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**INTRODUCTION TO  
APPLIED MATHEMATICS**



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**INTRODUCTION TO  
APPLIED  
MATHEMATICS**

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## PREFACE

It is now generally recognized that the mathematical equipment of the well-trained physicist or engineer of thirty years ago is no longer adequate for the physics and engineering of today. To understand wave mechanics it is not sufficient to master an old-fashioned treatment of vector analysis with its limitations to plane and space vectors with real coordinates and its emphasis on a visual realization of the basic concepts and relations. We must become familiar with multi-dimensional vectors with complex coordinates and with the matrices, or linear vector functions, which operate on these vectors. A thorough and detailed treatment of these vectors and matrices is given in Chapters 1 and 2 of this book. Chapter 3 is devoted to the concept of function-vectors and of the linear integral operators associated with them; it includes an account of the various Fourier series which are attached, each to a given orthonormal set of function-vectors. Chapter 4 gives a full account of the concept, so important from the practical point of view, of curvilinear coordinates of a vector and of a linear vector function. The next two chapters treat the solutions of Laplace's equation which may be derived by the method of separation of variables, particular attention being given to spherical harmonics and Bessel functions, and the application of the method of inversion to the determination of the electrostatic capacities of various condensers is considered in detail. Chapter 7 treats boundary-value problems and the associated Green's function, and Chapter 8, which is devoted to the Fredholm and Hilbert-Schmidt theory of integral equations, culminates with a proof of Rayleigh's principle. The last two chapters treat the calculus of variations (with particular emphasis on dynamical applications) and the operational calculus.

The subject matter of this book is that of a graduate course in applied mathematics which I have given for the past twenty years at Johns Hopkins. The course met three hours weekly for thirty weeks and was attended by graduate students in physics, engineering, chemistry, and mathematics. The content of the course has changed somewhat in the two decades during which it has been given, but the emphasis has always been on vectors and matrices, boundary-value problems, integral equations, and the calculus of variations with its applications to dynamics.

In the days when Ireland was known as the Land of Saints and Scholars, the usual inscription on a literary effort was "To the Glory of God and the Honor of Ireland." Now that my work at Hopkins is over and as I leave to teach mathematics in Brazil, I think it appropriate to close my preface with the inscription

To the Glory of God, Honor of Ireland,  
and  
Solidarity of the Americas.

FRANCIS D. MURNAGHAN

*Baltimore, Maryland*  
*May, 1948*

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# 1

## VECTORS AND MATRICES

### 1. Two-dimensional vectors

Let  $Oxy$  and  $O'x'y'$  be any two rectangular Cartesian reference frames in the  $(x, y)$ -plane, and let  $(l_1, m_1)$ ,  $(l_2, m_2)$  be the direction cosines of the positive  $x'$ - and  $y'$ -axes with respect to the  $Oxy$  reference frame. In other words, if  $\cos(xx')$  denotes the cosine of the angle between the positive  $x$ -axis and the positive  $x'$ -axis, and so on, let

$$l_1 = \cos(xx'), \quad m_1 = \cos(yx');$$

$$l_2 = \cos(xy'), \quad m_2 = \cos(yy').$$

If  $P$  is any point in the  $(x, y)$ -plane we denote its coordinates with respect to the  $Oxy$  reference frame by  $(x, y)$  and its coordinates with respect to the  $O'x'y'$  reference frame by  $(x', y')$ , and we use the symbol  $P:(x, y)$  to denote that  $P$  is the point

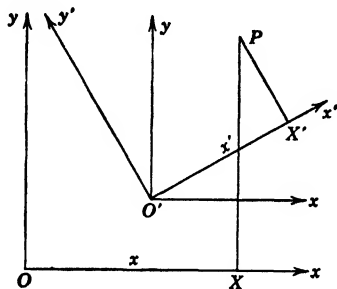


Fig. 1

whose coordinates with respect to the  $Oxy$  reference frame are  $(x, y)$ . If  $O':(x_0, y_0)$ , the coordinates of  $P$  with respect to the reference frame  $O'xy$  (i.e., the reference frame whose origin is  $O'$  and whose axes have the same directions as the axes of the  $Oxy$  reference frame) are  $(x - x_0, y - y_0)$ , and these coordinates are related to the coordinates  $(x', y')$  of  $P$  with respect to the  $O'x'y'$  reference frame as follows:

$$x - x_0 = l_1x' + l_2y'; \quad y - y_0 = m_1x' + m_2y'.$$

Let  $P + \Delta P$  be any point, other than  $P$ , in the  $(x, y)$ -plane, and denote the coordinates of  $P + \Delta P$  with respect to the  $Oxy$  reference frame by  $(x + \Delta x, y + \Delta y)$  and with respect to the  $O'x'y'$  reference frame by  $(x' + \Delta x', y' + \Delta y')$ . Thus the numbers  $(\Delta x, \Delta y)$  are the

projections of the line segment  $P \rightarrow P + \Delta P$  on the axes of the  $Oxy$  reference frame, and  $(\Delta x', \Delta y')$  are the projections of the *same* line segment on the axes of the  $O'x'y'$  reference frame. The coordinates of  $P + \Delta P$  with respect to the  $O'x'y'$  reference frame are

$$(x + \Delta x - x_0, \quad y + \Delta y - y_0)$$

and so

$$\begin{aligned} x + \Delta x - x_0 &= l_1(x' + \Delta x') + l_2(y' + \Delta y'); \\ y + \Delta y - y_0 &= m_1(x' + \Delta x') + m_2(y' + \Delta y'). \end{aligned}$$

On combining these with the two equations which furnished  $x - x_0$  and  $y - y_0$  in terms of  $x'$  and  $y'$  we obtain

$$\begin{aligned} \Delta x &= l_1 \Delta x' + l_2 \Delta y'; \\ \Delta y &= m_1 \Delta x' + m_2 \Delta y'. \end{aligned}$$

The projections  $\Delta x$  and  $\Delta y$  of the line segment  $P \rightarrow P + \Delta P$  on the axes of the  $Oxy$  reference frame do not *determine* the segment

$$P \rightarrow P + \Delta P$$

(in other words, there are many line segments  $P \rightarrow P + \Delta P$  which have the same projections on these axes; in fact the initial point  $P:(x, y)$  of the line segment may be chosen *arbitrarily*, and then the terminal point  $P + \Delta P:(x + \Delta x, y + \Delta y)$  is unambiguously determined by the numbers  $(\Delta x, \Delta y)$ ). All these line segments have the same *magnitude*

$$|\Delta P| = \{(\Delta x)^2 + (\Delta y)^2\}^{1/2}$$

and *direction*. We term the collection of *all* these equally long and equally directed line segments *a vector* (not many vectors), and we say that any one of the multitude of line segments (all of which have the same projections  $(\Delta x, \Delta y)$  on the axes of the  $Oxy$  reference frame) is a *representation* of the vector. The numbers  $(\Delta x, \Delta y)$  are termed the *coordinates* of the vector (of which the line segment  $P \rightarrow P + \Delta P$  is a representation) in (or with respect to) the  $Oxy$  reference frame. Similarly the numbers  $(\Delta x', \Delta y')$  are the coordinates of the same vector in the  $O'x'y'$  reference frame. In this terminology we have, then, the following fundamental result:

**The coordinates  $(\Delta x, \Delta y)$ , in the  $Oxy$  reference frame, of an arbitrary vector are connected with the coordinates  $(\Delta x', \Delta y')$ , in the  $O'x'y'$  reference frame, of the same vector by the formulas**

$$\Delta x = l_1 \Delta x' + l_2 \Delta y'; \quad \Delta y = m_1 \Delta x' + m_2 \Delta y'.$$

Here  $Oxy$ ,  $O'x'y'$  are *any* two rectangular Cartesian reference frames in the  $(x, y)$ -plane.

These formulas may be conveniently remembered by means of the following diagram:

	$\Delta x'$	$\Delta y'$
$\Delta x$	$l_1$	$l_2$
$\Delta y$	$m_1$	$m_2$

When the four direction cosines  $(l_1, m_1)$ ,  $(l_2, m_2)$  are written in this way we say that we have set up a *table of direction cosines*. To read from the table the expression for  $\Delta x$  we multiply each number in the first row of the table by the projection, in the  $O'x'y'$  reference frame, which is directly above this number, and we add the two products so obtained:

$$\Delta x = l_1 \Delta x' + l_2 \Delta y'.$$

Similarly  $\Delta y$  is obtained from the second row of the table:

$$\Delta y = m_1 \Delta x' + m_2 \Delta y'.$$

Now  $Oxy$  and  $O'x'y'$  were *any* two rectangular Cartesian reference frames in the  $(x, y)$ -plane. Hence the two reference frames may be interchanged. Since  $l_1 = \cos (xx')$ ,  $m_2 = \cos (yy')$  this interchange does not affect the value of  $l_1$  or the value of  $m_2$  (why?); on the other hand,  $m_1 = \cos (yx')$ ,  $l_2 = \cos (xy')$  so that the interchange of the roles of the reference frames interchanges  $m_1$  and  $l_2$  (why?). We thus obtain the new table of direction cosines:

	$\Delta x$	$\Delta y$
$\Delta x'$	$l_1$	$m_1$
$\Delta y'$	$l_2$	$m_2$

which is read as follows:

$$\Delta x' = l_1 \Delta x + m_1 \Delta y; \quad \Delta y' = l_2 \Delta x + m_2 \Delta y.$$

An equivalent statement of this result is as follows: The table of direction cosines

	$\Delta x'$	$\Delta y'$
$\Delta x$	$l_1$	$l_2$
$\Delta y$	$m_1$	$m_2$

may be read either by rows or by columns; thus

$$\begin{aligned}\Delta x &= l_1 \Delta x' + l_2 \Delta y'; & \Delta y &= m_1 \Delta x' + m_2 \Delta y'; \\ \Delta x' &= l_1 \Delta x + m_1 \Delta y; & \Delta y' &= l_2 \Delta x + m_2 \Delta y.\end{aligned}$$

### EXERCISES

1. Show that when  $Ox'y'$  is obtained by rotating  $Oxy$  through  $\pi$  then  $\Delta x' = -\Delta x$ ,  $\Delta y' = -\Delta y$ .

2. Show that when  $Ox'y'$  is obtained by rotating  $Oxy$  through  $\frac{\pi}{2}$  then  $\Delta x' = \Delta y$ ,  $\Delta y' = -\Delta x$ . What are the formulas for  $\Delta x'$  and  $\Delta y'$  when  $Ox'y'$  is obtained by rotating  $Oxy$  through  $-\frac{\pi}{2}$ ?

3. Show that when  $Ox'y'$  is obtained by rotating  $Oxy$  through  $\frac{\pi}{4}$  then

$$\Delta x' = 2^{-1/2}(\Delta x + \Delta y), \quad \Delta y' = 2^{-1/2}(-\Delta x + \Delta y).$$

We have been led to the consideration of plane vectors by considering line segments  $P \rightarrow P + \Delta P$  in the plane. We have seen that a vector is unambiguously determined by its coordinates  $(\Delta x, \Delta y)$  with respect to any given reference frame  $Oxy$  (its coordinates with respect to any other reference frame  $O'x'y'$  being furnished, in terms of  $\Delta x$  and  $\Delta y$ , by means of the table of direction cosines). Instead of regarding the representation of a vector (by any one of a multitude of line segments) as fundamental, we now adopt the point of view that the important thing is the manner in which the coordinates of the vector change when the reference frame is changed. Thus, instead of saying that "a vector is the collection of all line segments which have the same projections on the axes of any reference frame," we say that

**A vector is a collection of pairs of numbers  $(\Delta x, \Delta y)$ , it being understood that there is one pair for each reference frame and that the pairs for any two reference frames are connected with each other by means of the table of direction cosines which is defined by the two reference frames.**

Thus the pair for *any* given reference frame  $Oxy$  may be *arbitrarily* assigned, and then the vector is determined, the pair for *any* other reference frame  $O'x'y'$  being furnished by the formulas

$$\Delta x' = l_1 \Delta x + m_1 \Delta y; \quad \Delta y' = l_2 \Delta x + m_2 \Delta y.$$

The numbers  $(\Delta x, \Delta y)$  are termed the *coordinates* in the  $Oxy$  reference frame of the vector ( $\Delta x$  being the  $x$ -coordinate and  $\Delta y$  the  $y$ -coordi-

nate). Similarly  $\Delta x'$  is the  $x'$ -coordinate and  $\Delta y'$  the  $y'$ -coordinate of the vector.

*Remark.* This definition of a vector will undoubtedly seem somewhat vague and abstract at first, and the familiar definition may seem preferable: "A vector is that which possesses, in addition to the quality of magnitude, the quality of direction." Actually it is this "colloquial" definition that is vague. How can we tell when something "possesses the quality of direction"? The only answer is that it must have assigned to it, in each reference frame, a pair of numbers, and the various pairs, one in each reference frame, must be connected with each other in exactly the same way as are the projections of a line segment, i.e., by means of the table of direction cosines.

The simplest, and in some respects the most important, vector is the one obtained by setting  $(\Delta x, \Delta y) = (0, 0)$ . It follows (why?) that  $(\Delta x', \Delta y') = (0, 0)$  so that this particular vector has the property that its coordinates in all reference frames are the same. We term this vector the *zero vector*. Whenever we are in possession, in every coordinate reference frame, of a number *which is the same in each reference frame* we term this number a *scalar*.

*Remark.* Be sure that you understand clearly the difference between the terms "number" and "scalar." Every scalar is a number, but not all numbers are scalars. Thus the  $x$ -coordinate of any vector other than the zero vector is a number, but it is not a scalar. (Prove this.)

When we wish to denote a vector by a single symbol we shall use boldface type, and we shall indicate that  $\mathbf{v}$  is the vector whose coordinates in a given reference frame  $Oxy$  are  $(a, b)$ , say, by the notation

$$\mathbf{v} = v(a, b).$$

Thus

$$\mathbf{0} = v(0, 0)$$

(it being unnecessary to specify which coordinate reference frame we are using). We have, then, the following result:

**The zero vector  $\mathbf{0}$  is characterized by the fact that it is the only vector whose coordinates are scalars.**

*Note.* Despite the importance of the zero vector, it does not possess "the quality of direction"; in other words it does not single out any particular direction which it calls its own. If we attempt to represent it by a line segment the "two" ends of the segment coincide.

Every vector furnishes us with a scalar, namely, the common length of any line segment which is a representation of the vector. We term

this scalar the *magnitude* of the vector, and we denote the magnitude of  $\mathbf{v}$  by  $|\mathbf{v}|$  or simply by  $v$ ; thus if  $\mathbf{v} = v(\Delta x, \Delta y)$

$$|\mathbf{v}| = v = \{(\Delta x)^2 + (\Delta y)^2\}^{1/2}.$$

The fact that  $v$  is a scalar assures us that

$$(\Delta x')^2 + (\Delta y')^2 = (\Delta x)^2 + (\Delta y)^2,$$

and since  $\Delta x' = l_1 \Delta x + m_1 \Delta y$ ,  $\Delta y' = l_2 \Delta x + m_2 \Delta y$  this yields

$$\begin{aligned} (l_1^2 + l_2^2)(\Delta x)^2 + 2(l_1 m_1 + l_2 m_2) \Delta x \Delta y + (m_1^2 + m_2^2)(\Delta y)^2 \\ = (\Delta x)^2 + (\Delta y)^2. \end{aligned}$$

This relation must be true for arbitrary numbers  $(\Delta x, \Delta y)$ ; on setting  $\Delta x = 1$ ,  $\Delta y = 0$  we obtain  $l_1^2 + l_2^2 = 1$ , and on setting  $\Delta x = 0$ ,  $\Delta y = 1$  we obtain  $m_1^2 + m_2^2 = 1$ . Hence  $2(l_1 m_1 + l_2 m_2) \Delta x \Delta y = 0$ , and on setting  $\Delta x = 1$ ,  $\Delta y = 1$  we obtain  $l_1 m_1 + l_2 m_2 = 0$ . Thus

*The four direction cosines of the table of direction cosines are connected by the relations*

$$l_1^2 + l_2^2 = 1; \quad l_1 m_1 + l_2 m_2 = 0; \quad m_1^2 + m_2^2 = 1.$$

### EXERCISES

4. Show that the four direction cosines of the table of direction cosines are connected by the relations  $l_1^2 + m_1^2 = 1$ ,  $l_1 l_2 + m_1 m_2 = 0$ ,  $l_2^2 + m_2^2 = 1$ . *Hint.* Interchange the roles of the reference frames  $Oxy$ ,  $O'x'y'$ .

5. Show that if  $\theta$  is the angle from  $O'x$  to  $O'x'$  then the angle from  $O'y$  to  $O'y'$  is  $\theta - \frac{\pi}{2}$  (not  $\frac{\pi}{2} - \theta$ ) and that the table of direction cosines is

	$\Delta x'$	$\Delta y'$
$\Delta x$	cos $\theta$	-sin $\theta$
$\Delta y$	sin $\theta$	cos $\theta$

Verify the validity of the relations of Exercise 4. *Hint.* The angle from  $O'x$  to  $O'y'$  is  $\theta + \frac{\pi}{2}$ .

6. Show that the magnitude of the vector whose coordinates in the  $Oxy$  frame are  $(l_1, m_1)$  is unity. *Note.* We shall term any vector whose magnitude is unity a *unit vector*.

7. Show that the coordinates in the  $O'x'y'$  reference frame of the vector of Exercise 6 are  $(1, 0)$ .

8. Show that the coordinates in the  $O'x'y'$  reference frame of the vector whose coordinates in the  $Oxy$  reference frame are  $(0, 1)$  are  $(m_1, m_2)$ , and deduce that  $m_1^2 + m_2^2 = 1$ .

## 2. The scalar product of two plane vectors

If  $P: (x, y)$ , the vector of which the line segment  $O \rightarrow P$  is a representation is the vector  $\mathbf{v} = v(x, y)$ ; this vector is termed the *position vector* of the point  $P$  (with respect to the origin  $O$  of the  $Oxy$  reference frame). Conversely, when we are given any vector  $\mathbf{v}$ , there is an unambiguously determined point  $P$  which has  $\mathbf{v}$  as its position vector with respect to  $O$ . It is convenient, then, when specifying the coordinates of a vector in a given reference frame  $Oxy$ , to write these as the coordinates of the point  $P$ . When we write  $\mathbf{v} = v(x, y)$  we understand that “ $\mathbf{v}$  is the vector of which the line segment  $O \rightarrow P$ , where  $P: (x, y)$ , is a representation.”

Let, then,  $\mathbf{v}_1 = v(x_1, y_1)$ ,  $\mathbf{v}_2 = v(x_2, y_2)$  be any two plane vectors, and let us construct the vector  $\mathbf{v} = v(x_1 + x_2, y_1 + y_2)$ . What are the coordinates of  $\mathbf{v}$  in any other reference frame  $O'x'y'$ ? From the table of direction cosines we see that the  $x'$ -coordinate of  $\mathbf{v}$  is

$$l_1(x_1 + x_2) + m_1(y_1 + y_2) = (l_1x_1 + m_1y_1) + (l_1x_2 + m_1y_2).$$

In other words the  $x'$ -coordinate of  $\mathbf{v}$  is the sum of the  $x'$ -coordinate of  $\mathbf{v}_1$  and the  $x'$ -coordinate of  $\mathbf{v}_2$ . Similarly the  $y'$ -coordinate of  $\mathbf{v}$  is the sum of the  $y'$ -coordinate of  $\mathbf{v}_1$  and the  $y'$ -coordinate of  $\mathbf{v}_2$ . But how was  $\mathbf{v}$  defined? It was the vector whose  $x$ -coordinate was the sum of the  $x$ -coordinate of  $\mathbf{v}_1$  and the  $x$ -coordinate of  $\mathbf{v}_2$  and whose  $y$ -coordinate was the sum of the  $y$ -coordinate of  $\mathbf{v}_1$  and the  $y$ -coordinate of  $\mathbf{v}_2$ . In other words

*We obtain the same vector  $\mathbf{v}$  when we proceed in the  $O'x'y'$  reference frame as we have proceeded in the  $Oxy$  reference frame; the vector  $\mathbf{v}$  is independent of the reference frame used to define it.*

We express this fundamentally important result as follows:

$$v(x_1 + x_2, y_1 + y_2) \text{ is a vector.}$$

*Remark.* Be very sure that you understand what is meant by this abbreviated statement. Every pair of numbers, in particular the pair  $(x_1 + x_2, y_1 + y_2)$ , defines a vector. What we mean when we say that  $v(x_1 + x_2, y_1 + y_2)$  is a vector is not merely the platitide that we can construct a vector whose coordinates in the  $Oxy$  reference frame are  $(x_1 + x_2, y_1 + y_2)$ . We mean that this construction is *independent of the reference frame*; if we repeat the construction in any other reference frame we arrive at the *same* vector (the *coordinates* of the vector we arrive at being generally, and naturally, different).



We term the vector  $\mathbf{v} = v(x_1 + x_2, y_1 + y_2)$  the *sum* of the vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , and we write

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2.$$

It is at once evident that  $\mathbf{v}_2 + \mathbf{v}_1 = \mathbf{v}_1 + \mathbf{v}_2$  simply because  $x_2 + x_1 = x_1 + x_2$  and  $y_2 + y_1 = y_1 + y_2$ . If the line segment  $O \rightarrow P_1$  is a representation of  $\mathbf{v}_1$  and the line segment  $P_1 \rightarrow P$  a representation of  $\mathbf{v}_2$  then the line segment  $O \rightarrow P$  is a representation of  $\mathbf{v}$  (why?). This furnishes the *triangle construction* for the addition of vectors:

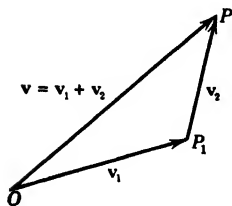


FIG. 2.

To add two plane vectors lay line segments which represent them end to end; then the line segment from the initial point of the first segment to the terminal point of the second segment is a representation of the sum of the two vectors.

If we have three vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  it is clear that

$$(\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3 = \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3).$$

In words: The addition of plane vectors is *associative*. We term the common sum  $(\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3, \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3)$  simply the sum of the three vectors  $\mathbf{v}_1, \mathbf{v}_2$ , and  $\mathbf{v}_3$ , and we denote this sum by the symbol  $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ . More generally if  $\mathbf{v}_1 = v(x_1, y_1), \dots, \mathbf{v}_n = v(x_n, y_n)$  are  $n$  plane vectors their sum is the vector

$$\mathbf{v}(x_1 + \dots + x_n, y_1 + \dots + y_n).$$

The geometrical construction of the sum of any number of plane vectors is the same as that of the sum of two plane vectors:

Lay representative segments of the vectors end to end; the segment from the initial point of the first segment to the terminal point of the last segment is a representation of the sum of the vectors.

### EXERCISES

1. Show that if  $c$  is any scalar and  $\mathbf{v} = v(x, y)$  any vector then  $v(cx, cy)$  is a vector. (What does this mean?) *Note.* The vector  $v(cx, cy)$  is termed the product of the vector  $\mathbf{v}$  by the scalar  $c$  and is denoted by the symbol  $c\mathbf{v}$ .

2. Show that the magnitude of  $c\mathbf{v}$  is  $|c|v$  where  $|c|$  is the absolute value of  $c$ ; i.e.,  $|c| = c$  if  $c \geq 0$ ,  $|c| = -c$  if  $c < 0$ .

3. Show that  $c\mathbf{v}$  has the same direction as  $\mathbf{v}$  if  $c > 0$  and the opposite direction to  $\mathbf{v}$  if  $c < 0$ . How is any representative segment of  $c\mathbf{v}$  related to any representative segment of  $\mathbf{v}$ ?

Now let  $\mathbf{v}_1 = v(x_1, y_1)$  and  $\mathbf{v}_2 = v(x_2, y_2)$  be any plane vectors, and let  $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$  be their sum. The squared magnitude of  $\mathbf{v}$  is a scalar which is furnished by the expression

$$\begin{aligned} v^2 &= (x_1 + x_2)^2 + (y_1 + y_2)^2 \\ &= (x_1^2 + y_1^2) + 2(x_1x_2 + y_1y_2) + (x_2^2 + y_2^2). \end{aligned}$$

Since  $x_1^2 + y_1^2 = v_1^2$  and  $x_2^2 + y_2^2 = v_2^2$  are scalars it follows (why?) that  $x_1x_2 + y_1y_2$  is a scalar. We term this scalar the *scalar product* of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , and we denote it by the symbol  $(\mathbf{v}_1|\mathbf{v}_2)$ :

$$(\mathbf{v}_1|\mathbf{v}_2) = x_1x_2 + y_1y_2.$$

It is clear that  $(\mathbf{v}_2|\mathbf{v}_1) = (\mathbf{v}_1|\mathbf{v}_2)$  simply because  $x_2x_1 = x_1x_2$  and  $y_2y_1 = y_1y_2$ ; in other words

*Scalar multiplication of plane vectors is commutative.*

The geometrical interpretation of  $(\mathbf{v}_1|\mathbf{v}_2)$  follows at once from the fact that  $(\mathbf{v}_1|\mathbf{v}_2)$  is a scalar; in fact the coordinates of  $\mathbf{v}_1$  in a reference frame whose positive  $x$ -axis has the direction of  $\mathbf{v}_1$  are  $(v_1, 0)$  so that

$$(\mathbf{v}_1|\mathbf{v}_2) = v_1x_2 + 0y_2 = v_1x_2 = v_1v_2 \cos \theta,$$

where  $\theta$  is the angle between the vectors  $v_1$  and  $v_2$ . Thus

**The scalar product of two plane vectors is the product of their magnitudes by the cosine of the angle between them.**

It follows that

*The scalar product of two plane vectors is zero when, and only when, the vectors are perpendicular (i.e., any representative segment of the first vector is perpendicular to any representative segment of the second) or when one of the two vectors is the zero vector.*

### EXERCISES

4. Show that  $(\mathbf{v}|\mathbf{v}) = v^2$ , i.e., that the scalar product of a vector by itself is the squared magnitude of the vector.

5. Show that  $(\mathbf{v}|\mathbf{v}) \geq 0$ , the equality holding when, and only when,  $\mathbf{v}$  is the zero vector.

6. Show that  $(\mathbf{v}_1|(\mathbf{v}_2 + \mathbf{v}_3)) = (\mathbf{v}_1|\mathbf{v}_2) + (\mathbf{v}_1|\mathbf{v}_3)$  and that  $((\mathbf{v}_1 + \mathbf{v}_2)|\mathbf{v}_3) = (\mathbf{v}_1|\mathbf{v}_3) + (\mathbf{v}_2|\mathbf{v}_3)$ . *Note.* The results of this exercise may be expressed as follows:

*Scalar multiplication of vectors is distributive with respect to addition.*

7. Show that  $((\mathbf{v}_1 + \mathbf{v}_2)|(\mathbf{v}_3 + \mathbf{v}_4)) = (\mathbf{v}_1|\mathbf{v}_3) + (\mathbf{v}_1|\mathbf{v}_4) + (\mathbf{v}_2|\mathbf{v}_3) + (\mathbf{v}_2|\mathbf{v}_4)$ .

8. Show that  $((\mathbf{v}_1 + \mathbf{v}_2)|(\mathbf{v}_1 + \mathbf{v}_2)) = (\mathbf{v}_1|\mathbf{v}_1) + 2(\mathbf{v}_1|\mathbf{v}_2) + (\mathbf{v}_2|\mathbf{v}_2)$ .

9. Deduce from the results of Exercises 4 and 8 the cosine law for plane triangles:  $c^2 = a^2 + b^2 - 2ab \cos C$ . *Hint.* If  $P_2 \rightarrow P_3$  and  $P_3 \rightarrow P_1$  are representations of

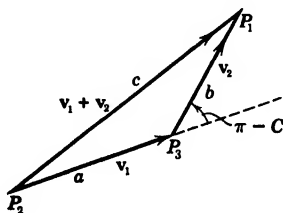


FIG. 3.

$\mathbf{v}_1$  and  $\mathbf{v}_2$ , respectively,  $P_2 \rightarrow P_1$  is a representation of  $\mathbf{v}_1 + \mathbf{v}_2$ , and the angle from  $\mathbf{v}_1$  to  $\mathbf{v}_2$  is  $\pi - C$  (not  $C$ , since the angle from  $\mathbf{v}_1$  to  $\mathbf{v}_2$  is the *exterior*, not the *interior*, angle of the triangle at  $P_3$ ). See Figure 3.

### 3. The alternating product of two plane vectors

If  $\theta$  is the angle from  $Ox$  to  $Ox'$ , the four cosines which occur in the table of direction cosines which is defined by the  $Oxy$  and  $O'x'y'$  reference frames are given by the formulas

$$\begin{aligned} l_1 &= \cos \theta, & m_1 &= \sin \theta; \\ l_2 &= -\sin \theta, & m_2 &= \cos \theta. \end{aligned}$$

Hence  $m_2 = l_1$ ,  $l_2 = -m_1$ . It follows that if  $\mathbf{v} = v(x, y)$  is any vector so also is  $v(-y, x)$ . In fact the coordinates, in the  $O'x'y'$  reference frame, of the vector whose coordinates in the  $Oxy$  reference frame are  $(-y, x)$  are

$$(-l_1y + m_1x, -l_2y + m_2x) = (-m_2y - l_2x, m_1y + l_1x) = (-y', x').$$

We term the vector  $v(-y, x)$  the *complement* of  $v(x, y)$ , and we denote it by  $\mathbf{v}^*$ . If  $P:(x, y)$  and  $P^*:(-y, x)$  the line segment  $O \rightarrow P^*$  has the same length as the line segment  $O \rightarrow P$ , and the angle from  $O \rightarrow P$  to  $O \rightarrow P^*$  is  $+\frac{\pi}{2}$ . Thus

*A representative segment of  $\mathbf{v}^*$  may be obtained from any representative segment of  $\mathbf{v}$  by rotating the latter through a right angle in the positive sense.*

### EXERCISES

1. Show that  $(\mathbf{v}^*)^* = -\mathbf{v}$  and that  $(\mathbf{v}_1 + \mathbf{v}_2)^* = \mathbf{v}_1^* + \mathbf{v}_2^*$ .
2. Show that  $(m\mathbf{v})^* = m\mathbf{v}^*$ .
3. Show that if  $\mathbf{u}_1 = v(l_1, m_1)$ ,  $\mathbf{u}_2 = v(l_2, m_2)$ , where  $l_1, m_1, l_2, m_2$  are the cosines of a table of direction cosines then  $\mathbf{u}_1$  and  $\mathbf{u}_2$  are unit vectors and  $\mathbf{u}_2 = \mathbf{u}_1^*$ .
4. What are the coordinates of the vectors  $\mathbf{u}_1$  and  $\mathbf{u}_2$  of Exercise 3 in the  $O'x'y'$  reference frame?
5. Show that  $(\mathbf{v}_1|\mathbf{v}_2) = (\mathbf{v}_1^*|\mathbf{v}_2^*)$ .

If  $\mathbf{v}_1 = v(x_1, y_1)$  and  $\mathbf{v}_2 = v(x_2, y_2)$  are any two plane vectors the number  $(\mathbf{v}_1^*|\mathbf{v}_2) = x_1y_2 - x_2y_1$  is a scalar (why?). Since

$$(\mathbf{v}_2^*|\mathbf{v}_1) = x_2y_1 - x_1y_2 = -(\mathbf{v}_1^*|\mathbf{v}_2)$$

this kind of multiplication of plane vectors is not commutative; it is, rather, anticommutative, i.e., an interchange of the two vectors involved in the product changes the sign of the product. We term  $(\mathbf{v}_1^*|\mathbf{v}_2) = x_1y_2 - x_2y_1$  the *alternating product* of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , and we denote this alternating product by the symbol  $(\mathbf{v}_1, \mathbf{v}_2)$ . Thus

$$(\mathbf{v}_1, \mathbf{v}_2) = x_1y_2 - x_2y_1 = -(\mathbf{v}_2, \mathbf{v}_1).$$

### EXERCISES

6. Show that  $(\mathbf{v}, \mathbf{v}) = 0$  and that  $(c_1\mathbf{v}_1, c_2\mathbf{v}_2) = c_1c_2(\mathbf{v}_1, \mathbf{v}_2)$ , where  $c_1, c_2$  are any scalars.

7. Show that  $(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_3) = (\mathbf{v}_1, \mathbf{v}_3) + (\mathbf{v}_2, \mathbf{v}_3)$  and that

$$(\mathbf{v}_1, \mathbf{v}_2 + \mathbf{v}_3) = (\mathbf{v}_1, \mathbf{v}_2) + (\mathbf{v}_1, \mathbf{v}_3).$$

8. Show that  $(\mathbf{v}_1, \mathbf{v}_2) = 0$  when, and only when, one of the two vectors  $\mathbf{v}_1, \mathbf{v}_2$  is a multiple of the other. *Hint.*  $(\mathbf{v}_1, \mathbf{v}_2) = (\mathbf{v}_1^*|\mathbf{v}_2)$ .

*Note.* When one of two vectors  $\mathbf{v}_1, \mathbf{v}_2$  is a multiple of the other the two vectors are said to be *linearly dependent*. If  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly dependent then either of them has the same direction as, or the opposite direction to, the other, or else one of the two vectors is the zero vector. The result of this exercise may be phrased as follows:

The equation  $(\mathbf{v}_1, \mathbf{v}_2) = 0$  is the criterion for linear dependence of the two vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ ; if  $(\mathbf{v}_1, \mathbf{v}_2) \neq 0$  the two vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly independent.

9. Show that if the line segments  $O \rightarrow P_1$ ,  $O \rightarrow P_2$  are representations of the vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , respectively, then  $(\mathbf{v}_1, \mathbf{v}_2)$  is the (signed) area of the parallelogram of which  $O \rightarrow P_1$  and  $O \rightarrow P_2$  are adjacent sides. *Hint.*

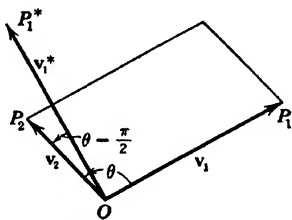


FIG. 5.

$$(\mathbf{v}_1, \mathbf{v}_2) = (\mathbf{v}_1^*|\mathbf{v}_2) = v_1^*v_2 \cos\left(\theta - \frac{\pi}{2}\right) = v_1v_2 \sin \theta,$$

where  $\theta$  is the angle from  $O \rightarrow P_1$  to  $O \rightarrow P_2$ .

10. Show that  $(\mathbf{v}_1, \mathbf{v}_2)$  is the determinant  $\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$ .

11. Show that if  $\mathbf{v}_1, \mathbf{v}_2$ , and  $\mathbf{v}_3$  are any three plane vectors then

$$(\mathbf{v}_2, \mathbf{v}_3)\mathbf{v}_1 + (\mathbf{v}_3, \mathbf{v}_1)\mathbf{v}_2 + (\mathbf{v}_1, \mathbf{v}_2)\mathbf{v}_3 = \mathbf{0}.$$

*Hint.* The first coordinate of the vector on the left is the three-rowed determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}, \text{ and the second coordinate is the three-rowed determinant } \begin{vmatrix} y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

It follows from the definition given in Exercise 8 that

Two plane vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly dependent when, and only when, two scalars  $c_1$  and  $c_2$ , not both zero, exist such that  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$  is the zero vector.

If we are given  $n$  plane vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  the vector

$$\mathbf{v} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n,$$

where  $(c_1, \dots, c_n)$  are any  $n$  scalars, is termed a *linear combination* of the  $n$  given vectors. If all the  $n$  scalars  $(c_1, \dots, c_n)$  are zero the linear combination is said to be *trivial*. In this terminology the definition of linear dependence of two plane vectors may be phrased as follows:

*Two plane vectors are linearly dependent when, and only when, there exists a nontrivial linear combination of the two vectors which is the zero vector.*

We extend the scope of this definition as follows:

*$n$  plane vectors are linearly dependent when, and only when, there exists a nontrivial linear combination of the  $n$  vectors which is the zero vector.*

### EXERCISES

12. Show that if a collection of  $n$  vectors is linearly dependent any collection of  $n + p$  vectors which contains the original collection is linearly dependent. *Hint.* If  $c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$  is a non-trivial linear combination of the vectors of the original collection,  $c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n + 0\mathbf{v}_{n+1} + \dots + 0\mathbf{v}_{n+p}$  is a non-trivial linear combination of vectors of the second collection, and

$$c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n + 0\mathbf{v}_{n+1} + \dots + 0\mathbf{v}_{n+p} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n.$$

13. Show that the only linearly dependent collection of one vector is the zero vector.

14. Show that there exist collections of two plane vectors which are *linearly independent*, i.e., not linearly dependent. *Hint.* If  $\mathbf{v}_1 = v(1, 0)$ ,  $\mathbf{v}_2 = v(0, 1)$ ,  $(\mathbf{v}_1, \mathbf{v}_2) = 1 \neq 0$ .

15. Show that all collections of three or more plane vectors are linearly dependent.

*Solution.* From the result of Exercise 12 it suffices to prove this result for the case  $n = 3$ . Let, then,  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  be any collection of three plane vectors. If  $(\mathbf{v}_2, \mathbf{v}_3) = 0$  the collection of two plane vectors  $\{\mathbf{v}_2, \mathbf{v}_3\}$  is linearly dependent, and hence the collection  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly dependent. If  $(\mathbf{v}_2, \mathbf{v}_3) \neq 0$  the collection  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly dependent by virtue of the result of Exercise 11. *Note.* The results of Exercises 14 and 15 are expressed by the statement that a *plane possesses two dimensions*.

It follows from the result of Exercise 11 that if  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is a linearly independent collection of two plane vectors then every plane vector  $\mathbf{v}$

may be expressed as a linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . In fact we are given that  $(\mathbf{v}_1, \mathbf{v}_2) \neq 0$  and so

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2,$$

where

$$c_1 = \frac{(\mathbf{v}, \mathbf{v}_2)}{(\mathbf{v}_1, \mathbf{v}_2)}; \quad c_2 = \frac{(\mathbf{v}_1, \mathbf{v})}{(\mathbf{v}_1, \mathbf{v}_2)}.$$

(To see this, replace  $\mathbf{v}_3$  by  $\mathbf{v}$  and use the alternating property of the products  $(\mathbf{v}_2, \mathbf{v})$  and  $(\mathbf{v}, \mathbf{v}_1)$ .) We express this result as follows: Every linearly independent pair of plane vectors is a *basis* for all plane vectors; we term the scalars  $c_1$  and  $c_2$  the *coordinates* of  $\mathbf{v}$  with respect to the basis  $\{\mathbf{v}_1, \mathbf{v}_2\}$ .

The simplest basis (from the point of view of computation) is one in which each of the basis vectors is a unit vector, the second being the complement of the first. Writing this basis as  $\{\mathbf{u}_1, \mathbf{u}_1^*\}$  we have

$$(\mathbf{u}_1, \mathbf{u}_1^*) = (\mathbf{u}_1^*|\mathbf{u}_1^*) = 1$$

and so

$$c_1 = (\mathbf{v}, \mathbf{u}_1^*) = (\mathbf{v}^*|\mathbf{u}_1^*) = (\mathbf{v}|\mathbf{u}_1) = (\mathbf{u}_1|\mathbf{v});$$

$$c_2 = (\mathbf{u}_1, \mathbf{v}) = (\mathbf{u}_1^*|\mathbf{v}) = (\mathbf{u}_2|\mathbf{v}), \quad \text{where } \mathbf{u}_2 = \mathbf{u}_1^*.$$

Thus we have the following rule:

*The coordinates of any plane vector with respect to the basis  $\{\mathbf{u}_1, \mathbf{u}_2\}$ , where  $\mathbf{u}_1$  and  $\mathbf{u}_2$  are unit vectors and  $\mathbf{u}_2 = \mathbf{u}_1^*$ , are  $(\mathbf{u}_1|\mathbf{v})$  and  $(\mathbf{u}_2|\mathbf{v})$ .*

### EXERCISES

16. Show that the vectors  $\mathbf{v}_1 = v(1, 1)$  and  $\mathbf{v}_2 = v(0, 1)$  constitute a basis for all plane vectors, and determine the coordinates of  $v(3, 2)$  with respect to this basis.

17. Show that  $x$  and  $y$  are the coordinates of  $v(x, y)$  with respect to the basis  $\mathbf{u}_1 = v(1, 0)$ ,  $\mathbf{u}_2 = \mathbf{u}_1^* = v(0, 1)$ .

18. Show that if the pair of vectors  $\{\mathbf{v}_1, \mathbf{v}_2\}$  constitutes a basis for all plane vectors so also does the pair  $\{\mathbf{v}_1^*, \mathbf{v}_2^*\}$ . *Hint.*  $(\mathbf{v}_1^*, \mathbf{v}_2^*) = (\mathbf{v}_1, \mathbf{v}_2)$ .

19. Show that the coordinates of  $\mathbf{v}^*$  with respect to the basis  $\{\mathbf{v}_1^*, \mathbf{v}_2^*\}$  are the same as the coordinates of  $\mathbf{v}$  with respect to the basis  $\{\mathbf{v}_1, \mathbf{v}_2\}$ .

### 4. Three-dimensional vectors

The theory of *three-dimensional* or *space* vectors is so similar to that of two-dimensional or plane vectors that it will suffice to run over (largely without proofs) the main points. If you have difficulty with any of the statements made, refer back to the corresponding statement for two-dimensional vectors.

You must first of all be clear about what is meant by a *right-handed*

rectangular Cartesian reference frame. If you hold out your right hand with your thumb, first finger, and second finger as nearly as possible at right angles to each other these will constitute a right-handed reference frame if the thumb has the direction of the positive  $x$ -axis, the first finger the direction of the positive  $y$ -axis, and the second finger the direction of the positive  $z$ -axis. An equivalent description of a right-handed reference frame is as follows: The point of an ordinary commercial screw will move in the direction of the positive  $z$ -axis when turned through a right angle from the positive  $x$ -axis towards the positive  $y$ -axis.

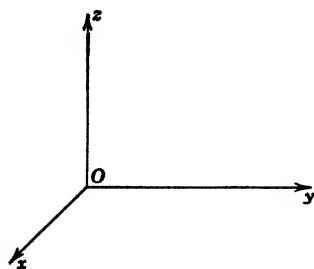


FIG. 6.

From now on we shall suppose that all our rectangular Cartesian reference frames are right handed. Let  $P \rightarrow P + \Delta P$  be any line segment, and denote its projections on the axes of any two reference frames  $Oxyz$  and  $O'x'y'z'$  by  $(\Delta x, \Delta y, \Delta z)$  and  $(\Delta x', \Delta y', \Delta z')$ , respectively. These two sets of projections are connected with one another by a *table of direction cosines*:

	$\Delta x'$	$\Delta y'$	$\Delta z'$
$\Delta x$	$l_1$	$l_2$	$l_3$
$\Delta y$	$m_1$	$m_2$	$m_3$
$\Delta z$	$n_1$	$n_2$	$n_3$

This table may be read either "by rows" or "by columns"; for example

$$\Delta y = m_1 \Delta x' + m_2 \Delta y' + m_3 \Delta z';$$

$$\Delta z' = l_3 \Delta x + m_3 \Delta y + n_3 \Delta z, \text{ and so on.}$$

A *three-dimensional vector* is a collection of sets of three numbers, there being one set attached to each reference frame, where the set attached to any reference frame  $Oxyz$  is connected with the set attached to any other reference frame through the table of direction cosines. Thus if  $(\Delta x', \Delta y', \Delta z')$  are the *coordinates* in the  $O'x'y'z'$  reference frame of the vector whose coordinates in the  $Oxyz$  reference frame are  $(\Delta x, \Delta y, \Delta z)$  we have

$$\Delta x' = l_1 \Delta x + m_1 \Delta y + n_1 \Delta z;$$

$$\Delta y' = l_2 \Delta x + m_2 \Delta y + n_2 \Delta z;$$

$$\Delta z' = l_3 \Delta x + m_3 \Delta y + n_3 \Delta z.$$

Attached to any vector  $\mathbf{v} = v(\Delta x, \Delta y, \Delta z)$  is a scalar, namely, its magnitude  $v$ :

$$v = \{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\}^{1/2}.$$

Since the relation

$$(\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

must hold for arbitrary values of  $(\Delta x, \Delta y, \Delta z)$  the nine direction cosines of the table of direction cosines must satisfy certain readily determined relations. On setting  $(\Delta x, \Delta y, \Delta z) = (1, 0, 0)$  we obtain  $l_1^2 + l_2^2 + l_3^2 = 1$ ; on setting  $(\Delta x, \Delta y, \Delta z) = (0, 1, 0)$  we obtain  $m_1^2 + m_2^2 + m_3^2 = 1$ , and on setting  $(\Delta x, \Delta y, \Delta z) = (0, 0, 1)$  we obtain  $n_1^2 + n_2^2 + n_3^2 = 1$ . On setting  $(\Delta x, \Delta y, \Delta z) = (0, 1, 1)$ , and using the relations already obtained, we find that  $m_1 n_1 + m_2 n_2 + m_3 n_3 = 0$ . Similarly  $n_1 l_1 + n_2 l_2 + n_3 l_3 = 0$ ,  $l_1 m_1 + l_2 m_2 + l_3 m_3 = 0$ . (Prove this.) Hence the nine cosines of the table of direction cosines satisfy the six relations:

$$\begin{aligned} l_1^2 + l_2^2 + l_3^2 &= 1; & m_1^2 + m_2^2 + m_3^2 &= 1; \\ n_1^2 + n_2^2 + n_3^2 &= 1; & m_1 n_1 + m_2 n_2 + m_3 n_3 &= 0; \\ n_1 l_1 + n_2 l_2 + n_3 l_3 &= 0; & l_1 m_1 + l_2 m_2 + l_3 m_3 &= 0. \end{aligned}$$

### EXERCISES

1. Show that the nine cosines of the table of direction cosines satisfy the six relations

$$\begin{aligned} l_1^2 + m_1^2 + n_1^2 &= 1; & l_2^2 + m_2^2 + n_2^2 &= 1; & l_3^2 + m_3^2 + n_3^2 &= 1; \\ l_2 l_3 + m_2 m_3 + n_2 n_3 &= 0; & l_3 l_1 + m_3 m_1 + n_3 n_1 &= 0; & l_1 l_2 + m_1 m_2 + n_1 n_2 &= 0. \end{aligned}$$

*Hint.* Interchange the roles of the two reference frames  $Oxyz, O'x'y'z'$ .

2. Show that the determinant

$$\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = 1.$$

*Solution.* This determinant has the same value as the determinant obtained by interchanging its rows and columns, and the product of these two equal determinants is 1 (by virtue of the relations satisfied by the nine direction cosines). Hence the determinant in question is  $\pm 1$ . Being a *continuous* function of the nine direction cosines it must be always +1 or always -1. Since it is +1 when the two reference frames coincide, it must be always +1.



3. Show that each element of the determinant of Exercise 2 is equal to its cofactor; e.g.,  $l_1 = m_2n_3 - m_3n_2$ ,  $l_2 = m_3n_1 - m_1n_3$ , and so on. *Hint.* The two equations  $l_1m_1 + l_2m_2 + l_3m_3 = 0$ ,  $l_1n_1 + l_2n_2 + l_3n_3 = 0$  yield

$$l_1 = k(m_2n_3 - m_3n_2), \quad l_2 = k(m_3n_1 - m_1n_3), \quad l_3 = k(m_1n_2 - m_2n_1),$$

where  $k$  is a factor of proportionality. On multiplying these equations by  $l_1$ ,  $l_2$ , and  $l_3$  respectively, and adding, we obtain, in view of the result of Exercise 2,  $k = 1$ .

If the line segment  $O \rightarrow P$  is a representation of a vector  $\mathbf{v}$  we have  $\mathbf{v} = v(x, y, z)$ , where  $P:(x, y, z)$ . The sum  $\mathbf{v}_1 + \mathbf{v}_2$  of two vectors  $\mathbf{v}_1 = v(x_1, y_1, z_1)$ ,  $\mathbf{v}_2 = v(x_2, y_2, z_2)$  is the vector

$$v(x_1 + x_2, y_1 + y_2, z_1 + z_2).$$

(Prove that this is a vector.) The product of the vector  $\mathbf{v} = v(x, y, z)$  by the scalar  $c$  is the vector  $c\mathbf{v} = v(cx, cy, cz)$ . (Prove that this is a vector.) Addition of vectors is commutative and associative. (What does this mean?) To obtain a representative segment of the sum of any number of vectors we have merely to lay representative segments of the individual vectors end to end; the line segment from the initial point of the first line segment to the terminal point of the last line segment is a representation of the sum of the vectors.

From a consideration of the squared magnitude of  $\mathbf{v}_1 + \mathbf{v}_2$  we obtain the *scalar product* of two vectors:

$$(\mathbf{v}_1|\mathbf{v}_2) = x_1x_2 + y_1y_2 + z_1z_2,$$

and we observe that  $(\mathbf{v}|\mathbf{v})$  is the squared magnitude of  $\mathbf{v}$ . On choosing a reference frame whose positive  $x$ -axis has the direction of  $\mathbf{v}_1$  we find that

$$(\mathbf{v}_1|\mathbf{v}_2) = v_1v_2 \cos \theta,$$

where  $\theta$  is the (unsigned) angle between  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .

### EXERCISES

4. Show that scalar multiplication of space vectors is commutative (what does this mean?) and distributive with respect to addition (what does this mean?).

5. Show that  $(\mathbf{v}_1|\mathbf{v}_2) = 0$  when, and only when,  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are perpendicular, or else when one or other of the two vectors is the zero vector.

6. Show that  $(c_1\mathbf{v}_1|c_2\mathbf{v}_2) = c_1c_2(\mathbf{v}_1|\mathbf{v}_2)$ , where  $c_1$  and  $c_2$  are scalars.

### 5. The vector or cross product of two space vectors

If  $\mathbf{v}_1 = v(x_1, y_1, z_1)$ ,  $\mathbf{v}_2 = v(x_2, y_2, z_2)$  are any two space vectors,  $v(y_1z_2 - y_2z_1, z_1x_2 - z_2x_1, x_1y_2 - x_2y_1)$  is a vector. In fact

## VECTOR OR CROSS PRODUCT OF TWO SPACE VECTORS 17

$$\begin{aligned}
 & l_1(y_1z_2 - y_2z_1) + m_1(z_1x_2 - z_2x_1) + n_1(x_1y_2 - x_2y_1) \\
 &= (m_2n_3 - m_3n_2)(y_1z_2 - y_2z_1) + (n_2l_3 - n_3l_2)(z_1x_2 - z_2x_1) \\
 &\quad + (l_2m_3 - l_3m_2)(x_1y_2 - x_2y_1)
 \end{aligned}$$

(see Exercise 3, p. 16)  $= y_1'z_2' - y_2'z_1'$  since  $y_1' = l_2x_1 + m_2y_1 + n_2z_1$ , etc. This vector is the *vector* or *cross product* of the vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ ; we denote it by the symbol  $(\mathbf{v}_1 \times \mathbf{v}_2)$ . If either  $\mathbf{v}_1$  or  $\mathbf{v}_2$  is the zero vector so also is  $(\mathbf{v}_1 \times \mathbf{v}_2)$ ; if neither  $\mathbf{v}_1$  or  $\mathbf{v}_2$  is the zero vector we choose the reference frame  $Oxyz$  so that  $Ox$  has the direction of  $\mathbf{v}_1$  while  $Oy$  lies in the plane determined by  $O \rightarrow P_1$  and  $O \rightarrow P_2$  (the direction of  $Oy$  being such that the  $y$ -coordinate of  $\mathbf{v}_2$  is positive). We have, then,

$$\begin{aligned}
 \mathbf{v}_1 &= v(v_1, 0, 0); \\
 \mathbf{v}_2 &= v(v_2 \cos \theta, v_2 \sin \theta, 0),
 \end{aligned}$$

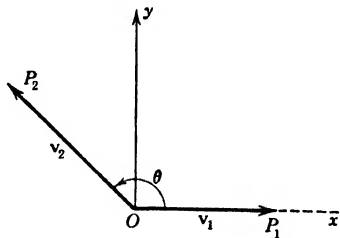


FIG. 7.

where  $\theta$  is the (unsigned) angle between  $O \rightarrow P_1$  and  $O \rightarrow P_2$  ( $\theta$  being measured so that  $0 \leq \theta \leq \pi$ ; if  $\theta = 0$  or  $\pi$  the line segments  $O \rightarrow P_1$  and  $O \rightarrow P_2$  do not determine a plane, and the  $y$ -axis may be taken to have *any* direction perpendicular to  $\mathbf{v}_1$ ). Hence

$$(\mathbf{v}_1 \times \mathbf{v}_2) = v(0, 0, v_1v_2 \sin \theta).$$

Thus the magnitude of  $(\mathbf{v}_1 \times \mathbf{v}_2)$  is  $v_1v_2 \sin \theta$ , and if  $(\mathbf{v}_1 \times \mathbf{v}_2)$  is not the zero vector the direction of  $(\mathbf{v}_1 \times \mathbf{v}_2)$  is that of the positive  $z$ -axis. In other words

*The magnitude of  $(\mathbf{v}_1 \times \mathbf{v}_2)$  is the area of the parallelogram of which  $O \rightarrow P_1$  and  $O \rightarrow P_2$  are adjacent sides; when  $(\mathbf{v}_1 \times \mathbf{v}_2)$  is not the zero vector the direction of  $(\mathbf{v}_1 \times \mathbf{v}_2)$  is that one of the two directions perpendicular to the plane containing  $O \rightarrow P_1$ ,  $O \rightarrow P_2$  along which the point of an ordinary commercial screw will move when the screw is turned through the angle (less than  $\pi$ ) between  $O \rightarrow P_1$  and  $O \rightarrow P_2$ .  $(\mathbf{v}_1 \times \mathbf{v}_2)$  is the zero vector when, and only when, one of the vectors  $\mathbf{v}_1, \mathbf{v}_2$  is a multiple of the other.*

### EXERCISES

1. Show that  $(\mathbf{v}_2 \times \mathbf{v}_1) = -(\mathbf{v}_1 \times \mathbf{v}_2)$ , i.e., that vector or cross multiplication of space vectors is anticommutative or alternating.
2. Show that  $(c_1\mathbf{v}_1 \times c_2\mathbf{v}_2) = c_1c_2(\mathbf{v}_1 \times \mathbf{v}_2)$ , where  $c_1$  and  $c_2$  are scalars.
3. Show that  $((\mathbf{v}_1 + \mathbf{v}_2) \times \mathbf{v}_3) = (\mathbf{v}_1 \times \mathbf{v}_3) + (\mathbf{v}_2 \times \mathbf{v}_3)$  and that

$$(\mathbf{v}_1 \times (\mathbf{v}_2 + \mathbf{v}_3)) = (\mathbf{v}_1 \times \mathbf{v}_2) + (\mathbf{v}_1 \times \mathbf{v}_3),$$

i.e., that vector (or cross) multiplication of space vectors is distributive with respect to addition.

4. Show that

$$((\mathbf{v}_1 + \mathbf{v}_2) \times (\mathbf{v}_3 + \mathbf{v}_4)) = (\mathbf{v}_1 \times \mathbf{v}_3) + (\mathbf{v}_1 \times \mathbf{v}_4) + (\mathbf{v}_2 \times \mathbf{v}_3) + (\mathbf{v}_2 \times \mathbf{v}_4).$$

### 6. The alternating scalar triple product; linear dependence of space vectors

Let  $\mathbf{v}_1 = v(x_1, y_1, z_1)$ ,  $\mathbf{v}_2 = v(x_2, y_2, z_2)$ , and  $\mathbf{v}_3 = v(x_3, y_3, z_3)$  be any three space vectors, and consider the scalar product  $(\mathbf{v}_1 | (\mathbf{v}_2 \times \mathbf{v}_3))$  of  $\mathbf{v}_1$  and the vector, or cross, product of  $\mathbf{v}_2$  and  $\mathbf{v}_3$ . This is the three-rowed determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = x_1(y_2z_3 - y_3z_2) + y_1(z_2x_3 - z_3x_2) + z_1(x_2y_3 - y_2x_3).$$

Hence it is *alternating* in the three vectors  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{v}_3$ , i.e., an interchange of *any* two of these vectors changes the sign of this scalar product. On the other hand an interchange of the  $|$  and  $\times$  in the symbol  $(\mathbf{v}_1 | (\mathbf{v}_2 \times \mathbf{v}_3))$  leaves the scalar product unaffected:

$$(\mathbf{v}_1 | (\mathbf{v}_2 \times \mathbf{v}_3)) = ((\mathbf{v}_1 \times \mathbf{v}_2) | \mathbf{v}_3).$$

We may, then, without fear of ambiguity omit both the  $|$  and the  $\times$  from the symbol. We do this and write it as follows:

$$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3).$$

We term this scalar the *alternating product* of the three vectors. Thus

$$\begin{aligned} (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= -(\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_2) = (\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_1) \\ &= -(\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3) = (\mathbf{v}_3, \mathbf{v}_1, \mathbf{v}_2) = -(\mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_1). \end{aligned}$$

The geometrical interpretation of  $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$  is clear; since the magnitude of  $(\mathbf{v}_2 \times \mathbf{v}_3)$  is the (unsigned) area of the parallelogram of which  $O \rightarrow P_2$  and  $O \rightarrow P_3$  are adjacent sides,  $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$  is the product of this area by  $v_1$  times the cosine of the angle between  $\mathbf{v}_1$  and  $(\mathbf{v}_2 \times \mathbf{v}_3)$ . It is, then, the signed volume of the box of which  $O \rightarrow P_1$ ,  $O \rightarrow P_2$  and  $O \rightarrow P_3$  are adjacent edges. This signed volume is positive when the angle between  $\mathbf{v}_1$  and  $(\mathbf{v}_2 \times \mathbf{v}_3)$  is acute and negative when this

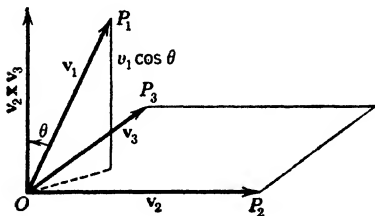


FIG. 8.

angle is obtuse. When  $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$  is positive we say that the three vectors  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{v}_3$ , taken in this order, are *positively oriented*; when  $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$  is negative we say that they are *negatively oriented*.

## EXERCISES

1. Show that  $(c_1\mathbf{v}_1, c_2\mathbf{v}_2, c_3\mathbf{v}_3) = c_1c_2c_3(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ , where  $c_1$ ,  $c_2$ , and  $c_3$  are any scalars.

2. Show that  $(\mathbf{v}_1 + \mathbf{w}_1, \mathbf{v}_2, \mathbf{v}_3) = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) + (\mathbf{w}_1, \mathbf{v}_2, \mathbf{v}_3)$ .

3. Show that  $(\mathbf{v}_1, \mathbf{v}_1, \mathbf{v}_2) = 0$ . *Hint.* Since  $(\mathbf{v}_1, \mathbf{v}_1, \mathbf{v}_2)$  is alternating

$$(\mathbf{v}_1, \mathbf{v}_1, \mathbf{v}_2) = -(\mathbf{v}_1, \mathbf{v}_1, \mathbf{v}_2).$$

4. Show that  $(\mathbf{v}_1, \mathbf{v}_2, c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = 0$ , where  $c_1$  and  $c_2$  are any two scalars.

We define linear dependence of space vectors exactly as in the case of plane vectors:

A set of  $n$  space vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is said to be linearly dependent if there exists a non-trivial (what does this mean?) linear combination  $c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$  of the space vectors which is the zero vector.

Thus when  $n = 1$  the only linearly dependent set is the set consisting of the zero vector. The criterion for linear dependence of sets of two vectors has been already obtained:

*The set of two space vectors  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is linearly dependent when, and only when,  $(\mathbf{v}_1 \times \mathbf{v}_2)$  is the zero vector.*

It is easy to see that  $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = 0$  if, and only if, the set of three space vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly dependent. In fact if this set is linearly dependent one of them,  $\mathbf{v}_3$  say, is a linear combination of the other two (why?):

$$\mathbf{v}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2.$$

Hence  $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = 0$  (why?). Conversely, let  $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = 0$ . If  $(\mathbf{v}_2 \times \mathbf{v}_3) = \mathbf{0}$  the set  $\{\mathbf{v}_2, \mathbf{v}_3\}$  of two space vectors is linearly dependent, and hence the set  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  of three space vectors is linearly dependent (why?). If  $(\mathbf{v}_2 \times \mathbf{v}_3)$  is not the zero vector the relation  $(\mathbf{v}_1 | (\mathbf{v}_2 \times \mathbf{v}_3)) = 0$  tells us that  $\mathbf{v}_1$  is perpendicular to  $(\mathbf{v}_2 \times \mathbf{v}_3)$  so that  $\mathbf{v}_1$  is a linear combination of  $\mathbf{v}_2$  and  $\mathbf{v}_3$  (why?). Hence the set  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly dependent. Since  $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = 1$ , where  $\mathbf{u}_1 = v(1, 0, 0)$ ,  $\mathbf{u}_2 = v(0, 1, 0)$ ,  $\mathbf{u}_3 = v(0, 0, 1)$ , there exist in space linearly independent sets of three vectors.

If  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  is any set of four space vectors we have

$$(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)\mathbf{v}_1 + (\mathbf{v}_3, \mathbf{v}_1, \mathbf{v}_4)\mathbf{v}_2 + (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4)\mathbf{v}_3 - (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\mathbf{v}_4 = \mathbf{0}.$$

In fact the  $x$ -coordinate of the vector on the left is the four-rowed determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

which is zero. Similarly the  $y$ - and  $z$ -coordinates of the vector on the left are zero. If, then, the set of three vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly independent, so that  $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \neq 0$ , we have

$$\mathbf{v}_4 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3,$$

where

$$c_1 = \frac{(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)}{(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)}; \quad c_2 = \frac{(\mathbf{v}_3, \mathbf{v}_1, \mathbf{v}_4)}{(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)}; \quad c_3 = \frac{(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4)}{(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)}.$$

Thus, while there exist linearly independent sets of three space vectors, every set of four or more space vectors is linearly dependent. We express this result by the statement that space is *three dimensional*.

We have just seen that every space vector is a linear combination of the vectors of a linearly independent set of three space vectors. In other words every linearly independent set of three space vectors is a *basis* for space vectors. We term the scalars  $c_1, c_2, c_3$  which occur in the linear combination  $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$  the *coordinates* of  $\mathbf{v}$  with respect to the basis  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ . The simplest bases, from the point of view of computation, are those in which the vectors of the linearly independent set of three vectors  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  are unit vectors such that  $(\mathbf{u}_1|\mathbf{u}_2) = 0$  and  $\mathbf{u}_3 = (\mathbf{u}_1 \times \mathbf{u}_2)$ . Then

$$(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = ((\mathbf{u}_1 \times \mathbf{u}_2)|\mathbf{u}_3) = (\mathbf{u}_3|\mathbf{u}_3) = 1$$

and  $(\mathbf{u}_2 \times \mathbf{u}_3) = \mathbf{u}_1$ ,  $(\mathbf{u}_3 \times \mathbf{u}_1) = \mathbf{u}_2$  (see Exercise 3, p. 16). Hence  $(\mathbf{u}_2, \mathbf{u}_3, \mathbf{v}) = (\mathbf{u}_1|\mathbf{v})$ ,  $(\mathbf{u}_3, \mathbf{u}_1, \mathbf{v}) = (\mathbf{u}_2|\mathbf{v})$  and  $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}) = (\mathbf{u}_3|\mathbf{v})$  so that

$$\mathbf{v} = (\mathbf{u}_1|\mathbf{v})\mathbf{u}_1 + (\mathbf{u}_2|\mathbf{v})\mathbf{u}_2 + (\mathbf{u}_3|\mathbf{v})\mathbf{u}_3.$$

Thus the coordinates of  $\mathbf{v}$  with respect to the basis  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  are  $(\mathbf{u}_1|\mathbf{v})$ ,  $(\mathbf{u}_2|\mathbf{v})$ , and  $(\mathbf{u}_3|\mathbf{v})$ .

### EXERCISES

5. Show that if  $\mathbf{u}_1 = v(1, 0, 0)$ ,  $\mathbf{u}_2 = v(0, 1, 0)$ ,  $\mathbf{u}_3 = v(0, 0, 1)$  the coordinates of  $\mathbf{v} = v(x, y, z)$  with respect to the basis  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  are  $(x, y, z)$ .
6. Show that the set of three space vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ , where  $\mathbf{v}_1 = v(-2, 0, 3)$ ,  $\mathbf{v}_2 = v(3, 5, 0)$ ,  $\mathbf{v}_3 = v(-7, -5, 6)$ , is linearly dependent, and express  $\mathbf{v}_1$  as a

linear combination of  $\mathbf{v}_2$  and  $\mathbf{v}_3$ . *Hint.* Write  $\mathbf{v}_1 = c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ , and solve for  $c_2$  and  $c_3$  the two equations obtained by equating two of the coordinates of  $\mathbf{v}_1$  and  $c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ .

### 7. The vector triple product

Let  $\mathbf{v}_1 = v(x_1, y_1, z_1)$ ,  $\mathbf{v}_2 = v(x_2, y_2, z_2)$ ,  $\mathbf{v}_3 = v(x_3, y_3, z_3)$  be any three space vectors. Then  $(\mathbf{v}_2 \times \mathbf{v}_3)$  is the vector  $v(y_2z_3 - y_3z_2, z_2x_3 - z_3x_2, x_2y_3 - x_3y_2)$  so that the  $x$ -coordinate of the vector  $(\mathbf{v}_1 \times (\mathbf{v}_2 \times \mathbf{v}_3))$  is  $y_1(x_2y_3 - x_3y_2) - z_1(z_2x_3 - z_3x_2) = (\mathbf{v}_1|\mathbf{v}_3)x_2 - (\mathbf{v}_1|\mathbf{v}_2)x_3$ . Similarly the  $y$ -coordinate of  $(\mathbf{v}_1 \times (\mathbf{v}_2 \times \mathbf{v}_3))$  is  $(\mathbf{v}_1|\mathbf{v}_3)y_2 - (\mathbf{v}_1|\mathbf{v}_2)y_3$  and the  $z$ -coordinate of  $(\mathbf{v}_1 \times (\mathbf{v}_2 \times \mathbf{v}_3))$  is  $(\mathbf{v}_1|\mathbf{v}_3)z_2 - (\mathbf{v}_1|\mathbf{v}_2)z_3$ . Hence

$$(\mathbf{v}_1 \times (\mathbf{v}_2 \times \mathbf{v}_3)) = (\mathbf{v}_1|\mathbf{v}_3)\mathbf{v}_2 - (\mathbf{v}_1|\mathbf{v}_2)\mathbf{v}_3.$$

A similar argument (give it) yields the result

$$((\mathbf{v}_1 \times \mathbf{v}_2) \times \mathbf{v}_3) = (\mathbf{v}_1|\mathbf{v}_3)\mathbf{v}_2 - (\mathbf{v}_2|\mathbf{v}_3)\mathbf{v}_1.$$

These results may be readily remembered as follows: The *vector triple product*  $(\mathbf{v}_1 \times (\mathbf{v}_2 \times \mathbf{v}_3))$  or  $((\mathbf{v}_1 \times \mathbf{v}_2) \times \mathbf{v}_3)$  is a linear combination of the two vectors which are first crossed, and the coefficient of the *middle* vector is the scalar product of the other two. The coefficient of the other vector (of the two which are first crossed) is the negative of the scalar product of the other two.

### EXERCISES

1. Show that  $((\mathbf{v}_1 \times \mathbf{v}_2) \times (\mathbf{v}_3 \times \mathbf{v}_4)) = (\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4)\mathbf{v}_2 - (\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)\mathbf{v}_1$ . *Hint.* Set  $(\mathbf{v}_3 \times \mathbf{v}_4) = \mathbf{w}$ , and expand  $((\mathbf{v}_1 \times \mathbf{v}_2) \times \mathbf{w})$ .

2. Show that  $((\mathbf{v}_1 \times \mathbf{v}_2) \times (\mathbf{v}_3 \times \mathbf{v}_4)) = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4)\mathbf{v}_3 - (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\mathbf{v}_4$ .

3. Show that  $((\mathbf{v}_1 \times \mathbf{v}_2)|(\mathbf{v}_3 \times \mathbf{v}_4)) = (\mathbf{v}_1|\mathbf{v}_3)(\mathbf{v}_2|\mathbf{v}_4) - (\mathbf{v}_1|\mathbf{v}_4)(\mathbf{v}_2|\mathbf{v}_3)$ . *Hint.*

$$((\mathbf{v}_1 \times \mathbf{v}_2)|(\mathbf{v}_3 \times \mathbf{v}_4)) = (((\mathbf{v}_1 \times \mathbf{v}_2) \times \mathbf{v}_3)|\mathbf{v}_4).$$

### 8. Scalar fields; the gradient vector

We first consider the case of the plane. If we have, attached to each point  $P:(x, y)$  of a collection of points (= point set) in the plane, a scalar  $f = f(P) = f(x, y)$  we say that  $f = f(P)$  is a *scalar field* defined over the given point set. If  $f = f(P)$  is defined over a neighborhood of  $P$  its increment  $\Delta f = f(P + \Delta P) - f(P)$ , where

$$P + \Delta P: (x + \Delta x, y + \Delta y),$$

is a scalar since the difference of two scalars is a scalar and both  $f(P + \Delta P)$  and  $f(P)$  are scalars. It follows that if  $f = f(P)$  is differentiable at  $P$  its differential  $df$  is a scalar. In fact  $\Delta f = df + \nu|\Delta P|$ ,

where  $\nu$  is null at  $|\Delta P| = 0$ , so that, since  $|\Delta P|$  is a scalar,  $\frac{df}{|\Delta P|} + \nu$  is a scalar. Since  $\frac{df}{|\Delta P|}$  is independent of the magnitude  $|\Delta P|$  of the vector  $v(\Delta x, \Delta y)$ , being dependent only on the direction of this vector, it follows that  $\frac{df}{|\Delta P|}$ , and hence  $df$ , is a scalar. In fact, on denoting by primes the values in any new reference frame  $O'x'y'$  of the various expressions involved, the fact that  $\frac{df}{|\Delta P|} + \nu$  is a scalar yields

$$\frac{(df)'}{|\Delta P|} + \nu' = \frac{df}{|\Delta P|} + \nu$$

and so  $\frac{(df)'}{|\Delta P|} - \frac{df}{|\Delta P|} = \nu - \nu'$  is null at  $|\Delta P| = 0$ . Being independent of  $|\Delta P|$  it must be zero (why?) and so  $\frac{(df)'}{|\Delta P|} - \frac{df}{|\Delta P|} = 0$ , or, equivalently,

$$(df)' = df.$$

In words:

$$df = f_x \Delta x + f_y \Delta y \text{ is a scalar.}$$

Since the direction of the vector  $v(\Delta x, \Delta y)$  is arbitrary it follows that  $v(f_x, f_y)$  is a vector. In fact the equation

$$f_{x'} \Delta x' + f_{y'} \Delta y' = f_x \Delta x + f_y \Delta y$$

yields (on setting  $\Delta y' = 0$  so that  $\Delta x = l_1 \Delta x'$ ,  $\Delta y = m_1 \Delta x'$  (why?))  $f_{x'} = l_1 f_x + m_1 f_y$  and, similarly,  $f_{y'} = l_2 f_x + m_2 f_y$ .

*Remark.* These equations which furnish  $f_{x'}$  and  $f_{y'}$  in terms of  $f_x$  and  $f_y$  might have been written down at once by using the law of composite differentiation for functions of two variables. We have preferred not to do this in order to point out the *intrinsic meaning* of the equations: They simply express the fact that  $df$  is a scalar when  $f$  is a scalar field. Incidentally we have proved the following useful converse of the scalar product theorem:

**If we have, attached to each reference frame, a pair of numbers  $(a, b)$  having the property that  $ax + by$  is a scalar for every vector  $v(x, y)$  then  $v(a, b)$  is a vector.** Since every plane vector is a linear

combination of the two vectors of a basis it is sufficient for the validity of this converse of the scalar product theorem to know that  $ax + by$  is a scalar for each of two linearly independent vectors

$$\mathbf{v}_1 = v(x_1, y_1), \quad \mathbf{v}_2 = v(x_2, y_2).$$

We term the vector  $v(f_x, f_y)$  the *gradient* of the (differentiable) scalar field  $f$ , and we denote it by  $\mathbf{grad} f$ .  $\mathbf{grad} f$  is a *vector field*; in other words we have, attached to each point  $P$  of our point set over which the scalar field  $f$  is granted to be differentiable, a vector. If we introduce the *differentiating operator*

$$\nabla = v\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

we may write

$$\mathbf{grad} f = \nabla f$$

or, simply,

$$\mathbf{grad} = \nabla.$$

When the symbol  $\nabla$  is used in this way it is sometimes called a *symbolic vector*; all that is meant by this statement is that the "coordinates"  $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$  of  $\nabla$  transform, when we pass from the reference frame  $Oxy$  to any other reference frame, in exactly the same way as do the coordinates of a vector.

The squared magnitude of  $\mathbf{grad} f$  is itself a scalar field which we shall denote by  $\Delta_1 f$ . In this notation, then, we have the following result:

*Associated with any differentiable scalar field  $f$  is a new scalar field*

$$\Delta_1 f = (\mathbf{grad} f | \mathbf{grad} f) = (f_x)^2 + (f_y)^2.$$

When we restrict our attention to points  $P(t) : (x(t), y(t))$  lying on a curve  $x = x(t)$ ,  $y = y(t)$ ,  $df$  is given by the formula

$$df = f_x dx + f_y dy = (\mathbf{grad} f | d\mathbf{P}),$$

where  $d\mathbf{P} = v(dx, dy)$  is the *vector element of arc* of the curve. That  $d\mathbf{P}$  actually is a vector is clear; in fact the equations

$$x = x_0 + l_1 x' + l_2 y'; \quad y = y_0 + m_1 x' + m_2 y'$$

yield at once

$$dx = l_1 dx' + l_2 dy'; \quad dy = m_1 dx' + m_2 dy'$$



which prove that  $v(dx, dy)$  is a vector. If our curve is a level curve of the scalar field  $f$ , i.e., a curve along which the point function  $f = f(P)$  is constant, we have  $df = 0$ . This furnishes the following important geometrical property of  $\mathbf{grad} f$ :

The gradient vector  $\mathbf{grad} f = v(f_x, f_y)$  of a scalar field  $f$  is perpendicular at each point  $P: (x, y)$  to the level curve of  $f$  which passes through  $P$ .

If  $\mathbf{u}$  is any unit vector the number  $(\mathbf{grad} f | \mathbf{u})$  is termed the directional derivative of  $f$  at  $P$  in the direction of  $\mathbf{u}$ . If  $\theta$  is the angle between  $\mathbf{u}$  and  $\mathbf{grad} f$  it follows that

The directional derivative of  $f$  at  $P$  in the direction of  $\mathbf{u} = |\mathbf{grad} f| \cos \theta$ .

In particular, when  $\theta = 0$ , we have the following useful fact: The magnitude of  $\mathbf{grad} f$  is the directional derivative of  $f$  at  $P$  in the direction of  $\mathbf{grad} f$ , i.e., in the direction of that one of the two directions perpendicular

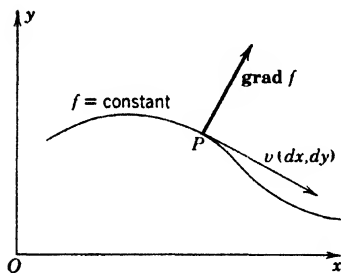


FIG. 9.

ular to the level curve of  $f$  which passes through  $P$  for which the directional derivative of  $f$  is positive (for the unit vector which has

the direction of  $\mathbf{grad} f$  is  $\frac{\mathbf{grad} f}{|\mathbf{grad} f|}$ . Since the magnitude of  $\mathbf{grad} f \geq 0$

it follows that the direction of  $\mathbf{grad} f$  (when  $\mathbf{grad} f$  is not the zero vector) is that direction normal to the level curve of  $f$  in which the directional derivative of  $f$  is positive).

*Note.* If  $s$  is the arc length along a curve through  $P$  which has  $\mathbf{u}$  as tangent vector at  $P$  (the direction of  $\mathbf{u}$  being that in which  $s$  increases) the function induced by  $f = f(P)$  on the curve may be written as a function of  $s$ , and the derivative of this function with respect to  $s$  is  $f_x x_s + f_y y_s = (\mathbf{grad} f | \mathbf{u})$ . For this reason the directional derivative of  $f$  in the direction of  $\mathbf{u}$  is often written in the form  $\frac{df}{ds}$ .

The directional derivative of  $f$  in a direction normal to a given curve through  $P$  is, similarly, often written in the form  $\frac{df}{dn}$ . Thus if  $\mathbf{n}$  is a unit vector normal to the curve

$$\frac{df}{dn} = (\mathbf{grad} f | \mathbf{n}).$$

**EXERCISES**

1. Determine  $\mathbf{grad} f$ , where  $f = (x^2 + y^2)^{1/2}$ . What are the level curves of this scalar field?
2. Determine the directional derivative of the scalar field of Exercise 1 in the direction of the  $x$ -axis and in the direction of the negative  $y$ -axis.
3. In what direction is the directional derivative of the scalar field of Exercise 1 greatest? least? zero?
4. Where does  $\mathbf{grad} (x^2 + y^2)^{1/2}$  fail to possess direction? Is there any level curve of the scalar field  $f = (x^2 + y^2)^{1/2}$  through such a point?
5. Repeat Exercise 1 for  $f = x$ . *Note.* Since  $f$  is a scalar field,  $f'$  is not  $x'$  but, rather,  $l_1x' + l_2y'$ . It is the *value*, not the *form*, of the point-function  $f = f(P)$  that is preserved under transformation from one reference frame to another.
6. If  $\mathbf{r} = v(O \rightarrow P)$  show that  $\frac{dr}{dn} = \cos \theta$ , where  $\theta$  is the angle between  $\mathbf{r}$  and  $\mathbf{n}$ .

The fact that  $\nabla = v\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$  is a symbolic vector assures us that if  $\mathbf{v} = v(f, g)$  is a differentiable vector field the expression

$$(\nabla|\mathbf{v}) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = f_x + g_y$$

is a scalar. (Prove this. *Hint.*  $f' = l_1f' + m_1g$ ,  $\frac{\partial}{\partial x'} = l_1\frac{\partial}{\partial x} + m_1\frac{\partial}{\partial y}$ . Hence  $f'_{x'} = l_1^2f_x + l_1m_1(g_x + f_y) + m_1^2g_y$ . Similarly  $g'_{y'} = l_2^2f_x +$  etc.) We term this scalar the *divergence* of the vector field  $\mathbf{v}$ , and we denote it by the symbol  $\text{div } \mathbf{v}$ :

$$\text{div } \mathbf{v} = (\nabla|\mathbf{v}) = f_x + g_y.$$

$\text{div } \mathbf{v}$  is a scalar field. A second such scalar field may be associated with the given (differentiable) vector field  $\mathbf{v} = v(f, g)$  as follows.  $\mathbf{v}^* = v(-g, f)$  is a vector field whose divergence is  $-g_x + f_y$ . We term the negative of  $\text{div } \mathbf{v}^*$  the *curl* of the vector field  $\mathbf{v}$ , and we denote it by the symbol  $\text{curl } \mathbf{v}$ :

$$\text{curl } \mathbf{v} = -\text{div } \mathbf{v}^* = g_x - f_y.$$

We may summarize these results as follows:

*Associated with any differentiable vector field  $\mathbf{v} = v(f, g)$  are two scalar fields:*

$$\text{div } \mathbf{v} = (\nabla|\mathbf{v}) = f_x + g_y;$$

$$\text{curl } \mathbf{v} = -\text{div } \mathbf{v}^* = -( \nabla|\mathbf{v}^*) = g_x - f_y.$$

Assuming that the partial derivatives of  $f$  and  $g$  are continuous, it is

easy to express the double integrals of  $\text{div } \mathbf{v}$  and of  $\text{curl } \mathbf{v}$  over a domain in the  $(x, y)$ -plane in terms of line integrals around the boundary of the domain. We suppose that the boundary of the domain  $D$  is a piece-

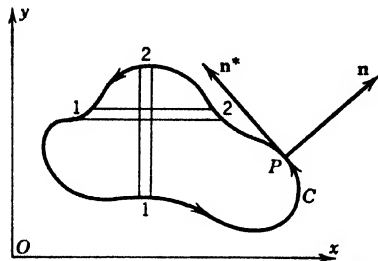


FIG. 10.

wise-smooth closed curve  $C$  (or that it is composed of several such). Applying the method of repeated integration we obtain

$$\begin{aligned}\int_D f_x dx dy &= \int_{+C} f dy \\ \int_D g_y dx dy &= \int_{-C} g dx \\ &= - \int_{+C} g dx,\end{aligned}$$

where the  $+C$  attached to the integral sign indicates that the integration around  $C$  is in the *positive sense* (i.e., that  $C$  is traced so that  $D$  lies to the left). Denoting by  $\mathbf{n}$  the unit normal to  $C$  drawn *away* from  $D$

we have, since the positively drawn tangent to  $C$  is  $\frac{\pi}{2}$  ahead of  $\mathbf{n}$ ,

$$v(dx, dy) = ds \mathbf{n}^*.$$

Hence  $ds \mathbf{n} = v(dy, -dx)$  so that

$$\begin{aligned}\int_D \text{div } \mathbf{v} dx dy &= \int_D (f_x + g_y) dx dy = \int_{+C} (f dy - g dx) \\ &= \int_{+C} (\mathbf{v} | \mathbf{n}) ds\end{aligned}$$

In words:

*The double integral of  $\text{div } \mathbf{v}$  over  $D$  is equal to the integral around the boundary  $C$  of  $D$ , in the positive sense, of  $(\mathbf{v} | \mathbf{n})$ , where  $\mathbf{n}$  is the outward normal to  $C$ , i.e., the normal which is drawn away from  $D$ .*

Since  $\text{curl } \mathbf{v} = -\text{div } \mathbf{v}^*$  we have

$$\int_D \text{curl } \mathbf{v} dx dy = - \int_{+C} (\mathbf{v}^* | \mathbf{n}) ds = \int_{+C} (\mathbf{v} | \mathbf{n}^*) ds = \int_{+C} (f dx + g dy)$$

since  $(\mathbf{v}^* | \mathbf{n}) = (\mathbf{v}^{**} | \mathbf{n}^*) = -(\mathbf{v} | \mathbf{n}^*)$ . In words:

**The double integral of  $\text{curl } \mathbf{v}$  over  $D$  is equal to the integral around the boundary  $C$  of  $D$ , in the positive sense, of  $(\mathbf{v} | \mathbf{n}^*) = (\mathbf{v} | \mathbf{t})$ , where  $\mathbf{t}$  is the unit tangent vector to  $C$  drawn in the positive sense. In other words the double integral of  $\text{curl } \mathbf{v}$  over  $D$  is the line integral  $\int_{+C} (\mathbf{v} | ds)$ ,**

where  $ds = ds \mathbf{t}$  is the vector element of arc. This important result is known as Stokes's theorem in the plane (after G. G. Stokes [1819-1903], an Irish mathematician).

EXERCISES

7. Show that the area of  $D = \frac{1}{2} \int_{+C} (x dy - y dx) = \frac{1}{2} \int_{+C} (\mathbf{r}|\mathbf{n}) ds$ . *Hint.*

Set  $\mathbf{v} = v(x, y) = \mathbf{r}$ .

8. Show that, if  $\mathbf{v}$  is the gradient of a scalar field  $f$ ,  $\text{div } \mathbf{v} = f_{xx} + f_{yy}$ . *Note.* We term the scalar field  $\text{div grad } f$  the *Laplacian* of  $f$  (after P. S. Laplace [1749-1827], a French mathematician), and we denote it by the symbol  $\Delta_2 f$ . Thus

$$\Delta_2 f = f_{xx} + f_{yy}.$$

9. Show that  $\int_D \Delta_2 f dx dy = \int_{+C} (\text{grad } f|\mathbf{n}) ds = \int_{+C} \frac{df}{dn} ds$ .

10. Show that  $\text{curl grad } f = 0$ , it being understood that  $f_{xy}$  is continuous.

11. Show that  $\int_D \{g \Delta_2 f + (\text{grad } g|\text{grad } f)\} dx dy = \int_{+C} g \frac{df}{dn} ds$ , where  $f$  and  $g$

are any two scalar fields which are such that  $f$  possesses continuous second derivatives and  $g$  a continuous gradient vector field. *Note.* A vector field is said to be continuous when each coordinate of the variable vector is continuous. *Hint.*

Set  $\mathbf{v} = g \text{ grad } f$  in the relation  $\int_D \text{div } \mathbf{v} dx dy = \int_{+C} (\mathbf{v}|\mathbf{n}) ds$ .

12. Show that if  $f$  and  $g$  are any two scalar fields which possess continuous second derivatives then

$$\int_D (g \Delta_2 f - f \Delta_2 g) dx dy = \int_{+C} \left( g \frac{df}{dn} - f \frac{dg}{dn} \right) ds.$$

*Note.* This important relation is known as Green's Lemma (after G. Green [1793-1841], an English mathematician). *Memorize it.*

13. Show that if  $\text{curl } \mathbf{v} = 0$  over a domain  $D$  the integral  $\int_C (\mathbf{v}|\mathbf{ds})$ , where  $C$  is any closed curve which, together with its interior, is covered by  $D$ , is zero.

14. Deduce from the result of Exercise 13 that, if  $\text{curl } \mathbf{v} = 0$  over a simply connected domain  $D$ ,  $\mathbf{v}$  is the gradient of a scalar field over  $D$ . *Note.* A domain  $D$  is said to be *simply connected* if the interior of any simple closed curve in the domain is covered by the domain (a simple closed curve being one which does not cross

itself). *Hint.* Set  $\int_{P_0}^P (\mathbf{v}|\mathbf{ds}) = f$ , and show that  $f$  is a point-function  $f = f(P)$ ,

i.e., that the value of  $f$  does not depend on the curve of integration from  $P_0$  to  $P$ . Then show that  $\text{grad } f = \mathbf{v}$ .

The discussion of space vectors is so similar to that of plane vectors

that it will suffice to give, without proofs, the main results. If  $f = f(P) = f(x, y, z)$  is a point-function which maintains its value (at each point  $P$ ) when we change from the reference frame  $Oxyz$  to any other reference frame  $O'x'y'z'$  we term  $f$  a *scalar field*. Thus, if  $f$  is a scalar field,

$$f = f'.$$

It follows that  $df = f_x \Delta x + f_y \Delta y + f_z \Delta z$  is a scalar ( $f$  being granted to be differentiable at  $P$ ). Hence  $v(f_x, f_y, f_z)$  is a vector (why?). We term this vector the *gradient* of the scalar field, and we denote it by the symbol **grad**  $f$ . The *level surfaces* of the scalar field  $f$  are those surfaces on each of which  $f$  has a constant value. At any point  $P$

$$df = (\mathbf{grad} f | d\mathbf{P}) = 0,$$

where  $d\mathbf{P} = v(dx, dy, dz)$  is an arbitrary vector in the tangent plane at  $P$  to the level surface of  $f$  which passes through  $P$ . Hence

**grad**  $f$  at  $\mathbf{P}$  is perpendicular to the level surface of  $f$  which passes through  $\mathbf{P}$ .

If  $\mathbf{u}$  is any unit vector (**grad**  $f | \mathbf{u}$ ) is the *directional derivative* of  $f$  in the direction of  $\mathbf{u}$ . If  $\mathbf{u}$  is normal to a given surface this directional derivative of  $f$  is frequently denoted by the symbol  $\frac{df}{dn}$ . Thus

$$\frac{df}{dn} = (\mathbf{grad} f | \mathbf{n}).$$

It follows that  $|\mathbf{grad} f|$  is the directional derivative of  $f$  in that one of the two directions at  $P$ , normal to the level surface of  $f$  through  $P$ , in which the directional derivative of  $f$  is positive (this direction being that of **grad**  $f$ ).

The vector differential operator

$$\nabla = v \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

enables us to associate with every differentiable vector field

$$\mathbf{v} = v(f, g, h)$$

a scalar field (known as the *divergence* of the vector field)

$$\operatorname{div} \mathbf{v} = (\nabla | \mathbf{v}) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = f_x + g_y + h_z,$$

and a vector field (known as the *curl* of the vector field)

$$\text{curl } \mathbf{v} = (\nabla \times \mathbf{v}) = v(h_y - g_z, f_z - h_x, g_x - f_y).$$

(Prove that this actually is a vector.) Upon applying the principle of repeated integration to the volume integrals  $\int_V f_x dx dy dz$ ,  $\int_V g_y dx dy dz$ ,  $\int_V h_z dx dy dz$  (it being understood that the integrands  $f_x$ ,  $g_y$ ,  $h_z$  are continuous) we obtain

$$\int_V f_x dx dy dz = \int_S f d(y, z);$$

$$\int_V g_y dx dy dz = \int_S g d(z, x);$$

$$\int_V h_z dx dy dz = \int_S h d(x, y),$$

where  $S$  is the boundary of the volume  $V$  of integration and  $v(d(y, z)$ ,  $d(z, x)$ ,  $d(x, y)$ ) is the *vector element of area*  $d\mathbf{S}$  of  $S$ ,  $S$  being so oriented,

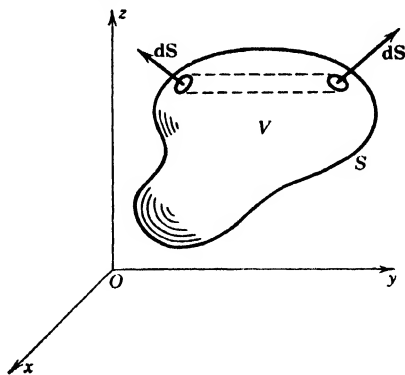


FIG. 11.

by the choice of the two parameters on it, that  $d\mathbf{S}$  has the direction of the normal to  $S$  which is drawn *away* from  $V$ . If  $\alpha$  and  $\beta$  are the two parameters on  $S$ ,  $d\mathbf{S}$  is the vector or cross product

$$d\mathbf{S} = (d_\alpha \mathbf{s} \times d_\beta \mathbf{s}),$$

where  $d_\alpha \mathbf{s} = d\alpha v(x_\alpha, y_\alpha, z_\alpha)$ ,  $d_\beta \mathbf{s} = d\beta v(x_\beta, y_\beta, z_\beta)$ . Upon adding together the three equations just written we obtain

$$\int_V \text{div } \mathbf{v} dx dy dz = \int_S (\mathbf{v} \cdot d\mathbf{S}) = \int_S (\mathbf{v} \cdot \mathbf{n}) dS,$$

where  $\mathbf{n}$  is the unit normal to  $S$  drawn away from  $V$ , and  $dS$  is the scalar element of area, i.e., the magnitude of  $d\mathbf{S}$ .

If  $x = x(\alpha, \beta)$ ,  $y = y(\alpha, \beta)$ ,  $z = z(\alpha, \beta)$  is any surface whose boundary is the closed curve  $C$  obtained by setting

$$\alpha = \alpha(t), \quad \beta = \beta(t)$$

the line integral  $\int_{+C} (\mathbf{v}|d\mathbf{s}) = \int_{+C} (f dx + g dy + h dz)$  (where  $\mathbf{v} = v(f, g, h)$ ) may be written in the form  $\int_{+\Gamma} (A d\alpha + B d\beta)$ , where  $\Gamma$

is the closed curve in the  $(\alpha, \beta)$ -plane that corresponds to  $C$ , and

$$\begin{aligned} A &= fx_\alpha + gy_\alpha + hz_\alpha; \\ B &= fx_\beta + gy_\beta + hz_\beta \end{aligned}$$

(the positive direction on  $C$  being determined by the positive direction on  $\Gamma$ ). Since  $f_\alpha = f_x x_\alpha + f_y y_\alpha + f_z z_\alpha$ , and so on, an easy calculation (perform it) shows that

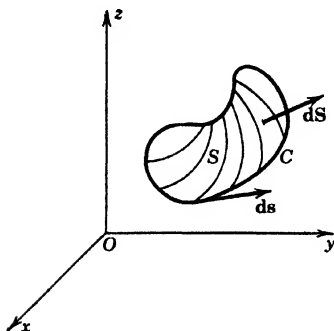


FIG. 12.

$$(B_\alpha - A_\beta) d\alpha d\beta = (\text{curl } \mathbf{v}|d\mathbf{S}).$$

It follows, then, on applying Stokes's theorem in the plane that

$$\int_{+C} (\mathbf{v}|d\mathbf{s}) = \int_S (\text{curl } \mathbf{v}|d\mathbf{S}).$$

This important result is known as *Stokes's theorem in space*.

### EXERCISES

15. Show that the divergence of the vector field  $\text{grad } f$  is the scalar field  $f_{xx} + f_{yy} + f_{zz}$ . *Note.* This scalar field is known as the *Laplacian* of the scalar field  $f$  and is denoted by the symbol  $\Delta_2 f$ :

$$\Delta_2 f = \text{div grad } f = f_{xx} + f_{yy} + f_{zz}.$$

16. Show that the Laplacian of any linear function of  $x$ ,  $y$ , and  $z$  is zero and that the Laplacian of the quadratic function  $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$  is zero if, and only if,  $a + b + c = 0$ .

17. Show that  $\text{div curl } \mathbf{v} = 0$  and that  $\text{curl grad } f = 0$ .

18. Show that  $\int_V \Delta_2 f dx dy dz = \int_S (\text{grad } f|d\mathbf{S}) = \int_S \frac{df}{dn} dS$ .

19. Show that  $\int_V \{g \Delta_2 f + (\text{grad } g| \text{grad } f)\} dx dy dz = \int_S g \frac{df}{dn} dS$ .

20. Show that 
$$\int_V (g \Delta_1 f - f \Delta_1 g) dx dy dz = \int_S \left( g \frac{df}{dn} - f \frac{dg}{dn} \right) dS.$$

21. Show that if  $\text{curl } \mathbf{v} = \mathbf{0}$  over a surface  $S$  the integral  $\int_C (\mathbf{v} | d\mathbf{s})$ , where  $C$  is the closed boundary of  $S$ , is zero.

22. Show that if  $\text{curl } \mathbf{v} = \mathbf{0}$  over a domain  $V$  then  $\mathbf{v}$  is the gradient of a scalar field over  $V$ . *Hint.* See Exercise 14, p. 27.

## 9. Vectors and matrices in $n$ -dimensional complex space

There are two directions in which it is important to extend the concept of plane and space vectors:

1. We pass from the *real* to the *complex* field; in other words we permit the coordinates of the vectors involved to be complex numbers.
2. We pass to spaces of higher dimensions; in other words we permit the number of coordinates of a vector to be more than three.

In each of these extensions our geometrical intuition fails us. In the very special case of one-dimensional complex vectors (where each vector is simply one complex number  $z = x + yi$ ) we could represent the vector by the line segment  $O \rightarrow P$ , where  $P: (x, y)$ . In other words we could regard our one-dimensional complex vectors as two-dimensional real vectors. For the (still very special) case of two-dimensional complex vectors we would require, for this method of *realization* of the vectors, a four-dimensional space so that we could no longer count on geometrical intuition as a guide. The best procedure is to lean on the algebra which formalized, in the cases already treated of plane and space vectors, our geometrical intuition. Once we decide to do this it becomes clear that the extension to the complex field (from the real field) is a *simplification* rather than a *complication*, for we can use the fundamental theorem of algebra which says that

*Every algebraic equation in the complex field has a solution.*

This theorem is, as you well know, not valid in the real field; for example there is no real number  $x$  for which  $x^2 + 1 = 0$ . The extension to the case of  $n$  dimensions (where  $n > 3$ ) is quite formal and does not present any *essential* complications (you have already seen that the theory of space vectors was, in most respects, the same as the theory of plane vectors). The main point of difference is that the number 2 no longer plays the very special role it occupies in the case  $n = 3$ , where  $2 = 1 + 1$  is also  $3 - 1$ . For example, we shall meet, in the theory of  $n$ -dimensional vectors, a vector or cross product of  $(n - 1)$  vectors rather than of two vectors.



Since we no longer have our geometrical intuition to guide us we cannot start, as we did when discussing real plane and space vectors, with coordinate reference frames and the table of "direction cosines" which describes the relative orientation of any two of these. We start, rather, with the concept of the squared magnitude of a vector. In the cases already treated this was the sum of the squares of the coordinates of the vector. The *essential* quality of this concept of magnitude is the following:

*The only vector whose magnitude is zero is the zero vector.*

When we pass from the real to the complex field this quality would be lost if we defined the squared magnitude to be the sum of the squares of the coordinates of the vector (for there is no essential difference in the complex field between a *sum* of two squares and a *difference* of two squares). For example if, in the case of two-dimensional complex vectors, we defined the magnitude  $v$  of the vector  $v(z, w)$  to be  $(z^2 + w^2)^{1/2}$  this magnitude would be zero for all vectors for which  $w = iz$ . What we have to do (in order to preserve the feature which we regard as essential, namely, that the only vector whose magnitude is zero is the zero vector, i.e., the vector *all* of whose coordinates are zero) is to use the "complex square," i.e., the squared modulus, of each of the coordinates rather than the "actual square" of each of the coordinates. Denoting the conjugate of any complex number by a superposed bar so that, if  $z = x + yi$ ,  $\bar{z} = x - yi$ , we define the magnitude of the two-dimensional complex vector  $v(z, w)$  by the formula

$$v = (\bar{z}z + \bar{w}w)^{1/2}.$$

It follows (prove this) that  $v = 0$  when, and only when,  $\mathbf{v}$  is the zero vector  $v(0, 0)$ .

We shall denote each of the  $n$  coordinates of an  $n$ -dimensional complex vector by the letter  $z$ , and we shall distinguish these coordinates from one another by a superscript. Thus we denote the  $j$ th coordinate by  $z^j$ ,  $j = 1, 2, \dots, n$ . (*Warning.* Be careful not to confuse  $z^2$  with the square of  $z$ ; if we have to write the square of  $z$  we shall write  $zz$ . In actual fact we shall need merely the complex square of  $z$  which we write as  $\bar{z}z$ .) Thus a complex  $n$ -dimensional vector  $\mathbf{v} = v(z^1, \dots, z^n)$  is an ordered set of  $n$  complex numbers  $z^1, \dots, z^n$  with which there is associated a non-negative number  $v$ , termed the magnitude of the vector, which is defined by the formula

$$v^2 = \bar{z}_1 z_1 + \dots + \bar{z}_n z_n.$$

What corresponds to a change of reference frame? This is merely a

homogeneous linear transformation

$$(z^1)' = u_1^1 z^1 + \cdots + u_n^1 z^n$$

$$(z^2)' = u_1^2 z^1 + \cdots + u_n^2 z^n$$

. . . . .

$$(z^n)' = u_1^n z^1 + \cdots + u_n^n z^n$$

which is such that  $v^2$  is a *scalar*, i.e., such that

$$\overline{(z^1)'}(z^1)' + \cdots + \overline{(z^n)'}(z^n)' = \overline{z^1}z^1 + \cdots + \overline{z^n}z^n.$$

It is essential to introduce a *condensed* notation. To do this we agree to denote summation symbols by Greek letters. A roman superscript will denote any one of the numbers 1 to  $n$ . Thus the homogeneous linear transformation above appears in the compact form

$$(z^j)' = u_\alpha^j z^\alpha.$$

In reading this be sure that you understand that this is any one of  $n$  equations of which the fifth, for example, is

$$(z^5)' = u_1^5 z^1 + \cdots + u_n^5 z^n.$$

The  $n^2$  numbers  $u_j^k$  are complex numbers; regarded as a single entity they constitute an  $n \times n$  matrix  $U$  of which the element in the  $j$ th row and  $k$ th column is  $u_k^j$ . (Note carefully that the superscript tells the row and the subscript the column.) If we regard the  $n$  complex numbers  $z^j$  as constituting an  $n \times 1$  matrix  $z$  (i.e., a matrix of  $n$  rows and one column) we may write the homogeneous linear transformation (which *defines* a transformation from one reference frame to another) in the compact form

$$z' = Uz.$$

Here the matrix product  $Uz$  follows the general rule of matrix multiplication:

**If  $A$  is a  $p \times q$  matrix (i.e., a matrix of  $p$  rows and  $q$  columns) and  $B$  a  $q \times r$  matrix then  $AB$  is a  $p \times r$  matrix  $C$ , where  $c_k^j = a_\alpha^j b_k^\alpha$  is the "product" of the  $j$ th row of  $A$  by the  $k$ th column of  $B$ . Note carefully that the product  $AB$  cannot be formed unless  $B$  has the same number of rows as  $A$  has columns.**

If  $A$  is a  $p \times q$  matrix we denote by  $A^*$  the  $q \times p$  matrix obtained by *transposing*  $A$  (i.e., interchanging its rows and columns) and replacing each element by its complex conjugate. Thus  $(a^*)_j^k = \bar{a}_k^j$ . In particular  $z^*$  is a  $1 \times n$  matrix (i.e., a matrix of one row and  $n$  columns)

whose coordinates are  $(\bar{z}^1, \dots, \bar{z}^n)$ . The squared magnitude of the vector  $v(z^1, \dots, z^n)$  is the  $1 \times 1$  matrix (= complex number)  $z^*z$ . In this notation, then, we define an  $n$ -dimensional complex vector as follows:

An  $n$ -dimensional complex vector  $\mathbf{v} = v(z^1, \dots, z^n)$  is the collection of all  $n \times 1$  complex matrices  $z$  which are connected with one another by the formula

$$z' = Uz,$$

where the  $n \times n$  complex matrix  $U$  is such that  $(z')^*z' = z^*z$ . The scalar  $z^*z$  is the squared magnitude  $v^2$  of  $\mathbf{v}$ .

### EXERCISES

1. Show that matrix addition is commutative and associative. (What does this mean?) *Note.* Matrix addition is defined only for matrices of the same type, i.e., of the same number of rows and columns. If  $A$  and  $B$  are two  $p \times q$  matrices,  $A + B = C$  is defined by the formula  $c_k^j = a_k^j + b_k^j$ .

2. Show that  $A(B + C) = AB + AC$ .

3. Show that when both  $AB$  and  $BA$  are defined  $A$  and  $B$  are square (i.e., possess the same number of rows and columns) and of the same dimension (the *dimension* of a square matrix being the number of its rows or columns).

4. Show that  $AB$  and  $BA$  are not in general the same.

5. Show that if  $A$  is an unrestricted variable square matrix of dimension  $n$  the one and only fixed matrix  $B$  for which  $AB = A$  is the matrix whose *diagonal* elements  $b_p^p$  are 1, all other elements being zero. Show that when this is the case  $BA = A$ . *Note.* In view of this result we term the  $n \times n$  matrix whose diagonal elements are 1 and whose non-diagonal elements are zero the  $n$ -dimensional *unit matrix*. We denote it by  $E_n$ .

6. Show that if  $AB$  exists so also does  $B^*A^*$  and  $(AB)^* = B^*A^*$ . *Note.* Remember this important result; note that the existence of  $AB$  does not in general guarantee the existence of  $A^*B^*$ .

7. Show that matrix multiplication is associative, i.e., that  $(AB)C = A(BC)$ . *Note.* The common value of these two products is denoted by  $ABC$ .

8. Denoting by  $e_k$  the  $n \times 1$  matrix for which the number in the  $k$ th row is 1, the numbers in all the other rows being zero, show that  $Ae_k$  is the  $n \times 1$  matrix furnished by the  $k$ th column of  $A$  ( $A$  being any  $n \times n$  matrix).

9. In the notation of Exercise 8 show that  $e_j^*A$  is the  $1 \times n$  matrix furnished by the  $j$ th row of  $A$ .

10. Show that  $e_j^*e_k$  is zero unless  $j = k$ , which case it is 1.

11. Show that  $e_j^*Ae_k = a_k^j$ .

12. Show that  $(B + C)^*A(B + C) = B^*AB + B^*AC + C^*AB + C^*AC$ .

The scalar nature of the squared magnitude  $v^2 = z^*z$  of the vector  $\mathbf{v} = v(z^1, \dots, z^n)$  leads naturally to the concept of the scalar product of a vector  $\mathbf{v}_2$  by a vector  $\mathbf{v}_1$ . If  $\mathbf{v}_1 = v(z_1^1, \dots, z_1^n)$ ,  $\mathbf{v}_2 = v(z_2^1, \dots, z_2^n)$  are any two vectors so also is  $\mathbf{v}_1 + \mathbf{v}_2 = v(z_1^1 + z_2^1, \dots,$

$z_1^n + z_2^n$  (prove this), and the squared magnitude of  $\mathbf{v}_1 + \mathbf{v}_2$  is

$$\begin{aligned}(z_1 + z_2)^*(z_1 + z_2) &= z_1^*z_1 + z_1^*z_2 + z_2^*z_1 + z_2^*z_2 \\ &= v_1^2 + z_1^*z_2 + z_2^*z_1 + v_2^2.\end{aligned}$$

Hence  $z_1^*z_2 + z_2^*z_1$  is a scalar (why?). If  $\mathbf{v} = v(z^1, \dots, z^n)$  is any vector then  $c\mathbf{v} = v(cz^1, \dots, cz^n)$  is a vector,  $c$  being any scalar. (Prove this.) In particular  $i\mathbf{v}$  is a vector if  $\mathbf{v}$  is and  $(iz)^* = -iz^*$  since  $\bar{i} = -i$ . Replacing, then,  $\mathbf{v}_2$  by  $i\mathbf{v}_2$  we see that  $i(z_1^*z_2 - z_2^*z_1)$  is a scalar, and it follows that  $z_1^*z_2 - z_2^*z_1$  is a scalar (why?). Hence  $z_1^*z_2 = \frac{1}{2}\{(z_1^*z_2 + z_2^*z_1) + (z_1^*z_2 - z_2^*z_1)\}$  is a scalar. We have, then, the following result:

**If  $\mathbf{v}_1 = v(z_1^1, \dots, z_1^n)$ ,  $\mathbf{v}_2 = v(z_2^1, \dots, z_2^n)$  are any two  $n$ -dimensional complex vectors,  $z_1^*z_2$  is a scalar.**

We term this scalar the *scalar product* of  $\mathbf{v}_2$  by  $\mathbf{v}_1$ , and we denote it by the symbol  $(\mathbf{v}_1|\mathbf{v}_2)$ .  $(\mathbf{v}_1|\mathbf{v}_2)$  is a  $1 \times 1$  matrix, i.e., a complex number; hence  $(\mathbf{v}_1|\mathbf{v}_2)^*$  is the conjugate complex number to  $(\mathbf{v}_1|\mathbf{v}_2)$  (why?). But  $(\mathbf{v}_1|\mathbf{v}_2)^* = (z_1^*z_2)^* = z_2^*z_1^{**} = z_2^*z_1 = (\mathbf{v}_2|\mathbf{v}_1)$ . In words:

**An interchange of the two vectors in a scalar product changes the scalar product into its conjugate complex.**

Thus scalar multiplication of complex vectors is not, in general, a commutative operation; it is commutative when, and only when, the scalar product is real. In particular when  $(\mathbf{v}_1|\mathbf{v}_2)$  is zero so also is  $(\mathbf{v}_2|\mathbf{v}_1)$ .

When  $(\mathbf{v}_1|\mathbf{v}_2) = 0$  we say that  $\mathbf{v}_2$  is *perpendicular* to  $\mathbf{v}_1$ ; if, then,  $\mathbf{v}_2$  is perpendicular to  $\mathbf{v}_1$ ,  $\mathbf{v}_1$  is perpendicular to  $\mathbf{v}_2$ , and we say, simply, that the two vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are perpendicular or *orthogonal*.

## EXERCISES

13. Show that  $((\mathbf{v}_1 + \mathbf{v}_2)|\mathbf{v}_3) = (\mathbf{v}_1|\mathbf{v}_3) + (\mathbf{v}_2|\mathbf{v}_3)$ .

14. Show that  $(c\mathbf{v}_1|\mathbf{v}_2) = \bar{c}(\mathbf{v}_1|\mathbf{v}_2)$ ;  $(\mathbf{v}_1|c\mathbf{v}_2) = c(\mathbf{v}_1|\mathbf{v}_2)$ , where  $c$  is any scalar. Note that when the scalar  $c$  is factored out from the first of the two vectors in the symbol  $(\mathbf{v}_1|\mathbf{v}_2)$  it is its *conjugate*  $\bar{c}$  that must be prefixed to the scalar product.

It is easy to determine the conditions imposed on the  $n \times n$  matrix  $U$  by the fact that  $(\mathbf{v}_1|\mathbf{v}_2)$  is a scalar. Since  $z_1' = Uz_1$ ,  $(z_1')^* = z_1^*U^*$  and so the relation  $(z_1')^*z_2' = z_1^*z_2$  yields  $z_1^*U^*Uz_2 = z_1^*z_2$ . On setting  $z_1 = e_j$ ,  $z_2 = e_k$  we find (see Exercises 10 and 11) that the number in the  $j$ th row and  $k$ th column of  $U^*U$  is zero unless  $j = k$  in which case it is unity. In other words

$U^*U$  is the  $n \times n$  unit matrix  $E_n$ .

It is clear, conversely, that if  $U^*U = E_n$  then  $z_1^*z_2$  is a scalar. We term any  $n \times n$  matrix which is such that  $U^*U = E_n$  a *unitary matrix*.

## EXERCISES

15. Show that  $\det U$  is a complex number of unit modulus. *Hint.*  $\det U^* = \overline{\det U}$ , and  $\det(U^*U) = \det U^* \det U$ .

16. Show that if  $z$  is any  $n \times 1$  matrix there is an unambiguously determinate  $n \times 1$  matrix  $w$  such that  $z = Uw$ . *Hint.*  $\det U \neq 0$ .

17. Show that if  $U^*z$  is the zero  $n \times 1$  matrix so also is  $z$ .

18. Show that  $UU^* = E_n$ . *Hint.* From  $U^*U = E_n$  we obtain  $U^*UU^* = U^*$  so that  $U^*(UU^* - E_n)$  is the zero  $n \times n$  matrix. Hence each column of  $UU^* - E_n$  is the zero  $n \times 1$  matrix. Hence  $UU^* = E_n$ . *Note.* The result of this exercise may be phrased as follows:

*If  $U$  is unitary so also is  $U^*$ .*

Let  $\mathbf{v}_1$  and  $\mathbf{v}_2$  be any two  $n$ -dimensional complex vectors, and let  $c$  be any real scalar. Then  $((\mathbf{v}_1 + c\mathbf{v}_2)|(\mathbf{v}_1 + c\mathbf{v}_2))$ , being the squared magnitude of  $\mathbf{v}_1 + c\mathbf{v}_2$ ,  $\geq 0$ . Hence

$$v_1^2 + c\{(\mathbf{v}_1|\mathbf{v}_2) + (\mathbf{v}_2|\mathbf{v}_1)\} + c^2v_2^2 \geq 0.$$

If the quadratic function of  $c$  on the left had two real zeros it would be negative in any neighborhood of each of these zeros, which is impossible. Hence  $\{(\mathbf{v}_1|\mathbf{v}_2) + (\mathbf{v}_2|\mathbf{v}_1)\}^2 \leq 4v_1^2v_2^2$ . If  $\mathbf{v}_2$  is multiplied by  $e^{-i\theta}$ , where  $\theta$  is the argument of  $(\mathbf{v}_1|\mathbf{v}_2)$ , the left-hand side of this inequality becomes  $4|(\mathbf{v}_1|\mathbf{v}_2)|^2$  since  $(\mathbf{v}_1|\mathbf{v}_2)$  becomes  $|(\mathbf{v}_1|\mathbf{v}_2)|$ , and when  $(\mathbf{v}_1|\mathbf{v}_2)$  is real it is equal to  $(\mathbf{v}_2|\mathbf{v}_1)$ . On the other hand the right-hand side of our inequality is unaffected (why?). Hence  $|(\mathbf{v}_1|\mathbf{v}_2)|^2 \leq v_1^2v_2^2$ , or, equivalently,

$$|(\mathbf{v}_1|\mathbf{v}_2)| \leq v_1v_2.$$

This inequality is known as Schwarz's inequality (after H. A. Schwarz [1845-1921], a German mathematician). Expressed in words:

**The modulus of the scalar product of any  $n$ -dimensional complex vector by any other is not greater than the product of the magnitudes of the two vectors.**

When the scalar product of two  $n$ -dimensional complex vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  is real [so that  $(\mathbf{v}_2|\mathbf{v}_1) = (\mathbf{v}_1|\mathbf{v}_2)$ ] we define the *angle  $\theta$  between the two vectors* by the formula

$$\cos \theta = \frac{(\mathbf{v}_1|\mathbf{v}_2)}{v_1v_2}$$

(it being assumed that neither of the vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  is the zero vector). It follows from Schwarz's inequality that the angle  $\theta$  defined in this way is always real. In the case of real  $n$ -dimensional vectors this definition of  $\theta$  enables us to introduce the concept of direction

cosines which played such an important role in the theory of real plane and space vectors.

**EXERCISES**

19. Show that if  $u_j$  is the vector  $v(u_j^1, \dots, u_j^n)$  then  $u_j$  is a unit vector and  $u_j$  is perpendicular to  $u_k$  if  $j \neq k$  (the  $u_j^p$  being the elements of a unitary matrix).

20. Show that if any, or all, of the columns of  $U$  are multiplied by complex numbers of unit modulus the new  $n \times n$  matrix so obtained is unitary.

21. Show that by multiplying one of the columns of  $U$  by a complex number of unit modulus we can obtain a unitary matrix whose determinant is unity. *Hint.*  $\det U$  is a complex number of unit modulus. *Note.* A square matrix whose determinant is 1 is said to be *unimodular*. We shall from now on restrict our linear homogeneous transformations  $z' = Uz$  to those whose matrices  $U$  are unimodular unitary matrices (just as in the case of real plane and space vectors where we restricted ourselves to reference frames for which the determinant of the table of direction cosines was  $+1$ ).

22. Show that each element of a unimodular unitary matrix is equal to the conjugate of its cofactor. *Hint.* The various equations  $\bar{u}_1^1 u_j^1 + \bar{u}_1^2 u_j^2 + \dots + \bar{u}_1^n u_j^n = 0, j \neq 1$ , tell us that  $\bar{u}_1^p = k$  times the cofactor of  $u_1^p$ . On multiplying these equations by  $u_1^1, \dots, u_1^n$ , respectively, and adding we find  $k = 1$ .

23. Show that the general unimodular unitary  $2 \times 2$  matrix is  $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ , where  $\bar{a}a + \bar{b}b = 1$ . What does this reduce to when the matrix is real? *Note.* On setting  $a = x_1 + x_2 i, b = x_3 + x_4 i$  the relation  $\bar{a}a + \bar{b}b = 1$  appears as  $(x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2 = 1$ . Thus

*Every  $2 \times 2$  unimodular unitary matrix is identified by a point on the unit sphere in four-dimensional space.*

This is the analogue of the fact that every complex number of unit modulus is identified by a point on the unit circle in the plane.

**10. The alternating product of  $n$  vectors; the vector or cross product of  $(n - 1)$  vectors**

Let  $\mathbf{v}_j = v(z_j^1, \dots, z_j^n), j = 1, \dots, n$ , be any set of  $n$   $n$ -dimensional complex vectors. The determinant of the  $n \times n$  matrix  $Z$ , whose  $j$ th column consists of the coordinates of  $\mathbf{v}_j$ , is a scalar which we shall denote by the symbol  $(\mathbf{v}_1, \dots, \mathbf{v}_n)$  and term the *alternating product* of the  $n$  vectors (in the indicated order). The reason for the adjective alternating is clear: An interchange of any two of the vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  changes the sign of  $\det Z$ . The proof of the fact that  $\det Z$  is a scalar is simple. Indeed since  $\mathbf{v}_j$  is a vector we have

$$z'_j = Uz_j,$$

where  $z_j$  is the  $n \times 1$  matrix whose elements are the coordinates of  $\mathbf{v}_j$ ; in other words  $z_j$  is the  $j$ th column of  $Z$ . The  $n$  equations obtained by letting  $j$  run over the range 1 to  $n$  may be compactly written as

$$Z' = UZ.$$

Hence  $\det Z' = \det U \det Z$ , and since  $\det U = 1$  it follows that

$$\det Z' = \det Z.$$

In other words  $\det Z$  is a scalar.

If  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  are any  $k$   $n$ -dimensional complex vectors and  $c^1, \dots, c^k$  are any  $k$  complex scalars the vector

$$\mathbf{v} = c^1\mathbf{v}_1 + \dots + c^k\mathbf{v}_k$$

is said to be a linear combination of the vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$ . If not all the scalars ( $c^1, \dots, c^k$ ) are zero the linear combination is said to be *nontrivial*; if all the scalars are zero the linear combination is *trivial*. If there is no non-trivial linear combination of the  $k$  vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  which is the zero vector, the set  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is said to be *linearly independent*; otherwise it is *linearly dependent*.

### EXERCISES

1. Show that  $(\mathbf{v}_1, \dots, \mathbf{v}_n) = \mathbf{0}$  if any two of the  $n$  vectors are the same.
2. Show that  $(\mathbf{v}_1, \dots, \mathbf{v}_n)$  is a linear function of each of the  $n$  vectors, i.e., that  $(\mathbf{v}_1, \dots, c\mathbf{v}_j, \dots, \mathbf{v}_n) = c(\mathbf{v}_1, \dots, \mathbf{v}_j, \dots, \mathbf{v}_n)$ ,  $c$  being any scalar, and that  $(\mathbf{v}_1, \dots, \mathbf{v}_j + \mathbf{w}_j, \dots, \mathbf{v}_n) = (\mathbf{v}_1, \dots, \mathbf{v}_j, \dots, \mathbf{v}_n) + (\mathbf{v}_1, \dots, \mathbf{w}_j, \dots, \mathbf{v}_n)$ .
3. Show that  $(\mathbf{v}_1, \dots, \mathbf{v}_n) = \mathbf{0}$  if any one of the vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  is a linear combination of the remaining vectors.
4. Show that  $(\mathbf{v}_1, \dots, \mathbf{v}_n) = \mathbf{0}$  if the set  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is linearly dependent.  
*Hint.* This is merely another way of phrasing Exercise 3.
5. Show that the set of  $n$  vectors  $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ , where the coordinates of  $\mathbf{u}_j$  are furnished by the  $j$ th column of a unitary matrix, is linearly independent.  
*Hint.*  $(\mathbf{u}_1, \dots, \mathbf{u}_n) = \det U \neq 0$ .
6. Show that if  $\{\mathbf{v}_1, \dots, \mathbf{v}_{n+1}\}$  is a set of  $n + 1$   $n$ -dimensional vectors then  $(\mathbf{v}_2, \dots, \mathbf{v}_{n+1})\mathbf{v}_1 - (\mathbf{v}_1, \mathbf{v}_3, \dots, \mathbf{v}_{n+1})\mathbf{v}_2 + (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4, \dots, \mathbf{v}_{n+1})\mathbf{v}_3 - \dots + (-1)^n(\mathbf{v}_1, \dots, \mathbf{v}_n)\mathbf{v}_{n+1} = \mathbf{0}$ . *Hint.* See Section 4.
7. Show that no set of more than  $n$   $n$ -dimensional complex vectors is linearly independent. *Note.* The results of Exercises 5 and 6 constitute the reason for terming the vectors " $n$ -dimensional."
8. Show that if  $(\mathbf{v}_1, \dots, \mathbf{v}_n) = \mathbf{0}$  the set  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is linearly dependent.  
*Hint.* If  $Z$  is the  $n \times n$  matrix whose  $j$ th column is furnished by the coordinates of  $\mathbf{v}_j$ ,  $\det Z = (\mathbf{v}_1, \dots, \mathbf{v}_n) = \mathbf{0}$ . Hence there exist a set of  $n$  numbers  $(c^1, \dots, c^n)$ , not all zero, such that  $c^\alpha z_{\alpha j} = 0$ . The numbers  $(c^1, \dots, c^n)$  are scalars since the equations  $c^\alpha z_{\alpha j} = 0$  imply that  $c^\alpha (z_{\alpha j})' = 0$ . (Prove this. *Hint.*  $(z_{kj})' = u_j^p z_k^\beta$ .) Hence the scalars  $(c^1, \dots, c^n)$  are such that  $c^\alpha \mathbf{v}_\alpha = \mathbf{0}$ . *Note.* On combining the results of Exercises 4 and 8 we obtain the following *criterion* for linear dependence of a set of  $n$  vectors:  
**The set  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  of  $n$  vectors in  $n$ -dimensional complex space is linearly dependent if, and only if,  $(\mathbf{v}_1, \dots, \mathbf{v}_n) = \mathbf{0}$ .**

9. Show that any set of  $n$  linearly independent vectors is a *basis* (what does this mean?) for  $n$ -dimensional complex vectors.

10. If  $\{\mathbf{v}_1, \dots, \mathbf{v}_4\}$  is a basis for four-dimensional complex vectors, determine the coordinates (what are these?) of an arbitrary vector  $\mathbf{v}$  with respect to this basis.

11. If  $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$  is a basis for  $n$ -dimensional complex vectors which is such that the matrix  $Z$ , whose  $j$ th column is furnished by the coordinates of  $\mathbf{u}_j$ , is unitary and unimodular, show that the coordinates of any vector  $\mathbf{v}$  with respect to the basis are  $c^j = (\mathbf{u}_j | \mathbf{v})$ ,  $j = 1, \dots, n$ . *Hint.* Exercise 22, p. 37.

Let, now,  $\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}\}$  be any set of  $(n-1)$   $n$ -dimensional complex vectors. If  $\mathbf{v} = v(z^1, \dots, z^n)$  is an arbitrary vector  $(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{v})$  is a scalar, and on expanding this  $n$ -rowed determinant in terms of the last column we obtain a homogeneous linear function of the coordinates of  $\mathbf{v}$  which is a scalar. The conjugates of the coefficients of this linear function are, then, the coordinates of a vector. The proof of this important *converse of the scalar product theorem* runs as follows: On denoting the coefficients of the homogeneous linear function by  $(a_1, \dots, a_n)$  we are given that

$$a_\alpha z^\alpha = (a_\alpha)' (z^\alpha)' = (a_\alpha)' u_\beta^\alpha z^\beta,$$

and since  $\mathbf{v} = v(z^1, \dots, z^n)$  is arbitrary this yields (how?)

$$a_k = (a_\alpha)' u_k^\alpha; \quad k = 1, \dots, n.$$

On multiplying these equations by  $\bar{u}_k^j$  and summing on  $k$  we obtain

$$(a_j)' = a_\beta \bar{u}_\beta^j.$$

On taking the conjugates of these equations and denoting  $\bar{a}_j$  by  $y^j$  we obtain

$$(y^j)' = u_\beta^j y^\beta$$

so that  $v(y^1, \dots, y^n)$  is a vector.

We denote the vector which is obtained in this way from the  $(n-1)$  vectors  $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$  by the symbol  $(\mathbf{v}_1 \times \mathbf{v}_2 \times \dots \times \mathbf{v}_{n-1})$ , and we term it the *vector* or *cross product* of the  $(n-1)$  vectors. Its coordinates are the conjugates of the cofactors of  $z^1, \dots, z^n$  in the matrix  $Z$  whose columns are furnished by the coordinates of the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n-1}, \mathbf{v}$ . It follows at once that

$(\mathbf{v}_1 \times \mathbf{v}_2 \times \dots \times \mathbf{v}_{n-1})$  is the zero vector when the set of  $(n-1)$  vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}\}$  is linearly dependent.

In fact when  $\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}\}$  is linearly dependent so also is  $\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{v}\}$ , where  $\mathbf{v}$  is an arbitrary vector (why?). Hence  $(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{v}) = ((\mathbf{v}_1 \times \mathbf{v}_2 \times \dots \times \mathbf{v}_{n-1}) | \mathbf{v}) = 0$ , where  $\mathbf{v}$  is an arbitrary vector. Hence  $(\mathbf{v}_1 \times \mathbf{v}_2 \times \dots \times \mathbf{v}_{n-1}) = \mathbf{0}$  (why?).



On the other hand if  $(\mathbf{v}_1 \times \mathbf{v}_2 \times \cdots \times \mathbf{v}_{n-1})$  is the zero vector the set  $\{\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}\}$  is linearly dependent. For  $(\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}, \mathbf{v}) = 0$ , where  $\mathbf{v}$  is arbitrary; hence there exists a non-trivial linear combination of  $\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}, \mathbf{v}$  which is the zero vector. If  $\{\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}\}$  were a linearly independent set the coefficient of  $\mathbf{v}$  in this non-trivial linear combination could not be zero (why?) so that an arbitrary vector  $\mathbf{v}$  would be a linear combination of the  $n - 1$  vectors  $\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}$ . But this would be absurd since there would then be no linearly independent set of  $n$  vectors (why?). We have, then, the following *criterion* for the linear dependence of sets of  $(n - 1)$  vectors.

**The set  $\{\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}\}$  of  $n - 1$   $n$ -dimensional complex vectors is linearly dependent when, and only when,  $(\mathbf{v}_1 \times \mathbf{v}_2 \times \cdots \times \mathbf{v}_{n-1})$  is the zero vector.**

*Note.* The case  $n = 2$  is somewhat special. Here  $(\mathbf{v}_1 \times \cdots \times \mathbf{v}_{n-1})$  is replaced by  $v(-\bar{z}_1^2, \bar{z}_1^1)$  which we may term the *complement* of  $\mathbf{v}_1$ . On the other hand the complement of  $\mathbf{v}_2$  is  $v(\bar{z}_2^2, -\bar{z}_2^1)$ . In fact to get the complement of  $\mathbf{v}_1$  we take the conjugates of the cofactors

of  $z^1$  and  $z^2$  in the matrix  $\begin{pmatrix} z_1^1 & z^1 \\ z_1^2 & z^2 \end{pmatrix}$  while to get the complement of  $\mathbf{v}_2$

we take the conjugates of the cofactors of  $z^1$  and  $z^2$  in the matrix  $\begin{pmatrix} z^1 & z_2^1 \\ z^2 & z_2^2 \end{pmatrix}$ . In this terminology  $(\mathbf{v}_1 \times \cdots \times \mathbf{v}_{n-1})$  is the *complement* of the (ordered) set of  $n - 1$  vectors  $\{\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}\}$  which we suppose to be linearly independent.

Let, now,  $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$  be any linearly independent set of  $n$   $n$ -dimensional complex vectors. There is associated with this set a second linearly independent set  $\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$  which is *reciprocal* to the first set in the following sense:

$$(\mathbf{w}_j | \mathbf{v}_i) = 1, \quad j = 1, \cdots, n;$$

$$(\mathbf{w}_j | \mathbf{v}_k) = 0, \quad j \neq k.$$

This set is constructed as follows: If  $n = 1$ ,  $\mathbf{w}_1 = \frac{\mathbf{v}_1}{v_1^2}$ . Note that

$$w_1 = \frac{1}{v_1} \text{ so that } \mathbf{v}_1 = \frac{\mathbf{w}_1}{w_1^2}.$$

If  $n = 2$ ,  $(\mathbf{v}_1, \mathbf{v}_2)\mathbf{w}_1 = v(\bar{z}_2^2, -\bar{z}_2^1)$ ,  $(\mathbf{v}_1, \mathbf{v}_2)\mathbf{w}_2 = v(-\bar{z}_1^2, \bar{z}_1^1)$ . It follows that  $(\mathbf{w}_1, \mathbf{w}_2)(\mathbf{v}_1, \mathbf{v}_2) = 1$  and the relation between the two pairs of vectors  $\{\mathbf{v}_1, \mathbf{v}_2\}$  and  $\{\mathbf{w}_1, \mathbf{w}_2\}$  is a reciprocal one.

If  $n > 2$ , let  $j_1, \cdots, j_n$  be an *even* permutation of the numbers  $1, 2, \cdots, n$ , and set  $(\mathbf{v}_1, \cdots, \mathbf{v}_n)\mathbf{w}_i = (\mathbf{v}_{j_i} \times \cdots \times \mathbf{v}_{j_n})$ . It is

clear that  $(\mathbf{w}_j|\mathbf{v}_j) = 1$  and that  $(\mathbf{w}_j|\mathbf{v}_k) = 0$ ,  $j \neq k$ . (Remember that when a scalar is factored out from the first vector in a scalar product the *conjugate* of the scalar must be prefixed to the scalar product.) The last  $n - 1$  of these relations tell us that  $\mathbf{v}_{j_1} = k(\mathbf{w}_{j_2} \times \cdots \times \mathbf{w}_{j_n})$ , and the first tells us that  $k = \frac{1}{(\mathbf{w}_1, \cdots, \mathbf{w}_n)}$ . Thus

$$\overline{(\mathbf{w}_1, \cdots, \mathbf{w}_n)}\mathbf{v}_{j_1} = (\mathbf{w}_{j_2} \times \cdots \times \mathbf{w}_{j_n}).$$

That  $(\mathbf{w}_1, \cdots, \mathbf{w}_n)$  is the reciprocal of  $(\mathbf{v}_1, \cdots, \mathbf{v}_n)$  is a consequence of the relations  $(\mathbf{w}_j|\mathbf{v}_j) = 1$ ,  $(\mathbf{w}_j|\mathbf{v}_k) = 0$ . In fact if  $Z$  and  $Y$  are the matrices whose columns are furnished by the vectors  $\mathbf{v}$  and  $\mathbf{w}$ , respectively, these relations say that  $Y^*Z$  is the unit  $n \times n$  matrix. Hence  $\det Y^* \det Z = 1$  and  $\det Z = (\mathbf{v}_1, \cdots, \mathbf{v}_n)$ ,  $\det Y^* = \overline{(\mathbf{w}_1, \cdots, \mathbf{w}_n)}$ .

### EXERCISES

12. Show that the coordinates of an arbitrary vector  $\mathbf{v}$  with respect to the basis  $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$  are  $c^j = (\mathbf{w}_j|\mathbf{v})$ ,  $j = 1, \cdots, n$ .

13. Show that if the coordinates of  $\mathbf{u}_j$  are the elements of the  $j$ th column of a unimodular unitary  $n \times n$  matrix the set  $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$  is *self-reciprocal*.

### REVIEW EXERCISES

1. Show that if  $\mathbf{v} = \mathbf{v}(t)$  is a vector whose coordinates  $z^1, \cdots, z^n$  are differentiable functions of a scalar parameter  $t$  then  $v(z_t^1, \cdots, z_t^n)$  is a vector ( $z_t^i$  denoting the derivative of  $z^i$  with respect to  $t$ ). *Note.* We term this vector the *derivative of v with respect to t*, and we denote it by the symbol  $\mathbf{v}_t$ .

2. Show that if  $\mathbf{v}_1 = \mathbf{v}_1(t)$ ,  $\mathbf{v}_2 = \mathbf{v}_2(t)$  are two differentiable vectors  $(\mathbf{v}_1|\mathbf{v}_2)_t = ((\mathbf{v}_1)_t|\mathbf{v}_2) + (\mathbf{v}_1|(\mathbf{v}_2)_t)$ .

3. Show that if  $\mathbf{v} = \mathbf{v}(t)$  is a differentiable vector and  $c = c(t)$  a differentiable scalar then  $(c\mathbf{v})_t = c_t\mathbf{v} + c\mathbf{v}_t$ .

4. Show that if  $\mathbf{v}$  is a real vector of constant magnitude ( $\neq 0$ ) then  $\mathbf{v}_t$  is perpendicular to  $\mathbf{v}$ , or else  $\mathbf{v}_t = \mathbf{0}$ .

5. If  $\mathbf{v} = v(f, g, h)$  is a three-dimensional vector field, possessing second derivatives, show that  $\Delta_2\mathbf{v} = v(\Delta_2f, \Delta_2g, \Delta_2h)$  is a vector field, where  $\Delta_2f = f_{xx} + f_{yy} + f_{zz}$ , etc.

6. Using the notation of Exercise 5 show that  $\text{curl curl } \mathbf{v} = \text{grad div } \mathbf{v} - \Delta_2\mathbf{v}$ . *Note.*  $\text{curl curl } \mathbf{v}$  is the vector triple product  $(\nabla \times (\nabla \times \mathbf{v}))$ . The result of this exercise may be written as follows:  $(\nabla \times (\nabla \times \mathbf{v})) = \nabla(\nabla|\mathbf{v}) - (\nabla|\nabla)\mathbf{v}$ .

7. Show that if  $U = U(t)$  is a real unitary matrix which is a differentiable function of a real scalar parameter  $t$  and if  $\mathbf{u}_j(t)$  is the vector whose coordinates are furnished by the  $j$ th column of  $U$  then  $(\mathbf{u}_j|(\mathbf{u}_j)_t) = 0$ ,  $(\mathbf{u}_j|(\mathbf{u}_k)_t) + (\mathbf{u}_k|(\mathbf{u}_j)_t) = 0$ . *Hint.*  $(\mathbf{u}_j|\mathbf{u}_j) = 1$ ;  $(\mathbf{u}_j|\mathbf{u}_k) = 0$ .

8. Prove the following *converse of the vector product theorem*: If  $(a, b, c)$  is a set of three numbers attached to any reference frame  $Oxyz$  which has the property that  $\mathbf{w} = v(bz - cy, cx - az, ay - bx)$  is a vector, where  $\mathbf{r} = v(x, y, z)$  is an arbitrary vector, then  $v(a, b, c)$  is a vector. *Hint.* If  $\mathbf{v}$  is an arbitrary vector  $(\mathbf{v}|\mathbf{w})$  is a scalar;

$(\mathbf{v}|\mathbf{w}) = af + bg + ch$ , where  $(\mathbf{r} \times \mathbf{v}) = v(f, g, h)$ . Since  $\mathbf{r}$  and  $\mathbf{v}$  are arbitrary vectors so also is  $(\mathbf{r} \times \mathbf{v})$  (why?). Hence  $v(a, b, c)$  is a vector.

9. Let  $\mathbf{u}_1 = \mathbf{u}_1(t)$ ,  $\mathbf{u}_2 = \mathbf{u}_2(t)$ ,  $\mathbf{u}_3 = \mathbf{u}_3(t)$  be a variable three-dimensional real unitary basis, and let  $\mathbf{v}$  be a vector whose coordinates with respect to this variable basis are constant. Calculate the coordinates of  $\mathbf{v}_t$  with respect to the variable basis. *Hint.* Let  $\mathbf{v} = f\mathbf{u}_1 + g\mathbf{u}_2 + h\mathbf{u}_3$ ; then  $\mathbf{v}_t = f(\mathbf{u}_1)_t + g(\mathbf{u}_2)_t + h(\mathbf{u}_3)_t$ .  $(\mathbf{v}_t|\mathbf{u}_1) = bh - cg$ ,  $(\mathbf{v}_t|\mathbf{u}_2) = cf - ah$ ,  $(\mathbf{v}_t|\mathbf{u}_3) = ag - bf$ , where  $a = ((\mathbf{u}_2)_t|\mathbf{u}_3) = -(\mathbf{u}_2|(\mathbf{u}_3)_t)$ ,  $b = ((\mathbf{u}_3)_t|\mathbf{u}_1) = -(\mathbf{u}_3|(\mathbf{u}_1)_t)$ ;  $c = ((\mathbf{u}_1)_t|\mathbf{u}_2) = -(\mathbf{u}_1|(\mathbf{u}_2)_t)$  (see Exercise 7). Since  $v(f, g, h)$  is an arbitrary vector it follows that  $v(a, b, c)$  is a vector. This vector is termed the *angular velocity* of the variable reference frame  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ . Denoting it by  $\boldsymbol{\omega}$  we have  $\mathbf{v}_t = (\boldsymbol{\omega} \times \mathbf{v})$ .

10. If the coordinates of  $\mathbf{v}$  with respect to the reference frame  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  of Exercise 9 are not constant calculate the coordinates of  $\mathbf{v}_t$  with respect to the reference frame  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ . *Answer.*  $f_t + bh - cg$ , etc.

*Note.* The vector  $v(f_t, g_t, h_t)$  is called the derivative of  $\mathbf{v}$  with respect to the variable reference frame  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ . Referring to  $\mathbf{v}_t$  as the *absolute* derivative of  $\mathbf{v}$  and to  $v(f_t, g_t, h_t)$  as the *relative* derivative of  $\mathbf{v}$  we have the following important relation:

**The absolute derivative of  $\mathbf{v}$  = The relative derivative of  $\mathbf{v}$  plus  $(\boldsymbol{\omega} \times \mathbf{v})$ .**

11. Calculate  $\mathbf{v}_{tt}$  (see Exercise 10).

*Solution.* On denoting the relative derivative of  $\mathbf{v}$  by  $\frac{\partial \mathbf{v}}{\partial t}$  we have

$$\begin{aligned} \mathbf{v}_t &= \frac{\partial \mathbf{v}}{\partial t} + (\boldsymbol{\omega} \times \mathbf{v}); & \mathbf{v}_{tt} &= \frac{\partial^2 \mathbf{v}}{\partial t^2} + \left( \frac{\partial \boldsymbol{\omega}}{\partial t} \times \mathbf{v} \right) + \left( \boldsymbol{\omega} \times \frac{\partial \mathbf{v}}{\partial t} \right) + (\boldsymbol{\omega} \times \mathbf{v}_t) \\ & & &= \frac{\partial^2 \mathbf{v}}{\partial t^2} + \left( \frac{\partial \boldsymbol{\omega}}{\partial t} \times \mathbf{v} \right) + 2 \left( \boldsymbol{\omega} \times \frac{\partial \mathbf{v}}{\partial t} \right) + (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{v})) \\ & & &= \frac{\partial^2 \mathbf{v}}{\partial t^2} + \left( \frac{\partial \boldsymbol{\omega}}{\partial t} \times \mathbf{v} \right) + 2 \left( \boldsymbol{\omega} \times \frac{\partial \mathbf{v}}{\partial t} \right) + (\boldsymbol{\omega}|\mathbf{v})\boldsymbol{\omega} - \omega^2 \mathbf{v}. \end{aligned}$$

*Note.* When  $\mathbf{v} = v(O \rightarrow P)$  is the position vector of a point  $P$  and  $t$  is the time variable,  $\mathbf{v}_{tt}$  is the (absolute) acceleration of  $P$ , and  $\frac{\partial^2 \mathbf{v}}{\partial t^2}$  is the *relative acceleration* of  $P$ , i.e., the acceleration of  $P$  relative to the moving reference frame  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ . The contribution  $2 \left( \boldsymbol{\omega} \times \frac{\partial \mathbf{v}}{\partial t} \right)$  to the absolute acceleration is known as the *Coriolis acceleration*.

12. Show that when  $\mathbf{u}_1 = v(\cos \theta, \sin \theta, 0)$ ,  $\mathbf{u}_2 = v(-\sin \theta, \cos \theta, 0)$ ,  $\mathbf{u}_3 = v(0, 0, 1)$  the angular velocity (see Exercise 9) is  $\boldsymbol{\omega} = \theta_t \mathbf{u}_3$ . Deduce that the absolute derivative of a vector  $\mathbf{v}$  is  $\frac{\partial \mathbf{v}}{\partial t} + \theta_t (\mathbf{u}_3 \times \mathbf{v})$ .

13. Show that when  $\mathbf{u}_1 = v(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ ,  $\mathbf{u}_2 = v(\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta)$ ,  $\mathbf{u}_3 = v(-\sin \phi, \cos \phi, 0)$ ,  $\boldsymbol{\omega} = \cos \theta \phi_t \mathbf{u}_1 - \sin \theta \phi_t \mathbf{u}_2 + \theta_t \mathbf{u}_3 = \phi_t v(0, 0, 1) + \theta_t \mathbf{u}_3$ . *Note.* This result may be expressed as follows:

*The angular velocity of a moving polar coordinate reference frame is the sum of  $\phi_t$  times a unit vector along the z-axis and  $\theta_t$  times a unit vector along the third polar coordinate axis.*

# 2

## LINEAR VECTOR FUNCTIONS

### 1. Quadratic and bilinear forms; linear vector functions

We shall introduce the concepts of quadratic and bilinear forms (a *form* being a *scalar* function of one or more vectors) by considering real three-dimensional vectors. The extension to  $n$ -dimensional complex vectors is simple and will cause no difficulty.

If  $\mathbf{v} = v(x, y, z)$  is any real three-dimensional vector its squared magnitude

$$v^2 = x^2 + y^2 + z^2$$

is a scalar quadratic function of the coordinates of  $\mathbf{v}$ . We say that it is a *quadratic form* in  $\mathbf{v}$ . If  $\mathbf{v}_1 = v(x_1, y_1, z_1)$  and  $\mathbf{v}_2 = v(x_2, y_2, z_2)$  are any two real three-dimensional space vectors the scalar product

$$(\mathbf{v}_1|\mathbf{v}_2) = x_1x_2 + y_1y_2 + z_1z_2$$

is a *bilinear form* in the two vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ ; by this we mean that it is a scalar which possesses the following properties: (a) If  $c$  is any scalar then  $(c\mathbf{v}_1|\mathbf{v}_2) = c(\mathbf{v}_1|\mathbf{v}_2) = (\mathbf{v}_1|c\mathbf{v}_2)$ . (b)  $((\mathbf{v}_1 + \mathbf{w}_1)|\mathbf{v}_2) = (\mathbf{v}_1|\mathbf{v}_2) + (\mathbf{w}_1|\mathbf{v}_2)$ ;  $(\mathbf{v}_1|(\mathbf{v}_2 + \mathbf{w}_2)) = (\mathbf{v}_1|\mathbf{v}_2) + (\mathbf{v}_1|\mathbf{w}_2)$ .

These two properties imply (prove this) that if  $c_1, d_1, c_2, d_2$  are any four scalars then

$$\begin{aligned} ((c_1\mathbf{v}_1 + d_1\mathbf{w}_1)|(c_2\mathbf{v}_2 + d_2\mathbf{w}_2)) &= c_1c_2(\mathbf{v}_1|\mathbf{v}_2) + c_1d_2(\mathbf{v}_1|\mathbf{w}_2) \\ &\quad + d_1c_2(\mathbf{w}_1|\mathbf{v}_2) + d_1d_2(\mathbf{w}_1|\mathbf{w}_2). \end{aligned}$$

If we denote by  $x$  the  $3 \times 1$  matrix whose elements are the coordinates of  $\mathbf{v}$ , the squared magnitude of  $\mathbf{v}$  appears in the form

$$v^2 = x^*x = x_\alpha^*x^\alpha$$

(where  $x^1 = x, x^2 = y, x^3 = z; x_1^* = x, x_2^* = y, x_3^* = z$ ). Similarly if we denote by  $y$  the  $3 \times 1$  matrix whose elements are the coordinates of  $\mathbf{w}$  the scalar product  $(\mathbf{w}|\mathbf{v})$  appears in the form

$$(\mathbf{w}|\mathbf{v}) = y^*x = y_\alpha^*x^\alpha.$$

This bilinear form is a member of the class of bilinear forms of the type

$$y^*Ax = y_\alpha^*a_\beta^\alpha x^\beta,$$

where  $A$  is a  $3 \times 3$  matrix of which  $a_k^j$  is the element in the  $j$ th row and  $k$ th column; in fact when  $A$  is the unit three-dimensional matrix  $y^*Ax$  reduces to  $y^*x$ . We now raise the question: What conditions are imposed on the matrix  $A$  by the requirement that  $y^*Ax$  be a scalar?

To answer this question we observe that since  $\mathbf{w} = v(y^1, y^2, y^3)$  is an arbitrary vector the elements of the  $3 \times 1$  matrix  $Ax$  are the coordinates of a vector (by the converse of the scalar product theorem). Thus the matrix  $A$  must be such that the elements of  $Ax$  are the coordinates of a vector, where  $\mathbf{v} = v(x^1, x^2, x^3)$  is an arbitrary vector. Conversely, if this is so,  $y^*Ax$  is a scalar. For this reason we say that the elements of  $A$  are the *coordinates*, in the reference frame  $Oxyz$  being used, of an *operator* (or *machine*) which feeds on vectors and produces vectors. We denote this operator by the symbol  $\mathbf{A}$ , and we denote by  $\mathbf{A}\mathbf{v}$  the vector whose coordinates in the  $Oxyz$  reference frame are the elements of the matrix  $Ax$ . It is at once evident (prove this) that

$$\mathbf{A}(c\mathbf{v}) = c\mathbf{A}\mathbf{v},$$

where  $c$  is any scalar;

$$\mathbf{A}(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{A}\mathbf{v}_1 + \mathbf{A}\mathbf{v}_2,$$

where  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are any two vectors.

We express these facts by the statement that  $\mathbf{A}$  is a *linear vector function*. Remember always that a linear vector function feeds on vectors to produce vectors; in other words the *independent variable* is a vector, and the *dependent variable* is a vector. We indicate by the symbol  $\mathbf{A}:A$  that the coordinates of  $\mathbf{A}$ , in the reference frame  $Oxyz$  being used, are the elements of the  $3 \times 3$  matrix  $A$ .

### EXERCISES

1. If  $\mathbf{A}:A$  and  $\mathbf{B}:B$  are linear vector functions show that the elements of  $A + B$  are the coordinates of a linear vector function (which we denote by  $\mathbf{A} + \mathbf{B}$  and term the *sum* of  $\mathbf{A}$  and  $\mathbf{B}$ ).

2. Show that the addition of two linear vector functions is commutative, i.e., that  $\mathbf{B} + \mathbf{A} = \mathbf{A} + \mathbf{B}$ .

3. Show that if  $\mathbf{A}:A$  is a linear vector function the elements of  $cA$ , where  $c$  is any scalar, are the coordinates of a linear vector function (which we denote by  $c\mathbf{A}$  and term the *product* of  $\mathbf{A}$  by  $c$ ).

4. Show that  $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$ .

5. Show that if  $\mathbf{A}: A$  and  $\mathbf{B}: B$  are linear vector functions then the elements of  $AB$  are the coordinates of a linear vector function (which we denote by  $\mathbf{AB}$  and term the product of  $\mathbf{B}$  by  $\mathbf{A}$ ).

6. Show that multiplication of linear vector functions is not, in general, commutative, i.e., that the product of  $\mathbf{A}$  by  $\mathbf{B}$  is not, in general, the same as the product of  $\mathbf{B}$  by  $\mathbf{A}$ .

It is easy to find the coordinates of a linear vector function  $\mathbf{A}$  in any new reference frame  $O'x'y'z'$ . Since  $y^*Ax$  is a scalar we have

$$y'^*A'x' = y^*Ax.$$

Here the coordinates  $(x^1)', (x^2)', (x^3)'$  of  $\mathbf{v}$  in the  $O'x'y'z'$  reference frame are connected with the coordinates  $x^1, x^2, x^3$  of  $\mathbf{v}$  in the  $Oxyz$  reference frame by means of the table of direction cosines which describes the relative orientation of the two reference frames. This table of direction cosines is a real unimodular unitary  $3 \times 3$  matrix which we shall denote by  $R$  and which we term a *rotation* matrix (since, when the two reference frames have a common origin, either of them may be transformed into the other by a rigid rotation). We have, then,

$$x = Rx', \quad y = Ry', \quad y^* = y'^*R^*$$

so that

$$y'^*A'x' = y'^*R^*ARx'.$$

It follows at once that

$$A' = R^*AR.$$

(To see this set  $\mathbf{w} = \mathbf{u}_j, \mathbf{v} = \mathbf{u}_k$ ; i.e.,  $\mathbf{w}$  is the vector whose  $j$ th coordinate in the  $O'x'y'z'$  reference frame is unity, all the other coordinates being zero, and  $\mathbf{v}$  is the vector whose  $k$ th coordinate in the  $O'x'y'z'$  reference frame is unity, all the other coordinates being zero.) We have, then, the following fundamental result:

**Under a change of reference frame the coordinates of a linear vector function  $\mathbf{A}$  undergo the transformation**

$$A' = R^*AR.$$

Here  $R$  is a rotation matrix:

$$R^*R = RR^* = E_3; \quad \det R = 1,$$

**and the change of reference frame is such that the coordinates of a vector undergo the transformation**

$$x = Rx'.$$

*Note.* Since  $R$  is real  $R^*$  is obtained from  $R$  by *transposing* it, i.e., by interchanging its rows and columns.

## EXERCISES

7. Show that when the roles of the reference frames  $Oxyz$  and  $O'x'y'z'$  are interchanged  $R$  is replaced by  $R^*$ , i.e., that  $x = R^*x'$ ,  $A = RA'R^*$ .

8. Show that the coordinates of the identity linear vector function  $E_3$ , i.e., the linear vector function for which  $E_3\mathbf{v} = \mathbf{v}$ , every  $\mathbf{v}$ , are scalars and are furnished by the elements of the three-dimensional unit matrix. *Hint.* If  $Ax = x$  for every  $x$  then  $A = E_3$  and  $R^*E_3R = R^*R = E_3$ .

9. Show that the zero linear vector function  $\mathbf{O}$ , i.e., the linear vector function for which  $\mathbf{O}\mathbf{v} = \mathbf{0}$ , every  $\mathbf{v}$ , are scalars and are furnished by the elements of the three-dimensional zero matrix.

10. Show that the only linear vector functions whose coordinates are scalars are scalar multiples of the identity linear vector function. *Hint.* The relation  $R^*AR = A$  is equivalent (why?) to  $AR = RA$ . Hence  $a_1'l_1 + a_2'm_1 + a_3'n_1 = l_1a_1 + l_2a_2 + l_3a_3$ , etc. Since  $m_1$  and  $n_1$  are independent of  $l_2$  and  $l_3$  we have  $a_1^2 = 0$ ,  $a_1^3 = 0$ . Similarly  $a_j^k = 0$ ,  $j \neq k$ . Equating the elements in the first row and second column of  $AR$  and of  $RA$  we find  $a_1^1 = a_2^2$ . Similarly  $a_1^1 = a_3^3$ . *Note.* A scalar multiple  $cE_3$  of the identity linear vector function is known as a *scalar linear vector function*; when an arbitrary vector  $\mathbf{v}$  is fed into a scalar linear vector function  $cE_3$  it is returned multiplied by the scalar  $c$ .

11. Show that the elements of the  $3 \times 3$  matrix  $A = xy^*$  are the coordinates of a linear vector function. *Hint.* If  $\mathbf{v}_3 = v(z^1, z^2, z^3)$  is an arbitrary vector the elements of the  $3 \times 1$  matrix  $Az = xy^*z = (\mathbf{v}_2|\mathbf{v}_3)x$  are the coordinates of the vector  $(\mathbf{v}_2|\mathbf{v}_3)\mathbf{v}_1$ . *Note.* The linear vector function  $\mathbf{A}: xy^*$  is termed the *direct product* of the vector  $\mathbf{v}_2 = v(y^1, y^2, y^3)$  by the vector  $\mathbf{v}_1 = v(x^1, x^2, x^3)$ .

## 2. Symmetric and alternating linear vector functions

If  $\mathbf{A}$  is any linear vector function the bilinear form  $(\mathbf{v}_2|\mathbf{A}\mathbf{v}_1)$  is a scalar,  $\mathbf{v}_1$  and  $\mathbf{v}_2$  being arbitrary vectors. On interchanging the vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  we obtain a second scalar  $(\mathbf{v}_1|\mathbf{A}\mathbf{v}_2)$  which may, or may not, be the same as the first. If the linear vector function  $\mathbf{A}$  is such that the bilinear form  $(\mathbf{v}_2|\mathbf{A}\mathbf{v}_1)$  is insensitive to an interchange of the vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  we say that it is *symmetric* or Hermitian (after C. Hermite [1822–1901], a French mathematician). Thus

The (real) linear vector function  $\mathbf{A}$  is symmetric, or Hermitian, if  $(\mathbf{v}_2|\mathbf{A}\mathbf{v}_1) = (\mathbf{v}_1|\mathbf{A}\mathbf{v}_2)$ ,  $\mathbf{v}_1$  and  $\mathbf{v}_2$  being any two real vectors.

It is easy to find the conditions imposed on the matrix  $A$  by the fact that  $\mathbf{A}: A$  is symmetric. We must have

$$y^*Ax = x^*Ay,$$

where  $x$  and  $y$  are any two (real)  $3 \times 1$  matrices. Since  $y^*Ax$  is a  $1 \times 1$  (real) matrix (i.e., a real number) it is equal to its "star":

$$y^*Ax = (y^*Ax)^* = x^*A^*y.$$

Hence

$$x^*A^*y = x^*Ay,$$

where  $x$  and  $y$  are any two real  $3 \times 1$  matrices. On setting  $x = e_j$ ,  $y = e_k$  we obtain

$$a^*_{kj} = a_{kj}; \quad j, k = 1, 2, 3;$$

i.e.,

$$A^* = A.$$

In words:

*The matrix whose elements are the coordinates of a symmetric real linear vector function is insensitive to an interchange of its rows and columns:*

$$a_j^k = a_{kj}; \quad j, k = 1, 2, 3.$$

### EXERCISES

1. Show that if  $A$  and  $B$  are any two symmetric linear vector functions so also is  $A + B$ .

2. Show that the elements of the  $3 \times 3$  matrix  $xy^* + x^*y$  are the coordinates of a symmetric linear vector function (the elements of the  $3 \times 1$  matrices  $x$  and  $y$  being coordinates of vectors).

If the linear vector function  $A$ , instead of being symmetric, is such that  $(v_2|Av_1)$  changes sign when the two vectors  $v_1$  and  $v_2$  are interchanged we say that  $A$  is *alternating* or *skew symmetric*. Thus

**The (real) linear vector function  $A$  is alternating, or skew symmetric, if  $(v_2|Av_1) = -(v_1|Av_2)$ ,  $v_1$  and  $v_2$  being any two (real) vectors.**

The same argument as that for symmetric linear vector functions (repeat it) shows that

*The elements of the  $3 \times 3$  matrix  $A$  are the coordinates of an alternating, or skew-symmetric, linear vector function if, and only if,*

$$A^* = -A.$$

It follows that the diagonal elements of  $A$  are zero (why?):

$$a_j^j = 0, \quad j = 1, 2, 3$$

while

$$a_{kj} = -a_j^k, \quad j \neq k.$$

### EXERCISES

3. Show that the elements of the  $3 \times 3$  matrix  $xy^* - x^*y$  are the coordinates of an alternating linear vector function (the elements of the  $3 \times 1$  matrices  $x$  and  $y$  being coordinates of vectors).

4. Show that if  $A: A$  is an alternating linear vector function then the elements of the  $3 \times 1$  matrix  $(a_2^2, a_3^1, a_1^2)$  are the coordinates of a vector. *Hint.* The coordinates of  $Av$ , where  $v = v(x, y, z)$ , are  $(a_2^1z - a_1^2y, a_1^2x - a_2^3z, a_2^3y - a_3^1x)$ , and since  $Av$  is a vector,  $v$  being an arbitrary vector, it follows by the converse of the vector product theorem (see Review Exercise 8, p. 41) that  $(a_2^2, a_3^1, a_1^2)$  are



the coordinates of a vector. *Note.* Remember this result; we shall denote by  $\mathbf{a}$  the vector  $v(a_x^3, a_x^2, a_x^1)$ , and we shall say that  $\mathbf{a}$  is the vector *associated with the alternating linear vector function*.

5. Show that if  $\mathbf{A} : xy^* - yx^*$ , where  $\mathbf{v} = v(x^1, x^2, x^3)$ ,  $\mathbf{w} = w(y^1, y^2, y^3)$  are arbitrary vectors, then the vector  $\mathbf{a}$  associated with  $\mathbf{A}$  is  $(\mathbf{v} \times \mathbf{w})$ .

6. Show that if  $\mathbf{a}$  is the vector associated with the alternating linear vector function  $\mathbf{A}$  then  $\mathbf{A}\mathbf{v} = (\mathbf{a} \times \mathbf{v})$ .

### 3. The gradient of a vector field; the divergence of a linear vector function field

On combining the following facts: (1) If  $\mathbf{v} = v(x^1, x^2, x^3)$ ,  $\mathbf{w} = v(y^1, y^2, y^3)$  are any two vectors then the direct product  $xy^*$  of  $\mathbf{w}$  by  $\mathbf{v}$  (see Exercise 11, p. 46) is a linear vector function; and (2) the differentiating operator  $\nabla = v\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$  is a symbolic vector (see p. 28) we obtain the important result that if  $\mathbf{v} = v(f, g, h)$  is a differentiable vector field then the elements of the  $3 \times 3$  matrix

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} (f, g, h) = \begin{bmatrix} f_x & g_x & h_x \\ f_y & g_y & h_y \\ f_z & g_z & h_z \end{bmatrix}$$

are the coordinates of a linear vector function which we term the *gradient* of the vector field  $\mathbf{v} = v(f, g, h)$  and denote by the symbol  $\mathbf{grad} \mathbf{v}$ . Thus

$$\mathbf{grad} \mathbf{v} : \nabla f^*,$$

where  $f^*$  is the  $1 \times 3$  matrix whose elements are the coordinates of  $\mathbf{v}$ . The adjoint of  $\mathbf{grad} \mathbf{v}$  is the linear vector function

$$(\mathbf{grad} \mathbf{v})^* : f \nabla^*,$$

where  $\nabla^*$  is the  $1 \times 3$  matrix  $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ . If  $d\mathbf{P} = v(dx, dy, dz)$  we have

$$(\mathbf{grad} \mathbf{v})^* d\mathbf{P} = d\mathbf{v} = v(df, dg, dh).$$

This relation serves, since  $d\mathbf{P}$  is an arbitrary vector, to verify that  $(\mathbf{grad} \mathbf{v})^* : f \nabla^*$  and hence  $\mathbf{grad} \mathbf{v} : \nabla f^*$  are linear vector functions (why?). On subtracting from  $(\mathbf{grad} \mathbf{v})^*$  its adjoint  $\mathbf{grad} \mathbf{v}$  we obtain the alter-

nating linear vector function whose coordinates are the elements of the matrix

$$\begin{bmatrix} 0 & f_y - g_x & f_z - h_x \\ g_x - f_y & 0 & g_z - h_y \\ h_x - f_z & h_y - g_z & 0 \end{bmatrix}.$$

The vector associated with this alternating linear vector function is **curl v**. (Verify this.) Note that  $\text{div } \mathbf{v}$  is the sum of the diagonal elements, or *trace*, of the matrix whose elements are the coordinates of the linear vector function **grad v** and that the determinant of this matrix is the Jacobian determinant of  $f$ ,  $g$ , and  $h$  with respect to  $x$ ,  $y$ , and  $z$ . We shall see shortly that if  $\mathbf{A}:A$  is any linear vector function then the trace of  $A$  and its determinant (and also the sum of its two-rowed diagonal minors) are scalars.

Let, now,  $\mathbf{A}:A$  be a differentiable linear vector function *field*. We mean by this that each element of the  $3 \times 3$  matrix  $A$  is a differentiable function of  $(x, y, z)$ ; thus to each point  $P:(x, y, z)$  of a three-dimensional domain there is attached a linear vector function  $\mathbf{A}$  and each of the coordinates of  $\mathbf{A}$  is a differentiable point-function. Since  $\mathbf{A}^*$  is a linear vector function,  $\mathbf{A}^*\mathbf{v}$  is a vector,  $\mathbf{v}$  being an arbitrary vector. Hence, on replacing  $\mathbf{v}$  by  $\nabla$ , we see that

$$v \left( \frac{\partial a_1^1}{\partial x} + \frac{\partial a_1^2}{\partial y} + \frac{\partial a_1^3}{\partial z}, \frac{\partial a_2^1}{\partial x} + \frac{\partial a_2^2}{\partial y} + \frac{\partial a_2^3}{\partial z}, \frac{\partial a_3^1}{\partial x} + \frac{\partial a_3^2}{\partial y} + \frac{\partial a_3^3}{\partial z} \right)$$

is a vector. Using the compact "Greek label for summation" notation we may express this result as follows:

If  $\mathbf{A}:A$  is any differentiable linear vector function field the elements of the  $3 \times 1$  matrix  $\left( \frac{\partial a_1^\alpha}{\partial x^\alpha}, \frac{\partial a_2^\alpha}{\partial x^\alpha}, \frac{\partial a_3^\alpha}{\partial x^\alpha} \right)$  are the coordinates of a vector. We

term this vector the *divergence* of the linear vector function field, and we denote it by the symbol  $\text{div } \mathbf{A}$ .

*Note.* The following rule of procedure may help you remember how to calculate the divergence of a linear vector function field: Imagine that the elements of each column of  $A$  are the coordinates of a vector, and calculate the divergence of each of the three "column vectors." The three numbers so obtained are the coordinates of  $\text{div } \mathbf{A}$ .

*Warning.* Never forget, when using this rule of procedure, the fiction involved. The elements of a column of  $A$  are not the coordinates of a vector (why?). Thus you should not be surprised by the

fact that the “divergence” of any of the “column vectors” of  $A$  is not a scalar.

### EXERCISES

1. Show that  $\text{div grad } \mathbf{v} = \Delta_2 \mathbf{v}$ .
2. Show that if  $c$  is any differentiable scalar field  $\text{grad } (c\mathbf{v}) = c \text{ grad } \mathbf{v} +$  the direct product of  $\mathbf{v}$  by  $\text{grad } c$ .
3. Show that  $\text{div } c\mathbf{A} = c \text{ div } \mathbf{A} + \mathbf{A}^* \text{ grad } c$ .

#### 4. Linear vector functions for $n$ -dimensional complex vectors

It will suffice to run over the main points since the extension from three-dimensional real vectors to  $n$ -dimensional complex vectors presents no essential difficulty. If  $A$  is an  $n \times n$  matrix, whose elements are complex numbers, which is such that the bilinear form

$$w^*Az = w_\alpha^* a_{\alpha\beta} z_\beta$$

is a scalar, where  $z$  and  $w$  are arbitrary  $n \times 1$  complex matrices whose elements are the coordinates of two arbitrary vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , we say that the elements of  $A$  are the coordinates, in the reference frame being used, of a *linear vector function*  $\mathbf{A}$ . The reason for the name is that the elements of the  $n \times 1$  matrix  $Az$  are the coordinates of a vector (prove this) which we denote by  $\mathbf{A}\mathbf{v}_1$ ; the scalar bilinear form  $w^*Az$  is, then, the scalar product  $(\mathbf{v}_2 | \mathbf{A}\mathbf{v}_1)$ . We shall use the notation  $\mathbf{A}:A$  to indicate that  $\mathbf{A}$  is the linear vector function whose coordinates, in the reference frame in use, are furnished by the elements of the  $n \times n$  matrix  $A$ . Then addition, multiplication by a scalar, and multiplication of linear vector functions are defined by the formulas

$$\mathbf{A} + \mathbf{B}:A + B; \quad c\mathbf{A}:cA; \quad \mathbf{AB}:AB.$$

(Prove that these formulas are *legitimate*, i.e., that they actually define linear vector functions. *Hint.* Remember that the *essential* or *intrinsic* quality of a linear vector function is the scalar nature of the bilinear form  $w^*Az = (\mathbf{v}_2 | \mathbf{A}\mathbf{v}_1)$  for arbitrary vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .) Under the change of reference frame  $z = Uz'$ , where  $U$  is an  $n \times n$  unimodular unitary matrix, the  $n \times n$  matrix  $A$  transforms according to the formula

$$A' = U^*AU.$$

(Prove this.)

Associated with any linear vector function  $\mathbf{A}$  is another linear vector function  $\mathbf{B}$  which is defined by the formula

$$(\mathbf{v}_2 | \mathbf{B}\mathbf{v}_1) = \overline{(\mathbf{v}_1 | \mathbf{A}\mathbf{v}_2)}.$$

In other words an interchange of the two vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  in the bilinear form  $(\mathbf{v}_1|\mathbf{A}\mathbf{v}_2)$ , when accompanied by a replacement of  $\mathbf{A}$  by  $\mathbf{B}$ , changes the value of the bilinear form into its conjugate.

The matrix  $B$  whose elements furnish the coordinates of  $\mathbf{B}$  is readily determined. We have

$$w^*Bz = \overline{z^*Aw};$$

now  $z^*Aw$  is a  $1 \times 1$  matrix and so its conjugate is its *star*, i.e.,

$$\overline{z^*Aw} = (z^*Aw)^* = w^*A^*z.$$

Hence

$$w^*Bz = w^*A^*z,$$

and since this relation must hold for arbitrary  $n \times 1$  matrices  $z$  and  $w$  we must have (prove this)

$$B = A^*.$$

We denote, accordingly, the linear vector function  $\mathbf{B}$  by the symbol  $\mathbf{A}^*$ , and we say that  $\mathbf{A}^*$  is the *adjoint* of  $\mathbf{A}$ . (Verify that  $\mathbf{A}^*: \mathbf{A}^*$  is a linear vector function.)

### EXERCISES

1. Show that  $(\mathbf{A} + \mathbf{B})^* = \mathbf{A}^* + \mathbf{B}^*$ :  $(c\mathbf{A})^* = \bar{c}\mathbf{A}^*$  (note carefully the  $\bar{c}$  rather than  $c$ );  $(\mathbf{A}\mathbf{B})^* = \mathbf{B}^*\mathbf{A}^*$ .
2. Show that the relation between  $\mathbf{A}$  and  $\mathbf{A}^*$  is a *partnership*, i.e., that  $(\mathbf{A}^*)^* = \mathbf{A}$ .
3. Show that  $(\mathbf{v}|\mathbf{A}^*\mathbf{v})$  is the conjugate of  $(\mathbf{v}|\mathbf{A}\mathbf{v})$ . *Hint.*  $(\mathbf{v}_1|\mathbf{A}^*\mathbf{v}_2)$  is the conjugate of  $(\mathbf{v}_2|\mathbf{A}\mathbf{v}_1)$  for any pair of vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ ; set  $\mathbf{v}_2 = \mathbf{v}_1 = \mathbf{v}$ .
4. Show that  $(\mathbf{A}^*\mathbf{v}_2|\mathbf{v}_1) = (\mathbf{v}_2|\mathbf{A}\mathbf{v}_1)$ . *Note.* This important result may be phrased as follows:

The linear vector function (or operator) may be transferred from one of the vectors  $\mathbf{v}_1, \mathbf{v}_2$  to the other in the symbol  $(\mathbf{v}_2|\mathbf{A}\mathbf{v}_1)$  if it is replaced by its adjoint.

A linear vector function  $\mathbf{A}$  is said to be *self-adjoint* or *Hermitian* (after C. Hermite [1822–1901], a French mathematician) if  $(\mathbf{v}_1|\mathbf{A}\mathbf{v}_2)$  shares with the scalar product  $(\mathbf{v}_1|\mathbf{v}_2)$  the property that an interchange of the two vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  changes it into its conjugate. It follows that

**If  $\mathbf{v}$  is any vector and  $\mathbf{A}$  is Hermitian,  $(\mathbf{v}|\mathbf{A}\mathbf{v})$  is real.**

In fact  $(\mathbf{v}|\mathbf{A}\mathbf{v})$  is equal to its conjugate, and a complex number which is equal to its conjugate is real.

It is easy to tell when a linear vector function is Hermitian. We are given that  $(\mathbf{v}_2|\mathbf{A}\mathbf{v}_1) = \overline{(\mathbf{v}_1|\mathbf{A}\mathbf{v}_2)}$ , and we know that  $(\mathbf{v}_1|\mathbf{A}\mathbf{v}_2) = (\mathbf{A}^*\mathbf{v}_1|\mathbf{v}_2) = \overline{(\mathbf{v}_2|\mathbf{A}^*\mathbf{v}_1)}$ . Hence  $(\mathbf{v}_2|\mathbf{A}\mathbf{v}_1) = (\mathbf{v}_2|\mathbf{A}^*\mathbf{v}_1)$ , and since  $\mathbf{v}_2$  is arbitrary this implies that  $\mathbf{A}\mathbf{v}_1 = \mathbf{A}^*\mathbf{v}_1$  (why?). Since  $\mathbf{v}_1$  is arbitrary this implies that  $\mathbf{A} = \mathbf{A}^*$  (why?). Thus

A linear vector function  $\mathbf{A}$  is Hermitian when, and only when,  $A = A^*$ .

In particular

A linear vector function for which  $A$  is real is Hermitian when, and only when,  $A$  is symmetric, i.e.,  $a_k^j = a_j^k$ ;  $j, k = 1, \dots, n$ . (Note that this symmetry relation is independent of the reference frame; in other words  $a_k^j = a_j^k$  implies  $a'_k{}^j = a'_j{}^k$ ;  $j, k = 1, \dots, n$ . This is evident since the definition of the Hermitian character of a linear vector was *intrinsic*; i.e., it made no mention of the reference frame in use. It may be readily verified since the relation  $A' = U^*AU$  yields  $A'^* = U^*A^*U$ . Hence, if  $A = A^*$ ,  $A' = A'^*$ .)

A linear vector function  $\mathbf{A}$  is said to be *unitary* if  $\mathbf{A}\mathbf{v}$  has the same magnitude as  $\mathbf{v}$  for every  $\mathbf{v}$ . Since the coordinates of  $\mathbf{A}\mathbf{v}$  are furnished by the  $n \times 1$  matrix  $Az$  the definition of a unitary linear vector function is contained in the formula

$$z^*A^*Az = z^*z; \quad z \text{ an arbitrary } n \times 1 \text{ matrix.}$$

On replacing  $z$ , in turn, by  $z + w$  and  $z + iw$  it follows that this implies the (apparently stronger) relation

$$w^*A^*Az = w^*z; \quad z \text{ and } w \text{ arbitrary } n \times 1 \text{ matrices.}$$

(Prove this.) On setting  $w = e_j$ ,  $z = e_k$  we see that  $A^*A$  is the unit  $n$ -dimensional matrix. Thus

A linear vector function  $\mathbf{A}$ :  $A$  is unitary when, and only when,  $A$  is a unitary matrix.

### EXERCISES

5. Show that if  $\mathbf{H}_1$  and  $\mathbf{H}_2$  are any two Hermitian linear vector functions so also is  $\mathbf{H}_1 + \mathbf{H}_2$ .

6. Show that if  $\mathbf{U}_1$  and  $\mathbf{U}_2$  are any two unitary linear vector functions so also are  $\mathbf{U}_1\mathbf{U}_2$  and  $\mathbf{U}_2\mathbf{U}_1$ .

7. Show that if  $\mathbf{U}$  is a unitary linear vector function so also is  $c\mathbf{U}$ , where  $c = e^{i\theta}$  is a scalar of unit modulus.

8. Show that if  $\mathbf{A}$  is either a Hermitian linear vector function or a unitary linear vector function then  $\mathbf{A}\mathbf{A}^* = \mathbf{A}^*\mathbf{A}$ .

9. Show that if  $\mathbf{A}$  is any linear vector function then both the linear vector functions  $\mathbf{A}\mathbf{A}^*$  and  $\mathbf{A}^*\mathbf{A}$  are Hermitian.

10. Show that if  $\mathbf{v}$  is any vector both  $(\mathbf{v}|\mathbf{A}\mathbf{A}^*\mathbf{v})$  and  $(\mathbf{v}|\mathbf{A}^*\mathbf{A}\mathbf{v})$  are nonnegative. *Hint.*  $(\mathbf{v}|\mathbf{A}\mathbf{A}^*\mathbf{v}) = (\mathbf{A}^*\mathbf{v}|\mathbf{A}^*\mathbf{v})$ ;  $(\mathbf{v}|\mathbf{A}^*\mathbf{A}\mathbf{v}) = (\mathbf{A}\mathbf{v}|\mathbf{A}\mathbf{v})$ . Note. A Hermitian linear function  $\mathbf{H}$  which is such that  $(\mathbf{v}|\mathbf{H}\mathbf{v}) \geq 0$  for every vector  $\mathbf{v}$  is termed *positive*. If the equality is valid only when  $\mathbf{v} = \mathbf{0}$  we say that  $\mathbf{H}$  is *definitely positive*.

11. Show that the linear vector functions  $\mathbf{A}\mathbf{A}^*$  and  $\mathbf{A}^*\mathbf{A}$  are definitely positive if the linear vector function  $\mathbf{A}$  is non-singular. Note. A linear vector function  $\mathbf{A}$  is said to be *singular* when there exists a vector  $\mathbf{v}$ , other than the zero vector, such that  $\mathbf{A}\mathbf{v} = \mathbf{0}$ . Thus  $\mathbf{A}$ :  $A$  is singular when, and only when,  $\det A = 0$ .

12. Show that  $\mathbf{A}^*$  is singular when, and only when,  $\mathbf{A}$  is singular.

13. Show that if  $\mathbf{A}$  is a real three-dimensional Hermitian linear vector function (so that  $\mathbf{A}$  is symmetric) and  $\mathbf{v} = v(x, y, z)$  then  $\mathbf{A}\mathbf{v} = \frac{1}{2} \text{grad} (\mathbf{v}|\mathbf{A}\mathbf{v})$ . *Note.* This result shows that  $\mathbf{A}\mathbf{v}$  is perpendicular to the level surface through  $P:(x, y, z)$  of the homogeneous quadratic function  $(\mathbf{v}|\mathbf{A}\mathbf{v})$ . Thus the Hermitian linear vector function  $\mathbf{A}$  may be regarded as an operator or machine that feeds on the vectors  $v(O \rightarrow P)$ , where  $P$  is any point of the quadric surface  $(\mathbf{v}|\mathbf{A}\mathbf{v}) = \text{constant}$ , and returns a vector which is perpendicular to the tangent plane at  $P$  to the quadric surface. Note that when  $\mathbf{A}$  is the identity linear vector function the quadric surface is a sphere.

14. Show that if  $(\mathbf{v}|\mathbf{A}\mathbf{v})$  is real,  $\mathbf{v}$  being an arbitrary complex vector, then  $\mathbf{A}$  is Hermitian. *Hint.* Since  $((\mathbf{v} + \mathbf{w})|\mathbf{A}(\mathbf{v} + \mathbf{w}))$  is real for every pair of vectors  $\mathbf{v}$  and  $\mathbf{w}$ ,  $(\mathbf{v}|\mathbf{A}\mathbf{w}) + (\mathbf{w}|\mathbf{A}\mathbf{v})$  is real. It follows that  $a_j^k + a_k^j$  is real and, on replacing  $\mathbf{w}$  by  $i\mathbf{w}$ , that  $i(a_j^k - a_k^j)$  is real; hence  $\mathbf{A} = \mathbf{A}^*$ . *Note.* It follows from this result that

A linear vector function is Hermitian when, and only when,  $(\mathbf{v}|\mathbf{A}\mathbf{v})$  is real,  $\mathbf{v}$  being an arbitrary vector.

This reality of the quadratic form  $(\mathbf{v}|\mathbf{A}\mathbf{v})$  makes Hermitian linear vector functions of special importance in theoretical physics.

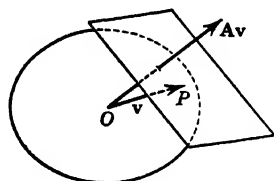


FIG. 13.

## 5. Characteristic vectors and characteristic numbers of a linear vector function

We now raise the following question: Given a linear vector function  $\mathbf{A}$  does there exist a vector  $\mathbf{v}$  such that  $\mathbf{A}\mathbf{v}$  is a scalar multiple of  $\mathbf{v}$ ?

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}; \quad \lambda \text{ a scalar.}$$

In the case where  $\mathbf{A}$  is a real three-dimensional linear vector function and the scalar multiplier  $\lambda$  is real the question may be phrased as follows: Does there exist a vector  $\mathbf{v}$  such that  $\mathbf{A}\mathbf{v}$  has the same direction as, or the opposite direction to, that of  $\mathbf{v}$ , or, equivalently (see Exercise 13, p. 53), does there exist a point  $P$  on a level surface of the quadratic form  $(\mathbf{v}|\mathbf{A}\mathbf{v})$  at which the tangent plane is perpendicular to  $v(O \rightarrow P)$ ?

*Note.* At this point the *simplification* (rather than *complication*) introduced by passing from the real field to the complex field shows up. We shall see shortly that every linear vector function in the complex field possesses at least one characteristic vector. In particular every real linear vector function possesses, *in the complex field*, at least one characteristic vector. In other words there exists a complex vector  $\mathbf{v}$  (other than the zero vector) and a complex scalar  $\lambda$  such that  $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ . But it may well happen that no such *real* vector  $\mathbf{v}$  and *real* scalar  $\lambda$  exist. If we insisted on working in the real field we would have to introduce the more complicated idea of a *characteristic plane* defined

as follows: If  $\mathbf{v}$  is any vector in the characteristic plane then  $\mathbf{Av}$  lies in the characteristic plane.

Equivalent to the definition of a characteristic plane is that of a characteristic pair of linearly independent vectors: A linearly independent pair of vectors  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is a characteristic pair of the linear vector function  $\mathbf{A}$  if both  $\mathbf{Av}_1$  and  $\mathbf{Av}_2$  are linear combinations of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .

We term any non-trivial vector (i.e., any vector other than the zero vector) which is such that  $\mathbf{Av}$  is a scalar multiple of  $\mathbf{v}$  a *characteristic vector* of  $\mathbf{A}$ , and we term the scalar multiple  $\lambda$  a *characteristic number* of  $\mathbf{A}$ . We say, further, that the characteristic vector  $\mathbf{v}$  of  $\mathbf{A}$  and the characteristic number  $\lambda$  of  $\mathbf{A}$  are *associated* with one another. It is clear that if  $\mathbf{v}$  is any characteristic vector of  $\mathbf{A}$ , with the associated characteristic number  $\lambda$ , then  $c\mathbf{v}$ , where  $c$  is any scalar other than zero, is a characteristic vector of  $\mathbf{A}$  with the *same* associated characteristic number  $\lambda$ . We use this flexible multiplying scalar  $c$  to arrange that the magnitude  $v$  of  $\mathbf{v}$  is unity; when this is done we say that the characteristic vector is *normalized*. We shall use the symbol  $\mathbf{u}$  to denote a normalized characteristic vector. Thus

**A normalized characteristic vector of  $\mathbf{A}$  is one for which**

$$\mathbf{Au} = \lambda\mathbf{u}; \quad (\mathbf{u}|\mathbf{u}) = u^2 = 1.$$

It is clear (prove this) that if  $\mathbf{u}$  is any normalized characteristic vector of  $\mathbf{A}$ , with the associated characteristic number  $\lambda$ , then  $e^{i\theta}\mathbf{u}$ , where  $e^{i\theta}$ ,  $\theta$  real, is any complex number of modulus 1, is a normalized characteristic vector of  $\mathbf{A}$  with the same associated characteristic number  $\lambda$ . This result is expressed as follows:

*Any normalized characteristic vector  $\mathbf{u}$  of a linear vector function  $\mathbf{A}$  is indeterminate to the extent of a phase factor.*

In the real field the corresponding result is as follows:

*Any normalized real characteristic vector of a real linear vector function is indeterminate to the extent of a reversal of direction (the multiplying scalar being now restricted to the values  $\pm 1$ ).*

The fact that every linear vector function possesses, in the complex field, at least one characteristic vector is easy to prove. On writing  $\lambda\mathbf{v}$  in the form  $\mathbf{Lv}$ , where  $\mathbf{L}$  is the scalar linear vector function defined by the formula

$$\mathbf{L} : \lambda\mathbf{E}_n,$$

the definition of a characteristic vector  $\mathbf{v}$  appears in the form

$$\mathbf{Av} = \mathbf{Lv}; \quad \mathbf{v} \neq \mathbf{0}$$

or, equivalently,

$$(\mathbf{A} - \mathbf{L})\mathbf{v} = \mathbf{0}; \quad \mathbf{v} \neq \mathbf{0}.$$

Hence the linear vector function  $\mathbf{A} - \mathbf{L}$  is singular; it follows that  
*If  $\lambda$  is a root of the algebraic equation of degree  $n$*

$$\det (A - \lambda E_n) = 0$$

then  $\lambda$  is a characteristic number of the linear vector function  $\mathbf{A}$ . If not all the cofactors of the matrix  $A - \lambda E_n$  are zero the ratios of the coordinates of an associated characteristic vector are those of the cofactors of a row of the matrix  $A - \lambda E_n$ . When the rank  $r$  of  $A - \lambda E_n < n - 1$ ,  $n - r$  of the coordinates of a characteristic vector of  $\mathbf{A}$  which is associated with  $\lambda$  may be chosen arbitrarily, and then the remaining  $r$  coordinates are linear combinations of these.

The algebraic equation which furnishes the characteristic numbers of the linear vector function  $\mathbf{A}$  may be written in the form

$$\lambda^n - I_1\lambda^{n-1} + I_2\lambda^{n-2} - \dots + (-1)^n I_n = 0,$$

where  $I_1 = a_{\alpha\alpha}$  is the sum of the diagonal elements (i.e., the trace of the matrix  $A$ );

$$I_2 = \sum_{\alpha < \beta} \begin{vmatrix} a_{\alpha\alpha} & a_{\beta\alpha} \\ a_{\alpha\beta} & a_{\beta\beta} \end{vmatrix} \text{ is the sum of the two-rowed principal (or diagonal minors) of } A;$$

⋮  
⋮  
⋮

$I_j$  is the sum of the  $j$ -rowed diagonal minors of  $A$ ;

⋮  
⋮  
⋮

$$I_n = \det A.$$

Since the characteristic numbers of a matrix are intrinsic, i.e., their definition is independent of the reference frame, the numbers  $I_1, \dots, I_n$  are scalars. This is easy to verify; in fact

$$\begin{aligned} \lambda^n - I_1'\lambda^{n-1} + \dots + (-1)^n I_n' &= \det (A' - \lambda E_n) = \det (U^*AU - \lambda E_n) \\ &= \det U^*(A - \lambda E_n)U = \det (A - \lambda E_n), \text{ (since } \det U^* \det U \\ &= \det U^*U = \det E_n = 1), = \lambda^n - I_1\lambda^{n-1} + \dots + (-1)^n I_n. \end{aligned}$$



Since this relation is an identity in  $\lambda$  (why?) we have (why?)

$$I_1' = I_1, \dots, I_n' = I_n.$$

In other words the numbers  $I_1, \dots, I_n$  are scalars. We term them the *invariants* of the linear vector function  $\mathbf{A}$ . The equation  $\lambda^n - I_1\lambda^{n-1} + \dots + (-1)^n I_n = 0$  is known as the *characteristic equation* of the linear vector function.

### EXERCISES

1. Show that each characteristic number  $\lambda$  of a linear vector function  $\mathbf{A}$  is the value of the quadratic form  $(\mathbf{u}|\mathbf{A}\mathbf{u})$ , where  $\mathbf{u}$  is any (normalized) characteristic vector associated with  $\lambda$ . *Hint.*  $(\mathbf{u}|\mathbf{A}\mathbf{u}) = (\mathbf{u}|\lambda\mathbf{u}) = \lambda(\mathbf{u}|\mathbf{u}) = \lambda$ .

2. Show that any characteristic number of a Hermitian linear vector function is real. *Note.* Remember this important result.

3. Show that if  $\lambda$  is a characteristic number, and  $\mathbf{u}$  an associated (normalized) characteristic vector, of any linear vector function  $\mathbf{A}$  then  $\bar{\lambda}\lambda = (\mathbf{A}\mathbf{u}|\mathbf{A}\mathbf{u}) = (\mathbf{u}|\mathbf{A}^*\mathbf{A}\mathbf{u})$ . *Hint.*  $\bar{\lambda}\lambda = (\lambda\mathbf{u}|\lambda\mathbf{u})$ .

4. Show that any characteristic number of a unitary linear vector function is of modulus unity. *Hint.* See Exercise 3 and remember that  $(\mathbf{U}\mathbf{u}|\mathbf{U}\mathbf{u}) = (\mathbf{u}|\mathbf{u}) = 1$ . *Note.* The following important case of this result is worthy of special mention:

*A real characteristic number of a unitary linear vector function is either 1 or -1.*

5. Show that a real three-dimensional unimodular unitary linear vector function, i.e., a *rotation*  $\mathbf{R}:R$ , has 1 as a characteristic number. *Hint.* One root of a third-degree algebraic equation is real; if only one is real the other two are conjugate complex numbers so that their product is positive. Furthermore the product of the three roots is positive, being  $\det R = 1$ .

6. Show that the invariants of a linear vector function are connected with its characteristic numbers as follows:

$I_1$  = the sum of the characteristic numbers;

$I_2$  = the sum of the products of the characteristic numbers taken two at a time;

.

.

.

$I_n$  = the product of the characteristic numbers.

*Note.* It is understood here that if a characteristic number is a multiple root of the characteristic equation it is repeated as often as it occurs. For example the characteristic equation of the identity linear vector function  $\mathbf{E}_n$  is  $(\lambda - 1)^n = 0$ ; thus  $\mathbf{E}_n$  has the one characteristic number 1 repeated  $n$  times. Hence  $I_1 = n$ ;  $I_2 = \frac{n(n-1)}{1 \cdot 2}$ , and so on.

7. Show that the linear vector function  $\mathbf{A}: \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  has the same characteris-

tic equation as the identity linear vector function  $\mathbf{E}_2$  but that while every vector (other than the zero vector) is a characteristic vector of  $\mathbf{E}_2$  the only characteristic vectors of  $\mathbf{A}$  are the non-zero scalar multiples of  $\mathbf{u}_1 = v(1, 0)$ .

8. Show that the characteristic numbers of  $\mathbf{A}^*$  are the conjugates of the characteristic numbers of  $\mathbf{A}$ . *Hint.* The invariants of  $\mathbf{A}^*$  are the conjugates of the invariants of  $\mathbf{A}$ .

9. Show that if  $\mathbf{u}$  is a characteristic vector of  $\mathbf{A}$ , associated with the characteristic number  $\lambda$ , and  $\mathbf{v}$  is a characteristic vector of  $\mathbf{A}^*$ , associated with the characteristic number  $\mu$ , then the two vectors  $\mathbf{u}$  and  $\mathbf{v}$  are perpendicular unless  $\bar{\mu} = \lambda$ . *Hint.*  $(\mathbf{v}|\mathbf{A}\mathbf{u}) = \lambda(\mathbf{v}|\mathbf{u})$ ;  $(\mathbf{v}|\mathbf{A}\mathbf{u}) = (\mathbf{A}^*\mathbf{v}|\mathbf{u}) = \bar{\mu}(\mathbf{v}|\mathbf{u})$ . Hence  $\lambda(\mathbf{v}|\mathbf{u}) = \bar{\mu}(\mathbf{v}|\mathbf{u})$  so that, if  $\bar{\mu} \neq \lambda$ ,  $(\mathbf{v}|\mathbf{u}) = 0$ .

10. Show that any two characteristic vectors of a Hermitian linear vector function  $\mathbf{A}$  which are associated with two different characteristic numbers of  $\mathbf{A}$  are perpendicular.

11. Show that if  $\lambda$  is a characteristic number of the linear vector function  $\mathbf{A}$ , with the associated characteristic vector  $\mathbf{u}$ , then  $\lambda^j$  is a characteristic number of  $\mathbf{A}^j$  with the same associated vector  $\mathbf{u}$ ,  $j = 2, 3, \dots$ . *Hint.*  $\mathbf{A}^2\mathbf{u} = \mathbf{A}(\mathbf{A}\mathbf{u}) = \mathbf{A}(\lambda\mathbf{u}) = \lambda\mathbf{A}\mathbf{u} = \lambda^2\mathbf{u}$ , and so on.

12. Show that if two linear vector functions  $\mathbf{A}$  and  $\mathbf{B}$  commute, i.e., if  $\mathbf{AB} = \mathbf{BA}$  and if  $\lambda$  is a characteristic number of  $\mathbf{A}$  with an associated characteristic vector  $\mathbf{u}$ , then  $\mathbf{B}\mathbf{u}$  is, if it is not the zero vector, a characteristic vector of  $\mathbf{A}$  which is associated with the characteristic number  $\lambda$ . *Hint.*  $\mathbf{A}\mathbf{B}\mathbf{u} = \mathbf{B}\mathbf{A}\mathbf{u} = \mathbf{B}(\lambda\mathbf{u}) = \lambda\mathbf{B}\mathbf{u}$ .

13. Show that if the vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  are characteristic vectors of the linear vector function  $\mathbf{A}$  which are associated with the same characteristic number  $\lambda$  of  $\mathbf{A}$  then any linear combination of the vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  is, if it is not the zero vector, a characteristic vector of  $\mathbf{A}$  which is associated with the characteristic number  $\lambda$ .

## 6. The canonical form of a linear vector function

We propose to prove in this section the following useful result: Let  $\mathbf{A}: A$  be any linear vector function; then there exists a unimodular unitary matrix  $U$  such that every element of  $A' = U^*AU$  which is below the diagonal is zero, i.e.,  $a'_{kj} = 0$  if  $j > k$ .

We term a matrix which is such that every element below the diagonal is zero a *triangular* matrix. In this terminology the theorem we wish to prove may be phrased as follows:

**There exists for every linear vector function a basis in which the matrix which presents the linear vector function is triangular.**

To prepare for the proof of this theorem we first observe that if  $\mathbf{v}_1$  is any vector other than the zero vector there exists a linearly independent set of  $n$  vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  containing  $\mathbf{v}_1$ . In fact  $\mathbf{v}_2$  is any vector which is not a multiple of  $\mathbf{v}_1$ ;  $\mathbf{v}_3$  is any vector which is not a linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , and so on till, finally,  $\mathbf{v}_n$  is any vector which is not a linear combination of  $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ . (How do we know, at each stage, that the vectors  $\mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n$  exist?) That

the set so constructed is linearly independent is clear; for if  $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$  we must have  $c_n = 0$  (for, otherwise,  $\mathbf{v}_n$  would be a linear combination of  $\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}$ ). Knowing that  $c_n = 0$  it follows, similarly, that  $c_{n-1} = 0$ , and so on to  $c_1 = 0$ . Next we observe that if  $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$  is any linearly independent set there exists a set of scalars  $c_j^k$ , where  $c_j^k = 0$  if  $k > j$ , such that the vectors

$$\mathbf{u}_1 = c_1^1\mathbf{v}_1;$$

$$\mathbf{u}_2 = c_2^1\mathbf{v}_1 + c_2^2\mathbf{v}_2;$$

.

.

.

$$\mathbf{u}_n = c_n^1\mathbf{v}_1 + \cdots + c_n^n\mathbf{v}_n$$

are each of unit magnitude and are mutually perpendicular:

$$(\mathbf{u}_j|\mathbf{u}_j) = 1, \quad j = 1, \cdots, n; \quad (\mathbf{u}_j|\mathbf{u}_k) = 0, \quad j \neq k.$$

To see this we set  $c_1^1 = \frac{1}{v_1}$  so that  $u_1 = 1$ . We then "pull off" from  $\mathbf{v}_2$  the "component" of  $\mathbf{v}_2$  along  $\mathbf{u}_1$ , and we multiply what is left by a scalar to make it of magnitude unity. Thus we set

$$\mathbf{u}_2 = \frac{1}{m_2} \{ \mathbf{v}_2 - (\mathbf{u}_1|\mathbf{v}_2)\mathbf{u}_1 \},$$

where  $m_2$  is the magnitude of  $\mathbf{v}_2 - (\mathbf{u}_1|\mathbf{v}_2)\mathbf{u}_1$ . That  $m_2 \neq 0$  follows from the fact that  $\mathbf{v}_2 - (\mathbf{u}_1|\mathbf{v}_2)\mathbf{u}_1$  is a non-trivial linear combination of  $\mathbf{v}_2$  and  $\mathbf{u}_1$ , i.e., a non-trivial linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . That  $(\mathbf{u}_1|\mathbf{u}_2) = 0$  is immediately evident. (Verify this.) To define  $\mathbf{u}_3$  we "pull off" from  $\mathbf{v}_3$  the components of  $\mathbf{v}_3$  along  $\mathbf{u}_1$  and  $\mathbf{u}_2$ , and we multiply what is left by a scalar to make it of magnitude unity. Thus we set

$$\mathbf{u}_3 = \frac{1}{m_3} \{ \mathbf{v}_3 - (\mathbf{u}_1|\mathbf{v}_3)\mathbf{u}_1 - (\mathbf{u}_2|\mathbf{v}_3)\mathbf{u}_2 \},$$

where  $m_3$  is the magnitude of  $\mathbf{v}_3 - (\mathbf{u}_1|\mathbf{v}_3)\mathbf{u}_1 - (\mathbf{u}_2|\mathbf{v}_3)\mathbf{u}_2$ . (How are we sure that  $m_3 \neq 0$ ?) It is clear that  $(\mathbf{u}_1|\mathbf{u}_3) = 0$ ,  $(\mathbf{u}_2|\mathbf{u}_3) = 0$ . (Verify this.) Proceeding in this way we prove the result stated since a linear combination of  $\mathbf{u}_1, \cdots, \mathbf{u}_j$  is a linear combination of  $\mathbf{v}_1, \cdots, \mathbf{v}_j$ ,  $j = 1, 2, \cdots, n$ .

*Note.* The process outlined here for obtaining a set of  $n$  mutually perpendicular unit vectors  $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$  each of which is an appropriate linear combination of the vectors of a given linearly independent set of  $n$  vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is known as the *orthogonalization process* of Schmidt (after E. Schmidt, a German mathematician).

Let, now,  $\mathbf{u}_1$  be any unit vector; then there exists a linearly independent set of  $n$  vectors  $\{\mathbf{u}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  of which  $\mathbf{u}_1$  is the first. Hence there exists a set of  $n$  mutually perpendicular unit vectors  $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$  of which  $\mathbf{u}_1$  is the first. The matrix  $U$  which is such that the elements in the  $j$ th column of  $U$  are the coordinates of  $\mathbf{u}_j$ ,  $j = 1, \dots, n$  is, then, unitary, and we may, without lack of generality, take it to be unimodular (because any one of the vectors  $\mathbf{u}_j$  may be multiplied by a scalar of modulus unity).

Returning now to our linear vector function  $\mathbf{A}$ , let  $\mathbf{u}_1$  be a normalized characteristic vector of  $\mathbf{A}$  associated with the characteristic number  $\lambda_1$  so that

$$\mathbf{A}\mathbf{u}_1 = \lambda_1\mathbf{u}_1.$$

If  $U_1$  is any unimodular unitary  $n \times n$  matrix which is such that the elements of its first column  $x_1$  are the coordinates of  $\mathbf{u}_1$  we have

$$x_1 = U_1e_1; \quad \mathbf{A}x_1 = \lambda_1x_1,$$

where  $e_1$  is the  $n \times 1$  matrix  $(1, 0, 0, \dots, 0)$  (why?). Hence

$$\mathbf{A}U_1e_1 = \lambda_1U_1e_1$$

or, equivalently (why?),

$$U_1^*\mathbf{A}U_1 = \lambda_1e_1.$$

In other words the first column of  $U_1^*\mathbf{A}U_1$  is the  $n \times 1$  matrix  $(\lambda_1, 0, 0, \dots, 0)$ . If we delete the first row and first column of  $U_1^*\mathbf{A}U_1$  we obtain an  $(n - 1) \times (n - 1)$  matrix  $B$  to which we may apply the argument just given; thus there exists an  $(n - 1) \times (n - 1)$  unimodular unitary matrix  $V$  such that the first column of  $V^*BV$  is the  $(n - 1) \times 1$  matrix  $(\lambda_2, 0, \dots, 0)$ . Now

$$U_2 = \begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix}$$

is an  $n \times n$  unimodular unitary matrix (prove this), and

$$U_2^*U_1^*\mathbf{A}U_1U_2 = \begin{pmatrix} 1 & 0 \\ 0 & V^* \end{pmatrix} \begin{pmatrix} \lambda_1 & * \\ 0 & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} \lambda_1 & * \\ 0 & V^*BV \end{pmatrix}$$

(where the \* above the  $B$  in the middle matrix indicates a  $1 \times (n - 1)$  matrix whose elements we do not need to know). Thus all the elements of the first column of  $U_2^*U_1^*AU_1U_2$  except the first are zero, and all the elements of the second column of this matrix are zero except the first two. Continuing this process and denoting by  $U$  the unimodular unitary  $n \times n$  matrix

$$U = U_1U_2 \cdots U_{n-1}$$

(show that  $U$  is unimodular and unitary) we see that  $U^*AU$  is triangular. This triangular matrix

$$A' = U^*AU$$

is said to be the *canonical form* of the linear vector function  $\mathbf{A}$ .

When the canonical form of a linear vector function  $\mathbf{A}$  is not only triangular but *diagonal*, i.e., when all the elements of  $A$  above the diagonal (in addition to those below the diagonal) are zero, we say that  $\mathbf{A}$  is *normal*. The following criterion enables us to tell, without having to determine the canonical form, whether a given linear vector function is normal or not:

*A linear vector function is normal when, and only when, it commutes with its adjoint, i.e., when, and only when,*

$$\mathbf{A}^*\mathbf{A} = \mathbf{A}\mathbf{A}^*$$

*or, equivalently (why?), when*

$$A^*A = AA^*.$$

The *necessity* of this commuting property is evident; in fact when  $\mathbf{A}$  is normal its canonical form is diagonal. Hence, in the basis in which  $\mathbf{A}$  is presented in its canonical form,  $\mathbf{A}^*$  is also diagonal (why?). Since any two diagonal matrices commute (prove this) it follows that  $\mathbf{A}$  and  $\mathbf{A}^*$  commute. (Show that the commuting property of any two linear vector functions is an *intrinsic* property, i.e., if it is valid in *any one* reference frame it is valid in *all* reference frames.) To prove the *sufficiency* of the commuting property we observe that the element in the first row and first column of  $\mathbf{A}^*\mathbf{A}$  is  $a^*_\alpha{}^1 a_{1\alpha} = \bar{a}_1{}^\alpha a_{1\alpha}$  and that, similarly, the element in the first row and first column of  $\mathbf{A}\mathbf{A}^*$  is  $a_\alpha{}^1 \bar{a}_\alpha{}^1$ . On equating these expressions and observing that ( $A$  being supposed to be the canonical form of  $\mathbf{A}$ )  $a_{1j} = 0$  if  $j > 1$  we obtain

$$a_2{}^1 \bar{a}_2{}^1 + \cdots + a_n{}^1 \bar{a}_n{}^1 = 0.$$

Since each of the terms in the sum on the left is non-negative (why?)

it follows that  $a_k^1 = 0$ ,  $k > 1$ . In words: All elements of the first row of  $A$  are zero except the first.

Equating now the elements in the second row and second column of  $A^*A$  and  $AA^*$  we find that all elements of the second row of  $A$  are zero except the second, and so on. Hence  $A$  is diagonal so that  $A$  is normal.

We may summarize the important results of this section as follows:

**Every linear vector function is essentially triangular; i.e., there exists a basis or reference frame in which the matrix, whose elements are the coordinates of the linear vector function in this reference frame, is triangular. The normal linear functions are those which are essentially diagonal; i.e., there exists a reference frame in which the matrix which presents the linear vector function is diagonal. They are distinguished from non-normal linear vector functions by the fact that any normal linear vector function commutes with its adjoint while no non-normal linear vector function does this.**

The process of determining the canonical (i.e., diagonal) form of a normal linear vector function is termed *diagonalization* of the (normal) linear vector function. It is clear (prove this) that

**The diagonal elements of the canonical form of a linear vector function  $A$  are the characteristic numbers of  $A$ .** *Hint.* Show that if the diagonal elements of the canonical (triangular) form  $A$  of  $A$  are  $\lambda_1, \dots, \lambda_n$  then  $\det(A - \lambda E_n) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$ . In particular

**The diagonal elements of the diagonal form of a normal linear vector function  $A$  are the characteristic numbers of  $A$ . Furthermore each vector of the basis in which  $A$  is diagonal is a characteristic vector of  $A$ . (Prove this.)**

## EXERCISES

1. Show that every Hermitian linear vector function is normal, and use this fact to verify that every characteristic number of a Hermitian linear vector function is real.

2. Show that every unitary linear vector function is normal, and use this fact to verify that every characteristic number of a unitary linear vector function is of unit modulus.

3. Show that every normal linear vector function  $A$  possesses a set of  $n$  unit and mutually perpendicular characteristic vectors  $u_1, \dots, u_n$ .

4. Show that if  $U$  is the unitary matrix of which the elements in the  $j$ th column are the coordinates of  $u_j$  (see Exercise 3) then  $AU = U\Lambda$ , where  $\Lambda$  is the diagonal matrix whose diagonal elements are the characteristic numbers  $(\lambda_1, \dots, \lambda_n)$  of  $A$ . Deduce that  $U^*AU = \Lambda$ .

5. Show that if  $v$  is a characteristic vector of a real linear vector function  $A$ ,

the real and imaginary parts  $\mathbf{v}_1$  and  $\mathbf{v}_2$  of  $\mathbf{v}$  are such that  $A\mathbf{v}_1$  and  $A\mathbf{v}_2$  are linear combinations of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .

*Solution.* That  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are real vectors is clear. In fact if  $z$  is the  $n \times 1$  complex matrix whose elements are the coordinates of  $\mathbf{v}$  and we set  $z = x + yi$ , where  $x$  and  $y$  are real  $n \times 1$  matrices, the equation  $z' = Rz$  implies, in view of the reality of  $R$ , the two equations  $x' = Rx$ ,  $y' = Ry$ . If  $\lambda = \lambda_1 + \lambda_2 i$  is the characteristic number of  $A$  which is associated with the characteristic vector  $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 i$  the equation  $Az = \lambda z$  yields, in view of the reality of  $A$ , the two equations  $Ax = \lambda_1 x - \lambda_2 y$ ;  $Ay = \lambda_2 x + \lambda_1 y$ . Hence  $A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1 - \lambda_2 \mathbf{v}_2$ ;  $A\mathbf{v}_2 = \lambda_2 \mathbf{v}_1 + \lambda_1 \mathbf{v}_2$ .

6. Show that if  $\lambda$  is a non-real characteristic number of a real linear vector function  $A$  the two vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  (see Exercise 5) are linearly independent. *Hint.* If  $\mathbf{v}_1$  and  $\mathbf{v}_2$  were linearly dependent one of them, being a non-zero multiple of  $\mathbf{v}$ , would be a real characteristic vector of  $A$  associated with the non-real characteristic number  $\lambda$ . This is absurd in view of the reality of  $A$ . *Note.* On combining the results of Exercises 5 and 6 we obtain the following theorem:

**Associated with each non-real characteristic number of a real linear vector function is a linearly independent characteristic pair of (real) vectors, i.e., a characteristic plane.**

7. Show that there is associated with each real characteristic number of a real linear vector function a real characteristic vector. *Hint.* If  $\lambda_2 = 0$  (see Exercise 5) so that  $\lambda_1 = \lambda$ ,  $\mathbf{v}_1$  and  $\mathbf{v}_2$  satisfy the relations  $A\mathbf{v}_1 = \lambda\mathbf{v}_1$ ,  $A\mathbf{v}_2 = \lambda\mathbf{v}_2$ .

8. Show that every real Hermitian linear vector function  $A$  possesses, in the real field, a diagonal canonical form; i.e., there exists a rotation matrix  $R$  such that  $A' = R^*AR$  is diagonal. *Hint.* The characteristic numbers of  $A$  are real, and the orthogonalization process of Schmidt may be carried through in the real field. (Prove this.) *Note.* The reality of the characteristic numbers of  $A$  is an essential part of the argument. Thus it is not true that every real normal matrix possesses in the real field a diagonal canonical form (if it did the diagonal elements of the canonical form would be real and these are the characteristic numbers of  $A$ ).

9. Show that the  $2 \times 2$  rotation matrix

$$R_2 = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}; \quad 0 \leq \theta < 2\pi$$

possesses, in the real field, a diagonal canonical form when, and only when,  $\theta = 0$  or  $\pi$ . What is the canonical form of  $R_2$  in the complex field?

10. Show that the  $2 \times 2$  reflexion matrix

$$\begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$$

possesses, for every  $\theta$ , in the real field a canonical diagonal form which is the same for every value of  $\theta$ .

11. Show that every real linear vector function possesses, in the real field, a nearly triangular canonical form (where we mean by the statement that a matrix  $A$  is nearly triangular that  $a_j^k = 0$  if  $k > j + 1$ ). *Hint.* Follow the argument used in the complex field modified in the light of the note to Exercise 6.

12. Show that every normal real linear vector function possesses in the real field a nearly diagonal canonical form, where we mean by the statement that a matrix  $A$  is nearly diagonal that  $a_j^k = 0$  if  $k > j + 1$  and if  $j > k + 1$ .

13. Show that every positive linear vector function  $A$  possesses an unambiguously determinate positive square root defined by the relation  $A = B^2$ .

*Solution.* If the canonical diagonal form of  $A$  has the diagonal elements  $(\lambda_1, \dots, \lambda_n)$  then the diagonal matrix  $B$  whose diagonal elements are  $(\lambda_1^{1/2}, \dots, \lambda_n^{1/2})$  is the canonical form (in the same reference frame) of a linear vector function  $B$  which is such that  $B^2 = A$ . The uniqueness follows from the fact that when  $B$  is in canonical form so also is  $A$  so that the canonical form of  $B$  is unambiguously determined (in view of the positive nature of  $B$ ). It follows that if  $B_1: B_1$  and  $B_2: B_2$  were two determinations of  $B$  then  $B_2 = U^*B_1U$  (why?), and this implies, since  $B_2^2 = A$ ,  $B_1^2 = A$ , that  $A = U^*A U$ . Hence  $U A = A U$ . If  $B_1$  is the canonical diagonal form of  $B_1$ ,  $A$  is the canonical diagonal form of  $A$ , and it follows that if the characteristic numbers of  $A$  are all different that  $U$  is diagonal. (Prove this.) Hence  $B_2 = U^*B_1U = B_1U^*U = B_1$ . (Remember that diagonal matrices commute.) If the characteristic numbers of  $A$  are not all different  $U$  breaks up into diagonal blocks (each block being associated with a set of equal characteristic numbers of  $A$ ), and the relation  $U A = A U$  implies the relation  $U B_1 = B_1 U$ . (Prove this. *Hint.* To each set of equal characteristic numbers of  $A$  corresponds a set of the same number of equal characteristic numbers of  $B_1$ .) Hence, again,  $B_2 = B_1$ .

14. Show that every linear vector function  $A$  can be written in the polar form  $A = P U$ , where  $P$  is an unambiguously determinate positive linear vector function and  $U$  an unambiguously determinate unitary linear vector function. *Hint.*  $AA^*$  is positive and so possesses a positive square root defined by  $AA^* = P^2$ . On setting  $U = P^{-1}A$  it follows that  $U U^* = E_n$  so that  $U$  is unitary. That  $P$  and hence  $U$  are unambiguously determinate follows from the fact that if  $A = P U$  then  $P^2 = AA^*$ . *Note.* When  $n = 1$  this result is the polar factorization  $z = re^{i\theta}$  of complex numbers.

15. Show that every linear vector function  $A$  may be written in the polar form  $A = V Q$ , where  $Q$  is an unambiguously determinate positive linear vector function and  $V$  an unambiguously determinate unitary linear vector function. *Hint.*  $A^*A = Q^2$ .

16. Show that the  $V$  of Exercise 15 is the same as the  $U$  of Exercise 14. *Hint.*  $A = V Q = (V Q V^*) V = P V$ , where  $P = V Q V^*$  is a positive linear vector function.

17. Show that the two polar factors  $P$  and  $U$  of  $A$  (see Exercise 14) commute when, and only when,  $A$  is normal.

18. Show that when  $A$  is Hermitian its polar factor  $U$  is such that  $U^2 = E_n$ .

19. Show that every real linear vector function  $A$  may be factored in the form  $A = R P$ , where  $P$  is a positive real linear vector function and  $R$  is either a rotation or a reflexion.

20. Show that the factor  $R$  of Exercise 19 may be restricted to be a rotation if it is not insisted that  $P$  be positive. *Hint.* Write  $R P = R F F P$ , where  $F$  is the linear vector function for which, in the reference frame in which  $P$  is diagonal,  $F$  is the diagonal matrix whose diagonal elements are  $(-1, 1, \dots, 1)$ . *Note.* In the classical theory of elasticity a homogeneous linear deformation is described by a real three-dimensional linear vector function  $A$ . If  $A$  transforms the arbitrary (real) vector  $v$  into the vector  $w$ :

$$A v = w$$

we may write this in the form

$$R(Pv) = w,$$



where  $\mathbf{P}$  is a real Hermitian (i.e., a symmetric) linear vector function and  $\mathbf{R}$  is a rotation. Thus  $\mathbf{R}$  transforms  $\mathbf{P}\mathbf{v}$  into  $\mathbf{w}$ . The symmetric linear vector function  $\mathbf{P}$  is said to describe a *pure deformation*, and the result  $\mathbf{A} = \mathbf{R}\mathbf{P}$  is phrased as follows:

*Every homogeneous linear deformation is the product of a pure deformation by a rotation.*

The linear vector function  $\frac{1}{2}(\mathbf{A}^*\mathbf{A} - \mathbf{E}_3) = \frac{1}{2}(\mathbf{P}^2 - \mathbf{E}_3)$  is known as the *deformation, or strain, tensor*.

# 3

## FUNCTION-VECTORS; FOURIER SERIES

### 1. Function-vectors

Let  $f(x)$  and  $g(x)$  be two complex-valued functions of the real interval variable  $a \leq x \leq b$ . We assume that the real and imaginary parts of  $f$  and  $g$  are piecewise-continuous functions of the interval variable  $[a, b]$ ; in other words if  $f(x) = f_1(x) + f_2(x)i$ ,  $g(x) = g_1(x) + g_2(x)i$  there exists a net  $a = x_0 < x_1 < \dots < x_n = b$  on  $[a, b]$  such that each of the functions  $f_1(x)$ ,  $f_2(x)$ ,  $g_1(x)$ ,  $g_2(x)$  is continuous over each cell of the net. We denote by  $\bar{g}(x)$  the conjugate complex of  $g(x)$ :

$$\bar{g}(x) = g_1(x) - g_2(x)i$$

and consider the integral

$$\int_a^b \bar{g}(x)f(x) dx.$$

This integral has many points of similarity with the scalar product  $(\mathbf{v}|\mathbf{u})$  of a complex vector  $\mathbf{u}$  by a complex vector  $\mathbf{v}$ . We regard, then, each of the complex-valued functions  $f(x)$  and  $g(x)$  of the real interval variable  $[a, b]$  as a vector, and we denote them, when so regarded, by the symbols  $\mathbf{f}$  and  $\mathbf{g}$ , respectively. We denote the integral  $\int_a^b \bar{g}(x)f(x)dx$  by the symbol  $(\mathbf{g}|\mathbf{f})$ , and we term this integral the *scalar product* of the *function-vector*  $\mathbf{f}$  by the function-vector  $\mathbf{g}$ . Just as for complex vectors, we term the scalar product of a function-vector  $\mathbf{f}$  by itself:

$$(\mathbf{f}|\mathbf{f}) = \int_a^b \bar{f}(x)f(x) dx$$

the *squared magnitude* of the function-vector  $\mathbf{f}$ . When  $(\mathbf{f}|\mathbf{f}) = 1$  the function-vector  $\mathbf{f}$  is termed a *unit vector*. It is clear that  $(\mathbf{f}|\mathbf{f})$  is real and  $\geq 0$  (why?), and it is easy to see that if the equality holds  $f(x)$  must be zero at any point of  $[a, b]$  at which it is continuous. In fact if  $f(x)$  is continuous and different from zero at a point  $t$  of  $[a, b]$  so also is  $\bar{f}f$  and so there exists a neighborhood of  $t$  over which  $\bar{f}f > 0$  (why?);

the integral of  $\bar{f}f$  over this neighborhood  $>0$  (why?) and so  $(\mathbf{f}|\mathbf{f}) > 0$  (why?). In the case of complex vectors the squared magnitude of a vector was zero when, and only when, the vector was the zero vector. In order that this should remain valid for function-vectors we define the zero function-vector  $\mathbf{0}$  as follows:

*If the piecewise-continuous function  $f(x)$  of the interval variable  $[a, b]$  is zero at all points of  $[a, b]$  at which it is continuous the function-vector  $\mathbf{f}$  is the zero function-vector.*

We use the notation  $\mathbf{f}:f(x)$  to indicate that  $\mathbf{f}$  is the function-vector defined by the piecewise-continuous, complex-valued function  $f(x)$  of the real interval variable  $[a, b]$  (it is unnecessary to specify this interval variable since all the functions involved are supposed to be functions of the same interval variable  $[a, b]$ ). We term any constant function  $c$  a *scalar*; then multiplication of a function-vector  $\mathbf{f}$  by a scalar  $c$  is defined by the formula

$$c\mathbf{f}:cf(x),$$

and the addition of two function-vectors  $\mathbf{f}$  and  $\mathbf{g}$  is defined by the formula

$$\mathbf{f} + \mathbf{g}:f(x) + g(x).$$

On combining these we obtain the general *linear combination*

$$c^1\mathbf{f}_1 + c^2\mathbf{f}_2:c^1f_1(x) + c^2f_2(x)$$

of two function-vectors. Two function-vectors are *equal* when their difference is the zero vector. In other words

*The two function-vectors  $\mathbf{f}:f(x)$  and  $\mathbf{g}:g(x)$  are the same if, and only if, the two functions  $f(x)$  and  $g(x)$  have the same value at those points of  $[a, b]$  at which  $f(x) - g(x)$  is continuous.*

It follows (prove this) that equality of function-vectors, defined in this way, is a *transitive* relation; i.e., the two relations  $\mathbf{f} = \mathbf{g}$ ,  $\mathbf{g} = \mathbf{h}$  imply the relation  $\mathbf{f} = \mathbf{h}$ . (In your proof consider the case where  $f - g$  and  $g - h$  are discontinuous and  $f - h$  continuous.)

It is easy to verify (do this) that the scalar product  $(\mathbf{g}|\mathbf{f})$  of a function-vector  $\mathbf{f}$  by a function-vector  $\mathbf{g}$  possesses the following properties of the scalar product  $(\mathbf{v}|\mathbf{u})$  of a complex vector  $\mathbf{u}$  by a complex vector  $\mathbf{v}$ :

- a.  $(\mathbf{f}|\mathbf{g}) = (\overline{\mathbf{g}}|\mathbf{f})$ ;
- b.  $(\mathbf{g}|c\mathbf{f}) = c(\mathbf{g}|\mathbf{f})$ ;  $(c\mathbf{g}|\mathbf{f}) = \bar{c}(\mathbf{g}|\mathbf{f})$ ,

where  $c$  is any scalar;

- c.  $((\mathbf{g}_1 + \mathbf{g}_2)|\mathbf{f}) = (\mathbf{g}_1|\mathbf{f}) + (\mathbf{g}_2|\mathbf{f})$ ;  $(\mathbf{g}|(\mathbf{f}_1 + \mathbf{f}_2)) = (\mathbf{g}|\mathbf{f}_1) + (\mathbf{g}|\mathbf{f}_2)$ .

## EXERCISES

1. Show that  $((d_1\mathbf{g}_1 + d_2\mathbf{g}_2)|(c_1\mathbf{f}_1 + c_2\mathbf{f}_2)) = \bar{d}_1c_1(\mathbf{g}_1|\mathbf{f}_1) + \bar{d}_1c_2(\mathbf{g}_1|\mathbf{f}_2) + \bar{d}_2c_1(\mathbf{g}_2|\mathbf{f}_1) + \bar{d}_2c_2(\mathbf{g}_2|\mathbf{f}_2)$ .

2. Show that the squared magnitude of  $\mathbf{f} + \mathbf{g} = (\mathbf{f}|\mathbf{f}) + (\mathbf{g}|\mathbf{g}) +$  twice the real part of  $(\mathbf{f}|\mathbf{g})$ .

3. Show that *Schwarz's inequality*

$$|(\mathbf{f}|\mathbf{g})|^2 \leq (\mathbf{f}|\mathbf{f})(\mathbf{g}|\mathbf{g})$$

is valid for function-vectors. *Hint.* The proof of Schwarz's inequality for complex vectors is valid. *Note.* When  $(\mathbf{f}|\mathbf{g})$  is real we term the (real) angle  $\theta$  defined by the formula

$$\cos \theta = \frac{(\mathbf{f}|\mathbf{g})}{(\mathbf{f}|\mathbf{f})^{1/2}(\mathbf{g}|\mathbf{g})^{1/2}}$$

(it being understood that neither  $\mathbf{f}$  nor  $\mathbf{g}$  is the zero function-vector) the angle between the two function-vectors  $\mathbf{f}$  and  $\mathbf{g}$ . When  $(\mathbf{f}|\mathbf{g}) = 0$  the two function-vectors  $\mathbf{f}$  and  $\mathbf{g}$  are said to be *perpendicular* or *orthogonal*.

4. Show that the squared magnitude of  $\mathbf{f} + \mathbf{g}$  is less than or equal to the square of  $(\mathbf{f}|\mathbf{f})^{1/2} + (\mathbf{g}|\mathbf{g})^{1/2}$ ; in other words show that

**The magnitude of  $\mathbf{f} + \mathbf{g}$  is less than or equal to the sum of the magnitudes of  $\mathbf{f}$  and  $\mathbf{g}$ .**

*Note.* This is the *triangle inequality* for function-vectors.

5. Show that if  $c = e^{i\theta}$ ,  $\theta$  real, is any complex number of unit modulus then  $c\mathbf{f}$  has the same squared magnitude as  $\mathbf{f}$ .

Complex-valued functions of two or more variables may also be regarded as vectors. It will suffice to illustrate what we mean by considering functions of three variables. The real and imaginary parts  $f_1$  and  $f_2$  of  $f = f_1 + f_2i$  are *point-functions*:

$$f_1 = f_1(x, y, z) = f_1(P); \quad f_2 = f_2(x, y, z) = f_2(P); \quad P: (x, y, z),$$

and we suppose them defined and continuous over a three-dimensional interval

$$a_1 \leq x \leq b_1; \quad a_2 \leq y \leq b_2; \quad a_3 \leq z \leq b_3.$$

The scalar product  $(\mathbf{g}|\mathbf{f})$  of  $\mathbf{f}$  by a second such vector-function  $\mathbf{g}$  is the integral (triple) of  $\bar{g}(P)f(P)$  over this interval. This scalar product possesses (verify this) the properties mentioned above (where we were dealing with complex-valued functions of a *single* variable  $x$ ). The three-dimensional interval may be replaced by any three-dimensional domain; if this domain is unbounded the squares of  $f_1(P)$  and  $f_2(P)$  must be integrable over it.

## 2. Orthonormal sets of function-vectors

Let us consider the function-vectors

$$\mathbf{f}_n: e^{in\pi x}, \quad n = 0, \pm 1, \pm 2, \dots; \quad -\pi \leq x \leq \pi.$$

These constitute a *sequence* of function-vectors of which the first is  $\mathbf{g}_1 = \mathbf{f}_0:1$ , the second  $\mathbf{g}_2 = \mathbf{f}_1:e^{ix}$ , the third  $\mathbf{g}_3 = \mathbf{f}_{-1}:e^{-ix}$ , and so on:

$$\mathbf{g}_1 = \mathbf{f}_0:1; \quad \mathbf{g}_{2j} = \mathbf{f}_j:e^{ijx}; \quad \mathbf{g}_{2j+1} = \mathbf{f}_{-j}:e^{-ijx}; \quad j = 1, 2, \dots$$

It will prove more convenient, however, to indicate this sequence by

$$\{\mathbf{f}_n\}, n = 0, \pm 1, \pm 2, \dots$$

rather than by

$$\{\mathbf{g}_n\}, n = 1, 2, 3, \dots$$

It is clear that all the function-vectors of the sequence have the same squared magnitude  $2\pi$  (prove this) and that the scalar product of any function-vector of the sequence by any other is zero (prove this). If we set  $\mathbf{u}_n = (2\pi)^{-1/2}\mathbf{f}_n$ ,  $n = 0, \pm 1, \pm 2, \dots$  we obtain a sequence of function-vectors possessing the following two properties:

a. Each member of the sequence is a *unit vector*:

$$(\mathbf{u}_n|\mathbf{u}_n) = 1; \quad n = 0, \pm 1, \dots$$

b. Any two members of the sequence are *perpendicular*:

$$(\mathbf{u}_j|\mathbf{u}_k) = 0; \quad j \neq k = 0, \pm 1, \dots$$

We term any set of function-vectors possessing these two properties an *orthonormal set*. The particular orthonormal set defined by the formula:

$$\mathbf{u}_n: (2\pi)^{-1/2}e^{inx}; \quad -\pi \leq x \leq \pi; \quad n = 0, \pm 1, \pm 2, \dots$$

is known as the *exponential orthonormal set* over  $[-\pi, \pi]$ .

The definitions of *linear dependence* and of *linear independence* of a (finite) set of function-vectors is the same as for  $n$ -dimensional complex vectors. Thus the set of  $n$  function-vectors  $\mathbf{f}_1, \dots, \mathbf{f}_n$  is linearly independent if the only linear combination

$$c^a\mathbf{f}_a = c^1\mathbf{f}_1 + \dots + c^n\mathbf{f}_n$$

of them which is the zero function-vector is the trivial one (for which  $c^j = 0$ ,  $j = 1, \dots, n$ ). It is clear that if the  $n$  vectors  $\mathbf{f}_1, \dots, \mathbf{f}_n$  constitute an orthonormal set then  $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$  is a linearly independent set. In fact

$$((c^1\mathbf{f}_1 + \dots + c^n\mathbf{f}_n)|\mathbf{f}_j) = \bar{c}^j$$

(why?) so that, if  $c^1\mathbf{f}_1 + \dots + c^n\mathbf{f}_n = \mathbf{0}$ ,  $c^j = 0$ ,  $j = 1, \dots, n$ . If  $\{\mathbf{u}_n\}$  is any *orthonormal sequence* (finite or infinite) we term the numbers

$$f^j = (\mathbf{u}_j|\mathbf{f}); \quad j = 1, 2, \dots,$$

the *Fourier coefficients* of the vector  $\mathbf{f}$  with respect to the orthonormal sequence (after J. B. J. Fourier [1768–1830], a French mathematician), and we term the vectors  $f^j \mathbf{u}_j = (\mathbf{u}_j | \mathbf{f}) \mathbf{u}_j$  the *components* of the vector  $\mathbf{f}$  with respect to the orthonormal sequence ( $f^j \mathbf{u}_j$  being the component along  $\mathbf{u}_j$ ).

With this understanding the orthogonalization process of Schmidt may be applied to sequences, finite or infinite, of function-vectors (it being understood that if the sequence is finite it is linearly independent while if it is infinite any finite subsequence selected from it is linearly independent). As an illustration let us consider the sequence of *power function-vectors*

$$\mathbf{f}_n : x^{n-1}; \quad n = 1, 2, \dots; \quad -1 \leq x \leq 1.$$

Any finite sequence selected from this infinite sequence of function-vectors is a linearly independent set (for the only polynomial which is zero over  $[-1, 1]$  is the zero polynomial, all of whose coefficients are zero). Since the squared magnitude of  $\mathbf{f}_1 : 1$  is 2 the first step in the orthogonalization process consists in setting

$$\mathbf{u}_1 = 2^{-1/2} \mathbf{f}_1 : 2^{-1/2}; \quad -1 \leq x \leq 1.$$

Since  $2^{-1/2}$  is an even function of  $x$  (being a constant function)  $(\mathbf{u}_1 | \mathbf{g}) = 0$  if  $\mathbf{g}$  is any linear combination of the evenly labelled function-vectors  $\mathbf{f}_2, \mathbf{f}_4, \dots$  (because the function which defines  $\mathbf{g}$ , being a linear combination of  $x, x^3, \dots$ , is an *odd* polynomial). In particular,  $(\mathbf{u}_1 | \mathbf{f}_2) = 0$  so that the component of  $\mathbf{f}_2$  along  $\mathbf{u}_1$  is the zero vector. The second step in the orthogonalization process, then, is to set  $\mathbf{u}_2$  equal to the product of  $\mathbf{f}_2$  by the reciprocal of the magnitude of  $\mathbf{f}_2$ . Since

$$\mathbf{f}_2 : x, \quad (\mathbf{f}_2 | \mathbf{f}_2) = \int_{-1}^1 x^2 dx = \frac{2}{3} \text{ and so } \mathbf{u}_2 = \left(\frac{3}{2}\right)^{1/2} \mathbf{f}_2 : \left(\frac{3}{2}\right)^{1/2} x. \text{ Since } \mathbf{f}_3 : x^2,$$

$$(\mathbf{u}_1 | \mathbf{f}_3) = 2^{-1/2} \int_{-1}^1 x^2 dx = \frac{1}{3} (2^{1/2}); \quad (\mathbf{u}_2 | \mathbf{f}_3) = 0 \text{ (why?) and so}$$

$$\mathbf{u}_3 = \frac{1}{m_3} \left\{ \mathbf{f}_3 - \frac{1}{3} (2^{1/2}) \mathbf{u}_1 \right\},$$

where  $m_3$  is the magnitude of  $\mathbf{f}_3 - \frac{1}{3} (2^{1/2}) \mathbf{u}_1$ . Since  $\mathbf{f}_3 - \frac{1}{3} (2^{1/2}) \mathbf{u}_1 : x^2 - \frac{1}{3}$  we have  $m_3^2 = \int_{-1}^1 \left(x^2 - \frac{1}{3}\right)^2 dx = \frac{8}{45}$  so that  $\frac{1}{m_3} = \left(\frac{3}{2}\right) \left(\frac{5}{2}\right)^{1/2}$ .

Hence

$$\mathbf{u}_3 = \frac{3}{2} \left(\frac{5}{2}\right)^{1/2} \left\{ \mathbf{f}_3 - \frac{1}{3} (2^{1/2}) \mathbf{u}_1 \right\}$$

$$= \left(\frac{5}{2}\right)^{1/2} \left\{ \frac{3}{2} \mathbf{f}_3 - \frac{1}{2} \mathbf{f}_1 \right\} : \left(\frac{5}{2}\right)^{1/2} \left(\frac{3}{2} x^2 - \frac{1}{2}\right).$$

Continuing this process (carry out the calculations) we find

$$\mathbf{u}_4 = \left(\frac{7}{8}\right)^{1/2} \left\{ \frac{5}{8} \mathbf{f}_4 - \frac{3}{2} \mathbf{f}_2 \right\} : \left(\frac{7}{8}\right)^{1/2} \left( \frac{5}{2} x^3 - \frac{3}{2} x \right);$$

$$\mathbf{u}_5 = \left(\frac{9}{8}\right)^{1/2} \left\{ \frac{35}{8} \mathbf{f}_5 - \frac{1}{4} 5 \mathbf{f}_3 + \frac{3}{8} \mathbf{f}_1 \right\} : \left(\frac{9}{8}\right)^{1/2} \left( \frac{35}{8} x^4 - \frac{1}{4} 5 x^2 + \frac{3}{8} \right).$$

The function-vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots$ , obtained in this way constitute an orthonormal infinite sequence.  $\mathbf{u}_{n+1}$ : the product of  $\left(\frac{2n+1}{2}\right)^{1/2}$

by a polynomial of degree  $n$  which is known as the *Legendre polynomial* of degree  $n$  (after A. M. Legendre [1752–1833], a French mathematician) and is denoted by the symbol  $P_n(x)$ ,  $n = 0, 1, 2, \dots$ . Thus we have shown that  $P_0(x) = 1$ ;  $P_1(x) = x$ ;  $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$ ;  $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$ ;  $P_4(x) = \frac{35}{8}x^4 - \frac{1}{4}5x^2 + \frac{3}{8}$ . We shall see later how to obtain  $P_n(x)$  as a particular solution of a linear second order differential equation, and it will appear that, for every  $n = 0, 1, 2, \dots$ ,  $P_n(1) = 1$ .

### EXERCISES

1. Show that  $P_3(x) = \frac{63}{8}x^3 - \frac{35}{4}x^2 + \frac{15}{8}x$ .

2. Show that  $x^0 = P_0$ ,  $x^1 = P_1$ ,  $x^2 = \frac{2}{3}P_2 + \frac{1}{3}P_0$ ,  $x^3 = \frac{2}{5}P_3 + \frac{3}{5}P_1$ .

3. Show that  $\int_{-1}^1 P_j(x)P_k(x) dx = 0$ ,  $j \neq k = 0, 1, 2, \dots$ . *Hint.*  $\mathbf{u}_{j+1}$ :

a multiple of  $P_j$ .

4. Calculate the Fourier coefficients of  $\mathbf{f}:x$  with respect to the exponential orthonormal set over  $[-\pi, \pi]$ .

*Solution.* Denoting the Fourier coefficients by  $f^n$ ,  $n = 0, \pm 1, \pm 2, \dots$ , we have  $f^n = (2\pi)^{-1/2} \int_{-\pi}^{\pi} e^{-ins} x dx$ . Hence  $f^0 = 0$  (why?):  $f^n = (2\pi)^{1/2} (-1)^n \frac{i}{n}$ ,  $n = \pm 1, \pm 2, \dots$ .

5. Calculate the Fourier coefficients of  $\mathbf{f}:x^2$  with respect to the exponential orthonormal set over  $[-\pi, \pi]$ .

*Answer.*  $f^0 = (2\pi)^{1/2} \frac{\pi^2}{3}$ ;  $f^n = \frac{2(2\pi)^{1/2} (-1)^n}{n^2}$ ;  $n = \pm 1, \pm 2, \dots$ .

### 3. Bessel's inequality

Let  $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$  be any finite orthonormal set, and let  $\mathbf{f}$  be any function-vector. We ask the following question: What linear combination  $c^{\alpha} \mathbf{u}_{\alpha} = c^1 \mathbf{u}_1 + \dots + c^n \mathbf{u}_n$  of the vectors of the orthonormal set best approximates  $\mathbf{f}$  in the sense that the squared magnitude of  $\mathbf{f} - c^{\alpha} \mathbf{u}_{\alpha}$  is least?

To answer this question we observe that the squared magnitude of  $\mathbf{f} - c^{\alpha} \mathbf{u}_{\alpha}$  is the scalar product  $((\mathbf{f} - c^{\alpha} \mathbf{u}_{\alpha}) | (\mathbf{f} - c^{\beta} \mathbf{u}_{\beta}))$  (remember that  $c^{\beta} \mathbf{u}_{\beta}$  means precisely the same, namely,  $c^1 \mathbf{u}_1 + \dots + c^n \mathbf{u}_n$ , as does

$c^\alpha \mathbf{u}_\alpha$ ). On developing this scalar product we see that the squared magnitude of  $\mathbf{f} - c^\alpha \mathbf{u}_\alpha = (\mathbf{f}|\mathbf{f}) - c^\beta \overline{f^\beta} - c^\alpha f_\alpha + \overline{c^\alpha} c^\alpha$ . (Remember that  $(\mathbf{u}_j|\mathbf{f}) = f^j$ ,  $(\mathbf{f}|\mathbf{u}_k) = \overline{f^k}$ ,  $(\mathbf{u}_j|\mathbf{u}_k) = 0$ ,  $j \neq k$ ,  $(\mathbf{u}_j|\mathbf{u}_j) = 1$ .) Hence the squared magnitude of  $\mathbf{f} - c^\alpha \mathbf{u}_\alpha = (\mathbf{f}|\mathbf{f}) - \overline{f^\alpha} f^\alpha + (f^\alpha - c^\alpha)(\overline{f^\alpha} - \overline{c^\alpha})$  so that the squared magnitude of  $\mathbf{f} - c^\alpha \mathbf{u}_\alpha$  is least when  $c^j = f^j$ ,  $j = 1, \dots, n$ , this least value being  $(\mathbf{f}|\mathbf{f}) - \overline{f^\alpha} f^\alpha$  (because the sum  $(f^\alpha - c^\alpha)(\overline{f^\alpha} - \overline{c^\alpha}) \geq 0$ ). We have, then, the following result:

The linear combination of the vectors of the orthonormal set  $(\mathbf{u}_1, \dots, \mathbf{u}_n)$  which best approximates a given vector  $\mathbf{f}$ , in the sense that the squared magnitude of the difference between  $\mathbf{f}$  and the linear combination is least, is

$$f^\alpha \mathbf{u}_\alpha = f^1 \mathbf{u}_1 + \dots + f^n \mathbf{u}_n,$$

where  $f^j = (\mathbf{u}_j|\mathbf{f})$  is the Fourier coefficient of  $\mathbf{f}$  with respect to  $\mathbf{u}_j$ . Furthermore the squared magnitude of

$$\mathbf{f} - f^\alpha \mathbf{u}_\alpha = (\mathbf{f}|\mathbf{f}) - \overline{f^\alpha} f^\alpha = (\mathbf{f}|\mathbf{f}) - (\overline{f^1} f^1 + \dots + \overline{f^n} f^n)$$

so that

$$(\mathbf{f}|\mathbf{f}) \geq \overline{f^\alpha} f^\alpha = (\overline{f^1} f^1 + \dots + \overline{f^n} f^n).$$

Let, now,  $\{\mathbf{u}_n\}$  be an infinite orthonormal sequence. It follows from the result just proved that, no matter what is the integer  $n$ ,  $(\mathbf{f}|\mathbf{f}) \geq \overline{f^1} f^1 + \dots + \overline{f^n} f^n$ . Hence (since each of the numbers  $\overline{f^j} f^j$  is real and non-negative):

The series  $\sum_1^\infty \overline{f^j} f^j$  of non-negative terms converges to a sum which is less than or equal to the squared magnitude of  $\mathbf{f}$ :

$$\sum_1^\infty \overline{f^j} f^j \leq (\mathbf{f}|\mathbf{f}).$$

This result is known as *Bessel's inequality* (after F. W. Bessel [1784-1846], a German mathematician).

For  $n$ -dimensional complex vectors the relation corresponding to Bessel's inequality is the *equality*

$$\sum_1^n \overline{v^j} v^j = (\mathbf{v}|\mathbf{v}),$$



where  $v^j = (\mathbf{u}_j | \mathbf{v})$ . In fact the vectors  $\mathbf{u}_j$ ,  $j = 1, \dots, n$ , constitute a basis for the  $n$ -dimensional complex vector space so that

$$\mathbf{v} = \sum_1^n (\mathbf{u}_j | \mathbf{v}) \mathbf{u}_j \text{ (why?)} = \sum_1^n v^j \mathbf{u}_j. \text{ Hence (why?)}$$

$$(\mathbf{v} | \mathbf{v}) = \sum_1^n \overline{v^j} v^j.$$

For some orthonormal infinite sets (in particular for the exponential orthonormal set) it happens that the equality in Bessel's inequality is valid for every function vector  $\mathbf{f}: f(x)$  (where  $f(x)$  is a piecewise-continuous function of  $[a, b]$ ). We term such orthonormal sets *complete* for the following reason:

*The only function-vector which is perpendicular to all the vectors of a complete orthonormal set is the zero function-vector; in other words a complete orthonormal set cannot be enlarged by the addition of a vector which does not belong to it.*

The proof of this statement is immediate: If  $\mathbf{f}$  is perpendicular to all the vectors of a complete orthonormal set all its Fourier coefficients

with respect to this orthonormal set are zero. Hence  $(\mathbf{f} | \mathbf{f}) = \sum_1^\infty \overline{f^j} f^j$

= 0 so that  $\mathbf{f}$  is the zero vector.

### EXERCISE

1. Show that if  $\{\mathbf{u}_n\}$  is a complete orthonormal set  $(\mathbf{g} | \mathbf{f}) = \sum_1^\infty \overline{g^j} f^j$ . *Hint.*

The relation  $((\mathbf{g} + \mathbf{f}) | (\mathbf{g} + \mathbf{f})) = \sum_1^\infty \overline{(g^j + f^j)} (g^j + f^j)$  yields  $(\mathbf{g} | \mathbf{f}) + (\mathbf{f} | \mathbf{g}) =$

$\sum_1^\infty (\overline{g^j} f^j + \overline{f^j} g^j)$ . On replacing  $\mathbf{f}$  by  $i\mathbf{f}$  we obtain  $(\mathbf{g} | \mathbf{f}) - (\mathbf{f} | \mathbf{g}) = \sum_1^\infty (\overline{g^j} f^j - \overline{f^j} g^j)$ ;

hence  $(\mathbf{g} | \mathbf{f}) = \sum_1^\infty \overline{g^j} f^j$ . *Note.* The result of this exercise is known as the *Parseval identity* (after M. A. Parseval, a French mathematician).

If  $\{\mathbf{f}_n\}$  is a sequence of function-vectors we say that this sequence

has the *limit*  $\mathbf{f}$  if the squared magnitude of  $\mathbf{f} - \mathbf{f}_n$  may be made arbitrarily small by merely making  $n$  sufficiently large, i.e., if the squared magnitude of  $\mathbf{f} - \mathbf{f}_n$  is null at  $n = \infty$ . It is immediately evident that the limit  $\mathbf{f}$  of a *convergent sequence*  $\{\mathbf{f}_n\}$ , i.e., of a sequence which possesses a limit, is unambiguously determinate. In fact if  $\mathbf{g}$  were a second limit the squared magnitude of  $\mathbf{f} - \mathbf{g} = (\mathbf{f} - \mathbf{f}_n) + (\mathbf{f}_n - \mathbf{g})$  would be null at  $n = \infty$  (why?). (*Hint.* Remember the triangle inequality.) Being independent of  $n$  it must be zero (why?). Hence  $\mathbf{f} = \mathbf{g}$  (why?).

**EXERCISES**

2. If  $\{\mathbf{f}_n\}$  is a convergent sequence of function-vectors with limit  $\mathbf{f}$  show that the sequence  $(\mathbf{g}|\mathbf{f}_n)$  of complex numbers,  $\mathbf{g}$  being any function-vector, is convergent with limit  $(\mathbf{g}|\mathbf{f})$ . *Hint.* On using Schwarz's inequality it is clear that the sequence  $(\mathbf{g}|\mathbf{f} - \mathbf{f}_n)$  of complex numbers is null at  $n = \infty$ ; this is only another way of saying that the sequence  $(\mathbf{g}|\mathbf{f}_n)$  is convergent with the limit  $(\mathbf{g}|\mathbf{f})$ .

3. Show that if  $\mathbf{1}$  is the function-vector defined by the unit constant function 1 then  $(\mathbf{1}|\mathbf{f}) = \int_a^b f(x) dx$ .

4. Show that if  $\{\mathbf{f}_n\}$  is a convergent sequence of function-vectors with limit  $\mathbf{f}$  then the sequence  $\int_a^b \mathbf{f}_n(x) dx$  of complex numbers converges to the limit  $\int_a^b \mathbf{f} dx$ .

5. Show that if  $[x_0, x]$  is any subinterval of  $[a, b]$  the sequence  $\int_{x_0}^x \mathbf{f}_n(t) dt$  of complex numbers converges to the number  $\int_{x_0}^x \mathbf{f}(t) dt$ ,  $\{\mathbf{f}_n\}$  being a convergent sequence of function-vectors with the limit  $\mathbf{f}$ . *Hint.* Replace the function-vector  $\mathbf{1}$  of Exercise 3 by the function-vector  $\mathbf{g}$  defined by  $g(t) = 1$  if  $x_0 \leq t \leq x$ ,  $g(t) = 0$  if  $a \leq t < x_0$  and if  $x < t \leq b$ .

6. Show that the convergence of the sequence  $\int_{x_0}^x \mathbf{f}_n(t) dt$  of Exercise 5 to its limit  $\int_{x_0}^x \mathbf{f}(t) dt$  is uniform with respect to  $x$  over  $[a, b]$ . *Hint.*  $|(\mathbf{g}|\mathbf{f} - \mathbf{f}_n)|^2 \leq (\mathbf{g}|\mathbf{g})(\mathbf{f} - \mathbf{f}_n|\mathbf{f} - \mathbf{f}_n) = (x - x_0)(\mathbf{f} - \mathbf{f}_n|\mathbf{f} - \mathbf{f}_n) \leq (b - a)(\mathbf{f} - \mathbf{f}_n|\mathbf{f} - \mathbf{f}_n)$ . *Note.* On combining the results of Exercises 4-6 we obtain the following important theorem:

*If a sequence of function-vectors  $\{\mathbf{f}_n\}$  converges to the function-vector  $\mathbf{f}$  the sequence of complex numbers  $\int_{x_0}^x \mathbf{f}_n(t) dt$ , where  $a \leq x_0 < x \leq b$ , converges to the complex number  $\int_{x_0}^x \mathbf{f}(t) dt$ , the convergence being uniform with respect to  $x$  over  $[a, b]$ .*

7. Show that if  $g(x)$  is any piecewise-continuous function a continuous function  $f(x)$  may be determined, once any positive number  $\epsilon$  is given, such that the squared magnitude of  $\mathbf{g} - \mathbf{f}$ :  $g(x) - f(x)$  is not greater than  $\epsilon$ . *Hint.* Consider first the

case where  $g(x)$  has only one point of discontinuity  $x_1$ , and set  $f(x) = g(x)$  if  $x$  is not covered by the interval  $[x_1 - \delta, x_1 + \delta]$ , where  $\delta > 0$  will be determined later; over  $[x_1 - \delta, x_1 + \delta]$ ,  $f(x)$  may be any continuous function which takes the same values as  $g(x)$  at the points  $x_1 - \delta$  and  $x_1 + \delta$ . If  $x_1 = a$  replace  $[x_1 - \delta, x_1 + \delta]$  by  $[a, a + \delta]$ , and if  $x_1 = b$  replace  $[x_1 - \delta, x_1 + \delta]$  by  $[b - \delta, b]$ . Both  $g(x)$  and  $f(x)$  are bounded over  $[a, b]$  (why?) and so there exists a number  $M$  such that  $|g(x) - f(x)| \leq M$  over  $[a, b]$ . Hence  $(\mathbf{g} - \mathbf{f}) | (\mathbf{g} - \mathbf{f}) \leq 2M^2\delta$  (why?), and all we have to do is to choose  $\delta$  so that  $2M^2\delta \leq \epsilon$ . If  $g(x)$  has  $n$  points of discontinuity  $(x_1, \dots, x_n)$  set  $f(x) = g(x)$  if  $x$  is not covered by any of the  $n$  intervals  $[x_j - \delta, x_j + \delta]$ ,  $j = 1, \dots, n$ , and choose  $\delta$  so that  $2nM^2\delta \leq \epsilon$ .

8. Show that the infinite orthonormal sequence  $\{\mathbf{u}_n\}$  is complete if Bessel's inequality is an equality for every function-vector  $\mathbf{f}: f(x)$  for which  $f(x)$  is continuous. *Hint.* Let  $\mathbf{g}: g(x)$  be any function-vector (so that  $g(x)$  is piecewise-continuous over  $[a, b]$ ), and let  $\mathbf{f}: f(x)$  be a function-vector for which  $f(x)$  is continuous over  $[a, b]$ , and the squared magnitude of  $\mathbf{g} - \mathbf{f}$  is  $\leq \epsilon$ . If  $\mathbf{s}_n = \sum_1^n f^j \mathbf{u}_j$  we

are granted that the squared magnitude of  $\mathbf{f} - \mathbf{s}_n$  is null at  $n = \infty$  (why?). Hence, by the triangle inequality, the squared magnitude of  $\mathbf{g} - \mathbf{s}_n = (\mathbf{g} - \mathbf{f}) + (\mathbf{f} - \mathbf{s}_n)$

is null at  $n = \infty$ . If  $\mathbf{s}'_n = \sum_1^n g^j \mathbf{u}_j$ , the squared magnitude of  $\mathbf{g} - \mathbf{s}'_n$  is not greater

than the squared magnitude of  $\mathbf{g} - \mathbf{s}_n$  (why?); hence the squared magnitude of  $\mathbf{g} - \mathbf{s}'_n$  is null at  $n = \infty$ . In other words Bessel's inequality reduces to an equality for the arbitrary function-vector  $\mathbf{g}: g(x)$ , where  $g(x)$  is piecewise-continuous.

#### 4. Fourier series

Let  $\{\mathbf{u}_n\}$  be a complete infinite orthonormal sequence, and let  $\mathbf{f}$  be any function-vector (so that  $f(x)$  is a piecewise-continuous function of the interval variable  $[a, b]$  which is attached to the orthonormal sequence). The Fourier coefficients of  $\mathbf{f}$  with respect to the orthonormal sequence are defined by the formula

$$f^j = (\mathbf{u}_j | \mathbf{f}).$$

The linear combination of the first  $n$  members of the orthonormal sequence which best approximates (in what sense?) the function-vector  $\mathbf{f}$  is

$$\sum_1^n f^j \mathbf{u}_j = f^1 \mathbf{u}_1 + \dots + f^n \mathbf{u}_n.$$

If we denote this function-vector by  $\mathbf{s}_n$  it is easy to see that the sequence  $\{\mathbf{s}_n\}$  of function-vectors has at  $n = \infty$  the limit  $\mathbf{f}$  (what does this mean?). In fact the squared magnitude of  $\mathbf{f} - \mathbf{s}_n$  is  $(\mathbf{f} | \mathbf{f}) - \sum_1^n f^j f^j$

(prove this), and this is null at  $n = \infty$  (since the orthonormal sequence  $\{\mathbf{u}_n\}$  is complete). We say, then, that  $\mathbf{f}$  is the *sum* of the *infinite series* of function-vectors  $f^i \mathbf{u}_i$ , and we write

$$\mathbf{f} = \sum_1^{\infty} f^i \mathbf{u}_i = f^1 \mathbf{u}_1 + f^2 \mathbf{u}_2 + \cdots + f^n \mathbf{u}_n + \cdots .$$

In other words

*Every complete orthonormal sequence  $\{\mathbf{u}_n\}$  is such that the sequence of function-vectors  $\{\mathbf{s}_n\}$ , where*

$$\mathbf{s}_n = \sum_1^n f^i \mathbf{u}_i = f^1 \mathbf{u}_1 + \cdots + f^n \mathbf{u}_n,$$

*converges to the (arbitrary) function-vector  $\mathbf{f}$  whose Fourier coefficients with respect to the orthonormal sequence are*

$$f^j = (\mathbf{u}_j | \mathbf{f}); \quad j = 1, 2, \cdots .$$

We term the series  $\sum_1^{\infty} f^i \mathbf{u}_i(x) = f^1 \mathbf{u}_1(x) + \cdots + f^n \mathbf{u}_n(x) + \cdots$  the

*Fourier series* of the function  $f(x)$  with respect to the given orthonormal set, and we express the convergence of the sequence  $\{\mathbf{s}_n\}$  of function-vectors by the statement that the Fourier series of an arbitrary (piecewise-continuous) function  $f(x)$  with respect to any complete orthonormal sequence *converges in the mean* to  $f(x)$ . The theorem of the note to Exercise 6 of the preceding section may, then, be phrased as follows:

**If the Fourier series of any piecewise-continuous function-vector  $\mathbf{f}$  with respect to any complete orthonormal set is integrated term by term over any subinterval  $[x_0, x]$  of the basic interval  $[a, b]$  the resulting series converges to the limit  $\int_{x_0}^x f(t) dt$ , the convergence being uniform with respect to  $x$  over  $[a, b]$ .**

The fact that the exponential orthonormal set is complete is proved in a course in advanced calculus, and we shall not repeat the proof here but shall merely recall the *guiding idea* of the proof (which is due to L. Féjér, a Hungarian mathematician). Let  $f(x)$  be any function which is continuous over  $[-\pi, \pi]$ , and denote by  $s_n$  the sum of the  $2n + 1$  "central" terms of the Fourier series of  $f(x)$  with respect to the exponential orthonormal set over  $[-\pi, \pi]$ :

$$s_n(x) = \sum_{-n}^n f^j u_j(x) = (2\pi)^{-1/2} \sum_{-n}^n f^j e^{ijx}; \quad n = 0, \pm 1, \dots$$

Denote by  $S_{n-1}(x)$  the *average* of the  $n$  functions  $s_0, s_1, \dots, s_{n-1}$ :

$$S_{n-1}(x) = \frac{s_0(x) + s_1(x) + \dots + s_{n-1}(x)}{n}; \quad n = 1, 2, \dots$$

It is then proved that  $f(x)$  may be approximated arbitrarily closely, uniformly over  $[-\pi, \pi]$ , by  $S_{n-1}$  by merely making  $n$  large enough. In other words if  $\epsilon$  is any positive number a positive integer  $N$  can be found such that  $|f(x) - S_{n-1}(x)| \leq \epsilon$  for every  $x$  covered by the interval  $[-\pi, \pi]$  provided only that  $n \geq N$ . Now  $S_{n-1}: S_{n-1}(x)$  is a linear combination of the  $2n - 1$  "central" function-vectors  $u_j$ ,  $j = 0, \pm 1, \dots, \pm(n - 1)$ , of the exponential orthonormal set, and the squared magnitude of  $f - S_{n-1}$  is null at  $n = \infty$  (being  $\leq 2\pi\epsilon^2$  (why?)). Hence the squared magnitude of  $f - s_{n-1}$  is null at  $n = \infty$  (why?); in other words Bessel's inequality reduces to an equality for the arbitrary continuous function-vector  $f$ . Hence (see Exercise 8, p. 74) the exponential orthonormal set is complete.

It follows at once from the result that  $f(x)$  may be approximated arbitrarily closely, uniformly over  $[-\pi, \pi]$ , by  $S_{n-1}$  that  $f(x)$  may be approximated arbitrarily closely, uniformly over  $[-\pi, \pi]$ , by a *polynomial function*. In fact  $e^{ijx}$  is defined by an everywhere convergent power series which, accordingly, converges uniformly over  $[-\pi, \pi]$ . This is only another way of saying that  $e^{ijx}$  is approximated arbitrarily closely, uniformly over  $[-\pi, \pi]$ , by the partial sums of its defining power series and these partial sums are polynomials. Now  $S_{n-1}(x)$  is a linear combination of the  $2n - 1$  functions  $e^{ijx}$ ,  $j = 0, \pm 1, \dots, \pm(n - 1)$ , and hence  $S_{n-1}(x)$  may be approximated arbitrarily closely, uniformly over  $[-\pi, \pi]$ , by a polynomial (why?). Hence  $f(x)$  may be approximated arbitrarily closely, uniformly over  $[-\pi, \pi]$ , by a polynomial (why?). Let, now,  $[a, b]$  be *any* closed interval, and let  $f(x)$  be any function which is continuous over  $[a, b]$ ; the substitution

$$x = \frac{b-a}{2\pi} \xi + \frac{a+b}{2}$$

transforms the interval  $a \leq x \leq b$  into the interval  $-\pi \leq \xi \leq \pi$  (prove this) and the function  $f(x)$  into the function  $\phi(\xi) = f(x)$  which is continuous over  $[-\pi, \pi]$  (why?). Hence  $\phi(\xi)$  may be approximated arbitrarily closely, uniformly over  $[-\pi, \pi]$ , by a polynomial function

of  $\xi$ . Since a polynomial function of  $\xi$  may be written as a polynomial function of  $x$  this proves the following important theorem (which is due to K. Weierstrass [1815–1897], a German mathematician):

**Every continuous function of a closed interval variable may be approximated arbitrarily closely, uniformly over the closed interval, by a polynomial.**

It follows at once from this theorem that the power-function orthonormal set (attached to the closed interval  $[-1, 1]$ ) is complete. In fact any polynomial is a finite linear combination of the functions  $u_j(x)$  which define the power-function orthonormal sequence  $\{u_n\}$ . Since the squared magnitude of the difference between any continuous function-vector  $\mathbf{f}$  and such a linear combination is null at  $n = \infty$  (why?) it follows that the power-function orthonormal set is complete (why?). Thus

*The Fourier series of every piecewise-continuous function of the interval variable  $[-1, 1]$  with respect to the power-function orthonormal set converges in the mean to the function.*

*Warning.* Be very sure that you understand clearly that this result says nothing about what happens at any specified point  $x$  of the interval  $[-1, 1]$ . If we denote the sum of the first  $n$  terms of the Fourier series of  $f(x)$  by  $s_n(x)$  it is true that the sequence of function-vectors  $\{s_n\}$  converges to the function-vector  $\mathbf{f}$  (what does this mean?), but this tells us nothing about the convergence of the sequence of complex numbers  $\{s_n(x)\}$ .

## EXERCISES

1. Show that the only continuous function of the closed interval variable  $[-1, 1]$  which is such that  $\int_{-1}^{+1} P_n(x) f(x) dx = 0$ ,  $n = 0, 1, 2, \dots$ , is the zero constant function ( $P_n(x)$  being the Legendre polynomial of degree  $n$ ).

2. State and prove the theorem concerning piecewise-continuous functions which corresponds to that of Exercise 1 concerning continuous functions.

When the complex-valued function  $f(x)$  of the closed-interval variable  $[-\pi, \pi]$  is real it is convenient to write the Fourier series of  $f(x)$  with respect to the exponential orthonormal set over  $[-\pi, \pi]$  in a real form. Since  $u_j(x) = (2\pi)^{-1/2} e^{ijx}$  we have  $u_{-j}(x) = \overline{u_j(x)}$  and so  $f^{-j} = \overline{f^j}$  (why?). Hence  $f^{-j} u_{-j}(x) = \overline{f^j u_j(x)}$  and so the sum  $\sum_{-(n-1)}^{n-1} f^j u_j(x)$  may be written as

$$\begin{aligned}
 f^0 u_0(x) + \sum_1^{n-1} \{f^{-j} u_{-j}(x) + f^j u_j(x)\} \\
 = f^0 u_0(x) + \sum_1^{n-1} 2 \{\text{real part of } f^j u_j(x)\}.
 \end{aligned}$$

Since  $u_j(x) = (2\pi)^{-1/2} e^{ijx} = (2\pi)^{-1/2} (\cos jx + i \sin jx)$  we have

$$f^j = (\mathbf{u}_j | \mathbf{f}) = (2\pi)^{-1/2} \left\{ \int_{-\pi}^{\pi} \cos jt f(t) dt - i \int_{-\pi}^{\pi} \sin jt f(t) dt \right\}.$$

Hence  $f^0 u_0(x) = (2\pi)^{-1} \int_{-\pi}^{\pi} f(t) dt$  while  $f^{-j} u_{-j}(x) + f^j u_j(x)$ , i.e., twice the real part of  $f^j u_j(x)$ ,

$$= \pi^{-1} \left\{ \left( \int_{-\pi}^{\pi} \cos jt f(t) dt \right) \cos jx + \left( \int_{-\pi}^{\pi} \sin jt f(t) dt \right) \sin jx \right\}.$$

Adopting the notations

$$a_0 = \pi^{-1} \int_{-\pi}^{\pi} f(t) dt; \quad a_j = \pi^{-1} \int_{-\pi}^{\pi} \cos jt f(t) dt; \quad b_j = \pi^{-1} \int_{-\pi}^{\pi} \sin jt f(t) dt; \\ j = 1, 2, \dots,$$

the Fourier series of  $f(x)$  appears as

$$\frac{a_0}{2} + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx).$$

The numbers  $a_0, a_1, \dots$  are known as the *Fourier cosine-coefficients* and the numbers  $b_n$  are known as the *Fourier sine-coefficients* of the real-valued function  $f(x)$ . We indicate that a function  $f(x)$  has Fourier cosine-coefficients  $a_n, n = 0, 1, 2, \dots$ , and Fourier sine-coefficients  $b_n, n = 1, 2, \dots$ , by the notation:

$$f(x) \sim \frac{a_0}{2} + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx); \quad -\pi \leq x \leq \pi.$$

*Suggestion.* In calculating the Fourier cosine- and sine-coefficients of a given real-valued function  $f(x)$  it is best to first calculate  $f^j$  and to use the relation

$$(2\pi)^{1/2} f^j = \pi(a_j - ib_j)$$

or, equivalently,

$$a_j - ib_j = \left(\frac{2}{\pi}\right)^{1/2} f_j.$$

For example we have seen (Exercise 4, p. 70) that, for  $f(x) = x$ ,  $f_j = \frac{(2\pi)^{1/2}(-1)^j i}{j}$ ; hence  $a_j = 0$ ,  $b_j = \frac{2(-1)^{j-1}}{j}$  so that

$$\frac{x}{2} \sim \frac{\sin x}{1} - \frac{\sin 2x}{2} + \cdots + (-1)^{n-1} \frac{\sin nx}{n} + \cdots$$

### EXERCISES

3. Show that the Fourier cosine-coefficients of an odd function and the Fourier sine-coefficients of an even function are all zero.

4. Show that

$$\frac{x^2}{4} \sim \frac{\pi^2}{12} - \left\{ \frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \cdots + (-1)^{n-1} \frac{\cos nx}{n^2} + \cdots \right\}$$

5. Show that if the Fourier series of a continuous function  $f(x)$  with respect to the exponential orthonormal set converges uniformly over  $[-\pi, \pi]$  its sum is  $f(x)$  over  $[-\pi, \pi]$ . *Hint.* Since the Fourier series may be integrated term by term (after multiplication by  $\bar{u}_j(x)$ ) its sum  $s(x)$  has the same Fourier coefficients as  $f(x)$  (why?). Since  $f(x)$  and  $s(x)$  are both continuous over  $[-\pi, \pi]$  (why?)  $s(x) - f(x) = 0$  over  $[-\pi, \pi]$  (why?).

6. Show that the symbol  $\sim$  in Exercise 4 may be replaced by  $=$ .

7. Show, by integrating the Fourier series for  $\frac{x}{2}$  from 0 to  $x$ , that  $\frac{x^2}{4} = (1 - \cos x) - \frac{1}{2^2}(1 - \cos 2x) + \cdots + \frac{(-1)^{n-1}}{n^2}(1 - \cos nx) + \cdots$ , and

deduce (by comparing this result with that of Exercise 6) that  $\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$ .

8. Determine the Fourier series, with respect to the exponential orthonormal set over  $[-\pi, \pi]$ , of the odd function which = 1 if  $0 < x \leq \pi$ . (Give your answer in real form.)

$$\text{Answer. } \frac{4}{\pi} \left( \frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots + \frac{\sin (2n+1)x}{2n+1} + \cdots \right).$$

9. Determine the Fourier series, with respect to the exponential orthonormal set over  $[-\pi, \pi]$ , of  $|x|$ . (Give your answer in real form.) *Hint.* Integrate term by term the Fourier series of Exercise 8 and deduce from the result of Exercise 7

that  $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$ .

$$\text{Answer. } \frac{\pi}{2} - \frac{4}{\pi} \left( \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \cdots + \frac{\cos (2n+1)x}{(2n+1)^2} + \cdots \right).$$



10. Show that the sum of the Fourier series of Exercise 9 is  $|x|$ . *Hint.* Use the theorem of Exercise 5.

### 5. Linear integral operators

The analogue for function-vectors of the linear vector function concept for  $n$ -dimensional complex vectors may be described as follows. Let

$K \begin{pmatrix} x \\ t \end{pmatrix}$  be a continuous complex-valued function of the two-dimensional interval

$$a \leq x \leq b; \quad a \leq t \leq b;$$

if  $\mathbf{f}: f(x)$  is any function-vector attached to the interval  $[a, b]$  the function of  $x$

$$\int_a^b K \begin{pmatrix} x \\ t \end{pmatrix} f(t) dt$$

is a continuous function of the interval variable  $[a, b]$ . (Prove this.) It defines, then, a function-vector which we denote by the symbol  $\mathbf{Kf}$ .  $\mathbf{K}$  is an *integral operator* which feeds on function-vectors  $\mathbf{f}$  to produce function-vectors  $\mathbf{Kf}$ ; and we say that  $\mathbf{Kf}$  is the result of *operating upon*  $\mathbf{f}$  by the integral operator  $\mathbf{K}$ . The following properties are immediate consequences of the definition of  $\mathbf{Kf}$ . (Prove them.)

1. If  $c$  is any scalar, i.e., any constant function,

$$\mathbf{K}(c\mathbf{f}) = c(\mathbf{Kf})$$

2. If  $\mathbf{f}$  and  $\mathbf{g}$  are any two function-vectors

$$\mathbf{K}(\mathbf{f} + \mathbf{g}) = \mathbf{Kf} + \mathbf{Kg}.$$

It follows that if  $c^1\mathbf{f}_1 + c^2\mathbf{f}_2$  is any linear combination of any two function-vectors  $\mathbf{f}_1$  and  $\mathbf{f}_2$  then

$$\mathbf{K}(c^1\mathbf{f}_1 + c^2\mathbf{f}_2) = c^1\mathbf{Kf}_1 + c^2\mathbf{Kf}_2.$$

If  $c^\alpha\mathbf{f}_\alpha = c^1\mathbf{f}_1 + \cdots + c^n\mathbf{f}_n$  is any linear combination of any set of  $n$  function-vectors we have

$$\mathbf{K}(c^\alpha\mathbf{f}_\alpha) = c^\alpha(\mathbf{Kf}_\alpha) = c^1(\mathbf{Kf}_1) + \cdots + c^n(\mathbf{Kf}_n).$$

We express this result by saying that the integral operator  $\mathbf{K}$  is a *linear integral operator*.

If  $K \begin{pmatrix} x \\ t \end{pmatrix}$  is the zero constant function (over the two-dimensional interval  $a \leq x \leq b, a \leq t \leq b$ )  $\mathbf{Kf}$  is the (continuous) zero vector no

matter what is the vector  $\mathbf{f}$ . Conversely if  $\mathbf{K}\mathbf{f}$  is the zero vector (no matter what is the vector  $\mathbf{f}$ )  $K \begin{pmatrix} x \\ t \end{pmatrix}$  must be the zero constant function.

In fact if  $K \begin{pmatrix} x_1 \\ t_1 \end{pmatrix} \neq 0$  there is a (one-dimensional) neighborhood of the point  $t_1$  over which  $K \begin{pmatrix} x_1 \\ t \end{pmatrix}$  is one-signed. Choosing  $f(t)$  to be zero outside this neighborhood and to be  $= K \begin{pmatrix} x_1 \\ t \end{pmatrix}$  over this neighborhood we have  $\int_a^b K \begin{pmatrix} x_1 \\ t \end{pmatrix} f(t) dt > 0$ . In view of the continuity of

$$\int_a^b K \begin{pmatrix} x \\ t \end{pmatrix} f(t) dt$$

it follows (how?) that  $\mathbf{K}\mathbf{f}$  is not the zero vector. We term the linear integral operator which transforms every function-vector  $\mathbf{f}$  into the zero function-vector  $\mathbf{0}$  the *zero operator*, and we denote it by the symbol  $\mathbf{O}$ :

$$\mathbf{O}\mathbf{f} = \mathbf{0}; \text{ every } \mathbf{f}.$$

Multiplication of an operator  $\mathbf{K}: K \begin{pmatrix} x \\ t \end{pmatrix}$  by a scalar  $c$  is defined by the formula  $c\mathbf{K}: cK \begin{pmatrix} x \\ t \end{pmatrix}$ , and the addition of two linear integral operators  $\mathbf{K}_1$  and  $\mathbf{K}_2$  is defined by the formula

$$\mathbf{K}_1 + \mathbf{K}_2: K_1 \begin{pmatrix} x \\ t \end{pmatrix} + K_2 \begin{pmatrix} x \\ t \end{pmatrix}.$$

Two operators  $\mathbf{K}_1$  and  $\mathbf{K}_2$  are *equal* when, and only when, their difference (what is this?) is the zero operator. It follows at once from these definitions (prove this) that

1.  $(c\mathbf{K})\mathbf{f} = c\mathbf{K}\mathbf{f} = \mathbf{K}(c\mathbf{f});$
2.  $(\mathbf{K}_1 + \mathbf{K}_2)\mathbf{f} = \mathbf{K}_1\mathbf{f} + \mathbf{K}_2\mathbf{f}.$

Finally the multiplication of a linear integral operator  $\mathbf{K}_2$  by a linear integral operator  $\mathbf{K}_1$  is defined by the formula

$$(\mathbf{K}_1\mathbf{K}_2)f = \mathbf{K}_1(\mathbf{K}_2f) : \int_a^b K_1 \begin{pmatrix} x \\ t_1 \end{pmatrix} \left\{ \int_a^b K_2 \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} f(t_2) dt_2 \right\} dt_1.$$

In view of the continuity of  $K_1 \begin{pmatrix} x \\ t \end{pmatrix}$  and  $K_2 \begin{pmatrix} x \\ t \end{pmatrix}$  and of the piecewise continuity of  $f(t)$  the order of the repeated integration may be interchanged. We find, then, that

$$\mathbf{K}_1\mathbf{K}_2 : \int_a^b K_1 \begin{pmatrix} x \\ t_1 \end{pmatrix} K_2 \begin{pmatrix} t_1 \\ t \end{pmatrix} dt_1.$$

It follows at once that the multiplication of linear integral operators is *associative* since

$$(\mathbf{K}_1\mathbf{K}_2)\mathbf{K}_3 : \int_a^b \int_a^b \left\{ K_1 \begin{pmatrix} x \\ t_1 \end{pmatrix} K_2 \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \right\} K_3 \begin{pmatrix} t_2 \\ t \end{pmatrix} dt_1 dt_2$$

$$\mathbf{K}_1(\mathbf{K}_2\mathbf{K}_3) : \int_a^b \int_a^b K_1 \begin{pmatrix} x \\ t_1 \end{pmatrix} \left\{ K_2 \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} K_3 \begin{pmatrix} t_2 \\ t \end{pmatrix} \right\} dt_1 dt_2.$$

We denote the various *powers* of a linear integral operator  $\mathbf{K} : K \begin{pmatrix} x \\ t \end{pmatrix}$  by the symbol  $\mathbf{K}^n$ ; thus

$$\mathbf{K}^2 = \mathbf{K}\mathbf{K}; \quad \mathbf{K}^3 = \mathbf{K}\mathbf{K}^2 = \mathbf{K}^2\mathbf{K} = \mathbf{K}\mathbf{K}\mathbf{K},$$

and so on.

*Warning.* Do not fall into the mistake of thinking that

$$\mathbf{K}^2 : \left\{ K \begin{pmatrix} x \\ t \end{pmatrix} \right\}^2.$$

In multiplying  $\mathbf{K}$  by  $\mathbf{K}$  we must proceed as in matrix multiplication where we understand that the upper variable  $x$  in the symbol  $K \begin{pmatrix} x \\ t \end{pmatrix}$  tells the row and the lower the column. Thus

$$\mathbf{K}^2 = \mathbf{K}\mathbf{K} : \int_a^b K \begin{pmatrix} x \\ t_1 \end{pmatrix} K \begin{pmatrix} t_1 \\ t \end{pmatrix} dt_1.$$

EXERCISES

1. Show that if  $r$  and  $s$  are two positive integers whose sum is  $n$  then  $\mathbf{K}^n = \mathbf{K}^r \mathbf{K}^s$ . In other words  $\mathbf{K}^n = \mathbf{K} \mathbf{K}^{n-1} = \mathbf{K}^2 \mathbf{K}^{n-2} = \dots = \mathbf{K}^{n-1} \mathbf{K}$ .

2. Show that if  $r_1, r_2, \dots, r_p$  are  $p$  positive integers whose sum is  $n$  then  $\mathbf{K}^n = \mathbf{K}^{r_1} \mathbf{K}^{r_2} \dots \mathbf{K}^{r_p}$ .

Associated with each linear integral operator  $\mathbf{K}$  and each pair of function-vectors  $\mathbf{f}$  and  $\mathbf{g}$  is the *bilinear form*

$$(\mathbf{g}|\mathbf{K}\mathbf{f}) = \int_a^b \int_a^b \bar{g}(x) K \begin{pmatrix} x \\ t \end{pmatrix} f(t) dt dx.$$

Exactly as for linear vector functions of  $n$ -dimensional complex vectors we see that this form is linear (in the sense of the scalar product of complex vectors) in each of the vectors  $\mathbf{f}$  and  $\mathbf{g}$ :

$$((\mathbf{g}_1 + \mathbf{g}_2)|\mathbf{K}\mathbf{f}) = (\mathbf{g}_1|\mathbf{K}\mathbf{f}) + (\mathbf{g}_2|\mathbf{K}\mathbf{f});$$

$$(c\mathbf{g}|\mathbf{K}\mathbf{f}) = \bar{c}(\mathbf{g}|\mathbf{K}\mathbf{f})$$

$$(\mathbf{g}|\mathbf{K}(c\mathbf{f})) = c(\mathbf{g}|\mathbf{K}\mathbf{f});$$

$$(\mathbf{g}|\mathbf{K}(\mathbf{f}_1 + \mathbf{f}_2)) = (\mathbf{g}|\mathbf{K}\mathbf{f}_1) + (\mathbf{g}|\mathbf{K}\mathbf{f}_2).$$

When  $\mathbf{g} = \mathbf{f}$  we obtain the *quadratic form*

$$(\mathbf{f}|\mathbf{K}\mathbf{f}) = \int_a^b \int_a^b \bar{f}(x) K \begin{pmatrix} x \\ t \end{pmatrix} f(t) dt dx.$$

Associated with each linear integral operator  $\mathbf{K}$  is another linear operator which we term the *adjoint* of  $\mathbf{K}$  and denote by the symbol  $\mathbf{K}^*$ ; it is defined by the formula

$$(\mathbf{g}|\mathbf{K}\mathbf{f}) = (\mathbf{K}^*\mathbf{g}|\mathbf{f}); \text{ every } \mathbf{f} \text{ and } \mathbf{g}.$$

Since  $(\mathbf{K}\mathbf{g}|\mathbf{f}) = \overline{(\mathbf{f}|\mathbf{K}\mathbf{g})} = \overline{(\mathbf{K}^*\mathbf{f}|\mathbf{g})} = (\mathbf{g}|\mathbf{K}^*\mathbf{f})$  we may phrase the definition of the adjoint linear integral operator  $\mathbf{K}^*$  of  $\mathbf{K}$  as follows:

*The bilinear form  $(\mathbf{g}|\mathbf{K}\mathbf{f})$ , or  $(\mathbf{K}\mathbf{g}|\mathbf{f})$ , is unaffected when the linear integral operator  $\mathbf{K}$  is transferred from one of the two function-vectors  $\mathbf{f}$ ,  $\mathbf{g}$  to the other provided that when this is done  $\mathbf{K}$  is replaced by its adjoint  $\mathbf{K}^*$ .*

EXERCISES

3. Show that  $(\mathbf{g}|\mathbf{K}_1\mathbf{f}) = (\mathbf{g}|\mathbf{K}_2\mathbf{f})$  for every pair of function-vectors  $\mathbf{f}$  and  $\mathbf{g}$  when, and only when,  $\mathbf{K}_1 = \mathbf{K}_2$ .

4. Show that if  $\mathbf{K}: K \begin{pmatrix} x \\ t \end{pmatrix}$  then  $\mathbf{K}^*: K^* \begin{pmatrix} x \\ t \end{pmatrix} = \overline{K \begin{pmatrix} t \\ x \end{pmatrix}}$ . *Hint.*  $\overline{(g|\mathbf{K}f)} = (f|\mathbf{K}^*g)$ .

5. Show that  $(f|\mathbf{K}_1f) = (f|\mathbf{K}_2f)$  for every function-vector  $f$  when, and only when,  $\mathbf{K}_1 = \mathbf{K}_2$ . *Hint.* Replace  $f$ , in turn, by  $f + g$  and  $if + g$ , and use the result of Exercise 3.

6. Show that  $(\mathbf{K}_1\mathbf{K}_2)^* = \mathbf{K}_2^*\mathbf{K}_1^*$ . *Hint.*  $(g|\mathbf{K}_1\mathbf{K}_2f) = (\mathbf{K}_1^*g|\mathbf{K}_2f) = (\mathbf{K}_2^*\mathbf{K}_1^*g|f)$ .

A linear integral operator  $\mathbf{K}$  is said to be *self-adjoint* or *Hermitian* when  $\mathbf{K}^* = \mathbf{K}$ . In other words

$\mathbf{K}$  is Hermitian when, and only when, the bilinear form  $(g|\mathbf{K}f)$  is insensitive to a transfer of the operator  $\mathbf{K}$  from the vector  $f$  to the vector  $g$ :

$$(g|\mathbf{K}f) = (\mathbf{K}g|f).$$

An equivalent formulation is the following (prove this): The linear integral operator  $\mathbf{K}$  is Hermitian when, and only when, an interchange of the two function vectors  $f, g$  in the bilinear form  $(g|\mathbf{K}f)$  changes this bilinear form into its conjugate complex.

### EXERCISES

7. Show that when  $\mathbf{K}$  is Hermitian the quadratic form  $(f|\mathbf{K}f)$  is real,  $f$  being any function-vector.

8. Show that the property of Exercise 7 is characteristic of Hermitian linear integral operators, i.e., that it is not possessed by any linear integral operator which is not Hermitian. *Hint.*  $(f|\mathbf{K}f) = (\mathbf{K}f|f)$  (since  $(f|\mathbf{K}f)$  is real)  $= (f|\mathbf{K}^*f)$ . Hence (see Exercise 5)  $\mathbf{K}^* = \mathbf{K}$ .

If the Hermitian linear integral operator  $\mathbf{K}$  is such that  $(f|\mathbf{K}f) \geq 0$ , every  $f$ , we term it *positive*; if the equality is valid only when  $f$  is the zero function-vector,  $\mathbf{K}$  is said to be *definitely positive*.

### EXERCISES

9. Show that the linear integral operators  $\mathbf{K}\mathbf{K}^*$  and  $\mathbf{K}^*\mathbf{K}$  are positive. *Hint.*  $(f|\mathbf{K}\mathbf{K}^*f) = (\mathbf{K}^*f|\mathbf{K}^*f)$ .

10. Show that if  $\mathbf{K}$  is Hermitian  $\mathbf{K}^2$  is positive.

If  $\mathbf{K}$  is a linear integral operator the equation

$$\mathbf{u} = \lambda\mathbf{K}\mathbf{u},$$

i.e.,

$$u(x) = \lambda \int_a^b K \begin{pmatrix} x \\ t \end{pmatrix} u(t) dt; \quad \lambda \text{ a constant,}$$

is known as a *homogeneous linear integral equation*. Any solution of the equation  $\mathbf{u} = \lambda\mathbf{K}\mathbf{u}$ , other than the zero function-vector, is a *character-*

istic vector of the linear integral operator  $\mathbf{K}$ , and the scalar constant  $\lambda$  is a characteristic number of  $\mathbf{K}$ . Just as for  $n$ -dimensional complex vectors there is no lack of generality in taking  $\mathbf{u}$  to be a unit vector; then  $\lambda$  is the reciprocal of the quadratic form  $(\mathbf{u}|\mathbf{K}\mathbf{u})$  (why?).

*Note.* In the case of  $n$ -dimensional complex vectors the characteristic numbers of a linear vector function were defined by the formula  $\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$  rather than, as here, by the formula  $\mathbf{K}\mathbf{u} = \frac{1}{\lambda}\mathbf{u}$ . In other words the characteristic numbers of a linear integral operator are the analogues of the reciprocals of the characteristic numbers of a linear vector function.

EXERCISES

11. Show that the characteristic numbers of a Hermitian linear integral operator are all real.

12. Show that if  $\mathbf{u}$  is a characteristic vector of  $\mathbf{K}$ , associated with the characteristic number  $\lambda$ , and  $\mathbf{v}$  is a characteristic vector of  $\mathbf{K}^*$ , associated with the characteristic number  $\mu$ , then  $(\mathbf{u}|\mathbf{v}) = 0$  if  $\mu \neq \bar{\lambda}$ . *Hint.*  $\mathbf{u} = \lambda\mathbf{K}\mathbf{u}$ ;  $\mathbf{v} = \mu\mathbf{K}^*\mathbf{v}$ . Hence  $(\mathbf{u}|\mathbf{v}) = \bar{\lambda}(\mathbf{K}\mathbf{u}|\mathbf{v}) = \mu(\mathbf{u}|\mathbf{K}^*\mathbf{v})$ . Since  $(\mathbf{K}\mathbf{u}|\mathbf{v}) = (\mathbf{u}|\mathbf{K}^*\mathbf{v})$  the inequality  $\lambda \neq \mu$  forces the relation  $(\mathbf{u}|\mathbf{v}) = 0$ .

13. Show that any two characteristic vectors of a Hermitian operator which are associated with different characteristic numbers are perpendicular.

14. Show that the characteristic numbers of a positive linear integral operator are non-negative.

15. Show that  $\mathbf{K} : xt$  has only one characteristic number, and determine this characteristic number and the associated characteristic vector.

*Solution.*  $u(x) = \lambda \int_a^b xt u(t) dt$ . Setting  $\int_a^b t u(t) dt = c$  we have  $u(x) = \lambda cx$  and so  $c = \int_a^b x u(x) dx = \lambda c \frac{(b^3 - a^3)}{3}$ . Since  $c \neq 0$  (otherwise  $\mathbf{u}$  would be the zero vector)  $\lambda = \frac{3}{(b^3 - a^3)}$ . The associated characteristic vector is  $\mathbf{u} : kx$ , where  $k$

is a constant. If  $\mathbf{u}$  is normalized (i.e., if  $(\mathbf{u}|\mathbf{u}) = 1$ ) and real,  $k = \pm \left( \frac{3}{(b^3 - a^3)} \right)^{1/2}$ .

16. Show that  $\mathbf{K} : x + t$  has two real characteristic numbers, and determine these for the interval  $[-1, 1]$ . *Hint.*  $u(x) = \lambda \left\{ x \int_{-1}^{+1} u(t) dt + \int_{-1}^1 t u(t) dt \right\} = \lambda(c_1x + c_2)$ , say. Then  $c_1 = \int_{-1}^{+1} u(x) dx = 2\lambda c_2$ ,  $c_2 = \int_{-1}^{+1} x u(x) dx = \frac{2}{3}\lambda c_1$ . Hence  $\lambda = \pm \frac{1}{2}(3^{1/2})$ .

17. Determine the two characteristic numbers of  $\mathbf{K} : x^2 + t^2$  for the interval  $[0, 1]$ .

18. Show how to determine the characteristic numbers of  $\mathbf{K}$  when  $K \begin{pmatrix} x \\ t \end{pmatrix} =$

$\sum_1^n \bar{q}_j(t) p_j(x)$  is the sum of  $n$  terms each of which is the product of a function of  $x$  by a function of  $t$ . *Note.* A linear integral operator of this type is said to be *separable*.

19. Show that  $\mathbf{K}: \bar{q}(t) p(x)$  has at most one characteristic number and that it has no characteristic number if  $(q|p) = \int_a^b \bar{q}(t) p(t) dt = 0$ .

20. Determine the conditions under which  $\mathbf{K}: \bar{q}_1(t) p_1(x) + \bar{q}_2(t) p_2(x)$  does not possess a characteristic number.

# 4

## CURVILINEAR COORDINATES

### 1. Plane polar coordinates

The simplest and most familiar example of plane *curvilinear coordinates* is the system of *plane polar coordinates*. If  $P:(x, y)$  is any point in the plane other than  $O:(0, 0)$  its plane polar coordinates are

$r = |OP|$ ;  $\theta =$  the angle from  $O \rightarrow I$  to  $O \rightarrow P$ , where  $I:(1, 0)$ .

The connection between the plane polar coordinates  $(r, \theta)$  of  $P$  and the rectangular Cartesian coordinates  $(x, y)$  of  $P$  is furnished by the equations

$$x = r \cos \theta; \quad y = r \sin \theta.$$

The point  $O:(0, 0)$  is a *singular point* of the polar coordinate system. It is characterized by the fact that  $r = 0$  while  $\theta$  is *indeterminate*. If  $r \neq 0$  the polar coordinates are furnished in terms of the Cartesian coordinates by the formula

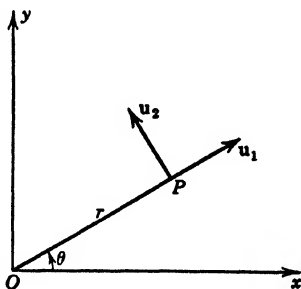


FIG. 14.

$$r = (x^2 + y^2)^{1/2}; \quad \theta = \arctan \frac{y}{x}, \quad x \neq 0; \quad \theta = \pm \frac{\pi}{2} \text{ if } x = 0$$

(it being understood that the determination of  $\theta$  which corresponds to the quadrant in which  $P$  lies is chosen). The *polar coordinate lines* are obtained by holding first  $\theta$  constant and then  $r$  constant in the equations

$$x = r \cos \theta; \quad y = r \sin \theta.$$

Thus the first family of polar coordinate lines is the family of rays from  $O$  towards  $P$  and the second family of polar coordinate lines is the



family of concentric circles whose common center is at  $O$ . The directions of the two polar coordinate lines which pass through any point  $P:(x, y)$ , other than  $O:(0, 0)$ , are those of the vectors

$$\mathbf{r}_r = v(x_r, y_r) = v(\cos \theta, \sin \theta);$$

$$\mathbf{r}_\theta = v(x_\theta, y_\theta) = v(-r \sin \theta, r \cos \theta),$$

where  $\mathbf{r}$  is the position vector  $v(x, y)$  of the point  $P$ . We denote the unit vectors which have the directions of  $\mathbf{r}_r$  and  $\mathbf{r}_\theta$ , respectively, by  $\mathbf{u}_1$  and  $\mathbf{u}_2$  so that

$$\mathbf{u}_1 = v(\cos \theta, \sin \theta) = \mathbf{r}_r; \quad \mathbf{u}_2 = v(-\sin \theta, \cos \theta) = \frac{1}{r} \mathbf{r}_\theta.$$

Thus  $\mathbf{u}_2$  is the complement  $\mathbf{u}_1^*$  of  $\mathbf{u}_1$ .

*Note.* The polar coordinates  $(r, \theta)$  are chosen in this order so that  $\mathbf{u}_2$  may be the complement of  $\mathbf{u}_1$ ; if we had taken them in the order  $(\theta, r)$ ,  $\mathbf{u}_1$  would have turned out to be the complement of  $\mathbf{u}_2$ . The proper order in which to choose the polar coordinates is determined by the fact that the alternating product  $(\mathbf{u}_1, \mathbf{u}_2) = (\mathbf{u}_1^* | \mathbf{u}_2) = 1$  must be positive. Since this alternating product is the product of the two-rowed determinant

$$J = (\mathbf{r}_r, \mathbf{r}_\theta) = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r$$

by  $\frac{1}{r}$  we have  $J > 0$ . This two-rowed determinant is known as the *Jacobian determinant* of the pair of (dependent) variables  $(x, y)$  with respect to the pair of (independent) variables  $(r, \theta)$  (after C. G. Jacobi [1804–1851], a German mathematician). We shall denote this Jacobian determinant of  $(x, y)$  with respect to  $(r, \theta)$  by the symbol  $\left| \begin{matrix} (x, y) \\ (r, \theta) \end{matrix} \right|$ ; the symbol  $\frac{(x, y)}{(r, \theta)}$ , without the vertical bars, indicates the  $2 \times 2$  Jacobian matrix

$$\begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix}$$

(note that the upper letters tell the rows and the lower the columns). Thus the rule determining the order in which the polar coordinates are chosen may be phrased as follows:

**The polar coordinates  $(r, \theta)$  are ordered so that the Jacobian determinant**

$$J = \left| \begin{array}{c} (x, y) \\ (r, \theta) \end{array} \right| > 0.$$

The rectangular Cartesian reference frame whose axes have the directions of the polar coordinate lines through  $P$ , i.e., the directions of  $\mathbf{u}_1$  and  $\mathbf{u}_2 = \mathbf{u}_1^*$ , is termed the *polar-coordinate reference frame* which is *attached to the point  $P$* . Since the vectors  $\mathbf{u}_1$  and  $\mathbf{u}_2$  vary as the point  $P$  is changed, the polar-coordinate reference frame is termed a *moving reference frame*. If  $\mathbf{v} = v^r \mathbf{u}_1 + v^\theta \mathbf{u}_2$  is any vector the *scalars* ( $v^r, v^\theta$ ) are termed the *coordinates*, with respect to the polar-coordinate reference frame, of the vector  $\mathbf{v}$ .

*Warning.* Be sure that you appreciate fully the fact that the vectors  $\mathbf{u}_1$  and  $\mathbf{u}_2 = \mathbf{u}_1^*$  depend on the point  $P$  to which the polar-coordinate reference frame is attached. Thus if  $\mathbf{v}$  is a constant vector field (i.e., if the coordinates of  $\mathbf{v}$  with respect to a *fixed* Cartesian reference frame are constant point-functions) the *polar coordinates*  $v^r$  and  $v^\theta$  of  $\mathbf{v}$  will not be, in general, constant point-functions.

One of the first questions we have to answer when we plan to use polar-coordinate reference frames is the following:

*What are the polar coordinates of the gradient of a scalar field  $f$ ?*

To answer this question we observe that if  $f$  is any differentiable function of  $(r, \theta)$ , not necessarily a scalar (so that  $f$  may be, for example, one of the Cartesian coordinates of a vector field) we have

$$f_x = f_r r_x + f_\theta \theta_x; \quad f_y = f_r r_y + f_\theta \theta_y.$$

In other words the  $1 \times 2$  matrix  $(f_x, f_y)$  is the product of the  $2 \times 2$  matrix  $\begin{pmatrix} r & \theta \\ x & y \end{pmatrix}$  by the  $1 \times 2$  matrix  $(f_r, f_\theta)$ .

When the rectangular Cartesian reference frame  $Oxy$  is so chosen that  $\theta = 0$  we have  $\begin{pmatrix} x & y \\ r & \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}$ . Now the matrix  $\begin{pmatrix} r & \theta \\ x & y \end{pmatrix}$  is the reciprocal of  $\begin{pmatrix} x & y \\ r & \theta \end{pmatrix}$ . (Prove this. *Hint.*  $x_r r_x + x_\theta \theta_x$  is the derivative of  $x$  with respect to  $x$  through the intermediary variables  $(r, \theta)$ ; hence it has the value 1.  $x_r r_y + x_\theta \theta_y$  is the derivative of  $x$  with respect to  $y$  through the intermediary variables  $(r, \theta)$ ; hence it has the value zero.)

Hence, when  $\theta = 0$ ,  $\begin{pmatrix} r & \theta \\ x & y \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{r} \end{bmatrix}$ . Thus if  $\theta = 0$ , i.e., if  $O \rightarrow I$

has the direction of  $\mathbf{u}_1$ , the  $1 \times 2$  matrix  $(f_x, f_y) = \left(f_r, \frac{1}{r} f_\theta\right)$ .

In particular, when  $f$  is a scalar field, we have the following important result:

**The polar coordinates of  $\text{grad } f$  are  $\left(f_r, \frac{1}{r} f_\theta\right)$ .**

### EXERCISES

1. Show that the polar coordinates of  $\text{grad } r$  are  $(1, 0)$  and that the polar coordinates of  $\text{grad } \theta$  are  $\left(0, \frac{1}{r}\right)$ .

2. Show that the polar coordinates of the vector element of arc  $d\mathbf{s} = v(dx, dy)$  are  $(dr, r d\theta)$ . *Hint.* If  $f$  is any scalar field  $df = (\text{grad } f|d\mathbf{s})$  and  $df = f_r dr + f_\theta d\theta = f_r dr + \frac{1}{r} f_\theta (r d\theta)$ .

Let, now,  $\mathbf{v}$  be any vector field whose polar coordinates are  $(v^r, v^\theta)$ :

$$\mathbf{v} = v^r \mathbf{u}_1 + v^\theta \mathbf{u}_2.$$

This relation tells us that the coordinates of  $\mathbf{v}$  in the fixed Cartesian reference frame  $Oxy$  are the elements of the product of the  $2 \times 1$  matrix  $(v^r, v^\theta)$  by the  $2 \times 2$  rotation matrix

$$R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

(because the elements of the first (second) column of  $R$  are the coordinates in the  $Oxy$  reference frame of  $\mathbf{u}_1$  ( $\mathbf{u}_2$ )). Let us denote by  $\rho$  the  $2 \times 1$  matrix whose elements are the polar coordinates  $(v^r, v^\theta)$  of the vector field  $\mathbf{v}$  and by  $\xi$  the  $2 \times 1$  matrix whose elements are the Cartesian coordinates  $(v^x, v^y)$  of the vector field  $\mathbf{v}$ ; then

$$\xi = R\rho.$$

The Cartesian reference frame  $Oxy$  being supposed chosen so that  $\theta = 0$ ,  $R$  is the  $2 \times 2$  unit matrix  $E_2$ , and  $\xi = \rho$  (a relation which only restates the definition of the polar coordinates of  $\mathbf{v}$  as the Cartesian coordinates of  $\mathbf{v}$  in the reference frame  $Oxy$  for which  $\theta = 0$ ). Let us calculate the divergence of the vector field  $\mathbf{v}$ , performing the calculation in the reference frame  $Oxy$  for which  $\theta = 0$ . Since, in this reference frame, the derivative of any point-function with respect to  $x$  is the same as its derivative with respect to  $r$  and since  $R$  is independent

of  $r$  we have

$$\xi_x = R\rho_r = \rho_r,$$

where  $\xi_x$ , for example, is the  $2 \times 1$  matrix  $\begin{pmatrix} v_x^x \\ v_{v_x}^x \end{pmatrix}$ , (the subscript  $x$  indicating differentiation with respect to  $x$ ). The reason that  $R\rho_r = \rho_r$  is that, when  $\theta = 0$ ,  $R$  is the  $2 \times 2$  unit matrix  $E_2$ . Since, when  $\theta = 0$ , differentiation with respect to  $y$  is equivalent to differentiation with respect to  $\theta$  followed by division by  $r$ , and since, when  $\theta = 0$ ,

$$R_\theta = \begin{pmatrix} -\sin \theta & -\cos \theta \\ \cos \theta & -\sin \theta \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

we have

$$\xi_v = \frac{1}{r} (R_\theta \rho + R\rho_\theta) = \frac{1}{r} \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rho + \rho_\theta \right\}.$$

The first element of  $\xi_x$  is  $(v^r)_r$ , and the second element of  $\xi_v$  is  $\frac{1}{r} \{v^r + (v^\theta)_\theta\}$ . The sum of these is the expression for the divergence of the vector field calculated in the Cartesian reference frame  $Oxy$  for which  $\theta = 0$ . Since  $\text{div } \mathbf{v}$  is a scalar it does not matter in what reference frame it is calculated. We term, then, this sum the divergence of  $\mathbf{v}$  in plane polar coordinates. Thus

**The divergence of the vector field  $\mathbf{v}$  in plane polar coordinates is**

$$(v^r)_r + \frac{1}{r} v^r + \frac{1}{r} (v^\theta)_\theta.$$

### EXERCISES

3. Express the Laplacian  $\Delta_2 f$  of a scalar field  $f$  in plane polar coordinates. *Hint.*

$\Delta_2 f = \text{div grad } f$ ; set  $\mathbf{v} = f_r, v^\theta = \frac{1}{r} f_\theta$  in the expression for  $\text{div } \mathbf{v}$ .

$$\text{Answer. } f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta\theta}.$$

4. Obtain the expression in plane polar coordinates for  $\text{curl } \mathbf{v}$ . *Hint.*  $\text{curl } \mathbf{v} = -\text{div } \mathbf{v}^*$ . Since  $\mathbf{v} = v^r \mathbf{u}_1 + v^\theta \mathbf{u}_2, \mathbf{v}^* = v^r \mathbf{u}_2 - v^\theta \mathbf{u}_1$ . Hence  $v^{*r} = -v^\theta, v^{*\theta} = v^r$ .

$$\text{Answer. } (v^\theta)_r + \frac{1}{r} v^\theta - \frac{1}{r} (v^r)_\theta.$$

5. Obtain the expression in plane polar coordinates for the direct product of the symbolic vector  $\nabla$  by the vector field  $\mathbf{v}$ ; in other words express the linear vector function

$$\mathbf{A} : \begin{pmatrix} (v^r)_x & (v^r)_y \\ (v^\theta)_x & (v^\theta)_y \end{pmatrix}$$

in plane polar coordinates. *Hint.* The columns of the matrix  $A$  of this linear vector function are  $\xi_x$  and  $\xi_y$ .

$$\text{Answer. } \begin{bmatrix} (v^r)_r & -\frac{1}{r}v^\theta + \frac{1}{r}(v^r)_\theta \\ (v^\theta)_r & \frac{1}{r}v^r + \frac{1}{r}(v^\theta)_\theta \end{bmatrix}.$$

6. Find the plane polar coordinates of the linear vector function whose Cartesian coordinates are the elements of the matrix of second derivatives

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

of a scalar field. *Hint.* Set  $\mathbf{v} = \text{grad } f$  in Exercise 5.

$$\text{Answer. } \begin{bmatrix} f_{rr} & \frac{1}{r}f_{r\theta} - \frac{1}{r^2}f_\theta \\ \frac{1}{r}f_{r\theta} - \frac{1}{r^2}f_\theta & \frac{1}{r}f_r + \frac{1}{r^2}f_{\theta\theta} \end{bmatrix}.$$

7. Find the plane polar coordinates of the differential vector  $d\mathbf{v} = v(dv^r, dv^\theta)$ . *Hint.*  $d\mathbf{v}$  is the result of operating on  $d\mathbf{s} = v(dx, dy)$  by the linear vector function of Exercise 5. See Exercise 2 for the plane polar coordinates of  $d\mathbf{s}$ .

$$\text{Answer. } (dv^r - v^\theta d\theta, dv^\theta + v^r d\theta).$$

*Note.* Since  $(-v^\theta, v^r)$  are the polar coordinates of the complement  $\mathbf{v}^*$  of  $\mathbf{v}$  (see Exercise 4) it follows that  $(dv^r, dv^\theta)$  are the plane polar coordinates of a vector, namely, the vector  $d\mathbf{v} - d\theta \cdot \mathbf{v}^*$ . This vector (which we shall denote by the symbol  $\delta\mathbf{v}$ ) is the *relative differential* of the vector field  $\mathbf{v}$  (with respect to the moving polar coordinate reference frame), the vector  $d\mathbf{v}$  being the *absolute differential* of the vector field  $\mathbf{v}$ . The relation between the absolute differential and the relative differential of the vector field  $\mathbf{v}$  is, accordingly, given by the formula

$$d\mathbf{v} = \delta\mathbf{v} + d\theta \cdot \mathbf{v}^*.$$

8. Show that the plane polar coordinates of the (absolute) velocity of  $P: (x, y)$  are  $(r_t, r\theta_t)$ . *Hint.* Here  $\mathbf{v}$  is the position vector field  $\mathbf{r} = v(x, y)$ , and its plane polar coordinates are  $v^r = r, v^\theta = 0$ . The absolute velocity of  $P$  is  $\mathbf{r}_t = \frac{d\mathbf{r}}{dt}$ .

9. Show that the plane polar coordinates of the (absolute) acceleration of  $P: (x, y)$  are  $(r_{tt} - r(\theta_t)^2, r\theta_{tt} + 2r_t\theta_t)$ .

Let, now,  $\mathbf{A}: A$  be any linear vector function. When our fixed Cartesian reference frame is the one for which  $\theta = 0$  we term the elements of  $A$  the plane polar coordinates of the linear vector function  $\mathbf{A}$ . We shall denote these plane polar coordinates of  $\mathbf{A}$  as follows:

$$\begin{pmatrix} \widehat{r\mathbf{r}} & \widehat{r\theta} \\ \widehat{\theta\mathbf{r}} & \widehat{\theta\theta} \end{pmatrix}.$$

Note that the first letter in a symbol such as  $\widehat{r\theta}$  tells the row while the second tells the column; thus  $\widehat{r\theta}$  is the element in the  $r$ th (i.e., the first) row and in the  $\theta$ th (i.e., the second) column. On denoting by  $A'$  the  $2 \times 2$  matrix whose elements are the plane polar coordinates of the linear vector function  $\mathbf{A}$  and by  $A$  the  $2 \times 2$  matrix whose elements are the coordinates of  $\mathbf{A}$  in the fixed Cartesian reference frame  $Oxy$  we have

$$A = RA'R^*$$

(in fact the elements of  $A\xi$  are the coordinates in the reference frame  $Oxy$  of the vector whose plane polar coordinates are  $A'\rho$ ; hence  $A\xi = RA'\rho$ , and since  $\xi = R\rho$  and  $\rho$  is an arbitrary  $2 \times 1$  matrix it follows that  $AR = RA'$ ; since  $RR^* = E_2$  it follows that  $A = RA'R^*$ ). We proceed to calculate  $\text{div } \mathbf{A}$  in plane polar coordinates, and as a first step we calculate  $A_x$  in the reference frame  $Oxy$  for which  $\theta = 0$ ; since for this reference frame both  $R$  and  $R^*$  are the  $2 \times 2$  unit matrix, and since differentiation with respect to  $x$  is equivalent to differentiation with respect to  $r$  (so that both  $R_x$  and  $R^*_x$  are each the  $2 \times 2$  zero matrix) we obtain

$$A_x = A'_r.$$

Since differentiation with respect to  $y$  is equivalent to differentiation with respect to  $\theta$  followed by division by  $r$  we have

$$R_y = \begin{bmatrix} 0 & -\frac{1}{r} \\ \frac{1}{r} & 0 \end{bmatrix}; \quad R^*_y = \begin{bmatrix} 0 & \frac{1}{r} \\ -\frac{1}{r} & 0 \end{bmatrix},$$

$$\text{and } A_y = \begin{bmatrix} 0 & -\frac{1}{r} \\ \frac{1}{r} & 0 \end{bmatrix} A' + \frac{1}{r} A'_\theta + A' \begin{bmatrix} 0 & \frac{1}{r} \\ -\frac{1}{r} & 0 \end{bmatrix}. \quad \text{Adding the first}$$

row of  $A_x$  to the second row of  $A_y$  we obtain  $\text{div } \mathbf{A}$  in plane polar coordinates. We find that the plane polar coordinates of  $\text{div } \mathbf{A}$  are the elements of the  $1 \times 2$  matrix:

$$\left( (\widehat{rr})_r + \frac{\widehat{rr}}{r} + \frac{1}{r} (\widehat{\theta r})_\theta - \frac{\widehat{\theta\theta}}{r}, \quad (\widehat{r\theta})_r + \frac{\widehat{r\theta}}{r} + \frac{1}{r} (\widehat{\theta\theta})_\theta + \frac{\widehat{\theta r}}{r} \right).$$

## EXERCISE

10. Find the plane polar coordinates of the Laplacian  $\Delta_2 \mathbf{v}$  of a vector field  $\mathbf{v}$ .  
*Hint.* Take  $\mathbf{A}$  to be the adjoint of the linear vector function of Exercises 5.

*Answer.* The first plane polar coordinate of  $\Delta_2 \mathbf{v}$  is  $(v^r)_{rr} + \frac{1}{r} (v^r)_r + \frac{1}{r^2} (v^r)_{\theta\theta}$   
 $-\frac{1}{r^2} v^r - \frac{2}{r^2} (v^\theta)_\theta$ ; the second plane polar coordinate of  $\Delta_2 \mathbf{v}$  is  $(v^\theta)_{rr} + \frac{1}{r} (v^\theta)_r +$   
 $\frac{1}{r^2} (v^\theta)_{\theta\theta} - \frac{1}{r^2} v^\theta + \frac{2}{r^2} (v^r)_\theta$ .

## 2. Orthogonal curvilinear coordinates in the plane

The general theory of orthogonal curvilinear coordinates in the plane is so similar to that of plane polar coordinates (which are a special case) that you should experience no difficulty with it, and we shall merely run over the main points.

The position vector  $\mathbf{r} = v(O \rightarrow P)$  is a differentiable function of two independent variables which we shall denote by  $(\alpha, \beta)$ . Thus we have two equations

$$x = x(\alpha, \beta); \quad y = y(\alpha, \beta),$$

and we obtain two families of curvilinear coordinate lines by holding first  $\beta$  constant and then  $\alpha$  constant in the equation

$$\mathbf{r} = \mathbf{r}(\alpha, \beta).$$

We assume that the two vectors

$$\mathbf{r}_\alpha = v(x_\alpha, y_\alpha); \quad \mathbf{r}_\beta = v(x_\beta, y_\beta)$$

are linearly independent, i.e., that the Jacobian determinant

$$J = (\mathbf{r}_\alpha, \mathbf{r}_\beta) = \begin{vmatrix} x & y \\ \alpha & \beta \end{vmatrix} = \begin{vmatrix} x_\alpha & x_\beta \\ y_\alpha & y_\beta \end{vmatrix}$$

is different from zero, and we agree that the curvilinear coordinates  $(\alpha, \beta)$  are so ordered that  $J > 0$ . Furthermore we assume that the coordinate lines  $\alpha = \text{constant}$ ,  $\beta = \text{constant}$  intersect at right angles wherever they meet. Thus

$$(\mathbf{r}_\alpha | \mathbf{r}_\beta) = 0.$$

We denote by  $\mathbf{u}_1$  and  $\mathbf{u}_2$  the unit vectors which have the directions of  $\mathbf{r}_\alpha$  and  $\mathbf{r}_\beta$ , respectively, and we write

$$\mathbf{u}_1 = h_1 \mathbf{r}_\alpha; \quad \mathbf{u}_2 = h_2 \mathbf{r}_\beta$$

so that  $h_1$  and  $h_2$  are the reciprocals of the magnitudes of  $\mathbf{r}_\alpha$  and  $\mathbf{r}_\beta$ , respectively. The  $2 \times 2$  matrix whose columns are furnished by the coordinates of  $\mathbf{u}_1$  and  $\mathbf{u}_2$  is a *rotation matrix*

$$R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

where  $\theta$  is the inclination of  $\mathbf{u}_1$  to the positive  $x$ -axis. The coordinates of a vector  $\mathbf{v}$  in the rectangular Cartesian reference frame for which  $\theta = 0$  (i.e., in the rectangular Cartesian reference frame whose positive  $x$ -axis has the direction of  $\mathbf{u}_1$ ) are known as the *curvilinear coordinates* of  $\mathbf{v}$ . We denote these curvilinear coordinates of  $\mathbf{v}$  by  $(v^\alpha, v^\beta)$  so that  $v^\alpha = (\mathbf{u}_1 | \mathbf{v})$ ,  $v^\beta = (\mathbf{u}_2 | \mathbf{v})$ , and

$$\mathbf{v} = v^\alpha \mathbf{u}_1 + v^\beta \mathbf{u}_2.$$

If  $f$  is any differentiable function of  $\alpha$  and  $\beta$  (not necessarily a scalar) the  $1 \times 2$  matrix  $(f_x, f_y)$  is the product of the  $2 \times 2$  matrix  $\frac{(\alpha, \beta)}{(x, y)}$  by the  $1 \times 2$  matrix  $(f_\alpha, f_\beta)$ . When the fixed Cartesian reference frame  $Oxy$  is such that  $\theta = 0$  we have

$$\frac{(x, y)}{(\alpha, \beta)} = \begin{bmatrix} \frac{1}{h_1} & 0 \\ 0 & \frac{1}{h_2} \end{bmatrix}$$

(why?) and so  $\frac{(\alpha, \beta)}{(x, y)}$ , which is the reciprocal of  $\frac{(x, y)}{(\alpha, \beta)}$  (why?), is  $\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$ . Hence, for this particular reference frame,

$$(f_x, f_y) = (h_1 f_\alpha, h_2 f_\beta).$$

In particular

**The curvilinear coordinates of grad  $f$ , where  $f$  is a differentiable scalar field, are  $(h_1 f_\alpha, h_2 f_\beta)$ .**

Observe that, when  $\theta = 0$ ,  $x_\alpha = \frac{1}{h_1}$ ,  $y_\alpha = 0$ ,  $x_\beta = 0$ ,  $y_\beta = \frac{1}{h_2}$ . *Note.*

The functions  $h_1$  and  $h_2$  are most easily remembered as follows. Since

$\mathbf{r}_\alpha = \frac{1}{h_1} \mathbf{u}_1$ ,  $\mathbf{r}_\beta = \frac{1}{h_2} \mathbf{u}_2$  we have

$$d\mathbf{s} = v(dx, dy) = r_\alpha d\alpha + r_\beta d\beta = \frac{d\alpha}{h_1} \mathbf{u}_1 + \frac{d\beta}{h_2} \mathbf{u}_2.$$



Hence  $(ds)^2$ , the squared magnitude of  $ds$ , is given by the formula

$$(ds)^2 = (ds|ds) = \frac{(d\alpha)^2}{h_1^2} + \frac{(d\beta)^2}{h_2^2}.$$

Thus

$h_1$  and  $h_2$  are the (positive) square roots of the reciprocals of the coefficients of  $(d\alpha)^2$  and  $(d\beta)^2$  in the expression for  $(ds)^2$ .

### EXERCISES

1. Show that the curvilinear coordinates of  $\mathbf{grad} \alpha$  are  $(h_1, 0)$  and that the curvilinear coordinates of  $\mathbf{grad} \beta$  are  $(0, h_2)$ .

2. Show that the curvilinear coordinates of the vector element of arc  $ds = v(dx, dy)$  are  $\left(\frac{d\alpha}{h_1}, \frac{d\beta}{h_2}\right)$ .

Let, now,  $\mathbf{v}$  be any vector field, and denote by  $\xi$  the  $2 \times 1$  matrix whose elements are the Cartesian coordinates of  $\mathbf{v}$  and by  $a$  the  $2 \times 1$  matrix whose elements are the curvilinear coordinates of  $\mathbf{v}$ . Then

$$\xi = Ra.$$

Choosing, as always, the fixed Cartesian reference frame  $Oxy$  so that  $\theta = 0$  we have

$$R_\alpha = \begin{pmatrix} 0 & -\theta_\alpha \\ \theta_\alpha & 0 \end{pmatrix}; \quad R_\beta = \begin{pmatrix} 0 & -\theta_\beta \\ \theta_\beta & 0 \end{pmatrix}$$

and so

$$\begin{aligned} \xi_x &= h_1\theta_\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} a + h_1a_\alpha; \\ \xi_y &= h_2\theta_\beta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} a + h_2a_\beta. \end{aligned}$$

The quantities  $(\theta_\alpha, \theta_\beta)$  are readily expressed in terms of  $h_1, h_2$  and their derivatives with respect to  $\alpha$  and  $\beta$ . In fact, on differentiating the relation  $-\sin \theta = h_2x_\beta$  with respect to  $\alpha$  and then setting  $\theta = 0$  (so that  $x_\beta = 0$ ), we obtain

$$\theta_\alpha = -h_2x_{\alpha\beta}.$$

On differentiating the relation  $(x_\alpha)^2 + (y_\alpha)^2 = \frac{1}{h_1^2}$  with respect to  $\beta$

and then setting  $\theta = 0$  (so that  $x_\alpha = \frac{1}{h_1}, y_\alpha = 0$ ) we obtain

$$x_{\alpha\beta} = -\frac{1}{h_1^2}(h_1)_\beta.$$

Hence  $\theta_\alpha = \frac{h_2}{h_1^2}(h_1)_\beta$ , and, similarly,  $\theta_\beta = -\frac{h_1}{h_2^2}(h_2)_\alpha$ . The divergence of the vector field  $\mathbf{v}$  is obtained by adding the first element of  $\xi_x$  to the second element of  $\xi_y$ . Thus the expression in (orthogonal) curvilinear coordinates for  $\text{div } \mathbf{v}$  is

$$\begin{aligned} & h_1(v^\alpha)_\alpha - h_1\theta_\alpha v^\beta + h_2\theta_\beta v^\alpha + h_2(v^\beta)_\beta \\ &= h_1(v^\alpha)_\alpha - \frac{h_2}{h_1}(h_1)_\beta v^\beta - \frac{h_1}{h_2}(h_2)_\alpha v^\alpha + h_2(v^\beta)_\beta \\ &= h_1 h_2 \left\{ \left( \frac{v^\alpha}{h_2} \right)_\alpha + \left( \frac{v^\beta}{h_1} \right)_\beta \right\}. \end{aligned}$$

### EXERCISES

3. Express the Laplacian  $\Delta_1 f$  of a scalar field  $f$  in orthogonal curvilinear coordinates.

$$\text{Answer. } h_1 h_2 \left\{ \left( \frac{h_1}{h_2} f_\alpha \right)_\alpha + \left( \frac{h_2}{h_1} f_\beta \right)_\beta \right\}.$$

*Note.* Remember this important result.

4. Obtain the expression for  $\text{curl } \mathbf{v}$  in orthogonal curvilinear coordinates.

$$\text{Answer. } h_1 h_2 \left\{ \left( \frac{v^\beta}{h_2} \right)_\alpha - \left( \frac{v^\alpha}{h_1} \right)_\beta \right\}.$$

5. Obtain the expression for the linear vector function  $\mathbf{A}$ :  $\begin{pmatrix} (v^x)_x & (v^x)_y \\ (v^y)_x & (v^y)_y \end{pmatrix}$  in orthogonal curvilinear coordinates.

$$\text{Answer. } \begin{pmatrix} h_1(v^\alpha)_\alpha - \frac{h_2}{h_1}(h_1)_\beta v^\beta & h_2(v^\alpha)_\beta + \frac{h_1}{h_2}(h_2)_\alpha v^\beta \\ h_1(v^\beta)_\alpha + \frac{h_2}{h_1}(h_1)_\beta v^\alpha & h_2(v^\beta)_\beta - \frac{h_1}{h_2}(h_2)_\alpha v^\alpha \end{pmatrix}.$$

6. If  $f$  is a scalar field find the expression in orthogonal curvilinear coordinates of the linear vector function whose Cartesian coordinates are the elements of the matrix of second derivatives  $\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$ .

$$\text{Answer. } \begin{pmatrix} h_1^2 f_{\alpha\alpha} + h_1(h_1)_\alpha f_\alpha - \frac{h_2^2}{h_1}(h_1)_\beta f_\beta & h_1 h_2 f_{\alpha\beta} + h_1(h_2)_\alpha f_\beta + h_2(h_1)_\beta f_\alpha \\ h_1 h_2 f_{\alpha\beta} + h_1(h_2)_\alpha f_\beta + h_2(h_1)_\beta f_\alpha & h_2^2 f_{\beta\beta} + h_2(h_2)_\beta f_\beta - \frac{h_1^2}{h_2}(h_2)_\alpha f_\alpha \end{pmatrix}.$$

7. Find the orthogonal curvilinear coordinates of the (absolute) differential vector  $dv$ .

$$\text{Answer. } \begin{bmatrix} dv^\alpha - \left[ h_1 \left( \frac{1}{h_2} \right)_\alpha d\beta - h_2 \left( \frac{1}{h_1} \right)_\beta d\alpha \right] v^\beta \\ dv^\beta + \left[ h_1 \left( \frac{1}{h_2} \right)_\alpha d\beta - h_2 \left( \frac{1}{h_1} \right)_\beta d\alpha \right] v^\alpha \end{bmatrix}.$$

8. Denoting the orthogonal curvilinear coordinates of a linear vector function  $\mathbf{A}$  as follows:

$$\begin{pmatrix} \widehat{\alpha\alpha} & \widehat{\alpha\beta} \\ \widehat{\beta\alpha} & \widehat{\beta\beta} \end{pmatrix}$$

determine the orthogonal curvilinear coordinates of the vector  $\text{div } \mathbf{A}$ .

*Solution.* On denoting by  $A$  the  $2 \times 2$  matrix whose elements are the coordinates of  $\mathbf{A}$  in the fixed Cartesian reference frame  $Oxy$  and by  $A'$  the  $2 \times 2$  matrix whose elements are the orthogonal curvilinear coordinates of  $\mathbf{A}$  we have  $A = RA'R^*$  and so

$$\begin{aligned} A_x &= R_x A' + A'_x + A'R^*_x \\ &= h_1 R_{\alpha} A' + h_1 A'_\alpha + h_1 A'R^*_{\alpha} \end{aligned}$$

( $\theta$  being set equal to zero after the differentiation). Here

$$\begin{aligned} R_\alpha &= \theta_\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \frac{h_2}{h_1^2} (h_1)_\beta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \\ R^*_\alpha &= \theta_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{h_2}{h_1^2} (h_1)_\beta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

Hence

$$A_x = \frac{h_2}{h_1} (h_1)_\beta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A' + h_1 A'_\alpha + \frac{h_2}{h_1} (h_1)_\beta A' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and, similarly,

$$A_y = -\frac{h_1}{h_2} (h_2)_\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A' + h_2 A'_\beta - \frac{h_1}{h_2} (h_2)_\alpha A' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The vector  $\text{div } \mathbf{A}$  is obtained by adding the first row of  $A_x$  to the second row of  $A_y$ . The first orthogonal curvilinear coordinate of  $\text{div } \mathbf{A}$  is, then,

$$h_1(\widehat{\alpha\alpha})_\alpha + h_2(\widehat{\beta\alpha})_\beta - \frac{h_2}{h_1} (h_1)_\beta (\widehat{\alpha\beta} + \widehat{\beta\alpha}) - \frac{h_1}{h_2} (h_2)_\alpha (\widehat{\alpha\alpha} - \widehat{\beta\beta})$$

while the second is

$$h_1(\widehat{\alpha\beta})_\alpha + h_2(\widehat{\beta\beta})_\beta + \frac{h_2}{h_1} (h_1)_\beta (\widehat{\alpha\alpha} - \widehat{\beta\beta}) - \frac{h_1}{h_2} (h_2)_\alpha (\widehat{\alpha\beta} + \widehat{\beta\alpha}).$$

These may be written as follows:

$$\text{First coordinate} = h_1 h_2 \left\{ \left( \frac{\widehat{\alpha\alpha}}{h_2} \right)_\alpha + \left( \frac{\widehat{\beta\alpha}}{h_1} \right)_\beta + \widehat{\alpha\beta} \left( \frac{1}{h_1} \right)_\beta - \widehat{\beta\beta} \left( \frac{1}{h_2} \right)_\alpha \right\},$$

$$\text{Second coordinate} = h_1 h_2 \left\{ \left( \frac{\alpha\beta}{h_2} \right)_\alpha + \left( \frac{\beta\beta}{h_1} \right)_\beta - \widehat{\alpha\alpha} \left( \frac{1}{h_1} \right)_\beta + \widehat{\beta\alpha} \left( \frac{1}{h_2} \right)_\alpha \right\}.$$

9. Show that  $\left( \frac{h_2}{h_1} \right) |(h_1)_\beta|$  is the curvature of the first of the two orthogonal coordinate lines ( $\beta = \text{constant}$ ) which passes through  $P$ . *Hint.* Since  $\sin \theta = h_1 y_\alpha$  we have, at  $\theta = 0$ ,  $\theta_\alpha = h_1 y_{\alpha\alpha}$ , and the curvature of the line  $\beta = \text{constant}$  is

$$\kappa_\alpha = \frac{|x_\alpha y_{\alpha\alpha} - y_\alpha x_{\alpha\alpha}|}{(x_\alpha^2 + y_\alpha^2)^{3/2}} = h_1^2 |y_{\alpha\alpha}|$$

since  $h_1 x_\alpha = 1$  and  $y_\alpha = 0$  at  $\theta = 0$ . Hence  $\kappa_\alpha = h_1 |\theta_\alpha| = \frac{h_2}{h_1} |(h_1)_\beta|$ .

### 3. Conjugate functions

A simple and important way of obtaining systems of orthogonal curvilinear coordinates in the plane is as follows.

Let  $z = x + yi$  be a function (in the complex variable sense) of  $\gamma = \alpha + \beta i$ . Then  $dz = z_\gamma d\gamma$  so that

$$(ds)^2 = \overline{dz} dz = \bar{z}_\gamma z_\gamma d\bar{\gamma} d\gamma,$$

or, equivalently,

$$(dx)^2 + (dy)^2 = |z_\gamma|^2 \{ (d\alpha)^2 + (d\beta)^2 \}.$$

Hence  $\alpha$  and  $\beta$  furnish a system of orthogonal curvilinear coordinates which is *special* in the following sense:

$$h_2 = h_1 = \frac{1}{|z_\gamma|} = |\gamma_*|.$$

We shall denote the common value of  $h_1$  and  $h_2$  simply by  $h$ . Since for any system of orthogonal curvilinear coordinates the Jacobian determinant

$$J = \left| \frac{(x, y)}{(\alpha, \beta)} \right| = \frac{1}{h_1 h_2}$$

we have here  $J = \frac{1}{h^2}$ . The functions  $\alpha = \alpha(x, y)$ ,  $\beta = \beta(x, y)$  are termed *conjugate functions*; their level curves intersect at right angles (since these level curves are the coordinate lines of the system of orthogonal curvilinear coordinates furnished by the equations  $x = x(\alpha, \beta)$ ,  $y = y(\alpha, \beta)$ ). We term the special system of orthogonal curvilinear coordinates which is furnished by a pair of conjugate functions a *conjugate system*, and we shall refer to the coordinates in a conjugate system of a vector or of a linear vector function as the

*conjugate coordinates* of the vector or linear vector function. The following results (which we set as exercises) follow from the results for any system of orthogonal curvilinear coordinates by setting  $h_1 = h_2 = h$ .

### EXERCISES

1. Show that the expression for the gradient of a scalar field  $f$  in a conjugate system is  $\text{grad } f = v(hf_\alpha, hf_\beta)$ .

2. Show that the conjugate coordinates of the vector element of arc  $ds = v(dx, dy)$  are  $\frac{1}{h}(d\alpha, d\beta)$ .

3. Show that the expression for the divergence of a vector field  $\mathbf{v}$  in a conjugate system is  $h\{(v^\alpha)_\alpha + (v^\beta)_\beta\} - v^\alpha h_\alpha - v^\beta h_\beta$ .

4. Show that the expression for the Laplacian  $\Delta_2 f$  of a scalar field  $f$  in a conjugate system is

$$\Delta_2 f = h^2(f_{\alpha\alpha} + f_{\beta\beta}).$$

*Note.* Pay particular attention to the remarkable simplicity of the expression for  $\Delta_2 f$  when  $h_1 = h_2$ .

5. Show that in a conjugate system

$$\text{curl } \mathbf{v} = h^2 \left\{ \left( \frac{v^\beta}{h} \right)_\alpha - \left( \frac{v^\alpha}{h} \right)_\beta \right\} = h \{ (v^\beta)_\alpha - (v^\alpha)_\beta \} + v^\alpha h_\beta - v^\beta h_\alpha.$$

6. Obtain the expression for the linear vector function

$$\mathbf{A} : \begin{pmatrix} (v^x)_x & (v^x)_y \\ (v^y)_x & (v^y)_y \end{pmatrix}$$

in a conjugate system.

$$\text{Answer. } \begin{pmatrix} h(v^\alpha)_\alpha - h_\beta v^\beta & h(v^\alpha)_\beta + h_\alpha v^\beta \\ h(v^\beta)_\alpha + h_\beta v^\alpha & h(v^\beta)_\beta - h_\alpha v^\alpha \end{pmatrix}.$$

7. Obtain the expression for the linear vector function

$$\mathbf{A} : \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix},$$

where  $f$  is a scalar field, in a conjugate system.

$$\text{Answer. } \begin{pmatrix} h^2 f_{\alpha\alpha} + h h_\alpha f_\alpha - h h_\beta f_\beta & h^2 f_{\alpha\beta} + h h_\alpha f_\beta + h h_\beta f_\alpha \\ h^2 f_{\alpha\beta} + h h_\alpha f_\beta + h h_\beta f_\alpha & h^2 f_{\beta\beta} + h h_\beta f_\beta - h h_\alpha f_\alpha \end{pmatrix}.$$

8. Obtain the conjugate coordinates of the (absolute) differential vector  $d\mathbf{v}$ .

$$\text{Answer. } \left( dv^\alpha - \frac{1}{h} [h_\beta d\alpha - h_\alpha d\beta] v^\beta; dv^\beta + \frac{1}{h} [h_\alpha d\alpha - h_\beta d\beta] v^\alpha \right).$$

9. Obtain the conjugate coordinates of the divergence of a linear vector function

**A.**

$$\begin{aligned} \text{Answer. } & h\{(\widehat{\alpha\alpha})_\alpha + (\widehat{\beta\alpha})_\beta\} - h_\alpha(\widehat{\alpha\alpha} - \widehat{\beta\beta}) - h_\beta(\widehat{\alpha\beta} + \widehat{\beta\alpha}); \\ & h\{(\widehat{\alpha\beta})_\alpha + (\widehat{\beta\beta})_\beta\} - h_\alpha(\widehat{\alpha\beta} + \widehat{\beta\alpha}) + h_\beta(\widehat{\alpha\alpha} - \widehat{\beta\beta}). \end{aligned}$$

10. Show that the curvatures of the coordinate lines of a conjugate system are  $|h_\theta|$  and  $|h_\alpha|$ , respectively.

**Examples of conjugate systems**

1.  $z = e^\gamma$ ;  $\gamma = \log z$ . Here  $\alpha = \log r$ ,  $\beta = \theta$  so that the coordinate lines are those of the system of plane polar coordinates. The difference between this conjugate system and plane polar coordinates is that the first coordinate is  $\log r$  instead of  $r$ . For this conjugate system  $h = e^{-\alpha} = \frac{1}{r}$ .

2.  $z = \gamma^{3/2}$ ;  $\gamma = z^2$ . Here  $\alpha = x^2 - y^2$ ,  $\beta = 2xy$  so that the coordinate lines are two orthogonal families of rectangular hyperbolas. The point  $O:(0, 0)$  at which  $\gamma_s = 0$  is a *singular point* (what does this mean?) of the conjugate system. For this conjugate system  $h = 2|\gamma|^{1/2} = 2r$ .

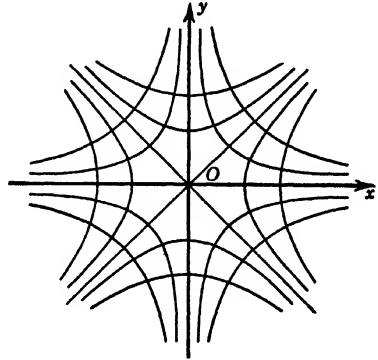


FIG. 15.

3.  $z = \gamma^2$ ;  $\gamma = z^{1/2}$ . Here  $x = \alpha^2 - \beta^2$ ,  $y = 2\alpha\beta$  so that the first system of coordinate lines is the family of parabolas  $x = \frac{y^2}{4\beta^2} - \beta^2$ ,

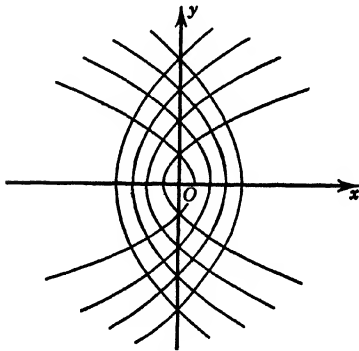


FIG. 16.

$\beta \neq 0$ , all of which have the origin  $O:(0, 0)$  as focus. (Prove this.)  $\beta = 0$  is the positive  $x$ -axis (why not the entire  $x$ -axis?). The second system of coordinate lines is the perpendicular family of parabolas  $x = \alpha^2 - \frac{y^2}{4\alpha^2}$ ,  $\alpha \neq 0$ , all of which have  $O:(0, 0)$  as focus. (Prove this.)  $\alpha = 0$  is the negative  $x$ -axis (why?). For this conjugate system  $h = \frac{1}{2|\gamma|} = \frac{1}{2r^{1/2}}$ . The point

$O:(0, 0)$ , at which  $z_\gamma = 0$ , is a singular point of the system.

4.  $z = \cos \gamma$ ;  $\gamma = \arccos z$ . Here  $x = \cos \alpha \cosh \beta$ ,  $y = -\sin \alpha \sinh \beta$ . The first system of coordinate lines is the family of ellipses

$$\frac{x^2}{\cosh^2 \beta} + \frac{y^2}{\sinh^2 \beta} = 1; \quad \beta \neq 0$$

all of which have their foci at the points  $(\pm 1, 0)$ . (Prove this.)  $\beta = 0$  is the segment  $(-1, 0) \rightarrow (1, 0)$  of the  $x$ -axis (why?). The second family of coordinate lines is the family of hyperbolas

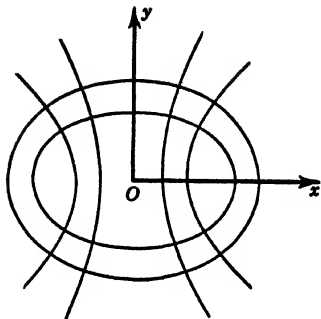


FIG. 17.

$$\frac{x^2}{\cos^2 \alpha} - \frac{y^2}{\sin^2 \alpha} = 1; \quad \sin \alpha \neq 0$$

all of which have their foci at the points  $(\pm 1, 0)$ . (Prove this.)  $\alpha = 0$  is the ray  $(1, 0) \rightarrow \infty$  on the  $x$ -axis, and  $\alpha = \pi$  is the ray  $(-1, 0) \rightarrow -\infty$  on the  $x$ -axis.  $\alpha$  is the interval variable  $0 \leq \alpha < 2\pi$  and  $\beta$  is the unrestricted variable. For

this conjugate system  $h = |\sin \gamma|^{-1}$ . Expressed in terms of the variables  $x$  and  $y$  we have

$$h = |\gamma_z| = \left| \frac{1}{(1 - z^2)^{1/2}} \right| = \frac{1}{(r_1 r_2)^{1/2}}$$

where  $r_1$  and  $r_2$  are the distances of the point  $P$  from the common foci  $(\pm 1, 0)$  of the two systems of conjugate coordinate lines.

### EXERCISES

11. Discuss the conjugate system furnished by the reciprocal function  $z = \frac{1}{\gamma}$ .
12. Discuss the conjugate system furnished by the linear fractional function  $\gamma = \frac{z - c}{z + c}$ . (Assume that  $c$  is real and positive.) Show that the coordinate lines of this conjugate system are all circles.

### 4. Orthogonal curvilinear coordinates in space

The simplest and most important instances of orthogonal curvilinear coordinates in space are the systems of *cylindrical coordinates* and of *space polar coordinates*. *Cylindrical coordinates* are obtained by adding to a system of plane polar coordinates in the  $(x, y)$ -plane a Cartesian coordinate  $z$ . Since we wish to reserve the symbols  $r$  and  $\theta$  for two of the coordinates of the system of space polar coordinates we shall now denote the plane polar coordinates in the  $(x, y)$ -plane by  $(\rho, \phi)$ ; thus the system of cylindrical coordinates is defined by the formulas

$$x = \rho \cos \phi; \quad y = \rho \sin \phi; \quad z = z.$$

Here  $\rho$  is the positive variable  $\rho > 0$ ,  $\phi$  is the interval variable  $0 \leq \phi < 2\pi$ , and  $z$  is the unrestricted variable.  $\phi$  is an angle measured in the  $(x, y)$ -plane; regarded as an angle in space it is a *dihedral angle*, i.e., an angle between two half-planes. These half-planes intersect in the  $z$ -axis, and  $\phi$  is the angle from the half-plane which contains the positive  $x$ -axis to the half-plane which contains the point  $P:(x, y, z)$ . The points of the axis  $\rho = 0$  of the system of cylindrical coordinates are singular points of the system (why?). Since

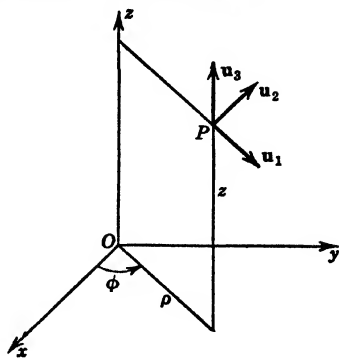


FIG. 18.

$$\mathbf{r}_\rho = v(\cos \phi, \sin \phi, 0); \quad \mathbf{r}_\phi = v(-\rho \sin \phi, \rho \cos \phi, 0); \quad \mathbf{r}_z = v(0, 0, 1)$$

the three unit vectors which determine the cylindrical coordinate reference frame at  $P$  are

$$\mathbf{u}_1 = v(\cos \phi, \sin \phi, 0); \quad \mathbf{u}_2 = v(-\sin \phi, \cos \phi, 0); \quad \mathbf{u}_3 = v(0, 0, 1),$$

and it is clear (prove this) that the three unit vectors  $\mathbf{u}_1$ ,  $\mathbf{u}_2$ , and  $\mathbf{u}_3$  are mutually perpendicular. The three cylindrical coordinates are

ordered that the alternating product  $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$  is positive; in other words the  $3 \times 3$  matrix  $R$  whose columns are furnished by the coordinates of the vectors  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$  in the  $Oxyz$  reference frame is a rotation matrix, i.e., an orthogonal matrix whose determinant is 1. Since

$$\begin{aligned} d\mathbf{s} &= v(dx, dy, dz) = \mathbf{r}_\rho d\rho + \mathbf{r}_\phi d\phi \\ &\quad + \mathbf{r}_z dz = (d\rho)\mathbf{u}_1 + (\rho d\phi)\mathbf{u}_2 + (dz)\mathbf{u}_3 \end{aligned}$$

the squared magnitude  $(ds)^2$  of  $d\mathbf{s}$  is

$$(ds)^2 = (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2.$$

The *space polar coordinates* of  $P:(x, y, z)$  are  $r, \theta, \phi$ , where  $r = |OP|$ ,  $\theta$  is the (unsigned) angle ( $\leq \pi$ ) between the positive  $z$ -axis and  $O \rightarrow P$ , and  $\phi$  is the angular coordinate of the system of cylindrical coordinates. The connection between space polar coordinates and cylindrical

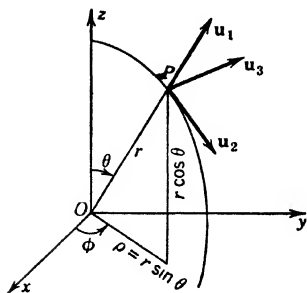


FIG. 19.



coordinates is furnished by the equations

$$z = r \cos \theta; \quad \rho = r \sin \theta$$

and so

$$\begin{aligned} x &= \rho \cos \phi = r \sin \theta \cos \phi; & y &= \rho \sin \phi = r \sin \theta \sin \phi; \\ z &= r \cos \theta, \end{aligned}$$

where  $r$  is the positive variable,  $\theta$  is the interval variable  $0 \leq \theta \leq \pi$ , and  $\phi$  is the interval variable  $0 \leq \phi < 2\pi$ . Notice carefully the difference between the two angular coordinates  $\theta$  and  $\phi$ ;  $\theta$  is an angle between two rays while  $\phi$  is a dihedral angle. Since

$$\begin{aligned} \mathbf{r}_r &= v(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta); \\ \mathbf{r}_\theta &= v(r \cos \theta \cos \phi, r \cos \theta \sin \phi, -r \sin \theta); \\ \mathbf{r}_\phi &= v(-r \sin \theta \sin \phi, r \sin \theta \cos \phi, 0) \end{aligned}$$

the three unit vectors  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$  which determine the space polar coordinate reference frame at  $P$  are

$$\begin{aligned} \mathbf{u}_1 &= v(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta); \\ \mathbf{u}_2 &= v(\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta); \\ \mathbf{u}_3 &= v(-\sin \phi, \cos \phi, 0) \end{aligned}$$

(notice that we pass from  $\mathbf{u}_1$  to  $\mathbf{u}_2$  by replacing  $\theta$  by  $\theta + \pi$ ). Since  $d\mathbf{s} = v(dx, dy, dz) = (dr)\mathbf{u}_1 + (r d\theta)\mathbf{u}_2 + (r \sin \theta d\phi)\mathbf{u}_3$ , and since  $\mathbf{u}_1, \mathbf{u}_2$ , and  $\mathbf{u}_3$  are mutually perpendicular (prove this) we have

$$(ds)^2 = (d\mathbf{s}|d\mathbf{s}) = (dr)^2 + r^2(d\theta)^2 + r^2 \sin^2 \theta (d\phi)^2.$$

The points of the *polar axis* (i.e., those points for which either  $r = 0$  or  $\theta = 0$  or  $\pi$ ) are singular points of the space polar coordinate system. Finally the order in which the space polar coordinates are chosen is such that the alternating product  $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$  is positive (being in fact 1). In other words the  $3 \times 3$  matrix  $R$  whose columns are furnished by the coordinates of the vectors  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$  in the  $Oxy$  reference frame is a rotation matrix (what does this mean?).

Cylindrical and space polar coordinates are particular instances of the general concept of a system of orthogonal curvilinear coordinates in space. These are furnished by a vector formula

$$\mathbf{r} = \mathbf{r}(\alpha, \beta, \gamma)$$

i.e., by three equations

$$x = x(\alpha, \beta, \gamma); \quad y = y(\alpha, \beta, \gamma); \quad z = z(\alpha, \beta, \gamma).$$

We denote the magnitudes of the vectors  $\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma$  by  $\frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}$ , respectively (it being granted that none of these vectors is the zero vector), and we define the three unit vectors  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$  as follows:

$$\mathbf{u}_1 = h_1 \mathbf{r}_\alpha; \quad \mathbf{u}_2 = h_2 \mathbf{r}_\beta; \quad \mathbf{u}_3 = h_3 \mathbf{r}_\gamma.$$

We assume that the three unit vectors  $\mathbf{u}_1, \mathbf{u}_2,$  and  $\mathbf{u}_3$  are mutually perpendicular and that the curvilinear coordinates  $\alpha, \beta, \gamma$  are so ordered that the alternating product  $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$  is positive. (Prove that the value of  $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$  is then 1.) The rectangular Cartesian reference frame whose axes have the direction of the unit vectors  $\mathbf{u}_1, \mathbf{u}_2,$  and  $\mathbf{u}_3$  is said to be the *curvilinear coordinate reference frame* which is attached to the point  $P$ . The coordinates  $(v^\alpha, v^\beta, v^\gamma)$  of any vector  $\mathbf{v}$  in this Cartesian reference frame are termed the *curvilinear coordinates* of  $\mathbf{v}$ ; thus

$$\mathbf{v} = v^\alpha \mathbf{u}_1 + v^\beta \mathbf{u}_2 + v^\gamma \mathbf{u}_3.$$

If, then, we denote by  $\xi$  the  $3 \times 1$  matrix whose elements are the coordinates  $(v^\alpha, v^\beta, v^\gamma)$  of  $\mathbf{v}$  in the fixed Cartesian reference frame  $Oxyz$  and by  $a$  the  $3 \times 1$  matrix whose elements  $(v^\alpha, v^\beta, v^\gamma)$  are the curvilinear coordinates of  $\mathbf{v}$  we have

$$\xi = Ra,$$

where  $R$  is the  $3 \times 3$  rotation matrix the elements of whose columns are the coordinates, in the fixed Cartesian reference frame  $Oxyz$ , of the unit vectors  $\mathbf{u}_1, \mathbf{u}_2$  and  $\mathbf{u}_3$ .

If  $f(\alpha, \beta, \gamma)$  is any differentiable function of  $(\alpha, \beta, \gamma)$  the  $1 \times 3$  matrix  $(f_x, f_y, f_z)$  is the product of the  $3 \times 3$  matrix  $\frac{(\alpha, \beta, \gamma)}{(x, y, z)}$  by the  $1 \times 3$  matrix  $(f_\alpha, f_\beta, f_\gamma)$ . When the fixed Cartesian reference frame  $Oxyz$  is so chosen that  $R$  is the  $3 \times 3$  unit matrix  $E_3$  we have

$$\frac{(x, y, z)}{(\alpha, \beta, \gamma)} = \begin{bmatrix} \frac{1}{h_1} & 0 & 0 \\ 0 & \frac{1}{h_2} & 0 \\ 0 & 0 & \frac{1}{h_3} \end{bmatrix}$$

and so (why?)

$$\frac{(\alpha, \beta, \gamma)}{(x, y, z)} = \begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{bmatrix}.$$

Hence

$$(f_x, f_y, f_z) = (h_1 f_\alpha, h_2 f_\beta, h_3 f_\gamma).$$

In particular

The curvilinear coordinates of  $\text{grad } f$ , where  $f$  is a scalar field, are  $(h_1 f_\alpha, h_2 f_\beta, h_3 f_\gamma)$ .

Observe that for the particular fixed Cartesian reference frame  $Oxyz$  which furnishes the curvilinear coordinate reference frame at  $P$

$$\begin{aligned} x_\alpha &= \frac{1}{h_1}, y_\alpha = 0, z_\alpha = 0; & x_\beta &= 0, y_\beta = \frac{1}{h_2}, z_\beta = 0; \\ x_\gamma &= 0, y_\gamma = 0, z_\gamma = \frac{1}{h_3}. \end{aligned}$$

Since the vector  $ds = v(dx, dy, dz) = \left(\frac{d\alpha}{h_1}\right)u_1 + \left(\frac{d\beta}{h_2}\right)u_2 + \left(\frac{d\gamma}{h_3}\right)u_3$  we have

$$(ds)^2 = (ds|ds) = \frac{(d\alpha)^2}{h_1^2} + \frac{(d\beta)^2}{h_2^2} + \frac{(d\gamma)^2}{h_3^2}.$$

This formula provides the easiest way of remembering the expressions for the functions  $h_1, h_2$ , and  $h_3$ . Thus

$$\text{For cylindrical coordinates } (h_1, h_2, h_3) = \left(1, \frac{1}{\rho}, 1\right);$$

$$\text{For space polar coordinates } (h_1, h_2, h_3) = \left(1, \frac{1}{r}, \frac{1}{r \sin \theta}\right).$$

### EXERCISES

1. Show that the cylindrical coordinates of  $\text{grad } f$  are  $\left(f_\rho, \frac{1}{\rho} f_\phi, f_z\right)$ .
2. Show that the space polar coordinates of  $\text{grad } f$  are  $\left(f_r, \frac{1}{r} f_\theta, \frac{1}{r \sin \theta} f_\phi\right)$ .

In order to evaluate the divergence of a vector field in orthogonal curvilinear coordinates we must evaluate the derivatives of the elements of  $R$  with respect to  $(x, y, z)$ , where  $Oxyz$  is the fixed Cartesian

reference frame for which  $R$  is the  $3 \times 3$  unit matrix  $E_3$ . (Why, then, is the derivative of  $R$  with respect to  $x$ , for example, not the zero matrix? *Hint.*  $R$  is  $E_3$  at  $P$  but not at every point  $P + \Delta P$  of a neighborhood of  $P$ .) Since  $RR^* = E_3$  (at every point) we obtain, on differentiating this relation with respect to  $x$  and evaluating the derivative at  $P$ , where  $R = E_3 = R^*$ ,

$$R_x + R^*_x = O.$$

Thus  $R_x$  is a skew-symmetric matrix (when evaluated at  $P$ ). In other words if  $r_{kj}$  denotes the element in the  $j$ th row and  $k$ th column of  $R$  we have

$$(r_{ij})_x = 0; \quad (r_i^k)_x = -(r_k^i)_x; \quad k \neq j = 1, 2, 3.$$

It is clear that these relations remain valid if  $x$  is replaced by  $y$  or by  $z$  (why?) or by  $\alpha$ ,  $\beta$ , or  $\gamma$  (why?). Furthermore  $r_2^3$ , for example,  $= h_2 z_\beta$  and so  $(r_2^3)_x = h_1 (r_2^3)_\alpha = h_1 h_2 z_{\beta\alpha}$  at  $P$  (since  $z_\beta = 0$  at  $P$ ). The symmetry of  $z_{\beta\alpha}$  in the labels  $\beta$  and  $\alpha$  tells us that

$$(r_1^3)_y = (r_2^3)_x.$$

It follows that each of the three derivatives

$$(r_2^3)_x, (r_3^1)_y, (r_1^2)_z$$

is zero at  $P$ . In fact

$$(r_2^3)_x = -(r_3^2)_x = -(r_1^2)_z = (r_2^1)_z = (r_3^1)_y = -(r_1^3)_y = -(r_2^3)_x$$

so that  $(r_2^3)_x = 0$ ; similarly  $(r_3^1)_y = 0$ ,  $(r_1^2)_z = 0$ . To calculate the various derivatives of the type  $(r_2^3)_x$  we observe that since  $r_2^3 = h_2 z_\beta$ ,  $(r_2^3)_\gamma = h_2 z_{\beta\gamma}$  at  $P$  (why?). On differentiating with respect to  $\beta$  the relation

$$\frac{1}{h_3^2} = (x_\gamma)^2 + (y_\gamma)^2 + (z_\gamma)^2$$

and evaluating at  $P$  we obtain

$$+ \frac{1}{h_3^3} (h_3)_\beta = \frac{1}{h_3} z_{\gamma\beta}.$$

Hence  $z_{\gamma\beta} = \left(\frac{1}{h_3}\right)_\beta$  so that  $(r_2^3)_\gamma = h_2 \left(\frac{1}{h_3}\right)_\beta$ . Thus

$$\begin{aligned}
 R_x = h_1 R_\alpha &= h_1 \begin{bmatrix} 0 & h_2 \left(\frac{1}{h_1}\right)_\beta & h_3 \left(\frac{1}{h_1}\right)_\gamma \\ -h_2 \left(\frac{1}{h_1}\right)_\beta & 0 & 0 \\ -h_3 \left(\frac{1}{h_1}\right)_\gamma & 0 & 0 \end{bmatrix}; \\
 R_y = h_2 R_\beta &= h_2 \begin{bmatrix} 0 & -h_1 \left(\frac{1}{h_2}\right)_\alpha & 0 \\ h_1 \left(\frac{1}{h_2}\right)_\alpha & 0 & h_3 \left(\frac{1}{h_2}\right)_\gamma \\ 0 & -h_3 \left(\frac{1}{h_2}\right)_\gamma & 0 \end{bmatrix} \\
 R_z = h_3 R_\gamma &= h_3 \begin{bmatrix} 0 & 0 & -h_1 \left(\frac{1}{h_3}\right)_\alpha \\ 0 & 0 & -h_2 \left(\frac{1}{h_3}\right)_\beta \\ h_1 \left(\frac{1}{h_3}\right)_\alpha & h_2 \left(\frac{1}{h_3}\right)_\beta & 0 \end{bmatrix}.
 \end{aligned}$$

*Note.* The elements of the first column of  $R_x$  are the coordinates of the curvature vector at  $P$  of the first coordinate curve through  $P$  (since the curvature vector is the derivative, with respect to arc length, of the unit tangent vector).

Since  $\xi = Ra$  we have, at  $P$ ,

$$\xi_x = R_x a + h_1 a_\alpha; \quad \xi_y = R_y a + h_2 a_\beta; \quad \xi_z = R_z a + h_3 a_\gamma.$$

The divergence of the vector field  $\mathbf{v}$  is obtained by adding together the first row of  $\xi_x$ , the second row of  $\xi_y$ , and the third row of  $\xi_z$ . We obtain

$$\begin{aligned}
 \operatorname{div} \mathbf{v} &= h_1 (v^\alpha)_\alpha + h_1 h_2 \left(\frac{1}{h_1}\right)_\beta v^\beta + h_1 h_3 \left(\frac{1}{h_1}\right)_\gamma v^\gamma \\
 &+ h_1 h_2 \left(\frac{1}{h_2}\right)_\alpha v^\alpha + h_2 (v^\beta)_\beta + h_2 h_3 \left(\frac{1}{h_2}\right)_\gamma v^\gamma \\
 &+ h_1 h_3 \left(\frac{1}{h_3}\right)_\alpha v^\alpha + h_2 h_3 \left(\frac{1}{h_3}\right)_\beta v^\beta + h_3 (v^\gamma)_\gamma
 \end{aligned}$$

$$= h_1 h_2 h_3 \left\{ \left( \frac{v^\alpha}{h_2 h_3} \right)_\alpha + \left( \frac{v^\beta}{h_3 h_1} \right)_\beta + \left( \frac{v^\gamma}{h_1 h_2} \right)_\gamma \right\}.$$

## EXERCISES

3. Show that  $\text{div } \mathbf{v}$  is given, in cylindrical coordinates, by the expression

$$(v^\rho)_\rho + \frac{1}{\rho} v^\rho + \frac{1}{\rho} (v^\phi)_\phi + (v^z)_z.$$

4. Show that  $\text{div } \mathbf{v}$  is given, in space polar coordinates, by the expression

$$(v^r)_r + \frac{2}{r} v^r + \frac{1}{r} (v^\theta)_\theta + \frac{\cot \theta}{r} v^\theta + \frac{1}{r \sin \theta} (v^\phi)_\phi.$$

5. Show that the expression for the Laplacian  $\Delta_2 f$  of a scalar field  $f$  in orthogonal curvilinear coordinates is

$$\Delta_2 f = h_1 h_2 h_3 \left\{ \left( \frac{h_1 f_\alpha}{h_2 h_3} \right)_\alpha + \left( \frac{h_2 f_\beta}{h_3 h_1} \right)_\beta + \left( \frac{h_3 f_\gamma}{h_1 h_2} \right)_\gamma \right\}.$$

*Hint.*  $\Delta_2 f = \text{div grad } f$ ; the curvilinear coordinates of  $\text{grad } f$  are  $(h_1 f_\alpha, h_2 f_\beta, h_3 f_\gamma)$ .

*Note.* Remember the result of this exercise; it is one of the most useful formulas in applied mathematics.

6. Obtain the expression for  $\Delta_2 f$  in cylindrical coordinates.

$$\text{Answer. } f_{\rho\rho} + \frac{1}{\rho} f_\rho + \frac{1}{\rho^2} f_{\phi\phi} + f_{zz}.$$

7. Obtain the expression for  $\Delta_2 f$  in space polar coordinates.

$$\text{Answer. } f_{rr} + \frac{2}{r} f_r + \frac{1}{r^2} f_{\theta\theta} + \frac{\cot \theta}{r^2} f_\theta + \frac{1}{r^2 \sin^2 \theta} f_{\phi\phi}.$$

8. Obtain the expression for  $\text{curl } \mathbf{v}$  in orthogonal curvilinear coordinates. *Hint.* The first coordinate of  $\text{curl } \mathbf{v}$  is obtained by subtracting the second row of  $\xi_z$  from the third row of  $\xi_y$ .

*Answer.* The first coordinate of  $\text{curl } \mathbf{v}$  is  $h_2 h_3 \left\{ \left( \frac{v^\gamma}{h_3} \right)_\beta - \left( \frac{v^\beta}{h_2} \right)_\gamma \right\}$ ; the second coordinate is  $h_3 h_1 \left\{ \left( \frac{v^\alpha}{h_1} \right)_\gamma - \left( \frac{v^\gamma}{h_3} \right)_\alpha \right\}$ ; and the third coordinate is  $h_1 h_2 \left\{ \left( \frac{v^\beta}{h_2} \right)_\alpha - \left( \frac{v^\alpha}{h_1} \right)_\beta \right\}$ .

9. Determine the cylindrical coordinates of  $\text{curl } \mathbf{v}$ .

$$\text{Answer. } \left( \frac{1}{\rho} (v^\phi)_\phi - (v^\phi)_z; (v^\rho)_z - (v^z)_\rho; (v^\phi)_\rho - \frac{1}{\rho} (v^\rho)_\phi + \frac{1}{\rho} v^\phi \right).$$

10. Determine the space polar coordinates of  $\text{curl } \mathbf{v}$ .

$$\text{Answer. } \left( \frac{1}{r} (v^\phi)_\theta - \frac{1}{r \sin \theta} (v^\theta)_\phi + \frac{\cot \theta}{r} v^\phi; \frac{1}{r \sin \theta} (v^\phi)_\rho - (v^\phi)_r - \frac{1}{r} v^\phi, \right. \\ \left. (v^\theta)_r - \frac{1}{r} (v^r)_\theta + \frac{1}{r} v^\theta \right).$$

11. Obtain the orthogonal curvilinear coordinates of the linear vector function

$$\mathbf{A} : \begin{bmatrix} (v^x)_x & (v^x)_y & (v^x)_z \\ (v^y)_x & (v^y)_y & (v^y)_z \\ (v^z)_x & (v^z)_y & (v^z)_z \end{bmatrix}.$$

*Hint.* The columns of the desired matrix are  $\xi_x$ ,  $\xi_y$ , and  $\xi_z$ .

$$\text{Answer.} \quad \begin{bmatrix} h_1(v^\alpha)_\alpha + h_1h_2\left(\frac{1}{h_1}\right)_\beta v^\beta + h_1h_3\left(\frac{1}{h_1}\right)_\gamma v^\gamma \\ h_1(v^\beta)_\alpha - h_1h_2\left(\frac{1}{h_1}\right)_\beta v^\alpha \\ h_1(v^\gamma)_\alpha - h_1h_3\left(\frac{1}{h_1}\right)_\gamma v^\alpha \end{bmatrix}$$

$$\left. \begin{array}{ll} h_2(v^\alpha)_\beta - h_2h_1\left(\frac{1}{h_2}\right)_\alpha v^\beta & h_3(v^\alpha)_\gamma - h_3h_1\left(\frac{1}{h_3}\right)_\alpha v^\gamma \\ h_2(v^\beta)_\beta + h_2h_3\left(\frac{1}{h_2}\right)_\gamma v^\gamma + h_2h_1\left(\frac{1}{h_2}\right)_\alpha v^\alpha & h_3(v^\beta)_\gamma - h_3h_2\left(\frac{1}{h_3}\right)_\beta v^\gamma \\ h_2(v^\gamma)_\beta - h_2h_3\left(\frac{1}{h_2}\right)_\gamma v^\beta & h_3(v^\gamma)_\gamma + h_3h_1\left(\frac{1}{h_3}\right)_\alpha v^\alpha + h_3h_2\left(\frac{1}{h_3}\right)_\beta v^\beta \end{array} \right]$$

12. Write out the cylindrical coordinates of the linear vector function of Exercise 11.

$$\text{Answer.} \quad \begin{bmatrix} (v^\rho)_\rho & \frac{1}{\rho}(v^\rho)_\phi - \frac{1}{\rho}v^\phi & (v^\rho)_z \\ (v^\phi)_\rho & \frac{1}{\rho}\{(v^\phi)_\phi + v^\rho\} & (v^\phi)_z \\ (v^z)_\rho & \frac{1}{\rho}(v^z)_\phi & (v^z)_z \end{bmatrix}.$$

13. Write out the space polar coordinates of the linear vector function of Exercise 11.

$$\text{Answer.} \quad \begin{bmatrix} (v^r)_r & \frac{1}{r}(v^r)_\theta - \frac{1}{r}v^\theta & \frac{1}{r \sin \theta}(v^r)_\phi - \frac{1}{r}v^\phi \\ (v^\theta)_r & \frac{1}{r}\{(v^\theta)_\theta + v^r\} & \frac{1}{r \sin \theta}(v^\theta)_\phi - \frac{\cot \theta}{r}v^\phi \\ (v^\phi)_r & \frac{1}{r}(v^\phi)_\theta & \frac{1}{r \sin \theta}(v^\phi)_\phi + \frac{1}{r}v^r + \frac{\cot \theta}{r}v^\theta \end{bmatrix}.$$

14. Obtain the orthogonal curvilinear coordinates of the linear vector function

$$\mathbf{A}: \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix},$$

where  $f$  is a scalar field. *Hint.* Set  $(v^\alpha, v^\beta, v^\gamma) = (h_1 f_\alpha, h_2 f_\beta, h_3 f_\gamma)$  in the answer to Exercise 11.

15. Obtain the orthogonal curvilinear coordinates of the vector field  $d\mathbf{v}$ . *Hint.*

Multiply the  $3 \times 1$  matrix  $\begin{pmatrix} d\alpha \\ d\beta \\ d\gamma \end{pmatrix}$  by the  $3 \times 3$  matrix given in the answer to Exercise 11.

*Answer.*  $d\mathbf{v}$  is the sum of the vector whose curvilinear coordinates are  $(dv^\alpha, dv^\beta, dv^\gamma)$  and the vector product of  $\mathbf{v}$  by the vector whose curvilinear coordinates are

$$\left( h_2 \left( \frac{1}{h_3} \right)_\beta d\gamma - h_1 \left( \frac{1}{h_2} \right)_\gamma d\beta, h_3 \left( \frac{1}{h_1} \right)_\gamma d\alpha - h_1 \left( \frac{1}{h_3} \right)_\alpha d\gamma, h_1 \left( \frac{1}{h_2} \right)_\alpha d\beta - h_2 \left( \frac{1}{h_1} \right)_\beta d\alpha \right).$$

If  $\mathbf{A}$  is any linear vector function we denote its orthogonal curvilinear coordinates (what does this mean?) as follows:

$$A' = \begin{bmatrix} \widehat{\alpha\alpha} & \widehat{\alpha\beta} & \widehat{\alpha\gamma} \\ \widehat{\beta\alpha} & \widehat{\beta\beta} & \widehat{\beta\gamma} \\ \widehat{\gamma\alpha} & \widehat{\gamma\beta} & \widehat{\gamma\gamma} \end{bmatrix}.$$

Then  $A = RA'R^*$  so that (at the point  $P$  where  $R$  is the unit  $3 \times 3$  matrix  $E_3$ )

$$A_x = R_x A' + A'_x + A'R^*_x.$$

From the expression given above for  $R_x$  it follows that the element in the first row and first column of  $A_x$  is

$$h_1(\widehat{\alpha\alpha})_\alpha + h_1 h_2 \left( \frac{1}{h_1} \right)_\beta (\widehat{\alpha\beta} + \widehat{\beta\alpha}) + h_1 h_3 \left( \frac{1}{h_1} \right)_\gamma (\widehat{\alpha\gamma} + \widehat{\gamma\alpha}).$$

Similarly the element in the second row and first column of  $A_y$  is

$$h_2(\widehat{\beta\alpha})_\beta + h_2 h_1 \left( \frac{1}{h_2} \right)_\alpha \widehat{\alpha\alpha} + h_2 h_3 \left( \frac{1}{h_2} \right)_\gamma \widehat{\gamma\alpha} - h_2 h_1 \left( \frac{1}{h_2} \right)_\alpha \widehat{\beta\beta},$$

and the element in the third row and first column of  $A_z$  is

$$h_3(\widehat{\gamma\alpha})_\gamma + h_3 h_1 \left( \frac{1}{h_3} \right)_\alpha \widehat{\alpha\alpha} + h_3 h_2 \left( \frac{1}{h_3} \right)_\beta \widehat{\beta\alpha} - h_3 h_1 \left( \frac{1}{h_3} \right)_\alpha \widehat{\gamma\gamma}.$$

On adding these three expressions together we obtain the first curvilinear coordinate of  $\text{div } \mathbf{A}$ . It may be written in the form



$$h_1 h_2 h_3 \left\{ \left( \frac{\widehat{\alpha\alpha}}{h_2 h_3} \right)_\alpha + \left( \frac{\widehat{\beta\alpha}}{h_3 h_1} \right)_\beta + \left( \frac{\widehat{\gamma\alpha}}{h_1 h_2} \right)_\gamma \right\} + h_1 h_2 \left( \frac{1}{h_1} \right)_\beta \widehat{\alpha\beta} \\ + h_1 h_3 \left( \frac{1}{h_1} \right)_\gamma \widehat{\alpha\gamma} - h_1 h_2 \left( \frac{1}{h_2} \right)_\alpha \widehat{\beta\beta} - h_1 h_3 \left( \frac{1}{h_3} \right)_\alpha \widehat{\gamma\gamma}.$$

The other two coordinates are obtained by permuting cyclically the letters  $\alpha, \beta, \gamma$  and the labels 1, 2, 3. *Note.* This result is of importance in the Theory of Elasticity.

### EXERCISES

16. Write out the cylindrical coordinates of  $\text{div } \mathbf{A}$ .

$$\text{Answer. } (\widehat{\rho\rho})_\rho + \frac{1}{\rho} \widehat{\rho\rho} + \frac{1}{\rho} (\widehat{\phi\rho})_\phi + (\widehat{z\rho})_z - \frac{1}{\rho} \widehat{\phi\phi}; (\widehat{\rho\phi})_\rho + \frac{1}{\rho} \widehat{\rho\phi} + \frac{1}{\rho} (\widehat{\phi\phi})_\phi + (\widehat{z\phi})_z \\ + \frac{1}{\rho} \widehat{\phi\rho}; (\widehat{\rho z})_\rho + \frac{1}{\rho} \widehat{\rho z} + \frac{1}{\rho} (\widehat{\phi z})_\phi + (\widehat{z z})_z.$$

17. Write out the space polar coordinates of  $\text{div } \mathbf{A}$ .

$$\text{Answer. } (\widehat{r r})_r + \frac{2}{r} \widehat{r r} + \frac{1}{r} (\widehat{\theta r})_\theta + \frac{\cot \theta}{r} \widehat{\theta r} + \frac{1}{r \sin \theta} (\widehat{\phi r})_\phi - \frac{1}{r} \widehat{\theta\theta} - \frac{1}{r} \widehat{\phi\phi}; \\ (\widehat{r\theta})_r + \frac{2}{r} \widehat{r\theta} + \frac{1}{r} (\widehat{\theta\theta})_\theta + \frac{\cot \theta}{r} \widehat{\theta\theta} + \frac{1}{r \sin \theta} (\widehat{\phi\theta})_\phi + \frac{1}{r} \widehat{\theta r} - \frac{\cot \theta}{r} \widehat{\phi\phi}; \\ (\widehat{r\phi})_r + \frac{2}{r} \widehat{r\phi} + \frac{1}{r} (\widehat{\theta\phi})_\theta + \frac{\cot \theta}{r} \widehat{\theta\phi} + \frac{1}{r \sin \theta} (\widehat{\phi\phi})_\phi + \frac{1}{r} \widehat{\phi r} + \frac{\cot \theta}{r} \widehat{\phi\theta}.$$

18. Denoting by  $\mathbf{E}$  the average  $\frac{1}{2}(\mathbf{A} + \mathbf{A}^*)$  of the linear vector function  $\mathbf{A}$  of Exercise 11 and its adjoint  $\mathbf{A}^*$  determine the orthogonal curvilinear coordinates of  $\mathbf{E}$ .

*Answer.* Denoting the desired curvilinear coordinates by  $\widehat{11}$ ,  $\widehat{12}$ , etc. we have

$$\widehat{11} = h_1 (v^\alpha)_\alpha + h_1 h_2 \left( \frac{1}{h_1} \right)_\beta v^\beta + h_1 h_3 \left( \frac{1}{h_1} \right)_\gamma v^\gamma; \\ \widehat{23} = \frac{1}{2} \left\{ h_2 (v^\gamma)_\beta + h_3 (v^\beta)_\gamma - h_2 h_3 \left( \frac{1}{h_3} \right)_\beta v^\gamma - h_2 h_3 \left( \frac{1}{h_2} \right)_\gamma v^\beta \right\} \\ = \frac{1}{2} \left\{ \frac{h_2}{h_3} (h_3 v^\gamma)_\beta + \frac{h_3}{h_2} (h_2 v^\beta)_\gamma \right\} = \widehat{32}.$$

The other coordinates are obtained by permuting cyclically the coordinates  $\alpha, \beta, \gamma$  and the labels 1, 2, 3. *Note.* This important result furnishes the orthogonal curvilinear coordinates of the strain tensor in the theory of elasticity.

19. Write out the cylindrical coordinates of the linear vector function  $\mathbf{E}$  of Exercise 18.

$$\text{Answer. } \widehat{11} = (v^\rho)_\rho; \widehat{22} = \frac{1}{\rho} \{(v^\phi)_\phi + v^\rho\}; \widehat{33} = (v^z)_z; \widehat{23} = \frac{1}{2} \left\{ \frac{1}{\rho} (v^z)_\phi + (v^\phi)_z \right\};$$

$$\widehat{31} = \frac{1}{2} \{(v^\rho)_z + (v^z)_\rho\}; \widehat{12} = \frac{1}{2} \left\{ (v^\theta)_\rho + \frac{1}{\rho} (v^\rho)_\phi - \frac{1}{\rho} v^\theta \right\}.$$

20. Write out the space polar coordinates of the linear vector function  $\mathbf{E}$  of Exercise 18.

$$\text{Answer. } \widehat{11} = (v^r)_r; \widehat{22} = \frac{1}{r} \{(v^\theta)_\theta + v^r\}; \widehat{33} = \frac{1}{r \sin \theta} (v^\phi)_\phi + \frac{1}{r} v^r + \frac{\cot \theta}{r} v^\theta;$$

$$\widehat{23} = \frac{1}{2r} \left\{ (v^\phi)_\theta + \frac{1}{\sin \theta} (v^\theta)_\phi - \cot \theta v^\phi \right\}; \widehat{31} = \frac{1}{2} \left\{ \frac{1}{r \sin \theta} (v^r)_\phi + (v^\phi)_r - \frac{1}{r} v^\phi \right\};$$

$$\widehat{12} = \frac{1}{2} \left\{ (v^\theta)_r + \frac{1}{r} (v^r)_\theta - \frac{1}{r} v^\theta \right\}.$$

21. Determine the cylindrical coordinates of  $\Delta_2 \mathbf{v}$ , the Laplacian of a vector field  $\mathbf{v}$ . *Hint.*  $\Delta_2 \mathbf{v}$  is the divergence of  $\mathbf{A}^*$  where  $\mathbf{A}$  is the linear vector function whose cylindrical coordinates are given in the answer to Exercise 12. See Exercise 16.

$$\text{Answer. } (v^\rho)_{\rho\rho} + \frac{1}{\rho} (v^\rho)_\rho + \frac{1}{\rho^2} (v^\rho)_{\phi\phi} + (v^\rho)_{zz} - \frac{2}{\rho^2} (v^\phi)_\phi - \frac{1}{\rho^2} v^\rho;$$

$$(v^\phi)_{\rho\rho} + \frac{1}{\rho} (v^\phi)_\rho + \frac{1}{\rho^2} (v^\phi)_{\phi\phi} + (v^\phi)_{zz} + \frac{2}{\rho^2} (v^\rho)_\phi - \frac{1}{\rho^2} v^\theta;$$

$$(v^z)_{\rho\rho} + \frac{1}{\rho} (v^z)_\rho + \frac{1}{\rho^2} (v^z)_{\phi\phi} + (v^z)_{zz}.$$

22. Determine the space polar coordinates of  $\Delta_2 \mathbf{v}$ , the Laplacian of a vector field  $\mathbf{v}$ .

$$\text{Answer. } (v^r)_{rr} + \frac{2}{r} (v^r)_r + \frac{1}{r^2} (v^r)_{\theta\theta} + \frac{\cot \theta}{r^2} (v^r)_\theta + \frac{1}{r^2 \sin^2 \theta} (v^r)_{\phi\phi} - \frac{2}{r^2} v^r - \frac{2}{r^2} (v^\theta)_\theta$$

$$- \frac{2 \cot \theta}{r^2} v^\theta - \frac{2}{r^2 \sin \theta} (v^\phi)_\phi;$$

$$(v^\theta)_{rr} + \frac{2}{r} (v^\theta)_r + \frac{1}{r^2} (v^\theta)_{\theta\theta} + \frac{\cot \theta}{r^2} (v^\theta)_\theta + \frac{1}{r^2 \sin^2 \theta} (v^\theta)_{\phi\phi} + \frac{2}{r^2} (v^r)_\theta$$

$$- \frac{\operatorname{cosec}^2 \theta}{r^2} v^\theta - \frac{2 \cos \theta}{r^2 \sin^2 \theta} (v^\phi)_\phi;$$

$$(v^\phi)_{rr} + \frac{2}{r} (v^\phi)_r + \frac{1}{r^2} (v^\phi)_{\theta\theta} + \frac{\cot \theta}{r^2} (v^\phi)_\theta + \frac{1}{r^2 \sin^2 \theta} (v^\phi)_{\phi\phi} + \frac{2}{r^2 \sin \theta} (v^r)_\phi$$

$$+ \frac{2 \cos \theta}{r^2 \sin^2 \theta} (v^\theta)_\phi - \frac{\operatorname{cosec}^2 \theta}{r^2} v^\phi.$$

# 5

## LAPLACE'S EQUATION

### 1. Problems in electrostatics

In an electrostatic field there are two fundamental vectors:

1. The *electric intensity* vector  $\mathbf{E}$ . This vector is such that the mechanical force which acts on a point charge  $e$  is given by the formula  $\mathbf{F} = e\mathbf{E}$ . The line integral  $\int_{C, P_0}^P (\mathbf{E} | d\mathbf{s})$ , along any curve  $C$  connecting any two points  $P_0$  and  $P$ , is independent of the curve  $C$  and so there exists a scalar point-function  $V(P)$  such that

$$\int_{P_0}^P (\mathbf{E} | d\mathbf{s}) = V(P_0) - V(P).$$

The function  $V(P)$  defined by this relation is indeterminate to the extent of an additive constant; if the electrostatic field is unbounded we determine this additive constant by arranging that  $V(P) \rightarrow 0$  as  $P \rightarrow \infty$  (it being assumed that  $V(P)$  has a limit as  $P \rightarrow \infty$ ). *Note.* The definition of  $V(P)$  by the formula given above rather than by the formula

$$\int_{P_0}^P (\mathbf{E} | d\mathbf{s}) = V(P) - V(P_0)$$

is attributable to a desire to have the direction of  $\mathbf{E}$  that of the *negative*, rather than of the *positive*, gradient of  $V$  so that  $V$  decreases (rather than increases) in the direction of  $\mathbf{E}$ . It follows from the definition of the point-function  $V(P)$  that

$$\mathbf{E} = -\text{grad } V.$$

Hence at any point  $P$ ,  $\mathbf{E}$  is perpendicular to the level surface of the point-function  $V$  which passes through  $P$ .  $V(P)$  is called the *potential*

of the electrostatic field, and a *conductor* is an *equipotential surface*, i.e., a level surface of the potential point-function.

2. The *displacement* vector  $\mathbf{D}$ . The displacement vector  $\mathbf{D}$  determines the distribution of charge in the electrostatic field as follows:

The surface integral  $\int_S (\mathbf{D} | d\mathbf{S})$  of  $\mathbf{D}$  over any closed surface  $S$  (where  $d\mathbf{S}$  has the direction of the *outward* normal to  $S$ ) furnishes the amount of electric charge inside (in the sense of: interior to or on) the surface. If  $S$  is the surface of a conductor and if  $\mathbf{n}$  is the outward unit normal to  $S$  then

$$D_n = (\mathbf{D} | \mathbf{n}) = \sigma$$

is the *surface density* of electric charge on  $S$ , the total charge on  $S$  being

$$\int_S \sigma dS = \int_S D_n dS = \int_S (\mathbf{D} | d\mathbf{S}).$$

*Note.* Be sure that you understand clearly the different roles of the two vectors  $\mathbf{E}$  and  $\mathbf{D}$ .  $\mathbf{E}$  serves to determine the mechanical force that would act upon a point charge at a given point  $P$  of the electrostatic field while  $\mathbf{D}$  serves to determine the distribution of electric charge in the field.  $\mathbf{E}$  is what may be termed a *line vector*, i.e., the kind of vector which is integrated *along a curve*, while  $\mathbf{D}$  is what may be called a *surface vector*, i.e., the kind of vector that is integrated *over a surface*.

Let  $W$  denote the volume of which a given closed surface  $S$  is the boundary (we use  $W$  rather than  $V$  since we are using  $V$  to denote the potential of the electrostatic field; we shall denote the element of volume  $dx dy dz$  by  $d\tau$ ). The relation

$$\int_S (\mathbf{D} | d\mathbf{S}) = \int_W \text{div } \mathbf{D} d\tau$$

tells us that the charge which is distributed over  $W$  is furnished by the volume integral  $\int_W \text{div } \mathbf{D} d\tau = \int_W \rho d\tau$ , where  $\rho = \text{div } \mathbf{D}$ . For this reason the point-function  $\rho = \text{div } \mathbf{D}$  is known as the *volume density* of the distribution of charge in the electrostatic field. At points free from charge  $\rho = 0$  so that

$$\text{div } \mathbf{D} = 0.$$

If we consider the electrostatic field due to a single point charge  $e$  located at the origin  $O$  of a system of space polar coordinates, reasons

of symmetry tell us that the polar coordinates of  $\mathbf{D}$  are  $(D^r, 0, 0)$ , where  $D^r$  is a function of  $r$  alone. Taking  $S$  as the surface of a sphere of radius  $r$  with center at  $O$  we have

$$\int_S (\mathbf{D} | d\mathbf{S}) = 4\pi r^2 D^r$$

and so (why?)

$$D^r = \frac{e}{4\pi r^2}.$$

The same reasons of symmetry tell us that the space polar coordinates of  $\mathbf{E}$  are  $(E^r, 0, 0)$ , where  $E^r$  is a function of  $r$  alone, and experience shows that  $E^r$  varies as  $\frac{1}{r^2}$ . The unit of electric charge is determined

so that, in empty space,  $E^r = \frac{e}{r^2}$  and so

$$D^r = \frac{1}{4\pi} E^r$$

or, equivalently (why?),

$$\mathbf{D} = \frac{1}{4\pi} \mathbf{E}.$$

We *postulate* that this relation is valid at points free from matter in general electrostatic fields and not merely for the electrostatic field due to a single point charge. Then on combining the relations

$$\operatorname{div} \mathbf{D} = \rho; \quad \mathbf{D} = \frac{1}{4\pi} \mathbf{E}; \quad \mathbf{E} = -\operatorname{grad} V$$

we obtain

$$\Delta_2 V = \operatorname{div} \operatorname{grad} V = -4\pi\rho.$$

This equation governing the potential of an electrostatic field is known as *Poisson's equation* (after S. D. Poisson [1781–1840], a French mathematician). At points where  $\rho = 0$ , i.e., where there is no volume density of electric charge, it reduces to

$$\Delta_2 V = 0.$$

This is known as *Laplace's equation* (after P. S. Laplace [1749–1827], a French mathematician). The problem of determining the electrostatic field due to a charged conductor may, then, be phrased as follows:

Determine a solution  $V$  of Laplace's equation  $\Delta_2 V = 0$  which takes an assigned (constant) value  $V_0$  over the surface of the conductor.

It is understood that  $V$  possesses a continuous matrix of second derivatives (and, hence, a continuous gradient) over the volume  $W$  bounded by  $S$  and that  $V$  is continuous over  $W + S$ ; furthermore, if  $W$  is the region exterior to  $S$ ,  $V$  must satisfy certain conditions, which we shall describe shortly, as the point of evaluation  $P$  of  $V \rightarrow \infty$ .

That this problem does not have more than one solution is an immediate consequence of the relation

$$\int_W \{g \Delta_2 f + (\mathbf{grad} g | \mathbf{grad} f)\} d\tau = \int_S g \frac{df}{dn} dS$$

(see Exercise 19, p. 30). When  $g = f$  this relation becomes

$$\int_W \{f \Delta_2 f + (\mathbf{grad} f | \mathbf{grad} f)\} d\tau = \int_S f \frac{df}{dn} dS.$$

If  $f$  is the difference of two hypothecated solutions of the conductor problem we have  $\Delta_2 f = 0$  over  $W$  (why?), and  $f = 0$  over  $S$  (why?). Hence, when  $W$  is the interior of  $S$ ,

$$\int_W (\mathbf{grad} f | \mathbf{grad} f) d\tau = 0.$$

Since  $(\mathbf{grad} f | \mathbf{grad} f)$  is continuous and  $\geq 0$  over  $W$  it follows that  $(\mathbf{grad} f | \mathbf{grad} f) = 0$  over  $W$  (why?) and hence that  $\mathbf{grad} f$  is the zero vector over  $W$ . Hence  $f$  is constant over  $W$ , and, being zero over  $S$ ,  $f$  must be zero over  $W$ . Hence the two hypothecated solutions of the conductor problem are one and the same.

*Note 1.* It is clear from the proof that it is not necessary for the validity of this *uniqueness theorem* that  $V$  be constant over  $S$  or that  $\Delta_2 V$  be zero over  $W$ . All that is required (in addition to the continuity assumptions) is that  $V$  have an assigned value at each point of  $S$  and that  $\Delta_2 V$  have an assigned value at each point of  $W$ . Then the difference  $f$  of two hypothecated solutions is zero over  $S$  and is such that  $\Delta_2 f$  is zero over  $W$ ; thus the proof proceeds as before.

*Note 2.* The uniqueness theorem limits the number of possible solutions to one, but it does not show that there is actually one solution. This *existence theorem* is known as the Dirichlet problem (after P. G. L. Dirichlet [1805–1859], a German mathematician). Its proof requires a detailed treatment, and we shall not enter into it here; a full discussion may be found in *Foundations of Potential Theory* by W. D.

**Kellogg.** In the problems we shall treat we shall actually furnish a solution of the conductor problem, and we only need the uniqueness theorem (proved above) to assure us that this solution is *the one and only* solution.

When  $W$  is the *interior* of the closed surface  $S$  the one and only solution of the conductor problem is  $V = V_0$ , where  $V_0$  is the assigned constant value of  $V$  on  $S$ . Hence we have the following theorem (which explains the *screening effect* of closed conductors):

*The electrostatic potential throughout the interior of a closed-conducting surface is constant.*

Since  $\mathbf{E} = -\text{grad } V$  an equivalent form of statement of this theorem is as follows:

*The electric intensity inside a closed conductor is zero.*

*Note.* If the region  $W$  is bounded by an exterior surface  $S'$  and an interior surface  $S$  (both surfaces being those of conductors) the electric intensity throughout  $W$  will be zero if, and only if,  $V'_0 = V_0$  i.e., if, and only if, both conductors are kept at the same potential. When  $V'_0 \neq V_0$  there will be a non-zero electric field throughout  $W$ , and the problem of determining  $V$  for this field is known as the *condenser problem*.

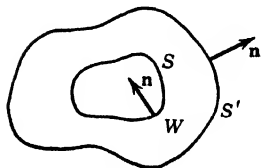


FIG. 20.

In view of the relation (see Exercise 18, p. 30)

$$\int_W \Delta_2 f \, d\tau = \int_S \frac{df}{dn} \, dS$$

we have, since  $\Delta_2 V = 0$  over  $W$ ,

$$\int_S \frac{dV}{dn} \, dS + \int_{S'} \frac{dV}{dn} \, dS = 0$$

(the unit normal vector  $\mathbf{n}$  being, in each of the two surface integrals, drawn away from  $W$ ). Since

$$\frac{dV}{dn} = (\text{grad } V | \mathbf{n}) = -(\mathbf{E} | \mathbf{n}) = -4\pi(\mathbf{D} | \mathbf{n}) = 4\pi\sigma$$

(where  $\sigma$  is the surface density of charge on the *inner* side of  $S'$  and on the *outer* side of  $S$ ) the relation just written may be put in the form

$$\int_S \sigma \, dS + \int_{S'} \sigma \, dS = 0.$$

In other words the total electric charge  $\int_S \sigma dS$  on the outer side of the conducting surface  $S$  is the negative of the total electric charge  $\int_{S'} \sigma dS$  on the inner side of the conducting surface  $S'$ . We term the absolute value of either of these charges the *charge* on the condenser, and we term the absolute value of the difference  $V_0 - V'_0$  between the potentials of  $S$  and  $S'$ , respectively, the *voltage* of the condenser. The ratio

$$C = \frac{\text{charge on condenser}}{\text{voltage on condenser}}$$

is known as the *capacity* of the condenser.

When  $W$  is the region *external* to a single conducting surface  $S$  the electric field throughout  $W$  will not be, in contrast with the *internal* field, zero unless  $V_0 = 0$ , i.e., unless the conductor is at zero potential. The reason for this is that the unbounded region  $W$  must first of all be replaced by a bounded region  $W'$  which may be conveniently obtained as the region between the inner conducting surface  $S$  and an outer conducting surface  $S'$  which is a sphere of sufficiently large radius  $r$  with its center located at any convenient origin  $O$ . We then investigate what happens as  $r \rightarrow \infty$ . In order to prove the uniqueness theorem we must now pay attention to the integral

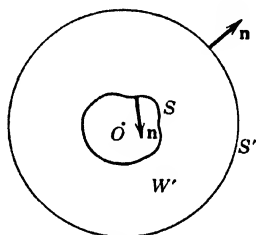


FIG. 21.

$$\int_{S'} f \frac{df}{dn} dS,$$

where  $f$  is the difference between two hypothesized solutions of the (external) conductor problem. If this integral has the limit zero at  $r = \infty$  the uniqueness theorem is valid since then the (improper) integral

$$\int_W (\text{grad } f | \text{grad } f) dr$$

is zero (the integral being improper since the region  $W$  is unbounded).

In order to make sure that the integral of  $f \frac{df}{dn}$  over  $S'$  is null at  $r = \infty$  we impose the following condition on the potential function  $V$ :

*Not only is  $V$  null at  $r = \infty$  but the product  $r \text{ grad } V$  is null at  $r = \infty$ .*



This condition is sufficient to ensure that the integral of  $V \frac{dV}{dn}$  over  $S'$  is null at  $r = \infty$ , and this implies (since both  $f$  and  $r \text{ grad } f$  are null at  $r = \infty$ ) that the integral of  $f \frac{df}{dn}$  over  $S'$  is null at  $r = \infty$ . In order to see that the conditions imposed on  $V$  near  $r = \infty$  are sufficient to make the integral of  $V \frac{dV}{dn}$  over  $S'$  null at  $r = \infty$  we first observe that

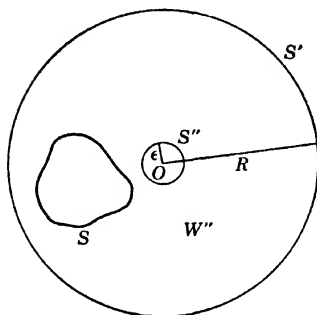


FIG. 22.

the function  $\frac{1}{r}$  satisfies Laplace's equation at all points save  $O$ . (Prove this. *Hint.* See Exercise 7, p. 109.) Let, then,  $O$  be any point exterior to the closed conductor  $S$ , and let  $W''$  be the region bounded by  $S$ , by a sphere  $S'$  of sufficiently large radius  $R$ , and by a sphere  $S''$  of sufficiently small radius  $\epsilon$  (both  $S'$  and  $S''$  having the same center  $O$ ). Applying to  $W''$  the relation (see Exercise 20, p. 31)

$$\int_{W''} \left\{ \frac{1}{r} \Delta_2 V - V \Delta_2 \left( \frac{1}{r} \right) \right\} d\tau = \int_{S+S'+S''} \left\{ \frac{1}{r} \frac{dV}{dn} - V \frac{d}{dn} \left( \frac{1}{r} \right) \right\} dS$$

we obtain

$$\int_{S+S'+S''} \left\{ \frac{1}{r} \frac{dV}{dn} - V \frac{d}{dn} \left( \frac{1}{r} \right) \right\} dS = 0$$

since

$$\Delta_2 V = 0, \quad \Delta_2 \left( \frac{1}{r} \right) = 0$$

over  $W''$ . Over  $S$ ,  $V$  has a constant value  $V_0$  and so

$$\int_S V \frac{d}{dn} \left( \frac{1}{r} \right) dS = V_0 \int_S \frac{d}{dn} \left( \frac{1}{r} \right) dS = 0$$

(since  $\Delta_2 \left( \frac{1}{r} \right) = 0$  over the interior of  $S$ . See Exercise 7, p. 109).

Furthermore, over  $S$ ,  $\frac{dV}{dn} = 4\pi\sigma$ , where  $\sigma$  is the surface density on the outer side of  $S$ . Hence

$$\int_S \left\{ \frac{1}{r} \frac{dV}{dn} - V \frac{d}{dn} \left( \frac{1}{r} \right) \right\} dS = 4\pi \int_S \frac{\sigma}{r} dS.$$

Over  $S'$ ,  $\frac{d}{dn} \left( \frac{1}{r} \right) = -\frac{1}{R^2}$  and so  $\int_{S'} V \frac{d}{dn} \left( \frac{1}{r} \right) dS = -\int_{S'} V \sin \theta d\theta d\phi$  (since  $dS = R^2 \sin \theta d\theta d\phi$  over  $S'$ ). Hence, since  $V$  is null at  $r = \infty$ ,  $\int_{S'} V \frac{d}{dn} \left( \frac{1}{r} \right) dS$  is null at  $R = \infty$ . Since  $r \mathbf{grad} V$  is, by hypothesis, null at  $r = \infty$ ,  $\int_{S'} \frac{1}{r} \frac{dV}{dn} dS$  is null at  $r = \infty$  (why?). Over  $S''$ ,  $dS = \epsilon^2 \sin \theta d\theta d\phi$  and so  $\int_{S''} \frac{1}{r} \frac{dV}{dn} dS$  is null at  $\epsilon = 0$ . On the other hand  $\int_{S''} V \frac{d}{dn} \left( \frac{1}{r} \right) dS$  has the limit  $4\pi V(O)$  at  $\epsilon = 0$ . (Prove this. *Hint.* Remember that  $\mathbf{n}$  has the direction of the inward drawn radius of  $S''$ ; hence, over  $S''$ ,  $\frac{d}{dn} \left( \frac{1}{r} \right) = \frac{1}{\epsilon^2}$ ). On letting  $\epsilon \rightarrow 0$  and  $R \rightarrow \infty$  we see that

$$4\pi \int_S \frac{\sigma}{r} dS - 4\pi V(O) = 0$$

or, equivalently,

$$V(O) = \int_S \frac{\sigma}{r} dS.$$

Since  $O$  is any point exterior to the conducting surface  $S$  this relation may be phrased as follows:

*The potential at any point  $P$  external to a conductor  $S$  is given by the surface integral*

$$V(P) = \int_S \frac{\sigma}{r} dS,$$

where  $\sigma$  is the surface density of charge on the outside of  $S$  and  $r = |QP|$  is the distance from the variable point of integration  $Q$  of integration on  $S$  to the point of evaluation  $P$  of  $V$ .

*Note 1.* This important result is easily remembered as follows: The element of the integral which furnishes  $V(P)$  may be regarded as the potential at  $P$  of the charge  $\sigma dS$  on the element of area  $dS$  at  $Q$ . The potential at  $P$  is the result of integrating over  $S$  the potentials of the charges  $\sigma dS$  on the various elements of area  $dS$  of  $S$ .

*Note 2.* It is clear that the result just proved is valid if there are

present several conductors  $S_1, S_2, \dots, S_n$ . Then

$$V(P) = \int_{S_1} \frac{\sigma}{r} dS + \dots + \int_{S_n} \frac{\sigma}{r} dS.$$

Let, now,  $O$  be any fixed origin, and denote  $|OP|$  by  $R$  (to distinguish it from  $r = |QP|$ , where  $Q$  is the variable point of integration on  $S$ ). Then it follows at once from the formula

$$V(P) = \int_S \frac{\sigma}{r} dS$$

that not only is  $V(P)$  null at  $R = \infty$  but also the product  $RV(P)$  has a limit at  $R = \infty$ , this limit being the charge  $\int_S \sigma dS$  on  $S$  (or, if several

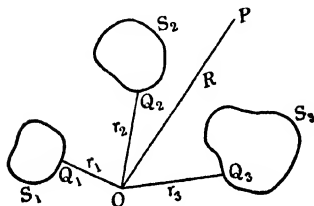


FIG. 23.

conductors  $S_1, \dots, S_n$  are present, the sum of the charges on these various conductors). In fact if we denote by  $d$  the absolute maximum of  $|OQ|$  over  $S$  we have  $|r - R| \leq d$  (why?) and so  $\frac{r}{R}$  has at  $R = \infty$  the limit 1, the convergence of  $\frac{r}{R}$  to its limit 1 being uniform over  $S$ .

(What does this mean?) Hence  $\frac{R}{r}$  has at  $R = \infty$  the limit 1, the convergence of  $\frac{R}{r}$  to its limit 1 being uniform over  $S$ . But

$$R V(P) - \int_S \sigma dS = \int_S \left( \frac{R}{r} - 1 \right) \sigma dS$$

and so  $R V(P) - \int_S \sigma dS$  is null at  $R = \infty$  (why?). In other words

$$R V(P) \text{ has at } R = \infty \text{ the limit } \int_S \sigma dS.$$

### EXERCISES

1. Show that if  $V(P)$  is the potential due to a point charge in the presence of one or more conductors then  $RV(P)$  has at  $R = \infty$  a limit equalling the total charge in the field, i.e., the sum of the point charge and the charges on the various conductors. *Hint.* A point charge  $e$  is characterized by the fact that near it  $V$  is unbounded. The product  $rV$ , where  $r$  is the distance from the point charge, is

granted to be bounded at  $r = 0$ , and the integral of  $\frac{dV}{dn}$  over any sufficiently small surface enclosing the point charge,  $\mathbf{n}$  being the outward drawn normal, is  $-4\pi e$ . Apply the argument of the text to a region obtained by removing from the region  $W''$  of the text a sphere of sufficiently small radius with center at the point charge, obtaining

$$V(P) = \int_{S_1} \frac{\sigma dS}{r} + \dots + \int_{S_n} \frac{\sigma dS}{r} + \frac{e}{r}$$

where, in the last term,  $r$  denotes the distance from the point charge to  $P$ .

2. Show that the potential function of the field of several point charges  $e_1, \dots, e_k$  in the presence of several conductors  $S_1, \dots, S_n$  is given by the formula

$$V(P) = \int_{S_1} \frac{\sigma dS}{r} + \dots + \int_{S_n} \frac{\sigma dS}{r} + \frac{e_1}{r_1} + \dots + \frac{e_k}{r_k}$$

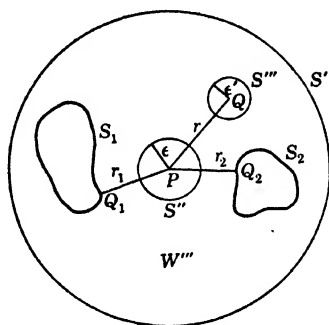


FIG. 24.

where  $r_j, j = 1, \dots, k$ , is the distance from the  $j$ th point charge to  $P$ .

*Warning.* Do not fall into the illusion of thinking that this formula solves the problem of determining the electrostatic field of several point charges in the presence of several conductors. The surface density of charge  $\sigma$  on the various conductors is unknown and can be determined, only after the problem has been solved, by means of the relation  $\frac{dV}{dn} = -4\pi\sigma$ .

*Note.* The part of the formula for  $V(P)$  which is known before the problem is solved, namely,  $\frac{e_1}{r_1} + \dots + \frac{e_k}{r_k}$ , is termed the *inducing potential*, and we shall

denote it by  $V'$ ; the remaining part, namely,  $\int_{S_1} \frac{\sigma dS}{r} + \dots + \int_{S_n} \frac{\sigma dS}{r}$ , is termed the *induced potential*, and we shall denote it by  $V''$ . Thus

$$V = V' + V'',$$

and the general problem of electrostatics may be phrased as follows: Given the inducing potential  $V'$ , determine the induced potential  $V''$  so that their sum  $V = V' + V''$ , which is the actual or total potential of the field, may satisfy the various conditions imposed on it. These conditions are

1.  $V$  must be a solution of Laplace's equation  $\Delta_2 V = 0$ .
2.  $V$  must have an assigned constant value on each conductor in the field.
3. The integral of  $\frac{dV}{dn}$  over any sufficiently small surface enclosing a point charge must be the product of this point charge by  $-4\pi$ .
4.  $V$  must be null at  $r = \infty$ , and  $r \text{ grad } V$  must be null at  $r = \infty$ .

3. Prove the uniqueness theorem for the field of several point charges in the presence of several conductors.

4. Show that the potential of a single point charge is  $\frac{e}{r}$ . *Note.* Although this is not the definition of the potential of a single point charge (what is the definition of this?) it was in order to have this relation that the hypothesis  $\mathbf{D} = \frac{1}{4\pi} \mathbf{E} = -\frac{1}{4\pi} \text{grad } V$  was adopted.

5. Show that the potential of a charged sphere is (at points outside the sphere) the same as if the total charge were concentrated at the center of the sphere. What is the potential of the charged sphere at points inside the sphere? What is the surface density of charge on the sphere?

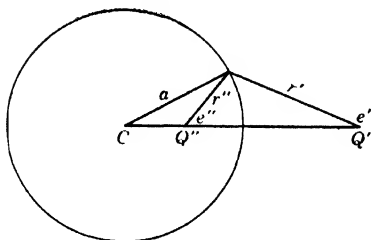


FIG. 25.

6. Show that one of the level surfaces of the potential function  $V = V(P)$  of two point charges of unlike sign is a sphere whose center is on the line joining the two point charges. *Hint.* If the point charges are  $e'$  and  $e''$  the level surface

$V = 0$  is such that  $\frac{r''}{r'} = -\frac{e''}{e'}$ , where  $r'$  and  $r''$  are the distances of a variable point  $P$  of the level surface from the charges  $e'$  and  $e''$ , respectively.

7. Show that if  $C$  is the center and  $a$  the radius of the sphere of Exercise 6 and if the point charges  $e'$  and  $e''$  are at  $Q'$  and  $Q''$ , respectively, then  $|CQ'| |CQ''| = a^2$ , and  $e'' = -\frac{a}{|CQ'|} e'$ . *Hint.* To obtain the relation  $e'' = -\frac{a}{|CQ'|} e'$  evaluate  $\frac{r''}{r'}$  at the point where the sphere intersects the ray  $C \rightarrow Q'$ .

8. Show that the induced potential of a point charge  $e'$  at a point  $Q'$  outside a conducting sphere of radius  $a$  which is *earthed*, i.e., maintained at potential zero, is the same as that of a point charge  $e'' = -\frac{ae'}{|CQ'|}$  located at the point  $Q''$ , where  $Q''$  is located on the ray  $C \rightarrow Q'$ , and such that  $|CQ'| |CQ''| = a^2$ . *Hint.* Show that  $V = \frac{e'}{r'} + \frac{e''}{r''}$  satisfies all the conditions imposed on  $V$ , and use the uniqueness theorem. *Note.* The point charge  $e''$  whose potential  $V'' = \frac{e''}{r''}$  is the induced potential is termed the *image* in the earthed sphere of the *inducing* point charge  $e'$  (whose potential is  $V' = \frac{e'}{r'}$ ). The point  $Q''$  at which the image of  $e'$  is located is termed the *image point* of  $Q'$  in the sphere.

9. Show that the capacity of a condenser formed by two concentric spheres of radii  $a$  and  $b > a$  is the reciprocal of  $\frac{1}{a} - \frac{1}{b}$ . *Hint.* The potential of the

electrostatic field between the two spherical conductors is of the form  $A + \frac{B}{r}$ , where  $r$  denotes the distance from the common center of the two spheres and the constants  $A$  and  $B$  are so determined that  $V(P)$  has assigned values on the two spherical conductors.

10. Show that the capacity of a sphere of radius  $a$  is  $a$ . *Note.* The capacity of a single conductor is the limit at  $r = \infty$  of the capacity of the condenser formed by the given conductor  $S$  and a sphere  $S'$  of sufficiently large radius. It is the ratio of the charge on the conductor to its potential. (Prove this.)

## 2. The method of separation of variables

The conductor problem may be divided into two parts:

1. We must solve Laplace's differential equation  $\Delta_2 V = 0$ .

2. We must *adjust* the solution of Laplace's equation which we have obtained to the *boundary conditions*; for example, we must make sure that  $V$  takes an assigned constant value over each of the given conductors, and, if the field is unbounded, we must make sure that  $V(P)$  behaves properly (what does this mean?) as  $r \rightarrow \infty$ .

Of these two parts the second is the more difficult and the more *individual*; the potential of *every* conductor problem must satisfy Laplace's equation, and what distinguishes one conductor problem from another is the shape and location of the one or more conductors involved. The boundary condition over a conductor is very stringent. Thus if the surface  $S$  of the conductor is given by

$$\mathbf{r} = \mathbf{r}(u, v)$$

then  $V = V(P) = V(x, y, z)$  becomes, over  $S$ , a function of the two independent variables, or parameters,  $u$  and  $v$ . This function must be a *constant* function of  $u$  and  $v$ . Let us suppose, for example, that  $u = y, v = z$  so that the conductor  $S$  (or, at least, a portion of it) is furnished by an equation of the form

$$x = x(y, z);$$

then  $V(x, y, z)$  must reduce, when  $x$  is replaced by  $x(y, z)$ , to a constant function of  $(y, z)$ . The simplest case would occur when  $x(y, z)$  is a constant function  $c_1$ , say, so that  $S$  (or, at least, a portion of  $S$ ) is part of a plane parallel to the  $x$ -plane. If  $V_0$  is the assigned potential of the conductor  $S$  the function

$$V(c_1, y, z) - V_0$$

must be zero, identically in  $y$  and  $z$ . In other words the function

$V(x, y, z) - V_0$  must have the property that when  $x$  is set equal to  $c_1$  the resulting function of  $y$  and  $z$  must be *identically* zero. One way of securing this result is to have  $V(x, y, z) - V_0$  the product of a function  $X$  of  $x$  alone by a function  $U(y, z)$  which does not involve  $x$ :

$$V(x, y, z) - V_0 = X(x) U(y, z).$$

If  $X(x)$  is such that  $X(c_1) = 0$  then  $V(x, y, z) - V_0$  is such that  $V(c_1, y, z)$  is identically zero. It is clear that this is not the only way of making sure that  $V - V_0$  is zero, identically in  $y$  and  $z$ , when  $x = c_1$ . In fact the sum of any number of terms of the type  $X(x)U(y, z)$  would do equally well,

$$V(x, y, z) - V_0 = X_1(x)U_1(y, z) + \cdots + X_n(x)U_n(y, z),$$

provided that each of the functions  $X_j(x)$ ,  $j = 1, \cdots, n$ , has the value zero when  $x = c_1$ ; and we may let  $n = \infty$  provided that convergence questions are attended to. Since  $V$  satisfies Laplace's equation so also does  $V - V_0$ . (Prove this.) Our first attempt, then, in trying to solve a conductor problem in which one of the conductors (or part of it) is a portion of a plane  $x = c_1$  is to seek for a solution of Laplace's equation of the form  $X(x)U(y, z)$ . Such a (very special) solution of Laplace's equation has the variable  $x$  *separated* from the other two variables  $y$  and  $z$ , and the method of obtaining such special solutions of Laplace's equation is referred to as *solving Laplace's equation by the method of separation of variables*.

The function  $X(x)$  which occurs in the solution of Laplace's equation by the method of separation of variables is, naturally, very special. On substituting  $XU$  for  $V$  in Laplace's equation we obtain

$$X_{xx}U + X(U_{yy} + U_{zz}) = 0$$

or, equivalently,

$$\frac{X_{xx}}{X} + \frac{U_{yy} + U_{zz}}{U} = 0; \quad V \neq 0.$$

Hence, at any point where  $V \neq 0$ ,  $\frac{X_{xx}}{X}$  which is, by hypothesis, a constant function of  $y$  and  $z$  is also a constant function of  $x$  (since it is the negative of  $\frac{1}{U}(U_{yy} + U_{zz})$ , and  $U$  is, by hypothesis, a constant function of  $x$ ). In other words

$$X_{xx} = kX; \quad k \text{ a constant.}$$

Hence  $X$  is a combination of exponential functions  $e^{pz}$ ,  $e^{-pz}$ , where  $p^2 = k$ ; we indicate this by the notation

$$X = e^{\pm pz}.$$

*Warning.* Do not think that the ambiguous sign  $\pm$  means that you must take *one* or *other* of the two signs  $+$  and  $-$ . It means that you may take any linear combination of what you obtain by each of these choices. Thus  $e^{\pm pz}$  is merely an abbreviation for  $C_1 e^{pz} + C_2 e^{-pz}$ , where  $C_1$  and  $C_2$  are arbitrary constants.

*Note 1.* Although  $V(P)$  is a real point-function so that  $X$  and hence  $k$  are real it is convenient to work in the complex field (for when  $k$  is negative  $p$  is a pure imaginary). Thus when  $k = -n^2$ , say, is negative we write

$$X = e^{\pm inx}$$

by which we mean that  $X$  is any *real* linear combination of the conjugate complex functions  $e^{inx}$  and  $e^{-inx}$ . Thus  $e^{\pm inx}$  is any linear combination of  $\cos nx$  and  $\sin nx$  with real coefficients. We indicate such a linear combination by the symbol

$$\left. \begin{array}{l} \cos nx \\ \sin nx \end{array} \right\}$$

so that  $e^{\pm inx}$  means exactly the same as  $\left. \begin{array}{l} \cos nx \\ \sin nx \end{array} \right\}$ , namely,  $C_1 \cos nx + C_2 \sin nx$ , where  $C_1, C_2$  are arbitrary real constants.

*Note 2.* When  $k = 0$  so that  $p = 0$ ,  $e^{\pm pz}$  is merely a constant multiple of  $e^{0x} = 1$ . In this case the differential equation governing  $X$  is simply  $X_{zz} = 0$  so that  $X$  is any (real) linear polynomial, i.e., any linear combination, with real coefficients, of the two functions 1 and  $x$ . We agree, then, to understand by the symbol  $e^{\pm 0x}$  any real linear polynomial. We indicate any such linear polynomial by the symbol  $\left. \begin{array}{l} x \\ 1 \end{array} \right\}$  so that

$$e^{\pm 0x} = \left. \begin{array}{l} x \\ 1 \end{array} \right\}.$$

### Example 1. The parallel plate condenser

We take it as granted that the plates of the condenser are so large that  $V$  may be assumed to be independent of  $y$  and  $z$  (the plates of the



condenser being parallel to the  $x$ -plane). This assumption is usually expressed by the statement that the plates of the condenser are *infinite*, but this form of statement is rather unfortunate since if the plates extended to infinity they would each have to be at potential zero. If the plates are reasonably large our assumption will be sufficiently accurate if we are not too near their edges, and the departure from reality of the assumption near the edges can be made negligible by suitable experimental arrangements (such as a guard ring). Since  $V$  is independent, by hypothesis, of  $y$  and  $z$  we have  $V = X$  so that  $V$  is a linear function of  $x$ :

$$V = C_1x + C_2.$$

The voltage of the condenser is, accordingly,  $|C_1|d$ , where  $d$  is the distance between the plates of the condenser. The surface density of electric charge is  $\pm \frac{C_1}{4\pi}$ . (Remember that  $\mathbf{n}$  has the direction of the positive or negative  $x$ -axis and that  $\sigma = -\frac{1}{4\pi} \frac{dV}{dn}$ .) Hence the charge on the positively charged plate is  $\frac{A|C_1|}{4\pi}$ , where  $A$  is the area of either plate of the condenser, so that the capacity of the condenser is

$$C = \frac{A}{4\pi d}.$$

*Note.* If the insulating material of the condenser has a *specific inductive capacity*  $\epsilon$  the relation between the two fundamental vectors  $\mathbf{D}$  and  $\mathbf{E}$  is

$$\mathbf{D} = \frac{\epsilon}{4\pi} \mathbf{E}$$

and so  $\sigma = D_n = -\frac{\epsilon}{4\pi} \frac{dV}{dn}$ . The capacity of a condenser whose insulating material has specific inductive capacity  $\epsilon$  is, then,  $\epsilon$  times the capacity of the same condenser when its insulating material has unit specific inductive capacity. Thus the capacity of a parallel plate condenser whose insulating material is of specific inductive capacity  $\epsilon$  is  $\frac{\epsilon A}{4\pi d}$ .

### Example 2. Separation of variables in cylindrical coordinates

Any solution of Laplace's equation  $\Delta_2 V = 0$  which is of the form

$$V = ZU,$$

where  $Z$  is a function of  $z$  alone while  $U$  is a function of  $\rho$  and  $\phi$ , is such that  $Z = e^{\pm imz}$ . In fact since, in cylindrical coordinates,  $\Delta_2 V = V_{\rho\rho} + \frac{1}{\rho} V_\rho + \frac{1}{\rho^2} V_{\phi\phi} + V_{zz}$  (see Exercise 6, p. 109) we have

$$Z \left( U_{\rho\rho} + \frac{1}{\rho^2} U_\rho + \frac{1}{\rho^2} U_{\phi\phi} \right) + UZ_{zz} = 0$$

so that, at all points where  $V \neq 0$ ,

$$\frac{Z_{zz}}{Z} = -\frac{1}{U} \left( U_{\rho\rho} + \frac{1}{\rho} U_\rho + \frac{1}{\rho^2} U_{\phi\phi} \right).$$

Hence  $\frac{Z_{zz}}{Z}$  is a constant which we may denote by  $\alpha^2$ ; we do not insist that  $\alpha$  be real so that the constant is not necessarily positive. Hence  $Z = e^{\pm\alpha z}$ . The function  $U$  of  $\rho$  and  $\phi$  satisfies the equation

$$U_{\rho\rho} + \frac{1}{\rho} U_\rho + \frac{1}{\rho^2} U_{\phi\phi} = -\alpha^2 U$$

(why?), and we seek solutions of this equation of the type

$$U = P\Phi,$$

where  $P$  is a function of  $\rho$  alone while  $\Phi$  is a function of  $\phi$  alone. We find that

$$\Phi \left( P_{\rho\rho} + \frac{1}{\rho} P_\rho + \alpha^2 P \right) + \frac{P}{\rho^2} \Phi_{\phi\phi} = 0$$

or, equivalently, that

$$\frac{\Phi_{\phi\phi}}{\Phi} = -\frac{\rho^2}{P} \left( P_{\rho\rho} + \frac{1}{\rho} P_\rho + \alpha^2 P \right).$$

Hence  $\frac{\Phi_{\phi\phi}}{\Phi}$  is a constant (why?), and we denote this constant by  $-m^2$  (the prefixed negative sign being used for reasons that will be immediately clear). Then  $\Phi = e^{\pm im\phi}$ . If we demand that  $V$  be a *uniform* point-function, i.e., that  $V$  be unambiguously determined by the point  $(x, y, z)$ ,  $\Phi$  must be a *periodic* function of  $\phi$  with period  $2\pi$  (because the angular coordinate  $\phi$  of the point  $(x, y, z)$  is indeterminate to the

extent of an arbitrary integral multiple of  $2\pi$ ). Hence the constant  $m$  must be an integer which may be taken, without loss of generality, to be non-negative since we are using both  $+m$  and  $-m$  (when  $m = 0$  only one solution, namely, 1 of the two, 1 and  $\phi$ , indicated by the symbol  $e^{\pm i m \phi}$ , is acceptable). This is the reason that we wrote our original constant in the form  $-m^2$ ; if we had written  $m^2$  we would have merely found that  $m$  must be an integral multiple of  $i$ . When a constant which arises in the discussion is *restricted* in this way we say that it is *quantized*; thus the constant  $m$  which occurs in the formula  $\Phi = e^{\pm i m \phi}$  is quantized to integral values by the condition that  $V$  be a *uniform* point-function. The remaining factor  $P$  in the expression

$$V = ZU = Z\Phi P$$

must satisfy the differential equation

$$P_{\rho\rho} + \frac{1}{\rho}P + \left(\alpha^2 - \frac{m^2}{\rho^2}\right)P = 0.$$

On introducing a new independent variable  $\xi$  defined by the formula  $\xi = \alpha\rho$  we have  $P_\rho = \alpha P_\xi$ ,  $P_{\rho\rho} = \alpha^2 P_{\xi\xi}$  so that

$$P_{\xi\xi} + \frac{1}{\xi}P_\xi + \left(1 - \frac{m^2}{\xi^2}\right)P = 0.$$

This is *Bessel's equation of order  $m$*  (after F. W. Bessel [1784–1846], a German astronomer). We shall discuss it in some detail later and note here only that its general solution is a linear combination of two functions  $J_m(\xi)$  and  $K_m(\xi)$ :

$$P = \left. \begin{array}{l} J_m(\xi) \\ K_m(\xi) \end{array} \right\} = \left. \begin{array}{l} J_m(\alpha\rho) \\ K_m(\alpha\rho) \end{array} \right\}.$$

Thus the solutions of Laplace's equation in which the cylindrical coordinates are (completely) separable are of the form

$$V = \left. \begin{array}{l} J_m(\alpha\rho) \\ K_m(\alpha\rho) \end{array} \right\} (e^{\pm i m \phi})(e^{\pm \alpha z}).$$

The condition that  $V$  be uniform quantizes the constant  $m$  to non-negative integral values.

### Example 3. Separation of variables in space polar coordinates

Laplace's equation takes, when written in space polar coordinates the form (see Exercise 7, p. 109)

$$V_{rr} + \frac{2}{r} V_r + \frac{1}{r^2} V_{\theta\theta} + \frac{\cot \theta}{r^2} V_\theta + \frac{1}{r^2 \sin^2 \theta} V_{\phi\phi} = 0.$$

Those solutions  $V = RU$ , where  $R$  is a function of  $r$  alone while  $U$  is a function of  $\theta$  and  $\phi$  (so that the radial variable  $r$  is separable from the angular variables  $\theta$  and  $\phi$ ), are such that

$$U \left( R_{rr} + \frac{2}{r} R_r \right) + \frac{R}{r^2} \left( U_{\theta\theta} + \cot \theta U_\theta + \frac{1}{\sin^2 \theta} U_{\phi\phi} \right) = 0.$$

Hence, at any point where  $V \neq 0$ ,

$$\frac{r^2}{R} \left( R_{rr} + \frac{2}{r} R_r \right) + \frac{1}{U} \left( U_{\theta\theta} + \cot \theta U_\theta + \frac{1}{\sin^2 \theta} U_{\phi\phi} \right) = 0$$

so that (why?)  $\frac{r^2}{R} \left( R_{rr} + \frac{2}{r} R_r \right)$  is a constant  $k$ , say. Thus  $R$  satisfies the differential equation

$$r^2 R_{rr} + 2r R_r = kR.$$

On setting  $R = r^n$  we see that  $n$  is one of the two roots of the quadratic equation

$$n(n+1) = k.$$

The sum of the two roots of this quadratic equation is  $-1$  (why?) so that if one root is  $n$  the other is  $-(n+1)$ . Instead of solving the quadratic equation (thus expressing  $n$  in terms of  $k$ ) it is simpler to leave  $n$  undetermined and use the relation  $n(n+1) = k$  to express  $k$  in terms of  $n$ . We shall see later that  $n$  is quantized (by the fact that  $V$  is continuous) so as to be a non-negative integer

$$n = 0, 1, 2, \dots,$$

but we shall not need this result just now. We have, then,

$$R = \left. \begin{array}{l} r^n \\ r^{-n-1} \end{array} \right\},$$

and  $U$  satisfies the differential equation

$$U_{\theta\theta} + \cot \theta U_\theta + \frac{1}{\sin^2 \theta} U_{\phi\phi} + n(n+1)U = 0.$$

It is important to notice that the solutions

$$V_i = r^n U; \quad V_e = r^{-n-1} U$$

of Laplace's equation, obtained in this way, are *homogeneous* functions of  $(x, y, z)$ ; in other words  $V_i(kx, ky, kz) = k^n V_i(x, y, z)$  so that  $V_i$  is a homogeneous function of degree  $n$  of  $(x, y, z)$  while  $V_e(kx, ky, kz) = k^{-n-1} V_e(x, y, z)$  so that  $V_e$  is a homogeneous function of degree  $-(n+1)$  of  $(x, y, z)$ . If  $n$  is a non-negative integer  $V_i$  is not null at  $r = \infty$  (unless it is the constant zero) and so it is only legitimate over a bounded region. On the other hand  $V_e$  is null at  $r = \infty$  as is also  $r \text{ grad } V_e$  so that  $V_e$  is acceptable as a potential function over an unbounded region. For these reasons  $V_i$  is termed an *internal harmonic* and  $V_e$  an *external harmonic* (any potential function being termed a *harmonic*). It is clear that if  $V$  is any homogeneous harmonic function of degree  $n$  then  $V_i = r^n U$ , where  $U$ , being a function of the ratios  $x:y:z$  (since it is a homogeneous function of degree zero of  $(x, y, z)$ ) is a function of the angular coordinates  $(\theta, \phi)$ . We have, then, the following important result: Associated with any internal harmonic  $V_i = r^n U$  of degree  $n$  is an external harmonic  $V_e = r^{-n-1} U = \frac{V_i}{r^{2n+1}}$ .

### Example 1

$V_i = 1$  is an internal harmonic of degree 0; hence  $V_e = \frac{1}{r}$  is an external harmonic of degree  $-1$ .

### Example 2

$x, y,$  and  $z$  are internal harmonics of degree 1; hence  $\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}$  are external harmonics of degree  $-2$ .

## EXERCISES

1. Show that the homogeneous polynomial function of degree 2,  $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ , is an internal harmonic if, and only if,  $a + b + c = 0$ .
2. Write down five linearly independent internal harmonics of degree 2, and determine the function  $U$  of  $\theta$  and  $\phi$  in each case.
3. Determine the function  $U$  of  $\theta$  and  $\phi$  for each of the internal harmonics of degree 1:  $x, y,$  and  $z$ .
4. Write down five linearly independent external harmonics of degree  $-3$ .

If the angular coordinate  $\phi$  may be separated from the angular coordinate  $\theta$  in the *surface harmonic*  $U$  we have  $U = M\Phi$  where  $M$  is a function  $\theta$  alone while  $\Phi$  is a function of  $\phi$  alone. Hence

$$\Phi(M_{\theta\theta} + \cot \theta M_{\theta}) + \frac{M}{\sin^2 \theta} \Phi_{\varphi\varphi} + n(n+1)M\Phi = 0$$

so that, at any point where  $U \neq 0$ ,

$$\frac{1}{\Phi} \Phi_{\varphi\varphi} = -\frac{\sin^2 \theta}{M} \{M_{\theta\theta} + \cot \theta M_{\theta} + n(n+1)M\}.$$

Hence  $\frac{1}{\Phi} \Phi_{\varphi\varphi}$  is a constant (why?) which we shall denote, as before, by the symbol  $-m^2$ . Then

$$\Phi = e^{\pm im\phi},$$

and  $m$  is quantized (by the requirement that  $V = RU = RM\Phi$  be a *uniform* function of position) so as to be a non-negative integer. The remaining factor  $M$  satisfies the following differential equation:

$$M_{\theta\theta} + \cot \theta M_{\theta} + \left\{n(n+1) - \frac{m^2}{\sin^2 \theta}\right\} M = 0.$$

On introducing the new independent variable  $\mu = \cos \theta$  we have

$$M_{\theta} = -\sin \theta M_{\mu};$$

$$M_{\theta\theta} = -\mu M_{\mu} + \sin^2 \theta M_{\mu\mu} = -\mu M_{\mu} + (1 - \mu^2)M_{\mu\mu},$$

and the differential equation governing  $M$  appears in the form

$$(1 - \mu^2)M_{\mu\mu} - 2\mu M_{\mu} + \left\{n(n+1) - \frac{m^2}{1 - \mu^2}\right\} M = 0.$$

The particular case of this equation that occurs when  $m = 0$  is known as *Legendre's equation* (after A. M. Legendre [1752–1833], a French mathematician), and we shall discuss it in some detail later. The general equation which occurs when  $m = 1, 2, 3, \dots$  is known as *Legendre's associated equation*, and its general solution is a linear combination of two functions  $P_n^m(\mu)$ ,  $Q_n^m(\mu)$  which we shall define later. Thus

$$M = \left. \begin{array}{l} P_n^m(\mu) \\ Q_n^m(\mu) \end{array} \right\}.$$

We see, then, that the solutions of Laplace's equation in which the space polar coordinates are (completely) separable are of the form

$$V = \left. \begin{array}{l} r^n \\ r^{-n-1} \end{array} \right\} \left. \begin{array}{l} P_n^m(\mu) \\ Q_n^m(\mu) \end{array} \right\} e^{\pm im\phi}.$$

Those particular harmonics that arise when  $m = 0$  (in which case  $e^{\pm im\phi}$  reduces to 1 (why?)) are independent of  $\phi$  and are known as *zonal harmonics*. They are of the form

$$V = \begin{matrix} r^n \\ r^{-n-1} \end{matrix} \left. \begin{matrix} P_n(\mu) \\ Q_n(\mu) \end{matrix} \right\},$$

where  $P_n(\mu)$ ,  $Q_n(\mu)$  stand for  $P_n^0(\mu)$  and  $Q_n^0(\mu)$ , respectively.

### EXERCISES

5. Show that the solutions of the wave equation

$$\Delta_2 W = \frac{1}{c^2} W_{tt}$$

( $c$  being a constant and  $W$  being a function of the three space variables  $x, y, z$  and the time variable  $t$ ) in which the time variable is separable from the space variables are of the form  $W = V e^{\pm i p c t}$ , where  $V = V(x, y, z)$  satisfies the equation

$$\Delta_2 V + p^2 V = 0.$$

6. Show that the solutions of the equation  $\Delta_2 V + p^2 V = 0$  of Exercise 5 in which the variables  $x, y, z$  are completely separable are of the form

$$V = e^{\pm i p (lx + my + nz)},$$

where  $l^2 + m^2 + n^2 = 1$ .

7. Show that the solutions of the wave equation of Exercise 5 in which the variables  $x, y, z, t$  are completely separable are of the form  $W = e^{\pm i p (lx + my + nz - ct)}$ , where  $l^2 + m^2 + n^2 = 1$ . *Note.* The level (moving) surfaces of  $W$  are the planes  $lx + my + nz = ct + \text{constant}$ , any one of which moves perpendicular to itself with velocity  $c$ . (Prove this.) The solution  $W$  of the wave equation given in this exercise is said, then, to be characteristic of *plane waves* whose *velocity of propagation* is  $c$ .

8. Show that the solutions of the equation  $\Delta_2 V + p^2 V = 0$  of Exercise 5 in which the *cylindrical* coordinate  $z$  is separable from the other two cylindrical coordinates  $(\rho, \phi)$  are of the form

$$V = U(\rho, \phi) e^{\pm i n z},$$

where  $U_{\rho\rho} + \frac{1}{\rho} U_{\rho} + \frac{1}{\rho^2} U_{\phi\phi} + (p^2 - n^2)U = 0$ .

9. Show that the solutions of the equation  $\Delta_2 V + p^2 V = 0$  in which the cylindrical coordinates  $(\rho, \phi, z)$  are completely separable are of the form

$$V = \left. \begin{matrix} J_m(\alpha\rho) \\ K_m(\alpha\rho) \end{matrix} \right\} e^{\pm im\phi} e^{\pm inz},$$

where  $\alpha = (p^2 - n^2)^{1/2}$  and  $m$  is quantized to be a non-negative integer if  $V$  is a uniform point-function.

10. Show that the solution of the wave equation  $\Delta_2 W = \frac{1}{c^2} W_{tt}$  (written in

cylindrical coordinates) in which the variables  $(\rho, \phi, z, t)$  are completely separable are of the form

$$W = \left. \begin{matrix} J_m(\alpha\rho) \\ K_m(\alpha\rho) \end{matrix} \right\} (e^{\pm im\phi})(e^{\pm inz})(e^{\pm ipt}),$$

where  $\alpha = (p^2 - n^2)^{1/2}$  and  $m$  is quantized to be a non-negative integer if  $W$  is a uniform function of position and time.

11. Show that the solutions of the equation  $\Delta_2 V + p^2 V = 0$  (written in space polar coordinates) in which the radial variable  $r$  is separable from the angular variables  $(\theta, \phi)$  are of the form

$$V = R(r)U(\theta, \phi),$$

where

$$\begin{aligned} r^2 R_{,rr} + 2r R_r + \{p^2 r^2 - n(n+1)\} R &= 0; \\ U_{\theta\theta} + \cot \theta U_\theta + \frac{1}{\sin^2 \theta} U_{\phi\phi} + n(n+1)U &= 0. \end{aligned}$$

12. Show that the substitution  $R = r^{-1/2} S$  transforms the equation of Exercise 11 governing  $R$  into the equation

$$r^2 S_{rr} + r S_r + \{p^2 r^2 - (n + \frac{1}{2})^2\} S = 0,$$

and deduce that

$$R = \left. \begin{matrix} r^{-1/2} J_{n+1/2}(pr) \\ r^{-1/2} K_{n+1/2}(pr) \end{matrix} \right\}.$$

13. Show that the solutions of the wave equation  $\Delta_2 W = \frac{1}{c^2} W_{tt}$  (written in space polar coordinates) in which the variables  $(r, \theta, \phi, t)$  are completely separable are of the form

$$W = \left. \begin{matrix} r^{-1/2} J_{n+1/2}(pr) \\ r^{-1/2} K_{n+1/2}(pr) \end{matrix} \right\} \left. \begin{matrix} P_n^m(\mu) \\ Q_n^m(\mu) \end{matrix} \right\} (e^{\pm im\phi})(e^{\pm ipt}),$$

where  $\mu = \cos \theta$  and  $m$  is quantized to be a non-negative integer by the condition that  $W$  be a uniform function of position and time.

14. Verify that  $W = \frac{e^{\pm i p(r-ct)}}{r}$  is a solution of the wave equation  $\Delta_2 W = \frac{1}{c^2} W_{tt}$ .

Note.  $x^{1/2} J_{1/2}(x)$  is a constant multiple of  $\sin x$  so that it follows, on setting  $n = 0$ ,  $m = 0$  in the result of Exercise 13, that  $\left(\frac{\sin pr}{r}\right) e^{\pm ipt}$  is a solution of the wave equation.  $x^{1/2} K_{1/2}(x)$  is a linear combination of  $\sin x$  and  $\cos x$  so that (again from the result of Exercise 13)  $\left(\frac{\cos pr}{r}\right) e^{\pm ipt}$  is a solution of the wave equation. Hence

$\left(\frac{e^{\pm i pr}}{r}\right) e^{\pm ipt}$  is a solution of the wave equation. The solution  $W$  of the wave equation given in this exercise is characteristic of *spherical waves* emanating from the origin, the velocity of propagation being  $c$  (the level (moving) surfaces of  $rW$  being the spheres  $r = ct + \text{constant}$ ).



### 3. Two-dimensional problems; conjugate functions

If the potential function  $V = V(P)$  of an electrostatic problem is independent of one of the coordinates,  $z$  say, of the point of evaluation  $P: (x, y, z)$  of  $V$ , the electrostatic problem is said to be *two-dimensional*. When the conductors are long cylinders whose axes are parallel to the  $z$ -axis the field is, at least approximately, two-dimensional provided that we do not approach too closely the ends of the cylindrical conductors or recede too far away from the conductors. The surface density  $\sigma$  of electric charge on any conductor in a two-dimensional problem, being furnished by the formula

$$\sigma = -\frac{1}{4\pi} \frac{dV}{dn},$$

is independent of  $z$ . If  $C$  is the curve of intersection of the cylindrical conductor and the  $z$ -plane the charge on any portion of length  $l$  of the cylinder (intercepted between two planes parallel to the  $z$ -plane) is given by

$$\int_S \sigma dS = l \int_C \sigma ds.$$

When  $l = 1$  we obtain  $\int_C \sigma ds$ , and we term this the *charge per unit length* or, briefly, the charge on the (cylindrical) conductor. Thus line integrals play in two-dimensional problems the role previously played (in three-dimensional problems) by surface integrals.

Laplace's equation reduces, for two-dimensional problems, to  $V_{xx} + V_{yy} = 0$ . When written in plane polar coordinates  $(r, \theta)$  this takes the form (see Exercise 3, p. 91)

$$V_{rr} + \frac{1}{r} V_r + \frac{1}{r^2} V_{\theta\theta} = 0.$$

The solutions of this equation which are symmetrical about the origin, (i.e., which are independent of  $\theta$ ) are of the form

$$V = A \log r + B.$$

(Prove this.) If  $C$  is a circle of radius  $\epsilon$  with center at the origin  $\frac{dV}{dn} = \frac{A}{\epsilon}$  over  $C$  so that the integral of  $-\frac{1}{4\pi} \frac{dV}{dn}$  over  $C$  is  $-\frac{A}{2}$ .

On denoting this integral by  $e$  we have

$$V = 2e \log \frac{1}{r} + B.$$

This is said to be the (logarithmic) potential of a *line charge of strength*  $e$  located at the origin.

The proof of the uniqueness theorem for a bounded (plane) region interior to a conductor (there being any number of other conductors and line charges in the field) is the same as that of the corresponding theorem for a bounded three-dimensional electrostatic field. (Repeat this proof.)

Since  $V$  is indeterminate to the extent of an additive constant there is no lack of generality in taking the potential of the enclosing conductor to be zero; when this is done the potential function is unambiguously determined by the potentials of the enclosed conductors and by the strengths and locations of the various line charges. The simplest problem of this type is that of the (cylindrical) condenser; here there is only one enclosed conductor, and there are no line charges. The *capacity (per unit length)* of the condenser is the quotient of the absolute value of the charge per unit length on either the enclosed or the enclosing conductor by the *voltage* of the condenser (what is this?).

Since Laplace's equation in a system of conjugate orthogonal curvilinear coordinates  $(\alpha, \beta)$  obtained by setting the complex variable  $z = x + iy$  equal to any analytic function of the complex variable  $\gamma = \alpha + i\beta$  is

$$V_{\alpha\alpha} + V_{\beta\beta} = 0$$

(see Exercise 4, p. 100), the solutions of Laplace's equation in which the coordinates  $\alpha$  and  $\beta$  are separable are of the form

$$V = e^{\pm ik\alpha} e^{\pm k\beta}; \quad k \text{ any (complex) constant}$$

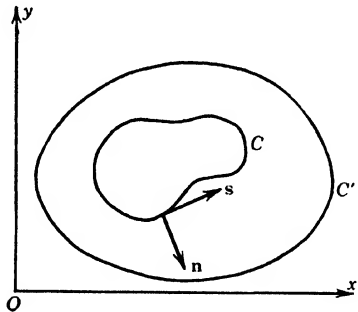


FIG. 26.

(see Section 2). In particular the solutions which are functions of  $\alpha$  alone are linear functions of  $\alpha$  (why?). Let us suppose that the two curves  $C$  and  $C'$ , which furnish the cross sections of the two conductors of our condenser, are level curves of the function  $\alpha = \alpha(x, y)$ ,  $C$  being the inner conductor. Since  $V$  satisfies the equation  $V_{xx} + V_{yy} = 0$

there exists a conjugate function  $W = W(x, y)$  such that  $V + iW$  is a function, in the complex variable sense, of  $z = x + iy$ . Since  $\text{grad } W = (\text{grad } V)^*$  we have  $\frac{dV}{dn} = \frac{dW}{ds}$ , where the direction  $\mathbf{s}$  is a quarter turn ahead of the direction  $\mathbf{n}$ . Hence the charge per unit length on the conductor  $C$  is

$$e = -\frac{1}{4\pi} \int_C \frac{dV}{dn} ds = -\frac{1}{4\pi} \int_C \frac{dW}{ds} ds = -\frac{\Delta W}{4\pi},$$

where  $\Delta W$  denotes the increment in  $W$  as we pass around  $C$  in the positive (= counterclockwise) sense. If  $V = A\alpha + B$ , the voltage of the condenser is  $|A\Delta\alpha|$ , where  $\Delta\alpha$  is the increment of  $\alpha$  as we pass from  $C$  to  $C'$ . Since  $W$  is the function of  $z = x + iy$  or, equivalently, of  $\gamma = \alpha + i\beta$ , which is conjugate to  $V$ , we have  $W = A\beta + B'$  (why?) so that  $\Delta W = A\Delta\beta$ . Hence

$$e = -\frac{A}{4\pi} \Delta\beta,$$

and the capacity (per unit length) of the cylindrical condenser is

$$C = \frac{1}{4\pi} \left| \frac{\Delta\beta}{\Delta\alpha} \right|.$$

The potential function of the field is

$$V = -\frac{4\pi e}{\Delta\beta} \alpha + B.$$

### Example 1. The concentric circular cylinder condenser

Here  $z = e^\gamma$ ,  $\gamma = \log z$ ,  $\alpha = \log r$ ,  $\beta = \theta$ . Hence  $\Delta\beta = 2\pi$ . If  $a$  and  $b > a$  are the radii of the circles  $C$  and  $C'$  we have  $\Delta\alpha = \log \frac{b}{a}$  and so the capacity is

$$C = \frac{1}{2 \log \frac{b}{a}}.$$

The potential function is

$$V = -2e \log r + B,$$

or, on determining the additive constant  $B$  so that the outer conductor

is at potential zero,

$$V = 2e \log \frac{b}{r}.$$

*Note.* It is worthy of notice that  $V$  is independent of  $a$ ; thus the potential function of a circular cylinder depends only on the charge and is the same as if the entire charge were concentrated, as a line charge, on the axis of the cylinder.

### Example 2. The confocal elliptical condenser

Here  $z = c \cosh \gamma$ ,  $x = c \cosh \alpha \cos \beta$ ,  $y = c \sinh \alpha \sin \beta$ . The curves  $\alpha = \text{constant}$  are the confocal ellipses

$$\frac{x^2}{c^2 \cosh^2 \alpha} + \frac{y^2}{c^2 \sinh^2 \alpha} = 1.$$

If the semimajor axes of the interior and exterior ellipses are  $a$  and  $a'$ , respectively, we have  $\Delta\alpha = \cosh^{-1} \left( \frac{a'}{c} \right) - \cosh^{-1} \left( \frac{a}{c} \right)$ . As we trace the interior ellipse  $C$  in the counterclockwise direction,  $\beta$  traces the interval  $[0, 2\pi]$  so that  $\Delta\beta = 2\pi$ . Hence the capacity of the elliptical condenser is

$$C = \frac{1}{2 \left( \cosh^{-1} \frac{a'}{c} - \cosh^{-1} \frac{a}{c} \right)}.$$

Since  $\cosh^{-1} u = \log \{u + (u^2 - 1)^{1/2}\}$  this may be put in the equivalent form

$$C = 1 \div 2 \log \left[ \frac{\frac{a'}{a} + \left\{ \left( \frac{a'}{a} \right)^2 - \frac{c^2}{a^2} \right\}^{1/2}}{1 + \left( 1 - \frac{c^2}{a^2} \right)^{1/2}} \right],$$

and as  $c \rightarrow 0$  this approaches  $1 \div 2 \log \left( \frac{a'}{a} \right)$ , i.e., the capacity, per unit length, of the concentric circular cylinder.

### Example 3. The eccentric circular cylinder condenser

Here  $\gamma = \log \left\{ \frac{z - c}{z + c} \right\}$ ,  $\alpha = \log \left( \frac{r_1}{r_2} \right)$ , where  $r_1$  and  $r_2$  are the distances of any point in the field from the points  $z = c$  and  $z = -c$ ,

respectively. Let both circles enclose the point  $z = c$ , and let the distances of the centers of the circles of radii  $a$  and  $b > a$ , respectively, from the point  $z = c$  be  $p$  and  $q$ , respectively. Furthermore let  $d$  be the distance between the centers of the two circles. Then  $q = p + d$ , and since the distances of the two centers from  $z = -c$  are  $\frac{a^2}{p}$  and  $\frac{b^2}{q}$ , respectively, we have also  $\frac{b^2}{q} = \frac{a^2}{p} + d$ . On eliminating  $q$

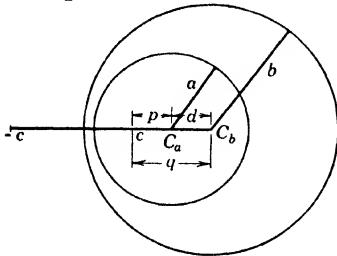


FIG. 27.

we obtain the quadratic equation

$$\left(\frac{p}{a}\right)^2 - \frac{(b^2 - a^2 - d^2)}{ad} \left(\frac{p}{a}\right) + 1 = 0$$

for  $\frac{p}{a}$ . On setting

$$\cosh \theta = \frac{b^2 - a^2 - d^2}{2ad}; \quad \theta > 0,$$

we have, since  $\frac{p}{a} < 1$ ,  $\frac{p}{a} = e^{-\theta}$ . Similarly (show this)

$$\frac{q}{b} = e^{-\phi} \text{ where } \cosh \phi = \frac{b^2 - a^2 + d^2}{2bd}; \quad \phi > 0.$$

For the inner circle we obtain (on evaluating  $\frac{r_1}{r_2}$  at the point where the circle intersects the segment from  $z = -c$  to  $z = +c$ )

$$\alpha = \log \frac{a - p}{\frac{a^2}{p} - a} = \log \frac{p}{a} = -\theta.$$

For the outer circle  $\alpha$  has the value  $-\phi$  so that

$$|\Delta\alpha| = \theta - \phi$$

(it being easy to verify, since  $b - a > d$  (why?), that  $\theta > \phi$ ). Since  $\beta$  is the angle from the ray  $-c \rightarrow z$  to the ray  $c \rightarrow z$  the increment  $\Delta\beta$  in  $\beta$  as we go around the inner circle in the positive sense is  $-2\pi$ . Hence the capacity of the eccentric circular cylinder condenser is

$$C = \frac{1}{2|\Delta\alpha|} = 1 \div 2 \left\{ \cosh^{-1} \frac{b^2 - a^2 - d^2}{2ad} - \cosh^{-1} \frac{b^2 - a^2 + d^2}{2bd} \right\}.$$

Let us examine the limit of  $C$  as  $b \rightarrow \infty$ ,  $a$  and  $p$  remaining fixed. From the two relations  $q = d + p$ ,  $b^2 = q \left( d + \frac{a^2}{p} \right)$  we obtain

$$b^2 = d^2 + a^2 + d \left( p + \frac{a^2}{p} \right) \text{ and so } \frac{b}{d} = \left\{ 1 + \frac{p + \frac{a^2}{p}}{d} + \frac{a^2}{d^2} \right\}^{\frac{1}{2}}$$

the limit 1 as  $b$ , and hence  $d \rightarrow \infty$ . Hence  $\frac{b^2 - a^2 + d^2}{2bd}$

$$= \frac{d + \frac{1}{2} \left( p + \frac{a^2}{p} \right)}{b} \text{ has the limit 1 as } b \rightarrow \infty \text{ and so } \cosh^{-1} \frac{b^2 - a^2 + d^2}{2bd}$$

has the limit 0 as  $b \rightarrow \infty$ . On the other hand  $\frac{b^2 - a^2 - d^2}{2ad}$  has the constant value  $\frac{1}{2} \left( \frac{p}{a} + \frac{a}{p} \right)$  and so

$C$  has the limit

$$1 \div 2 \cosh^{-1} \frac{1}{2} \left( \frac{p}{a} + \frac{a}{p} \right)$$

as  $b \rightarrow \infty$ . We term this the capacity per unit length of the circular cylinder in the presence of the infinite plane which bisects perpendicularly the join of the two points  $-c$  and  $+c$ . If  $\delta$  is the distance of the axis of the circular cylinder from this plane we have  $\frac{a^2}{p} = 2\delta - p$  so that  $\frac{1}{2} \left( \frac{a}{p} + \frac{p}{a} \right) = \frac{\delta}{a}$ , and the capacity per unit length of the circular cylinder in the presence of the infinite plane takes the simple form

$$C = 1 \div 2 \cosh^{-1} \frac{\delta}{a}$$

#### 4. The method of inversion

Let  $P$  be any point whose coordinates in a system of space polar coordinates are  $(r, \theta, \phi)$ ; the point  $P'$  whose coordinates in the same system of space polar coordinates are  $(r', \theta, \phi)$ , where  $r' = \frac{k^2}{r}$ ,  $k$  being any positive real constant, lies on the ray  $O \rightarrow P$  and is such that

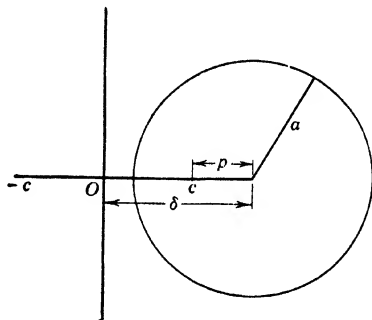


FIG. 28.

$$|OP||OP'| = k^2.$$

$P'$  is termed the *inverse* of  $P$  in the sphere of center  $O$  and radius  $k$  (which sphere is termed the *sphere of inversion*). It is clear that the relation between  $P'$  and  $P$  is a *partnership*: When  $P'$  is the inverse of  $P$ ,  $P$  is the inverse of  $P'$ . As  $P \rightarrow O$ ,  $P'$  approaches  $\infty$ , i.e., as  $r \rightarrow 0$ ,  $r'$  becomes arbitrarily large. We adopt the *convention* that there is only one point at infinity (whose neighborhoods consist of the points for which  $r$  is greater than a specified number  $R$ ), and then we can say that the inverse of the *center of inversion*  $O$  is the point at infinity.

*Note.* If the convention that there is only one point at infinity confuses you, think of the relationship between the North Pole and the South Pole on the earth. To one whose vision is restricted to a small neighborhood of the North Pole the statement that the meridians converge to a point, the South Pole, as we proceed southward along them, seems absurd. The South Pole bears to the North Pole exactly the same relationship that the point at infinity bears to the center of inversion in the theory of inversion.

Since  $r' = \frac{k^2}{r}$  we have  $dr' = -\left(\frac{k^2}{r^2}\right) dr$ . If, then,  $C$  is any curve having the vector element of arc  $ds$  whose space polar coordinates are  $(dr, r, d\theta, r \sin \theta d\phi)$ ,  $C$  will invert into a curve  $C'$  whose vector element of arc  $ds'$  has the space polar coordinates  $(dr', r' d\theta, r' \sin \theta d\phi) = \left(\frac{k^2}{r^2}\right) (-dr, r d\theta, r \sin \theta d\phi)$ . Hence the magnitudes of the two vector elements of arc are connected by the relation

$$ds' = \frac{k^2}{r^2} ds.$$

If we have two curves  $C_1$  and  $C_2$  intersecting at  $P$  their angle of intersection  $\alpha$  is furnished by the formula

$$d_1s d_2s \cos \alpha = d_1r d_2r + r^2 d_1\theta d_2\theta + r^2 \sin^2 \theta d_1\phi d_2\phi$$

(remember that  $r_1 = r_2 = r$ ,  $\theta_1 = \theta_2 = \theta$ ,  $\phi_1 = \phi_2 = \phi$  since the curves intersect at  $P$ . Why is  $d_1r$  not, necessarily, the same as  $d_2r$ ?). The inverse curves  $C_1'$  and  $C_2'$  intersect at  $P'$ , their angle of intersection  $\alpha'$  being furnished by the formula

$$\begin{aligned} d_1s' d_2s' \cos \alpha' &= d_1r' d_2r' + r'^2 d_1\theta d_2\theta + r'^2 \sin^2 \theta d_1\phi d_2\phi \\ &= \frac{k^4}{r^4} (d_1r d_2r + r^2 d_1\theta d_2\theta + r^2 \sin^2 \theta d_1\phi d_2\phi) \end{aligned}$$

$$= \frac{k^4}{r^4} d_1s d_2s \cos \alpha.$$

Since  $d_1s' = \left(\frac{k^2}{r^2}\right) d_1s$ ,  $d_2s' = \left(\frac{k^2}{r^2}\right) d_2s$  it follows that  $\cos \alpha' = \cos \alpha$ .

In words:

**The cosine of the angle of intersection of any two curves is unaltered by inversion.**

Since the scalar element of area of any surface  $S$  is the magnitude of the vector product  $(\mathbf{d}_1s \times \mathbf{d}_2s)$ , where the subscripts refer to the parameters on the surface, it follows (prove this) that

$$dS' = \frac{k^4}{r^4} dS.$$

Similarly, since the element of volume is the absolute value of the alternating product  $(\mathbf{d}_1s \mathbf{d}_2s \mathbf{d}_3s)$ , where the subscripts refer to the independent variables, or parameters, which describe the region whose element of volume is being calculated, we have

$$d\tau' = \frac{k^6}{r^6} d\tau.$$

(Prove this. *Hint.* The alternating product of any three vectors is a three-rowed determinant; the numerical, or absolute, value of this determinant is unaffected by a change of sign of the elements in its first row.)

It follows from the fact that  $P'$  is on the ray  $O \rightarrow P$  that

$$\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'} = \frac{r}{r'} = \frac{k^2}{(r')^2}$$

and so

$$x = \frac{k^2 x'}{(x')^2 + (y')^2 + (z')^2}; \quad y = \frac{k^2 y'}{(x')^2 + (y')^2 + (z')^2};$$

$$z = \frac{k^2 z'}{(x')^2 + (y')^2 + (z')^2}.$$

Hence the surface  $S$  whose equation is

$$A(x^2 + y^2 + z^2) + 2Bx + 2Cy + 2Dz + E = 0$$

inverts into the surface  $S'$  whose equation is

$$E(x'^2 + y'^2 + z'^2) + 2k^2(Bx' + Cy' + Dz') + Ak^4 = 0.$$



$S$  is a sphere or a plane (according as  $A$  is not zero or is zero), and  $S'$  is a sphere or a plane (according as  $E$  is not zero or is zero).  $S$  passes through the center of inversion  $O$  if  $E = 0$ , and  $S'$  passes through the center of inversion  $O$  if  $A = 0$ . A plane is, then, the inverse of a sphere which passes through  $O$ , and we express this result as follows:

**A plane is a sphere which passes through the point at infinity.**

In this terminology we may phrase our result as follows:

*Every sphere inverts into a sphere; the particular spheres which are planes invert into spheres through the center of inversion, or, equivalently, the particular spheres which pass through the center of inversion invert into planes.*

Since the cosine of the angle between any two directions is unaffected

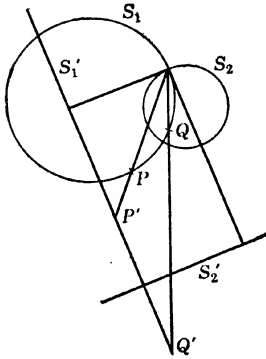


FIG. 29.

by inversion any two surfaces  $S_1$  and  $S_2$  which intersect at right angles invert into two surfaces  $S_1'$  and  $S_2'$  which intersect at right angles. Let  $S_1$  and  $S_2$  be two spheres which intersect at right angles and invert from any point of intersection of  $S_1$  and  $S_2$ ; then  $S_1'$  and  $S_2'$  are two planes which intersect at right angles. If, then,  $P'$  is any point on  $S_1'$  the image  $Q'$  of  $P'$  in  $S_2'$  lies on  $S_1'$ , and  $Q'$  has the property that any sphere through  $P'$  and  $Q'$  cuts  $S_2'$  at right angles; conversely, every sphere through  $P'$  which cuts  $S_2'$  at right angles passes through  $Q'$ . Hence, if  $P$  is any point on  $S_1$ ,  $S_1$  also contains a point  $Q$  such that every sphere

through  $P$  and  $Q$  cuts  $S_2$  at right angles, and, conversely, every sphere through  $P$  which cuts  $S_2$  at right angles passes through  $Q$ . Since every plane through  $P$  and the center  $C_2$  of  $S_2$  cuts  $S_2$  at right angles,  $Q$  lies on every plane through  $C_2$  and  $P$ ; in other words  $Q$  lies on the line through  $C_2$  and  $P$ . Since  $S_1$  cuts  $S_2$  at right angles the product  $|C_2P||C_2Q|$  is the square of the radius of  $S_2$  (why?); hence  $P$  and  $Q$  are a pair of inverse points with respect to the sphere  $S_2$ . Let us now consider any sphere  $S$  and a pair of inverse points  $P$  and  $Q$  with respect to  $S$ . Inverting from any center of inversion  $O$  we obtain a sphere  $S'$  and a pair of points  $P'$  and  $Q'$  which have the property that any sphere through  $P'$  and  $Q'$  cuts  $S'$  at right angles. Hence  $P'$  and  $Q'$  must be a pair of inverse points with respect to  $S'$ . We have, then, the following result:

**Any sphere  $S$  and a pair of inverse points with respect to  $S$  invert into a sphere  $S'$  and a pair of inverse points with respect to  $S'$ .**

Since the center of a sphere and the point at infinity are a pair of inverse points with respect to the sphere we have the following special case of this result:

**Any sphere  $S$  and its center  $C$  invert into a sphere  $S'$  and the inverse point of the center of inversion with respect to  $S'$ .**

For example, if  $S$  is a sphere of radius  $a$  and if  $S'$  is a plane (so that  $O$  is on  $S$ ), the center  $C$  of  $S$  inverts into a point whose distance from  $O$  is  $2\delta$ ,  $\delta$  being the distance of  $O$  from the plane. Hence  $2a\delta = k^2$ .

Laplace's equation in space polar coordinates is

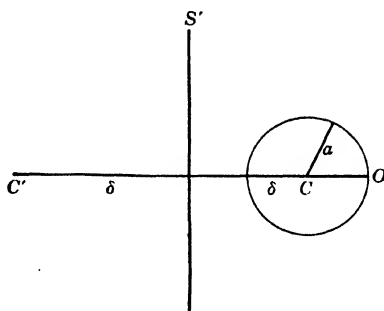


FIG. 30.

$$V_{rr} + \frac{2}{r} V_r + \frac{1}{r^2} V_{\theta\theta} + \frac{\cot \theta}{r^2} V_\theta + \frac{1}{r^2 \sin^2 \theta} V_{\phi\phi} = 0.$$

Replacing the independent variable  $r$  by  $r'$  by means of the substitution

$$r = \frac{k^2}{r'}; \quad r' = \frac{k^2}{r}$$

we have

$$V_r = -\frac{k^2}{r^2} V_{r'} = -\frac{r'^2}{k^2} V_{r'};$$

$$V_{rr} = \frac{2k^2}{r^3} V_{r'} + \frac{k^4}{r^4} V_{r'r'} = \frac{2r'^3}{k^4} V_{r'} + \frac{r'^4}{k^4} V_{r'r'}$$

and so

$$\Delta_2 V = \left(\frac{r'}{k}\right)^4 \left\{ V_{r'r'} + \frac{1}{r'^2} V_{\theta\theta} + \frac{\cot \theta}{r'^2} V_\theta + \frac{1}{r'^2 \sin^2 \theta} V_{\phi\phi} \right\}.$$

We now replace the dependent variable  $V$  by a new dependent variable  $W$  defined by the formula

$$W = \frac{rV}{k} = \frac{kV}{r'}; \quad r' \neq 0;$$

then

$$V = \frac{r'W}{k}; \quad V_r = \frac{W}{k} + \frac{r'W_{r'}}{k}; \quad V_{r'r'} = \frac{r'}{k} \left\{ W_{r'r'} + \frac{2}{r'} W_{r'} \right\}$$

and so

$$\Delta_2 V = \left(\frac{r'}{k}\right)^5 \Delta'_2 W,$$

where  $\Delta'_2$  is the Laplacian written in the space polar coordinates  $(r', \theta, \phi)$ :

$$\Delta'_2 W = W_{r'r'} + \frac{2}{r'} W_{r\theta} + \frac{1}{r'^2} W_{\theta\theta} + \frac{\cot \theta}{r'^2} W_\theta + \frac{1}{r'^2 \sin^2 \theta} W_{\phi\phi}.$$

Thus at any point  $P'$ , which is the inverse of a point  $P$  at which  $\Delta_2 V = 0$ , the function  $W$  satisfies the equation  $\Delta'_2 W = 0$ . Let us suppose, for a moment, that  $V$  is the electrostatic potential of a conductor  $S$  at potential  $V_0$ , there being no point charges present.

Then  $W = \frac{kV_0}{r'}$  over the inverse  $S'$  of  $S$ . Since  $\frac{kV_0}{r'}$  is, in general, variable over  $S'$  (why?)  $W$  cannot serve as the potential of  $S'$  regarded as a conductor. However, the function  $V'$  defined as follows:

$$V' = W - \frac{kV_0}{r'} = \frac{k(V - V_0)}{r'}$$

is constant over  $S'$ , its constant value being zero. Furthermore  $\Delta'_2 V' = 0$  at all points except the center  $O$  of inversion; at  $r' = 0$ ,  $W$  has the value  $\frac{e}{k}$  where  $e$  is the charge on  $S$  (since  $rV$  has at  $r = \infty$  the limit  $e$ ) and so  $V'$  is unbounded at  $r' = 0$  in the manner characteristic of the potential of a point charge  $= -kV_0$  at  $O$ . In other words  $W$  is the induced potential of the conducting surface  $S'$  supposed earthed (i.e., at potential zero) in the presence of an inducing point charge  $-kV_0$  located at the center of inversion.

*Note.* Be sure that you understand clearly that  $V$  is evaluated at  $P$  while  $W$  and  $V'$  are evaluated at  $P'$ . Thus in order to obtain the value of  $V'$  at  $P'$  we must evaluate  $V$  at  $P$  (not at  $P'$ ), subtract  $V_0$  from this value, and multiply the result of the subtraction by  $\frac{k}{r'}$ .

We obtain in this way from one electrostatic field another, and we say that either field is the *inverse* of the other. We started with the field of a conductor  $S$  at potential  $V_0$ ; its inverse was the field of an earthed conductor  $S'$  in the presence of an inducing point charge  $-kV_0$  located at the center of inversion. We could, however, have started with the second field and obtained the first. Thus, if  $W$  is the

induced potential of an earthed conductor  $S'$  in the presence of a point charge  $e'$ ,  $V = \frac{kW}{-}$  is the potential of the inverse  $S$  of  $S'$  (regarded

as a conductor at potential  $V_0 = -\frac{e'}{k}$ ), the center of inversion being

at the inducing charge  $e'$ . When we look at the matter in this way we say that we have *inverted away* the inducing charge. Since the charge  $e$  on  $S$  is  $kW(O)$  the capacity of  $S$  is the absolute value of  $\frac{k^2W(O)}{e'}$ . We have, then, the following rule which is very useful for

the determination of the capacities of certain conductors: Let  $W$  be the induced potential of an earthed conductor  $S'$  in the presence of a charge  $e'$  at  $O$ . Then the absolute value of  $\frac{k^2W(O)}{e'}$  is the capacity of the conductor  $S$  obtained by inverting  $S'$  with respect to a sphere of radius  $k$  whose center is at  $O$ .

If  $V$  is the potential of the field due to a single point charge  $e$  located at  $Q$ , where  $Q$  is any point other than the center of inversion  $O$ ,  $W$  is the field due to a single point charge  $e'$  located

at  $Q'$ . In fact  $V(P) = \frac{e}{|QP|}$ ,  $W(P') = \frac{|OP|e}{k|QP|}$ , and since the four points  $P, Q, Q', P'$  are concyclic the triangles  $OPQ$  and  $OQ'P'$  are similar so that  $\frac{|OP|}{|QP|} = \frac{|OQ'|}{|Q'P'|}$ .

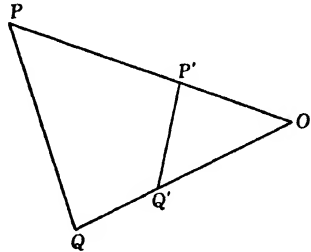


FIG. 31.

Hence

$$W(P') = \frac{e|OQ'|}{k|Q'P'|}$$

so that  $W$  is the potential of a point charge  $e' = \frac{e|OQ'|}{k} = \frac{ke}{|OQ|}$  located at  $Q'$ . The value of  $e'$  may be conveniently remembered by the following rule: The value of  $W$  at  $O = \frac{e}{k}$ .

When the point charge  $e$  is located at the center of inversion  $O$ ,  $V = \frac{e}{r}$  so that  $W = \frac{e}{k}$  is constant. Hence  $\text{grad}' W = 0$ . Thus the

inverse of the field due to a point charge, from a center of inversion located at the point charge, is the zero field.

If we have several point charges  $e_1, \dots, e_n$  located at points  $Q_1, \dots, Q_n$ , none of which is the center of inversion  $O$ ,  $W$  is the potential of the point charges  $e'_1, \dots, e'_n$ , where

$$e'_j = \frac{e_j |OQ'_j|}{k} = \frac{ke_j}{|OQ_j|}; \quad j = 1, \dots, n,$$

located at the points  $Q'_1, \dots, Q'_n$ , respectively. The value of  $W$

at  $O$  is  $\frac{1}{k} \sum_1^n e_j$ .

If one of the point charges  $e_1$ , say, is located at  $O$ ,  $W$  is (save for an additive constant) the potential of the point charges  $e'_2, \dots, e'_n$  located at the points  $Q'_2, \dots, Q'_n$ .

### Example 1. The earthed plane in the presence of a point charge

Let  $S$  be a sphere of radius  $a$  at potential  $V_0$ . Then  $V$  is (at points outside  $S$ ) the potential of a charge  $e = aV_0$  located at the center of  $S$ .

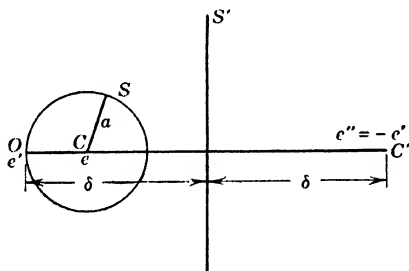


FIG. 32.

Inverting from a point on  $S$  we obtain the field due to an earthed plane conductor  $S'$  in the presence of an influencing charge  $e' = -kV_0$  at a distance  $\delta$  from the plane where  $2\delta a = k^2$ . The induced potential is that due to a point charge  $e''$  located at the image point of  $O$  in the plane (because this is the point into which the center of the sphere inverts); the connection between  $e''$  and the influencing charge  $e'$  follows from the relation

$$\frac{e}{k} = W(O) = \frac{e''}{2\delta}.$$

Hence  $e'' = \frac{2\delta e}{k} = \frac{ke}{a} = kV_0 = -e'$ ; in other words  $e''$  is the negative of the influencing point charge  $e' = -kV_0$  which is located at  $O$ . Thus

**The induced potential due to a point charge in the presence of a conducting plane is that of an equal, but oppositely signed, point charge located at the image point of the influencing charge in the plane.**

*Note.* Assuming this result known we can determine the capacity of a sphere. Inverting from the location  $O$  of the influencing charge (so as to invert this charge away) we obtain a sphere of radius  $a$  where  $2\delta a = k^2$ . Since the value of the induced potential  $W$  at  $O$  is  $\frac{e''}{2\delta} = -\frac{ae'}{k^2}$  the capacity of a sphere of radius  $a$  is  $a$  (why?).

The relation between the surface density of charge  $\sigma'$  on the earthed conductor  $S'$  and the surface density of charge  $\sigma$  on the conductor  $S$  follows readily from the relation

$$V' = \frac{k(V - V_0)}{r'}$$

Since  $V = V_0$  over  $S$  we have, over  $S'$ ,

$$V'_{r'} = \frac{k}{r'} V_{r'} = -\left(\frac{k}{r'}\right)^3 V_{r'};$$

$$\frac{1}{r'} V'_{\theta} = \frac{k}{r'^2} V_{\theta} = \left(\frac{k}{r'}\right)^3 \frac{1}{r} V_{\theta}; \quad \frac{1}{r' \sin \theta} V'_{\phi} = \left(\frac{k}{r'}\right)^3 \frac{1}{r \sin \theta} V_{\phi}.$$

Hence  $\mathbf{grad}' V'$  at any point  $P'$  of  $S'$  = the product of  $\mathbf{grad} V$  at the corresponding point  $P$  of  $S$  by  $\left(\frac{k}{r'}\right)^3$ . Since any direction  $(dr, r d\theta, r \sin \theta d\phi)$  inverts into a direction  $(-dr, r d\theta, r \sin \theta d\phi)$  it follows that  $\frac{dV'}{dn'}$  over  $S'$  is the product of  $\frac{dV}{dn}$  over  $S$  by  $\left(\frac{k}{r'}\right)^3$  (remember that  $\frac{dV}{dn} = (\mathbf{grad} V | \mathbf{n})$ ). Hence

$$\sigma' = \left(\frac{k}{r'}\right)^3 \sigma.$$

Since  $\sigma$  is constant for an isolated sphere at potential  $V_0$  it follows that the surface-density of charge on an earthed plane in the presence of an

influencing point charge varies inversely as the cube of the distance from the influencing charge. The factor of proportionality is easily determined by evaluating the density at the foot  $A'$  of the perpendicular from  $O$  on the plane. At this point  $\mathbf{n}'$  has the direction of  $A' \rightarrow O$ ; if  $e'$  is the influencing point charge  $V' = e' \left( \frac{1}{r'} - \frac{1}{r''} \right)$ , where  $r''$  is the distance from the image  $Q'$  of  $O$  in the plane to the point of evaluation  $P'$  of  $V'$ . Hence, at  $A'$ ,

$$\frac{dV'}{dn'} = e' \left( \frac{1}{\delta^2} + \frac{1}{\delta^2} \right) = \frac{2e'}{\delta^2},$$

where  $\delta$  is the distance  $|OA'|$  of the influencing point charge from the plane (remember that  $\mathbf{n}'$  has the direction of  $Q' \rightarrow A'$  so that, at  $A'$ ,

$$\frac{d}{dn'} \left( \frac{1}{r''} \right) = \left( \frac{1}{r''} \right)_{r''}.$$

Hence, at  $A'$ ,  $\sigma' = -\frac{e'}{2\pi\delta^2} = -\frac{e'\delta}{2\pi\delta^3}$ . It follows (why?) that at any point  $P'$  of the earthed plane

$$\sigma' = -\frac{e'\delta}{2\pi(r')^3},$$

where  $r' = |OP'|$ .

### Example 2. The earthed sphere in the presence of a point charge

Let  $S$  be again a sphere of radius  $a$  at potential  $V_0$ , and invert from a point outside  $S$ . Then  $S'$  is a sphere, and we have the solution of the problem of an earthed sphere in the presence of an influencing charge located at a point  $O$  outside  $S'$ . The induced potential is that of a point charge  $e''$  located at the image point  $Q'$  of  $O$  in  $S'$  (why?). Let  $a'$  be

the radius of  $S'$ , and let  $d'$  be the distance of  $O$  from the center of  $S'$ . Then the distance of the image point of  $O$  in  $S'$  from  $O$  is

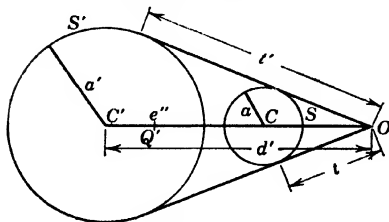


FIG. 33.

$d' - \left(\frac{a'^2}{d'}\right) = \frac{t'^2}{d'}$ , where  $t'$  is the length of the tangent from  $O$  to  $S'$ .

Hence  $W(O) = \frac{e' d'}{t'^2}$ , and since  $W(O) = \frac{e}{k}$  we have

$$e'' = \frac{e t'^2}{k d'}$$

If  $t$  is the length of the tangent from  $O$  to  $S$  we have  $t t' = k^2$  and  $\frac{a'}{a} = \frac{t'}{t} = \frac{t'^2}{k^2}$  so that

$$e'' = \frac{k e a'}{a d'} = k V_0 \frac{a'}{d'}$$

In words:

The point charge whose potential furnishes the induced potential  $W$  due to the influencing charge  $e' = -kV_0$  in the presence of an earthed conducting sphere of radius  $a'$  is the negative of the product of this influencing charge by  $\frac{a'}{d'}$ , where  $d'$  is the distance of the influencing charge from the center of the earthed sphere.

The surface density of charge on the earthed sphere varies inversely as the cube of the distance from the influencing charge (why?). The factor of proportionality is found in exactly the same way as for the earthed plane by evaluating the density at the point  $A'$  of the sphere which is nearest the influencing charge. The value of  $\sigma'$  at  $A'$  is  $-\frac{e'}{4\pi|OA'|^2} + \frac{e''}{4\pi|Q'A'|^2}$ . Since  $e'' = -e' \frac{a'}{d'}$  and  $\left|\frac{Q'A'}{OA'}\right| = \frac{a'}{d'}$  the value of  $\sigma'$  at  $A'$  is  $-\frac{e'}{4\pi|OA'|^2} \left\{1 + \frac{d'}{a'}\right\} = -\frac{e' t'^2}{4\pi a' |OA'|^3}$ , where  $t'^2 = d'^2 - a'^2$  is the square of the tangent from  $O$  to  $S'$  (remember that  $|OA'| = d' - a'$ ). Hence the density at any point  $P'$  of  $S'$  is given by the formula

$$\sigma' = -\frac{e' t'^2}{4\pi a'} \frac{1}{|OP'|^3}$$

As  $a' \rightarrow \infty$ ,  $\frac{t'^2}{a'} = (d' - a') \left(1 + \frac{d'}{a'}\right) = \delta \left(2 + \frac{\delta}{a'}\right) \rightarrow 2\delta$ , where  $\delta = d' - a'$  is the distance  $|OA'|$ , and we recover the formula



$$\sigma' = -\frac{e'\delta}{2\pi|OP|^3}$$

for the surface density of electric charge on the earthed plane conductor.

**Example 3. The capacity of a conductor formed by two spheres which intersect at right angles**

Let  $S'$  consist of two infinite planes intersecting at right angles, and let  $O$  be at distances  $\delta_1, \delta_2$  from the two planes. Then the induced potential  $W$  of the charge induced on  $S'$  by a charge  $-kV_0$  at  $O$  ( $S'$  being at potential zero) is the same as that due to the following three charges:

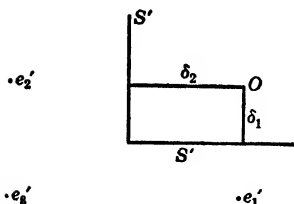


FIG. 34

1. a charge  $e'_1 = kV_0$  located at the image of  $O$  in the first plane;
2. a charge  $e'_2 = kV_0$  located at the

image of  $O$  in the second plane;

3. a charge  $e'_3 = -kV_0$  located at the "image" of  $O$  in the line of intersection of the two planes; this point is equally the image of the location of the charge 1 in the second plane and the image of the location of the charge 2 in the first plane.

(Prove this statement. *Hint.* Show that  $V' = W - \frac{kV_0}{r'}$  satisfies the conditions on  $V'$  at  $O$ , on  $S'$ , and at  $\infty$ , and use the uniqueness theorem.) Hence

$$W(O) = \frac{kV_0}{2} \left( \frac{1}{\delta_1} + \frac{1}{\delta_2} - \frac{1}{(\delta_1^2 + \delta_2^2)^{1/2}} \right).$$

On inverting from  $O$  we see that the potential function of a conductor  $S$  which consists of two spheres of radii  $a_1$  and  $a_2$  which intersect at right angles is that due to the following three charges:

1. a charge  $e_1 = \frac{k^2V_0}{2\delta_1} = a_1V_0$  located at the center of the first sphere;

2. a charge  $e_2 = \frac{k^2V_0}{2\delta_2} = a_2V_0$  located at the center of the second sphere;

3. a charge  $e_3 = -\frac{k^2V_0}{2(\delta_1^2 + \delta_2^2)^{1/2}} = -\frac{a_1a_2V_0}{(a_1^2 + a_2^2)^{1/2}}$  located at the

point  $Q_3$  which is equally the image of the center of the second sphere in the first and the image of the center of the first sphere in the second. In view of the relations  $2\delta_1 a_1 = k^2$ ,  $2\delta_2 a_2 = k^2$  we have

$$W(O) = \frac{V_0}{k} \left\{ a_1 + a_2 - \frac{a_1 a_2}{(a_1^2 + a_2^2)^{1/2}} \right\}.$$

Hence the capacity of the conductor  $S$  is

$$C = a_1 + a_2 - \frac{a_1 a_2}{(a_1^2 + a_2^2)^{1/2}}$$

(since the charge on  $S$  is  $kW(O)$ ).

The density of electrification at any point  $P$  of  $S$  may be readily determined as follows. The contribution to the density  $\sigma$  at any point

$P$  of  $S_1$  from the potential of the point charge  $e_1 = a_1 V_0$  is  $\frac{a_1 V_0}{4\pi a_1^2} = \frac{V_0}{4\pi a_1}$ .

The contribution to  $\sigma$  from the potential of the two point charges  $e_2$  and  $e_3$  (which together would maintain  $S_1$  at potential zero) is

$$- \frac{a_2 V_0 t_2^2}{4\pi a_1} \frac{1}{|C_2 P|^3}$$

Here  $t_2$  is the length of the tangent from the center  $C_2$  of  $S_2$  to  $S_1$  so that  $t_2 = a_2$  (why?). Hence the value of  $\sigma$  at any point  $P$  of  $S_1$  is

$$\sigma = \frac{V_0}{4\pi a_1} \left\{ 1 - \frac{a_2^3}{|C_2 P|^3} \right\}.$$

Similarly the value of  $\sigma$  at any point  $P$  of  $S_2$  is

$$\sigma = \frac{V_0}{4\pi a_2} \left\{ 1 - \frac{a_1^3}{|C_1 P|^3} \right\},$$

where  $C_1$  is the center of  $S_1$ . The density at any point of intersection of  $S_1$  and  $S_2$  is zero.

### EXERCISE

1. Show that the charge on the first sphere is

$$\frac{V_0}{2} \left\{ a_1 + a_2 + \frac{a_1^2 - a_2^2 - a_1 a_2}{(a_1^2 + a_2^2)^{1/2}} \right\}.$$

*Hint.* The potential of  $e_1$  contributes to the charge on  $S_1$  the same fraction of  $e_1$  that the area of the "exposed" portion of  $S_1$  is to the total area  $4\pi a_1^2$  of  $S_1$ . The potential of  $e_2$  contributes to the charge on  $S_1$  the same fraction of  $e_2$  that the area of the "unexposed" portion of  $S_2$  is to the total area  $4\pi a_2^2$  of  $S_2$ . The potential of  $e_3$  contributes to the charge on  $S_1$  one-half the charge of  $e_3$  (why?).

**Example 4. The capacity of a conductor formed by two spheres which touch externally**

Inverting from the point of contact we have the problem of two parallel earthed planes under the influence of a point charge  $e' = -kV_0$  located between them (at distances  $\delta_1$  and  $\delta_2$ , say, from the two planes). This problem is solved by two trains of images designed to keep the planes at zero potential. Take the  $x$ -axis of a system of rectangular

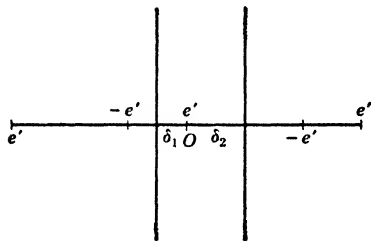


FIG. 35.

Cartesian coordinates, whose origin  $O$  is located at the inducing charge, perpendicular to the planes, and let the equation of the left-hand plane be  $x + \delta_1 = 0$  while that of the right-hand plane is  $x - \delta_2 = 0$ . The first train of images is obtained by starting with the image of  $e'$  in the left-hand plane. The first image of

this train is a charge  $-e'$  at the point  $(-2\delta_1, 0, 0)$ ; the second is a charge  $e'$  at the point  $(2(\delta_1 + \delta_2), 0, 0)$ ; the third is a charge  $-e'$  at the point  $(-4\delta_1 + 2\delta_2, 0, 0)$ ; the fourth is a charge  $e'$  at the point  $(4(\delta_1 + \delta_2), 0, 0)$ , and so on. All the images of the first train in the left-hand plane carry the charge  $-e'$ , and the common spacing between them is  $2(\delta_1 + \delta_2)$ , i.e., twice the distance between the parallel planes. All the images (of the first train) in the right-hand plane carry the charge  $e'$ , and the common spacing between them is again  $2(\delta_1 + \delta_2)$ . The images of the second train are obtained by interchanging the roles of the two planes (and  $\delta_1$  and  $\delta_2$ ). The contribution to  $W(O)$  from the images in the left-hand plane is

$$-e' \left\{ \frac{1}{2\delta_1} - \frac{1}{2(\delta_1 + \delta_2)} + \frac{1}{4\delta_1 + 2\delta_2} - \frac{1}{4(\delta_1 + \delta_2)} + \dots \right\}.$$

Note carefully that this series, while convergent (since it is an alternating series) is not absolutely convergent. Thus the order in which the terms appear is of importance. The charge on the left-hand sphere of the conductor  $S$  (the radii of the two spheres being  $a_1$  and  $a_2$ ) is, accordingly, the product of the series just written by  $k$ ; since  $2\delta_1 a_1 = k^2$ ,  $2\delta_2 a_2 = k^2$ ,  $e' = -kV_0$  this turns out to be

$$V_0 \left\{ a_1 - \frac{a_1 a_2}{a_1 + a_2} + \frac{a_1 a_2}{a_1 + 2a_2} - \frac{a_1 a_2}{2a_1 + 2a_2} + \frac{a_1 a_2}{2a_1 + 3a_2} - \dots \right\}$$

(the successive denominators being obtained by adding alternately  $a_1$  and  $a_2$  to the previous denominators). Since the charge on the second sphere  $S_2$  is obtained by interchanging  $a_1$  and  $a_2$  the capacity of the conductor  $S$  is

$$\left\{ a_1 - \frac{a_1 a_2}{a_1 + a_2} + \frac{a_1 a_2}{a_1 + 2a_2} - \dots \right\} + \left\{ a_2 - \frac{a_1 a_2}{a_1 + a_2} + \frac{a_1 a_2}{2a_1 + a_2} - \dots \right\}.$$

In the particular case when  $a_1 = a_2 = a$ , say, this reduces to

$$2a(1 - \frac{1}{2} + \frac{1}{3} - \dots) = 2a \log 2.$$

### 5. Ellipsoidal coordinates

Let  $a, b, c$  be any three unequal positive real numbers which we suppose arranged in descending order of magnitude

$$a > b > c > 0.$$

Then the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is an equation of an ellipsoid whose semiaxes are  $(a, b, c)$ . This ellipsoid is a member of the family of second-degree surfaces

$$\frac{x^2}{a^2 + t} + \frac{y^2}{b^2 + t} + \frac{z^2}{c^2 + t} = 1$$

obtained by assigning to the parameter  $t$  different fixed values. In order not to rule out the values  $-a^2, -b^2, -c^2$  as possible values of  $t$  we shall write the equation of the family in the form

$$F(t) = 0,$$

where

$$F(t) = (b^2 + t)(c^2 + t)x^2 + (c^2 + t)(a^2 + t)y^2 + (a^2 + t)(b^2 + t)z^2 - (a^2 + t)(b^2 + t)(c^2 + t).$$

When  $t = -c^2$  the corresponding surface is the plane  $z = 0$  (counted twice); when  $t = -b^2$  the corresponding surface is the plane  $y = 0$  (counted twice); and when  $t = -a^2$  the surface is the plane  $x = 0$  (counted twice). Let, now,  $P(x, y, z)$  be any point which does not lie on any of the coordinate planes so that none of the numbers  $x, y, z$

is zero. Since the term of highest degree in  $F(t)$  is  $-t^3$ ,  $F(t)$  is negative if  $t$  is sufficiently large. On the other hand  $F(-c^2)$  is positive since

$$F(-c^2) = (a^2 - c^2)(b^2 - c^2)z^2$$

and  $a^2 > c^2$ ,  $b^2 > c^2$ . Hence no matter what is the point  $(x, y, z)$ , as long as it does not lie on one of the coordinate planes, there is a value of  $t > -c^2$  for which  $F(t) = 0$ . We denote this value of  $t$  by  $\lambda$  and observe that each of the three numbers  $a^2 + \lambda$ ,  $b^2 + \lambda$ ,  $c^2 + \lambda$  is positive. Since the equation  $F(\lambda) = 0$  may be written in the form

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} = 1$$

$F(\lambda) = 0$  is an equation of an ellipsoid.

Since  $F(-b^2) = (c^2 - b^2)(a^2 - b^2)y^2 < 0$  there is a value of  $t$  in the open interval  $(-b^2, -c^2)$  for which  $F(t) = 0$ . We denote this value of  $t$  by  $\mu$  and observe that  $a^2 + \mu$  and  $b^2 + \mu$  are positive while  $c^2 + \mu$  is negative. Since the equation  $F(\mu) = 0$  may be written in the form

$$\frac{x^2}{a^2 + \mu} + \frac{y^2}{b^2 + \mu} + \frac{z^2}{c^2 + \mu} = 1$$

$F(\mu) = 0$  is an equation of a hyperboloid of one sheet.

Finally, since  $F(-a^2) = (b^2 - a^2)(c^2 - a^2)x^2 > 0$ , there is a value of  $t$  in the open interval  $(-a^2, -b^2)$  for which  $F(t) = 0$ . We denote this value of  $t$  by  $\nu$  and observe that  $a^2 + \nu$  is positive while  $b^2 + \nu$  and  $c^2 + \nu$  are negative. Since the equation  $F(\nu) = 0$  may be written in the form

$$\frac{x^2}{a^2 + \nu} + \frac{y^2}{b^2 + \nu} + \frac{z^2}{c^2 + \nu} = 1$$

$F(\nu) = 0$  is an equation of a hyperboloid of two sheets.

Associated, then, with any point  $P: (x, y, z)$  which does not lie on one of the coordinate planes are three numbers  $\lambda, \mu, \nu$  which are such that

$$\lambda > -c^2 > \mu > -b^2 > \nu > -a^2.$$

Thus each of the three symbols  $(\lambda, \mu, \nu)$  is a point-function; the level surfaces of the point-function

$$\lambda = \lambda(P) = \lambda(x, y, z)$$

are ellipsoids while the level surfaces of the point-function

$$\mu = \mu(P) = \mu(x, y, z)$$

are one-sheeted hyperboloids and the level surfaces of the point-function

$$\nu = \nu(P) = \nu(x, y, z)$$

are two-sheeted hyperboloids. The three numbers  $(\lambda, \mu, \nu)$  may be regarded as curvilinear coordinates of the point  $P$ ; we shall see shortly that they constitute a system of *orthogonal* curvilinear coordinates, and this system is known as the system of *ellipsoidal coordinates*.

When  $P: (x, y, z)$  is given,  $(\lambda, \mu, \nu)$  are the three zeros of the cubic polynomial  $F(t) = 0$ . In other words

$$\begin{aligned} \frac{x^2}{a^2 + t} + \frac{y^2}{b^2 + t} + \frac{z^2}{c^2 + t} - 1 &= \frac{F(t)}{(a^2 + t)(b^2 + t)(c^2 + t)} \\ &= \frac{(\lambda - t)(\mu - t)(\nu - t)}{(a^2 + t)(b^2 + t)(c^2 + t)}. \end{aligned}$$

$(\lambda, \mu, \nu)$  are the point-functions  $\lambda(x, y, z)$ ,  $\mu(x, y, z)$ ,  $\nu(x, y, z)$  referred to above, and the relation

$$\frac{x^2}{a^2 + t} + \frac{y^2}{b^2 + t} + \frac{z^2}{c^2 + t} - 1 = \frac{(\lambda - t)(\mu - t)(\nu - t)}{(a^2 + t)(b^2 + t)(c^2 + t)}$$

is an identity in the four independent variables  $x, y, z$ , and  $t$ . On differentiating this identity with respect to  $t$  and then setting  $t = \lambda$  we obtain

$$\frac{x^2}{(a^2 + \lambda)^2} + \frac{y^2}{(b^2 + \lambda)^2} + \frac{z^2}{(c^2 + \lambda)^2} = \frac{(\mu - \lambda)(\nu - \lambda)}{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}.$$

Similarly

$$\begin{aligned} \frac{x^2}{(a^2 + \mu)^2} + \frac{y^2}{(b^2 + \mu)^2} + \frac{z^2}{(c^2 + \mu)^2} &= \frac{(\lambda - \mu)(\nu - \mu)}{(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)}; \\ \frac{x^2}{(a^2 + \nu)^2} + \frac{y^2}{(b^2 + \nu)^2} + \frac{z^2}{(c^2 + \nu)^2} &= \frac{(\lambda - \nu)(\mu - \nu)}{(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)}. \end{aligned}$$

The equation  $\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} = 1$  is an identity in the three independent variables  $(x, y, z)$  ( $\lambda$  is here not a constant, but the point-function  $\lambda = \lambda(x, y, z)$ ). On differentiating this identity with respect to  $x$  we obtain

$$\left\{ \frac{x^2}{(a^2 + \lambda)^2} + \frac{y^2}{(b^2 + \lambda)^2} + \frac{z^2}{(c^2 + \lambda)^2} \right\} \lambda_x = \frac{2x}{a^2 + \lambda},$$

and there are similar equations for  $\lambda_y$  and  $\lambda_z$ . Thus  $\mathbf{grad} \lambda$  has the direction of the vector  $v\left(\frac{x}{a^2 + \lambda}, \frac{y}{b^2 + \lambda}, \frac{z}{c^2 + \lambda}\right)$ , and, similarly,  $\mathbf{grad} \mu$  has the direction of the vector  $v\left(\frac{x}{a^2 + \mu}, \frac{y}{b^2 + \mu}, \frac{z}{c^2 + \mu}\right)$ . It follows that the two vectors  $\mathbf{grad} \lambda$  and  $\mathbf{grad} \mu$  are perpendicular; in fact we obtain on subtracting the equation

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} = 1$$

from the equation

$$\frac{x^2}{a^2 + \mu} + \frac{y^2}{b^2 + \mu} + \frac{z^2}{c^2 + \mu} = 1$$

the equation

$$(\lambda - \mu) \left\{ \frac{x^2}{(a^2 + \lambda)(a^2 + \mu)} + \frac{y^2}{(b^2 + \lambda)(b^2 + \mu)} + \frac{z^2}{(c^2 + \lambda)(c^2 + \mu)} \right\} = 0.$$

Since  $\lambda - \mu > 0$  it follows that

$$\frac{x^2}{(a^2 + \lambda)(a^2 + \mu)} + \frac{y^2}{(b^2 + \lambda)(b^2 + \mu)} + \frac{z^2}{(c^2 + \lambda)(c^2 + \mu)} = 0.$$

Hence the vectors  $v\left(\frac{x}{a^2 + \lambda}, \frac{y}{b^2 + \lambda}, \frac{z}{c^2 + \lambda}\right)$ ,  $v\left(\frac{x}{a^2 + \mu}, \frac{y}{b^2 + \mu}, \frac{z}{c^2 + \mu}\right)$  are perpendicular so that the vectors  $\mathbf{grad} \lambda$  and  $\mathbf{grad} \mu$  are perpendicular. A repetition of this argument shows that the three vectors  $\mathbf{grad} \lambda$ ,  $\mathbf{grad} \mu$ ,  $\mathbf{grad} \nu$  are mutually perpendicular. In other words

**The system of curvilinear coordinates  $(\lambda, \mu, \nu)$  is a system of orthogonal curvilinear coordinates.**

Denoting by  $(h_1, h_2, h_3)$  the magnitudes of the vectors  $\mathbf{grad} \lambda$ ,  $\mathbf{grad} \mu$ ,  $\mathbf{grad} \nu$  the Jacobian matrix  $\begin{pmatrix} \lambda, \mu, \nu \\ x, y, z \end{pmatrix}$  is the product of a  $3 \times 3$  rotation matrix by the  $3 \times 3$  diagonal matrix

$$D = \begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{bmatrix}.$$

Hence the Jacobian matrix  $\frac{(x, y, z)}{(\lambda, \mu, \nu)}$ , being the reciprocal of the Jacobian matrix  $\frac{(\lambda, \mu, \nu)}{(x, y, z)}$ , is the product of the reciprocal of  $D$  by a  $3 \times 3$  rotation matrix. Hence the magnitude of the vector  $v(x_\lambda, y_\lambda, z_\lambda)$  is  $\frac{1}{h_1}$ , and so on. Thus the vector element of arc  $ds = v(dx, dy, dz)$  may be written in the form

$$ds = \frac{d\lambda}{h_1} \mathbf{u}_1 + \frac{d\mu}{h_2} \mathbf{u}_2 + \frac{d\nu}{h_3} \mathbf{u}_3,$$

where  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$  are a set of unit and mutually perpendicular vectors (namely, the unit vectors along the axes of the system of orthogonal curvilinear coordinates). Hence

$$(ds)^2 = \frac{(d\lambda)^2}{h_1^2} + \frac{(d\mu)^2}{h_2^2} + \frac{(d\nu)^2}{h_3^2}.$$

Since  $h_1$  is the magnitude of  $\text{grad } \lambda$  it follows from the equations

$$\left\{ \frac{x^2}{(a^2 + \lambda)^2} + \frac{y^2}{(b^2 + \lambda)^2} + \frac{z^2}{(c^2 + \lambda)^2} \right\} \lambda_x = \frac{2x}{a^2 + \lambda}, \text{ etc.},$$

$$\begin{aligned} \text{that } h_1^2 &= 4 \div \left\{ \frac{x^2}{(a^2 + \lambda)^2} + \frac{y^2}{(b^2 + \lambda)^2} + \frac{z^2}{(c^2 + \lambda)^2} \right\} \\ &= \frac{4(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}{(\mu - \lambda)(\nu - \lambda)}. \end{aligned}$$

It is convenient to denote the product  $(a^2 + t)(b^2 + t)(c^2 + t)$  by the symbol  $\Delta(t)$ ; then  $\Delta(\lambda)$  is positive,  $\Delta(\mu)$  is negative, and  $\Delta(\nu)$  is positive. (Prove this.) In this notation

$$h_1 = 2\{\Delta(\lambda)\}^{1/2}(\lambda - \mu)^{-1/2}(\lambda - \nu)^{-1/2};$$

$$h_2 = 2\{-\Delta(\mu)\}^{1/2}(\lambda - \mu)^{-1/2}(\mu - \nu)^{-1/2};$$

$$h_3 = 2\{\Delta(\nu)\}^{1/2}(\lambda - \nu)^{-1/2}(\mu - \nu)^{-1/2}.$$



The Laplacian  $\Delta_2 V$  of any doubly differentiable point-function  $V$  appears when expressed in orthogonal curvilinear coordinates in the form

$$h_1 h_2 h_3 \left\{ \left( \frac{h_1 V_\lambda}{h_2 h_3} \right)_\lambda + \left( \frac{h_2 V_\mu}{h_3 h_1} \right)_\mu + \left( \frac{h_3 V_\nu}{h_1 h_2} \right)_\nu \right\}$$

(see Exercise 5, p. 109). For the system of ellipsoidal coordinates this appears as

$$4 \left\{ \frac{\{\Delta(\lambda)\}^{1/2} \{\Delta(\lambda)\}^{1/2} V_\lambda}{(\lambda - \mu)(\lambda - \nu)} + \frac{\{-\Delta(\mu)\}^{1/2} \{-\Delta(\mu)\}^{1/2} V_\mu}{(\lambda - \mu)(\mu - \nu)} + \frac{\{\Delta(\nu)\}^{1/2} \{\Delta(\nu)\}^{1/2} V_\nu}{(\lambda - \nu)(\mu - \nu)} \right\}.$$

The solutions of Laplace's equation  $\Delta_2 V = 0$  which are functions of  $\lambda$  alone satisfy, then, the equation

$$\Delta(\lambda)^{1/2} \{\Delta(\lambda)\}^{1/2} V_\lambda = 0.$$

Let  $\xi = \xi(\lambda)$  be such that  $\xi_\lambda = -\{\Delta(\lambda)\}^{-1/2}$ ; then  $\lambda_\xi = -\{\Delta(\lambda)\}^{1/2}$  so that

$$\begin{aligned} -\{\Delta(\lambda)\}^{1/2} V_\lambda &= V_\xi; \\ -\{\Delta(\lambda)\}^{1/2} (V_\xi)_\lambda &= V_{\xi\xi}. \end{aligned}$$

Hence  $V_{\xi\xi} = 0$  so that  $V$  is a linear function of  $\xi$ :

$$V = A + B\xi,$$

where  $\xi$  is undetermined to the extent of an additive constant; we fix this additive constant by the requirement that  $\xi \rightarrow 0$  as  $\lambda \rightarrow \infty$ . Then

$$\xi = \int_\lambda^\infty \frac{du}{\{\Delta(u)\}^{1/2}} = \int_\lambda^\infty \frac{du}{\{(a^2 + u)(b^2 + u)(c^2 + u)\}^{1/2}}.$$

Thus  $\xi$  is an *elliptic integral*. Since  $\lambda, \mu, \nu$  are the three roots of the equation  $F(t) = 0$  we have

$$\lambda + \mu + \nu = x^2 + y^2 + z^2 = r^2.$$

Since  $\mu$  and  $\nu$  are bounded it follows that as  $P \rightarrow \infty$ ,  $\lambda \rightarrow \infty$  and so  $\xi \rightarrow 0$ . Since the potential function of an electrostatic field is null at  $\infty$  it follows that  $A = 0$ ; hence we have the following result: If the potential function  $V$  of an electrostatic field is a function of the ellipsoidal coordinate  $\lambda$  alone then

$$V = B\xi,$$

where

$$\xi = \int_{\lambda}^{\infty} \frac{du}{\{\Delta(u)\}^{1/2}} = \int_{\lambda}^{\infty} \frac{du}{\{(a^2 + u)(b^2 + u)(c^2 + u)\}^{1/2}}.$$

On writing  $\{(a^2 + u)(b^2 + u)(c^2 + u)\}^{-1/2}$  in the form

$$u^{-3/2} \left(1 + \frac{a^2}{u}\right)^{-1/2} \left(1 + \frac{b^2}{u}\right)^{-1/2} \left(1 + \frac{c^2}{u}\right)^{-1/2} = u^{-3/2} - \frac{1}{2}(a^2 + b^2 + c^2)u^{-5/2} + \dots$$

we see that  $\xi$  has, near  $\lambda = \infty$ , the development

$$\xi = 2\lambda^{-1/2} - \frac{1}{3}(a^2 + b^2 + c^2)\lambda^{-3/2} + \dots$$

Hence  $\lambda^{1/2}\xi$  has at  $\lambda = \infty$  the limit 2. Since  $\frac{\lambda}{r^2}$  has at  $r = \infty$  the limit 1 it follows that the limit of  $r\xi$  at  $r = \infty$  is 2.

Let, now,  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$  be the surface of a conductor at potential  $V_0$ . The function

$$V = B\xi$$

satisfies Laplace's equation outside the conductor and takes the constant value  $B \int_0^{\infty} \frac{du}{\{\Delta(u)\}^{1/2}}$  over the conductor. Hence the potential of the electrostatic field of the conductor is

$$V = V_0 \frac{\xi}{\int_0^{\infty} \frac{du}{\{\Delta(u)\}^{1/2}}}.$$

The charge  $e$  on the conductor, being the limit of  $rV$  at  $r = \infty$ , is

$$e = \frac{2V_0}{\int_0^{\infty} \frac{du}{\{\Delta(u)\}^{1/2}}}.$$

Hence the capacity of the ellipsoidal conductor is given by the formula

$$C = \frac{2}{\int_0^{\infty} \frac{du}{\{\Delta(u)\}^{1/2}}} = \frac{2}{\int_0^{\infty} \frac{du}{\{(a^2 + u)(b^2 + u)(c^2 + u)\}^{1/2}}}.$$

## EXERCISES

1. Show that the capacity of an *oblate spheroid* (i.e., an ellipsoid of revolution for which  $b = a > c$ ) is  $C = (a^2 - c^2)^{1/2} / \text{Arc cos } \frac{c}{a}$ .

*Hint.* Use the substitution  $c^2 + u = v^2$  to evaluate the integral

$$\int_0^\infty \frac{du}{(a^2 + u)(c^2 + u)^{1/2}}$$

2. Show that the capacity of a circular disc of radius  $a$  is  $\frac{2a}{\pi}$ . *Hint.* Let  $c \rightarrow 0$  in the result of Exercise 1.

3. Show that the capacity of a *prolate spheroid* (i.e., an ellipsoid of revolution for which  $b = c < a$ ) is

$$C = \frac{(a^2 - c^2)^{1/2}}{\log \left\{ \frac{a + (a^2 - c^2)^{1/2}}{c} \right\}}$$

4. Show that  $h_1$  is twice the perpendicular distance  $p$  from  $O: (0, 0, 0)$  to the tangent plane at  $P$  to the first coordinate surface,  $\lambda = \text{constant}$ , through  $P$ .

*Hint.* The surface  $\lambda = \text{constant}$  is the ellipsoid  $\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} = 1$ ;

$p$  is the reciprocal of the magnitude of the coefficient vector  $\mathbf{v} = v \left( \frac{x}{a^2 + \lambda}, \frac{y}{b^2 + \lambda}, \frac{z}{c^2 + \lambda} \right)$  of the tangent plane  $\left( \frac{x}{a^2 + \lambda} \right) X + \left( \frac{y}{b^2 + \lambda} \right) Y + \left( \frac{z}{c^2 + \lambda} \right) Z = 1$  to the level surface  $\lambda = \text{constant}$ . Since  $\text{grad } \lambda = 2p^2 \mathbf{v}$ ,  $h_1 = |\text{grad } \lambda| = 2p^2 \left( \frac{1}{p} \right) = 2p$ .

5. Show that the surface density of electric charge at any point  $P$  of the ellipsoidal conductor  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$  (supposed isolated and carrying a charge  $e$ ) is  $\frac{ep}{4\pi abc}$ , where  $p$  is the distance from the center of the ellipsoid to the

tangent plane at  $P$ . *Hint.*  $\sigma = -\frac{1}{4\pi} \frac{dV}{dn} = -\frac{h_1}{4\pi} V_\lambda = \frac{h_1}{4\pi} \{ \Delta(\lambda) \}_{\lambda=0}^{-1/2} V_\xi = \frac{h_1 B}{4\pi abc}$ ,

where  $B = \frac{1}{2}e$ . Since  $h_1 = 2p$  (see Exercise 4) it follows that  $\sigma = \frac{ep}{4\pi abc}$ .

6. Show that the surface density of electric charge at any point  $P$  not on the edge of a conducting circular plate of radius  $a$  (supposed isolated and carrying a charge  $e$ ) is  $\frac{e}{4\pi a(a^2 - r^2)^{1/2}}$ , where  $r = |OP| \neq a$  is the distance of  $P$  from the

center of the plate. *Hint.* For a prolate spheroid  $\frac{1}{p} = \left( \frac{x^2}{a^4} + \frac{y^2}{c^4} \right)^{1/2} = \frac{1}{c} \left\{ \frac{c^2(x^2 + y^2)}{a^4} + 1 - \frac{x^2 + y^2}{a^2} \right\}^{1/2}$ . Hence the limit of  $\frac{c}{p}$  as  $c \rightarrow 0$  is  $\left( 1 - \frac{x^2 + y^2}{a^2} \right)^{1/2}$ .

*Note.*  $\sigma$  is unbounded at  $r = a$ .

In dealing with problems such as that of determining the induced potential of an earthed conducting ellipsoid when subjected to a uniform electrostatic field we seek for solutions of Laplace's equation of the form

$$V = xU,$$

where  $U$  is a function of the ellipsoidal coordinate  $\lambda$  alone. Since  $V_x = U + xU_x$ ,  $V_{xx} = 2U_x + xU_{xx}$ ,  $V_{yy} = xU_{yy}$ ,  $V_{zz} = xU_{zz}$  we have

$$\Delta_2 V = x \Delta_2 U + 2U_x = x \Delta_2 U + 2U_x \lambda_x.$$

Since

$$\begin{aligned} \lambda_x &= \frac{2x}{a^2 + \lambda} \div \left\{ \frac{x^2}{(a^2 + \lambda)^2} + \frac{y^2}{(b^2 + \lambda)^2} + \frac{z^2}{(c^2 + \lambda)^2} \right\} \\ &= \frac{2\Delta(\lambda)}{(\lambda - \mu)(\lambda - \nu)} \frac{x}{a^2 + \lambda} \end{aligned}$$

we have, if  $V$  is a solution of Laplace's equation,

$$\Delta_2 U + \frac{4 \Delta(\lambda) U_x}{(\lambda - \mu)(\lambda - \nu)(a^2 + \lambda)} = 0.$$

Since  $U$  is, by hypothesis, a function of  $\lambda$  alone we have

$$\Delta_2 U = \frac{4 \{\Delta(\lambda)\}^{1/2}}{(\lambda - \mu)(\lambda - \nu)} \{\Delta(\lambda)^{1/2} U_x\}_\lambda$$

and so  $U$  must satisfy the equation

$$\{\Delta(\lambda)^{1/2} U_x\}_\lambda + \Delta(\lambda)^{1/2} \frac{U_x}{a^2 + \lambda} = 0.$$

Hence

$$\Delta(\lambda)^{1/2} (a^2 + \lambda) U_x = \text{a constant}$$

and so

$$U = D \int_\lambda^\infty \frac{du}{(a^2 + u) \{\Delta(u)\}^{1/2}} + E,$$

where  $D$  and  $E$  are constants of integration. The requirement that  $U$  be null at infinity (i.e., at  $\lambda = \infty$ ) makes  $E$  zero and so

$$U = D \int_\lambda^\infty \frac{du}{(a^2 + u) \{\Delta(u)\}^{1/2}} = D \int_\lambda^\infty \frac{du}{(a^2 + u)^{1/2} (b^2 + u)^{1/2} (c^2 + u)^{1/2}}.$$

**Example.** The potential of the electrostatic field due to an earthed conducting ellipsoid in the presence of a uniform field

We take, in the first instance, the uniform field to be a field of intensity  $E_x$  parallel to the  $x$ -axis, where  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$  is an equation of the ellipsoid. The inducing potential is, accordingly,  $-E_x x$ .

*Note.* This inducing potential is not null at infinity. We ignore this fact, however, and assume that in a sufficiently large neighborhood of the ellipsoidal conductor the potential of the inducing field is approximated, to a sufficient degree of accuracy, by  $-E_x x$ . The fact that this potential is not null at  $x = \pm \infty$  means simply that it is impossible to realize an electrostatic field which is uniform over the entire  $x$ -axis.

The function  $V = -E_x x + AxU$  is harmonic outside the ellipsoid, and the part  $AxU$  is null at  $\infty$ . In fact the development of  $(a^2 + u)^{-3/2}(b^2 + u)^{-1/2}(c^2 + u)^{-1/2}$  near  $u = \infty$  starts out with the term  $u^{-3/2}$  so that the development of  $U(\lambda)$  near  $\lambda = \infty$  starts out with the term  $\frac{2}{3}\lambda^{-3/2}$ . Since  $\frac{r}{\lambda^{3/2}}$  has the limit 1 at  $\lambda = \infty$  it follows that  $rU$  has the limit zero at  $r = \infty$ ; in fact  $r^2U$  is null at  $r = \infty$  so that  $xU$  is the potential of a distribution of electric charge for which the total charge is zero (since  $rxU$  is null at  $r = \infty$ ). In order that  $V$  may have the constant value zero over the conductor  $\lambda = 0$  we have merely to set

$$A = E_x \div \int_0^\infty \frac{du}{(a^2 + u)\{\Delta(u)\}^{1/2}}.$$

Hence, by the uniqueness theorem, the field outside the conductor has its potential furnished by the formula

$$V = -E_x x \left\{ 1 - \frac{\int_\lambda^\infty \frac{du}{(a^2 + u)^{3/2}(b^2 + u)^{1/2}(c^2 + u)^{1/2}}}{\int_0^\infty \frac{du}{(a^2 + u)^{3/2}(b^2 + u)^{1/2}(c^2 + u)^{1/2}}} \right\}.$$

The potential of the field outside the earthed conductor when the potential of the uniform inducing field is  $-E_x x - E_y y - E_z z$  is the sum of the potential just given and the two similar potentials

$$-E_{xy} \left\{ 1 - \frac{\int_{\lambda}^{\infty} \frac{du}{(a^2 + u)^{1/2}(b^2 + u)^{3/2}(c^2 + u)^{1/2}}}{\int_0^{\infty} \frac{du}{(a^2 + u)^{1/2}(b^2 + u)^{3/2}(c^2 + u)^{1/2}}} \right\};$$

$$-E_{zz} \left\{ 1 - \frac{\int_{\lambda}^{\infty} \frac{du}{(a^2 + u)^{1/2}(b^2 + u)^{1/2}(c^2 + u)^{3/2}}}{\int_0^{\infty} \frac{du}{(a^2 + u)^{1/2}(b^2 + u)^{1/2}(c^2 + u)^{3/2}}} \right\}.$$

If the conductor, instead of being earthed, is at potential  $V_0$  we simply add to the potential just given the potential of an isolated ellipsoidal conductor at potential  $V_0$ , namely,

$$V_0 \frac{\int_{\lambda}^{\infty} \frac{du}{\{\Delta(u)\}^{1/2}}}{\int_0^{\infty} \frac{du}{\{\Delta(u)\}^{1/2}}}.$$

## 6. Concluding remarks

The problem of determining the capacity of a conductor has been solved for very few conductors. No one has yet been able to determine the capacity of a conductor whose bounding surface is a cube or the capacity of a right circular cylinder. Recently progress has been made in the direction of determining limits between which the capacity of a conductor must lie. An interesting account of this progress may be found in the paper by G. Pólya: "Estimating Electrostatic Capacity," *American Mathematical Monthly*, **54**: 201-206, 1947. For instance the capacity of a cube of edge  $a$  lies between  $0.62a$  and  $0.72a$ .

# 6

## SPHERICAL HARMONICS AND BESSEL FUNCTIONS

### 1. The power series solution of Legendre's equation

We have seen in Chapter 5, Section 2 that the solutions of Laplace's equation  $\Delta_2 V = 0$  in which the space polar coordinates  $(r, \theta, \phi)$  are completely separable are of the form

$$V = \frac{r^n}{r^{n-1}} \left\{ \begin{array}{l} P_n^m(\mu) \\ Q_n^m(\mu) \end{array} \right\} e^{\pm im\phi},$$

where  $\mu = \cos \theta$  and  $m$  is quantized (by the requirement that  $V$  be a uniform point-function) to be a (non-negative) integer. The functions  $P_n^m(\mu)$ ,  $Q_n^m(\mu)$  are linearly independent solutions of the equation

$$(1 - \mu^2)M_{\mu\mu} - 2\mu M_\mu + \left\{ n(n+1) - \frac{m^2}{1 - \mu^2} \right\} M = 0.$$

We shall consider first the case  $m = 0$  (so that  $V$  is independent of  $\phi$ ), and we shall denote  $P_n^0(\mu)$ ,  $Q_n^0(\mu)$  simply by  $P_n(\mu)$ ,  $Q_n(\mu)$ , respectively. On denoting differentiation with respect to  $\mu$  by the symbol "D" the differential equation of which  $P_n(\mu)$  and  $Q_n(\mu)$  are linearly independent solutions is *Legendre's equation*:

$$(1 - \mu^2)D^2M - 2\mu DM + n(n+1)M = 0;$$

$\mu = \cos \theta$  is a *real* number which is restricted to lie in the interval  $[-1, 1]$ . It is better, in discussing the differential equation, to ignore this restriction and to regard  $\mu$  as a *complex* variable. Since Legendre's equation appears, when solved for  $D^2M$ , in the form

$$D^2M = \frac{2\mu}{1-\mu^2} DM - \frac{n(n+1)}{1-\mu^2} M$$

the point  $\mu = 0$  is a *regular* point of the differential equation (by which we mean, simply, that the coefficients  $\frac{2\mu}{(1-\mu^2)}$  and  $-\frac{n(n+1)}{(1-\mu^2)}$  are analytic (what does this mean?) at  $\mu = 0$ ). It follows from the general existence theorem for linear differential equations (in the complex field) that

*Legendre's equation possesses over a neighborhood of the point  $\mu = 0$  two linearly independent solutions each of which is analytic at  $\mu = 0$  and, hence, developable in a series of powers of  $\mu$ .*

*Note 1.* Since the coefficients  $\frac{2\mu}{(1-\mu^2)}$ ,  $-\frac{n(n+1)}{(1-\mu^2)}$  are analytic over the (open) circle  $|\mu| < 1$  the general existence theorem assures us that the two linearly independent solutions which are analytic at  $\mu = 0$  are analytic over the (open) circle  $|\mu| < 1$ .

*Note 2.* Since  $n(n+1)$  is unchanged when  $n$  is replaced by  $-n-1$  and since the average of  $n$  and  $-n-1$  is  $-\frac{1}{2}$  there is no lack of generality involved in the assumption that  $n \geq -\frac{1}{2}$ , and we shall make this assumption. In particular if  $n$  is an integer it is a *non-negative* integer.

In order to obtain the two linearly independent solutions of Legendre's equation which are analytic over the circle  $|\mu| < 1$  we set

$$M = c_0\mu^\alpha + c_1\mu^{\alpha+1} + \cdots + c_k\mu^{\alpha+k} + \cdots,$$

where  $c_0 \neq 0$  and  $\alpha$  is an as-yet-undetermined non-negative integer. On substituting this expression for  $M$  in Legendre's differential equation (remember that power series may be differentiated term by term) and collecting together like powers of  $\mu$  (this is permissible since power series converge absolutely) we obtain a power series whose sum is zero over the (open) circle  $|\mu| < 1$ , namely,

$$\begin{aligned} & \alpha(\alpha-1)c_0\mu^{\alpha-2} + (\alpha+1)\alpha c_1\mu^{\alpha-1} \\ & + \{(\alpha+2)(\alpha+1)c_2 + [n(n+1) - \alpha(\alpha+1)]c_0\}\mu^\alpha \\ & + \cdots + \{(\alpha+j+2)(\alpha+j+1)c_{\alpha+j+2} \\ & + [n(n+1) - (\alpha+j)(\alpha+j+1)]c_{\alpha+j}\}\mu^{\alpha+j} + \cdots \end{aligned}$$

Since every coefficient of this power series must be zero (why?) we obtain the following series of equations:



$$\alpha(\alpha - 1)c_0 = 0;$$

$$(\alpha + 1)\alpha c_1 = 0;$$

$$(\alpha + 2)(\alpha + 1)c_2 + \{n(n + 1) - \alpha(\alpha + 1)\}c_0 = 0;$$

$$(\alpha + j + 2)(\alpha + j + 1)c_{j+2} + \{n(n + 1) - (\alpha + j)(\alpha + j + 1)\}c_j = 0;$$

$j = 1, 2, \dots$

Since  $c_0 \neq 0$  we learn from the first of these equations that  $\alpha$  must be *either* 0 or 1. The second equation tells us nothing about  $c_1$  if  $\alpha = 0$  (in other words  $c_1$  is arbitrary if  $\alpha = 0$ ) while it tells us that  $c_1 = 0$  if  $\alpha = 1$ . Making the choice  $\alpha = 0$  the third of our series of equations tells us that

$$c_2 = -\frac{n(n + 1)}{1 \cdot 2} c_0.$$

The fourth of our equations tells us that

$$c_3 = -\frac{1}{2 \cdot 3} \{n(n + 1) - 2\}c_1 = -\frac{(n - 1)(n + 2)}{3!} c_1,$$

and so on.. In general, we find that  $c_{j+2}$  is a multiple of  $c_j$ :

$$c_{j+2} = -\frac{\{n(n + 1) - j(j + 1)\}}{(j + 1)(j + 2)} c_j = -\frac{(n - j)(n + j + 1)}{(j + 1)(j + 2)} c_j.$$

Thus

$$c_4 = -\frac{(n - 2)(n + 3)}{3 \cdot 4} c_2 = \frac{(n - 2)n(n + 1)(n + 3)}{4!} c_0;$$

$$c_5 = -\frac{(n - 3)(n + 4)}{4 \cdot 5} c_3 = \frac{(n - 3)(n - 1)(n + 2)(n + 4)}{5!} c_1,$$

and, generally,

$$c_{2j} = (-1)^j \frac{(n - 2j + 2)(n - 2j) \cdots n(n + 1)(n + 3) \cdots (n + 2j - 1)}{(2j)!} c_0;$$

$$c_{2j+1} = (-1)^j \frac{(n - 2j + 1)(n - 2j - 1) \cdots (n - 1)(n + 2)(n + 4) \cdots (n + 2j)}{(2j + 1)!} c_1;$$

$$j = 1, 2, \dots$$

In this way we obtain the solution

$$M = c_0 M_1 + c_1 M_2$$

of Legendre's equation, where

$$M_1 = 1 - \frac{n(n+1)}{2!} \mu^2 + \frac{(n-2)n(n+1)(n+3)}{4!} \mu^4 - \dots;$$

$$M_2 = \mu - \frac{(n-1)(n+2)}{3!} \mu^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} \mu^5$$

— . . . .

### EXERCISES

1. Show that the choice  $\alpha = 1$  furnishes a solution which is a multiple of  $M_2$ .
2. Show that the two solutions  $M_1$  and  $M_2$  are linearly independent. *Hint.*  $M_1(0) = 1, M_2(0) = 0$ .

If  $n$  is an integer it is non-negative (why?), and *one* of the two solutions  $M_1, M_2$  of Legendre's equation is a polynomial function of  $\mu$  (but *not both*). If  $n$  is not an integer no solution of Legendre's equation is a polynomial function of  $\mu$ , for, then, neither  $M_1$  nor  $M_2$  is a polynomial, and no linear combination of them can be a polynomial since  $M_1$  involves only even powers of  $\mu$  while  $M_2$  involves only odd powers of  $\mu$ . If  $n$  is an even integer,  $M_1$  is an *even* polynomial of degree  $n$ , i.e., a polynomial function of  $\mu$ , of degree  $n$ , involving only even powers of  $\mu$  while, if  $n$  is an odd integer,  $M_2$  is an *odd* polynomial of degree  $n$ , i.e., a polynomial function of  $\mu$ , of degree  $n$ , involving only odd powers of  $\mu$ .

### EXERCISE

3. Show that any polynomial solution of Legendre's equation is a multiple of  $M_1$  if  $n$  is an even integer or a multiple of  $M_2$  if  $n$  is an odd integer. *Hint.* Any solution is a linear combination of  $M_1$  and  $M_2$ , and  $M_1$  and  $M_2$  are not *both* polynomials.

When  $n$  is an integer we may obtain a polynomial function of  $\mu$  which is a solution of Legendre's equation by the following device. First construct the polynomial function of  $\mu$ , of degree  $2n$ ,  $(\mu^2 - 1)^n$ ; differentiate this  $n$  times with respect to  $\mu$ . The resulting polynomial of degree  $n$

$$D^n(\mu^2 - 1)^n$$

is a solution of Legendre's equation. To see this set

$$\xi = (\mu^2 - 1)^n;$$

then  $\log \xi = n \log (\mu^2 - 1)$  so that

$$\frac{D\xi}{\xi} = \frac{2n\mu}{\mu^2 - 1}$$

or, equivalently,

$$(\mu^2 - 1)D\xi - 2n\mu\xi = 0.$$

Differentiating this relation  $(n + 1)$  times we obtain

$$(\mu^2 - 1)D^{n+2}\xi + 2(n + 1)\mu D^{n+1}\xi + (n + 1)nD^n\xi - 2n\mu D^{n+1}\xi - 2(n + 1)nD^n\xi = 0;$$

i.e.,

$$(\mu^2 - 1)D^{n+2}\xi + 2\mu D^{n+1}\xi - (n + 1)nD^n\xi = 0.$$

Hence  $M = D^n\xi = D^n(\mu^2 - 1)^n$  satisfies Legendre's equation

$$(1 - \mu^2)D^2M - 2\mu DM + (n + 1)nM = 0.$$

It follows that, if  $n$  is even,  $D^n(\mu^2 - 1)^n$  is a multiple of  $M_1$  or, equivalently (why?), that  $M_1$  is a multiple of  $D^n(\mu^2 - 1)^n$  and that, if  $n$  is odd,  $M_2$  is a multiple of  $D^n(\mu^2 - 1)^n$ . On writing  $D^n(\mu^2 - 1)^n = D^n\{(\mu - 1)^n(\mu + 1)^n\} = n!(\mu + 1)^n +$  terms having  $(\mu - 1)$  as a factor, we see that  $D^n(\mu^2 - 1)^n$  has the value  $2^n n!$  at  $\mu = 1$ . It follows then that

*No polynomial solution of Legendre's equation is zero at  $\mu = 1$ .* This result enables us to *normalize* the polynomial solutions of Legendre's equation as follows: The normalized polynomial solutions of Legendre's equations are those solutions which have the value 1 at  $\mu = 1$ .

These normalized polynomial solutions of Legendre's equation (there being one for each non-negative integral value of  $n$ ) are known as *Legendre polynomials* and are denoted by the symbol  $P_n(\mu)$ ,  $n = 0, 1, 2, \dots$ .

### EXERCISES

4. Show that  $P_n(-1) = 1$ , if  $n$  is even, while  $P_n(-1) = -1$ , if  $n$  is odd. *Hint.*  $P_n$  is an even, or odd, polynomial according as  $n$  is even or odd.

5. Show that  $P_n(\mu) = \frac{1}{2^n(n!)} D^n(\mu^2 - 1)^n$ .

6. Show that  $P_0(\mu) = 1$ ,  $P_1(\mu) = \mu$ ,  $P_2(\mu) = \frac{3}{2}(\mu^2 - \frac{1}{2})$ ,  $P_3(\mu) = \frac{5}{2}(\mu^3 - \frac{3}{2}\mu)$ ,  $P_4(\mu) = \frac{35}{8}(\mu^4 - \frac{6}{7}\mu^2 + \frac{3}{8})$ ,  $P_5(\mu) = \frac{63}{8}(\mu^5 - \frac{15}{8}\mu^3 + \frac{5}{16}\mu)$ . *Hint.* Use the power series for  $M_1$  and  $M_2$  rather than the formula of Exercise 5.

7. Show that the coefficient of  $\mu^n$  in  $P_n(\mu)$  is

$$\frac{(2n!)}{2^n(n!)^2} = \frac{(2n-1)(2n-3)\cdots 5 \cdot 3}{n!}.$$

*Hint.* Use the formula of Exercise 5.

8. Show that  $P_n(\mu) = \frac{(2n-1)(2n-3)\cdots 5\cdot 3}{n!} \left\{ \mu^n - \frac{n(n-1)}{2(2n-1)} \mu^{n-2} + \frac{(n-2)n(n-1)(n-3)}{2\cdot 4(2n-1)(2n-3)} \mu^{n-4} - \cdots \right\}$ . *Hint.* Use the relation  $\frac{c_j}{c_{j+2}} = \frac{(n-j)(n+j+1)}{2(j+2)(j+1)}$  connecting the coefficients  $c_j$  and  $c_{j+2}$  of the general power series solution of Legendre's equation.

9. Show that  $P_6(\mu) = \frac{2\cdot 3\cdot 1}{1\cdot 6}(\mu^6 - \frac{1\cdot 5}{1\cdot 1}\mu^4 + \frac{5}{1\cdot 1}\mu^2 - \frac{5}{2\cdot 3}\mu)$ .

10. Show that  $P_{2n}(0) = \frac{(-1)^n (2n)!}{2^{2n} (n!)^2}$ .

11. Show that  $\lim_{\mu \rightarrow 0} \frac{P_{2n+1}(\mu)}{\mu} = \frac{(-1)^n (2n+2)!}{2^{2n+1} n!(n+1)!}$

12. Show that all the zeros of  $P_n(\mu)$  are real and simple and that they are covered by the open interval  $(-1, 1)$ . *Hint.*  $(\mu^2 - 1)^n$  has a zero of multiplicity  $n$  at  $\mu = -1$  and a zero of multiplicity  $n$  at  $\mu = +1$ . Hence  $D(\mu^2 - 1)^n$  has a zero of multiplicity  $(n-1)$  at  $\mu = -1$ , a (simple) zero in the open interval  $(-1, 1)$ , and a zero of multiplicity  $(n-1)$  at  $\mu = +1$ . Continuing this argument we see that  $D^n(\mu^2 - 1)^n$  has  $n$  simple zeros which are covered by the open interval  $(-1, 1)$ .

13. Show that any polynomial of degree  $n$  is a linear combination of the polynomials  $P_0 = 1, P_1, \dots, P_n$ .

14. Show that  $\mu^0 = P_0, \mu^1 = P_1, \mu^2 = \frac{2}{3}P_2 + \frac{1}{3}P_0, \mu^3 = \frac{2}{5}P_3 + \frac{3}{5}P_1, \mu^4 = \frac{8}{35}P_4 + \frac{7}{5}P_2 + \frac{1}{5}P_0, \mu^5 = \frac{8}{63}P_5 + \frac{8}{9}P_3 + \frac{2}{3}P_1$ . *Note.* The sum of the coefficients in each of these linear combinations must be 1 since all the  $P_n$  are 1 at  $\mu = 1$ .

## 2. The second solution of Legendre's equation when $n$ is an integer

When  $n$  is an even integer

$$M_1 = 1 - \frac{n(n+1)}{2!} \mu^2 + \frac{(n-2)n(n+1)(n+3)}{4!} \mu^4 - \dots$$

is a constant multiple of  $P_n(\mu)$  while, when  $n$  is an odd integer,

$$M_2 = \mu - \frac{(n-1)(n+2)}{3!} \mu^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} \mu^5 - \dots$$

is a constant multiple of  $P_n(\mu)$ . Let us denote by  $M$ , simply,  $M_2$  when  $n$  is an even integer and  $M_1$  when  $n$  is an odd integer. Then the function

$\frac{M}{P_n}$  is an odd function of  $\mu$  (why?), and it is easy to verify that the

derivative of this odd function is the product of  $\frac{1}{(1-\mu^2)P_n^2}$  by a non-zero constant. To do this we combine the two equations

$$(1 - \mu^2)D^2M - 2\mu DM + n(n+1)M = 0$$

$$(1 - \mu^2)D^2P_n - 2\mu DP_n + n(n+1)P_n = 0$$

as follows: Multiply the first equation by  $P_n$ , the second by  $-M$ , and add. We obtain

$$(1 - \mu^2)(P_n D^2 M - M D^2 P_n) - 2\mu(P_n D M - M D P_n) = 0.$$

Since  $P_n D^2 M - M D^2 P_n = D(P_n D M - M D P_n)$  and  $-2\mu = D(1 - \mu^2)$  it follows that  $(1 - \mu^2)(P_n D M - M D P_n)$  is a constant (why?). The constant cannot be zero since, if it were,  $P_n D M - M D P_n = P_n^2 D\left(\frac{M}{P_n}\right)$  would be zero, and this is absurd since  $M$  is not a constant multiple of  $P_n$  (why?). Hence

$$D\left(\frac{M}{P_n}\right) = \frac{C}{(1 - \mu^2)P_n^2(\mu)}; \quad C \neq 0.$$

Let the  $n$  (real and simple) zeros of  $P_n$  be  $\alpha_1, \alpha_2, \dots, \alpha_n$ . Then the analysis of  $\frac{1}{(1 - \mu^2)P_n^2(\mu)}$  into simple fractions is of the form

$$\frac{1}{(1 - \mu^2)P_n^2(\mu)} = \frac{\frac{1}{2}}{1 - \mu} + \frac{\frac{1}{2}}{1 + \mu} + \sum_1^n \left\{ \frac{A_j}{(\mu - \alpha_j)^2} + \frac{B_j}{\mu - \alpha_j} \right\}$$

(remember that the value of  $P_n^2(\mu)$  at each of the points  $\pm 1$  is 1). It is easy to see that  $B_j = 0, j = 1, 2, \dots, n$ . In fact  $|\alpha_j| < 1$  so that  $M$  is analytic at  $\mu = \alpha_j$ ; hence, since  $\frac{M}{P_n}$  does not have a logarithmic singularity at  $\mu = \alpha_j$ ,  $D\left(\frac{M}{P_n}\right)$  has no  $\frac{1}{\mu - \alpha_j}$  term in its development in a Laurent series about its isolated singularity  $\mu = \alpha_j$ . We have, then,

$$D\left(\frac{M}{P_n}\right) = C \left\{ \frac{\frac{1}{2}}{1 - \mu} + \frac{\frac{1}{2}}{1 + \mu} + \sum_1^n \frac{A_j}{(\mu - \alpha_j)^2} \right\}$$

and so

$$\frac{M}{P_n} = C \left\{ \frac{1}{2} \log \frac{1 + \mu}{1 - \mu} - \sum_1^n \frac{A_j}{\mu - \alpha_j} + \text{constant} \right\}.$$

Hence

$$\frac{M}{P_n} = C \left\{ \frac{1}{2} \log \frac{1 + \mu}{1 - \mu} + \frac{W_{n-1}(\mu)}{P_n} + \text{constant} \right\},$$

where  $W_{n-1}(\mu)$  is a polynomial function of  $\mu$  of degree not greater than  $n - 1$  (remember that  $P_n(\mu)$  is a constant multiple of  $(\mu - \alpha_1) \cdots (\mu - \alpha_n)$ ). Since  $\frac{1}{2} \log \frac{(1 + \mu)}{(1 - \mu)}$  is an odd function of  $\mu$  (prove this) as is also  $\frac{M}{P_n}$  (why?) it follows that  $\frac{W_{n-1}(\mu)}{P_n(\mu)} + \text{constant}$  is an odd function of  $\mu$ . The additive constant must, then, be zero; in fact it is  $-\frac{1}{2} \left\{ \frac{W_{n-1}(-\mu)}{P_n(-\mu)} + \frac{W_{n-1}(\mu)}{P_n(\mu)} \right\}$  which is the quotient of a polynomial of degree  $< n$  by a polynomial of degree  $n$ , and such a quotient is constant only when it is zero (why?). We have, then,

$$M = C \left\{ \frac{1}{2} P_n \log \frac{1 + \mu}{1 - \mu} + W_{n-1} \right\},$$

and it follows that when  $n$  is even  $W_{n-1}$  is an odd polynomial and that when  $n$  is odd  $W_{n-1}$  is an even polynomial (remember that  $\frac{1}{2} \log \frac{(1 + \mu)}{(1 - \mu)}$  is an odd function and that when  $n$  is even  $M$  is an odd function while when  $n$  is odd  $M$  is an even function). We denote by  $Q_n$  the function

$$Q_n = Q_n(\mu) = \frac{1}{2} P_n \log \frac{1 + \mu}{1 - \mu} + W_{n-1};$$

$Q_n$  is a solution of Legendre's equation which is not a constant multiple of  $P_n$ . The general solution of Legendre's equation (when  $n$  is a non-negative integer) is, accordingly, a linear combination of  $P_n$  and  $Q_n$ .

*Note.* The case  $n = 0$  is somewhat special since  $P_0 = 1$  has no zeros. Here

$$Q_0 = \frac{1}{2} \log \frac{1 + \mu}{1 - \mu},$$

and in order to fit this into the general formula

$$Q_n = \frac{1}{2} P_n \log \frac{1 + \mu}{1 - \mu} + W_{n-1}$$

we understand that  $W_{-1}$  is the zero constant function.

The polynomial function  $W_{n-1}$  is best determined by the fact that  $M$  is a constant multiple of  $\frac{1}{2}P_n \log \frac{1+\mu}{1-\mu} + W_{n-1} = P_n \left( \mu + \frac{\mu^3}{3} + \frac{\mu^5}{5} + \dots \right) + W_{n-1}$ . For example, when  $n = 1$ ,  $M = M_1 = 1 - \mu^2 - \frac{\mu^4}{3} + \dots$  is a constant multiple of  $\left\{ \mu^2 + \frac{\mu^4}{3} + \dots \right\} + W_0$ . Hence  $W_0 = -1$  (and the constant multiple is  $-1$ ). Thus

$$Q_1 = \frac{1}{2} \mu \log \frac{1+\mu}{1-\mu} - 1 = -1 + \mu^2 + \frac{\mu^4}{3} + \dots$$

When  $n = 2$ ,  $\frac{1}{2}P_2 \log \frac{1+\mu}{1-\mu} = \frac{3}{2} \left( \mu^2 - \frac{1}{3} \right) \left( \mu + \frac{\mu^3}{3} + \dots \right) = -\frac{1}{2} \mu + \frac{4}{3} \mu^3 + \dots$  and  $M = M_2 = \mu - \frac{2}{3} \mu^3 + \dots$ . Hence the constant multiple is  $-\frac{1}{2}$ , and  $W_1 = -\frac{3}{2} \mu$ . Hence

$$Q_2(\mu) = \frac{1}{2} P_2 \log \frac{1+\mu}{1-\mu} - \frac{3}{2} \mu = -2\mu + \frac{4}{3} \mu^3 + \dots$$

### EXERCISES

1. Verify that  $Q_1(\mu) = \frac{1}{2} \mu \log \frac{1+\mu}{1-\mu} - 1$  is a solution of Legendre's equation when  $n = 1$ .

2. Verify that  $Q_2(\mu) = \frac{3}{4}(\mu^2 - \frac{1}{3}) \log \frac{1+\mu}{1-\mu} - \frac{3}{2} \mu$  is a solution of Legendre's equation when  $n = 2$ .

3. Show that  $W_2(\mu) = \frac{2}{3} - \frac{5}{2} \mu^2$ .

4. Verify that  $Q_3(\mu) = \frac{5}{4}(\mu^3 - \frac{3}{5}\mu) \log \frac{1+\mu}{1-\mu} + \frac{2}{3} - \frac{5}{2} \mu^2$  is a solution of Legendre's equation when  $n = 3$ .

5. Obtain the power series development of the solutions of Legendre's equation near  $\mu = \infty$ . *Hint.* Set  $\mu = \frac{1}{\xi}$ , and show that Legendre's equation takes the form  $\xi^2(\xi^2 - 1)M_{\xi\xi} + 2\xi^3 M_{\xi} + n(n+1)M = 0$ . On setting  $M = c_0 \xi^\alpha + c_1 \xi^{\alpha+1} + \dots$  we find that  $\alpha = -n$  or  $n+1$ . The choice  $\alpha = -n$  gives a multiple of  $P_n$  while the choice  $\alpha = n+1$  gives a linear combination of  $P_n$  and  $Q_n$ .

*Answer.* The solution which is not a polynomial function of  $\mu = \frac{1}{\xi}$  is  $C_0 \left\{ \xi^{n+1} + \frac{(n+1)(n+2)}{2(2n+3)} \xi^{n+3} + \frac{(n+1)(n+2)(n+3)(n+4)}{(2)(4)(2n+3)(2n+5)} \xi^{n+5} + \dots \right\}$ .

6. Obtain the general formula for  $W_{n-1}$  where  $Q_n = \frac{1}{2}P_n \log \frac{1+\mu}{1-\mu} + W_{n-1}$ .

*Hint.* Consider the development of  $Q_n$  near  $\xi = 0$ , where  $\xi = \frac{1}{\mu}$ .  $Q_n$  differs from  $\frac{1}{2}P_n \left(\frac{1}{\xi}\right) \log \frac{1+\xi}{1-\xi} + W_{n-1} \left(\frac{1}{\xi}\right)$  by a constant multiple of  $P_n \left(\frac{1}{\xi}\right)$ . Since  $\frac{1}{2} \log \frac{1+\xi}{1-\xi} = \xi + \frac{\xi^3}{3} + \dots$  the lowest power of  $\xi$  in the development of  $\frac{1}{2}P_n \left(\frac{1}{\xi}\right) \log \frac{1+\xi}{1-\xi}$  is the  $-(n-1)$ st. Hence the lowest power of  $\xi$  in the development of  $\frac{1}{2}P_n \left(\frac{1}{\xi}\right) \log \frac{1+\xi}{1-\xi} + W_{n-1} \left(\frac{1}{\xi}\right)$  is the  $-(n-1)$ st. The solution given in Exercise 5 is a linear combination of  $\frac{1}{2}P_n \left(\frac{1}{\xi}\right) \log \frac{1+\xi}{1-\xi} + W_{n-1} \left(\frac{1}{\xi}\right)$  and  $P_n \left(\frac{1}{\xi}\right)$ , and the coefficient of  $P_n \left(\frac{1}{\xi}\right)$  in this linear combination must be zero since there is no term involving  $\left(\frac{1}{\xi}\right)^n$  in the solution in Exercise 5. Since this solution starts out with the term  $\xi^{n+1}$ ,  $W_{n-1} \left(\frac{1}{\xi}\right)$  is such as to cancel the negative power terms in  $\frac{1}{2}P_n \left(\frac{1}{\xi}\right) \log \frac{1+\xi}{1-\xi}$ . Use the formula for  $P_n(\mu)$  given in Exercise 8, p. 171.

*Answer.* 
$$W_{n-1}(\mu) = -\frac{(2n-1)(2n-3) \cdots (5)(3)}{n!} \left\{ \mu^{n-1} + \mu^{n-3} \left( \frac{1}{3} - \frac{n(n-1)}{2(2n-1)} \right) + \mu^{n-5} \left( \frac{1}{5} - \frac{n(n-1)}{2(2n-1)} \frac{1}{3} + \frac{(n-2)n(n-1)(n-3)}{(2)(4)(2n-1)(2n-3)} \right) + \cdots \right\}.$$

**3. Surface harmonics; the orthogonality relation**

We have seen that any solution of Laplace's equation  $\Delta_2 V = 0$ , in which the space polar coordinate  $r$  is separable from the angular coordinates  $(\theta, \phi)$ , is of the form

$$V = \left. \begin{matrix} r^n \\ r^{-n-1} \end{matrix} \right\} U,$$

where  $U$  is a function of  $(\theta, \phi)$  which satisfies the equation

$$\Delta_2^* U \equiv U_{\theta\theta} + \cot \theta U_\theta + \frac{1}{\sin^2 \theta} U_{\phi\phi} + n(n+1)U = 0.$$

We term  $U$  a *surface harmonic*, and when we wish to indicate the value of  $n$  with which  $U$  is associated we do this by means of a sub-



script. Thus if  $U_n$  is a surface harmonic then  $r^n U_n$  and  $r^{-n-1} U_n$  are solutions of Laplace's equation.

Let, now,  $\sum_0^{\infty} c_n r^n$  be a power series in  $r$  whose coefficients  $c_n$ ,  $n = 0$ ,

$1, 2, \dots$ , are functions of the angular coordinates  $(\theta, \phi)$ . If the sum of the power series is a solution  $V$  of Laplace's equation and if term by term differentiation is permissible for the differentiations involved in  $\Delta_2^* V$  then each coefficient  $c_n$  of the power series is a surface harmonic,  $c_n$  being associated with  $n$ . In fact, since

$$\Delta_2 V = \sum_0^{\infty} r^{n-2} \Delta_2^* c_n = 0,$$

$\Delta_2^* c_n = 0$ ,  $1, 2, \dots$ . An important instance of this result is the following: Denote by  $R$  the distance from a point  $(0, 0, h)$  on the positive  $z$ -axis to a variable point  $(x, y, z)$  in space. Then  $\frac{1}{R} = \{x^2 + y^2 + (z - h)^2\}^{-1/2}$  is a solution of Laplace's equation, as is at once seen by introducing a system of space polar coordinates whose origin is at  $(0, 0, h)$ . Since  $z = r \cos \theta = r\mu$  we have

$$\frac{1}{R} = (r^2 - 2r\mu h + h^2)^{-1/2} = \frac{1}{h} \left( 1 - 2\frac{r}{h}\mu + \frac{r^2}{h^2} \right)^{-1/2}.$$

Let  $r$  be so small that  $2\left(\frac{r}{h}\right)|\mu| + \left(\frac{r^2}{h^2}\right) < 1$ ; then, if  $\xi = 2\left(\frac{r}{h}\right)\mu - \left(\frac{r^2}{h^2}\right)$ ,  $|\xi| < 1$ , and

$$\frac{1}{R} = \frac{1}{h} (1 - \xi)^{-1/2} = \frac{1}{h} \left( 1 + \frac{1}{2}\xi + \frac{1 \cdot 3}{2 \cdot 4}\xi^2 + \dots \right).$$

The series

$$1 + \frac{1}{2} \left( 2\frac{r}{h}\mu - \frac{r^2}{h^2} \right) + \frac{1 \cdot 3}{2 \cdot 4} \left( 4\frac{r^2}{h^2}\mu^2 - 4\frac{r^3}{h^3}\mu + \frac{r^4}{h^4} \right) + \dots$$

obtained by expanding the various powers of  $\xi$  is absolutely convergent (because the series of absolute values is the series for  $\frac{1}{h} (1 - \eta)^{-1/2}$ ,

where  $\eta = 2\left(\frac{r}{h}\right)|\mu| + \left(\frac{r^2}{h^2}\right)$ , and any partial sum of this series is less

than  $\frac{1}{h}(1 - \eta)^{-\frac{1}{2}}$ . Hence the terms involving like powers of  $r$  may be collected, and we obtain

$$\frac{1}{R} = c_0 + c_1 r + c_2 r^2 + \dots,$$

where

$$c_0 = \frac{1}{h}, \quad c_1 = \frac{1}{h^2} \mu, \quad c_2 = \frac{3}{2h^3} \left( \mu^2 - \frac{1}{3} \right), \quad \text{and so on.}$$

Since the series for  $\frac{1}{R}$  may also be written as a power series in  $\mu$ , repeated term-by-term differentiation with respect to  $\mu$  is permissible. Hence each of the coefficients  $c_0, c_1, \dots, c_n, \dots$  of the series for  $\frac{1}{R}$  (as a power series in  $r$ ) is a surface harmonic; since these coefficients are independent of  $\phi$ ,  $c_n$  satisfies Legendre's equation

$$(1 - \mu^2)D^2c_n - 2\mu Dc_n + n(n+1)c_n = 0.$$

Since  $c_n$  is a polynomial function of  $\mu$  it is a constant multiple of  $P_n$ . When  $\mu = 1$  we have

$$\frac{1}{R} = \frac{1}{h-r} = \frac{1}{h} + \frac{r}{h^2} + \dots + \frac{r^n}{h^{n+1}} + \dots$$

so that the value of  $c_n$  when  $\mu = 1$  is  $\frac{1}{h^{n+1}}$ . Hence

$$c_n = \frac{P_n(\mu)}{h^{n+1}}$$

so that

$$\frac{1}{R} = \frac{P_0(\mu)}{h} + \frac{P_1(\mu)}{h^2} r + \dots + \frac{P_n(\mu)}{h^{n+1}} r^n + \dots$$

*Note.* It was precisely in order to have this simple formula that the Legendre polynomials were normalized by the requirement that they should all have the value 1 when  $\mu = 1$ .

### EXERCISE

1. Show that if  $r$  is so large that  $2 \left( \frac{h}{r} \right) |\mu| + \left( \frac{h^2}{r^2} \right) < 1$  then

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$$\frac{1}{R} = \frac{P_0(\mu)}{r} + \frac{hP_1(\mu)}{r^2} + \cdots + \frac{h^n P_n(\mu)}{r^{n+1}} + \cdots$$

Note.  $\frac{1}{R} = (r - he^{i\theta})^{-1/2}(r - he^{-i\theta})^{-1/2}$  so that the only singularities of  $\frac{1}{R}$ , regarded as a function of the complex variable  $r$ , are on the circumference  $|r| = h$ . Hence the development

$$\frac{1}{R} = \frac{P_0(\mu)}{h} + \cdots + \frac{P_n(\mu)}{h^{n+1}} r^n + \cdots$$

is valid over the interior of the circle (in the complex  $r$  plane)  $|r| < h$ , and the development

$$\frac{1}{R} = \frac{P_0(\mu)}{r} + \cdots + \frac{h^n P_n(\mu)}{r^{n+1}} + \cdots$$

is valid if  $|r| > h$ .

Let, now,  $j$  and  $k$  be any two different non-negative integers, and let  $U_j$ ,  $U_k$  be surface harmonics which are associated with  $j$  and  $k$ , respectively. Then  $V_j = r^j U_j$ ,  $V_k = r^k U_k$  are solutions of Laplace's equation so that

$$\int_S \left( V_j \frac{d}{dn} V_k - V_k \frac{d}{dn} V_j \right) dS = 0,$$

$S$  being any closed surface (see Exercise 20, p. 31). Taking  $S$  to be a sphere of radius  $a$  with center at  $O$ ,  $\frac{d}{dn} V_k = (V_k)_r$ ,  $\frac{d}{dn} V_j = (V_j)_r$ , and we obtain

$$a^{j+k+1}(k - j) \int U_j U_k d\omega = 0,$$

where  $d\omega = \sin \theta d\theta d\phi$  is the element of area of the unit sphere (i.e., the element of solid angle). Since  $k \neq j$  we have

$$\int U_j U_k d\omega = 0.$$

If  $U_j$  is a complex-valued function of  $\theta$  and  $\phi$  its conjugate complex,  $\bar{U}_j$ , is also a surface harmonic associated with the integer  $j$ . (Prove this.) Hence

$$\int \bar{U}_j U_k d\omega = 0.$$

We express this result as follows:

**Two surface harmonics  $U_j, U_k$  which are associated with different non-negative integers  $j$  and  $k$  are orthogonal over the unit sphere.**

An important special case of this result occurs when  $U_j$  and  $U_k$  are zonal surface harmonics, i.e., surface harmonics which are independent of  $\theta$ . In this case the integration with respect to  $\phi$  is trivial and  $\int U_j U_k d\omega$  reduces to  $2\pi \int_0^\pi U_j U_k \sin \theta d\theta = 2\pi \int_{-1}^1 U_j U_k d\mu$ . Thus, in particular, if  $P_j(\mu)$  and  $P_k(\mu), j \neq k$ , are Legendre polynomials (so that  $\bar{P}_j(\mu) = P_j(\mu)$ ) then

$$\int_{-1}^1 P_j(\mu) P_k(\mu) d\mu = 0.$$

We express this result as follows:

**Any two different Legendre polynomials are orthogonal over the interval  $[-1, 1]$ .**

We can construct from the sequence of Legendre polynomials  $\{P_n\}, n = 0, 1, 2, \dots$ , an orthonormal set as soon as we know the squared magnitude of the function-vector  $P_n$ , i.e., the value of the integral  $\int_{-1}^1 P_n^2(\mu) d\mu$ . To obtain this we consider the solution  $V = \frac{1}{R}$  of Laplace's equation, where  $R = |QP|, Q: (0, 0, h), P: (x, y, z)$  ( $P$  being distinct from  $Q$ ). If  $U_j$  is a surface harmonic associated with any non-negative integer  $j$ , both  $V$  and  $V_j = r^j U_j$  satisfy Laplace's equation over the region between a sphere  $S$  of sufficiently small radius  $\epsilon$  with center at  $Q$  and a sphere  $S'$  of radius  $a > h$  with center at  $O$ . Hence

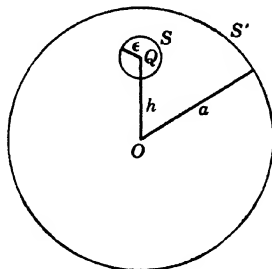


FIG. 36.

$$\int_S \left( V \frac{d}{dn} V_j - V_j \frac{d}{dn} V \right) dS + \int_{S'} \left( V \frac{d}{dn} V_j - V_j \frac{d}{dn} V \right) dS = 0.$$

Over  $S, V = \frac{1}{\epsilon} \frac{d}{dn} V = \frac{1}{\epsilon^2}, dS = \epsilon^2 d\omega$  so that the limit at  $\epsilon = 0$  of

$$\int_S \left( V \frac{d}{dn} V_j - V_j \frac{d}{dn} V \right) dS = -4\pi h^j (U_j)_{\theta=0}$$

(because the value of  $V_j$  at  $Q = h^j (U_j)_{\theta=0}$ ). Over  $S'$ ,

$$V = \sum_0^{\infty} \left( \frac{h^k}{a^{k+1}} \right) P_k(\mu),$$

$$\frac{d}{dn} V_i = ja^{j-1} \bar{U}_i, \quad V_i = a^j \bar{U}_i, \quad \frac{d}{dn} V = - \sum_0^{\infty} (k+1) \frac{h^k}{a^{k+2}} P_k(\mu),$$

$dS = a^2 d\omega$  and so

$$\begin{aligned} \int_{S'} \left( V \frac{d}{dn} V_i - V_i \frac{d}{dn} V \right) dS &= \int \{ hij P_j(\mu) \bar{U}_i + (j+1) h^i P_j(\mu) \bar{U}_i \} d\omega \\ &= (2j+1) h^i \int P_j(\mu) \bar{U}_i d\omega \end{aligned}$$

(remember that the integral of  $P_k(\mu) \bar{U}_i$  over the unit sphere,  $k \neq j$ , is zero). Hence

$$\int P_j(\mu) \bar{U}_i d\omega = \frac{4\pi}{2j+1} (\bar{U}_i)_{\theta=0}.$$

In particular when  $U_i = P_j(\mu)$  we have

$$\int P_j^2(\mu) d\omega = 2\pi \int_{-1}^1 P_j^2 d\mu$$

and so

$$\int_{-1}^1 P_j^2(\mu) d\mu = \frac{2}{2j+1}, \quad \text{since } (P_j)_{\theta=0} = P_j(1) = 1.$$

It follows that the function-vectors

$$\mathbf{u}_n : \left( \frac{2n+1}{2} \right)^{1/2} P_n(\mu); \quad -1 \leq \mu \leq 1$$

constitute an orthonormal set (see Chapter 3, Section 2) which we term the power-function orthonormal set. We have seen (see Chapter 3, Section 4) that this power-function orthonormal set is *complete*. In other words the only function which is continuous over the closed interval  $[-1, 1]$  and which is orthogonal to every vector of the power-function orthonormal set is the zero constant function. This fact serves to *quantize* to non-negative integral values the constant  $n$  which appears in Legendre's equation. In fact if  $k$  is any constant ( $\geq -\frac{1}{2}$ )

which is not a non-negative integer and if  $M$  is any solution of the corresponding Legendre equation

$$(1 - \mu^2)D^2M - 2\mu DM + k(k + 1)M = 0$$

over the *closed* interval  $[-1, 1]$  then  $r^k M$  is a solution of Laplace's equation, and the same argument which proved the orthogonality of  $P_j(\mu)$  and  $P_k(\mu)$ ,  $j \neq k$ , over  $[-1, 1]$  proves the orthogonality of  $M$  and  $P_j(\mu)$ , every  $j = 0, 1, 2, \dots$ . Hence  $M$  is the zero constant function. We have, then, the following fundamental result:

**If Legendre's equation**

$$(1 - \mu^2)D^2M - 2\mu DM + k(k + 1)M = 0$$

**possesses over  $[-1, 1]$  a solution which is not identically zero then  $k$  is a non-negative integer  $n$ , and  $M$  is a constant multiple of  $P_n$ .**

#### 4. The associated Legendre functions

The differential equation which  $P_n^m(\mu)$  and  $Q_n^m(\mu)$  must satisfy is

$$(1 - \mu^2)D^2M - 2\mu DM + \left\{ n(n + 1) - \frac{m^2}{1 - \mu^2} \right\} M = 0.$$

We proceed now to the proof of the following important theorem: If  $N$  is any solution of Legendre's equation

$$(1 - \mu^2)D^2N - 2\mu DN + n(n + 1)N = 0$$

then  $M = (1 - \mu^2)^{(m/2)} D^m N$  is a solution of Legendre's *associated* equation

$$(1 - \mu^2)D^2M - 2\mu DM + \left\{ n(n + 1) - \frac{m^2}{1 - \mu^2} \right\} M = 0.$$

To prove this theorem we calculate  $DM$  and  $D^2M$ :

$$DM = -m\mu(1 - \mu^2)^{(m/2)-1} D^m N + (1 - \mu^2)^{(m/2)} D^{m+1} N;$$

$$D^2M = -m(1 - \mu^2)^{(m/2)-1} D^m N + m(m - 2)\mu^2(1 - \mu^2)^{(m/2)-2} D^m N \\ - 2m\mu(1 - \mu^2)^{(m/2)-1} D^{m+1} N + (1 - \mu^2)^{(m/2)} D^{m+2} N.$$

On substituting these expressions in the left-hand side of Legendre's associated equation we obtain the product of  $(1 - \mu^2)^{(m/2)}$  by

$$(1 - \mu^2)D^{m+2}N - 2(m + 1)\mu D^{m+1}N + (n - m)(n + m + 1)N.$$

It follows, on differentiating Legendre's equation  $m$  times, that this expression is zero, which proves the theorem.

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In particular, when  $n$  is a non-negative integer and  $N = P_n(\mu)$ , we obtain a solution of Legendre's associated equation which we denote by  $P_n^m(\mu)$ :

$$P_n^m(\mu) = (1 - \mu^2)^{(m/2)} D^m P_n.$$

If  $m > n$ ,  $P_n^m(\mu) = 0$  (why?). The solution which is obtained from  $Q_n$  in this way is denoted by  $Q_n^m(\mu)$ :

$$Q_n^m(\mu) = (1 - \mu^2)^{(m/2)} D^m Q_n.$$

For each non-negative integral value of  $n$  we have  $n + 1$  values of  $m$ , namely,  $0, 1, \dots, n$  for which  $P_n^m(\mu)$  is not identically zero. With each of these values of  $m$ , except the value  $m = 0$ , we may associate two surface harmonics  $P_n^m(\mu)e^{im\phi}$  and  $P_n^m(\mu)e^{-im\phi}$  or, equivalently,  $P_n^m(\mu) \cos m\phi$  and  $P_n^m(\mu) \sin m\phi$ . Associated with the value  $m = 0$  we have only one surface harmonic, namely,  $P_n^0(\mu) = P_n(\mu)$ . Thus we have associated with any non-negative integral value of  $n$ ,  $2n + 1$  surface harmonics  $U_n$ , and for each of these  $V = r^n U_n$  is a polynomial function of  $x, y$ , and  $z$ .

### Examples

$$n = 0; \quad m = 0; \quad P_0^0(\mu) = 1; \quad P_0^0(\mu) \cos 0\phi = 1; \quad V = 1;$$

$$n = 1; \quad m = 0; \quad P_1^0(\mu) = \mu; \quad P_1^0(\mu) \cos 0\phi = \mu; \quad V = z;$$

$$m = 1; \quad P_1^1(\mu) = (1 - \mu^2)^{1/2};$$

$$P_1^1(\mu) \cos \phi = (1 - \mu^2)^{1/2} \cos \phi = \sin \theta \cos \phi; \quad V = x;$$

$$P_1^1(\mu) \sin \phi = (1 - \mu^2)^{1/2} \sin \phi = \sin \theta \sin \phi; \quad V = y;$$

$$n = 2; \quad m = 0; \quad P_2^0(\mu) = \frac{3}{2}(\mu^2 - \frac{1}{3}); \quad V = \frac{3}{2}(r^2\mu^2 - \frac{1}{3}r^2) \\ = \frac{1}{2}(2z^2 - x^2 - y^2);$$

$$m = 1; \quad P_2^1(\mu) = 3(1 - \mu^2)^{1/2}\mu;$$

$$P_2^1 \cos \phi = 3(1 - \mu^2)^{1/2}\mu \cos \phi; \quad V = 3xz;$$

$$P_2^1 \sin \phi = 3(1 - \mu^2)^{1/2}\mu \sin \phi; \quad V = 3yz;$$

$$m = 2; \quad P_2^2 = 3(1 - \mu^2);$$

$$P_2^2 \cos 2\phi = 3(1 - \mu^2) \cos 2\phi; \quad V = 3(x^2 - y^2)$$

$$P_2^2 \sin 2\phi = 3(1 - \mu^2) \sin 2\phi; \quad V = 3xy;$$

Let us denote  $P_n^m(\mu)e^{im\phi}$ , where  $m$  is any integer positive, negative, or zero, by  $U_n^m$ . Then  $U_n^m$  is a surface harmonic which is associated with the non-negative integer  $n$ . Hence, by the orthogonal property of surface harmonics,

$$\int U_{n_1}^{m_1} U_{n_2}^{m_2} d\omega = 0 \text{ if } n_1 \neq n_2.$$

When  $n_1 = n_2$  the integral is still zero if  $m_1 \neq m_2$  since

$$\int U_n^{m_1} U_n^{m_2} d\omega = \int_1^1 P_n^{m_1}(\mu) P_n^{m_2}(\mu) d\mu \int_0^{2\pi} e^{i(m_2 - m_1)\phi} d\phi.$$

When  $n_1 = n_2 = n$ , say, and  $m_1 = m_2 = m$ , say, we have

$$\int U_n^m U_n^m d\omega = 2\pi \int_{-1}^1 \{P_n^m(\mu)\}^2 d\mu.$$

On writing  $\{P_n^m(\mu)\}^2$  in the form  $\{(1 - \mu^2)^m D^m P_n\} D^m P_n$  and integrating by parts we obtain (since, when  $m > 0$ , the integrated part is zero at  $\mu = \pm 1$ )

$$\begin{aligned} & \int_{-1}^1 \{P_n^m(\mu)\}^2 d\mu \\ &= - \int_{-1}^1 D^{m-1} P_n \{(1 - \mu^2)^m D^{m+1} P_n - 2m\mu(1 - \mu^2)^{m-1} D^m P_n\} d\mu. \end{aligned}$$

On differentiating the equation

$$(1 - \mu^2)D^2 P_n - 2\mu D P_n + n(n+1)P_n = 0$$

$(m-1)$  times we obtain

$$(1 - \mu^2)D^{m+1} P_n - 2m\mu D^m P_n + (n+m)(n-m+1)D^{m-1} P_n = 0$$

so that

$$\begin{aligned} \int_{-1}^1 \{P_n^m(\mu)\}^2 d\mu &= (n+m)(n-m+1) \int_{-1}^1 (1 - \mu^2)^{m-1} \{D^{m-1} P_n\}^2 d\mu \\ &= (n+m)(n-m+1) \int_{-1}^1 \{P_n^{m-1}(\mu)\}^2 d\mu. \end{aligned}$$

On continuing this *reduction* until we reach  $\int_{-1}^1 \{P_n^0(\mu)\}^2 d\mu =$

$$\int_{-1}^1 P_n^2 d\mu = \frac{2}{2n+1} \text{ we find that}$$



$$\int_{-1}^1 \{P_n^m(\mu)\}^2 d\mu = (n+m)(n+m-1) \cdots (n+1) \cdot (n-m+1) \cdots n \cdot \frac{2}{2n+1}$$

$$= \frac{(n+m)!}{(n-m)!} \frac{2}{2n+1}$$

Hence  $\int \bar{U}_n^m U_n^m d\omega = \frac{(n+m)!}{(n-m)!} \frac{4\pi}{2n+1}$  so that the function-vectors

$$\mathbf{u}_n^m: \left\{ \frac{(2n+1)(n-m)!}{4\pi(n+m)!} \right\}^{1/2} U_n^m = \left\{ \frac{(2n+1)(n-m)!}{4\pi(n+m)!} \right\}^{1/2} P_n^m(\mu) e^{im\phi} = u_n^m;$$

$$-1 \leq \mu \leq 1; \quad 0 \leq \phi < 2\pi; \quad n = 0, 1, 2, \dots;$$

$$m = 0, \pm 1, \dots, \pm n$$

constitute an orthonormal set over the unit sphere. If  $f(\mu, \phi)$  is any function which is integrable over the unit sphere its Fourier coefficients with respect to the orthonormal set  $\{\mathbf{u}_n^m\}$  are given by the formula

$$c_n^m = (\mathbf{u}_n^m | \mathbf{f}) = \int \bar{u}_n^m f d\omega, \quad \text{where } \mathbf{f}: f(\mu, \phi).$$

The Fourier series of  $\mathbf{f}$  with respect to the orthonormal set  $\{\mathbf{u}_n^m\}$  is

$$\sum_{n=0}^{\infty} \sum_{m=-n}^n c_n^m \mathbf{u}_n^m.$$

This series is known as the *Laplace series* of the function  $f(\mu, \phi)$  of position on the unit sphere. Just as for the Fourier series of a continuous function  $f(\mu)$  with respect to the power-function orthonormal set, and of a continuous function  $f(\phi)$  with respect to the exponential orthonormal set, it is possible to show that if  $f(\mu, \phi)$  is continuous over the unit sphere it may be approximated arbitrarily closely, uniformly over the unit sphere, by a linear combination of a finite number of the functions  $u_n^m$  (this finite combination being an average of partial sums of the Fourier series of  $f(\mu, \phi)$ ). The proof of this fact is detailed, and we do not give it here. It may be found in *Spherical and Ellipsoidal Harmonics* by E. W. Hobson, Cambridge University Press. Taking this proof as granted it follows that the orthonormal set  $\{\mathbf{u}_n^m\}$  is complete (prove this), and this serves to quantize the integer  $m$  which occurs in the formula

$$V = \frac{r^n}{r^{n-1}} \left\{ \begin{matrix} P_n^m(\mu) \\ Q_n^m(\mu) \end{matrix} \right\} e^{\pm im\phi}$$

for those solutions in which the space polar coordinates are completely separable. In the first place if  $k$  is not a (non-negative) integer and we denote by  $M_k^m$  any solution of Legendre's associated equation over  $[-1, 1]$  then  $f(\mu, \phi) = M_k^m(\mu)e^{\pm im\phi}$  is orthogonal to every vector of the complete orthonormal set  $\{\mathbf{u}_n^m\}$  by virtue of the general principle of orthogonality of surface harmonics. Hence  $f(\mu, \phi) \equiv 0$ ; this quantizes  $k$  to the non-negative integers. If  $n$  is any non-negative integer and  $m$  is any integer greater than  $n$  let us denote by  $M_n^m$  any solution of Legendre's associated equation over  $[-1, 1]$ . Then  $f(\mu, \phi) = M_n^m(\mu)e^{\pm im\phi}$  is orthogonal to every member of the complete orthonormal set  $\{\mathbf{u}_n^m\}$  (why?) so that  $f(\mu, \phi) \equiv 0$ . Thus we have the following fundamental result:

**If Legendre's associated equation**

$$(1 - \mu^2)D^2M - 2\mu DM + \left\{ k(k+1) - \frac{m^2}{1 - \mu^2} \right\} M = 0$$

possesses over  $[-1, 1]$  a solution which is not identically zero then

1.  $k$  is a non-negative integer  $n$ ;
2.  $m$  is one of the  $n + 1$  non-negative integers  $0, 1, \dots, n$ .

*Note.* Pay particular attention to the *closed* nature of the interval  $[-1, 1]$  in the statement of this theorem.  $Q_n^m(\mu)$ , for example, is not a solution of Legendre's associated equation over  $[-1, 1]$  since  $Q_n^m(\mu)$  is not differentiable at the end-points  $\pm 1$  of this interval.

## EXERCISES

1. At what points of the unit sphere are  $P_1, P_1^1 \cos \phi$ , and  $P_1^1 \sin \phi$  zero?
2. At what points of the unit sphere are  $P_2, P_2^1 \cos \phi, P_2^1 \sin \phi, P_2^2 \cos 2\phi$ , and  $P_2^2 \sin 2\phi$  zero?

### 5. The power series solutions of Bessel's equation

We have seen in Chapter 5, Section 2 that the solutions of Laplace's equation  $\Delta_2 V = 0$  in which the cylindrical coordinates  $(\rho, \phi, z)$  are completely separable are of the form

$$V = \left. \begin{array}{l} J_m(\alpha\rho) \\ K_m(\alpha\rho) \end{array} \right\} e^{\pm im\phi} \left. \right\} e^{\pm \alpha z},$$

where  $m$  is quantized (by the requirement that  $V$  be a uniform function of position) to be a (non-negative) integer. The functions  $J_m(\alpha\rho)$ ,

$K_m(\alpha\rho)$  are linearly independent solutions of the equation (*Bessel's equation of order  $m$* ):

$$P_{\xi\xi} + \frac{1}{\xi}P_{\xi} + \left(1 - \frac{m^2}{\xi^2}\right)P = 0.$$

We shall consider first the case  $m = 0$  so that the equation we have to discuss is *Bessel's equation of zero order*

$$\xi P_{\xi\xi} + P_{\xi} + \xi P = 0.$$

We shall regard  $\xi$  as a complex variable, and on writing our differential equation in the form

$$P_{\xi\xi} = -\frac{1}{\xi}P_{\xi} - P$$

it is clear that the point  $\xi = 0$  is a *singular point* of the differential equation (i.e., that at least one of the coefficients (of  $P_{\xi}$  and  $P$ ) is not analytic at  $\xi = 0$ ). There is a general theorem on second-order linear differential equations in the complex field which says that if  $\xi = 0$  is an (isolated) singular point of the differential equation and if (1) the coefficient of  $P_{\xi}$ , when the equation is solved for  $P_{\xi\xi}$ , is either analytic at  $\xi = 0$  or has  $\xi = 0$  as a *simple pole* and if (2) the coefficient of  $P$ , when the equation is solved for  $P_{\xi\xi}$ , is either analytic at  $\xi = 0$  or has  $\xi = 0$  as a pole of either the first or second order then the differential equation has a solution which may be written, in a neighborhood of  $\xi = 0$ , as the product of a power series in  $\xi$  by a power  $\xi^{\alpha}$  of  $\xi$ ,  $\alpha$  being some complex number.

*Note.* The theorem just stated is part of a theorem known as *Fuchs's theorem* (after L. Fuchs [1833–1902], a German mathematician). A singularity of the type described is known as a *regular singular point*.

In order to determine a solution of Bessel's equation of zero order of the type described we set

$$P = \xi^{\alpha}(c_0 + c_1\xi + \cdots + c_n\xi^n + \cdots) = c_0\xi^{\alpha} + \cdots + c_n\xi^{\alpha+n} + \cdots,$$

where  $c_0 \neq 0$ . On substituting this expression in Bessel's equation we obtain a power series whose sum is zero over a neighborhood of  $\xi = 0$ . Since each coefficient of this power series must be zero (why?) we obtain the following series of equations:

$$c_0\alpha^2 = 0;$$

$$c_1(\alpha + 1)^2 = 0;$$

$$c_2(\alpha + 2)^2 + c_0 = 0;$$

.

.

.

$$c_j(\alpha + j)^2 + c_{j-2} = 0; \quad j = 2, 3, \dots$$

The first of these equations tells us that  $\alpha = 0$ , the second that  $c_1 = 0$ , and the remaining coefficients are determined by the formula

$$c_j = -\frac{c_{j-2}}{j^2}; \quad j = 2, 3, \dots$$

Hence  $c_{2i+1} = 0$  (why?),  $c_{2i} = (-1)^i \frac{c_0}{(2^2)(4^2) \dots (2j)^2}$ , and the solution is

$$c_0 \left[ 1 - \frac{\xi^2}{2^2} + \frac{\xi^4}{(2^2)(4^2)} - \dots + (-1)^i \frac{\xi^{2i}}{(2^2)(4^2) \dots (2j)^2} + \dots \right].$$

The function inside the brackets is known as *Bessel's function of zero order of the first kind* and is denoted by  $J_0(\xi)$ :

$$J_0(\xi) = 1 - \frac{\xi^2}{2^2} + \frac{\xi^4}{(2^2)(4^2)} - \dots = \sum_0^{\infty} (-1)^n \frac{\left(\frac{\xi}{2}\right)^{2n}}{n!n!}.$$

The power series which defines  $J_0(\xi)$  converges for every (finite) value of  $\xi$ . (Prove this.)

Since the quadratic equation which determined  $\alpha$ , namely  $\alpha^2 = 0$ , had  $\alpha = 0$  as a double root, Fuchs's theorem (*part* of which we have already quoted) assures us that Bessel's equation of zero order possesses a second solution of the form

$$P = J_0(\xi) \log \xi + w(\xi),$$

where  $w(\xi) = a_1\xi + a_2\xi^2 + \dots + a_n\xi^n + \dots$  is analytic, and zero, at  $\xi = 0$ . Upon substituting this expression for  $P$  in Bessel's equation of zero order we find that  $w(\xi)$  must satisfy the following differential equation:

$$\xi D^2w + Dw + \xi w = -2DJ_0,$$

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where  $D$  denotes differentiation with respect to  $\xi$ . Upon equating coefficients of like powers of  $\xi$  on both sides we obtain the following series of equations:

$$a_1 = -2c_1;$$

$$2^2 a_2 = -(2)(2)c_2;$$

$$3^2 a_3 + a_1 = -(2)(3)c_3;$$

.  
.  
.

$$j^2 a_j + a_{j-2} = -(2)(j)c_j = \frac{2}{j} c_{j-2}; \quad j = 3, 4, \dots,$$

where  $J_0(\xi) = 1 + c_1 \xi + c_2 \xi^2 + \dots$ . Since  $c_1, c_3, c_5, \dots$  are all zero it follows that  $a_1, a_3, a_5, \dots$  are all zero. The second of our series of equations tells us that  $a_2 = -c_2 = \frac{1}{2^2}$ ; the fourth that

$$a_4 = \frac{1}{4^2} \left( \frac{2}{4} c_2 - a_2 \right) = \frac{c_2}{4^2} \left( 1 + \frac{1}{2} \right) = -\frac{1}{(2^2)(4^2)} \left( 1 + \frac{1}{2} \right),$$

and so on. Thus

$$a_6 = \frac{1}{6^2} \left( \frac{2}{6} c_4 - a_4 \right) = \frac{1}{(2^2)(4^2)(6^2)} \left( 1 + \frac{1}{2} + \frac{1}{3} \right),$$

and, generally,

$$a_{2j} = \frac{(-1)^{j-1}}{(2^2)(4^2) \dots (2j)^2} \left\{ 1 + \frac{1}{2} + \dots + \frac{1}{j} \right\}; \quad j = 1, 2, \dots$$

This second solution of Bessel's equation of zero order which we have obtained is known as Bessel's function of zero order of the second kind and is denoted by the symbol  $K_0(\xi)$ :

$$K_0(\xi) = J_0(\xi) \log \xi + \frac{\xi^2}{2^2} - \frac{\xi^4}{(2^2)(4^2)} \left( 1 + \frac{1}{2} \right) + \frac{\xi^6}{(2^2)(4^2)(6^2)} \left( 1 + \frac{1}{2} + \frac{1}{3} \right) - \dots$$

Applying the same method to Bessel's equation of order  $m$

$$\xi^2 D^2 P + \xi D P + (\xi^2 - m^2) P = 0$$

we find first that  $\alpha = \pm m$ . In the case we are discussing  $m$  is quantized so as to be a (non-negative) integer, but it is useful (in other connections) to develop the solution of Bessel's equation "of order  $m$ " when  $m$  is not an integer. For instance the case where  $m$  is "an integer plus  $\frac{1}{2}$ " is important in connection with those solutions of the *wave equation* in which the radial coordinate  $r$  of a system of space polar coordinates is separable from the angular coordinates  $(\theta, \phi)$  (see Exercises 12 and 13, p. 135). We shall, therefore, ignore the fact that  $m$  is quantized to (non-negative) integral values and proceed on the assumption that  $m$  is any (positive) number. The two roots of the quadratic equation which determines  $\alpha$  are distinct since  $m \neq 0$ ; their difference has the numerical value  $2m$ , and if this is not an integer we obtain two linearly independent solutions of the differential equation by setting first  $\alpha = m$  and then  $\alpha = -m$ . We denote these solutions by  $J_m(\xi)$  and  $J_{-m}(\xi)$  so that  $J_m(\xi)$  is the product of a power series, which is different from zero at  $\xi = 0$ , by  $\xi^m$  while  $J_{-m}(\xi)$  is the product of a power series, which is different from zero at  $\xi = 0$ , by  $\xi^{-m}$ .

On substituting  $P = c_0\xi^\alpha + c_1\xi^{\alpha+1} + \dots + c_n\xi^{\alpha+n} + \dots$  in the differential equation  $\xi^2 D^2 P + \xi DP + (\xi^2 - m^2)P = 0$ , the equations which determine  $\alpha$  and the coefficients  $c_1, c_2, \dots$  (the latter in terms of  $c_0$ ) are

$$(\alpha^2 - m^2)c_0 = 0;$$

$$\{(\alpha + 1)^2 - m^2\}c_1 = 0;$$

$$\{(\alpha + j)^2 - m^2\}c_j + c_{j-2} = 0; \quad j = 2, 3, \dots$$

Since  $m > 0$  none of the coefficients of  $c_1, c_2, \dots$  in these equations is zero when  $\alpha = m$ . Hence  $c_1 = 0$ , and it follows at once that  $c_3, c_5, \dots, c_{2n+1}, \dots$  are all zero. Furthermore

$$c_2 = -\frac{c_0}{(\alpha + 2 + m)(\alpha + 2 - m)}; \quad c_4 = -\frac{c_2}{(\alpha + 4 + m)(\alpha + 4 - m)},$$

and so on. On setting  $\alpha = m$  we obtain

$$c_2 = -\frac{c_0}{2^2(m+1)(1)}; \quad c_4 = \frac{c_2}{(2^2)(m+2)^2} = \frac{c_0}{2^4(m+2)(m+1)2!};$$

$$\dots; \quad c_{2j} = (-1)^j \frac{c_0}{2^{2j}(m+j) \dots (m+1)(j)!}.$$

On replacing  $c_0$  by  $C_0 \div \Gamma(m+1)$  we obtain the formally simpler expressions

$$c_2 = \frac{C_0}{\Gamma(m+2)(1!)}; \quad c_4 = \frac{C_0}{\Gamma(m+3)(2!)}; \quad \dots; \\ c_{2j} = \frac{C_0}{\Gamma(m+j+1)(j!)}.$$

*Note.*  $\Gamma(\beta)$  is the *gamma function* which is defined for all complex numbers  $\beta$  whose real part is positive by the formula

$$\Gamma(\beta) = \int_0^{\infty} e^{-t} t^{\beta-1} dt.$$

When  $\beta$  is a positive integer we have

$$\Gamma(\beta) = (\beta - 1)!$$

(it being agreed that  $0! = \Gamma(1) = 1$ ). When  $\beta = \frac{1}{2}$  we have

$$\Gamma\left(\frac{1}{2}\right) = \pi^{1/2}.$$

We have, then, the following solution of Bessel's equation of order  $m$  (obtained by setting  $C_0 = \frac{1}{2^m}$ )

$$J_m(\xi) = \frac{1}{\Gamma(m+1)} \left(\frac{\xi}{2}\right)^m - \frac{1}{\Gamma(m+2)(1!)} \left(\frac{\xi}{2}\right)^{m+2} \\ + \frac{1}{\Gamma(m+3)(2!)} \left(\frac{\xi}{2}\right)^{m+4} - \dots \\ = \sum_0^{\infty} (-1)^n \frac{1}{\Gamma(m+n+1)(n!)} \left(\frac{\xi}{2}\right)^{m+2n}$$

When  $m$  is an integer  $\Gamma(m+p+1)$  may be replaced by  $(m+p)!$ . For example,

$$J_1(\xi) = \frac{\xi}{2} - \frac{\xi^3}{(2^2)4} + \frac{\xi^5}{(2^2)(4^2)(6)} - \frac{\xi^7}{(2^2)(4^2)(6^2)(8)} + \dots$$

### EXERCISE

1. Show that  $J_{1/2}(\xi) = \left(\frac{2}{\pi\xi}\right)^{1/2} \sin \xi$ . *Hint.* Remember that  $\Gamma\left(\frac{1}{2}\right) = \pi^{1/2}$ .

When  $2m$  is not an integer (so that none of the coefficients in the equations which serve to determine  $c_1, c_2, \dots$  is zero) we obtain a

second solution of Bessel's equation of order  $m$  on setting  $\alpha = -m$ . This second solution is

$$c_0 \xi^{-m} \left\{ 1 - \frac{\xi^2}{2^2(1-m)} + \frac{\xi^4}{(2^2)(4^2)(1-m)(2-m)(2!)} - \cdots \right\}.$$

If  $m < 1$  ( $\neq \frac{1}{2}$ ) we may set  $c_0 = \frac{2^m}{\Gamma(1-m)}$ , and we denote the solution so obtained by the symbol  $J_{-m}(\xi)$ :

$$\begin{aligned} J_{-m}(\xi) &= \frac{\left(\frac{\xi}{2}\right)^{-m}}{\Gamma(1-m)} - \frac{\left(\frac{\xi}{2}\right)^{-m+2}}{\Gamma(2-m)(1!)} + \frac{\left(\frac{\xi}{2}\right)^{-m+4}}{\Gamma(3-m)(2!)} - \cdots \\ &= \sum_0^{\infty} (-1)^n \frac{\left(\frac{\xi}{2}\right)^{-m+2n}}{\Gamma(n+1-m)(n!)}. \end{aligned}$$

The (apparently) exceptional case  $m = \frac{1}{2}$  falls under this formula. In this case the coefficient of  $c_1$  is zero, but this only means that  $c_1$  is arbitrary. On setting  $c_1 = 0$  we obtain a multiple of  $J_{-\frac{1}{2}}(\xi)$  (as defined above). If we do not set  $c_1 = 0$  we simply obtain a linear combination of  $J_{-\frac{1}{2}}(\xi)$  and  $J_{\frac{1}{2}}(\xi)$ .

### EXERCISES

2. Show that  $J_{-\frac{1}{2}}(\xi) = \left(\frac{2}{\pi\xi}\right)^{\frac{1}{2}} \cos \xi$ .

3. Show that the change of dependent variable  $P = \xi^{-\frac{1}{2}}w$  transforms Bessel's differential equation of order  $m$  into

$$D^2w + \left(1 + \frac{\frac{1}{4} - m^2}{\xi^2}\right)w = 0.$$

4. Deduce from the result of Exercise 3 that  $J_{\frac{1}{2}}(\xi)$  must be a constant multiple of  $\xi^{-\frac{1}{2}} \sin \xi$  and that  $J_{-\frac{1}{2}}(\xi)$  must be a constant multiple of  $\xi^{-\frac{1}{2}} \cos \xi$ . *Hint.*  $\xi^{\frac{1}{2}}J_{\frac{1}{2}}(\xi)$  is a linear combination of  $\cos \xi$  and  $\sin \xi$ . Since  $\xi^{\frac{1}{2}}J_{\frac{1}{2}}(\xi)$  is zero at  $\xi = 0$  it must be a constant multiple of  $\sin \xi$ . Similarly  $\xi^{\frac{1}{2}}J_{-\frac{1}{2}}(\xi)$  is a linear combination of  $\cos \xi$  and  $\sin \xi$ ; being an even function of  $\xi$  it must be a multiple of  $\cos \xi$ .

5. Verify that  $J_1(\xi) = -DJ_0(\xi)$ .

In order to obtain the second solution  $K_m(\xi)$  of Bessel's equation of order  $m$ , when  $m$  is an integer, we combine the two equations

$$\begin{aligned} \xi^2 D^2 P + \xi D P + (\xi^2 - m^2) P &= 0; \\ \xi^2 D^2 J_m + \xi D J_m + (\xi^2 - m^2) J_m &= 0 \end{aligned}$$



as follows: Multiply the first equation by  $J_m$ , the second by  $-P$ , and add; we obtain, on dividing through by  $\xi$ ,

$$\xi(J_m D^2 P - P D^2 J_m) + (J_m D P - P D J_m) = 0.$$

Since  $(J_m D^2 P - P D^2 J_m) = D(J_m D P - P D J_m)$  it follows that  $\xi(J_m D P - P D J_m)$  is a constant (why?). Hence

$$D\left(\frac{P}{J_m}\right) = \frac{C}{\xi J_m^2},$$

where the constant  $C$  is not zero since we take it as granted that  $P$  is not a constant multiple of  $J_m$ .  $\xi J_m^2$  is a power series in  $\xi$  which starts out with the  $\xi^{2m+1}$  term. Hence  $\frac{P}{J_m}$  is a constant times  $\log \xi$  plus a power series in  $\xi$  plus some terms involving negative powers of  $\xi$ , the lowest power of  $\xi$  being the  $-2m$ th power. Since it is shown, in the proof of Fuchs's theorem, that the constant which multiplies  $\log \xi$  is not zero we can take it, without lack of generality, to be 1 (why?). Hence

$$P = J_m \log \xi + w(\xi),$$

where  $w(\xi)$  is a power series in  $\xi$  plus the sum of a finite number of terms involving negative powers of  $\xi$ , the lowest power being the  $-m$ th. Upon substitution of this expression for  $P$  in Bessel's equation we obtain

$$\xi^2 D^2 w + \xi D w + (\xi^2 - m^2)w = -2\xi D J_m.$$

On setting  $w = a_0 \xi^{-m} + a_1 \xi^{-m+1} + \cdots + a_{m-1} \xi + \sum_{n=0}^{\infty} a_{m+n} \xi^n$  we

obtain a series of equations which serve to determine the coefficients  $a_0, a_1, \cdots$ . The first of these equations tells us nothing about  $a_0$ ; the second  $(-2m+1)a_1 = 0$  tells us that  $a_1 = 0$ ; the third  $2(-2m+2)a_2 + a_0 = 0$  tells us (provided that  $m > 1$ ) that  $a_2 = \frac{a_0}{2^2(m-1)}$ ; the fourth equation  $3(-2m+3)a_3 + a_1 = 0$  tells us that  $a_3 = 0$  (why?). In this way we see that all the coefficients  $a_{2j+1}$  are zero and that, if  $m > 2$ ,

$$a_4 = \frac{a_2}{4(2m-4)} = \frac{a_0}{(2^4)(m-1)(m-2)2!}$$

Similarly

$$a_6 = \frac{a_4}{(6)(2m-6)} = \frac{a_0}{2^6(m-1)(m-2)(m-3)3!} \text{ if } m > 3,$$

and, generally,

$$a_{2j} = \frac{a_0}{2^{2j}m(m-1) \cdots (m-j)j!}; \quad j = 1, 2, \dots, m-1.$$

When we reach the term  $a_{2m}$  its coefficient is zero, and we have the equation

$$a_{2m-2} = -2mc_0,$$

where  $J_m(\xi) = c_0\xi^m + c_2\xi^{m+2} + \dots$  so that  $c_0 = \frac{1}{(2^m)m!}$ . Hence

$$a_{2m-2} = -\frac{1}{(2^{m-1})(m-1)!}$$

so that (from the formula for  $a_{2j}$ , with  $j = m-1$ )

$$a_0 = 2^{2(m-1)}(m-1)!(m-1)!a_{2m-2} = -2^{m-1}(m-1)!.$$

The coefficient  $a_{2m}$  is undetermined (which is only another way of saying that  $P$  is indeterminate to the extent of an additive constant multiple of  $J_m(\xi)$ ). The remaining coefficients  $a_{2m+2}, a_{2m+4}, \dots$ , are determined by the series of equations

$$2(2m+2)a_{2m+2} + a_{2m} = -2(m+2)c_2;$$

$$4(2m+4)a_{2m+4} + a_{2m+2} = -2(m+4)c_4;$$

$$6(2m+6)a_{2m+6} + a_{2m+4} = -2(m+6)c_6;$$

and so on. The first of these equations yields

$$a_{2m+2} = \frac{-c_2}{2} \left( 1 + \frac{1}{m+1} \right) - \frac{a_{2m}}{4(m+1)}.$$

If we set

$$\frac{a_{2m}}{4(m+1)} = \frac{c_2}{2} \left( 1 + \frac{1}{2} + \dots + \frac{1}{m} \right)$$

we obtain the relatively simple formula

$$a_{2m+2} = -\frac{c_2}{2} \left( 1 + 1 + \frac{1}{2} + \dots + \frac{1}{m+1} \right).$$

Since  $\frac{c_2}{c_0} = -\frac{1}{4(m+1)}$  the definition of  $a_{2m}$  may be put in the form

$$a_{2m} = -\frac{c_0}{2} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{m} \right).$$

*Note.* The reason for the particular choice of the value of  $a_{2m}$  just made will be explained in the following section.

The second of our equations yields

$$a_{2m+4} = -\frac{c_4}{2} \left( \frac{1}{2} + \frac{1}{m+2} \right) - \frac{a_{2m+2}}{(2^2)(m+2)2}$$

and since, by virtue of the relation  $c_2 = -4(m+2)(2c_4)$ ,

$$\frac{a_{2m+2}}{2^2(m+2)(2)} = \frac{c_4}{2} \left( 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{m+1} \right)$$

we obtain  $a_{2m+4} = -\frac{c_4}{2} \left( 1 + \frac{1}{2} + 1 + \frac{1}{2} + \cdots + \frac{1}{m+2} \right)$ .

Continuing this argument we obtain the general formula

$$a_{2m+2j} = -\frac{c_{2j}}{2} \left\{ \left( 1 + \frac{1}{2} + \cdots + \frac{1}{j} \right) + \left( 1 + \frac{1}{2} + \cdots + \frac{1}{m+j} \right) \right\}.$$

We denote by  $K_m(\xi)$  the solution of Bessel's equation of order  $m$  obtained in this way; thus

$$\begin{aligned} K_m(\xi) = & J_m(\xi) \log \xi - \frac{1}{2} \left\{ (m-1)! \left( \frac{\xi}{2} \right)^{-m} + \frac{(m-2)!}{1!} \left( \frac{\xi}{2} \right)^{-m+2} \right. \\ & \left. + \cdots + \frac{1}{(m-1)!} \left( \frac{\xi}{2} \right)^{m-2} \right\} - \frac{c_0}{2} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{m} \right) \xi^m \\ & - \sum_1^{\infty} c_{2j} \left\{ \left( 1 + \frac{1}{2} + \cdots + \frac{1}{j} \right) + \left( 1 + \frac{1}{2} + \cdots + \frac{1}{m+j} \right) \right\} \xi^{m+2j}. \end{aligned}$$

where  $J_m(\xi) = c_0 \xi^m + c_2 \xi^{m+2} + \cdots + c_{2j} \xi^{m+2j} + \cdots$ , so that

$$c_{2j} = (-1)^j \div 2^{m+2j} j! m + j!.$$

### EXERCISE

6. Write out the expression for  $K_1(\xi)$ , and verify the relation

$$K_1(\xi) = -DK_0(\xi).$$

**6. The recurrence relations for Bessel functions**

The relations of Exercises 5 and 6 of the preceding section are special cases of the following general theorem:

$$J_{m+1}(\xi) = -\xi^m D \left\{ \frac{J_m(\xi)}{\xi^m} \right\}; \quad K_{m+1}(\xi) = -\xi^m D \left\{ \frac{K_m(\xi)}{\xi^m} \right\}.$$

To prove this theorem we observe that if  $P_m(\xi)$  is any solution of Bessel's equation of order  $m$  then

$$R_m(\xi) = \frac{P_m(\xi)}{\xi^m}$$

satisfies the differential equation

$$\xi D^2 R_m + (2m + 1) D R_m + \xi R_m = 0.$$

In fact  $D P_m = \xi^m D R_m + m \xi^{m-1} R_m$ ,  $D^2 R_m = \xi^m D^2 R_m + 2m \xi^{m-1} D R_m + m(m-1) \xi^{m-2} R_m$  so that

$$\begin{aligned} \xi^2 D^2 P_m + \xi D P_m + (\xi^2 - m^2) P_m \\ = \xi^{m+1} \{ \xi D^2 R_m + (2m + 1) D R_m + \xi R_m \}, \end{aligned}$$

which proves the statement. Set, now,  $\zeta = \frac{1}{2} \xi^2$ , and denote differentiation with respect to  $\zeta$  by the symbol  $\delta$  so that

$$D R_m = \xi \delta R_m; \quad D^2 R_m = \delta R_m + \xi^2 \delta^2 R_m = \delta R_m + 2\zeta \delta^2 R_m.$$

It follows that

$$2\zeta \delta^2 R_m + 2(m+1) \delta R_m + R_m = 0,$$

and on differentiating this equation with respect to  $\zeta$  we obtain

$$2\zeta \delta^3 R_m + 2(m+2) \delta^2 R_m + \delta R_m = 0.$$

In words:

$\delta R_m$  is a solution of the equation which  $R_{m+1}$  satisfies.

Returning to the original dependent variable  $P_m$  and independent variable  $\xi$  this result appears as follows:

If  $P_m$  is any solution of Bessel's equation of order  $m$  then  $\left(\frac{1}{\xi}\right) D \left(\frac{P_m}{\xi^m}\right)$  is the quotient of a solution of Bessel's equation of order  $m+1$  by  $\xi^{m+1}$ .

An equivalent statement is the following:

If  $P_m(\xi)$  is any solution of Bessel's equation of order  $m$  then  $\xi^m D \left(\frac{P_m}{\xi^m}\right)$  is a solution of Bessel's equation of order  $m+1$ .

When  $P_m(\xi)$  is  $J_m(\xi)$ ,  $\frac{P_m}{\xi^m}$  is a power series and so  $\xi^m D \left( \frac{P_m}{\xi^m} \right)$  is the product of a power series by  $\xi^m$ . Hence  $\xi^m D \left( \frac{P_m}{\xi^m} \right)$ , which we know to be a linear combination of  $J_{m+1}(\xi)$  and the second solution,  $K_{m+1}(\xi)$  or  $J_{-(m+1)}(\xi)$  as the case may be, cannot involve this second solution. Hence  $\xi^m D \left( \frac{P_m}{\xi^m} \right)$  is a constant multiple of  $J_{m+1}(\xi)$ . On checking the derivative of the second term of  $\frac{J_m(\xi)}{\xi^m}$  against the first term of  $J_{m+1}(\xi)$  we find that the constant multiple is  $-1$ . Hence

$$J_{m+1}(\xi) = -\xi^m D \left\{ \frac{J_m(\xi)}{\xi^m} \right\},$$

or, equivalently,

$$\frac{J_{m+1}(\xi)}{\xi^{m+1}} = -\frac{1}{\xi} D \left\{ \frac{J_m(\xi)}{\xi^m} \right\}.$$

When  $P_m$  is  $K_m(\xi)$ ,  $\xi^m D \left( \frac{K_m(\xi)}{\xi^m} \right)$  is a linear combination of  $J_{m+1}(\xi)$  and  $K_{m+1}(\xi)$ . Since the coefficient of  $\log \xi$  in  $\xi^m D \left\{ \frac{K_m(\xi)}{\xi^m} \right\}$  is  $\xi^m D \left\{ \frac{J_m(\xi)}{\xi^m} \right\} = -J_{m+1}(\xi)$  and since the coefficient of  $\log \xi$  in  $K_{m+1}(\xi)$  is  $J_{m+1}(\xi)$  the coefficient of  $K_{m+1}(\xi)$  is the linear combination of  $J_{m+1}(\xi)$  and  $K_{m+1}(\xi)$  which furnishes  $\xi^m D \left\{ \frac{K_m(\xi)}{\xi^m} \right\}$  is  $-1$ :

$$\xi^m D \left\{ \frac{K_m(\xi)}{\xi^m} \right\} = A J_{m+1}(\xi) - K_{m+1}(\xi).$$

On comparing the coefficients of  $\xi^{m+1}$  in this equation we find that  $A = 0$ . In fact  $\frac{K_m(\xi)}{\xi^m} = (c_0 + c_2 \xi^2 + \dots) \log \xi + \dots - \frac{1}{2} c_2 \left\{ 1 + 1 + \frac{1}{2} + \dots + \frac{1}{m+1} \right\} \xi^2 + \dots$  so that the coefficient of  $\xi$  in  $D \left\{ \frac{K_m(\xi)}{\xi^m} \right\}$  is  $-c_2 \left\{ 1 + \frac{1}{2} + \dots + \frac{1}{m+1} \right\}$ . However, the  $c_0$  for  $J_{m+1} = \frac{1}{2^{m+1}(m+1)!} = -2c_2$  so that the coefficient of  $\xi^{m+1}$  in  $K_{m+1}(\xi) = -c_2 \left\{ 1 + \frac{1}{2} + \dots + \frac{1}{m+1} \right\}$ . Hence  $A = 0$ .

Thus

$$K_{m+1}(\xi) = -\xi^m D \left\{ \frac{K_m(\xi)}{\xi^m} \right\},$$

or, equivalently,

$$\frac{K_{m+1}(\xi)}{\xi^{m+1}} = -\frac{1}{\xi} D \left\{ \frac{K_m(\xi)}{\xi^m} \right\}.$$

*Note.* It was in order to have this relation that the particular choice  $-\frac{c_0}{2} \left( 1 + \frac{1}{2} + \dots + \frac{1}{m} \right)$  for the constant  $a_{2m}$  that occurred when we were defining  $K_m(\xi)$  was made.

### EXERCISES

1. Show that  $J_{3/2}(\xi) = \left( \frac{2}{\pi\xi^3} \right)^{1/2} (\sin \xi - \xi \cos \xi)$ .
2. Show that if  $P_m$  is any solution of Bessel's equation of order  $m$  then  $\left( \frac{1}{\xi^m} \right) D(\xi^m P_m)$  is a solution of Bessel's equation of order  $(m-1)$ .
3. Show that  $\left( \frac{1}{\xi^m} \right) D(\xi^m J_m) = J_{m-1}$  or, equivalently, that  $\xi^{m-1} J_{m-1}(\xi) = \left( \frac{1}{\xi} \right) D(\xi^m J_m)$ . *Note.* When  $m$  is half an odd integer, so that  $2m$  is an integer without  $m$  being an integer, the second solution  $J_{-m}(\xi)$  of Bessel's equation may be obtained from  $J_{1/2}(\xi)$  by repeated application of the formula  $J_{m-1} = \xi^{-m} D(\xi^m J_m)$ . Thus  $J_{-1/2}(\xi) = \xi^{-1/2} D\{\xi^{1/2} J_{1/2}(\xi)\}$ ;  $J_{-3/2}(\xi) = \xi^{1/2} D\{\xi^{-1/2} J_{-1/2}(\xi)\}$ , and so on.
4. Show that  $J_{-3/2}(\xi) = -\left( \frac{2}{\pi\xi} \right)^{1/2} \left\{ \sin \xi + \frac{\cos \xi}{\xi} \right\}$ .
5. Show that  $\left( \frac{1}{\xi^m} \right) D(\xi^m K_m) = K_{m-1}$  or, equivalently, that  $\xi^{m-1} K_{m-1}(\xi) = \left( \frac{1}{\xi} \right) D(\xi^m K_m)$ . *Hint.* Check the coefficients of  $\xi^{m-1}$  on both sides of the relation  $\left( \frac{1}{\xi^m} \right) D(\xi^m K_m) = A J_{m-1} + K_{m-1}$  to show that  $A = 0$ .
6. Show that  $\xi\{J_{m+1}(\xi) + J_{m-1}(\xi)\} = 2mJ_m(\xi)$ . *Hint.* Eliminate  $DJ_m$  from the two relations  $J_{m+1}(\xi) = -\xi^m D \left\{ \frac{J_m(\xi)}{\xi^m} \right\}$ ;  $J_{m-1}(\xi) = \left( \frac{1}{\xi^m} \right) D\{\xi^m J_m(\xi)\}$ .
7. Show that  $\xi\{K_{m+1}(\xi) + K_{m-1}(\xi)\} = 2mK_m(\xi)$ .
8. Show that  $DJ_m(\xi) = \frac{1}{2}\{J_{m+1}(\xi) + J_{m-1}(\xi)\}$ .
9. Show that  $J_{-1}(\xi) = -J_1(\xi)$  and, generally, that if  $m$  is an integer  $J_{-m}(\xi) = (-1)^m J_m(\xi)$ .

### 7. The zeros of the Bessel functions of the first kind

The function  $J_m(\xi)$  satisfies the differential equation

$$\xi^2 D^2 J_m(\xi) + \xi D J_m(\xi) + (\xi^2 - m^2) J_m(\xi) = 0.$$

Let  $\alpha$  and  $\beta$  be any two complex constants, and consider the two functions  $f(\xi)$ ,  $g(\xi)$  defined as follows:

$$f(\xi) = J_m(\alpha\xi); \quad g(\xi) = J_m(\beta\xi); \quad \beta \neq \alpha.$$

Since the operators  $\xi D$  and  $\xi^2 D^2$  are unaffected when  $\xi$  is replaced by any multiple of itself we have

$$\xi^2 D^2 f + \xi D f + (\alpha^2 \xi^2 - m^2) f = 0;$$

$$\xi^2 D^2 g + \xi D g + (\beta^2 \xi^2 - m^2) g = 0.$$

On dividing each of these equations by  $\xi$  they appear as follows:

$$D(\xi D f) + \left( \alpha^2 \xi - \frac{m^2}{\xi} \right) f = 0;$$

$$D(\xi D g) + \left( \beta^2 \xi - \frac{m^2}{\xi} \right) g = 0.$$

Upon multiplying the first of these two equations by  $g$ , the second by  $-f$ , and adding we obtain

$$(g D f - f D g) + \xi (g D^2 f - f D^2 g) + (\alpha^2 - \beta^2) \xi f g = 0,$$

or, equivalently, since  $(g D^2 f - f D^2 g) = D(g D f - f D g)$ ,

$$D\{\xi(g D f - f D g)\} + (\alpha^2 - \beta^2) \xi f g = 0.$$

Upon integrating this relation over the interval  $[0, 1]$ , provided that  $\xi(g D f - f D g)$  exists over  $[0, 1]$ , we obtain

$$\xi(g D f - f D g) \Big|_0^1 = (\beta^2 - \alpha^2) \int_0^1 \xi f(\xi) g(\xi) d\xi.$$

From the series expansion  $J_m(\xi) = c_0 \xi^m \left\{ 1 - \frac{\xi^2}{4(m+1)} + \dots \right\}$ ,

where  $c_0 = \frac{1}{2^m(m!)}$ , it is easy to verify that

$\xi(g D f - f D g) = \xi\{\alpha J_m(\beta\xi) J'_m(\alpha\xi) - \beta J_m(\alpha\xi) J'_m(\beta\xi)\}$ , where  $J'_m(\alpha\xi)$ ,  $J'_m(\beta\xi)$  denote the values of  $D J_m(\xi)$  when  $\xi$  is replaced by  $\alpha\xi$  and  $\beta\xi$ ,

respectively, starts out with the term involving  $\xi^{2m+2}$ , unless  $\beta = -\alpha$ , in which case it starts out with a higher power term. (Prove this.) Hence  $\xi(gDf - fDg)$  is zero at  $\xi = 0$  if  $m > -1$ , and we shall suppose this to be the case. Let, now,  $\alpha$  and  $\beta$  be any two distinct zeros of  $J_m(\xi)$  (assuming that  $J_m(\xi)$  has two distinct zeros). Then both  $g$  and  $f$  are zero at  $\xi = 1$  so that

$$(\beta^2 - \alpha^2) \int_0^1 \xi f(\xi)g(\xi)d\xi = 0.$$

It follows from this relation that  $J_m(\xi)$  has no non-real zeros if  $m > -1$ . In fact it is evident, since  $J_m(\xi)$  is the product of  $\xi^m$  by an *alternating* power series in  $\xi^2$ , that  $J_m(\xi)$  has no purely imaginary non-real zeros. If, then,  $\alpha$  is a non-real zero of  $J_m(\xi)$ ,  $\bar{\alpha} \neq \alpha$  (since  $\alpha$  is non-real) and  $\bar{\alpha} \neq -\alpha$  (since  $\alpha$  is not purely imaginary). Hence  $\bar{\alpha}^2 - \alpha^2 \neq 0$ . Now, since the coefficients of the power series multiplying  $\xi^m$  in the definition of  $J_m(\xi)$  are real, it follows that  $\bar{\alpha}$  is a zero of  $J_m$  when  $\alpha$  is; in fact  $J_m(\bar{\alpha})$  is the conjugate of  $J_m(\alpha)$ . On setting, then,  $\beta = \bar{\alpha}$  we have, since  $\beta^2 \neq \alpha^2$ ,

$$\int_0^1 \xi f(\xi)g(\xi) d\xi = 0.$$

Since  $f(\xi) = J_m(\alpha\xi)$ ,  $g(\xi) = J_m(\beta\xi) = J_m(\bar{\alpha}\xi)$  we have  $g(\xi) = \bar{f}(\xi)$ , and the relation  $\int_0^1 \xi f(\xi)g(\xi) d\xi = 0$  is absurd since the integrand  $\xi f(\xi)g(\xi)$  is non-negative and not identically zero. We have, then, the following fundamental result:

**If  $m > -1$ ,  $J_m(\xi)$  has no non-real zeros.**

Furthermore if  $\alpha_1$  and  $\alpha_2$  are any two (necessarily real) distinct positive zeros of  $J_m(\xi)$  the two function-vectors

$$v_1 : v_1 = \xi^{1/2} J_m(\alpha_1 \xi); \quad v_2 : v_2 = \xi^{1/2} J_m(\alpha_2 \xi); \quad 0 \leq \xi \leq 1$$

are orthogonal. (We confine our attention to positive zeros since  $J_m(-\xi)$  is a constant multiple of  $J_m(\xi)$ .)

*Note.* The same argument is applicable to the zeros of  $DJ_m$  if  $m \geq 0$ . It is necessary for the argument to go through that  $m$  be non-negative (and not merely  $> -1$ ) since, on writing

$$J_m(\xi) = c_0 \xi^m + c_2 \xi^{m+2} + \dots,$$

we have

$$DJ_m(\xi) = mc_0 \xi^{m-1} + (m+2)c_2 \xi^{m+1} + \dots$$

so that  $DJ_m(\xi)$  can have a pure imaginary zero if  $m < 0$ . Let  $\xi = i\eta$ ,  $\eta$  real, be a hypothesized zero of  $DJ_m(\xi)$ ; then



$$m + \frac{m+2}{m+1} \left(\frac{\eta}{2}\right)^2 + \frac{(m+4)}{1(2)(m+1)(m+2)} \left(\frac{\eta}{2}\right)^4 + \dots = 0.$$

The function of  $\eta$  on the left is monotone-increasing (why? remember that  $m > -1$ ). Hence if  $m < 0$  it has precisely one positive zero. Since it is an even function it has precisely two zeros. Thus,

**When  $-1 < m < 0$ ,  $DJ_m$  has two purely imaginary zeros and no other complex zeros. If  $\beta_1$  and  $\beta_2$  are any two positive zeros of  $DJ_m(\xi)$  the two function-vectors**

$$w_1 : w_1 = \xi^{1/2} J_m(\beta_1 \xi); \quad w_2 : w_2 = \xi^{1/2} J_m(\beta_2 \xi); \quad 0 \leq \xi \leq 1$$

**are orthogonal.**

Having shown that  $J_m(\xi)$  has, when  $m > -1$ , no non-real zeros it only remains to discuss the situation as regards real zeros. We shall confine our attention to (non-negative) integral values of  $m$  and shall prove the following result:

**$J_m(\xi)$  has an infinity of real zeros.**

It is sufficient to prove this theorem for  $J_0(\xi)$  in view of the recurrence relations. For example the two relations

$$J_1(\xi) = -DJ_0(\xi); \quad J_0(\xi) = \frac{1}{\xi} D\{ \xi J_1(\xi) \}$$

tell us that (1) between any two zeros of  $J_0(\xi)$  there lies at least one zero of  $J_1(\xi)$ ; and (2) between any two zeros of  $\xi J_1(\xi)$  there lies at least one zero of  $\xi J_0(\xi)$ . Since  $J_0(\xi)$  is an even function and  $J_1(\xi)$  is an odd function we may confine our attention to non-negative values of  $\xi$ . Then statement 2 above may be phrased as follows: (2') between any two non-negative zeros of  $J_1(\xi)$  there lies at least one zero of  $J_0(\xi)$ .

*Note.* We shall understand from now on by the phrase "a zero of  $J_m$ " a non-negative zero of  $J_m$ .

Assuming for the moment that  $J_m(\xi)$  possesses zeros we know, since  $J_m(\xi)$  is an analytic function of  $\xi$ , that each of these zeros is isolated (what does this mean?). Hence in any circle with center at  $\xi = 0$  there are only a finite number of zeros of  $J_m(\xi)$  (why?). Hence the zeros of  $J_m(\xi)$  can be counted, and it is convenient to do the counting according to size. We shall see shortly that  $J_0(\xi)$  has an infinity of simple real zeros, and we denote these by  $\alpha_1, \alpha_2, \dots$ , where

$$0 < \alpha_1 < \alpha_2 < \dots$$

It follows, on combining the statements 1 and 2' with the fact that

$\xi = 0$  is a zero of  $J_1$ , that  $J_1$  has an infinity of zeros which we denote by  $0, \beta_1, \beta_2, \dots$  and which are such that

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots$$

We express this result as follows:

**The zeros of  $J_0(\xi)$  and  $J_1(\xi)$  interlace each other.**

On using, in a similar manner, the two relations

$$J_2(\xi) = -\xi D \left\{ \frac{J_1(\xi)}{\xi} \right\}; \quad J_1(\xi) = \frac{1}{\xi^2} D \{ \xi^2 J_2(\xi) \}$$

we see that  $J_2(\xi)$  has an infinity of zeros  $0, \gamma_1, \gamma_2, \dots$ , which are such that

$$0 < \beta_1 < \gamma_1 < \beta_2 < \gamma_2 < \dots$$

In other words the zeros (other than  $\xi = 0$ ) of  $J_1(\xi)$  and  $J_2(\xi)$  interlace each other, the first positive zero of  $J_1(\xi)$  being less than the first positive zero of  $J_2(\xi)$ . In general,

**The zeros, other than  $\xi = 0$ , of  $J_m(\xi)$  and  $J_{m+1}(\xi)$  interlace each other, the first positive zero of  $J_m(\xi)$  being less than the first positive zero of  $J_{m+1}(\xi)$ .**

In order to prove that  $J_0(\xi)$  has an infinity of positive zeros we set  $w = \xi^{1/2} J_0(\xi)$ . Then (see Exercise 3, p. 191)  $w$  satisfies the equation

$$D^2 w + \left( 1 + \frac{1}{4\xi^2} \right) w = 0.$$

We combine this with the equation

$$D^2 v + v = 0$$

of which  $v = \sin(\xi - a)$ ,  $a$  any number, is a solution, as follows: Multiply the equation satisfied by  $w$  by  $v$  and the equation satisfied by  $v$  by  $-w$ , and add to obtain

$$(vD^2 w - wD^2 v) + \frac{vw}{4\xi^2} = 0.$$

Since  $(vD^2 w - wD^2 v) = D(vDw - wDv)$  we obtain, on integrating over the interval  $[a, a + \pi]$ ,  $a$  being any positive number,

$$(vDw - wDv) \Big|_a^{a+\pi} = -\frac{1}{4} \int_a^{a+\pi} \frac{v(\xi)w(\xi)}{\xi^2} d\xi.$$

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Since  $v(\xi) = 0$  when  $\xi = a$  and when  $\xi = (a + \pi)$  and since  $Dv(\xi) = \cos(\xi - a)$  is 1 when  $\xi = a$  and  $-1$  when  $\xi = a + \pi$ , we obtain

$$w(a + \pi) + w(a) = -\frac{1}{4} \int_a^{a+\pi} \frac{v(\xi)w(\xi)}{\xi^2} d\xi.$$

Since  $v(\xi)/\xi^2$  is non-negative over the interval  $[a, a + \pi]$  it follows that  $w(\xi)$  changes sign over this interval. In fact if  $w(\xi)$  were non-negative over the interval we would have the contradiction that  $w(a + \pi) + w(a)$  is negative (remember that  $\frac{vw}{\xi^2}$  is not the zero constant function over  $[a, a + \pi]$ ). Similarly  $w(\xi)$  cannot be non-positive over  $[a, a + \pi]$ . Since  $w(\xi)$  is a continuous function of  $\xi$  (why?) it follows that  $w(\xi)$  has at least one zero in the interval  $(a, a + \pi)$ . Thus, since  $w(\xi) = \xi^{1/2}J_0(\xi)$ ,

**Every open interval of length  $\pi$  on the positive  $x$ -axis contains at least one zero of  $J_0(\xi)$ .**

On denoting by  $\alpha_1, \alpha_2, \dots, \alpha_n, \dots$  the positive zeros of  $J_0(\xi)$ , arranged in order of magnitude, the function vectors  $\{\mathbf{v}_n\}$  defined as follows

$$\mathbf{v}_n : v_n(\xi) = \xi^{1/2}J_0(\alpha_n \xi); \quad 0 \leq \xi \leq 1$$

are mutually perpendicular. To obtain from this sequence of mutually perpendicular vectors an orthonormal set we must determine the squared magnitude  $\int_0^1 \xi J_0^2(\alpha_n \xi) d\xi$  of  $\mathbf{v}_n$ . For this purpose we consider the relation

$$\xi(gDf - fDg)\Big|_0^1 = (\beta^2 - \alpha^2) \int_0^1 \xi f(\xi)g(\xi) d\xi,$$

where

$$f = J_0(\alpha\xi), \quad g = J_0(\beta\xi); \quad \alpha \text{ and } \beta \text{ any two real numbers.}$$

Written out in full this relation appears in the form

$$\alpha J_0(\beta)DJ_0(\alpha) - \beta J_0(\alpha)DJ_0(\beta) = (\beta^2 - \alpha^2) \int_0^1 \xi f(\xi)g(\xi) d\xi,$$

and on differentiating this with respect to  $\beta$  and then setting  $\beta = \alpha$  we obtain

$$\alpha \{DJ_0(\alpha)\}^2 - J_0(\alpha)DJ_0(\alpha) - \alpha J_0(\alpha)D^2J_0(\alpha) = 2\alpha \int_0^1 \xi \{J_0(\alpha\xi)\}^2 d\xi.$$

On replacing  $\alpha D^2 J_0(\alpha)$  by  $-DJ_0(\alpha) - \alpha J_0(\alpha)$  we obtain

$$\alpha \{DJ_0(\alpha)\}^2 + \{J_0(\alpha)\}^2 = 2\alpha \int_0^1 \xi \{J_0(\alpha\xi)\}^2 d\xi.$$

Let, now,  $\alpha$  be a (positive) zero of  $J_0(\xi)$ ; then

$$\int_0^1 \xi \{J_0(\alpha\xi)\}^2 d\xi = \frac{1}{2} \{DJ_0(\alpha)\}^2.$$

Thus the function-vectors  $\{\mathbf{u}_n\}$  defined by

$$\mathbf{u}_n : u_n(\xi) = \frac{2^{1/2} \xi^{1/2} J_0(\alpha_n \xi)}{DJ_0(\alpha_n)}; \quad 0 \leq \xi \leq 1.$$

constitute an orthonormal set. We shall prove later (by means of a general result concerning integral equations) that this orthonormal set is *complete*.

*Note.* The relation  $\int_0^1 \xi \{J_0(\alpha\xi)\}^2 d\xi = \frac{1}{2} \{DJ_0(\alpha)\}^2$  assures us that  $DJ_0(\alpha) \neq 0$  (why?). Hence the zeros  $\alpha_1, \dots, \alpha_n, \dots$  of  $J_0(\xi)$  are all simple.

### EXERCISES

1. Show that if  $\alpha \neq 0$  is a zero of  $J_m(\xi)$ , where  $m > -1$ , then

$$\int_0^1 \xi \{J_m(\alpha\xi)\}^2 d\xi = \frac{1}{2} \{DJ_m(\alpha)\}^2,$$

and deduce that the positive zeros of  $J_m(\xi)$  are all simple.

2. Show that if  $\beta$  is a positive zero of  $J_1(\xi)$  (or, equivalently, of  $DJ_0(\xi)$ ) then

$$\int_0^1 \xi \{J_0(\beta\xi)\}^2 d\xi = \frac{1}{2} \{J_0(\beta)\}^2.$$

3. Show that if  $\beta_1, \beta_2, \dots, \beta_n, \dots$  are the positive zeros of  $J_1(\xi)$  then the function-vectors  $\{\mathbf{u}_n\}$  defined by

$$\mathbf{u}_n : u_n(\xi) = \frac{2^{1/2} \xi^{1/2} J_0(\beta_n \xi)}{J_0(\beta_n)}; \quad 0 \leq \xi \leq 1,$$

constitute an orthonormal set.

4. Show that if  $\gamma_1, \gamma_2, \dots, \gamma_n, \dots$  are the positive zeros of  $DJ_m(\xi)$ ,  $m$  any positive integer, then the function-vectors  $\{\mathbf{u}_n\}$  defined by

$$\mathbf{u}_n : u_n(\xi) = \frac{2^{1/2} \xi^{1/2} J_m(\gamma_n \xi)}{\left(1 - \frac{m^2}{\gamma_n^2}\right)^{1/2}} J_m(\gamma_n)$$

constitute an orthonormal set.

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5. Deduce from the proof of the result of Exercise 4 that the smallest positive zero of  $DJ_m(\xi)$  is greater than  $m$ .

6. Show that, using the notation of Exercise 4,  $\{J_m^2(\gamma_n)\}$  is a monotone decreasing sequence ( $m$  fixed,  $n = 1, 2, 3, \dots$ ). *Hint.* On setting  $F(\xi) = J_m^2(\xi) + \frac{\{DJ_m(\xi)\}^2}{1 - \frac{m^2}{\xi^2}}$  we obtain

$$DF(\xi) = -\frac{2}{\xi} \frac{\{DJ_m(\xi)\}^2}{\left(1 - \frac{m^2}{\xi^2}\right)^2}$$

Hence  $F(\xi)$  is a monotone decreasing function of  $\xi$  over  $\xi > 0$ . At the points  $\gamma_1, \gamma_2, \dots, \gamma_n, \dots$ ,  $F(\xi) = J_m^2(\xi)$ . *Note.* This result tells us that the absolute values of the successive maximum and minimum values of  $J_m(\xi)$  steadily diminish as  $\xi > 0$  increases.

# 7

## BOUNDARY-VALUE PROBLEMS

### 1. Self-adjoint linear differential operators

We have seen (Exercise 5, p. 134) that the solutions of the wave equation

$$\Delta_2 V = \frac{1}{c^2} V_{tt}$$

which depend only on  $x$  and  $t$ , where  $x$  is one of the rectangular Cartesian coordinates of a point in space, and for which the variables  $x$  and  $t$  are separable are of the form

$$V = ue^{\pm i p c t}.$$

Here  $p$  is a constant and  $u = u(x)$  is a function of  $x$  alone which satisfies the equation

$$D^2u + \lambda u = 0; \quad \lambda = p^2,$$

where  $D$  denotes differentiation with respect to  $x$ . (Prove this.) This equation governs the (free) vibrations of a tightly stretched string or wire,  $x$  being a coordinate along the string and  $u = u(x)$  being the amplitude of the (transverse) displacement of the point whose coordinate is  $x$ . If we denote the left-hand side of the differential equation which  $u$  must satisfy by  $L(u) + \lambda u$  so that

$$L(u) \equiv D^2u,$$

$L$  is a *linear differential operator*; in other words

$L(cu) = cL(u)$ ;  $c$  any constant;

$L(u + v) = L(u) + L(v)$ ;  $u$  and  $v$  any two functions possessing  
second derivatives.

We term  $L$  a linear differential operator of the *second order* to indicate

that the highest order derivative that occurs in  $L$  is the second, and we write

$$L = D^2.$$

$L$  is not the most general linear differential operator of the second order, this being the linear differential operator

$$pD^2 + qD + r,$$

where  $p$ ,  $q$ , and  $r$  are functions of  $x$  which we assume to be continuous over some interval  $[a, b]$ . The particular operator  $L = D^2$  is special in the sense that its *coefficients* are constant functions of  $x$ :

$$p = 1; \quad q = 0; \quad r = 0;$$

but this is not the feature of  $L = D^2$  to which we wish to direct your attention.  $L = D^2$  arose from a consideration of special solutions of the wave equation  $\Delta_2 V = \frac{1}{c^2} V_u$ . Now those solutions of the wave equation in which the time variable is separable from the space variables are of the form

$$V = Ue^{\pm i p c t},$$

where  $p$  is a constant and the function  $U$  of the space variables satisfies the equation

$$\Delta_2 U + \lambda U = 0; \quad \lambda = p^2.$$

The Laplacian linear operator  $\Delta_2$  possesses the following remarkable property:

$$\int_R (u \Delta_2 v - v \Delta_2 u) d\tau = \int_S \left( u \frac{dv}{dn} - v \frac{du}{dn} \right) dS; \quad S \text{ the boundary of the region } R$$

(see Exercise 20, p. 31). When  $u$  and  $v$  are functions of  $x$  alone  $\Delta_2$  reduces to  $D^2$ , and the relation just written reduces, when applied to a cylinder whose axis is parallel to the  $x$ -axis, to

$$\int_a^x (uD^2v - vD^2u) dx = (uDv - vDu) \Big|_a^x$$

which is only another way of writing the relation

$$(uD^2v - vD^2u) = D(uDv - vDu).$$

Now, when  $L = D^2$ , we have  $uLv - vLu = (uD^2v - vD^2u)$  and so

$$uLv - vLu = D(uDv - vDu).$$

Thus

The linear differential operator  $L = D^2$  possesses the property that  $uLv - vLu$  is the derivative of an expression  $uDv - vDu$  which involves only the first derivatives of  $u$  and  $v$  and which is a linear differential operator on either of the two functions  $u$  and  $v$ , the other being regarded as a given fixed function.

If  $L = pD^2 + qD + r$  is any linear differential operator of the second kind we can obtain a relation similar to, but not as special as, the relation  $uLv - vLu = D(uDv - vDu)$  which the particular operator  $L = D^2$  satisfies. In fact a simple application of the process of integration by parts yields

$$\begin{aligned} \int^x uqD(v) dx &= qw - \int^x vD(qu) dx; \\ \int^x upD^2(v) dx &= puDv - \int^x DvD(pu) dx \\ &= puDv - vD(pu) + \int^x vD^2(pu) dx. \end{aligned}$$

Hence  $\int^x uL(v) dx = puDv - vD(pu) + qw + \int^x vM(u) dx$ , or, equivalently,

$$\int^x \{uL(v) - vM(u)\} dx = p(uDv - vDu) + (q - Dp)uw,$$

where  $M(u) = D^2(pu) - D(qu) + ru$  so that  $M$  is the linear differential operator

$$M = pD^2 + (2Dp - q)D + D^2p - Dq + r.$$

We say that the linear differential operator  $M$  is the *adjoint* of the linear differential operator  $L$ , and the relationship between an operator  $L$  and its adjoint  $M$  may be written as follows:

$$uL(v) - vM(u) = D\{p(uDv - vDu) + (q - Dp)uw\}.$$

From the alternating character of this relationship (i.e., from the fact that an interchange of  $u$  and  $v$  followed by a change of sign is equivalent to an interchange of the two operators  $L$  and  $M$ ) it is clear that the relation between  $L$  and  $M$  is a *partnership*:



When  $M$  is the adjoint of  $L$  then  $L$  is the adjoint of  $M$ . (Verify this. *Hint.* You have merely to show that  $2Dp - (2Dp - q) = q$  and that  $D^2p - D(2Dp - q) + (D^2p - Dq + r) = r$ .)

When the two mutually adjoint operators  $L$  and  $M$  coincide we say that  $L$  is a *self-adjoint* operator. This is the property of the special operator  $L = D^2$  to which we wish to direct your attention. This special operator is self-adjoint. (Prove this.) The quality of a linear differential operator of being self-adjoint is the analogue of the quality of a real linear vector function of being *symmetric* or of the quality of a complex linear vector function of being *Hermitian*. For this reason self-adjoint operators are also termed *Hermitian*. The criterion by which we can recognize self-adjoint linear differential operators of the second order is clear. We must have  $2Dp - q = q$  or, equivalently,  $q = Dp$ , and this *necessary* condition is *sufficient* since, then,  $D^2p - Dq + r = r$ . We have, then, the following result:

**The linear differential operator of the second order  $L = pD^2 + qD + r$  is self-adjoint when, and only when,  $q = Dp$ , i.e., when it is of the form  $L = D(pD) + r$ .**

For a self-adjoint operator of the second order  $L$  we have

$$\{uLv - vLu\} = D\{p(uDv - vDu)\}.$$

An operator  $L = pD^2 + qD + r$  which is not self-adjoint may be made self-adjoint by multiplying it by a factor  $\mu$  which is never zero (it being understood that  $\mu L$  is the operator  $\mu pD^2 + \mu qD + \mu r$ ). We have only to determine  $\mu$  so that  $\mu q = D(\mu p)$ . On dividing this relation by  $\mu p$  and integrating we see that

$$\log(\mu p) = \int^x \frac{q}{p} dx.$$

### EXERCISES

1. Show that the operator  $L = (1 - \mu^2)D^2 - 2\mu D + n(n + 1) - \frac{m^2}{1 - \mu^2}$  which occurs in the theory of Legendre's functions is self-adjoint ( $\mu$  is the independent variable).

2. Show that the operator  $L = \xi^2 D^2 + \xi D + (\xi^2 - m^2)$  which occurs in the theory of Bessel's functions is not self-adjoint but that the operator  $\frac{1}{\xi}L$  is self-adjoint ( $\xi$  is the independent variable).

3. Show that any operator of the form  $D^2 + r$  is self-adjoint.

4. Show that the operator adjoint to the third-order linear differential operator  $pD^3 + qD^2 + rD + s$  is furnished by the formula  $M(u) = -D^3(pu) + D^2(qu)$

–  $D(\tau u) + su$ , and deduce that no third-order linear differential operator is self-adjoint. *Hint.* Self-adjointness would require  $-p = p$ , i.e.,  $p = 0$ .

5. Show that the operator adjoint to the fourth-order differential operator  $p_0D^4 + p_1D^3 + p_2D^2 + p_3D + p_4$  is furnished by the formula

$$M(u) = D^4(p_0u) - D^3(p_1u) + D^2(p_2u) - D(p_3u) + p_4u,$$

and deduce the conditions for adjointness.

*Answer.*  $p_1 = 2Dp_0; p_3 = Dp_2 - D^2p_4$ .

6. Show that a linear differential operator of the fourth order is self-adjoint if, and only if, it is of the form  $L = D^2(pD^2) + D(qD) + r$ , and verify the relation

$$uL(v) - vL(u) = D\{p(uD^2v - vD^2u) + (Dp)(uD^2v - vD^2u) - p(DuD^2v - DvD^2u) + q(uDv - vDu)\}.$$

*Hint.*  $p = p_0; q = p_2 - D^2p_0; r = p_4$ .

## 2. The boundary conditions; regular boundary-value problems

In the case of the free vibrations of a tightly stretched string the amplitude  $u = u(x)$  of the transverse displacement of any point  $x$  of the string must not only satisfy the differential equation

$$L(u) + \lambda u \equiv D^2u + \lambda u = 0; \quad \lambda = p^2$$

but also  $u(x)$  must be zero at the (fixed) ends of the string. Taking our origin at the mid-point of the string (whose length we denote by  $l$ ) we must have

$$u\left(\pm \frac{l}{2}\right) = 0.$$

These conditions which must be satisfied by a solution of the differential equation  $L(u) + \lambda u = 0$  so that it may be an acceptable solution of the problem are known as *boundary conditions*, and the problem of determining a function  $u = u(x)$  which does the following *two* things:

1.  $u = u(x)$  satisfies the differential equation  $L(u) + \lambda u = 0$ ;
2.  $u = u(x)$  satisfies the boundary conditions,

is known as a *boundary-value problem*. The zero constant function is evidently a solution of the boundary-value problem, but we disregard this *trivial* solution; when we speak, then, of a solution of the boundary-value problem we shall always mean a *non-trivial* solution, i.e., a solution which is not the zero constant function.

The boundary conditions serve to *quantize* the constant  $\lambda$  which occurs in  $L(u) + \lambda u$ . In fact the general solution of  $L(u) + p^2u = 0$  is

$$u = e^{\pm ipx} = \begin{cases} \cos px \\ \sin px \end{cases} = A \cos px + B \sin px,$$

where  $A$  and  $B$  are undetermined constants. On substitution of the values  $\pm \frac{l}{2}$  for  $x$  we obtain the two equations

$$A \cos \frac{pl}{2} + B \sin \frac{pl}{2} = 0; \quad A \cos \frac{pl}{2} - B \sin \frac{pl}{2} = 0$$

so that  $A \cos \frac{pl}{2} = 0$ ,  $B \sin \frac{pl}{2} = 0$ . Since not both  $A$  and  $B$  are zero

(why?) we must have either  $\cos \frac{pl}{2} = 0$  or  $\sin \frac{pl}{2} = 0$ . Hence  $pl$  must

be an integral multiple of  $\pi$  so that  $p$  must be real. Since  $p$  enters the problem only through its square there is no lack of generality in taking  $p$  to be non-negative. It must actually be positive since, if  $p = 0$ ,  $A = 0$ , and  $u$  is the trivial solution  $u \equiv 0$ . Hence  $p$  is quantized

to the numbers  $\frac{n\pi}{l}$ ,  $n = 1, 2, 3, \dots$ , or, equivalently,  $\lambda$  is quantized

to the values  $\frac{n^2\pi^2}{l^2}$ ,  $n = 1, 2, 3, \dots$ . We term the values to which  $\lambda$

is quantized the *characteristic numbers* of the boundary-value problem. Thus

**A characteristic number of a boundary-value problem is a value of  $\lambda$  for which the boundary-value problem, supposed homogeneous, possesses a non-trivial solution.**

*Note.* A boundary-value problem is said to be *homogeneous* when it is insensitive to multiplication of the unknown function  $u$  by any constant. The differential equation  $L(u) + \lambda u = 0$  must be homogeneous, and the boundary conditions must be homogeneous. A linear differential equation such as  $D^2u + \lambda u = 1$  is not homogeneous (why?), and a boundary condition such as  $u\left(\frac{l}{2}\right) = 5$  is not homogeneous (why?).

When  $\lambda$  is a characteristic number we term any (non-trivial) solution of the boundary-value problem a *characteristic function* which is said to be *associated* with the characteristic number  $\lambda$ . Thus, in the case of the vibrating string, when  $\lambda = \frac{n^2\pi^2}{l^2}$ ,  $u = \cos \frac{n\pi x}{l}$  is a characteristic

function when  $n$  is an odd integer, and  $u = \sin \frac{n\pi x}{l}$  is a characteristic function when  $n$  is an even integer. When  $n$  is an odd integer  $2k + 1$ ,

say, the amplitude  $u = u(x)$  of the transverse vibration is zero at the points  $x = \pm(2j + 1)\frac{l}{2}$ ,  $j = 0, 1, 2, \dots, k$ , and when  $n$  is an even integer  $2k$ , say, the amplitude is zero at the points  $x = \pm 2j\frac{l}{2}$ ,  $j = 0, 1, \dots, k$ . The  $n - 1$  points  $\pm(2j + 1)\frac{l}{2}$ ,  $j = 0, \dots, k - 1$  when  $n = 2k + 1$  is odd and  $0, \pm 2j\frac{l}{2}$ ,  $j = 1, \dots, k - 1$  when  $n = 2k$  is even are known as *nodes*. Thus the string vibrates in  $n$  separate parts which are separated from one another by the nodes.

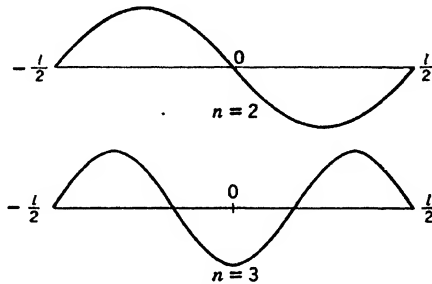


FIG. 37.

We shall confine our attention for the present to second-order differential equations, and we shall suppose that the linear differential operator which is furnished by the differential equation is self-adjoint. An undetermined constant  $\lambda$  will appear in the differential equation, and one of the first problems we have to answer is the following:

*How is the constant  $\lambda$  quantized by the boundary conditions?*

An equivalent formulation of this problem is the following:

*What are the characteristic numbers of the boundary-value problem?*

We shall write our differential equation in the form

$$L(u) + \lambda su = 0;$$

here

$$L(u) = D\{pDu\} + ru,$$

and  $s = s(x)$  is a function of  $x$  which we shall assume (for reasons that will shortly be clear) to be non-negative. The functions  $p$ ,  $r$ , and  $s$  are taken to be continuous over a given interval  $[a, b]$ , and  $p$  is granted to be differentiable over this interval. The boundary conditions involve

the values of  $u$  and  $Du$  at the points  $a$  and  $b$  in a *linear* and *homogeneous* manner. We write them as follows:

$$B_1(u): \alpha_1 u(a) + \alpha'_1 Du(a) + \beta_1 u(b) + \beta'_1 Du(b) = 0;$$

$$B_2(u): \alpha_2 u(a) + \alpha'_2 Du(a) + \beta_2 u(b) + \beta'_2 Du(b) = 0.$$

Thus in the case of the vibrating string for which  $B_1(u): u(a) = 0$  and  $B_2(u): u(b) = 0$  (where  $a = -\frac{l}{2}$ ,  $b = \frac{l}{2}$ ) we have  $(\alpha_1, \alpha'_1, \beta_1, \beta'_1) = (1, 0, 0, 0)$  and  $(\alpha_2, \alpha'_2, \beta_2, \beta'_2) = (0, 0, 1, 0)$ . The vibrating string is an instance of a *special* type of boundary-value problem which is of great importance. For this type of boundary-value problem  $\beta_1 = 0$ ,  $\beta'_1 = 0$  (so that the end-point  $b$  is not mentioned in the statement of the boundary condition  $B_1$ ), and  $\alpha_2 = 0$ ,  $\alpha'_2 = 0$  (so that the end-point  $a$  is not mentioned in the statement of the boundary condition  $B_2$ ). We say that this type of boundary-value problem is *unmixed* (by which we mean, simply, that the end-points  $a$  and  $b$  of the interval  $[a, b]$  are kept separated from one another in the statement of the boundary conditions).

Let us suppose that we have two functions  $u = u(x)$  and  $v = v(x)$  which satisfy the boundary conditions  $B_1$  and  $B_2$ :

$$B_1(u) = 0; \quad B_2(u) = 0; \quad B_1(v) = 0; \quad B_2(v) = 0.$$

Whether or not  $u = u(x)$  and  $v = v(x)$  satisfy the differential equation which is part of the boundary-value problem is, for the moment, of no concern to us. We denote by  $p_{23}, p_{31}, \dots, p_{34}$  the two-rowed determinants which may be obtained from the  $2 \times 4$  matrix

$$\begin{pmatrix} \alpha_1 & \alpha'_1 & \beta_1 & \beta'_1 \\ \alpha_2 & \alpha'_2 & \beta_2 & \beta'_2 \end{pmatrix}$$

by ignoring two of its columns (the subscripts on the  $p$ 's indicate the columns we do *not* ignore and also the order in which these appear in the two-rowed determinant in question). Thus

$$p_{23} = \alpha'_1 \beta_2 - \alpha'_2 \beta_1; \quad p_{31} = \beta_1 \alpha_2 - \beta_2 \alpha_1; \quad p_{12} = \alpha_1 \alpha'_2 - \alpha_2 \alpha'_1;$$

$$p_{14} = \alpha_1 \beta'_2 - \alpha_2 \beta'_1; \quad p_{24} = \alpha'_1 \beta'_2 - \alpha'_2 \beta'_1; \quad p_{34} = \beta_1 \beta'_2 - \beta_2 \beta'_1.$$

In the same way we denote by  $\pi_{23}, \pi_{31}, \dots, \pi_{34}$  the two-rowed determinants obtained from the  $2 \times 4$  matrix

$$\begin{pmatrix} u(a) & Du(a) & u(b) & Du(b) \\ v(a) & Dv(a) & v(b) & Dv(b) \end{pmatrix}$$

by ignoring two of its columns so that

$$\begin{aligned} \pi_{23} &= Du(a)v(b) - Dv(a)u(b); & \pi_{31} &= u(b)v(a) - v(b)u(a); \\ \pi_{12} &= u(a)Dv(a) - v(a)Du(a); & \pi_{14} &= u(a)Dv(b) - v(a)Du(b); \\ \pi_{24} &= Du(a)Dv(b) - Dv(a)Du(b); & \pi_{34} &= u(b)Dv(b) - v(b)Du(b). \end{aligned}$$

Then it is easy to see that the ratios

$$p_{23}:p_{31}:p_{12}:p_{14}:p_{24}:p_{34}$$

are the same as the ratios

$$\pi_{14}:\pi_{24}:\pi_{34}:\pi_{23}:\pi_{31}:\pi_{12}.$$

(Note the rule:  $\pi_{jk}$  corresponds to  $p_{lm}$  if  $(lmjk)$  is an *even* permutation of 1234.) To prove this we eliminate  $u(a)$  from the two relations  $B_1(u) = 0, B_2(u) = 0$  obtaining

$$p_{12}Du(a) + p_{13}u(b) + p_{14}Dv(b) = 0.$$

Similarly  $p_{12}Dv(a) + p_{13}v(b) + p_{14}Dv(b) = 0$ , and on eliminating  $p_{12}$  from these two relations we obtain

$$p_{13}\pi_{23} + p_{14}\pi_{24} = 0.$$

Since  $p_{31} = -p_{13}$  it follows that  $p_{31}:p_{14} = \pi_{24}:\pi_{23}$ , and the equality of the other ratios may be similarly proved.

The boundary-value problems for which

$$p(a)\{u(a)Dv(a) - v(a)Du(a)\} = p(b)\{u(b)Dv(b) - v(b)Du(b)\}$$

(where  $u = u(x), v = v(x)$  are any two functions which satisfy the two boundary conditions and where  $p = p(x)$  is the coefficient of  $D^2$  in the self-adjoint operator  $L$ ) are particularly important and simple. We term such boundary-value problems *regular*, and we shall confine our attention to regular boundary-value problems. Since

$$u(a)Dv(a) - v(a)Du(a) = \pi_{12}, \quad u(b)Dv(b) - v(b)Du(b) = \pi_{34}$$

and since  $\pi_{12}:\pi_{34} = p_{34}:p_{12}$  we have the following result:

**The boundary-value problem**

$$L(u) + \lambda su = 0; \quad B_1(u) = 0; \quad B_2(u) = 0$$

is regular when, and only when,

$$p(a)p_{34} = p(b)p_{12}.$$

Since  $p_{12}$  and  $p_{34}$  are each zero for an unmixed boundary-value problem (why?) we see that

**Every unmixed boundary-value problem is regular.**

The characteristic feature of regular boundary-value problems which makes them particularly simple is the following: Since  $uL(v) - vL(u) = D\{p(uDv - vDu)\}$  it follows that

$$\int_a^b \{uL(v) - vL(u)\} dx = p(uDv - vDu) \Big|_a^b.$$

Hence if  $u$  and  $v$  are any two functions which possess second derivatives which are continuous (or piecewise-continuous) over  $[a, b]$  and which satisfy the boundary conditions  $B_1$  and  $B_2$  then

$$\int_a^b \{uL(v) - vL(u)\} dx = 0$$

if, and only if, the boundary-value problem is regular. *Note carefully* that  $u$  and  $v$  are any two functions satisfying the boundary conditions for which  $L(u)$  and  $L(v)$  are piecewise-continuous over  $[a, b]$ . It is by no means implied that  $u$  and  $v$  are solutions of the boundary-value problem.

**EXERCISES**

1. Show that if  $u$  and  $v$  are two solutions of the boundary-value problem (supposed regular) corresponding to different characteristic numbers  $\lambda$  and  $\mu$  then

$$\int_a^b suv dx = 0.$$

*Hint.*  $L(u) + \lambda su = 0$ ,  $L(v) + \mu sv = 0$ . Hence  $uL(v) - vL(u) = (\lambda - \mu)sv$ .

2. Show that the relation  $\int_{-\frac{l}{2}}^{\frac{l}{2}} \left. \begin{matrix} \sin \frac{n\pi x}{l} \\ \cos \frac{m\pi x}{l} \end{matrix} \right\} \frac{\sin \frac{m\pi x}{l}}{\cos \frac{n\pi x}{l}} dx = 0$ ,  $n \neq m = 1, 2, \dots$

is a consequence of the fact that the problem of the vibrating string is a regular boundary-value problem.

3. Show that the characteristic numbers of a regular boundary-value problem are real (the coefficient functions  $p$ ,  $\tau$ , and  $s \geq 0$  which occur in the differential equation being real as are also the coefficients  $\alpha$  and  $\beta$  which occur in the boundary conditions). *Hint.* If  $u = u(x)$  is a (non-trivial) solution of the boundary-value problem which corresponds to a characteristic number  $\lambda$  then  $v = \bar{u}(x)$  is a solution of the boundary-value problem which corresponds to the characteristic number

$\mu = \bar{\lambda}$ . If  $\lambda$  is not real  $\mu \neq \lambda$  and so  $\int_a^b s\bar{u}u dx = 0$ . Since  $s$  is by hypothesis

non-negative (and not identically zero) the relation  $\int_a^b s\bar{u}u dx = 0$  is absurd (why?).

Hence if a characteristic number exists at all it must be real.

4. Show that if a regular boundary-value problem has two distinct characteristic

numbers  $\lambda$  and  $\mu$  (so that  $\lambda$  and  $\mu$  are real) and if  $u$  and  $v$  are solutions of the boundary-value problem which are associated with the characteristic numbers  $\lambda$  and  $\mu$ , respectively, then  $\int_a^b suv dx = 0$ . *Note.* This important result may be phrased as follows:

If a regular boundary-value problem has two distinct (real) characteristic numbers  $\lambda$  and  $\mu$  and if  $u$  and  $v$  are solutions of the boundary-value problem which are associated with  $\lambda$  and  $\mu$ , respectively, then the function-vectors  $v_1$  and  $v_2$  defined by

$$v_1(x) = s^{1/2}u(x); \quad v_2(x) = s^{1/2}v(x); \quad a \leq x \leq b$$

are orthogonal.

It follows that if a regular boundary-value problem has an infinite sequence of distinct characteristic numbers  $\lambda_1, \lambda_2, \dots, \lambda_n, \dots$  there is associated with the boundary-value problem an orthonormal set. One of the main results which we shall prove in the next chapter is the following:

Every regular boundary-value problem possesses an infinite sequence of characteristic numbers. If all, or all but a finite number, of these characteristic numbers are positive the corresponding orthonormal sequence is complete.

### 3. The Green's function of a regular boundary-value problem for which zero is not a characteristic number

We know that the characteristic numbers of the boundary-value problem of the vibrating string

$$(D^2 + \lambda)u = 0; \quad u\left(-\frac{l}{2}\right) = 0; \quad u\left(\frac{l}{2}\right) = 0$$

are the numbers  $\lambda = \frac{n^2\pi^2}{l^2}$ ,  $n = 1, 2, \dots$ . Thus the boundary-value problem of the vibrating string does not have zero as a characteristic number; in other words the boundary-value problem

$$D^2u = 0; \quad u\left(-\frac{l}{2}\right) = 0; \quad u\left(\frac{l}{2}\right) = 0$$

does not have any solution other than the trivial one  $u \equiv 0$ . An important class of boundary-value problems shares this property with the boundary-value problem of the vibrating string. No problem of the class has zero as a characteristic number. We shall treat this class of boundary-value problems first and shall consider later those boundary-value problems which have zero as a characteristic number.

Assuming, then, that zero is not a characteristic number of our boundary-value problem

$$L(u) + \lambda su = 0; \quad B_1(u) = 0; \quad B_2(u) = 0$$



we know that the boundary-value problem

$$L(u) = 0; \quad B_1(u) = 0; \quad B_2(u) = 0$$

has no solution (other than the trivial one  $u \equiv 0$ ). In other words the conditions imposed on  $u$  are too stringent to permit the existence of a (non-trivial) solution. We lighten these conditions in the slightest possible manner as follows: We still demand that  $u = u(x)$  satisfy the boundary conditions  $B_1(u) = 0$ ,  $B_2(u) = 0$ , and we demand that it satisfy the differential equation  $L(u) = 0$  at every point *except one* of the interval  $[a, b]$ . At this point (which is an arbitrarily selected point  $t$  of the interval) we do not demand that  $u$  possess a second derivative or even a first derivative so that the operator  $L$  becomes devoid of meaning at the point  $t$  for the (non-trivial) solution (of the lightened (or weakened) boundary problem) which we are attempting to determine. This solution is a function of the interval variable  $a \leq x \leq b$  which depends on the accessory variable or parameter  $t$ . We shall see shortly that it is unambiguously determinate if we impose a certain condition concerning the permissible discontinuity of the first derivative at the point  $t$  (no discontinuity in the function itself being permitted). This unambiguously determinate solution of the lightened boundary-value problem

$$L(u) = 0; \quad x \neq t; \quad B_1(u) = 0; \quad B_2(u) = 0$$

is known as the *Green's function* of the original boundary-value problem

$$L(u) + \lambda su = 0; \quad B_1(u) = 0; \quad B_2(u) = 0$$

(after G. Green [1793–1841], an English mathematician). We shall denote it by the symbol

$$G \begin{pmatrix} x \\ t \end{pmatrix}.$$

The upper symbol  $x$  indicates the variable of which  $G \begin{pmatrix} x \\ t \end{pmatrix}$  is a function, and the lower symbol  $t$  indicates the point of the interval  $[a, b]$  at which a discontinuity in the derivative  $DG$  of  $G$  is permitted.

*Note.* We use the symbol  $G \begin{pmatrix} x \\ t \end{pmatrix}$  in anticipation of the result, which we shall shortly prove, that there is closely associated with  $G \begin{pmatrix} x \\ t \end{pmatrix}$  a

linear *integral* operator  $\mathbf{K} : K \begin{pmatrix} x \\ t \end{pmatrix}$  which serves to replace the linear *differential* operator  $L + \lambda s$  and the boundary conditions  $B_1$  and  $B_2$ .

The fact that we have selected a point  $t$  of the interval  $[a, b]$ , at which the linear differential operator is devoid of meaning when applied to  $G \begin{pmatrix} x \\ t \end{pmatrix}$ , forces us to consider *two* intervals  $[a, t]$  and  $[t, b]$  rather than the single interval  $[a, b]$ . The function  $G \begin{pmatrix} x \\ t \end{pmatrix}$  must be a solution of the equation  $L(G) = 0$  over each of the two intervals, but it need not be the *same* solution. In other words, if  $u_1 = u_1(x)$  and  $u_2 = u_2(x)$  are two linearly independent solutions of the equation  $L(u) = 0$ , the Green's function which we are seeking must be a linear combination of  $u_1$  and  $u_2$  over each of the intervals  $[a, t]$ ,  $[t, b]$ , but it need not be the *same* linear combination over these two intervals. We denote by  $G_l \begin{pmatrix} x \\ t \end{pmatrix}$  the linear combination

$$G_l \begin{pmatrix} x \\ t \end{pmatrix} = c_1 u_1(x) + c_2 u_2(x)$$

over the left interval  $[a, t]$ , and we denote by  $G_r \begin{pmatrix} x \\ t \end{pmatrix}$  the linear combination

$$G_r \begin{pmatrix} x \\ t \end{pmatrix} = c_3 u_1(x) + c_4 u_2(x)$$

over the right interval  $[t, b]$ . *Warning.* Do not fall into the mistake of thinking that there are *two* Green's functions  $G_l \begin{pmatrix} x \\ t \end{pmatrix}$  and  $G_r \begin{pmatrix} x \\ t \end{pmatrix}$ ;  $G_l \begin{pmatrix} x \\ t \end{pmatrix}$  is a function of the interval variable  $[a, t]$  while  $G_r \begin{pmatrix} x \\ t \end{pmatrix}$  is a function of the interval variable  $[t, b]$ . The Green's function  $G \begin{pmatrix} x \\ t \end{pmatrix}$  of our boundary-value problem is a function of the interval variable  $[a, b]$  which is defined as follows:

$$G\left(\begin{matrix} x \\ t \end{matrix}\right) = G_l\left(\begin{matrix} x \\ t \end{matrix}\right) \text{ over } [a, t], \text{ i.e., if } x \leq t;$$

$$G\left(\begin{matrix} x \\ t \end{matrix}\right) = G_r\left(\begin{matrix} x \\ t \end{matrix}\right) \text{ over } [t, b], \text{ i.e., if } x \geq t.$$

The two functions  $G_l\left(\begin{matrix} x \\ t \end{matrix}\right)$  and  $G_r\left(\begin{matrix} x \\ t \end{matrix}\right)$  must *fit together* at the point  $t$  which is common to their intervals of definition  $[a, t]$  and  $[t, b]$ , respectively. This establishes one linear, homogeneous relation between the four coefficients  $(c_1, c_2, c_3, c_4)$  of the two linear combinations of  $u_1$  and  $u_2$  which serve to define  $G_l\left(\begin{matrix} x \\ t \end{matrix}\right)$  and  $G_r\left(\begin{matrix} x \\ t \end{matrix}\right)$ ; this relation is

$$c_1u_1(t) + c_2u_2(t) = c_3u_1(t) + c_4u_2(t).$$

The fact that  $G\left(\begin{matrix} x \\ t \end{matrix}\right)$  must satisfy the two boundary conditions

$$B_1(G) = 0; \quad B_2(G) = 0$$

imposes two more linear homogeneous relations between the four coefficients,  $(c_1, c_2, c_3, c_4)$ , namely,

$$\alpha_1 G_l\left(\begin{matrix} a \\ t \end{matrix}\right) + \alpha'_1 D G_l\left(\begin{matrix} a \\ t \end{matrix}\right) + \beta_1 G_r\left(\begin{matrix} b \\ t \end{matrix}\right) + \beta'_1 D G_r\left(\begin{matrix} b \\ t \end{matrix}\right) = 0;$$

$$\alpha_2 G_l\left(\begin{matrix} a \\ t \end{matrix}\right) + \alpha'_2 D G_l\left(\begin{matrix} a \\ t \end{matrix}\right) + \beta_2 G_r\left(\begin{matrix} b \\ t \end{matrix}\right) + \beta'_2 D G_r\left(\begin{matrix} b \\ t \end{matrix}\right) = 0.$$

We suppose that these three linear homogeneous relations between the four coefficients  $(c_1, c_2, c_3, c_4)$  serve to determine the ratios of these four coefficients. Then  $G\left(\begin{matrix} x \\ t \end{matrix}\right)$  is determined up to a multiplicative constant.

### Example

For the boundary-value problem whose solution determines the free vibrations of a tightly stretched string  $L = D^2$  so that we may set  $u_1 = 1$ ,  $u_2 = x$ . Hence  $G_l\left(\begin{matrix} x \\ t \end{matrix}\right) = c_1 + c_2x$ ;  $G_r\left(\begin{matrix} x \\ t \end{matrix}\right) = c_3 + c_4x$ , and

the *fitting together* relation  $G_l \left( \begin{smallmatrix} t \\ t \end{smallmatrix} \right) = G_r \left( \begin{smallmatrix} t \\ t \end{smallmatrix} \right)$  yields  $c_1 + c_2 t = c_3 + c_4 t$ .

The boundary condition  $B_1(G) = 0$  is  $G_l \left( \begin{smallmatrix} -l/2 \\ t \end{smallmatrix} \right) = 0$  so that  $c_1 - c_2 \frac{l}{2} = 0$ ; the boundary condition  $B_2(G) = 0$  is  $G_r \left( \begin{smallmatrix} l/2 \\ t \end{smallmatrix} \right) = 0$  so that  $c_3 + c_4 \frac{l}{2} = 0$ . Hence  $c_1 = \frac{l}{2} c_2$ ;  $c_3 = -\frac{l}{2} c_4$ ,  $\left(\frac{l}{2} + t\right) c_2 = \left(-\frac{l}{2} + t\right) c_4$  so that  $c_1 : c_2 : c_3 : c_4 = \frac{l}{2} \left(-\frac{l}{2} + t\right) : -\frac{l}{2} + t : -\frac{l}{2} \left(\frac{l}{2} + t\right) : \frac{l}{2} + t$ . Thus

$$G_l \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) = c \left(-\frac{l}{2} + t\right) \left(\frac{l}{2} + x\right); \quad G_r \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) = c \left(\frac{l}{2} + t\right) \left(-\frac{l}{2} + x\right),$$

where  $c$  is an undetermined constant.

The undetermined multiplicative constant  $c$  is fixed by the following requirement:

$$p(t) \left\{ DG_l \left( \begin{smallmatrix} t \\ t \end{smallmatrix} \right) - DG_r \left( \begin{smallmatrix} t \\ t \end{smallmatrix} \right) \right\} = 1.$$

The reason for this (at present artificial looking) requirement will be clear in the next paragraph. It requires that  $p(t) \left\{ DG_l \left( \begin{smallmatrix} t \\ t \end{smallmatrix} \right) - DG_r \left( \begin{smallmatrix} t \\ t \end{smallmatrix} \right) \right\}$  be different from zero over  $[a, b]$ , and we shall make this assumption.

In the case of the vibrating string

$$DG_l \left( \begin{smallmatrix} t \\ t \end{smallmatrix} \right) = c_2 = c \left(-\frac{l}{2} + t\right); \quad DG_r \left( \begin{smallmatrix} t \\ t \end{smallmatrix} \right) = c_4 = c \left(\frac{l}{2} + t\right)$$

so that  $DG_l \left( \begin{smallmatrix} t \\ t \end{smallmatrix} \right) - DG_r \left( \begin{smallmatrix} t \\ t \end{smallmatrix} \right) = -cl$ . Since  $p(t) = 1$ ,  $c = -\frac{1}{l}$ . Hence the Green's function for the vibrating string is

$$G \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) = \begin{cases} G_l \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) = l \left(\frac{1}{2} - \frac{t}{l}\right) \left(\frac{1}{2} + \frac{x}{l}\right); & -\frac{l}{2} \leq x \leq t; \\ G_r \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) = l \left(\frac{1}{2} + \frac{t}{l}\right) \left(\frac{1}{2} - \frac{x}{l}\right); & t \leq x \leq \frac{l}{2} \end{cases}$$

Thus the graph of the Green's function for the vibrating string is a broken line connecting the two ends  $\left(-\frac{l}{2}, 0\right), \left(\frac{l}{2}, 0\right)$  of the string, the

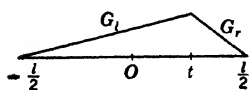


FIG. 38

slope of the left part of the broken line being greater than the slope of the right part by 1.

The Green's function of a regular self-adjoint boundary-value problem possesses a remarkable property:

$G\left(\begin{matrix} x \\ t \end{matrix}\right)$  is a symmetric function of the two variables  $(x, t)$ ; i.e.,

$$G\left(\begin{matrix} t \\ x \end{matrix}\right) = G\left(\begin{matrix} x \\ t \end{matrix}\right).$$

In order to prove this let  $t_1$  and  $t_2 > t_1$  be any two points of the open interval  $(a, b)$ , and consider the two functions  $u(x) = G\left(\begin{matrix} x \\ t_1 \end{matrix}\right)$ ,  $v(x) = G\left(\begin{matrix} x \\ t_2 \end{matrix}\right)$ . Then we have to prove that  $u(t_2) = v(t_1)$ . Since both  $u$  and  $v$  satisfy the boundary conditions  $B_1$  and  $B_2$  (why?) and since the boundary value problem is regular we know that

$$p(uDv - vDu)\Big|_a^b = 0.$$

Furthermore, since the boundary-value problem is self-adjoint, we know that, at any point  $x$  where  $u$  and  $v$  possess second derivatives,

$$D\{p(uDv - vDu)\} = uL(v) - vL(u) = 0$$

(why? Remember that  $L\left\{G\left(\begin{matrix} x \\ t \end{matrix}\right)\right\} = 0$  for every  $t$ ). Since  $u$  does not possess a first derivative at  $x = t_1$  and since  $v$  does not possess a first derivative at  $x = t_2$  this relation is meaningless at the points  $t_1$  and  $t_2$ ; however, it is true over the open interval  $(t_1, t_2)$  and the half-open intervals  $[a, t_1)$  and  $(t_2, b]$ , and on integrating it over these intervals we obtain

$$p(uDv - vDu)\Big|_a^{t_1^-} + \Big|_{t_1^+}^{t_2^-} + \Big|_{t_2^+}^b = 0,$$

where we mean by a symbol  $t_1^+$ , for example, that we take the limit of  $p(uDv - vDu)$  as  $x$  approaches  $t_1$  through values greater than  $t_1$ .

Now

$$p(uDv - vDu) \Big|_a^{t_1-} + \Big|_{t_1+}^{t_2-} + \Big|_{t_2+}^b = p(uDv - vDu) \Big|_a^b + \Big|_{t_1+}^{t_1-} + \Big|_{t_1+}^{t_2-}$$

since each side means the same thing, namely, the sum of the values of  $p(uDv - vDu)$  at the points  $t_1-$ ,  $t_2-$ ,  $b$  less the sum of the values of  $p(uDv - vDu)$  at the points  $a$ ,  $t_1+$ , and  $t_2+$ . The first of the three terms in the expression on the right is zero (why?); since  $p$ ,  $u$ ,  $v$ , and  $Dv$  are continuous at  $x = t_1$  the second term =  $-p(t_1)v(t_1) \left\{ Du \Big|_{t_1+}^{t_1-} \right\}$  and since  $p$ ,  $u$ ,  $v$ , and  $Du$  are continuous at  $x = t_2$  the third term =  $p(t_2)u(t_2) \left\{ Dv \Big|_{t_2+}^{t_2-} \right\}$ . Hence we shall have  $u(t_2) = v(t_1)$  if, and only if,

$$p(t_2)D(v) \Big|_{t_2+}^{t_1-} = p(t_1)D(u) \Big|_{t_1+}^{t_1-}$$

or, equivalently,

$$p(x)DG \left( \begin{matrix} x \\ t_2 \end{matrix} \right) \Big|_{t_2+}^{t_1-} = p(x)DG \left( \begin{matrix} x \\ t_1 \end{matrix} \right) \Big|_{t_1+}^{t_1-}$$

We arrange the multiplicative constant which remained undetermined in the definition of  $G \left( \begin{matrix} x \\ t \end{matrix} \right)$  so that this equality is valid; in other words we determine this multiplicative constant (which is a function of  $t$ ) so that  $p(x)DG \left( \begin{matrix} x \\ t \end{matrix} \right) \Big|_{t+}^{t-}$  is independent of  $t$ . We choose it to be 1

(note that this choice makes  $p(t)DG_t \left( \begin{matrix} t \\ t \end{matrix} \right)$  greater than  $p(t)DG_t \left( \begin{matrix} t \\ t \end{matrix} \right)$ ).

We have proved, then, the following result: If  $a < t_1 < t_2 < b$ ,

$$G \left( \begin{matrix} t_1 \\ t_2 \end{matrix} \right) = G \left( \begin{matrix} t_2 \\ t_1 \end{matrix} \right).$$

Since this relation is equivalent to the relation

$$G_1 \left( \begin{matrix} t_1 \\ t_2 \end{matrix} \right) = G_1 \left( \begin{matrix} t_2 \\ t_1 \end{matrix} \right)$$

and since  $G_1 \left( \begin{matrix} t_1 \\ t_2 \end{matrix} \right)$  is a continuous function of  $t_1$  over  $[a, t_2]$  it follows that

$G_r \begin{pmatrix} t_2 \\ t_1 \end{pmatrix}$  is a continuous function of  $t_1$  over  $[a, t_2]$  and that  $G_l \begin{pmatrix} a \\ t_2 \end{pmatrix} = G_r \begin{pmatrix} t_2 \\ a \end{pmatrix}$ ; i.e.,  $G \begin{pmatrix} a \\ t_2 \end{pmatrix} = G \begin{pmatrix} t_2 \\ a \end{pmatrix}$ . Similarly  $G \begin{pmatrix} t_1 \\ b \end{pmatrix} = G \begin{pmatrix} b \\ t_1 \end{pmatrix}$  and  $G \begin{pmatrix} a \\ b \end{pmatrix} = G \begin{pmatrix} b \\ a \end{pmatrix}$ . Thus

$$G \begin{pmatrix} x \\ t \end{pmatrix} = G \begin{pmatrix} t \\ x \end{pmatrix}; \quad a \leq x \leq b; \quad a \leq t \leq b.$$

*Warning.* Be careful, in checking the Green's function of a boundary-value problem for symmetry, that you do not thoughtlessly merely interchange  $x$  and  $t$  in either  $G_l \begin{pmatrix} x \\ t \end{pmatrix}$  or  $G_r \begin{pmatrix} x \\ t \end{pmatrix}$ . Neither  $G_l \begin{pmatrix} x \\ t \end{pmatrix}$  nor  $G_r \begin{pmatrix} x \\ t \end{pmatrix}$  is, in general, symmetric; the symmetry of  $G \begin{pmatrix} x \\ t \end{pmatrix}$  is expressed by the relation

$$G_l \begin{pmatrix} x \\ t \end{pmatrix} = G_r \begin{pmatrix} t \\ x \end{pmatrix}.$$

Thus once you have  $G_l \begin{pmatrix} x \\ t \end{pmatrix}$  you may write down  $G_r \begin{pmatrix} x \\ t \end{pmatrix}$  by merely interchanging the variables  $(x, t)$  in the expression for  $G_l \begin{pmatrix} x \\ t \end{pmatrix}$ . However, this is merely a convenient memory rule since, in order to determine  $G_l \begin{pmatrix} x \\ t \end{pmatrix}$ , you have to simultaneously determine  $G_r \begin{pmatrix} x \\ t \end{pmatrix}$ .

#### 4. The Green's function as a linear integral operator

Let  $G \begin{pmatrix} x \\ t \end{pmatrix}$  be the Green's function of a regular self-adjoint boundary-value problem (for which zero is not a characteristic number), and let  $h = h(x)$  be any function which is continuous over  $[a, b]$ . We regard the function  $h$  of the interval variable  $[a, b]$  as a function-vector  $\mathbf{h}$ , and we regard the function  $G \begin{pmatrix} x \\ t \end{pmatrix}$  of the square variable  $a \leq x \leq b$ ,

$a \leq t \leq b$  as a linear integral operator  $\mathbf{G}$ . Then

$$\mathbf{G}h: \int_a^b G \begin{pmatrix} x \\ t \end{pmatrix} h(t) dt = k(x); \quad a \leq x \leq b$$

is a new function-vector  $\mathbf{k}$  which is in important respects *smoother* than  $\mathbf{h}$ . Thus  $k(x)$  is not only continuous but differentiable over  $[a, b]$  while  $h(x)$  was not assumed to be differentiable over  $[a, b]$ . To show that  $k(x)$  is differentiable over  $[a, b]$  we write its definition in the form

$$\begin{aligned} k(x) &= \int_a^x G \begin{pmatrix} x \\ t \end{pmatrix} h(t) dt + \int_x^b G \begin{pmatrix} x \\ t \end{pmatrix} h(t) dt \\ &= \int_a^x G_r \begin{pmatrix} x \\ t \end{pmatrix} h(t) dt + \int_x^b G_l \begin{pmatrix} x \\ t \end{pmatrix} h(t) dt. \end{aligned}$$

Since both  $G_r \begin{pmatrix} x \\ t \end{pmatrix}$  and  $G_l \begin{pmatrix} x \\ t \end{pmatrix}$  are differentiable (with respect to  $x$ ) over  $[a, b]$  it follows by the rule for differentiating an integral that  $k(x)$  is differentiable over  $[a, b]$  with

$$\begin{aligned} Dk &= G_r \begin{pmatrix} x \\ x \end{pmatrix} h(x) + \int_a^x DG_r \begin{pmatrix} x \\ t \end{pmatrix} h(t) dt - G_l \begin{pmatrix} x \\ x \end{pmatrix} h(x) \\ &\quad + \int_x^b DG_l \begin{pmatrix} x \\ t \end{pmatrix} h(t) dt \\ &= \int_a^x DG_r \begin{pmatrix} x \\ t \end{pmatrix} h(t) dt + \int_x^b DG_l \begin{pmatrix} x \\ t \end{pmatrix} h(t) dt \end{aligned}$$

since  $G_r \begin{pmatrix} x \\ x \end{pmatrix} = G_l \begin{pmatrix} x \\ x \end{pmatrix}$ . Thus  $Dk$  may be written in the form

$$Dk = \int_a^b DG \begin{pmatrix} x \\ t \end{pmatrix} h(t) dt.$$

*Warning.* Do not fall into the error of trying to obtain this result "cheaply" by merely differentiating, under the sign, the integral  $\int_a^b G \begin{pmatrix} x \\ t \end{pmatrix} h(t) dt$  which furnishes  $k(x)$ . This is not a legitimate procedure since  $G \begin{pmatrix} x \\ t \end{pmatrix}$  is not differentiable over  $[a, b]$  (although  $G_l$  and  $G_r$



are differentiable over their intervals of definition). The fact that the incorrect procedure would furnish the correct result is due to the (accidental) facts that  $G_l \left( \begin{smallmatrix} x \\ x \end{smallmatrix} \right) = G_r \left( \begin{smallmatrix} x \\ x \end{smallmatrix} \right)$  and that  $h$  is continuous at  $x$ .

It follows by repeating the argument that  $k(x)$  is not only differentiable over  $[a, b]$  but that it possesses a continuous *second* derivative over  $[a, b]$ . However, this second derivative is *not* furnished by the formula

$$D^2k = \int_a^b D^2G \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) h(t) dt. \quad \text{In fact}$$

$$Dk = \int_a^x DG_r \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) h(t) dt + \int_x^b DG_l \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) h(t) dt$$

so that

$$\begin{aligned} D^2k &= DG_r \left( \begin{smallmatrix} x \\ x \end{smallmatrix} \right) h(x) + \int_a^x D^2G_r \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) h(t) dt - DG_l \left( \begin{smallmatrix} x \\ x \end{smallmatrix} \right) h(x) \\ &\quad + \int_x^b D^2G_l \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) h(t) dt \\ &= \left\{ DG_r \left( \begin{smallmatrix} x \\ x \end{smallmatrix} \right) - DG_l \left( \begin{smallmatrix} x \\ x \end{smallmatrix} \right) \right\} h(x) + \int_a^b D^2G \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) h(t) dt. \end{aligned}$$

Hence  $p(x)D^2k = -h(x) + \int_a^b p(x)D^2G \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) h(t) dt$ , by virtue of the relation

$$pDG \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) \Big|_{t+}^{t-} = 1.$$

On combining this result with the two relations

$$Dp(x)Dk = \int_a^b Dp(x)DG \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) h(t) dt;$$

$$r(x)k = \int_a^b r(x)G \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) h(t) dt$$

(note that  $t$  is the variable of integration so that  $x$  is constant in the integration) we obtain, since  $L = pD^2 + (Dp)D + r$ ,

$$\begin{aligned} Lk &= -h + \int_a^b LG \begin{pmatrix} x \\ t \end{pmatrix} h(t) dt \\ &= -h \end{aligned}$$

since  $LG \begin{pmatrix} x \\ t \end{pmatrix} = 0$  for every point  $P:(x, t)$  of the square  $a \leq x \leq b$ ,  $a \leq t \leq b$  except the points of the diagonal  $x = t$  of this square. In view of the definition  $\mathbf{k} = \mathbf{G}\mathbf{h}$  we may write this result as follows:

$$L\mathbf{G}\mathbf{h} = -\mathbf{h}.$$

In words:

**The linear integral operator  $\mathbf{G}$  followed by the linear differential operator  $L$  is equivalent to multiplication by  $-1$ .**

It is clear that the function-vector  $\mathbf{k} = \mathbf{G}\mathbf{h}$  satisfies the boundary conditions  $B_1(\mathbf{k}) = 0$ ,  $B_2(\mathbf{k}) = 0$  (it being indifferent whether or not the function-vector  $\mathbf{h}$  satisfies these boundary conditions). In fact

$$\begin{aligned} B_1[\mathbf{G}(\mathbf{h})] &= \alpha_1 \int_a^b G \begin{pmatrix} a \\ t \end{pmatrix} h(t) dt + \alpha'_1 \int_a^b DG \begin{pmatrix} a \\ t \end{pmatrix} h(t) dt \\ &\quad + \beta_1 \int_a^b G \begin{pmatrix} b \\ t \end{pmatrix} h(t) dt + \beta'_1 \int_a^b DG \begin{pmatrix} b \\ t \end{pmatrix} h(t) dt \\ &= \int_a^b [B_1 G] h(t) dt = 0, \end{aligned}$$

and, similarly,  $B_2(\mathbf{k}) = 0$ . Thus

**The linear integral operator  $\mathbf{G}$  not only smooths out the arbitrary continuous function-vector  $\mathbf{h}$  so that the linear differential operator  $L$  can feed on it but it also adjusts  $\mathbf{k}$  to the boundary conditions. When  $L$  feeds on  $\mathbf{k} = \mathbf{G}\mathbf{h}$  it returns to us the original function-vector  $\mathbf{h}$  changed in sign.**

Conversely let  $k(x)$  be a function which satisfies the boundary conditions  $B_1$  and  $B_2$  and which possesses a continuous second derivative over  $[a, b]$ . On denoting  $Lk$  by  $-h$  we have, on combining the two relations

$$L(k) = -h; \quad L(G) = 0$$

in the manner with which you are now familiar,

$$kL(G) - GL(k) = Gh.$$

Since both  $k$  and  $G$  satisfy the boundary conditions  $B_1$  and  $B_2$  we have

$p(kDG - GDk) \Big|_a^b = 0$  (since the boundary-value problem is regular).

Hence on integrating the relation

$$\begin{aligned} G \begin{pmatrix} x \\ t \end{pmatrix} h(x) &= kL \left\{ G \begin{pmatrix} x \\ t \end{pmatrix} \right\} - G \begin{pmatrix} x \\ t \end{pmatrix} L(k) \\ &= D \left\{ p(x) \left[ kDG \begin{pmatrix} x \\ t \end{pmatrix} - G \begin{pmatrix} x \\ t \end{pmatrix} Dk \right] \right\} \end{aligned}$$

over the intervals  $[a, t]$  and  $[t, b]$  and adding the results we obtain

$$\int_a^b G \begin{pmatrix} x \\ t \end{pmatrix} h(x) dx = k(t)$$

since

$$p(x) DG \begin{pmatrix} x \\ t \end{pmatrix} \Big|_{t+}^{t-} = 1.$$

On interchanging the roles of the variables  $x$  and  $t$  we obtain

$$\int_a^b G \begin{pmatrix} t \\ x \end{pmatrix} h(t) dt = k(x),$$

and, by virtue of the symmetry of  $G \begin{pmatrix} x \\ t \end{pmatrix}$ , this is equivalent to

$$\int_a^b G \begin{pmatrix} x \\ t \end{pmatrix} h(t) dt = k(x);$$

i.e.,

$$Gh = k.$$

Thus not only is  $LGh = -h$  but  $GLk = -k$ . Combining these results we have the following fundamental theorem:

The linear integral operator  $G$  and the linear differential operator  $L$  combine, in either order, to yield a simple change in sign.  $G$  feeds on any continuous vector  $h$  and returns a vector  $k$  which possesses a continuous second derivative and which satisfies the boundary conditions  $B_1$  and  $B_2$ .  $L$  feeds on function-vectors  $k$  which possess continuous second derivatives over  $[a, b]$ ; if, in addition,  $k$  satisfies

the boundary conditions  $B_1$  and  $B_2$ , it is of the form  $Gh$ , where  $h = -Lk$ . In other words either of the two relations

$$Lk = -h; \quad k = Gh$$

is equivalent to the other.

We indicate this theorem by the symbolism

$$LG = GL = -1.$$

*Warning.* In using this symbolism be sure that you understand the difference between the vectors  $h$  and  $k$  on which  $G$  and  $L$ , respectively, operate.  $h$  is any continuous vector while  $k$  must not only possess a continuous second derivative but must also satisfy the boundary conditions  $B_1$  and  $B_2$ .

### 5. The equivalence of the boundary-value problem and an integral equation

We can now easily show that any solution of our boundary-value problem is a solution of an integral equation and, conversely, that any solution of this integral equation is a solution of the boundary-value problem. In proving this theorem it is just as easy to deal with a *non-homogeneous* boundary problem

$$L(y) + \lambda sy = f; \quad B_1(y) = 0; \quad B_2(y) = 0$$

(where  $f = f(x)$  is a given continuous function of  $x$  and  $y = y(x)$  is the sought for function) as with the (associated) homogeneous boundary-value problem

$$L(u) + \lambda su = 0; \quad B_1(u) = 0; \quad B_2(u) = 0,$$

and we shall do this. We first suppose that  $y = y(x)$  is a solution of the non-homogeneous boundary-value problem. Then  $L(y) = -\lambda sy + f$  so that

$$GL(y) = -\lambda G(sy) + Gf.$$

Since  $GL(y) = -y$  we have

$$y = \lambda Ky + F,$$

where  $K: G \begin{pmatrix} x \\ t \end{pmatrix} s(t)$  and  $F: - \int_a^b G \begin{pmatrix} x \\ t \end{pmatrix} f(t) dt$ . Thus  $y = y(x)$  is a solution of the following *integral equation*

$$y(x) = \lambda \int_a^b K \begin{pmatrix} x \\ t \end{pmatrix} y(t) dt + F(x),$$

where  $K \begin{pmatrix} x \\ t \end{pmatrix} = G \begin{pmatrix} x \\ t \end{pmatrix} s(t)$ ;  $F(x) = - \int_a^b G \begin{pmatrix} x \\ t \end{pmatrix} f(t) dt$ . The function  $K \begin{pmatrix} x \\ t \end{pmatrix}$  of the square variable  $a \leq x \leq b$ ,  $a \leq t \leq b$  is termed the *kernel* of the integral equation. Since  $s(x)$  is non-negative we may replace the integral equation by an integral equation whose kernel is symmetric. All we have to do is to introduce the new unknown

$$z = z(x) = \{s(x)\}^{1/2} y(x);$$

then the function-vector  $\mathbf{z}$  satisfies the integral equation

$$\mathbf{z} = \lambda \mathbf{H} \mathbf{z} + \mathbf{F}_1,$$

where  $\mathbf{H} : G \begin{pmatrix} x \\ t \end{pmatrix} \{s(x)s(t)\}^{1/2}$  and  $\mathbf{F}_1 : \{s(x)\}^{1/2} F(x)$ . (Prove this.) Since  $G \begin{pmatrix} x \\ t \end{pmatrix}$  is symmetric it is clear that  $H \begin{pmatrix} x \\ t \end{pmatrix}$  is symmetric.

Conversely, let  $y = y(x)$  be any solution of the integral equation  $y = \lambda \mathbf{K} y + \mathbf{F}$ . Since  $\lambda \mathbf{K} y$  and  $\mathbf{F}$  are continuous (why?) so also is  $y$  (why?). Since

$$\mathbf{K} y : \int_a^b K \begin{pmatrix} x \\ t \end{pmatrix} y(t) dt = \int_a^b G \begin{pmatrix} x \\ t \end{pmatrix} s(t) y(t) dt$$

we have

$$\mathbf{L} \mathbf{K} y = \mathbf{L} G(sy) = -sy$$

and since  $\mathbf{F} = -\mathbf{G}f$  we have  $\mathbf{L}(\mathbf{F}) = \mathbf{f}$ . Hence

$$\mathbf{L}(y) = -\lambda sy + \mathbf{f}$$

so that  $y = y(x)$  satisfies the differential equation

$$\mathbf{L}(y) + \lambda s(x)y(x) = f(x).$$

Furthermore  $\mathbf{K} y = \mathbf{G}(sy)$  and  $\mathbf{F} = -\mathbf{G}f$  each satisfies the boundary conditions  $B_1$  and  $B_2$ . Hence  $y = \lambda \mathbf{K} y + \mathbf{F}$  satisfies the boundary conditions  $B_1$  and  $B_2$ . This completes the proof of the *equivalence theorem*:

**Each solution of the boundary-value problem is a solution of the integral equation, and each solution of the integral equation is a solution of the boundary-value problem.**

We shall discuss in detail in Chapter 8 the integral equation

$$y(x) = \lambda \int_a^b K \begin{pmatrix} x \\ t \end{pmatrix} y(t) dt + F(x)$$

or, equivalently,

$$y = \lambda Ky + F.$$

The equivalence theorem assures us that in solving this integral equation we solve the boundary-value problem: every solution of the integral equation is a solution of the boundary-value problem, and no solution of the boundary-value problem escapes us. The essential advantage of the integral equation point of view is that the boundary conditions are *automatically* cared for; once the Green's function has been constructed from a knowledge of these boundary conditions we do not have to worry about them any more since *every* function-vector of the form  $Gh$  satisfies them.

We carry over the terms *characteristic number* and *characteristic function* from the boundary-value problem to the integral equation.

Thus a characteristic number of the integral equation of which  $K \begin{pmatrix} x \\ t \end{pmatrix}$  is the kernel is a value of  $\lambda$  for which the homogeneous integral equation  $u = \lambda Ku$  possesses a non-trivial solution, i.e., a solution other than the zero function-vector  $u: u(x) \equiv 0$ . Any such non-trivial solution is a *characteristic function* or *characteristic vector* which is said to be *associated* with the characteristic number in question.

### EXERCISE

1. Show that when  $\lambda$  is a characteristic number of the boundary-value problem then the non-homogeneous boundary-value problem does not, in general, possess a solution. *Hint.* If  $u$  is a characteristic vector associated with  $\lambda$  so also is  $\bar{u}$ ; combine the two equations  $L(\bar{u}) + \lambda s\bar{u} = 0$ ,  $L(y) + \lambda sy = f$  to obtain  $\bar{u}L(y) - yL(\bar{u}) = \bar{u}f$ . Upon integrating and using the fact that the boundary-value problem is regular we obtain  $(u|f) = 0$  (remember that  $\bar{u}$  and  $y$  both satisfy the boundary conditions). *Note.* The important result of this exercise may be phrased as follows:

**In order that the non-homogeneous boundary-value problem may possess a solution when  $\lambda$  is a characteristic number the function-vector  $f$  must be orthogonal to every characteristic vector  $u$  which is associated with  $\lambda$ .**

We shall see in the next chapter that this necessary condition is sufficient.

### 6. The Green's function when zero is a characteristic number

The boundary-value problem

$$(D^2 + \lambda)u = 0, \quad Du \left( -\frac{l}{2} \right) = 0; \quad Du \left( \frac{l}{2} \right) = 0$$

has  $\lambda = 0$  as a characteristic number. In fact  $u(x) = 1$  is a non-trivial solution of the boundary-value problem

$$D^2u = 0, \quad Du\left(-\frac{t}{2}\right) = 0, \quad Du\left(\frac{t}{2}\right) = 0.$$

The construction of the Green's function of a boundary-value problem which has zero as a characteristic number is as follows. Let  $\mathbf{v}$  be a real characteristic vector which is associated with the characteristic number zero. The assumption that  $\mathbf{v}$  is real is merely a simplifying assumption which involves no loss in generality; in fact if  $\mathbf{v}$  is any characteristic vector which is associated with the characteristic number zero so also are  $\bar{\mathbf{v}}$  and  $\mathbf{v} + \bar{\mathbf{v}}$ , and  $\mathbf{v} + \bar{\mathbf{v}}$  is a real vector. Since any multiple of  $\mathbf{v}$  is also a characteristic vector which is associated with the characteristic number zero we may assume, without any lack of generality, that  $s^{1/2}\mathbf{v}$  is a unit vector. We consider the non-homogeneous boundary-value problem

$$L(z) = s(x)v(x)v(t); \quad B_1(z) = 0, \quad B_2(z) = 0.$$

Since  $s(x)v(x)v(t)$  is not, in general, orthogonal to  $v(x)$  (why?) this non-homogeneous boundary-value problem does not, in general, possess a solution (see Exercise 1, p. 229). We seek solutions of the differential equation over  $[a, t]$  and over  $[t, b]$ , and we patch these together to form the desired Green's function:

$$G\left(\begin{matrix} x \\ t \end{matrix}\right) = \begin{cases} G_l\left(\begin{matrix} x \\ t \end{matrix}\right) & \text{if } a \leq x \leq t; \\ G_r\left(\begin{matrix} x \\ t \end{matrix}\right) & \text{if } t \leq x \leq b; \end{cases} \quad B_1(G) = 0; \quad B_2(G) = 0.$$

On combining in the usual way (what is this?) the equations

$$L(v) = 0; \quad L(G) = s(x)v(x)v(t)$$

we obtain

$$vL(G) - GL(v) = s(x)v^2(x)v(t),$$

and on integrating this over the intervals  $[a, t]$ ,  $[t, b]$  and adding we find (since both  $v$  and  $G$  satisfy the boundary conditions  $B_1$  and  $B_2$  and since the boundary-value problem is self-adjoint and regular)

$$pDG\Big|_{t+}^{t-} = 1$$

at those points where  $v(t) \neq 0$ . Thus we do not have to use our last undetermined constant to ensure the validity of this relation; we use this constant to make the integral  $\int_a^b G\left(\begin{smallmatrix} x \\ t \end{smallmatrix}\right) s(x)v(x) dx$  zero since this will ensure the symmetry of  $G\left(\begin{smallmatrix} x \\ t \end{smallmatrix}\right)$ . In fact if  $a < t_1 < t_2 < b$  and we denote  $G\left(\begin{smallmatrix} x \\ t_1 \end{smallmatrix}\right)$  by  $z_1$  and  $G\left(\begin{smallmatrix} x \\ t_2 \end{smallmatrix}\right)$  by  $z_2$  we find, on combining the two equations

$$L(z_1) = s(x)v(x)v(t_1); \quad L(z_2) = s(x)v(x)v(t_2),$$

that  $z_2L(z_1) - z_1L(z_2) = s(x)v(x)\{v(t_1)z_2 - v(t_2)z_1\}$ . Upon integrating this relation over the intervals  $[a, t_1]$ ,  $[t_1, t_2]$ ,  $[t_2, b]$  and adding we obtain

$$z_2(t_1) - z_1(t_2) = v(t_1) \int_a^{t_1} G\left(\begin{smallmatrix} x \\ t_2 \end{smallmatrix}\right) s(x)v(x) dx - v(t_2) \int_a^{t_2} G\left(\begin{smallmatrix} x \\ t_1 \end{smallmatrix}\right) s(x)v(x) dx \\ = 0.$$

Hence  $G\left(\begin{smallmatrix} t_1 \\ t_2 \end{smallmatrix}\right) = G\left(\begin{smallmatrix} t_2 \\ t_1 \end{smallmatrix}\right)$  so that  $G\left(\begin{smallmatrix} x \\ t \end{smallmatrix}\right)$  is symmetric in  $x$  and  $t$  over  $[a, b]$  (why?).

Let, now,  $k(x)$  be any function possessing a continuous second derivative which satisfies the boundary conditions  $B_1$  and  $B_2$ , and set  $L(k) = -h$ . From the two equations

$$Lk = -h; \quad LG\left(\begin{smallmatrix} x \\ t \end{smallmatrix}\right) = s(x)v(x)v(t),$$

we obtain (how?)

$$k(t) = v(t) \int_a^b s(x)v(x)k(x) dx + \int_a^b G\left(\begin{smallmatrix} x \\ t \end{smallmatrix}\right) h(x) dx,$$

and this may be written, in view of the symmetry of  $G\left(\begin{smallmatrix} x \\ t \end{smallmatrix}\right)$ , in the form

$$k(x) = v(x) \int_a^b s(t)v(t)k(t) dt + \int_a^b G\left(\begin{smallmatrix} x \\ t \end{smallmatrix}\right) h(t) dt$$

or, equivalently,

$$\mathbf{k} = \mathbf{Gh} + \alpha\mathbf{v}; \quad \alpha = \int_a^b s(t)v(t)k(t) dt.$$



It follows (how?) that any solution of the boundary-value problem

$$L(y) + \lambda sy = f; \quad B_1(y) = 0; \quad B_2(y) = 0$$

satisfies the integral equation

$$y(x) = \lambda \int_a^b G \left( \begin{matrix} x \\ t \end{matrix} \right) s(t)y(t) dt - \int_a^b G \left( \begin{matrix} x \\ t \end{matrix} \right) f(t) dt + v(x) \int_a^b s(t)v(t)y(t) dt.$$

Since  $v$  is a solution of the boundary-value problem

$$L(v) = 0; \quad B_1(v) = 0; \quad B_2(v) = 0,$$

a necessary condition for the boundary-value problem

$$L(y) + \lambda sy = f; \quad B_1(y) = 0; \quad B_2(y) = 0$$

to have a solution is that  $\mathbf{v}$  be orthogonal to  $\mathbf{f} - \lambda s\mathbf{y}$ . Thus

$$\lambda \int_a^b s(t)y(t)v(t) dt = \int_a^b f(t)v(t) dt.$$

Hence, if  $\lambda \neq 0$ ,  $y(x)$  must satisfy the integral equation

$$y(x) = \lambda \int_a^b G \left( \begin{matrix} x \\ t \end{matrix} \right) s(t)y(t) dt - \int_a^b G \left( \begin{matrix} x \\ t \end{matrix} \right) f(t) dt + \frac{v(x)}{\lambda} \int_a^b f(t)v(t) dt.$$

Conversely it follows (exactly as in the case where  $\lambda = 0$  was not a characteristic number) that any solution of this integral equation is a solution of the boundary-value problem

$$L(y) + \lambda sy = f; \quad B_1(y) = 0; \quad B_2(y) = 0.$$

(Prove this. Remember that  $L(v) = 0$ .)

**Example. The Green's function for the boundary-value problem**

$$(D^2 + \lambda)u = 0; \quad Du \left( -\frac{l}{2} \right) = 0; \quad Du \left( \frac{l}{2} \right) = 0.$$

Here  $\lambda = 0$  is a characteristic number.  $v(x)$  is a constant function, and to make  $\mathbf{v}$  a unit vector we set  $v(x) = l^{-1/2}$ . The Green's function satisfies the equation  $D^2G = \frac{1}{l}$  and so  $G_l \left( \begin{matrix} x \\ t \end{matrix} \right) = \frac{x^2}{2l} + c_1 + c_2x$ ;

$G_r \left( \begin{matrix} x \\ t \end{matrix} \right) = \frac{x^2}{2l} + c_3 + c_4x$ . The boundary conditions yield  $c_2 = \frac{1}{2}$ ,

$c_4 = -\frac{1}{2}$ , and the "fitting together" condition  $G_l\left(\frac{t}{t}\right) = G_r\left(\frac{t}{t}\right)$  yields  $c_1 + \frac{1}{2}t = c_3 - \frac{1}{2}t = c$ , say. Hence

$$G_l\left(\frac{x}{t}\right) = \frac{x^2}{2l} + c + \frac{1}{2}(x - t); \quad G_r\left(\frac{x}{t}\right) = \frac{x^2}{2l} + c + \frac{1}{2}(t - x).$$

The remaining constant  $c$  is determined by the relation

$$\int_a^b G\left(\frac{x}{t}\right) dx = 0.$$

We find  $c = \frac{l}{12} + \frac{t^2}{2l}$  so that

$$G\left(\frac{x}{t}\right) = \begin{cases} \frac{l}{2l}(x^2 + t^2) + \frac{1}{2}(x - t) + \frac{l}{12}; & -\frac{l}{2} \leq x \leq t; \\ \frac{l}{2l}(x^2 + t^2) + \frac{1}{2}(t - x) + \frac{l}{12}; & t \leq x \leq \frac{l}{2}. \end{cases}$$

### 7. Bessel's and Legendre's boundary-value problems

Bessel's equation of zero order is

$$\xi D^2 u + Du + \xi u = 0.$$

On setting  $\xi = \alpha x$  this appears as

$$xD^2 u + Du + \alpha^2 x u = 0,$$

where, now,  $D$  denotes differentiation with respect to  $x$  (rather than differentiation with respect to  $\xi$ , as before). On introducing the notation

$$L = xD^2 + D = D(xD); \quad \lambda = \alpha^2; \quad s(x) = x$$

the differential equation part of our boundary-value problem is

$$L(u) + \lambda s u = 0.$$

The interval  $[a, b]$  is  $[0, 1]$ , and we note at once that  $p = x$  is zero at  $x = 0$ . In other words the end-point  $a = 0$  of the interval is a *singular* point of the self-adjoint linear differential operator  $L$ . This introduces a certain novelty in the statement of the boundary conditions. We shall consider only *unmixed* boundary-value problems, and we choose the boundary condition  $B_2$  which refers to the end  $b = 1$  of the interval

to be simply  $u = 0$ . Hence the value of  $p(uDv - vDu)$  at  $b = 1$  is zero ( $u$  and  $v$  being any two functions which satisfy the boundary condition  $B_2$ ). In order, then, that the boundary-value problem be regular the boundary condition  $B_1$  which refers to the end  $a = 0$  must be such that  $p(uDv - vDu)$  be zero at  $x = 0$ . Since  $p = x$  is zero at  $x = 0$  it is not necessary to prescribe a relation between  $u$  and  $Du$  at  $x = 0$ ; it is sufficient (but not necessary) that  $u$  and  $Du$  be defined over  $[0, 1]$  for then  $x(uDv - vDu)$  will certainly be zero at  $x = 0$ . Since  $L$  is self-adjoint, and  $s = x$  is non-negative, the boundary-value problem, being regular, has no non-real characteristic numbers. The general solution of  $L(u) \equiv D(xDu) = 0$  is  $u = c_1 + c_2 \log x$ . Hence

$G_l \begin{pmatrix} x \\ t \end{pmatrix} = c_1$  (why?),  $G_r \begin{pmatrix} x \\ t \end{pmatrix} = c_1 \log x$  (why?), and the fitting together condition  $G_l \begin{pmatrix} t \\ t \end{pmatrix} = G_r \begin{pmatrix} t \\ t \end{pmatrix}$  yields  $c_1 = c_2 \log t$ . Since  $DG_l \begin{pmatrix} x \\ t \end{pmatrix} = 0$ ,  $DG_r \begin{pmatrix} x \\ t \end{pmatrix} = \frac{c_2}{x}$  the condition

$$pDG \begin{pmatrix} x \\ t \end{pmatrix} \Big|_{t+}^{t-} = 1$$

yields  $c_2 = -1$ . Hence

$$G \begin{pmatrix} x \\ t \end{pmatrix} = \begin{cases} -\log t; & 0 \leq x < t; \\ -\log x; & t \leq x \leq 1. \end{cases}$$

The point  $O:(0, 0)$  is a singular point of  $G \begin{pmatrix} x \\ t \end{pmatrix}$ , and  $G \begin{pmatrix} x \\ t \end{pmatrix}$  is not bounded over the square  $0 \leq x \leq 1, 0 \leq t \leq 1$ . Nevertheless the general theory of the previous sections is applicable since if  $h$  is any function which is continuous over  $(0, 1)$  the integral  $\int_0^1 G \begin{pmatrix} x \\ t \end{pmatrix} h(t) dt$  exists and possesses a continuous second derivative over  $[0, 1]$  (it being understood that the integral  $\int_0^1 G \begin{pmatrix} 0 \\ t \end{pmatrix} h(t) dt$  is an *improper* integral, its value being the limit of  $\int_0^1 G \begin{pmatrix} x \\ t \end{pmatrix} h(t) dt$  as  $x \rightarrow 0$  through positive values).

For Legendre's boundary-value problem we have

$$L = D\{(1 - \mu^2)D\}; \quad \lambda = n(n + 1), \quad s(\mu) = 1.$$

The underlying interval is  $[a, b] = [-1, 1]$ , and the end-points of this interval are singular points of the self-adjoint linear differential operator  $L$ . The boundary conditions state simply the *existence* of  $u$  and  $Du$  at the end-points of the interval  $[-1, 1]$ . In view of the fact that  $p = 1 - \mu^2$  is zero at these end-points the (apparently vague) boundary conditions are strong enough to make the boundary-value problem regular. Since  $L = D\{(1 - \mu^2)D\}$ ,  $\lambda = 0$  is a characteristic number,  $v = a$  constant being an associated characteristic function. Choosing the constant so that  $s^{1/2}v = v$  is a unit vector we have  $v(x) = 2^{-1/2}$ . The differential equation which serves to determine the Green's function is

$$D\{(1 - \mu^2)DG\} = \frac{1}{2},$$

and a particular solution of this is  $-\frac{1}{4} \log(1 - \mu^2)$ . (Derive this result (mere verification is not enough). *Hint.* The solution of the associated homogeneous differential equation is  $c_1 + c_2 \log \frac{1 + \mu}{1 - \mu}$ .

Use the method of variation of constants.) Hence

$$G_l \left( \begin{matrix} \mu \\ t \end{matrix} \right) = c_1 + c_2 \log \frac{1 + \mu}{1 - \mu} - \frac{1}{4} \log(1 - \mu^2);$$

$$G_r \left( \begin{matrix} \mu \\ t \end{matrix} \right) = c_3 + c_4 \log \frac{1 + \mu}{1 - \mu} - \frac{1}{4} \log(1 - \mu^2).$$

The boundary conditions yield  $c_2 = \frac{1}{4}$  (since the coefficient of  $\log(1 + \mu)$  in  $G_l \left( \begin{matrix} \mu \\ t \end{matrix} \right)$  must be zero) and, similarly,  $c_4 = -\frac{1}{4}$ . The fitting

together condition  $G_l \left( \begin{matrix} \mu \\ t \end{matrix} \right) = G_r \left( \begin{matrix} \mu \\ t \end{matrix} \right)$  yields  $c_3 = c_1 + \frac{1}{2} \log \frac{1 + t}{1 - t}$ . Thus

$$G_l \left( \begin{matrix} \mu \\ t \end{matrix} \right) = c - \frac{1}{2} \log(1 - \mu);$$

$$G_r \left( \begin{matrix} \mu \\ t \end{matrix} \right) = c + \frac{1}{2} \log \frac{1 + t}{1 - t} - \frac{1}{2} \log(1 + \mu),$$

where we have denoted  $c_1$  simply by  $c$ . Determining the constant  $c$  by the relation  $\int_{-1}^1 G \binom{\mu}{t} dt = 0$  we find

$$c = \log 2 - \frac{1}{2} - \frac{1}{2} \log (1 + t).$$

(Prove this. *Hint.*  $\int^x \log x dx = x(\log x - 1)$ .) Hence

$$G \binom{\mu}{t} = \begin{cases} \log 2 - \frac{1}{2} - \frac{1}{2} \log (1 + t)(1 - \mu); & -1 \leq \mu \leq t; \\ \log 2 - \frac{1}{2} - \frac{1}{2} \log (1 - t)(1 + \mu); & t \leq \mu \leq 1. \end{cases}$$

### 8. The Green's function for boundary-value problems of higher order

In the theory of the vibrations of beams we are confronted with a boundary-value problem of the fourth order such as

$$(D^4 + \lambda)u = 0; \quad u(0) = 0, \quad Du(0) = 0, \quad D^2u(l) = 0, \quad D^3u(l) = 0$$

(this is the boundary-value problem which governs the free vibrations of a beam which is built in at one end and free at the other). Here the operator  $L = D^4$  is self-adjoint:

$$uL(v) - vL(u) = D\{(uD^3v - vD^3u) - (DuD^2v - DvD^2u)\}.$$

The boundary-value problem is *regular*; in other words if  $u$  and  $v$  are any two functions which satisfy the *four* boundary conditions  $B_1$ ,  $B_2$ ,  $B_3$ , and  $B_4$  and which possess fourth-order derivatives then  $(uD^3v - vD^3u) - (DuD^2v - DvD^2u)$  has the same value at both ends 0 and  $l$  of the interval  $[0, l]$ .  $\lambda = 0$  is not a characteristic number. (Prove this.)

The Green's function  $G \binom{x}{t}$  must possess a continuous second derivative over  $[0, l]$ , but a discontinuity in the third derivative is permitted at the point  $t$ . The amount of the discontinuity is furnished by the formula

$$D^3G \Big|_{t+}^{t-} = 1.$$

Exactly as in the case of a boundary-value problem of the second order it follows that  $G \binom{x}{t}$  is symmetric. (Prove this.) The actual determination of  $G \binom{x}{t}$  runs as follows:

$$G_l \left( \begin{matrix} x \\ t \end{matrix} \right) = c_1 x^2 + c_2 x^3 \text{ (why?); } G_r \left( \begin{matrix} x \\ t \end{matrix} \right) = c_3 + c_4(x - l)$$

(the expression for  $G_r \left( \begin{matrix} x \\ t \end{matrix} \right)$  follows from the fact that any cubic polynomial function of  $x$  may be written as a cubic polynomial function of  $x - l$ ). Since  $D^3 G_l \left( \begin{matrix} x \\ t \end{matrix} \right) = 6c_2$ ,  $D^3 G_r \left( \begin{matrix} x \\ t \end{matrix} \right) = 0$  the relation  $D^3 G \left. \vphantom{D^3 G} \right|_{t+}^{t-} = D^3 G_l \left( \begin{matrix} t \\ t \end{matrix} \right) - D^3 G_r \left( \begin{matrix} t \\ t \end{matrix} \right) = 1$  yields  $c_2 = \frac{1}{6}$ . Continuity of  $D^2 G$  at  $x = t$  yields  $2c_1 + 6c_2 t = 0$  so that  $c_1 = -\frac{t}{2}$ . Continuity of  $DG$  at  $x = t$  yields  $c_4 = 2c_1 t + 3c_2 t^2 = -\frac{1}{2} t^2$ , and continuity of  $G$  at  $x = t$  yields  $c_3 = -\frac{1}{2} t^2 l + \frac{1}{6} t^3$ . Thus

$$G \left( \begin{matrix} x \\ t \end{matrix} \right) = \begin{cases} -\frac{t}{2} x^2 + \frac{1}{6} x^3; & 0 \leq x \leq t; \\ -\frac{t^2}{2} x + \frac{1}{6} t^3; & t \leq x \leq l. \end{cases}$$

### EXERCISES

1. Determine the Green's function for the boundary-value problem.

$$(D^4 + \lambda)u = 0; \quad u \left( -\frac{l}{2} \right) = 0; \quad Du \left( -\frac{l}{2} \right) = 0; \quad u \left( \frac{l}{2} \right) = 0; \quad Du \left( \frac{l}{2} \right) = 0.$$

*Note.* This is the boundary-value problem governing the free vibrations of a beam which is "built in" at both ends.

2. What is the definition of regularity for the general self-adjoint boundary-value problem of the fourth order?

*Answer.*  $p(uD^3v - vD^3u) - p(DuD^2v - DvD^2u) + Dp(uD^2v - vD^2u) + q(uDv - vDu)$ , where  $L = D^2(pD^2) + D(qD) + r$ , must have the same value at both ends of the interval.

3. Show that if the discontinuity in  $D^3G$  for the general self-adjoint regular boundary-value problem of the fourth order for which  $L = D^2(pD^2) + D(qD) + r$  is determined by the formula

$$pD^3G \left( \begin{matrix} x \\ t \end{matrix} \right) \left. \vphantom{pD^3G} \right|_{t+}^{t-} = 1$$

then  $G \left( \begin{matrix} x \\ t \end{matrix} \right)$  is symmetric.

### 9. Boundary-value problems in more than one dimension

The argument being the same for two, three, or more independent variables we shall leave unspecified the number  $n$  of these independent variables. Let  $P$  be any  $n \times n$  matrix whose elements are differentiable functions of the  $n$  variables  $(x_1, \dots, x_n)$ , and let  $u$  be a function of position which possesses continuous second derivatives  $u_{rs}$ . The matrix  $D_2u$  of these second derivatives is symmetric, and the trace (i.e., the sum of the diagonal elements) of the product  $PD_2u$  is given by the formula

$$[PD_2u] = \sum_{r=1}^n \sum_{s=1}^n p_{rs}u_{rs}.$$

(Prove this.) If  $q$  is a  $1 \times n$  matrix whose elements  $q_r$  are continuous functions of position the product  $q \text{ grad } u$  (where  $\text{grad } u$  is the  $n \times 1$  matrix whose elements are  $u_r$ ,  $r = 1, \dots, n$ ) is furnished by the formula

$$\sum_1^n q_r u_r.$$

The linear differential operator  $L$  which we have to consider is

$$L(u) \equiv [PD_2u] + q \text{ grad } u + ru,$$

where  $r$  is a continuous function of position. The boundary conditions are of the form

$$B(u): \alpha u + \beta \frac{du}{dn} = 0 \text{ over } S,$$

where  $S$  is a closed  $(n - 1)$ -dimensional spread in the  $n$ -dimensional space. The function  $u = u(P)$  must satisfy these boundary conditions and, in addition, must satisfy the partial differential equation

$$L(u) + \lambda u = 0$$

over the  $n$ -dimensional region  $R$  whose boundary is  $S$ .

The conditions that are imposed on the linear differential operator  $L$  by the requirement that it be *self-adjoint* are found in much the same way as in the case  $n = 1$  which we have already treated in detail. Since

$$u p_{rs} v_{rs} = (u p_{rs} v_s)_r - \{v(u p_{rs})_r\}_s + v(u p_{rs})_{rs};$$

$$u q_r v_r = (u p_r v)_r - v(u q_r)_r$$

the integral of  $uL(v)$  over  $R$  may be replaced by an integral over the boundary  $S$  of  $R$  plus the integral of  $vM(u)$  over  $R$ , where  $M(u)$  is the linear differential operator

$$M(u) \equiv \sum_{r=1}^n \sum_{s=1}^n (up_{rs})_{rs} - \sum_{r=1}^n (uq_r)_r + ru.$$

(Remember that if  $f$  is any differentiable function of position the integral of  $f_r$  over  $R$  is equivalent to an integral over  $S$ . See Chapter 1, Section 8; and observe that the results given there for spaces of two and three dimensions may be extended to spaces of  $n$  dimensions,  $n = 4, 5, \dots$ )

We term  $M$  the *adjoint* operator to  $L$ ; its characteristic feature is that the integral  $\int_R \{uL(v) - vM(u)\} d\tau$  of  $uL(v) - vM(u)$  over  $R$  ( $d\tau$  being the  $n$ -dimensional element of content) is equivalent to an integral over the boundary  $S$  of  $R$ . The operator  $L$  is said to be *self-adjoint* when  $M$  coincides with  $L$ . Since  $M(u)$  may be written in the form

$$\sum_{r=1}^n \sum_{s=1}^n \{p_{rs}u_{rs} + 2u_r(p_{rs})_s + u(p_{rs})_{rs}\} - \sum_{r=1}^n \{(q_r)_r u + u_r q_r\} + ru$$

(prove this) a *necessary* condition for self-adjointness is

$$\sum_{s=1}^n (p_{rs})_s = q_r,$$

and it is at once clear that this necessary condition is *sufficient*. Thus

**The linear differential operator**

$$L = [PD_2] + q \text{ grad} + r$$

(where  $[PD_2]$  indicates the trace of the product of the matrix  $D_2$  of second derivatives by the matrix  $P$ ) is self-adjoint if, and only if,

$$q = \text{div } P.$$

In particular if  $P$  is a constant matrix the condition for self-adjointness is simply  $q = 0$ .

**EXERCISE**

1. Show that the Laplacian differential operator  $\Delta_2$  is self-adjoint.

When the operator  $L$  is self-adjoint the vector whose integral over  $S$



is equivalent to the integral of  $\{uL(v) - vL(u)\}$  over  $R$  takes a simple form.  $uL(v) - vL(u)$  reduces to  $\sum_{r=1}^n \left\{ u \sum_{s=1}^n p_{rs} v_s - v \sum_{s=1}^n p_{rs} u_s \right\}$ , and the integral of  $uL(v) - vL(u)$  over  $R$  is the same as the integral of the vector whose  $r$ th coordinate is  $\left\{ u \sum_{s=1}^n p_{rs} v_s - v \sum_{s=1}^n p_{rs} u_s \right\}$ , i.e., the vector  $u(\mathbf{P} \text{ grad } v) - v(\mathbf{P} \text{ grad } u)$ , over  $S$ . If  $\mathbf{n}$  is the unit vector normal to  $S$  (drawn away from  $R$ ) and if  $\mathbf{Pn}$  is the vector  $\mathbf{N}$  we have

$$\int_R \{uL(v) - vL(u)\} d\tau = \int_S \{u(\text{grad } v|\mathbf{N}) - v(\text{grad } u|\mathbf{N})\} dS.$$

In the particular case where  $P$  is the unit matrix (so that  $L$  is the  $n$ -dimensional Laplacian)  $\mathbf{N} = \mathbf{n}$ , and we recover the familiar formula

$$\int_R (u \Delta_2 v - v \Delta_2 u) d\tau = \int_S \left( u \frac{dv}{dn} - v \frac{du}{dn} \right) dS.$$

The boundary-value problem is said to be *regular* if the integral over  $S$  is zero for any two functions  $u$  and  $v$  which satisfy the boundary conditions. In particular, if the boundary condition is  $u = 0$ , or  $(\text{grad } u|\mathbf{N}) = 0$ , the problem is regular. The values of  $\lambda$  for which the boundary-value problem

$$L(u) + \lambda su = 0; \quad B(u) = 0$$

possesses a non-trivial solution (what does this mean?) are termed the characteristic numbers of the boundary-value problem. When  $\lambda = 0$  is not a characteristic number we define the *Green's function* of the boundary-value problem by the following specifications (the question whether any function exists which meets the specifications being a fundamentally important one which must be settled): Let  $T: (t_1, \dots, t_n)$  be any fixed point of  $R$ . Then the Green's function of the problem

is a function of position  $G \left( \begin{smallmatrix} P \\ T \end{smallmatrix} \right)$ ,  $P$  being the point of evaluation of  $G$ ,

which depends on the location of the point  $T$  and which satisfies the (partial) differential equation  $L(G) = 0$  at every point of  $R$ , except  $T$ ;

$G \left( \begin{smallmatrix} P \\ T \end{smallmatrix} \right)$  also satisfies the boundary conditions.  $G \left( \begin{smallmatrix} P \\ T \end{smallmatrix} \right)$  need not be bounded at  $T$ , but if  $S'$  is a sphere ( $n$ -dimensional) with center at  $T$

and of radius  $\epsilon$  the integral of  $\mathbf{G}\mathbf{v}$  over  $S'$  is null at  $\epsilon = 0$ ,  $\mathbf{v}$  being any vector which is bounded at  $T$ . Furthermore the integral of  $(\mathbf{grad} G|\mathbf{N})$  over  $S'$  has the limit 1 at  $\epsilon = 0$ . Here  $\mathbf{N}$  is the vector  $\mathbf{Pn}$  where  $\mathbf{n}$  is the unit vector normal to  $S'$  (drawn away from the region  $R'$  bounded by  $S$  and  $S'$ ).

The question as to the existence of this Green's function is a difficult one about which we shall make only a few remarks. In the one-dimensional case the function  $p(x)$  dominated the situation, and it was not permitted to change sign over the interval  $[a, b]$  (any point at which  $p(x)$  is zero being a singular point of the differential operator  $L$ ). When  $n = 2$  the coefficient function  $p(x)$  is replaced by the *coefficient matrix*

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

which may be taken, without loss of generality, to be symmetric (why?). The differential operator  $L$  is said to be *elliptic* when  $P$  is either *positively* or *negatively definite*, and (since  $L$  may be multiplied through by  $-1$ ) we may restrict ourself to the case where  $P$  is *positively* definite, by which we mean that  $\mathbf{v}^*P\mathbf{v} > 0$  for every non-zero, two-dimensional vector  $\mathbf{v}$ . If the matrix  $P$  is indefinite (so that  $\mathbf{v}^*P\mathbf{v} > 0$  for some vectors  $\mathbf{v}$  and  $\mathbf{v}^*P\mathbf{v} < 0$  for some vectors  $\mathbf{v}$ ) then the differential operator  $L$  is said to be *hyperbolic*. The Green's function does not exist for hyperbolic differential operators while it does exist (under suitable conditions) for elliptic differential operators. When  $n > 2$  the same distinction and difference in behavior as regards the existence of the Green's function between elliptic and hyperbolic differential operators exist. The simplest case of an elliptic differential operator arises when  $P$  is the unit matrix. When  $n = 2$  we have, then,

$$Lu \equiv u_{xx} + u_{yy} = \Delta_2 u,$$

and when  $n = 3$  we have

$$Lu \equiv u_{xx} + u_{yy} + u_{zz} = \Delta_3 u$$

so that  $L$  is, in either case, the Laplacian operator. In these cases the Green's function exists and takes the following forms:

1.  $n = 2$ .  $G = \frac{1}{2\pi} \log \frac{1}{r} + V(P)$ , where  $r = |TP|$  and  $V$  is a solution of Laplace's equation over  $R$ . Since  $P$  is the unit matrix  $\mathbf{N} = \mathbf{n}$  so

that  $(\mathbf{grad} G|\mathbf{N}) = \frac{dG}{dn} = \frac{1}{2\pi\epsilon} + \frac{dV}{dn}$ . Since  $\frac{dV}{dn}$  is continuous at  $T$  (why?) the integral of  $(\mathbf{grad} G|\mathbf{N})$  over  $S'$  has the limit 1 at  $\epsilon = 0$ . Furthermore the integral of  $G\mathbf{v}$ , where  $\mathbf{v}$  is bounded at  $T$ , over  $S'$  is null at  $\epsilon = 0$ . (Prove this.) If the boundary condition is  $u = 0$  the function  $V(P)$  must =  $\frac{1}{2\pi} \log r$  over  $S$ . In other words,  $V(P)$  is the potential of the electrostatic field of the charge induced on the earthed conductor  $S$  by a (line) charge of strength  $\frac{1}{4\pi}$  at  $T$  and  $G$  is the potential (inducing plus induced) of the field. This is not a "proof" of the existence of  $G\left(\begin{smallmatrix} P \\ T \end{smallmatrix}\right)$ , but it makes the existence of  $G$  plausible.

2.  $n = 3$ .  $G = \frac{1}{4\pi r} + V(P)$ , where  $r = |TP|$  and  $V(P)$  is a solution of Laplace's equation over  $R$ . The integral of  $(\mathbf{grad} G|\mathbf{N}) = \frac{dG}{dn}$  over  $S'$  has again the limit 1 at  $\epsilon = 0$ ; and the integral of  $G\mathbf{v}$  where  $\mathbf{v}$  is bounded at  $T$ , over  $S'$  is null at  $\epsilon = 0$  (prove this);  $G$  is the potential of the field of a point charge  $\frac{1}{4\pi}$  at  $T$  in the presence of the earthed conductor  $S$ .

Assuming the existence of the Green's function (so that  $L$  is granted to be not hyperbolic at each point of  $R$ ) we proceed exactly as in the one-dimensional case. On combining the two equations

$$Lu + \lambda su = 0; \quad L(G) = 0$$

in the usual manner we obtain

$$uL(G) - GL(u) = \lambda Gsu,$$

and on integrating this relation over the region  $R'$  between  $S$  and  $S'$  we obtain

$$\int_{R'} \{uL(G) - GL(u)\} d\tau = \lambda \int_{R'} G\left(\begin{smallmatrix} P \\ T \end{smallmatrix}\right) s(P)u(P) d\tau$$

( $P$  being the variable point of integration). The integral on the left may be replaced by an integral over  $S$  plus an integral over  $S'$ , the integrand being, in each case,  $u(\mathbf{grad} G|\mathbf{N}) - G(\mathbf{grad} u|\mathbf{N})$ . The integral over  $S$  is zero since the boundary-value problem is, by hypothe-

sis, regular. The integral of  $G(\text{grad } u|\mathbf{N})$  over  $S'$  is null at  $\epsilon = 0$  since  $\text{grad } u$  is bounded at  $T$ , and the integral of  $u(\text{grad } G|\mathbf{N})$  over  $S'$  has the limit  $u(T)$  at  $\epsilon = 0$  (why?). Hence

$$u(T) = \lambda \int_R G \left( \begin{matrix} P \\ T \end{matrix} \right) s(P) u(P) d\tau.$$

On interchanging the roles of the points  $P$  and  $T$  this may be written in the form

$$u(P) = \lambda \int_R G \left( \begin{matrix} T \\ P \end{matrix} \right) s(T) u(T) d\tau$$

(where  $P$  is now the point of evaluation of  $u$  and  $T$  denotes the variable point of integration).

If  $T_1$  and  $T_2$  are any two points of  $R$  we consider the two functions

$$u_1 = G \left( \begin{matrix} P \\ T_1 \end{matrix} \right); \quad u_2 = G \left( \begin{matrix} P \\ T_2 \end{matrix} \right)$$

and denote by  $R''$  the region bounded by  $S, S'_1$ , and  $S'_2$  where  $S'_1$  and  $S'_2$  are spheres of radii  $\epsilon_1, \epsilon_2$  with centers at  $T_1$  and  $T_2$ , respectively.

On combining the two equations

$$L(u_1) = 0; \quad L(u_2) = 0$$

we obtain the relation

$$u_2 L(u_1) - u_1 L(u_2) = 0$$

which is valid over  $R''$ . On integrating this relation over  $R''$  and replacing the integral over  $R''$  by the sum of three integrals over  $S, S'_1$ , and  $S'_2$ , respectively, we obtain, on letting  $\epsilon_1$  and  $\epsilon_2$  tend, independently, to zero (and using the fact that the boundary-value problem is regular),

$$u_2(T_1) = u_1(T_2)$$

or, equivalently,

$$G \left( \begin{matrix} T_1 \\ T_2 \end{matrix} \right) = G \left( \begin{matrix} T_2 \\ T_1 \end{matrix} \right).$$

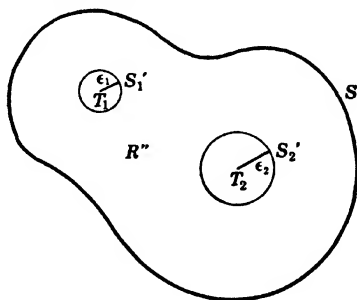


FIG. 39.

Thus

The Green's function  $G \begin{pmatrix} P \\ T \end{pmatrix}$  is a symmetric function of the two points  $P$  and  $T$ .

It follows that

$$u(P) = \lambda \int_R G \begin{pmatrix} P \\ T \end{pmatrix} s(T) u(T) d\tau$$

or, on regarding  $u(P)$  over  $R$  as a function-vector  $\mathbf{u}$ ,

$$\mathbf{u} = \lambda \mathbf{K} \mathbf{u},$$

where  $\mathbf{K}$  is the linear integral operator  $\mathbf{K}: G \begin{pmatrix} P \\ T \end{pmatrix} s(T)$ . Hence

Every solution of the homogeneous boundary-value problem is a solution of the integral equation

$$\mathbf{u} = \lambda \mathbf{K} \mathbf{u} = \lambda \mathbf{G}(s\mathbf{u}).$$

When  $s$  is non-negative this integral equation may be replaced, on setting  $s^{1/2} \mathbf{u} = \mathbf{v}$ , by the integral equation

$$\mathbf{v} = \lambda \mathbf{H} \mathbf{v},$$

where the linear integral operator  $\mathbf{H}: G \begin{pmatrix} P \\ T \end{pmatrix} \{s(P)s(T)\}^{1/2}$  is symmetric.

### EXERCISE

2. Show that if  $Lk = -h$  then  $\mathbf{k} = \mathbf{G}h$ : i.e., that  $\mathbf{G}L = -1$ .

In order to prove the *equivalence* of the boundary-value problem and the integral-equation problem we have merely to show that if  $\mathbf{k} = \mathbf{G}h$  then  $Lk = -h$ , i.e., that  $\mathbf{L}\mathbf{G} = -1$ . In performing the differentiations of  $\mathbf{G}h$  involved in calculating  $\mathbf{L}\mathbf{G}h$  we have to pay attention to the fact that  $\mathbf{G}h$  is an *improper* integral. Owing to the fact that the limit of the integral of the product of  $G$  by any function which is bounded at  $P$  over  $S'$  (where  $S'$  is a sphere of radius  $\epsilon$  with center at  $P$ ) is null at  $\epsilon = 0$  the first derivatives of  $\mathbf{k} = \mathbf{G}h$  are obtained by *differentiating under the sign*:

$$k_r = \int_R G_r \begin{pmatrix} P \\ T \end{pmatrix} h(T) d\tau.$$

However, when we pass to the second derivatives, we have to take

account of the fact that the integral of  $(\mathbf{grad} G|N)$  over  $S'$  has the limit 1 at  $\epsilon = 0$ . The integral over  $S'$  arises from an *inner* bounding surface of the various improper integrals involved and this introduces a minus sign. The end result is that

$$Lk = -h(P).$$

This suffices to prove the equivalence of the integral equation problem and the boundary-value problem. (Show this.)

*Warning.* Be sure that you understand clearly that you do not have any integral equation at all until you have the Green's function. Thus you must be sure that the Green's function actually exists before you attempt to apply the method of integral equations. For example, it would be absurd to attempt to deal with hyperbolic partial differential operators by methods which depend on the existence of the Green's function.

# 8

## INTEGRAL EQUATIONS

### 1. The Fredholm determinant

The integral equation we propose to consider is of the form

$$y(x) = \lambda \int_a^b K \begin{pmatrix} x \\ t \end{pmatrix} y(t) dt + f(x);$$

i.e.,

$$y = \lambda \mathbf{K}y + \mathbf{f}.$$

The function  $K \begin{pmatrix} x \\ t \end{pmatrix}$  is granted to be continuous over the square  $a \leq x \leq b, a \leq t \leq b$ .  $f = f(x)$  is a given function which is granted to be continuous over  $[a, b]$ , and the problem is to determine the unknown function  $y = y(x)$ . Since  $\mathbf{K}y$  is continuous (why?)  $y = y(x)$ , if the integral equation has a solution at all, must be continuous. When the given function  $f = f(x)$  is the zero constant function the integral equation is said to be *homogeneous*; if  $f = f(x)$  is not the zero constant function, the homogeneous integral equation

$$\mathbf{u} = \lambda \mathbf{K}\mathbf{u}$$

is termed the *homogeneous equation associated with the non-homogeneous integral equation*

$$y = \lambda \mathbf{K}y + \mathbf{f}.$$

### EXERCISES

1. Show that the difference of any two solutions of the non-homogeneous equation  $y = \lambda \mathbf{K}y + \mathbf{f}$  is a solution of the associated homogeneous equation  $\mathbf{u} = \lambda \mathbf{K}\mathbf{u}$ .

2. Show that if  $\mathbf{u}$  is any solution of the homogeneous integral equation  $\mathbf{u} = \lambda \mathbf{K}\mathbf{u}$  so also is  $c\mathbf{u}$  where  $c$  is a constant and that if  $\mathbf{u}_1, \mathbf{u}_2$  are any two solutions of the homogeneous integral equation so also is  $\mathbf{u}_1 + \mathbf{u}_2$ .

3. Deduce from the results of Exercise 2 that if  $u_1, u_2$  are any two solutions of the homogeneous integral equation  $u = \lambda Ku$  so also is  $c_1 u_1 + c_2 u_2$ , where  $c_1, c_2$  are any constants. *Note.* In view of the result of this exercise the homogeneous integral equation  $u = \lambda Ku$  and the non-homogeneous integral equation  $y = \lambda Ky + f$  are said to be *linear*.

We may find an approximation to a solution of the homogeneous equation  $u = \lambda Ku$  (provided always that this equation actually possesses a solution, other than the obvious and trivial solution  $u \equiv 0$ ) as follows: Construct a net

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

on the interval  $[a, b]$ , and replace the integral  $\int_a^b K \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) u(t) dt$  by the approximating sum

$$\sum = K \left( \begin{smallmatrix} x \\ t_1 \end{smallmatrix} \right) u(t_1) \Delta_1 t + \cdots + K \left( \begin{smallmatrix} x \\ t_n \end{smallmatrix} \right) u(t_n) \Delta_n t; \quad \Delta_p(t) = t_p - t_{p-1}.$$

On evaluating this approximating sum at the  $n$  points  $t_1, \cdots, t_n$  we obtain  $n$  linear homogeneous combinations of the expressions  $u(t_1), \cdots, u(t_n)$ :

$$\sum^j = K \left( \begin{smallmatrix} t_j \\ t_1 \end{smallmatrix} \right) u(t_1) \Delta_1 t + \cdots + K \left( \begin{smallmatrix} t_j \\ t_n \end{smallmatrix} \right) u(t_n) \Delta_n t.$$

(On setting

$$u(t_p) = u^p; \quad K \left( \begin{smallmatrix} t_p \\ t_q \end{smallmatrix} \right) \Delta_q t = K_q^p$$

the sum  $\sum^j$  is the  $j$ th element of the  $n \times 1$  matrix  $Ku$ , where  $K$  is the  $n \times n$  matrix of which the element in the  $p$ th row and  $q$ th column is  $K_q^p$  and  $u$  is the  $n \times 1$  matrix whose  $p$ th element is  $u^p$ . Then the elements of the  $n \times 1$  matrix  $u$  will furnish an approximation to the values of the sought-for function-vector  $u$  at the points  $t_1, \cdots, t_n$  if

$$u = \lambda Ku$$

or, equivalently,

$$(E_n - \lambda K)u = 0; \quad E_n \text{ the } n \times n \text{ unit matrix.}$$

In order that this equation may have a solution other than the trivial one  $u^p = 0, p = 1, \cdots, n$ , the matrix  $E_n - \lambda K$  must be singular:

$$\det (E_n - \lambda K) = 0.$$



Each element of the matrix  $E_n - \lambda K$  is obtained by subtracting an element of  $\lambda K$  from the corresponding element of  $E_n$ . Hence  $\det(E_n - \lambda K)$  may be analyzed into the sum of  $2^n$  determinants in each of which there are a certain number of columns from  $E_n$  and a certain number of columns from  $-\lambda K$ . Thus any one of the  $2^n$  determinants is a principal minor of  $-\lambda K$  (the determinant of  $E_n$ , namely, 1, and the determinant of  $-\lambda K$  being included). For example, when  $n = 3$ , the determinant of  $E_3 - \lambda K$  is the sum of eight determinants of which the first is

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1;$$

the second

$$-\lambda \begin{vmatrix} K_1^1 & 0 & 0 \\ K_1^2 & 1 & 0 \\ K_1^3 & 0 & 1 \end{vmatrix} = -\lambda K_1^1;$$

and so on, the last being

$$-\lambda^3 \begin{vmatrix} K_1^1 & K_2^1 & K_3^1 \\ K_1^2 & K_2^2 & K_3^2 \\ K_1^3 & K_2^3 & K_3^3 \end{vmatrix} = -\lambda^3 \det K.$$

On collecting these eight determinants together we obtain

$$\det(E_3 - \lambda K) = 1 - \lambda(K_1^1 + K_2^2 + K_3^3) + \lambda^2(K_{23}^{23} + K_{31}^{31} + K_{12}^{12}) - \lambda^3 \det K,$$

where

$$K_{23}^{23} = \begin{vmatrix} K_2^2 & K_3^2 \\ K_2^3 & K_3^3 \end{vmatrix}, \text{ and so on.}$$

In general, we have

$$\det(E_n - \lambda K) = 1 - \lambda K_{\alpha}^{\alpha} + \lambda^2 (K_{\alpha\beta}^{\alpha\beta})^* - \dots + (-1)^n \lambda^n \det K,$$

where a symbol such as  $(K_{\alpha\beta}^{\alpha\beta})^*$  indicates a summation of two-rowed principal minors of  $K$ , the summation being over all pairs of numbers selected from the set  $1, \dots, n$ . Since  $K_{rs}^{rs} = 0$  if  $r = s$  and since  $K_{sr}^{sr} = K_{rs}^{rs}$  the summation  $(K_{\alpha\beta}^{\alpha\beta})^*$  is one-half the summation  $K_{\alpha\beta}^{\alpha\beta}$ , where, in this summation,  $\alpha$  and  $\beta$  range, independently of each other, over the numbers  $1, \dots, n$ . Similarly

$$(K_{\alpha\beta\gamma}{}^{\alpha\beta\gamma})^* = \frac{1}{3!} K_{\alpha\beta\gamma}{}^{\alpha\beta\gamma},$$

and so on. Hence we may write  $\det(E_n - \lambda K)$  in the form

$$\det(E_n - \lambda K) =$$

$$1 - \lambda K_{\alpha}{}^{\alpha} + \frac{1}{2!} \lambda^2 K_{\alpha\beta}{}^{\alpha\beta} - \cdots + \frac{(-1)^n}{n!} \lambda^n K_{\alpha_1 \dots \alpha_n}{}^{\alpha_1 \dots \alpha_n}$$

(where  $K_{j_1 \dots j_n}{}^{j_1 \dots j_n} = \det K$  when the  $n$  letters  $j_1, \dots, j_n$  constitute a permutation of the  $n$  numbers  $(1, \dots, n)$ ). As  $n$  increases

indefinitely  $K_{\alpha}{}^{\alpha}$  approaches  $\int_a^b K \binom{t}{t} dt$  arbitrarily closely,  $K_{\alpha\beta}{}^{\alpha\beta}$

approaches  $\int_a^b \int_a^b K \begin{pmatrix} t_1 & t_2 \\ t_1 & t_2 \end{pmatrix} dt_1 dt_2$  arbitrarily closely, and so on.

We are led in this way to a consideration of the following power series in  $\lambda$ :

$$D(\lambda) = 1 - A_1 \lambda + \frac{1}{2!} A_2 \lambda^2 - \cdots + \frac{(-1)^p}{p!} A_p \lambda^p + \cdots,$$

where

$$A_p = \int_a^b \cdots \int_a^b K \begin{pmatrix} t_1 & \cdots & t_p \\ t_1 & \cdots & t_p \end{pmatrix} dt_1 dt_2 \cdots dt_p.$$

We shall see shortly that this power series in  $\lambda$  converges for every (finite) value of  $\lambda$ . It is known as *Fredholm's determinant* (after E. I. Fredholm, a Swedish mathematician). The power series  $D(\lambda)$  is called a *determinant* since it was suggested to Fredholm by the determinant of the matrix  $E_n - \lambda K$ , and it plays, for the integral equation, the role played by the determinant of the matrix  $E_n - \lambda K$  for the system of  $n$  homogeneous equations

$$u = \lambda K u.$$

In order to show that  $D(\lambda)$  converges for every (finite) value of  $\lambda$  we first make an appraisal of a  $p$ -rowed determinant. At first we confine ourselves to the case where the column vectors  $\mathbf{u}_1, \dots, \mathbf{u}_p$ , of the matrix whose determinant we wish to appraise are *unit vectors*. The elements of the matrix need not be real so that the vectors  $\mathbf{u}_1, \dots, \mathbf{u}_p$  are  $p$ -dimensional complex vectors. The determinant of the matrix is a complex variable which is a continuous function of the elements of the matrix. Hence the modulus (i.e., absolute value) of the determinant of the matrix is a continuous function of the (real) parameters

which identify the points of the  $p$ -dimensional (in the complex sense) unit sphere whose coordinates are the coordinates of the vectors  $\mathbf{u}_1, \dots, \mathbf{u}_p$ . This unit sphere may be regarded as the  $2p$ -dimensional real unit sphere, and since this is closed (what does this mean?) the modulus of the determinant of our matrix has an absolute maximum (why?). Now the determinant of the matrix whose column vectors are  $\mathbf{u}_1, \dots, \mathbf{u}_p$  is the scalar product of  $\mathbf{u}_1$  by the vector product  $(\mathbf{u}_2 \times \dots \times \mathbf{u}_p)$  (see Chapter 1, Section 10). Let us denote, for a moment,  $(\mathbf{u}_2 \times \dots \times \mathbf{u}_p)$  by  $\mathbf{v}_1$  so that the determinant we are interested in is  $(\mathbf{v}_1 | \mathbf{u}_1)$ . We know, from Schwarz's inequality (see Chapter 1, Section 10) that  $(\mathbf{v}_1 | \mathbf{u}_1) \leq v_1$ . The equality is valid when  $\mathbf{u}_1 = \frac{\mathbf{v}_1}{v_1}$  (why?), and it is easy to see that the equality is *not* valid unless

$\mathbf{u}_1$  is the product of  $\frac{\mathbf{v}_1}{v_1}$  by a complex number of unit modulus. (*Note.*

We have no concern with the case where  $v_1 = 0$  since, then, the determinant in which we are interested is zero (why?).) In fact if  $(\mathbf{v}_1 | \mathbf{u}_1) = v_1$  the vector  $\mathbf{v}_1 - (\mathbf{u}_1 | \mathbf{v}_1)\mathbf{u}_1$  is the zero vector since its squared magnitude is  $((\mathbf{v}_1 - (\mathbf{u}_1 | \mathbf{v}_1)\mathbf{u}_1) | (\mathbf{v}_1 - (\mathbf{u}_1 | \mathbf{v}_1)\mathbf{u}_1)) = v_1^2 - 2(\mathbf{u}_1 | \mathbf{v}_1)(\mathbf{v}_1 | \mathbf{u}_1) + (\mathbf{u}_1 | \mathbf{v}_1)(\mathbf{v}_1 | \mathbf{u}_1) = v_1^2 - (\mathbf{u}_1 | \mathbf{v}_1)(\mathbf{u}_1 | \mathbf{v}_1) = 0$ . Thus

*When the modulus of the determinant of the matrix whose column vectors are the unit vectors  $\mathbf{u}_1, \dots, \mathbf{u}_p$  attains its absolute maximum,  $\mathbf{u}_1$  is the product of  $(\mathbf{u}_2 \times \dots \times \mathbf{u}_p)$  by a constant; in other words  $\mathbf{u}_1$  is perpendicular to each of the vectors  $\mathbf{u}_2, \dots, \mathbf{u}_p$ .*

Since the modulus of the determinant in question is a symmetric function of the vectors  $\mathbf{u}_1, \dots, \mathbf{u}_p$  (why?) this result is valid for any one of the vectors  $\mathbf{u}_1, \dots, \mathbf{u}_p$  and not merely for  $\mathbf{u}_1$ . Hence the  $p$  vectors  $\mathbf{u}_1, \dots, \mathbf{u}_p$  are *mutually* perpendicular and so, since these vectors are all unit vectors, the matrix whose column vectors they are is unitary (why?). Hence the modulus of the determinant of the matrix is 1 (see Exercise 15, p. 36). We have, then, the following important result:

**The absolute maximum of the modulus of the determinant of a  $p \times p$  matrix all of whose column vectors are unit vectors is 1 (the matrix being unitary when its modulus is 1).**

It is easy to extend this result to the case of any  $p \times p$  matrix (none of whose column vectors  $\mathbf{v}_1, \dots, \mathbf{v}_p$  is the zero vector). In fact if we set  $\mathbf{v}_j = v_j \mathbf{u}_j, j = 1, \dots, p$ , the vectors  $\mathbf{u}_1, \dots, \mathbf{u}_p$  are all unit vectors, and the determinant of the matrix whose column vectors are  $\mathbf{v}_1, \dots, \mathbf{v}_p$  is the product of the determinant of the matrix whose column vectors are  $\mathbf{u}_1, \dots, \mathbf{u}_p$  by  $v_1 v_2 \dots v_p$ . Hence if the magni-

tudes  $v_1, \dots, v_p$  are assigned we know that the modulus of the determinant of the matrix is not greater than  $v_1 \dots v_p$ . Thus

**The modulus of the determinant of a  $p \times p$  matrix whose column vectors have magnitudes  $v_1, v_2, \dots, v_p$  is not greater than the product of these magnitudes, and it is equal to the product of these magnitudes when, and only when, the  $p$  vectors are mutually perpendicular.**

This theorem is known as *Hadamard's theorem* (after J. Hadamard, a French mathematician).

If the modulus of each element of our matrix is less than, or at most equal to, a given number  $M$  then each of the column vectors  $\mathbf{v}_1 \dots, \mathbf{v}_p$  has a magnitude which is less than, or at most equal to,  $p^{1/2}M$  (why?) so that the modulus of the determinant  $\leq p^{p/2}M^p$ . It is this corollary to Hadamard's theorem that shows at once that the power series which defines  $D(\lambda)$  is convergent for every finite value of  $\lambda$ . In fact we are

granted that  $K \begin{pmatrix} x \\ t \end{pmatrix}$  is continuous over the square  $a \leq x \leq b, a \leq t$

$\leq b$ . Hence the modulus of  $K \begin{pmatrix} x \\ t \end{pmatrix}$  has an absolute maximum,  $M$  say,

over this square. Hence the modulus of  $K \begin{pmatrix} t_1 \dots t_p \\ t_1 \dots t_p \end{pmatrix}$  is not greater

than  $p^{p/2}M^p$  and so the modulus of  $A_p$ , where

$$A_p = \int_a^b \dots \int_a^b K \begin{pmatrix} t_1 \dots t_p \\ t_1 \dots t_p \end{pmatrix} dt_1 \dots dt_p,$$

is not greater than  $p^{p/2}M^p(b-a)^p$ . Hence the power series whose sum is  $D(\lambda)$  is dominated by (what does this mean?) the following power series:

$$1 + M(b-a)|\lambda| + \frac{2}{2!} M^2(b-a)^2|\lambda|^2 + \dots + \frac{p^{p/2}M^p(b-a)^p|\lambda|^p}{p!} + \dots$$

The ratio of the  $(p+1)$ st term of this series to the  $p$ th is

$$\frac{p^{p/2}M(b-a)|\lambda|}{(p-1) \frac{p-1}{2} p} = \left(1 + \frac{1}{p-1}\right) \frac{p-1}{2} \frac{M(b-a)|\lambda|}{p^{1/2}}$$

and since  $\lim_{p \rightarrow \infty} \left(1 + \frac{1}{p-1}\right)^{\frac{p-1}{2}} = e^{1/2}$  the ratio of the  $(p+1)$ st term of our dominating series to the  $p$ th has at  $p = \infty$  the limit zero (no matter what is the value of  $|\lambda|$ ). Hence the dominating series converges for every value of  $|\lambda|$ , and it follows (why?) that the power series whose sum is  $D(\lambda)$  is convergent for every (finite) value of  $\lambda$ . Thus

**$D(\lambda)$  is an integral function of the complex variable  $\lambda$ ; in other words it is an analytic function of  $\lambda$  at every point (other than  $\lambda = \infty$ ) of the complex  $\lambda$ -plane.**

*Suggestion.* Try to grasp how remarkable this result is. The only information you were given about  $K \begin{pmatrix} x \\ t \end{pmatrix}$  was that it was continuous and, therefore, bounded over the square  $a \leq x \leq b, a \leq t \leq b$ . Nevertheless this mere *boundedness* (you did not have to use its *continuity*) coupled with its integrability is enough to ensure the convergence of the power series, whose sum is  $D(\lambda)$ , for every (finite) value of  $\lambda$ .

## 2. The Fredholm minors

Fredholm was led, by his examination of the system of  $n$  linear equations  $\mathbf{u} = \lambda \mathbf{K}\mathbf{u}$ , to other power series in  $\lambda$  which are closely associated with  $D(\lambda)$ . Since we shall not have to use anything more than the definition of these power series we shall merely give this definition and shall content ourselves with the remark that they play for the integral equation

$$\mathbf{u} = \lambda \mathbf{K}\mathbf{u}$$

the role played by the cofactors and lower-order minors of the matrix  $E_n - \lambda K$  for the system of equations

$$u = \lambda K u.$$

In order to abbreviate the writing we adopt the following notation:

We denote by  $\begin{bmatrix} t_1 & \cdots & t_p \\ t_1 & \cdots & t_p \end{bmatrix}$  what we have previously denoted by

$K \begin{pmatrix} t_1 & \cdots & t_p \\ t_1 & \cdots & t_p \end{pmatrix}$ . Thus  $\begin{bmatrix} t_1 & \cdots & t_p \\ t_1 & \cdots & t_p \end{bmatrix}$  is the determinant of the  $p \times p$

matrix of which the element in the  $r$ th row and  $s$ th column is  $K \begin{pmatrix} t_r \\ t_s \end{pmatrix}$ .

More generally, we understand by the symbol

$$\begin{bmatrix} x_1 & \cdots & x_p \\ t_1 & \cdots & t_p \end{bmatrix}$$

the determinant of the matrix of which the element in the  $r$ th row and  $s$ th column is  $K \begin{pmatrix} x_r \\ t_s \end{pmatrix}$ . In particular  $\begin{bmatrix} x \\ t \end{bmatrix} = K \begin{pmatrix} x \\ t \end{pmatrix}$ . Remember the general rule

*In the symbol  $\begin{bmatrix} x_1 & \cdots & x_p \\ t_1 & \cdots & t_p \end{bmatrix}$  the upper letters tell the row, and the lower the column, of the matrix involved.*

Furthermore a *single* integral sign will be used to denote multiple, or repeated, integration. The integrations are all over the basic interval  $[a, b]$ , and the variables of integration are those symbols which occur *twice, once above and once below*, in the integrand. In this abbreviated notation  $A_p$  appears, for example, as follows:

$$A_p = \int \begin{bmatrix} t_1 & \cdots & t_p \\ t_1 & \cdots & t_p \end{bmatrix} dt_1 \cdots dt_p$$

(the limits are not attached to the integral sign since the lower limit is *always a* and the upper limit *always b*). We now introduce the symbol

$B_p \begin{pmatrix} x_1 & \cdots & x_q \\ t_1 & \cdots & t_q \end{pmatrix}$  by means of the following definition:

$$B_p \begin{pmatrix} x_1 & \cdots & x_q \\ t_1 & \cdots & t_q \end{pmatrix} = \int \begin{bmatrix} x_1 & \cdots & x_q & \tau_1 & \cdots & \tau_p \\ t_1 & \cdots & t_q & \tau_1 & \cdots & \tau_p \end{bmatrix} d\tau_1 \cdots d\tau_p.$$

The subscript  $p$  attached to  $B$  tells the number of integrations involved in its definition. We shall agree, then, that  $B_0 \begin{pmatrix} x_1 & \cdots & x_q \\ t_1 & \cdots & t_q \end{pmatrix}$  indicates the determinant  $\begin{bmatrix} x_1 & \cdots & x_q \\ t_1 & \cdots & t_q \end{bmatrix}$ .  $B_p \begin{pmatrix} x_1 & \cdots & x_q \\ t_1 & \cdots & t_q \end{pmatrix}$  is a function of the  $q$  points  $P_1: (x_1, t_1), \cdots, P_q: (x_q, t_q)$  of the square  $a \leq x \leq b, a \leq t \leq b$ . When  $q = 1$  we obtain the functions  $B_p \begin{pmatrix} x \\ t \end{pmatrix}$  of the single point

$P: (x, t)$ , and we use these to define the *simple Fredholm minor*  $D \begin{pmatrix} x & \\ & t \end{pmatrix} \lambda$  as follows:

$$D \begin{pmatrix} x & \\ & t \end{pmatrix} \lambda = B_0 \begin{pmatrix} x \\ t \end{pmatrix} - B_1 \begin{pmatrix} x \\ t \end{pmatrix} \lambda + \cdots + (-1)^{p-1} \frac{B_{p-1} \begin{pmatrix} x \\ t \end{pmatrix}}{(p-1)!} \lambda^{p-1} + \cdots$$

The *generalized Fredholm minor*  $D \begin{pmatrix} x_1 \cdots x_q & \\ & t_1 \cdots t_q \end{pmatrix} \lambda$  is defined in the same way:  $D \begin{pmatrix} x_1 \cdots x_q & \\ & t_1 \cdots t_q \end{pmatrix} \lambda = B_0 \begin{pmatrix} x_1 \cdots x_q \\ t_1 \cdots t_q \end{pmatrix} - B_1 \begin{pmatrix} x_1 \cdots x_q \\ t_1 \cdots t_q \end{pmatrix} \lambda + \cdots + (-1)^{p-1} \frac{B_{p-1} \begin{pmatrix} x_1 \cdots x_q \\ t_1 \cdots t_q \end{pmatrix}}{(p-1)!} \lambda^{p-1} + \cdots$ . The same proof

as that given for  $D(\lambda)$  shows that the Fredholm minors (simple and generalized) are integral analytic functions of the complex variable  $\lambda$ ;

in other words the power series which serve to define  $D \begin{pmatrix} x_1 \cdots x_q & \\ & t_1 \cdots t_q \end{pmatrix} \lambda$ ,  $q = 1, 2, \dots$ , converge for every (finite) value of  $\lambda$ . In fact the

modulus of  $\begin{bmatrix} x_1 \cdots x_q & \tau_1 \cdots \tau_p \\ t_1 \cdots t_q & \tau_1 \cdots \tau_p \end{bmatrix}$  is not greater than  $(q+p)^{\frac{q+p}{2}} M^{q+p}$

and so  $\left| B_p \begin{pmatrix} x_1 \cdots x_q \\ t_1 \cdots t_q \end{pmatrix} \right| \leq (q+p)^{\frac{q+p}{2}} M^{q+p} (b-a)^p$ . Hence the

power series whose sum is  $D \begin{pmatrix} x_1 \cdots x_q & \\ & t_1 \cdots t_q \end{pmatrix} \lambda$  is dominated by the power

series whose  $(p+1)$ st term is  $(q+p)^{\frac{q+p}{2}} M^{q+p} \frac{(b-a)^p}{p!} |\lambda|^p$ , and the ratio of the  $(p+1)$ st term of this series to the  $p$ th is

$$\frac{(q+p)^{\frac{q+p}{2}} M (b-a) |\lambda|}{(q+p-1)^{\frac{q+p-1}{2}} p} = \left( 1 + \frac{1}{q+p-1} \right)^{\frac{q+p-1}{2}} M (b-a) |\lambda| \frac{(q+p)^{\frac{1}{2}}}{p}$$

The limit of this ratio at  $p = \infty$  is zero (why?) no matter what are the values of  $q, M$ , or  $\lambda$  and so the dominating power series converges for every value of  $|\lambda|$ . In other words

$D \begin{pmatrix} x_1 & \cdots & x_q \\ t_1 & \cdots & t_q \end{pmatrix} \lambda$  is an analytic function of  $\lambda$  at every (finite) point of the complex  $\lambda$ -plane.

*Note.* The convergence of the dominating series shows (since the coefficients of this series do not depend on the points  $P_1: (x_1, t_1), \dots, P_q: (x_q, t_q)$ ) that the convergence of the power series in  $\lambda$ , whose sum is  $D \begin{pmatrix} x_1 & \cdots & x_q \\ t_1 & \cdots & t_q \end{pmatrix} \lambda$ , is uniform with respect to the points  $P_1, \dots, P_q$  over the square  $a \leq x \leq b, a \leq t \leq b$  (what does this mean?). Since the coefficients of this series are continuous functions of the points  $P_1, \dots, P_q$ ,  $D \begin{pmatrix} x_1 & \cdots & x_q \\ t_1 & \cdots & t_q \end{pmatrix} \lambda$  is, then, a continuous function of these points (why?).

## EXERCISES

1. Show that  $A_1 = \int B_0 \begin{pmatrix} \tau \\ \tau \end{pmatrix} d\tau$ ;  $A_2 = \int B_1 \begin{pmatrix} \tau \\ \tau \end{pmatrix} d\tau$  and, generally, that  $A_{p+1} = \int B_p \begin{pmatrix} \tau \\ \tau \end{pmatrix} d\tau$ .
2. Show that  $\int D \begin{pmatrix} \tau \\ \tau \end{pmatrix} \lambda d\tau = A_1 - A_2\lambda + A_3 \frac{\lambda^2}{2!} - \dots = -D'(\lambda)$ , where the prime denotes differentiation with respect to  $\lambda$ . Justify term-by-term integration of the series for  $D \begin{pmatrix} \tau \\ \tau \end{pmatrix} \lambda$ .
3. Show that  $\int B_p \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_1 & \tau_2 \end{pmatrix} d\tau_1 d\tau_2 = A_{p+2}$  and, generally, that  $\int B_p \begin{pmatrix} \tau_1 & \cdots & \tau_q \\ \tau_1 & \cdots & \tau_q \end{pmatrix} d\tau_1 \cdots d\tau_q = A_{p+q}$ .
4. Show that  $\int D \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_1 & \tau_2 \end{pmatrix} \lambda d\tau_1 d\tau_2 = A_2 - A_3\lambda + \frac{A_4}{2!}\lambda^2 - \dots = D''(\lambda)$ .
5. Show that  $\int D \begin{pmatrix} \tau_1 & \cdots & \tau_q \\ \tau_1 & \cdots & \tau_q \end{pmatrix} \lambda d\tau_1 \cdots d\tau_q = A_q - A_{q+1}\lambda + \dots = (-1)^q D^{(q)}(\lambda)$ , where  $D^{(q)}(\lambda)$  denotes the  $q$ th derivative of  $D(\lambda)$  with respect to  $\lambda$ .
6. Show that  $B_{p+1} \begin{pmatrix} x_1 & \cdots & x_q \\ t_1 & \cdots & t_q \end{pmatrix} = \int B_p \begin{pmatrix} x_1 & \cdots & x_q \\ t_1 & \cdots & t_q \end{pmatrix} \tau d\tau$ ,  $p = 0, 1, 2, \dots$ .
7. Show that  $\int D \begin{pmatrix} x_1 & \cdots & x_q \\ t_1 & \cdots & t_q \end{pmatrix} \tau \lambda d\tau = -D' \begin{pmatrix} x_1 & \cdots & x_q \\ t_1 & \cdots & t_q \end{pmatrix} \lambda$ .



8. Show that  $B_{p+r} \begin{pmatrix} x_1 \cdots x_q \\ t_1 \cdots t_q \end{pmatrix} = \int B_p \begin{pmatrix} x_1 \cdots x_q & \tau_1 \cdots \tau_r \\ t_1 \cdots t_q & \tau_1 \cdots \tau_r \end{pmatrix} dt_1 \cdots dt_r$ .

9. Show that  $\int D \begin{pmatrix} x_1 \cdots x_q & \tau_1 \cdots \tau_r \\ t_1 \cdots t_q & \tau_1 \cdots \tau_r \end{pmatrix} dt_1 \cdots dt_r =$   
 $(-1)^r D^{(r)} \begin{pmatrix} x_1 \cdots x_q & \\ t_1 \cdots t_q & \lambda \end{pmatrix}$ .

10. Show that if  $\lambda_0$  is a simple zero of  $D(\lambda)$  then  $D \begin{pmatrix} x \\ t \\ \lambda_0 \end{pmatrix}$  is not the zero constant function over the square  $a \leq x \leq b$ ,  $a \leq t \leq b$ . *Hint.*  $D'(\lambda_0) \neq 0$  (see Exercise 2).

11. Show that if  $\lambda_0$  is a zero of multiplicity  $p$  of  $D(\lambda)$  then  $D \begin{pmatrix} x_1 \cdots x_p \\ t_1 \cdots t_p \\ \lambda_0 \end{pmatrix}$  is not identically zero, i.e., is not zero for every choice of the points  $P_1: (x_1, t_1), \dots, P_p: (x_p, t_p)$ . *Hint.* See Exercise 5.

12. Show that the symbol  $B_n \begin{pmatrix} x_1 \cdots x_q \\ t_1 \cdots t_q \end{pmatrix}$  is *alternating* in the upper variables  $x$  and in the lower variables  $t$ ; i.e., that an interchange of any two of the symbols  $x_1, \dots, x_q$  or any two of the symbols  $t_1, \dots, t_q$  changes the sign of  $B_n \begin{pmatrix} x_1 \cdots x_q \\ t_1 \cdots t_q \end{pmatrix}$ .

13. Show that  $D \begin{pmatrix} x_1 \cdots x_q \\ t_1 \cdots t_q \\ \lambda \end{pmatrix}$  is alternating in the upper variables  $x$  and in the lower variables  $t$ .

14. Show that  $D \begin{pmatrix} x_1 \cdots x_q \\ t_1 \cdots t_q \\ \lambda \end{pmatrix}$  is zero if any two of the  $x$ 's, or any two of the  $t$ 's, are equal.

15. Show that  $B_n \begin{pmatrix} x \\ t \end{pmatrix} = \begin{bmatrix} x \\ t \end{bmatrix} A_n - n \int \begin{bmatrix} x \\ \tau \end{bmatrix} B_{n-1} \begin{pmatrix} \tau \end{pmatrix} dt$ . *Hint.* On expanding  $\begin{bmatrix} x & \tau_1 \cdots \tau_n \\ t & \tau_1 \cdots \tau_n \end{bmatrix}$  in terms of the first row we obtain  $\begin{bmatrix} x \\ t \end{bmatrix} \begin{bmatrix} \tau_1 \cdots \tau_n \\ \tau_1 \cdots \tau_n \end{bmatrix} - \begin{bmatrix} x \\ \tau_1 \end{bmatrix} \begin{bmatrix} \tau_1 \cdots \tau_n \\ t \tau_2 \cdots \tau_n \end{bmatrix} + \begin{bmatrix} x \\ \tau_2 \end{bmatrix} \begin{bmatrix} \tau_1 \cdots \tau_n \\ t \tau_1 \tau_3 \cdots \tau_n \end{bmatrix} - \dots = \begin{bmatrix} x \\ t \end{bmatrix} \begin{bmatrix} \tau_1 \cdots \tau_n \\ \tau_1 \cdots \tau_n \end{bmatrix} - \begin{bmatrix} x \\ \tau_1 \end{bmatrix} \begin{bmatrix} \tau_1 \cdots \tau_n \\ t \tau_2 \cdots \tau_n \end{bmatrix} - \dots$  (in view of the alternating character of the symbol  $\begin{bmatrix} \tau_1 \cdots \tau_n \\ t_1 \cdots t_n \end{bmatrix}$  in its upper (i.e., row) labels). Hence  $B_n \begin{pmatrix} x \\ t \end{pmatrix} = \begin{bmatrix} x \\ t \end{bmatrix} A_n - n \int \begin{bmatrix} x \\ \tau \end{bmatrix} B_{n-1} \begin{pmatrix} \tau \end{pmatrix} dt$  since the  $n$  negatively signed terms all integrate to the same thing, namely,  $\int \begin{bmatrix} x \\ \tau \end{bmatrix} B_{n-1} \begin{pmatrix} \tau \end{pmatrix} dt$ .

16. Show that  $B_n \begin{pmatrix} x \\ t \end{pmatrix} = \begin{bmatrix} x \\ t \end{bmatrix} A_n - n \int B_{n-1} \begin{pmatrix} x \\ \tau \end{pmatrix} \begin{bmatrix} \tau \\ t \end{bmatrix} d\tau$ . *Hint.* Expand the determinant  $\begin{bmatrix} x & \tau_1 & \cdots & \tau_n \\ t & \tau_1 & \cdots & \tau_n \end{bmatrix}$  in terms of the first column.

17. Show that  $B_n \begin{pmatrix} x_1 & x_2 \\ t_1 & t_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ t_1 \end{pmatrix} B_n \begin{pmatrix} x_2 \\ t_2 \end{pmatrix} - \begin{bmatrix} x_1 \\ t_2 \end{bmatrix} B_n \begin{pmatrix} x_2 \\ t_1 \end{pmatrix} - n \int \begin{bmatrix} x_1 \\ \tau \end{bmatrix} B_{n-1} \begin{pmatrix} \tau & x_2 \\ t_1 & t_2 \end{pmatrix} d\tau$ .

18. Show that  $B_n \begin{pmatrix} x_1 & x_2 \\ t_1 & t_2 \end{pmatrix} = \begin{bmatrix} x_1 \\ t_1 \end{bmatrix} B_n \begin{pmatrix} x_2 \\ t_2 \end{pmatrix} - \begin{bmatrix} x_2 \\ t_1 \end{bmatrix} B_n \begin{pmatrix} x_1 \\ t_2 \end{pmatrix} - n \int B_{n-1} \begin{pmatrix} x_1 & x_2 \\ \tau & t_1 \end{pmatrix} \begin{bmatrix} \tau \\ t_1 \end{bmatrix} d\tau$ .

19. Show that  $B_n \begin{pmatrix} x_1 \cdots x_q \\ t_1 \cdots t_q \end{pmatrix} = \sum (-1)^{p-1} \begin{bmatrix} x_1 \\ t_p \end{bmatrix} B_n \begin{pmatrix} x_2 \cdots x_q \\ t_1 \cdots t_{p-1} t_{p+1} \cdots t_q \end{pmatrix} - n \int \begin{bmatrix} x_1 \\ \tau \end{bmatrix} B_{n-1} \begin{pmatrix} \tau & x_2 \cdots x_q \\ t_1 & t_2 \cdots t_q \end{pmatrix} d\tau$ .

20. Show that  $B_n \begin{pmatrix} x_1 \cdots x_q \\ t_1 \cdots t_q \end{pmatrix} = \sum_1^q (-1)^{p-1} \begin{bmatrix} x_p \\ t_1 \end{bmatrix} B_n \begin{pmatrix} x_1 \cdots x_{p-1} x_{p+1} \cdots x_q \\ t_2 \cdots t_q \end{pmatrix} - n \int B_{n-1} \begin{pmatrix} x_1 \cdots x_q \\ \tau & t_2 \cdots t_q \end{pmatrix} \begin{bmatrix} \tau \\ t_1 \end{bmatrix} d\tau$ .

It follows from the result of Exercise 15 that

$$D \begin{pmatrix} x & \lambda \\ t & \lambda \end{pmatrix} = \begin{bmatrix} x \\ t \end{bmatrix} D(\lambda) + \lambda \int \begin{bmatrix} x \\ \tau \end{bmatrix} D \begin{pmatrix} \tau & \lambda \\ t & \lambda \end{pmatrix} d\tau.$$

In fact the coefficient of  $(-1)^n \frac{\lambda^n}{n!}$  on the left is  $B_n \begin{pmatrix} x \\ t \end{pmatrix}$  while the coefficient of  $(-1)^n \frac{\lambda^n}{n!}$  on the right is  $\begin{bmatrix} x \\ t \end{bmatrix} A_n - n \int \begin{bmatrix} x \\ \tau \end{bmatrix} B_{n-1} \begin{pmatrix} \tau \\ t \end{pmatrix} d\tau$ . In the same way it follows from the result of Exercise 16 that

$$D \begin{pmatrix} x & \lambda \\ t & \lambda \end{pmatrix} = \begin{bmatrix} x \\ t \end{bmatrix} D(\lambda) + \lambda \int D \begin{pmatrix} x & \lambda \\ \tau & \lambda \end{pmatrix} \begin{bmatrix} \tau \\ t \end{bmatrix} d\tau.$$

The two relations

$$D \begin{pmatrix} x & \lambda \\ t & \lambda \end{pmatrix} = \begin{bmatrix} x \\ t \end{bmatrix} D(\lambda) + \lambda \int \begin{bmatrix} x \\ \tau \end{bmatrix} D \begin{pmatrix} \tau & \lambda \\ t & \lambda \end{pmatrix} d\tau;$$

$$D \begin{pmatrix} x \\ t \end{pmatrix} \lambda = \begin{bmatrix} x \\ t \end{bmatrix} D(\lambda) + \lambda \int D \begin{pmatrix} x \\ \tau \end{pmatrix} \lambda \begin{bmatrix} \tau \\ t \end{bmatrix} d\tau$$

are known as the *Fredholm relations*. If we consider the two linear integral operators

$$\mathbf{K} : \begin{bmatrix} x \\ t \end{bmatrix} = K \begin{pmatrix} x \\ t \end{pmatrix}; \quad \mathbf{D} : D \begin{pmatrix} x \\ t \end{pmatrix}$$

they state that the two linear integral operators  $\mathbf{K}$  and  $\mathbf{D}$  commute:

$$\mathbf{KD} = \mathbf{DK} = \frac{1}{\lambda} \{\mathbf{D} - D(\lambda)\mathbf{K}\}.$$

(we assume that  $\lambda \neq 0$  since when  $\lambda = 0$  we have no integral equation).

We obtain in the same way from the result of Exercise 19 the following *generalized Fredholm relation*:

$$D \begin{pmatrix} x_1 \cdots x_q \\ t_1 \cdots t_q \end{pmatrix} \lambda = \sum_1^q (-1)^{p-1} \begin{bmatrix} x_1 \\ t_p \end{bmatrix} D \begin{pmatrix} x_2 \cdots \cdots x_q \\ t_1 \cdots t_{p-1} t_{p+1} \cdots t_q \end{pmatrix} \lambda \\ + \lambda \int \begin{bmatrix} x_1 \\ \tau \end{bmatrix} D \begin{pmatrix} \tau x_2 \cdots x_q \\ t_1 t_2 \cdots t_q \end{pmatrix} \lambda d\tau.$$

From the result of Exercise 20 we obtain the second generalized Fredholm relation

$$D \begin{pmatrix} x_1 \cdots x_q \\ t_1 \cdots t_q \end{pmatrix} \lambda = \sum_1^q (-1)^{p-1} \begin{bmatrix} x_p \\ t_1 \end{bmatrix} D \begin{pmatrix} x_1 \cdots x_{p-1} x_{p+1} \cdots x_q \\ t_1 \cdots \cdots t_q \end{pmatrix} \lambda \\ + \lambda \int D \begin{pmatrix} x_1 \cdots x_q \\ \tau t_2 \cdots t_q \end{pmatrix} \lambda \begin{bmatrix} \tau \\ t_1 \end{bmatrix} d\tau.$$

*Suggestion.* In a first reading these various Fredholm relations will doubtless seem forbidding, complicated, and confusing. They are, however, the central core of the subject of integral equations, and you must try to master them. Content yourself at first with the simple Fredholm relations which merely express the commutativity of the two linear integral operators  $\mathbf{K}$  and  $\mathbf{D}$ , the common value of  $\lambda\mathbf{KD}$  and  $\lambda\mathbf{DK}$  being  $\mathbf{D} - D(\lambda)\mathbf{K}$ . You will see in the next section how this result enables you to solve in a very simple manner the non-homogeneous integral equation

$$\mathbf{y} = \lambda\mathbf{K}\mathbf{y} + \mathbf{f}$$

when  $D(\lambda) \neq 0$  and the homogeneous equation  $\mathbf{u} = \lambda \mathbf{K} \mathbf{u}$  when  $D(\lambda) = 0$  provided that  $\mathbf{D}$  is not the zero linear integral operator. It is only in the "exceptional" case where  $\mathbf{D}$  is the zero linear integral operator that you will have to appeal to the generalized Fredholm relations.

### 3. The solution of the integral equation when $D(\lambda) \neq 0$

If we *assume* the existence of a continuous solution of the integral equation (non-homogeneous or homogeneous)

$$\mathbf{y} = \lambda \mathbf{K} \mathbf{y} + \mathbf{f}$$

it follows at once from the first simple Fredholm relation that this solution is unambiguously determinate. In fact

$$\mathbf{D} \mathbf{y} = \lambda \mathbf{D} \mathbf{K} \mathbf{y} + \mathbf{D} \mathbf{f} = (\mathbf{D} - D(\lambda) \mathbf{K}) \mathbf{y} + \mathbf{D} \mathbf{f}$$

so that  $D(\lambda) \mathbf{K} \mathbf{y} = \mathbf{D} \mathbf{f}$ . Hence if  $D(\lambda) \neq 0$  we have  $\mathbf{K} \mathbf{y} = \frac{1}{D(\lambda)} \mathbf{D} \mathbf{f}$  so that

$$\mathbf{y} = \frac{\lambda}{D(\lambda)} \mathbf{D} \mathbf{f} + \mathbf{f}.$$

The second simple Fredholm relation enables us to verify that the function-vector  $\mathbf{y}$  furnished by this formula actually is a solution of the integral equation, and thus our *assumption* that the integral equation actually possesses a solution is validated. In fact it follows from the relation

$$\mathbf{y} = \frac{\lambda}{D(\lambda)} \mathbf{D} \mathbf{f} + \mathbf{f}$$

that

$$\begin{aligned} \mathbf{K} \mathbf{y} &= \frac{\lambda}{D(\lambda)} \mathbf{K} \mathbf{D} \mathbf{f} + \mathbf{K} \mathbf{f} \\ &= \frac{\{\mathbf{D} - D(\lambda) \mathbf{K}\} \mathbf{f}}{D(\lambda)} + \mathbf{K} \mathbf{f} = \frac{1}{D(\lambda)} \mathbf{D} \mathbf{f}. \end{aligned}$$

Hence  $\mathbf{y} = \lambda \mathbf{K} \mathbf{y} + \mathbf{f}$ . We have, then, the following fundamental result:

**When  $D(\lambda) \neq 0$  the linear integral equation  $\mathbf{y} = \lambda \mathbf{K} \mathbf{y} + \mathbf{f}$  possesses one and only one continuous solution**

$$\mathbf{y} = \lambda \mathbf{\Gamma} \mathbf{f} + \mathbf{f}; \quad \mathbf{\Gamma} = \frac{1}{D(\lambda)} \mathbf{D}$$

The linear integral operator  $\Gamma: \Gamma \begin{pmatrix} x & \\ t & \lambda \end{pmatrix} = \frac{D \begin{pmatrix} x & \\ t & \lambda \end{pmatrix}}{D(\lambda)}$  which furnishes, in this way, the unambiguously determinate solution of the integral equation is known as the *resolving operator*, and  $\Gamma \begin{pmatrix} x & \\ t & \lambda \end{pmatrix}$  is known as the *resolving kernel*.

In particular we see that when  $f$  is the zero constant function, so that the integral equation is homogeneous, the one and only solution of the integral equation is the zero constant function. In other words

When  $D(\lambda) \neq 0$  the only solution of the homogeneous integral equation

$$\mathbf{u} = \lambda \mathbf{K} \mathbf{u}$$

is the trivial one  $\mathbf{u} \equiv \mathbf{0}$ . Thus, if  $D(\lambda) \neq 0$ ,  $\lambda$  is not a characteristic number of the integral equation. We may phrase this result as follows:

**If the integral equation possesses characteristic numbers these will be found amongst the zeros of the Fredholm determinant  $D(\lambda)$ .**

### EXERCISE

1. Show that if  $D(\lambda) = 0$  the integral equation  $\mathbf{y} = \lambda \mathbf{K} \mathbf{y} + \mathbf{f}$  does not possess a solution unless  $D\mathbf{f} = \mathbf{0}$ .

#### 4. The solution of the homogeneous integral equation when $D(\lambda) = 0$

Let  $\lambda = \lambda_0$  be a zero of  $D(\lambda)$ . Then

$$\lambda_0 \mathbf{D} \mathbf{K} = \lambda_0 \mathbf{K} \mathbf{D} = \mathbf{D}.$$

Hence  $u(x) = D \begin{pmatrix} x & \\ t & \lambda_0 \end{pmatrix}$  is a solution, no matter what is the value of  $t$ , of the homogeneous integral equation

$$\mathbf{u} = \lambda_0 \mathbf{K} \mathbf{u}.$$

If, then,  $D \begin{pmatrix} x & \\ t & \lambda_0 \end{pmatrix}$  is not identically zero,  $\lambda_0$  is a characteristic number of the integral equation. This will certainly be true if  $\lambda_0$  is a simple zero of  $D(\lambda)$  (see Exercise 10, p. 256), but it may also be true if  $\lambda_0$  is a multiple zero of  $D(\lambda)$ . Let  $\lambda_0$  be a zero of multiplicity  $p$  of  $D(\lambda)$ ; then

$D \begin{pmatrix} x_1 \cdots x_p & \\ t_1 \cdots t_p & \lambda_0 \end{pmatrix}$  is not identically zero (see Exercise 11, p. 256). Let

$q$  be the positive integer  $\leq p$  which is such that  $D \begin{pmatrix} x_1 \cdots x_q & \lambda_0 \\ t_1 \cdots t_q \end{pmatrix}$  is not identically zero while  $D \begin{pmatrix} x_1 \cdots x_r & \lambda_0 \\ t_1 \cdots t_r \end{pmatrix}$  is identically zero if  $r < q$ , and let the points  $\pi_1: (\xi_1, \tau_1), \dots, \pi_q: (\xi_q, \tau_q)$  be such that the number  $D \begin{pmatrix} \xi_1 \cdots \xi_q & \lambda_0 \\ \tau_1 \cdots \tau_q \end{pmatrix}$  is not zero. Then the functions

$$u_1(x) = \frac{D \begin{pmatrix} x \ \xi_2 \cdots \xi_q & \lambda_0 \\ \tau_1 \ \tau_2 \cdots \tau_q \end{pmatrix}}{D \begin{pmatrix} \xi_1 \cdots \xi_q & \lambda_0 \\ \tau_1 \cdots \tau_q \end{pmatrix}}; \quad \dots; \quad u_q(x) = \frac{D \begin{pmatrix} \xi_1 \cdots \xi_{q-1} x & \lambda_0 \\ \tau_1 \cdots \tau_q \end{pmatrix}}{D \begin{pmatrix} \xi_1 \cdots \xi_q & \lambda_0 \\ \tau_1 \cdots \tau_q \end{pmatrix}}$$

are all solutions of the homogeneous integral equation  $\mathbf{u} = \lambda \mathbf{K} \mathbf{u}$ . To prove this we observe that, in view of the definition of  $q$ , the first generalized Fredholm relation yields

$$D \begin{pmatrix} x \ \xi_2 \cdots \xi_q & \lambda_0 \\ \tau_1 \cdots \tau_q \end{pmatrix} = \lambda_0 \int_{[\tau]}^{[x]} D \begin{pmatrix} \tau \ \xi_2 \cdots \xi_q & \lambda_0 \\ \tau_1 \cdots \tau_q \end{pmatrix} d\tau,$$

and on dividing this equation by  $D \begin{pmatrix} \xi_1 \cdots \xi_q & \lambda_0 \\ \tau_1 \cdots \tau_q \end{pmatrix}$  we obtain

$$\mathbf{u}_1 = \lambda_0 \mathbf{K} \mathbf{u}_1$$

so that  $u_1(x)$  is a solution of the homogeneous integral equation.  $u_1(x)$  has the value 1 when  $x = \xi_1$  so that  $\lambda_0$  is a characteristic number (why?). Thus we have the following result:

*Every zero of  $D(\lambda)$  is a characteristic number of the integral equation.*

On combining this result with the result given at the end of Section 3 we see that

**The zeros of  $D(\lambda)$  are all characteristic numbers of the integral equation and there are no others; in other words the characteristic numbers of the integral equation are precisely the zeros of  $D(\lambda)$ .**

Since  $u_2(x) = \frac{D \begin{pmatrix} x & \xi_1 & \xi_2 & \cdots & \xi_q \\ \tau_1 & \cdots & \cdots & \cdots & \tau_q \end{pmatrix}}{D \begin{pmatrix} \xi_1 & \cdots & \xi_q \\ \tau_1 & \cdots & \tau_q \end{pmatrix}} \lambda_0$  (why?) the same argu-

ment as that given for  $u_1(x)$  shows that  $u_2(x)$  is a solution of the homogeneous integral equation. Similarly  $u_3(x), \cdots, u_q(x)$  are also solutions of the homogeneous integral equation. Hence any linear combination

$$\mathbf{u} = c_1 \mathbf{u}_1 + \cdots + c_q \mathbf{u}_q$$

of the function-vectors  $\mathbf{u}_1, \cdots, \mathbf{u}_q$  is a solution of the equation  $\mathbf{u} = \lambda_0 \mathbf{K} \mathbf{u}$ . The  $q$  function-vectors  $\mathbf{u}_1, \cdots, \mathbf{u}_q$  are linearly independent. In fact if  $c_1 \mathbf{u}_1 + \cdots + c_q \mathbf{u}_q$  is the zero vector it follows on setting  $x = \xi_1$  that  $c_1 = 0$  (since the functions  $u_2(x), \cdots, u_q(x)$  are all zero at  $x = \xi_1$  (see Exercise 14, p. 256)). Similarly  $c_2 = 0, \cdots, c_q = 0$ .

We now proceed to prove, conversely, that any function-vector which satisfies the homogeneous integral equation  $\mathbf{u} = \lambda_0 \mathbf{K} \mathbf{u}$  is a linear combination of the function-vectors  $\mathbf{u}_1, \cdots, \mathbf{u}_q$ . Let, then,  $\mathbf{v}$  be any function-vector which satisfies the equation  $\mathbf{u} = \lambda_0 \mathbf{K} \mathbf{u}$ , and consider the function-vector

$$\mathbf{w} = \mathbf{v} - \sum_1^q v^p \mathbf{u}_p; \quad v^p = v(\xi_p); \quad p = 1, \cdots, q.$$

Since  $\mathbf{v}$  satisfies the equation  $\mathbf{u} = \lambda_0 \mathbf{K} \mathbf{u}$  we have

$$v^p = \lambda_0 \int K \begin{pmatrix} \xi_p \\ \tau \end{pmatrix} v(\tau) d\tau$$

and so

$$\mathbf{w} = \mathbf{v} - \lambda_0 \int \left\{ \sum_1^q K \begin{pmatrix} \xi_p \\ \tau \end{pmatrix} \mathbf{u}_p \right\} v(\tau) d\tau.$$

From the definition of the functions  $u_p(x)$  the sum  $\sum_1^q -K \begin{pmatrix} \xi_p \\ \tau \end{pmatrix} u_p(x)$  is, by virtue of the second generalized Fredholm relation, the quotient by  $D \begin{pmatrix} \xi_1 & \cdots & \xi_q \\ \tau_1 & \cdots & \tau_q \end{pmatrix} \lambda_0$  of

$$D \begin{pmatrix} x & \xi_1 & \cdots & \xi_q \\ \tau & \tau_1 & \cdots & \tau_q \end{pmatrix} \lambda_0 - K \begin{pmatrix} x \\ \tau \end{pmatrix} D \begin{pmatrix} \xi_1 & \cdots & \xi_q \\ \tau_1 & \cdots & \tau_q \end{pmatrix} \\ - \lambda_0 \int D \begin{pmatrix} x & \xi_1 & \cdots & \xi_q \\ \alpha & \tau_1 & \cdots & \tau_q \end{pmatrix} \begin{bmatrix} \alpha \\ \tau \end{bmatrix} d\alpha.$$

Hence

$$w = v + \lambda_0 \int \frac{D \begin{pmatrix} x & \xi_1 & \cdots & \xi_q \\ \tau & \tau_1 & \cdots & \tau_q \end{pmatrix} \lambda_0}{D \begin{pmatrix} \xi_1 & \cdots & \xi_q \\ \tau_1 & \cdots & \tau_q \end{pmatrix} \lambda_0} v(\tau) d\tau - \lambda_0 \int K \begin{pmatrix} x \\ \tau \end{pmatrix} v(\tau) d\tau \\ - \lambda_0^2 \int \frac{D \begin{pmatrix} x & \xi_1 & \cdots & \xi_q \\ \alpha & \tau_1 & \cdots & \tau_q \end{pmatrix} \lambda_0}{D \begin{pmatrix} \xi_1 & \cdots & \xi_q \\ \tau_1 & \cdots & \tau_q \end{pmatrix} \lambda_0} \begin{bmatrix} \alpha \\ \tau \end{bmatrix} v(\tau) d\alpha d\tau.$$

Since  $\mathbf{v}$  is, by hypothesis, a solution of the homogeneous integral equation the first and third of the terms on the right cancel each other.

Since  $\lambda_0 \int \begin{bmatrix} \alpha \\ \tau \end{bmatrix} v(\tau) d\tau = v(\alpha)$  the second and fourth terms on the right also cancel (why?) and so  $w$  is the zero constant function, or, equivalently,  $\mathbf{v}$  is a linear combination of the function-vectors  $\mathbf{u}_1, \cdots, \mathbf{u}_q$ . We have, then, the following fundamental result:

If  $\lambda = \lambda_0$  is a zero of multiplicity  $p$  of  $D(\lambda)$  there exists a positive integer  $q \leq p$ , which we term the index of the characteristic number  $\lambda_0$ , such that every solution of the homogeneous integral equation (i.e., every characteristic vector) which is associated with the characteristic number  $\lambda_0$  is a linear combination of  $q$  such characteristic vectors.

### 5. Necessary and sufficient conditions for the solution of the non-homogeneous integral equation when $\lambda$ is a characteristic number

We have already seen (Exercise 1, p. 260) that a necessary condition for a solution of the non-homogeneous equation  $\mathbf{y} = \lambda \mathbf{K}\mathbf{y} + \mathbf{f}$  when  $D(\lambda) = 0$ , i.e., when  $\lambda$  is a characteristic number, is  $D\mathbf{f} = \mathbf{0}$ . This condition is vacuous when  $\mathbf{D}$  is the zero linear integral operator i.e., when  $\lambda = \lambda_0$  is a characteristic number of index  $> 1$ . Let  $\lambda = \lambda_0$  be a characteristic number of index  $q$ , and let  $(\xi_1, \tau_1), \cdots, (\xi_q, \tau_q)$  be



such that  $D \begin{pmatrix} \xi_1 & \cdots & \xi_q \\ \tau_1 & \cdots & \tau_q \end{pmatrix} \lambda_0 \neq 0$ . On multiplying the relation  $y = \lambda_0 \mathbf{K}y + \mathbf{f}$  by  $D \begin{pmatrix} \xi_1 & \cdots & \xi_q \\ x & \tau_2 & \cdots & \tau_q \end{pmatrix} \lambda_0$  and integrating with respect to  $x$  we obtain

$$\begin{aligned} \int D \begin{pmatrix} \xi_1 & \cdots & \xi_q \\ x & \tau_2 & \cdots & \tau_q \end{pmatrix} \lambda_0 y(x) dx \\ = \lambda_0 \int \left[ \begin{matrix} x \\ \tau \end{matrix} \right] y(\tau) D \begin{pmatrix} \xi_1 & \cdots & \xi_q \\ x & \tau_2 & \cdots & \tau_q \end{pmatrix} \lambda_0 d\tau dx \\ + \int D \begin{pmatrix} \xi_1 & \cdots & \xi_q \\ x & \tau_2 & \cdots & \tau_q \end{pmatrix} \lambda_0 f(x) dx. \end{aligned}$$

By virtue of the second generalized Fredholm relation

$$\lambda_0 \int D \begin{pmatrix} \xi_1 & \cdots & \xi_q \\ x & \tau_2 & \cdots & \tau_q \end{pmatrix} \lambda_0 \left[ \begin{matrix} x \\ \tau \end{matrix} \right] dx = D \begin{pmatrix} \xi_1 & \cdots & \xi_q \\ \tau & \tau_2 & \cdots & \tau_q \end{pmatrix} \lambda_0$$

(remember that  $q$  is the index of  $\lambda_0$ ) and so the first term on the right cancels the term on the left. Thus a necessary condition for the existence of a solution of the non-homogeneous equation is

$$\int D \begin{pmatrix} \xi_1 & \cdots & \xi_q \\ x & \tau_2 & \cdots & \tau_q \end{pmatrix} \lambda_0 f(x) dx = 0$$

On denoting the quotient of  $D \begin{pmatrix} \xi_1 & \cdots & \xi_q \\ x & \tau_2 & \cdots & \tau_q \end{pmatrix} \lambda_0$  by  $D \begin{pmatrix} \xi_1 & \cdots & \xi_q \\ \tau_1 & \cdots & \tau_q \end{pmatrix} \lambda_0$  by  $\bar{v}_1(x)$  this necessary condition may be written as follows:

$$(\mathbf{v}_1 | \mathbf{f}) = 0.$$

In words:

*The function-vector  $\mathbf{f}$  must be perpendicular to the function-vector  $\mathbf{v}_1$ .*

Since  $D \begin{pmatrix} \xi_1 & \cdots & \xi_q \\ \tau_1 & x & \tau_3 & \cdots & \tau_q \end{pmatrix} \lambda_0 = -D \begin{pmatrix} \xi_1 & \cdots & \xi_q \\ x & \tau_1 & \tau_3 & \cdots & \tau_q \end{pmatrix} \lambda_0$

the same argument as that for the function  $v_1(x)$  shows that  $\mathbf{f}$  must be perpendicular to the function-vector

$$\mathbf{v}_2 : v_2(x), \quad \text{where } \bar{v}_2(x) = \frac{D \begin{pmatrix} \xi_1 & \cdots & \xi_q \\ \tau_1 & x & \tau_3 & \cdots & \tau_q \end{pmatrix} \lambda_0}{D \begin{pmatrix} \xi_1 & \cdots & \xi_q \\ \tau_1 & \cdots & \tau_q \end{pmatrix} \lambda_0}.$$

Proceeding in this way we see that, in order that the non-homogeneous integral equation

$$\mathbf{y} = \lambda_0 \mathbf{K}\mathbf{y} + \mathbf{f}$$

may possess a solution when  $\lambda_0$  is a characteristic number of index  $q$ , it is necessary that the function-vector  $\mathbf{f}$  be perpendicular to each of the function-vectors  $\mathbf{v}_1, \dots, \mathbf{v}_q$ , where  $\bar{v}_p(x)$  is the quotient of  $D\left(\begin{matrix} \xi_1 & \dots & \xi_q \\ \tau_1 & \dots & \tau_q \end{matrix} \lambda_0\right)$  by  $D\left(\begin{matrix} \xi_1 & \dots & \xi_q \\ \tau_1 & \dots & \tau_q \end{matrix}\right)$ .

### EXERCISE

1. Show that the function-vectors  $\mathbf{v}_1, \dots, \mathbf{v}_q$  are linearly independent. *Hint.* When  $x = \tau_1$ ,  $v_1(x) = 1$ ,  $v_2(x) = 0, \dots, v_q(x) = 0$ .

It is easy to see that when the function-vector  $\mathbf{f}$  is perpendicular to each of the  $q$  function-vectors  $\mathbf{v}_1, \dots, \mathbf{v}_q$  the non-homogeneous integral equation  $\mathbf{y} = \lambda \mathbf{K}\mathbf{y} + \mathbf{f}$  possesses a solution. Let us denote by  $P\left(\begin{matrix} x \\ t \end{matrix}\right)$  the function

$$P\left(\begin{matrix} x \\ t \end{matrix}\right) = \frac{D\left(\begin{matrix} x & \xi_1 & \dots & \xi_q \\ t & \tau_1 & \dots & \tau_q \end{matrix} \lambda_0\right)}{D\left(\begin{matrix} \xi_1 & \dots & \xi_q \\ \tau_1 & \dots & \tau_q \end{matrix} \lambda_0\right)},$$

and let  $\mathbf{P}$  be the linear integral operator

$$\mathbf{P}: P\left(\begin{matrix} x \\ t \end{matrix}\right).$$

The first generalized Fredholm relation tells us that

$$P\left(\begin{matrix} x \\ t \end{matrix}\right) = K\left(\begin{matrix} x \\ t \end{matrix}\right) - K\left(\begin{matrix} x \\ \tau_1 \end{matrix}\right) \bar{v}_1(t) - \dots - K\left(\begin{matrix} x \\ \tau_q \end{matrix}\right) \bar{v}_q(t) \\ + \lambda \int \left[ \begin{matrix} x \\ \tau \end{matrix} \right] P\left(\begin{matrix} \tau \\ t \end{matrix}\right) d\tau.$$

Since  $\mathbf{f}$  is perpendicular to each of the function-vectors  $\mathbf{v}_1, \dots, \mathbf{v}_q$  it follows that

$$\mathbf{P}\mathbf{f} = \mathbf{K}\mathbf{f} + \lambda \mathbf{K}\mathbf{P}\mathbf{f} = \mathbf{K}\mathbf{z},$$

where  $\mathbf{z} = \mathbf{f} + \lambda \mathbf{P}\mathbf{f}$ . Hence  $\mathbf{z} = \lambda \mathbf{K}\mathbf{z} + \mathbf{f}$  so that  $\mathbf{z}$  is a solution of the

non-homogeneous linear integral equation  $\mathbf{y} = \lambda \mathbf{K}\mathbf{y} + \mathbf{f}$ . If  $\mathbf{y}$  is any other solution of this equation  $\mathbf{y} - \mathbf{z}$  is a solution of the homogeneous equation  $\mathbf{u} = \lambda \mathbf{K}\mathbf{u}$  (why?). Hence  $\mathbf{y} - \mathbf{z}$  is a linear combination of the function-vectors  $\mathbf{u}_1, \dots, \mathbf{u}_q$ . Thus the general solution of the non-homogeneous linear integral equation is

$$\mathbf{y} = \mathbf{z} + c_1 \mathbf{u}_1 + \dots + c_q \mathbf{u}_q,$$

where the  $c_1, \dots, c_q$  are arbitrary constants.

We have now finished the *theory* of the homogeneous integral equation  $\mathbf{u} = \lambda \mathbf{K}\mathbf{u}$  and of the non-homogeneous integral equation  $\mathbf{y} = \lambda \mathbf{K}\mathbf{y} + \mathbf{f}$ . The net result is as follows:

If  $D(\lambda) \neq 0$  the integral equation, homogeneous or non-homogeneous, possesses the unambiguously determinate solution  $\mathbf{y} = \Gamma \mathbf{f} + \mathbf{f}$ ,

$$\text{where } \Gamma : \Gamma \begin{pmatrix} x \\ t \end{pmatrix} \lambda = \frac{D \begin{pmatrix} x \\ t \end{pmatrix} \lambda}{D(\lambda)}.$$

The unambiguously determinate solution of the homogeneous integral equation is, accordingly, the zero vector. If we reject this solution of the homogeneous integral equation as trivial we have the following contrast between the non-homogeneous integral equation and the homogeneous integral equation:

**When  $D(\lambda) \neq 0$  the non-homogeneous integral equation possesses an unambiguously determinate solution while the homogeneous equation does not possess a (non-trivial) solution.**

On the other hand the situation is reversed when  $D(\lambda) = 0$ .  $\lambda$  has an index  $q$  which is not greater than its multiplicity (as a zero of  $D(\lambda)$ ), and the general solution of the homogeneous equation is a linear combination of  $q$  linearly independent vectors  $\mathbf{u}_1, \dots, \mathbf{u}_q$ . The non-homogeneous integral equation does not now, in general, possess a solution. If, however,  $\mathbf{f}$  is orthogonal to  $q$  linearly independent vectors  $\mathbf{v}_1, \dots, \mathbf{v}_q$  the non-homogeneous equation does possess the solution  $\mathbf{z} = \mathbf{f} + \lambda \mathbf{P}\mathbf{f}$ , and the general solution of the non-homogeneous equation is obtained by adding to  $\mathbf{z}$  an arbitrary linear combination of the vectors  $\mathbf{u}_1, \dots, \mathbf{u}_q$ .

## 6. Adjoint integral equations; Hermitian integral equations

If  $\mathbf{K}^* : \mathbf{K}^* \begin{pmatrix} x \\ t \end{pmatrix} = \bar{\mathbf{K}} \begin{pmatrix} t \\ x \end{pmatrix}$  is the adjoint linear integral operator to  $\mathbf{K}$  (see Exercise 4, p. 84), we say that the integral equation  $\mathbf{y} = \lambda \mathbf{K}^* \mathbf{y} + \mathbf{f}$  is

the *adjoint* of (or is *adjoint to*) the integral equation  $y = \lambda Ky + f$ . It follows that the relationship between two adjoint integral equations is a *partnership*: *Either* of the two equations is the adjoint of the other. If we indicate the various quantities that refer to the adjoint integral equation by attaching a star to the symbol for the corresponding quantity for its partner, it is at once clear that  $\begin{bmatrix} x_1 \cdots x_q \\ t_1 \cdots t_q \end{bmatrix}^*$  is the

conjugate complex of  $\begin{bmatrix} t_1 \cdots t_q \\ x_1 \cdots x_q \end{bmatrix}$  (why?). Hence  $A_q^*$  is the conjugate complex of  $A_q$  (why?) and so  $D^*(\lambda)$  is the power series whose coefficients are the conjugate complexes of the coefficients of  $D(\lambda)$ . Denoting the conjugate complex of  $\lambda$  by  $\lambda^*$  (rather than by the usual  $\bar{\lambda}$ ) it follows that  $D^*(\lambda^*)$  is the conjugate complex of  $D(\lambda)$ . In particular  $D^*(\lambda^*) = 0$  when  $D(\lambda) = 0$  and  $D(\lambda) = 0$  when  $D^*(\lambda^*) = 0$ . Hence

**The characteristic numbers of the adjoint integral equation are the conjugate complexes of the characteristic numbers of its partner.**

The same argument that showed that  $A_p^*$  is the conjugate complex of  $A_p$  proves that  $B_p^* \begin{pmatrix} x_1 \cdots x_q \\ t_1 \cdots t_q \end{pmatrix}$  is the conjugate complex of  $B_p \begin{pmatrix} t_1 \cdots t_q \\ x_1 \cdots x_q \end{pmatrix}$ . It follows (why?) that  $D^* \begin{pmatrix} x_1 \cdots x_q & \lambda^* \\ t_1 \cdots t_q \end{pmatrix}$  is the conjugate complex of  $D \begin{pmatrix} t_1 \cdots t_q & \lambda \\ x_1 \cdots x_q \end{pmatrix}$ . Hence the index of the characteristic number  $\lambda^*$  of the adjoint integral equation is the same as the index of the characteristic number  $\lambda$  of its partner. The functions

$$u_1^*(x) = \frac{D^* \begin{pmatrix} x \tau_2 \cdots \tau_q & \lambda_0^* \\ \xi_1 \cdots \xi_q \end{pmatrix}}{D^* \begin{pmatrix} \tau_1 \cdots \tau_q & \lambda_0^* \\ \xi_1 \cdots \xi_q \end{pmatrix}}, \dots,$$

$$u_q^*(x) = \frac{D^* \begin{pmatrix} \tau_1 \cdots \tau_{q-1} x & \lambda_0^* \\ \xi_1 \cdots \xi_q \end{pmatrix}}{D^* \begin{pmatrix} \tau_1 \cdots \tau_q & \lambda_0^* \\ \xi_1 \cdots \xi_q \end{pmatrix}}$$

are such that  $\bar{u}_1^*(x) = \frac{D \begin{pmatrix} \xi_1 & \cdots & \xi_q \\ x & \tau_2 & \cdots & \tau_q \end{pmatrix}}{D \begin{pmatrix} \xi_1 & \cdots & \xi_q \\ \tau_1 & \cdots & \tau_q \end{pmatrix}}$ , etc. In other words

$u_1^*(x) = v_1(x), \cdots, u_q^*(x) = v_q(x)$ . Thus the necessary and sufficient condition that the non-homogeneous equation  $\mathbf{y} = \lambda_0 \mathbf{K}\mathbf{y} + \mathbf{f}$  should possess a solution when  $\lambda_0$  is a characteristic number of index  $q$  may be phrased as follows:

**f must be orthogonal to every characteristic vector of the adjoint homogeneous equation  $\mathbf{u} = \lambda \mathbf{K}^* \mathbf{u}$  which is associated with the characteristic number  $\lambda_0^*$  of this equation.**

Let, now,  $\lambda_m$  and  $\lambda_n$  be two distinct characteristic numbers of the equation  $\mathbf{u} = \lambda \mathbf{K}\mathbf{u}$ . We have the following remarkable theorem:

**Every characteristic vector  $\mathbf{u}$  of the equation  $\mathbf{u} = \lambda \mathbf{K}\mathbf{u}$  which is associated with the characteristic number  $\lambda_m$  is orthogonal to every characteristic vector  $\mathbf{u}^*$  of the adjoint equation  $\mathbf{u} = \lambda \mathbf{K}^* \mathbf{u}$  which is associated with the characteristic number  $\lambda_n^*$ .**

To prove this theorem we observe that the relation  $\mathbf{u} = \lambda_m \mathbf{K}\mathbf{u}$  implies that  $(\mathbf{u}^* | \mathbf{u}) = \lambda_m (\mathbf{u}^* | \mathbf{K}\mathbf{u})$  and that the relation  $\mathbf{u}^* = \lambda_n^* \mathbf{K}^* \mathbf{u}^*$  implies that  $(\mathbf{u}^* | \mathbf{u}) = \lambda_n (\mathbf{K}^* \mathbf{u}^* | \mathbf{u})$ . Since  $(\mathbf{K}^* \mathbf{u}^* | \mathbf{u}) = (\mathbf{u}^* | \mathbf{K}\mathbf{u})$  (why?) it follows that

$$(\lambda_m - \lambda_n)(\mathbf{u}^* | \mathbf{K}\mathbf{u}) = 0.$$

Since  $\lambda_m - \lambda_n \neq 0$  it follows that  $(\mathbf{u}^* | \mathbf{K}\mathbf{u}) = 0$ . Hence  $(\mathbf{u}^* | \mathbf{u}) = 0$  which proves the theorem.

When the integral equation  $\mathbf{y} = \lambda \mathbf{K}\mathbf{y} + \mathbf{f}$  is Hermitian (i.e., self-adjoint) all its characteristic numbers are real. In fact the relation  $\mathbf{u} = \lambda \mathbf{K}\mathbf{u}$  implies that  $(\mathbf{u} | \mathbf{u}) = \lambda (\mathbf{u} | \mathbf{K}\mathbf{u})$  and both  $(\mathbf{u} | \mathbf{u})$  and  $(\mathbf{u} | \mathbf{K}\mathbf{u})$  are real (why?). Since  $(\mathbf{u} | \mathbf{u}) \neq 0$  (why?)  $(\mathbf{u} | \mathbf{K}\mathbf{u}) \neq 0$  and so  $\lambda = \frac{(\mathbf{u} | \mathbf{u})}{(\mathbf{u} | \mathbf{K}\mathbf{u})}$

is real. We shall denote a self-adjoint (or Hermitian) linear integral operator by the symbol  $\mathbf{H}$ , and, on applying to a Hermitian integral equation the result just derived for two adjoint integral equations, we have the following fundamental theorem:

**Let  $\lambda_m$  and  $\lambda_n$  be two distinct characteristic numbers (necessarily real) of the Hermitian integral equation  $\mathbf{u} = \lambda \mathbf{H}\mathbf{u}$ . Then any characteristic vector associated with  $\lambda_m$  is orthogonal to any characteristic vector associated with  $\lambda_n$ .**

If  $\lambda_m$  has an index  $q > 1$  the characteristic vectors  $\mathbf{u}_1, \cdots, \mathbf{u}_q$  associated with  $\lambda_m$  are not necessarily mutually orthogonal, but, on

applying to them Schmidt's orthogonalization process (see Chapter 3, Section 2) we obtain a set of  $q$  mutually orthogonal unit characteristic vectors which are associated with the characteristic number  $\lambda_m$ . Dealing in this way with each of the characteristic numbers we obtain an orthonormal sequence. (Remember that since the zeros of an analytic function are isolated they can be counted, there being only a finite number of them whose modulus is less than any given positive number.) In counting the members of the orthonormal sequence we attach to each the characteristic number with which it is associated (so that each characteristic number is repeated if its index is greater than 1, the number of times it appears being equal to its index).

The important results of the preceding paragraph would be vacuous if our Hermitian integral equation did not possess any characteristic numbers. We shall prove, however, in the next section the remarkable theorem that every Hermitian integral equation possesses at least one characteristic number. Anticipating this result we summarize the important results of the preceding paragraphs as follows:

**Associated with every Hermitian integral equation is an orthonormal sequence of function-vectors each of which is a characteristic vector of the integral equation; every characteristic number of the integral equation is real, and every characteristic vector of the integral equation is a linear combination of members of the orthonormal sequence.**

### 7. The existence of a characteristic number for a Hermitian integral equation

In order to show that a given integral equation possesses a characteristic number it suffices to prove that its Fredholm determinant  $D(\lambda)$  possesses a zero (why?). If  $D(\lambda)$  did not possess a zero the resolving kernel

$$\Gamma \begin{pmatrix} x & \\ t & \lambda \end{pmatrix} = \frac{D \begin{pmatrix} x & \\ t & \lambda \end{pmatrix}}{D(\lambda)}$$

would be an integral analytic function of  $\lambda$  (i.e., it would be analytic at every finite point  $\lambda$  in the complex  $\lambda$ -plane), for both  $D \begin{pmatrix} x & \\ t & \lambda \end{pmatrix}$  and  $D(\lambda)$  are integral functions of  $\lambda$ . Hence if the integral equation  $\mathbf{u} = \lambda \mathbf{K} \mathbf{u}$  does not possess a characteristic number the development of  $\Gamma \begin{pmatrix} x & \\ t & \lambda \end{pmatrix}$  as a power series in  $\lambda$  is convergent for every value of  $\lambda$ .

We propose to show that this cannot be the case if  $\mathbf{K} = \mathbf{H}$  is a Hermitian linear integral operator.

The development of  $\Gamma \begin{pmatrix} x \\ t \quad \lambda \end{pmatrix}$  as a power series in  $\lambda$  is readily found, whether  $\mathbf{K}$  is Hermitian or not, provided that  $|\lambda|$  is sufficiently small. When  $\lambda = 0$ ,  $\mathbf{y} = \mathbf{f}$ , and on setting  $\mathbf{y} = \mathbf{f}$  in the right-hand side of the integral equation  $\mathbf{y} = \lambda \mathbf{K}\mathbf{y} + \mathbf{f}$  we obtain  $\mathbf{y} = \mathbf{f} + \lambda \mathbf{K}\mathbf{f}$ . Since  $\mathbf{y} = \mathbf{f} + \lambda \mathbf{K}\mathbf{f}$  the power series for  $\Gamma \begin{pmatrix} x \\ t \quad \lambda \end{pmatrix}$  must start out with the term  $K \begin{pmatrix} x \\ t \end{pmatrix}$ .

On substituting the expression  $\mathbf{f} + \lambda \mathbf{K}\mathbf{f}$  for  $\mathbf{y}$  in the right-hand side of the integral equation we obtain  $\mathbf{y} = \mathbf{f} + \lambda \mathbf{K}\mathbf{f} + \lambda^2 \mathbf{K}^2 \mathbf{f}$ . Proceeding in this way we obtain the successive *approximations*

$$\mathbf{y}_0 = \mathbf{f}; \quad \mathbf{y}_1 = \mathbf{f} + \lambda \mathbf{K}\mathbf{f}; \quad \dots; \quad \mathbf{y}_n = \mathbf{f} + \lambda \mathbf{K}\mathbf{f} + \dots + \lambda^n \mathbf{K}^n \mathbf{f}$$

to the sought-for solution of the integral equation. The question immediately arises: Do these approximations converge, and, if they do, is their limit a solution of the integral equation? The answer to both parts of this question is "yes" provided that  $|\lambda|$  is sufficiently small, and we proceed to show this.

In the first place we observe that the various powers  $\mathbf{K}^n$  of the linear integral operator  $\mathbf{K}$  obey the exponent law:  $\mathbf{K}^{m+n} = \mathbf{K}^m \mathbf{K}^n$  (why?). In particular

$$\mathbf{K}^n = \mathbf{K} \mathbf{K}^{n-1}$$

so that  $K^n \begin{pmatrix} x \\ t \end{pmatrix} = \int K \begin{pmatrix} x \\ \tau \end{pmatrix} K^{n-1} \begin{pmatrix} \tau \\ t \end{pmatrix} d\tau$ . On regarding  $\bar{K} \begin{pmatrix} x \\ t \end{pmatrix}$  and

$K^{n-1} \begin{pmatrix} \tau \\ t \end{pmatrix}$  as function-vectors over the interval  $a \leq \tau \leq b$  it follows

that  $K^n \begin{pmatrix} x \\ t \end{pmatrix}$  is the scalar product of the second of these function-vectors by the first. Hence, by Schwarz's inequality,

$$\left| K^n \begin{pmatrix} x \\ t \end{pmatrix} \right|^2 \leq \int K \begin{pmatrix} x \\ \tau \end{pmatrix} \bar{K} \begin{pmatrix} x \\ \tau \end{pmatrix} d\tau \int \bar{K}^{(n-1)} \begin{pmatrix} \tau \\ t \end{pmatrix} K^{n-1} \begin{pmatrix} \tau \\ t \end{pmatrix} d\tau.$$

Let us denote by  $G_n^2$  the integral of  $\left| K^n \begin{pmatrix} x \\ t \end{pmatrix} \right|^2 = K^{n*} \begin{pmatrix} t \\ x \end{pmatrix} K^n \begin{pmatrix} x \\ t \end{pmatrix}$  over the square  $a \leq x \leq b; a \leq t \leq b$ . Then it follows, on integrating the inequality just written over this square, that

$$G_n^2 \leq G_1^2 G_{n-1}^2, \quad n = 2, 3, \dots$$

Thus  $G_2^2 \leq G_1^4$ ,  $G_3^2 \leq G_1^2 G_2^2 \leq G_1^6$ , and, generally,

$$G_n^2 \leq G_1^{2n}; \quad n = 1, 2, \dots$$

On writing  $\mathbf{K}^n$  in the form  $\mathbf{K}\mathbf{K}^{n-2}\mathbf{K}$ ,  $K^n \begin{pmatrix} x \\ t \end{pmatrix}$  appears as

$$\int K \begin{pmatrix} x \\ \tau_1 \end{pmatrix} K^{n-2} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} K \begin{pmatrix} \tau_2 \\ t \end{pmatrix} d\tau_1 d\tau_2.$$

On regarding  $K^{n-2} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}$  and  $\bar{K} \begin{pmatrix} x \\ \tau_1 \end{pmatrix} \bar{K} \begin{pmatrix} \tau_2 \\ t \end{pmatrix}$  as function-vectors (two dimensional) over the square  $a \leq \tau_1 \leq b$ ,  $a \leq \tau_2 \leq b$ ,  $K^n \begin{pmatrix} x \\ t \end{pmatrix}$  is the scalar product of the first of these two function-vectors by the second. Hence (why?)

$$\begin{aligned} \left| K^n \begin{pmatrix} x \\ t \end{pmatrix} \right|^2 &\leq \\ \int \bar{K}^{n-2} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} K^{n-2} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} d\tau_1 d\tau_2 \int K \begin{pmatrix} x \\ \tau_1 \end{pmatrix} K \begin{pmatrix} \tau_2 \\ t \end{pmatrix} \bar{K} \begin{pmatrix} x \\ \tau_1 \end{pmatrix} \bar{K} \begin{pmatrix} \tau_2 \\ t \end{pmatrix} d\tau_1 d\tau_2 \\ &= G_{n-2}^2 C^2 \leq C^2 G_1^{2n-4}, \end{aligned}$$

where  $C^2$  stands for the non-negative number  $\int K \begin{pmatrix} x \\ \tau_1 \end{pmatrix} K \begin{pmatrix} \tau_2 \\ t \end{pmatrix} \bar{K} \begin{pmatrix} x \\ \tau_1 \end{pmatrix} \bar{K} \begin{pmatrix} \tau_2 \\ t \end{pmatrix} d\tau_1 d\tau_2$ . Thus  $\left| K^n \begin{pmatrix} x \\ t \end{pmatrix} \right| \leq C G_1^{n-2}$  and so  $\int K^n \begin{pmatrix} x \\ t \end{pmatrix} f(t) dt \leq C G_1^{n-2} N(b-a)$ , where  $|f(x)| \leq N$  over  $a \leq x \leq b$ . It follows that the series  $f + \lambda \mathbf{K}f + \dots + \lambda^n \mathbf{K}^n f + \dots$  converges at every point  $x$  of  $[a, b]$  if  $|\lambda| < \frac{1}{G_1}$ , the convergence being uniform with respect to  $x$  over  $[a, b]$ . On denoting the sum of the series by  $\mathbf{z}$  we have  $\mathbf{K}\mathbf{z} = \mathbf{K}f + \lambda \mathbf{K}^2 f + \dots + \lambda^n \mathbf{K}^{n+1} f + \dots$  (term-by-term integration being permitted by the uniformity of the convergence of the series  $f + \lambda \mathbf{K}f + \dots + \lambda^n \mathbf{K}^n f + \dots$ ). Thus  $\lambda \mathbf{K}\mathbf{z} = \lambda \mathbf{K}f + \lambda^2 \mathbf{K}^2 f + \dots + \lambda^{n+1} \mathbf{K}^{n+1} f + \dots = \mathbf{z} - f$  so that  $\mathbf{z} = f + \lambda \mathbf{K}f + \dots + \lambda^n \mathbf{K}^n f + \dots$  is a solution of the equation  $\mathbf{y} = f + \lambda \mathbf{K}\mathbf{y}$ . We have, then, the following result:



If  $|\lambda| < \frac{1}{G_1}$ ,  $\lambda$  is not a characteristic number, and the unambiguously determinate solution of the integral equation  $y = \lambda \mathbf{K}y + \mathbf{f}$  is given by the formula

$$y = \lambda \Gamma \mathbf{f} + \mathbf{f},$$

where

$$\Gamma = \mathbf{K} + \lambda \mathbf{K}^2 + \cdots + \lambda^{n-1} \mathbf{K}^n + \cdots$$

We have already seen that if  $D(\lambda) \neq 0$  the unambiguously determinate solution of the integral equation is

$$y = \frac{\lambda}{D(\lambda)} \mathbf{Df} + \mathbf{f}$$

and so

$$\Gamma \mathbf{f} = \frac{1}{D(\lambda)} \mathbf{Df}; \quad |\lambda| < \frac{1}{G_1}$$

On setting  $\tilde{f}(t) = \Gamma \begin{pmatrix} x \\ t \quad \lambda \end{pmatrix} - \frac{1}{D(\lambda)} D \begin{pmatrix} x \\ t \quad \lambda \end{pmatrix}$  we obtain  $(\tilde{f}|\mathbf{f}) = 0$ , every  $x$ . Hence

$$\frac{D \begin{pmatrix} x \\ t \quad \lambda \end{pmatrix}}{D(\lambda)} = \Gamma \begin{pmatrix} x \\ t \quad \lambda \end{pmatrix} = K \begin{pmatrix} x \\ t \end{pmatrix} + \lambda K^2 \begin{pmatrix} x \\ t \end{pmatrix} + \cdots + \lambda^{n-1} K^n \begin{pmatrix} x \\ t \end{pmatrix} + \cdots$$

If  $D(\lambda)$  has no zeros the power series expansion of  $\frac{D \begin{pmatrix} x \\ t \quad \lambda \end{pmatrix}}{D(\lambda)}$  which is

completely determined by the values of  $\frac{D \begin{pmatrix} x \\ t \quad \lambda \end{pmatrix}}{D(\lambda)}$  over any neighborhood of  $\lambda = 0$  must converge for every (finite) value of  $\lambda$ . In other words the series  $K \begin{pmatrix} x \\ t \end{pmatrix} + \lambda K^2 \begin{pmatrix} x \\ t \end{pmatrix} + \cdots + \lambda^{n-1} K^n \begin{pmatrix} x \\ t \end{pmatrix} + \cdots$  must converge for every (finite) value of  $\lambda$ . In order to prove, then, that every Hermitian integral equation  $y = \lambda \mathbf{H}y + \mathbf{f}$  has a characteristic constant it is sufficient to show that the series

$$H \begin{pmatrix} x \\ t \end{pmatrix} + \lambda H^2 \begin{pmatrix} x \\ t \end{pmatrix} + \dots + \lambda^{n-1} H^n \begin{pmatrix} x \\ t \end{pmatrix} + \dots$$

has a *finite* radius of convergence.

*Note.* The solution of any integral equation  $\mathbf{y} = \lambda \mathbf{K}\mathbf{y} + \mathbf{f}$ , namely,

$$\mathbf{y} = \mathbf{f} + \lambda \mathbf{K}\mathbf{f} + \dots + \lambda^n \mathbf{K}^n \mathbf{f} + \dots$$

which is valid if  $|\lambda|$  is sufficiently small (in particular if  $|\lambda| < \frac{1}{G_1}$ ) is known as the *Neumann solution* (after C. Neumann [1832–1925], a German mathematician). It suffers from the serious fault that it is valid only when  $|\lambda|$  is *sufficiently* small. By way of contrast compare the Fredholm solution

$$\mathbf{y} = \frac{\lambda}{D(\lambda)} \mathbf{D}\mathbf{f} + \mathbf{f}$$

which is valid for every  $\lambda$  which is not a zero of  $D(\lambda)$ .

It is clear from the definition of a Hermitian linear integral operator that if  $\mathbf{H}$  is Hermitian so also are  $\mathbf{H}^2, \mathbf{H}^3, \dots, \mathbf{H}^n$ . In fact  $(\mathbf{u}|\mathbf{H}^2\mathbf{v}) = (\mathbf{H}\mathbf{u}|\mathbf{H}\mathbf{v}) = (\mathbf{H}^2\mathbf{u}|\mathbf{v})$  so that  $\mathbf{H}^2$  is Hermitian (why?). Similarly  $\mathbf{H}^3$  is Hermitian, and so on. Since  $H^{2n} \begin{pmatrix} x \\ x \end{pmatrix} = \int H^n \begin{pmatrix} x \\ \tau \end{pmatrix} H^n \begin{pmatrix} \tau \\ x \end{pmatrix} d\tau = \int \bar{H}^n \begin{pmatrix} \tau \\ x \end{pmatrix} H^n \begin{pmatrix} \tau \\ x \end{pmatrix} d\tau$  it follows that  $H^{2n} \begin{pmatrix} x \\ x \end{pmatrix}$  is real and non-negative. Furthermore  $\int H^{2n} \begin{pmatrix} x \\ x \end{pmatrix} dx > 0$  since if  $\int H^{2n} \begin{pmatrix} x \\ x \end{pmatrix} dx$  were zero  $H^n \begin{pmatrix} t \\ x \end{pmatrix}$  would be identically zero (why?), and no power of  $\mathbf{H}$  is the zero operator. In fact if  $H^{2n} \begin{pmatrix} x \\ t \end{pmatrix}$  were identically zero  $H^n$  would be identically zero (why?), and if  $\mathbf{H}^{2n+1}$  were identically zero  $\mathbf{H}^{2n+2} = \mathbf{H}^{2n+1}\mathbf{H}$  would be identically zero, and this would force  $\mathbf{H}^{n+1}$  to be identically zero. Thus the lowest power of  $\mathbf{H}$  which is identically zero is neither even nor odd; in other words no power of  $\mathbf{H}$  is identically zero (it being understood that  $\mathbf{H}$  is not identically zero). Let us denote by  $p_{2n}$  the positive number  $p_{2n} = \int H^{2n} \begin{pmatrix} x \\ x \end{pmatrix} dx$  (so that  $p_{2n}$

is the squared magnitude of the two-dimensional function-vector  $\mathbf{w}_n : H^n \begin{pmatrix} t \\ x \end{pmatrix}$ ; it follows from the relation

$$p_{2n} = \int H^{n-1} \begin{pmatrix} x \\ \tau \end{pmatrix} H^{n+1} \begin{pmatrix} \tau \\ x \end{pmatrix} d\tau dx; \quad n = 2, 3, \dots,$$

that  $p_{2n}$  is the scalar product of the (two-dimensional) function-vector  $\mathbf{w}_{n+1} : H^{n+1} \begin{pmatrix} \tau \\ x \end{pmatrix}$  by the (two-dimensional) function-vector  $\mathbf{w}_{n-1} : H^{n-1} \begin{pmatrix} \tau \\ x \end{pmatrix}$ . Hence, by Schwarz's inequality,

$$p_{2n}^2 \leq p_{2n-2} p_{2n+2}; \quad n = 2, 3, \dots$$

In other words the sequence of positive numbers  $\left\{ \frac{p_{2n+2}}{p_{2n}} \right\}$ ,  $n = 1, 2, \dots$ , is monotone-increasing. Hence the ratio of the  $(n+1)$ st term of the series

$$p_2 + \mu p_4 + \dots + \mu^{n-1} p_{2n} + \dots; \quad \mu > 0$$

to the  $n$ th term  $> \mu \frac{p_4}{p_2}$ . The series is, accordingly, divergent if  $\mu > \frac{p_2}{p_4}$ . It follows that the series

$$H^2 \begin{pmatrix} x \\ x \end{pmatrix} + \mu H^4 \begin{pmatrix} x \\ x \end{pmatrix} + \dots + \mu^{n-1} H^{2n} \begin{pmatrix} x \\ x \end{pmatrix} + \dots$$

does not converge (uniformly with respect to  $x$ ) if  $\mu > \frac{p_2}{p_4}$ , for if it did the series

$$p_2 + \mu p_4 + \dots + \mu^{n-1} p_{2n} + \dots$$

obtained by term-by-term integration would be convergent. Hence the integral equation whose kernel is  $H^2 \begin{pmatrix} x \\ t \end{pmatrix}$  possesses a characteristic number, for if it did not the series

$$H^2 \begin{pmatrix} x \\ t \end{pmatrix} + \mu H^4 \begin{pmatrix} x \\ t \end{pmatrix} + \dots + \mu^{n-1} H^{2n} \begin{pmatrix} x \\ t \end{pmatrix} + \dots$$

would converge, uniformly over the square  $a \leq x \leq b$ ,  $a \leq t \leq b$ , for every (finite) value of  $\mu$ .

Since  $(\mathbf{u}|\mathbf{H}^2\mathbf{u}) = (\mathbf{H}\mathbf{u}|\mathbf{H}\mathbf{u})$  all characteristic numbers of  $\mathbf{H}^2$  are positive (why?). Denote, then, the characteristic number of  $\mathbf{H}^2$ , whose existence we have just proved, by  $\mu > 0$ . Let  $\mathbf{u}$  be a characteristic vector of  $\mathbf{H}^2$  which is associated with  $\mu$ , and construct the two vectors

$$\mathbf{v}_1 = \mathbf{u} + \mu^{1/2}\mathbf{H}\mathbf{u}; \quad \mathbf{v}_2 = \mathbf{u} - \mu^{1/2}\mathbf{H}\mathbf{u}.$$

Since  $\mathbf{v}_1 + \mathbf{v}_2 = 2\mathbf{u}$  not both the vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  can be the zero vector (why?). Furthermore

$$\mathbf{H}\mathbf{v}_1 = \mathbf{H}\mathbf{u} + \mu^{1/2}\mathbf{H}^2\mathbf{u}; \quad \mathbf{H}\mathbf{v}_2 = \mathbf{H}\mathbf{u} - \mu^{1/2}\mathbf{H}^2\mathbf{u}$$

so that, since  $\mathbf{u} = \mu\mathbf{H}^2\mathbf{u}$ ,

$$\mu^{1/2}\mathbf{H}\mathbf{v}_1 = \mathbf{u} + \mu^{1/2}\mathbf{H}\mathbf{u} = \mathbf{v}_1; \quad -\mu^{1/2}\mathbf{H}\mathbf{v}_2 = \mathbf{u} - \mu^{1/2}\mathbf{H}\mathbf{u} = \mathbf{v}_2.$$

Hence either  $\mu^{1/2}$  or  $-\mu^{1/2}$  (possibly both) is a characteristic number of  $\mathbf{H}$ . This completes the proof of our principal theorem:

**Every Hermitian integral equation possesses at least one characteristic number.**

Incidentally we have proved that every characteristic vector of  $\mathbf{H}^2$  is a linear combination of characteristic vectors of  $\mathbf{H}$  (since  $2\mathbf{u} = \mathbf{v}_1 + \mathbf{v}_2$  and each of the vectors  $\mathbf{v}_1, \mathbf{v}_2$  is either the zero vector or else is a characteristic vector of  $\mathbf{H}$ ). It is clear that if  $\lambda$  is a characteristic number of  $\mathbf{H}$  then  $\lambda^2$  is a characteristic number of  $\mathbf{H}^2$ , any characteristic vector of  $\mathbf{H}$  which is associated with  $\lambda$  being a characteristic vector of  $\mathbf{H}^2$  which is associated with  $\lambda^2$ . In fact the relation  $\mathbf{u} = \lambda(\mathbf{H}\mathbf{u})$  implies  $\mathbf{u} = \lambda\mathbf{H}(\lambda(\mathbf{H}\mathbf{u})) = \lambda^2\mathbf{H}^2\mathbf{u}$ . Thus the characteristic numbers of  $\mathbf{H}^2$  are precisely the squares of the characteristic numbers of  $\mathbf{H}$ .

### EXERCISES

1. Show that if  $\lambda$  is a characteristic number of  $\mathbf{H}$  of index  $q$  then  $\lambda^2$  is a characteristic number of  $\mathbf{H}^2$  of index  $\geq q$ .

2. Show that if  $\lambda$  is a characteristic number of  $\mathbf{H}$  of index  $q_+$  and  $-\lambda$  a characteristic number of  $\mathbf{H}$  of index  $q_-$  (it being understood that  $q_- = 0$  if  $-\lambda$  is *not* a characteristic number of  $\mathbf{H}$ ) then  $\lambda^2$  is a characteristic number of  $\mathbf{H}^2$  of index  $q_+ + q_-$ .

3. Show that if  $\{\mathbf{u}_n\}$  is an orthonormal sequence determined by  $\mathbf{H}$  then  $\{\mathbf{u}_n\}$  is an orthonormal sequence determined by  $\mathbf{H}^2$ .

4. Show that the modulus of the characteristic number of  $\mathbf{H}$  whose modulus is least  $\leq \left(\frac{p_2}{p_4}\right)^{1/2}$ . *Hint.* There is at least one singular point of an analytic function on the circumference of its circle of convergence.

5. Show that if  $\lambda$  is a characteristic number of the integral equation  $\mathbf{y} = \lambda \mathbf{K}^n \mathbf{y} + \mathbf{f}$  then one of the  $n$ th roots of  $\lambda$  is a characteristic number of  $\mathbf{K}$  and that every characteristic vector of  $\mathbf{K}^n$  is a linear combination of characteristic vectors of  $\mathbf{K}$ .  
*Hint.* Let  $\mathbf{u}$  be a characteristic vector of  $\mathbf{K}^n$  which is associated with  $\lambda$ , and let

$$\epsilon_p = |\lambda|^{1/n} e^{ip\theta}, \quad p = 1, \dots, n, \quad n \text{ be the } n \text{th roots of } \lambda. \quad \text{Set } \mathbf{v}_j = \mathbf{u} + \sum_{p=1}^{n-1} \epsilon_j^p \mathbf{K}^p \mathbf{u},$$

$j = 1, \dots, n$ . Then the function-vector  $\mathbf{v}_j$  satisfies the equation  $\mathbf{v}_j = \epsilon_j \mathbf{K} \mathbf{v}_j$ ,

$$\text{and } \sum_{j=1}^n \mathbf{v}_j = n\mathbf{u}.$$

6. Show that the characteristic number  $\lambda$  of the integral equation  $\mathbf{y} = \lambda \mathbf{H} \mathbf{y} + \mathbf{f}$  whose modulus is least is such that  $\frac{1}{G_1} \leq |\lambda| \leq \left(\frac{p_2}{p_4}\right)^{1/2}$ , where  $G_1^2$  is the squared magnitude of the (two-dimensional) function-vector  $\mathbf{H}: H \begin{pmatrix} x \\ t \end{pmatrix}$ .

7. Show that when  $\mathbf{H}: xt, [a, b] = [-1, 1]$  then  $D(\lambda) = 1 - \frac{2}{3}\lambda$ . Show that  $\mathbf{H}^2: \frac{2}{3}xt$  so that  $\mathbf{H}^2 = \frac{2}{3}\mathbf{H}$  and  $\mathbf{H}^4 = \frac{8}{27}\mathbf{H}$ . Show, further, that  $p_2 = \frac{4}{9}$ ,  $p_4 = \frac{8}{27}$ , and verify that the inequality  $|\lambda| \leq \left(\frac{p_2}{p_4}\right)^{1/2}$  reduces to an equality in this case.

8. Show that, for the Hermitian linear integral operator of Exercise 7, the inequality  $\frac{1}{G_1} \leq |\lambda|$  reduces to an equality.

9. Show that when  $\mathbf{H}: x + t, [a, b] = [-1, 1]$  then  $D(\lambda) = 1 - \frac{4}{3}\lambda^2$ ,  $\mathbf{H}^2: 2xt + \frac{2}{3}$ ,  $\mathbf{H}^4: \frac{8}{3}xt + \frac{8}{9}$ ,  $p_2 = \frac{8}{3}$ ,  $p_4 = \frac{32}{9}$ ,  $G_1^2 = \frac{8}{3}$ . Verify that the inequality of Exercise 6 is  $\frac{1}{2} \left(\frac{8}{3}\right)^{1/2} \leq \frac{1}{2} (3)^{1/2} \leq \frac{1}{2} (3)^{1/2}$ .

## 8. Separable linear integral operators

When  $K \begin{pmatrix} x \\ t \end{pmatrix}$  is of the form  $p_1(x)\bar{q}_1(t) + \dots + p_n(x)\bar{q}_n(t)$  the linear

integral operator  $\mathbf{K}: K \begin{pmatrix} x \\ t \end{pmatrix}$  is said to be *separable*. There is no lack of

generality in assuming that the  $n$  function-vectors  $\mathbf{p}_1, \dots, \mathbf{p}_n$  are linearly independent (why?), and we shall make this hypothesis. On denoting  $(\mathbf{q}_j | \mathbf{y})$  by  $c^j$ ,  $j = 1, \dots, n$ , the integral equation  $\mathbf{y} = \lambda \mathbf{K} \mathbf{y} + \mathbf{f}$  is equivalent to the equation

$$y(x) = \lambda \{c^1 p_1(x) + \dots + c^n p_n(x)\} + f(x)$$

and so  $c^j = (\mathbf{q}_j | \mathbf{y})$  is given by the formula

$$c^j = \lambda \{ (q_j | p_1) c^1 + \cdots + (q_j | p_n) c^n \} + (q_j | f).$$

On denoting by  $A$  the  $n \times n$  matrix of which the element in the  $j$ th row and  $m$ th column is  $(q_j | p_m)$  we have

$$c = \lambda A c + d,$$

where  $c$  is the  $n \times 1$  matrix of which the element in the  $j$ th row is  $c^j$  and  $d$  is the  $n \times 1$  matrix of which the element in the  $j$ th row is  $(q_j | f)$ . In order that the homogeneous integral equation may have a (non-trivial) solution it is necessary that the matrix  $E_n - \lambda A$  be singular. Conversely if  $E_n - \lambda A$  is singular there exists a non-zero  $n \times 1$  matrix  $c$  such that  $c = \lambda A c$ . On setting

$$u(x) = \lambda \sum_{\alpha=1}^n c^\alpha p_\alpha(x)$$

it follows that

$$\mathbf{K} \mathbf{u} : \lambda \sum_{\beta=1}^n p_\beta(x) \sum_{\alpha=1}^n (q_\beta | p_\alpha) c^\alpha = \lambda \sum_{\beta=1}^n (A c)^\beta p_\beta(x) = \sum_{\beta=1}^n c^\beta p_\beta(x)$$

so that  $\lambda \mathbf{K} \mathbf{u} = \mathbf{u}$ . Since the function-vectors  $p_1, \cdots, p_n$  are, by hypothesis, linearly independent,  $\mathbf{u}$  is not the zero vector. Hence  $\lambda$  is a characteristic number of the separable kernel  $K \begin{pmatrix} x \\ t \end{pmatrix}$ . Thus

**The characteristic numbers of a separable kernel are finite in number; they are the zeros of the determinant of the  $n \times n$  matrix  $E_n - \lambda A$  where  $A_m^j = (q_j | p_m)$ .**

## EXERCISES

1. Show that  $\mathbf{K} : p(x)\bar{q}(t)$  does not have a characteristic number if  $(q|p) = 0$  and that if  $(q|p) \neq 0$  it has as its only characteristic number the reciprocal of  $(q|p)$ .

2. Determine the one and only characteristic number of the Hermitian linear integral operator  $\mathbf{H} : xt$ ,  $[a, b] = [-1, 1]$ . Check your answer with the result of Exercise 7, p. 276.

3. Determine the two characteristic numbers of  $\mathbf{H} : x + t$ ,  $[a, b] = [-1, 1]$ . Check your answer with the result of Exercise 8, p. 276.

4. Show that the necessary and sufficient conditions that  $\mathbf{K} : p_1(x)\bar{q}_1(t) + p_2(x)\bar{q}_2(t)$ , where the function-vectors  $p_1$  and  $p_2$  are linearly independent, should have no characteristic numbers are  $(q_1|p_1) + (q_2|p_2) = 0$ ,  $\det \begin{pmatrix} (q_1|p_1) & (q_1|p_2) \\ (q_2|p_1) & (q_2|p_2) \end{pmatrix}$

= 0.

### 9. The bilinear formula

Let  $\mathbf{H} : H \begin{pmatrix} x \\ t \end{pmatrix}$  be any Hermitian linear integral operator, and let  $\{u_n\}$  be the orthonormal sequence (finite or infinite) which it determines. If we hold  $t$  fixed  $H \begin{pmatrix} x \\ t \end{pmatrix}$  may be regarded as a one-dimensional function-vector whose  $j$ th Fourier coefficient with respect to the orthonormal sequence  $\{u_n\}$  is

$$\int \bar{u}_j(x) H \begin{pmatrix} x \\ t \end{pmatrix} dt.$$

This is the conjugate of  $\int H \begin{pmatrix} t \\ x \end{pmatrix} u_j(x) dx = \frac{u_j(t)}{\lambda_j}$ . Thus (since the characteristic numbers of  $\mathbf{H}$  are real) the Fourier coefficients of  $H \begin{pmatrix} x \\ t \end{pmatrix}$ , regarded as a one-dimensional function-vector by the device of holding  $t$  fixed, are  $\frac{\bar{u}_j(t)}{\lambda_j}$ . Hence the Fourier series of  $H \begin{pmatrix} x \\ t \end{pmatrix}$ ,  $t$  fixed, is

$$\frac{\bar{u}_1(t)u_1(x)}{\lambda_1} + \dots + \frac{\bar{u}_n(t)u_n(x)}{\lambda_n} + \dots$$

This Fourier series may, or may not, converge, and, if it converges, its sum may, or may not, be  $H \begin{pmatrix} x \\ t \end{pmatrix}$ . We proceed to prove the following fundamental theorem:

**If the Fourier series of  $H \begin{pmatrix} x \\ t \end{pmatrix}$  converges over the square  $a \leq x \leq b$ ,  $a \leq t \leq b$  and if its convergence is uniform with respect to  $x$  for every fixed  $t$ ,  $a \leq t \leq b$ , its sum is  $H \begin{pmatrix} x \\ t \end{pmatrix}$ .**

As a first step in the proof of this theorem we shall prove that if the Fourier series of  $H \begin{pmatrix} x \\ t \end{pmatrix}$  converges uniformly (regarded as a function of the two variables  $x$  and  $t$ ) over the square  $a \leq x \leq b$ ,  $a \leq t \leq b$  then

its sum is  $H \begin{pmatrix} x \\ t \end{pmatrix}$ . To do this we denote by  $\mathbf{H}'$  the linear integral operator

$$\mathbf{H}': H' \begin{pmatrix} x \\ t \end{pmatrix} = H \begin{pmatrix} x \\ t \end{pmatrix} - \sum_1^{\infty} \frac{\bar{u}_n(t)u_n(x)}{\lambda_n}.$$

The assumed uniform continuity of the series  $\sum_1^{\infty} \frac{\bar{u}_n(t)u_n(x)}{\lambda_n}$  over the square  $a \leq x \leq b$ ,  $a \leq t \leq b$ , coupled with the continuity of  $H \begin{pmatrix} x \\ t \end{pmatrix}$

over this square, assures us that  $H' \begin{pmatrix} x \\ t \end{pmatrix}$  is continuous over this square.

$\mathbf{H}'$  is Hermitian (why?) so that, if  $H' \begin{pmatrix} x \\ t \end{pmatrix}$  is not identically zero, it possesses a characteristic number  $\mu$ , say, and an associated characteristic vector  $\mathbf{v}$ , say. Since the infinite series which appears in the definition of  $H' \begin{pmatrix} x \\ t \end{pmatrix}$  converges uniformly in  $t$  for every fixed  $x$ ,  $\mathbf{H}'\mathbf{u}_j$  may be obtained by term-by-term integration. By virtue of the orthonormal character of the sequence  $\{\mathbf{u}_n\}$  we find that

$$\mathbf{H}'\mathbf{u}_j = \mathbf{H}\mathbf{u}_j - \frac{\mathbf{u}_j}{\lambda_j} = 0 \text{ (why?); } j = 1, 2, \dots$$

Since  $\mathbf{v} = \mu\mathbf{H}'\mathbf{v}$  it follows that  $\mathbf{v}$  is orthogonal to each and every member of the orthonormal sequence  $\{\mathbf{u}_n\}$ . In fact  $(\mathbf{u}_j|\mathbf{v}) = \mu(\mathbf{u}_j|\mathbf{H}'\mathbf{v}) = \mu(\mathbf{H}'\mathbf{u}_j|\mathbf{v})$  (why?) = 0. Hence, on evaluating  $\mathbf{H}'\mathbf{v}$  by term-by-term integration,  $\mathbf{H}'\mathbf{v} = \mathbf{H}\mathbf{v}$  so that  $\mathbf{v}$  is a characteristic vector of  $\mathbf{H}$ . But this is absurd since  $\mathbf{v}$  is orthogonal to every characteristic vector of  $\mathbf{H}$  (why?). Hence  $H' \begin{pmatrix} x \\ t \end{pmatrix}$  must be identically zero or, equivalently,

$$H \begin{pmatrix} x \\ t \end{pmatrix} = \sum_1^{\infty} \frac{\bar{u}_n(t)u_n(x)}{\lambda_n}.$$

This fundamental result is known as the *bilinear formula*. Our preliminary result may, then, be stated as follows:



The bilinear formula is valid when the series  $\sum_1^{\infty} \frac{\bar{u}_n(t)u_n(x)}{\lambda_n}$  converges uniformly over the square  $a \leq x \leq b, a \leq t \leq b$ .

When the series  $\sum \frac{\bar{u}_n(t)u_n(x)}{\lambda_n}$  is a finite one, i.e., when  $\mathbf{H}$  has only a finite number of characteristic numbers, the question of convergence does not arise. Thus

Every Hermitian kernel which possesses only a finite number of characteristic numbers may be written in the form

$$H \begin{pmatrix} x \\ t \end{pmatrix} = \frac{\bar{u}_1(t)u_1(x)}{\lambda_1} + \cdots + \frac{\bar{u}_n(t)u_n(x)}{\lambda_n}.$$

In particular every such Hermitian kernel is separable.

*Note.* This result assures us that the Hermitian linear operator  $\mathbf{H}:G \begin{pmatrix} x \\ t \end{pmatrix} \{s(x)s(t)\}^{1/2}$ , which occurs when we look at a regular self-adjoint boundary-value problem from the integral equation point of view, always possesses an *infinite number* of characteristic numbers. In fact, since the functions  $u_1(x), \cdots, u_n(x), \cdots$  are differentiable over  $[a, b]$  (why?),  $H \begin{pmatrix} x \\ t \end{pmatrix}$  would be differentiable over  $[a, b]$  if it possessed only a finite number of characteristic numbers (why?). But  $H \begin{pmatrix} x \\ t \end{pmatrix}$  is *not* differentiable over  $[a, b]$  (why?). Hence it must possess an infinite number of characteristic numbers.

The Fourier series of  $H^2 \begin{pmatrix} x \\ t \end{pmatrix}$  is

$$\sum_1^{\infty} \frac{\bar{u}_n(t)u_n(x)}{\lambda_n^2}$$

(see Exercise 2, p. 275). This series is uniformly convergent over the square  $a \leq x \leq b, a \leq t \leq b$ , and the bilinear formula

$$H^2 \begin{pmatrix} x \\ t \end{pmatrix} = \sum_1^{\infty} \frac{\bar{u}_n(t)u_n(x)}{\lambda_n^2}$$

is valid. In order to prove this we first consider the Fourier series

$$\sum_1^{\infty} \frac{\bar{u}_n(t)u_n(x)}{\lambda_n^4}$$

of  $H^4\left(\begin{smallmatrix} x \\ t \end{smallmatrix}\right)$  (why is this the Fourier series of  $H^4\left(\begin{smallmatrix} x \\ t \end{smallmatrix}\right)$ ?). In order to examine the convergence of this series we consider the absolute value of the sum  $\sum_{N+1}^{N+p} \frac{\bar{u}_n(t)u_n(x)}{\lambda_n^4}$ ; this is not greater than  $\frac{1}{\lambda_{N+1}^2} \sum_{N+1}^{N+p} \frac{|\bar{u}_n(t)u_n(x)|}{\lambda_n^2}$

since  $\lambda_{N+1}^2 \leq \lambda_{N+2}^2 \leq \dots \leq \lambda_{N+p}^2$ . Now the sum  $\sum_{N+1}^{N+p} \frac{|\bar{u}_n(t)u_n(x)|}{\lambda_n^2}$  may be regarded as the scalar product of the two real  $p$ -dimensional vectors

$$v\left(\frac{|u_{N+1}(t)|}{\lambda_{N+1}}, \dots, \frac{|u_{N+p}(t)|}{\lambda_{N+p}}\right); \quad v\left(\frac{|u_{N+1}(x)|}{\lambda_{N+1}}, \dots, \frac{|u_{N+p}(x)|}{\lambda_{N+p}}\right).$$

Hence, by Schwarz's inequality,  $\sum_{N+1}^{N+p} \frac{|\bar{u}_n(t)u_n(x)|}{\lambda_n^2}$  is not greater than the product of the magnitudes of these two  $p$ -dimensional vectors. Now since the Fourier coefficients of  $H\left(\begin{smallmatrix} x \\ t \end{smallmatrix}\right)$ ,  $t$  fixed, are  $\frac{\bar{u}_p(t)}{\lambda_p}$  it follows from Bessel's inequality that

$$\sum_1^{\infty} \frac{\bar{u}_p(t)u_p(t)}{\lambda_p^2} \leq \int \bar{H}\left(\begin{smallmatrix} x \\ t \end{smallmatrix}\right) H\left(\begin{smallmatrix} x \\ t \end{smallmatrix}\right) dx = H^2\left(\begin{smallmatrix} t \\ t \end{smallmatrix}\right).$$

Since  $H^2\left(\begin{smallmatrix} x \\ t \end{smallmatrix}\right)$  is a continuous function of the square variable  $a \leq x \leq b$ ,  $a \leq t \leq b$ ,  $H^2\left(\begin{smallmatrix} t \\ t \end{smallmatrix}\right)$  is bounded over  $[a, b]$ . Thus there exists a number  $M$ , say, such that  $\sum_1^{\infty} \frac{\bar{u}_p(t)u_p(t)}{\lambda_p^2} \leq M$  over  $[a, b]$ ,  $M$  being independent of  $t$ . Hence the squared magnitude of each of the vectors

$$v \left( \frac{|u_{N+1}(t)|}{\lambda_{N+1}}, \dots, \frac{|u_{N+p}(t)|}{\lambda_{N+p}} \right); \quad v \left( \frac{|u_{N+1}(x)|}{\lambda_{N+1}}, \dots, \frac{|u_{N+p}(x)|}{\lambda_{N+p}} \right) \leq M$$

and so

$$\sum_{N+1}^{N+p} \frac{|\bar{u}_n(t)u_n(x)|}{\lambda_n^2} \leq M^2.$$

Thus

$$\left| \sum_{N+1}^{N+p} \frac{\bar{u}_n(t)u_n(x)}{\lambda_n^4} \right| \leq \frac{M^2}{\lambda_{N+1}^2},$$

and since  $\lambda_{N+1}^2$  may be made arbitrarily large by making  $N$  sufficiently large the uniform convergence over the square  $a \leq x \leq b$ ,  $a \leq t \leq b$  of the series

$$\sum_1^{\infty} \frac{\bar{u}_n(t)u_n(x)}{\lambda_n^4}$$

has been demonstrated; this guarantees the validity of the bilinear formula for  $H^4 \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right)$ :

$$H^4 \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) = \sum_1^{\infty} \frac{\bar{u}_n(t)u_n(x)}{\lambda_n^4}.$$

It is not so easy to prove the uniform convergence of the Fourier series of  $H^2 \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right)$  over the square  $a \leq x \leq b$ ,  $a \leq t \leq b$ , but it is easy to show that this Fourier series converges uniformly in  $x$  for every fixed  $t$  (and uniformly in  $t$  for every fixed  $x$ ). This together with the validity of the bilinear formula for  $H^4 \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right)$  enables us to prove the validity of the bilinear formula for  $H^2 \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right)$ , and after (not before) this validity of the bilinear formula for  $H^2 \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right)$  is known we can prove that the

Fourier series of  $H^2\left(\begin{smallmatrix} x \\ t \end{smallmatrix}\right)$  actually does converge uniformly over the square  $a \leq x \leq b, a \leq t \leq b$ . To see that the Fourier series of  $H^2\left(\begin{smallmatrix} x \\ t \end{smallmatrix}\right)$  converges uniformly in either of the two variables  $(x, t)$  when the other is fixed we consider the sum

$$\sum_{N+1}^{N+p} \frac{\bar{u}_n(t)u_n(x)}{\lambda_n^2}.$$

This may be regarded as the scalar product of the  $p$ -dimensional vector  $v\left(\frac{u_{N+1}(x)}{\lambda_{N+1}}, \dots, \frac{u_{N+p}(x)}{\lambda_{N+p}}\right)$  by the  $p$ -dimensional vector  $v\left(\frac{u_{N+1}(t)}{\lambda_{N+1}}, \dots, \frac{u_{N+p}(t)}{\lambda_{N+p}}\right)$ . Hence, by Schwarz's inequality, the absolute value of the sum  $\sum_{N+1}^{N+p} \frac{\bar{u}_n(t)u_n(x)}{\lambda_n^2}$  is not greater than the product of the magnitudes of these two vectors. In order to prove the uniform convergence with respect to  $x, t$  being fixed, of the series  $\sum_1^{\infty} \frac{\bar{u}_n(t)u_n(x)}{\lambda_n^2}$  we observe that the squared magnitude of the vector  $v\left(\frac{u_{N+1}(x)}{\lambda_{N+1}}, \dots, \frac{u_{N+p}(x)}{\lambda_{N+p}}\right) \leq \sum_1^{\infty} \frac{\bar{u}_n(x)u_n(x)}{\lambda_n^2}$  (why?)  $\leq M$  (where  $M$  is independent of  $x$ ). Also, in view of the convergence of the series  $\sum_1^{\infty} \frac{\bar{u}_n(t)u_n(t)}{\lambda_n^2}$ , the magnitude of the vector  $v\left(\frac{u_{N+1}(t)}{\lambda_{N+1}}, \dots, \frac{u_{N+p}(t)}{\lambda_{N+p}}\right)$  may be made arbitrarily small by making  $N$  sufficiently large (the choice of  $N$  depending on  $t$  but *not* on  $x$ ). Thus the absolute value of the sum  $\sum_{N+1}^{N+p} \frac{\bar{u}_n(t)u_n(x)}{\lambda_n^2}$  may be made arbitrarily small by merely making  $N$  sufficiently large, the choice of  $N$  depending on  $t$  but not on  $x$ . In other words the series

$\sum_1^{\infty} \frac{\bar{u}_n(t)u_n(x)}{\lambda_n^2}$  is convergent over the square  $a \leq x \leq b, a \leq t \leq b$ , the convergence being uniform with respect to  $x$  for every fixed  $t$ . Similarly (or by observing that  $\sum_1^{\infty} \frac{\bar{u}_n(t)u_n(x)}{\lambda_n^2}$  is Hermitian) (what does this mean?) the convergence is uniform with respect to  $t, x$  being fixed.

In order to prove the validity of the bilinear formula for  $H^2$  we set

$$H^2_n \begin{pmatrix} x \\ t \end{pmatrix} = H^2 \begin{pmatrix} x \\ t \end{pmatrix} - \sum_1^n \frac{\bar{u}_p(t)u_p(x)}{\lambda_p^2}; \quad n = 1, 2, \dots;$$

$$H^2_{\infty} \begin{pmatrix} x \\ t \end{pmatrix} = H^2 \begin{pmatrix} x \\ t \end{pmatrix} - \sum_1^{\infty} \frac{\bar{u}_p(t)u_p(x)}{\lambda_p^2}$$

so that

$$H^2_{\infty} \begin{pmatrix} x \\ t \end{pmatrix} = H^2_n \begin{pmatrix} x \\ t \end{pmatrix} - R^2_n \begin{pmatrix} x \\ t \end{pmatrix},$$

where

$$R^2_n \begin{pmatrix} x \\ t \end{pmatrix} = \sum_{n+1}^{\infty} \frac{\bar{u}_p(t)u_p(x)}{\lambda_p^2}.$$

Thus  $R^2_n \begin{pmatrix} x \\ t \end{pmatrix}$  is null at  $n = \infty$  uniformly with respect to  $x$  for any fixed  $t$ . Since  $H^2_{\infty} \begin{pmatrix} x \\ t \end{pmatrix}$  is a continuous function of  $x$ , for any fixed  $t$ , and since  $R^2_n \begin{pmatrix} x \\ t \end{pmatrix}$  is null at  $n = \infty$ , uniformly with respect to  $x$ , it follows that  $H^2_n \begin{pmatrix} x \\ t \end{pmatrix}$  is a bounded function of  $x$ , the boundedness being uniform with respect to  $n$ ; in other words there exists a constant  $C$  such that

$$\left| H^2_n \begin{pmatrix} x \\ t \end{pmatrix} \right| \leq C; \quad t \text{ fixed, } x \text{ and } n \text{ arbitrary}$$

( $C$  may well depend on  $t$ , but it is independent of  $x$  and  $n$ ). Let us

calculate the squared magnitude of  $H^2_n \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right)$  regarded as a function of  $x$ ,  $t$  being fixed. On taking account of the relations

$$\int \bar{H}^2 \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) H^2 \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) dx = \int H^2 \left( \begin{smallmatrix} t \\ x \end{smallmatrix} \right) H^2 \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) dx = H^4 \left( \begin{smallmatrix} t \\ t \end{smallmatrix} \right);$$

$$\int \bar{H}^2 \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) u_p(x) dx = \int H^2 \left( \begin{smallmatrix} t \\ x \end{smallmatrix} \right) u_p(x) dx = \frac{u_p(t)}{\lambda_p^2};$$

(u<sub>p</sub>|u<sub>p</sub>) = 1; (u<sub>p</sub>|u<sub>q</sub>) = 0; p ≠ q

we obtain

$$\int \bar{H}^2_n \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) H^2_n \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) dx = H^4 \left( \begin{smallmatrix} t \\ t \end{smallmatrix} \right) - \sum_1^n \frac{\bar{u}_p(t) u_p(t)}{\lambda_p^4}.$$

Since the bilinear formula is valid for  $H^4$  it follows that the squared magnitude of  $H^2_n \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right)$ ,  $t$  fixed, is null at  $n = \infty$ .

We next appraise the squared magnitude of  $H^2_\infty \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right)$ . Since  $H^2_\infty \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) = H^2_n \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) - R^2_n \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right)$  this squared magnitude =  $\int \bar{H}^2_n \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) H^2_n \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) dx - \int \bar{H}^2_n \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) R^2_n \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) dx - \int H^2_n \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) \bar{R}^2_n \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) dx + \int \bar{R}^2_n \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) R^2_n \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) dx$ . Since  $R^2_n \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right)$  is null at  $n = \infty$ , uniformly with respect to  $x$ , and since  $\left| H^2_n \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) \right|$  is bounded, uniformly with respect to  $x$  and  $n$ , it follows that the squared magnitude of  $H^2_\infty \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right)$ ,  $t$  fixed, is null at  $n = \infty$ . (Prove this.) Being independent of  $n$  it must be zero (why?) and so  $H^2_\infty \left( \begin{smallmatrix} x \\ t \end{smallmatrix} \right) = 0$  for every  $x$  and every  $t$ . In other

words

**The bilinear formula is valid for  $H^2$ .**

A repetition of this argument (applied to  $\mathbf{H}$  rather than  $\mathbf{H}^2$ ) proves our main theorem. We set

$$H_n \begin{pmatrix} x \\ t \end{pmatrix} = H \begin{pmatrix} x \\ t \end{pmatrix} - \sum_1^n \frac{\bar{u}_p(t) u_p(x)}{\lambda_p}; \quad n = 1, 2, 3, \dots;$$

$$H_\infty \begin{pmatrix} x \\ t \end{pmatrix} = H \begin{pmatrix} x \\ t \end{pmatrix} - \sum_1^\infty \frac{\bar{u}_p(t) u_p(x)}{\lambda_p}.$$

*Note.* We cannot *prove* the convergence of the series  $\sum_1^\infty \frac{\bar{u}_p(t) u_p(x)}{\lambda_p}$  (the convergence being uniform with respect to  $x$  for every fixed  $t$ ) as we could for the series  $\sum_1^\infty \frac{\bar{u}_p(t) u_p(x)}{\lambda_p^2}$ . This must now be granted,

but from this point on the argument is precisely the same (except for the fact that we lean upon the validity of the bilinear formula for  $\mathbf{H}^2$  rather than, as before, on the validity of the bilinear formula for  $\mathbf{H}^4$ ). We have, then, proved the following fundamental result:

If the Fourier series of  $H \begin{pmatrix} x \\ t \end{pmatrix}$  converges uniformly in  $x$  for every fixed  $t$  the bilinear formula is valid:

$$H \begin{pmatrix} x \\ t \end{pmatrix} = \sum_1^\infty \frac{\bar{u}_n(t) u_n(x)}{\lambda_n}.$$

The bilinear formula is valid for  $\mathbf{H}^2$  no matter what is the (Hermitian) linear integral operator  $\mathbf{H}$ :

$$H^2 \begin{pmatrix} x \\ t \end{pmatrix} = \sum_1^\infty \frac{\bar{u}_n(t) u_n(x)}{\lambda_n^2}.$$

As a first application of this result we prove the following remarkable theorem: Every function-vector  $\mathbf{k}$  of the form  $\mathbf{H}\mathbf{h}$ , where  $\mathbf{h}$  is any continuous function-vector, possesses a uniformly convergent Fourier series, with respect to the orthonormal sequence  $\{\mathbf{u}_n\}$  determined by  $\mathbf{H}$ , and the sum of this Fourier series is  $h(x)$ .

Since  $(\mathbf{u}_j | \mathbf{k}) = (\mathbf{u}_j | \mathbf{H}\mathbf{h}) = (\mathbf{H}\mathbf{u}_j | \mathbf{k}) = \frac{1}{\lambda_j} (\mathbf{u}_j | \mathbf{h})$  the Fourier coefficients  $k^j$  of  $\mathbf{k}$  are given, in terms of the Fourier coefficients  $h^j$  of  $\mathbf{h}$ , by means of the formula

$$k^j = \frac{1}{\lambda_j} h^j; \quad j = 1, 2, \dots$$

Thus the Fourier series of  $k(x)$  is

$$\sum_1^{\infty} \frac{h^j}{\lambda_j} u_j(x).$$

Since the series  $\sum_1^{\infty} \bar{h}^j h^j$  converges (why?) and since  $\sum_1^{\infty} \frac{\bar{u}_j(x) u_j(x)}{\lambda_j^2}$  is less than a fixed number  $M$  (by fixed we mean independent of  $x$ ) it follows, by the same argument which proved the uniform convergence in  $x$  (for every fixed  $t$ ) of the bilinear formula for  $\mathbf{H}^2$ , that the Fourier series for  $k(x)$  converges uniformly in  $x$ . It remains only to prove that its sum is  $k(x)$ . To do this we set  $\mathbf{v} = \mathbf{k} - \sum_1^{\infty} \frac{h^j}{\lambda_j} \mathbf{u}_j$ . Since the uniform

convergence of the series  $\sum_1^{\infty} \frac{h^j}{\lambda_j} u_j(x)$  permits term-by-term integration (after multiplication by  $\bar{u}_j(x)$ ) it follows (by virtue of the orthonormal quality of the sequence  $\{\mathbf{u}_n\}$ ) that  $(\mathbf{u}_j | \mathbf{v}) = (\mathbf{u}_j | \mathbf{k}) - \frac{h^j}{\lambda_j} = 0$ . Hence  $\mathbf{v}$  is orthogonal to each member of the orthonormal sequence  $\{\mathbf{u}_n\}$ . In view of the uniform convergence in  $t$ , for every fixed  $x$ , of the bilinear formula for  $\mathbf{H}^2$  it follows that  $\mathbf{H}^2 \mathbf{v} = \mathbf{0}$ . (Prove this.) Hence  $(\mathbf{H}^2 \mathbf{v} | \mathbf{v}) = 0$  and so, since  $(\mathbf{H}^2 \mathbf{v} | \mathbf{v}) = (\mathbf{H} \mathbf{v} | \mathbf{H} \mathbf{v})$ ,  $\mathbf{H} \mathbf{v} = \mathbf{0}$ . Since  $\mathbf{k} = \mathbf{H} \mathbf{h}$  it follows that  $(\mathbf{k} | \mathbf{v}) = (\mathbf{H} \mathbf{h} | \mathbf{v}) = (\mathbf{h} | \mathbf{H} \mathbf{v}) = 0$ . Hence (on again using term-by-term integration)  $(\mathbf{v} | \mathbf{v}) = 0$  so that  $v(x) \equiv 0$ . In other words the sum of the Fourier series of  $k(x)$  is  $k(x)$ .

Let, now,  $\mathbf{y}$  be a solution of the integral equation  $\mathbf{y} = \lambda \mathbf{H} \mathbf{y} + \mathbf{f}$ . Since  $\frac{\mathbf{y} - \mathbf{f}}{\lambda}$  is of the form  $\mathbf{H} \mathbf{y}$  it possesses a uniformly convergent Fourier series

$$\frac{\mathbf{y} - \mathbf{f}}{\lambda} = \sum_1^{\infty} c^p \mathbf{u}_p.$$

On multiplying this equation scalarly by  $\mathbf{u}_p$  we find



$$\frac{y^p - f^p}{\lambda} = c^p; \quad p = 1, 2, \dots$$

Since  $c^p$  is the  $p$ th Fourier coefficient of  $\mathbf{H}y$  we have  $c^p = \frac{y^p}{\lambda_p}$  and so

$$y^p \left( \frac{1}{\lambda} - \frac{1}{\lambda_p} \right) = \frac{f^p}{\lambda}. \quad \text{Hence, if } \lambda \neq \lambda_p,$$

$$c^p = \frac{f^p}{\lambda \lambda_p \left( \frac{1}{\lambda} - \frac{1}{\lambda_p} \right)} = \frac{f^p}{\lambda_p - \lambda}$$

so that  $y = f + \lambda \sum_1^{\infty} \frac{f^p}{\lambda_p - \lambda} u_p(x)$ . This form of the solution of a

Hermitian integral equation is known as the *Hilbert-Schmidt* solution (after the German mathematicians D. Hilbert [1862-1943] and E. Schmidt). It shows clearly the effect of *resonance* as  $\lambda$  approaches one of the characteristic numbers  $\lambda_p$  of the integral equation. When  $\lambda = \lambda_p$  we must have  $f^p = 0$ ; in other words it is necessary, for the existence of a solution of the non-homogeneous equation, that  $f$  be orthogonal to every characteristic vector associated with  $\lambda_p$ . When these necessary conditions are satisfied, the  $y^p$  (and hence the  $c^p$ ) which are associated with  $\lambda_p$  are arbitrary, but the  $c^p$  which are associated with the other characteristic numbers are determined by the formula  $c^p = \frac{f^p}{\lambda_p - \lambda}$ .

#### 10. The completeness of the orthonormal sequence $\{u_n\}$ defined by a regular boundary-value problem

We know that, if  $\mathbf{H}$  is any Hermitian linear integral operator, its "square"  $\mathbf{H}^2$  is furnished by the bilinear formula

$$\mathbf{H}^2: H^2 \begin{pmatrix} x \\ t \end{pmatrix} = \sum_1^{\infty} \frac{\bar{u}_n(t) u_n(x)}{\lambda_n^2},$$

where the series converges uniformly in  $x$  for every fixed  $t$  (and uniformly in  $t$  for every fixed  $x$ ). Since term-by-term integration (after multiplication by any continuous function  $v(t)$ ) is legitimate (why?) and since the sequence  $\{u_n\}$  is orthonormal it follows that if the function-vector  $\mathbf{v}$  is orthogonal to every vector of the orthonormal sequence then  $\mathbf{H}^2\mathbf{v} = \mathbf{0}$ . (Prove this.) Hence  $(\mathbf{H}^2\mathbf{v}|\mathbf{v}) = (\mathbf{H}\mathbf{v}|\mathbf{H}\mathbf{v})$

= 0 so that  $\mathbf{H}\mathbf{v} = \mathbf{0}$  (why?). If the linear integral operator  $\mathbf{H}$  is such that  $\mathbf{H}\mathbf{v} = \mathbf{0}$  only when  $\mathbf{v}$  is the zero vector it follows that the orthonormal sequence  $\{\mathbf{u}_n\}$  is complete (what does this mean?). We have, then, the following theorem:

**If the function-vector  $\mathbf{v}$  is orthogonal to every vector of the orthonormal sequence  $\{\mathbf{u}_n\}$  determined by the Hermitian linear integral operator  $\mathbf{H}$  then  $\mathbf{H}\mathbf{v} = \mathbf{0}$ ; if  $\mathbf{H}$  is such that  $\mathbf{H}\mathbf{v} = \mathbf{0}$  only when  $\mathbf{v}$  is the zero vector then the orthonormal sequence  $\{\mathbf{u}_n\}$  is complete.**

We know that a regular self-adjoint boundary-value problem

$$Lu + \lambda su = 0; \quad B_1(u) = 0, \quad B_2(u) = 0$$

for which  $s(x)$  is non-negative is equivalent to the integral equation problem

$$\mathbf{v} = \lambda \mathbf{H}\mathbf{v},$$

where  $v(x) = \{s(x)\}^{1/2}u$  and  $\mathbf{H}:G\begin{pmatrix} x \\ t \end{pmatrix} \{s(x)s(t)\}^{1/2}$ . Since  $LG = -1$  it

is clear that  $\mathbf{H}\mathbf{u}$  is zero only when  $\mathbf{u}$  is the zero vector. In fact  $\mathbf{H}\mathbf{u} = \{s(x)\}^{1/2}\mathbf{G}\mathbf{v}$  so that  $\mathbf{G}\mathbf{v} = \{s(x)\}^{-1/2}\mathbf{H}\mathbf{u}$ . If, then,  $\mathbf{H}\mathbf{u} = \mathbf{0}$  we have  $\mathbf{G}\mathbf{v} = \mathbf{0}$  and so  $LG\mathbf{v}$ , i.e.,  $-\mathbf{v}$ , is the zero vector. Hence  $\mathbf{u} = \{s(x)\}^{-1/2}\mathbf{v}$  is the zero vector. *Notc.* We take it as granted that if  $s(x)$  is ever zero the points at which it is zero are either finite in number or else they may be covered by a sequence of intervals the sum of whose lengths is arbitrarily small. The fact that  $\mathbf{G}\mathbf{v}$  may fail to be defined at these points is, therefore, of no consequence (why?). We have, then, the important result that

**The orthonormal sequence  $\{\mathbf{u}_n\}$  determined by any regular self-adjoint boundary-value problem is complete.**

## 11. Non-negative and positive linear integral operators

A Hermitian linear integral operator  $\mathbf{H}$  is said to be non-negative if the real number  $(\mathbf{v}|\mathbf{H}\mathbf{v})$  is non-negative for every vector  $\mathbf{v}$ ; if this number is positive when  $\mathbf{v}$  is not the zero vector  $\mathbf{H}$  is said to be *positive*. It is clear that the characteristic numbers of a non-negative linear integral operator are positive; in fact if  $\lambda_p$  is any characteristic number of  $\mathbf{H}$  and  $\mathbf{u}_p$  a characteristic vector of  $\mathbf{H}$  associated with  $\lambda_p$  we have  $\mathbf{u}_p = \lambda_p \mathbf{H}\mathbf{u}_p$  so that  $(\mathbf{u}_p|\mathbf{u}_p) = \lambda_p(\mathbf{u}_p|\mathbf{H}\mathbf{u}_p)$ . Hence  $(\mathbf{u}_p|\mathbf{H}\mathbf{u}_p) > 0$  (why?) and  $\lambda_p > 0$ . The converse of this theorem is true:

**If all the characteristic numbers of a Hermitian linear integral operator  $\mathbf{H}$  are positive,  $\mathbf{H}$  is non-negative; if the orthonormal sequence  $\{\mathbf{u}_n\}$  is complete,  $\mathbf{H}$  is positive.**

In fact if  $\mathbf{v}$  is any vector  $\mathbf{H}\mathbf{v}$  possesses a uniformly convergent Fourier series  $\sum_1^{\infty} \frac{\nu^p}{\lambda_p} u_p(x)$  and so  $(\mathbf{v}|\mathbf{H}\mathbf{v}) = \sum_1^{\infty} \frac{\bar{\nu}_p \nu_p}{\lambda_p} \geq 0$ . If the orthonormal sequence  $\{u_n\}$  is complete not all the Fourier coefficients  $\nu^p$  of  $\mathbf{v}$  are zero if  $\mathbf{v}$  is not the zero vector; hence  $(\mathbf{v}|\mathbf{H}\mathbf{v}) > 0$  if  $\mathbf{v}$  is not the zero vector so that  $\mathbf{H}$  is positive.

The "diagonal values"  $H \begin{pmatrix} x \\ x \end{pmatrix}$  of any Hermitian kernel are real (why?); if  $\mathbf{H}$  is non-negative  $H \begin{pmatrix} x \\ x \end{pmatrix}$  is non-negative over the closed interval  $a \leq x \leq b$ . To see this set  $H \begin{pmatrix} x \\ t \end{pmatrix} = R \begin{pmatrix} x \\ t \end{pmatrix} + iS \begin{pmatrix} x \\ t \end{pmatrix}$ , where both  $R$  and  $S$  are real linear integral operators. Then  $H \begin{pmatrix} x \\ x \end{pmatrix} = R \begin{pmatrix} x \\ x \end{pmatrix}$  and if  $\mathbf{v}$  is any real vector  $(\mathbf{v}|\mathbf{H}\mathbf{v}) = (\mathbf{v}|\mathbf{R}\mathbf{v})$  (why?). If  $R \begin{pmatrix} x \\ x \end{pmatrix}$  were negative at an interior point  $\xi$  of the interval  $a \leq x \leq b$ ,  $R \begin{pmatrix} x \\ t \end{pmatrix}$  would, by virtue of its continuity, be negative over a two-dimensional interval  $\xi - \delta \leq x \leq \xi + \delta$ ,  $\xi - \delta \leq t \leq \xi + \delta$  centered at  $(\xi, \xi)$ . Choosing  $v(x) = 0$  outside the open interval  $(\xi - \delta, \xi + \delta)$  and positive over this interval,  $(\mathbf{v}|\mathbf{R}\mathbf{v}) < 0$  (why?) which is absurd since  $\mathbf{H}$  is, by hypothesis, non-negative and  $(\mathbf{v}|\mathbf{R}\mathbf{v}) = (\mathbf{v}|\mathbf{H}\mathbf{v})$  since  $\mathbf{v}$  is a real vector. Hence  $H \begin{pmatrix} x \\ x \end{pmatrix}$  is non-negative over the open interval  $(a, b)$ ; by virtue of the continuity of  $H \begin{pmatrix} x \\ x \end{pmatrix}$  it follows (why?) that  $H \begin{pmatrix} x \\ x \end{pmatrix}$  is non-negative over the closed interval  $[a, b]$ . This result enables us to prove the following important theorem:

**Every non-negative kernel  $H \begin{pmatrix} x \\ t \end{pmatrix}$  is furnished by the bilinear formula**

$$H \begin{pmatrix} x \\ t \end{pmatrix} = \sum_1^{\infty} \frac{\bar{u}_n(t) u_n(x)}{\lambda_n}$$

where the infinite series (if it is infinite) converges uniformly over the square  $a \leq x \leq b, a \leq t \leq b$ .

To prove this theorem we set

$$H_p \begin{pmatrix} x \\ t \end{pmatrix} = H \begin{pmatrix} x \\ t \end{pmatrix} - \sum_1^p \frac{\bar{u}_n(t)u_n(x)}{\lambda_n}.$$

It is clear that  $H_p$  is Hermitian (why?) and that  $H_p u_n = 0, n = 1, \dots, p$ . (Prove this.) Let  $\mu$  be any characteristic number of  $H_p$ , and let  $v$  be a characteristic vector of  $H_p$  which is associated with  $\mu$ . Since  $v = \mu H_p v$  it follows that  $(u_n | v) = 0, n = 1, \dots, p$ . (Prove this. Use the fact that  $H_p u_n = 0$ .) Hence  $H_p v = H v$  (prove this) so that  $v$  is a characteristic vector, and  $\mu$  a characteristic number, of  $H$ . Note that  $\mu$  is one of the characteristic numbers  $\lambda_{p+1}, \lambda_{p+2}, \dots$  since  $v$  is orthogonal to each of the vectors  $u_1, \dots, u_p$ . Since all the characteristic numbers of  $H_p$  are positive,  $H_p$  is a non-negative linear integral operator. Hence  $H_p \begin{pmatrix} x \\ x \end{pmatrix} \geq 0, p = 1, 2, \dots$  so that

$$\sum_1^p \frac{\bar{u}_n(x)u_n(x)}{\lambda_n} \leq H \begin{pmatrix} x \\ x \end{pmatrix} \leq M, \text{ say, where } M \text{ is independent of } x.$$

Hence the series  $\sum_1^\infty \frac{\bar{u}_n(x)u_n(x)}{\lambda_n}$  converges to a sum  $\leq M$ . On regard-

ing the sum  $\sum_{N+1}^{N+p} \frac{\bar{u}_n(t)u_n(x)}{\lambda_n}$  as the scalar product of the  $p$ -dimensional vector  $v \left( \frac{u_{N+1}(x)}{(\lambda_{N+1})^{1/2}}, \dots, \frac{u_{N+p}(x)}{(\lambda_{N+p})^{1/2}} \right)$  by the  $p$ -dimensional vector  $v \left( \frac{u_{N+1}(t)}{(\lambda_{N+1})^{1/2}}, \dots, \frac{u_{N+p}(t)}{(\lambda_{N+p})^{1/2}} \right)$  it follows, on applying Schwarz's inequality,

that the bilinear formula  $\sum_1^\infty \frac{\bar{u}_n(t)u_n(x)}{\lambda_n}$  converges uniformly in  $x$

for every fixed  $t$ . Hence its sum is  $H \begin{pmatrix} x \\ t \end{pmatrix}$ :

$$H \begin{pmatrix} x \\ t \end{pmatrix} = \sum_1^\infty \frac{\bar{u}_n(t)u_n(x)}{\lambda_n},$$

and, in particular

$$H \begin{pmatrix} x \\ x \end{pmatrix} = \sum_1^{\infty} \frac{\bar{u}_n(x) u_n(x)}{\lambda_n}.$$

The convergence of the series  $\sum_1^{\infty} \frac{\bar{u}_n(x) u_n(x)}{\lambda_n}$  is uniform with respect to  $x$  for the following two reasons: (1) Each term of the series is positive; and (2) the sum of the series is continuous. In fact let us denote by  $R_n(x)$  the remainder after  $n$  terms:

$$R_n(x) = H \begin{pmatrix} x \\ x \end{pmatrix} - \sum_1^n \frac{\bar{u}_j(x) u_j(x)}{\lambda_j}.$$

Then  $R_n(x)$  is non-negative and continuous over  $[a, b]$ ; denote by  $x_n$  a point of  $[a, b]$  at which  $R_n(x)$  attains its absolute maximum over  $[a, b]$ , and let  $\xi$  be an accumulation point (what does this mean?) of the

sequence  $\{x_n\}$ . Since  $\sum_1^{\infty} \frac{\bar{u}_n(\xi) u_n(\xi)}{\lambda_n}$  converges we can determine  $N$

such that  $0 \leq R_n(\xi) \leq \epsilon$ , where  $\epsilon$  is any given positive number, if  $n \geq N$ . Hence  $0 \leq R_n(x_m) \leq 2\epsilon$ , if  $n \geq N$ , for arbitrarily large values of  $m$ . Hence there exists an integer  $m$  for which  $R_m(x_m) \leq 2\epsilon$ . Hence  $R_m(x) \leq 2\epsilon$  over  $[a, b]$  (why?). Hence  $R_q(x) \leq 2\epsilon$  if  $q \geq m$  (why?). In

other words the series  $\sum_1^{\infty} \frac{\bar{u}_n(x) u_n(x)}{\lambda_n}$  converges uniformly over  $[a, b]$ .

Since the modulus of  $\sum_{N+1}^{N+p} \frac{\bar{u}_n(t) u_n(x)}{\lambda_n}$  is not greater than the product

of the magnitudes of the two  $p$ -dimensional vectors  $v \left( \frac{u_{N+1}(x)}{(\lambda_{N+1})^{1/2}}, \dots, \frac{u_{N+p}(x)}{(\lambda_{N+p})^{1/2}} \right)$ ,  $v \left( \frac{u_{N+1}(t)}{(\lambda_{N+1})^{1/2}}, \dots, \frac{u_{N+p}(t)}{(\lambda_{N+p})^{1/2}} \right)$  it follows (show this) that the

series  $\sum_1^{\infty} \frac{\bar{u}_n(t) u_n(x)}{\lambda_n}$  converges uniformly over the square  $a \leq x \leq b$ ,

$a \leq t \leq b$ . Note carefully that this is a stronger statement than this series converges uniformly in  $x$  for every fixed  $t$ .

The uniform convergence of the series  $\sum_1^{\infty} \frac{\bar{u}_n(x)u_n(x)}{\lambda_n}$  permits term-by-term integration of the relation

$$H\left(\begin{matrix} x \\ x \end{matrix}\right) = \sum_1^{\infty} \frac{\bar{u}_n(x)u_n(x)}{\lambda_n}$$

and so

$$\int H\left(\begin{matrix} x \\ x \end{matrix}\right) dx = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_n} + \cdots$$

In words:

The series formed by the reciprocals of the characteristic numbers of a non-negative linear integral operator converges to the value

$$\int H\left(\begin{matrix} x \\ x \end{matrix}\right) dx.$$

Since the characteristic numbers  $\lambda_1, \lambda_2, \cdots$  are all positive it follows that  $\frac{1}{\lambda_1} \leq \int H\left(\begin{matrix} x \\ x \end{matrix}\right) dx$ . An equivalent statement is the following:

**The smallest characteristic number of a non-negative linear integral operator is not less than the reciprocal of  $\int H\left(\begin{matrix} x \\ x \end{matrix}\right) dx$ .**

If  $\mathbf{H}$  is any Hermitian linear integral operator the squares of its characteristic numbers are the characteristic numbers of  $\mathbf{H}^2$ . Hence  $\mathbf{H}^2$  is non-negative (why?), and

$$\int H^2\left(\begin{matrix} x \\ x \end{matrix}\right) dx = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \cdots + \frac{1}{\lambda_n^2} + \cdots$$

Thus

**The modulus of any characteristic number of any Hermitian kernel is not less than the positive square root of the reciprocal of  $\int H^2\left(\begin{matrix} x \\ x \end{matrix}\right) dx$ .**

### Example

The boundary-value problem of the vibrating string is equivalent to the Hermitian integral equation whose kernel is the Green's function

$$G \begin{pmatrix} x \\ t \end{pmatrix} = \begin{cases} \frac{1}{l} \left( \frac{l}{2} - t \right) \left( \frac{l}{2} + x \right); & -\frac{l}{2} \leq x \leq t; \\ \frac{1}{l} \left( \frac{l}{2} + t \right) \left( \frac{l}{2} - x \right); & t \leq x \leq \frac{l}{2}. \end{cases}$$

The characteristic numbers of the boundary-value problem (or equivalently of the integral equation) are all positive:

$$\lambda_n = \frac{n^2 \pi^2}{l^2}.$$

Since  $H \begin{pmatrix} x \\ x \end{pmatrix} = \frac{1}{l} \left( \frac{l^2}{4} - x^2 \right)$ ,  $\int H \begin{pmatrix} x \\ x \end{pmatrix} dx = \frac{l^2}{6}$ . Hence  $\lambda_1 \geq \frac{6}{l^2}$ . Actually  $\lambda_1 = \frac{\pi^2}{l^2}$ .

## 12. Rayleigh's principle

Let us consider the boundary-value problem which is involved in the discussion of the transverse vibrations of a string, not necessarily uniformly dense, which is tightly stretched between two fixed points.

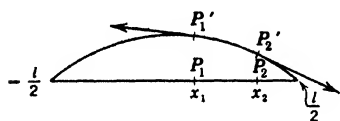


FIG. 40.

If  $P_1: (x_1, 0)$ ,  $P_2: (x_2, 0)$  are any two points of the string when it is in its equilibrium position we denote by  $P_1': (x_1, y_1)$ ,  $P_2': (x_2, y_2)$  the positions, at any time  $t$  during the vibration, of these two points of the string. If  $\rho$

is the (linear) density of the string when in its equilibrium position the principle of conservation of mass yields  $\rho' ds = \rho dx$ , where  $\rho'$  is the linear density and  $s$  the arc length, at the time  $t$ . The transverse component of the force acting at the time  $t$  on the portion of the string between  $P_1'$  and  $P_2'$  is

$$\int_{s_1}^{s_2} \rho' y_{tt} ds = \int_{x_1}^{x_2} \rho y_{tt} dx.$$

This force is exerted by the tensions of the remaining portions of the string at the points  $P_1'$  and  $P_2'$ . We suppose the tension  $P$  when the string is in its equilibrium position so great that the additional tension due to its slight extension when vibrating may be neglected (this is what we imply when we refer to the string as *tightly stretched*). Then the transverse component of the force due to the tension at  $P_2'$  is

$P \frac{dy}{ds} (P_2')$ , and we suppose the string so slightly bent away from its equilibrium position (along the  $x$ -axis) that  $\frac{dy}{ds}$  may be replaced by  $y_x$ ; this amounts to replacing  $s_x = (1 + y_x^2)^{1/2}$  by 1. The transverse component of the force acting on the portion  $P_1'P_2'$  of the string is, accordingly,  $P\{y_x(x_2) - y_x(x_1)\} = Py_{xx}(\xi) \Delta x$  where  $x_1 < \xi < x_2$  and  $\Delta x = x_2 - x_1$ . Hence

$$Py_{xx}(\xi) \Delta x = \int_{x_1}^{x_2} \rho y_u dx = \rho(\xi') y_u(\xi') \Delta x,$$

where  $x_1 < \xi' < x_2$ . Thus

$$Py_{xx}(\xi) = \rho(\xi') y_u(\xi'),$$

and on setting  $x_1 = x$  and letting  $\Delta x \rightarrow 0$  we obtain, by virtue of the assumed continuity of  $\rho$  and of the second derivatives  $y_{xx}$ ,  $y_u$ , the equation

$$y_{xx} = \frac{\rho}{P} y_u = \frac{1}{c^2} y_u; \quad c^2 = \frac{P}{\rho}.$$

This is the equation we have discussed in detail in the particular case where  $\rho$  is a constant function of  $x$  (so that the string is uniformly dense). In the more general situation of a non-uniform string we find, on setting  $y = ue^{\pm ipP^{1/2}t}$ , that

$$u_{xx} + \rho p^2 u = 0.$$

Taking the string to be of length  $l$  and the origin to be at its mid-point (when it is in its equilibrium position) the boundary conditions are  $u\left(-\frac{l}{2}\right) = 0$ ,  $u\left(\frac{l}{2}\right) = 0$ . On setting  $p^2 = \lambda$  we have the self-adjoint equation

$$Lu + \lambda su = 0,$$

where  $L \equiv D^2$ ,  $s = \rho > 0$ .

The *kinetic energy* of the vibrating string is

$$T = \frac{1}{2} \int_{-\frac{l}{2}}^{\frac{l}{2}} \rho y_t^2 dx.$$

Hence the time rate of change of  $T$  is



$$T_t = \int_{-\frac{l}{2}}^{\frac{l}{2}} \rho y_t y_{tt} dx = P \int_{-\frac{l}{2}}^{\frac{l}{2}} y_t y_{xx} dx.$$

On integrating by parts and using the fact that  $y_t = 0$  at both ends of the string we obtain

$$T_t = -P \int_{-\frac{l}{2}}^{\frac{l}{2}} y_x y_{xt} dx$$

If, then, we set

$$V = \frac{1}{2}P \int_{-\frac{l}{2}}^{\frac{l}{2}} y_x^2 dx$$

we have  $V_t + T_t = 0$  so that  $V + T$  is constant. By virtue of the principle of conservation of energy we term  $V$  the *potential energy* of the vibrating string. On integrating by parts and using the fact that  $y = 0$  at both ends of the string we obtain

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} y_x^2 dx = - \int_{-\frac{l}{2}}^{\frac{l}{2}} y y_{xx} dx.$$

Thus the additive constant in the potential energy has been fixed so that the potential energy is zero when the string is in its equilibrium position. We have, then, the following result:

*The kinetic energy of the vibrating string is  $T = \frac{1}{2} \int_{-\frac{l}{2}}^{\frac{l}{2}} \rho y_t^2 dx$ , and the potential energy of the vibrating string is  $V = \frac{P}{2} \int_{-\frac{l}{2}}^{\frac{l}{2}} y_x^2 dx = -\frac{P}{2} \int_{-\frac{l}{2}}^{\frac{l}{2}} y y_{xx} dx$ .*

When the string is executing a *normal* vibration

$$y = u e^{\pm i p P^{1/2} t} = u(\alpha e^{i p P^{1/2} t} + \bar{\alpha} e^{-i p P^{1/2} t}); \quad p \neq 0$$

we have  $y_t = i p P^{1/2} u(\alpha e^{i p P^{1/2} t} - \bar{\alpha} e^{-i p P^{1/2} t})$  so that

$$y_t^2 = -p^2 P u^2 (\alpha^2 e^{2i p P^{1/2} t} - 2\alpha\bar{\alpha} + \bar{\alpha}^2 e^{-2i p P^{1/2} t}).$$

Hence the average value of  $T$  with respect to  $t$  over a cycle is

$$\alpha \bar{a} p^2 P \int_{-1/2}^{1/2} \rho u^2 dx$$

(the time average of  $e^{2i\rho P^{1/2}t}$  and of  $e^{-2i\rho P^{1/2}t}$  over a cycle being zero since  $p \neq 0$ ). A similar calculation shows that the time average of  $V = \frac{1}{2}P \int_{-1/2}^{1/2} y_x^2 dx$  over a cycle of a normal vibration is  $\alpha \bar{a} P \int_{-1/2}^{1/2} u_x^2 dx$ . An integration by parts yields, since  $u = 0$  at each end of the vibrating string,  $\int_{-1/2}^{1/2} u_x^2 dx = - \int_{-1/2}^{1/2} uu_{xx} dx = p^2 \int_{-1/2}^{1/2} \rho u^2 dx$  since  $u_{xx} + p^2 \rho u = 0$ . Thus

**The time average, over a cycle, of the kinetic energy in a normal vibration is the same as the time average, over a cycle, of the potential energy.**

This result merely expresses in terms of the physical concepts of kinetic and potential energy the fact that if  $\mathbf{u}$  is a characteristic vector of the boundary-value problem which is associated with the characteristic number  $\lambda$  the ratio of  $-(\mathbf{u}|\mathbf{Lu}) = - \int_{-1/2}^{1/2} uu_{xx} dx$  to  $\int_{-1/2}^{1/2} \rho u^2 dx$  is  $\lambda$ . Since  $s = \rho$  the vector  $\mathbf{v} = \rho^{1/2}\mathbf{u}$  satisfies the Hermitian integral equation  $\mathbf{v} = \lambda \mathbf{H}\mathbf{v}$ , where  $\mathbf{H}: G \begin{pmatrix} x \\ t \end{pmatrix} \rho^{1/2}(x)\rho^{1/2}(t)$ . It is clear from the expression for  $\mathbf{H}$  that

$$\mathbf{H}\mathbf{v} = \rho^{1/2}\mathbf{G}(\rho\mathbf{u})$$

and so

$$(\mathbf{v}|\mathbf{H}\mathbf{v}) = (\rho^{1/2}\mathbf{u}|\rho^{1/2}\mathbf{G}(\rho\mathbf{u})) = (\rho\mathbf{u}|\mathbf{G}(\rho\mathbf{u})) = (\mathbf{h}|\mathbf{G}\mathbf{h}),$$

where  $\mathbf{h} = \rho\mathbf{u} = \rho^{1/2}\mathbf{v}$ . If  $\mathbf{v}$  is any continuous vector the vector  $\mathbf{H}\mathbf{v} = \rho^{1/2}\mathbf{G}\mathbf{h}$  possesses a uniformly convergent Fourier series

$$\mathbf{H}\mathbf{v}: \sum_1^{\infty} \frac{\nu^p}{\lambda_p} \nu_p(x)$$

with respect to the orthonormal sequence  $\{\nu_n\}$  determined by  $\mathbf{H}$ . Hence

$$(\mathbf{h}|\mathbf{G}\mathbf{h}) = (\mathbf{v}|\mathbf{H}\mathbf{v}) = \sum_1^{\infty} \frac{\bar{\nu}^p \nu^p}{\lambda_p}$$

On setting  $\mathbf{k} = \mathbf{G}\mathbf{h} = \rho^{-1/2}\mathbf{H}\mathbf{v}$ , so that  $\mathbf{k}$  satisfies the boundary conditions, we have  $\mathbf{L}\mathbf{k} = -\mathbf{h}$ , since  $\mathbf{L}\mathbf{G} = -1$ . Hence  $(\mathbf{h}|\mathbf{G}\mathbf{h}) = -(\mathbf{L}\mathbf{k}|\mathbf{k})$

so that  $(\mathbf{h}|\mathbf{Gh}) = \int_{-1/2}^{1/2} k_x^2 dx$  (since  $\mathbf{Lk} : k_{xx}$  and  $k = 0$  at both ends of the string). On the other hand  $(\rho\mathbf{Gh}|\mathbf{Gh}) = (\mathbf{Hv}|\mathbf{Hv}) = \sum_1^{\infty} \frac{\bar{v}^p v^p}{\lambda_p^2}$  since the orthonormal sequence  $\{\mathbf{v}_n\}$  is complete. Hence

$$(\rho\mathbf{k}|\mathbf{k}) = \sum_1^{\infty} \frac{\bar{v}^p v^p}{\lambda_p^2}.$$

On combining the two relations

$$(\mathbf{h}|\mathbf{Gh}) = \sum_1^{\infty} \frac{\bar{v}^p v^p}{\lambda_p}; \quad (\rho\mathbf{k}|\mathbf{k}) = \sum_1^{\infty} \frac{\bar{v}^p v^p}{\lambda_p^2}$$

so as to eliminate  $\bar{v}^1 v^1$  we obtain

$$(\mathbf{h}|\mathbf{Gh}) - \lambda_1(\rho\mathbf{k}|\mathbf{k}) = \sum_2^{\infty} \frac{\bar{v}^p v^p}{\lambda_p} \left(1 - \frac{\lambda_1}{\lambda_p}\right).$$

Since  $(\mathbf{v}|\mathbf{Hv}) = (\mathbf{h}|\mathbf{Gh}) = \int_{-1/2}^{1/2} k_x^2 dx$  the kernel  $\mathbf{H}$  is non-negative so that the characteristic numbers  $\lambda_1, \dots, \lambda_n, \dots$  are all positive.  $\lambda_1$  being the smallest of these it follows that

$$(\mathbf{h}|\mathbf{Gh}) - \lambda_1(\rho\mathbf{k}|\mathbf{k}) \geq 0$$

or, equivalently,

$$\lambda_1 \leq \frac{(\mathbf{h}|\mathbf{Gh})}{(\rho\mathbf{k}|\mathbf{k})} = \frac{-(\mathbf{Lk}|\mathbf{k})}{(\rho\mathbf{k}|\mathbf{k})}.$$

In this formula lies the content of *Rayleigh's principle* (after J. W. Strutt, Lord Rayleigh [1842–1919], an English applied mathematician), which may be phrased as follows:

Let  $Lu + \lambda\rho u = 0$ ,  $B_1(u) = 0$ ,  $B_2(u) = 0$  be a regular self-adjoint boundary-value problem whose associated linear integral operator  $\mathbf{H} : G \begin{pmatrix} x \\ t \end{pmatrix} \rho^{1/2}(x)\rho^{1/2}(t)$  is non-negative. Then the least characteristic number  $\lambda_1$  of  $\mathbf{H}$  is not greater than the quotient of  $-(\mathbf{Lk}|\mathbf{k})$  by  $(\rho\mathbf{k}|\mathbf{k})$ , where  $\mathbf{k}$  is any vector which is such that  $\mathbf{Lk}$  is continuous and which satisfies the boundary conditions.

*Note.* Rayleigh phrased his principle in terms of the physical concepts of kinetic and potential energy. The numerator of our fraction, namely,  $-(\mathbf{Lk}|\mathbf{k})$  was termed by him a multiple of the *average potential energy*, and the denominator was termed the same multiple of the *average kinetic energy* of the mechanical system whose vibrations are governed by the boundary-value problem. In actual computation it is simpler to start with the explicit formula

$$\lambda_1 \leq \frac{-(\mathbf{Lk}|\mathbf{k})}{(\rho\mathbf{k}|\mathbf{k})}$$

In order to determine a function  $k(x)$  which satisfies the boundary conditions you may set  $\mathbf{k} = \mathbf{Gh}$ , where  $\mathbf{h}$  is any continuous vector; then  $\mathbf{Lk} = -\mathbf{h}$ , and  $k(x)$  may be determined by integrating this equation and determining the constants of integration so that  $\mathbf{k}$  satisfies the boundary conditions.

Rayleigh's principle furnishes a method of approximating the least characteristic number of the non-negative linear integral operator  $\mathbf{H}$ . On setting

$$\lambda_1 = \frac{-(\mathbf{Lk}|\mathbf{k})}{(\rho\mathbf{k}|\mathbf{k})}$$

we obtain an approximation to  $\lambda_1$  which is too large by the amount

$$\sum_2^{\infty} \frac{\bar{v}^p v^p}{\lambda_p} \left(1 - \frac{\lambda_1}{\lambda_p}\right) \div (\rho\mathbf{k}|\mathbf{k}).$$

Here the numbers  $v^p$  are the Fourier

coefficients of  $\mathbf{h}$ , where  $\mathbf{k} = \mathbf{Gh}$ . We can use  $\rho\mathbf{k}$  as a new  $\mathbf{h}$  and thus obtain a new approximation which is better than the original one. To see this set  $\mathbf{k}_1 = \mathbf{Gh}_1 = \mathbf{G}(\rho\mathbf{u}) = \rho^{-1/2}\mathbf{Hv}$ ;  $\mathbf{k}_2 = \mathbf{Gh}_2 = \mathbf{G}(\rho\mathbf{k}_1) = \rho^{-1/2}\mathbf{H}(\rho^{1/2}\mathbf{k}_1) = \rho^{-1/2}\mathbf{H}^2\mathbf{v}$ . Then  $\mathbf{H}^2\mathbf{v}$  has the uniformly convergent Fourier series

$$\mathbf{H}^2\mathbf{v} : \sum_1^{\infty} \frac{v^p}{\lambda_p^2} v_p(x)$$

so that

$$(\mathbf{Hv}|\mathbf{H}^2\mathbf{v}) = \sum_1^{\infty} \frac{\bar{v}^p v^p}{\lambda_p^3} = (\mathbf{h}_2|\mathbf{Gh}_2) = -(\mathbf{Lk}_2|\mathbf{k}_2);$$

$$(\mathbf{H}^2\mathbf{v}|\mathbf{H}^2\mathbf{v}) = \sum_1^{\infty} \frac{\bar{v}^p v^p}{\lambda_p^4} = (\rho\mathbf{k}_2|\mathbf{k}_2).$$

The error in the approximation to  $\lambda_1$  obtained by applying Rayleigh's principle using the function-vector  $\mathbf{k}_2$  is

$$\left\{ \sum_2^{\infty} \frac{\bar{v}^p v^p}{\lambda_p^3} \left( 1 - \frac{\lambda_1}{\lambda_p} \right) \right\} \div \sum_1^{\infty} \frac{\bar{v}^p v^p}{\lambda_p^4}.$$

The error when we used  $\mathbf{k}_1$  was

$$\left\{ \sum_2^{\infty} \frac{\bar{v}^p v^p}{\lambda_p} \left( 1 - \frac{\lambda_1}{\lambda_p} \right) \right\} \div \sum_1^{\infty} \frac{\bar{v}^p v^p}{\lambda_p^2},$$

and on subtracting the former of these expressions from the latter we obtain a positive result. (Show this.)

### Example 1. The uniform vibrating string

The differential equation governing the transverse vibrations of a tightly stretched uniform vibrating string is

$$y_{xx} = \frac{1}{c^2} y u,$$

where  $c = \left(\frac{P}{\rho}\right)^{1/2}$  is a constant. In our first discussion of this problem we separated the variables  $x$  and  $t$  by making the substitution  $y = u e^{\pm i p c t}$ , and we found that the characteristic numbers were  $\lambda_n = p_n^2 = \frac{n^2 \pi^2}{l^2}$ ,  $n = 1, 2, \dots$ ,  $l$  being the length of the string. In order to

be able to compare the solution of the problem of the uniform vibrating string with that of a non-uniform vibrating string we now separate the variables  $x$  and  $t$  by making the substitution  $y = u e^{\pm i k P^{1/2} t}$  (so that  $k = p/\rho^{1/2}$ ), and we write our differential equation in the form  $u_{xx} + \lambda u = 0$ ,  $\lambda = k^2$ , instead of  $u_{xx} + \lambda u = 0$  as in the first discussion. The new characteristic numbers are, then, the quotients of the original characteristic numbers by  $\rho$ :

$$\lambda_n = \frac{n^2 \pi^2}{\rho l^2}; \quad n = 1, 2, \dots$$

The smallest characteristic number is  $\frac{\pi^2}{\rho l^2}$ , and we wish to see how good an approximation to this number Rayleigh's principle furnishes. Starting with  $u(x) = 1$ ,  $h(x) = \rho$ ,  $\mathbf{k}$  is determined by means of the equation  $\mathbf{Lk} = -\rho$ . Since Rayleigh's principle furnishes the desired

approximation to  $\lambda_1$  as the quotient of two homogeneous quadratic forms in  $\mathbf{k}$  we may multiply  $\mathbf{k}$  by any convenient (constant) factor. Setting, then,  $h(x) = 1$  rather than  $h(x) = \rho$  we have  $D^2k = -1$ ,  $k(x) = -\frac{1}{2}x^2 + c_1x + c_2$ . Since  $k(x) = 0$  when  $x = -\frac{l}{2}$  and when  $x = \frac{l}{2}$  we obtain  $k(x) = \frac{1}{2}\left(\frac{l^2}{4} - x^2\right)$ . (Verify this.) Hence  $Dk(x) = -x$  so that  $-(L\mathbf{k}|\mathbf{k}) = -(D^2\mathbf{k}|\mathbf{k}) = \int_{-l/2}^{l/2} (Dk)^2 dx = \frac{l^3}{12}$ . Also  $\int_{-l/2}^{l/2} \rho k^2 dx = \frac{\rho l^5}{120}$ . (Check this.) Thus the approximation to  $\lambda_1 = \frac{\pi^2}{\rho l^2}$  furnished by Rayleigh's principle (using  $h = 1$ ) is  $\frac{10}{\rho l^2}$ . Since  $\pi^2 = 9.8696$  the approximation is extraordinarily good (the error being less than  $1\frac{1}{2}\%$ ).

To obtain a better approximation we set  $\mathbf{h}_2: l^2 - 4x^2$  so that  $D^2\mathbf{k}_2: 4x^2 - l^2$ ;  $k_2(x) = \frac{x^4}{3} - \frac{l^2x^2}{2} + c_1x + c_2$ . Since  $k_2(x) = 0$  at  $x = \pm \frac{l}{2}$ ,  $c_1 = 0$ , and we obtain  $k_2(x) = \frac{x^4}{3} - \frac{l^2x^2}{2} + \frac{5l^4}{48}$ . Hence  $D\mathbf{k}_2: \frac{4}{3}x^3 - l^2x$ ,  $\int_{-l/2}^{l/2} (Dk_2)^2 dx = \frac{17l^7}{315}$ . (Check this.)  $\int_{-l/2}^{l/2} \rho k_2^2 dx = \frac{31\rho l^9}{70(9^2)}$ . (Check this.) Hence the approximation to  $\lambda_1 = \frac{\pi^2}{\rho l^2}$  furnished by Rayleigh's principle, using  $\mathbf{h}: l^2 - 4x^2$ , is  $\frac{306}{31\rho l^2} = \frac{9.871}{\rho l^2}$ . This is in excess of the true value by less than one fiftieth of one per cent.

**Example 2. The vibrating beam**

The differential equation governing the transverse vibrations of a beam is

$$D^4y + \frac{1}{c^2} yu = 0; \quad c^2 = \frac{EI}{\rho}$$

where  $D$  denotes differentiation with respect to  $x$ ,  $E$  is Young's modulus,  $I$  is the moment of inertia of a cross section of the beam about its central axis, and  $\rho$  is the volume density. On setting  $y = ue^{\pm i\nu(EI)^{1/4}x}$  (on the assumption that  $EI$  is a constant) we obtain the equation

$$Lu + \lambda \rho u = 0; \quad \lambda = p^2; \quad L = -D^4.$$

The kinetic energy of the vibrating beam is

$$T = \frac{A}{2} \int \rho y_t^2 dx,$$

where  $A$  is the cross-sectional area (supposed constant) of the beam. Thus  $T_t = A \int \rho y_t y_{tt} dx = -AEI \int y_t D^4 y dx$ . On integrating by parts and assuming that either  $y_t$  or  $D^3 y$  is zero at each end of the beam we obtain  $T_t = AEI \int y_{tx} D^3 y dx$ . Again integrating by parts and assuming that either  $y_{tx}$  or  $D^2 y$  is zero at each end of the beam we obtain

$$T_t = -AEI \int y_{xxt} D^2 y dx.$$

If, then, we set

$$V = \frac{1}{2} AEI \int (D^2 y)^2 dx$$

so that

$$V_t = AEI \int (D^2 y) y_{xxt} dx$$

we see that  $T + V = \text{constant}$  so that we term  $V$  the *potential energy* of the beam (the additive constant being fixed by the fact that  $V$  is zero when the beam is in its equilibrium position, in which  $D^2 y \equiv 0$ ). An integration by parts, coupled with the assumption that either  $Dy$  or  $D^2 y$  is zero at each end of the beam, shows that

$$V = -\frac{1}{2} AEI \int Dy D^3 y dx.$$

A second integration by parts, coupled with the assumption that either  $y$  or  $D^3 y$  is zero at each end of the beam, yields

$$V = \frac{1}{2} AEI \int y D^4 y dx.$$

Thus we have the following result:

*The kinetic energy of the vibrating beam is  $T = \frac{1}{2} A \int \rho y_t^2 dx$ , and the potential energy of the vibrating beam is  $\frac{AEI}{2} \int (D^2 y)^2 dx = \frac{AEI}{2} \int y D^4 y dx$ .*

The same argument as that for the vibrating string (repeat this argument) shows that when the beam is executing a *normal vibration*  $y = ue^{\pm ip(El)^{1/2}t} = u(\alpha e^{ip(El)^{1/2}t} + \bar{\alpha} e^{-ip(El)^{1/2}t})$  the average value of the kinetic energy over a cycle is  $\alpha\bar{\alpha}AEIp^2 \int \rho u^2 dx$  and that the average value of the potential energy over a cycle is  $\alpha\bar{\alpha}AEI \int (D^2u)^2 dx$ . A repeated integration by parts (coupled with the assumptions already made concerning the values of  $u$ ,  $Du$ ,  $D^2u$ , and  $D^3u$  at the ends of the beam) shows that the average value of the potential energy of the beam over a cycle (of a normal vibration) is  $\alpha\bar{\alpha}AEI \int u D^4u dx$ . In view of the fact that  $D^4u = \lambda\rho u = p^2\rho u$  it follows that Rayleigh's principle is valid for the vibrating beam:

*In a normal vibration the time averages (over a cycle) of the kinetic and potential energies of the vibrating beam are equal.*

An equivalent form of statement of this result is as follows:

**In any normal vibration of the beam, associated with a characteristic number  $\lambda$ , we have**

$$\lambda = \frac{-(u|Lu)}{(u|\rho u)}.$$

Since  $-(u|Lu) = (u|D^4u) = (D^2u|D^2u)$  the characteristic numbers of the boundary-value problem of the vibrating beam are all positive. Hence the linear integral operator  $H$  of the associated integral equation is non-negative and so the method of approximation to the least characteristic number which has been explained in detail for the vibrating string is applicable without any change to the vibrating beam.

We treat by this method the case of a uniform beam which is "built-in" at one end and free at the other. Placing our origin at the built-in end the boundary conditions are  $u(0) = 0$ ,  $Du(0) = 0$ ,  $D^2u(l) = 0$ ,  $D^3u(l) = 0$ . The equation which determines the characteristic numbers is easily determined. On setting  $\lambda\rho = n^4$  we have to solve the equation  $D^4u = n^4u$  and adjust the solution to the boundary conditions. The general solution of the differential equation is  $u = c_1 \cosh nx + c_2 \sinh nx + c_3 \cos nx + c_4 \sin nx$ , and the boundary conditions at  $x = 0$  yield  $c_1 + c_3 = 0$ ,  $c_2 + c_4 = 0$  (since  $n \neq 0$ ). (Verify that  $\lambda = 0$  is not a characteristic number.) Hence  $u = c_1(\cosh nx - \cos nx) + c_2(\sinh nx - \sin nx)$ . The boundary conditions at  $x = l$  yield the two equations



$$(\cosh nl + \cos nl)c_1 + (\sinh nl + \sin nl)c_2 = 0;$$

$$(\sinh nl - \sin nl)c_1 + (\cosh nl + \cos nl)c_2 = 0.$$

The determinant of these two equations must be zero (why?) and so  $n$  must be a zero of  $\cosh nl \cos nl + 1$ . The first four positive zeros of the function  $\cosh \alpha \cos \alpha + 1$  are  $\alpha_1 = 1.8751$ ,  $\alpha_2 = 4.6941$ ,  $\alpha_3 = 7.8548$ ,  $\alpha_4 = 10.9955$  (the remaining zeros being furnished to a high degree of accuracy by the formula  $\alpha_n = (2n - 1)\frac{\pi}{2}$ ). Hence the smallest characteristic number of our boundary-value problem is

$$\lambda_1 = \frac{n_1^4}{\rho} = \frac{\alpha_1^4}{\rho l^4} = \frac{12.362}{\rho l^4}.$$

We obtain the first approximation to this, by means of Rayleigh's principle, by setting  $\mathbf{k} = \mathbf{Gh}$ , where  $h(x) = 1$ . Then  $D^4k(x) = 1$ ,

$k(x) = \frac{x^4}{24} + c_1 + c_2x + c_3x^2 + c_4x^3$ . The boundary conditions at

$x = 0$  yield  $c_1 = 0$ ,  $c_2 = 0$ . The boundary conditions at  $x = l$  yield

$c_3 = \frac{l^2}{4}$ ,  $c_4 = -\frac{l}{6}$  (check this) so that  $k(x) = \frac{l^2x^2}{4} - \frac{lx^3}{6} + \frac{x^4}{24}$ . Hence

$-(\mathbf{k}|\mathbf{Lk}) = (\mathbf{k}|D^4\mathbf{k}) = \frac{l^5}{20}$ ;  $(\rho\mathbf{k}|\mathbf{k}) = \rho \int_0^l k^2 dx = \frac{13}{40(9^2)} \rho l^9$ . Hence the

approximation furnished by Rayleigh's principle to  $\lambda_1 = \frac{12.362}{\rho l^4}$  is

$\frac{162}{13\rho l^4} = \frac{12.462}{\rho l^4}$  which is in excess by less than 1%.

# 9

## THE CALCULUS OF VARIATIONS

### 1. The variation of a line integral

If  $y = y(x)$  is a differentiable function of a single independent variable  $x$ , any value of  $x$  at which  $dy = y_x dx$  is zero is termed a *stationary point* of  $y = y(x)$ . Here the stationary point (i.e., the value of the independent variable  $x$  at which the dependent variable  $y$  is stationary) is a number. In the *calculus of variations* we seek for stationary points of a dependent variable or function, but the independent variable is no longer a *collection of numbers*; it is, rather, a collection of geometrical entities, e.g., a collection of curves. Attached to each "value" of the independent variable (for example, to each curve of the collection) is a number. Thus the *dependent* variable is still, as it was in elementary differential calculus, a collection of numbers. We shall confine our attention at first to the case where the independent variable is a collection of curves and where the dependent variable is obtained by integrating along the curve a given function of *position* and of *direction* (the direction at any point of the curve along which the given function is being integrated being furnished by the tangent vector to the curve); we suppose, then, that each curve of the family of curves which constitutes the independent variable is *smooth*, i.e., that each of the coordinates of any point  $P$  of the curve possesses a continuous derivative with respect to the independent variable, or parameter, which names, or identifies, each point of the curve.

We denote the coordinates of  $P$  by  $(x^1, x^2, \dots, x^n)$  so that our curves, each of which is a "value" of the independent variable, lie in a space of  $n$  dimensions. We denote the independent variable along any curve of our family by  $\tau$ , and we suppose that the various members of our family of curves are identified by the values of an accessory variable  $\alpha$ . In other words the equations of the curve  $C_\alpha$  of our family are of the form

$$x^j = x^j(\tau, \alpha); \quad j = 1, \dots, n.$$

As  $\tau$  varies,  $\alpha$  remaining fixed, the point  $P: (x^1, \dots, x^n)$  traces out the curve  $C_\alpha$ ; when  $\alpha$  is changed we obtain a new member of the family of curves. We denote the initial point of  $C_\alpha$  by  $P_0$  and the final point by  $P_1$  so that  $P_0: (x_0^1, \dots, x_0^n)$  and  $P_1: (x_1^1, \dots, x_1^n)$ , where

$$x_0^j = x^j(\tau_0, \alpha); \quad x_1^j = x^j(\tau_1, \alpha); \quad j = 1, \dots, n.$$

The variables  $\tau$  and  $\alpha$  are taken to be independent (since  $\tau$  must be capable of varying when  $\alpha$  is held constant), but this does not prevent

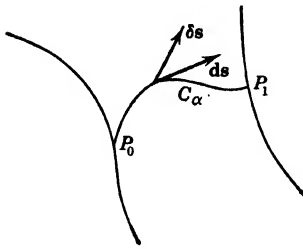


FIG. 41.

the possibility that  $\tau_0$  and  $\tau_1$  (the initial and final values of the parameter  $\tau$ ) may vary with  $\alpha$ . We assume that the functions  $x^j = x^j(\tau, \alpha)$ ,  $j = 1, \dots, n$ , not only possess continuous first derivatives with respect to  $\tau$  and  $\alpha$  but that the second mixed derivatives  $x_{\tau\alpha}$ ,  $x_{\alpha\tau}$  exist and are equal (continuity of these derivatives is sufficient to ensure this). Each of the functions  $x^j(\tau, \alpha)$  possesses two partial differentials:

1.  $dx^j = (x^j)_\tau d\tau$ ; the vector  $ds = v(dx^1, \dots, dx^n)$  is the vector element of arc of the curve  $C_\alpha$  obtained by holding  $\alpha$  fixed in the equations  $x^j = x^j(\tau, \alpha)$ .

2.  $\delta x^j = (x^j)_\alpha d\alpha$ ; the vector  $\delta s = v(\delta x^1, \dots, \delta x^n)$  is the vector element of arc of the curve obtained by holding  $\tau$  fixed (and regarding  $\alpha$  as variable) in the equations  $x^j = x^j(\tau, \alpha)$ ,  $j = 1, \dots, n$ .

*Note.* In the formulas furnishing  $dx^j$  and  $\delta x^j$ ,  $d\tau$  and  $d\alpha$  are arbitrary numbers (since they are differentials of independent variables).

The coordinates  $\delta x^j$ ,  $j = 1, \dots, n$ , of  $\delta s$  are known as the *variations* of the coordinates  $x^j$ ,  $j = 1, \dots, n$ , of  $P$ . In general if  $f(\tau, \alpha)$  is any function of  $\tau$  and  $\alpha$  we denote by  $df$  the partial differential of  $f$  with respect to  $\tau$ :

$$df = f_\tau d\tau$$

and by  $\delta f$  the partial differential of  $f$  with respect to  $\alpha$ , i.e., the *variation* of  $f$ :

$$\delta f = f_\alpha d\alpha.$$

Since the coordinates of the end-points  $P_0$  and  $P_1$  of  $C_\alpha$  may involve  $\alpha$ , not only explicitly but also implicitly through the initial and final values  $\tau_0$  and  $\tau_1$  of the parameter  $\tau$ , we have

$$\left. \begin{aligned} \delta(x_0^j) &= x_\tau^j(\tau_0, \alpha) \delta\tau_0 + x_\alpha^j(\tau_0, \alpha) d\alpha \\ \delta(x_1^j) &= x_\tau^j(\tau_1, \alpha) \delta\tau_1 + x_\alpha^j(\tau_1, \alpha) d\alpha \end{aligned} \right\} j = 1, \dots, n.$$

Since  $(\delta x^j)_{\tau=\tau_0} = x_\alpha^j(\tau_0, \alpha) d\alpha$  it follows that  $\delta(x_0^j)$  is not, in general, the same as  $(\delta x^j)_{\tau=\tau_0}$  and, similarly, that  $\delta(x_1^j)$  is not, in general, the same as  $(\delta x^j)_{\tau=\tau_1}$ . In other words

*The order of substitution of  $\tau_0$  or  $\tau_1$  for  $\tau$  and of variation (i.e., of differentiation with respect to  $\alpha$ ) cannot be interchanged.*

*Note.* The equations  $x_0^j = x^j(\tau_0, \alpha)$ ,  $j = 1, \dots, n$ , are those of the curve traced out by the end-points  $P_0$  of the curves  $C_\alpha$  of the family of curves which constitutes our independent variable. The vector  $\delta\mathbf{s}_0 = v(\delta(x_0^1), \dots, \delta(x_0^n))$  is the vector element of arc of this curve. Similarly the vector  $\delta\mathbf{s}_1 = v(\delta(x_1^1), \dots, \delta(x_1^n))$  is the vector element of arc of the curve traced out by the end-points  $P_1$  of the various curves  $C_\alpha$ . If  $P_0$  is a fixed point  $\delta\mathbf{s}_0$  is the zero vector, and if  $P_1$  is a fixed point  $\delta\mathbf{s}_1$  is the zero vector.

The dependent variable which we wish to consider is of the form

$$I(\alpha) = \int_{\tau_0}^{\tau_1} F(x, x_\tau) d\tau,$$

where we denote by  $F(x, x_\tau)$  a given function of the  $n$  coordinates  $(x^1, \dots, x^n)$  of  $P$  and of the  $n$  derivatives  $(x_\tau^1, \dots, x_\tau^n)$  of these  $n$  coordinates with respect to  $\tau$ ; we take it as granted that  $F(x, x_\tau)$  possesses continuous derivatives with respect to each of the  $2n$  variables  $x$  and  $x_\tau$ . We require that  $I(\alpha)$  be independent of the particular parameter  $\tau$  chosen to represent the points  $P:(x^1, \dots, x^n)$  of the curve  $C_\alpha$  along which  $F(x, x_\tau)$  is being integrated. The only changes of parameter which we permit are those of the form

$$\tau = \tau(u),$$

where  $\tau_u$  is continuous and non-negative over  $[u_0, u_1]$  where  $\tau_0 = \tau(u_0)$ ,  $\tau_1 = \tau(u_1)$ . When such a change of parameter is made our integral  $I(\alpha)$  takes the form

$$I(\alpha) = \int_{u_0}^{u_1} F(x, x_\tau) \tau_u du,$$

and our requirement that  $I(\alpha)$  be the same when the parameter  $u$  is used as when the parameter  $\tau$  is used implies that

$$\int_{u_0}^{u_1} F(x, x_u) du = \int_{u_0}^{u_1} F(x, x_\tau) \tau_u du.$$

This equality must hold for all intervals  $[u_0, u_1]$  and so, since each of the two integrands is continuous,

$$F(x, x_u) = F(x, x_\tau)\tau_u$$

or, equivalently,

$$F(x, kx_\tau) = kF(x, x_\tau); \quad k = \tau_u > 0.$$

In other words

**When each of the  $n$  derivatives  $x_\tau$  is multiplied by one and the same positive number  $k$  the function  $F$  is multiplied by  $k$ .**

We express this requirement which must be satisfied by  $F(x, x_\tau)$  by saying that  $F(x, x_\tau)$  must be *positively homogeneous* in the  $n$  variables  $x^1, \dots, x^n$ .

*Note.* If we are given an integral of the form

$$I(\alpha) = \int_{t_0}^{t_1} L(x, x_t) dt,$$

where  $L$  is not positively homogeneous of degree one in the  $n$  variables  $x^1, \dots, x^n$ , we may bring it under the scope of our discussion by regarding  $t$ , not as a mere naming parameter of points on the curve  $C_\alpha$ , but as one of the *coordinates* of the point  $P$ . Thus the curve  $C_\alpha$  is now regarded as a curve in a space of  $n + 1$ , rather than  $n$ , dimensions. On writing, as before,  $t = t(\tau)$ , where  $t_\tau > 0$ ,  $I(\alpha)$  takes the form

$$\int_{t_0}^{t_1} F(x, x_\tau) d\tau,$$

where, now,  $x$  stands for the  $n + 1$  coordinates  $(x^1, \dots, x^n, t)$  and  $F(x, x_\tau) = L(x, x_t)t_\tau = L\left(x, \frac{x_\tau}{t_\tau}\right)t_\tau$ . Since  $L\left(x, \frac{x_\tau}{t_\tau}\right)$  is homogeneous of degree zero (why?) in the  $n + 1$  variables  $(x^1, \dots, x^n, t_\tau)$  it is clear that  $F$  is homogeneous (and, hence, positively homogeneous) of degree one in these  $n + 1$  variables. We shall suppose, from now on, that this change of integrand from  $L(x, x_t)$  to  $F(x, x_\tau) = L(x, x_t)t_\tau$  is made (at the expense of increasing by one the dimension of the space in which our curves  $C_\alpha$ , which constitute our independent variable, lie) so that our integrand  $F(x, x_\tau)$  is always positively homogeneous of degree one in the variables  $x_\tau$ .

Our problem is the determination of the curve  $C_\alpha$  which is such that the integral  $I(\alpha)$  is stationary. In other words  $\delta I$  must be zero for arbitrary values of  $\delta x^j, j = 1, \dots, n$ , along the curve  $C_\alpha$ . Be sure that you understand precisely what is meant by this and that you

appreciate how strict a requirement it is upon the curve  $C_\alpha$ . Once the family of curves  $C_\alpha$ :

$$C_\alpha: x^j = x^j(\tau, \alpha); \quad j = 1, \dots, n$$

has been chosen  $\delta x^j = x^j_\alpha d\alpha$  is determined along  $C_\alpha$ . We require that no matter what is the family of curves into which the particular curve we are trying to track down is "imbedded" the variation  $\delta I$  of  $I(\alpha)$  will be zero. In order to find what conditions are imposed upon the curve we are seeking by this requirement we must first calculate  $\delta I$  for any curve of any family which contains the curve. Since  $\alpha$  enters  $I(\alpha)$  in three ways:

1. through the upper limit  $\tau_1 = \tau_1(\alpha)$  of the integral which defines  $I(\alpha)$ ;

2. through the lower limit  $\tau_0 = \tau_0(\alpha)$  of this integral; and

3. through the integrand  $F(x, x_\tau)$  of this integral,

$\delta I$  will be the sum of three parts:

1.  $F_1 \delta \tau_1$ , where  $F_1$  denotes the value of the integrand  $F(x, x_\tau)$  when  $\tau = \tau_1$ ;

2.  $-F_0 \delta \tau_0$ , where  $F_0$  denotes the value of this integrand when  $\tau = \tau_0$ ; and

3.  $\int_{\tau_0}^{\tau_1} \delta F d\tau$ . Here  $\delta F$  must be obtained by the Rule of Composite

Differentiation since  $F(x, x_\tau)$  involves  $\alpha$  through the  $n$  variables  $x^j(\tau, \alpha)$  and the  $n$  derivatives  $x^j_\tau(\tau, \alpha)$ ,  $j = 1, \dots, n$ , of these variables. In applying this Rule of Composite Differentiation we regard the  $2n$  variables  $x^j, x^j_\tau$ ,  $j = 1, \dots, n$ , as independent variables.

*Warning.* Do not have a confused feeling that since both  $x^j(\tau, \alpha)$  and  $x^j_\tau(\tau, \alpha)$ ,  $j = 1, \dots, n$ , are functions of only two independent variables they *must* be, in some way, dependent. The Rule of Composite Differentiation brutally ignores this; it says: Proceed *as if* the  $2n$  variables  $x^j$  and  $x^j_\tau$  are independent, and you will get the correct result.

It is convenient to condense our notation, and this can be done without sacrifice of clarity. According to the Rule of Composite Differentiation we have

$$\delta F = \sum_{j=1}^n F_{x^j} \delta x^j + \sum_{j=1}^n F_{x^j_\tau} \delta x^j_\tau.$$

We shall omit the summation signs and the label  $j$  attached to  $x$  and  $x_\tau$ ; thus we write simply

$$\delta F = F_x \delta x + F_{x_r} \delta x_r.$$

Since the symbol  $x$  must carry a label before it can be evaluated (for you must know which coordinate, of the point  $P$ , you are talking about) the omission of the label is a clear warning. This omission is equivalent to the instruction:

*Insert every label from 1 to  $n$ , and add the results obtained.*

*Note.* This instruction applies to terms such as  $F_x \delta x$  and  $F_{x_r} \delta x_r$  where the label is omitted in each of two factors occurring in the term.

Thus  $F_x \delta x$  is  $\sum_{j=1}^n F_{x_j} \delta x^j$  and  $F_{x_r} \delta x_r$  is  $\sum_{j=1}^n F_{x_{j_r}} \delta x^j_r$ . Whenever a symbol such as  $F_{x_r}$ , in which the omitted label is only omitted once, appears we understand that this implies the whole set of  $n$  numbers ( $F_{x_{1_r}}, \dots, F_{x_{n_r}}$ ). Thus we shall very shortly have to consider the *vector*

$$p: F_{x_r}.$$

We understand by this equation of definition that the  $n$  numbers  $F_{x_{j_r}}$ ,  $j = 1, \dots, n$ , are the coordinates of a vector  $v(p_1, \dots, p_n)$ , where

$$p_j = F_{x_{j_r}}; \quad j = 1, \dots, n.$$

The general principle for translating our *stenographic notation* may be stated briefly as follows: If an omitted label occurs twice in a term insert the label and sum over all values of the label from 1 to  $n$ ; if an omitted label occurs once in a term insert the label and evaluate, but do *not* add.

In evaluating the three contributions to  $\delta I$  we start with 3:

$$\int_{\tau_0}^{\tau_1} \delta F \, d\tau = \int_{\tau_0}^{\tau_1} (F_x \delta x + F_{x_r} \delta x_r) \, d\tau.$$

Since  $\delta x_r = x_{r\alpha} d\alpha = x_{\alpha r} d\alpha$  (why?) =  $(\delta x)_r$  (since  $d\alpha$  is independent of  $\tau$ ) the term  $\int_{\tau_0}^{\tau_1} (F_{x_r} \delta x_r) \, d\tau$  may be integrated by parts. On introducing the notation

$$p = F_{x_r}$$

we obtain  $\int_{\tau_0}^{\tau_1} (p \delta x_r) \, d\tau = \int_{\tau_0}^{\tau_1} p (\delta x)_r \, d\tau = p \delta x \Big|_{\tau_0}^{\tau_1} - \int_{\tau_0}^{\tau_1} p_r \delta x \, d\tau$ . Hence the contribution 3 to  $\delta I$  is

$$p \delta x \Big|_{\tau_0}^{\tau_1} + \int_{\tau_0}^{\tau_1} (F_x - p_r) \delta x \, d\tau.$$

On adding to this the contributions 1 and 2 to  $\delta I$  we obtain

$$\delta I = \{F_1 \delta \tau_1 + (p \delta x)_{\tau=\tau_1}\} - \{F_0 \delta \tau_0 + (p \delta x)_{\tau=\tau_0}\} + \int_{\tau_0}^{\tau_1} (F_x - p_r) \delta x \, d\tau.$$

Since  $F$  is homogeneous of degree one in the  $n$  variables  $x_r$ , we have

$$F = \sum_{j=1}^n F_{x_j} x_j = p x_r,$$

and since

$$\delta(x_1) = (x_r)_1 \delta \tau_1 + (\delta x)_{\tau=\tau_1}$$

we have

$$F_1 \delta \tau_1 + (p \delta x)_{\tau=\tau_1} = p_1 \{(x_r)_1 \delta \tau_1 + (\delta x)_{\tau=\tau_1}\} = p_1 \delta(x_1),$$

and, similarly,

$$F_0 \delta \tau_0 + (p \delta x)_{\tau=\tau_0} = p_0 \delta(x_0).$$

Hence

$$\delta I = p_1 \delta(x_1) - p_0 \delta(x_0) + \int_{\tau_0}^{\tau_1} (F_x - p_r) \delta x \, d\tau.$$

### EXERCISES

1. Show that  $F(x, x_r) = \{(x_r)^2 + (y_r)^2 + (z_r)^2\}^{1/2}$  is positively homogeneous of degree one in  $x_r = (x_r, y_r, z_r)$ . Show, further, that it is not homogeneous of degree one in the variables  $(x_r, y_r, z_r)$ . Calculate the vector  $\mathbf{p} = v(p_x, p_y, p_z) = v(F_{x_r}, F_{y_r}, F_{z_r})$ , and verify that  $F = p x_r = p_x x_r + p_y y_r + p_z z_r$ .

2. Show that the coordinates of the vector  $\mathbf{p}$  of Exercise 1 are subject to the relation  $\phi = p_x^2 + p_y^2 + p_z^2 - 1 = 0$ .

3. Denoting the (three) dependent variables  $(x^1, x^2, x^3)$  by  $(r, \theta, \phi)$  repeat Exercise 1 for the function

$$F = \{(r_r)^2 + r^2(\theta_r)^2 + r^2 \sin^2 \theta (\phi_r)^2\}^{1/2}.$$

$$\text{Answer. } p_r = \frac{r_r}{F}; p_\theta = \frac{r^2 \theta_r}{F}; p_\phi = \frac{r^2 \sin^2 \theta \phi_r}{F}.$$

4. Show that the coordinates of the vector  $\mathbf{p}$  of Exercise 3 are subject to the relation  $\phi(x, p) = p_r^2 + \frac{1}{r^2} p_\theta^2 + \frac{1}{r^2 \sin^2 \theta} p_\phi^2 - 1 = 0$ .

5. Show that each coordinate  $F_{x_r}$  of the vector  $\mathbf{p}$  is positively homogeneous of degree zero in the  $n$  variables  $x$ . *Hint.* Differentiate with respect to  $x_r$  the relation

$$F(x, kx_r) = kF(x, x_r).$$

6. Deduce from the result of Exercise 5 that each coordinate of  $\mathbf{p}$ :  $F_{x_r}$  is a function of the  $n-1$  ratios  $x^1_r : \dots : x^n_r$  (it being understood that not all the  $n$  derivatives  $x_r$  are zero). *Hint.* If  $x^i_r \neq 0$  set  $k = 1/|x^i_r|$  in the relation  $p(x, kx_r) = p(x, x_r)$ .



## 2. The Euler-Lagrange equations

Let  $C$  be a curve which is such that  $\delta I = 0$ , no matter what family of curves  $C_\alpha$  containing  $C$  is used to define  $\delta I$ , provided only that the end points  $P_0$  and  $P_1$  are fixed, i.e., the same for every curve of the family of curves  $C_\alpha$ . We term  $C$  an *extremal* of the calculus of variations problem. Since  $\delta(x_0) = 0$ ,  $\delta(x_1) = 0$

$$\delta I = \int_{\tau_0}^{\tau_1} \{F_x - p_r\} \delta x \, d\tau.$$

In order to determine the extremals  $C$  we impose conditions on the integrand function  $F(x, x_r)$  which are strong enough to ensure that  $F_x$  and  $p_r$  are continuous along  $C$ . Since  $p_r = F_{x_r x} x_r + F_{x_r x_r} x_{r\tau}$  we suppose that  $F$  possesses continuous second derivatives with respect to the  $2n$  variables  $x, x_r$  and that, along  $C$ , the functions  $x = x(\tau, \alpha)$  possess continuous second derivatives with respect to  $\tau$ . The fact that  $F_x - p_r$  is continuous along  $C$  makes it certain that if a single one of the  $n$  expressions  $F_{x_i} - (p_i)_r$ ,  $j = 1, \dots, n$ , is different from zero at a single point  $\tau = \xi$ , say, of  $C$  then  $C$  is *not* an extremal. In fact  $F_{x_i} - (p_i)_r$ , being continuous, has the same sign over an interval  $[\xi - \delta, \xi + \delta]$  centered at  $\xi$ , and we may set  $\delta x^k = 0$ ,  $k \neq j$ ,  $\delta x^j = 0$  at any point  $\tau$  not covered by  $(\xi - \delta, \xi + \delta)$  while  $\delta x_j > 0$  over the open interval  $(\xi - \delta, \xi + \delta)$ . For any family  $C_\alpha$  for which the coordinates of  $\delta s = v(\delta x^1, \dots, \delta x^n)$  have these values along  $C$

$$\delta I = \int_{-\delta}^{\xi+\delta} \{F_{x_j} - (p_j)_r\} \delta x^j \, d\tau \neq 0 \text{ (why?)}$$

so that  $C$  is not an extremal. Thus the following  $n$  equations

$$F_x - p_r = 0$$

must be satisfied along  $C$  if  $C$  is to be an extremal. It is clear that if these equations are satisfied  $C$  is an extremal since the equations ( $3n$  in all)

$$\delta(x_0) = 0; \quad \delta(x_1) = 0; \quad F_x - p_r = 0$$

ensure that  $\delta I = 0$ . Thus we have the following fundamental result:

**The curve  $C$  is an extremal if, and only if, along  $C$**

$$F_x - p_r = 0.$$

*Note.* In these equations  $x$  may be regarded as a function of the single variable  $\tau$  rather than as a function of the two independent

variables  $\tau$  and  $\alpha$ , for the particular value of  $\alpha$  which picks out  $\mathcal{C}$  may be substituted for  $\alpha$  before, rather than after, the differentiations with respect to  $x$  and  $\tau$  (remember that the variables  $\tau$  and  $\alpha$  are independent).

The  $n$  equations

$$F_x - p_x = 0$$

are known as the *Euler-Lagrange* equations (after L. Euler [1707–1783], a Swiss mathematician, and J. L. Lagrange [1736–1813], a French mathematician). Since  $p = F_{x\tau}$ ,  $(p_j)_\tau = F_{x_i\tau x_i\tau} + F_{x_i\tau x_i\tau\tau}$  so that the Euler-Lagrange equations are, in general, differential equations of the second order for the unknown functions  $x$  of the single independent variable  $\tau$ . However, when  $F$  is a linear function of the variables  $x_\tau$  so that  $F_{x_\tau x_\tau}$  is the zero  $n \times n$  matrix, the Euler-Lagrange equations reduce to first-order differential equations (unless  $F$  is independent of the variables  $x$ , in which case every curve is an extremal, the Euler-Lagrange equations being then trivial since  $F_x$  is the zero  $n \times 1$  matrix and  $F_{x_\tau x}$  is the zero  $n \times n$  matrix).

### EXERCISES

1. Show that if  $F$  is a constant function of  $x^j$  then  $p_j$  is constant along an extremal.

2. Show that  $(F_x - p_x)x_\tau = 0$ , and deduce that the  $n$  Euler-Lagrange equations are not independent. *Hint.*  $F_\tau = F_{x_\tau x_\tau} + F_{x_\tau x_\tau\tau}$ , and, since  $F = px_\tau$ ,  $F_\tau = p_x x_\tau + p x_{\tau\tau}$ . Since  $F_{x_\tau} = p$  we have  $F_{x_\tau x_\tau} = p_x x_\tau$ .

3. Show that if  $F = \{(x_\tau)^2 + (y_\tau)^2 + (z_\tau)^2\}^{1/2}$  then  $\left(\frac{x_\tau}{F}, \frac{y_\tau}{F}, \frac{z_\tau}{F}\right)$  are constant along an extremal. Deduce that  $dx:dy:dz$  are constant along an extremal so that the extremals are straight lines. *Note.* Since  $\int_{r_0}^{r_1} F d\tau$  is the arc-length integral this exercise shows that curves of stationary length are straight lines.

When the variables  $x$  are not independent but are connected by one or more relations of the form  $\phi(x) = 0$ ,  $\phi$  being a differentiable function of the  $n$  variables  $x$ , the Euler-Lagrange equations take a slightly different form. We must now have

$$(F_x - p_x)\delta x = 0,$$

where the vector  $\delta s = v(\delta x^1, \dots, \delta x^n)$  is any vector tangent to the spread  $\phi(x) = 0$ ; in other words  $(F_x - p_x)\delta x = 0$  for all vectors  $\delta s$  which satisfy the relation  $\phi_x \delta x = 0$ . If, then,  $\lambda$  is an undetermined multiplier we must have, for all such vectors  $\delta s$ ,

$$(F_x - p_r + \lambda \phi_x) \delta x = 0.$$

Choosing  $\lambda$  so that the coefficient of one of the coordinates of  $\delta s$  is zero the coefficients of the remaining coordinates must also be zero since these coordinates may be arbitrarily chosen. If there is more than one relation  $\phi = 0$  we choose the corresponding  $\lambda$ 's so that the coefficients of those coordinates of  $\delta s$  which cannot be assigned arbitrary values are all zero. It follows that the coefficients of all the coordinates of  $\delta s$  are zero; hence, when there is only one relation  $\phi = 0$ ,

$$p_r = F_x + \lambda \phi_x.$$

If the coordinates  $x$  are subject to two constraints  $\phi = 0$ ,  $\psi = 0$  we have

$$p_r = F_x + \lambda \phi_x + \mu \psi_x,$$

and so on. These equations are known as the Euler-Lagrange equations for *constrained extremals*, and the unknown multipliers  $\lambda$ ,  $\mu$ ,  $\dots$  are known as *Lagrange's undetermined multipliers*.

**Example.** Determine the extremal curves for the arc-length integral on the sphere  $x^2 + y^2 + z^2 - a^2 = 0$

Here  $F = \{(x_\tau)^2 + (y_\tau)^2 + (z_\tau)^2\}^{1/2}$ . Choosing as our parameter  $\tau$  the arc length  $s$  along the extremal we have  $F = 1$  along the extremal and

$$p_x = x_s; \quad p_y = y_s; \quad p_z = z_s.$$

Since  $F$  is independent of  $x$ ,  $y$ , and  $z$  the Euler-Lagrange equations are

$$p_s = \lambda \phi_x;$$

i.e.,

$$x_{ss} = 2\lambda x; \quad y_{ss} = 2\lambda y; \quad z_{ss} = 2\lambda z.$$

On eliminating the undetermined multiplier  $\lambda$  we obtain

$$yz_{ss} - zy_{ss} = 0; \quad zx_{ss} - xz_{ss} = 0; \quad xy_{ss} - yx_{ss} = 0$$

so that

$$yz_s - zy_s = c_1; \quad zx_s - xz_s = c_2; \quad xy_s - yx_s = c_3,$$

where  $c_1$ ,  $c_2$ , and  $c_3$  are constants of integration. Hence  $c_1x + c_2y + c_3z = 0$  so that the extremal is part of the intersection of the plane  $c_1x + c_2y + c_3z = 0$  with the sphere  $x^2 + y^2 + z^2 - a^2 = 0$ ; in other words it is part of an arc of a great circle of the sphere.

### 3. The Hamiltonian canonical equations

The coordinates of the vector  $\mathbf{p}: F_{x_r}$  are homogeneous functions of degree zero of the  $n$  variables  $x_r$ . Hence (see Exercise 6, p. 311) the

$n$  functions  $p_j$ ,  $j = 1, \dots, n$ , are functions of the  $n - 1$  ratios  $x^1 : x^2 : \dots : x^n$ . There is, then, a relation

$$\phi(x, p) = 0$$

connecting the coordinates of the vector  $\mathbf{p}$ . This relation is an identity in the  $2n$  variables  $x, x_r$ , and, on differentiating it with respect to these variables, we obtain the  $2n$  equations

$$\phi_x + \phi_p p_x = 0; \quad \phi_p p_{x_r} = 0.$$

Since the coordinates of  $\mathbf{p}$  are homogeneous functions of degree zero of the  $n$  variables  $x_r$  we have the  $n$  equations

$$(p_j)_{x_r} x_r = 0$$

or, equivalently,

$$F_{x^i x^r} x_r = 0.$$

Hence the  $n \times n$  matrix

$$M = (F_{x^i x^r})$$

is singular. We assume that it is of rank  $n - 1$ , i.e., that at least one of the elements of its cofactor matrix is different from zero. If, then,  $\mathbf{v} = v(v^1, \dots, v^n)$  is any vector which is such that  $\mathbf{M}\mathbf{v} = \mathbf{0}$  (so that  $\mathbf{v}$  is a characteristic vector of  $\mathbf{M}$  which is associated with the characteristic number zero)  $\mathbf{v}$  is determined as far as direction is concerned, for the coordinates of  $\mathbf{v}$  are proportional to the cofactors of any column of  $M$ . It follows, therefore, from the two (vector) equations

$$F_{x^i x^r} x_r = 0; \quad \phi_p p_{x_r} = 0$$

(since  $(p_j)_{x_r} = F_{x^i x^r}$  and since  $M$  is symmetric, that the two vectors  $x_r$  and  $\phi_p$  have the same direction:

$$x_r = \lambda \phi_p$$

( $\lambda$  being an undetermined multiplier about which the only thing we know is that it is a homogeneous function of degree one of the  $n$  variables  $x_r$ , for  $\phi_p$  is a homogeneous function of degree zero of these variables;  $\lambda$  will, in general, involve the variables  $x$ ).

On using the relation  $x_r = \lambda \phi_p$  in the relation  $\phi_x + \phi_p p_x = 0$  we obtain  $\lambda \phi_x + p_x x_r = 0$ . Since  $p_x = F$ ,  $p_x x_r = F_x$  and so  $F_x = -\lambda \phi_x$ . We have, then, the following relations which are identities in the  $2n$  independent variables  $x$  and  $x_r$ :

$$x_r = \lambda \phi_p; \quad F_x = -\lambda \phi_x.$$

Since  $p_r = F_x$  along an extremal it follows that

Along an extremal

$$x_\tau = \lambda \phi_p; \quad p_\tau = -\lambda \phi_x$$

or, equivalently,

$$\frac{dx}{\phi_p} = \frac{dp}{-\phi_x} = \lambda d\tau.$$

These are the famous *canonical equations* of Hamilton (after W. R. Hamilton [1805–1865], an Irish mathematician). They express the remarkable fact that if the extremal curves  $x = x(\tau)$  are represented in the  $2n$ -dimensional  $(x, p)$  space by means of equations

$$x = x(\tau); \quad p = p(\tau)$$

the curves so obtained are extremal curves of the integral

$$\int p dx = \int (px_\tau + 0p_\tau) d\tau$$

subject to the constraint  $\phi(x, p) = 0$ . In fact if we denote by  $F^*$  the linear homogeneous function  $px_\tau$  of the  $2n$  variables  $(x_\tau, p_\tau)$  we have

$$F^*_{x_\tau} = 0; \quad F^*_{p_\tau} = x_\tau; \quad F^*_{x_\tau} = p; \quad F^*_{p_\tau} = 0$$

so that the equations of the constrained extremals are

$$p_\tau = 0 + \lambda^* \phi_x; \quad 0 = x_\tau + \lambda^* \phi_p,$$

and, on replacing  $\lambda^*$  by  $-\lambda$ , we recover the Hamilton canonical equations. This result is remarkable since the space in which the extremal is represented is  $2n$ -dimensional (rather than  $n$ -dimensional as in the case of the Euler-Lagrange equations); thus the class of “comparison curves”

$$x = x(\tau, \alpha); \quad p = p(\tau, \alpha)$$

in the  $2n$ -dimensional space is much more extensive than the class of comparison curves

$$x = x(\tau, \alpha)$$

in the  $n$ -dimensional space. Nevertheless the conditions which must be satisfied by the extremals in the  $n$ -dimensional space (namely, the Euler-Lagrange equations) are strong enough to ensure that  $\delta I = 0$  for the curves in the  $2n$ -dimensional space (the end-points being fixed).

From the mathematical point of view the essential simplification introduced by the use of the Hamilton canonical equations, as opposed to the Euler-Lagrange equations, is that the canonical equations consist of  $2n$  differential equations of the *first order* while the Euler-

Lagrange equations consist, in general, of  $n$  differential equations of the second order.

#### 4. The equations of mechanics; ignorable coordinates

Let  $T$  and  $V$  denote, respectively, the kinetic and potential energies of a mechanical system. If the mechanical system has  $n$  *degrees of freedom* the rectangular Cartesian coordinates of any particle of the system are functions of  $n$  *generalized coordinates*  $(x^1, \dots, x^n)$  and, possibly, of the time. Thus if  $\xi$  denotes one of these Cartesian coordinates

$$\xi = \xi(x, t).$$

When the *constraints* that define the mechanical system are *fixed*  $\xi$  will not involve  $t$  explicitly so that

$$\xi = \xi(x).$$

$\xi$  will, of course, vary, in general, with  $t$  through the fact that the generalized coordinates  $x$  vary, in general, with  $t$ ; in other words  $\xi$  depends *implicitly* on  $t$  even when it does not depend *explicitly* on  $t$ . It follows from the formula

$$\frac{d\xi}{dt} = \xi_x x_t + \xi_t$$

that the squared velocity of the particle whose rectangular Cartesian coordinates are  $(\xi, \eta, \zeta)$  is a quadratic function of the *coordinate velocities*  $x_t$ . Hence the kinetic energy  $T$  is a quadratic function of these coordinate velocities:

$$T = T_2 + T_1 + T_0.$$

Here  $T_2$  is a homogeneous quadratic function of the coordinate velocities  $x_t$ ,  $T_1$  is a homogeneous linear function of the coordinate velocities, and  $T_0$  is a constant function of these coordinate velocities. When the constraints which define the mechanical system are fixed the linear and constant functions  $T_1$  and  $T_0$  are not present, and  $T = T_2$  is a homogeneous quadratic function of the coordinate velocities  $x_t$ . Using Greek labels to indicate summation from 1 to  $n$  we write

$$T_2 = \frac{1}{2} g_{\alpha\beta} x^\alpha x^\beta; \quad T_1 = h_\alpha x^\alpha; \quad g_{pq} = g_{qp}.$$

For example, for a system with two degrees of freedom,

$$T_2 = \frac{1}{2} \{g_{11}(x^1)^2 + 2g_{12}x^1x^2 + g_{22}(x^2)^2\};$$

$$T_1 = h_1x^1 + h_2x^2.$$

The coefficients  $g_{pq}$  and  $h_i$  are, in the general case, functions of the generalized coordinates  $x$  and of  $t$ . When the constraints are fixed the coefficients  $g_{pq}$  do not involve  $t$  explicitly; in this case

$$T = T_2 = \frac{1}{2}g_{\alpha\beta}x^\alpha x^\beta t,$$

where the  $g_{pq}$  are functions of the generalized coordinates  $x$ , and  $g_{qp} = g_{pq}$ ,  $p, q = 1, 2, \dots, n$ .

### EXERCISES

1. Show that for a particle of mass  $m$  whose generalized coordinates are plane polar coordinates  $(r, \theta)$ ,  $T = T_2 = \frac{m}{2} \{(r_t)^2 + r^2(\theta_t)^2\}$ .

2. Show that for a particle of mass  $m$  whose generalized coordinates are space polar coordinates  $(r, \theta, \phi)$

$$T = T_2 = \frac{m}{2} \{(r_t)^2 + r^2(\theta_t)^2 + r^2 \sin^2\theta(\phi_t)^2\}.$$

The potential energy  $V$  of a mechanical system is a function of the generalized coordinates  $x$  of the system, and it may, possibly, also involve the time variable explicitly:

$$V = V(x, t).$$

In a *virtual displacement* of the mechanical system the *virtual work* of the forces acting on the system is given by the expression

$$-\delta V = -V_x \delta x.$$

We say that a mechanical system is a *natural* one when the following conditions are met:

1. The constraints which serve to define the mechanical system do not involve  $t$  explicitly so that

$$T = T_2 = \frac{1}{2}g_{\alpha\beta}x^\alpha x^\beta t; \quad g_{pq} \text{ independent of } t.$$

2. The potential energy  $V$  does not involve  $t$  explicitly:

$$V = V(x).$$

The function  $L = T - V$  is known as the *Lagrangian function* of the mechanical system. It is a function of the generalized coordinates  $x$ , the generalized coordinate velocities  $x_t$  and, if the mechanical system is not a natural one, of the time  $t$ :

$$L = L(x, t, x_t).$$

On the other hand if the mechanical system is a natural one  $L$  does not involve  $t$  explicitly:

$$L = L(x, x_t).$$

The laws governing the motion of our mechanical system may now be stated as follows:

The motion of the mechanical system is such that the curve in the  $n$ -dimensional generalized-coordinate space:  $x = x(t)$  is an extremal of the integral  $I = \int_{t_0}^{t_1} L dt$ .

In order to bring this integral under the scope of our previous discussion we regard  $t$  as one of the coordinates of the mechanical system; setting  $t = t(\tau)$ , where  $t_\tau > 0$ , we have

$$I = \int_{\tau_0}^{\tau_1} F d\tau; \quad F = Lt_\tau.$$

The  $(n + 1)$ -dimensional vector

$$\mathbf{p}: (F_{x_r}, F_{t_\tau})$$

is known as the *momentum vector*. The first  $n$  coordinates of  $\mathbf{p}$  are furnished by the formula

$$p_j = F_{x_j t_\tau} = L_{x_j t_\tau} = L_{x_j}; \quad j = 1, 2, \dots, n,$$

while the time coordinate of the momentum vector is given by the formula

$$\begin{aligned} p_{n+1} &= F_{t_\tau} = L_{t_\tau t_\tau} + L \\ &= -L_{x_i} x_t + L \left( \text{for } (x_t)_{t_\tau} = -\frac{x_t}{t_\tau} \right). \end{aligned}$$

Since  $L = T - V$ ,  $L_{x_i} = T_{x_i}$  and since  $T = T_2 + T_1 + T_0$  we have

$$T_{x_i} x_t = 2T_2 + T_1$$

and so

$$p_{n+1} = -(T_2 - T_0) - V.$$

For a natural mechanical system, then,

$$p_{n+1} = -T_2 - V = -T - V.$$

The functional relation  $\phi(x, p) = 0$  which connects the  $(n + 1)$  coordinates  $(x, t)$  and the  $(n + 1)$  coordinates  $(p_1, \dots, p_n, p_{n+1})$  of the momentum vector  $\mathbf{p}$  is readily found as follows. Since  $L_{x_i} = T_{x_i}$  we have

$$p_j = g_{j\alpha} x_t^\alpha + h_j; \quad j = 1, \dots, n.$$



We take it as granted that the matrix  $g_{pq}$  (i.e., the matrix of the coefficients of the homogeneous quadratic form  $T_2$ ) is non-singular. For a natural mechanical system  $T_2 = T$  is the kinetic energy of the system, and the non-singularity of the matrix  $g_{pq}$  is assured by the fact that  $T$  is a *positive quadratic form* in the coordinate velocities  $x_i$  (in other words  $T$  is never negative and is zero only when all the coordinate velocities are zero). Let, then,  $g^{pq}$  be the matrix which is the reciprocal of the matrix  $g_{pq}$ ; since the matrix  $g_{pq}$  is symmetric so also is the matrix  $g^{pq}$ . The relations

$$g_{i\alpha}x^\alpha = p_i - h_i$$

are, accordingly, equivalent to the relations

$$x^k = g^{k\beta}(p_\beta - h_\beta); \quad k = 1, \dots, n.$$

On substitution of these expressions in the relation

$$p_{n+1} = L - L_{x_i}x_i = T_0 - T_2 - V = T_0 - \frac{1}{2}g_{\alpha\beta}x^\alpha x^\beta - V$$

we obtain

$$p_{n+1} + \frac{1}{2}g^{\alpha\beta}(p_\alpha - h_\alpha)(p_\beta - h_\beta) + V - T_0 = 0.$$

Thus the functional relation that connects the  $2n + 2$  quantities  $x^j$ ,  $t$ ,  $p_j$ ,  $p_{n+1}$ ,  $j = 1, \dots, n$ , is

$$\phi(x, p) \equiv p_{n+1} + \frac{1}{2}g^{\alpha\beta}(p_\alpha - h_\alpha)(p_\beta - h_\beta) + V - T_0 = 0.$$

The Hamilton canonical equations are, accordingly, since  $\phi_{p_{n+1}} = 1$ ,

$$\frac{dx}{\phi_p} = \frac{dt}{1} = \frac{dp}{-\phi_x} = \frac{dp_{n+1}}{-\phi_t} = \lambda d\tau.$$

We shall confine our attention from now on to *natural* mechanical systems (which are characterized by the fact that the function  $\phi(x, p)$  does not involve  $t$  explicitly). The *time component*  $p_{n+1}$  of the momentum vector is constant along an extremal (why?) and since  $p_{n+1} + T + V = 0$  it follows that  $T + V$  is constant along an extremal. We denote the constant value of  $T + V$  by  $E$ , and we term  $E$  the *energy* of the mechanical system.  $E$  may vary from extremal to extremal but along any extremal (i.e., for any particular motion of the mechanical system) it is constant. *Warning.* Be very sure that you understand that the concept of energy is only properly applicable to *natural* mechanical systems. For a non-natural mechanical system you may term  $T$  the kinetic energy, and  $V$  the potential energy, of the system, but  $T + V$  will not, in general, be constant for any given motion of the

system so that there is no particular point in giving names to the symbols  $T$  and  $V$ .

For a natural mechanical system the function  $\phi$  is of the form

$$p_{n+1} + H(x, p) = 0,$$

where  $H = \frac{1}{2}g^{\alpha\beta}p_\alpha p_\beta + V(x)$ . From now on we understand by the symbol  $x$  the  $n$  numbers  $(x^1, \dots, x^n)$  and not the  $(n+1)$  numbers  $(x^1, \dots, x^n, t)$ ; similarly we understand by the symbol  $p$  the  $n$  numbers  $(p_1, \dots, p_n)$  and not the  $(n+1)$  numbers  $(p_1, \dots, p_n, p_{n+1})$ .  $H$  is obtained by expressing  $T + V$  as a function of the  $2n$  variables  $(x, p)$ , rather than as a function of the  $2n$  variables  $(x, x_i)$ , and we term  $H(x, p)$  the *Hamiltonian function* (or, simply, the *Hamiltonian*) of the (natural) mechanical system. The function  $L (= T - V)$  of the  $2n$  variables  $(x, x_i)$  is known as the *Lagrangian function* (or, simply, as the *Lagrangian*) of the (natural) mechanical system. Along an extremal  $H(x, p)$  has the constant value  $E$  and  $p_{n+1} = -E$ . The Hamiltonian equations may be written in the form

$$\frac{dx}{H_p} = \frac{dp}{-H_x} = dt; \quad dp_{n+1} = 0.$$

Since  $t$  does not appear in any of the  $2n$  equal ratios  $\frac{dx}{H_p}, \frac{dp}{-H_x}$  we may confine our attention (at the beginning) to the  $2n - 1$  equations

$$\frac{dx}{H_p} = \frac{dp}{-H_x}.$$

If we are successful in integrating these, i.e., in obtaining  $x$  and  $p$  as functions of a parameter  $\tau$  (and initial values of  $x$  and  $p$ ), then  $t$  may be obtained by a quadrature (what does this mean?) from the relation

$$dt = \frac{dx}{H_p} \left( \text{or, } dt = \frac{dp}{-H_x} \right).$$

$p_{n+1}$  does not have to be found since  $p_{n+1} = -E = -H(x_0, p_0)$ . When we proceed in this way we say that we have *ignored* (at the beginning of our solution) the time coordinate  $t$ , and we refer to  $t$  as an *ignorable coordinate*. It is clear that the same procedure is applicable to any coordinate  $x$  which does not occur explicitly in the function  $\phi$ . We term any such coordinate of the mechanical system an *ignorable coordinate*:

**DEFINITION.** *An ignorable coordinate of a mechanical system is one that does not appear explicitly in the function  $\phi(x, t, p, p_{n+1})$ .*

Associated with each ignorable coordinate  $x$  is a *momentum integral*; by this we mean simply that the corresponding coordinate of the momentum vector  $\mathbf{p}$  is constant along an extremal. (Prove this.) Thus

A natural mechanical system is one for which  $t$  is an ignorable coordinate; the associated momentum integral is the energy integral (the time component of the  $(n + 1)$ -dimensional momentum vector being the negative of the energy of the system).

### EXERCISES

3. Show that for a particle of mass  $m$ , whose generalized coordinates are plane polar coordinates  $(r, \theta)$  and for which  $V = V(r, \theta)$ ,  $H = \frac{1}{2m} \left\{ (p_r)^2 + \frac{1}{r^2} (p_\theta)^2 \right\} + V(r, \theta)$ .

4. Show that if, in Exercise 3,  $V = V(r)$  does not involve  $\theta$  explicitly then  $\theta$  is an ignorable coordinate and  $p_\theta = mr^2\dot{\theta}$  is constant along an extremal.

5. On denoting by  $h$  the constant value of  $p_\theta$  in Exercise 4 show that the *path* of the particle may be obtained by first solving the equation

$$\frac{dr}{H^*_{,r}} = \frac{dp_r}{-H^*},$$

where  $H^* = \frac{1}{2m} p_r^2 + \frac{h^2}{2mr^2} + V(r)$ , for  $p_r$  as a function of  $r$  and then integrating the equation  $\frac{dr}{p_r/m} = \frac{d\theta}{h/mr^2}$ . *Note.* The result of this exercise shows that the

procedure of ignoring  $\theta$  is equivalent to the addition of  $\frac{h^2}{2mr^2}$  to the potential energy, i.e., to the addition of a radial force of amount  $\frac{h^2}{mr^3} = mr\dot{\theta}^2$  to the force acting on the particle. This is the explanation of the phenomenon of *centrifugal force*.

6. Show that for a particle of mass  $m$ , whose generalized coordinates are space polar coordinates, the angular coordinate  $\phi$  is ignorable if  $V = V(r, \theta)$  is independent of  $\phi$ . What is the corresponding momentum integral?

7. Show that the motion of the mass particle of Exercise 6 takes place in a plane through the origin. *Hint.* Choose the polar axis so that it lies in the plane containing the initial direction of motion (so that the initial value of  $\phi_i$  is zero), and deduce from the *angular-momentum* integral  $p_\phi = \text{constant}$  that  $\phi$  is constant along any extremal.

### 5. The determination of the Lagrangian from the Hamiltonian; the principle of Maupertuis

When we are given a problem in the calculus of variations with an integrand function  $F(x, x_r)$  which is positively homogeneous of degree one in the variables  $x_r$  we know how to find the function  $\phi(x, p)$  which appears in the Hamilton canonical equations

$$\frac{dx}{\phi_p} = \frac{dp}{-\phi_x} = \lambda d\tau$$

of the extremals. If the  $n \times n$  matrix  $F_{x_\tau x_\tau}$  is of rank  $n - 1$  there is only one (independent) relation  $\phi(x, p) = 0$  connecting the  $2n$  variables  $x$  and  $p$ , and this is obtained by eliminating the  $n - 1$  ratios  $x^1_\tau : \dots : x^n_\tau$  from the  $n$  functions  $p$  of these ratios. The force of the parenthetical word independent is simply this: Any function of  $\phi$  which is zero when  $\phi$  is zero (e.g.,  $\phi^2$ ) will do just as well as  $\phi$ . However, this merely amounts to a change of the undetermined multiplier  $\lambda$ ; if  $\phi$  is replaced by  $f(\phi)$ ,  $\lambda$  is replaced by  $f_\phi \lambda$ . (Prove this.) For a mechanical system where  $F = L(x, x_t) t_\tau$  the relation  $\phi(x, p) = 0$  could be "solved for  $p_{n+1}$ "; in other words we could write  $\phi(x, p) = 0$  in the form

$$p_{n+1} + H(x, t, p_1, \dots, p_n) = 0,$$

and then the function  $H$  (the Hamiltonian of the mechanical system, natural or not) was unambiguously determinate. The problem we wish to solve now is the following:

*Given the Hamiltonian function  $H$  determine the Lagrangian function  $L$  to which it corresponds.*

For the general problem of the calculus of variations this problem may be stated as follows:

*Given the function  $\phi(x, p)$  determine the function  $F(x, x_\tau)$  to which  $\phi(x, p)$  corresponds (it being understood that the same function  $F$  will correspond to any differentiable function of  $\phi$ ).*

In order to solve this problem we make the following hypothesis:

We assume that the  $(n + 1) \times (n + 1)$  matrix

$$\begin{pmatrix} \phi_{pp} & \phi_p \\ \phi_p & 0 \end{pmatrix}$$

is not singular for every point  $(x, p)$  on the  $2n - 1$  dimensional spread  $\phi(x, p) = 0$  in the  $2n$ -dimensional coordinate-momentum space (in which the coordinates of any point are the  $2n$  numbers  $x$  and  $p$ ).

To see the force of this hypothesis consider a mechanical system (natural or not). Here  $\phi = p_{n+1} + H(x, t, p)$  and our coordinate space is  $n + 1$  dimensional; hence our matrix is the  $(n + 2) \times (n + 2)$  matrix

$$\begin{bmatrix} H_{pp} & 0 & H_p \\ 0 & 0 & 1 \\ H_p & 1 & 0 \end{bmatrix}$$

whose determinant is  $-\det(H_{pp})$ . Hence our hypothesis amounts to the assumption that the  $n \times n$  matrix  $H_{pp}$  is non-singular or, equivalently, that the matrix of the coefficients of the homogeneous quadratic form  $T_2$  in the  $n$  variables  $x_i$  is non-singular. (Prove this.) For a natural mechanical system our hypothesis amounts (since  $T_2 = T$ ) to the assumption that the matrix of the coefficients of the kinetic energy (expressed either as a function of the  $n$  coordinate velocities or as a function of the  $n$  momenta  $p$ ) is non-singular.

In the neighborhood, then, of any point  $(x, p)$  of the spread  $\phi(x, p) = 0$  at which the  $(n + 1) \times (n + 1)$  matrix

$$\begin{pmatrix} \phi_{pp} & \phi_p \\ \phi_p & 0 \end{pmatrix}$$

is non-singular the  $n + 1$  equations

$$\lambda \phi_p = x_r; \quad \phi(x, p) = 0$$

may be solved, in an unambiguous manner, for  $p$  and  $\lambda$  as functions of  $x_r$  provided that  $\lambda \neq 0$ . In fact the Jacobian matrix of the  $n + 1$  functions  $(\lambda \phi_p, \phi)$  with respect to the  $n + 1$  variables  $p, \lambda$  is

$$\begin{pmatrix} \lambda \phi_{pp} & \phi_p \\ \phi_p & 0 \end{pmatrix},$$

and the determinant of this matrix is readily found (on dividing the first  $n$  rows by  $\lambda$  and then multiplying the last column by  $\lambda$ ) to be the product of the determinant of the matrix

$$\begin{pmatrix} \phi_{pp} & \phi_p \\ \phi_p & 0 \end{pmatrix}$$

by  $\lambda^{n-1}$ . Since  $\lambda$  is a continuous function of the  $n$  variables  $x_r$  (the matrix  $\begin{pmatrix} \phi_{pp} & \phi_p \\ \phi_p & 0 \end{pmatrix}$  being assumed to be a continuous function of the  $n$  variables  $p$ ) and since  $\lambda$  cannot be zero, it cannot change sign; we agree, merely for the sake of convenience, that  $\lambda > 0$ . If the  $x_r$  are multiplied by a common factor  $k (> 0)$ ,  $p$  and  $k\lambda$  will be a solution of the new equations (why?). In view of the unambiguously determinate nature of the solution of the equations for  $p$  and  $\lambda$  it follows that

**The  $n$  functions  $p$  of  $x_r$  are positively homogeneous of degree zero and  $\lambda$  is positively homogeneous of degree one in the  $n$  variables  $x_r$ .**

We propose to show that the Hamilton canonical equations

$$\frac{dx}{\phi_p} = \frac{dp}{-\phi_x} = \lambda d\tau$$

are the equations (in the  $2n$ -dimensional  $(x, p)$ -space) of the extremals of the integral

$$I = \int_{\tau_0}^{\tau_1} F(x, x_\tau) d\tau,$$

where  $F = x_\tau p$ . To do this we first observe that since the functions  $p$  are positively homogeneous of degree zero the function  $F = x_\tau p$  is positively homogeneous of degree one. Furthermore

$$F_{x_\tau} = p + x_\tau p_{x_\tau} = p.$$

In fact the relation  $\phi(x, p) = 0$  is an identity in the  $n$  variables  $x_\tau$ , and on differentiating this relation with respect to  $x_\tau$  we obtain

$\sum_{j=1}^n \phi_{p_j} (p_j)_{x_\tau} = 0$ ; since  $x_\tau = \lambda \phi_p$  it follows that  $x_\tau p_{x_\tau} = \sum_{j=1}^n x'_\tau (p_j)_{x_\tau} = 0$ . It remains only to show that  $p_x = F_x$  or, equivalently, since  $p_x = -\lambda \phi_x$ , that  $F_x = -\lambda \phi_x$ . From the definition of  $F$  it follows that

$$F_x = x_\tau p_x,$$

and on differentiating the relation  $\phi(x, p) = 0$  with respect to  $x$  we obtain

$$\phi_x + \phi_p p_x = 0.$$

On multiplying this equation by  $\lambda$  and replacing  $\lambda \phi_p$  by  $x_\tau$  we obtain

$$\lambda \phi_x + x_\tau p_x = 0.$$

Hence  $F_x = -\lambda \phi_x$ . This completes the proof of our theorem. The procedure to be followed in order to determine  $F$  when  $\phi$  is given may be formulated as follows:

**Solve for  $p$  and  $\lambda$  the equations  $\lambda \phi_p = x_\tau$ ,  $\phi(x, p) = 0$ . Then  $F = x_\tau p$ .**

**Example 1. The general mechanical system**

Here  $\phi = p_{n+1} + H(x, t, p)$ , where  $H$  is a quadratic function of  $p_1, \dots, p_n$ :

$$H = H_2 + H_1 + H_0 + V = \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta + k^\alpha p_\alpha + H_0 + V, \text{ say.}$$

Hence

$$x'_\tau = \lambda H_{p_i} = \lambda (g^{i\alpha} p_\alpha + k^i); \quad t_\tau = \lambda$$

so that  $x^j_t = H_{p_j}$  and

$$g^{ja} p_a = x^j_t - k^j; \quad j = 1, \dots, n.$$

Our assumption concerning the non-singularity of the matrix  $\begin{pmatrix} \phi_{pp} & \phi_p \\ \phi_p & 0 \end{pmatrix}$  tells us that the matrix  $g^{pa}$  is non-singular. Denoting its reciprocal

by  $g_{pa}$  we have  $p_j = g_{ja}(x^j_t - k^j)$ . The function  $F = px_\tau = \sum_{j=1}^n p_j x^j_t$

$$+ p_{n+1}t_\tau = t_\tau \left\{ \sum_{j=1}^n p_j x^j_t + p_{n+1} \right\}, \text{ and since } p_{n+1} = -H = -H_2 - H_1$$

$$- H_0 - V \text{ and } \sum_{j=1}^n p_j x^j_t = \sum_{j=1}^n p_j H_{p_j} = 2H_2 + H_1 \text{ we have}$$

$$F = (H_2 - H_0 - V)t_\tau$$

so that the Lagrangian function is

$$L = F \div t_\tau = H_2 - H_0 - V.$$

For a natural mechanical system  $H = H_2 + V$  so that  $L = H_2 - V = H - 2V$ .

### Example 2. The Lagrangian function for a natural mechanical system in which the time is ignored

The canonical equations are

$$\frac{dx}{H_p} = \frac{dp}{-H_x},$$

where  $H = \frac{1}{2}g^{\alpha\beta}p_\alpha p_\beta + V(x)$  and  $\phi \equiv H - E$ ,  $E = H(x_0, p_0)$  being the energy of the motion. The matrix whose non-singularity must be assured is

$$\begin{pmatrix} H_{pp} & H_p \\ H_p & 0 \end{pmatrix}.$$

On subtracting from the last column of this matrix the sum of the products of the first column by  $p_1$ , the second by  $p_2$ ,  $\dots$ , and the  $n$ th by  $p_n$  it is clear that its determinant is the product of the determinant of the matrix  $g^{pa}$  by  $-g^{\alpha\beta}p_\alpha p_\beta$ . Assuming, then, that  $g^{\alpha\beta}p_\alpha p_\beta \neq 0$  (which will certainly be true if  $p$  is not the zero vector since

$H_2 = T$  is the kinetic energy of the system) we may proceed to determine  $F$ . Our equations are

$$\lambda g^{j\alpha} p_\alpha = x_\tau; \quad j = 1, \dots, n;$$

$$\frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta + V = E.$$

From the first  $n$  of these equations we obtain

$$p_j = \frac{1}{\lambda} g_{ja} x^\alpha_\tau; \quad j = 1, \dots, n,$$

and on substituting these expressions in our remaining equation we find

$$g_{\alpha\beta} x^\alpha_\tau x^\beta_\tau = 2\lambda^2(E - V).$$

The desired function  $F$  is given by the formula

$$F = p x_\tau = \frac{1}{\lambda} g_{\alpha\beta} x^\alpha_\tau x^\beta_\tau,$$

and since

$$\lambda = \left\{ \frac{g_{\alpha\beta} x^\alpha_\tau x^\beta_\tau}{2(E - V)} \right\}^{1/2}$$

we obtain

$$F = \{2(E - V) g_{\alpha\beta} x^\alpha_\tau x^\beta_\tau\}^{1/2}.$$

This is the mathematical expression of the **principle of Maupertuis** (after P. Maupertuis [1698–1759], a French applied mathematician). This principle says that a natural mechanical system moving with energy  $E$  moves in such a way that the curves  $x = x(\tau)$  are extremals of the integral

$$\{2(E - V)\}^{1/2} \{g_{\alpha\beta} x^\alpha_\tau x^\beta_\tau\}^{1/2}.$$

If we regard the coefficients  $g_{pq}$  of the quadratic form which furnishes the kinetic energy of the system as furnishing a method for measuring arc lengths in the  $n$ -dimensional coordinate (i.e.,  $x$ -) space we may set

$$g_{\alpha\beta} x^\alpha_\tau x^\beta_\tau = (s_\tau)^2,$$

where  $s$  denotes arc length in the coordinate space; when this is done the integral whose extremals describe the motion of the mechanical system appears in the form

$$\int_{s_0}^{s_1} \{2(E - V)\}^{1/2} ds.$$



If the mechanical system is moving freely, i.e., without being subjected to any force field,  $V$  is constant, and the integral whose extremals furnish the motion of the mechanical system is the arc-length integral

$\int_{s_0}^{s_1} ds$  (the constant factor  $\{2(E - V)\}^{1/2}$  being without significance).

When a force field is present we may introduce the parameter  $\tau$  defined by the formula

$$s_\tau = \{2(E - V)\}^{-1/2},$$

and then the integral whose extremals furnish the motion of the mechanical system is

$$\int_{\tau_0}^{\tau_1} 1 d\tau.$$

The analogy with Fermat's *principle of least time* in optics (after P. de Fermat [1601-1665], a French mathematician) is evident; the expression  $\{2(E - V)\}^{1/2}$ , the reciprocal of  $s_\tau$ , plays the role of the *index of refraction* in optics. This analogy between the dynamics of a natural mechanical system and optics has found its culmination in the modern theory of *wave mechanics*.

## 6. The action integral; the principle of least action

Once the extremals of a problem in the calculus of variations have been found the integral of  $F(x, x_\tau)$  along an extremal is an unambiguously determined function of the end-points  $P_0$  and  $P$  of the extremal (it being understood that  $P$  is sufficiently near to  $P_0$  so that the existence and uniqueness theorems of ordinary differential equations may be appealed to to assure us that there is one, and only one, extremal through the two points  $P_0$  and  $P$ ). We shall denote the value of the integral of  $F(x, x_\tau)$  along the unique extremal  $C$  connecting the points  $P_0: (x_0^1, \dots, x_0^n)$  and  $P: (x^1, \dots, x^n)$  by the symbol  $S(x, x_0)$ :

$$S(x, x_0) = \int_{\tau_0}^{\tau} F(x, x_\tau) d\tau.$$

We have already seen that the variation of  $I(\alpha)$  reduces, when the curve along which  $F$  is being integrated is an extremal, to

$$\delta I = p_1 \delta(x_1) - p_0 \delta(x_0)$$

and so

$$S_x = p; \quad S_{x_0} = -p_0.$$

Since  $\phi(x, p) = 0$  it follows that the function  $S$  must satisfy the two partial differential equations of the first order

$$\phi(x, S_x) = 0; \quad \phi(x_0, -S_{x_0}) = 0.$$

For a mechanical system the first of these equations takes the form

$$S_t + H(x, t, S_x) = 0.$$

This is known as the *Hamilton-Jacobi* equation of the mechanical system. When the system is a natural one the Hamilton-Jacobi equation simplifies to

$$S_t + H(x, S_x) = 0,$$

where  $H(x, S_x)$  has the constant value  $E$  along an extremal. We may proceed one step towards obtaining a solution of the Hamilton-Jacobi equation by writing

$$S = -E(t - t_0) + A(x, x_0),$$

where the *action function*  $A(x)$  satisfies the partial differential equation

$$H(x, A_x) = E.$$

Since  $F = px_\tau$  and since, for a natural mechanical system,  $p_{n+1}$  has the constant value  $-E$  along an extremal, it is clear that

$$S(x, x_0) = \int_{\tau_0}^{\tau} \left( \sum_1^n p_j x_\tau^j \right) d\tau - E(t - t_0)$$

and so

$$\begin{aligned} A(x, x_0) &= \int_{\tau_0}^{\tau_1} \left\{ \sum_{j=1}^n p_j x_\tau^j \right\} d\tau = \int_{t_0}^{t_1} \left( \sum_{j=1}^n p_j x_t^j \right) dt \\ &= 2 \int_{t_0}^{t_1} T dt. \end{aligned}$$

Since the variation of  $I(\alpha)$  is zero when the curve of integration is an extremal and the end-points are fixed we have the following fundamental result for natural mechanical systems:

**The motion of a natural mechanical system is furnished by the extremals of the action integral**

$$A(x, x_0) = 2 \int_{t_0}^{t_1} T dt$$

**provided that all comparison curves are traversed with the same energy  $E$  and that the time of passage  $t_1 - t_0$  is the same for all.**

This is known as the **principle of least action**.

### 7. Liouville's theorem

Hamilton's canonical equations are a special system of ordinary differential equations of the first order. The general system of ordinary differential equations of the first order may be written in the form

$$\frac{dx^1}{u^1} = \cdots = \frac{dx^n}{u^n} = d\tau,$$

where each of the symbols  $u^j$ ,  $j = 1, \cdots, n$ , denotes a function (which we assume to be differentiable, with continuous derivatives) of the  $n$  variables  $(x^1, \cdots, x^n)$ . If the initial point  $P_0$  of a *solution curve* or *path* of the differential equations has the coordinates  $(x_0^1, \cdots, x_0^n)$ , the coordinates  $(x^1, \cdots, x^n)$  of any point  $P$  of the path are given by equations of the form

$$x = x(\tau, x_0),$$

where  $x(\tau, x_0)$  reduces to  $x_0$  when  $\tau = \tau_0$ . Let  $V_0$  be any  $n$ -dimensional region of initial points  $x_0$ , and let us consider the following question:

Does there exist a point-function  $\rho = \rho(x)$  which is such that the integral  $\int_V \rho d(x)$  is independent of  $\tau$  (no matter what is the  $n$ -dimensional region  $V_0$ ):

$$\int_V \rho d(x) = \int_{V_0} \rho_0 d(x_0)?$$

Here  $V$  is the collection of points  $x$  obtained by assigning to  $\tau$  a given value (the same for all initial points  $P_0: (x_0)$  of  $V_0$ ). If such a function

$\rho = \rho(x)$  exists we say that the integral  $\int_V \rho d(x)$  is an *invariant*

*integral* ( $n$ -dimensional) of the system of differential equations  $\frac{dx}{u} = d\tau$ ,

and we say that the point-function  $\rho = \rho(x)$  is a *density function*, or *multiplier*, of this system of differential equations.

*Note.* We have tacitly supposed that when  $V_0$  is an  $n$ -dimensional region so also is  $V$ . This will be true if the Jacobian matrix  $\frac{(x)}{(x_0)}$  is non-singular. This Jacobian matrix reduces to the  $n \times n$  unit matrix when  $\tau = \tau_0$ . Hence  $\frac{(x)}{(x_0)}$  is non-singular (its determinant being, in fact, positive) if  $\tau - \tau_0$  is sufficiently small. We shall suppose that this is the case.

In order to determine the conditions that must be imposed on the integrand  $\rho = \rho(x)$  of the integral  $I = \int_V \rho d(x)$  in order that  $I$  may be an invariant integral we must calculate  $I_\tau$  and set it equal to zero. In order to facilitate the differentiation we change the variables of integration from  $x$  to  $x_0$  so that the region of integration is transformed from  $V$  to  $V_0$ :

$$I = \int_V \rho d(x) = \int_{V_0} \rho \left| \frac{(x)}{(x_0)} \right| d(x_0),$$

where  $\left| \frac{(x)}{(x_0)} \right|$  is the determinant of the Jacobian matrix  $\frac{(x)}{(x_0)}$ , so that  $\left| \frac{(x)}{(x_0)} \right|$  is positive. Hence  $I_\tau$  is the integral over  $V_0$  of the derivative with respect to  $\tau$  of  $\rho \left| \frac{(x)}{(x_0)} \right|$ . Since the determinant of any matrix is the alternating product of the row vectors of the matrix the derivative of an  $n$ -rowed determinant with respect to any variable is the sum of  $n$  determinants each of which is obtained by replacing one row of the matrix, whose determinant we wish to differentiate, by its derivative and then taking the determinant of the resulting matrix. Thus, if  $n = 3$ , and

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = d\tau,$$

we have

$$\left| \frac{(x, y, z)}{(x_0, y_0, z_0)} \right|_\tau = \left| \frac{(u, y, z)}{(x_0, y_0, z_0)} \right| + \left| \frac{(x, v, z)}{(x_0, y_0, z_0)} \right| + \left| \frac{(x, y, w)}{(x_0, y_0, z_0)} \right|.$$

Since  $\frac{(u, y, z)}{(x_0, y_0, z_0)} = \frac{(u, y, z)}{(x, y, z)} \frac{(x, y, z)}{(x_0, y_0, z_0)}$  we have

$$\left| \frac{(u, y, z)}{(x_0, y_0, z_0)} \right| = u_x \left| \frac{(x, y, z)}{(x_0, y_0, z_0)} \right|.$$

Similarly  $\left| \frac{(x, v, z)}{(x_0, y_0, z_0)} \right| = v_y \left| \frac{(x, y, z)}{(x_0, y_0, z_0)} \right|$  and

$$\left| \frac{(x, y, w)}{(x_0, y_0, z_0)} \right| = w_z \left| \frac{(x, y, z)}{(x_0, y_0, z_0)} \right|$$

so that  $\left| \frac{(x, y, z)}{(x_0, y_0, z_0)} \right|_\tau = (u_x + v_y + w_z) \left| \frac{(x, y, z)}{(x_0, y_0, z_0)} \right|$ . Similarly, if

$$\frac{dx^1}{u^1} = \frac{dx^2}{u^2} = \cdots = \frac{dx^n}{u^n} = d\tau,$$

$$\left| \frac{(x)}{(x_0)} \right|_\tau = \{(u^1)_{x^1} + \cdots + (u^n)_{x^n}\} \left| \frac{(x)}{(x_0)} \right|.$$

Hence

$$I_\tau = \int_{V_0} [\rho_\tau + \rho \{(u^1)_{x^1} + \cdots + (u^n)_{x^n}\}] \left| \frac{(x)}{(x_0)} \right| d(x_0).$$

Thus the necessary and sufficient condition that  $I$  be an invariant ( $n$ -dimensional) integral is

$$\rho_\tau + \rho \{(u^1)_{x^1} + \cdots + (u^n)_{x^n}\} = 0.$$

Since  $\rho_\tau = \rho_x x_\tau = \rho_x u$  this condition may be put in the more compact form

$$(\rho u)_x = 0.$$

(Here we understand, of course, by the symbol  $(\rho u)_x$  the sum  $\sum_{j=1}^n (\rho u^j)_{x^j}$ .)

We have, then, the following important result:

**In order that  $\rho = \rho(x)$  be a density function, or multiplier, of the system of differential equations  $\frac{dx}{u} = d\tau$  it is necessary, and sufficient, that  $(\rho u)_x = 0$ .**

It follows at once that any constant function is a density function, or multiplier, of the Hamilton canonical equations. In fact the sum  $(\rho u)_x$  splits, for the canonical equations, into the two sums  $(\rho \phi_p)_x$  and  $(-\rho \phi_x)_p$ , and these cancel each other if  $\rho$  is a constant function of the  $2n + 2$  variables  $x, t, p, p_{n+1}$ . Thus

The points in the  $(2n + 2)$ -dimensional  $(x, t, p, p_{n+1})$ -space which represent any mechanical system move in such a way that the integral  $\int dx \cdots dt dp_1 \cdots dp_n dp_{n+1}$  is invariant.

For natural mechanical systems we obtain in the same way the following theorem, known as *Liouville's theorem* (after J. Liouville [1809-1882], a French mathematician):

**The points in the  $2n$ -dimensional  $(x, p)$ -space which represent any natural mechanical system move in such a way that the integral**

$\int dx^1 \cdots dx^n dp_1 \cdots dp_n$  is invariant; in other words the points move like the particles of an incompressible fluid, the volume of any portion of the fluid remaining unchanged during the motion.

Any point-function  $f(x)$  which remains invariant along a solution curve  $x = x(\tau, x_0)$  of the system of differential equations  $\frac{dx}{u} = d\tau$  is termed an *invariant function* or *first integral* of this system of differential equations. Since  $f_\tau = f_x u$  we have the following result:

The necessary and sufficient condition that  $f = f(x)$  should be an invariant function of the system of differential equations  $\frac{dx}{u} = d\tau$  is that  $f$  should satisfy the first-order partial differential equation

$$uf_x = \sum_{j=1}^n u^j f_{x^j} = 0.$$

For the Hamilton canonical equations the equation  $uf_x = 0$  takes the form

$$\sum_{j=1}^n (\phi_{p_j} f_{x^j} - \phi_{x^j} f_{p_j}) + f_t - \phi_t f_{p_{n+1}} = 0.$$

If  $f$  is a function of  $(x, t, p_1, \cdots, p_n)$ , so that  $f_{p_{n+1}} = 0$ , this reduces to

$$\sum_{j=1}^n (\phi_{p_j} f_{x^j} - \phi_{x^j} f_{p_j}) + f_t = 0.$$

For a natural mechanical system the equation  $uf_x = 0$  takes the form

$$\sum_{j=1}^n (H_{p_j} f_{x^j} - H_{x^j} f_{p_j}) = 0.$$

### EXERCISES

1. Show that the quotient of any two density functions is an invariant function.

*Hint.* The equations  $(\rho_1)_\tau + \rho_1 \sum_{j=1}^n (u^j)_{x^j} = 0$ ,  $(\rho_2)_\tau + \rho_2 \sum_{j=1}^n (u^j)_{x^j} = 0$  yield

$\rho_2(\rho_1)_\tau - \rho_1(\rho_2)_\tau = 0$  or, equivalently,  $\left(\frac{\rho_2}{\rho_1}\right)_\tau = 0$ ,  $\rho_1 \neq 0$ .

2. Show that any density function for the Hamilton canonical equations is an invariant function for these equations.

3. Verify that  $\phi(x, p)$  is an invariant function for the canonical equations

$$\frac{dx}{\phi_p} = \frac{dp}{-\phi_x} = \lambda d\tau.$$

4. Verify that  $H(x, p)$  is an invariant function for the canonical equations (of a natural mechanical system)  $\frac{dx}{H_p} = \frac{dp}{-H_x}$ .

# 10

## THE OPERATIONAL CALCULUS

### 1. The Laplace transformation

Let  $f(t) = f_1(t) + if_2(t)$  be any piecewise-continuous complex-valued function of the non-negative real variable  $t \geq 0$ . The assumption of piecewise continuity assures us that the limits

$$f(t + 0) = \lim_{\delta \rightarrow 0} f(t + \delta); \quad \delta > 0; \quad t \geq 0$$

$$f(t - 0) = \lim_{\delta \rightarrow 0} f(t - \delta); \quad \delta > 0; \quad t > 0$$

exist. We shall denote  $f(0 + 0)$  simply by  $f(0)$ ; in other words

*We understand by the symbol  $f(0)$  the limit of  $f(t)$  as  $t$  tends to zero through positive values.*

When convenient we extend the definition of  $f(t)$  so that it becomes a function of the unrestricted real variable by the convention that  $f(t) = 0$  if  $t < 0$ .

Let  $z_0 = z_0 + iy_0$  be a value of the complex variable  $z = x + iy$  which is such that the integral  $\int_0^{\infty} e^{-zt}f(t) dt$  exists. Then the following fundamental theorem is true:

**The integral  $\int_0^{\infty} e^{-zt}f(t) dt$  exists for every  $z$  for which  $x > x_0$ , and this integral is an analytic function of the complex variable  $z$  over the half-plane  $x > x_0$ .**

If we set  $P_0: (x_0, y_0)$ ,  $P: (x, y)$  we may phrase this theorem as follows:

*The mere existence of the integral  $\int_0^{\infty} e^{-zt}f(t) dt$  at any point  $P_0$  is sufficient to ensure the existence and analyticity of this integral at any point  $P$  which lies to the right of  $P_0$ .*

In order to prove this theorem consider the function



$$g(s) = \int_0^s e^{-st} f(t) dt; \quad s \geq 0;$$

$g(s)$  is a continuous function of the non-negative variable  $s$ , and, being defined at  $s = \infty$ , it is bounded (why?). Hence the integral

$$\int_0^{\infty} e^{-(s-x_0)t} g(t) dt; \quad x > x_0$$

is absolutely convergent; in fact if  $M$  is an upper bound of  $|g(s)|$  we have

$$\begin{aligned} \int_a^b e^{-(s-x_0)t} g(t) dt &\leq M \int_a^b e^{-(s-x_0)t} dt \text{ (why?)} \\ &= \frac{M}{x - x_0} \{e^{-(s-x_0)a} - e^{-(s-x_0)b}\}, \end{aligned}$$

and this is arbitrarily small if  $a$  (and hence  $b > a$ ) is sufficiently large.

(Note that if  $x - x_0 \geq \delta$ ,  $\left| \int_a^b e^{-(s-x_0)t} g(t) dt \right| < \frac{M}{\delta} e^{-\delta a}$  so that the convergence of the integral  $\int_0^{\infty} e^{-(s-x_0)t} g(t) dt$  is uniform, with respect to  $z$ , over any point set for which  $x - x_0 \geq \delta$ .) It follows that the integral

$$\int_0^{\infty} e^{-zt} f(t) dt; \quad x > x_0$$

is absolutely convergent; in fact on writing  $e^{-zt} f(t) dt$  in the form  $\{e^{-(s-x_0)t}\} e^{-z_0 t} f(t) dt$  and integrating by parts we obtain, since  $e^{-st} f(t) dt = dg$ ,

$$\int_0^{\infty} e^{-zt} f(t) dt = (z - z_0) \int_0^{\infty} e^{-(s-x_0)t} g(t) dt.$$

That the function of  $z$  defined by the absolutely convergent integral  $\int_0^{\infty} e^{-zt} f(t) dt$  is analytic at any point  $P$  to the right of  $P_0$  follows from

the fact that the convergence of the integral  $\int_0^{\infty} e^{-(s-x_0)t} g(t) dt$  is uniform with respect to  $z$  over any collection of points  $P$  for which  $x - x_0 \geq \delta$ ,  $\delta$  being an arbitrary, fixed positive number. In fact each member of the sequence of functions

$$G_n(z) = \int_0^n e^{-(s-x_0)t} g(t) dt; \quad n = 1, 2, \dots$$

is everywhere analytic (why?), and this sequence of analytic functions converges, uniformly with respect to  $z$  over the half-plane  $x - x_0 \geq \delta$ , to  $\int_0^\infty e^{-(s-s_0)t} g(t) dt$ . Hence, by one of the fundamental theorems of analytic-function theory, the uniform limit  $\int_0^\infty e^{-(s-s_0)t} g(t) dt$  of the sequence of analytic functions  $G_n(z)$  is itself analytic over the half-plane  $x - x_0 \geq \delta$  or, equivalently, since  $\delta$  is arbitrary, over the half-plane  $x > x_0$ .

We term the function  $\int_0^\infty e^{-st} f(t) dt$  the *Laplace transform* of the function  $f(t)$ , and we denote it by  $Lf$ . Thus the *Laplace transformation* transforms complex-valued functions (which we take to be piecewise-continuous) of the *real* non-negative variable  $t$  into functions of a complex variable  $z$  which are analytic over a half-plane  $x > x_0$ . It is clear that the *Laplace operator*  $L$  is a linear operator; in other words

$$L(cf) = cLf; \quad c \text{ any complex constant};$$

$$L(f_1 + f_2) = Lf_1 + Lf_2.$$

(Prove this.) In stating the second relation it is understood that both  $Lf_1$  and  $Lf_2$  are defined at a common point  $z_0 = x_0 + iy_0$ ; then the relation is valid over the half-plane  $x > x_0$ .

The Laplace transform of  $f(t)$  is a special case of what is known as the *Fourier transform* of a complex-valued function of the unrestricted real variable  $t$ . If  $\phi(t)$  is such a function (which we take to be piecewise-continuous) which is absolutely integrable over  $(-\infty, \infty)$  the integral

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-i\omega t} \phi(t) dt$$

exists for every  $y$ , and the function of  $y$  which it defines is termed the *Fourier transform* of  $\phi(t)$ . It follows that if  $\phi(t)$  is the complex-valued piecewise-continuous function of the unrestricted real variable  $t$  which is defined as follows:

$$\phi(t) = (2\pi)^{1/2} e^{-st} f(t); \quad t \geq 0$$

$$\phi(t) = 0; \quad t < 0$$

then  $Lf$  is the *Fourier transform* of  $\phi(t)$  ( $Lf$  being regarded as a function of the unrestricted real variable  $y$  by the device of holding  $x$  fixed). (Prove this.) There is a fundamental theorem on *Fourier transforms* which runs as follows: Let  $\phi(t)$  be piecewise-continuous and absolutely

integrable over  $(-\infty, \infty)$ , and let  $\psi(y)$  be the Fourier transform of  $\phi(t)$ ; then

$$\int_0^x \phi(t) dt = (2\pi)^{-1/2} \int_{-(\infty)}^{(\infty)} \psi(y) \frac{e^{iyx} - 1}{iy} dy$$

(where the symbol  $\int_{-(\infty)}^{(\infty)}$  denotes the limit at  $a = \infty$  of  $\int_{-a}^a$ ; in other words  $\int_{-(\infty)}^{(\infty)}$  is the *Cauchy*, or *principal*, value of the, not necessarily convergent, improper integral  $\int_{-\infty}^{\infty}$  (what does this mean?)).

*Note.* The *Fourier integral theorem* states that at every point where the Fourier series of  $\phi(t)$  converges to the sum  $\phi(t)$  the relation

$$\phi(t) = (2\pi)^{-1/2} \int_{-(\infty)}^{(\infty)} \psi(y) e^{iyt} dy$$

is valid. The theorem we have just stated says that despite the fact that the relation which expresses the Fourier integral theorem is not necessarily valid even at the points of continuity of  $\phi(t)$  the relation obtained by integrating, over the interval  $[0, x]$ , the Fourier integral theorem relation (the integration being done on the right under the  $\int_{-(\infty)}^{(\infty)}$  sign) is valid for every value of  $x$ .

It follows from this theorem that  $\int_0^x \phi(t) dt$  is unambiguously determined by the Fourier transform  $\psi(y)$  of  $\phi(t)$ . Since, at any point of continuity of  $\phi(x)$ ,  $\phi(x)$  is the derivative with respect to  $x$  of  $\int_0^x \phi(t) dt$  we see that

$\phi(t)$  is unambiguously determined at any point where it is continuous by its Fourier transform  $\psi(y)$ .

Since the Laplace transform  $Lf$  may be regarded as the Fourier transform of  $\phi(t)$ , where

$$\phi(t) = (2\pi)^{1/2} e^{-xt} f(t); \quad t \geq 0$$

$$\phi(t) = 0; \quad t < 0,$$

it follows that

**Any piecewise-continuous function  $f(t)$  which possesses a Laplace transform  $Lf$  is unambiguously determined, at its points of continuity, by this Laplace transform.**

We symbolize this fundamental uniqueness theorem as follows:

$$f \rightleftharpoons Lf.$$

In the work which follows it is necessary not only to "cross" from  $f$  to  $Lf$ , i.e., to determine the Laplace transform of a given function  $f(t)$  but also to "cross" from  $Lf$  to  $f$ , i.e., given a function  $Lf$  of  $z$  to find the function  $f$  of which it is the Laplace transform. Our uniqueness theorem assures us that the sought for function  $f(t)$  is unambiguously determinate at any value of  $t$  at which it is continuous.

### EXERCISES

1. Show that the Laplace transform of the *unit function*, i.e., the function  $f(t) = 1, t \geq 0$ , is  $\frac{1}{z}, x > 0$ :

$$L(1) = \frac{1}{z}, \quad x > 0.$$

2. Show that if  $Lf = \phi(x), x > x_0$ , then  $L(e^{ct}f) = \phi(z - c)$  where  $R(z - c)$  (i.e., the *real part* of  $z - c$ )  $> x_0$ . *Note.* This important result may be phrased as follows:

**Multiplication by an exponential function of  $t$  is reflected in the complex plane by a translation of the origin.**

3. Show that  $L(e^{ct}) = \frac{1}{z - c}, R(z - c) > 0$ .

4. Show that  $L(\cos \alpha t) = \frac{z}{z^2 + \alpha^2}, x > |\text{Im}\alpha|$  (where  $\text{Im}\alpha$  denotes the imaginary part of the complex number  $\alpha$ ). *Hint.*  $\cos \alpha t = \frac{1}{2}(e^{i\alpha t} + e^{-i\alpha t})$ .

5. Show that  $L(\sin \alpha t) = \frac{\alpha}{z^2 + \alpha^2}, x > |\text{Im}\alpha|$ .

6. Determine  $L(\cosh \alpha t)$  and  $L(\sinh \alpha t)$ .

$$\text{Answer. } \frac{z}{z^2 + \alpha^2}, x > |\text{Re}\alpha|; \quad \frac{\alpha}{z^2 - \alpha^2}, x > |\text{Re}\alpha|.$$

7. Show that if  $Rm > 0, L(t^{m-1}) = \frac{\Gamma(m)}{z^m}, x > 0$ , where  $\Gamma(m) = \int_0^\infty e^{-t} t^{m-1} dt$ .

*Hint.* If  $z = x > 0, L(t^{m-1}) = \frac{\Gamma(m)}{x^m}$ . (Use the substitution  $xt = \tau$ .) The unambiguously determinate function of  $z$  which is analytic over the half-plane  $x > 0$  and which takes the values  $\frac{\Gamma(m)}{x^m}$  when  $z = x > 0$  is  $\frac{\Gamma(m)}{z^m}$ .

8. Show that  $f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{zt} (Lf) dy$  where the integration is along any line  $x = \text{constant}$  in the half-plane over which  $Lf$  is analytic and the relation is valid at all points  $t$  where the Fourier series of  $f(t)$ , for an interval  $[a, b]$  which covers  $t$ , converges to  $f(t)$ . *Hint.* This is merely a restatement of the Fourier integral theorem. *Note.* The result of this exercise furnishes a useful method of "crossing back" from  $Lf$  to  $f$ , i.e., of determining  $f$  when its Fourier transform is given. The

integral which furnishes  $f(t)$  is known as the *Bromwich integral* (after T. J. Bromwich [1875–1929], an English mathematician). In evaluating the Bromwich integral the path of integration may be deformed (by virtue of Cauchy's theorem) provided no singular points of  $Lf$  appear on the path during the deformation.

9. Show that multiplication by  $t$  is reflected in the complex  $z$ -plane by differentiation with respect to  $z$  followed by a change in sign:

$$L(tf) = -(Lf)_z; \quad x > x_0.$$

*Solution.* Setting  $g(s) = \int_0^s e^{-st}f(t) dt$  we have  $Lf = (z - z_0) \int_0^\infty e^{-(z-z_0)t} g(t) dt$ ,  $x > x_0$ . An easy calculation yields

$$\frac{\Delta Lf}{\Delta z} = \int_0^\infty e^{-(z-z_0)t} e^{-(\Delta z)t} g(t) dt + (z - z_0) \int_0^\infty e^{-(z-z_0)t} \frac{e^{-(\Delta z)t} - 1}{\Delta z} g(t) dt$$

and so

$$(Lf)_z = \int_0^\infty e^{-(z-z_0)t} g(t) dt - (z - z_0) \int_0^\infty e^{-(z-z_0)t} t g(t) dt$$

(why?). Upon integrating the second integral on the right by parts we obtain

$$(Lf)_z = - \int_0^\infty e^{-(z-z_0)t} t g(t) dt = - \int_0^\infty e^{-zt} f(t) dt$$

which proves the result of the exercise.

10. Use the result of Exercise 9 to verify that  $L(t) = \frac{1}{z^2}$ ,  $x > 0$ .

11. Show that if  $p$  is any positive integer  $L(t^p f) = (-1)^p \delta^p(Lf)$ ,  $x > x_0$ , where  $\delta$  denotes differentiation with respect to  $z$ .

12. Use the result of Exercise 11 to verify that if  $p$  is any positive integer  $L(t^p) = \frac{p!}{z^{p+1}}$ ,  $x > 0$ .

13. Show that  $L(te^{\alpha t}) = \frac{1}{(z - \alpha)^2}$ ,  $x > R(\alpha)$ .

14. Show that  $L(t \cos \alpha t) = \frac{z^2 + \alpha^2}{(z^2 + \alpha^2)^2}$ ,  $x > |\operatorname{Im} \alpha|$  and that  $L(t \sin \alpha t) = \frac{2\alpha z}{(z^2 + \alpha^2)^2}$ ,  $x > |\operatorname{Im} \alpha|$ .

15. Show that  $L(t^{1/2}) = \frac{\pi^{1/2}}{2z^{3/2}}$ ,  $x > 0$ ;  $L(t^{-1/2}) = \left(\frac{\pi}{z}\right)^{1/2}$ ,  $x > 0$ . *Hint.* Use the result of Exercise 7 and the facts that  $\Gamma(\frac{1}{2}) = \pi^{1/2}$ ,  $\Gamma(\frac{3}{2}) = \frac{1}{2}\pi^{1/2}$ .

16. Show that  $Lf$  is null at  $x = \infty$ .

*Solution.* Since the integral which defines  $Lf$  converges absolutely if  $x > x_0$  we can determine a number  $T$  such that the absolute value of  $\int_T^\infty e^{-st}f(t) dt$  is arbitrarily small, the choice of  $T$  being independent of  $z$  provided that  $x \geq x_0 + \delta$ , where  $\delta$  is any positive number. On writing  $\int_0^T$  in the form  $\int_0^{\delta'} + \int_{\delta'}^T$  it becomes

clear that the absolute value of  $\int_0^T e^{-xt} f(t) dt$  can be made arbitrarily small by making  $x$  sufficiently large. (Prove this. *Hint.*  $|e^{-xt}| = e^{-xt} \leq 1$  so that, if  $x > 0$ ,  $\left| \int_0^{\delta'} e^{-xt} f(t) dt \right| \leq M\delta'$ , where  $M$  is the maximum of  $|f(t)|$  over  $[0, T]$ . Over  $[\delta', T]$ ,  $|e^{-xt}| \leq e^{-x\delta'}$  so that  $\left| \int_{\delta'}^T e^{-xt} f(t) dt \right| < MT e^{-x\delta'}$  which is null at  $x = \infty$ .) Hence  $Lf = \left\{ \int_0^T + \int_T^\infty \right\} e^{-xt} f(t) dt$  is null at  $x = \infty$ . *Note.* The result of this exercise has the following important implication:

**Any analytic function of  $z$  which is not null at  $x = \infty$  fails to be the Laplace transform of any function of  $t$ .**

17. Determine the Laplace transform of  $\frac{\sin \alpha t}{t}$ . *Hint.* Since multiplication by  $t$  is reflected by differentiation with respect to  $z$ , followed by a change in sign, and since all Laplace transforms are null at  $x = \infty$ , division by  $t$  is reflected (provided the resulting function has a Laplace transform at all) by integration from  $z$  to  $x = \infty$ . Thus  $L\left(\frac{\sin \alpha t}{t}\right) = \alpha \int_z^{x=\infty} \frac{dz}{z^2 + \alpha^2} = \frac{\pi}{2} - \text{Arc tan } \frac{z}{\alpha} = \text{Arc tan } \frac{\alpha}{z}$ .

18. Determine  $L(\cos^2 \omega t)$ . *Hint.* Set  $\cos^2 \omega t = \frac{1}{2}(1 + \cos 2\omega t)$ .

$$\text{Answer. } \frac{z^2 + 2\omega^2}{z(z^2 + 4\omega^2)}, x > |\text{Im}2\omega|.$$

## 2. Determination of the function whose Laplace transform is a given proper rational fraction

It follows at once from the result of Exercise 16 of the preceding section that no improper rational fraction can be the Laplace transform of any function of  $t$ ; for an improper rational fraction is not null at  $z = \infty$ , and, consequently, it is not null at  $x = \infty$  (why?). On the other hand every proper rational fraction is the Laplace transform of a function of  $t$ ; this function is a linear combination of exponential functions of  $t$  each multiplied by a polynomial function of  $t$ . To see this we have merely to note that every proper rational fraction may be analyzed into simple fractions each of the type  $\frac{A_p}{(z - \alpha)^p}$  and  $A_p \frac{t^{p-1}}{(p - 1)!} e^{\alpha t}$  has

$\frac{A_p}{(z - \alpha)^p}$  as its Laplace transform. In actually determining the function of  $t$  which has a given proper rational fraction as its Laplace transform it is frequently convenient not to try to analyze the fraction completely but to use such known results as that which says that the Laplace transform of  $\frac{1}{\alpha} \sin \alpha t$  is  $\frac{1}{z^2 + \alpha^2}$ ,  $x > |\text{Im}\alpha|$ . The following examples will illustrate the method.

**Example 1.**  $Lf = \frac{1}{(z^2 + \alpha^2)^2}, x > |\text{Im}\alpha|$

Denoting differentiation with respect to  $z$  by  $\delta$  we have

$$\delta \left( \frac{z}{z^2 + \alpha^2} \right) = \frac{1}{z^2 + \alpha^2} - \frac{2z^2}{(z^2 + \alpha^2)^2} = \frac{2\alpha^2}{(z^2 + \alpha^2)^2} - \frac{1}{z^2 + \alpha^2}.$$

Hence  $f(t) = \frac{1}{2\alpha^3} \sin \alpha t - \frac{t}{2\alpha^2} \cos \alpha t.$

**Example 2.**  $Lf = \frac{1}{z(z + \alpha)}, x > 0, x > -R\alpha$

$$\frac{1}{z(z + \alpha)} = \frac{A}{z} + \frac{B}{z + \alpha} \quad \text{On multiplying by } z \text{ and setting } z = 0$$

we find  $A = \frac{1}{\alpha}$ ; on multiplying by  $z + \alpha$  and setting  $z = -\alpha$  we find

$$B = -\frac{1}{\alpha}. \quad \text{Hence (why?) } f(t) = \frac{1}{\alpha} (1 - e^{-\alpha t}).$$

**Example 3.**  $Lf = \frac{1}{z(z^2 + \alpha^2)}, x > |\text{Im}\alpha|$

$$\frac{1}{z(z^2 + \alpha^2)} = \frac{1}{\alpha^2 z} - \frac{z}{\alpha^2(z^2 + \alpha^2)}. \quad \text{Hence } f(t) = \frac{1}{\alpha^2} (1 - \cos \alpha t).$$

### EXERCISES

1. Determine the function of which  $\frac{z}{(z - \beta)^2 + \alpha^2}$ , where  $R(z - \beta) > |\text{Im}\alpha|$  and  $\alpha \neq 0$ , is the Laplace transform. *Hint.* Write  $\frac{z}{(z - \beta)^2 + \alpha^2}$  in the form

$$\frac{z - \beta}{(z - \beta)^2 + \alpha^2} + \frac{\beta}{(z - \beta)^2 + \alpha^2}. \quad \text{Answer. } e^{\beta t} \left\{ \cos \alpha t + \frac{\beta}{\alpha} \sin \alpha t \right\}.$$

2. What is the function whose Laplace transform is  $\frac{z}{(z - \beta)^2}$ ;  $R(z - \beta) > 0$ .  
*Answer.*  $e^{\beta t}(1 + \beta t).$

3. Determine the function of which  $\frac{1}{(z - \beta)^2 + \alpha^2}$ , where  $\alpha \neq 0$  and  $R(z - \beta) > |\text{Im}\alpha|$ , is the Laplace transform. *Answer.*  $\frac{1}{\alpha} e^{\beta t} \sin \alpha t.$

4. Show that if  $Lf$  is a proper rational fraction this proper rational fraction has near  $z = \infty$  the power series development

$$Lf = \frac{f(0)}{z} + \frac{Df(0)}{z^2} + \dots + \frac{D^{n-1}f(0)}{z^n} + \dots,$$

where  $D$  denotes differentiation with respect to  $t$ .

*Solution.* The result is evidently true if  $Lf = \frac{1}{z - \alpha}$  for then  $f = e^{\alpha t}$  and  $\frac{1}{z - \alpha}$  has, near  $z = \infty$ , the power series development  $\frac{1}{z} + \frac{\alpha}{z^2} + \dots + \frac{\alpha^{n-1}}{z^n} + \dots$ .

It is also true if  $Lf = \frac{1}{(z - \alpha)^p}$ , where  $p$  is any positive integer. In fact  $f(t)$  is then  $\frac{t^{p-1}}{(p-1)!} e^{\alpha t}$  so that  $D^k f = \frac{e^{\alpha t}}{(p-1)!} (D + \alpha)^k t^{p-1-k}$ . Hence  $D^k f(0) = 0$  if

$k < p - 1$  while  $D^{p-1} f(0) = 1$ ,  $D^p f(0) = p\alpha$ ,  $D^{p+1} f(0) = \frac{1}{2!} (p+1)p\alpha^2$ , and so on.

Since  $\frac{1}{(z - \alpha)^p}$  has near  $z = \infty$  the power series development

$$\frac{1}{z^p} + \frac{p\alpha}{z^{p+1}} + \frac{(p+1)p\alpha^2}{2!z^{p+2}} + \dots,$$

this shows that the statement of the exercise is true when  $Lf = \frac{1}{(z - \alpha)^p}$ . Since any proper rational fraction may be analyzed into a sum of fractions each of the form  $\frac{A_p}{(z - \alpha)^p}$ , where  $A_p$  is constant, the statement of the exercise is true for any proper rational fraction (why?).

### 3. The convolution process

Let  $f(t)$  and  $g(t)$  be any two piecewise-continuous functions of the non-negative variable  $t \geq 0$ . Consider the interval  $[0, t]$ , and form the product of the value of  $f(t)$  at any point  $\tau$  of this interval by the value of  $g(t)$  at the "complementary" point  $t - \tau$ . The integral of this product from 0 to  $t$  is a function of  $t$  which we term the *convolution* of  $f$  and  $g$  and which we denote by the symbol  $f * g$ :

$$f * g = \int_0^t f(\tau)g(t - \tau) d\tau.$$

On replacing the variable of integration  $\tau$  by  $\sigma = t - \tau$  we find that

$$f * g = \int_t^0 f(t - \sigma)g(\sigma)(-d\sigma) = \int_0^t g(\sigma)f(t - \sigma) d\sigma = g * f.$$

Thus

**The process of convolution is commutative:**

$$f * g = g * f.$$

It is easy to see that the convolution process is a *smoothing* one in the following sense:  $f * g$  is a continuous function of  $t$  despite the fact



that neither  $f$  nor  $g$  is necessarily continuous (they are assumed to be only piecewise-continuous). To prove this we observe that

$$\Delta(f * g) = \int_t^{t+\Delta t} f(\tau)g(t + \Delta t - \tau) d\tau + \int_0^t f(\tau) \Delta_t g(t - \tau) d\tau.$$

The first of these two integrals is null at  $\Delta t = 0$  (why?). To see that the second integral is also null at  $\Delta t = 0$  we cover the interval  $[0, t]$  by a net which is such that the points at which  $g(t - \tau)$ , regarded as a function of  $\tau$ , is discontinuous are interior points of cells of the net. Over each of the remaining cells of the net  $\Delta_t g(t - \tau)$  is arbitrarily small (uniformly with respect to  $\tau$ ), and since the sum of the lengths of the cells over which  $g(t - \tau)$  is not continuous is arbitrarily small it follows that  $\int_0^t f(\tau) \Delta_t g(t - \tau) d\tau$  is null at  $\Delta t = 0$ . This proves that

$f * g$  is a continuous function of the non-negative variable  $t \geq 0$ .

Let now  $f$  and  $g$  be such that  $Lf$  and  $Lg$  exist at a given point  $z_0 = x_0 + iy_0$  of the complex plane. Then the following important theorem is true:

$L(f * g)$  exists at any point  $P : (x, y)$  which lies to the right of  $P_0 : (x_0, y_0)$ , and

$$L(f * g) = (Lf)(Lg).$$

In other words

*Convolution is reflected in the complex  $z$ -plane by ordinary multiplication.*

The proof of the existence of  $L(f * g)$  at  $P$  is simple in view of the absolute convergence of the integrals which define  $Lf$  and  $Lg$  at  $P$ . In fact the absolute convergence at  $P$  of the integral

$$\int_0^\infty e^{-zt} dt \int_0^t f(\tau)g(t - \tau) d\tau$$

is assured if the integral  $\int_0^T e^{-xt} dt \int_0^t |f(\tau)g(t - \tau)| d\tau$  is a bounded function of  $T$  (why?). In view of the piecewise-continuous nature of the functions  $f$  and  $g$  the order of integration may be reversed in the integral last written, and when this is done it appears in the form

$$\int_0^T |f(\tau)| d\tau \int_\tau^T e^{-xt} |g(t - \tau)| dt.$$

On making the substitution  $t = \tau + \sigma$  this may be written as follows:

$$\int_0^T e^{-x\tau} |f(\tau)| d\tau \int_0^{T-\tau} e^{-x\sigma} |g(\sigma)| d\sigma,$$

and this is not greater than  $\int_0^{\infty} e^{-z\tau} |f(\tau)| d\tau \int_0^{\infty} e^{-z\sigma} |g(\sigma)| d\sigma$  (why?).

Thus the integral which defines  $L(f * g)$  converges absolutely at  $P$ . In order to show that  $L(f * g) = (Lf)(Lg)$  we observe that, in view of the absolute convergence of the integral

$$\int_0^{\infty} e^{-zt} dt \int_0^t f(\tau)g(t - \tau) d\tau,$$

it is legitimate to interchange the order of integration. Indeed if we extend the definition of  $g(t)$  by setting  $g(t) = 0$  if  $t < 0$  we may write the integral last written in the form

$$\int_0^{\infty} e^{-zt} dt \int_0^{\infty} f(\tau)g(t - \tau) d\tau$$

(why?), and it then appears as the sum of the double series  $\sum_0^{\infty} \sum_0^{\infty} a_{mn}$ ,

where

$$a_{mn} = \int_m^{m+1} e^{-zt} dt \int_n^{n+1} f(\tau)g(t - \tau) d\tau.$$

This series is absolutely convergent (since the integral is absolutely convergent) and so we may sum first with respect to  $m$  and then with respect to  $n$  (instead of first with respect to  $n$  and then with respect to  $m$ ). This is equivalent to interchanging the order of the integrations with respect to  $t$  and  $\tau$ . Thus

$$L(f * g) = \int_0^{\infty} f(\tau) d\tau \int_0^{\infty} e^{-z\sigma} g(\sigma) d\sigma.$$

On making the substitution  $t = \tau + \sigma$  and observing that  $g(\sigma) = 0$  if  $\sigma < 0$  we obtain

$$\begin{aligned} L(f * g) &= \int_0^{\infty} e^{-z\tau} f(\tau) d\tau \int_0^{\infty} e^{-z\sigma} g(\sigma) d\sigma \\ &= (Lf)(Lg). \end{aligned}$$

Convolution with the unit function is equivalent to integration over the interval  $[0, t]$ :

$$f * 1 = \int_0^t f(\tau) d\tau.$$

Since  $L(1) = \frac{1}{z}$  we have

$$L \left\{ \int_0^t f(\tau) d\tau \right\} = \frac{Lf}{z}$$

In words:

Integration with respect to  $t$  over the interval  $[0, t]$  is reflected in the complex  $z$ -plane by multiplication by the reciprocal function  $\frac{1}{z}$ .

If  $f(t)$  is differentiable with a piecewise-continuous derivative  $Df(t)$ , we have

$$f(t) = f(0) + \int_0^t Df(\tau) d\tau.$$

If, then,  $Df$  possesses at  $z_0 = x_0 + iy_0$  a Laplace transform we have

$$\frac{L(Df)}{z} = L\{f(t) - f(0)\} = Lf - \frac{f(0)}{z}; \quad x > x_0$$

or, equivalently,

$$L(Df) = zLf - f(0); \quad x > x_0.$$

In words:

Differentiation with respect to  $t$  is reflected in the complex  $z$ -plane by multiplication by  $z$  followed by the adjustment  $-f(0)$ . In particular, for those differentiable functions for which  $f(0) = 0$ , differentiation with respect to  $t$  is reflected in the complex  $z$ -plane by multiplication by  $z$ .

If  $f(t)$  possesses a second derivative  $D^2f$  which has at  $P_0: (x_0, y_0)$  a Laplace transform then

$$\begin{aligned} L(D^2f) &= zL(Df) - Df(0) \\ &= z^2Lf - zf(0) - Df(0); \quad x > x_0, \end{aligned}$$

and so on. In general if  $f(t)$  possesses a  $p$ th order derivative ( $p = 1, 2, \dots$ ) which has at  $P_0: (x_0, y_0)$  a Laplace transform then

$$L(D^p f) = z^p Lf - z^{p-1}f(0) - z^{p-2}Df(0) - \dots - D^{p-1}f(0); \quad x > x_0.$$

### EXERCISES

1. Deduce that  $L(\sin at) = \frac{\alpha}{z^2 + \alpha^2}$ ,  $x > |\operatorname{Im}\alpha|$ , from the fact that  $L(\cos at) = \frac{z}{z^2 + \alpha^2}$ ,  $x > |\operatorname{Im}\alpha|$ . *Hint.*  $\sin at = \alpha(\cos at * 1)$ .

2. Obtain the function whose Laplace transform is  $\frac{1}{(z^2 + \alpha^2)^2}$  by evaluating  $\frac{1}{\alpha^2} (\sin at * \sin at)$ .

3. Show that  $f * g$  is linear (what does this mean?) in each of the functions  $f$  and  $g$ .

4. Show that convolution is an associative process, i.e., that  $(f * g) * h = f * (g * h)$ . *Hint.* Use the fact that a function is unambiguously determined at its points of continuity by its Laplace transform and that ordinary multiplication of complex numbers is an associative process. *Note.* We denote the common value of  $(f * g) * h$  and  $f * (g * h)$  simply by  $f * g * h$ .

5. If  $Df$  exists and is piecewise-continuous show that, at any value of  $t$  at which  $g$  is continuous,

$$D(f * g) = f(0)g + (Df * g).$$

*Hint.*  $f(t) = f(0) + (Df * 1)$ . Hence  $f * g = (f(0) * g) + Df * 1 * g = f(0)(g * 1) + Df * g * 1$ . Since  $Df * g$  is everywhere continuous and since convolution with the unit function is equivalent to integration over the interval  $[0, t]$ , it follows (in view of the assumed continuity of  $g$ ) that  $D(f * g) = f(0)g + (Df * g)$ .

#### 4. The operational solution of an $n$ th order linear differential equation with constant coefficients

Denoting differentiation with respect to the independent variable  $t$  by  $D$ , the equation we wish to solve is  $p(D)y = f$ , where

$$p(D) = c_0 D^n + c_1 D^{n-1} + \cdots + c_n$$

is a polynomial *differential operator* with constant coefficients. The function  $f = f(t)$  is a given piecewise-continuous function which possesses (we assume) at some point  $z_0 = x_0 + iy_0$  of the complex  $z$ -plane a Laplace transform. We are only interested in the values of  $f$  and  $y$  for non-negative values of  $t$  and so we agree that if  $t < 0$  both  $f(t)$  and  $y(t)$  are zero. We take as granted the existence and uniqueness theorem of linear differential equations which assures us that there is an unambiguously determinate solution which has assigned *initial values*  $y(0), \cdots, D^{n-1}y(0)$  for itself and its derivatives up to the  $(n - 1)$ st order, inclusive. In order to determine this solution we *assume* that its  $n$ th derivative possesses at some point  $z_0 = x_0 + iy_0$  of the complex  $z$ -plane a Laplace transform, and this point  $z_0$  may be taken, without loss of generality, to be the same as the point at which  $f$  has been granted to possess a Laplace transform (why?). If, then,  $z$  is any point of the complex  $z$ -plane for which  $x > x_0$ , the following relations are valid

$$L(Dy) = zLy - y(0);$$

$$L(D^2y) = z^2Ly - zy(0) - Dy(0);$$

. . .

$$L(D^ny) = z^nLy - z^{n-1}y(0) \cdots - D^{n-1}y(0).$$

On substituting these expressions in the relation  $L\{p(D)y\} = Lf$ , we obtain

$$p(z)Ly = p_{n-1}(z)y(0) + p_{n-2}(z)Dy(0) + \cdots + p_0D^{n-1}y(0) + Lf,$$

where the polynomial functions of  $z$ ,  $(p_{n-1}(z), \cdots, p_0)$ , are furnished by the following formulas:

$$p_{n-1}(z) = c_0z^{n-1} + c_1z^{n-2} + \cdots + c_{n-1};$$

$$p_{n-2}(z) = c_0z^{n-2} + c_1z^{n-3} + \cdots + c_{n-2};$$

...

$$p_1(z) = c_0z + c_1;$$

$$p_0 = c_0.$$

*Note.* The following observation makes it easy to remember these formulas:

$p_{n-j}$ ,  $j = 1, 2, \cdots, n$ , is the polynomial part of the quotient of  $p(z)$  by  $z^j$ , where  $p(z) = c_0z^n + c_1z^{n-1} + \cdots + c_n$ .

It is clear from the relation

$$p(z)Ly = p_{n-1}(z)y(0) + \cdots + p_0D^{n-1}y(0) + Lf$$

that the simplest initial conditions (from the point of view of determining, by the method we are following, the unambiguously determinate solution) are

$$y(0) = 0, Dy(0) = 0, \cdots, D^{n-1}y(0) = 0.$$

These initial conditions are those for which  $y(t)$  simulates as closely as possible, near  $t = 0$ , the zero constant function. We term the solution furnished by these initial conditions the *rest solution* of the differential equation, and we denote this solution by  $r(t)$ . Thus

**The rest solution  $r = r(t)$  of the differential equation  $p(D)y = f$  is that solution for which  $r(0), Dr(0), \cdots, D^{n-1}r(0)$  are all zero.**

It follows (why?) that

$$p(z)Lr = Lf$$

or, equivalently, that

$$Lr = \frac{1}{p(z)}Lf; \quad p(z) \neq 0.$$

In order to make sure that the condition  $p(z) \neq 0$  is fulfilled we agree that the point  $z = x + iy$  of the complex  $z$ -plane lies to the right of

each zero of the  $n$ th degree polynomial  $p(z)$ . Then  $\frac{1}{p(z)}$  is a proper rational fraction so that there exists a function  $q = q(t)$  which has  $\frac{1}{p(z)}$  as its Laplace transform. Hence

$$Lr = (Lq)(Lf)$$

so that

$$r = q * f$$

or, equivalently,

$$r(t) = \int_0^t q(\tau)f(t - \tau) d\tau.$$

In other words

*The determination of the rest solution  $r = r(t)$  has been made to rest on the determination of the function  $q = q(t)$  whose Laplace transform is  $\frac{1}{p(z)}$ .*

Once the rest solution  $r = r(t)$  has been determined the solution  $y = y(t)$  for which the initial values of  $y, Dy, \dots, D^{n-1}y$  are not all zero may be found as follows. We have

$$p(z)Ly = p_{n-1}(z)y(0) + \dots + p_0D^{n-1}y(0) + Lf$$

so that

$$Ly = \frac{p_{n-1}(z)}{p(z)}y(0) + \dots + \frac{p_0}{p(z)}D^{n-1}y(0) + \frac{Lf}{p(z)}.$$

Each of the expressions  $\frac{p_{n-1}(z)}{p(z)}, \dots, \frac{p_0}{p(z)}$  is a proper rational fraction and so there exist functions  $s_{n-1}(t), \dots, s_0(t)$  which are such that

$$Ls_{n-1} = \frac{p_{n-1}(z)}{p(z)}, \dots, Ls_0 = \frac{p_0}{p(z)}.$$

These functions are readily expressible in terms of the function  $q = q(t)$ . To see this we first observe that  $q(t)$  is zero, together with its derivatives up to the  $(n-2)$ nd, inclusive, at  $t = 0$ . In fact the development of  $\frac{1}{p(z)}$  near  $z = \infty$  is

$$\frac{1}{p(z)} = \frac{1}{c_0z^n} - \frac{c_1}{c_0^2z^{n+1}} + \dots$$

Hence (why?)

$$q(0) = 0, Dq(0) = 0, \dots, D^{n-2}q(0) = 0;$$

$$D^{n-1}q(0) = \frac{1}{c_0}; \quad D^n q(0) = \frac{-c_1}{c_0^2}; \quad \dots$$

Since  $Ls_0 = \frac{p_0}{p(z)} = \frac{c_0}{p(z)}$  we have  $s_0(t) = c_0 q_0(t)$ , and since  $Ls_1 = \frac{c_0 z + c_1}{p(z)}$  we have

$$\begin{aligned} s_1(t) &= c_0 Dq(t) + c_1 q(t) \quad (\text{why?}) \\ &= (c_0 D + c_1)q(t). \end{aligned}$$

Continuing in this way we see that

$$\begin{aligned} s_j(t) &= c_0 D^j q(t) + \dots + c_j q(t); \quad j = 0, 1, \dots, n-1, \\ &= (c_0 D^j + c_1 D^{j-1} + \dots + c_j)q(t). \end{aligned}$$

The desired (general) solution of the differential equation

$$p(D)y = f$$

is

$$y = s_{n-1}y(0) + \dots + s_0 D^{n-1}y(0) + r$$

or, equivalently,

$$y(t) = s_{n-1}(t)y(0) + \dots + s_0(t)D^{n-1}y(0) + q * f,$$

where

$$\begin{aligned} s_{n-1}(t) &= c_0 D^{n-1}q(t) + \dots + c_{n-1}q(t) \\ &= (c_0 D^{n-1} + c_1 D^{n-2} + \dots + c_{n-1})q(t); \\ s_{n-2}(t) &= c_0 D^{n-2}q(t) + \dots + c_{n-2}q(t) \\ &= (c_0 D^{n-2} + c_1 D^{n-3} + \dots + c_{n-2})q(t); \\ &\cdot \\ &\cdot \\ &\cdot \end{aligned}$$

$$s_0(t) = c_0 q(t).$$

It is easy to see that the function  $q = q(t)$  is a solution of the *associated homogeneous equation*

$$p(D)u = 0.$$

In fact, since  $q(0) = 0, Dq(0) = 0, \dots, D^{n-2}q(0) = 0, D^{n-1}q(0) = \frac{1}{c_0}$ , we have

$$Lq = \frac{1}{p(z)}; \quad L(Dq) = \frac{z}{p(z)}; \quad \dots, \quad L(D^{n-1}q) = \frac{z^{n-1}}{p(z)};$$

$$L(D^n q) = \frac{z^n}{p(z)} - \frac{1}{c_0}.$$

Hence

$$L\{p(D)q\} = c_0 L(D^n q) + \dots + c_n L(q)$$

$$= \frac{p(z)}{p(z)} - 1 = 0,$$

and, since a continuous function is unambiguously determined by its Laplace transform, it follows that  $p(D)q = 0$ . Thus

$q = q(t)$  is that solution of the homogeneous equation  $p(D)u = 0$  which is zero together with its derivatives up to the  $(n - 2)$ nd inclusive at  $t = 0$  while its  $(n - 1)$ st derivative has the value  $\frac{1}{c_0}$  at  $t = 0$ .

*Note.* The only rest solution of the homogeneous equation, i.e., the only solution of the homogeneous equation which vanishes together with all its derivatives, up to the  $(n - 1)$ st, inclusive, at  $t = 0$ , is the zero constant function (why?). Thus  $q = q(t)$  may be regarded as the solution of the homogeneous equation which is as nearly as possible, near  $t = 0$ , the rest solution (i.e., the *dead* solution) without being dead and which is normalized by the requirement that  $c_0 D^{n-1}q(0) = 1$ .

**Example 1.**  $(D^2 + n^2)y = f$

Here  $p(z) = z^2 + n^2$  so that  $q(t) = \frac{1}{n} \sin nt$  if  $n \neq 0$ ; if  $n = 0$ ,  $q(t) = t$ . When  $n \neq 0$  we have  $s_0(t) = \frac{1}{n} \sin nt$ ;  $s_1(t) = \cos nt$  since  $c_0 = 1$ ,  $c_1 = 0$ . Hence  $y(t) = \cos nt y(0) + \frac{1}{n} \sin nt Dy(0) + \frac{1}{n} (\sin nt * f)$ . If  $n = 0$ ,  $s_0(t) = t$ ,  $s_1(t) = 1$  and  $y(t) = y(0) + tDy(0) + (t * f)$ .

**Example 2.**  $(D^2 + n^2)y = \cos \omega t$

Here  $p(z) = z^2 + n^2$  so that, if  $n \neq 0$ ,  $q(t) = \frac{1}{n} \sin nt$  while, if  $n = 0$ ,  $q(t) = t$ . We shall treat separately the two cases  $n \neq 0$  and  $n = 0$ .



Case 1.  $n \neq 0$ .

Since  $c_0 = 1$ ,  $c_1 = 0$ ,  $c_2 = n^2$ , we have

$$s_1 = c_0 Dq + c_1 = \cos nt; \quad s_0 = c_0 q = \frac{1}{n} \sin nt.$$

Furthermore  $r = \cos \omega t * \frac{1}{n} \sin nt$ .  $r$  is best obtained from the rela-

tion  $Lr = \frac{z}{(z^2 + n^2)(z^2 + \omega^2)}$ ; if  $\omega^2 \neq n^2$  we obtain, on analyzing  $Lr$

into  $\frac{1}{n^2 - \omega^2} \left\{ \frac{z}{z^2 + \omega^2} - \frac{z}{z^2 + n^2} \right\}$ ,  $r = \frac{1}{n^2 - \omega^2} (\cos \omega t - \cos nt)$ . If

$\omega^2 = n^2$ ,  $r = \frac{t}{2n} \sin nt$ . Thus we have the two following subcases:

Case 1a.  $\omega^2 \neq n^2$ .

$$y = \frac{1}{n^2 - \omega^2} (\cos \omega t - \cos nt) + (\cos nt)y(0) + \left( \frac{1}{n} \sin nt \right) Dy(0).$$

Case 1b.  $\omega^2 = n^2$ .

$$y = \frac{t}{2n} \sin nt + (\cos nt)y(0) + \left( \frac{1}{n} \sin nt \right) Dy(0).$$

Case 2.  $n = 0$ .

Here  $Lr = \frac{z}{z^2(z^2 + \omega^2)} = \frac{1}{z(z^2 + \omega^2)}$ . If  $\omega \neq 0$  we have  $Lr =$

$\frac{1}{\omega^2 z} - \frac{z}{\omega^2(z^2 + \omega^2)}$  so that  $r(t) = \frac{1}{\omega^2} (1 - \cos \omega t)$ . If  $\omega = 0$ ,  $r(t) = \frac{1}{2} t^2$ .

Since  $q(t) = t$ , we have  $s_1(t) = 1$ ,  $s_0(t) = t$ . We have the two following subcases:

Case 2a.  $\omega \neq 0$ .

$$y = \frac{1}{\omega^2} (1 - \cos \omega t) + y(0) + tDy(0).$$

Case 2b.  $\omega = 0$ .

$$y = \frac{1}{2} t^2 + y(0) + tDy(0).$$

*Note.* This example covers the theory of the forced oscillations (including resonance effects) of a simple undamped *harmonic oscillator*.

## EXERCISES

1. Show that each of the functions  $s_0(t), \dots, s_{n-1}(t)$  is a solution of the homogeneous equation  $p(D)u = 0$ . *Hint.* Since  $q(t)$  is a solution of the homogeneous equation so also is  $D^j q(t)$ ,  $j = 1, 2, \dots$ .

2. Show that  $s_{n-1}(0) = 1, Ds_{n-1}(0) = 0, \dots, D^{n-1}s_{n-1}(0) = 0$ . *Hint.*  $s_{n-1}(0) = c_0 D^{n-1}q(0) = 1$ ;  $Ds_{n-1}(0) = c_0 D^n q(0) + c_1 D^{n-1}q(0) = 0$ , since  $p(D)q = 0$ ;  $D^2s_{n-1}(0) = c_0 D^{n+1}q(0) + c_1 D^n q(0) + c_2 D^{n-1}q(0) = 0$ , since  $D\{p(D)q\} = 0$ , and so on.

3. Show that  $s_{n-2}(0) = 0, Ds_{n-2}(0) = 1, D^2s_{n-2}(0) = 0, \dots, D^{n-1}s_{n-2}(0) = 0$ .

4. Determine the initial values of  $s_{n-j}(t), Ds_{n-j}(t), \dots, D^{n-1}s_{n-j}(t)$ ,  $j = 3, 4, \dots, n$ .

5. Verify that the initial values of  $r, Dr, \dots, D^{n-1}r$  are all zero. *Hint.* Since  $r = q * f$ ,  $r(0) = 0$  (any convolution being zero at  $t = 0$  (why?)). Since  $q(0) = 0$ ,  $Dr = Dq * f$  (see Exercise 5, p. 347). Hence  $Dr(0) = 0$  (why?). Similarly  $D^2r(0) = 0, \dots, D^{n-1}r(0) = 0$ . *Question.* Is this verification necessary?

*Answer.* Yes.

6. Verify that  $p(D)r = f$ . *Hint.*  $r = q * f$ ;  $Dr = Dq * f$ ;  $\dots, D^{n-1}r = D^{n-1}q * f$ ;  $D^n r = D^{n-1}q(0)f + (D^n q * f) = \frac{1}{c_0}f + (D^n q * f)$ . Hence  $p(D)r = f + (p(D)q * f) = f$  (why?). *Question.* Is this verification necessary?

7. Verify that  $y = s_{n-1}y(0) + \dots + s_0 D^{n-1}y(0) + r$  is the unambiguously determinate solution of  $p(D)y = f$  for which  $y, Dy, \dots, D^{n-1}y$  have the assigned initial values  $y(0), Dy(0), \dots, D^{n-1}y(0)$ .

## 5. The principle of superposition

The *principle of superposition* furnishes a method for determining the *rest solution* of the differential equation  $p(D)y = f$  from the *rest solution* of the differential equation  $p(D)y = 1$ . In other words

The principle of superposition enables us to determine the rest solution of the differential equation  $p(D)y = f$ , where  $f = f(t)$  is an arbitrary piecewise-continuous "applied impulse" (which possesses at some point  $z_0 = x_0 + iy_0$  of the complex  $z$ -plane a Laplace transform), once we have solved the problem of determining the rest solution of the simpler differential equation  $p(D)y = 1$ , where 1 denotes the unit-applied impulse defined by  $1(t) = 1, t \geq 0, 1(t) = 0, t < 0$ .

*Warning.* Be very clear from the beginning that the principle of superposition applies only to rest solutions. Never attempt to apply it to solutions which are not rest solutions.

We shall denote the rest solution corresponding to the applied unit impulse by  $r_1(t)$  (the subscript 1 may help to focus your attention on the fact that it is not the general rest solution but the particular rest solution obtained when  $f(t) = 1, t \geq 0$ ). Since  $L(1) = \frac{1}{z}$ , we have

$$Lr_1 = \frac{1}{zp(z)}.$$

Hence

$$\begin{aligned} Lr &= \frac{Lf}{p(z)} = (zLf)(Lr_1) \\ &= \{L(Df) + f(0)\}(Lr_1). \end{aligned}$$

Thus (why?)

$$r = f(0)r_1 + Df * r_1;$$

i.e.,

$$r(t) = f(0)r_1(t) + \int_0^t Df(\tau)r_1(t - \tau) d\tau.$$

This is the *principle of superposition* which has been variously attributed to Maxwell, Boltzmann, and others. (J. C. Maxwell [1831–1879] was an English applied mathematician; L. Boltzmann [1844–1906] was an Austrian applied mathematician). We formulate the principle of superposition as follows:

**The rest solution  $r = r(t)$  of the equation  $p(D)y = f$  may be obtained from the rest solution  $r_1 = r_1(t)$  of the equation  $p(D)y = 1$  by the formula**

$$r(t) = f(0)r_1(t) + \int_0^t Df(\tau)r_1(t - \tau) d\tau.$$

### EXERCISES

1. Show that  $r_1(t) = \int_0^t q(\tau) d\tau$ . *Hint.*  $Lr_1 = \frac{1}{zp(z)}$ ;  $L\left\{\int_0^t q(\tau) d\tau\right\} = L(1 * q) = \frac{1}{zp(z)}$ .

2. Show that  $r = D(f * r_1)$ . *Hint.* Show that  $r$  and  $D(f * r_1)$  have the same Laplace transform. *Note.* The various different formulas

$$r = q * f; \quad r = f(0)r_1 + (Df * r_1); \quad r = D(f * r_1)$$

are merely different versions of the principle of superposition.

### 6. The Heaviside expansion formula

The determination of the function  $q = q(t)$  on which the solution of the non-homogeneous equation  $p(D)y = f$  has been made to rest involves only the analysis of  $\frac{1}{p(z)}$  into simple fractions. Similarly the determination of the rest solution  $r_1 = r_1(t)$  of the non-homogeneous equation

$p(D)y = 1$  involves only the analysis of  $\frac{1}{zp(z)}$  into simple fractions.

In the case where  $p(z)$  has  $n$  simple zeros it is easy to derive a simple explicit formula for  $q(t)$ ; similarly in the case where  $p(z)$  has  $n$  simple zeros, none of which is zero, it is easy to give a simple explicit formula for  $r_1(t)$ . In fact, if we denote the  $n$  simple zeros of  $p(z)$  by  $\alpha_1, \dots, \alpha_n$ , we have

$$\frac{1}{p(z)} = \frac{A_1}{z - \alpha_1} + \dots + \frac{A_n}{z - \alpha_n}.$$

Since  $p(\alpha_j) = 0$ , the development of  $p(z)$  near  $z = \alpha_j$  starts out with the term  $\{\delta p(\alpha_j)\}(z - \alpha_j)$ , where  $\delta$  indicates differentiation with respect to  $z$ ; the fact that  $\alpha_j$  is a simple zero of  $p(z)$  assures us that  $\delta p(\alpha_j) \neq 0$ . Hence the negative-power part of the Laurent development of  $\frac{1}{p(z)}$  near its isolated singularity  $z = \alpha_j$  is  $\frac{1}{\{\delta p(\alpha_j)\}(z - \alpha_j)}$ .

Since each of the fractions  $\frac{A_k}{z - \alpha_k}$ ,  $k \neq j$ , is analytic at  $z = \alpha_j$  it follows

that  $A_j = \frac{1}{\delta p(\alpha_j)}$ ,  $j = 1, \dots, n$ . Thus

$$\frac{1}{p(z)} = \sum_{j=1}^n \frac{1}{\{\delta p(\alpha_j)\}(z - \alpha_j)}$$

and so

$$q(t) = \sum_{j=1}^n \frac{e^{\alpha_j t}}{\delta p(\alpha_j)}.$$

Similarly, provided that none of the zeros  $\alpha_1, \dots, \alpha_n$ , of  $p(z)$  is zero,

$$\frac{1}{zp(z)} = \frac{1}{p(0)z} + \sum_{j=1}^n \frac{1}{\{\alpha_j \delta p(\alpha_j)\}(z - \alpha_j)}$$

and so

$$r_1(t) = \frac{1}{p(0)} + \sum_{j=1}^n \frac{e^{\alpha_j t}}{\alpha_j \delta p(\alpha_j)}.$$

This result is known as the *Heaviside expansion formula* (after O. Heaviside [1850–1925], an English electrical engineer). It is possible to derive similar (but more complicated) formulas for  $q(t)$  in the case where  $p(z)$  has repeated zeros and for  $r_1(t)$  in the case where  $p(z)$  has

repeated zeros or where one or more of the zeros of  $p(z)$  are zero; but this is not worth while. To determine  $q(t)$  or  $r_1(t)$  in any given problem we adopt the following simple rule of procedure:

To determine  $q(t)$  analyze  $\frac{1}{p(z)}$  into simple fractions, and read off from these the function of  $t$  which has  $\frac{1}{p(z)}$  as its Laplace transform:

To determine  $r_1(t)$  either integrate  $q(t)$  from 0 to  $t$  or analyze  $\frac{1}{zp(z)}$  into simple fractions, and read off from these the function of  $t$  which has  $\frac{1}{zp(z)}$  as its Laplace transform.

### EXERCISES

1. Determine the general solution of  $(D^2 + 3D + 2)u = 0$ . *Hint.*  $q(t) = e^{-t} - e^{-2t}$ ,  $s_1(t) = (D + 3)q = 2e^{-t} - e^{-2t}$ ,  $s_0(t) = q(t)$ .

*Answer.*  $u(t) = (2e^{-t} - e^{-2t})u(0) + (e^{-t} - e^{-2t})Du(0)$ .

2. Find the general solution of  $D(D + 1)^2u = 0$ .

*Answer.*  $u(t) = u(0) + (2 - 2e^{-t} - te^{-t})Du(0) + (1 - e^{-t} - te^{-t})D^2u(0)$ .

3. Find the rest solution of  $D(D - 1)y = t^2$ . *Hint.*  $L(r) = \frac{2}{z^2(z - 1)}$ .

*Answer.*  $r(t) = 2e^t - \frac{1}{3}t^3 - t^2 - 2t - 2$ .

4. Find the rest solution of  $(D^2 + 1)y = t \cos 2t$ . *Hint.* The Laplace transform of  $t \cos 2t = -\delta \left\{ \frac{z}{z^2 + 4} \right\} = \frac{1}{z^2 + 4} - \frac{8}{(z^2 + 4)^2}$ .

*Answer.*  $r(t) = -\frac{5}{9} \sin t + \frac{4}{9} \sin 2t - \frac{1}{3}t \cos 2t$ .

5. Find the rest solution of  $(D^2 + 1)^2y = \sin t$ .

*Answer.*  $r(t) = \frac{1}{8}(3 - t^2) \sin t - \frac{3}{8}t \cos t$ .

6. Find the rest solution of  $(D^2 + 1)^2y = t \sin t$ .

*Answer.*  $r(t) = \frac{1}{24}\{3t - t^3\} \sin t - 3t^2 \cos t$ .

### 7. The operational solution of a system of linear differential equations with constant coefficients

The system of linear differential equations may be written in the form

$$P(D)y = f,$$

where  $y = v(y^1, \dots, y^m)$  is a vector dependent variable,  $f = v(f^1, \dots, f^m)$  is a given vector function of the one independent variable  $t$ , and  $P(D)$  is an  $m \times m$  matrix whose elements  $P_r^s(D)$  are given polynomial functions of the differential operator  $D$  (which indicates differentiation with respect to the independent variable  $t$ ). If the highest-order derivative (i.e., the highest power of  $D$ ) that appears

amongst the various elements  $P_r^s(D)$  of  $P(D)$  is the  $n$ th we may write  $P(D)$  in the form

$$P(D) = C_0 D^n + C_1 D^{n-1} + \cdots + C_n$$

where  $C_0, C_1, \cdots, C_n$  are  $m \times m$  matrices of which the first,  $C_0$ , is not the zero matrix. At this point an *essential* difference between the theory of the operational solution of a system of  $m$  differential equations for  $m$  unknowns ( $m > 1$ ) and the theory of the operational solution of a single differential equation appears. For a single differential equation  $C_0 = c_0$  is a  $1 \times 1$  matrix, i.e., an ordinary number. Since  $c_0 \neq 0$  it possesses a reciprocal; in other words every  $1 \times 1$  matrix, other than the zero  $1 \times 1$  matrix, is non-singular. Matters are very different when  $m > 1$ , i.e., when we are considering a system of differential equations.  $C_0$  is still, by hypothesis, not the zero matrix, but it may well be singular; in other words its *determinant* may be zero. The particular case that occurs when  $\det C_0 \neq 0$  may be treated in a manner that is almost a verbal repetition of the treatment already given for the case of a single differential equation. When  $\det C_0 \neq 0$  we say that the system of differential equations is *normal* and when  $\det C_0 = 0$  we say that the system is *abnormal*. Thus every system for which  $m = 1$ , i.e., every system consisting of a single differential equation, is normal.

The Laplace-transform operation may be applied to matrices of any number of rows and columns as well as to ordinary complex-valued functions (which may be regarded as  $1 \times 1$  matrices). To obtain the Laplace transform of a matrix we simply form the Laplace transform of each element of the matrix. In the same way the convolution process may be applied to matrices. If, for example,  $R(t)$  is an  $m \times m$  matrix and  $f(t)$  an  $m \times 1$  matrix (where each element of the matrices  $R(t)$  and  $f(t)$  is a piecewise-continuous function of  $t$  which assumes the value zero when  $t < 0$ ) we form  $R * f$  as follows: We first form the matrix product  $R(t - \tau)f(\tau)$ , and then we integrate this product with respect to  $\tau$  from 0 to  $t$ :

$$R * f = \int_0^t R(t - \tau)f(\tau) d\tau.$$

On replacing the variable  $\tau$  of integration by  $t - \sigma$  it is clear that an equivalent definition of  $R * f$  is

$$R * f = \int_0^t R(\sigma)f(t - \sigma) d\sigma.$$

Either of the products  $R(t - \tau)f(\tau)$ ,  $R(\sigma)f(t - \sigma)$  is an  $m \times 1$  matrix, and we understand by the integral of either of these matrices the matrix obtained by integrating each of its elements. *Warning.* Do not be slipshod concerning the order in which matrix products are formed and write  $f * R$  instead of  $R * f$ . Since  $f$  is an  $m \times 1$  matrix and  $R$  an  $m \times m$  matrix the product  $f(\tau)R(t - \tau)$  cannot be formed; even if it could, matrix multiplication is not, in general, commutative and so convolution of matrices is not, in general, commutative.

We now proceed to determine the solution of the system of  $m$  ordinary differential equations

$$P(D)y = f$$

which satisfies appropriately formulated initial conditions. We assume that the vector  $f$  and the  $n$ th derivative of the vector  $y$  possess, at a common point  $z_0 = x_0 + iy_0$  of the complex  $z$ -plane, Laplace transforms, and we subject the equation  $P(D)y = f$  to the Laplace transformation. It follows from the definition of the Laplace transform of a vector that

$$\begin{aligned} L(Dy) &= zLy - y(0); \\ L(D^2y) &= z^2Ly - zy(0) - Dy(0); \\ &\cdot \\ &\cdot \\ &\cdot \\ L(D^ny) &= z^nLy - z^{n-1}y(0) \cdot \cdot \cdot - D^{n-1}y(0), \end{aligned}$$

and so

$$\begin{aligned} L\{P(D)y\} &= L\{C_0D^ny + \cdot \cdot \cdot + C_ny\} \\ &= P(z)Ly - P_{n-1}(z)y(0) - \cdot \cdot \cdot - P_0D^{n-1}y(0), \end{aligned}$$

where the  $m \times m$  matrices  $P_{n-1}(z)$ ,  $\cdot \cdot \cdot$ ,  $P_0(z)$  are polynomial functions of  $z$  (each of the degree indicated by its subscript) which are furnished by the formulas

$$\begin{aligned} P_{n-1}(z) &= C_0z^{n-1} + \cdot \cdot \cdot + C_{n-1}; \\ P_{n-2}(z) &= C_0z^{n-2} + \cdot \cdot \cdot + C_{n-2}; \\ &\cdot \\ &\cdot \\ &\cdot \\ P_0(z) &= C_0. \end{aligned}$$

These formulas are easily remembered if you note that  $P_j(z)$  is the polynomial part of the quotient of  $P(z)$  by  $z^{n-j}$ ,  $j = 0, 1, \dots, n-1$ . We have, then, the relation

$P(z)Ly = P_{n-1}(z)y(0) + \dots + P_0D^{n-1}y(0) + Lf$ . The expression  $P_{n-1}(z)y(0) + \dots + P_0D^{n-1}y(0)$  is an  $m \times 1$  matrix which is a polynomial function of  $z$  of degree  $\leq n-1$ . The coefficient of  $z^{n-1}$  in this polynomial function is  $C_0y(0)$ ; the coefficient of  $z^{n-2}$  is  $C_1y(0) + C_0Dy(0)$ , and so on. Hence this polynomial function of  $z$  will vanish identically if, and only if, the initial values of  $y$  and its derivatives up to the  $(n-1)$ st inclusive are adjusted to the equations

$$C_0y(0) = 0;$$

$$C_1y(0) + C_0Dy(0) = 0;$$

...

$$C_{n-1}y(0) + C_{n-2}Dy(0) + \dots + C_0D^{n-1}y(0) = 0.$$

We shall term any solution of our system of differential equations for which the initial values of the solution and of its derivatives are adjusted to these equations a *rest solution*, and we shall denote such a rest solution by  $r = r(t)$ . It follows that

$$P(z)Lr = Lf.$$

There is no lack of generality in assuming that the matrix  $P(z)$  is not singular for every value of  $z$ . In fact if we denote by  $\text{adj } P$  the adjoint matrix of  $P(z)$ , i.e., the matrix obtained by interchanging the rows and columns of the cofactor matrix of  $P(z)$ , we see, on applying to the given differential equation,  $P(D)y = f$ , the matrix differential operator  $\text{adj } P(D)$ , that

$$\det P(D)y = (\text{adj } P(D))f.$$

If, then,  $\det P(z)$  is independent of  $z$  each coordinate of  $y$  is furnished by a linear algebraic equation so that we have left the province of differential equation theory. It may well be that one of the linear algebraic equations which furnish the coordinates of  $y$  is inconsistent; in this case the original system of differential equations is inconsistent. On the other hand it may happen that all the equations furnishing the coordinates of  $y$  are indeterminate; in this case the original system is indeterminate (one or more of the differential equations being a linear combination of the rest). In any event we take it as granted that  $\det P(z)$  is a polynomial function of  $z$  of degree  $\geq 1$  and so  $P(z)$  is



singular for only a finite number of values of  $z$ . Choosing  $z$  so that the point  $z = x + iy$  lies to the right of each of the zeros of  $\det P(z)$  we are assured that  $P(z)$  possesses a reciprocal  $P^{-1}(z)$ , and, then, the equation  $P(z)Lr = Lf$  is equivalent to the equation

$$Lr = P^{-1}(z)Lf.$$

Since a matrix is unambiguously determined by its Laplace transform it follows that our system of differential equations does not possess more than one rest solution. We can, then, appropriately speak of *the* rest solution of the given system.

When the system of differential equations is a normal one the rest solution is the one which vanishes together with its derivatives up to the  $(n - 1)$ st inclusive at  $t = 0$ . In fact since  $C_0$  is non-singular the equation  $C_0y(0) = \mathbf{0}$  yields  $y(0) = \mathbf{0}$ ; the next equation  $C_0Dy(0) + C_1y(0) = \mathbf{0}$  reduces, then, to  $C_0Dy(0) = \mathbf{0}$ , and this yields  $Dy(0) = \mathbf{0}$ , and so on. Thus

**The rest solution of a normal system of linear differential equations is that solution which vanishes, together with its derivatives up to the  $(n - 1)$ st, inclusive, at  $t = 0$ .**

We have seen that the Laplace transform of the rest solution of our system of differential equations (assuming that the  $n$ th derivative of the rest solution possesses a Laplace transform) is furnished, whether the system is normal or abnormal, by the formula

$$Lr = P^{-1}(z)Lf.$$

If each element of  $P^{-1}(z)$  is a proper rational fraction there exists an  $m \times m$  matrix  $Q(t)$  whose Laplace transform is  $P^{-1}(z)$  and so

$$Lr = (LQ)(Lf).$$

It follows, since the process of convolution is linear in each of its factors, that

$$r = Q * f$$

(prove this) or, equivalently, that

$$r(t) = \int_0^t Q(t - \tau)f(\tau) d\tau = \int_0^t Q(\sigma)f(t - \sigma) d\sigma.$$

It may happen that not every element of  $P^{-1}(z)$  is a proper rational fraction but that every element of  $\frac{1}{z}P^{-1}(z)$  is a proper rational fraction.

There exists, then, an  $m \times m$  matrix  $R(t)$  whose Laplace transform is  $\frac{1}{z} P^{-1}(z)$ , and

$$L(\mathbf{r}) = (LR)(zL\mathbf{f}) = (LR)\{L(D\mathbf{f}) + \mathbf{f}(0)\}$$

(it being granted that the vector  $\mathbf{f}(t)$  possesses a piecewise-continuous derivative  $D\mathbf{f}$  which has a Laplace transform). We obtain, then, the *principle of superposition* for any system of differential equations, normal or abnormal, which is such that each element of  $\frac{1}{z} P^{-1}(z)$  is a proper rational fraction:

$$\mathbf{r} = R\mathbf{f}(0) + R * D\mathbf{f},$$

or, equivalently,

$$\mathbf{r}(t) = R(t)\mathbf{f}(0) + \int_0^t R(t - \tau)D\mathbf{f}(\tau) d\tau.$$

Corresponding to the unit impulse for a single differential equation we have an  $m \times m$  *matrix unit impulse* for a system of differential equations. The first column of this matrix unit impulse corresponds to the situation where we apply a unit impulse to the first equation of our system and zero impulses to the other equations. The second column corresponds to the situation where we apply a unit impulse to the second equation of our system and zero impulses to the other equations, and so on. Denoting this matrix unit impulse by  $E_m$  its Laplace transform is  $\frac{1}{z} E_m$ . Since  $LR = \frac{1}{z} P^{-1}(z)$  it follows that

**$R(t)$  is the matrix of rest solutions corresponding to the matrix unit impulse 1.**

Thus the first column of  $R(t)$  is the rest solution of our system of differential equations corresponding to the situation where we apply a unit impulse to the first of our differential equations and zero impulses to the rest, and so on. If, then, it happens that not every element of  $\frac{1}{z} P^{-1}(z)$  is a proper rational fraction we know that there is no set of rest solutions of our system of differential equations corresponding to the matrix unit impulse. (We may take it as granted from the general theory of systems of linear differential equations with constant coefficients that every solution of such a system is such that its  $n$ th derivative possesses a Laplace transform, the applied impulses being constant.)

*Note.* We shall assume from now on that our system of differential equations is such that each element of  $\frac{1}{z}P^{-1}(z)$  is a proper rational fraction. If this is not the case but each element of  $\frac{1}{z^2}P^{-1}(z)$  is a proper rational fraction we could introduce the  $m \times m$  matrix  $S(t)$  whose Laplace transform is  $\frac{1}{z^2}P^{-1}(z)$ , and then

$$Lr = (LS)(z^2Lf).$$

If  $f$  possesses a second derivative which has a Laplace transform,  $z^2Lf = L(D^2f) + zf(0) + Df(0)$ , and since  $zLS = L(DS) + S(0)$  we obtain

$$Lr = (LS)L(D^2f) + L(DS)f(0) + S(0)f(0) + (LS)Df(0).$$

Since every Laplace transform is null at  $z = \infty$  we must have  $S(0)f(0) = 0$  and if this condition is satisfied

$$r = DSf(0) + SDf(0) + S * D^2f,$$

or, equivalently,

$$r(t) = DS(t)f(0) + S(t)Df(0) + \int_0^t S(t - \tau)D^2f(\tau) d\tau.$$

This formula may be regarded as the extension of the principle of superposition to those systems for which not every element of  $\frac{1}{z}P^{-1}(z)$  is a proper rational fraction. Note that  $S(0)$  is not the zero matrix since the expansion of  $\frac{1}{z^2}P^{-1}(z)$  near  $z = \infty$  starts out with the term  $\frac{S(0)}{z}$ , and we have assumed that not every element of  $\frac{1}{z}P^{-1}(z)$  is a proper rational fraction. Hence the condition  $S(0)f(0) = 0$  on the applied impulse, which must be satisfied if a rest solution exists, is not vacuous.

It is easy to verify that the vector function  $r = r(t)$  furnished by the formula

$$r = Rf(0) + R * Df$$

actually is the unambiguously determinate rest solution of the (vector) differential equation  $P(D)y = f$ . (We assume from now on that each element of the matrix  $\frac{1}{z}P^{-1}(z)$  is a proper rational fraction and  $R(t)$  is

the  $m \times m$  matrix whose Laplace transform is  $\frac{1}{z}P^{-1}(z)$ . In fact  $\frac{1}{z}P^{-1}(z)$  has near  $z = \infty$  the development  $\frac{R(0)}{z} + \frac{DR(0)}{z^2} + \dots + \frac{D^{j-1}R(0)}{z^j} + \dots$  so that  $P^{-1}(z)$  has near  $z = \infty$  the development

$$P^{-1}(z) = R(0) + \frac{DR(0)}{z} + \dots + \frac{D^{j-1}R(0)}{z^{j-1}} + \dots$$

On multiplying both sides of this equation on the left by the matrix  $P(z) = C_0z^n + C_1z^{n-1} + \dots + C_n$  and equating coefficients of like powers of  $z$  we obtain the series of equations

$$\begin{aligned} C_0R(0) &= 0; \\ C_0DR(0) + C_1R(0) &= 0; \\ &\vdots \\ &\vdots \\ &\vdots \\ C_0D^{n-1}R(0) + \dots + C_{n-1}R(0) &= 0. \end{aligned}$$

Thus

Each column vector of the  $m \times m$  matrix  $R(t)$  satisfies the initial conditions appropriate to a rest solution of the equation  $P(D)y = f$ . In particular, when the differential equation is normal, each column vector of  $R(t)$  vanishes together with its derivatives up to the  $(n-1)$ st, inclusive, at  $t = 0$ .

This fact, combined with the relation  $P(D)R = E_m$  (which is a consequence of the relation  $L\{P(D)R\} = \frac{E_m}{z}$  (why?)), assures us that  $\mathbf{r}$  is the rest solution of  $P(D)y = f$ . In fact

$$D\mathbf{r} = DR\mathbf{f}(0) + R(0)D\mathbf{f} + DR * D\mathbf{f}$$

(see Exercise 5, p. 347);

$$D^2\mathbf{r} = D^2R\mathbf{f}(0) + DR(0)D\mathbf{f} + R(0)D^2\mathbf{f} + D^2R * D\mathbf{f},$$

and so on. Hence

$$\begin{aligned} P(D)\mathbf{r} &= P(D)R\mathbf{f}(0) + (P(D)R * D\mathbf{f}) \\ &\quad + (C_0D^{n-1} + \dots + C_{n-1})R(0)D\mathbf{f} + \dots + C_0R(0)D^n\mathbf{f} \\ &= \mathbf{f}(0) + (E_m * D\mathbf{f}) \\ &= \mathbf{f}(0) + \int_0^t D\mathbf{f}(\tau) d\tau = \mathbf{f}(t). \end{aligned}$$

Thus  $\mathbf{r} = \mathbf{r}(t)$  is a solution of the differential equation. Since every convolution is zero at  $t = 0$  we have

$$\mathbf{r}(0) = R(0)\mathbf{f}(0);$$

$$D\mathbf{r}(0) = DR(0)\mathbf{f}(0) + R(0)D\mathbf{f}(0);$$

$$D^2\mathbf{r}(0) = D^2R(0)\mathbf{f}(0) + DR(0)D\mathbf{f}(0) + R(0)D^2\mathbf{f}(0),$$

and so on. Hence  $\mathbf{r} = \mathbf{r}(t)$  satisfies the initial conditions appropriate to the rest solution. (Prove this.) We may summarize as follows the principle of superposition for a system of differential equations or, what is the same thing, for a *vector* differential equation:

Under the hypothesis that each element of  $\frac{1}{z}P^{-1}(z)$  is a proper rational fraction the vector differential equation

$$P(D)\mathbf{y} = \mathbf{f}$$

possesses an unambiguously determinate rest solution  $\mathbf{r} = \mathbf{r}(t)$  which is furnished by the formula

$$\mathbf{r}(t) = R(t)\mathbf{f}(0) + \int_0^t R(t - \tau)D\mathbf{f}(\tau) d\tau$$

(it being assumed that  $\mathbf{f}$  possesses a piecewise-continuous derivative which has at some point of the complex  $z$ -plane a Laplace transform).

Here  $R(t)$  is the  $m \times m$  matrix whose Laplace transform is  $\frac{1}{z}P^{-1}(z)$ ; it satisfies the matrix differential equation

$$P(D)R = E_m,$$

and each column vector of  $R$  satisfies the initial conditions appropriate to a rest solution of the equation  $P(D)\mathbf{y} = \mathbf{f}$ .

It remains only to find the solution of the vector differential equation  $P(D)\mathbf{y} = \mathbf{f}$  which, together with its derivatives up to the  $(n - 1)$ st, inclusive, assumes appropriately assigned initial values. When our vector differential equation is normal these initial values may be assigned arbitrarily, but when it is abnormal this is not the case. In fact if we multiply the equation

$$P^{-1}(z) = R(0) + \frac{DR(0)}{z} + \cdots + \frac{D^{n-1}R(0)}{z^{n-1}} + \cdots \text{ on the right}$$

by  $P(z)$  and equate coefficients of like powers of  $z$  we obtain the following relations:

$$\begin{aligned}
 R(0)C_0 &= 0; \\
 DR(0)C_0 + R(0)C_1 &= 0; \\
 &\cdot \\
 &\cdot \\
 &\cdot \\
 D^{n-1}R(0)C_0 + \cdots + R(0)C_{n-1} &= 0; \\
 D^n R(0)C_0 + \cdots + R(0)C_n &= E_m; \\
 D^{n+1}R(0)C_0 + \cdots + DR(0)C_n &= 0; \\
 &\cdot \\
 &\cdot \\
 &\cdot
 \end{aligned}$$

If, then, we multiply the equation  $P(D)\mathbf{y} = \mathbf{f}$  on the *left* by  $R(0)$  we see that any solution of this equation must satisfy the following vector differential equation of order  $n - 1$ :

$$R(0)\{C_1 D^{n-1} + \cdots + C_n\}\mathbf{y} = R(0)\mathbf{f}.$$

Hence the initial values of  $\mathbf{y}$  and of its derivatives up to the  $(n - 1)$ st, inclusive, must satisfy the relation

$$R(0)\{C_1 D^{n-1}\mathbf{y}(0) + \cdots + C_n \mathbf{y}(0)\} = R(0)\mathbf{f}(0).$$

(When the system of differential equations is normal this relation is vacuous since  $R(0)$  is, then, the zero  $m \times m$  matrix.) When  $n > 1$  further such relations connecting  $\mathbf{y}(0), \cdots, D^{n-1}\mathbf{y}(0)$  may be obtained as follows: On differentiating the equation of order  $n - 1$  obtained above with respect to  $t$  and adding the result to the product of the original (vector) differential equation by  $DR(0)$  we obtain, by virtue of the relation  $DR(0)C_0 + R(0)C_1 = 0$ ,

$$\begin{aligned}
 \{DR(0)C_1 + R(0)C_2\}D^{n-1}\mathbf{y} + \cdots + DR(0)C_n \mathbf{y} \\
 = DR(0)\mathbf{f} + R(0)D\mathbf{f}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 \{DR(0)C_1 + R(0)C_2\}D^{n-1}\mathbf{y}(0) + \cdots + DR(0)C_n \mathbf{y}(0) \\
 = DR(0)\mathbf{f}(0) + R(0)D\mathbf{f}(0).
 \end{aligned}$$

(When our vector differential equation is normal this relation is vacuous (why?). We shall not write down the various relations connect-

ing  $\mathbf{y}(0), \dots, D^{n-1}\mathbf{y}(0)$  which may be obtained in this way since the manner of deriving them is clear.

We assume that the initial values of  $\mathbf{y}, \dots, D^{n-1}\mathbf{y}$  have been *properly* assigned if the differential equation is abnormal and *arbitrarily* assigned if the differential equation is normal, and we wish to determine the solution of  $P(D)\mathbf{y} = \mathbf{f}$  which satisfies these initial conditions. We observe that the matrix  $Q(t) = DR(t)$  satisfies the matrix differential equation

$$P(D)Q = 0.$$

In other words

*Each column vector of the matrix  $Q(t)$  is a solution of the homogeneous vector differential equation*

$$P(D)\mathbf{u} = 0.$$

To prove this it suffices to show (why?) that the Laplace transform of  $P(D)Q$  is the zero matrix. We have

$$\begin{aligned} L\{P(D)Q\} &= L\{P(D)DR\} \\ &= L\{D\{P(D)R\}\} = zL\{P(D)R\} - P(D)R(0). \end{aligned}$$

Since  $P(D)R = E_m$  we have  $L\{P(D)R\} = \frac{1}{z}E_m$  and  $P(D)R(0) = E_m$  (why?). Hence  $L\{P(D)Q\} = 0$ .

### EXERCISES

1. Show that  $\mathbf{r} = R(0)\mathbf{f} + Q * \mathbf{f}$ .

*Solution.* On writing  $R * \mathbf{f}$  in the form  $\int_0^t R(\tau)\mathbf{f}(t - \tau) d\tau$  we obtain

$$\begin{aligned} D(R * \mathbf{f}) &= R(t)\mathbf{f}(0) + \int_0^t R(\tau)D\mathbf{f}(t - \tau) d\tau \\ &= R(t)\mathbf{f}(0) + R * D\mathbf{f} = \mathbf{r}. \end{aligned}$$

On rewriting  $R * \mathbf{f}$  in the form  $\int_0^t R(t - \tau)\mathbf{f}(\tau) d\tau$  we obtain

$$\begin{aligned} \mathbf{r} = D(R * \mathbf{f}) &= R(0)\mathbf{f}(t) + \int_0^t DR(t - \tau)\mathbf{f}(\tau) d\tau \\ &= R(0)\mathbf{f}(t) + DR * \mathbf{f} \\ &= R(0)\mathbf{f} + Q * \mathbf{f}. \end{aligned}$$

2. Show that when the vector differential equation is normal  $\mathbf{r} = Q * \mathbf{f}$ . *Hint.*  $R(0) = 0$ .

Since  $Q(t)$  is a solution of the homogeneous matrix differential equation

$$P(D)U = 0$$

so also are the various matrices  $DQ, D^2Q, \dots$ . In fact

$$P(D)(DQ) = D\{P(D)Q\} = 0,$$

and so on. Hence the matrix

$$S_{n-1} = D^{n-1}QC_0 + D^{n-2}QC_1 + \dots + QC_{n-1}$$

is a solution of the matrix differential equation

$$P(D)U = 0.$$

In other words each column vector of the matrix  $S_{n-1}$  is a solution of the homogeneous vector differential equation

$$P(D)\mathbf{u} = \mathbf{0}.$$

Furthermore

$$\begin{aligned} S_{n-1}(0) &= D^n R(0)C_0 + \dots + DR(0)C_{n-1} \\ &= E_m - R(0)C_n, \end{aligned}$$

and  $DS_{n-1}(0) = -DR(0)C_n$ ;  $D^2S_{n-1}(0) = -D^2R(0)C_n$ , and so on. Similarly the matrix

$$S_{n-2} = D^{n-2}QC_0 + \dots + QC_{n-2}$$

is a solution of the homogeneous matrix differential equation

$$P(D)U = 0,$$

and  $S_{n-2}(0) = -R(0)C_{n-1}$ ,  $DS_{n-2}(0) = E_m - R(0)C_n - DR(0)C_{n-1}$ ,  $D^2S_{n-2}(0) = -DR(0)C_n - D^2R(0)C_{n-1}$ , and so forth. Proceeding in this way we construct the matrices  $S_k$ ,  $k = n-1, \dots, 0$ , ending with  $S_0$  where  $S_0 = QC_0$ , and we set up the vector

$$\mathbf{y} = \mathbf{r} + S_{n-1}\mathbf{y}(0) + S_{n-2}D\mathbf{y}(0) + \dots + S_0D^{n-1}\mathbf{y}(0),$$

where  $\mathbf{r}$  is the rest solution of the vector differential equation

$$P(D)\mathbf{y} = \mathbf{f}.$$

Each of the vectors  $S_{n-1}\mathbf{y}(0), S_{n-2}D\mathbf{y}(0), \dots, S_0D^{n-1}\mathbf{y}(0)$  is a solution of the homogeneous vector differential equation



$$P(D)\mathbf{u} = \mathbf{0}$$

(why?) and so  $\mathbf{y}$  is a solution of the vector differential equation

$$P(D)\mathbf{y} = \mathbf{f}$$

(why?). Since  $\mathbf{r} = R\mathbf{f}(0) + R * D\mathbf{f}$  we have  $\mathbf{r}(0) = R(0)\mathbf{f}(0)$  and so the value of  $\mathbf{y}$  at  $t = 0$  is

$$R(0)\mathbf{f}(0) + \mathbf{y}(0) - R(0)C_n\mathbf{y}(0) - R(0)C_{n-1}D\mathbf{y}(0) - \cdots -$$

$R(0)C_1D^{n-1}\mathbf{y}(0) = \mathbf{y}(0)$  (since the initial values of  $\mathbf{y}, \cdots, D^{n-1}\mathbf{y}$  are assumed to have been properly assigned). Furthermore

$$D\mathbf{r} = DR\mathbf{f}(0) + DR * D\mathbf{f} + R(0)D\mathbf{f}$$

so that  $D\mathbf{r}(0) = DR(0)\mathbf{f}(0) + R(0)D\mathbf{f}(0)$ . Hence the value at  $t = 0$  of  $D\mathbf{y}$  is

$DR(0)\mathbf{f}(0) + R(0)D\mathbf{f}(0) - DR(0)C_n\mathbf{y}(0) + D\mathbf{y}(0) - \{R(0)C_n + DR(0)C_{n-1}\}D\mathbf{y}(0) - \cdots - \{R(0)C_2 + DR(0)C_1\}D^{n-1}\mathbf{y}(0) = D\mathbf{y}(0)$ . Continuing the argument we see that, whether the vector differential equation

$$P(D)\mathbf{y} = \mathbf{f}$$

is normal or abnormal, the solution which takes on arbitrarily assigned, or properly assigned, initial values, as the case may be, is

$$\mathbf{y} = \mathbf{r} + S_{n-1}\mathbf{y}(0) + S_{n-2}D\mathbf{y}(0) + \cdots + S_0D^{n-1}\mathbf{y}(0),$$

where

$$\mathbf{r} = R\mathbf{f}(0) + R * D\mathbf{f};$$

$$S_{n-1} = D^{n-1}QC_0 + D^{n-2}QC_1 + \cdots + QC_{n-1};$$

$$S_{n-2} = D^{n-2}QC_0 + \cdots + QC_{n-2};$$

.

.

.

$$S_0 = QC_0.$$

*Note.* For a normal vector differential equation the initial values of  $R$  and of its derivatives up to the  $(n - 1)$ st, inclusive, are zero while the initial value of  $D^n R$  is  $C_0^{-1}$ . Hence the initial values of  $Q = DR$  and of its derivatives up to the  $(n - 2)$ nd, inclusive, are zero while the initial value of  $D^{n-1}Q$  is  $C_0^{-1}$ . Thus

$$S_{n-1}(0) = E_m, S_{n-1}(0) = 0, \dots, D^{n-1}S_{n-1}(0) = 0;$$

$$S_{n-2}(0) = 0, DS_{n-2}(0) = E_m, \dots, D^{n-1}S_{n-2}(0) = 0;$$

.

.

.

$$S_0(0) = 0, \dots \dots \dots, D^{n-1}S_0(0) = E_m.$$

Furthermore since  $R(0) = 0, L(Q) = L(DR) = zL(R) = P^{-1}(z)$ . Since the general formula for  $y$  is fairly complicated we think it well to reformulate the result in the important special cases  $n = 1$  and  $n = 2$ .

*Case 1.  $n = 1$ .* This is the case of a *first-order* vector differential equation. The solution is

$$y(t) = r(t) + QC_0y(0).$$

In the normal case the vector  $y(0)$  may be assigned arbitrarily while in the abnormal case we must have

$$R(0)C_1y(0) = R(0)f(0).$$

Furthermore  $Q(0)C_0 = E_m - R(0)C_1$ .

*Case 2.  $n = 2$ .* This is the case of a *second-order* vector differential equation. It is of fundamental importance in the theory of the vibrations of mechanical and electric systems. The solution is

$$y(t) = r(t) + (DQC_0 + QC_1)y(0) + QC_0Dy(0).$$

In the normal case the vectors  $y(0)$  and  $Dy(0)$  may be assigned arbitrarily while in the abnormal case they are subject to the relations

$$R(0)\{C_1Dy(0) + C_2y(0)\} = R(0)f(0);$$

$$\{DR(0)C_1 + R(0)C_2\}Dy(0) + DR(0)C_2y(0) \\ = DR(0)f(0) + R(0)Df(0)$$

which may be written in the equivalent form

$$-DR(0)C_0Dy(0) + R(0)C_2y(0) = R(0)f(0);$$

$$Dy(0) - D^2R(0)C_0Dy(0) + DR(0)C_2y(0) = DR(0)f(0) + R(0)Df(0).$$

### Example 1

$$(D - 1)x - 2y = t;$$

$$-2x + (D - 1)y = t.$$

Here  $n = 1$  and  $C_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . Thus the vector differential equation is normal, and  $x(0)$ ,  $y(0)$  may be assigned arbitrarily. Let us set  $x(0) = 2$ ,  $y(0) = 4$ . We have

$$P(z) = \begin{pmatrix} z-1 & -2 \\ -2 & z-1 \end{pmatrix}, \det P(z) = z^2 - 2z - 3 = (z-3)(z+1)$$

so that

$$\begin{aligned} P^{-1}(z) &= \begin{bmatrix} \frac{z-1}{(z-3)(z+1)} & \frac{2}{(z-3)(z+1)} \\ \frac{2}{(z-3)(z+1)} & \frac{z-1}{(z-3)(z+1)} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2(z-3)} + \frac{1}{2(z+1)} & \frac{1}{2(z-3)} - \frac{1}{2(z+1)} \\ \frac{1}{2(z-3)} - \frac{1}{2(z+1)} & \frac{1}{2(z-3)} + \frac{1}{2(z+1)} \end{bmatrix}. \end{aligned}$$

Since  $L(Q) = P^{-1}(z)$  (why?) we have

$$Q(t) = \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{2}e^{3t} - \frac{1}{2}e^{-t} \\ \frac{1}{2}e^{3t} - \frac{1}{2}e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{pmatrix}.$$

Since  $\mathbf{f} = v(t, t)$  we have  $Q(t-\tau)\mathbf{f}(\tau) = v(e^{3(t-\tau)\tau}, e^{3(t-\tau)\tau})$  so that

$$\begin{aligned} \mathbf{r} &= Q * \mathbf{f} = \int_0^t Q(t-\tau)\mathbf{f}(\tau) d\tau \\ &= v\left(\frac{e^{3t}}{9} - \frac{t}{3} - \frac{1}{9}, \frac{1}{9}e^{3t} - \frac{t}{3} - \frac{1}{9}\right). \end{aligned}$$

Since  $C_0 = E_2$  the desired solution is obtained by adding to  $\mathbf{r}$  the result of operating on  $v(2, 4)$  by  $Q(t)$ , namely,  $v(3e^{3t} - e^{-t}, 3e^{3t} + e^{-t})$ . Hence

$$x = \frac{28e^{3t}}{9} - e^{-t} - \frac{t}{3} - \frac{1}{9}; \quad y = \frac{28e^{3t}}{9} + e^{-t} - \frac{t}{3} - \frac{1}{9}$$

### Example 2

$$\begin{aligned} (2D - 1)x + (3D - 2)y &= te^t; \\ (2D + 1)x + (3D + 2)y &= te^{2t}. \end{aligned}$$

This is an abnormal first-order vector differential equation since the matrix  $C_0 = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$  is singular. Since  $P(z) = \begin{pmatrix} 2z - 1 & 3z - 2 \\ 2z + 1 & 3z + 2 \end{pmatrix}$  we have  $\det P(z) = 2z$  and so

$$P^{-1}(z) = \begin{bmatrix} \frac{3}{2} + \frac{1}{z} & -\frac{3}{2} + \frac{1}{z} \\ -1 - \frac{1}{2z} & 1 - \frac{1}{2z} \end{bmatrix}.$$

Hence

$$L(R) = \frac{P^{-1}(z)}{z} = \begin{bmatrix} \frac{3}{2z} + \frac{1}{z^2} & -\frac{3}{2z} + \frac{1}{z^2} \\ -\frac{1}{z} - \frac{1}{2z^2} & \frac{1}{z} - \frac{1}{2z^2} \end{bmatrix}$$

so that

$$R(t) = \begin{pmatrix} \frac{3}{2} + t & -\frac{3}{2} + t \\ -1 - \frac{1}{2}t & 1 - \frac{1}{2}t \end{pmatrix}.$$

Hence  $Q(t) = \begin{pmatrix} 1 & 1 \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$ . Since

$$R(0) = \begin{pmatrix} \frac{3}{2} & -\frac{3}{2} \\ -1 & 1 \end{pmatrix}, C_1 = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}$$

we have  $R(0)C_1 = \begin{pmatrix} -3 & -6 \\ 2 & 4 \end{pmatrix}$  so that the initial values  $x(0), y(0)$  of  $x$  and  $y$  must satisfy the relation  $-3x(0) - 6y(0) = 0$ , or, equivalently,  $x(0) + 2y(0) = 0$ . Since  $Q(t - \tau)f(\tau) = v(\tau(e^\tau + e^{2\tau}), -\frac{1}{2}\tau(e^\tau + e^{2\tau}))$  we have  $Q * f = v(\frac{5}{4} + e^t(t - 1) + \frac{1}{4}e^{2t}(2t - 1), -\frac{5}{8} - \frac{1}{2}e^t(t - 1) - \frac{1}{8}e^{2t}(2t - 1))$ . Hence  $r = R(0)f + Q * f = v(\frac{5}{4} + \frac{1}{2}e^t(5t - 2) - \frac{1}{4}e^{2t}(4t + 1), -\frac{5}{8} - \frac{1}{2}e^t(3t - 1) + \frac{1}{8}e^{2t}(6t + 1))$ . Since  $Q(t)C_0 = \begin{pmatrix} 4 & 6 \\ -2 & -3 \end{pmatrix}$  the general solution is obtained by adding  $v(4x(0) + 6y(0), -2x(0) - 3y(0))$  to  $r(t)$ . Taking into account the relation connecting  $x(0)$  and  $y(0)$  we find

$$\begin{aligned} x &= \frac{5}{4} + \frac{1}{2}e^t(5t - 2) - \frac{1}{4}e^{2t}(4t + 1) + x(0); \\ y &= -\frac{5}{8} - \frac{1}{2}e^t(3t - 1) + \frac{1}{8}e^{2t}(6t + 1) + y(0). \end{aligned}$$

*Note.* In this example  $f(0) = \mathbf{0}$  so that, even though  $R(0) \neq \mathbf{0}$ ,  $\mathbf{r}(0) = \mathbf{0}$  since  $\mathbf{r}(0) = R(0)f(0)$  (why?). Hence  $\mathbf{y}$  is found by adding  $v(x(0), y(0))$  to  $\mathbf{r}$ .

### Example 3

$$\begin{aligned}(2D^2 - D + 9)x - (D^2 + D + 3)y &= \mathbf{0}; \\ (2D^2 + D + 7)x - (D^2 - D + 5)y &= \mathbf{0}.\end{aligned}$$

Here  $C_0 = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}$  so that the second-order vector differential equation is abnormal.  $P(z) = \begin{pmatrix} 2z^2 - z + 9 & -(z^2 + z + 3) \\ 2z^2 + z + 7 & -(z^2 - z + 5) \end{pmatrix}$ ,  $\det P(z) = 6z^3 - 6z^2 + 24z - 24 = 6(z-1)(z^2+4)$ . Hence

$$\begin{aligned}P^{-1}(z) &= \frac{1}{6} \begin{bmatrix} \frac{-(z^2 - z + 5)}{(z-1)(z^2+4)} & \frac{z^2 + z + 3}{(z-1)(z^2+4)} \\ \frac{-(2z^2 + z + 7)}{(z-1)(z^2+4)} & \frac{2z^2 - z + 9}{(z-1)(z^2+4)} \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} \frac{1}{z^2+4} - \frac{1}{z-1} & \frac{1}{z^2+4} + \frac{1}{z-1} \\ -\frac{1}{z^2+4} - \frac{2}{z-1} & -\frac{1}{z^2+4} + \frac{2}{z-1} \end{bmatrix}.\end{aligned}$$

Since each element of  $P^{-1}(z)$  is a proper rational fraction we have  $R(0) = \mathbf{0}$  (since  $P^{-1}(z) = R(0) + \frac{DR(0)}{z} + \dots$ ) and so  $L(Q) = P^{-1}(z)$  (why?). Hence

$$Q(t) = \frac{1}{6} \begin{pmatrix} \frac{1}{2} \sin 2t - e^t & \frac{1}{2} \sin 2t + e^t \\ -\frac{1}{2} \sin 2t - 2e^t & -\frac{1}{2} \sin 2t + 2e^t \end{pmatrix}$$

and so

$$R(t) = \frac{1}{6} \begin{pmatrix} -\frac{1}{4}(\cos 2t - 1) - (e^t - 1) & -\frac{1}{4}(\cos 2t - 1) + e^t - 1 \\ \frac{1}{4}(\cos 2t - 1) - 2(e^t - 1) & \frac{1}{4}(\cos 2t - 1) + 2(e^t - 1) \end{pmatrix}.$$

Since  $DR(0) = Q(0) = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}$ ,  $C_1 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$ ,  $C_2 = \begin{pmatrix} 9 & -3 \\ 7 & -5 \end{pmatrix}$

we have  $DR(0)C_1 = \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}$ ,  $DR(0)C_2 = \begin{pmatrix} -2 & -2 \\ -4 & -4 \end{pmatrix}$ . Hence  $x(0)$ ,  $y(0)$ ,  $Dx(0)$ ,  $Dy(0)$  are connected by the relation

$$-2x(0) - 2y(0) + 2Dx(0) + 2Dy(0) = 0$$

or, equivalently,

$$x(0) + y(0) - Dx(0) - Dy(0) = 0.$$

Since  $\mathbf{f}$  is the zero vector,  $\mathbf{r}(t)$  is the zero vector (why?). Again

$$QC_0 = \frac{1}{8} \begin{pmatrix} 2 \sin 2t & -\sin 2t \\ -2 \sin 2t & \sin 2t \end{pmatrix};$$

$$DQ = \frac{1}{8} \begin{pmatrix} \cos 2t - e^t & \cos 2t + e^t \\ -\cos 2t - 2e^t & -\cos 2t + 2e^t \end{pmatrix};$$

$$DQC_0 = \frac{1}{8} \begin{pmatrix} 4 \cos 2t & -2 \cos 2t \\ -4 \cos 2t & 2 \cos 2t \end{pmatrix}; \quad QC_1 = \frac{1}{8} \begin{pmatrix} 2e^t & 2e^t \\ 4e^t & 4e^t \end{pmatrix}$$

so that

$$DQC_0 + QC_1 = \frac{1}{8} \begin{pmatrix} 2 \cos 2t + e^t & -\cos 2t + e^t \\ -2 \cos 2t + 2e^t & \cos 2t + 2e^t \end{pmatrix}.$$

Hence

$$x(t) = \frac{1}{8} \{ (2 \cos 2t + e^t)x(0) + (e^t - \cos 2t)y(0) \\ + \frac{1}{8} \{ 2 \sin 2t Dx(0) - \sin 2t Dy(0) \};$$

$$y(t) = \frac{1}{8} \{ (2e^t - 2 \cos 2t)x(0) + (2e^t + \cos 2t)y(0) \\ + \frac{1}{8} \{ -2 \sin 2t Dx(0) + \sin 2t Dy(0) \}.$$

### EXERCISES

3.  $(D^2 - 3D + 2)x + (D - 1)y = 0$ ;  $-(D - 1)x + (D^2 - 5D + 4)y = 0$ ;  
 $x(0) = 0$ ,  $Dx(0) = 0$ ;  $y(0) = 1$ ,  $Dy(0) = 0$ .

4. Solve the homogeneous equations

$$(D^2 + 1)x + (D^2 - 2D)y = 0;$$

$$(D^2 + D)x + D^2y = 0.$$

5. Find the rest solution of the equations

$$(3D + 2)x + Dy = 1;$$

$$Dx + (4D + 3)y = 0.$$

6. Find the solution of the homogeneous equations

$$(3D + 2)x + Dy = 0;$$

$$Dx + (4D + 3)y = 0$$

having assigned initial values  $x(0)$ ,  $y(0)$ .

### 8. The determination of functions from their Laplace transforms

When we are given a function of the complex variable  $z$  which is a proper rational fraction we have seen how to determine the function of which it is the Laplace transform by analyzing the given proper rational fraction into simple fractions. We have seen also that the development, near  $z = \infty$ , of the proper rational fraction is

$$\frac{f(0)}{z} + \frac{Df(0)}{z^2} + \cdots + \frac{D^{n-1}f(0)}{z^n} + \cdots,$$

where  $f = f(t)$  is the function of which the given proper rational fraction is the Laplace transform. We now proceed to prove a general theorem of which this result is a special case.

Let  $\phi(z)$  be a function of the complex variable, and let  $\alpha$  be a complex number whose real part is positive which is such that  $z^\alpha \phi(z)$  is analytic at  $z = \infty$  so that, near  $z = \infty$ ,

$$\phi(z) = \frac{1}{z^\alpha} \sum_0^\infty \frac{c_n}{z^n}.$$

Then the power series  $\sum_0^\infty \frac{c_n}{\Gamma(n + \alpha)} t^n$  converges for every  $t$ , and the function

$$f(t) = t^{\alpha-1} \sum_0^\infty \frac{c_n}{\Gamma(n + \alpha)} t^n; \quad t \geq 0$$

has  $\phi(z)$  as its Laplace transform at points  $z$  whose real part  $x$  is sufficiently large.

*Note.* The case where  $\phi(z)$  is a proper rational fraction is cared for by the particular case of this theorem which occurs when  $\alpha = 1$ . The proper rational fraction has, near  $z = \infty$ , a development

$$\phi(z) = \sum_0^\infty \frac{c_n}{z^{n+1}},$$

and the function  $f(t)$  of which it is the Laplace transform is

$$f(t) = \sum_0^{\infty} \frac{c_n}{\Gamma(n+1)} t^n = \sum_0^{\infty} \frac{c_n}{n!} t^n.$$

The series whose sum is  $f(t)$  converges for every  $t$  and  $c_n = D^n f(0)$ .

In order to prove our theorem we first observe that we are granted

that the series  $\sum_0^{\infty} \frac{c_n}{z^n}$  converges if  $|z|$  is sufficiently large. Hence if

$x_0 > 0$  is a sufficiently large positive real number the series  $\sum_0^{\infty} \left(\frac{c_n}{x_0^n}\right)$

converges. Since the terms of any convergent series constitute a bounded variable we know that there exists a positive number  $M$ , independent of  $n$ , such that

$$|c_n| \leq Mx_0^n.$$

Since  $\frac{\Gamma(n+\alpha)}{\Gamma(n+1+\alpha)} = n-1+\alpha$  it follows from the ratio test that the series

$$\sum_0^{\infty} \frac{Mx_0^n}{\Gamma(n+\alpha)} t^n$$

is convergent for every value of  $t$ . Hence, by the comparison test, the series

$$\sum_0^{\infty} \frac{c_n}{\Gamma(n+\alpha)} t^n$$

is convergent for every value of  $t$ . We consider the function  $f(t)$  which is defined as follows:

$$\begin{aligned} f(t) &= t^{\alpha-1} \sum_0^{\infty} \frac{c_n}{\Gamma(n+\alpha)} t^n = \sum_0^{\infty} \frac{c_n}{\Gamma(n+\alpha)} t^{n+\alpha-1}; \quad \text{if } t > 0; \\ &= 0; \quad \text{if } t < 0, \end{aligned}$$

and we propose to show that the Laplace transform of  $f(t)$  is  $\phi(z)$ .

*Note.* If the real part of  $\alpha \geq 1$ ,  $f(0) = 0$  while if the real part of  $\alpha$



lies between 0 and 1,  $f(0)$  is not defined ( $f(t)$  being not bounded at  $t = 0$ ). Nevertheless the Laplace transform of  $f(t)$  exists in the following sense:

$$L(f) = \int_{+0}^{\infty} e^{-st}f(t) dt = \lim_{\delta \rightarrow 0} \int_{\delta}^{\infty} e^{-st}f(t) dt; \quad \delta > 0.$$

In order to show that  $f(t)$  possesses a Laplace transform and that this Laplace transform is  $\phi(z)$  we consider the integral

$$\int_0^T e^{-xt}f(t) dt = \int_0^T e^{-xt} \left\{ \sum_0^{\infty} \frac{c_n}{\Gamma(n + \alpha)} t^{n+\alpha-1} \right\} dt, \quad x > x_0.$$

Since power series permit term-by-term integration over any (finite) subinterval of their interval of convergence (this process of term-by-term integration remaining valid if the series is multiplied by any continuous function) we have

$$\begin{aligned} \int_0^T e^{-xt}f(t) dt &= \sum_0^{\infty} \frac{c_n}{\Gamma(n + \alpha)} \int_0^T e^{-xt}t^{n+\alpha-1} dt \\ &= \sum_0^{\infty} \frac{c_n}{\Gamma(n + \alpha)} \left\{ \int_0^{\infty} - \int_T^{\infty} e^{-xt}t^{n+\alpha-1} dt \right\}. \end{aligned}$$

Now the infinite series  $\sum_0^{\infty} \frac{c_n}{\Gamma(n + \alpha)} \int_0^{\infty} e^{-xt}t^{n+\alpha-1} dt = \sum_0^{\infty} \frac{c_n}{x^{n+\alpha}}$  converges (why?). Hence the infinite series

$$\sum_0^{\infty} \left\{ \frac{c_n}{\Gamma(n + \alpha)} \int_T^{\infty} e^{-xt}t^{n+\alpha-1} dt \right\}$$

converges with the limit

$$\sum_0^{\infty} \frac{c_n}{x^{n+\alpha}} - \int_0^T e^{-xt}f(t) dt.$$

If, then, we can show that

$$\sum_n^{\infty} \left\{ \frac{c_n}{\Gamma(n + \alpha)} \int_T^{\infty} e^{-xt}t^{n+\alpha-1} dt \right\}$$

has at  $T = \infty$  the limit zero we shall have proved the existence of the (improper) integral  $\int_0^\infty e^{-xt}f(t) dt$  with the value  $\sum_0^\infty \frac{c_n}{x^{n+\alpha}}$ . In other

words we shall have proved the equality of the two functions  $L(f)$

$$= \int_0^\infty e^{-xt}f(t) dt \text{ and } \phi(z) = \sum_0^\infty \frac{c_n}{z^{n+\alpha}}$$

of  $z$  at all points  $x$  of the real axis which lie to the right of  $x_0$ . This implies (why?) that the Laplace transform of  $f$  is  $\phi(z)$  at all points  $z$  to the right of  $x_0$ .

Let us apply repeated integration by parts to the integral

$$\int_T^\infty e^{-xt}t^{n+\alpha-1} dt.$$

When  $n = 1$  we obtain

$$\begin{aligned} \int_T^\infty e^{-xt}t^\alpha dt &= \frac{e^{-xT}T^\alpha}{x} + \frac{\alpha}{x} \int_T^\infty e^{-xt}t^{\alpha-1} dt \\ &= \frac{\Gamma(\alpha + 1)e^{-xT}T^{\alpha-1}}{x^2} \left\{ \frac{xT}{\Gamma(\alpha + 1)} + \beta \right\}, \end{aligned}$$

where  $\beta = \frac{x \int_T^\infty e^{-xt}t^{\alpha-1} dt}{\Gamma(\alpha)e^{-xT}T^{\alpha-1}}$ . When  $n = 2$  we obtain

$$\begin{aligned} \int_T^\infty e^{-xt}t^{\alpha+1} dt &= \frac{e^{-xT}T^{\alpha+1}}{x} + \frac{\alpha + 1}{x} \int_T^\infty e^{-xt}t^\alpha dt \\ &= \frac{\Gamma(\alpha + 2)e^{-xT}T^{\alpha-1}}{x^3} \left\{ \frac{(xT)^2}{\Gamma(\alpha + 2)} + \frac{xT}{\Gamma(\alpha + 1)} + \beta \right\}, \end{aligned}$$

and, generally,

$$\begin{aligned} \int_T^\infty e^{-xt}t^{n+\alpha-1} dt &= \frac{\Gamma(\alpha + n)e^{-xT}T^{\alpha-1}}{x^{n+1}} \left\{ \frac{(xT)^n}{\Gamma(\alpha + n)} + \frac{(xT)^{n-1}}{\Gamma(\alpha + n - 1)} \right. \\ &\quad \left. + \dots + \frac{xT}{\Gamma(\alpha + 1)} + \beta \right\}. \end{aligned}$$

If we denote by  $r_n$  the remainder after  $n$  terms of the convergent series

$\sum_0^\infty \frac{c_n}{x^{n+1}}$  we have to examine the behavior at  $T = \infty$  of the series

$$e^{-xT}T^{\alpha-1} \left[ (r_0 - r_1)\beta + \sum_1^{\infty} (r_n - r_{n+1}) \left\{ \frac{(xT)^n}{\Gamma(\alpha + n)} + \frac{(xT)^{n-1}}{\Gamma(\alpha + n - 1)} + \cdots + \frac{xT}{\Gamma(\alpha + 1)} + \beta \right\} \right],$$

where  $r_0 = \sum_0^{\infty} \frac{c_n}{x^n}$ . The sum of  $p$  terms of the infinite series which multiplies  $e^{-xT}T^{\alpha-1}$  is

$$\begin{aligned} & (r_0 - r_1)\beta + (r_1 - r_2) \left\{ \frac{xT}{\Gamma(\alpha + 1)} + \beta \right\} + \cdots \\ & + (r_{p-1} - r_p) \left\{ \frac{(xT)^{p-1}}{\Gamma(\alpha + p - 1)} + \cdots + \frac{xT}{\Gamma(\alpha + 1)} + \beta \right\} \\ & = r_0\beta + r_1 \frac{xT}{\Gamma(\alpha + 1)} + \cdots + r_{p-1} \frac{(xT)^{p-1}}{\Gamma(\alpha + p - 1)} \\ & \quad - r_p \left\{ \frac{(xT)^{p-1}}{\Gamma(\alpha + p - 1)} + \cdots + \frac{xT}{\Gamma(\alpha + 1)} + \beta \right\}. \end{aligned}$$

The series  $\beta + \frac{xT}{\Gamma(\alpha + 1)} + \cdots + \frac{(xT)^{n-1}}{\Gamma(\alpha + n - 1)} + \cdots$  is convergent for every value of  $T$  (prove this by means of the ratio test), and  $\lim_{p \rightarrow \infty} r_p = 0$  (why?). Hence the sum of the series

$$r_0\beta + \sum_1^{\infty} (r_n - r_{n-1}) \left\{ \frac{(xT)^n}{\Gamma(\alpha + n)} + \cdots + \frac{xT}{\Gamma(\alpha + 1)} + \beta \right\}$$

is the same as the sum of the series  $r_0\beta + \sum_2^{\infty} r_{n-1} \frac{(xT)^{n-1}}{\Gamma(\alpha + n - 1)}$ . Let

$\epsilon$  be any assigned positive number, and determine the positive integer  $N$  such that  $|r_n| \leq \epsilon$  is  $n \geq N$ . Since  $\{r_n\}$  is a null sequence it is bounded; in other words there exists a fixed number  $C$  such that

$|r_n| < C$  for every  $n$ . Writing  $\sum_2^{\infty} = \sum_2^N + \sum_N^{\infty}$  we obtain

$$\left| \sum_2^{\infty} r_{n-1} \frac{(xT)^{n-1}}{\Gamma(\alpha + n - 1)} \right| \leq C \sum_2^N \frac{(xT)^{n-1}}{|\Gamma(\alpha + n - 1)|} + \epsilon \sum_{N+1}^{\infty} \frac{(xT)^{n-1}}{|\Gamma(\alpha + n - 1)|}.$$

Since  $\lim_{T \rightarrow \infty} e^{-xT} T^k = 0$ , every  $k$ , it will follow that  $e^{-xT} T^{\alpha-1} \sum_2^{\infty} \tau_{n-1} \frac{(xT)^{n-1}}{\Gamma(\alpha+n-1)}$  can be made arbitrarily small (by making  $T$  sufficiently large) if we can show that  $e^{-xT} T^{\alpha-1} \sum_{N+1}^{\infty} \frac{(xT)^{n-1}}{|\Gamma(\alpha+n-1)|}$  is bounded at  $T = \infty$ , uniformly with respect to  $N$ . To show this we observe that

$$\begin{aligned} \sum_{N+1}^{\infty} \frac{(xT)^{n-1}}{|\Gamma(\alpha+n-1)|} &= \frac{(xT)^N}{|\Gamma(\alpha+N)|} + \frac{(xT)^{N+1}}{|\Gamma(\alpha+N+1)|} + \dots \\ &\leq \frac{xT}{|\Gamma(\alpha+1)|} + \frac{x^2 T^2}{|\Gamma(\alpha+2)|} + \dots \\ &= \frac{1}{|\Gamma(\alpha)|} \left\{ \frac{xT}{|\alpha|} + \frac{x^2 T^2}{|(\alpha+1)\alpha|} + \dots \right\} \end{aligned}$$

so that  $\left| T^{\alpha-1} \sum_{N+1}^{\infty} \frac{(xT)^{n-1}}{\Gamma(\alpha+n-1)} \right| \leq \frac{1}{|x^{\alpha-1}\Gamma(\alpha)|} \left\{ \frac{|xT|^\alpha}{|\alpha|} + \frac{|xT|^{\alpha+1}}{|\alpha(\alpha+1)|} + \dots \right\}$ . Denote  $xT$  by  $\xi$  and  $R\alpha$  by  $\alpha_1$  (so that  $\alpha_1 > 0$ ). Then  $|\alpha| \geq \alpha_1$ ,  $|\alpha+1| > \alpha_1+1$ , and so on, and so on, and  $|xT|^\alpha = (xT)^{\alpha_1}$ ,  $|xT|^{\alpha+1} = (xT)^{\alpha_1+1}$ , and so on. Hence

$$\left| T^{\alpha-1} \sum_{N+1}^{\infty} \frac{(xT)^{n-1}}{\Gamma(\alpha+n-1)} \right| \leq \frac{1}{x^{\alpha_1-1}|\Gamma(\alpha)|} \left\{ \frac{\xi^{\alpha_1}}{\alpha_1} + \frac{\xi^{\alpha_1+1}}{\alpha_1(\alpha_1+1)} + \dots \right\}.$$

On denoting by  $s$  the sum of the series  $\frac{\xi^{\alpha_1}}{\alpha_1} + \frac{\xi^{\alpha_1+1}}{\alpha_1(\alpha_1+1)} + \dots$  we have  $s_\xi - s = \xi^{\alpha_1-1}$ , and since  $s$  is zero when  $\xi = 0$  it follows that  $s = e^\xi \int_0^\xi e^{-t} t^{\alpha_1-1} dt < e^\xi \Gamma(\alpha_1)$  (why?). Hence

$$\left| e^{-xT} T^{\alpha-1} \sum_{N+1}^{\infty} \frac{(xT)^{n-1}}{\Gamma(\alpha+n-1)} \right| < \frac{\Gamma(\alpha_1)}{x^{\alpha_1-1}|\Gamma(\alpha)|},$$

$N = 1, 2, \dots, T$  an arbitrary positive number. Hence, since  $e^{-xT} T^{\alpha-1} \beta \rightarrow 0$  as  $T \rightarrow \infty$ ,

$$\lim_{T \rightarrow \infty} e^{-zT} T^{\alpha-1} \left[ (r_0 - r_1)\beta + \sum_1^{\infty} (r_n - r_{n+1}) \left\{ \frac{(xT)^n}{\Gamma(\alpha + n)} + \dots + \beta \right\} \right] = 0,$$

or, equivalently,

$$\lim_{T \rightarrow \infty} \sum_0^{\infty} \frac{c_n}{\Gamma(\alpha + n)} \int_T^{\infty} e^{-zt} t^{\alpha-1} dt = 0.$$

This proves our main theorem:

The function  $\phi(z) = \sum_0^{\infty} \frac{c_n}{z^{\alpha+n}}$  is the Laplace transform of the sum  $f(t)$  of the everywhere-convergent infinite series  $\sum_0^{\infty} \frac{c_n}{\Gamma(n + \alpha)} t^{\alpha-1}$ , it being understood that the series defining  $\phi(z)$  converges if  $|z|$  is sufficiently large and that the real part of the complex number  $\alpha$  is positive.

### Example 1

Set  $\phi(z) = \frac{1}{z} e^{-\frac{1}{z}} = \sum_0^{\infty} \frac{(-1)^n}{n!} \frac{1}{z^{n+1}}$ . The function of which  $\phi(z)$  is the Laplace transform is the sum  $f(t)$  of the everywhere-convergent series  $\sum_0^{\infty} \frac{(-1)^n}{n!n!} t^n$ . On setting  $t = \left(\frac{\tau}{2}\right)^2$  we have

$$f(t) = \sum_0^{\infty} \frac{(-1)^n \tau^{2n}}{n!n! 2^{2n}} = 1 - \frac{\tau^2}{2^2} + \frac{\tau^4}{(2^2)(4^2)} - \dots = J_0(\tau).$$

In other words  $f(t) = J_0(2t^{1/2})$ . Hence

$$L\{J_0(2t^{1/2})\} = \frac{e^{-\frac{1}{z}}}{z}, \quad x > 0 \text{ (why?)}$$

### Example 2

Set  $\phi(z) = \frac{1}{z^\alpha} e^{-\frac{1}{z}}$ , where the real part of  $\alpha$  is positive. Then the function of which  $\phi(z)$  is the Laplace transform is the sum  $f(t)$  of the

everywhere convergent series  $\sum_0^{\infty} (-1)^n \frac{1}{n! \Gamma(n + \alpha)} t^{n+\alpha-1}$ . On setting

$t = \left(\frac{\tau}{2}\right)^2$  we have

$$\begin{aligned} f(t) &= \sum_0^{\infty} (-1)^n \frac{1}{n! \Gamma(n + \alpha)} \frac{\tau^{2n+2\alpha-2}}{2^{2n+2\alpha-2}} \\ &= \left(\frac{\tau}{2}\right)^{\alpha-1} \sum_0^{\infty} (-1)^n \frac{1}{\Gamma(n + \alpha)n!} \left(\frac{\tau}{2}\right)^{\alpha-1+2n} \\ &= \left(\frac{\tau}{2}\right)^{\alpha-1} J_{\alpha-1}(\tau). \end{aligned}$$

In other words  $L\{t^{\frac{\alpha-1}{2}} J_{\alpha-1}(2t^{1/2})\} = \frac{e^{-\frac{1}{z}}}{z^{\alpha}}$ ,  $R(\alpha) > 0$ ,  $x > 0$ .

### Example 3

On setting  $\alpha = \frac{1}{2}$  in our general theorem we have the following important special case of this theorem:

The function of which  $\frac{1}{z^{1/2}} \left( c_0 + \frac{c_1}{z} + \dots + \frac{c_{n-1}}{z^{n-1}} + \dots \right)$  is the Laplace transform is the sum  $f(t)$  of the everywhere-convergent series

$$\begin{aligned} \frac{1}{\Gamma(\frac{1}{2})t^{1/2}} \left\{ c_0 - \frac{c_1 t}{\frac{1}{2}} + \frac{c_2 t^2}{\frac{1}{2} \cdot \frac{3}{2}} + \dots \right\} \\ = \pi(t)^{-1/2} \left\{ c_0 + \frac{c_1}{1} (2t) + \frac{c_2}{(1)(3)} (2t)^2 + \dots \right\}. \end{aligned}$$

If we choose the coefficients  $c_0, c_1, c_2, \dots$  so that the expression in parentheses is  $\cosh t^{1/2}$ , i.e., if we set  $c_0 = 1$ ,  $c_1 = \frac{1}{(2^2)1!}$ ,  $c_2 = \frac{1}{(2^4)2!}$ ,  $c_3 = \frac{1}{(2^6)3!}$ ,  $\dots$ , the function  $\phi(z) = \frac{1}{z^{1/2}} \left( 1 + \frac{1}{(2^2)z} + \frac{1}{(2^4)2!z^2} + \dots \right)$  turns out to be  $\frac{e^{4z}}{z^{1/2}}$ . Hence  $L\left(\frac{\cosh t^{1/2}}{t^{1/2}}\right) = \frac{\Gamma(\frac{1}{2})}{z^{1/2}} e^{\frac{1}{4z}} = \left(\frac{\pi}{z}\right)^{1/2} e^{\frac{1}{4z}}$ .

### EXERCISES

1. Show that  $L(\sinh t^{1/2}) = \frac{\pi^{1/2}}{2z^{3/2}} e^{\frac{1}{4z}}$ .

2. Show that  $L\{J_0(t)\} = (1+z^2)^{-1/2}$ . *Hint.* Write  $(1+z^2)^{-1/2}$  as  $\frac{1}{z} \left(1 + \frac{1}{z^2}\right)^{-1/2}$
- $$= \frac{1}{z} - \frac{1}{2z^3} + \frac{(1)(3)}{(2)(4)} \frac{1}{z^5} - \dots$$
3. Show that  $L\left\{\frac{\cos t^{1/2}}{t^{1/2}}\right\} = \left(\frac{\pi}{z}\right)^{1/2} e^{-\frac{1}{4z}}$ ,  $x > 0$ .
4. Show that  $L\{\sin t^{1/2}\} = \frac{\pi^{1/2}}{2z^{3/2}} e^{-\frac{1}{4z}}$ ,  $x > 0$ .
5. Show that  $\frac{1}{z(1+z^2)^{3/2}} = L(\operatorname{erf} t^{1/2})$ ,  $x > 0$ , where  $\operatorname{erf} x$  is the *error function* defined by  $\operatorname{erf} x = \frac{2}{\pi^{1/2}} \int_0^x e^{-t^2} dt$ . *Hint.*  $\alpha = \frac{3}{2}$  and  $\Gamma(\frac{3}{2}) = \frac{1}{2}\pi^{1/2}$ .

### REVIEW EXERCISES

In each of the following exercises where you have to determine a Laplace transform indicate the values of  $z$  for which the result is valid.

- Show that  $L(\sin \alpha t - \alpha t \cos \alpha t) = \frac{2\alpha^3}{(z^2 + \alpha^2)^2}$ .
- Show that  $L(\cos \alpha t \cosh \alpha t) = \frac{z^3}{z^4 + 4\alpha^4}$ .
- Show that  $L(\sinh \alpha t - \sin \alpha t) = \frac{2\alpha^3}{z^4 - \alpha^4}$ .
- Show that a particular solution of the differential equation  $(D^2 - n^2)x = f(t)$  is  $x = \frac{1}{n} \int_0^t f(\tau) \sinh n(t - \tau) d\tau$ .
- Determine the particular solution of the differential equation of Exercise 4 which is zero when  $x = 0$  and when  $x = l$ .
- Show that  $L\{J_1(t)\} = 1 - \frac{z}{(1-z^2)^{3/2}}$ . *Hint.*  $J_1(t) = DJ_0(t)$  and  $L\{J_0(t)\} = (1+z^2)^{-1/2}$  (see Exercise 2, at top of page).
- Show that  $L\{tJ_1(t)\} = (1+z^2)^{-3/2}$ .
- Determine the rest solution of  $(D^2 + aD)x = te^{-t}$ .
- Determine the rest solution of  $(D^2 + n^2)x = a \sin nt$ .
- Determine the rest solution of  $(D^2 + 1)^2x = \sin t$ .
- Determine the rest solution of the vector differential equation

$$(D^2 - 4)x - (D + 2)y + (D - 2)z = \sin 2t;$$

$$2Dx - (D^2 - 3)y + (D^2 - 4)z = 0;$$

$$(D - 2)x - y + (D^2 - 4)z = 0.$$

12. Determine the solution of the homogeneous vector differential equation

$$(D^2 - 3D + 2)x + (D - 1)y = 0;$$

$$-(D - 1)x + (D^2 - 5D + 4)y = 0;$$

for which  $x(0) = 0$ ,  $y(0) = 1$ ,  $Dx(0) = 0$ ,  $Dy(0) = 0$ .

13. Show that  $L\{J_0(t)\} = (1 + z^2)^{-1/2}$  by applying the Laplace transformation to the equation  $tD^2u + Du + tu = 0$ . *Note.* Any linear differential equation whose coefficients are linear functions of the independent variable transforms, under the Laplace transformation, into a first-order differential equation. The solutions of the original differential equation which are "lost" on applying the Laplace transformation are those which do not possess a Laplace transform (e.g.,  $K_0(t)$  in the case of Bessel's equation of zero order).





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