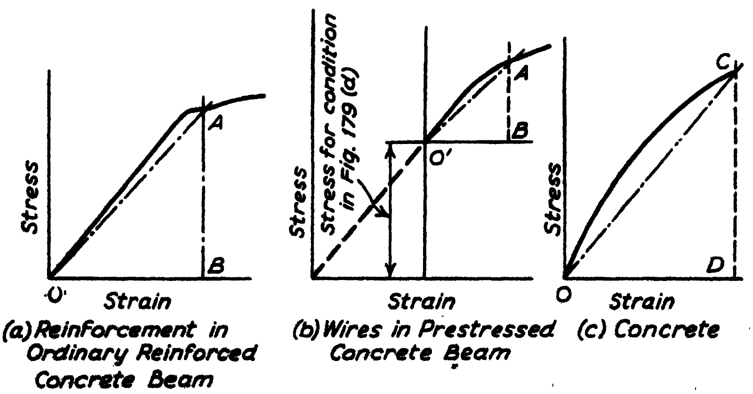


FIG. 180.

Since strain is proportional to the distance from the neutral axis,

$$\frac{t_y}{E'_s(1 - n_1)} = \frac{c'}{E'_c n_1},$$

from which
$$E'_c = \frac{c'E'_s(1 - n_1)}{t_y n_1} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It has been shown by tests that equation (1) applies when the reinforcement has a yield-point stress of 50,000 lb. per square inch and E'_s is about 30×10^6 lb. per square inch. The same result may be obtained with mild steel having a yield-point stress of, say, 36,000 lb. per square inch if E'_s is $\frac{30 \times 10^6 \times 36{,}000}{50{,}000}$, that is about 22×10^6 lb. per square inch, the decrease in E'_s being due to a slight yield giving the same extension at 36,000 lb. as for the high-tensile steel at 50,000 lb. per square inch. Note that E'_s as defined in *Fig.* 180(*a*) is not the true modulus of elasticity.

Substituting $t_y = 50{,}000$ and $E'_s = 30 \times 10^6$ in. equation (4) gives, for case 1, $E'_c = 440c_c$, and for case 2, $E'_c = 480c_c$. The relation between E'_c and c_c accounts for the omission of the term m (the modular ratio) from the expression for the ultimate moment of resistance of a reinforced concrete beam, but this simplification does not apply to a prestressed concrete beam, as is seen later. Further research may modify slightly the numerical relationship, but since the difference between specified strengths and actual strengths of concrete provides a margin, especially for high-quality concrete, it is reasonable to base calculations of the ultimate resistance of the concrete on the preceding assumptions, which are repeated in the table in *Fig.* 181 and are extended to give $R' = \frac{M'}{bd^2c_c}$ in terms of the strength of test cylinders and also of test cubes, which is the strength commonly specified in Britain. The cylinder strength is used in the analysis because many plastic theories are based on this strength. In the conversion it is assumed that the cylinder strengths of ordinary concretes are 0·8 of the cube strengths.

Reinforcement.—For balanced design the percentage of reinforcement p_s is obtained by equating the total tension at failure, $0{\cdot}01p_sbdt_y$, to the total compression at failure, $\alpha bn_1dc'$, and substituting t_y from (4) to give

$$p_s = \frac{100\alpha n_1^2}{m'(1 - n_1)} = \frac{50n_1^2}{zm'(1 - n_1)} \quad . \quad . \quad . \quad . \quad (5)$$

The curves in *Fig.* 182, which applies to reinforced concrete and prestressed concrete beams, relate p_s and n_1 for various values of zm', where for reinforced concrete $z = \frac{1}{2\alpha}$. If $\alpha = 0{\cdot}5$, as in the standard method of design (triangular distribution), z equals unity.

In an under-reinforced beam, that is when p_s is less than is required for balanced design, the stress in the concrete will be low when the steel begins to yield and little plasticity may be developed. The distribution of stress may therefore be triangular, that is $\gamma = 0{\cdot}33$, and the lever-arm for the calculation of the amount of reinforcement is $d\left(1 - \frac{n_1}{3}\right)$ as in ordinary design.

Prestressed Concrete Beams.

The assumption that a prestressed concrete beam cracks on the tensile side before failure is made in addition to the assumptions enumerated for reinforced concrete beams. The additional notation is

β, the ratio of the stress in the wires for the condition of loading giving the distribution in *Fig.* 179(*d*) to the stress in the wires at ultimate load.

F, the ratio of the average strain in the concrete around the wires to the greatest strain in the concrete adjacent to or around the wires.

The value of F, as described in Chapter VI, is generally about 0·67 for an end-anchored prestressed beam subjected to a uniformly-distributed load, and 0·5 for a

Type of beam		F	α	β	z	E'_s lb. per sq. in.	E'_c lb. per sq. in.	m'	$\frac{c'}{c_c}$	zm'	t_y lb. per sq. in.	n_1	γ	R'	
Reinforced concrete	Case No. 1.	1·0	1·0	0	0·5	30×10^6	$440c_c$	$\frac{68,000}{c_c}$	0·85	$\frac{34,000}{c_c}$	50,000	0·537	0·5	$0{\cdot}333c_c$	$0{\cdot}264c_u$
	Case No. 2.	1·0	0·8	0	0·625	30×10^6	$480c_c$	$\frac{62,500}{c_c}$	1·0	$\frac{39,100}{c_c}$	50,000	0·556	0·45	$0{\cdot}333c_c$	$0{\cdot}264c_u$
Pre-stressed concrete	End-anchored: Load at midspan	0·5	1·0	0·5	0·5	20×10^6	$440c_c$	$\frac{45,500}{c_c}$	0·85	$\frac{22,750}{c_c}$	175,000	0·181*	0·5	$0{\cdot}134c_c$*	$0{\cdot}107c_u$*
		0·5	0·8	0·75	1·25	30×10^6	$480c_c$	$\frac{62,500}{c_c}$	1·0	$\frac{78,125}{c_c}$	200,000	0·384	0·45	$0{\cdot}254c_c$	$0{\cdot}203c_u$
	End-anchored: Uniformly-distributed load	0·67	1·0	0·5	0·67	20×10^6	$440c_c$	$\frac{45,500}{c_c}$	0·85	$\frac{30,200}{c_c}$	175,000	0·227*	0·5	$0{\cdot}171c_c$*	$0{\cdot}138c_u$*
		0·67	0·8	0·75	1·67	30×10^6	$480c_c$	$\frac{62,500}{c_c}$	1·0	$\frac{104,150}{c_c}$	200,000	0·455	0·45	$0{\cdot}290c_c$	$0{\cdot}232c_u$
	Concrete-gripped	1·0	1·0	0·75	2·0	20×10^6	$440c_c$	$\frac{45,500}{c_c}$	0·85	$\frac{91,000}{c_c}$	175,000	0·470	0·5	$0{\cdot}305c_c$	$0{\cdot}244c_u$
		1·0	1·0	0·75	2·0	20×10^6	$440c_c$	$\frac{45,500}{c_c}$	0·85	$\frac{91,000}{c_c}$	200,000	0·437	0·5	$0{\cdot}290c_c$	$0{\cdot}232c_u$
		1·0	0·8	0·75	2·5	30×10^6	$480c_c$	$\frac{62,500}{c_c}$	1·0	$\frac{156,250}{c_c}$	175,000	0·588	0·45	$0{\cdot}346c_c$	$0{\cdot}275c_u$
		1·0	0·8	0·75	2·5	30×10^6	$480c_c$	$\frac{62,500}{c_c}$	1·0	$\frac{156,250}{c_c}$	200,000	0·555	0·45	$0{\cdot}334c_c$	$0{\cdot}267c_u$

* These values show that a small value of β causes an excessive rise in the position of the neutral axis and consequently a small value of R'.

FIG. 181.—FACTORS FOR ULTIMATE MOMENT OF RESISTANCE OF BEAMS OF BALANCED DESIGN.

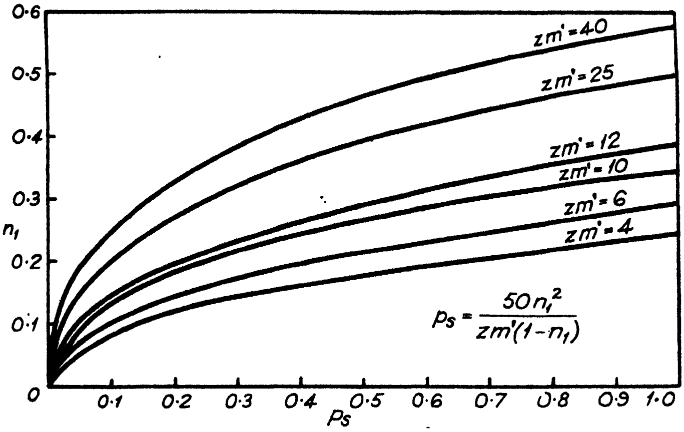


FIG. 182.

similar beam subjected to a load concentrated at mid-span. Slightly lower values may result if the beam is not properly prestressed. The value of β depends mainly on the amount of the initial tensioning and the stress in the wires at failure of the beam. The reduction in tension in the wires due to creep and shrinkage is generally less than the increase in tension due to bending which occurs due to the loading necessary to produce the condition in *Fig.* 179(*d*). A more accurate calculation of the stress in the wires under this condition can be made, but it is generally sufficiently accurate to assume that the stress in the wires at this stage slightly exceeds the initial tension so that, in a properly prestressed beam of balanced design, β is about 0·75.

The value of E_s', which in this case is $\frac{\text{OA}}{\text{O'B}}$ in *Fig.* 180(*b*), may be, say, 20×10^6 lb. per square inch for the type of wire used in prestressed concrete and at the stress in the wire for the condition in *Fig.* 179(*d*). As in the case of a reinforced concrete beam, the strain at the top of the beam is $\frac{c'}{E_c'}$ at failure. Since strain is directly proportional to the distance from the neutral axis, the strain in the concrete around the wires is $\frac{c'}{E_c'} \cdot \frac{1 - n_1}{n_1}$ at failure. For the condition in *Fig.* 179(*d*) there is no strain in the concrete around the wires, but the strain in the wires is considerable since, as already explained, it is approximately equal to that due to the initial stretching of the wires. In a concrete-gripped prestressed beam at failure, the increase in the strain in the wires, which equals the virtual strain in the cracked concrete around the wires, is that which occurs when the stress in the wires increases from βt_y to t_y, that is $\frac{t_y - \beta t_y}{E_s'}$ where E_s' is as in *Fig.* 180(*b*). Equating the strains at failure,

$$\frac{t_y(1 - \beta)}{E_s'} = \frac{c'}{E_c'} \cdot \frac{(1 - n_1)}{n_1}.$$

Therefore
$$t_y = \frac{c'm'(1 - n_1)}{n_1(1 - \beta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

In an end-anchored prestressed beam, in which the wires are not grouted, the strain throughout the whole length of the wires equals the average strain in the concrete around the wires, which is F times the greatest strain in the concrete, that is

$$t_y = \frac{Fc'm'(1 - n_1)}{n_1(1 - \beta)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6b)$$

T

The table in *Fig.* 181 gives the values of n_1, and R' (in terms of c_c and c_u) for reasonable values of F, α, β, γ, and E'_s for two assumed values of t_y. The values of n_1 are calculated from

$$n_1 = \frac{1}{1 + \frac{t_y}{2\alpha z m' c'}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which is derived by substituting $z = \frac{F}{2\alpha(1 - \beta)}$ in equation (6*b*) and transposing. The value of p_s for a balanced design is calculated by substituting n_1 from (7) in (5), and the curves in *Fig.* 182 relate n_1 to p_s for various values of zm'.

Plastic Deformation of Reinforced Concrete Beams.

The assumptions are that strain is proportional to the distance from the neutral axis and that all changes of slope, or of radius of curvature, can be related to, and expressed in terms of, the deformation of the concrete on the compressive side of the neutral axis. Referring to *Fig.* 29 in Chapter II, it follows from the first assumption that the slope at P for the plastic condition is as for the ordinary elastic case, and is $\int \frac{c'}{E'_c n_1 d} . dx$ if $n = n_1 d$. From equation (1) of this appendix, for the plastic condition

$$c' = \frac{M'}{\alpha b d^2 (n_1 - \gamma n_1^2)}$$

and the slope of the member is therefore given by

$$\int \frac{M'}{E'_c \alpha b d^3 (n_1^2 - \gamma n_1^3)} . dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Let $$I' = \alpha b d^3 (n_1^2 - \gamma n_1^3) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Then the slope is $$\int \frac{M'}{E'_c I'} . dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The common methods of determining slopes, deflections, distribution of moments, etc., in accordance with the elastic theory, which takes into account variation in the moment of inertia, can therefore be used for the plastic theory provided that the variation of the value of $E'_c I'$ throughout the length of the member is taken into account. The variation of n_1 due to changes in the value of E'_c and E'_s can be derived from the equations given in a preceding section of this appendix. Tests of reinforced concrete beams show that, provided the reinforcement does not yield, the position of the neutral axis remains almost unchanged under increasing load. It is seen from equation (5) that if n_1 and E'_s are constant, since p_s is also constant, $E'_c \alpha$ must be constant, and from (9) it is seen that the value of $E_c I'$ is governed by the term $n_1^2 - \gamma n_1^3$ which varies only slightly for reasonable extreme values of γ.

Redistribution of the bending moments in continuous members due to plasticity of the concrete only may not therefore be very great except where plasticity extends over a considerable portion of the length of the member. The constancy of $E'_c \alpha$ is probably only roughly true if the load is maintained for a long period, the value of E'_c being then much less than its value at failure after a short period of loading.

Redistribution of Bending Moments due to Plasticity of Concrete and Yield of Reinforcement.

The method of determining the influence of plasticity on the distribution of the bending moments in continuous members is best demonstrated by an example. *Fig.* 183 shows a beam continuous over two equal spans l of 60 ft. each and carrying a uniformly-distributed load causing a maximum free bending moment M_F of 20 units at the middle of each span. For this condition with two spans, the bending moment M_B at support B is also 20 units. Assume that EI is unity throughout the

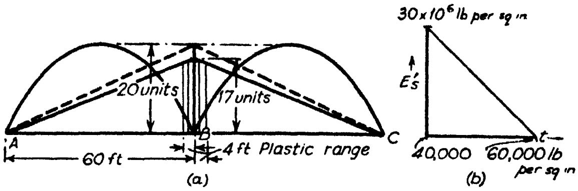


FIG. 183.

beam before plasticity develops. With A as the origin of x, the slope at B is $\frac{1}{l}\int\frac{Mx}{EI}.dx$ and the slope at B due to M_B alone is

$$\frac{M_B}{EI}\cdot\frac{l}{2}\cdot\frac{2l}{3} = 20 \times 30 \times 40 = 24{,}000.$$

If no plasticity develops the resultant bending moment at mid-span is 10 units. If the beam is over-reinforced so that the concrete crushes before the steel commences to yield, the value of I throughout, say, the end 4 ft. of the beam on either side of B (see *Fig.* 183) might be reduced from $0{\cdot}58\alpha bd^3$ to $0{\cdot}5\alpha bd^3$, thereby causing an increase in the value of $\int\frac{Mx}{EI}.dx$ of $20 \times 4 \times 58\left(\frac{0{\cdot}58}{0{\cdot}5} - 1\right) = 740$. This reduction, which is based on extremely high assumptions, is small compared with 24,000, and it is clear therefore that plasticity in the concrete can assist very little in adjusting the bending moments at mid-span and the support unless the state of plasticity extends over a very long length at the support.

It can be shown that yielding of the reinforcement is very much more effective. From the shape of the bending-moment diagram it is also clear that sufficient reinforcement must be provided at midspan to prevent any yield occurring there before sufficient yield has taken place at the support appreciably to reduce the bending moment at the support. Yield over a considerable length at the middle of the span increases the value of $\int\frac{Mx}{EI}.dx$ for the free bending moment at a faster rate than in the case of the bending moment at the support since the greatest bending at the support extends over only a short length of beam unless the support is very wide. Yield due to the plasticity of the concrete only at midspan has the same effect and causes an increase in the bending moment at the support.

Redistribution of Bending Moment due to Yield of Steel only.

It is best to assume values of the bending moments at the support and then to check them by calculation. The assumptions made are: (1) The maximum free bending moment at midspan is 20 units. (2) The bending moment at the support before plasticity develops is also 20 units. (3) The values of α, b, d, γ, and the crushing strength of the concrete are such that the concrete crushes when n_1 is less than 0·2, and the steel weakens appreciably and may be considered to have reached its full tensile strength at a stress of 53,000 lb. per square inch; for this stress p_s is 0·6 per cent. when n_1 is 0·20 and M_B is 17 units. (4) The value of E'_s varies as shown in *Fig.* 183. (5) The value of E'_c just before the concrete crushes is $2{\cdot}5 \times 10^6$ lb. per square inch. (6) Sufficient reinforcement is provided at midspan to prevent premature yield at this point.

As a first approximation assume the bending moment at the support to be 17 units when reduced by plasticity just before failure. The conditions to be satisfied at the support are: (1) p_s must be not less than 0·6 per cent. to avoid excessive tensile stress in the reinforcement when $n_1 = 0{\cdot}2$. (2) n_1 must be not less than 0·2, which depends on the values of p_s and m'. The value of m' depends on E'_s, which in turn depends on p_s.

x	E_s' lb. per sq. in.	m'	n_1	$n_1^2 - yn_1^2$	$\frac{M}{EI}$ (plastic) / $\frac{M}{EI}$ (elastic)	Factor for increase of $\frac{M}{EI}$
59·5	10×10^6	4·0	0·20	0·035	2·50	1·50
58·5	17×10^6	6·8	0·24	0·055	1·64	0·64
57·5	24×10^6	9·6	0·28	0·075	1·18	0·18
56·5	30×10^6	12·0	0·32	0·090	1·0	0

FIG. 184.

From *Fig.* 183, E_s' is 10×10^6 lb. per square inch when t_y is 53,000 lb. per square inch, and $m' = \frac{E_s'}{E_c'} = 4$. From the curves in *Fig.* 182, when p_s is 0·6 and m' is 4, n_1 is about 0·2, which satisfies the foregoing condition.

It can therefore be assumed that, under conditions of ultimate load, the reinforcement at the support develops a stress of 53,000 lb. per square inch. The bending-moment diagrams to the left of B in *Fig.* 183 must now be divided into strips of convenient widths starting from B and working towards A, until the bending moment is reduced to, say, $\frac{40{,}000}{53{,}000}$ of the greatest bending moment, that is until the tensile stress is, say, 40,000 lb. per square inch, at which stress yield might not occur in mild steel reinforcement. The increase in the value of $\frac{Mx}{EI}$ due to yield in this length of the beam can now be determined conveniently by tabulation and summation as in the tables in *Figs.* 184 and 185, wherein it is assumed that p_s is 0·6 per cent. throughout, and that any slight increase in p_s required to ensure that no yield occurs at midspan will not affect appreciably the value of I.

Bending moment at support = 17 units.

$\frac{M}{EI}$-values are for elastic condition and increases due to yield of steel.

	$\frac{M}{EI}$	x	$\frac{Mx}{EI}$	Factor for increase due to yield	Increase of $\frac{Mx}{EI}$
Bending moments at support	16·75	59·5	1000	1·50	1500
	14·50	58·5	850	0·64	545
	13·00	57·5	750	0·18	135
	Total increase of $\frac{M}{EI}$ for support moments				2180
Free bending moment	0·5	59·5	30	1·50	45
	1·5	58·5	88	0·64	66
	2·5	57·5	144	0·18	26
	Total increase of $\frac{M}{EI}$ for free moments				137

At ultimate load after yield, total $\frac{Mx}{EI}$ for the free moment is 24,000 + 137 = 24,137.

The total value of $\frac{Mx}{EI}$ for the support moments is $(17 \times 30 \times 40) + 2180 = 22{,}480$.

FIG. 185.

The value of E'_s varies with the tensile stress in accordance with *Fig.* 183, and the tensile stress is assumed to vary with M (that is it is sufficiently accurate to assume that the lever arm is constant). Therefore $m' = \frac{E_s}{2 \cdot 5 \times 10^6}$. The value of n_1 is obtained from *Fig.* 182, and in the term $(n_1^2 - \gamma n_1^3)$ it is assumed that γ has the mean value of 0·33 and 0·5. The value of $(n_1^2 - \gamma n_1^3)$ is proportional to I', which for comparative purposes is assumed to be unity over the non-yielded length of the beam, that is $I' = 1$ when $(n_1^2 - \gamma n_1^3)$ is 0·09.

It is seen from the tables in *Figs.* 184 and 185 that the values of $\frac{Mx}{EI}$ for the free bending moment and for the assumed reduced bending moment at the support are not equal, but equalisation would occur in a second trial with $M_B = 18$ units. In practice the width of the support is an important factor and assists in reducing large moments at the support. Redistribution of sway moments in frames can be dealt with in a similar manner, and the influence of the yield of the reinforcement would be greater on account of the more gradual rate of reduction of the maximum bending moments.

In members, such as a column or a prestressed beam, subjected to longitudinal compression, the influence of the additional stresses must be taken into account, as must also the effect on the position of the neutral axis of the occurrence of cracks under the ultimate load, but a procedure similar to the foregoing can be adopted. Redistribution of moments will not generally be great except when yield of the steel is not concentrated in a short length of the member and can occur without the concrete crushing.

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