ELEMENTS
OF
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BY

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PREFACE TO THIRD EDITION

In presenting this long overdue revision the authors wish to express their appreciation of the continued use of the book. No radical changes have been introduced in the arrangement of the text but effort has been exerted to bring each topic up-to-date without making it too complex and difficult for the student and ordinary practicing engineer. As in other editions, catalogue material and data that are subject to rapid change have been excluded, as such material and pictorial illustrations can best be presented by the teacher. New subject matter has been added where necessary but without increasing materially the size of the book. The notation has been changed so as to be more in line with modern practice. Some obsolete sections have been eliminated. The authors will be grateful, as heretofore for criticisms and corrections that will make the book accurate and more useful.

D. S. K.
J. H. B.

Ithaca, N. Y., Sept., 1935.

PREFACE TO THE SECOND EDITION

The cordial reception accorded to this book by teachers, and the somewhat extended use of it by practical men, have encouraged the authors to issue it in revised form. In making this revision, certain changes in arrangement have been made, based upon experience in the use of the book, and a chapter on the fundamental principles of balancing has been added. It is hoped that the changes and additions that have been made will extend the usefulness of the work, by bringing it into accord with modern practice. The authors are greatly indebted to Professor C. D. Albert, Professor of Machine Design at Cornell University, and to Professor E. F. Garner of the same institution, for much helpful criticism in making the revision, and for assistance in reading the proof sheets.

D. S. K.
J. H. B.

Ithaca, N. Y., Jan., 1923.

PREFACE TO THE FIRST EDITION

THIS book is the outgrowth of the experience of the authors in teaching Machine Design to engineering students in Sibley College, Cornell University. It presupposes a knowledge of Mechanism and Mechanics of Engineering. While the former subject is a logical part of Machine Design, it may be, and usually is, for convenience, treated separately and in advance of that portion of the subject which treats of the proportioning of machine parts so that they will withstand the loads applied. The same logical order is usually followed in actual designing, as it is, ordinarily, necessary and convenient to outline the mechanism before proportioning the various members.

With the mechanism determined, the remainder of the work of designing a machine consists of two distinct parts:

(*a*) Consideration of the energy changes in the machine, and the maximum forces resulting therefrom.

(*b*) Proportioning the various parts to withstand these forces.

This logical procedure, and the fundamental principles underlying the first part (*a*), are seldom made clear to the student, in works of this character; and such information as is given on energy transformation in machines is, in general, that relating to special cases or types. A thorough understanding of these general principles is, however, in most cases, essential to successful design, since a consideration of the machine as a *whole* necessarily precedes consideration of details. A very brief discussion of typical energy and force problems is given, therefore, in Chapter II, in the hope of making this important matter somewhat clearer to the beginner.

While the treatment presented presupposes a knowledge of Mechanics of Materials, a brief discussion of the more important straining actions is given in Chapter III, partly to make the application of the various formulæ to engineering problems somewhat more definite, and partly to present such rational theory as is of assistance in selecting working stresses and factors of safety. This discussion serves also to show why certain equations have been selected in preference to others, and also to collect in concise form the more important equations relating to stress and strain with which the designer needs to be familiar.

The general principles of lubrication and efficiency are discussed in Chapter IV. Both of these are of prime importance to the engineer; and while the discussion is necessarily brief it is believed that the fundamental principles are fully covered.

The remainder of the book is devoted to the discussion of some of the more important machine details, with a view of showing how the theoretical considerations and equations discussed in the first part of the work are applied and modified in practice. The treatise is, in no sense, a handbook, neither is it a manual for the drafting room; but is a discussion of the fundamental principles of design, and only such practical data have been collected as are needed to verify or modify logical theory. It is hoped that the illustrative numerical examples which are introduced throughout the work may, in conjunction with the analytical methods given, suggest proper treatment of practical problems in design. The treatment of all topics is necessarily brief, as it was desired to obtain a text book which could be conveniently covered in one college year and yet present the salient features of the subject needed by the student as a preparation and basis for more advanced work. While intended primarily for engineering students, it is hoped that it may also prove of some interest to the practicing designer. It has been the endeavor, in the preparation of the book, not only to develop rational analytical treatment, with due regard to constructive considerations and other practical limitations, but to reduce the analysis to such forms and terms that definite numerical results can be obtained in concrete problems.

Considerable of the matter contained in the book has already been published, specially for the use of students in Sibley College, under the title of "Special Topics on the Design of Machine Elements," by John H. Barr, and also in "Elements of Machine Design," Part I, by the Authors. The writers have availed themselves freely of the work of many others in the field, for which due credit is given in the text.

The authors are especially indebted to Professor G. F. Blessing of Swarthmore College, Professors W. N. Barnard, L. A. Darling, and C. D. Albert of Sibley College, Cornell University, all of whom have given instruction in the course at various times, and also to Mr. A. J. Briggs, for many helpful suggestions and criticisms. They will be very grateful for further suggestions or criticisms which will improve the book.

D. S. K.

ITHACA, N. Y., June, 1909. J. H. B.

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MACHINE DESIGN

CHAPTER I

INTRODUCTORY

1. The purpose of machinery is to transform energy obtained directly or indirectly from natural sources into useful work for human needs. Useful work involves both *motion* and *force;* hence the basis of machine design is the laws that govern motion and force.

The term useful work carries with it the idea of definite *motion* and definite *force*, for work itself is always of a definite or measurable character. An examination of any machine will show that its parts are so put together as to give definite *constrained* motion suitable for the work to be done. The constrainment of motion is determined by the moving parts, the stationary frame, and the nature of the connections between them.

Mechanics is the science which treats of the relative motions of bodies, solid, liquid, or gaseous, and of the forces acting upon them.

Mechanics of machinery is that portion of pure mechanics which is involved in the design, construction, and operation of machinery. It has been noted that the consideration of a machine involves *constrained* motion; hence that portion of pure mechanics is mostly needed in machine design which deals with stationary structures and constrained motion.

Machine design, therefore, may be defined as the practical application of mechanics of machinery to the design and construction of machines. It involves the determination of the structure, form, size, and relation of the various parts of a machine, in advance of its construction. The laws of mechanics of machinery give us the underlying principles on which machine action rests, but their practical application depends upon many modifying conditions. In some problems of machine design it is difficult, if not impossible, to apply the laws of mechanics with accurate results, and recourse must often be had to judgment and experience.

A mechanism is a combination of material bodies so connected that motion of any member involves definite, relative, constrained motion of the other members. A mechanism or combination of mechanisms which is constructed not only for modifying motion but also for the transmission of definite forces and for the performance of useful work is called a **machine.** A machine consists of one or more mechanisms; a *mechanism,* however, is not necessarily a machine. Many mechanisms transmit no energy except that required to overcome their own frictional resistance, and are used only to modify motion, as in the case of most engineering instruments, watches, models, etc.

A brief reflection will show that the same mechanism will serve for different machines (see any treatise on kinematics), and, within limits, the design of the mechanism for a given machine may usually be carried out, so far as motion is concerned, with little regard to the amount of energy to be transmitted. This, of course, does not apply to such mechanisms as centrifugal governors, or, in general, where inertia or other kinetic actions affect constrainment of motion. Except for such limitations as those just noted, the design of any machine may be divided into two main parts:

(**1**) Design of the mechanism to give the required motion.

(**2**) Proportioning of the parts so that they will carry the necessary loads due to transmitting the energy, without undue distortion or practical departure from the required constrained motion.

(1) The design or selection of the mechanism for a machine is governed by the manner in which the energy is supplied and the character of the work to be done, for energy may be supplied in one form of motion and the work may have to be done with quite a different one. If mechanisms already exist which will accomplish the desired result, the problem is one of selection and arrangement of parts. But if a new type of machine is to be built, or a new mechanism is desired, the solution of the motion problem borders on, or may indeed be of the nature of, invention. Although it is true that usually the mechanism and the relative proportions of its parts can be designed to suit the work to be done, without reference to the energy transmitted, in general it is necessary to know something about the energy transmitted before any *definite dimensions* of the parts of the mechanism can be fixed, and frequently before the nature of the mechanism is determined. Furthermore, the methods and available facilities of construction control the design to a large extent. Thus, in designing a steam engine, the size of the cylinder must be fixed before the length of crank and connecting-rod can be determined,

and, in general, even though the mechanism can be treated apart from the energy problem, it is necessary to keep the latter constantly in mind.

(2) The problem of proportioning the various parts of a machine, so that they will carry their loads without excessive or undue deformation, may conveniently be divided into two parts:

(a) Solution, as a whole, of the energy and force problem in the mechanism.

(b) Assigning of dimensions to the various parts, based on the forces acting upon them.

(a) When the type and proportions of the mechanism have been fixed, the relative velocity of any point in the mechanism may be found. If, then, the energy which the mechanism must transmit is known, it is generally possible to find the forces acting at any point, since the law of conservation of energy underlies all machines; or, the product of velocity and force is constant throughout the train. If the forces acting on a machine member and the manner in which it is connected are known, these may serve as a basis for the assigning of definite dimensions to the part. A fuller discussion of this important principle is given in Chapter III.

(b) If an accurate theoretical analysis of the forces acting upon a machine member can be made, the form and size of the member may be satisfactorily determined by the application of rational formulas based upon the laws of mechanics. And even when these forces can be determined with only a fair degree of accuracy, a satisfactory solution of the size and form of the member can often be obtained by the application of rational formulas or of semi-rational formulas—that is, rational formulas with coefficients that have been determined experimentally or by practical experience. Thus, a machine member subjected to a simple tension within known limits can be intelligently proportioned by the use of the well-known rational tension formulas. But in many cases the forces acting on a machine member are very complex, the theoretical design is not always clear, and our knowledge of materials and their laws is limited in many respects. Recourse must therefore often be had to judgment or to empirical data, the result of experience. Even when the conditions are clear, theoretical design must always be tempered with practical modification and by constructive considerations, etc. The logical method of proportioning machine elements where theory is applicable is, therefore, as follows:

(a) Make as close an analysis as possible of all forces acting, and proportion parts according to theoretical principles.

(b) Modify such design by judgment and by a consideration of the practical production of the part.

For details and unimportant parts, judgment and empirical data are commonly the best guides.

Summing up, then, the logical steps in the design of a machine are as follows:

(I) Selection of the mechanism.
(II) Solution of the energy and force problem.
(III) Design of the various machine members, with a view to preventing undue distortion or breakage under the loads carried.
(IV) Specification and drawing.

The design of all machine members involves the selection of the material best fitted to withstand the forces applied to it and the conditions surrounding it. Ordinarily this is not a difficult problem, but it should be borne in mind that in many cases it may be very difficult, demanding an extensive knowledge of the properties of materials of construction and the several uses to which they have been successfully applied in the past. Strength and rigidity are usually basic requirements, but durability, resistance to corrosion, magnetic qualities, antifriction qualities, and others may be of importance in deciding just what material should be employed.

The actual design of a machine, furthermore, involves many considerations other than the properties of materials and the computation of strength and rigidity. Thus, proper provision must be made for lubrication; convenience and safety in operation may be important; the facilities for fabricating the machine should always be taken into account; and considerations involving special tools and fixtures quite frequently affect the design. Again, the possibility of using standard parts or parts and material on hand may modify the design somewhat. The cost of the machine is almost always an important factor and is never to be lost sight of by the designer. Care must also be used that the completed machine be accessible for adjustment and repairs.

This last feature of machine design is closely connected with the problem of assembling the machine. Obviously, the designer must be sure that the machine can be assembled with a minimum of difficulty. In complex machines, such as typewriters, automobiles, and cash registers, this may require considerable forethought, especially if the machines are to be made in large quantities and a high degree of division of labor is the practice in the assembly. In such machines the use of so-called " unit assembly " has become quite common. A well-designed automobile, for instance, is made up of a number of self-contained units, each of which can be assembled as an independent structure. Thus, the motor is one unit, the gear case another, the rear axle with its differential another. The actual assembly of the automobile consists, therefore, of the assembling of a number of units, some of which may contain many pieces and may be quite complex in themselves. Obviously, such a method of construction simplifies repairs, if the design is such that unit assemblies can be removed without disturbing many of the adjacent parts or other units. This principle, though it is coming into extended use in the production of automobiles, typewriters, and other machines that are made in large quantities, has not received the consideration that its importance deserves. An extended discussion of these several important practical factors that affect the design of a machine is beyond the limits of this work, but the student of design will profit by a careful consideration of them.

Specification and drawing are necessary and important adjuncts to the process of design; they are powerful aids to the designer's mental process and are the best means of showing the workman what is to be done to construct the machine in question, and also of making a record of what has actually been done. Specification and drawing are not, in themselves, machine design, however, as machines may be designed and built without any drawings. They are, nevertheless, an indispensable part of the designer's equipment. Very often written specifications accompanying the drawings are not only useful but even necessary. In fact, the highest skill on the part of the designer is often needed to specify clearly and fully, in writing, just what is to be done, as the writing of specifications presupposes the most intimate knowledge of theory of design and selection of materials.

From the foregoing it is seen that the part of machine design included in mechanism can be and generally is, for convenience, taught as a separate subject, and the student is expected to have a knowledge of mechanism, mechanical drawing, mechanics of engineering, and materials of engineering as a preparation for the work contained in this book. The chapters that follow deal, therefore, with the solution of the energy and force problem and the design of machine elements.

CHAPTER II

THE ENERGY AND FORCE PROBLEM

2. From the law of conservation of energy it is known that energy can be transformed or dissipated but not destroyed. Therefore, all the energy supplied to any machine must be expended as either useful or lost work. Since frictional resistances, and frequently other losses, occur in all machines, the useful work done must always be less than the energy received. The useful work delivered, divided by the energy received, is called the **efficiency** of the machine. This factor is different for different machines and is evidently a fraction, or less than unity. In the discussion which follows in this chapter, frictional losses are neglected, unless otherwise stated.

A kinematic cycle is made by a machine when its moving parts start from any given set of simultaneous positions, pass through all positions possible for them to occupy, and ultimately return to their original positions.

The energy received by a machine during a kinematic cycle may or may not be equal to the work done plus frictional losses. Thus, the energy supplied during a number of cycles may be stored in some heavy moving part and then be given out during some succeeding kinematic cycle, as in a punching machine with a heavy flywheel.

An energy cycle is made by a machine when its moving parts start from any given set of simultaneous energy conditions, pass through a series of energy changes, and ultimately return to their original energy conditions.

Thus the complete mechanism of a four-stroke gas engine makes one kinematic cycle every two revolutions of the crankshaft. The slider-crank mechanism of the engine, considered separately, makes a complete kinematic cycle every revolution of the crank. The engine makes one energy cycle every two revolutions of the crank. If a punching machine, driven by a belt and running continuously, punches a hole every fourth stroke of the punch, it will be making a complete kinematic cycle every stroke and a complete energy cycle every four strokes.

Therefore, during a kinematic cycle,

$$\textbf{Energy received} = \textbf{Useful work} + \textbf{Lost work} \pm \textbf{Stored energy}$$

And during an energy cycle,

Energy received = Useful Work + Lost work

Generally speaking, the useful work to be done and also the character of the source of energy are known, and the problem of design is, therefore, to select the mechanism which will transform the motion of the source of energy into the required motion, to determine the capacity of the driving device, and to proportion the machine members.

The proportions of any machine part depend, as regards strength and rigidity, on the maximum force it must carry; and this maximum force may be due to the direct action of the driving device, or it may result from the inertia effect of some member which has a capacity for storing energy, and in such a case may be greatly in excess of any direct force that the driving device may deliver. Before this maximum force can be determined for any member, it is therefore necessary to make a complete solution of the energy problem, including the determination of the driving device.

A knowledge of the *quantity* of energy required to do the desired work during a complete energy cycle is not always sufficient information upon which to base the design of the machine or the capacity of its driving device.

A machine may receive energy at either a **uniform** or **variable rate** and may be called upon to do work at either a uniform or variable rate. **Power,** or *rate of doing work,* being the product obtained by multiplying together simultaneous values of velocity and force, it follows that in making any energy transformations both the force and the velocity factors must be kept in mind. Although the mechanism chosen may transform the motion of the source of energy into the *desired motion,* it may not necessarily so modify the *energy* as to give a distribution of *force,* at the point where work is being done, which exactly or even approximately fulfills the required conditions. Again, some of the moving machine parts may have to be very heavy in order to carry the required loads, and during one part of the cycle they may absorb energy, thus reducing the operating force, whereas at another part of the cycle they may give up energy, thus increasing the operating force. Such a condition may make an entirely different distribution of the forces acting on the members of the mechanism from that which would occur were the parts light or the motion of the machine very slow, and may materially modify the design.

If it is predetermined that some device is to be used for storing energy when the effort is in excess, and for giving it out when the effort is deficient, the *capacity* of the driving device need only be such as will supply, during the energy cycle, an amount of energy equal to the useful work and lost work during that cycle. But in many machines such devices are not desirable and in many others they cannot be applied.

Two examples may be noted. (a) In many machines under **continuous operation,** where flywheels are not desirable, it is found that, if the driving device is proportioned so as to supply energy at a uniform rate equal to the *average* rate required throughout the energy cycle, the force at the operating point is sometimes greater and sometimes less than that required. If simultaneous values of the force and velocity at the working point are multiplied together, their product is the rate at which work will be done at the point considered. The maximum product thus obtained will be the **maximum rate** at which work will be done and also at which energy must be supplied by the source. It is evident that the capacity of the driving device will be greater in such a case than if based on the average rate of energy required per energy cycle. If the driving device under the above conditions should be too large or expensive, as it is likely to be in large work, recourse must be had to a different mechanism or to the use of flywheels or other means of storing and redistributing energy. (b) Again, consider any **hoisting mechanism.** Not only must the driving device supply, during the cycle of operations (the raising of the load), energy equal to the work done, but it must also be able to *start* and *sustain* the load at any point. It is evident that the *torque* of the driving device on the hoisting drum must be at least equal to that of the load, and if the torque of the driving device should be variable, its minimum torque must be equal to that of the load when referred to the same shaft.* If this minimum torque should be small compared to the maximum, the driving device chosen might have to be excessively large and this condition might preclude the use of the driving device first selected.

In any of these cases, after the form and capacity of the driving device have been determined, the **maximum force** that may come on any member may also be determined.

It is to be noted that the choice of mechanism and the capacity of the driving device are governed largely by the relative manner in which energy is to be received and work done, and it may be well to enumerate the combinations that can occur, before applying the above principles to the discussion of illustrative problems.

In any machine *under continuous operation* energy may be received and work may be done in one of the following ways:

* In certain hoisting devices friction is utilized to sustain the load or prevent overhauling; this statement does not apply broadly to such devices.

(a) Energy may be received at a constant rate and work be done at a constant rate.

(b) Energy may be received at a constant rate and work be done at a variable rate.

(c) Energy may be received at a variable rate and work be done at a constant rate.

(d) Energy may be received at a variable rate and work be done at a variable rate.

3. Case (a). As an example of a machine in which energy is received at a constant rate and work done at a constant rate, consider a steam turbine driving a centrifugal pump raising water to a fixed level. Evidently the rate at which energy is supplied must just equal the rate at which work is done, plus frictional and other losses, for any given period, and the capacity of the turbine is very easily determined.

4. Case (b). As an example of this case (energy received at a constant rate and work done at a variable rate) consider a machine for punching holes in boiler plate. Here the driving belt can supply energy at a constant rate, while the useful work, which is of considerable magnitude, is delivered intermittently. If the driving belt were designed with sufficient capacity to force the punch through the plate by direct pull, it would have to be very large. The machine runs idly a large portion of the time, while the plate is being shifted, and in a machine of this kind a device for storing energy, such as a flywheel, can be used to advantage. The total capacity of the driving belt need only be sufficient to supply, during the energy cycle, an amount of energy equal to the useful work plus the lost work. When a hole is punched the velocity of the wheel is reduced, the wheel giving up stored energy. During the time that the machine is running idly the belt can store up energy in the flywheel by bringing its velocity up to normal. The maximum *force* that may be transmitted by the machine members will be based on the maximum force at the tool and will be transmitted only by the members that lie between the tool and the flywheel.

As a second example of these conditions, take the design of a small shaping machine. Here the useful work is done during the forward stroke of the ram. During the return stroke frictional resistances only are to be overcome. The resistance of the cut during the forward stroke is uniform, and the speed of cutting is limited by the character of the metal to be cut. During the return stroke, however, the velocity may be greatly increased, the limiting velocity depending on the mass of the moving parts, as these should be brought to rest at the end of the stroke without shock. The machine is driven by a belt which *can* supply energy at a maximum uniform rate. As noted above, the work is done at a variable rate, and if a flywheel is used the conditions are identical with the preceding problem. It is usually not desirable to use a flywheel as its inertia interferes with starting and stopping the machine readily. The problem, therefore, is to design the driving belt and proportion the machine members on the basis of the *maximum continuous* pull that the belt may be able to exert.

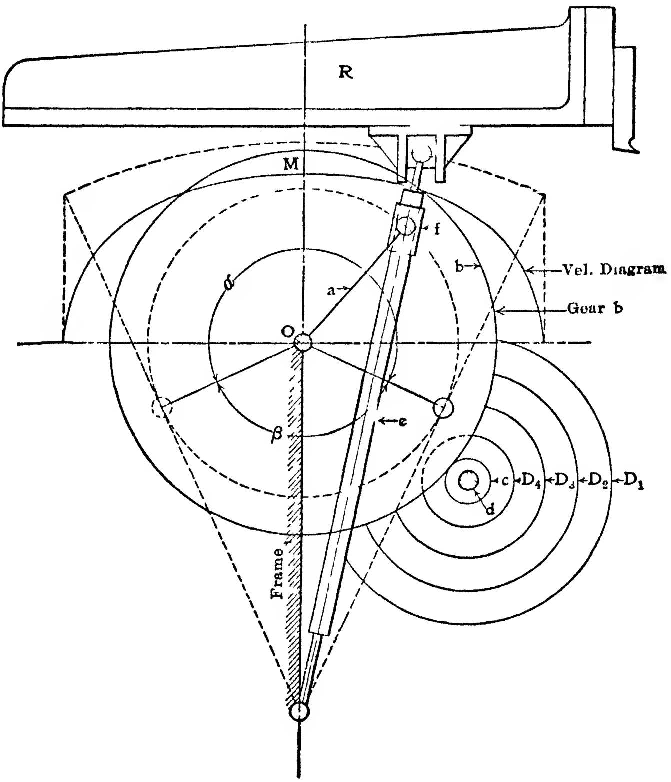


Fig. 1.

Numerous mechanisms have been devised to meet these conditions. Suppose that a mechanism such as shown in Fig. 1 has been selected. The maximum length of stroke is fixed by the work to be done, and the minimum length of stroke should be 3 or 4 in. Continuous rotary motion is imparted to the crank *a* through the gear *b* of which it forms a part. The gear *b* is in turn driven by the pinion *c* which is rigidly attached to the shaft *d*. On the other end of *d* is a stepped pulley having diameters D_1 D_2 D_3 D_4. On the countershaft overhead is a mating stepped pulley so placed that, when the belt is on the largest step of the machine D_1, it is also on the smallest step of the countershaft pulley. The crankpin on a is adjustable and can be moved from the outer position as shown toward the center of the crank, so that the vibrator e can be made to give the ram R any length of stroke from the maximum (20 in. in this example) to a minimum of 3 or 4 in. The range of velocity of the tool for any length of stroke must be such that it can be lowered to the cutting velocity of hard cast iron or tool steel and raised to the economical cutting velocity of brass. With the pin in its extreme outer position and the belt on the large step D_1 the speed of the ram will be a maximum for that position of the belt. As the crank is drawn toward the center (the belt remaining in its original position) the velocity of the ram is obviously decreased. If now the belt is shifted to a smaller step as D_2, the velocity of the ram will be increased, so that at any stroke variable speed may be obtained to suit the metal to be cut.

The mechanism transforms the uniform rotary motion of the line shaft into the required reciprocating motion. Consider the crankpin at its extreme outward position and the belt on D_1. The velocity diagram for full forward stroke under these conditions is shown, the ordinates of the diagram * representing the velocity of the ram to the scale that the crank length represents the uniform velocity of the crankpin. The diagram for the backward stroke is not drawn since it is not needed in the solution of the energy problem; but it should in general be drawn to make sure that the change in velocity at the extreme ends of the stroke is not excessive. If the belt supplies energy at a constant rate the force which it can deliver at the tool will vary inversely as its cutting velocity. The cutting resistance, however, is uniform, so that though the mechanism produces the desired transformation in *motion* it may not give the distribution of *force* desired.

To design the driving device (or belt) for such a mechanism, the operating conditions of the machine when the belt has both its maximum and minimum velocity must be investigated. The maximum pull which a belt can give is $T_1 - T_2$, where T_1 is the allowable tension on the tight side of the belt. (See Chapter XV.) The *power* † that a belt can give out is therefore $V(T_1 - T_2)$, where V is the velocity of the belt. Since $T_1 - T_2$ has, at all moderate belt speeds, a constant maximum value for a given belt, the power that a belt can deliver

* For a full discussion of these so-called *quick-return* mechanism and the methods of drawing velocity diagrams see "Kinematics of Machinery" by Barr and Wood, "Machine Design" by Smith and Marx and "Kinematics of Machinery" by Albert and Rogers.

† A full discussion of the power transmitted by belting is given in Chapter XV.

will vary directly with its velocity. The belt receives its energy from a shaft running at constant speed, and when the belt is on the smallest step of the countershaft cone it will also be on the largest step D_1 of the machine cone and will in consequence be running at its lowest velocity, under which condition its capacity for delivering energy is a minimum.

The maximum power required for small machine tools is approximately constant at all speeds, for, since the heating effect which governs the cutting capacity of the tool is proportional to the work done, it follows that as the cutting speed is increased the resistance of the cut must be decreased, and *vice versa,* thus keeping their product approximately constant. If then the belt is designed to have sufficient capacity when the ram is making full stroke and the belt is on D_1, and hence at the lowest belt velocity, it will have excess capacity when in any other position. If a softer metal is to be cut the velocity of the ram may be increased, but this can be done only by shifting the belt to a position where its velocity and hence its capacity will be greater.

As before noted, the effect of moving the crankpin inward, the belt remaining in the same position, is to decrease the average velocity of the ram. Therefore as the stroke is made shorter the velocity of the crank, to maintain a given cutting speed, must be increased by shifting the belt to a smaller step of the machine cone. The other limiting condition is when the ram is making its shortest stroke and giving a cutting velocity high enough for the softest metal to be worked. The belt should then be on the smallest diameter D_4, and hence at its highest speed.

An inspection of the velocity diagram when the ram is making full stroke shows that its velocity is a maximum when the ram is in mid position. Neglecting friction and inertia, which here are small, the force exerted on the ram will be a minimum where the velocity of the ram is a maximum at any given belt velocity, because, for a given belt pull, since no flywheel is used, force at belt $\times$ velocity of belt = force at tool $\times$ velocity of tool. If, therefore, with the ram making full stroke, the capacity of the belt when running on D_1 is made great enough to give a force at mid position of the ram equal to the required cutting force, it will have excess capacity at any other position; and if this condition does not give too large a belt the driving device will be satisfactory. The maximum *force* that any member may have to sustain will be based on the maximum torque of the belt, which will occur when it is running at D_1; for since the inertia forces are small this torque will be transmitted directly to the members, and the resulting stresses may be easily computed.

Example.

Let the greatest resistance of cut = 800 lb
" " maximum stroke of ram = 20 in.
" " minimum stroke of ram = 4 "
" " maximum length of crank = $6\frac{1}{2}$ "
" " minimum length of crank = $1\frac{1}{2}$ "
" " maximum cutting speed on shortest stroke and highest belt speed = 60 ft per min.
" " maximum cutting speed on full stroke and lowest belt speed = 25 ft per min.

Then, in general,

$$\frac{\text{Linear velocity of crank}}{\text{Length of crank}} = \frac{\text{Maximum linear velocity of ram}}{\text{Maximum ordinate of diagram *}}$$

Hence, in this example when the ram is making full stroke at lowest speed,

$$\text{Linear velocity of crank} = \frac{25 \text{ ft} \times 6\frac{1}{2}}{7} = 23.5 \text{ ft per min}$$

$$\therefore \text{ rpm of crank} = \frac{23.5 \times 12}{2 \times \pi \times 6\frac{1}{2}} = 6.9$$

In a similar way, when the ram is making the shortest stroke at highest speed,

$$\text{Linear velocity of crank} = 42.5 \text{ ft per min}$$

Therefore, $$\text{rpm of crank} = \frac{42.5 \times 12}{2 \times \pi \times 1\frac{1}{2}} = 54.1$$

Let the gear ratio be 8 to 1. Then the minimum and maximum rpm of shaft d = 55.2 and 432.8, respectively. A 14-in. pulley is a convenient diameter for D_1.

$$\therefore \text{ Velocity of belt on low speed} = \frac{14 \times \pi \times 55.2}{12} = 204 \text{ ft per min}$$

If the efficiency of the machine be 85 per cent, the maximum rate of doing work at this position of belt is the cutting resistance multiplied by the maximum velocity of the ram, divided by the efficiency, or

$$\frac{800}{.85} \times 25 = 23{,}500 \text{ ft-lb per min}$$

$$\therefore \text{ Effective pull at belt} = \frac{23{,}500}{204} = 115 \text{ lb approximately.}$$

* In the mechanism here chosen the position of the ram for maximum velocity can be located by inspection and the value of the velocity determined without drawing the complete diagram. In general, however, the diagram must be drawn in order to locate the maximum ordinate.

The effective pull of single-ply belt per inch of width may be taken at 40 to 45 lb.

$$\therefore \text{Width of belt} = \frac{115}{45} = 2\tfrac{1}{2} \text{ in., nearly}$$

If the cone pulleys on machine and countershaft are alike, as is usual in metal-working tools, then

$$\frac{D_1}{D_4} = \sqrt{\frac{\text{Max rpm of machine cone}}{\text{Min rpm of machine cone}}}$$

$$\therefore D_4 = D_1\sqrt{\frac{\text{Min rpm of machine cone}}{\text{Max rpm of machine cone}}}$$

and hence, in the example if $D_1 = 14$, $D_4 = 14\sqrt{\frac{55.2}{432.8}} = 5$ in., nearly.

The **maximum force** that may be applied to any member will be based on the maximum torque of the driving belt, which occurs when the belt is on D_1, the largest step of the machine cone. The difference in this respect between this case and the punching machine discussed above should be noted, for, while the driving mechanisms of both can deliver energy at a uniform rate and while both do work at a variable rate, the maximum load is applied in entirely different ways.

During the complete energy * cycle of the machine the total work done, neglecting friction, is equal to the length of stroke multiplied by the uniform resistance of the cut, or

$$800 \times \frac{20}{12} = 1333 \text{ ft-lb}$$

For every cycle of the machine the shaft d makes 8 revolutions; hence the amount of energy that the belt could deliver if work were done uniformly during one cycle is

$$8 \times \frac{14 \times \pi}{12} \times 115 = 3370 \text{ ft-lb}$$

The **capacity** of the belt is therefore two and one-half times as great as it would need to be if a device for equalizing the energy, such as a flywheel, had been used. If a small machine is belt-driven, as the one under discussion, this added first cost is not serious. But when the power needed is great, or in such cases as direct driving by electric motor, the additional cost of a driving device so greatly in excess of average requirements needs to be carefully considered. This, in fact, is one of the most important elements to be considered in fixing the size of

* The kinematic and energy cycle are, in this example, simultaneous.

motors needed for direct-driven machine tools, sometimes making it desirable to introduce a flywheel to reduce the size of motor.

5. Case (c). One of the best examples of Case (c), where energy is

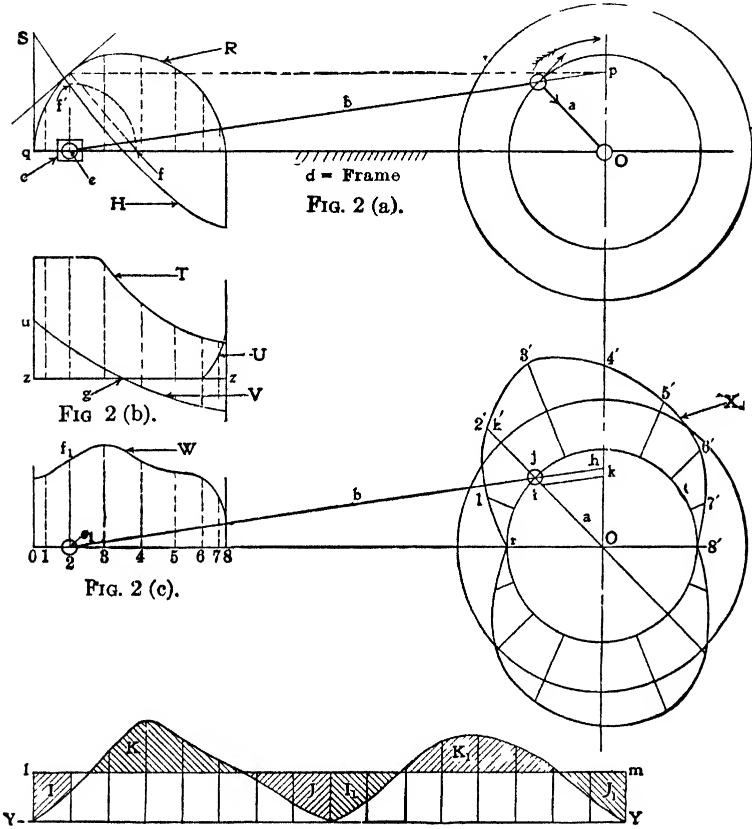


Fig. 2 (a).

Fig 2 (b).

Fig. 2 (c).

Fig. 2 (d).

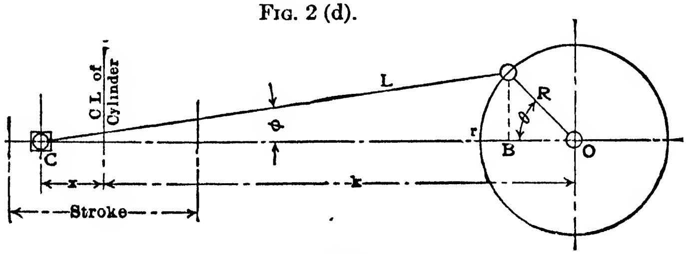


Fig. 3.

received at a variable rate and work is performed at a uniform rate, is found in the reciprocating **steam engine,** and since this machine is of such great importance to the engineer it will be discussed somewhat in detail. Here the energy is supplied in the form of steam pressure, and after cutoff occurs and the steam expands in the cylinder the pressure falls from the " initial " or boiler pressure to somewhat above exhaust or atmospheric pressure. The energy is therefore supplied at a **varying rate.** But the engine is required to deliver energy at the driving belt at a **uniform rate.** The mechanism used will produce the required transformation of the reciprocating motion of the piston into the rotary motion of the crankshaft. But the distribution of the driving force in the form of torque or tangential effort will not be uniform; it will be a maximum somewhere near the position at which the crank is at right angles to the connecting-rod, and it will become zero when the crank is on the dead center. The turning effort will therefore sometimes be greater and sometimes less than the resisting effort of the driving belt, and the machine will stop unless a redistributing device, such as a flywheel, is used. The reciprocating parts, such as the piston and crosshead, and also the connecting-rod, are heavy, and their maximum velocity is considerable; hence the forces due to their inertia cannot be neglected.

Referring to Fig. 2 (a), the crank a is required to rotate around the center O with uniform velocity and to give a uniform force at the driving belt. The moment at the driving belt is equal to the average moment at the crankpin, hence the equivalent uniform force at the crankpin may be derived from that at the belt. This required driving force at the crankpin may be plotted radially from the crank circle as a base, forming a radial diagram of the required force at the pin, as shown by circle S. The crosshead C moves at a varying rate of speed. If the velocity of the crankpin be represented by the length of the crank, the intercept Op made by the connecting-rod on the vertical through O will represent the simultaneous velocity of the crosshead to the same scale. These intercepts may be plotted at the corresponding positions of the crosshead, thus outlining the curve whose ordinates represent the velocity of the crosshead at any point.

The forces which act upon the piston and which must be transmitted to the crank are:

(1) The **steam pressure** which is represented at any point by the ordinates of the curve T, Fig. 2 (b).

(2) The **back pressure** * on the other side of the piston, acting against the steam pressure, and represented by the exhaust pressure line zz and the compression curve U.

(3) The **inertia forces** due to accelerating and retarding the heavy reciprocating parts.

* This generally amounts to 2 or 3 lb per sq in. above atmospheric pressure in non-condensing engines.

During the first part of the stroke these inertia forces tend to reduce the effective pressure transmitted to the crankpin, and during the latter part they increase the effective force on the rod. They can be represented graphically by such a curve as *V*. The first two curves can be found by the well-known methods of drawing indicator cards, and the third can be found either by mathematical deduction or by graphic methods * based on the velocity diagram. It is believed that the analytical method is the most satisfactory, and such a method is presented in a succeeding article.

If the acceleration is known, the force necessary to produce the acceleration is also known, since accelerating force = mass × acceleration, and the force at any point (reduced to pounds per square inch of piston) may be plotted as shown by curve *V*, Fig. 2 (b). When the reciprocating parts reach their maximum velocity their acceleration is zero, hence the curve of acceleration forces crosses the axis at a point *g* corresponding to the point of maximum velocity. This point is very nearly at the position where the crank and the connecting-rod are at right angles, and the error introduced by assuming this to be so is small with ordinary ratios of crank to connecting-rod length. Beyond *g* the reciprocating parts are retarded, hence the inertia forces increase the effective crankpin pressure from that point on. The compression curve (*U*) tends to decrease the effective pressure on the piston and hence its ordinates must be subtracted from the forward pressure. The algebraic sum of the curves *T*, *U*, and *V* will give a resultant pressure curve *W*, Fig. 2 (c), whose ordinates at any point represent the effective pressure acting at the crosshead pin. This effective pressure is transmitted to the crank by the connecting-rod *b*. The pressure of the rod against the crankpin may be resolved into two components, one **tangential** to the crank circle and tending to produce rotative motion, and one **radial** along the crank tending to produce compression or tension in the crank and friction in the main bearing. Only the tangential force can do useful work. If friction be neglected the rate at which work is done by this force at the crank must equal the rate at which work is being done at the piston. Now the curves *R* and *W*, Fig. 2 (a) and 2 (c) respectively, give the simultaneous values of velocity and force at every point of the stroke. If such simultaneous values be multiplied together and divided by the uniform velocity of the crank (all in the proper units) the quotient is the tangential force at the pin, and this may be plotted radially on the crank circle as a base thus

* For a full discussion of this matter, see "Kinematics of Machinery" by J. H. Barr and E. H. Wood, page 70 and "Kinematics of Machinery" by Albert and Rogers, pages 74 to 86.

giving what is called a radial crank-effort diagram, Fig. 2 (c), Curve X.

It is easier to find these values of the tangential force graphically. It will be remembered that the ordinates of the velocity diagram (R), as drawn in Fig. 2 (a), represent the velocity of the crosshead to the same scale as the length of the crank represents the velocity of the crankpin. In Fig. 2 (c), the connecting-rod extended, if necessary, cuts the perpendicular through O in the point h. Therefore Oh = velocity of crosshead when Oj = velocity of crankpin. Neglecting friction, the rate of work at the crankpin is equal to the rate of work at the crosshead; hence the velocity of the crankpin multiplied by the force at the crankpin is equal to the velocity of the crosshead multiplied by the force at the crosshead, or the tangential force $\times\ Oj = e_1f_1 \times Oh$.

$$\therefore \text{ Tangential force } = \frac{e_1f_1 \times Oh}{Oj}$$

Lay off $Oi = e_1f_1$ and draw ik parallel to b. Then

$$\frac{Ok}{Oh} = \frac{Oi}{Oj}$$

Therefore,

$$Ok = \frac{Oi \times Oh}{Oj} = \frac{e_1f_1 \times Oh}{Oj} = \text{tangential force}$$

Therefore Ok may be laid off radially from j as an ordinate of the required curve as jk'. The construction for the return stroke is performed in a similar manner.

It will be noted that the distribution of force, as represented by this diagram, is less uniform than the original curve of pressure at the crosshead. By the conditions of the problem, however, the mechanism must produce a uniform turning effort at the driving belt or such as would be given by a crank-effort diagram like S, Fig. 2 (a). A flywheel must therefore be used to store energy when the crank effort is in excess and to give out energy when the crank effort is deficient. Fig. 2 (d) shows the crank-effort diagram rectified with rectangular ordinates equal to the radial ordinates of curve X. The base YY is equal to the circumference of the crank circle, and the ordinates of the line lm are equal to the ordinates of the required uniform crank-effort curve S. Since the abscissas represent *space* and the ordinates represent *force*, the areas I, K, J, I_1, K_1, etc., represent *work*. The work represented by $K + K_1$ is that which the flywheel must absorb, and the area represented by $I + J + I_1 + J_1$ that which it must give up in one revolution. Manifestly $I + J + I_1 + J_1$ must equal $K + K_1$. A full discussion of the design of the flywheel will be given in a later chapter.

The **maximum force** that may come upon the crosshead can be seen from an inspection of the force diagram W. It is to be noted in this regard that, if the engine is designed for variable cutoff, an indicator diagram at late cutoff should be drawn for the purpose of locating this maximum force, as an earlier cutoff will not give the maximum value. The method of analysis developed above will enable the designer to determine the maximum straining action on any member of the mechanism.

The graphical methods of finding the inertia curves, though convenient, are open to criticism on account of their inaccuracy, because the tangents or sub-normals to the curve, on which these graphic methods depend, are difficult to construct with accuracy and are at some points indeterminate. In general, therefore, it is thought that the following method or some similar one is more satisfactory.

Referring to Fig. 3 (page 15)

Let a = acceleration of reciprocating parts in ft/sec/sec.
R = length of crank in feet.
L = length of connecting-rod in feet.
N = rpm.
θ and ϕ = angles made with center line by the crank and connecting-rod respectively at any position measured from the crank position Or.
k = distance from center of crankshaft to mid position of crosshead in feet.
x = displacement of crosshead from mid position in feet.
n = L/R.
v = velocity of crosshead in ft/sec for any displacement x.
t = time elapsed in seconds.
ω = angular velocity in radians per second.

Then
$$x + k = OB + BC = R\cos\theta + L\cos\phi$$

But
$$L\cos\phi = \sqrt{L^2 - R^2\sin^2\theta} = R\sqrt{\frac{L^2}{R^2} - \sin^2\theta}$$
$$= R\sqrt{n^2 - \sin^2\theta}$$
$$\therefore x + k = R(\cos\theta + \sqrt{n^2 - \sin^2\theta}) \tag{1}$$

Expanding the radical by the binomial theorem and omitting all terms beyond the second (which can be done without appreciable error * with the limiting proportions ordinarily used), equation (1) becomes

$$x + k = R\left[\cos\theta + \left(n - \frac{\sin^2\theta}{2n}\right)\right] \qquad (2)$$

Now x = the distance moved through by the crosshead, from mid stroke and velocity at $x = dx/dt$; and therefore differentiating (2) with reference to t

$$v = \frac{dx}{dt} = -R\left(\sin\theta + \frac{\sin 2\theta}{2n}\right)\frac{d\theta}{dt} \qquad (3)$$

The acceleration =

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -R\left(\cos\theta + \frac{\cos 2\theta}{n}\right)\left(\frac{d\theta}{dt}\right)^2 \qquad (4)$$

but $\frac{d\theta}{dt}$ = angular velocity in radians per second = $\frac{2\pi N}{60}$

hence

$$a = -\left(\frac{2\pi N}{60}\right)^2 R\left(\cos\theta + \frac{\cos 2\theta}{n}\right) \qquad (5)$$

which is the general expression for acceleration of the reciprocating parts.

If the weight of the reciprocating parts be called W, from mechanics it is known that the force necessary to produce an acceleration (a) is

$$P = \frac{W}{g}a$$

where g = 32.2 in English units; therefore,

$$P = -\frac{W}{g}R\left(\frac{2\pi N}{60}\right)^2\left(\cos\theta + \frac{\cos 2\theta}{n}\right) \qquad (6)$$

where R is in feet. Or reducing,

$$P = -\frac{WrN^2}{35{,}200}\left(\cos\theta + \frac{\cos 2\theta}{n}\right) \qquad (7)$$

where r is in inches.

When the solution of the above expression gives a negative result the force of inertia is acting away from the crank, and when positive, toward the crank. It is also to be noted that the expression $\frac{W}{g}R\left(\frac{2\pi N}{60}\right)^2$

* This error is less than one-quarter of one per cent of the acceleration when $\frac{L}{R} = 4$ and still less when $\frac{L}{R} > 4$.

is the centrifugal force of a weight equal to that of the reciprocating parts concentrated at the crankpin since centrifugal force in general is equal to $WR\omega^2/g$.

By means of equation (7) all points on the acceleration curve could be found and plotted. In general, however, the exact characteristics of the curve are not essential and it is sufficient to make the three most simple solutions as follows, and a curve drawn through the three points thus located is sufficiently accurate for all ordinary purposes. In cases of extremely high speed with small ratios of connecting-rod to crank, a more accurate determination of the curve may be desired.

When $\theta = 0$,

$$P = -\left(\frac{WrN^2}{35,200}\right)\left(1 + \frac{1}{n}\right) \tag{8}$$

When $\theta = 180°$,

$$P = +\left(\frac{WrN^2}{35,200}\right)\left(1 - \frac{1}{n}\right) \tag{9}$$

When $\theta = 90°$ or $270°$,*

$$P = \left(\frac{WrN^2}{35,200}\right)\left(\frac{1}{n}\right) \tag{10}$$

It will be noted that the motion of the connecting-rod is complex, the end attached to the crosshead having a reciprocating motion, while the end attached to the crankpin has a motion of rotation around the axis of the shaft. The exact effect of the inertia of the rod upon the crank-effort diagram is not readily computed. It is found, however, that if one-half † the connecting-rod is considered as having a reciprocating motion and if one-half of its weight is added to the weight of the parts that have a true reciprocating motion, the results obtained are sufficiently accurate for most cases.

In vertical engines with heavy unbalanced reciprocating parts, the effect of gravity upon these unbalanced parts may affect somewhat the crank-effort diagram, and it may be necessary to consider this disturbing factor. In horizontal engines the effect of gravity upon the connecting-rod, so far as its influence upon the crank-effort diagram is concerned, is usually neglected.

If the inertia forces are to be combined with the steam pressure, as shown graphically in Fig. 2 (b), they must be reduced to **pounds per square inch of piston** to give correct diagrams.

An **example** may serve to make these points clearer. Let it be

* The piston is *not* at half stroke.

† See "Influence of the Connecting Rod upon Engine Forces" by Sanford A. Moss, *Trans.* A.S.M.E., vol. 26, page 367.

required to design a steam engine to deliver 150 hp with the following data:

Steam pressure = 90 lb gage. Cutoff at $\frac{3}{8}$ stroke.

Ratio of crank to connecting-rod = 1 to 5.

Piston speed = strokes per minute multiplied by length of stroke = 640 ft.

Here something must be known about the size of cylinder necessary, before definite dimensions are assigned to the various members. Let a theoretical indicator card be drawn as in Fig. 2 (b), neglecting for the present the inertia curve V since this only tends to redistribute the energy and does not affect its quantity. The distance zz represents the piston travel, and the ordinates of the curve T represent piston pressures; therefore the area between zz and the curve T represents the work done by the steam pressure during the stroke. In a similar way the area under curve U represents the work of compression due to back pressure. The difference of these areas is the net work done per stroke of piston, and the mean ordinate corresponding to this area represents to the proper scale the average pressure per square inch on the piston during stroke. In the example given, zz = 2 in. Area under T minus area under U = 1.75 sq in. Therefore, mean ordinate = 1.75/2 = 0.875 in. The scale of pressures taken is 1 in. = 70 lb. Therefore mean pressure during stroke = 70 × 0.875 = 62 lb, nearly.

Let A = area of piston.
P = mean effective pressure per square inch.
L = length of stroke in feet.
N = number of revolutions per minute.
HP = horsepower required.

Then

$$HP = \frac{2PLAN}{33{,}000}$$

Here P, $N \times L$ and HP are known. Whence

$$A = \frac{HP \times 33{,}000}{P \times 2NL} = \frac{150 \times 33{,}000}{62 \times 640} = 132 \text{ sq in.}$$

or a diameter of cylinder of 13 in.

If the stroke be taken at about twice the diameter of the cylinder, or say 24 in., the proportions will be good.

Hence since $2L \times N = 640$, $N = 160$ rpm. The mechanism can now be laid out to scale. This has been done in Fig. 2 (a and c),* the space scale being 1 in. = 1 ft.

* Reduced in reproduction about one-half.

As before stated, the location of the three points, namely, where θ is respectively 0°, 180°, and 90° or 270° (Fig. 3), is sufficient to locate the inertia curve. In the above example W = 3.5 lb sq in., n = 5, and N = 160.

The general expression for the inertia force is, for $\theta = 0$,

$$P = \frac{WrN^2}{35{,}200}\left(1 + \frac{1}{n}\right) = C\left(1 + \frac{1}{n}\right)$$

where C is a constant and here equal to $\frac{3.5 \times 12 \times 160^2}{35{,}200} = 30.5$. Therefore,

When $\theta = 0°$,

$$P = 30.5(1 + \tfrac{1}{5}) = 36.6 \text{ lb per sq in.}$$

When $\theta = 90°$,

$$P = 30.5(\tfrac{1}{5}) = 6.1 \text{ lb per sq in.}$$

When $\theta = 180°$,

$$P = 30.5(1 - \tfrac{1}{5}) = 24.4 \text{ lb per sq in.}$$

These values serve to locate the curve as in Fig. 2 (b).

The resultant of TU and V, curve W, Fig. 2 (c), can now be drawn and the crank-effort diagram X plotted. The crank-effort curve can be rectified as in Fig. 2 (d) and the mean ordinate Yl drawn. The area $I + J = K$ will be proportional to the energy to be absorbed and delivered by the flywheel. One inch of ordinate here = 70 lb per sq in. of piston and 1 in. of abscissa = 1 ft; therefore 1 sq in. of area = 70 ft-lb.

The area of K = 0.5 sq in. and area of piston = 132 sq in. Hence, if E = energy to be absorbed:

$E = 0.5 \times 70 \times 132 = 4620$ ft-lb on which the design of the flywheel can be based.

The maximum pressure that can occur on the piston is the initial or boiler pressure, as the ordinates of W are at all points less than those of T. Hence, when running, the parts will be subjected to less load than in starting up, when full boiler pressure may be applied before inertia forces become noticeable.

6. Case (d). A good example of energy supplied at a varying rate and work done at a varying rate is founded in **a direct-driven air compressor.** Here the varying steam pressure in the steam cylinder is opposed by a varying air pressure in the air cylinder as shown in Fig. 4 (a). The areas of the cylinders are, for simplicity, assumed to be equal. The steam cylinder takes steam at 80 lb. pressure, and the air compressor cylinder delivers air at 100 lb. pressure. The **efficiency** of the system shown is taken at 80 per cent, and hence the area of the compressor card is 80 per cent of the steam card.* If both the pistons are rigidly attached to the same rod it is evident that the maximum steam pressure will occur where the air pressure is a minimum. If, however, each cylinder is independently connected to a common shaft

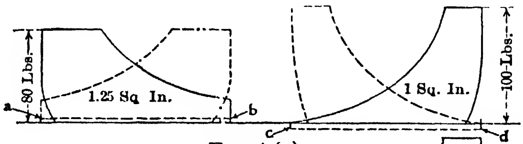


FIG. 4 (a).

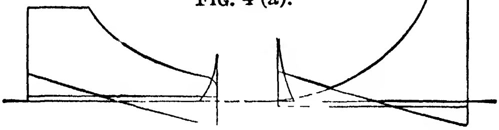


FIG. 4 (b).

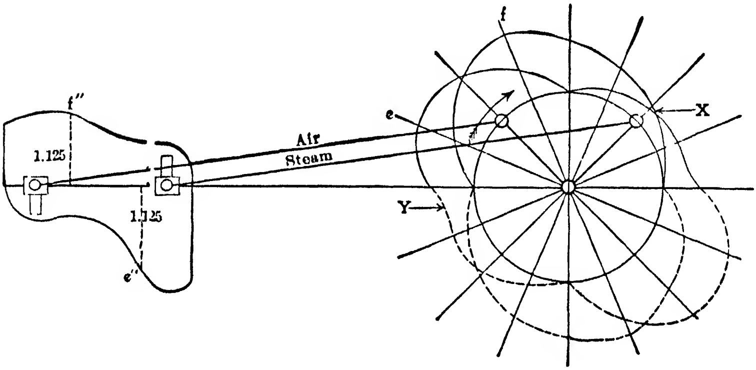


FIG. 4 (c).

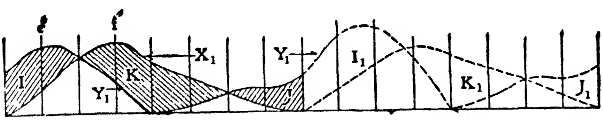


FIG. 4 (d).

by means of a crank and connecting-rod mechanism, the maximum and minimum pressures of the cards may be made to coincide more closely

* In the general case, where the cylinders are of different diameter and area, the diagrams which represent pounds per square inch of piston area would not have a ratio equal to the efficiency. The mean effective pressure of the air cylinder multiplied by the area of the air cylinder, divided by the mean effective pressure of the steam cylinder multiplied by the area of the steam cylinder, would, in this instance, equal the efficiency.

by placing the crankpins at the proper angular distance apart. In other words, the mechanism may be so designed that energy will be delivered at the working point more nearly at the rate required by the work to be done. The loss by friction, etc., is about 20 per cent. Part of this is lost on the steam side and part on the air-compressor side. It can be assumed, without great error, that the losses can be evenly divided between the two slider-crank chains and also that the loss is at a uniform rate throughout the stroke. Thus the loss on the steam side can be represented by the line *ab*, Fig. 4 (a), which *reduces* the effective pressure at every point by a fixed amount. In a similar way ordinates to the line *cd increase* the effective resistance of the air diagram. The area of the diagrams modified in this way will be equal and all energy supplied will be accounted for.

Since the moving parts of both slider-crank chains will be heavy, the effect of inertia cannot be neglected. In Fig. 4 (b) the air and steam cards are shown with the inertia curve, the friction line, and the compression curves in their correct relationship. Fig. 4 (*c*) shows the resultant pressure curves, the curve of air pressures being plotted below the base line for convenience. The crank-effort curve of the steam cylinder is represented by X, and the resisting crank-effort curve of the air cylinder is represented by Y. The cranks are here placed 90° apart, the steam crank being in advance, a common arrangement in practice. It is evident, however, that this is not the most advantageous angle, for if the point *e* on the air curve is made to correspond with *f* on the steam curve, Fig. 4 (c), the excess and deficiency of effort will be still further reduced. This would place the cranks at 45° apart. This is even more clearly shown in Fig. 4 (d), on the rectified curve of crank effort. Here the area $K + K_1$ is the amount of energy to be absorbed, and $I + J + I_1 + J_1$ the amount to be given up by the flywheel during one revolution. In the steam slider-crank mechanism the greatest pressure is, as before, that due to the initial steam pressure, while on the air side it will be that due to the terminal air pressure.

In the four cases discussed above the action of the machine has in all instances been supposed to be continuous, and all machines which operate continuously will belong to one of these classes. Where the action of the machine is **intermittent** or **irregular,** these general solutions will not always hold, and the design of the machine cannot be based on the energy given or received, but will depend on the **maximum force** or **maximum torque** or, in other words, on the **mechanical advantage** which the motor must possess. Thus the motor on an automobile has a certain maximum capacity for delivering power. On a level road it can propel the car at a high rate of speed, the engine making only a few turns to every revolution of the wheels. But on a steep hill the gears must be shifted so that the engine has a greater mechanical advantage, and gives a greater torque on the axle, the engine making many revolutions to every one of the wheels. Another example of this is the case of hoisting mechanisms already discussed briefly (see Article 2). An engine or a motor might be capable of giving out energy at a rate equal to that required to lift the load in a given time, and it might be able, running continuously, to raise the load to the required height. But its ability to *start* and *sustain* the load at any point will depend on whether it has a *mechanical advantage* at that point and *not* on its *capacity*.

In deep mine hoisting the winding drum is frequently made conical in form. When the hoisting cage is at the bottom of the shaft, the rope is at the smaller end of the drum, and the torque on the drum shaft is equal to the weight of the loaded cage plus the weight of the rope multiplied by the radius of the small end of the drum. In a deep mine the weight of the rope forms a large part of the load, when hoisting from the bottom of the shaft. When the rope is all wound in and the loaded cage is at the surface, the torque on the drum shaft is equal to the weight of the loaded cage multiplied by the radius of the larger end of the drum. Obviously the radii of the larger and smaller end can be so chosen as to make the torque on the shaft equal when the loaded cage is either at the bottom of the shaft or at the surface. During the period of hoisting, however, the radius on which the rope is wound is constantly increasing, while the weight of the load is constantly decreasing because of the lessening length of the hoisting rope. With a truly conical drum, this results in a varying torque on the drum shaft, which is often considerably greater when the load is part way up than it is when the load is at either of the extreme positions. It will be noted, however, that the **minimum** torque of the engine or motor must always exceed the **maximum** torque of the load, when referred to the same shaft, if it is to maintain and start the load at all positions. It is possible, of course, to make a drum of such shape that the radius will vary so as to maintain a constant torque, but the longitudinal cross-section of the surface of such a drum is usually a complex curve which is difficult to construct. The mathematical discussion of this problem is beyond the scope of this treatise, but the general principle must be observed in designing all hoisting engines and similar machines which act intermittently and slowly, and where devices for storing and redistributing energy are undesirable or impossible.

7. Redistribution of Energy and Inertia Effects. Devices for storing and redistributing energy are very common in transmission systems. Thus, in hydraulic distribution, the excess supply of power is stored in an **accumulator,** and given out again when the supply is deficient. In electrical distribution a **storage battery** is sometimes used for the same purpose. In transmission of power by compressed air a large **reservoir** is sometimes employed as a storehouse of energy. In a single machine, the redistribution is effected by compressing a gas, by using a spring, or by accelerating and retarding some heavy moving part. Thus in the steam engine the piston compresses steam in the clearance space at the end of its stroke, and the energy so absorbed is returned to it during the next stroke. Again, when the energy supplied by the steam is in excess of the effort required, the flywheel absorbs the excess and thereby has its velocity (and hence its kinetic energy) increased. When the effort is in excess, the wheel gives up the stored energy at the expense of its velocity.

It does not necessarily follow, however, that all heavy moving parts simply *redistribute* the absorbed energy as *useful* work, as the action may be a positive source of loss. In Fig. 5 let A be the platen of a large

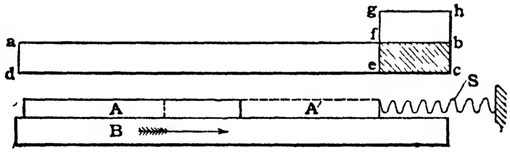


Fig. 5.

planing machine, and suppose it to be making its return stroke, moving from left to right. The force just necessary to slowly move the platen may be represented by the vertical ordinates of the diagram *abcd.* Suppose now, that a greater force is applied, in order to hasten the operation, so that at the position A', the platen has been accelerated till its kinetic energy is equal to the rectangle *eghc.* Evidently the platen will not stop at the end of the stroke if the actuating force be removed at A', as the work of friction during the remainder of the stroke is less than the stored energy. If, therefore, the " return " belt is removed at A' and the " driving " belt applied, the latter will slip upon the driving pulley till the excess of energy is absorbed and dissipated as heat. If the point A' has been properly chosen the platen will just stop at the end of the stroke and the energy absorbed by the belt will equal the area *fghb.* If a spring, S, were fitted to the machine, so that the work of compression from the position A' to the end of the stroke just equaled the excess kinetic energy of the platen, at that position, the return belt could be thrown off at A', and the platen would stop at the end of the stroke. The energy stored in the spring would then be returned to the platen on the forward stroke. This latter action is identical with that of compression in the steam-engine cylinder, Fig. 2, the energy under the curve U being returned to the reciprocating parts on the next stroke. It is to be noted in this last case, that even if the work of compression is not quite equal to the energy to be absorbed during the latter part of the stroke, there is no loss of energy (friction neglected), as what is not absorbed by compression is absorbed at the crankpin in useful effort.

CHAPTER III

STRAINING ACTIONS IN MACHINE ELEMENTS

8. Nature of Forces Acting in Machines. From the foregoing chapter it is clear that machine members which transmit energy are subjected to forces of a varying character and intensity. Since the various parts of a machine must be constrained to move in fixed paths it is important that they should neither break nor be distorted appreciably under the loads carried; that is, the members must be not only *strong* but also *stiff*. The proportioning of machine elements as dictated by various methods of loading is therefore most important, and will be considered in this chapter.

The forces acting on a machine element may be one or several of the following:

(a) The **useful load** due to the energy transmitted.

(b) Forces due to **frictional resistances.**

(c) The **weight** of the part itself or of other parts.

(d) **Inertia forces** due to change of velocity.

(e) **Centrifugal or inertia forces** due to change in direction of motion.

(f) Forces due to change of **temperature.**

(g) **Magnetic attractions,** as in electrical machinery.

(h) **Vibratory forces** due to lack of balance in moving parts, etc.

These forces or loads may be applied to a machine in several ways. They may act steadily in one direction; they may act intermittently in one direction, or they may be applied first in one direction and then in the reverse; they may be applied gradually, or suddenly in the nature of a shock.

A **steady** or **dead load** is one which is always applied steadily in the same direction. Such a load induces stresses that do not vary either in character or in magnitude. A **live load** is one which is alternately applied and removed. Such a load induces stresses that vary in magnitude or that vary in both magnitude and character. A live load imposed suddenly but without initial velocity is called a **suddenly applied load.** When a live load is applied with initial velocity, as a blow from a falling body, the member is subjected to **impact.**

9. Nature of Straining Actions, Stress and Strain. Since all materials of construction are more or less elastic, a machine element must change its form to some extent whenever subjected to a load. This change of form may be very small and temporary; it may be a permanent distortion; or if the load applied be sufficiently heavy the element may even be ruptured. Such change of form, whether temporary or permanent, is called a **strain.** When a machine member is thus distorted under a load certain molecular reactions, equal and opposite to the load applied, are set up within the material and resist the deformation. **Stress** is the term applied to this internal reaction and is to be clearly distinguished from **strain,** stress being in the nature of a force and strain being a dimension.

The character of the straining action and of the stress which results from a given load depends upon the direction and point of application of the load (or forces), and upon the form, the position, and the arrangement of the supports of the member. A given load may produce tension, compression, shearing, flexure, or torsion, or a combination of these. Of course tension and compression cannot both exist at the same time between the same set of molecules. Flexure is a combination of tensile and compressive stresses between different sets of molecules; or, as it is often expressed, in different fibers * of the same body. Torsion is a special form of shearing stress.

The stresses due to tension, compression, and flexure are essentially molecular actions normal to the planes separating adjacent sets of interacting molecules; that is, the stresses increase or decrease the distances between these molecules along lines connecting them.

The primary straining effect of shearing and torsional actions is a displacement of adjacent molecules, tangentially to the planes separating such molecules. In uniform shear the interacting molecules move or are strained relatively with a rectilinear translation. In torsional action the adjacent molecules each side of a plane of stress have a relative motion or strain about an axis. A brief reflection will show that in reality only two kinds of strain exist, namely, elongation (contraction if negative) and shearing. In a similar way only two corresponding kinds of stress are met with, namely, normal or direct, and tangential or shearing.

Machine members are often subjected to combinations of these simple stresses, as flexure and torsion. Such stresses, called **compound stresses,** will be more fully treated later.

When a load is applied to a piece of material the strain or deforma-

* It should be noted that the term fiber is used in a conventional sense when discussing homogeneous metals, such as iron and steel.

tion which results is a function of the load and of the character of the material involved. In general, for a given loading the deformation is different for different materials but constant in its relation to stress for any one material. These relations have been determined experimentally for all the ordinary materials used in engineering, and works on mechanics of materials treat of the subject fully. Enough will be inserted here to make the discussion complete.

If a bar of metal is tested under an increasing tensile load and the strain caused by each successive load is accurately observed, the relation between the unit stress and the unit strain can be shown graphically as at *Oade*, Fig. 6; such a diagram is called a **stress-strain** diagram.

If axes *OX* and *OY* are chosen and the stresses plotted as ordinates and strains as abscissas, it will be found that up to a certain point as *a*, either in tension or compression, the curve so formed is sensibly a straight line; that is, **stress is proportional to strain.** Further, if at any point below *a* the stress is released, the piece returns to its original shape. But above *a* this relation ceases, strain usually increasing * faster than stress. The point *a* is called the **elastic limit** and is fairly well defined in the ductile materials. In testing ductile materials, such as mild steel and soft brass, the stress-strain diagram usually shows a sharp break shortly after passing the elastic limit, small increases of stress resulting in much greater increases in the strain. The point where this change occurs is called the **yield point** and is indicated on Fig. 6 by *y*. The application of the stress between the points *a* and *y* results in an increasing but very small permanent deformation of the piece tested. Beyond the point *y*, the permanent deformation is more marked, small increases of stress causing large increases of strain. If the stress is released at *y*, the stress-strain diagram does not retrace itself but will be along a line *yC* very nearly parallel to *Oa*. The distance *OC* is the permanent deformation. Similar results will be found for any point beyond *y*. The yield point is quite easy to locate in making a test of a specimen, but the elastic limit is not easy to locate with great accuracy. Usually, how-

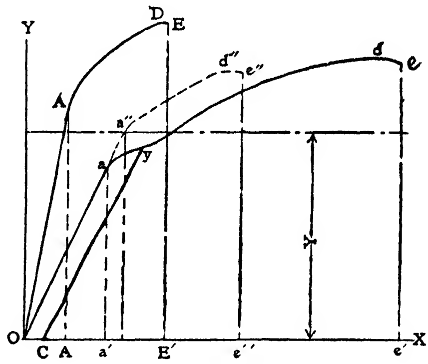


Fig. 6.

* Ordinary rubber is an exception to this general rule, strain decreasing as stress increases.

ever, they are so close together that the difference in their values is negligible, and commercially the yield point is frequently used instead of the elastic limit in specifying the properties of ductile materials. Cast iron and the *brittle* materials have no well-defined elastic limit and little permanent elongation.

If at any point on the curve below *a* the stress be divided by the strain a ratio is obtained which is constant for all points below *a*. This ratio is called the **modulus or coefficient of elasticity.** If, therefore, this modulus of elasticity is known for a given material, the strain corresponding to any given stress may be calculated, providing it does not exceed the value corresponding to the point *a*. The value of the modulus of elasticity for tension, or the **tension modulus,** is practically the same as the modulus for compression. The modulus of elasticity in shearing action is different from that in tension for the same material, and is called the **shear modulus.** It is measured by dividing the shearing stress by the relative twist of the specimen. When the term modulus of elasticity is used without further specification, reference is made to the tension modulus. The modulus of elasticity is a measure of the **stiffness,** or rigidity, of the material, and not of its strength. It should be noted that the stiffness of a strong material may be no higher than that of a relatively weak material. Thus, the coefficient of elasticity of all grades of steel, from the softest to the hardest, is about the same—30,000,000. For a given intensity of stress within the elastic limit, therefore, the corresponding strain is about the same for all grades of this material. If, therefore, it is desired to replace a steel machine member with a more rigid one of the same dimensions, nothing is gained by using a harder grade of the same material. Either the unit stress must be reduced by using larger dimensions or a material must be employed that has a higher modulus.

If sufficient tensile stress is applied to a test piece its elongation increases until finally it "necks down" at its weakest point and rupture occurs. The load per unit of original area under which a bar breaks is called its **ultimate strength,** and the corresponding stress or reaction per unit of original area is called the **ultimate stress.** Similar phenomena are observed when a piece is tested in compression or torsion, etc.

It is evident that the working stress of a machine member must be less than the elastic limit if the piece is to retain permanency of form. The stress at which a member is designed to be operated is called the **working stress,** and the ratio of the ultimate stress to the working stress is called the **factor of safety.** It is to be especially noted that the working stress in the member must not only be kept below the value where permanent deformation takes place, but must also be so low that the resulting *strain,* whatever it may be, shall be so small as not to destroy the proper alignment of the piece, or cause unnecessary friction through distortion. A machine member may be amply strong enough to carry the load with perfect safety, and yet distort so badly under the load as to render it unfit for the service desired. Both *strength* and *stiffness* should therefore be kept in mind in designing a mach ne part, as sometimes one and sometimes the other will dictate the form and dimensions to be used. A short discussion will now be given of the relations which exist between load, stress, and strain for the cases most often met and of their bearing on the selection of the form and s ze of a machine member. In this discussion it will be assumed that the load is a dead load applied without shock, and the modifying effect of suddenly applied and repeated loads will be considered after the fundamental relations between load and stress are established.

10. Tension. Let s_t be the stress in the section, P the load, and A the area of cross-section. The relation which exists between them in simple tension is

$$s_t = \frac{P}{A} \tag{A}$$

If l be the length of the member and Δ the total elongation, the unit elongation or unit strain will be Δ/l. Hence, if E be the coefficient of elasticity,

$$E = \frac{\text{Unit stress}}{\text{Unit strain}} = \frac{P}{A} \div \frac{\Delta}{l} = \frac{Pl}{\Delta A} \tag{B}$$

If, then, a tension member is to be designed to join two machine parts, the formula for strength dictates a piece of uniform cross-section without regard to any particular form. Hence the most convenient or cheapest form would be used, avoiding thin, wide sections where concentrated stress at the edge might cause undue elongation.

Suppose it is required to hold the two surfaces within certain limits, as in machine tools where accuracy is desired. If the tension member is long it may yield more than is desirable, though the working stress may be well below the elastic limit and a greater area may be necessary to reduce Δ to the desired value.

Example. Let $P = 20{,}000$ lb, let the allowable stress $s_t = 10{,}000$ lb, let $E = 30{,}000{,}000$, let $l = 40$ in., and let it be required to keep Δ within 0.001 in. If the design is based on allowable stress alone,

$$A = \frac{P}{s_t} = \frac{20{,}000}{10{,}000} = 2 \text{ sq in.}$$

But for $\Delta = 0.001$,

$$A = \frac{Pl}{\Delta E} = \frac{20{,}000 \times 40}{0.001 \times 30{,}000{,}000} = 26 \text{ sq in.}$$

In general, therefore, where tension members are of any considerable length and distortion under load is of importance, they should be checked as in the foregoing.

11. Compression. If the member under consideration be subjected to compression, the remarks of the last paragraph apply equally well if the member can be considered a short column, i.e., one whose length is not greater than six times its least diameter. If longer than this it must be considered as a long column and the conditions governing its design will be more fully treated hereafter. (See Article 23.)

12. Shear. If the member is subjected to simple shear the expressions for the relations existing between the stress, area, and load are similar to those for tension, or

$$s_s = \frac{P}{A} \qquad (C)$$

13. Torsion. If the member is subjected to a torsional stress, the following relations exist:

Let P = load applied in pounds.
a = arm of load in inches.
d = diameter of circular shaft in inches.
b = side of solid square member in inches.
s_s = shearing stress in pounds per square inch at outer fiber.
c = distance from neutral axis to outer fiber in inches.
l = length of member in inches.
θ = angle of deformation in radians.
T = twisting moment applied to member in inch-pounds.
E_s = transverse coefficient of elasticity.
J = polar moment of inertia.

Then for torsional strength of solid circular shafts,

$$Pa = T = s_s \frac{J}{c} = \frac{s_s \pi d^3}{16} \qquad (D)$$

and for torsional strength of solid square members of side b,

$$T = 0.208 b^3 s_s \qquad (E)$$

For a hollow circular section whose outside and inside diameters are d_1 and d_2 respectively,

$$T = s_s \frac{J}{c} = \frac{s_s \pi (d_1^4 - d_2^4)}{16 d_1} \qquad (F)$$

For deformation under stress for a solid circular section, which is the most common case,

$$\theta = \frac{Tl}{E_s J}$$

where J is the polar moment of inertia. Hence, for solid circular sections,

$$\theta = \frac{32Tl}{\pi E_s d^4} \qquad (G)$$

and for a hollow circular section,

$$\theta = \frac{32Tl}{\pi E_s (d_1^4 - d_2^4)} \qquad (H)$$

Examination of equations (D) and (F) shows that for circular sections torsional strength is proportional to the third power of the outer diameter. Equations (G) and (H) show that torsional deformation is

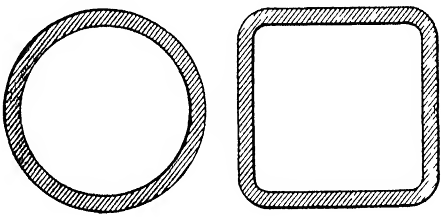


FIG. 7.

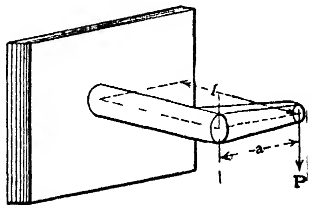


FIG 8.

inversely proportional to the fourth power of the outer diameter, hence torsional stiffness is directly proportional to the fourth power of the outer diameter.

For a given amount of material that section in which this material is distributed farthest from the gravity axis will be strongest and stiffest as long as the walls of the section do not become so thin and weak as to yield locally from other causes. The hollow circular and hollow rectangular sections, commonly called the "box section," Fig. 7, are best adapted, therefore, to resist torsional strains. The box section is peculiarly useful in machine construction, as many machine members must carry a combination of stresses. Machine frames may be subjected to tension, compression, or shearing, combined with torsion, and the box section, while equally good for simple stresses, is, as has been noted, vastly superior in torsion. Furthermore, the box section is well adapted to resist combined flexure and torsion. The flat sides of a box section also afford facilities for attaching auxiliary parts, and its appearance is one of strength and stability. The thickness of the walls being less in hollow than in solid forms insures a better quality of metal in castings and also more skin surface, where the greatest strength of cast iron lies. An advantage not to be overlooked in some lines of work is the ease with which hollow sections can be strengthened by increasing the thickness of the walls by changing the core without changing the external dimensions. The cost of pattern work is about the same, in general, for hollow sections as for I or other sections, while the work in the foundry is, in general, a little greater.

Example. A circular cast-iron boring bar 60 in. long carries a solid circular boring head 60 in. in diameter. The bar is subjected to a torsional moment of 60,000 in.-lb which is applied at one end. It is desired to keep the torsional deflection of the tool below $\frac{1}{32}$ in. when the bar is transmitting power through its entire length, in order to prevent chattering of the tool. What should be the diameter of the bar if the working stress be taken as 3000 lb. per sq in. and E_s be taken as 6,000,000.

For torsional strength from formula (D),

$$d^3 = \frac{60{,}000 \times 16}{3000 \times \pi} = 100$$

$$\therefore \; d = 4.6 \text{ in.}$$

For torsional stiffness,

$$\theta^* = \frac{\frac{1}{32}}{30} = \frac{1}{960}$$

since θ is in radians and the length of an arc $= r\theta$, where r = radius.

$$\therefore \text{ From } G,\; d^4 = \frac{32 \times 60{,}000 \times 60}{\pi \times 6{,}000{,}000 \times \frac{1}{960}} = 5870$$

hence $d = 8.8$ in. It is evident that the shaft will be amply *strong* if designed for stiffness; therefore the last value would be used.

If the section is made hollow less metal can be used. Then either the inside or outside diameter or the ratio between them can be assumed. Let

$$\frac{d_2}{d_1} = \frac{3}{4} \quad \text{or} \quad d_2 = \frac{3d_1}{4}$$

whence,

$$d_2^4 = \frac{81d_1^4}{256} \quad \text{and} \quad d_1^4 - d_2^4 = \frac{175}{256}\,d_1^4$$

* The angular deflection in radians $= \frac{2\pi}{360}$ (angular deflection in degrees); hence, the angular deflection in degrees = 57.296 (angular deflection in radians).

Substituting in H,

$$d_1^4 - d_2^4 = \frac{175}{256} d_1^4 = \frac{32 \times 60{,}000 \times 60}{\pi \times 6{,}000{,}000 \times \frac{1}{960}}$$

$$\therefore \quad d_1^4 = 8550 \quad \text{and} \quad d_1 = 9.6 \text{ in.}$$

hence $d_2 = 7.2$ in.

The area of the hollow shaft = 31.67 sq in.; that of the solid shaft = 60.84 sq in., so that with a small increase in diameter one-half the metal secures, by using the hollow section, the same stiffness.

14. Compound Stresses. In the simple loadings just discussed only one form of stress is brought on the member and the design of the cross-section can be safely based on this stress. When, however, the loads applied induce stresses of several kinds, it is no longer possible in general to base the design on any one stress, but regard must be had to the *combination of stresses* that may occur. In many cases one or more of the stresses are so small, or their action is such, that they may be neglected in designing the member, though they should always be borne in mind. The stress on which the design of the member is based may be called the **predominating or primary stress,** and it may be a simple stress or a combination of simple stresses. The latter will be called a **compound stress.**

15. Flexure. When a beam is subjected to simple bending the principal stresses that are induced are (a) a tension on one side of the neutral axis, (b) a compression on the other side of the neutral axis, and (c) a shearing stress which acts on every section of the beam at right angles to the tension and compression. Generally speaking, the shearing stress is small compared with the tension or compression and can often be neglected. It must never be forgotten, however, and where the beam is designed to withstand the bending moment only, care should be exercised that the sections which are subjected to a small bending moment are not made so small as to yield under shear. The predominating stress in general will be the tension or compression, depending on the material and the form of section.

In determining the stresses in machine members, it is customary to consider the member as stationary at the moment when the particular forces that induce the stresses are acting, whether the member is actually stationary or not. It will be helpful, in the discussions that follow, to remember that for static equilibrium the sum of all the vertical forces must equal zero, the sum of all the horizontal forces must equal zero, and the sum of all the applied moments about any point must also equal zero. These principles are of great importance in determining unknown values of forces and reactions acting upon the member.

The bending moment at any section of a beam is the algebraic sum of all the moments of the external forces acting on either side of the section under consideration. The shear at any section of a beam is the algebraic sum of all external forces acting on either side of the section, and where the shearing force passes through a zero value the bending moment is a maximum or a minimum.

When a beam is subjected to simple flexure:

Let M = bending moment at any section in inch-pounds.
I = moment of inertia of section in biquadratic inches.
c = distance from neutral axis to outermost fiber in inches.
Δ = deflection at any point in inches.
P = load applied in pounds.
s = maximum stress at outer fiber in pounds per square inch.
E = coefficient of elasticity.

Then for strength, in general, within the elastic limit,

$$M = \frac{sI}{c} * \qquad (J)$$

Every beam when loaded deflects somewhat, depending on the shape of its cross-section, the material, the way in which it is supported, and the load applied. The curve assumed by a beam loaded within the elastic limit is called the elastic curve and is of course different for different combinations of the above conditions. The general equation of the elastic curve, whatever the shape of the beam may be, or the manner in which it is loaded and supported, is

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

To find the particular equation for any case, M must be expressed in terms of x and the expression integrated twice. The ordinate y, which is the deflection, can then be found for any value of x, and its greatest value is the maximum deflection. This integration has been performed for all the cases usually met with in practice, and the results are tabulated in Table I. It is to be noted that this tabulation is for beams of *uniform section* and for stresses within the elastic limit. Here,

* The expression I/c is sometimes called the modulus of the section and is generally indicated by the letter Z. It should be noted, however, that this expression is applicable only to symmetrical sections, as c may have two values for other sections.

sI/c is termed the resisting moment.

as in other classes of machine members, the design of the part may be based on strength or stiffness, depending on the conditions, and in general both should be considered.

Example. A steel I-beam 20 ft long and supported at the ends is used as a track for a crane trolley carrying 4000 lb. Select a standard rolled I-beam that will carry the load with a deflection of not more than $\frac{3}{16}$ in. at the center and a maximum stress of not more than 8000 lb.

From Table I,

$$\Delta = \frac{3}{16} = \frac{Pl^3}{48EI} = \frac{4000 \times 240^3}{48 \times 30{,}000{,}000 \times I}$$

whence

$$I = \frac{4000 \times 240^3 \times 16}{48 \times 30{,}000{,}000 \times 3} = 205$$

From handbooks on structural shapes it is found that the moment of inertia of a 12-in. I-beam weighing 31.5 lb per ft is 215.8. Let such a beam be chosen. Then from formula (J), the stress

$$s = \frac{Mc}{I} = \frac{2000 \times 10 \times 12 \times 6}{215.8} = 6700 \text{ lb, nearly}$$

The section therefore is satisfactory.

16. Beams of Uniform Strength. The values in Table I refer to beams of uniform cross-section. In nearly all cases the bending moment, which is usually the basis of design, varies, and if, therefore, the beam is made strong enough at its most strained section and uniform in cross-section throughout its length it will have an excess of material at every other section.* Sometimes it is desirable to have the cross-section uniform; in other cases the metal can be so distributed that every section shall have the necessary strength to resist the bending moment and no more. In the latter cases the shearing stress must be looked after carefully. Table II gives a few of the forms most usually met, and an example may make their application clear.

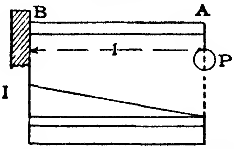
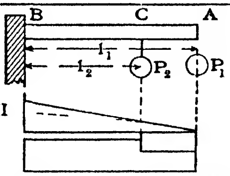
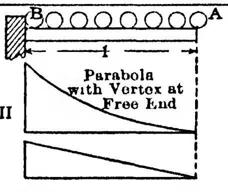
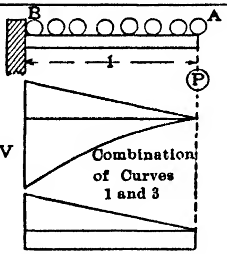
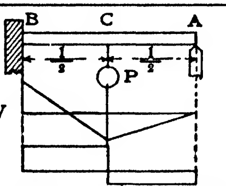
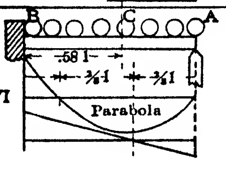
Example. A cantilever of rectangular section 30 in. long carries at its outer end a load of 1000 lb. It is to have a uniform thickness. What is its vertical outline so as to have uniform strength?

Let the thickness $= b$ and the variable height $= y$. Then the moment at any section at a distance x (Fig. I, Table 2) is Px, and this

* This of course does not cover the possible case where the effect of shearing or other stresses may exceed that due to flexure.

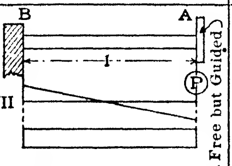
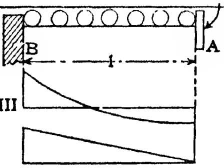
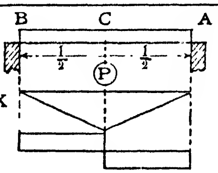
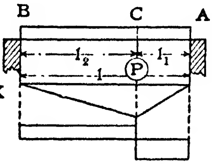
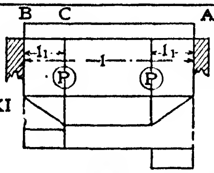
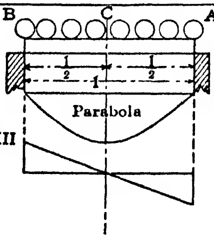
TABLE I

Beams of Uniform Section

Diagram of Loads, Bending Moments and Shear		Greatest Bending Moment M	Location of M	Greatest Deflection $\triangle$	Location of $\triangle$	Maximum Shearing Force	Section where Shear is Max
I 	Cantilever	Pl	B	$\frac{Pl^3}{3EI}$	A	P	Any
II 	Cantilever	$P_1 l_1 + P_2 l_2$	B		A	$P_1 + P_2$	From C to B
III 	Cantilever	$\frac{wl^2}{2}$	B	$\frac{Wl^3}{8EI}$	A	$wl = W$	B
IV 	Cantilever	$\frac{wl^2}{2} + Pl$	B	$\frac{l^3}{EI}\left[\frac{P}{3} + \frac{wl}{8}\right]$	A	$wl + P$	B
V 	Fixed at B Supported at A	$-\frac{5}{32} Pl \rightarrow$ $+\frac{3}{16} Pl \rightarrow$	C B	$\frac{Pl^3}{107EI}$ At	0.451 l from A	$\frac{11}{16}$ P $\frac{5}{16}$ P	B to C C to A
VI 	Fixed at B Supported at A	$\frac{9wl^2}{128} \rightarrow$ $+\frac{wl^2}{8} \rightarrow$	C B	$\frac{wl^3}{184EI}$	C	$\frac{3wl}{8}$ $\frac{5wl}{8}$	A B

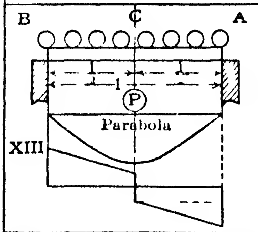
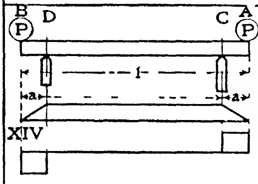
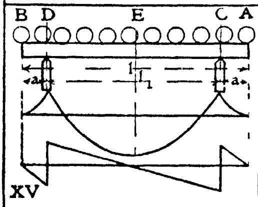
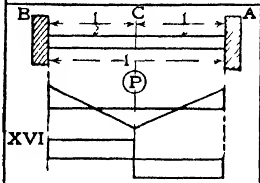
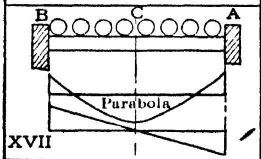
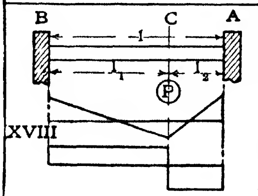
w = load per unit length. W = total distributed load. P = concentrated load.

TABLE I—*Continued*

Diagram of Loads, Bending Moments and Shear		Greatest Bending Moment M	Location of M.	Greatest Deflection Δ	Location of Δ	Maximum Shearing Force	Section where Shear is Max.
VII 	One End Fixed Other Free but Guided	$\frac{Pl}{2}$	A and B	$\frac{Pl^3}{12EI}$	A	P	Equal in All
VIII 	One End Fixed Other Free but Guided.	$\frac{wl^2}{3}$ $-\frac{wl^2}{6}$ 0	At B At A At 0.423l from B	$\frac{Wl^3}{24EI}$	A	W	B
IX 	Supported at both Ends Load Central.	$\frac{Pl}{4}$	C	$\frac{Pl^3}{48EI}$	C	$\frac{P}{2}$ $-\frac{P}{2}$	B to C C to A
X 	Supported at both Ends, Load not Central.	$\frac{Pl_1l_2}{l}$	C	$\frac{Pl_1^2l_2^2}{3lEI}$ Not Max.	At C	$\frac{Pl_1}{l}$ $-\frac{Pl_2}{l}$	B to C C to A
XI 	Supported at both Ends, Two Symmetrical Loads.	Pl_1	C to D	$\frac{Pl_1}{24EI}\times\left[3l^2-4l_1^2\right]$	Centre	P	B to C D to A
XII 	Supported at both Ends, Uniform Load.	$\frac{wl^2}{8}$	Centre	$\frac{5Wl^3}{384EI}$	Centre	$\frac{wl}{2}$	A or B

w = load per unit length. W = total distributed load. P = concentrated load.

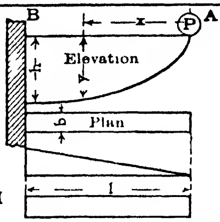
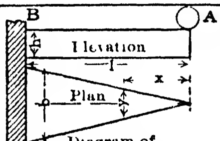
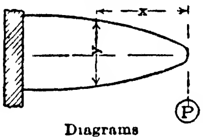
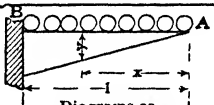
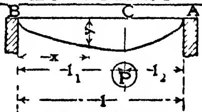
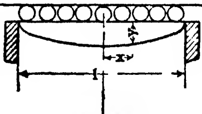
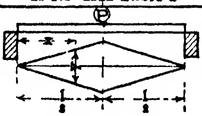
TABLE I—*Continued*

Diagram of Loads Bending Moments and Shear		Greatest Bending Moment M	Location of M	Greatest Deflection Δ	Location of Δ	Maximum Shearing Force	Section where Shear is Max
XIII 	Supported at both Ends	$[P+\frac{wl}{2}]\frac{l}{4}$	C	$(P+\frac{5}{8}W)\times\frac{l^3}{48EI}$	C	$\frac{P}{2}+\frac{W}{2}$	A or B
XIV 	Two Symmetrical Loads Overhanging Supports	Pa	D to C			P	in a
XV 	Uniform Load Symmetrical Supports	$\frac{W}{l}\left[\frac{l_1^2}{8}-\frac{a^2}{2}\right]$ $+\frac{Wa^2}{2l}$	→ E C and D			$\frac{W}{2}-wa$	C and D
XVI 	Fixed at both ends One Load at Centre	$\frac{Pl}{8}$	at A B. and C.	$\frac{Pl^3}{192EI}$	C	$\frac{P}{2}$ $-\frac{P}{2}$	B to C C to A
XVII 	Fixed at both ends Uniform Load	$\frac{wl^2}{12}$ $\frac{wl^2}{24}$	A or B C	$\frac{wl^4}{384EI}$	C	$\frac{wl}{2}$	A or B
XVIII 	Fixed at both ends Uniform Load	$\frac{Pl_1l_2^2}{l^2}$ $\frac{2Pl_1^2l_2^2}{l^3}$ $\frac{Pl_2l_1^2}{l^2}$	at B at C at A	$\frac{2Pl_1^2l_2^2}{3EI(l+2l_1^2)}$			

w = load per unit length. *W* = total distributed load. *P* = concentrated load.

TABLE II

Beams of Uniform Strength

	Outline of Beam	Greatest Bending Moment M	Location of M	Greatest Deflection Δ	Location of Δ	Maximum Shearing Stress	Section where Shear is Max
I 	Parabola with Vertex at P Rectangular Section Equation $y^2 = \frac{6Px}{bs}$	Pl	B	$\frac{8Pl^3}{Ebh^3}$	A	P	any
II  Diagram of Moments and Shear same as I	Rectangular Section, Equation of Triangular Plan $y = \frac{6P}{h^2 s}x$	Pl	B	$\frac{6Pl^3}{Ebh^3}$	A	P	any
III  Diagrams as in I	Circular Section of variable Diameter $y^3 = \frac{32P}{\pi s}x$	Pl				P	any
IV  Diagrams as in No III Table I	Breadth b Uniform $y^2 = \frac{3w}{sb}x^2$	$\frac{wl^2}{2}$	B			W	B
V  Diagrams same as No X Table I	Breadth, Uniform Two Parabolas $y^2 = \frac{6Pl_1}{bsl}x$ $y^2 = \frac{6Pl_2}{bsl}x$	$\frac{Pl_1l_2}{l}$	C	when $l_1 = l_2$ $\Delta = \frac{Pl^3}{2Ebh^3}$	C		
VI  Diagrams same as No XII Table I	Breadth, Uniform Elliptical Outline $y^2 = \frac{3w}{4sb}\left[l^2 - 4x^2\right]$	$\frac{wl^2}{8}$	Centre				
VII  Diagrams same as No IX Table I	Depth, Uniform $y = \frac{6P}{2h^2 s}x$	$\frac{Pl}{4}$	Centre	$\frac{3Pl^3}{8Ebh^3}$			

w = load per unit length. W = total distributed load P = concentrated load.

must be equal to the resisting moment of the section at each point; hence

$$Px = \frac{sI}{c} = \frac{sby^2}{6} \quad \text{or} \quad y^2 = \frac{6Px}{sb}$$

which is the equation of a parabola whose vertex is at the outer end of the beam. In the problem assumed, let $b = 1.5$ in. and let $s = 4000$ lb. Then when $x = 30$, $y = h = 5.5$ in. In a similar way other points may be found or the curve may be laid out by graphical method. The shearing load at any point is P, and hence the shearing stress increases as the cross-section of the beam decreases. When $x = 0$, $y = 0$, and in general when x is small, y is very small; therefore the outer end of the member must be modified so as to carry the shearing stress safely. Reference will be made to this again under the section dealing with machine attachments (see Chapter XIX). It is to be especially noted that these theoretical shapes are based on certain assumptions and unless these are observed in the design, the theoretical outlines do not apply. Thus in the cantilever example above, if the thickness of the beam is not kept uniform the outline for uniform strength is *not* a parabola. *The mistake of using a parabola when the thickness is not uniform is often made* when I or T sections are used instead of uniform thickness or depth. It is evident that, whatever may be the form of section adopted, by means of the bending moment and shearing load the correct depth of section can be found for a number of points and a curve plotted that will answer the requirements of uniform strength.

17. Combined Flexure and Torsion. Let the force P, Fig. 8, act upon a rod with an arm a at a distance from the support equal to l. Then the stresses induced in the section close to the support are

(a) Tension and compression stresses due to the bending moment Pl.
(b) Shear stress due to the twisting moment Pa.
(c) Shear stress due to bending.

The shear stress due to bending is zero where the tension and compression stresses due to bending are a maximum and can be neglected; the predominating stress therefore is that due to the combined action of the bending and twisting moments.

It can be shown that if a bar or rod is subjected to a longitudinal tensile or compressive stress and at the same time to a shearing stress at right angles to its length, the combination of these stresses may produce similar stresses greater than either and acting along planes other than those along which the original stresses act.*

* See any standard treatise on mechanics.

If s_t be the tensile or compressive stress and s_s the shearing stress applied to the bar at right angles to s_t, then the maximum tensile or compressive stress due to s_t and s_s is given by the following equation:

$$\text{max. } s'_t = \text{max } s'_c = \tfrac{1}{2}[s_t + \sqrt{s_t^2 + 4s_s^2}] \qquad (1)$$

and the maximum shearing stress s'_s due to s_t and s_s is

$$\text{max. } s'_s = \tfrac{1}{2}\sqrt{s_t^2 + 4s_s^2} \qquad (2)$$

It is evident that the numerical value of max. s'_t will always exceed that of max. s'_s, and therefore if the material used has approximately the same tensile and shearing strength the design can be safely based on (1). But should the allowable shearing strength of the material be less than the allowable tensile strength, as is usual, it may happen that the shearing stress as found by (2) would dictate a larger section than that dictated by formula (1).

If the tensile stress is due to a bending moment and the shearing stress is due to a twisting moment the values of s_t and s_s can be found from equations (J) and (D) respectively and max. s'_t and max. s'_s obtained as above in equations (1) and (2) respectively.

Example. A certain section of a circular steel shaft is subjected to a bending moment M of 10,000 in-lb and a twisting moment T of 60,000 in-lb. The allowable tensile stress is 10,000 lb per sq in. and the allowable shearing stress is 8000 lb per sq in. It is required to design the cross-section of the shaft.

From (J),

$$s_t = \frac{Mc}{I} = \frac{32M}{\pi d^3} = \frac{32 \times 10{,}000}{\pi \times d^3} = \frac{101{,}910}{d^3}$$

and from (D),

$$s_s = \frac{Tc}{J} = \frac{16T}{\pi d^3} = \frac{16 \times 60{,}000}{\pi d^3} = \frac{305{,}730}{d^3}$$

hence from (1)

$$\text{max. } s'_t = \frac{360{,}915}{d^3}$$

and since the allowable tensile stress = 10,000

$$\therefore\ d^3 = \frac{360{,}915}{10{,}000}$$

or

$$d = 3.3 \text{ in.}$$

From (2),

$$\text{max. } s'_s = \frac{309{,}960}{d^3}$$

and since the allowable shearing stress = 8000,

$$d^3 = \frac{309{,}960}{8000}$$

$$\therefore\ d = 3.4 \text{ in.}$$

or $\frac{1}{10}$ in. greater than that given by (1). It is evident that the last value should be taken.

Equations (1) and (2) are general and applicable to any and all sections, but for circular shafts operating under conditions that produce both bending and twisting it has been found convenient to make use of what may be called an **equivalent or ideal bending moment** which may be derived from equation (1) as follows:

Let M_e = the equivalent bending moment which will produce the same maximum tensile or compressive stress as will be produced by the combined action of M and T.

M = the bending moment producing the tensile or compressive stress s_t.

T = the twisting moment producing the shearing stress s_s.

r = radius of shaft.

From (J),

$$M = \frac{s_t I}{r} \quad \text{and} \quad M_e = \frac{s'_t I}{r}$$

and from (D),

$$T = \frac{s_s J}{r} = \frac{2 s_s I}{r}$$

(Since $J = 2I$ for circular or other sections for which the moments of inertia about two perpendicular axes are equal.)

Multiply equation (1) through by I/r, whence

$$\frac{s'_t I}{r} = \frac{1}{2}\left[\frac{s_t I}{r} + \sqrt{\frac{s_t I^2}{r^2} + \frac{4 s_s{}^2 I^2}{r^2}}\right] = M_e$$

$$\therefore\ \frac{s'_t I}{r} = M_e = \tfrac{1}{2}M + \tfrac{1}{2}\sqrt{M^2 + T^2} \qquad (K)$$

In a similar manner an *equivalent twisting* moment can be deduced from (2), thus

$$\frac{2 s'_s I}{r} = T_e = \sqrt{M^2 + T^2} \qquad (K_1)$$

The quantities M and T are usually large, and the numerical work involved in solving (K) and (K_1) can be simplified by transforming them into the equivalent forms:

$$M_e = \frac{M}{2}\left[1 + \sqrt{1 + \left(\frac{T}{M}\right)^2}\right] \qquad (K_2)$$

$$T_e = M\sqrt{1 + \left(\frac{T}{M}\right)^2} \qquad (K_3)$$

It is to be especially noted that M_e and T_e are equivalent moments in a numerical sense only; that is, if a bending moment M and a twisting moment T are applied to a shaft, producing a tensile stress s_t and a shearing stress s_s respectively, then M_e is a bending moment which will give a stress equal to the maximum resultant tensile or compressive stress max. s'_t, and T_e is a twisting moment which will give a stress equal to the maximum resultant shearing stress max. s'_s, reference being made to the same section.

The application of these equations to the investigation of any existing shaft subjected to a bending moment M and a twisting moment T is obvious, and it remains to consider their application to the design of new shafts. It has been noted that the greater numerical value given by equation (1) does not necessarily indicate that a larger section will result from its adoption than from the use of equation (2). For the same reasons the greater numerical value of M_e, obtained from (K), may not give a larger section than would be obtained from T_e by applying (K_1). It is necessary, therefore, to determine under what conditions each should be used for designing in order that the *maximum* diameter of shaft shall be found in all cases.

From (J),

$$M_e = \frac{s'_t I}{r} = \frac{s'_t \pi d^3}{32}$$

whence

$$d^3 = \frac{32 M_e}{\pi s'_t} = \frac{16}{\pi} \times \frac{2M_e}{s'_t} \qquad (3)$$

In a similar way from (E),

$$d^3 = \frac{16}{\pi} \times \frac{T_e}{s'_s} \qquad (4)$$

Since in any given problem M and T are always known, M_e and T_e can always be found from (K) and (K_1) or (K_2) and (K_3), and since the allowable values of s'_t and s'_s can always be assigned, the diameter of the shaft d can always be determined from both equations (3) and (4) and the larger value selected as in the problem previously solved. It is desirable, however, to know, for any set of conditions, whether equation (3) or equation (4) will give the greater value of d without the necessity of solving both equations.

It is evident that in order that equations (3) and (4) may give the *same* diameter of shaft $2M_e/s'_t$ must equal T_e/s'_s or

$$\frac{2M_e}{T_e} = \frac{s'_t}{s'_s}$$

and that for conditions other than these, either equation (3) or equation (4) may give the greater diameter. It is therefore necessary to investigate the relations existing between $2M_e/T_e$ and s'_t/s'_s for three sets of conditions:

(1) When equations (3) and (4) will give equal values of d.
(2) When equation (3) will give the greatest value of d.
(3) When equation (4) will give the greatest value of d.

(1) It has already been shown that equations (3) and (4) will give equal values of d when

$$\frac{s'_t}{s'_s} = \frac{2M_e}{T_e}$$

or if s'_t/s'_s be called y, then

$$y = \frac{2M_e}{T_e} = \frac{M + \sqrt{M^2 + T^2}}{\sqrt{M^2 + T^2}} \tag{5}$$

is the equation of a curve which expresses all the simultaneous values of s'_t/s'_s and $2M_e/T_e$ for which equations (3) and (4) will give equal values of d. The values of either M or T in equation (5) may vary from zero to infinity, and the most convenient manner of plotting *simultaneous* values of M and T is to plot their *ratio.* If, also for simplicity, $x = T/M$, equation (5) becomes

$$y = \frac{2M_e}{T_e} = \frac{1 + \sqrt{1 + x^2}}{\sqrt{1 + x^2}} \tag{6}$$

which is the equation of a curve expressing all the simultaneous values of y (or s'_t/s'_s) and x (or T/M), for which equations (K_2) and (K_3) will give equal diameters of shaft.

It is desirable, before plotting, the curve to examine the limits between which x and y may vary. It is clear that for $M = 0$, $x = \infty$.

and for $T = 0$, $x = 0$; hence the limits of x are 0 and ∞. Using these same limits for M and T in equation (5), it is found that

when $$M = 0,\ y = 1 \quad \text{and} \quad M_e = \frac{T_e}{2}$$

and when $$T = 0,\ y = 2 \quad \text{and} \quad M_e = T_e$$

That is, for all materials where the ratio of allowable tensile stress to shearing stress lies between 1 and 2, there are always simultaneous values of M and T for which equations (3) and (4) will give equal values

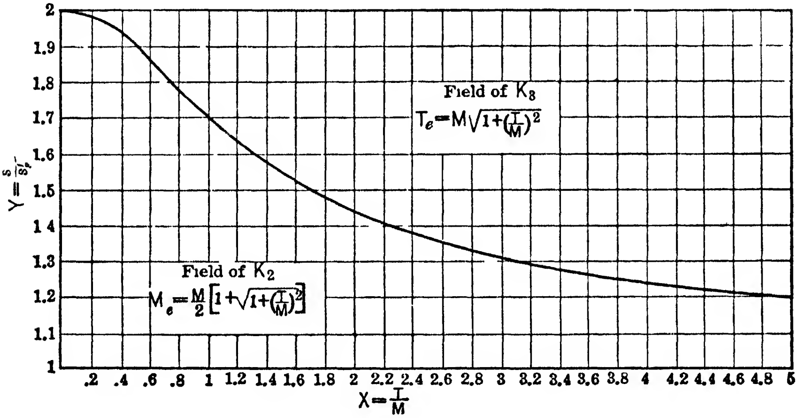


Fig. 9.

of d. The curve showing these simultaneous values is shown in Fig. 9 and has been plotted from equation (6).

(2) If for any given value of s'_t/s'_s within the limits 1 and 2, a ratio T/M be taken less than the simultaneous value given by the curve (or, in other words, if the coordinates chosen intersect below the curve), equation (3) will give the largest value of d, for values of T/M can be decreased only by making M greater relatively to T, and an examination of (K) and (K_1) shows that increasing M increases (K) more rapidly than it does (K_1). Hence, in such cases, (K) or (K_2) applies, and equation (3), which is based upon them, will give the largest value.

Furthermore, for values of s'_t/s'_s equal to or *less* than unity, equation (3) will also give the largest value of d, for it has just been shown that M_e can never be less than $T_e/2$ and equals this value only when $M = 0$. *For all finite values of M, therefore, M_e must be greater than $T_e/2$*; and it is evident from equations (3) and (4) that for values of

$s'_t = s'_s$ and $M_e > T_e/2$ equation (3), which is based upon (K) or (K_2) will give the greater value of d.

(3) In a similar manner it can be shown that for all simultaneous values of s'_t/s'_s and T/M which intersect above the curve and within the limits $y = 1$ and $y = 2$, or for all materials where s'_t/s'_s, is greater than 2, equation (4), which is based upon K_1 (or K_3) will give the greatest value of d.

Summary.—Equations (K_2) and (K_3) are the most convenient forms of equivalent moments and will be used in this work. It is to be particularly noted that they are applicable only to circular sections. Equation (K_2) should be used where the simultaneous values of s'_t/s'_s and T/M intersect below the curve, as they always do whenever $s'_t/s'_s < 1$. Equation (K_3) should be used where the simultaneous values of s'_t/s'_s and T/M intersect above the curve, as they always do whenever $s'_t/s'_s > 2$.

Example 1. An engine cylinder is 16 in. by 24 in. (piston 16 in. in diameter and stroke of 24 in.), steam pressure = 100 lb per sq in. The center of the crankpin overhangs the center of the main journal by 15 in. (measured parallel to the axis of shaft). Assume that the pressure on the crankpin may be equal to 100 lb unbalanced pressure per square inch of the piston when the connecting-rod is perpendicular to the crank radius. Allowing 8000 lb as the maximum allowable normal stress and 6400 as the maximum allowable shearing stress, compute the diameter of the shaft.

Area of piston = 200 sq in.; radius of crank (arm of maximum twisting moment) r = 12 in.; arm of bending moment a = 15 in.

$$\therefore\ T = 200 \times 100 \times 12 = 240{,}000 \text{ in-lb}$$

Also

$$M = 200 \times 100 \times 15 = 300{,}000$$

$$x = \frac{T}{M} = \frac{12}{15} = 0.8 \quad \text{and} \quad y = \frac{s'_t}{s'_s} = \frac{8000}{6400} = 1.25$$

By referring to Fig. 9, it is seen that for $y = 1.25$ and $x = 0.8$ the ordinates intersect below the curve; hence (K_2) should be used.

From (K_2),

$$M_e = \frac{300{,}000}{2}\left[1 + \sqrt{1 + (0.8)^2}\right] = 342{,}000 \text{ in-lb}$$

From (3),

$$d^3 = \frac{32 \times 342{,}000}{\pi \times 8000} = 436$$

$$\therefore\ d = 7.58 \text{ in.}$$

Example 2. A circular steel shaft is subjected to a twisting moment of 250,000 in-lb and a bending moment of 62,500 in-lb. The allowable tensile stress is 8000 lb per sq in. and the allowable shearing stress 5600 lb. Determine the diameter of the shaft.

Here

$$y = \frac{s'_t}{s'_s} = \frac{8000}{5600} = 1.43 \quad \text{and} \quad x = \frac{250{,}000}{62{,}500} = 4$$

From the curve, Fig. 9, it is seen that for $y = s'_t/s'_s = 1.43$ and $x = 4$, the intersection of the ordinates falls above the curve; hence (K_3) should be used. Then

$$T_e = 62{,}500[\sqrt{1 + 4^2}] = 257{,}500$$

$$d^3 = \frac{16 \times 257{,}500}{\pi \times 5600} = 234.25$$

$$\therefore\ d = 6.16 \text{ in.}$$

Suppose, however, that (K_2) should be used. Then

$$M_e = \frac{62{,}500}{2}[1 + \sqrt{1 + 4^2}] = 160{,}000$$

$$d^3 = \frac{32 \times 160{,}000}{\pi \times 8000} = 203.75$$

$$\therefore\ d = 5.87$$

or 0.29 in. less than the value given by (K_2).

18. Other Formulas. Equation (K) is sometimes transformed into an *equivalent twisting moment.* Since in general

$$M = \frac{s_t I}{r} \quad \text{and} \quad T = \frac{2s_s I}{r}$$

for an equal intensity of stress (that is, $s_t = s_s$) $T = 2M$ for the same section. If therefore, it is considered more convenient to use an equivalent twisting moment instead of an equivalent bending moment it is allowable to substitute for M_e (the bending moment, equivalent to the combined bending and twisting moment), $\frac{1}{2}T_e$ (a *twisting moment* equivalent to the combined bending and twisting moments), provided that the same allowable *direct* stress is used with T_e in solving for the diameter of shaft.

$$\therefore\ T_e = 2M_e = M + \sqrt{M^2 + T^2} \qquad (K_4)$$

Equations (K_2), (K_3), and (K_4) are all different forms of Rankine's formula for combined bending and twisting. Other authorities give slightly different coefficients. Thus Grashof gives

$$M_e = \tfrac{3}{8}M + \tfrac{5}{8}\sqrt{M^2 + T^2} \tag{7}$$

and others give

$$M_e = 0.35M + 0.65\sqrt{M^2 + T^2} \tag{8}$$

The diameter of shaft given by equations (7) and (8) will not differ much from that given by (K_2), for any set of conditions, except where the bending moment is very small. At the limit where the bending moment M is equal to zero, Grashof's formula gives a value of M_e, 25 per cent greater than that given by (K_2). But it may be noted that in general for all materials whose shearing strength is less than their tensile strength (and this is true of most materials used in engineering) that when M is small or, in other words, when the shearing stress predominates, it is safer to use (K_3) in preference to (K_2). It will be found that, for the range where equations (7) and (8) give values greater than (K_2), these values will still be less than those obtained from (K_3) or at least not enough greater to warrant the use of a different formula in place of (K_3). Take for example steel, where

$$\frac{s_t}{s_s} = 1.25 \quad \text{and} \quad x = 10$$

which is down close to the limit where Grashof's formula gives the greatest value compared to (K_2). Expressing d^3 in terms of (M) as in equations (3) and (4),

From (K_2),

$$d = 3.83\sqrt[3]{\frac{M}{s}}$$

From (K_3),

$$d = 4.00\sqrt[3]{\frac{M}{s}}$$

From Grashof's formula,

$$d = 4.07\sqrt[3]{\frac{M}{s}}$$

from which it is seen that the difference between d as determined by (K_3) and by Grashof's formula is negligible. The same evidently applies to equation (8), which differs but little from Grashof's. As the value of x decreases, the difference between these equivalent *bending* moments decreases, and any variation is more than covered by the factor of safety which must be used.

19. Combined Torsion and Compression. Propeller shafts of steamers and vertical shafts carrying considerable weight are subjected to combined **twist** and **thrust.** The span, or distance between bearings, is frequently so small that the shaft may be considered as subjected to simple compression, so far as the action of the thrust is concerned.

The intensity of this compressive stress in such cases is

$$s_c = \frac{4P}{\pi d^2}$$

in which P = the thrust, and d = the diameter of the (solid circular) shaft.

If T = the twisting moment on the shaft, $r = \frac{1}{2}d$ = the radius of the shaft, J = the polar moment of inertia = $2I$ (=2 times the rectangular moment of inertia), and s_s = the intensity of shearing stress due to T, then

$$T = \frac{s_s J}{r} = \frac{4 s_s I}{d}. \quad \therefore \; s_s = \frac{Td}{4I} = \frac{16T}{\pi d^3}$$

for solid circular shafts.

The resultant maximum stresses are those due to the combined actions of a normal stress (compression) and a tangential stress (shear) as in the case of combined bending and twisting (Article 17); hence equations (1) and (2) of the preceding article apply and may be used to find the maximum compressive or maximum shearing stress; or if s_c be the compressive stress due to P, s_s be the shearing stress due to T, s'_c the **maximum resultant compressive stress,** and s'_s the **maximum resultant shearing stress,** then

$$s'_c = \tfrac{1}{2}s_c + \tfrac{1}{2}\sqrt{s_c^2 + 4s_s^2} \qquad \text{and} \quad s'_s = \tfrac{1}{2}\sqrt{s_c^2 + 4s_s^2}$$

or

$$s'_c = \frac{2P}{\pi d^2} + \frac{1}{2}\sqrt{\frac{16P^2}{\pi^2 d^4} + \frac{4(16)^2T^2}{\pi^2 d^6}} \quad \text{and} \quad s'_s = \frac{1}{2}\sqrt{\frac{16P^2}{\pi^2 d^4} + \frac{4(16)^2T^2}{\pi^2 d^6}}$$

or

$$s'_c = \frac{2P}{\pi d^2}\left[1 + \sqrt{1 + \left(\frac{8T}{dP}\right)^2}\,\right] \qquad (L)$$

and

$$s'_s = \frac{2P}{\pi d^2}\sqrt{1 + \left(\frac{8T}{dP}\right)^2} \qquad (L_1)$$

It is difficult to find the value of d for a given value of s'_c or s'_s from the above equations, and it is much more convenient to assume a trial diameter d and then check for the values of s'_c and s'_s to see that they do not exceed the allowable compressive and shearing stresses of the material under consideration.

If the span of the shaft between bearings is so great that the shaft must be considered as a column likely to buckle, the trial diameter of the shaft may be taken so as to bring the mean compressive stress s_c well below the allowable value, and after solving for s'_c and s'_s the shaft may also be checked as a long column (Article 23). In steel shafting it is necessary usually to apply equation (L) only, but it is well to check the shearing stress s'_s against the allowable stress by applying (L_1).

20. Flexure Combined with Direct Stress. If a short straight prism be acted on by a force P at a distance from its gravity axis O equal to a, the stresses induced in the section will be:

(a) A **uniformly distributed stress** due to the load P and equal to P/A per unit area. This will be tensile or compressive, depending on the direction of P.

(b) A **flexural stress** due to the bending moment Pa. This flexural stress will be a tensile stress on one side of the gravity axis which is at right angles to a, and compressive on the other.

The vertical portion of the clamp in Fig. 10 (a) may be considered to be such a prism. Let XY be any section of this prism. If the direct stress induced in the section by the load P is tensile, then the flexural stress on the side toward the load is tensile. If the direct stress induced is compressive, the flexural stress on the side toward the load is compressive. The maximum stress will be the greatest algebraic sum of these combined stresses at the outer fibers at X or Y. The distribution of these stresses for both cases is shown graphically in Fig. 10 (b), where tensile stresses are plotted above the line UV and the compressive stress below, the ordinates under rs representing the flexural stresses and those under mn the direct stresses. An inspection will show where the algebraic sum is greatest. In the case shown the *combined* compressive or *combined* tensile stresses at X are the greatest which may come on the section, depending on the direction of P. This is not necessarily so, as a brief reflection will show that if O be located near enough to X the reverse of the above conditions may exist. The form of section and location of the gravity axis should be fixed with reference to the relative tensile and compressive strength of the material used.

Let s_1 = the direct stress due to P.
s_2 = the tensile or compressive stress due to Pa.
s' = the maximum stress in the section at X or Y.

Then from formula (A),

$$s_1 = \frac{P}{A}$$

and from formula (J),

$$s_2 = \frac{Pac}{I}$$

Therefore,

$$s' = s_1 + s_2 = \frac{P}{A} + \frac{Pac}{I} \, *$$

where c is the distance from O to the outer fiber at either X or Y, depending on which is under consideration.

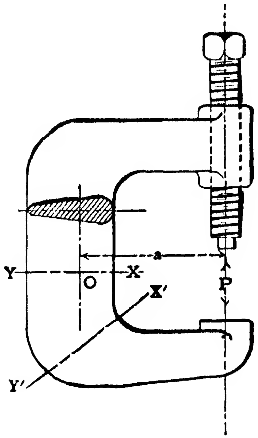


Fig. 10 (a).

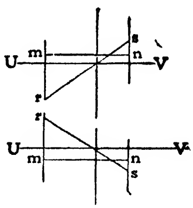


Fig. 10 (b).

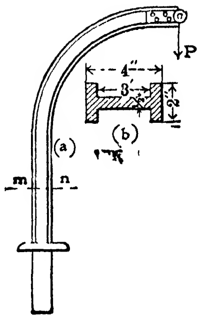


Fig. 11.

It is obvious that tensile or compression stresses, other than these caused by the force which creates the bending moment, may be induced in such a section. The most general form of this equation is, therefore,

$$s' = s_1 + s_2 = \frac{P}{A} + \frac{Mc}{I} \qquad (M)$$

where P is the sum of all tensile and compressive forces acting upon the section, and M is the sum of the bending moments applied. It is assumed that all bending moments act in the same plane with reference to the section.

If the material used is equally strong in tension and compression the gravity axis should not be far from central, but where cast iron is used

* See any work on mechanics.

it is advantageous to distribute the metal more toward the tension side, thus drawing the gravity axis toward that side. This increases c on the compression side, and hence increases the compressive stress. It decreases c on the tension side, and hence decreases the tensile stress. Cast iron is much stronger in compression than in tension, and therefore a greater moment can be withstood by a given cross-sectional area when distributed in this manner.

It is not practicable, in general, to solve equation (M) for the direct determination of the dimensions of a cross-section to sustain a given eccentric load P with an assigned intensity of stress s', because A, I, and c are all functions of the required dimensions; and with any but the simplest sections complicated functions result. With solid, square, or circular sections, or in general where only one dimension is unknown, it is possible to reduce (M) to a form which can be solved; but the algebraic expression is a troublesome cubic equation. The practical way is to assume a trial section, and check this for P or s'.

Example. A small crane (Fig. 11) has a clear swing of 28 in. The lower part of the crane is a straight prism. The radius of curvature of the upper part is large compared with the depth of the cross-section, and the moment arms on the sections of the curved part are less than that applied to those of the lower portion. Equation (M) can be applied, therefore, with safety. The section at mn is shown by Fig. 11 (b). Find the load corresponding to a maximum fiber stress (compression) of 9000 lb per sq in. at n.

$$s' = \frac{P}{A} + \frac{Pac}{I} \qquad \therefore \; P = \frac{s'AI}{I + Aac}$$

$$a = 28 + 2 = 30, \quad A = 2 \times 4 - 1.5 \times 3 = 3.5$$

$$I = \tfrac{1}{12}(2 \times 64 - 1.5 \times 27) = 7.3$$

$$\therefore \; P = \frac{9000 \times 3.5 \times 7.3}{7.3 + 3.5 \times 30 \times 2} = 1060 \text{ lb}$$

For many years equation (M) was the only one in use for all cases of combined bending and direct stress. It is applicable with accuracy, however, only to straight, short prisms and does not give accurate results for a curved member. It does not apply, for instance, to such a member as is illustrated in Fig. 12, or to such sections as $X'Y'$ in Fig. 10. Even now, when the theory of curved beams is better understood, this equation continues to be much used, partly because of its simplicity and partly because, through long-continued use, such low allowable stresses have been established in connection with it as to give safe results. The use of this equation for curved members, however, is always attended with more or less uncertainty, as in general, it indicates much lower stresses than those given by the modern theory which is to be described. This is particularly true of curved members whose outlines have not been fixed, to some extent, by experience and usage.

21. Flexure and Direct Stress in Curved Beams. Figure 12 shows a clamp similar to that shown in Fig. 10, but of such a curved form as to make the application of equation (M) of doubtful accuracy. It is acted upon by a load, P at a distance a from the gravity axis, O. Let R be the radius of curvature passing through O and having a center at C;

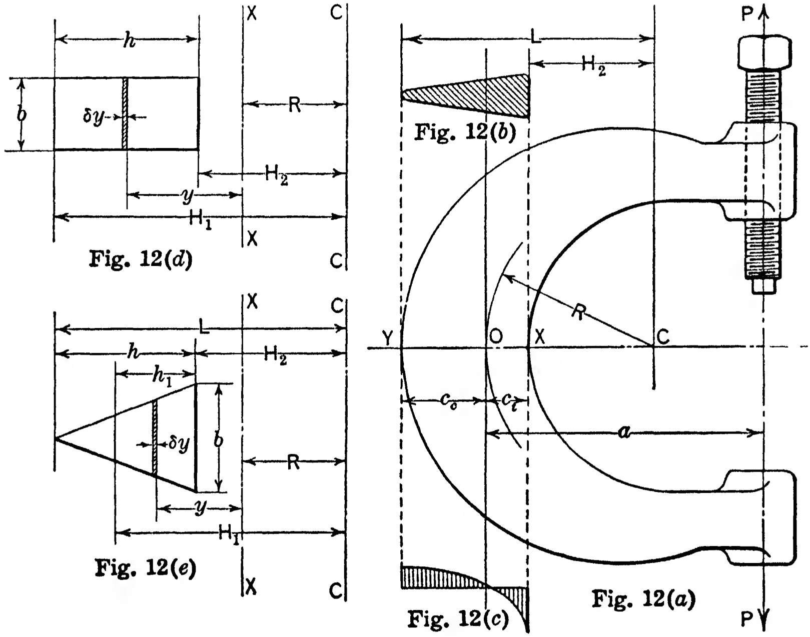


FIG. 12.

it is desired to find the maximum stresses in the section XY. As in a straight prism with an eccentric load, the unit normal stress at any point of the cross-section is considered to be the sum of a unit direct stress and a unit bending stress. The unit bending stress *does not vary uniformly*, as in a straight prism, but is distributed somewhat as shown in Fig. 12 (c). The maximum tension s'_t in the fibers at x is

$$s'_t = \frac{P}{A}\left[1 + \frac{a}{R - c_t}\left(\frac{c_t}{R} \times \frac{A'}{A' - A} - 1\right)\right]^* \qquad (M_1)$$

* See "Strength of Materials," by Morley, page 378.

where a is the moment arm of the force P, c_t the distance from the gravity axis to the outer fiber at X and A' a derived area whose value is

$$A' = R\int \frac{\delta A}{R - y} \text{*}$$

This area may be computed analytically for regular geometric figures and can be derived graphically for any section. Thus, for a rectangle whose dimension parallel to the neutral axis is b, and whose length at right angles to that axis is h,

$$A' = R\int_{\frac{h}{2}}^{-\frac{h}{2}} \frac{b\,\delta y}{R - y} = Rb \log_e\left[\frac{2R + h}{2R - h}\right]$$

the integration being performed with reference to the gravity axis and y being the distance of any infinitesimal area from this axis.

In punching and shearing machines and similar constructions, the main frame is often a casting of curved outline and of a complex cross-section. Usually these cross-sections can be subdivided into rectangles and triangles whose gravity axes do not, in general, coincide with the gravity axis of the entire section. In such cases it is very convenient to have the value of A' for each subdivision expressed in terms of its dimensions, and the distance of the cross-section from the center of curvature. Thus, in Fig. 12 (d), let the rectangular figure be considered as a portion of a larger cross-section whose gravity axis is XX, the center of curvature of this axis being in the line CC. Then

$$A' = Rb \log_e\left[\frac{H_1}{H_2}\right]$$

and, in a similar manner, for a triangle or trapezoid, Fig. 12 (e),

$$A' = R\frac{b}{h}\left[L \log_e\left(\frac{H_1}{H_2}\right) - h_1\right]$$

Example. In Fig. 12 (b) the cross-section may be considered as a triangle. Let $b = 2$ in., $h = 6$ in., $R = 8$ in., and $a = 13$ in. Whence $H_2 = 6$ in., $L = 12$ in., and $c_t = 2$ in. If $P = 2000$ lb, what is the maximum tensile stress on the inner fiber at X?

Here

$$A' = 8 \times \tfrac{2}{6}[12 \log_e (\tfrac{12}{6}) - 6] = 6.18$$

Whence by (M_1),

$$s'_t = \frac{2000}{6}\left[1 + \frac{13}{8 - 2}\left(\frac{2}{8} \times \frac{6.18}{6.18 - 6} - 1\right)\right] = 5800 \text{ lb}$$

* y is positive if measured from the gravity axis toward and negative if measured away from the center of curvature.

By equation (M),

$$s'_t = \frac{2000}{6} + \frac{2000 \times 13 \times 2}{12} = 4666 \text{ lb}$$

The maximum compressive stress at Y may be found by replacing c_t in equation (M_1) with c_c, using a minus value for the latter, and proceeding as in the foregoing example.

22. Graphic Derivation of A′. Let the solid outline of Fig. 12 (f) * represent the cross-section of a curved member for which it is desired to find the value of A'. Let C be the center of curvature of the member and let OO' be the gravity axis of the section. At any distance y

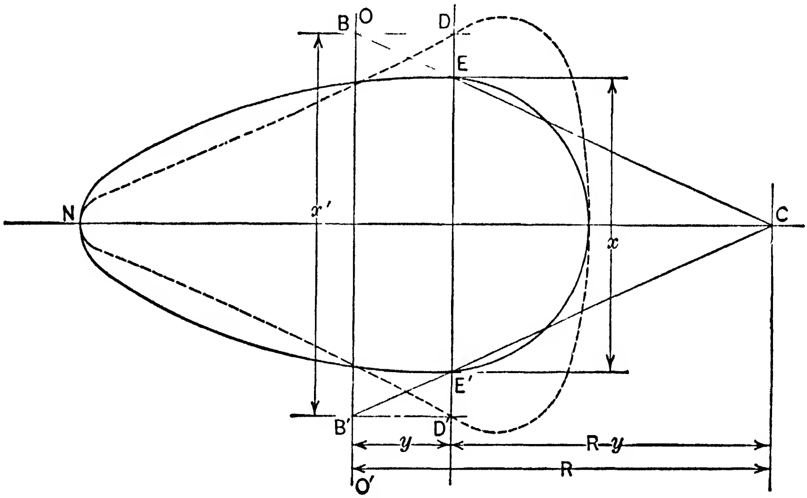


FIG. 12 (f).

from the gravity axis, draw a line parallel to it, intersecting the outline of the cross-section at E and E'. From C draw straight lines through E and E' cutting O, O' in B and B'. From B and B' draw lines parallel to NC intersecting EE' at D and D'. Taking other distances on both sides of OO' repeat this construction until enough points, such as D and D', are obtained to outline the figure shown by the broken line. The area inclosed by this broken line represents

$$R \int \frac{\delta A}{R - y} = A'$$

as can be seen by the following:

* See also "Strength of Materials," by Morley.

Let x = the distance between E and E'.
x' = the distance between D and D'.

Then, from similar triangles,

$$\frac{x'}{R} = \frac{x}{R - y}. \qquad \therefore\ x' = R\frac{x}{R - y}$$

But the area inclosed by the broken line is $\int x'\,\delta y$ and, substituting the value of x' from the foregoing, this area is

$$R\int\frac{x\,\delta y}{R - y} = R\int\frac{\delta A}{R - y} = A'$$

23. Stresses in Columns or Long Struts. When a short bar is subjected to an **axial compressive load** the stress induced in each section is simple compression (see Article 11), and the value of the stress s_c is given by formula (A) or

$$s_c = \frac{P}{A}$$

If, however, the bar is **more than four to six times** as long as its least diameter, the above equation does not apply, as the bar will, if so proportioned, deflect laterally under the load and will ultimately break under a compound stress due to compression and lateral bending. Such a member is called a **column.**

Theoretical equations for the design of columns were first developed by Euler. Other formulas were later developed experimentally by Hodgkinson and Tredgold. Gordon and Rankine have also proposed equations for the design of this class of members. The student is referred to any good treatise on the mechanics of materials for a fuller discussion of these expressions than can be given in this work.

Let l = the length of the column in inches.
ρ = the least radius of gyration of cross-section = $\sqrt{I/A}$.
I = the least moment of inertia of cross-section.
A = the area of the cross-section in square inches.
m = a coefficient depending upon the end conditions.
n = the factor of safety.
P_c = the *failure* load on the column in pounds.
s'_c = the *mean* intensity of stress under the *failure* load, or the *unit* failure load, = $P_c \div A$

s_c = the crushing strength of the material, or unit stress at the yield point. This is the maximum intensity of stress in the column when the mean intensity of stress is s'_c.

P = the *working* load on the column in pounds, $P_c \div n$.

s' = the *mean* intensity of *working* stress, or *unit* working load $= s'_c \div n = P \div A$.

s = the intensity of *working* stress in the column ($= s_c \div n$). This is the maximum intensity of stress in the column when the mean intensity of stress is s'.

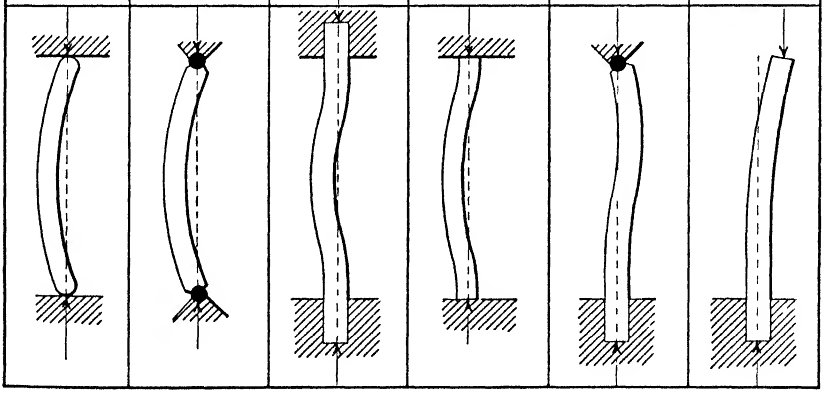
Then **Euler's formula** for long columns is

$$P_c = m\frac{\pi^2 EI}{l^2} \tag{1}$$

It is to be especially noted that Euler's equation is rational and deduced from the theory of elasticity. The coefficient m is also rational and applicable to other forms of column formulas, though, as will be shown later, the Euler equation is strictly applicable only to very long columns.

Table III

	Values of $\underline{m}$ for Different End Conditions				
Case I	Case II	Case III	Case IV	Case V	Case VI
Round Ends Both Ends Free $m = 1$	Pin Ends Both Ends Free but Guided $m = 1$	Fixed Ends $m = 4$	Square or Flat Ends $m = 1$ to 4	One End Fixed the other Guided $m = 2.05$	Fixed at One End, Free at Other $m = \frac{1}{4}$



The relation of the factor m to the form of the ends of the column, and the manner in which the load is applied, should be carefully noted. These relations are shown in Table III. " Round ends " are those that are free to turn upon the surfaces which transmit the load "Pin-ended" columns permit rotation of the ends in one plane and are virtually equivalent to round-ended columns as far as carrying strength in one plane is concerned Round-ended columns are seldom used in machines, pin-ended columns are quite common A column with fixed ends is one whose ends are "built in" or restrained, so that the tangent to the elastic curve remains vertical at the ends when a lateral deflection occurs "Square" or "flat-ended" columns have flat ends that simply abut on the plane surfaces that transmit the load, without being fixed Professor Merriman says that the strength of such a column approximates that of a column with fixed ends when it is short, and that of a pin-ended column when it is long In the discussion that follows, the weight of the column itself is neglected When the column is placed in a vertical position the increase in stress due to the weight is usually small compared to other stresses If, however, the column is placed horizontally, the stress due to its weight may be considerable and should be taken into account

Very short compression members, of ductile material, fail under stresses corresponding to, or only slightly in excess of, the apparent elastic limit, or yield point, for when this stress is reached the metal flows, although it does not actually break Very long columns may approximate the resistance as given by Euler's formula Columns of lengths intermediate between compression members which yield by simple crushing and those which fail by pure flexure are weaker than the former and stronger than the latter If a column is initially exactly straight, perfectly homogeneous, and subjected to an absolutely concentric load (that is, if it is an ideal column) there seems to be no reason why its strength should diminish rapidly with an increase of length, other conditions remaining the same

However, even an ideal very long column would reach the condition of unstable equilibrium when subjected to a certain critical load (the greatest load consistent with stability) If the load is increased beyond this limit and a deflection is caused in any way, the deflection will increase until the stress due to flexure produces failure of the column. If a deflection is caused while the column is under a load less than this greatest load consistent with stability, the elasticity of the material tends to make the column regain its normal form Initial defects in the form or structure of a column or eccentric application of load tend to produce such a deflection, hence long struts fail under smaller loads than short struts of similar material and cross-section, for the ideal conditions are not realized in practice Or, in other words, for equal safety under a given load long columns must have a greater cross-section, and lower mean stress.* Even in columns of moderate length, if of ductile material, the flow at the yield point causes buckling.

Merriman says that if the length of a compression member be only from four to six times its least "diameter," it may be treated as one which will yield by simple compression. Johnson gives limits within which the Euler formula should not be applied as $l \div \rho = 150$ for pin-ended, and $= 200$ for square-ended columns. Other authorities give somewhat different limits; but nearly all agree that most of the columns in ordinary structures and machines are intermediate between simple compression members and those to which Euler's formulas apply. A great many column formulas have been proposed. A graphical representation of several of them is shown in Fig. 13. In this diagram,

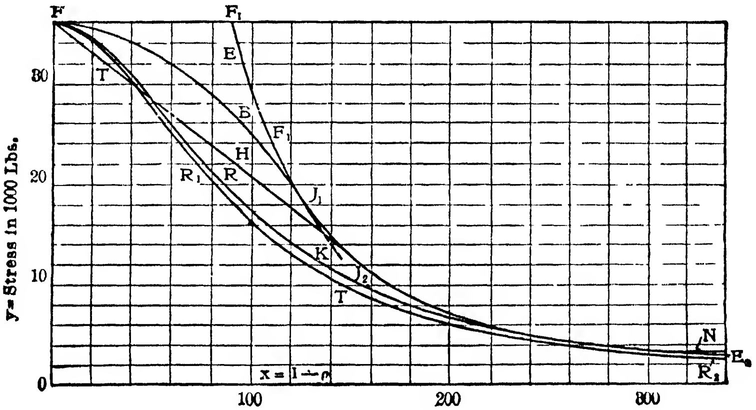


Fig. 13.

abscissas represent ratios of the length of column to the least radius of gyration of the cross-section, and the ordinates represent the nominal (mean) intensity of compressive stress. Or

$$x = l \div \rho = l \div \sqrt{I \div A} \quad \text{and} \quad y = s'_c = P_c \div A$$

The diagram is drawn for the ultimate resistance of pin-ended columns of steel having a yield-point strength in compression of 36,000 lb per sq in., and a modulus of elasticity, E, of 29,400,000. The value of s'_c is 36,000 for a very short compression member, and it is evident that a long column could not be expected to have a greater strength; hence no formula should be used which would give a value of s'_c in excess

* Owing to the flexure of the long column, the stress is not uniform across the section. The maximum intensity of stress must be kept within the compressive strength of the material; hence the mean stress is less than for shorter compression members, in which the mean stress is more nearly equal to the maximum.

of the crushing resistance s_c. Referring to the diagram, it will appear that the Euler formula (represented by the curve EE_1E_2) cannot apply to pin-ended columns (of this particular material) in which $l \div \rho < 90$. If columns with a ratio of l to ρ less than this limit yielded by simple crushing, and those with a greater ratio of l to ρ followed Euler's formula, the straight line FF_1 and the curve $F_1E_1E_2$ would give the laws for all lengths of columns. It is not reasonable to expect such an abrupt change of law in passing this limit ($l \div \rho = 90$); and, as already stated, columns of moderate length fail under a mean stress considerably less than the simple crushing resistance of the material; or the strength of columns is inversely as some function of the length divided by the "least diameter."

Mr. Thomas H. Johnson has developed a formula which is based on the assumption that the strength of the column may be taken inversely as $l \div \rho$. This expression is

$$s'_c = \frac{P_c}{A} = s_c - k\frac{l}{\rho} \tag{2}$$

in which the coefficient k has the value

$$k = \frac{s_c}{3}\sqrt{\frac{4s_c}{3m\pi^2E}}$$

This formula is represented by the straight line THJ_2 in Fig. 13. For this reason it has become known as a **straight-line formula.** It will be noted that this line is tangent to the Euler curve at J_2, and the equation of the latter is to be used, should the columns exceed the length corresponding to this point of tangency ($l \div \rho > 150$). This expression is quite simple, after k has been determined. It is very convenient in making a large number of computations for columns of any one material, and it is employed in structural work to a considerable extent. It does not appear to have any advantage, on the ground of simplicity, when some particular value of k does not apply to several computations.

An expression for the safe mean stress s' corresponding to the safe load P and the maximum stress s may be obtained by dividing equation (2) by a factor of safety n, thus

$$s' = \frac{P}{A} = \frac{s_c}{n} - \frac{kl}{n\rho} = s - \frac{s}{3}\sqrt{\frac{4s_c}{3m\pi^2E}}\frac{l}{\rho} \tag{3}$$

The **Rankine** or **Gordon** formula has been extensively used for columns. It may be expressed as follows:

$$s'_c = \frac{P_c}{A} = \frac{s_c}{1 + \frac{1}{m}\beta\left(\frac{l}{\rho}\right)^2} \tag{4}$$

The above formula is based upon experiments on the *breaking* strength of columns. The coefficient β is purely empirical, and this fact limits its usefulness, for it leaves much uncertainty as to how this coefficient should be modified for materials different from those which have been actually tested as columns.

An expression for the safe mean stress s' may be obtained by dividing equation (4) by a factor of safety n, thus

$$s' = \frac{P}{A} = \frac{s}{1 + \frac{1}{m}\beta\left(\frac{l}{\rho}\right)^2} \qquad (5)$$

The *form* of the Rankine expression is rational, but the coefficient β is not. Merriman recommends the following values of β/m:

Material	Both Ends Round	Both Ends Fixed	One End Round, One Fixed
Timber..................	$\frac{1}{750}$	$\frac{1}{3,000}$	$\frac{1.95}{3,000}$
Cast iron................	$\frac{1}{1250}$	$\frac{1}{5,000}$	$\frac{1.95}{5,000}$
Wrought iron............	$\frac{1}{900}$	$\frac{1}{36,000}$	$\frac{1\ 95}{36,000}$
Steel....................	$\frac{1}{6250}$	$\frac{1}{25,000}$	$\frac{1\ 95}{25,000}$

Professor Merriman says, in his " Mechanics of Materials " (Tenth Edition, page 129): " Several attempts have been made to establish a formula for columns which shall be theoretically correct. . . . The most successful attempt is that of **Ritter**, who, in 1873, proposed the formula

$$s' = \frac{P}{A} = \frac{s}{1 + \frac{s_e}{m\pi^2 E}\left(\frac{l}{\rho}\right)^2} \qquad (6)$$

" The form of this formula is the same as that of Rankine's formula, . . . but it deserves a special name because it completes the deduction of the latter formula by finding for β a value which is closely correct when the stress s does not exceed the elastic limit s_e." The

Merriman notation is changed to agree with that previously used in this article. For ultimate strength, this formula might be written:

$$s'_c = \frac{P_c}{A} = \frac{s_c}{1 + \frac{s_c}{m\pi^2 E}\left(\frac{l}{\rho}\right)^2} \tag{7}$$

but the first form [equation (6)] is the more important. The curve R_1TR_2 (Fig. 13) is the graphical representation of the last expression, equation (7).*

Merriman gives the Euler formula for safe load and stress thus:

$$s' = \frac{P}{A} = s\frac{m\pi^2 E\rho^2}{s_c l^2} \tag{8}$$

Failure occurs if $s \geqq s_c$. The Ritter formula equation (6) reduces to this last expression for columns so long that the term unity in the denominator is negligible; strictly speaking, this is so only when $l \div \rho =$ infinity. Professor Merriman also shows, mathematically, that the two curves, EE_1E_2 and R_1TR_2 are tangent to each other when $l \div \rho =$ infinity. If $l \div \rho = 0$, the Ritter formula reduces to $s' = P \div A$, which is the ordinary formula for short compression members.

It will be noted from Fig. 13 that the Ritter and Rankine formulas agree very closely for the material taken for illustration; but the fact that the curve of the latter crosses the Euler curve near the right-hand limit of the diagram indicates that its constant β is not theoretically correct. The facts that the Ritter formula is rational in form, that it gives correct values at the limits $l \div \rho = \infty$ and $l \div \rho = 0$, and that it covers the entire range of length of columns, have led to its extended use in machine design. It will be noted from Fig. 13 that it gives values well on the safe side as compared with other formulas, but these values do not agree closely with the actual results of experimentation.

The late **Professor J. B. Johnson** derived a formula from the results of the very careful experiments of Considère and Tetmajer. His formula is:

$$s'_c = \frac{P_c}{A} = s_c\left[1 - \frac{s_c l^2}{4m\pi^2 E\rho^2}\right] \tag{N_1}$$

The curve FBJ_1 (Fig. 13) represents this expression. This curve is a parabola tangent to the Euler curve, and with its vertex in the axis of ordinates at F, the direct crushing stress of the material. For columns

* Professor Merriman developed equation (7) independently, but later than Ritter. He gives Ritter sole credit for the formula in the 1897 edition of his "Mechanics of Materials."

having $l \div \rho$ greater than the value corresponding to the point of tangency J_1 (should such be used), the Euler formula is to be employed. The formula of Professor Johnson's is empirical, but it agrees remarkably well with carefully conducted experiments on *breaking* loads. Johnson's * theory that the strength of short columns is limited by the yield point of the material has been corroborated by tests on large columns at the U. S. Bureau of Standards. Tests made by Johnson himself, and others made elsewhere, agree remarkably well with this formula.

An expression for the safe load and stress may be obtained by dividing equation (N_1) by a factor of safety n, thus

$$s' = \frac{P}{A} = s\left[1 - \frac{s_c l^2}{4m\pi^2 E\rho^2}\right] \qquad (N_2)$$

Or, if

$$k = \frac{s_c l^2}{m\pi^2 E}$$

the equation may be written

$$s' = \frac{P}{A} = s\left[1 - \frac{k}{4\rho^2}\right] \qquad (N)$$

and equation (N_1) may be written

$$s'_c = \frac{P_c}{A} = s_c\left[1 - \frac{k}{4\rho^2}\right]$$

which, as will be seen, are very convenient forms for solving.

The Euler equation may be written,

$$s' = \frac{P}{A} = s\frac{\rho^2}{k} \quad \text{or} \quad s'_c = \frac{P_c}{A} = s_c\frac{\rho^2}{k}$$

The Euler and Johnson formulas therefore will give equal values when

$$\frac{k}{\rho^2} = 2$$

That is, the Euler equation should be used when $\frac{k}{\rho^2} \overline{\gtrless}\ 2$, and the Johnson equation should be used when $\frac{k}{\rho^2} \overline{\lessgtr}\ 2$.

24. Conclusions. The following deductions may be drawn from the foregoing. The Euler equation applies with accuracy to very long columns, which, in general, are not frequently found in machine design. There appears to be no particular justification, either theoretical

* See Johnson's "Materials of Construction," 1918 edition, page 20. Also G. B. Upton's "Materials of Construction," page 95.

or practical, for the use of the T. H. Johnson equation. The Rankine equation, though empirical, has long been in use for designing bridges and other structures where cast-iron columns and those made up of steel structural forms are employed. The empirical constants that have been developed and tested practically in this class of work will, no doubt, perpetuate the use of this equation in this field until some more rational equation has been fully tested and approved by practice. The Ritter equation, though rational, gives values considerably on the safe side. This disagreement with other well-tried equations is not excessive for wrought iron and steel, but the difference for cast iron and wood is so great that, as Professor Merriman remarks, "Ritter's formula cannot be regarded as satisfactory." The J. B. Johnson formula, although also empirical and though giving the smallest dimensions of column for a given failure load, appears to agree more closely with actual experiments than any other, within the range to which it applies. Taken in connection with the Euler equation, it probably offers as accurate and simple a solution of the design of columns, as used in machines, as exists at present, and it is therefore adopted in this work.

Example. The connecting-rod of a steam engine is 5 ft long and is subjected to a load of 25,000 lb. If the maximum allowable stress is 6000 lb per sq in., determine the diameter of a circular section at the middle of the rod. Take $E = 30{,}000{,}000$, and the elastic limit $s_e = 36{,}000$ lb per sq in. The rod may be considered a pin-ended column. Hence $m = 1$.

If the rod were designed as a short column, the required area would be

$$A = \frac{25{,}000}{6000} = 4.16 \text{ sq in. or a diameter of } 2\tfrac{5}{16} \text{ in.}$$

and it is evident that for a long column the diameter must be greater than this. Assume $2\frac{3}{4}$ in. as a trial diameter. Then,

$$A = 5.94, \quad \rho^2 = 0.473, \quad l = 60 \text{ in.}$$

whence

$$k = 0.437 \quad \text{and} \quad \frac{k}{\rho^2} = 0.913$$

and the Johnson formula applies. Then, in equation (N),

$$s' = \frac{P}{A} = 6000\left[1 - \frac{0.437}{4 \times 0.473}\right] = 4614 \text{ lb per sq in.}$$

$$\therefore\ P = 4614 \times 5.94 = 27{,}406$$

which is somewhat more than the applied load, and the section will fulfill the requirements.

By the Ritter equation,

$$s^1 = 3136 \text{ lb}$$

and

$$P = 3136 \times 5.94 = 18{,}637$$

which is considerably lower than the applied load and indicates that if the Ritter equation is followed the cross-section must be increased to have sufficient carrying capacity.

In the foregoing discussion and example it has been assumed that the column will fail on the concave side, where the stress is a maximum, and this will be true of ductile materials. In cast-iron columns, failure may occur on the convex side through tension, since cast iron is much weaker in tension than in compression. The value of the unit stress on the convex side, using the J. B. Johnson equation is

$$s_t = \frac{P}{A}\left[\frac{2k - 4\rho^2}{4\rho^2 - k}\right] \tag{N_3}$$

where $k = s_c l^2/m\pi^2 E$ as before. If s_t is positive, the stress is tensile; if negative, the stress is compressive.

25. Other Column Formulas. The most extended experience with columns has been in structural work and even though columns made of rolled shapes are uncommon in machine design brief reference to some of the modifications of column formulas as appearing in structural work may help to visualize the complexity of the problem. As stated in the foregoing, the **straight-line** type of formula has been much used in structural work, and Prof. J. E. Boyd * cites the following simplifications of this equation:

(1) The Chicago building laws for structural steel specify

$$\frac{P}{A} = 16{,}000 - 70\frac{l}{\rho} \tag{9}$$

The maximum value of P/A is fixed at 14,000 lb per sq in., and the value of l/ρ is not to exceed 180.

(2) The American Railway Engineering Association recommends

$$\frac{P}{A} = 15{,}000 - 50\frac{l}{\rho} \tag{10}$$

The maximum value of P/A is not to exceed 12,500 lb per sq in., and l/ρ is not to exceed 100 for main structures nor 120 for wind and stay bracing.

* For a fuller discussion of column formulas see "Strength of Materials" by James E. Boyd, fourth edition, page 374.

(3) The American Bridge Company uses

$$\frac{P}{A} = 19{,}000 - 100\frac{l}{\rho} \tag{11}$$

with a maximum value of $P/A = 13{,}000$ for values of l/ρ not greater than 120 and

$$\frac{P}{A} = 13{,}000 - 50\frac{l}{\rho} \tag{12}$$

for values of l/ρ between 120 and 200.

In a similar manner the American Institute of Steel Construction has recommended the following simplification of the Rankine formula

$$\frac{P}{A} = \frac{18{,}000}{1 + \dfrac{l^2}{18{,}000\rho^2}} \tag{13}$$

The maximum value of P/A is set at 15,000 lb per sq in.; l/ρ is not to exceed 120 for main members nor 200 for secondary members. It should be remembered that round-ended columns are not used in structures.

26. Eccentric Loading of Long Columns. In the preceding discussion of columns it has been assumed that the load has been applied axially. This is obviously the best way of applying the load, but cases often occur where it must be applied at a distance a from the axis of the column. In such a case the column is said to carry an **eccentric** load, and the arm a is called the **eccentricity.** If the length of the column be less than four or six times its least diameter, that is, if the ratio l/p be less than about 25, the member may be treated by the method outlined in paragraph (20) and formula (M) will apply or

$$s = \frac{P}{A} + \frac{Pac}{I}$$

If, however, the column be longer than four to six times its least diameter, it can no longer be assumed that the direct stress P/A due to the load is uniformly distributed over the section, as it has been shown by the discussion on long columns that such is not the case.

In addition, if the load is applied eccentrically, it is obvious that the column will deflect somewhat more than it would if the load were applied axially. This will have the effect of adding to the original lever arm a an additional amount, α, due to this deflection.

The stresses, therefore, acting on an eccentrically loaded column are:

(a) A **compressive stress** s_1, such as would be induced if the load were axial.

(**b**) A **flexural stress** s_2, due to the eccentricity and proportional to the bending moment $P(a + \alpha)$

For the first, from the J B Johnson equation (N),

$$s_1 = \frac{P}{A}\left[\frac{4\rho^2}{4\rho^2 - k}\right]$$

and for the second from (J),

$$s_2 = \frac{P(a + \alpha)c}{I} = \frac{P(a + \alpha)c}{A\rho^2}$$

Therefore, the maximum compressive stress in the section is

$$s = s_1 + s_2 = \frac{P}{A}\left[\frac{4\rho^2}{4\rho^2 - k} + \frac{(a + \alpha)c}{\rho^2}\right] \qquad (O)$$

The stress induced on the convex side of an eccentrically loaded column may be either tensile or compressive, but will always be less than the stress on the concave side For materials whose elastic strength is about the same in either tension or compression, the stress on the convex side is of no importance If, however, the column is made of a material such as cast iron whose tensile strength is much less than its compressive strength, the character and magnitude of the stress on the convex side should be investigated If c' be the distance from the neutral axis to the outer fiber on the convex side, then the magnitude of the stress on the convex side is

$$s = \frac{P}{A}\left[\frac{2k - 4\rho^2}{4\rho^2 - k} + \frac{(a + \alpha)c'}{\rho^2}\right] \qquad (O')$$

If s is positive the stress is tensile, if s is negative the stress is compressive

For columns whose ratio of l/ρ is less than 100, and working stresses such as must be used in machine design, the deflection α may be neglected For columns longer than this, or where the stress is necessarily high, α can be determined by the theory of elasticity For a full discussion of the manner of computation see Merriman's "Mechanics of Materials" For the ordinary problems of machine design this refinement may be omitted

Example 1. A circular wooden pole 8 ft 3 in high is required to carry a transformer weighing 2400 lb, with an eccentricity of 10 in What must be the diameter at the middle in order that the stress due to this load shall not exceed 500 lb per sq in? Let s_c = 3000 lb per sq in and E = 1,500,000 Also $m = \frac{1}{4}$ (See Table III)

Assume a diameter of 8 in. Then.

$$\rho^2 = 4, \quad A = 50.26 \quad \text{and} \quad k = \frac{3000 \times 99^2}{\frac{1}{4} \times \pi^2 \times 1{,}500{,}000} = 8 \text{ and } \frac{k}{\rho^2} = 2$$

whence Johnson's formula applies and

$$s = \frac{2400}{50.26}\left[\frac{4 \times 4}{(4 \times 4) - 8} + \frac{10 \times 4}{4}\right] = 573 \text{ lb}$$

If the excess is considered too great, a second approximation must be made.

Example 2. It should be noted that when $k/\rho^2 > 2$ the Johnson formula does not apply and the Euler equation should be used. Thus, in the foregoing example, let l = 300 in. and P = 900 lb, all other values remaining the same. Whence

$$k = \frac{3000 \times 300^2}{\frac{1}{4} \times \pi^2 \times 1{,}500{,}000} = 72 \quad \text{and} \quad \frac{k}{\rho^2} = 18$$

Then, from the Euler equation and equation (J), neglecting the deflection of the pole,

$$s = \frac{P}{A\rho_2}[k + ac] = \frac{900}{50.26 \times 4}[72 + 10 \times 4] = 501 \text{ lb}$$

27. Stress Due to Change of Temperature. Practically all metals expand when heated, and contract again when cooled The amount which a bar expands per unit of length, for a rise of 1° in temperature, is called its **coefficient of linear expansion,** and will be denoted by C. The following table gives values of C for various substances for 1° F:

Hard steel	$C = 0.0000074$
Soft steel	$C = 0.0000065$
Cast iron	$C = 0.0000062$
Wrought iron	$C = 0.0000068$

If a bar of metal is held at the ends, so as to prevent it from expanding or contracting, stresses are produced in it which are called **temperature stresses,** the effect being the same as though the bar had been compressed, or elongated, an amount corresponding to its expansion or contraction due to the change in temperature.

Let t = change in temperature in degrees.
s = stress induced per unit area.

Since

$$E = \frac{\text{Stress}}{\text{Strain}} = \frac{s}{Ct} \qquad \therefore s = CtE$$

Example. A bar of wrought iron 2 in. square is raised to a temperature 100° above its normal. If held so that it cannot expand, what stress will be induced in it, and what force must oppose it to prevent expansion?

Let $E = 30{,}000{,}000$.

$$s = CtE = 0.0000068 \times 100 \times 30{,}000{,}000 = 20{,}400 \text{ lb per sq in.}$$

and the total opposing force P will be

$$P = 20{,}400 \times 4 = 81{,}600 \text{ lb}$$

28. Resilience. In all the previous discussions on the various straining actions to which a member may be subjected, it has been assumed that the load was a simple dead load and applied without initial velocity or impulse. But, as already pointed out, the load may be applied impulsively; or it may be applied in any way, and removed and applied again and again repeatedly. The application of a load in an impulsive manner, or the repeated application of a load, does not affect the *character* of the straining action, but does affect the *magnitude* of the stress or strain. In order to discuss more clearly the effect of impulsive loading it will be necessary to consider the straining effect of a load somewhat more fully; the discussion of repeated loads will be given in a succeeding section.

If a material is distorted by a straining action, it is capable of doing a certain amount of work as it recovers its original form. If the deformation does not exceed the elastic strain, the amount of this work is equal to the work done upon the material in producing such deformation. If the material is strained beyond the elastic limit, it returns work only equal to that expended in producing elastic deformation; and the energy required to cause the plastic deformation, or set, is not recovered, as it is not stored but has been expended in producing such permanent change of form. Ordinary springs illustrate the first case; the shaping of ductile metals by forging, rolling, wire-drawing, etc., are processes in which nearly all the energy is expended in producing permanent deformation.

The work done in straining a member is called the **work of deformation.** If the strain produced is equal to the deformation at the true elastic limit, the energy expended is called **elastic resilience.*** If the piece is ruptured, the energy expended in breaking it is called **total work of deformation.** If *Oade* (Fig. 6) is the stress-strain diagram for a given material, the area *Oaa′* represents the elastic resilience, and *Oadee′* represents the total work of deformation *per cubic inch of the material.*

* When the term resilience is used without qualifying context, elastic resilience is to be understood.

In such materials as have well-marked elastic limits (proportionality between stress and strain through a definite range) the line Oa is a sensibly straight line, and the elastic resilience $Oaa' = \frac{1}{2}aa' \times Oa'$; *or, the elastic resilience equals the elastic strain* (Oa') *multiplied by one-half the elastic stress* ($\frac{1}{2}aa'$). *The area Oadee'* equals the base (Oe') multiplied by the mean ordinate (y) of the curve $Oade$; or, if the quotient of this mean ordinate of the curve divided by the maximum ordinate be called k, the work of deformation equals the ultimate strain multiplied by k times the maximum stress. It is evident that for a straining action beyond the elastic limit, $k > \frac{1}{2}$ and $k < 1$.

The curve $OADEE'$ represents the stress-strain diagram of a material having higher elastic and ultimate strength than the former. The greater inclination of the elastic line (OA) with the axis of strain (OX) shows, in the second case, a higher modulus of elasticity, as this modulus equals the elastic stress divided by the elastic strain. In the first case

$$E_1 = \frac{aa'}{Oa'}$$

in the second,

$$E_2 = \frac{AA'}{OA'}$$

The stress-strain diagram $OADEE'$ shows that of two materials one may have both the higher elastic and ultimate strength, and still have less elastic resilience and less total work of deformation. If the curve $Oa''d''e''$ is the stress-strain diagram of a third material (having a modulus of elasticity similar to the first) it appears that this third material possesses greater elastic resilience, but less total work of deformation than the first.

A comparison of these illustrative stress-strain diagrams (for quite different materials) also shows that, *for a given stress*, the more ductile, less rigid material may have the greater resilience. Hence, when a member must absorb considerable energy, as in a severe shock, a comparatively weak yielding material may be safer than a stronger, stiffer material. This is frequently recognized in drawing specifications. The principle is similar to that involved in the use of springs to avoid undue stress from shock. In fact, springs differ from the so-called rigid members only in the *degree* of distortion under load, or in having much greater resilience for a given maximum load.

If a material is strained beyond its elastic limit, as to a' [Fig. 14 (a)], upon removal of the load it will be found to have such a permanent set as OO'. Upon again applying load, its elastic curve will be $O'a'$; but beyond the point a' its stress-strain diagram will fall in with the curve which would have been produced by continuing the first test (i.e., $a'de$). Similarly, if loaded to a'', the permanent set is OO'', and upon again applying load, the stress-strain diagram becomes $O''a''de$. The elastic limit a'' of the overstrained material is evidently higher than the original elastic limit, a; and the original total work of deformation, $Oade$, is considerably greater than the total work of deformation of the overstrained material, $O''a''de$. The effects of strain beyond the elastic limit are thus seen to be:

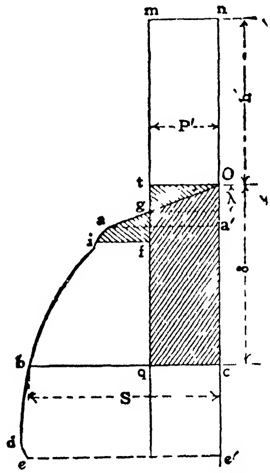


FIG. 14 (b).

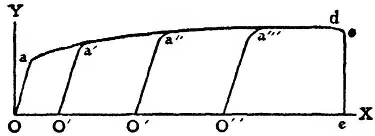


FIG. 14 (a).

I. Elevation of the elastic strength and increase of the elastic resilience.

II. Reduction of the total work of deformation.

These facts have an important influence on resistance to repeated shock. The above-noted elevation of the elastic limit by overstraining can usually be largely or wholly removed by annealing.

29. Suddenly Applied Load, Impact, Shock. It will perhaps be well to first consider the general case of a load impinging on the member, with an initial velocity, this velocity (v) corresponding to a free fall through the height h. For simplicity, the discussion will be confined to a load producing a tensile stress; but the formulas will apply equally well to uniform compressive and shearing stresses, and (1), (2), (3), and (4) apply directly to torsion and flexure.

P = static value of load applied to member.
h = height corresponding to velocity with which load is applied.
δ = total distortion of member due to impulsive load.

s = maximum *intensity* of resulting stress.
A = area of cross-section of the member.
$S = sA$ = maximum value of stress due to sudden or impulsive action.
λ = total distortion of member due to static load P.
k = a constant; its value is $\frac{1}{2}$ if E. L. is not passed; but if E. L. is exceeded $k > \frac{1}{2}$ and $k < 1$.

The energy to be absorbed by the member due to the impulsive application of the load is $P(h + \delta)$; the work of deformation is $kS\delta$. (See preceding article, Resilience.)

Case I.—Maximum Stress within Elastic Limit.

$$P(h + \delta) = kS\delta = \tfrac{1}{2}S\delta \tag{1}$$

$$\delta : \lambda :: S : P. \quad \therefore \delta = \frac{S\lambda}{P} \tag{2}$$

$$S = \frac{2Ph}{\delta} + 2P = \frac{2P^2h}{S\lambda} + 2P \tag{3}$$

$$\therefore S^2 = \frac{2P^2h}{\lambda} + 2PS. \quad \therefore S = P\left(1 + \sqrt{1 + \frac{2h}{\lambda}}\right) \tag{4}$$

$$s = \frac{S}{A} = \frac{P}{A}\left(1 + \sqrt{1 + 2\frac{h}{\lambda}}\right) \tag{5}$$

$$\delta = \frac{S\lambda}{P} = \lambda\left(1 + \sqrt{1 + 2\frac{h}{\lambda}}\right) \tag{6}$$

If L = length of the member,

$$\lambda = L \times \text{strain} = L \times \frac{\text{stress}}{E} = \frac{LP}{AE} \tag{7}$$

As λ is small for metals (except in the forms of springs) a moderate impinging velocity may produce very severe stress. It will be evident that λ and δ are directly proportional to the length of the member; hence the stress produced by a given velocity of impact (height h) is reduced by using as long a member as possible.

If the load is applied instantaneously, but without initial velocity, $h = 0$; whence

$$S = P(1 + \sqrt{1 + 0}) = 2P \tag{4'}$$

$$s = \frac{S}{A} = \frac{2P}{A} \tag{5'}$$

$$\delta = \lambda(1 + \sqrt{1 + 0}) = 2\lambda \tag{6'}$$

It should be noted that in determining these expressions for suddenly applied and impact loads the yielding of members supporting the bodies subjected to these loads is neglected. These expressions therefore give results that are on the safe side.

Case II.—Maximum Stress beyond the Elastic Limit. If the maximum stress exceeds the elastic limit, the constant k of equation (1) is between $\frac{1}{2}$ and 1 (see Article 28, Resilience), and its exact value cannot be determined in the absence of the stress-strain diagram for the particular material. Thus (Fig. 14 (b)), $P(h + \delta)$, is represented by the rectangle *mncq*; and this area must equal the area *Oabc*; the latter being greater than the elastic resilience, *Oaa′*, and less than the total work of deformation *Oadee′*, in this illustration.

When the stress-strain diagram is known, the following problems can readily be solved:

(a) Determination of the velocity of impinging of a given load (or corresponding value of h) to produce a given stress, or strain.

(b) Determination of the load which will produce any particular stress, or strain, when impinging with a given velocity.

(c) Determination of the stress, or strain, produced by a given load impinging with a given velocity.

Let the work of deformation corresponding to the known stress, or strain, in (a) and (b), be called $R = kS\delta$. Since the stress-strain diagram is for stress per unit of sectional area and strain per unit of length of the member, let P' be the load per unit of sectional area; h' the height due to the velocity of impinging divided by the total acting length of the member; δ' the distortion per unit of length of the member due to impulsive load; and R' the resilience for unit of volume, or the modulus of resilience.

(a) $$P'(h' + \delta') = ks\delta' = R'. \quad \therefore h' = \frac{R'}{P'} - \delta' \tag{8}$$

(b) $$P' = \frac{R'}{h' + \delta'} \tag{9}$$

(c) The solution of this problem is not quite so definite, in the general case, as the preceding; but it can be easily accomplished, graphically, with sufficient accuracy. Draw the line *gq* (Fig. 14 (b)) (indefinitely), parallel to *Oe′*, and at a distance from it equal to P'; take out the area *fig* = *gOt*. Whatever the value of δ', the shaded area *OcqfigO* = $P'\delta'$; hence the unshaded area under the stress-strain curve must equal $P'h'$. A few trials will suffice to locate the limiting line *bqc* which will give *fibqf* = *mnOt* = $P'h'$.

The case in which the maximum stress is within the elastic limit is by far the most important, as it is almost always desired to keep the maximum intensity of stress, $S \div A$, within the elastic limit, especially as every overstrain (beyond this limit) raises the elastic limit and decreases the total resilience (see Fig. 14 (a)). The effect of a shock which strains a member beyond the elastic limit is to reduce its margin of safety for subsequent similar loads, because of reduction in its ultimate resilience. Numerous successive reductions of the total resilience by such actions may finally cause the member to break under a load which it has often previously sustained.

No doubt many failures can be accounted for by the effects just discussed; but there is another and quite different kind of deterioration of material, which is treated in the following article.

Dr. Thurston has shown that the prolonged application of a dead load may produce rupture, in time, with an intensity of stress considerably below the ordinary static ultimate strength but above the elastic stress. It is well known that an appreciable time is necessary for a ductile metal to flow, as it does flow when its section is changed under stress; hence, a test piece will show greater apparent strength when the load is applied quickly than when it is applied more slowly, provided the application of load is not so rapid as to become impulsive.

The kind of failure which is the subject of the next topic is due to a real, permanent deterioration of the metal, and to entirely different causes from those mentioned above.

30. On the Peculiar Action of Live Loads. Fatigue of Metals. It has been found by experience and experiment that materials which are subjected to continuous variation of load cannot be depended upon to resist as great stress as they will carry if the load is applied but once, or only a few times. When the load is suddenly applied, and frequently repeated, the decline of strength or of the power of endurance may perhaps be ascribed, in part at least, to the elevation of the elastic limit and reduction of the ultimate resilience, as discussed in Article 29. But apart from this cause, with repeated loads, even in the absence of appreciable shock, a decided deterioration of the material very frequently occurs. This effect has been called the **fatigue of materials,** although some authorities restrict this term to the kind of deterioration already referred to as the simple result of a decrease of resilience. The term fatigue implies a weakening of the material due to a general change of structure. It was formerly supposed that the repeated variation of stress caused such change of the general structure, possibly owing to slight departure from perfect elasticity under stress much below that ordinarily designated as the elastic limit. The crystalline appearance of the fracture sustained this view; but numerous tests of pieces from a member ruptured in this way (taken as near as possible to the break), fail to show such crystalline fracture, and it is difficult to reconcile the normal appearance and behavior of such test pieces with the theory of general change of structure.

The theory that fatigue is due to crystallization has therefore been generally discarded. All metals are crystalline in structure. Investigation, under the microscope, of the structure of metals that have been subjected to repeated stress reveals microscopic cracks *within the crystals of the material.* These **micro-flaws** or **slip-lines** begin to appear with a comparatively small number of repeated stresses, and possibly start with minute imperfections in the crystals. If a cycle of stress is repeated a great many times, these micro-flaws tend to extend across the section under variation of stress and may, in time, reduce the net sound section so greatly that the intensity of stress in the fibers that remain intact becomes equal to the normal breaking stress of the material. Professor Johnson suggested "**the gradual fracture of metals**" as a more appropriate term than "fatigue." However, the latter term is widely used and will be adopted here. The theory of "gradual fracture" through the extension of micro-flaws seems to accord with the observed facts more closely than the older theory of general change of structure.

The theory of the subject is, as yet, too incomplete to permit of derivation of rational formulas to account for the effects of repeated live loads; and if the "micro-flaw" theory is correct, it is not probable that such rational analysis can ever be satisfactorily applied.

All the formulas that have been derived for computation of breaking strength under known variations of load, or stress, are empirical ones which have been adjusted to fit the experimentally determined facts.

Experiment has shown that the breaking strength under repeated loading, or the "carrying strength," is a function of the magnitude of the variation of stress and of the number of repetitions, of such varying stress. Furthermore, this function is different for different materials; and authentic observations are on record which go to show that, as between different materials, the one with the higher static breaking strength does not always possess the greater endurance under repeated loading. In general, however, the carrying strength under repeated loads is a function of the static strength. Thus it appears that the raising of the elastic limit by heat treatment does not materially affect the endurance limit. This appears to be reasonable since the relation of strain to stress under the elastic limit is constant for ferrous materials and endurance is a function of strain rather than stress. (See "The Fatigue of Materials" by Moore and Kommers, page 160.)

The allowable working stress usually depends upon: (a) The number of applications of the load This should be considered as indefinite, or practically infinite, in many machine members (b) The range of load This is frequently either from zero to a maximum, or between equal plus and minus values (c) The static breaking strength or the elastic strength

The first systematic experiments upon the effect of repeated loading were conducted by **Wohler** (1859 to 1870) He found, for example, that a bar of wrought iron, subjected to tensile stress varying from zero to the maximum, was ruptured by:

800 repetitions from 0 to 52,800 lb per sq in
107,000 repetitions from 0 to 48,000 lb per sq in
450,000 repetitions from 0 to 39,000 lb per sq in
10,140,000 repetitions from 0 to 35,000 lb per sq in

Wohler's experiments, which are the most extensive that have been conducted along these lines, have been corroborated by the work of Spangenberg, Bauschinger, and others In this country the work of Prof H F Moore is noteworthy The most important deductions that can be drawn from these experiments, within the field to which they apply, are:

(1) The number of repetitions of stress necessary to produce rupture depends, within certain limits, upon the range of unit stress, and not upon the maximum unit stress (By range of stress is meant the algebraic difference between the maximum and minimum unit stresses applied)

(2) If the stresses are reversed (that is, alternately tensile and compressive) unit stresses considerably below the elastic limit will produce rupture if repeated a great many times

It was found, for example, that the stress could be varied from zero up to something less than the elastic limit an indefinite number of times (several millions) before rupture occurred, but with complete reversal of stress, or alternate equal and opposite stresses (tension and compression), it could be broken, by a sufficient number of applications, when the maximum stress was only about one-half to two-thirds the stress at the elastic limit

A number of efforts have been made to deduce, from the experiments of Wohler, formulas which could be applied to the design of machine members. One of the best of these formulas is that of Professor Johnson as it is easily applied to all cases that will arise; it is simpler than most of those previously proposed

Two formulas which have been very generally accepted for computing the probable carrying strength, according to the work of Wöhler, are: **Launhardt's** for varying stress of one kind only, and **Weyrauch's** for stress which changes sign.

Suppose a material to have a static ultimate strength s_u of 60,000 lb per sq in. If the minimum unit stress be plotted as a straight line, *AOB* (Fig. 15), the locus of the maximum unit stress, from the Launhardt formula, is the broken curve from *B* to *D*. That is, for example, when the minimum tensile stress is 12,500, the maximum tensile carrying stress would be about 40,000; or the material could be expected to stand an indefinite number of loadings if the range of stress did not exceed 12,500 to 40,000 lb per sq in. in tension. In a similar way,

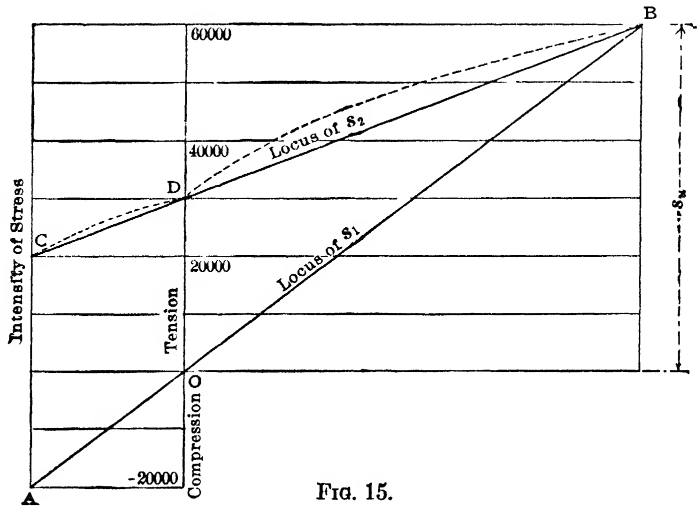


Fig. 15.

the broken curve from *D* to *C* is the locus of maximum tension, from the Weyrauch formula, when the locus of minimum stress (negative tension, or compression) is the straight line *AO*. It will appear that the straight line *CDB* agrees fairly well with these two curves. Inasmuch as it seems unreasonable to expect an abrupt change of law when the minimum stress passes through zero, and as there is no rational basis for the Launhardt and Weyrauch formulas, it appears reasonable to adopt the upper straight line as the locus of the maximum stress. Owing to the discrepancies in the observations (which must be expected from the probable cause of the deterioration of the metal), this straight line may be accepted as representing the law as accurately as could be expected of any empirical line. These are, in substance, the reasons given by Professor Johnson for basing his formula on the straight line *CDB*.

For full discussion and derivation of the following formula,* see Johnson's "Materials of Construction," seventh edition, page 783.

Let s_2 = carrying strength or maximum intensity of stress that will cause failure.

s_1 = minimum intensity of stress.

$\frac{s_1}{s_2}$ = ratio of the minimum to the maximum intensity of the varying stress.

s_u = ultimate (static) intensity of stress.

Then, in general,

$$s_2 = \frac{\frac{1}{2}s_u}{1 - \frac{1}{2}\frac{s_1}{s_2}} = \frac{s_u}{2 - \frac{s_1}{s_2}} \qquad (P)$$

As the expressions contain the *ratio* of the minimum to maximum intensities of stress, instead of their *difference*, they are applicable when the area of cross-section of the member is unknown; for whatever this area, the ratio of the stresses is the same as the ratio of the loads producing these stresses. In substituting values of s_1 and s_2, care must be taken to use proper signs; thus, if tension is taken as positive, compression is negative; or, if the stress varies between tension and compression s_2 is positive and s_1 is negative.

For dead load $s_1 = s_2$.

$$\therefore s_2 = \frac{\frac{1}{2}s_u}{1 - \frac{1}{2}\frac{s_2}{s_2}} = \frac{\frac{1}{2}s_u}{\frac{1}{2}} = s_u \qquad (1)$$

For repeated load when $s_1 = 0$

$$\frac{s_1}{s_2} = 0.$$

$$\therefore s_2 = \frac{\frac{1}{2}s_u}{1 - 0} = \frac{1}{2}s_u \qquad (2)$$

For complete reversal of load $s_1 = - s_2$.

$$\therefore s_2 = \frac{\frac{1}{2}s_u}{1 - \frac{1}{2}\frac{- s_2}{+ s_2}} = \frac{\frac{1}{2}s_u}{1 + \frac{1}{2}} = \frac{1}{3}s_u \qquad (3)$$

* Developed also independently by John Goodman. See "Mechanics Applied to Engineering."

The three special cases (1), (2), and (3), are those most commonly met with in designing, but the general expression (P) should not be lost sight of

If unit stresses be plotted as ordinates and the corresponding number of repetitions causing rupture be plotted as abscissas, the curve so outlined drops quite rapidly at the beginning, but at about 3,000,000 repetitions of the stress begins to be so nearly parallel to the axis as to suggest that a limit had been reached where the corresponding stress might be applied indefinitely without causing rupture Earlier writers on mechanics of materials assumed that such a limit could be reached after a few million repetitions and gave the name of **endurance limit** to the corresponding stress This theory, it will be noted, is in accord with the deductions that have been drawn from Wohler experiments In all probability, however, this curve becomes parallel to the axis only for zero unit stress, and recent studies of the subject would indicate that there is no stress range above zero for which any material will withstand an infinite number of repetitions of stress

Many machine members are subjected to a much greater number of repeated stresses than is indicated in the range of present experimentation Several attempts have been made to extend this present experimental basis to cover such cases In the second edition of this book an equation of this kind suggested by Moore and Seely is recorded, but the proposers of the equation note that it can be considered tentative only because of lack of conclusive data In " The Fatigue of Metals " by Moore and Kommers, definite limitations to the use of this equation are presented, and it is omitted from this work Furthermore, it should be noted that the number of stress cycles to which a machine member may be subjected is usually a matter of conjecture or judgment, and the careful designer can provide for this possibility as he does for other unknown contingencies by means of a factor of safety To the experienced designer this may not be a difficult matter, but the following suggestion may be helpful to the beginner

Professor Kommers has suggested a reasonable method of extending the Johnson equation beyond the field of experimentation on which it is based This method assumes that the character of the stress-repetition curve does not change materially when extended to represent a very large number of repetitions, that is, it assumes that the plot of the logarithm of the stress to the logarithm of repetitions remains a straight line when extended beyond the field of experimentation It assumes also that, when the stress is reduced 9 per cent, the number of repetitions necessary to cause rupture is doubled, as stated by Moore and Seely. Thus, if the stress is progressively reduced, 9 per cent until its value is

51.7 per cent of the original stress, the number of repetitions can be made 128 times the original number. This corresponds to a factor of safety of 1.934 on the original stress and of 128 on the original number of repetitions. Professor Kommers has shown that the general relation which exists between the ratio r of the number of repetitions and the corresponding ratio c of the induced stresses is

$$0.136 \log r = \log c \tag{4}$$

The Johnson formula is based on stresses that caused rupture in from 4,000,000 to 10,000,000 repetitions, and Professor Kommers recommends that 5,000,000 repetitions be assumed as a safe value to be used with the Johnson equation.

If, then, it is desired to apply this equation to a larger number of repetitions, the ratio of the larger number to 5,000,000, gives r of equation (4) from which c may be computed. It should be remembered, again, that accurate experimental data on the effect of very large numbers of repetitions of stress are lacking. It is known also that imperfect working or faulty heat treatment of the material, particularly of steel, greatly affects the life of the member. A liberal allowance in the way of a factor of safety must be made, therefore, in computing working stresses. For convenience, a table of cycles of stress with the corresponding factors, c, is subjoined.

TABLE IV

Number of Repetitions of Stress	Factor c	Number of Repetitions of Stress	Factor c
5,000,000	1.00	8,000,000,000	2 72
10,000,000	1.10	9,000,000,000	2.77
50,000,000	1.37	10,000,000,000	2 81
100,000,000	1.53	20,000,000,000	3.08
500,000,000	1.87	30,000 000 000	3.26
1 000 000,000	2.06	40,000,000,000	3.39
2,000,000,000	2.26	50,000,000,000	3 49
3,000,000,000	2.39	60,000,000,000	3.59
4,000,000,000	2.48	70,000,000,000	3.67
5,000,000,000	2.56	80,000,000,000	3.72
6,000,000,000	2.62	90,000,000,000	3.78
7,000,000,000	2.68	100,000,000,000	3.85

Example. A bar of steel whose ultimate *static* strength is 70,000 lb per sq in. is to be subjected to 2,000,000,000 stress cycles, where the minimum stress is one-half the maximum. What should be the value of the carrying strength or maximum stress, neglecting shock and other uncertainties of operation, to avoid failure below this number of repetitions.

Since the stress will be proportional to the load, $s_1 = s_2/2$. Hence, substituting in equation (P),

$$s_2 = \frac{\frac{1}{2} \times 70{,}000}{1 - \frac{1}{2}\frac{s_2}{2s_2}} = 46{,}666 \text{ lb}$$

for 5,000,000 stress cycles. From Table IV, $c = 2.26$. Hence, for 2,000,000,000 stress cycles, the carrying strength is

$$\frac{47{,}000}{2.26} = 20{,}640 \text{ lb}$$

It should be noted that if this member were designed for 5,000,000 stress cycles the carrying strength would be 46,666 lb. This is considerably above the elastic limit of most materials, and even if there were no uncertainties as to the conditions under which the member is to operate a **factor of safety** should be used to reduce the maximum stress well below the elastic limit.

31. Other Formulas for Repeated Stresses. The Johnson equation assumes a fixed value for the ratio of the endurance limit for $s_1 = 0$ and s_u which may or may not be true for all materials. Moore and Kommers * have suggested the equation

$$s_2 = \frac{1.5s'}{1 - 0.5r} \quad \text{or} \quad \frac{s_2}{s'} = \frac{3}{2 - r}$$

where s' = the endurance limit for complete reversal of stress and $r = s_1/s_2$. s' is to be determined expermentally for various metals. If $r = 0$, $s_2/s' = 1.5$ as in the Johnson formula.

Howell, on the basis of tests made at the University of Illinois, has suggested the formula

$$\frac{s_2}{s'} = \frac{r + 3}{2} \quad \text{or} \quad s_2 = s'\left(\frac{r + 3}{2}\right)$$

Since all these equations rest upon incomplete information, and as in any case the working stress must be kept well below s_2, the Moore and Kommers and the Howell equations would not appear to possess manifest advantages over the Johnson formula. They may be of use with certain metals differing from steel in their characteristics.

* See "The Fatigue of Metals" by Moore and Kommers, page 185.

32. Repeated Stresses in Torsion. Experimental data on repeated stresses in shear are very scanty Moore and Kommers record the results of a few tests made by McAdam at Annapolis and by Moore and Jasper at the University of Illinois These would indicate that the stress range for torsional shear does not follow the Johnson formula but is much more constant in character Moore and Kommers conclude that for repeated torsional shearing stresses, particularly for stresses below the elastic limit, the assumption that the stress range is constant involves no great error They suggest the following formula:

$$s_2 = 2s' + s_1 \quad \text{or} \quad s_2 = \frac{2s'}{1 - \dfrac{s_1}{s_2}}$$

where s' is the endurance limit for complete reversal of stress in torsion and $\frac{s_1}{s_2}$ is a minus quantity if the stress is completely reversed. If the range of stress and s' are known, s_2 may be computed. More experimental data are needed to verify the assumption on which this equation is based.

33. Other Considerations. Wohler's experiments were conducted at comparatively low speeds, 60 to 80 cycles per minute. Other experimenters have conducted experiments at much higher rates; but the results reported are so contradictory as to throw little light upon the influence of the rate at which the stresses are repeated upon the theory of fatigue.

Messrs Stanton and Bairstow have shown that the form and finish of the member under stress may greatly affect its life, under repeated loading Sharp corners and abrupt changes in form should be avoided. These experimenters estimate the following relative values of strength for the several shapes listed:

Rounded fillets	100
Standard screw threads	70
Sharp corners	50

These general conclusions are borne out by all practical experience. If such weak shapes cannot be avoided, ample allowance should be made for them by a reduction in the stress.

As already noted in Article 27, ductile metals will slowly elongate when a steady dead load is applied to them even though the intensity of stress may be well below the ultimate strength or even the elastic limit Seely defines the **creep limit** as follows: "the creep limit for a material at a given temperature is the maximum unit stress that can be developed in the material during a specified length of time without causing more than a specified deformation; a time period of 100,000 hours and a deformation of 1 per cent of the gage length has been used rather widely." For the ordinary problems in machine design both the stresses and temperatures involved render consideration of creep unnecessary. But where higher temperatures are encountered, as in steam turbines, heated pressure vessels, and the like, this phenomenon deserves careful consideration for it naturally is accelerated by increased temperature. For a fuller discussion see "The Fatigue of Metals" by Moore and Kommers, page 44.

34. The Factor of Safety. The preceding paragraphs (Articles 9 to 32) have considered the effect that different methods of applying the load will have on a member, and the relations which exist between a given *dead* load and the resulting stress and strain. It has been shown in Article 29 that if the load is applied suddenly the resulting stress and strain theoretically will be twice as great as for a dead load. And finally in Article 30 it has been shown that the maximum stress that can with safety be induced repeatedly in a member will depend on the range of stress. It would seem as though a member designed in accordance with these logical theories would be satisfactory. But it must be remembered that these theories are not absolute; that the information regarding the characteristics of materials is still very incomplete; that flaws and hidden defects always exist; and finally that there is always danger of accidental overloading.

Most of the formulas of mechanics which are applicable to the design of machine members are based on theoretical treatment of the stresses induced by the action of given forces within the elastic limit upon the member under consideration; and the theoretical conclusions so reached are amply verified by practical experiment. When, therefore, the conditions under which the member is to work can be analyzed, and the laws of mechanics applied to its design, such methods as those outlined in this chapter are perfectly rational, if intelligent allowance is made for contingencies. Many machine members, however, are subjected to such a complicated system of stress that analysis cannot be strictly applied, and less satisfactory approximations or assumptions are unavoidable in the present state of knowledge. Under these circumstances, the designer must either base the design on the *predominating stress*, if there is one, allowing such a margin or factor of safety as experience or experiment may show, to provide for the minor uncertain stresses; or, if the case considered be beyond such treatment, recourse must be had to empirical methods or judgment. (See Article 1.)

In most problems of design, as has been noted, rigidity is of greater importance than strength, and usually a member is strong enough if it has sufficient rigidity. Nevertheless, the matter of strength cannot be neglected, particularly where there may be danger to life and limb.

In addition, it is generally essential that a machine member be not only strong enough to avoid breaking under the regular maximum working load, but also that it shall not receive a permanent set, for a machine member ordinarily becomes useless if it takes such set after it has been given the required form. Frequently a temporary strain, even considerably below that corresponding to the elastic limit, would seriously impair the accuracy of operation; and in such cases the member often requires great excess of strength to secure sufficient rigidity. It follows, therefore, from these considerations that if the design of a machine member were based on the maximum allowable stress or carrying strength, as indicated by Wohler's experiments (such stress being modified by the theory of suddenly applied loading, should it be present), there would be no margin to allow for the uncertainties and unknown defects enumerated above; and often there would be no assurance that the elastic limit would not be exceeded. Therefore, although stresses fixed in accordance with these theories form a good basis, they must in general be reduced by means of a **factor of safety** so that the *working stress* is low enough, in comparison to the carrying strength, to provide for these uncertainties.

The factor of safety is usually defined as the quotient of the ultimate static strength divided by the working stress. A consideration of Wohler's experiments shows that such a definition is misleading. A factor of safety of 2, for instance, might be perfectly safe for a dead load; but for a repeated load with stress in one direction it would leave no margin at all for contingencies. The **apparent factor of safety** would seem to be a better term, and, inasmuch as the working stress should never exceed the elastic limit, it would be simpler and more logical to base the design of machine members carrying steady loads upon the elastic limit itself, and the design of members carrying live loads upon the carrying strength, as fixed by Johnson's equation. It is universal practice, however, to base the factor of safety upon the ultimate strength, and this conventional method will be adhered to here.

In steady or dead loading, the apparent factor of safety may be considered as consisting of two factors a and e, the first being used to insure the reduction of the working stress to a value below the elastic limit, and the second or **real factor of safety** providing for the unknown conditions and contingencies under which the member must work. For carbon steels that are not heat treated, the ratio of the ultimate strength to the elastic limit is about 2. Cast iron has no real elastic limit, but the stress-strain curve of cast iron approximates a straight line up to one-third or one-half of the ultimate strength. A factor of from 2 to 3 is sufficient, therefore, to insure elastic action for cast iron. For timber a factor of 3 may be safely assumed. For other materials and other kinds of steel, the designer should consult such tables as Table VII, where the elastic limits of such materials are listed.

For the real factor of safety, values of $1\frac{1}{2}$ to 2 for mild steel and wrought iron, 2 to $2\frac{1}{2}$ for cast iron and other brittle materials, and 2 to 3 for timber may be taken as average values, as found in practice and verified by experience. Hence, *average* values of the apparent factor of safety for steady loading are 3 to 4 for ductile materials, 4 to 5 for brittle materials, and 7 for timber.

In a similar manner, a clearer understanding of the apparent factor of safety for live loads may be obtained by considering this factor as made up of four factors,* namely: a factor b for obtaining the carrying strength for about 5,000,000 cycles of stress, as indicated by equation (P); a factor c to allow for a greater number of cycles if such be contemplated; a factor d to provide for shock if it is to be provided for; and lastly, a real factor of safety e, as before, to provide for uncertainties.

For repeated loading from zero to a maximum, Johnson's equation indicates that the stress should not exceed one-half the ultimate strength, hence, for this case b would be 2. For reversed stresses of equal intensity, b would have a value of 3. The factor c may be taken from Table IV. It was shown in Article 29 that, theoretically, shock doubles the stress due to steady loading. This effect is considerably modified, no doubt, by the yielding of the supports of the member, and unless the construction is very rigid the factor d may be safely taken as 1.5. The real factor of safety to provide for uncertainties may be taken, as before, as $1\frac{1}{2}$ to 2 for ductile materials, 2 to $2\frac{1}{2}$ for brittle metals, and 2 to 3 for timber.

It should be noted that where the range of the repeated stress and the number of repetitions are low the apparent factor of safety ($b \times c \times e$) or ($b \times c \times d \times e$) may give a value less than the apparent factor of safety ($a \times e$) for a dead load. Although such a result may not be in error so far as strength is concerned it may be in error so far as rigidity and the margin of safety are concerned. The apparent factor of safety for a live load should never be taken as low as the safe minimum for a dead load.

Example. The shaft of a Doble water wheel, running under a high head and liable to shock because of suddenly varying water pressure, is

* "See Factors of Safety and Allowable Stress" by C. D. Albert, *American Machinist*, Vol. 57, page 54. Also see "Factor of Safety and Working Stresses" by C. R. Soderberg, *Trans.* A.S.M.E., vol. 52.

to make 250 rpm for 12 hours a day, 300 days in the year, for an expected life of 20 years. The material is to be steel having an ultimate strength of 70,000 lb per sq in. What is the maximum allowable fiber stress.

Here the stress is reversed tension and compression, shearing being neglected for simplicity; hence, from Johnson's equation, $b = 3$. The shaft will make approximately 1,000,000,000 revolutions during its expected life; hence, $c = 2$. The shock factor d may be taken as 1.5, and the real factor of safety, to cover uncertainties, may be safely assumed as 1.5. Hence, the apparent factor of safety is

$$b \times c \times d \times e = 3 \times 2 \times 1.5 \times 1.5 = 13.5$$

and the allowable stress is $70{,}000 \div 13.5 = 5180$ lb per sq in.

Few machine members are required to withstand as many stress cycles as the shaft in the foregoing example, the majority probably being required to withstand less than half that number. Table V contains factors of safety which agree quite closely with the foregoing theory and which also agree quite well with such factors as are used in practice. They are, of course, average values and must be applied with judgment; but, in the absence of trained judgment or as an aid to its development, they may be found useful.

TABLE V

FACTORS OF SAFETY

Character of Material	Dead Load	Repeated Stress in One Direction, 500,000,000 Cycles		Repeated Reversed Stress, 500,000,000 Cycles	
		Gradually Applied Load	Suddenly Applied Load	Gradually Applied Load	Suddenly Applied Load
Wrought iron, steel, or other ductile metals. . .	3 to 4	6	10	8	12
Cast iron or other brittle metals.	4 to 5	7	12	10	20
Timber	7	10	15	15	20

It may be observed that an increased factor of safety may not always, for cast metals, give a stronger member. If the increased dimensions give sections so thick that sponginess results, the gain in strength may be negative; and when internal pressure, such as is found in hydraulic work, is to be withstood, it is often necessary to do with a smaller factor of safety to insure soundness.

The factor of safety has been called the "factor of ignorance," and, as it is too often applied, it is perhaps little else. Thus very often it is specified that all the members of a machine shall be designed with a certain fixed factor of safety without regard to the conditions under which the various members may have to act. A factor of safety applied in this manner is, generally speaking, a factor of ignorance. It is probable that the factor of safety will always retain an element of uncertainty, for it can hardly be hoped that the powers of analysis will ever permit the prediction of the exact effect of every possible straining action, due to regular service and accident. Neither can it be expected that the methods of manufacture, and inspection, will become so perfect as to eliminate or measure precisely every possible defect in materials or workmanship. But a careful study of the conditions of each particular case and a proper attention to the effects which *may* be weighed (at least approximately) should, with the knowledge now to be had, enable the designer to make a fairly accurate application of the factor of safety, an intelligent choice of which is the most important part of design. Tables VII and VIII contain values of the ultimate strength and elastic limits of the materials most used in engineering. They also are average values such as the designer must use in the absence of exact information regarding the material to be employed, and in general such exact information is lacking.

It should be remembered likewise that the foregoing discussion is based upon experiments on ordinary untreated steel. In recent years great advances have been made in the art of heat treatment and also in the production of alloy steels of markedly different qualities as compared to ordinary steel. In these new alloys and in heat-treated steels the relation of the elastic limit to the ultimate strength may vary considerably as compared to ordinary steel, the elastic limit in many cases being proportionately much higher. This must be taken into consideration by the designer if he is working with these new materials. The National Metals Handbook and the S.A.E. Handbook give full information as to the characteristics of engineering materials.

35. Limitations of Analytical Methods. The equations presented in the foregoing have been verified by many laboratory experiments and they have been widely and successfully used in actual practice. Yet they have certain limitations that should be observed. First, they assume that machine members conform in physical proportions to the simple beams and other shapes on which they are based, an assumption that may be far from true. Second, they assume that the material of the member is homogeneous and free from internal stresses, which may be far from the actual conditions. Third, even though they may give a good idea of the maximum stress due to the load at the most greatly strained point they do not convey a convincing picture of the *distribution* of stress over the most dangerous area. And lastly, for many complex structures the values of the stress that they indicate may be suggestive only. If extreme care must be used in the design of a few important members or if a very large number of a given piece of complicated form is to be made supplementary methods of stress determination are often helpful. Only the briefest mention of these methods can be made here.* In one method a model of brittle material such as pottery-plaster is made exactly like the piece that is to be manufactured and a simple test piece is made from the same material. Both are tested to destruction. The test of the simple test specimen gives approximately the ultimate strength of the material, and the test of the model gives some idea of the load which produces this ultimate stress in the model. From these test values a fair idea can be obtained of the maximum allowable stress in the actual machine part.

Another method much used for examining bodies that are stressed in one plane is to expose a transparent model of the part to be examined to polarized light, the model being subjected to the same kind of stress as the original part. The transparent model may be made of glass, celluloid, or transparent Bakelite. The light after passing through the model falls upon a screen. When such a model is thus exposed it takes on color in its various sections depending upon the state of strain in its material. If the distribution of stress is uniform the color will be uniform; if the stress is variable the colors will vary accordingly. Such projected figures can of course be photographed, and they are highly instructive in showing the distribution of stress, particularly around changes of outline and in irregular sections. Figure 145 shows a photograph of a pair of celluloid gear teeth. Note particularly the compressive stresses at the point of contact of the teeth. See also Article 162.

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* For a fuller discussion see "Advanced Mechanics" by F. B. Seely.

TABLE VI

	Character of Stress or Strain	Formula
A	Stress in tension or compression	$s=\frac{P}{A}$
B	Strain in tension or compression	$\Delta=\frac{Pl}{AE}$
C	Stress in shear	$s_s=\frac{P}{A}$
D	Torsional stress, solid circular shaft	$Pa=T=\frac{s_s\pi d^3}{16}$
E	Torsional stress, solid square shaft	$Pa=T=0.208b^3s_s$
F	Torsional stress, hollow circular shaft	$Pa=T=\frac{s_s\pi(d_1^4-d_2^4)}{16d_1}$
G	Torsional strain, solid circular shaft	$\theta=\frac{32Tl}{\pi E_s d^4}$
H	Torsional strain, hollow circular shaft	$\theta=\frac{32Tl}{\pi E_s(d_1^4-d_2^4)}$
I	Deflection in bending	See Table I
J	Stress due to flexure	$M=\frac{sI}{c}$ See Table I
K_2	Combined bending and twisting	$M_e=\frac{1}{2}M\left[1+\sqrt{1+\frac{T^2}{M^2}}\right]$
K_3	Combined bending and twisting	$T_e=M\sqrt{1+\frac{T^2}{M^2}}$
L	Combined torsion and compression	$s'_c=\frac{2P}{\pi d^2}\left[1+\sqrt{1+\left(\frac{8T}{dP}\right)^2}\right]$
L_1	Combined torsion and compression	$s'_s=\frac{2P}{\pi d^2}\sqrt{1+\left(\frac{8T}{dP}\right)^2}$
M	Combined flexure and direct stress	$s'=\frac{P}{A}+\frac{Mc}{I}$
M_1	Flexure and direct stress in curved members	$s_t=\frac{P}{A}\left[1+\frac{a}{R-c_t}\left(\frac{c_t}{R}\times\frac{A'}{A'-A}-1\right)\right]$
N_2	Long column	$s'=\frac{P}{A}=s\left[1-\frac{s_c}{4m\pi^2E}\left(\frac{l}{\rho}\right)^2\right]$
O	Eccentrically loaded column	$s'=s_1+s_2=\frac{P}{A}\left[\frac{4\rho^2}{4\rho^2-k}+\frac{ac}{\rho^2}\right]$
P	Repeated stress	$s_2=\frac{su}{2-\frac{s_1}{s_2}}$

TABLE VII

Ultimate and Elastic Strengths[1]

Material	Ultimate Strength			Elastic Limit			Direct Coefficient of Elasticity E	Transverse Coefficient of Elasticity, E_s
	Tension	Compression	Shear	Tension	Compression	Shear		
Cast iron (common)	20,000	75,000	22,000	*a*	*a*	*a*	10,000 000	4,000,000
Cast iron (best)	24,000	100,000	27 000	*a*	*a*	*a*	15 000,000	6 000,000
Malleable iron (common)	40,000	*b*		16,000			22,000,000	10,000,000
Malleable iron (best)	52,000	*b*		23 000			22,000 000	10,000 000
Wrought iron	55 000	*b*	40,000	30,000	30 000	20,000	28,000 000	10,000,000
Steel, common, cold rolled	80,000	*b*	60,000	60,000	60,000	36 000	30,000,000	12 000,000
Steel 0 10 per cent carbon (rolled)	50,000	*b*	35,000	30,000	30,000	18 000	30 000,000	12,000,000
Steel 0 20 per cent carbon (rolled)	60,000	*b*	45,000	35,000	35,000	21,000	30,000,000	12,000,000
Steel 0 40 per cent carbon (rolled)	90,000	*b*	65,000	50 000	50 000	30,000	30,000 000	12,000,000
Steel 0 60 per cent carbon (rolled)	115,000	*b*	85,000	60,000	60,000	36,000	30,000,000	12,000 000
Steel 0 80 per cent carbon (rolled)	135,000	*b*	100,000	70,000	70,000	42,000	30,000,000	12,000,000
Steel 1 00 per cent carbon (rolled annealed)	150,000	*b*	115,000	80 000	80,000	48,000	30,000,000	12,000 000
Steel 1 00 per cent carbon (rolled, heat treated)	200 000	*b*	150,000	160,000	160,000	96,000	30 000,000	12,000,000
Nickel steel 3 5 per cent nickel	90,000	*b*	65,000	50,000	50 000	30,000	30,000,000	12,000,000
Iron or soft steel wire, unannealed	85,000	*b*		70,000			30,000,000	
High-carbon steel wire, unannealed	200,000	*b*		150,000				
Aluminum castings	15,000	12 000		6,500			10 000,000	

[1] For more detailed data on metals, alloys, heat treatment, etc , see the National Metals Handbook of the American Society for Steel Treating and the S A E Handbook

(*a*) Cast iron has no well-defined elastic limit (*b*) For ductile materials in compression, the elastic limit is practically the ultimate strength

TABLE VIII

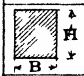
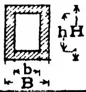
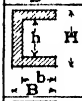
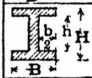
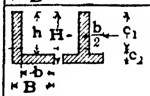
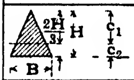
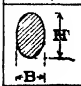
AVERAGE VALUES FOR STRENGTH * AND DUCTILITY OF COPPER ALLOYS TAKEN BY PERMISSION FROM MATERIALS OF ENGINEERING, BY H F MOORE

Alloy	Approximate Composition, Per Cent	Weight per Cubic Inch	Strength in Tension per Square Inch: Proportional Limit	Strength in Tension per Square Inch: Ultimate Strength	Elongation in 2 In Per Cent
Gun metal castings	Copper 88, tin 10, zinc 2	0 31	10,000	35 000	16
Phosphor-bronze castings	Copper 95, tin 4 9, phosphorus trace	0 32	16,000	32 000	6
Rolled metal	Copper 95, tin 4 9, phosphorus trace	0 32	40,000	65,000	30
Soft gear bronze castings	Copper 88, tin 10, lead 2	0 32	18,000	32,000	7
Red brass castings	Copper 83, tin 4, lead 6, zinc 7	0 31	16,000	30,000	17
Manganese-bronze castings	Copper 60, iron 1 5, zinc 38 5, manganese trace	0 30	30,000	70,000	25
Rolled metal	Copper 60, iron 1 5, zinc 38 5, manganese trace	0 30	45 000	75 000	25
Tobin bronze castings	Copper 58 tin 2, zinc 40	0 29	25 000	60 000	
Rolled metal	Copper 58, tin 2, zinc 40	0 29	54 000	79,000	35
Delta metal castings	Copper 65, zinc 30, iron 5			45,000	10
Rolled metal	Copper 65 zinc 30, iron 5			65,000	17
Aluminum-bronze castings	Copper 90 aluminum 10	0 27	25,000	60 000	25
Rolled metal	Copper 90, aluminum 10	0 27	30 000	70,000	30
Cold-drawn	Copper 90, aluminum 10	0 27	80,000	90,000	10
Alloyed aluminum castings	Aluminum 92, copper 8	0 10	15 000	20 000	2
Rolled metal	Aluminum 92, copper 8	0 10	17,000	29 000	15

* Data are lacking for compression values, but the compressive strength may be taken as equal to the strength at the proportional limit in tension

TABLE IX

Properties of Sections

Shape of Section	Moment of Inertia I	Modulus of Section. $\frac{I}{c}$	Square of Radius of Gyration. $\rho^2=\frac{I}{A}$	Polar Moment of Inertia J
	$\frac{\pi d^4}{64}=.049d^4$	$\frac{\pi D^3}{32}=.098D^3$	$\frac{D^2}{16}$	$\frac{\pi D^4}{32}$
	$\frac{\pi}{64}\left[D^4-d^4\right]$	$\frac{\pi}{32}\left[\frac{D^4-d^4}{D}\right]$	$\frac{D^2+d^2}{16}$	$\frac{\pi(D^4-d^4)}{32}$
	$\frac{BH^3}{12}$	$\frac{BH^2}{6}$	$\frac{H^2}{12}$	$\frac{BH(B^2+H^2)}{12}$
	$\frac{1}{12}\left[BH^3-bh^3\right]$	$\frac{1}{6H}\left[BH^3-bh^3\right]$	$\frac{1}{12}\left[\frac{BH^3-bh^3}{BH-bh}\right]$	
	$\frac{1}{12}\left[BH^3-bh^3\right]$	$\frac{1}{6H}\left[BH^3-bh^3\right]$	$\frac{1}{12}\left[\frac{BH^3-bh^3}{BH\quad bh}\right]$	
	$\frac{1}{12}\left[BH^3-bh^3\right]$	$\frac{1}{6H}\left[BH^3-bh^3\right]$	$\frac{1}{12}\left[\frac{BH^3-bh^3}{BH-bh}\right]$	
	$I=\frac{(BH^2-bh^2)^2-4BHbh(H-h)^2}{12(BH-bh)}$		$\frac{I}{c_1}=\frac{(BH^2-bh^2)^2-4BHbh(H-h)^2}{6(BH^2-bh)}$ $\frac{I}{c_2}=\frac{(BH^2-bh^2)^2-4BHbh(H-h)^2}{6(BH^2-2bhH+bh^2)}$	
	$\frac{1}{12}\left[bH^3+Bh^3\right]$	$\frac{1}{6H}\left[bH^3+Bh^3\right]$	$\frac{bH^3+Bh^3}{12(bH+Bh)}$	
	$\frac{BH^3}{36}$	$\frac{I}{c_1}=\frac{BH^2}{24}$ $\frac{I}{c_2}=\frac{BH^2}{12}$	$\frac{H^2}{18}$	
	$\frac{\pi BH^3}{64}$	$\frac{\pi BH^2}{32}$	$\frac{H^2}{16}$	$\frac{\pi(BH^3+HB^3)}{64}$

CHAPTER IV

GENERAL THEORY OF FRICTION, LUBRICATION, AND EFFICIENCY

36. Friction in General. When two solid surfaces are held in contact by any appreciable force, any effort tending to move them relatively to each other is met by a resisting force acting tangentially to the surface of separation of the two bodies. This resistance to relative motion is due to the interlocking of the minute depressions and elevations which exist even in the smoothest surfaces and will, of course, vary with different properties of materials and different qualities of finish. Thus, unsurfaced cast iron will show a very great resistance to relative motion, but two hardened and ground surfaces of steel will move over each other with much greater ease. If the two surfaces are very carefully fitted together without any foreign matter between, they will, in the case of many substances, *adhere* firmly together, which still further increases the resistance to relative motion. If oils or lubricants of any kind are interposed between the surfaces, the resistance to relative motion is, to a considerable extent, overcome.

This tendency to resist relative motion is sometimes a desirable feature and sometimes not. In bearing and rubbing surfaces generally, such frictional resistances result in loss of power and should be reduced to a minimum; in friction clutches, brake straps, keys, screw fastenings, etc., frictional resistance is of great utility and every effort is made to insure its presence. The laws of friction, and the manner of their application, therefore, are of prime importance to the engineer.

These laws are at present rather imperfectly understood, though considerable experimental work has been done. It has been found that many of the older theories based on experimental work are true only for the range of conditions covered by the experiments, and that conditions different from these show entirely different results.

The ratio of frictional resistance F to the normal load P is called the **coefficient of friction**; or if this ratio be denoted by f, then *for flat surfaces*

$$f = \frac{F}{P} \quad \text{or} \quad F = fP$$

On circular surfaces, such as journals and bearings, the distribution of the normal pressure is variable and dependent on the manner in which the surfaces are fitted together. In such cases it is customary for convenience to define the coefficient of friction as the frictional resistance, F, at the surface of the journal divided by the load P normal to the axis of the journal. The **coefficient of friction for circular surfaces** will be denoted by μ. Hence

$$\mu = \frac{F}{P} \quad \text{or} \quad F = \mu P$$

The *intensity* of normal pressure on circular surfaces, as before stated, is difficult of accurate determination and it is therefore customary to use as the normal pressure the **intensity of pressure per unit of projected area.** Or, if d = diameter of shaft, and l = length of bearing, then the intensity of pressure per unit of projected area

$$p = \frac{P}{dl}$$

The energy absorbed by frictional resistance is transformed into heat which is carried away by conduction and radiation to the air, or, in certain kinds of bearings, by water circulation or other means. The **work of friction** is often therefore an important factor in the design of rubbing surfaces. For flat plates the foot-pounds of energy absorbed per minute is $E = fPV$, where V the velocity is in feet per minute and P is in pounds. For circular surfaces, if N be the number of revolutions per minute, and d the diameter of shaft in inches,

$$*E = \text{Frictional resistance} \times \text{Velocity} = (\mu P) \times \left(\frac{\pi d N}{12}\right) = 0.2618 \mu d N P$$

If then f or μ be known, for any pair of rubbing surfaces, the frictional resistance and the energy absorbed for any load P may be calculated. Values of f and μ have been obtained experimentally for many of the materials and conditions met with in engineering, but the data so far available are still incomplete.

The consideration of the laws of friction, as applied to machinery, naturally divides itself into two parts:

(a) **Friction of dry or unlubricated surfaces.**

(b) **Friction of lubricated surfaces.**

* For other forms of surfaces, see Kent's "Mechanical Engineer's Pocketbook," also Thurston's "Friction and Lost Work," page 40.

37. Friction of Unlubricated Surfaces. The experiments of Morin, Rennie, Coulomb,* and many others, furnish the following laws for dry or very slightly lubricated surfaces.

(1) The frictional resistance is approximately proportional to the normal load.

(2) The frictional resistance is approximately independent of the extent of the surfaces.

(3) The frictional resistance, except at very low speeds, decreases as the velocity increases.

It was formerly supposed that an abrupt change took place in the value of f when the body passed from a state of motion to one of rest. It seems now, however, that while the coefficient of rest is in general greater than that of motion, the change in value is gradual and the value at rest is not far different from that at very slow motion. As the velocity increases, the value of f materially decreases and this must be taken account of in designing machinery where friction is involved. Unfortunately the information regarding high or even moderate speeds is also very incomplete.

The following values of f must, in view of the incomplete information, and also because of variations which come with slight changes of conditions, be looked on as approximate values only. Unless it is positively known that the surfaces will be kept free from even slight contamination by oily substances, these values must be used with judgment.

Coefficients of Friction (f) for Dry or Slightly Lubricated Surfaces

Wood on wood—	Static or very low velocity	0.3 to 0.5
Wood on metals—	" " "	0.2 to 0.6
Leather on metals—	" " "	0.3 to 0.6
Leather on wood—	" " "	0.3 to 0.5
Metal on metal—	" " " (average)	0.3
Cast iron on steel—Velocity	= 440 ft per min	0.32
" " — "	= 2640 " "	0.2
" " — "	= 5280 " "	0.06

There are no experimental data giving the decrease in the value of f at high speeds, for combinations such as wood or leather on metals. The data for cast iron on steel will, however, serve as a rough guide to what may be expected to occur. It is to be particularly noted that, in designing brake shoes or other friction machinery where great velocities are involved, allowance must be made for the decrease in the value of the coefficient.

* See "Lubrication and Lubricants," Archbutt and Deely, for a full discussion of these points.

38. Dry Rolling Friction. It has been found that, when a curved body rolls upon a plane or curved surface, the so-called frictional resistance due to the rolling action is much less than that due to sliding, for the same load.

If P = the load;

F = the horizontal force required at the axis of a circular body to produce and sustain uniform motion; and

r = radius of rolling body, it has been found that

$$F = \frac{kP}{r}$$

where k is a coefficient to be determined experimentally. If r be expressed in inches k is found to have a value of about 0.02 for iron or steel rolling on iron or steel.

Neither the coefficient k nor the exact theory of rolling friction is at present very accurately known. The most important use of rolling

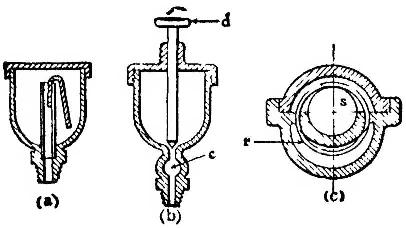


Fig. 16.

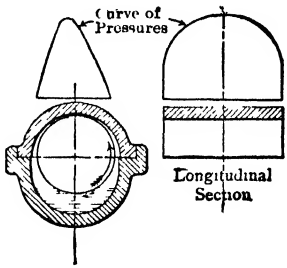


Fig. 17.

friction is, as far as the present discussion is concerned, in connection with roller bearings for shafting, and a fuller discussion of these will be given later.

39. Friction of Lubricated Surfaces. When a lubricant is interposed between a pair of rubbing surfaces, the frictional resistance is materially reduced because the surfaces are wholly or partially separated from each other by the lubricant. The lubricant may be fed to the surfaces in a number of ways. If the motion is intermittent, and other conditions will allow, a simple oil hole leading to the rubbing surfaces is often used. If the motion is continuous, some form of oil cup which will give a continuous supply is better. Fig. 16 (a) shows a cup of the simpler type where a wick of cotton or wool draws up the oil by capillary attraction and feeds it slowly into the oil hole. This is sometimes called siphon feed. Fig 16 (b) shows a so-called sight feed cup where the oil falling by gravity from the cup can be seen as it passes the hole e and the flow can be regulated by the screw *d*. Centrifugal action is also used to some extent to feed oil to rotating parts. Sometimes an opening is made in the bearing so that a pad saturated with lubricant can be kept pressed up against the moving surface, thus lubricating the whole length of the journal continuously. For heavy lubricants, such as greases, where very heavy pressures are carried on the rubbing surfaces, so-called compression cups are often used and are constructed so as to force the lubricant in between the surfaces. Fig. 16 (c) shows a "ring oiled" bearing. The ring *r* running loose on the shaft *s* dips into the pocket below the shaft. The friction of the ring on the shaft causes it to rotate and draw up oil from the pocket. Sometimes chains are used instead of solid rings. For the most efficient lubrication the journal itself runs in a bath of oil (Fig. 17) or is flooded with oil supplied under pressure. The relative merits of these various methods of supplying the lubricant will be more apparent after a discussion of the general laws of lubrication.

The effect of friction, and the efficiency of lubrication of so-called lubricated surfaces, may conveniently be treated under three heads:

(a) Static friction of lubricated surfaces.

(b) Friction of imperfectly lubricated surfaces.

(c) Friction of perfectly lubricated surfaces.

40. Static Friction and Lubrication. When a pair of lubricated surfaces are pressed together by a load, the pressure tends to expel the lubricant slowly from between the surfaces. Experiments and experience show that it is very difficult, even with limited areas and heavy pressures, to expel the lubricant completely. If ordinary machinery, however, is allowed to stand at rest for a short period of time, this action is sufficient to expel so much of the lubricant that may have been between the surfaces while running as to allow the metallic surfaces to come more or less into contact. The static coefficient of friction of lubricated surfaces is hence very much higher than that of surfaces which move even very slowly, for it will be seen presently that even at low velocities the surfaces tend to draw in the lubricant by their motion. It is a well-known fact that heavy machinery always offers a great resistance to starting after lying idle a short time, and often the rubbing surfaces, if not oiled before starting, will abrade each other before the lubricating action due to running begins to take effect. The materials, therefore, for the rubbing surfaces of heavy machinery should be carefully chosen for their anti-friction qualities, and oil grooves should be carefully provided so that lubricant can be applied as near the point of greatest pressure as possible before motion begins.

The coefficient of static friction for lubricated surfaces is not very accurately known, it varies somewhat with the pressure and character of the lubricant. A fair average value for metal surfaces and pressures ranging from 75 to 500 lb per sq in. is 0.15.*

41. Imperfect Lubrication. When one lubricated surface slides over another, the moving surface, even at low velocities, tends to carry the lubricant, if properly applied, in between the surfaces. Thus the layer of oil which touches the surface of a journal adheres to it and is carried along under the bearing. This layer in turn tends to carry along the layer which next adjoins it, because the viscosity of the lubricant opposes the shearing action which results between layers on account of the action of the moving surface of the journal. In plane sliding surfaces the lubricant is generally applied to the stationary surface and tends to cling to it in spite of the tendency of the slider to rub it off. The action of the sliding surfaces in drawing in the lubricant is similar to that of the rotating journal, but in a much less marked degree, as would naturally be expected. If the velocity of rubbing be very low, or the pressure very high, or the supply of lubricant limited, the quantity of lubricant that is carried in is very small and the surfaces in contact are very slightly lubricated and may even be in actual metallic contact. The materials, therefore, for the rubbing surfaces of slow-moving machinery should also be carefully chosen for their anti-friction qualities, as even after the machinery has been successfully set in motion metallic contact may occur between them.

If the velocity of rubbing and the supply of lubricant be increased, the load remaining the same, more and more lubricant is thrust between the surfaces by the action noted above till, at a point depending on the pressure, velocity of rubbing, and viscosity of the lubricant, the metallic surfaces are completely separated and the friction becomes only that due to the fluid friction of the lubricant itself. This last state is known as **perfect lubrication.** The formation of this separating film with increasing speed is probably gradual and the character of the contact most probably passes through a gradual change, from contact which is nearly metallic through successive states of partially fluid contact to complete fluid separation. The exact point at which perfect lubrication occurs for any given load, velocity, and lubricant is not accurately known, but what data are available will be given in connection with the discussion of perfect lubrication which follows. It is known, however, that perfect lubrication cannot be obtained without a plentiful supply of the lubricant, as where a journal runs in an oil bath, or is flooded with the lubricant from a continuous supply. It is impossible or inconvenient, however, to lubricate the greater part of the rubbing surfaces of machines

* See Thurston's "Friction and Lost Work," pages 316–317.

in this manner, and, therefore, all surfaces lubricated by such means as simple oil holes, oil cups, oily pads, etc., where the supply of lubricant is in any way restricted, must be considered as imperfectly lubricated.

As already noted, the exact condition which will exist between such surfaces depends on the pressure, the velocity of rubbing, the supply and character of the lubricant, and the temperature of the bearing as affecting the viscosity of the oil. Naturally where so many variables exist, experimental results are very discordant, and although an immense amount of work has been done, the results serve only to emphasize the great variation in conditions with change of these variables. It is evident, for instance, that if velocity and pressure remain constant, almost any condition may be produced from metallic contact to perfect lubrication simply by varying the supply of lubricant. The law of variation of the coefficient of friction, with either varying pressure or velocity, is also found to be modified by the rate at which oil is supplied. The generally accepted theories for imperfectly lubricated bearings running under average conditions, i e , at normal temperature, and with good oil supply from cups or pads, are as follows:*

(a) Starting from rest with constant load, the coefficient of friction first increases slightly with increasing velocity and then decreases, rapidly until at a velocity somewhere below 200 ft per min (and depending upon the oil supply) a minimum value is reached (see Fig. 18).† With further increase of velocity the coefficient increases till the temperature affects the viscosity of the lubricant to such an extent that abrasion and failure occur.

(b) With constant velocity and very light loads (see Fig. 19) the coefficient of friction is very high. As the load is increased, the coefficient decreases very rapidly at first, and then more slowly till pressures of about 100 to 200 lb per sq in. are obtained, when the coefficient again slowly increases.

(c) The law of variation of friction with temperature is very complex and not well defined. Its general characteristics, however, may be expressed as follows: every combination of pressure and velocity requires a lubricant of a certain viscosity for best results. At high speeds and light loads, a light, thin oil will be readily drawn in between the bearings, and its fluid friction, which constitutes the greater part of the resistance in such cases, will be less than that of a heavier oil. Increasing the temperature of a lubricant decreases its viscosity and, in

* See Archbutt and Deeley, page 58, and Thurston's "Friction and Lost Work," pages 296–312.

† It is to be noted that this discussion and the coefficients given refer to circular bearings and friction of rotation.

the above case, therefore, would cause a decrease in friction. With heavier loads and lower velocities usually met with in machines, an increase of temperature decreases the viscosity and may, owing to the expulsion of the lubricant, give an increase in friction.

Care should be used therefore to obtain an oil suited to the conditions, for sometimes a change of lubricant is sufficient to cause great trouble, or, on the other hand, to reduce the temperature of a bearing that is heating. The failure of imperfectly lubricated bearings generally results from the lowering of the viscosity by increased temperature, so that the oil film is no longer maintained and metallic contact and abrasion ensue.

From the foregoing it is evident that the coefficient of friction for imperfect lubrication will necessarily be a variable quantity. Figs. 18

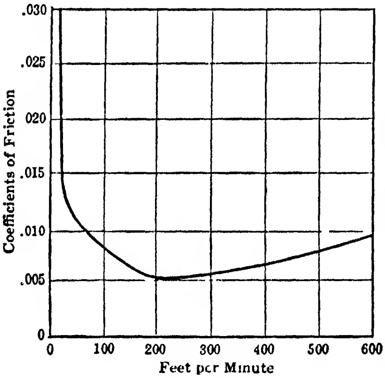


Fig. 18.

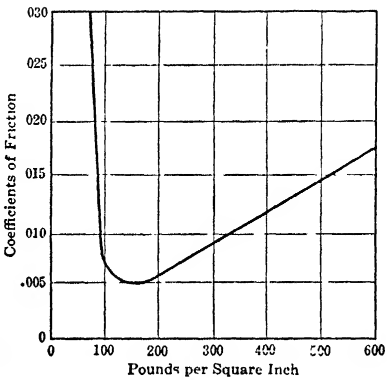


Fig. 19.

and 19 show the variation of μ for varying velocities and pressures. With good lubrication and moderate velocity it may be as low as 0.005, and again with low velocity and poor lubrication it may rise to 0.05 or more. When the velocity is exceedingly low, the coefficient approaches that of static friction of lubricated surfaces, the average value of which is 0.15. A fair average range for pressures from 50 to 500 lb, and velocities from 50 to 500 ft per min, is from 0.02 to 0.008 and, for purposes of design of ordinary machinery, may be taken at 0.015. It is to be noted that with imperfectly lubricated surfaces and low velocities the coefficient of friction is less dependent on the character of the lubricant and more dependent on the character of the rubbing surfaces. The curves Figs. 18 and 19 are composite curves taken from a number of actual experimental results. They are not to be considered as giving exact values of the coefficient μ, but serve to show graphically the general laws by which it varies. In interpreting such curves as Fig. 19

it must be kept in mind that, while the coefficient is decreasing or increasing, the actual *frictional resistance* may not be changing in like manner. The frictional resistance is the product of the load and the coefficient of friction. If, for instance, the coefficient *decreases* as fast as the load *increases*, the frictional resistance will remain constant. The curves, however, show where best results may be expected when designing new machinery, and throw some light on proposed changes in running speed of machinery already installed. They also indicate the complexity of the relation which exists between velocity, pressure, and the coefficient of friction. When it is considered that the temperature also greatly affects these relations, it is evident that a statement of these relations for imperfect lubrication, in the form of a general law or mathematical expression is impracticable, and all such expressions are misleading.

42. Perfect Lubrication. It has been shown in the last article that any rotating journal will, by means of the molecular attraction between it and the lubricant, combined with the viscosity of the lubricant, draw more or less of the lubricant in between the journal and bearing, the amount so drawn in depending on the velocity and pressure. If the journal be allowed to run in an oil bath, or is otherwise plentifully supplied with oil well distributed lengthwise of the journal, and the velocity be high enough for the pressure carried, it is found that this action is so marked that the rubbing surfaces are completely separated by a thin film of lubricant and the friction becomes only that due to the fluid friction of the lubricant itself.

Mr. Beaucamp Tower, experimenting with journal friction (see *Proceedings* of the Institution of Mechanical Engineers, 1883) found that, with a journal and bearing arranged as in Fig. 17, the above action was so marked as to form a **film of oil under pressure** such that the load was completely **fluid borne.** The distribution of the pressure in this film was found to be as indicated by the diagrams above the cross-sections, rising to a maximum at the middle and falling to zero at the edges of the bearing. Mr. Tower succeeded in this way in carrying a load of 625 lb per sq in. of projected area at a velocity of 471 ft per min. With a load of about 330 lb per sq in., and a velocity of about 150 ft per min, a maximum oil pressure of 625 lb was found near the middle point of the bearing. It has been proved mathematically, and verified experimentally, that the situation which exists in a bearing running under these conditions is as follows: the journal, being slightly smaller than the bore of the bearing, tends to be crowded back from the side where the lubricant is carried in, as shown in an exaggerated manner in Fig. 35, giving a wedging effect. The pressure is consequently greatest at a point a little more than halfway beyond the center of loading where the distance between surfaces is least.

The exact relation that must exist between velocity and pressure to allow a perfect oil film to form is not known, nor is it likely that exact limits can ever be set. Enough is known, however, to serve as a general guide for average conditions. Professor H. F. Moore found that for circular journals the limiting values of velocity and pressure at which the film would just form could be expressed by the equation $p = 7.47\sqrt{V}$, where p is in pounds per square inch of projected area and V is in feet per minute. Moore's experiments were carefully conducted,

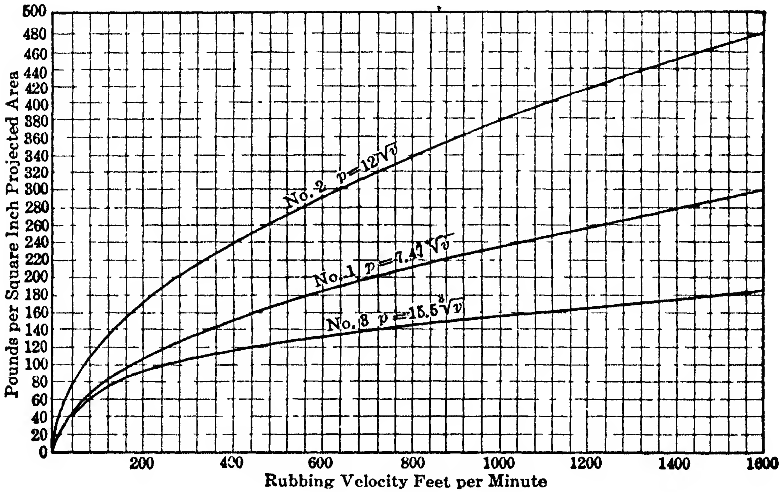


FIG. 20.

but the maximum speed used was only 140 ft per min. Curve 1 in Fig. 20 represents Moore's equation extended to 1600 ft per min. In Tower's experiments sets of simultaneous low values appear where the frictional resistance is a minimum, indicating that the film was at least well formed. These relations can be expressed approximately by the equation $p = 12\sqrt{V}$, and Curve 2 (Fig. 20) shows these relations extended beyond the range of Tower's experiments. Tower kept his experimental bearing at a temperature of 90° by artificial means, and hence a second set of limiting values appears at higher speeds and pressures, where lubrication failed through squeezing out of the lubricant. These upper values correspond roughly to the equation $p = 20\sqrt{V}$, and Professor Guido H. Marx reports the same values from an examination of the work of Stribeck. Professor Marx, after a study of the several series of experiments, is of the opinion that the equation $p = 10\sqrt{V}$ can be safely assumed to express these limitations up to velocities of 500 ft, which is about the maximum velocity attained by Tower in the majority of his experiments The curves of Fig 20 would appear to indicate that this deduction is reasonable, Tower finding stable conditions of the oil film at lower values than those indicated by Curve 2. And though there is no experimental basis for extending Moore's curve, it does appear to corroborate other results

Professor Marx also states that, for speeds above 500 ft per min the equation $p = 30\sqrt[3]{V}$ agrees quite closely with such experimental data as are available There would appear to be no reason for a marked change in these relations at 500 ft per min, and in all probability more extended investigations will develop a single equation that will express these relations accurately over the entire range The General Electric Company has used the equation * $p = 15.5\sqrt[3]{V}$ with success, in the design of motor and generator bearings This curve is shown also in Fig 20 If Professor Marx's equation, $p = 30\sqrt[3]{V}$, represents limiting conditions, the equation of the *General Electric Company* would correspond to a factor of safety of nearly 2, which would not seem to be excessive

The greater portion of our experimental data concerning perfect lubrication is obtained from the work of Tower, Lasche, Stribeck, and Kingsbury Tower's experiments cover a range of velocity from 105 to 471 ft per min and a range of loading from 100 to 625 lb per sq in of projected area His results are very concordant and conclusive and show that the laws of friction for perfectly lubricated surfaces are quite definite, the coefficient of friction varying as the square root of the velocity and inversely as the pressure, very nearly Thus, for olive oil the relation is expressed very closely by

$$\mu = 0.038\frac{\sqrt{V}}{p}$$

and for values obtained with rapeseed oil, as given in Tables X and XI.

$$\mu = 0.03\frac{\sqrt{V}}{p}, \text{ very closely}$$

It follows also from these relations that, for any fixed velocity and temperature, the product of μ and p will be a constant That is, the frictional resistance is practically constant with change of load for fixed velocity and temperature of operation This was found to be the actual result in the experiments, a variation of pressure from 100 to 500

* See "Bearings," by L P Alford, page 81.

lb per sq in. not appreciably affecting the frictional resistance, as shown in Table XI.

Tables X and XI will serve to show the remarkable regularity of the results, and the low values of the coefficient of friction as compared with imperfectly lubricated surfaces. Much lower values have been attained in more highly perfected testing machines under more ideal conditions, but such low values must not be considered as attainable under ordinary working conditions. There is no good reason, however, why such coefficients as are given in Table X cannot be obtained in well-constructed machinery.

TABLE X

BATH OF RAPESEED OIL

Load in Lb per Sq In	Coefficients of Friction for Speeds as Below							
	105 Ft per Min	157 Ft per Min	209 Ft per Min	262 Ft per Min	314 Ft per Min	366 Ft per Min	419 Ft per Min	471 Ft per Min
573		0 00102	0 00108	0 00118	0 00126	0 00132	0 00139	
520		0 00095	0 00105	0 00115	0 00125	0 00133	0 00142	0 00148
415		0 00093	0 00107	0 00119	0 00130	0 00140	0 00149	0 00158
363		0 00084	0 00096	0 00110	0 00122	0 00134	0 00147	0 00155
258	0 00107	0 00139	0 00162	0 00178	0 00195	0 00213	0 00227	0 00243
153	0 00162	0 00200	0 00239	0 00267	0 00300	0 00334	0 00367	0 00396
100	0 00277	0 00357	0 00423	0 00503	0 00576	0 00619	0 00663	0 00714

TABLE XI

BATH OF RAPESEED OIL

Load in Lb per Sq In. of Projected Area	Frictional resistance in pounds per square inch of projected area of bearing surface = μp, for velocities in feet per minute as below Temperature = 90° F.							
	105 Ft	157 Ft	209 Ft	262 Ft	314 Ft	366 Ft	419 Ft	471 Ft
573		0 583	0 62	0 678	0 721	0 758	0 794	Seized
520		0 496	0 546	0 597	0 648	0 691	0 735	0 771
415		0 386	0 445	0 495	0 539	0 582	0 619	0 655
363		0 306	0 35	0 401	0 444	0 488	0 532	0 561
258	0 277	0 357	0 416	0 459	0 503	0 547	0 583	0 626
153	0 248	0 306	0 364	0 408	0 459	0 510	0 561	0 605
100	0 277	0 357	0 423	0 503	0 576	0 619	0 663	0 714

Tower's experiments at different temperatures show that the coefficient of friction, for the above range of pressure and velocities, decreases as the temperature increases. His principal experiments, from which Tables X and XI were taken, were conducted at 90° F and without artificial means of cooling the bearing. The difference between the coefficients of friction obtained at 90° F and those obtained at temperatures as high as are usually allowed in practice can be neglected, as far as designing is concerned, especially since those at 90° are on the safe side. Tables X and XI may, therefore, be taken as representing fairly well the relation existing between pressure, velocity, and frictional resistance for this range, which fortunately covers the most usual conditions in practice. It is to be noted that, at the greatest pressure and highest velocity, the bearing seized, indicating that with such velocity a lower pressure must be assigned, if a perfect oil film is to be maintained, or that with this greatest load a lower velocity must be assigned, if the bearing is to radiate the heat of friction.

Mr. Axel Pederson has shown that the entire range of the coefficient of friction, as found by Tower's experiment, including temperature variation, can be expressed by the formula

$$\mu = \frac{2.3\sqrt{V}}{p(t-32)} \tag{1}$$

where t is the temperature in Fahrenheit degrees attained by the bearing. For a temperature of 90°, by the equation,

$$\mu = 0.0396\frac{\sqrt{V}}{p}$$

which agrees closely with the value given in the foregoing for olive oil. Pederson's equation gives values of the coefficient that agree remarkably well with those given by Tower to illustrate the variation with temperature when lard oil is used as a lubricant.

It is to be especially noted that, within the limits of pressure where a perfect oil film will form, the frictional resistance, for a given velocity, is practically constant and independent of the pressure. (See Table XI.)

The frictional resistance, and coefficient of friction, for bearings running at velocities of over 2000 ft per min with perfect lubrication, have been quite fully determined by Wm. O. Lasche.* The experimental work was very extensive; the results were very conclusive and should be carefully read by designers of high-speed machinery. A discussion of these experiments is beyond the scope of this treatise, but a few of the most important results will be considered. Lasche found that at these

* *Traction and Transmission*, January, 1903.

high velocities the coefficient of friction was practically independent of the velocity, but varied inversely with the pressure as in the Tower experiments, and also varied inversely with the temperature. He found that, if p be the bearing pressure in pounds per square inch, and t the temperature of the bearing in Fahrenheit degrees, then

$$\mu p(t - 32) = 51.2 \tag{2}$$

or

$$\mu p = \frac{51.2}{(t - 32)} \tag{3}$$

For velocities between 500 and 2000 ft per min the coefficient of friction varies about as the fifth root of the velocity, as shown by the experiments of Stribeck. As far as designing is concerned, the difference between the coefficients for this range, and those found by Lasche for the higher velocities, may be neglected, and Lasche's equation may be applied, without serious error, to all velocities above 500 ft per min.

Tower's experiments were conducted with a half bearing, so arranged that the oil film could force the bearing and journal apart as far as it was able. It will be clear, however, that if the journal is closely confined by the bearing on all sides, and if the difference between the bore of the bearing and the diameter of the shaft is very small, the thickness of the oil film and the consequent frictional resistance will be affected thereby. The most important application of the laws of friction is in connection with the design of bearings and is more fully discussed in the next chapter.

43. Summary. From the foregoing discussion the following statements may be made:

(**a**) The friction of imperfectly lubricated surfaces depends partly on the character of the surfaces themselves, and in a greater degree on the character and amount of the lubricant supplied.

(**b**) The load that can be successfully carried on an imperfectly lubricated surface will vary greatly with the amount of lubricant supplied, and must be kept very low where this supply is restricted.

(**c**) The friction of perfectly lubricated surfaces depends very little on the character of the rubbing surfaces, but depends largely on the character of the lubricant.

(**d**) The coefficient of friction of perfectly lubricated surfaces varies inversely with the load when the temperature and velocity remain constant; that is, the frictional resistance per unit area is a constant for these conditions, hence,

(e) The frictional resistance of perfectly lubricated surfaces is, within the ordinary limits, independent of the intensity of pressure and dependent only on the velocity.

(f) The coefficient of friction of perfectly lubricated surfaces varies inversely with the temperature of operation.

(g) The coefficient of friction of perfectly lubricated surfaces, for any given pressure and temperature, varies very nearly as the **square root** of the velocity for velocities up to 500 ft per min; approximately as the **fifth root** of the velocity for velocities between 500 and 2000 ft per min; and is practically **independent** of the velocity for values above 2000 ft per min.

44. Efficiency. It has been noted that not all the energy supplied to a machine is transformed into useful work, but that some of it is always lost in overcoming frictional resistances and doing useless work. There are many ways in which energy losses may occur in machines, and a careful distinction must be made between certain of these ways in order to get a clear definition of the term efficiency. Thus, the steam engine receives its supply of heat in the form of steam under pressure. A considerable portion of the heat so received is lost by condensation of steam on the cooler cylinder walls, and some escapes by radiation without doing any work whatever on the piston. Of the energy actually applied to the piston, part is transformed into useful work at the driving belt, and part is lost in overcoming the frictional resistances just discussed at the various constraining surfaces.

The gas engine is subject to similar losses; a large part of the heat of combustion escaping to the jacket water or to the atmosphere by radiation, and doing no work on the piston, while only a part of the energy actually applied to the piston reappears as useful work. Hydraulic and electric machinery has similar elements of loss. The first class of these energy losses might be called **leakage losses,** as they are of the same character as losses by actual leakage of the medium which is used to transmit the energy. The losses in the machine itself are known as **frictional losses** and are common to all machines; no machine can transform all the energy supplied into useful work, but must lose some of it in friction or other wasteful resistances.

Efficiency has been defined (Article 2) as the ratio of useful work to energy supplied; and from the above it appears that a machine may have two efficiencies, depending on whether reference is had to total energy supplied, or only to that portion of the total energy which the machine transforms into useful and useless work. These efficiencies are respectively known as the **absolute efficiency** and the **mechanical efficiency.** Thus, if a gas engine is supplied with 1000 thermal units, and transforms 200 units into useful work, and 50 units into the useless work of friction, its absolute efficiency is $\frac{200}{1000} = 0.20$, and the mechanical efficiency is $\frac{200}{250} = 0.80$ The consideration of absolute efficiency is beyond the scope of this work, for the design of many machines it does not need to be considered, but the mechanical efficiency can seldom be neglected, since, in general, the amount of work to be done is fixed, and the source of energy must supply enough energy in excess of this amount to compensate for the frictional losses of the machine

The mechanical efficiency of any train of mechanism is the continued product of the efficiencies * of all the several pairs of constraining surfaces in the train at which frictional losses occur Let any machine have n pairs of such surfaces, and let their respective efficiencies be $e, e_1, e_2, e_3, e_4,$ ——— e_n Let E be the mechanical efficiency of the whole machine, and let K be the total amount of energy available for transformation into either useful or useless work Then, the amount of energy which the first pair of constraining surfaces delivers to the second is $K \times e$, and the amount which the second delivers to the third is $Ke \times e_1$, and so on, until the amount of energy delivered by the last element (or the work done) is $K(e \times e_1 \times e_2 ——— e_n)$ But the mechanical efficiency of the train is

$$E = \frac{\text{Work done}}{\text{Energy supplied}} = \frac{K(e \times e_1 \times e_2 ———— e_n)}{K}$$
$$= (e \times e_1 \times e_2 ——— e_n)$$

A machine may consist of several trains of mechanism. If these several trains are arranged in series so that the energy passes from one to another consecutively, the efficiency of the whole machine, by reasoning similar to that in the last paragraph, is the continued product of the efficiencies of the several trains of mechanism If, however, the trains are arranged in parallel so that the total energy is transmitted simultaneously through several trains of mechanism, each train transmitting only a portion of the energy, the above reasoning for the efficiency of the whole machine does not hold If the amount of energy supplied to each train is known, the amount of work which it will deliver can be

* It may be noted in passing that the term efficiency is used in a number of ways other than as the ratio of work done to energy expended Thus the strength of a riveted joint, compared to the strength of the original unpunched plate, is called the *efficiency* of the joint, when what really is meant is its *relative strength* Again, in an air compressor, the ratio of the air actually discharged per stroke, to the whole amount raised to the required pressure per stroke, is called the volumetric efficiency It is evident that such efficiencies are of a different character from those discussed above and do not enter into the calculations of the efficiency of the machine, as a whole, in the manner indicated above

computed as above The sum of all the work, delivered by all the trains, divided by the total energy supplied, will be the efficiency of the whole machine

If, therefore, the efficiencies of the several constraining surfaces of a machine are known, the mechanical efficiency of the whole machine can be calculated The mechanical efficiency of any machine element is, however, a variable quantity, for the coefficient of friction of any pair of constraining surfaces will vary with the lubricant and its method of application, the temperature, the alignment of the surfaces, the velocity of rubbing, and the bearing pressure Furthermore, when all other conditions are constant, the same pair of constraining surfaces will have an entirely different efficiency for the same amount of power transmitted, depending on the manner in which the load is applied Thus, consider a simple wheel and axle driven by a belt on the periphery of the wheel With a given diameter of wheel, the transmission of a given amount of power will bring a certain definite frictional load on the bearings If, however, the diameter of the wheel is doubled, the belt speed is increased in a like ratio, and the belt tension will, for the same power transmitted, be one-half of the former value, and, as a consequence, the frictional resistance at the bearings will be reduced to one-half the original value, the revolutions remaining constant

In general, therefore, it is impossible to calculate precisely from the analysis of a design what the mechanical efficiency will be, particularly if the mechanism is at all complicated, though a reasonable approximation is possible If machines of a similar type have been built, it is far more accurate to base the design of new ones on efficiency tests made on those already in existence Such tests have been made for all standard machines, and the recorded results form a valuable basis for the design of new machines of like characteristics But when a machine of a new type is to be designed, and no recorded tests are to be had that will give any information as to the probable efficiency, an estimate must often be made and the efficiency calculated as outlined above In general, a close approximation can be made, and the making of such estimates is a great aid to the development of that judgment in such matters, which comes only with experience In such cases a knowledge of the efficiencies of various machine elements becomes necessary If the coefficient of friction for any constraining surface could be accurately determined, it would be possible to calculate its efficiency with some degree of certainty But, as before noted, this quantity varies with the velocity of rubbing, with changes in bearing pressures, etc , and such methods of computation are necessarily cumbersome and to be attempted only where a very close estimate is required.

The following are rough average values of the efficiencies of the most common elements For more accurate values the student is referred to the respective discussions of these various elements which follow:

Common bearing, singly	96–98
Common bearing, long lines of shafting	95
Roller bearing	98
Ball bearings	99
Spur gear cast teeth, including bearings	93
Spur gear cut teeth, including bearings	96
Bevel gear cast teeth, including bearings	92
Bevel gear cut teeth, including bearings	95
Worm gear, varies with thread angle, see Article 59	
Belting	96–98
Pin-connected chains, as used on bicycles	95–97
High grade transmission chains	97–99

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CHAPTER V

CONSTRAINING SURFACES

45. General Considerations. As the various members of a machine must move with definite relative motion, they must be retained in correct position by **constraining surfaces.** Thus a shaft is held in position by bearings which locate its axis of rotation, and by collars which prevent motion endwise. The relative motion of a pair of constrained members may be that of **sliding,** as an engine crosshead and its guide; **rotation,** as a shaft journal and its bearing; **rolling,** as in roller and ball bearings; or a combination of some of these as illustrated in certain forms of cams, where both sliding and rolling exist. Dry metallic surfaces, under any appreciable load, even though smoothly machined, will not slide over each other without abrasion. It is necessary, therefore, to keep rubbing surfaces separated by a thin film of some kind of lubricant, and the whole subject of the design of constraining surfaces is closely connected with the theory of lubrication.*

It has been noted in Chapter IV that, when bath or flooded lubrication is maintained, the friction between two rubbing surfaces is independent of the *character* of the material of which the surfaces are composed; but when the surfaces are "imperfectly" lubricated the frictional resistance depends somewhat on the metals used. Experience has shown that *like* metals usually do not rub together well. Thus, steel on steel (except when hardened), steel on wrought iron, bronze on bronze, babbitt metal on babbitt metal, or cast iron on cast iron, are poor combinations unless the velocity is low and the pressure light. If two rubbing surfaces of cast iron can be run together for some time without cutting they take on hard glazed surfaces which will run well together. This is well illustrated in slide valves and pistons of steam engines. Care must be exercised that the surfaces are well lubricated when first put in service. Both soft steel and wrought iron will run well on hardened steel, and hardened steel may be run on hardened steel at very high pressures and velocities, if the surfaces are ground true and polished. Steel and wrought iron will run very well on brass or bronze. The alloys of copper, tin, zinc, antimony, lead, etc., commonly known

* See Chapter IV.

as **anti-friction** or **babbitt metals,** run extremely well with steel or wrought-iron journals.* Innumerable alloys of this kind are upon the market under different names. They can be made of any degree of hardness, depending largely upon the proportion of antimony used. Very hard alloys of this kind are sometimes known as **white brass.** In using babbitt metal for heavy pressures, care should be exercised that the particular alloy selected is hard enough so as not to flow under the applied pressure. Other materials, such as wood, are sometimes used for rubbing surfaces. An innovation is the use of a compound of rubber in pump bearings and similar places where muddy and gritty water is to be handled. Water is used as a lubricant for these bearings. The conditions which influence the selection of materials for rubbing surfaces, and the practical considerations governing their application, will be more fully discussed in connection with the several forms of constraining surfaces.

The most common forms of motion in machines are rectilinear translation and rotation; therefore the most important forms of constraining surfaces are

(a) Sliding surfaces, for the constrainment of rectilinear motion.

(b) Journals and bearings, for the constrainment of motion of rotation.

SLIDING SURFACES

46. Forms of Sliding Pairs. The stationary member of a pair of surfaces, which have relative sliding motion, is usually called the **guide,** the moving part has various names depending on the service, as the **ram** of a shaping machine, the **table** of a planing machine, or the **crosshead** of an engine. The general term **sliding member** will be used here to denote the moving member. Sliding pairs may be classified by the degree of **lateral constrainment** afforded the slider by the guides, and this may be:

(a) Partial lateral constrainment.

(b) Complete lateral constrainment.

In either case the rubbing surfaces of the guide and sliding member may be either **square, angular,** or **circular.** Thus Fig. 21 shows a form of angular guide much used on planing machines, and Fig. 22 shows a set of square guides for a similar purpose. In each the lateral constrainment is only partial, the tendency of the platen to rise being resisted by gravity. Figure 23 (a) shows the crosshead of a steam

* See Kent's "Mechanical Engineers' Pocket Book" for detailed analysis and properties of some of the best known alloys. See also the National Metals Handbook of the American Society for Steel Testing.

engine with an angular guide. Here, lateral constrainment is complete. Figure 23 (b) is also a steam-engine crosshead with circular guiding surfaces. This form of surface may be considered a special form of the angular type. If the circular guiding surfaces have a common center at O, the crosshead is prevented from rotating around O only by the connecting-rod; and as long as it is so held from rotating the lateral constrainment is complete. If the surfaces have different

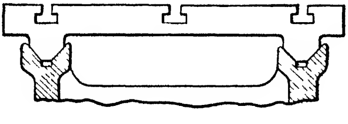


Fig. 21.

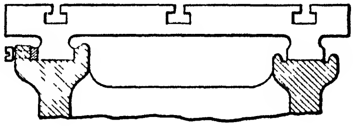
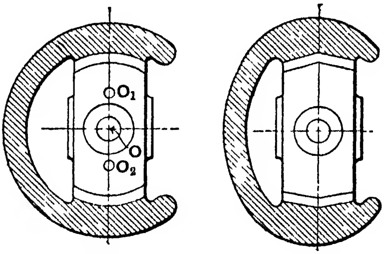


Fig 22.

centers as O_1O_2, it is obvious that rotation cannot take place. Figures 24 (a) and 24 (b) show square and angular guides where constrainment is complete.

The characteristic which distinguishes the square guide from the angular one is that in the square guide two sets of adjustments must be made to compensate for wear, whereas in the angular guide one set only is needed. Thus in Fig. 24 (a), vertical wear must be compensated for by lowering the piece A; lateral wear is taken up by the set screws C



(b) Fig. 23. (a)

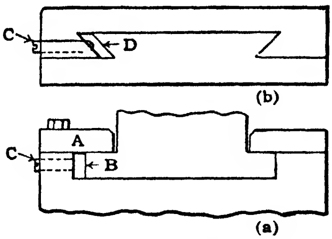


Fig. 24.

which press against the wearing strip or gib B. In Fig. 24 (b) both lateral and vertical wear are compensated for by the set screws C which press upon the gib D. Sometimes D is made tapering and provided with a screw adjustment so that it can be moved endwise, thus compensating for wear. In such cases the set screws C are omitted.

As to the relative merits of square and angular guiding surfaces, it may be said, in general, that square surfaces are easier to machine and fit than the angular ones. There are many places, however, such as the cross slides of lathe carriages, where the angular guide is much more convenient. In places such as lathe beds the V guides commonly used have the advantage of automatically taking up lost motion, no matter how badly they are worn. But, as a rule, the bearing surfaces of such V guides are very small and wear soon begins to be apparent, especially as the wear from the carriage is usually concentrated on a short portion of the bed. There is a tendency among manufacturers to discard the V guide in favor of flat surfaces. A combination of V and flat guides is also often used.

47. General Principles. If a short block, Fig. 25, slides backward and forward upon another member *B*, carrying a fixed load *P*, it is evident that, if the material in *A* and *B* were homogeneous and the velocity were uniform throughout the stroke, the frictional resistance and consequent wear would be practically uniform over the entire surface of *B*. These conditions are difficult to attain and seldom occur in practice. Since *A* must be stopped and started at each end of the stroke, it follows that the velocity cannot be uniform, although in some machines such as plate planers this condition is approximated. Usually, however, the velocity varies from zero at the beginning to a maximum somewhere near the middle of the stroke, as in engine crossheads, shaping machines, etc. Again, the load *P* may vary greatly. Thus in the steam engine, the normal pressure *P* between the crosshead and its guide is zero at each end of the stroke and a maximum near mid-stroke. The velocity of the crosshead also varies from zero at each end of the stroke to a maximum near mid-stroke. Ordinarily the greatest frictional resistance * and wear will occur near mid-stroke, because both velocity and normal pressure between the bearing surfaces are greatest at this position. If the crosshead could be made the same length as the guide, the unit bearing pressure, at the middle of the stroke, would be practically uniform over the entire surface, and would be small compared to the unit normal pressure attained when the crosshead is short. For positions of the crosshead near mid-stroke the wear would be approximately equal over the entire surface, and much less than when the crosshead is very short, but still theoretically greater than at the end position, when both velocity and normal pressure are zero. It has been found by experience that, when the sliding block and guide are made the same length, the wear, even under varying load and velocity, is very small and more uniform over the entire contact surfaces.

It is seldom possible, however, to make the sliding member the same length as the guide. Thus, in lathe carriages, the rams of shaping machines, and the tables of planing machines, the sliding member is, in some machines, shorter than the guide, and in other machines longer. In most cases of this kind the wear is likely to be greater on one part of

* See Article 41.

the guide, or sliding member, than on another. Thus in a shaping machine the ram seldom operates at full stroke, and the wear on the back end of the ram is very small, the result being that, when appreciable wear takes place on the forward end of the ram and the guides are readjusted to compensate for the same, the back end of the ram will not pass through the guides at all, hence the adjustment must be somewhat slack, and accurate work cannot be done. In other machines the excessive local wear comes on the guide, and a similar result occurs. Professor Sweet * corrected this difficulty, in certain machines which he built, by reducing the wearing surface on that portion of the sliding member or guide, as the case may be, where the tendency to wear is least. He suggested the following convenient method of laying out the wearing strips on the surface of a sliding member. Figure 26 shows a sliding surface such as is found on the ram of a shaping machine, where

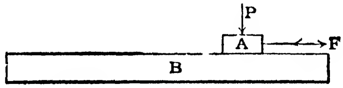


FIG. 25.

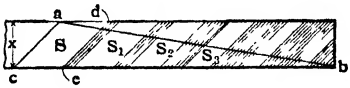


FIG. 26.

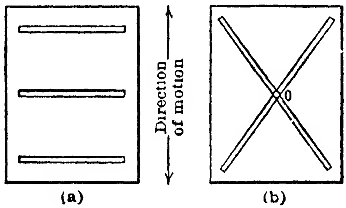


FIG. 27.

little wear occurs on the back or right-hand end, as here shown. The shaded portions represent the parts of the surface which have been relieved, leaving the wearing strips S, S_1, and S_2, etc. To lay off the surface, draw the diagonal *ab* across the surface to be relieved. From *a* draw the line *ac*, making any convenient angle with the horizontal. Lay off *ce* equal to the width of the face x. Draw *de* parallel to *ac* and take the vertical distance above the point of intersection of *ab* and *de* for the first gap, and the corresponding vertical distance below the point of intersection for the first wearing strip, repeating this operation to the end of the surface. Similar wearing strips should be cut in the opposite direction on the other member, if it is comparatively long; but where a short block slides in a long guide, the guide only need be relieved.

48. Bearing Pressures on Sliding Surfaces. It is noted in Article 40 that the tendency of a loaded flat surface to expel the lubricant is resisted to a certain degree by the viscosity of the lubricant and by its power to adhere to the stationary member. This resisting power is

* Professor Sweet embodied some of his experience along this line in a little book called "Things That Are Usually Wrong," which will well repay reading.

much less marked in sliding surfaces than in rotating surfaces, as here the motion is intermittent. It is difficult therefore to lubricate sliding surfaces as efficiently as rotating surfaces, and, in general, they must be considered as "imperfectly" lubricated surfaces. The unit bearing pressure that can be sustained by sliding surfaces is, therefore, much less than can be borne by rotating journals. Further, it is difficult to obtain initially true sliding surfaces and, as noted in the foregoing, very difficult to maintain their accuracy under service. The sliding part, and also the guides themselves should, therefore, be designed for rigidity; in fact, considerations of strength seldom need to be taken into account, but the guides should be so stiff that localized pressure will not occur. It is not surprising, in view of these considerations, that the allowable bearing pressures as fixed by practice vary greatly, even with similar classes of work. Owing to the difficulties of lubrication and compensation for wear, it may be stated, as a general principle, that the bearing pressure must be kept so low that wear is inappreciable, if accurate surfaces are to be maintained.

The following are average values of bearing pressures for different forms of sliding surfaces, as fixed by practice:

Crossheads,* stationary slow-speed engines ..	30 lb to 50 lb
Crossheads, stationary high-speed engines. .	10 lb to 30 lb
Crossheads, marine engines............. ..	50 lb to 75 lb

49. Lubrication of Sliding Surfaces. Sliding surfaces are very difficult to lubricate efficiently on account of the "wiping" action of the sliding member and because the relative motion is not continuous and in the same direction as in rotating bearings. In high-speed engines, bath lubrication is commonly obtained by enclosing the running parts, and allowing them to run in what practically amounts to an oil bath.

Where this cannot be done, care must be exercised in the manner in which the lubricant is supplied. If possible, when the guide is horizontal, the lubricant should be supplied near the middle of the guide. The oil grooves in the moving part should also be given careful consideration. From the theory of lubrication it is evident that the oil channels on all constraining surfaces should be *at right angles* to the direction of motion, wherever the velocity is great enough to draw lubricant between the surfaces. If made otherwise their effect is to relieve any tendency to form a pressure film. The grooves in crossheads, and other sliding members, should, therefore, be made as in Fig. 27 (a) and not as in 27 (b). In either case the grooves should be stopped some distance

* See *Trans.* A.S.M.E., vol. 18, page 753

from the edge of the surface so as not to facilitate the escape of the oil. When the load is so heavy that forced lubrication must be used, the system of grooves shown in Fig. 27 (b) is correct, the oil being forced in at *O*. Care should also be taken that the outer edges of the slider, and the edges of the oil grooves, are chamfered so as to assist the entrance of the lubricant. If the edges are square and sharp their scraping effect may seriously impair the lubrication. Where the guiding surfaces are very long, as in planing machines, oiling devices, such as rollers dipping in an oil pocket, placed at intervals along the guides, are very effective. In certain forms of thrust bearings perfect lubrication is obtained under flat sliding surfaces by a peculiar arrangement of the rubbing surfaces which is fully discussed in Article 66.

JOURNALS AND BEARINGS

Bearings

50. Forms of Bearings. The part of a machine frame, or other member, which constrains a rotating member, such as a shaft, is known as a **bearing.** That portion of the rotating member which engages with the bearing is known as a **journal.** Journals are necessarily circular in all cross-sections, but their profile may be cylindrical, conical, spherical, or even more complex in form, as thrust bearings. (See Art. 61.)

One or more of the following considerations affect the design of the bearing proper:

(**a**) Rigidity, in order that the alignment may not be seriously affected by deflection.

(**b**) Strength, to resist rupture under the greatest loads.

(**c**) Adjustment, to compensate for wear.

(**d**) Formation and maintenance of an oil film.

(**e**) Automatic adjustment, to insure alignment.

(**a and b**). The inside diameter or bore of the bearing, and also its length, are fixed by the dimensions of the journal which engages with it; and the required strength and rigidity may be secured by a proper distribution of metal in accordance with the general principles discussed in Chapter III, which apply to all forms of bearings, as far as strength and stiffness are concerned.

Usually the question of strength does not enter into the design of the main part of the bearing. If, however, the cap *A*, Fig. 28, should be called upon to carry the load as it often is in practice, its dimensions should, in general, be checked for strength, and its design should be such that stiffness is secured. The exact distribution of the pressure over a bearing is not known; * but the assumption that the cap is a beam loaded at the center and of a length equal to the distance between the cap bolts will give dimensions on the safe side for strength and deflection. The greatest bending moment and deflection for such beams are given in Case IX, Table 1. It is impossible to adjust the cap bolts so as to be sure that the load is uniformly distributed among them, and the uncertainty of the initial stress due to screwing up the nuts makes the problem more difficult. For this reason the cap bolts should be designed to carry more than the apparent load. If only two bolts are used each should be designed for two-thirds of the total load; if four are used each should be able to carry one-third of the load with an apparent stress of not more than 6000 lb per sq in., or each bolt may be assumed to carry its proportionate share of the load and allowance made in the assumed stress for the initial load due to screwing up.

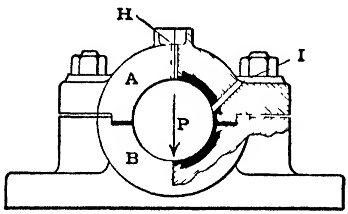


Fig. 28.

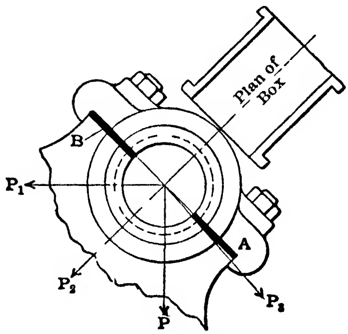


Fig. 29.

The last three items, **c**, **d**, and **e**, affect the *form* of the bearing. Consider first **c** and **d**. It is evident that the metal of the bearing will wear away most rapidly in the *line of greatest pressure*, hence adjustment for wear should also be along this line. It follows also that the bearing should be parted at *right angles* to the line of greatest pressure. Thus, if the load on the shaft be a simple vertical load P, as in Fig. 28, wear will take place only on the bottom half of the bearing. If this wear is so small as not to interfere with the alignment of the shaft, or if all the bearings on the shaft wear uniformly, adjustment may be made by lowering the cap A. If the shaft must occupy a fixed position relative to the frame of the machine, alignment must be maintained by raising the lower bearing surface. Where this is desirable the lower wearing surface is usually made separate from the pillow-block, as in Fig. 30, thus allowing the bearing to remain fixed in position, while the wearing

* See Article 36.

part may be raised to compensate for the wear. If the load P, Fig. 28, be in a upward direction, all necessary adjustment may be made by means of the cap.

It was shown in Articles 41 and 42 that a journal will automatically tend to form a film of lubricant between itself and the bearing. If the conditions under which the lubricant is supplied are correct, fluid pressure may thus be created between the journal and bearing, *provided the surface of the bearing is continuous for some distance on each side of the line of action of the load.* The greatest pressure will be found near this line of action. It is evident that the bearing shown in Fig. 28 fulfills both these requirements for vertical load either upward or downward; but is unsuited for lateral pressure from the standpoints both of adjustment for wear and lubrication.

Suppose, however, that the journal carries a heavy vertical load P (Fig. 29), and is subjected at the same time to a heavy horizontal belt

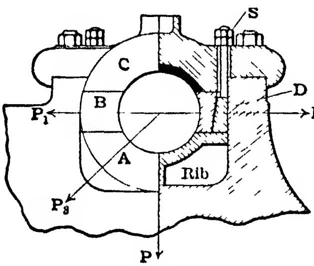


FIG. 30.

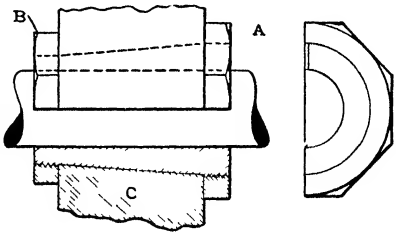


FIG. 31.

pull P_1. The resultant of these forces is P_2, and the arrangement of parts shown in Fig. 29 is correct for motion of rotation in either direction. If P_1 be reversed in direction the resultant of P_1 and P_2 will be P_3, and the arrangement is not correct for adjustment against wear, and very defective as far as lubrication is concerned, as the surface is broken near the point where the greatest film pressure should exist. Bearings of this form are often used in steam-engine work, and in such cases the force P_1 due to the steam pressure on the piston, is continually reversed in direction. Another adjustment for a similar case is shown in Fig. 30. Here the shoe or bottom " brass " can be raised up by introducing thin " shims," or liners, underneath it; lateral wear can be taken up by setting out the " cheek pieces " B, by means of the wedges D. Provision is thus made, by this arrangement, for taking up wear in all directions and keeping the shaft accurately aligned and located. For horizontal pressures in either direction the resultant P_3 passes close to the point at which the bearing is parted; and hence the best conditions for lubrication do not exist. Pressure films more or less perfect, depending on the oil supply, will form on the lower shoe, but the continual reversing of the lateral pressure, P, hardly allows time for the formation of pressure films on the cheeks. These reversals in pressure, however, allow the lubricant to be *carried* by the shaft, first under one cheek, and then under the other, thus lubricating them effectively.

Sometimes a bearing consists of a conical bushing split at some convenient place, as shown in Fig. 31. By releasing the nut A, and screwing up on B, the bushing may be forced into the frame C, thus closing the bore of the bushing slightly and compensating for wear. It is obvious that, once the bore of the bushing is worn eccentrically, no amount of taking up can rectify its shape; in fact, taking up wear in this manner tends to destroy the fit of the journal in the bearing. Occasionally the journal itself is made conical, and adjustment for wear is made by moving the shaft endwise. The application of such bearings is limited to short shafts, such as machine-tool spindles.

Machine bearings are made in many forms, depending on the location and service. The bearings are sometimes split into three pieces, and various other means of compensating for wear are used, but the fundamental principles outlined above, regarding the point where the bearing should be parted, apply to all forms.

Consider the last item (e, automatic adjustment). In long lines of shafting, which tend rapidly to get out of alignment, it is desirable that the bearing be so constructed as to adjust itself automatically to the changing position of the shaft, in order to avoid localized pressure, which would result in heating. In fast-running machinery, also, such as countershafts, dynamos, and motors, where perfect alignment is necessary, self-adjusting bearings have been found almost essential. Figure 32 shows a bearing of this kind as used in dynamo and motor bearings. The sleeve A has a spherical surface turned upon the outside, the center of the surface being at O. This surface engages with a similar surface bored in the outer casing B. The sleeve may swivel in any direction, but the center line of the shaft must always pass through O. If a shaft has only two bearings of this kind it is evident that perfect alignment can be secured, within the

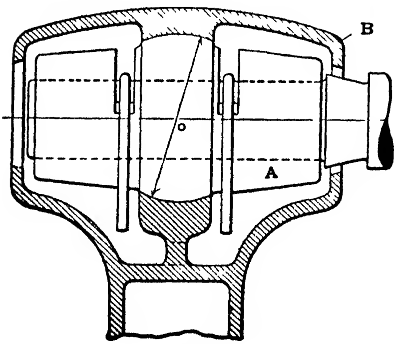


Fig. 32.

range of motion of the sleeves. Similar devices are used in long shafting, where many bearings are required. It is obvious that the fundamental principles regarding adjustment for wear, and maintenance of the oil film, apply to all bearings of this form also.

In constructing large and heavy revolving parts, it is not always possible to make the gravity axis of the structure coincide with the geometric axis, and when such a construction runs at high speed it tends to rotate around the gravity axis rather than the geometric axis, thus setting up vibrations. To obviate this difficulty, the bearings of high-speed machinery, such as steam turbines, are sometimes made up of several concentric tubes with very small clearance between them, the tubes being mounted in a rigid outer shell. These small clearances are filled with oil, thus forming an adjustable resistance to small lateral movements of the shaft and permitting the entire structure to find its true center of rotation.

51. Practical Construction of Bearings. It was shown in Article 45 that metals like brass, bronze, and the white alloys make excellent bearing surfaces for wrought-iron or steel journals, on account of their anti-friction and good wearing qualities. It is to be noted that even with perfect lubrication, where the character of the rubbing surfaces is less important *once the oil film is established,* care must be exercised in the selection of the material for the bearing surface, in order that abrasion may not occur *before* the film is formed or if the film should fail. A further advantage in having the bearing surface softer than the journal is that it is very desirable to have the journal maintain its form against wear, which it is more likely to do when rubbing against a soft surface than when rubbing against one harder than itself. The bearing itself should be rigid, so as to insure proper alignment of the shaft. Rigidity, against even moderate pressure, could not ordinarily be attained if the entire bearing member were made of the white alloys, and economy prohibits the use of brass and bronze for the entire bearing. It is customary, therefore, to make the main body of the bearing of cast iron (or sometimes a steel casting), and to fit into it wearing surfaces of the softer metals. These wearing surfaces may be either rigidly attached to the main castings or may be removable. In Fig. 28 is shown a bearing of the type commonly used for heavy shafts when the babbitt-metal lining is rigidly attached by means of dovetail shaped recesses, into which the babbitt is poured in a molten state. The necessary shrinkage due to cooling, which would leave the lining loose in the recesses, is usually overcome by hammering the babbitt, when cold, till it again fills the recesses, and then boring the babbitt to size. For cheap work the lining is often cast to size on a metal mandrel and no further work put upon it, but for all good work the bore of the lining is cast small enough to allow of hammering or peening, and then boring to a smooth surface Figure 29 shows removable linings of brass or bronze which are circular in section, and are prevented from turning when in place by the parting piece *B* This parting piece, or "liner," also permits taking up wear by reducing its thickness as occasion requires Figure 30 shows an arrangement of wearing surfaces common on horizontal steam-engine bearings The cap *C* is babbitted with some form of cheap metal since there is no wear upon it, all the pressure being either downward or sidewise The "quarter boxes" *B*, and the lower box or shoe *A*, may be of brass or bronze or of cast iron lined with babbitt Where there is danger of the boxes breaking, through pounding by the shaft, and where it is desired to use a babbitt metal, they may be made of brass or bronze and babbitt-lined When cast-iron wearing surfaces are used, and compensation for wear is important, as in the case of machine tools, it is customary to make the wearing surfaces removable, as indicated in Fig 29 For less accurate work the bearing surface is part of the main casting itself, machined to the required size Hardened-steel bearing surfaces are obtained by making circular shells or "bushings," of the required internal diameter, and of sufficient thickness to insure strength These bushings are forced into openings in the main casting and no provision for taking up wear is made If the forcing operation closes the bore of the bushing, it is "lapped" out with emery and oil to the required size Where the bearing must work under water, as a propeller shaft or the lower bearing of a vertical turbine water wheel, a lining of lignum vitae or other hard wood is often used The surrounding water furnishes the only lubricant necessary. A detailed description of the many arrangements of bearing surfaces is beyond the scope of this treatise

If the bearing must work under trying conditions, as on shipboard or in a heated room, and there is some question as to whether the heat of friction will be dissipated by radiation, the bearing is cast hollow so that water may be circulated around it thus carrying off the heat and maintaining the lubrication. In an emergency, water may be allowed to run over the outside of the bearing, accomplishing the same purpose. High-grade marine work, and large stationary-engine installations, are often equipped with a complete system of water circulation on the most important bearings.

Quite frequently the conditions of service render the wear so slight, and the adjustment to compensate for wear so unimportant, that the bearing can be made in one piece The bearings are then called **solid bearings,** and are usually lined with thin cylindrical **bushings,** of brass, babbitt metal, or some other bearing material, driven into the circular bore of the bearing. Usually the bushing is driven tight enough to prevent turning under the action of the journal, but sometimes pins or set screws are used to insure against any turning of the bushing in the bearing. Excessive wear is compensated for by replacing the worn bushing with a new one.

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52. Theoretical Design of Journals. The considerations affecting the design of any journal are one or more of the following:

(a) Strength to resist rupture.
(b) Rigidity, or stiffness, to prevent undue yielding.
(c) Maintenance of form against wear.
(d) Maintenance of lubrication.
(e) Radiation of the heat due to frictional resistance.

The first two considerations, strength and rigidity, are covered by the general principles laid down in Chapter III, and are more fully considered in Chapter VII, where the special problems in connection with shafts are discussed. Economy of material dictates that the shaft be of the *minimum* diameter consistent with the applied bending and twisting moments.

The third consideration (c) particularly affects such journals as those on the spindles of grinding machines and machine tools generally, where the accuracy of the product depends on the accuracy of the journals. Usually, in such cases, the wearing surface must be so great, in order to reduce the wear to an inappreciable amount, that the consideration of strength does not enter into the computations.

The considerations (d) and (e) are closely correlated. It was shown in Articles 41 and 42 that, if the unit bearing pressure on the journal is not too great, the lubricant, because of its viscosity, may be drawn in between the journal and the bearing, thereby reducing the frictional resistance. This frictional resistance can never be reduced to zero even with perfect lubrication. The energy thus absorbed appears as heat, and is radiated to the surrounding air by the metallic surfaces of the bearing, the temperature of which rises till the rate of radiation equals that at which heat is being generated. In well-designed machinery the temperature of the bearing should not exceed 150° F. The raising of the temperature of the bearing has a tendency to lower the viscosity of the lubricant, and if the bearing becomes too hot, the lubricant becomes so thin that the pressure squeezes it out completely, and failure of the bearing by abrasion occurs. It is evident, therefore, that a journal of given dimensions may carry a given load very satisfactorily under certain conditions, and fail absolutely under others, the same lubricant being used in each case. The consideration of the proper radiation of the heat generated is, therefore, most important. It may be assumed, without serious error, that the rate of radiation of heat is proportional to the projected area of the bearing. The number of heat units which will be radiated from a unit of surface, at any given difference in temperature between the bearing and the surrounding air, is a fixed quantity for any set of conditions; and if the heat of friction per unit area is greater than can be radiated at the desired bearing temperature, the temperature of the bearing must rise till equilibrium is obtained. It follows, therefore, that for any desired bearing temperature the work of friction per unit of projected area of bearing must not exceed the rate of radiation per unit of projected area, or

$$\mu p V = K \quad \text{or} \quad pV = \frac{K}{\mu} \tag{1}$$

where μ is the coefficient of friction, p the load in pounds per unit of projected area, V the velocity of rubbing in feet per minute, and K the rate of radiation per unit of projected area in foot-pounds per minute, to be determined experimentally.

It is to be especially noted that, if μ be considered as constant, increasing the diameter of a journal (the number of revolutions and the total load remaining constant) does not materially affect the development or dissipation of heat, since the velocity of rubbing is increased in the same ratio as radiating surface is increased. If, however, the bearing be lengthened, the radiating surface is increased and the work of friction remains unchanged, with the same total load as before. This last statement, true for imperfectly lubricated surfaces, is only approximately true for bearings with perfect lubrication, as will be seen presently.

53. Radiating Capacity of Bearings. The amount of heat which will be radiated from a bearing has been experimentally determined by Lasche.* The curves shown in Fig. 33 are those shown in his Fig. 57, transformed into English units, and with the scale of radiation further modified so as to read in foot-pounds per square inch of projected area per minute, instead of per square inch of actual bearing surface. Curve 1 represents actual exper mental results, with bearings of the usual proportions, in still air. Curve 2 is for bearings which are connected to large iron masses, or which are ventilated by air currents. Curve 3 was calculated from theoretical considerations. It gives the radiation

* See *Traction and Transmission*, January, 1903, page 52. See also "Performance of Oil-Ring Bearings" by G. B. Karelitz, *Trans.* A.S.M.E., 1930.

from a very thin bearing or sleeve and indicates that radiation is more effective as the bearing becomes thicker, as might be expected, for metal is a better conductor of heat than air, and hence the thick bearing more easily carries the heat away to a greater radiating surface. The

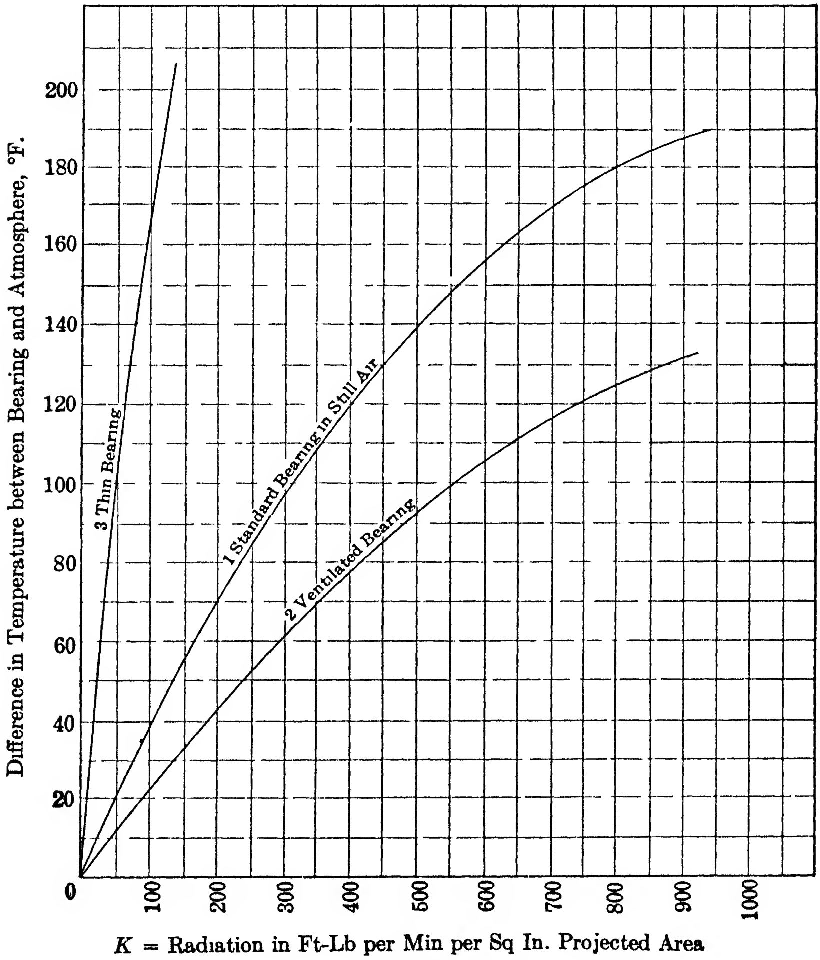

FIG. 33.

values obtained from these curves may therefore be used for K in equation (1). Lasche points out that, though these experiments represent only a limited variety of conditions, they are probably on the safe side and will serve at least as a very useful check in designing.

If, in designing a journal, the value of μ can be determined, equation

(1) and Fig. 33 give the relations which must exist between the velocity and pressure in order that the safe bearing temperature may not be exceeded; or if the pressure and velocity are fixed by other circumstances, Fig. 33 indicates whether radiation must be assisted by artificial means, such as water circulation or currents of air.

If the work of friction cannot be radiated by the bearing and connected parts, cooling oil may be pumped into the bearing under pressure

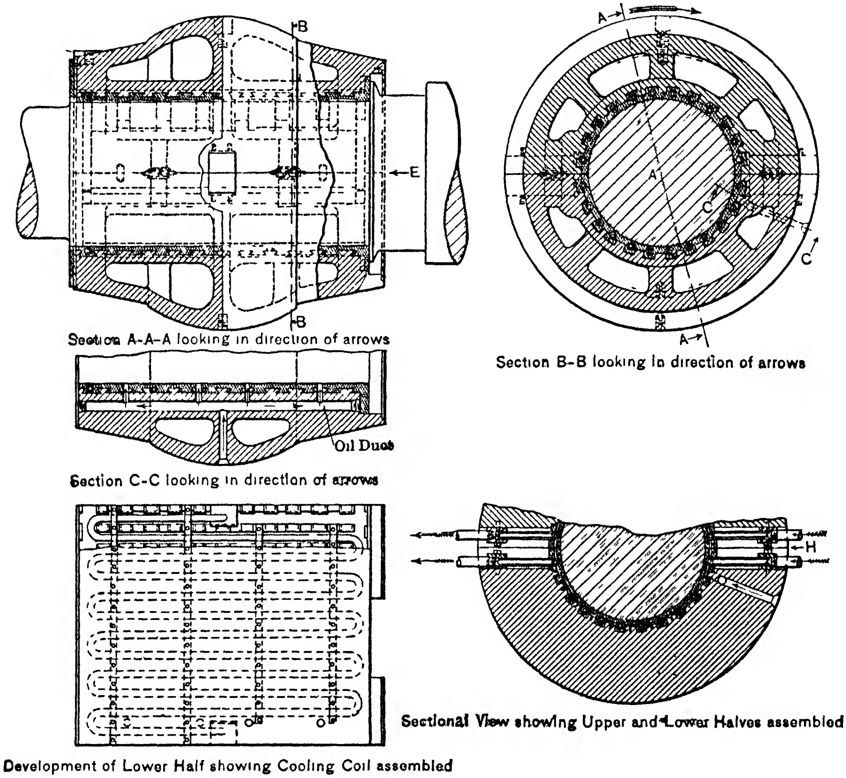

Fig. 34.—Turbo-generator Bearing.

thus, carrying away the excess heat by oil circulation. This method has been used even where flooded lubrication was not necessary, because of excessive bearing pressure. Care should be taken in such cases that the oil is delivered to the areas where the heat is generated, in order to reduce the necessary amount of oil to a minimum. Since the specific heat of oil is known, or can be determined, it is possible to compute the amount of cooling oil that must be delivered to a given bearing if the work of friction is known.

The most efficient method of carrying away the excess heat from a bearing is by a water jacket, as described in Article 51. Water circulation is usually obtained by casting water passages directly in the cap and bearing proper, through which water can be circulated under pressure.

It is obvious that these water passages should lie as close to the rubbing surfaces as possible. Figure 34 shows a bearing constructed by the General Electric Company for large turbo-generators. In this bearing a copper cooling coil is embedded in the babbitt lining, thus bringing the circulating water very close to the source of heat. Provision is made also for circulating the oil so as to insure perfect film lubrication. This bearing, therefore, represents very advanced practice for conditions where a large amount of heat is to be dissipated. A publication of the General Electric Company states that, with good water jacketing, one gallon of water will carry away from 3 to 5 horsepower (127.3 to 212.2 thermal units per minute), and where the jacketing is carefully arranged, as in Fig. 34, from 10 to 15 horsepower can be carried away by a gallon of water. For bearings cooled by forced circulation and cooling of the oil, from 0.05 to 0.15 gallon of oil per minute for each square inch of bearing surface represents average practice.

54. Imperfectly Lubricated Journals. It has been shown in Articles 41 and 42 that the value of μ, for imperfectly lubricated surfaces, is a very variable quantity, even for the same simultaneous values of velocity and pressure. Not only does it vary with velocity, pressure, and temperature, but the regularity of the oil supply (over which the designer has little control) affects it much more seriously. Furthermore, bearings running under the same nominal load and velocity give widely different values of frictional resistance and temperature rise, depending on whether the load is constant or intermittent, or whether the motion is steady or vibratory, etc. Notwithstanding this, equation (1) may be made to serve as a useful check in doubtful cases by assuming a safe value of μ.

The assumption is sometimes made that μ is a constant; and formulas of the form

$$pV = \frac{K}{\mu} = C$$

where C is a constant that has been determined from practice, are much used. Thus if p be expressed in pounds per square inch of projected area and V in feet per minute, Mr. Fred W. Taylor * gives for mill work $C = 24{,}000$, and says that $C = 12{,}000$ is not safe for cast-iron bearings

* *Trans.* A.S.M.E., vol. 27.

with ordinary lubrication. If the rise of temperature in the bearing be taken as 75° and μ be taken as 0.015, which is ordinarily a safe value, then from Curve 1, Fig. 33, $K = 222$, whence

$$C = \frac{K}{\mu} = \frac{222}{0.015} = 15{,}000$$

From Curve 2, $K = 384$, whence for ventilated bearings

$$C = \frac{384}{0.015} = 25{,}600$$

These values agree with Mr. Taylor's limits better than would be expected.

All formulas of this empirical form must be considered, so far as imperfectly lubricated journals are concerned, as applying only to the conditions and range for which they have been found true, and for which μ is apparently constant. This is more evident when the wide variation of the value of such constants as determined by practice is considered. Thus, Mr. H. G. Reist gives, as the practice of the General Electric Company on generator bearings with ordinary ring lubrication, a limiting value of $C = 50{,}000$ for bearing pressures from 30 to 80 lb per sq in. Mr. H. P. Been gives the practice of one of the largest Corliss engine builders as $C = 60{,}000$ to 78,000 for bearing pressures not higher than 140 lb per sq in.

The great variation in these values of C is no more than might be expected in view of the foregoing, and also in view of the difference in lubrication and in radiating capacities of bearings, due to material, form, and location. Therefore, although these coefficients may form a guide, and although doubtful cases may be checked for heating by equation (1), care should be exercised that the bearing pressure is kept within the limits which will admit of good lubrication. The allowable bearing pressures as fixed by practice for various classes of machines are given in the following table, and it may be noted that these are *more accurately known than the values of μ, or the values of the coefficient of radiation, K.*

Economy in the use of material and the importance of minimizing the work of friction suggest that the diameter of the journal shall be as small as is consistent with strength and stiffness. With the diameter of the journal determined by these considerations, it is evident that the length of the journal must be such that the bearing pressure is within the allowable limit. It may be, however, that the length of the journal thus determined will be so great that localized pressure may result;

or it may be that the type of machine will not allow space enough for such a length of bearing In such cases the diameter must be made larger and the length may be correspondingly decreased.

TABLE XII

BEARING PRESSURES FOR VARIOUS CLASSES OF BEARINGS

Class of Bearing and Condition of Operation		Allowable Bearing Pressure in Pounds per Square Inch
Bearings for very slow speed as in turntables in bridge work		7000 to 9000
Bearings for slow speed and intermittent load as in punch presses		3000 to 4000
Locomotive wrist pins		3000 to 4000
Locomotive crankpins		1500 to 1700
Locomotive driving journals		190 to 220
Railway car axles		300 to 325
Marine engine main bearings	Naval practice	275 to 400
	Merchant practice	400 to 500
Marine engine crankpins		400 to 500
Stationary high-speed engine main bearings	For dead load *	60 to 120
	For steam load	150 to 250
Stationary high-speed engine crankpins	Overhung crank	900 to 1500
	Center crank	400 to 600
Stationary high-speed engine wrist pins		1000 to 1800
Stationary slow-speed engine main bearings	For dead load *	80 to 140
	For steam load	200 to 400
Stationary slow-speed engine crankpins		800 to 1300
Stationary slow-speed engine wrist pins		1000 to 1500
Gas engines, main bearings		500 to 700
Gas engines, crankpins		1500 to 1800
Gas engines, wrist pins		1500 to 2000
Heavy line shaft brass or babbitt lining		100 to 150
Light line shaft cast-iron bearing surfaces		15 to 25
Generator and dynamo bearings		30 to 80

* In horizontal engines the dead load which consists of the weight of the shaft, flywheels, etc governs the design of the lower bearing surfaces while the steam load governs the design of the quarter brasses In a vertical engine the sum of the dead load and the live load comes upon the lower bearing surface

Although practice shows wide variations, it is found that the ratio of the length of the journal to its diameter (l/d) is fairly well defined for any given class of machinery. It often occurs, therefore, that, when journals are designed with the ratio as fixed by practice, they have an excess of strength while barely satisfying the conditions as to bearing pressure.

The following are average values of l/d as found in good practice:

TABLE XIII

Type of Bearing	Values of l/d
Marine engine, main bearings	1 to 1 5
Marine engine, crankpins	1 to 1 5
Stationary engine, main journals	1 5 to 2 5
Stationary engine, crankpins	1
Stationary engine, crosshead pins	1 to 1 5
Ordinary heavy shafting with fixed bearings	2 to 3
Ordinary shafting with self-adjusting bearings	3 to 4
Generator bearings	3

55. Summary. From the foregoing the following statements may be made regarding **imperfectly lubricated journals**:

(**a**) The minimum diameter of a journal is fixed by the considerations of **strength** and **stiffness** under the loads applied.

(**b**) The smaller the diameter of the journal for a given coefficient of friction, the less is the work of friction and consequent liability to heating.

(**c**) The tendency of the bearing to heat, other things being equal, is not materially affected by changing the diameter of the journal, but is reduced by increasing the length.

(**d**) The projected area of the journal must be such that the bearing pressure will be kept within the allowable limits for the particular conditions; and the ratio of length to diameter must not be so great that severe localization of bearing pressure can result. These considerations may require a larger bearing than the previous requirements alone would demand.

(**e**) The work of friction, per unit area, must not exceed the rate of radiation, per unit area, for the allowable bearing temperature.

56. Perfectly Lubricated Journals. It was shown in Article 42 that if a journal is supplied with sufficient lubricant, of proper viscosity, and proper distribution axially is provided, the journal itself may draw in the lubricant till a film is formed under such a pressure that the load will be entirely fluid-borne. With any given set of conditions, therefore, and perfect lubrication, a definite journal velocity will permit the carrying of a definite load per unit area upon the journal, and once the relation is established between the load, velocity, temperature, and coefficient of friction, it is constant, and not unstable, as in the case of imperfectly lubricated surfaces.

It was further shown that the following statements are true regarding perfectly lubricated surfaces:

(a) The friction of perfectly lubricated surfaces for a given velocity depends very little on the materials which form the rubbing surfaces, but does depend largely on the character of the lubricant.

(b) The **frictional resistance** of perfectly lubricated journals for any given velocity is, within the limits of pressure under which the oil film may be maintained, independent of the pressure (that is, μp = a constant).

(c) The coefficient of friction for perfectly lubricated bearings varies inversely with the temperature of the bearing.

Perfect lubrication can be attained in horizontal bearings in a limited number of ways. The first is by bath lubrication as used by Tower

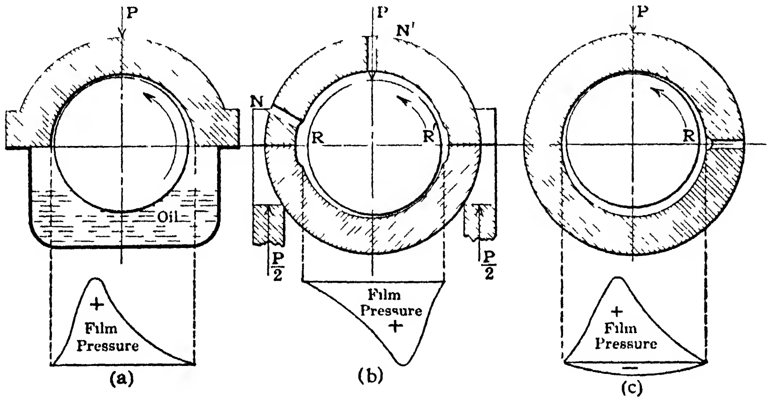

Fig. 35.

and illustrated in Fig. 35 (a), the load P resting upon the bearing as indicated and the shaft thrusting an oil film between itself and the bearing. The general character of the pressure curves is as indicated, the pressure falling to zero at the lower edges of the bearing both at ends and sides. (See also Fig. 17.) In the second and more usual case the load on the shaft is carried on the lower half of the bearing, as in Fig. 35 (b). Oil either is supplied plentifully at the point N on the *on* side where the pressure is zero or is applied plentifully at N' and conducted by ample oil grooves to the longitudinal distributing channels at R. The cap usually has liberal oil grooves cut diagonally in its surface so that no film is likely to form and the pressure curve is approximately as shown. In the third case, Fig. 35 (c), the bearing is a solid sleeve, the oil pressure curve that supports the load being approximately as shown, but there may be a negative pressure on the lower side of the shaft. Oil is supplied at R, the point of zero pressure, or it is sometimes fed to the lower side from the end of the bearing. In each and every case there must be a plentiful supply of the lubricant to obtain a complete fluid film between shaft and bearing. It will be noted that in each case in Fig. 35 once the film is formed the journal is no longer concentric but is somewhat nearer the bearing slightly past the line of the load, thus reducing the clearance at that point and marking the place of greatest pressure. In the illustrations of Fig. 35 the clearance between shaft and bearing is greatly exaggerated for clearness.

Since the load is entirely fluid-borne the laws of fluid friction should apply, and this is found to be the fact. Much work has been done upon the mathematical theory of lubrication, beginning with the masterly work of Osborne Reynolds,* who verified Tower's experimental work mathematically. These mathematical deductions, though they have illuminated the general theory of lubrication, are quite complex, and in general their results have not been reduced to such forms as to make them readily applicable by practical designers. A brief bibliography of the more important contributions of this sort is given at the end of this chapter. More recent experimental work has, however, aided greatly in determining values of the coefficient of friction and also in determining the effect of certain variables that are met with in bearing design.

From the theory of perfect lubrication (Article 37) it will be clear that the coefficient of friction in perfect lubrication will vary directly with the viscosity of the lubricant, the dimensions of the journal, and the number of revolutions of the shaft. It will vary inversely with the load and the clearance between journal and bearing. It is most convenient to express this last as the clearance per unit of diameter or c/d. Then if d = diameter, l = length, N = number of revolutions per minute, z = the absolute viscosity of the lubricant, and p the load per square inch $= P/ld$,

$$\mu \text{ varies as } \frac{zNdl}{P\frac{c}{d}} \text{ or as } \frac{zNdl}{pdl\frac{c}{d}} \text{ or as } \frac{zN}{p\frac{c}{d}} \text{ or as } \frac{zN}{p}\left(\frac{d}{c}\right) \qquad (2)$$

For any given bearing c/d is a constant, and if therefore experimental values of μ and zN/p are plotted as in Fig. 36 a characteristic curve is established for all bearings of the same proportions. It will be noted that beyond the critical point M where the film first forms the locus of μ

* *Philosophical Transactions,* Part 1, 1886.

is a straight line. If all other conditions are kept constant and the clearance is varied then the simultaneous values for different ratios of c/d are as shown in Fig. 37. These diagrams are reproduced from the work of Messrs. S. A. McKee and T. R. McKee,* upon small bearings

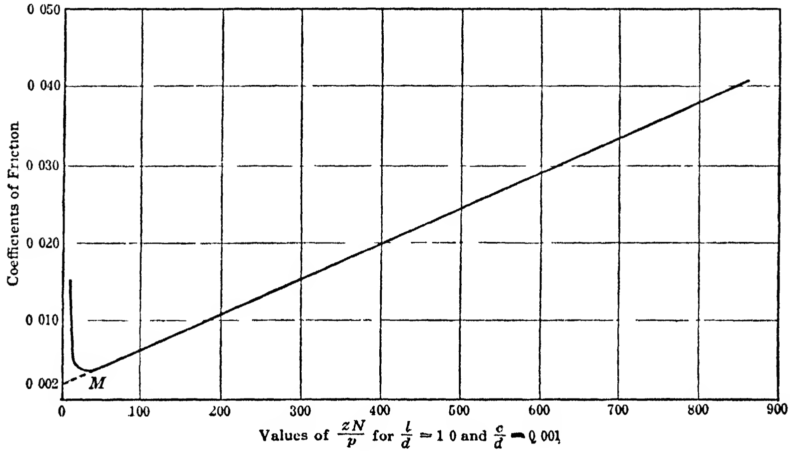

FIG. 36.

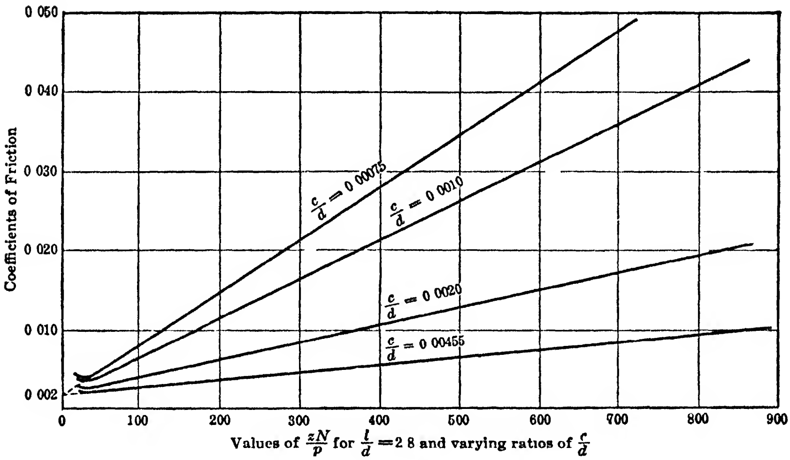

FIG. 37.

in which the ratios of l/d were 0.25; 0.50; 0.75; 1.00; and 2.8. The effect of end leakage was quite marked with ratios of l/d below unity, but above that value the effect of length was small. Thus the line for

* See *Trans.* A.S.M.E., 1929, page 161.

$c/d = 0.001$ in Fig 36 gives practically the same values of the coefficient as the corresponding line in Fig 37 Yet the value of l/d is unity in Fig 36 and 2 8 in Fig 37

The Messrs McKee also determined the constants for equation (2) when d/c is also a variable to be

$$\mu = 0.002 + \frac{473}{10^{10}}\left(\frac{zN}{p}\right)\frac{d}{c} \tag{3}$$

Although these results are very helpful in visualizing the effect of several factors upon the coefficient of friction, as the Messrs McKee state they should not be taken too literally The bearings experimented upon were small, carefully finished, and comparatively lightly loaded, thus eliminating the disturbing factors usually found in larger bearings such as distortion of the shaft and roughness and inaccuracies in the bearing The speed of rubbing was also moderate Furthermore, the temperature was kept constant by circulating water through the bearing, thus controlling the viscosity of the lubricant Criticism of similar character applies to most of the theoretical discussions based upon experimental work Many experiments have been conducted as these were under ideal conditions and upon complete cylindrical bearings with unbroken surfaces such as Fig 35 (c), which is not the most common form met with in practice Within the range of the experimental work on which they are based the results are no doubt significant and illustrate how in any given case, especially where a large number of bearings are to be built, quite accurate predictions as to performance can be made It may be well therefore to consider some of the results of practical experience in designing perfectly lubricated bearings

Referring to Article 42, it has been noted that the relation between the coefficient of friction of perfectly lubricated surfaces, the unit load, the velocity and the temperature, for velocities up to 500 ft per min, can be expressed closely by the equation

$$\mu = \frac{2.3\sqrt{V}}{p(t - 32)} \tag{4}$$

and for velocities above 500 ft per min the relation between the coefficient of friction, the unit load, and the temperature can be expressed closely by the equation

$$\mu = \frac{51.2}{p(t - 32)} \tag{5}$$

these relations being practically independent of the velocity.

Equations (4) and (5) may be written

$$\mu p V = \frac{2.3 V^{1/2}}{t - 32} \tag{6}$$

and

$$\mu p V = \frac{51.2V}{t - 32} \tag{7}$$

where V is the velocity of rubbing in feet per minute. Since $\mu p V$ is the frictional loss per unit of projected area in foot pounds per minute, equations (6) and (7) may be used to compute the heat that a perfectly lubricated journal must radiate per square inch of projected area per minute, if a temperature of t is to be maintained.

For the relation between unit pressure and velocity in feet per minute that will insure the formation of an oil film, the practice of the General Electric Company, as expressed in equation * (8) which follows, may be safely adopted, if it be remembered that the expression includes a factor of safety of 2.

$$p = 15.5\sqrt[3]{V} \tag{8}$$

The radiating capacity of bearings may be taken from Fig. 33.

The equations listed in this section established the relations that must exist between *the unit pressure* on the bearing and the velocity and temperature, if perfect lubrication is to be maintained. It should be carefully noted, however, that, in general, these equations are not sufficient for the design of bearings. The *minimum diameter* of a bearing is fixed by the bending and twisting moments applied to it, and there is no definite relation between these moments and the direct load on the bearing, each case being an independent problem. It should be remembered also that the relations between diameter and length of bearing established in Article 56 apply to perfectly lubricated bearings. Aside also from considerations of strength and rigidity, it is important to determine the minimum diameter of the bearing in order to reduce frictional resistance to a minimum, for it should be noted that with perfect lubrication the product μp for any velocity is a constant quantity. It follows, therefore, that for any given total load P, the unit bearing pressure should be kept as high as possible, provided it does not exceed the maximum allowable value for the given conditions. For if the unit bearing pressure is decreased, by increasing either the diameter or length of the bearing, the coefficient of friction is correspondingly increased; hence the total frictional resistance μP is also increased. Care should,

* See also Article 42 and Fig. 20.

of course, be exercised in any case that the heat of friction is properly carried away. The examples that follow in Article 59 may serve to make the application of all of the foregoing theory more definite.

57. The Limiting Values of Pressure and Velocity. The limiting values of the pressure and velocity under which a perfect oil film can be maintained, at high velocities, have not been fully determined. In Lasche's experiments a load of 213 lb per sq in. of projected area was carried at a velocity of 1968 ft per min. In Kingsbury's * experiments, loads from 80 to 86 lb per sq in. were repeatedly carried at velocities up to 1990 ft per min. In both Kingsbury's and Lasche's work either the oil circulation, or the bearing itself, was artificially cooled, thus materially assisting the radiation.

The values given by these experiments were obtained on experimental machines and may be looked upon as limiting values. Successful practice in the design of steam turbine bearings gives velocities ranging from 1800 to 3000 ft per min, with pressures inversely as the velocity ranging from 80 to 50 lb per sq in. Where the pressure is as high as 90 lb per sq in., it is found that the velocity must be kept below 1800 ft † per min. The empirical equation $pV = 150{,}000$ is much used for this class of work, and gives values agreeing with those just quoted. It is evident that with these high velocities the radiation must be assisted. Thus let $V = 2000$ and $p = 75$ in accordance with the empirical rule just given, and let it be required to keep the temperature t at 150° F or a temperature of say 75° F above the atmosphere.

Then by equation (7) the frictional work is

$$\mu p V = \frac{51.2 \times 2000}{(150 - 32)} = 867 \text{ ft-lb per min}$$

whereas the bearing alone, if connected to a heavy iron frame, will, from Curve 2, Fig. 33, radiate only 384 ft-lb per min. Since the specific heat of both water and oil are known, the supply of either necessary to carry off the excess heat of friction can be calculated.

58. Bearing Clearance and Film Thickness. In Tower's experiments the bearing was free to move away from the journal as the oil film was built up, and the film, therefore, could assume the full thickness consistent with the conditions. It will be obvious, however, that if the bearing surrounds the journal completely and fits it very closely, the full film thickness may not be formed. It is important, therefore, that sufficient clearance be allowed between the bearing and the journal for this purpose. Small bearings are sometimes fitted very closely and

* *Trans.* A.S M.E., vol. 27, page 425.

† See "Steam Turbines," by Frank Foster, page 181.

yet operate well if liberally oiled. In such cases the film pressure is probably great enough to spring the cap or stretch the bolts sufficiently to permit of lubrication, whereas in large bearings such results would not be attained. It has been shown experimentally and verified mathematically that the friction increases as the clearance is decreased. Careful designers, therefore, specify what the clearance shall be and do not trust to chance. The following table shows the practice of the General Electric Company in this respect and shows also the allowable variation or tolerance from the nominal dimensions:

59. Examples of Journal Design. Journals generally form an integral part of a shaft or spindle, and the determination of the stresses acting upon them is a part of the solution of the stresses in the shaft itself. It is desirable, however, to point out some of the special features of journal design.

The actual distribution of pressure over the length of a journal is not known; but there is every reason to believe that the distribution

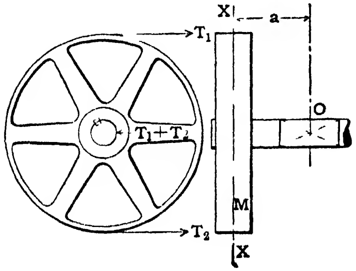

Fig. 38.

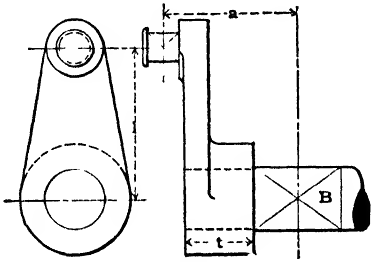

Fig. 39.

is fairly uniform. Thus bearings, as a rule, wear quite uniformly over their entire length, where fair alignment is maintained. It is customary, in the absence of exact data, to assume for computations as to strength and rigidity that the load on the journal is concentrated at the middle of its length. This assumption is on the safe side, and will sometimes give shaft diameters excessively large so far as strength is concerned.

The following examples (a, b, c, and d) illustrate the most important cases of journal design.

Example (a). This case is illustrated in Fig. 38. Here the center of the bearing is fixed at O, by the construction of the machine. The center line of the pulley M is also fixed at XX, by the location which the belt must occupy, so that the pulley overhangs the bearing by the distance a. The diameter of the pulley d is 40 in., a = 10 in., the pull on the tight side of the belt is 500 lb, and the pull on the slack side is 300

TABLE XIV

Clearances and Limits for Journals and Bearings

Nominal Dimensions in Inches	Journal		Bearing						Axle Linings for Railway Motors	
			Horizontal		Vertical		Step			
	Maximum Diameter Inch	Allowable Variation Below Maximum Diameter	Minimum Bore Inch	Allowable Variation above Minimum Bore	Minimum Bore Inch	Allowable Variation above Minimum Bore	Minimum Bore Inch	Allowable Variation above Minimum Bore	Minimum Bore Inch	Allowable Variation above Minimum Bore
⅜	0.375	0.0005	0.377	0.001	0.376	0.001	0.3755	0.0005	0.380	0.004
½	0.500	0.0005	0.502	0.001	0.501	0.001	0.5005	0.0005	0.505	0.004
⅝	0.625	0.0005	0.627	0.001	0.626	0.001	0.6255	0.0005	0.630	0.004
¾	0.750	0.0005	0.752	0.001	0.751	0.001	0.7505	0.0005	0.755	0.004
⅞	0.875	0.0005	0.877	0.001	0.876	0.001	0.8755	0.0005	0.880	0.004
1	1.000	0.0005	1.002	0.001	1.001	0.001	1.0005	0.0005	1.005	0.004
1⅛	1.125	0.0005	1.128	0.001	1.127	0.001	1.126	0.0005	0.130	0.004
1¼	1.250	0.0005	1.253	0.001	1.252	0.001	1.251	0.0005	1.255	0.004
1½	1.500	0.0005	1.505	0.001	1.502	0.001	1.501	0.0005	1.505	0.004
1¾	1.750	0.0005	1.753	0.001	1.752	0.001	1.751	0.0005	1.755	0.004
2	2.000	0.0005	2.003	0.001	2.002	0.001	2.001	0.0005	2.005	0.004
2¼	2.250	0.0005	2.253	0.001	2.252	0.001	2.251	0.0005	2.255	0.004
2½	2.500	0.0005	2.503	0.001	2.503	0.001	2.501	0.0005	2.505	0.004
2¾	2.750	0.0005	2.751	0.002	2.753	0.002	2.7515	0.0005	2.755	0.004
3	3.000	0.0005	3.004	0.002	3.003	0.002	3.0015	0.0005	3.005	0.004
3½	3.500	0.001	3.504	0.002	3.504	0.002	3.5015	0.0005	3.507	0.004
4	4.000	0.001	4.005	0.002	4.004	0.002	4.002	0.001	4.007	0.004
4½	4.500	0.001	4.505	0.002	4.504	0.002	4.502	0.001	4.509	0.004
5	5.000	0.001	5.006	0.002	5.005	0.002	5.0025	0.001	5.009	0.004
5½	5.500	0.001	5.507	0.002	5.505	0.002	5.503	0.001	5.511	0.004
6	6.000	0.001	6.009	0.002	6.00	0.002	6.003	0.001	6.011	0.004
7	7.000	0.001	7.011	0.002	7.006	0.002	7.0035	0.001	7.012	0.004
8	8.000	0.001	8.012	0.003	8.006	0.003	8.004	0.002	8.013	0.004
9	9.000	0.001	9.013	0.004	9.007	0.004	9.0045	0.002		
10	10.000	0.0015	10.014	0.005	10.007	0.005	10.005	0.002		
11	11.000	0.0015	11.015	0.005	11.008	0.005	11.0055	0.002		
12	12.000	0.0015	12.016	0.005	12.008	0.005	12.006	0.002		
13	13.000	0.0015	13.016	0.005	13.009	0.005	13.0065	0.002		
14	14.000	0.0015	14.016	0.005	14.009	0.005	14.007	0.002		
15	15.000	0.0015	15.016	0.005	15.010	0.005	15.0075	0.002		
16	16.000	0.0015	16.016	0.005	16.010	0.005	16.008	0.002		
17	17.000	0.0015	17.018	0.005	17.011	0.005	17.008	0.002		
18	18.000	0.0015	18.018	0.005	18.011	0.005	18.008	0.002		
19	19.000	0.0015	19.018	0.005	19.012	0.005	19.008	0.002		
20	20.000	0.0015	20.018	0.005	20.012	0.005	20.008	0.002		
22	22.000	0.002	22.020	0.008	22.013	0.005	22.008	0.002		
24	24.000	0.002	24.020	0.008	24.013	0.005	24.008	0.002		
26	26.000	0.003	26.020	0.008						
28	28.000	0.003	28.022	0.008						
30	30.000	0.003	30.022	0.008						
32	32.000	0.003	32.024	0.010						
34	34.000	0.003	34.024	0.010						
36	36.000	0.003	36.024	0.010						

lb. It is required to determine the dimensions of the journal. Lubrication is to be by ordinary oil cup.

The stresses induced in the journal are torsional stress, due to the twisting moment $(T_1 - T_2)\frac{d}{2}$, and flexural stress due to the bending moment $(T_1 + T_2)a$. The journal is subjected to combined bending and twisting.

Formula (K_2) or (K_3) (page 93), therefore, applies.

The bending moment

$$M = (T_1 + T_2)a = (500 + 300)10 = 8000$$

The twisting moment

$$T = (T_1 - T_2)\frac{d}{2} = (500 - 300)20 = 4000$$

Hence

$$\frac{T}{M} = \frac{4000}{8000} = \frac{1}{2}$$

and taking

$$\frac{s_t}{s_s} = 1.25$$

it is found from Fig. 9 that equation (K_2) applies.

$$\therefore\ M_e = \frac{M}{2}\left[1 + \sqrt{1 + \left(\frac{T}{M}\right)^2}\right] = \frac{8000}{2}\ [1 + \sqrt{1 + (\tfrac{1}{2})^2}] = 8480 \text{ in-lb}$$

From equation (J), page 93,

$$M_e = \frac{sI}{c} = \frac{s\pi d^3}{32}$$

or

$$d^3 = \frac{32M_e}{\pi s} = \frac{32 \times 8480}{\pi \times 10{,}000} = 8.63$$

$$\therefore\ d = 2.05 \text{ in.,} \quad \text{or say} \quad 2\tfrac{1}{4} \text{ in.}$$

If the length of the bearing be taken at 7 in. (see Table XIII), the bearing pressure will be

$$\frac{T_1 + T_2}{2\frac{1}{4} \times 7} = \frac{500 + 300}{15.75} = 50 \text{ lb, nearly}$$

which is a safe value.

If the number of revolutions be 300 per min, and μ be taken as 0.015, the work of friction per unit of projected area will be

$$50 \times 0.015 \times 300 \times \frac{\pi \times 2.25}{12} = 133 \text{ ft-lb per min}$$

From Curve 1, Fig. 33, it is seen that to radiate this amount of energy the temperature of the bearing will rise about 50° above the surrounding air. This is a safe value and the design is satisfactory.

Example (b). Let the line of action of the load pass through the center line of the journal, as in the steam-engine crankpin in Fig. 39. Let the length of the crank be 18 in., and the total maximum pressure on the crankpin be 25,000 lb. What should be the dimensions of the crankpin in order to be safe against rupture and overheating?

Referring to Table XIII, it is seen that journals of this character are short compared to their diameter, and hence are usually strong enough and stiff enough if designed for a bearing pressure low enough to prevent overheating. Let l/d be taken as 1.25. From Table XII, it is seen that 900 lb per sq in. may be safely carried on this type of pin. If d be the diameter of the pin and l the length, then the projected area of the pin is $l \times d = 1.25d \times d = 1.25d^2$. Whence

$$1.25d^2 \times 900 = 25{,}000$$

or

$$d^2 = 22.2$$

$$\therefore\ d = 4.7 \quad \text{or} \quad \text{say 5 in.}$$

and

$$l = 5 \times 1.25 = 6.25 \text{ in.}$$

The pin may now be checked for strength. In a short pin of this kind it is more accurate to assume the load uniformly distributed along the pin, than to assume it as concentrated at the middle. The pin may, therefore, be considered as a cantilever uniformly loaded with a load $W = 25{,}000$.

Whence, from Table I, case 3, the maximum bending moment

$$M = \frac{Wl}{2} = \frac{25{,}000 \times 6.25}{2} = 78{,}125 \text{ in-lb}$$

Therefore, from equation (J), page 93,

$$M = \frac{s_t I}{c} \text{ or } s_t = \frac{Mc}{I} = \frac{32M}{\pi d^3} \text{ or } s_t = \frac{32 \times 78{,}125}{\pi \times 5^3} = 6400 \text{ lb, nearly,}$$

which is a safe value.

In a similar way the pin may be checked for deflection, if desired, by means of case 3, Table I.

Example (c). Sometimes the location of the bearing is dependent on the diameter of the shaft, which is unknown, and in such case a tentative method must be adopted. Thus in Fig. 39 neither the length of the bearing B, nor the thickness of the crank hub t, can be definitely decided upon till something is known about the diameter of the journal. The diameter must therefore be assumed, and then checked by the equations which apply. Usually a close estimate can be made from existing machines of similar type. In the case of the steam-engine shaft, for example, it is known that the main journal is frequently about one-half the diameter of the cylinder. The data taken in example (**b**) correspond to a cylinder diameter of about 18 in., and the journal diameter may therefore be assumed as 9 in. From Table XIII, the length of the journal may be taken as 20 in. The length of the hub should be at least 8 in., for this diameter. The boss under the pin may be taken as $\frac{3}{8}$ in. in height, and since the pin, from case (**b**), is 6.25 in. long, the total distance from the middle of the crankpin to the middle of the bearing may be assumed as $21\frac{1}{2}$ in. The projected area of the journal is $9 \times 20 = 180$ sq in., which gives a bearing pressure of $25{,}000 \div 180 = 140$ lb per sq in.; and from Table XII, it is seen that this is a safe value as far as the load due to steam pressure is concerned. If the shaft also carries a heavy flywheel this must be taken into account (see Chapter VII).

The stresses induced in the journal are of the same character as in case (**a**). Taking the length of crank $l = 18$ in., and the pressure on the pin $= 25{,}000$ as before, then the bending moment

$$M = 25{,}000 \times 21\tfrac{1}{2} = 537{,}500 \text{ in-lb}$$

the twisting moment

$$T = 25{,}000 \times 18 = 450{,}000 \text{ in-lb}$$

whence

$$\frac{T}{M} = 0.8375$$

and taking $s_t/s_s = 1.25$, equation (K_2) is found by Fig. 9 to apply to the case.

$$\therefore\ M_e = \frac{M}{2}\left[1 + \sqrt{1 + \left(\frac{T}{M}\right)^2}\right] = \frac{537{,}500}{2}[1 + \sqrt{1 + 0.8375^2}]$$

$$= 616{,}500 \text{ in-lb}$$

From (J) [as in Example (a)],

$$s' = \frac{32M_t}{\pi d^3} = \frac{32 \times 616{,}500}{\pi \times 9^3} = 8600$$

which is a safe value and the design is satisfactory, so far as strength is concerned.

If ordinary lubrication is used and if the velocity of the rubbing surfaces is determined, the rise in temperature may be computed as in Example (a).

Example (d). The bending and twisting moments applied to a certain journal indicate that a diameter of 4 in. is necessary to provide the requisite strength and rigidity. The conditions of service permit the journal to be 10 in. in length. The total *load* on the bearing is 4000 lb and the rotative speed 1500 rpm. Is perfect lubrication possible? And, if so, how many thermal units must be carried away by either the lubricant or jacket water in order that the temperature of the bearing shall not exceed 150° F when the room temperature is 75° F?

Here $V = 1570.8$ ft per min, and $p = 100$ lb, and from (8) it is found that for this velocity the allowable bearing pressure, for a factor of safety of 2, is 180 lb; hence, perfect lubrication is easily possible.

The work of friction per square inch of projected area is, from equation (7)

$$\mu p V = \frac{51.2V}{t - 32} = \frac{51.2 \times 1570.8}{150 - 32}$$

$$= 681.56 \text{ ft-lb per min}$$

The difference in temperature between bearing and atmosphere is 150° − 75° = 75°. From Curve (2), Fig. 33, it is seen that with this difference a standard bearing will radiate about 225 ft-lb per min, leaving 681.56 − 225 = 455.56 ft-lb to be carried away either by circulating cooled lubrication oil or by means of a water-jacket.

Since $p = 100$ and $V = 1570$

$$\mu = \frac{681.56}{100 \times 1570} = 0.0043$$

which agrees fairly well with the values of Table X.

Checking by equation (3) of Article 56 and selecting the allowable clearance shown in Table XIV, $c = 0.006$. Since the pressure is not excessive and the velocity fairly high a medium oil in which $z = 15$ may be selected.

Then $$\mu = 0.002 + \frac{473}{10^{10}}\left(\frac{15 \times 1500}{100}\right)\frac{4}{0.006} = 0.009$$

This is considerably higher than the value found from the work of Tower, Lasche, and Stribeck, but it should be noted that the coefficient of friction varies inversely with the temperature. The McKee experiments were all conducted at a temperature of about 77° F, with the aid of water circulation. Had higher temperatures been used, the value of the coefficient would no doubt have been much smaller than those reported, and this should be considered in interpreting and using their results.

Furthermore, it should be noted that the coefficient of friction varies directly with the viscosity of the oil, and this again varies with the temperature. A direct solution for the coefficient of friction is therefore not usually possible, but as in most cases of machine design assumptions must be made and checked by the equations until a satisfactory value is obtained.

60. Lubrication of Journals. The **point of application of the lubricant** is of utmost importance, and the method of supplying the lubricant to the journal sometimes materially affects the design of the bearing. The most common methods of feeding lubricants to rubbing surfaces as given in Article 39 apply fully to journals and may be classified as follows:

Imperfect Lubrication	Common oil hole. Common wick or siphon feed cup. Common drop sight feed cup. Oily pad against journal. Ring or chain feed. Centrifugal oiler. Compression grease cup.
Perfect Lubrication	Bath lubrication. Flooded lubrication. Forced lubrication.

In flooded lubrication (sometimes erroneously called forced lubrication), the oil is supplied to the bearing under a low pressure which insures that the journal is always flooded at the point of application, as in bath lubrication, but it does not force the lubricant between the surfaces. In forced lubrication the oil is supplied at a pressure in *excess of the film pressure* at the point of application, and is thus forced in between the surfaces, no reliance being placed on the tendency of the journal to draw in the lubricant. The compression grease cup, while supplying the lubricant under slight pressure, gives only imperfect lubrication, as the supply of lubricant is not copious as in forced lubrication.

In applying any of these methods of lubrication, therefore, except the compression grease cup and forced lubrication, care should be exercised that the point of application is in the region of lowest pressure and at the place where the journal will naturally draw in the lubricant. Thus, in Fig. 28, if the pressure is always downward, lubricant can be supplied at *H* for motion in either direction. If the pressure were upward, an oil hole at *H* would not only be useless for supplying lubricant but would actually be fatal to good lubrication, as any tendency for a pressure film to form would be destroyed by relief of the pressure at the hole. In such a case the lubricant should be supplied from underneath, or if the direction of rotation were anti-clockwise an oil hole, as shown at *I*, would be good design. In forced lubrication the point of application should be the region of greatest bearing pressure, and the hydraulic pressure under which the oil is supplied should be *greater* than the maximum bearing pressure.

While the decreased friction due to perfect lubrication is evident, it does not follow that an effort should be made to design every bearing so as to secure this advantage. In some places a simple oil hole is sufficient, in others a constant supply from a wick feed will suffice, while again, with greater speeds, a ring oiling device is necessary. In many modern power installations, with either steam turbines or reciprocating engines, very complete apparatus for supplying flooded lubrication will be found. The bearings are constructed so as to catch all the oil, as it leaves the journal, and pipes convey it to a central receiver. A pump continually circulates the oil to the various bearings, and in the best installations the oil is filtered and cooled during the circuit. The same results are obtained by flooded lubrication as with bath lubrication Forced lubrication is resorted to only where the bearing pressures are excessive and beyond those which can be supported by the natural action of the film formed by rotation of the journal. (See Article 42.)

The **location and character of the oil grooves** deserve special attention. If the velocity of the journal is so low as to draw in little lubricant the oil grooves should be so cut as to allow the lubricant to flow near the region of greatest pressure. Grooves, or scores on the journal itself, have been found helpful in drawing in the lubricant under such circumstances; especially where the lubricant is heavy. But where the velocity is above 25 ft per min (see Fig. 18), and for ordinary pressures, care should be used that no oil grooves are cut that will tend to prevent the formation of the pressure film. If the lubricant is delivered at *H*, Fig. 28, and the pressure is downward, oil grooves of any kind running from *H* which will distribute the oil over the surface of the journal, are allowable so long as they terminate at a little distance from the edge of the bearing. If the oil is delivered at *I*, and the pressure is either downward or upward, the grooves should be cut *at right angles to the direction* *of motion*, so as to distribute the oil along the entire length of the bearing If cut diagonally they will extend under the journal toward the region of greatest oil pressure, thus relieving any tendency to the formation of a pressure film, and the lubrication will not be as good as it would be if no grooves were present

The sharp edges of all oil grooves should be carefully removed to facilitate the passage of the oil under the journal The sharp edges of the bearings themselves should also be filed or scraped away for the same reason Where one bearing surface encircles nearly one-half of the shaft, as in Fig 28, the surfaces should be relieved for some little distance from the parting line to help the wedging action of the oil and to insure the journal against side pressure due to springing of the bearing under the load. A bearing which binds sidewise will not lubricate properly

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CHAPTER VI

CONSTRAINING SURFACES—*Continued*

THRUST BEARINGS

61. General Considerations. When a shaft is subjected to a heavy end thrust, either from the weight of the parts carried or on account of the power transmitted, the simple collars which are used to prevent end thrust in ordinary shafting will not suffice, and bearings of special form, known as **thrust bearings,** must be provided. If the bearing is designed so that the thrust is taken on the end of the shaft it is called a **step-bearing** or **footstep-bearing.** If the thrust bearing must be placed at some distance from the end of the shaft it is called a **collar bearing.**

62. Step-Bearings. If the motion of rotation is very slow, as is the case in swinging cranes and similar work, a simple cast-iron step, as shown in Fig. 40, will meet the requirements, even if the pressure is heavy. If, however, the velocity is high, this simple arrangement will not give good results, even when the pressure per unit area is low. It may be assumed, without great error, that the unit pressure between the faces of a newly fitted step-bearing is uniform at all points. The velocity of rubbing, however, is a maximum at the outer edge, and, theoretically, it is zero at the geometric center of the pivot. Since the wear is proportional to the product of pressure and velocity, it follows that the surface will wear unevenly, the greater wear taking place at the outer edge. This will bring a concentrated pressure at other points, and heating and cutting may result. It is always advisable in heavy work, for this reason, to remove the wearing surface near the center, where the motion is slowest, and where eventually the greatest concentration of pressure is likely to be produced (see Fig. 40). Decreasing the bearing pressure by increasing the surface, is effective within limits, since the area increases as the square of the diameter while the velocity of rubbing increases directly as the

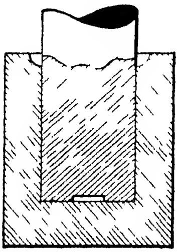


FIG. 40.

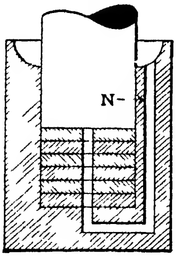


FIG. 41.

diameter. Increasing the radius, however, increases the average moment arm of the frictional resistance, and hence increases the lost energy. It is often better, therefore, to carry a higher bearing pressure, and thus keep the diameter of the pivot small.

If a number of discs are placed between the step, or pivot, and the bearing (Fig. 41), they have the effect of reducing the relative velocity between adjacent surfaces; and if the rotative velocity of the pivot is high, they are very useful as a safeguard against cutting; for, if abrasion should begin between any pair of discs, motion will cease at that point till the lubrication becomes effective again. These washers are usually made alternately of steel and brass, or some other metal, and the upper and lower washers are fastened to the shaft and bearing respectively. An oil hole passes through the center of the washers, and radial grooves cut across the faces permit a flow of oil between the surfaces, centrifugal action assisting the lubrication. If the top of the bearing is connected to the bottom by an oil passage, as shown at N (Fig. 41), the centrifugal action will set up a continuous circulation of the oil, making the lubrication effective. The unit pressure between washers is the same as between the shaft and the first washer, but the relative motion between the surfaces is decreased and the wear thus reduced. A combination of hardened and ground steel washers, alternating with brass or bronze washers, makes an effective bearing. Sometimes the washers are made lenticular in shape, as shown in Fig. 42, in order to allow the shaft automatically to adjust its alignment. For very light work the shaft sometimes rests on a pair of hardened steel buttons, or a hardened steel ball which runs between hardened steel surfaces is introduced. In the submerged step-bearings of water turbines, the shaft, which is often capped with bronze, rests on a lignum vitae step and lubrication is effected by the surrounding water.

If the outline of a step-bearing be made that of a **tractrix** * (Fig. 43) it is found that the tendency to wear in an axial direction is uniform at all points; in fact, if two homogeneous flat surfaces are rotated together they tend to wear into the form of a tractrix, as has been proven by experiment. This is, therefore, the correct shape, theoretically, for all step-bearings; but on account of the difficulty and expense of machining the surfaces it is seldom used. The tractrix has been called **Schiele's antifriction curve,** after the discoverer of the above property. This is a misnomer, however, for the friction of a tractrix-shaped step is much higher than that of a plain pivot.

It is evident that the rubbing surfaces of all the step-bearings which have been discussed can be submerged in an oil bath. The lubrication

* See Church's "Mechanics," page 181.

thus obtained is not to be confused with that obtained on horizontal rotating bearings discussed formerly While centrifugal force does drive the oil from the center to the outside, there is little action on the part of the surfaces themselves tending, on account of its viscosity, to

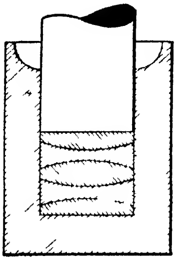


Fig 42

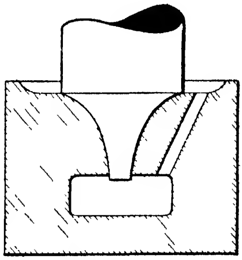


Fig 43

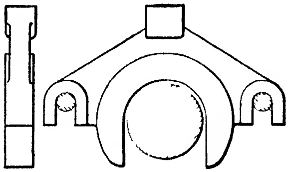
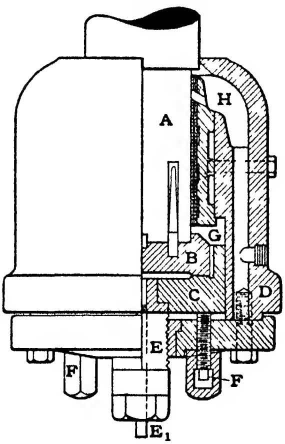


Fig 44

draw the lubricant between them, as in horizontal bearings Such lubrication cannot therefore be looked on as perfect lubrication although giving excellent results The experiments of Beaucamp Tower * on a steel foot step, 3 in in diameter, gives considerable information on this subject It was found that a single diametral oil groove was better than more, and pressures up to 160 lb per sq in were successfully carried at 128 rpm The foot step was freely lubricated, and rested directly on the bearing, no washers being interposed At 240 lb per sq in the bearing seized.



THRUST BEARING OF CURTISS VERTICAL STEAM TURBINE

Fig 45.

If under heavy loads the maintenance of lubrication is important, the lubricant should be supplied at the center of the step-bearing under a pressure such that the metallic surfaces are forced apart and the load is fluid-borne. Fig 45 shows a form of the step-bearing once used on the Curtiss steam turbine. The vertical shaft A, which supports the heavy rotating parts of both turbine and generator, is carried on the disc B which rotates with it The lower disc C can be adjusted vertically, by means of the screw E, and is prevented from rocking on E by the screws F. Oil is forced between the discs through the central pipe E_1, forcing the

* *Trans* Institution of Mechanical Engineers, 1891, page 111.

discs apart and escaping into the cavity *G*. The load is thus completely fluid-borne and perfect lubrication is maintained. The oil passes from *G* upward through the guide bearing escaping at *H*.

63. Collar Thrust Bearings. When the thrust bearing must be placed at some distance from the end of the shaft, the shaft is provided with collars integral with itself, which bear against the resisting surfaces as shown in Fig. 46, which illustrates a thrust bearing as used for marine work. In cheap work, or where the load is small, a single collar is sometimes used. Occasionally a series of washers, as in Fig. 41, are interposed between the collar and the bearing ring. The objection to this form of single-collar bearing for heavy loads is that the large diameter necessary to obtain a practical bearing pressure increases the work of friction, due to the increased velocity, and the difference between the

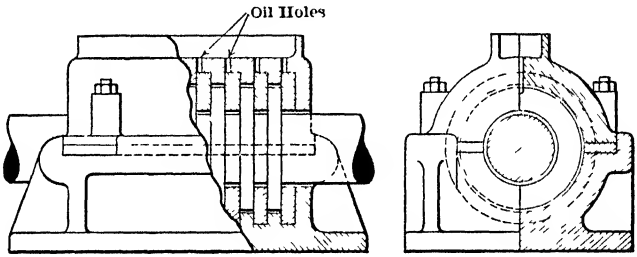


Fig. 46.

rubbing velocities of the ring at the shaft and at its outer diameter results in unequal wear. The outer diameter of the ring, or collar, is usually, therefore, not more than one and one-half times the diameter of the shaft, which limits the width of face of the collar even in large shafts to a few inches; and the necessary area is obtained by using a number of rings.

In small or cheap work, the bearing surfaces of the thrust block are sometimes made integral with the bearing proper; but usually they are made detachable. Thus the main casting of the block may be of cast iron, and the bearing rings of brass are inserted and held in place by radial grooves cut in the block. These rings must be scraped until each collar on the shaft bears properly against its mating ring, so that the thrust is uniformly distributed. The most modern practice in marine work is to make the bearing rings horseshoe-shaped, as in Fig. 44, so that each ring can be withdrawn without disturbing any other portion of the bearing or shaft. Occasionally the horseshoe collars are adjustable along the shaft so as to be more easily brought to a proper bearing.

In first-class work each horseshoe has its own independent water circulation, to that local heating may be prevented, and the lower part of the bearing constitutes an oil bath into which the collars dip. This oil bath also has a water circulation for cooling the oil.

64. Friction and Efficiency of Thrust Bearings. If P be the total uniformly distributed load on a flat circular pivot of radius r_1 and μ be the coefficient of friction, then the equivalent frictional radius R_f * $= \frac{2}{3}r_1$, and the **frictional moment** resisting rotation is

$$M = \tfrac{2}{3}\mu P r_1 * \qquad (1)$$

The energy, E, lost per minute in friction is the product of the rubbing velocity, V, and the frictional force, F, and if r_1 be in inches and P in pounds, then

$$E = FV = (\mu P)\left(\frac{2\pi R_f N}{12}\right) = (\mu P) \times \frac{2}{3}\left(\frac{2\pi r_1 N}{12}\right) = 0.349\mu P r_1 N \qquad (2)$$

where N is the number of revolutions per minute.

In a similar manner, the equivalent frictional radius for a thrust collar with outside radius r_1 and inner radius r_2 is $\frac{2}{3}\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)$, and the frictional moment is

$$M = \mu P \frac{2}{3}\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right) \qquad (3)$$

whence

$$E = (\mu P) \times \frac{2}{3}\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)\frac{2\pi N}{12} = 0.349\mu P N\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right) \qquad (4)$$

The efficiency of a thrust bearing cannot always be expressed as a function of the power transmitted. Thus, for a vertical shaft carrying a heavy load of gears, the frictional resistance of the step has little to do with the power transmitted. For the thrust bearing of a steamship the frictional moment and energy loss are directly proportional to the driving force P. In either, however, the frictional moment or the energy loss must be added to the turning moment or the energy supplied, as the case may be.

* See Church's "Mechanics," page 180.

The following coefficients of friction are taken from Tower's experiments:

TABLE XV

Pressures in Pounds per Unit Area	Coefficients of Friction of Flat Pivots for the Revolutions per Minute as Given Below				
	50 rpm	128 rpm	194 rpm	290 rpm	353 rpm
20	0 0196	0 0080	0 0102	0 0178	0 0167
40	0 0147	0 0054	0 0061	0 0107	0 0096
80	0 0181	0 0063	0 0045	0 0064	0 0063
120	0 0221	0 0083	0 0052	0 0048	0 0053
140	....	0 0093	0 0062	0 0046	0 0054

At 50 and 128 rpm, the oil supply was restricted; but at the other velocities the bearing was flooded. In all cases the coefficient increased at revolutions below 40 rpm, which was probably due to the decrease of the centrifugal force (the bearing being oiled from the center). This would seem to indicate that devices for reducing relative rubbing velocity, such as multiple washers (Article 62), may be carried to an extreme, causing more friction than a plain flat pivot where centrifugal action is effective. In the case of thrust collars, such as shown in Fig. 44, running in an oil bath, the surfaces themselves tend to draw in lubricant in a way similar to that of the ordinary journal. The coefficients of friction for this class of thrust should therefore be as low at least as those given above.

65. Bearing Pressures on Ordinary Thrust Bearings. Where the velocity of rubbing is very low and wear is not important, as in swinging cranes, very heavy unit loads may be put upon pivot bearings, especially if they rotate in an oil bath. Where the velocity is high, or even moderate, and wear is important, much lower pressures must be carried with imperfect lubrication, than on ordinary bearings, running at the same velocity. With forced lubrication, as in the step-bearing shown in Fig. 45, it is evident that very heavy pressure may be maintained. If, on the other hand, too many collars are used on a collar thrust bearing, in an effort to keep the bearing pressure down to a low value, there is danger that all the collars will not bear simultaneously. The following are average values of bearing pressures, for thrust bearings, as found in practice:

TABLE XVI

Mean Velocity in Feet per Minute	Character of Lubrication	Bearing Pressure in Pounds per Square Inch
Very slow as in hand cranes	Bath as in Fig 40	2000 to 3000
Up to 50 ft	Bath as in Fig 41	200
50 to 125	Bath as in Fig 41	150
125 to 200	Bath as in Fig 41	100
200 to 500	Bath as in Fig 41	50
500 to 800	Thrust bearing and bath lubrication as in Fig 46	75 to 50

Example. Design the thrust journal for a steamship having the following data, and estimate the frictional loss in the thrust bearing:

Speed in knots (1 knot = 6080 ft per hr)	15
Indicated horsepower of one engine	5000
Inside diameter of thrust collars	14 in
Outside diameter of thrust collars	21 in
Allowable pressure per square inch of surface	40 lb
Revolutions of the shaft per minute	120

Owing to frictional losses in the engine, propeller, and shaft only about two-thirds of the indicated power is delivered to the thrust block The pressure against the thrust block multiplied by the distance through which the ship moves per minute must equal the energy delivered by the propellor to the block per minute, or if P be the thrust, S the speed of the ship in knots per hour, and the indicated horsepower be denoted by i h p, then

$$\frac{2}{3} \times \text{i h p} \times 33{,}000 = \frac{P \times S \times 6080}{60}$$

or

$$P = \frac{2 \times \text{i h p} \times 33{,}000 \times 60}{3 \times S \times 6080} = \frac{\text{i h p} \times 217}{S}$$

Hence, in the above example,

$$P = \frac{5000 \times 217}{15} = 72{,}300$$

The area of each thrust collar $= \frac{\pi}{4}(21^2 - 14^2) = 192$ sq in. Therefore, the total allowable pressure on each collar $= 192 \times 40 = 7680$ and the number of collars $= 72{,}300 \div 7680 = 9.5$ or, say, 10

If the bearing runs in an oil bath, the coefficient of friction will not be more than 0.01 under the worst ordinary conditions. Therefore from (4),

$$E = \mu P \times \frac{2}{3}\left[\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right]\frac{2\pi N}{12}$$

$$= 0.349\mu PN\left[\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right]$$

$$= 0.349 \times 0.01 \times 72{,}300 \times 120\left[\frac{10.5^3 - 7^3}{10.5^2 - 7^2}\right]$$

$$= 405{,}000 \text{ ft-lb per min or } 12.3 \text{ hp}$$

66. Collar Thrust Bearings with Perfect Lubrication. It will be evident that in collar thrust bearings, such as have been described, perfect oil films will not form naturally, even when the collars dip into an oil bath. In a few instances, forced lubrication has been successfully applied to such rings by constructing oil passages from the rubbing surfaces to a hole bored axially in the center of the shaft and forcing oil through these passages under a pressure sufficient to separate the surfaces when loaded. Perfect film lubrication of collar thrust bearings is attained, however, in the Kingsbury, the Michell,* and the Reist thrust bearings.

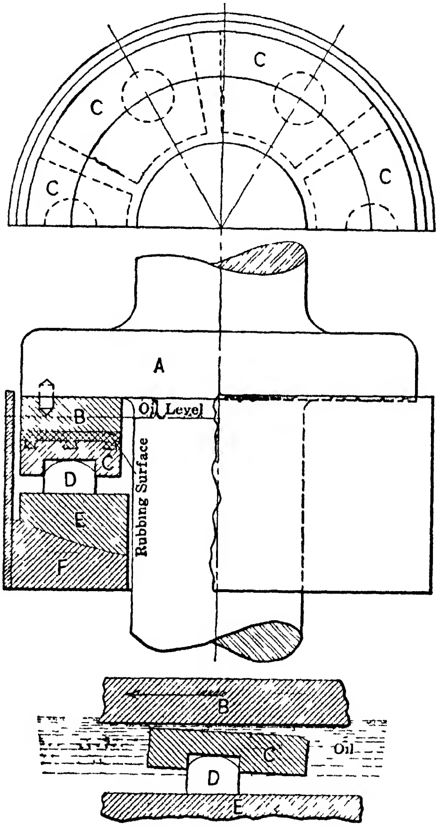


Fig. 47.—Diagrammatic Arrangement of Kingsbury Bearing.

The Kingsbury bearing, as applied to vertical shafts, is shown diagrammatically in Fig. 47. The thrust collar, *A*, transmits the load to the removable runner, *B*, which rotates with it. The ring *B* is carried on a number of segmental shoes *C* which have bab-

* Kingsbury and Michell developed this type of bearing independently about the same time.

bitted wearing surfaces; these shoes are supported and held in position by cylindrical supports with spherical supporting surfaces on which the shoes may rock freely within the limits permitted by the engaging wearing surfaces. These supports are placed slightly back of the center of the shoes, in the direction of rotation. This causes the shoe to incline slightly when at rest and thus offers a wedge-shaped opening between the rubbing surfaces, in which a perfect oil film can form in the same manner as in a rotating journal. The supporting ring, *E*, engages the base plate, *F*, through a spherical surface, so that alignment is automatic. In actual construction the supports, *D*, rest on flat wedges so that the clearance between the shoes, *C*, and the runner *B* can be adjusted. These details are omitted from Fig. 47 for the sake of clearness. The wearing surfaces run in an oil bath.

These bearings have operated very successfully under both vertical and horizontal loading. They have been applied to large vertical water wheels, a load of 350 lb per sq in. being carried successfully at a mean speed of 900 ft per min with a coefficient of friction as low as 0.001. They have been extensively applied in connection with steam turbine installations, where a pressure of 500 lb per sq in. has been successfully carried at a velocity of 4500 ft per min. Much higher velocities and pressures have been carried on experimental installations, and apparently these bearings have a great overload capacity. Other experiments * have shown that they are also very efficient at low speeds; a bearing erected in 1911 carried successfully a load of 980 lb per sq in. at a speed of 126 ft per min.

The Reist thrust bearing, shown diagrammatically in Fig. 48, is based upon a different theory. The thrust collar on the shaft (not shown in the drawing) transmits the load to the runner, *A*, which moves with the collar. The runner rests upon a stationary steel ring, *B*, which has a babbitted wearing surface. This ring is comparatively thin and is parted at one point by a narrow saw-cut, so as to eliminate any tendency to dish with change in temperature. It rests upon a nest of helical springs, which in turn are supported by the base, *C*. The ring, *B*, is prevented from turning by pins or dowels, *D*, fixed in the base, *C*. The springs are designed to close about $\frac{1}{16}$ in. under the assigned load; any undue pressure at any point will, however, compress the springs at that point a greater amount and the bearing will thus automatically align itself and compensate also for any inaccuracies in finish. Oil grooves are always provided in the stationary ring, *B*, and sometimes in the runner, *A*. When both rings are grooved, the number in the runner is different from that in the stationary ring.

* *Trans.* A.S.M.E., vol. 41, page 685.

These bearings have also been operated very successfully under both horizontal and vertical loading. The usual load is from 300 to 400 lb per sq in., and a coefficient of friction as low as 0.0018 has been recorded. Like the Kingsbury and Michell bearings, they are lubricated by an

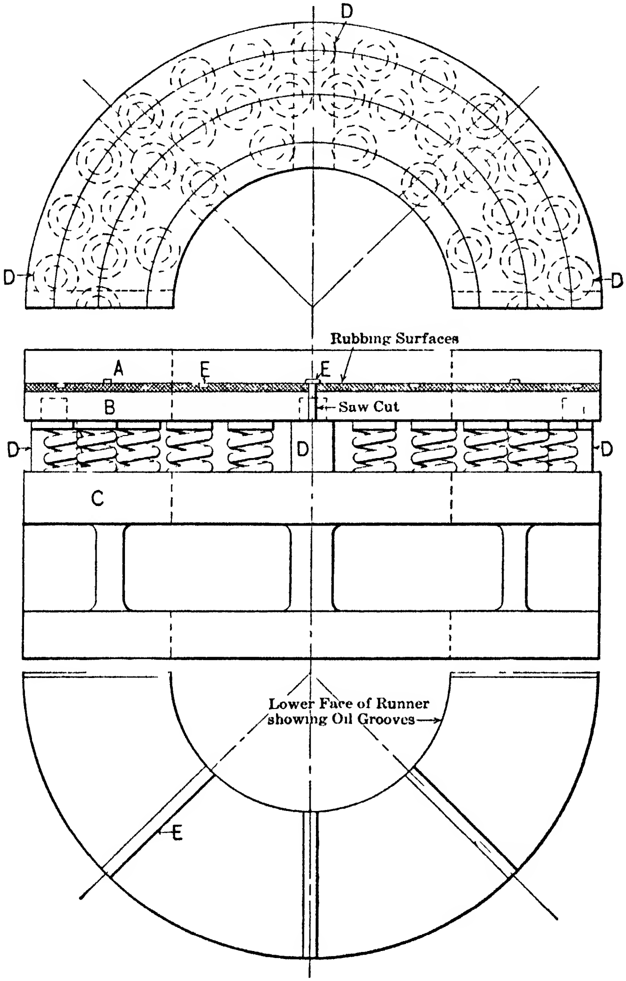


Fig. 48.—Diagrammatic Arrangement of Riest Bearing.

oil bath. Thrust bearings of the Kingsbury, Michell, and Reist types will, no doubt, supersede the old multi-collar type wherever frictional loss is an important factor. Michell has applied this principle to bearings for supporting radial loads, but except possibly for some very special case there would not appear to be any advantage in such an application.

ROLLER AND BALL BEARINGS

67. General Consideration of Rolling. It was noted in Article 38 that the resistance due to rolling friction was much less than that due to sliding friction, for a given load. The application of this principle to very heavy loads and low speeds, as the moving of heavy bodies on rollers, is of great antiquity; but only in recent years have mechanics been able to produce surfaces of such a character as to carry even very light loads at high speeds on either roller or ball bearings. At present, however, bearings of this character can be obtained which will run well under very severe conditions.

When a curved surface rolls upon any other surface with which it theoretically makes line or point contact, the two surfaces tend mutually to deform each other, the amount of deformation depending on the character and hardness of the materials forming the surfaces, and the intensity of the load sustained. If the surfaces of both members are very hard, and the load is very light, the deformation is negligible and true rolling can be practically attained. When, however, any appreciable load is to be carried, the mutual deformation of the surfaces destroys the theoretical line or point contact and the load is borne on a small surface. This occurs even when the surfaces are very hard, and the action, instead of being that of pure rolling, is a combination of rolling and sliding.* The true theory of this action, which is very complex, has not been fully demonstrated and is beyond the scope of this treatise. It can readily be seen that it is closely connected with the elastic properties of materials, on which much research work has been done. Undoubtedly the work of this character which is of most value in the design of roller or ball bearings, is that of Professor Stribeck, whose masterly report has been translated into English by Mr. Henry Hess,† and to this translation reference will be made hereafter.

If the intensity of pressure be such that the elastic limit of the materials is exceeded, permanent deformation will occur. In roller or ball bearings, this may result in the destruction of the surfaces either by flaking off locally, or by simply crushing out of shape. In either case continued action of this character is destructive to the bearing. Experiments on either balls or rollers to determine the ultimate crushing load are, therefore, misleading and useless as far as the

* The student may demonstrate this action by rolling a round lead pencil on a piece of soft rubber under pressure.

† See *Trans.* A.S.M.E., vol. 29, pages 367 and 420.

design of such bearings is concerned. It appears from experiment and experience that bearings of this character can be constructed to carry fairly heavy loads at high speeds for a long period of time, provided the intensity of pressure is not too great. It will be evident also that both balls and ball races are subjected to repeated stress as they rotate (Article 28) and therefore the working life will be a function of the speed of rotation for a given load. Or as the load is increased the speed must be decreased for a given length of life. It is obvious from the foregoing that the materials used in such bearings must be homogeneous, and of uniform hardness. The success of the modern ball and roller bearing has been made possible by improved materials and workmanship rather than by new theories.

Referring to Fig. 50, it is evident that when two adjacent rollers or balls, *A* and *B*, touch each other, the directions of motion of the common points of contact are in opposite directions. It is often stated that this results in considerable frictional loss; and sometimes small intermediate balls, or rollers, are used as shown at *C*, Fig. 50 to obviate the supposed loss. Such intermediate balls or rollers must be kept in place by a cage such as *E*, Fig. 50, and this cage will give rise to a greater frictional loss than that which it is expected to remedy. A brief reflection will show that very little pressure can possibly exist between *A* and *B*. The only pressures that can be exerted by the guiding surfaces upon the balls or rollers are in a radial direction or normal to the surfaces, and these have no component tending to force the adjacent rollers or balls together. Sometimes the rollers or balls are separated by a guiding cage (see Fig. 49), and if any appreciable pressure could exist between adjacent rollers or balls the same would necessarily exist between them and this guiding cage. This theory is not borne out by experience, as these cages, in well-built roller bearings, do not wear appreciably. The frictional loss from this source is undoubtedly very small.

The friction of roller and ball bearings while at rest is very small, and this is a very important point in the design of heavy, slow-moving machinery where, with ordinary sliding bearings, it often takes a much greater effort to start the machinery from rest than to maintain motion at full speed.

Roller Bearings

68. Forms of Roller Bearings. Roller bearings, in common with the ordinary bearing, are classified as **radial** or **thrust** bearings, according to the manner in which the load is sustained. A typical form of construction of roller bearings for radial loading is shown in Fig. 49. A shell of hardened steel, *B*, surrounds the shaft, *A*, and is secured firmly to it.

The rollers, *C*, bear against this shell, *B*, and against an outer shell, *D*, which is secured to the bearing proper, *E*. Both rollers and shells are usually made of high-carbon steel hardened and ground, or of mild steel case-hardened. The rollers are held parallel with the axis of the shaft by means of a cage, *F*, which is made of brass or other soft material. Some form of cage is necessary in all roller bearings on account of the

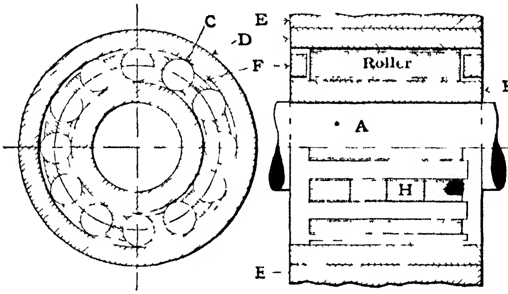


Fig 49.

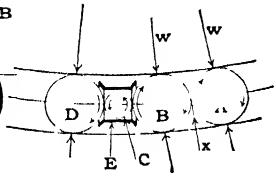


Fig 50.

tendency of the rollers to twist out of line with the shaft, thus replacing the theoretical line contact with point contact, and also causing an end pressure and cramping on the rollers. This tendency to end thrust is sometimes provided for by putting a small ball at each end of the roller to act as a thrust bearing. If the axis of the roller is not parallel to that of the shaft, it cannot make line contact with the shaft unless it assumes

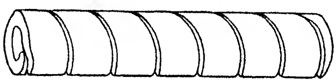


Fig. 51.

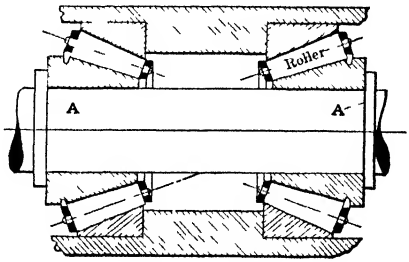


Fig 52.

a helical form. If the surfaces which confine the roller are accurately made, and the clearance is very small, as it should be, the roller cannot get out of parallelism with the shaft without being *bent* into a helical form. If the rollers are hardened this may result in fracturing them, especially if they are relatively long. To obviate this trouble the rollers are sometimes made in short lengths, as shown at *H*, in Fig. 49, or the roller is made flexible, as illustrated by the Hyatt roller shown in Fig.

51. This roller is made by winding steel strip helically upon a mandrel, thus making a hollow flexible roller. It is to be especially noted that neither of these methods will compensate for inaccurate workmanship. For continuous line contact the outer and inner shells must be machined with great accuracy and placed in very accurate alignment, and the rollers must be guided so as to remain perfectly parallel to the shaft. These conditions are difficult to obtain initially, and almost impossible to maintain with great accuracy under continuous service. The rollers in bearings for radial loading may be cylindrical or they may be conical, as in the Grant bearing shown in Fig 52 or in the Timken bearing so widely used on automobiles. The construction here shown permits of adjustment for wear, which is difficult to obtain where the roller is cylindrical.

If the direction of the load to be carried is axial, roller thrust bearings of the form shown in Fig. 53 are often used. The shaft, A, carries a

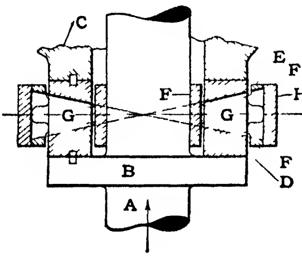


FIG. 53.

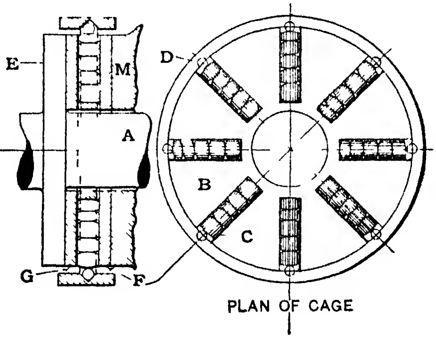


FIG 54

thrust collar, B, and the thrust is taken on the frame of the machine by a corresponding collar, C. A hardened steel ring, D, is attached to B and rotates with it, while a similar ring, E, is fastened to the stationary part, C. The conical rollers, G, roll between these rings, carrying with them the cage, F. A thrust ring, H, prevents the rollers from moving radially outward. The apex angle of the roller should not exceed 15°, and usually is kept down to 6° or 7° to prevent serious end pressure against this retaining ring. It is evident that, where the roller is conical in form, *the apex of the cone lying in the center line of the shaft,* the velocity of any point in its perpphery is proportional to its distance from the axis of the shaft, and, theoretically, true rolling will be obtained.

Bearings of this character with conical rollers are expensive to make in an accurate manner, and a simpler form, as shown in Fig. 54, is sometimes used. Here the rollers are cylindrical in form and are made in short lengths so as to reduce relative slipping. The outer rollers rotate faster than the inner rollers, and the lengths and arrangement of the rollers are such that ridges are not worn in the seat.

Space does not permit of discussion of the many forms of roller bearings on the market; but their fundamental principles are the same, and the student is referred to current trade catalogues for variations in methods of construction.

69. Allowable Bearing Pressures. It is evident that the bearing pressure in roller bearings must not be great enough to stress the material of either roller or bearing surface beyond the elastic limit, but theoretical considerations are of little service in the actual designing of such bearings. The most reliable experimental data bearing on the subject are the results of Stribeck's work. In roller bearings under radial pressure, the quantity equivalent to the projected area of the ordinary bearing, so far as carrying capacity is concerned, is considered as the product of length and diameter of a single roller, multiplied by one-fifth the total number of rollers in the bearing. Thus, according to Stribeck, for cylindrical bearings, if

N = total number of rollers;
W = total load on bearing in pounds;
w = load on one roller in pounds;
d = diameter of roller in inches (mean diameter for conical rollers);
l = length of roller in inches;
k = a constant to be determined experimentally,

then

$$w = kld \tag{1}$$

and

$$W = kld\frac{N}{5} \tag{2}$$

From Stribeck's * experiments k has a value of 550 for unhardened rollers and bearing surfaces and 1000 for hardened surfaces.

In thrust bearings, the load may be considered as distributed over the total number of rollers. Bearings of the type shown in Fig. 54 have been constructed to carry a load of 156,000 lb at 250 rpm.

The manufacture of roller bearings is highly specialized, and the experience of manufacturers is the best guide in selecting such bearings. The Timken Roller Bearing Company † which has had wide experience

* See *Trans.* A.S.M.E., vol. 27, page 444.

† See "The Design of the Timken Bearing," *Timken Journal.*

in this field has established standards of selection based upon 500 rpm. The life expectancy is provided for by one factor and the conditions of service are provided for by another. Then the catalogue rating required is given by the equation:

$$\text{Required rating} = \frac{\text{Load on bearing} \times \text{Life factor} \times \text{Application factor}}{\text{Speed factor}}$$

In the actual use of these factors the life factor and the application factor are combined into a service factor.

Ball Bearings

70. Theoretical Considerations. Let the ball *A*, Fig. 55 (b), roll along the circular path * *B*, with pure rolling motion, making point contact with the path. Let the path *B* be parallel to the plane *CD*, and suppose also that the ball as it rolls remains a fixed distance from this

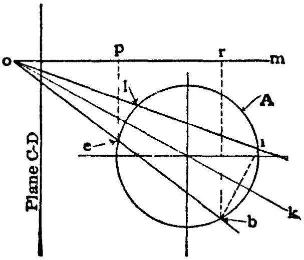


Fig. 55 (a)

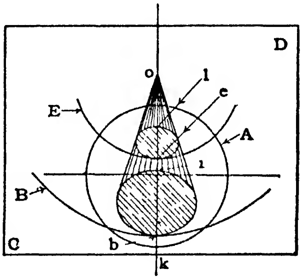


Fig. 55 (b).

same plane. Then it is evident that, if *A* rolls with pure rolling motion along *B*, it will rotate around some one of its diameters, at right angles to *B*, as an axis, and will make contact with *B* along the edges of such a disc as would be cut from it by a plane passing through the point of contact *b* perpendicular to the diameter around which the ball rotates. Thus the ball may rotate around *Ok* as an axis, and roll along the edges of the disc *bi*. It is clear, however, that the ball can rotate around only one diameter at a time, and preserve true rolling contact with *B*. If the ball has two concentric paths of contact as *B* and *E*, Fig. 55 (b) whose points of contact with the ball are *b* and *e* [Fig. 55 (a)] respectively, then it must roll along two discs *bi* and *el*, and these discs must have a common axis of rotation *Ok* perpendicular to their planes and passing through the center of the ball. Further, the discs must be so placed that

* The guiding surfaces of ball bearings are almost invariably circular in form.

the lines *il* and *be* intersect on the line *om*, passing through the common center of *B* and *E*; for then

$$\frac{pe}{el} = \frac{rb}{bi}$$

or the circumferences of the rolling discs are proportional to the circumferences of the paths of contact, and true rolling may be attained. It is not possible to have more than two points of contact between the ball and one of its guiding surfaces, with pure rolling, as the proportionality given above is not true for any other points on the line *ob* except those given. The above principles are fundamental and apply to all ball bearings with circular guiding surfaces.

71. Spinning. Usually one of the guiding members is fixed and the other rotates, the friction between the moving member and the ball causing the latter to roll. If the load carried is so small that no distortion of the surfaces takes place, and true point contact exists, this frictional force will act tangentially to the outer circumference of the disc of contact and be parallel to its plane. Such theoretical conditions never exist in practice, as the surfaces of contact are deformed, even under light loads, and the load is carried on a small area instead of a point. The frictional force rotating the ball is, hence, indeterminate and in general has components which tend to rotate the ball about other axes than the one which will give pure rolling motion. It is clear that inaccurate workmanship will give the same result. This action is known as **spinning** and is necessarily accompanied by friction.

72. Forms of Ball Bearings. Ball bearings are divided into three types, according to the character of the load and the way it is sustained by the bearing:

(**a**) Radial bearings, for loads acting at right angles to the shaft.

(**b**) Thrust bearings, for loads acting parallel to the axis of the shaft.

(**c**) Angular bearings, for taking loads both perpendicular and parallel to the axis of the shaft.

Each of these types may be either a **two-point, three-point,** or **four-point** bearing, depending on the number of points of contact made by the ball on the guiding surfaces.

73. Radial Bearings. Figures 56 (a) and (b) show a two-point radial bearing. The race *B* is secured to the shaft, *A*, while the race *F* is secured to the other member, *C*. Either *A* or *C* may be the rotating part. In order to place the balls in the raceway an opening is often cut in the side of one of the races, as shown at *E*, and the opening then closed with a filling piece as shown. If the race *F* is stationary this filling piece can be located on the unloaded side and no wear brought upon it. If *B* is stationary the opening must be cut in it, and the same care used in locating the filling piece with reference to the load. If both the shaft *A* and hub *C* rotate, this cannot be accomplished, and the full load is brought upon this filling piece, thus decreasing the capacity of the bearing to sustain a load, on account of the break in the surface of the race. If the velocity of the rotating member is high this break in continuity of the race is destructive to the bearing.

If about half the total number of balls necessary to fill the race completely is used, each race may be made of one solid piece. In such cases the bearing is assembled by moving the inner race over eccentrically to the outer race, filling in the balls and then distributing them.

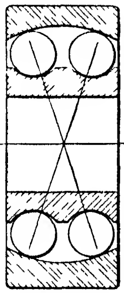


FIG. 57.

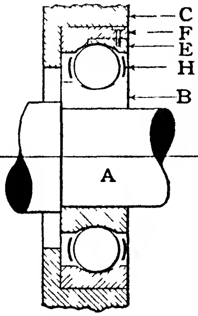


FIG. 56 (b).

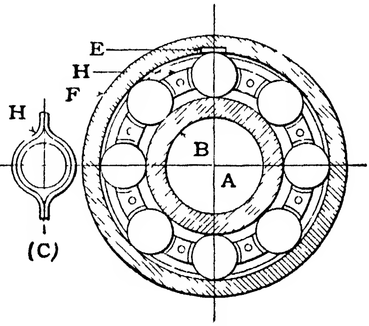


FIG. 56 (a).

It is desirable of course to have as many balls in the bearing as possible; at the same time it is undesirable to have adjacent balls rub against each other (see Article 63 and Fig. 50). The balls are therefore held apart by a light metallic cage *H* made in two circular sections which can be put in place after the balls are placed in the race. This cage lessens the number of balls that can be inserted, but this is compensated for by using balls of greater diameter and hence greater carrying capacity.

In general it has been found that a single row of balls capable of carrying the load is better than two or more because of the difficulty in distributing the load. However, in view of the more highly developed manufacturing processes of today this is less true than formerly and bearings with two rows of balls are quite common. Figure 57 illustrates a so-called **self-aligning** bearing with two rows of balls. The outer race is spherical in form and common to both rows, thus allowing a limited amount of deflection of the shaft without serious disturbance of running conditions.

The carrying capacity of radial ball bearings, according to Stribeck's experiments, is not affected materially by velocity, within reasonable limits, so long as the velocity of rotation is uniform; but sharp variations of velocity at high speed reduce the capacity.

74. Thrust Bearings. Figure 58 illustrates a four-point thrust bearing. Here there is no difficulty in filling in the balls when the races are solid. In Fig. 58 the angles ϕ and ϕ' are equal, but this is not necessary as it is evident that any line drawn through O and intersecting the ball circle will locate a pair of rolling discs which will roll on B, without interfering with the pair shown which may roll on A.

The surfaces C and D are sometimes made both flat and parallel. It is difficult, however, to obtain absolute parallelism, initially, between C, D and the ball races, and much more difficult to maintain this parallelism under running conditions. An error in alignment, either from

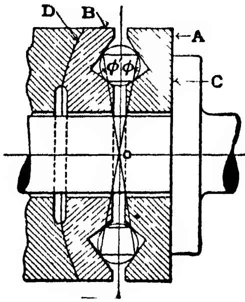


FIG. 58.

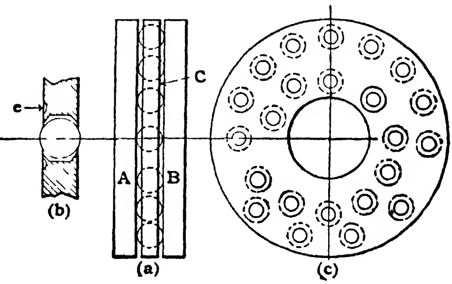


FIG. 59.

poor workmanship or deflection under load, of less than one-thousandth of an inch will cause concentrated loading of the balls on one side. If possible, therefore, such bearings should be seated on spherical surfaces, as shown at D, thus allowing the races to adjust themselves correctly. Mr. Henry Hess states that speed is an important factor in such bearings and he gives 1500 rpm as a maximum.

A simple form of ball thrust bearing is shown in Fig. 59. Here the balls run against flat hardened surfaces A and B, and are kept in position by a cage C made of some soft alloy. The cage may be made to retain the ball loosely by drilling the openings for the balls almost through as shown in Fig. 59 (b), inserting the ball and then closing down the upper edge a little with a set as shown at e, Fig. 59 (b).

75. Angular Bearings. If possible, radial loads should be supported by radial bearings, axial loads by thrust bearings, and angular bearings should be avoided. Radial bearings should not sustain heavy axial loads, and thrust bearings should not be loaded radially. For light loads the angular bearing will sustain pressure in either of these directions.

There are innumerable forms of angular bearings. Figure 60 (a), (b), (c), and (d) may be taken as typical of two-, three-, and four-point angular bearings with pure rolling action. The races can be made continuous in all cases, and are often adjustable. This last feature, though sometimes necessary and often claimed to be an advantage, is really a detriment as it puts the bearing at the mercy of an unskilled person. Properly designed ball bearings do not wear appreciably, and if wear does take place it will occur on the loaded side only; and adjustment cannot compensate for this, but only hastens the failure of the bearing.

It is evident that all the arrangements shown in Fig. 60 fulfill the requirements for pure rolling contact as outlined in Article 69. The path of the ball is not so definitely determined at a, Fig 60, as in the

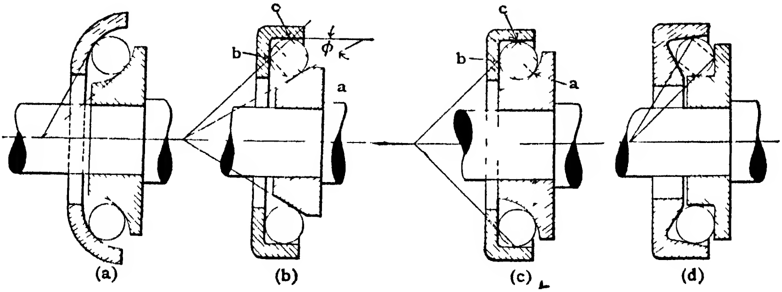


FIG. 60.

other forms. For this reason the radius of the ball races, in order to prevent wedging of the ball, should not be greater than three-quarters the diameter of the ball. For the same reason the angle ϕ in Fig. 60 (b) should not be less than about 25°. In Figs. 60 (b) and 60 (c) the point a may, theoretically, be anywhere, as long as it lies between the discs which roll on the outer raceway. It should be so placed, however, as nearly to equalize the loads at b and c.

There are many cases where both radial and thrust loads must be withstood and where it is not convenient or desirable to install a thrust bearing. For such purposes the New Departure Manufacturing Company recommends a bearing such as shown in Fig. 61 (a). The radii of the ball races are not much greater than the diameter of the ball, and the shoulders of the races on the thrust sides are higher than in the simple radial bearing. In theory they are like Fig. 60 (a). They can be used for combinations of thrust and radial loading. Most usually they are applied in pairs as shown in Fig. 61 (b). In order to accommodate different ratios of thrust and radial pressure these bearings are manufactured with three different angles of contact between balls and races. They are particularly adapted to **preloading,** which will be described.

76. Preloading of Ball Bearings. The principle of preloading was largely an outgrowth of an effort to apply ball bearings to the spindles of machine tools where great precision is essential. All bearings have an initial " give " or " softness " when put under load, the ordinary journal, from the nature of its construction if well made, having greater initial resistance. Ball bearings from the fact of point contact of the ball and race naturally " give " more proportionally when first loaded. Preloading consists of applying an initial load to the bearings before any pressure from the work to be done is put upon them. Figure 61 (b) shows a preloading arrangement as applied by the New Departure

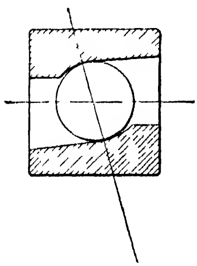


Fig. 61 (a).

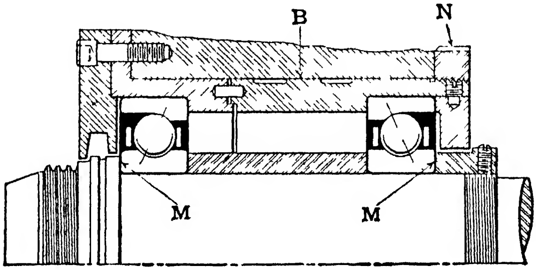


Fig. 61 (b).

Manufacturing Company to a lathe spindle. If the sleeve *B* is moved to the right by the preloading nut *N* the outside races will be forced apart and load applied to the balls, motion on the part of the inside races being resisted by the shoulders of the spindle at *M* and *M*. This action also forces the balls outwardly and thus centers the spindle more firmly and takes up any initial softness in a radial direction.

In the practice of the S. K. F. Company the inner race is made slightly thinner than the outer race, thus allowing it to crowd over and load the ball by pressure against the outer race. Two such bearings are placed side by side and the inner races pressed together. The exact amount of loading is predetermined during the process of manufacture and therefore remains constant until wear occurs.

77. Allowable Load. The allowable load that may be put upon a ball bearing will depend on the following:

(a) The character of the materials forming the balls and races.
(b) The shape of the raceways.
(c) The diameter of the balls.
(d) The velocity of rubbing.

(**a**) Ball bearings fail by overstressing the material of the raceways or balls. If the stress induced is far beyond the elastic limit, and often repeated, the surfaces will flake off and failure will occur. Experiments on the *crushing* strength of balls or races are useless and misleading, as the life of the bearing depends upon the *elastic* and not the *crushing* strength. Evidently none but hard materials can be used for appreciable loads, and these must be homogeneous in texture. Case-hardened materials are of doubtful value for severe service. For most trying circumstances special steels and alloys are much used.

(**b**) Theoretically, a ball supports the load on a point, but practically the unavoidable distortion of the material increases the point to a small surface. It can be demonstrated mathematically, and is evident on reflection, that a greater bearing surface will be formed for a given distortion of ball and ball race the more closely the cross-section of the ball race corresponds to the cross-section of the ball. On the other hand, and as a direct consequence of this increase of surface, it is found that the friction increases as the cross-section of the races approaches the cross-section of the ball, a result to be expected.

It is almost impossible to machine and adjust ball bearings of three- or four-point contact so that the load is uniformly distributed at the various points of contact. It is borne out by experiment, and it is well known, that two-point bearings can carry heavier loads than any other form for a given diameter of ball.

(**c**) The allowable load which a ball can carry varies with the square of the diameter.

These statements have been proved experimentally by Stribeck, who found that the carrying capacity of a ball could be expressed by

$$w = kd^2 \tag{1}$$

where w = greatest load on one ball in pounds.
k = a constant depending on the material and shape of ball races.
d = diameter of ball in inches.

Stribeck showed that the total load that may be carried on a single-row ball bearing is equal to one-fifth of the allowable load on one ball multiplied by the number of balls. If, therefore, W be the total load in pounds on one row of balls, and N the total number of balls,

$$W = w\frac{N}{5} = kd^2\frac{N}{5} \tag{2}$$

For hardened-steel races made of good quality of steel:

k = 450 to 750 for flat or conical races, three- or four-point contact;
k = 1500 for two-point contact and raceways, whose radius of curvature equals $\frac{2}{3}d$.

With more perfect materials Stribeck states that these values may be increased 50 per cent. Since his experiments were performed great progress has been made both in the quality of steel used for ball bearings and also in the accuracy with which the surfaces are finished. It will be noted that the load is not uniformly distributed over the balls that are under load at any time. It is greatest upon the ball directly under the load and decreases as the ball moves away from this position.

(d) As noted in Article 67 the contact surfaces of both balls and races are subjected to repeated compressive stresses which in ordinary operation are very high. The working life of the bearing before the surface flakes will depend upon the speed of rotation. For a given life of the bearing the load must be reduced as the speed is increased and it may be increased as the speed is decreased.

It is possible to write a rational equation for the carrying capacity of ball bearings in terms of these variables, but the coefficients must be determined experimentally. Manufacturers have developed tabulated statements of the carrying capacities of their product in terms of rotative speed, and these form the most reliable sources of information on this point. Manufacturers' literature also contains extended information concerning the mounting of ball bearings since proper mounting has much to do with their successful operation.

78. Practical Considerations. It is clear that in order to insure an even distribution of load, initially, the workmanship on both balls and races must be very accurate; and in order to maintain this distribution the material must be uniform in quality and hardness throughout. It is also found that, for best results, the surfaces must be highly polished and free from scratches. The bearing must be kept free from acid and rust, and provision must be made for excluding dust and grit and for retaining a supply of lubricant, the function of the lubricant being largely to prevent rusting and to reduce the frictional loss due to the small amount of unavoidable sliding. As has been stated, modern manufacturers have succeeded in producing special steels for ball bearings that are homogeneous to a high degree. The accuracy and finish of the surfaces are also remarkable. Mr. Brunner * states that the dimensions of the balls may be held to an accuracy of 0.000025 in.

* *Trans.* A.S.M.E., vol. 54, M.S.P. pages 54–3.

The **coefficient of friction** of ball bearings is virtually independent of the load as would be expected. It will range from 0.0005 to 0.003 with an average around 0.001 to 0.002. From Table X it will be seen that this is about the range of perfectly lubricated cylindrical bearings *when in full operation.* The ball bearing has a great advantage in starting and in places like line shafting where perfect lubrication is not attainable.

Ball bearings, like roller bearings, are highly specialized products and should therefore be selected for important work with the advice of the manufacturer not only as to carrying capacity but also as to the manner of lubrication, mounting, etc. Trade catalogues of reputable manufacturers contain full information for most general purposes.

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CHAPTER VII

AXLES, SHAFTS, AND SHAFT COUPLINGS

79. General. The terms **axle, shaft,** and **spindle** are applied somewhat indiscriminately to machine members which are so constrained by journals and bearings as to admit of motion of rotation. These rotating members may be subjected to simple torsion or bending, or to combinations of torsion and bending. Shear, also, usually exists as in loaded beams. Rotating members may be classified roughly as follows, according to the predominating stress (see Article 14), or to the particular purpose for which they are intended.

(**a**) **Axles,** loaded transversely and subjected principally to bending.

(**b**) **Shafts,** subjected to torsion or combined torsion and bending.

(**c**) **Spindles,** or short shafts which directly carry a tool for actually doing work, and which as a consequence must have accurate motion.

The axles of railway freight cars are good examples of case (**a**); transmission shafting in factories, or the shafts of steam engines are good examples of (**b**); lathe and milling-machine spindles illustrate (**c**).

Considerations of strength seldom enter into the design of spindles. In these members torsional and flexural stiffness and accuracy of form in the bearings are, usually, the most important considerations. When the spindle is designed with these latter requirements in view, there is usually an excess of strength against rupture. The discussions given in Article 13 apply in this case, and it will not be considered further here.

80. Axles. Let A (Fig. 62) be an axle which carries the loads P_1, P_2, and P_3, but is not subjected to any torsional stress except that due to negligible bearing friction. Suppose the axle to be supported by the bearings N and N. The distribution of the bearing reactions is indeterminate, as explained in Article 50, and the assumption is usually made that they are concentrated at the middle of the bearings, as indicated. This assumption is on the safe side, so far as the strength of the shaft is concerned, as the slightest deflection of the shaft tends to concentrate the reaction at the inner edge of the bearing. The axle can, therefore, be treated as a simple beam (Article 15). If the load P_2, were zero, and the loads P_1 and P_3 were equal and symmetrically placed (which is the most usual condition, as in car axles), the case would be identical with Case XIV of Table I. It will be instructive, however, to make a solution of the general case given above.

The principal stress to which the axle is subjected is simple bending. Shear also exists in every section; but from the general theory of beams (Article 15) it is known that, usually, this latter may be neglected in the body of the shaft. If, however, the shaft is short, and consequently need not be large to withstand the applied bending moment, the section of the bearing at XX should be checked for shearing stress. The **dangerous section** of the shaft will be where the bending moment is a maximum, and hence it is necessary to determine this maximum moment, which also involves the determination of the unknown reactions. The reactions may be determined mathematically by taking moments around R_2. Then,

$$R_1 l = P_1 l_1 + P_2 l_2 + P_3 l_3$$

$$\therefore \; R_1 = \frac{P_1 l_1 + P_2 l_2 + P_3 l_3}{l} = 1640 \text{ lb}$$

and

$$R_2 = P_1 + P_2 + P_3 - R_1 = 1860 \text{ lb}$$

The bending moment at any section is the algebraic sum of the moments of the external forces on either side of the plane considered. Thus the bending moment at $P_2 = M_2 = R_1(l - l_2) - P_1(l_1 - l_2)$, and this value may be used in equation (J) of Table VI ($M_2 = sI/c$) to determine the stress for a given cross-section or to determine the cross-section for an assumed stress.

A graphical solution is often more convenient as it shows at once where the *maximum* bending moment is located. In Fig. 62 denote the forces P_1, P_2, and P_3 thus: ab, bc, cd. Draw the force diagram taking 1 in. equal to 1200 lb. To this scale take AE equal to R_1 and ED equal to R_2. At E draw a horizontal line to some convenient distance, here taken as 2 in., and locate the pole O. This procedure it will be noted insures that the closing line of the moment diagram eo is horizontal, which is convenient for purposes that follow. Lay off $AB = P_1$, $BC = P_2$, and $CD = P_3$, and draw AO, BO, CO, and DO. It will be noted that these forces are drawn consecutively downward since they act in that direction and their sum AD must equal the sum of the reactions or upward forces. From any point on ab in the moment diagram draw oa and ob parallel, respectively, to OA and OB in the force diagram. From the intersection of ob and bc draw oc parallel to OC, and in similar manner draw od. Join the intersection of oa

and *ea* with the intersection of *od* and *de*, thus locating the closing string *oe*. Since *OE* has already been located in the force diagram, *oe* should be parallel to it, in this case horizontal. It should be noted that it is not necessary to determine the reactions mathematically, and no matter where the pole is taken a line through *O* parallel to *oe* will locate the point *E*, thus determining the value of the reactions graphically.

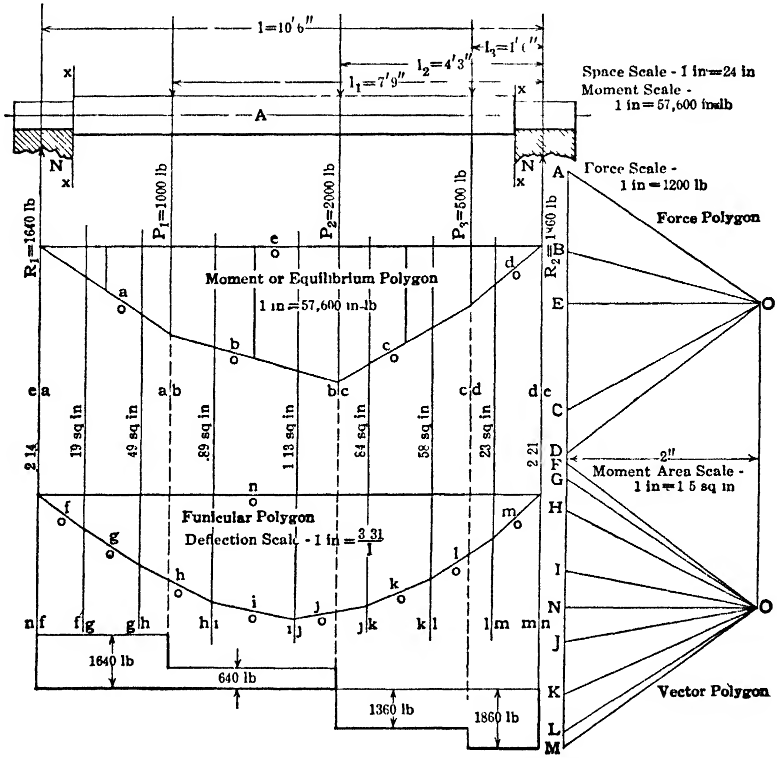

FIG. 62.

The vertical ordinates of the moment polygon are proportional to the bending moments at any point. The numerical value of any bending moment is the continued product of the force scale, the space scale, the pole distance, and the ordinate of the moment diagram at the point under consideration. Thus the force scale is 1 in. = 1200 lb; the space scale is 1 in. = 24 in.; the pole distance = 2 in. Hence the scale of the moment diagram is

$$1200 \times 24 \times 2 = 57{,}600 \text{ in-lb}$$

The maximum bending moment occurs under P_2 where the intercept on the moment diagram is 1.4 in. Its value is therefore

$$57{,}600 \times 1.4 = 81{,}000 \text{ in-lb}$$

If the stress is limited to 10,000 lb, then from equation (5), page 93,

$$M = s\frac{I}{c} = 81{,}000 = 10{,}000 \times 0.098d^3$$

whence

$$d^3 = \frac{81{,}000}{10{,}000 \times 0.098}$$

$$d = 4.4 \text{ in.} \quad \text{or} \quad \text{say, } 4.5 \text{ in.}$$

Article 14 discusses the general theory of deflection in beams, and Table I shows the location and value of the deflection in simple cases. For more complex loading graphic methods are often helpful. In Fig. 62 the moment diagram is divided into a number of small parts so as to obtain a smoother deflection curve. The verticals through the centers of gravity of these smaller divisions are designated at the bottom of the diagram as *fg*, *gh*, *hi*, etc.; and the areas in square inches are shown on these lines directly below the moment diagram. The imaginary reactions *nf* and *mn* are also indicated. A force scale of 1 in. = 1.5 sq in. has been assumed. To this scale *FN* is laid off equal to the left-hand reaction and *NM* equal to the right-hand reaction. The pole *O* is taken 2 in. on a perpendicular from *N*, thus again insuring a horizontal closing line *no*. The deflection diagram is constructed in the same manner as the moment diagram. The deflection scale is computed by multiplying together the moment scale, the linear scale squared, the moment area scale, and the pole distance. In Fig. 62 the moment scale is 1 in. = 57,600 in-lb; the linear scale is 1 in. = 24 in., the moment area scale is 1 in. = 1.5 sq in., and the polar distance = 2 in.

Hence the deflection scale $= \dfrac{57{,}600 \times 24^2 \times 1.5 \times 2}{IE}$ (see Article 14).

If $E = 30{,}000{,}000$ the scale $= \dfrac{3.31}{I}$ per inch of ordinate.

For a 4.5-in. shaft $I = 0.049d^4 = 20.1$. The maximum deflection occurs at *ij*, and the ordinate scales 1.3 in., hence the

Maximum deflection $= \dfrac{1.3 \times 3.31}{20.1} = 0.21$ in., which is somewhat large.

Practice limits the deflection of shafting to 0.01 in. per ft of length. The allowable deflection in this case is therefore $0.01 \times 10.5 = 0.1$ in. For this deflection since $I = 0.049d^4$

$$0.1 = \frac{1.3 \times 3.31}{I} = \frac{1.3 \times 3.31}{0.049d^4}$$

whence
$$d^4 = \frac{1.3 \times 3.31}{0.049 \times 0.1} = 878$$

and
$$d = 5.4 \text{ in.} \quad \text{or} \quad \text{say, } 5.5 \text{ in.}$$

The shear diagram shown at the bottom of Fig. 62 indicates that, as would be expected, the shear can be neglected. If the bending load on such a shaft is uniformly distributed it may be considered as a number of concentrated loads and the same procedure followed as in the foregoing.

It should be particularly noted that in this example I is a constant quantity since the diameter of the shaft is constant throughout its length. If the diameter varies irregularly then the deflection diagram must be modified by dividing its ordinates by the value of I which belongs to each diameter. This as can be seen results in a diagram with a serrated outline.*

81. Shafts Subjected Principally to Torsion. The fundamental relations existing in a shaft which is subjected to torsion only have been fully discussed in Article 13, and for such cases or where other stresses such as those due to bending, are negligible, Article 13 is applicable. Shafts subjected to pure torsion rarely occur in practice, as bending is almost always present, being due to the weight of the shaft itself, and to the weight of pulleys which it supports, as well as to belt pulls, etc. There are many cases, however, where the torsional stress is predominant, or where it is difficult if not impossible to compute the bending effect. Thus, in long factory shafting, where the power is supplied to the shaft at one point, and is given off in small increments at short intervals all along the shaft, the bending due to the pull of the belts is small. This is especially true if care is exercised to place the pulleys as close to the bearings as possible.

If the shaft is of considerable length, the angular distortion is of importance, and it may often occur that a shaft having sufficient torsional *strength* will not have proper torsional *stiffness*. If the power is applied at one end of the shaft, and taken off at the other end, computations for both strength and stiffness are easily made and may be of

* See "Vibration Problems in Engineering" by S. Timoshenko, page 64.

service. In nearly all cases, however, power is delivered in varying quantities all along the shaft, and such computations are not only difficult to make but would indicate that the diameter of the shaft should vary at different parts of its length. This would be undesirable, as it is important that shafting, hangers, etc., should, as far as possible, be uniform and interchangeable for convenience and economy; and the practice of reducing the diameter of the shaft as it extends from the driving point is confined to larger shafting (say over 3 in. in diameter). The design of shafts subjected principally to torsion, therefore, is usually based on the formula for torsional strength, modified by practical coefficients which experience has shown will provide for stiffness against torsion and bending.

Referring to equation *D*, Article 13,

$$d^3 = \frac{16T}{\pi s_s} \quad \text{or} \quad d = \sqrt[3]{\frac{16T}{\pi s_s}} \qquad (1)$$

If P be the equivalent force applied at the periphery of the shaft, so that $T = Pr$, where r is the radius of the shaft in inches; and if N be the number of revolutions of the shaft per minute; then the horsepower transmitted will be

$$\text{Hp} = \frac{2Pr\pi N}{33{,}000 \times 12} = \frac{2T\pi N}{33{,}000 \times 12}$$

or

$$T = \frac{33{,}000 \times 12 \times \text{Hp}}{2\pi N} = \frac{63{,}024\ \text{Hp}}{N}$$

Substituting this value of T in (1) above,

$$d = \sqrt[3]{\frac{321{,}400}{s_s} \times \frac{\text{Hp}}{N}} = k\sqrt[3]{\frac{\text{Hp}}{N}} \qquad (2)$$

where k is a constant depending on the stress assigned. If shearing stress alone were to be considered, s_s, might be taken as high as 9000 lb per sq in., for steel shafting. In order to secure stiffness, and to provide for the indeterminate bending in line shafts, it is customary to assume a lower stress (or higher factor of safety), depending on the material used, and the service for which the shaft is intended. The larger and more important the shaft, the lower should be the working stress, as the failure of a head shaft or shaft of a prime mover is accompanied by great inconvenience and expense. The following factors of safety are indicated by successful practice:

For head shafts.	15
For line shafts carrying pulleys .	10
For small short shafts, countershafts, etc. . . .	7

For steel shafting, the allowable stress for the above factors would be about 4000, 6000, and 8500, respectively, whence

For head shafts,

$$d = 4.3\sqrt[3]{\frac{\text{Hp}}{N}} \tag{3}$$

For line shafting carrying pulleys,

$$d = 3.75\sqrt[3]{\frac{\text{Hp}}{N}} \tag{4}$$

For small short shafts, countershafts, etc.,

$$d = 3.36\sqrt[3]{\frac{\text{Hp}}{N}} \tag{5}$$

The Code for Design of Transmission Shafting of the A.S.M.E. recommends for shafts in simple flexure:

16,000 lb per sq in. for " commercial steel " shafting without allowance for keyways.

12,000 lb per sq in. for " commercial steel " shafting with allowance for keyways.

60 per cent of the elastic limit in tension but not more than 36 per cent of the ultimate tensile strength for shafting purchased under definite physical specifications.

For shafts subjected to simple torsion:

8000 lb per sq in. for " commercial steel " shafting without allowance for keyways.

6000 lb per sq in. for " commercial steel " shafting with allowance for keyways.

30 per cent of the elastic limit in tension but not more than 18 per cent of the ultimate strength for shafting steel purchased under definite physical specifications.

It must be borne in mind, however, that a universal rule cannot be laid down for any class of shafting; and cases will arise that need further consideration than that given by the foregoing equations and discussion. For example, in the span of shafting where the power is applied by a large belt the bending action may be excessive, and this particular span may have to be of a larger diameter than the remainder of the shaft. It is to be especially noted that a shaft carrying a transverse load which applies a bending moment to the shaft is subjected to a reversed stress as it rotates. If, in addition, the twisting moment varies in magnitude, the factors of safety owing to complete or partial reversal of stress (see Articles 30 to 35) must be high, and this accounts for the low stresses allowable in such shafts.

82. Shafts Subjected to Torsion and Bending. In engine shafts, head shafts driven by heavy belts, and many others, the torsional stress is not predominant and may, in fact, be less than that due to bending. A full discussion of the relations which exist in this case has been given in Article 17 and it remains to show the application of this discussion to actual designs.

From Article 17 (equations K, K_1 and Fig. 9), it appears that, if the bending and twisting moments can be determined for any section, the theoretical diameter of the shaft at that section can be found. Usually the twisting moment can be determined without difficulty, but the bending moment is often difficult to determine, and sometimes the designer must be content with an approximation. One of the greatest sources of uncertainty is the location of the reactions at the bearings. Usually, as already pointed out, the safe assumption is made that these reactions are concentrated at the center line of the bearing. When the shaft is of appreciable length (15 or 20 diameters), the error is small; but in such cases as the crankshafts of multiple-cylinder engines, where the distance between the centers of bearings is only four or five diameters, or less, it is evident that the assumption is in the direction of excessive safety.

In line shafting, particularly with the usual swivel bearings, the error from this source is small, and at first sight the conditions of such shafting would appear to approximate those of a continuous beam. Although such an assumption might be safely made when the shafting has been put in perfect alignment, it would not be safe as a general principle, as perfect alignment, even under best conditions, is of short duration, and bending stresses soon appear as a result of lack thereof. It appears, therefore, that, in this case, the safest procedure would be to treat each span as if disconnected at the bearing, when computing bending moments.

A typical example of combined twisting and bending is the engine shaft shown in Fig. 63 (a), the data taken being those of the example in Case (c), Article 5. Here the shaft is supported by the bearings at the points X and X', as indicated, and carries a heavy generator spider at Y. The weight of this spider, and that of the shaft itself, with the probable magnetic pull which may occur when the shaft wears downward a little, is estimated at 22,000 lb. The maximum pressure (P) on the crank-pin, due to the steam pressure, is 25,000 lb. This force is a maximum

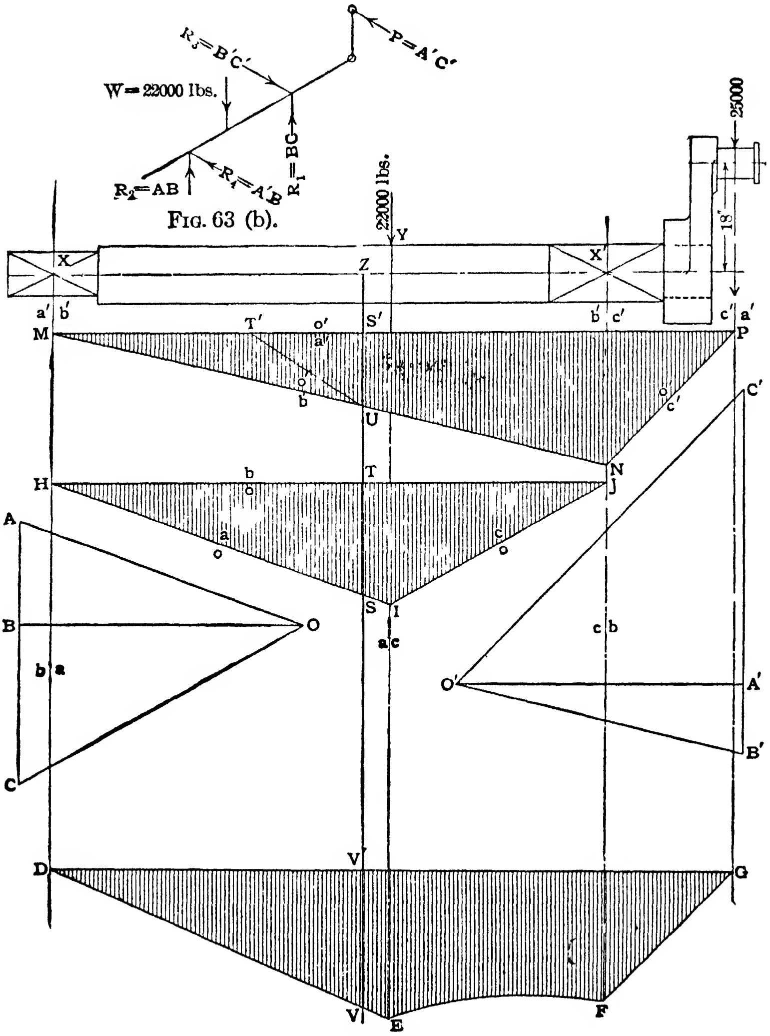

FIG. 63 (b).

FIG. 63 (a).

when the crank is about vertical, and, at that position, it exerts a twisting moment on the shaft from the crank to the point Y * where power

* The reinforcing effect of the hub of the spider is neglected.

is delivered, and also a bending moment on the shaft in a horizontal direction The weight of the generator, etc, exerts a simple bending moment in a downward direction, and at right angles to that induced by P. Figure 63 (b) shows, isometrically, the direction and point of application of the various forces and reactions, and it is required to find the maximum *equivalent* bending moment on the shaft.

It was shown in Example (c), Article 59, how a tentative solution could be made for the diameter and length of the main journal, thus fixing the distance of its center line from the center line of the crankpin at $21\frac{1}{2}$ in Other data fix the distance between bearings as 7 ft 9 in

Graphical Analysis is here very convenient, and the order of procedure will be as follows:

(a) Find the bending moment due to the steam pressure P.

(b) Find the bending moment due to the dead load W

(c) Combine these bending moments to find the maximum resultant bending moment

(a) Consider, for convenience, that the force P, and all the reactions due to it, have been rotated into the plane of the paper so that P is represented as acting vertically Draw the force * diagram, $O'B'A'C'$ for force P, and the reactions due to it, to a convenient scale, here taken as 8000 lb per in, taking O' on a horizontal line through A', thus making the closing string of the moment diagram also horizontal, which is convenient for later work Draw the moment diagram, MNP for force P, and the reactions due to it, as shown The space scale is $\frac{3}{4}$ in to 1 ft or 1 in = 16 in.

(b) In a similar manner construct the force diagram $ABCO$, for force W, and the corresponding moment diagram $HTJI$, for the force W, *making the pole distance* = OB, taken here as 3 in †

(c) To combine the bending moments at any section, as Z, take the intercept ST, on HIJ, and lay it off as $S'T'$ on the diagram MNP. The distance $T'U$ is proportional to the combined bending moments and may be used as an ordinate VV' in the diagram of combined bending moments $DGFE$

It often occurs that the shaft carries a heavy flywheel at Y, instead of a generator, and a heavy belt may also run on the wheel It is evident that the resultant force due to the weight of the wheel and the pull of the belt can be determined, both in magnitude and direction In general the direction of this force will not be vertical, but will make an angle, φ, less than 90° with the direction of the force P In such a case

* See also Article 80

† Reduced to one-half size in cut.

the moments may be combined by the triangle of forces, taking into consideration the angle ϕ.

The numerical value of any moment is the continued product of the ordinate which represents it, the pole distance, the scale of the space diagram, and the scale of the force diagram. Thus the maximum bending moment, which occurs at

$$Y = 1\tfrac{5}{16} \text{ in.} \times 3 \times \tfrac{16}{1} \times \tfrac{8000}{1} = 485{,}400 \text{ in-lb}$$

The twisting moment is seen by inspection to be uniform over the whole length of the shaft which it affects. Its numerical value is, as before, 25,000 × 18 = 450,000 in-lb; and these two moments may be combined, to determine the safe diameter of the main part of the shaft according to the methods of Article 17. Since the shaft is of mild steel in which the ratio of the allowable tensile stress to the allowable shearing stress (s_t/s_s) is approximately 2 it will be advisable to use the equivalent twisting moment $T_e = \sqrt{M^2 + T^2}$.

The methods outlined in the foregoing are clearly applicable to any shaft which has not more than two points of support, since in such cases the reactions can readily be found. If the shaft has more than two points of support it becomes a "continuous" beam, a discussion of which is beyond the limits of this book. In practice it is customary in long small transmission shafts to neglect the effect of continuity and treat each span of the shaft as though it were a simple beam, this procedure being on the safe side.

The equivalent bending and twisting moment formulas given in Article 17, namely,

$$M_e = \tfrac{1}{2}M + \tfrac{1}{2}\sqrt{M^2 + T^2} \qquad (K)$$

and

$$T_e = \sqrt{M^2 + T^2} \qquad (K_1)$$

are those in most general use and are commonly referred to as based upon the theory of failure through **maximum stress** in tension, compression, or shear. There are, however, other theories of failure under combined stress, such as the **maximum strain** theory which assumes that failure takes place when the *strain* or deformation reaches a certain value. These theories are discussed in all good treatises on mechanics of materials. The A.S.M.E. Code for Design of Transmission Shafting, to which reference has been made, reviews these theories as they apply to shafting and concludes that equation (K_1) is on the whole the safest for *steel* shafting. In arriving at this conclusion the assumption is made that the allowable stress in tension is twice the allowable elastic stress in shear. For most mild steels this assumption is approximately true, and the discussion in Article 17 and the curve of Fig. 9 bear out the conclusions reached in the Code. However, it should be particularly noted that this conclusion is universally true only when s_t/s_s is equal to 2 as shown in Fig. 9. For other values of s_t/s_s equation (K) may give a greater diameter for the shaft, and the conclusions of the Code do not apply to other materials or even to wrought iron.

As noted in Article 81, the factor of safety that should be used in the design of a shaft depends upon its relative importance. The A.S.M.E. Code to which reference has been made recommends that the maximum shearing stress in combined stresses be limited as follows:

8000 lb per sq in. for "commercial steel" shafting without allowance for keyways.

6000 lb per sq in. for "commercial steel" with allowance for keyways.

30 per cent of the elastic limit in tension, but not more than 18 per cent of the ultimate tensile strength for shafting steel bought under definite physical specifications.

The Code also quite properly notes that suddenly applied loads or shock may be applied to the shaft by either or both of the bending and twisting moments and recommends that independent shock and fatigue factors be applied to these moments as the conditions may dictate. These factors are defined as follows:

K_t = numerical combined shock and fatigue factor to be applied to the computed torsional moment.

K_m = numerical combined shock and fatigue factor to be applied to the computed bending moment.

Since in all cases K_t and K_m are equal to or greater than unity, the effect of multiplying the bending moment or twisting moment by these factors is to increase their numerical value and consequently the diameter of the shaft. The recommended values of K_t and K_m are as follows:

	K_m	K_t
Stationary shaft:		
Gradually applied load . .	1.0	1.0
Suddenly applied load	1.5 to 2.0	1.5 to 2.0
Rotating shafts:		
Gradually applied or steady load	1.5	1.0
Suddenly applied loads, minor shock	1.5 to 2.0	1.0 to 1.5
Suddenly applied loads, heavy shock	2.0 to 3.0	1.5 to 3.0

These values, of course, should be considered as suggestive only, for conditions vary widely and in each case the result should be checked by the basic discussion of Article 30.

83. Shafts Subjected to Torsion and Compression. The general equations for combined stress of this sort are given in Article 18, and reference may be made to that section.

84. Effect of Keyways on Strength and Rigidity of Shafting. It is evident that keyways must decrease both the strength and the rigidity of shafting, but conclusive data as to the influence of keyways are lacking, practically the only information available being the results of tests conducted by Professor H. F. Moore and reported in *Bulletin* 42 of the Experiment Station of the University of Illinois. These tests were conducted on shafting varying from $1\frac{1}{4}$ to $2\frac{1}{4}$ in. in diameter; the keyways were of the usual proportions as found in average practice. A few tests were made on shafts having Woodruff keyways of usual proportions, the results not differing markedly from those obtained with the ordinary keyways.

Although the number of experiments was not very great and the range of shaft diameters was comparatively limited, certain results were clearly developed. It appears that the ultimate strength of shafting, either under torsion or under torsion and bending, is not materially affected by ordinary keyways. The strength at the elastic limit, however, is lowered, and Professor Moore states that the relative strength or ratio of the elastic strength of a shaft having a keyway to the elastic strength of a solid shaft of the same diameter is expressed by the equation

$$e = 1.0 - 0.2w - 1.1h \tag{1}$$

where e is the relative strength, w the ratio of width of keyway to shaft diameter, and h the ratio of the depth of keyway to the shaft diameter. Tests of shafts having keyways much longer than ordinarily used did not show any marked diminution of strength as compared to shafts having keyway of ordinary length. A test of a single shaft having two keyways 90° apart showed a reduction of elastic strength nearly three times as great as that in a similar shaft having one keyway only.

It was found that keyways reduced the rigidity of the shaft; that is, the angle of twist was increased *in those portions of the shaft where the keyways were cut.* Professor Moore states that the relative rigidity, k/r, or ratio of angle of twist of that portion of the shaft that carries the keyway to a similar length of the solid shaft is expressed by the equation

$$k = 1 + 0.4w + 0.7h \tag{2}$$

where w and h have the same significance as in (1); Professor Moore states that since equation (1) depends entirely upon experiments of limited range it should not be used for work far outside the limits of these experiments.

85. Torsional Stiffness and Deflection of Shafting. When a shaft has considerable length, the matter of torsional stiffness is important. A rule, common in practice, is to limit the twist in the shaft to one degree for every 20 diameters in length. Another rule limits the twisting to 0.075 degree for every foot in length. The **lateral deflection** of the shaft should not exceed $\frac{1}{100}$ in. per foot of length, to insure proper contact at the bearings. Theoretical considerations, however, do not enter so largely into the spacing of bearings of line shafting, as does the construction of the framework to which the bearings are fastened. Care should be exercised in laying out such structures, that provision is made for fastening the hangers close enough together to avoid excessive deflection. For the average range of velocities found in practice the following formula * can be used for ordinary small shafting:

$$L = 7\sqrt[3]{d^2} \text{ for shaft without pulleys} \quad (1)$$

$$L = 5\sqrt[3]{d^2} \text{ for shaft carrying pulleys} \quad (2)$$

where L = distance between hangers in feet and d = diameter of shaft in inches. In large and important shafts that carry heavy loads the deflection can usually be computed by the formulas of Table I or the graphic methods of this chapter. For small shafts that are in effect continuous beams each span may be considered as a simple beam without serious error, and that will be on the side of safety.

If T be the twisting moment in foot-pounds applied to a shaft, then the power transmitted at N revolutions per minute is $2T\pi N$; from which it appears that, the greater the velocity of the shaft, the smaller is the required turning moment, for a given amount of power transmitted.

86. Critical Speed of Shafting. If a slightly deflected shaft is rotated, centrifugal force, acting on the eccentric mass of the shaft, tends to equalize the forces which hold the shaft deflected in one plane and to **whirl** the shaft as a whole around the axis of rotation. At low speeds the action of centrifugal force is small, and the deflecting force will hold the shaft deflected in its plane. As the effect of centrifugal force increases with the velocity, while the effect of the deflecting force is constant, it is clear that as the speed is increased the centrifugal force

* See also Kent's "Mechanical Engineers' Pocket Book."

will, at some speed, balance the effect of the deflecting force, and the shaft will become unstable. Beyond this speed the shaft will whirl about the central axis. For a given diameter and length of shaft there is one definite speed within which it will maintain a stable condition with a given deflection.

In high-speed machines, such as modern steam turbines and certain types of rotating electrical apparatus, the critical speed has become a problem of great importance. It is practically impossible to obtain truly homogeneous materials and very difficult to balance perfectly a rotating body, so that its gravity axis, or axis of mass, coincides with its geometric axis. As the body begins to rotate the gravity axis will rotate in a very small circle around the geometric axis, creating a centrifugal force. This centrifugal force, acting radially through the gravity axis and rotating with it, will bend the shaft until this tendency is balanced by the elastic action of the shaft. If the speed of rotation is low, the vibrations caused by the unbalanced mass may not be serious. If the speed be increased, however, the shaft will assume a condition of unstable equilibrium and violent vibrations may ensue. The speed at which this unstable equilibrium occurs is called the **critical speed.** If the speed is increased beyond this point the body will rotate around its gravity axis, the shaft whirling in a bowed form to permit this action; the vibrations will die away and the system will run smoothly. No satisfactory explanation has been given of the manner in which the gravity axis passes from the outside of the bow of the shaft to the inside of the bow at the critical speed. Theoretically, also, the deflection of the shaft at the critical speed becomes infinite; practically this does not occur, perhaps, as has been suggested, because of the resistance of the air, and possibly because in practice the critical speed is passed through quickly. Many machines have been successfully operated above the critical speed, the vibrations due to unbalance being obviously less than at speeds below the critical speed. The actual working speed should, in any case, be 15 or 20 per cent above or below the critical speed. De Laval, recognizing this phenomenon, purposely designed his turbines with a small or flexible shaft, so that the rotor could readily find its gravity axis, and his rotative speeds were far above the critical speed of the rotating system.

This phenomenon can be demonstrated readily by mounting a small wooden disc, say $\frac{3}{4}$ in. thick and 8 in. in diameter, eccentrically, as shown somewhat exaggerated in Fig. 64 (a), upon a piece of circular dowel rod $\frac{1}{4}$ in. in diameter and about 3 ft in length. One end of the rod should be made conical so that it may be fitted into a small hole in the floor. If the rod is rotated very slowly between the palms of the hand the entire system will rotate around the geometric axis of the rod. If the speed is increased, centrifugal action will carry the center of gravity of the disc *outside* the axis of the rod as shown in Fig. 64 (a), a critical speed will be reached, the system will become unstable and "whirl" as a whole around the axis AB. If the speed is still further increased the center of gravity of the disc will pass back *inside* the rod as shown in Fig. 64 (b) and the entire system will rotate around its gravity axis, which in this case will be close to the gravity axis of the disc, the shaft bending as it rotates as in Fig. 64 (b). So long as this

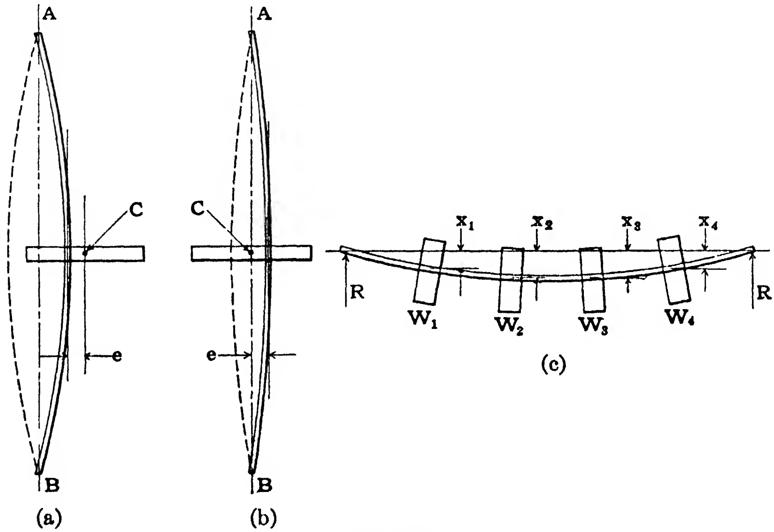

FIG. 64.

higher speed is maintained the entire system will rotate quietly and smoothly.

The mathematical determination of the critical speed of rotating bodies is somewhat involved and beyond the scope of this book. Reference will be made, however, to two of the most common cases. It can be shown * that for a single disc mounted in the center of a vertical shaft the critical speed is

$$N_c = \frac{30}{\pi}\sqrt{\frac{48 \times E \times I \times g \times 12}{W \cdot l^3}} = \frac{720}{\pi}\sqrt{\frac{E \times I \times 32.2}{Wl^3}}$$

where E = modulus of elasticity,

I = moment of inertia of the cross-section,

* See "Vibration Problems in Engineering" by S. Timoshenko, page 60. See also "Vibration Prevention in Engineering" by A. L. Kimball, page 63.

l = the length of the span in inches,
N_c = the revolutions per minute,
W = the weight of the disc in pounds.

The more usual case is a horizontal shaft carrying a series of weights W_1, W_2, W_3, etc., as shown in Fig. 64 (c), these weights being deflected from the true axis by distances x_1, x_2, x_3, etc. Then it can be shown that the critical speed of such a system in revolutions per minute is

$$N_c = \frac{30\sqrt{g \times 12}}{\pi}\left[\frac{\Sigma Wx}{\Sigma Wx^2}\right]^{1/2}$$

or for the case shown in Fig. 64 (c)

$$N_c = \frac{30\sqrt{g \times 12}}{\pi}\left[\frac{W_1x_1 + W_2x_2 + W_3x_3 + W_4x_4}{W_1x_1^2 + W_2x_2^2 + W_3x_3^2 + W_4x_4^2}\right]^{1/2}$$

It is to be noted that if x is expressed in inches g must be reduced to inches as indicated. The solution of these equations depends upon the determination of the several deflections under the loads. The methods of finding these deflections is explained in Article 80. The foregoing discussion assumes that the bearings offer rigid support to the shaft. In actual practice this is far from being true, especially in heavy turbine or electric rotors in which a certain degree of elasticity must be expected. Mr. A. L. Kimball states that the lowering of critical speed in such cases " is frequently as much as 25 per cent and may in extreme cases be over 50 per cent."

Example. What is the critical speed of the 5.5-in. shaft of Article 80 when carrying the loads shown in Fig. 62?

Here the deflection scale = $3.31/I$ per inch and $I = 0.049 \times 5.5^4 = 44.8$, whence the deflection scale = 0.0738 per inch. The deflection intercepts under the three loads scale 0.85 in., 1.2 in., and 0.6 in., respectively, whence the deflections at these places are $x_1 = 0.738 \times 0.85 = 0.063$; $x_2 = 0.0738 \times 1.2 = 0.088$; and $x_3 = 0.0738 \times 0.62 = 0.046$. Then

$$N_c = \frac{30\sqrt{g \times 12}}{\pi}\left[\frac{1000 \times 0.063 + 2000 \times 0.088 + 500 \times 0.046}{100 \times 0.63^2 + 2000 \times 0.088^2 + 500 \times 0.046^2}\right]^{1/2}$$

or

$$N_c = 647.4 \text{ rpm}$$

87. Practical Considerations; Hollow Shafting, etc. Shafts may be and often are made of cast iron, bronze, and other material. By far the greater number, however, are made of machinery steel, so called, or steel containing 0.20 to 0.40 per cent of carbon. Large shafts are forged or rolled hot and finished by turning in a lathe. Large and important shafts are often made of alloy steel such as nickel steel. Most commercial shafting up to about 3 in. is made by **cold rolling** or **cold drawing,** though turned shafting can still be obtained in small sizes. The effect of cold rolling or cold drawing of steel is of course to overstress the metal and elevate both the elastic limit and the ultimate strength. The elevation of the elastic limit may be as great as 40 per cent of the original value. This naturally increases the elastic or shock-resisting quality (see Article 28). Cold rolling, it should be noted, is not simply a surface effect but in bars of comparatively small size the effect penetrates deeply into the bar. Cold-rolled or cold-drawn shafting is made in sizes from $\frac{1}{8}$-in. diameter to 1-in. diameter in increments of $\frac{1}{32}$ in., and from 1-in. to 3-in. in increments of $\frac{1}{16}$ in. Beyond this size commercial shafting is finished by turning in a lathe. Cold-rolled and cold-drawn shafting is very straight and accurate to diametral size. Because of internal stresses, however, it deforms * badly if the surface is cut at any place. For this and other reasons pulleys and couplings for small shafting are usually fastened by clamping of some sort.

The use of hollow shafts not only reduces the weight for a given strength, but the removal of the metal from the core of a steel shaft (or of the ingot from which it is made) very greatly increases its reliability under repeated application of stress.

Shortly after a steel ingot is cast, the exterior solidifies and becomes comparatively cool while the internal portion is still fluid. The subsequent contraction, during complete cooling, is much less in the exterior walls than it is in the hotter interior mass. Unless the interior is " fed," during this period, it will be less dense than the outer portions and shrinkage cavities are apt to be present near the center of the ingot. Numerous expedients have been adopted to reduce this evil, among which is " fluid compression," or subjecting the ingot to heavy pressure immediately after it is poured. The difficulty is not entirely overcome by such means, however, as the walls of large ingots become too rigid to yield to the pressure before the interior is entirely solidified. The external walls " freeze," after which the internal shrinkage is made up by metal flowing from the upper portion toward the bottom as long as any of it remains fluid. This leaves a shrinkage cavity at the upper end of the ingot. Gas liberated during cooling collects in this cavity also. The result of these two actions is to form what is called the " pipe," which frequently extends to a considerable depth. The top

* See Article 13.

end of the ingot is cut off and remelted, but this does not insure removal of all of the pipe, and it also involves much expense. If the portion cut off is not sufficient to remove all of the pipe, a piece rolled or forged from the ingot contains a flaw near the center which is drawn out into a long crack if the ingot is worked into a long piece. The rolling and forging may squeeze the sides of the cavity together so that it is not easily detected at any section, but as this work is done at a temperature much below that corresponding to welding, the defect is not removed. This flaw is more or less irregular or ragged; hence its form is favorable to starting a fracture, under variations of stress, which may finally extend far enough to cause rupture.

If the ingot is bored out, the pipe is effectually removed, and the metal remaining is superior to that of a solid shaft. It will be evident that casting a hollow ingot is not the equivalent of boring out one which was cast solid; for if the ingot is cast hollow the outer and inner walls cool before the intermediate mass does, and the shrinkage effect takes place in the latter. In fact, a shaft made from a hollow ingot is worse than the solid shaft, in the respect that the former has the defective material nearer the outer fibers where the stress is greater.

COUPLINGS AND CLUTCHES

88. General Description. Couplings are machine members which fasten together the ends of two shafts, so that rotary motion of one causes rotary motion of the other. Where the connection is to be broken only at rare intervals, as in making of repairs, the couplings are generally constructed so that they must be partially or wholly dismantled to separate the shafts. Such couplings are known as **permanent couplings.** When it is desired to disengage the shafts at will, the coupling is of a different construction and is generally known as a **clutch coupling.*** The use of clutches is not, however, confined to securing together the ends of shafting, but they are much used for engaging and disengaging pulleys at will, in connection with the shafts on which they are placed. For this service clutches making use of friction are much used, and this particular type is discussed in Chapter XVII.

Couplings should be placed near a bearing, so as to bring the joint in the shaft near a supported point, and should be placed on the side of the bearing farthest away from the point where power is applied, so that when the shaft is disconnected the running part is supported near the end.

* See *Trans.* A.S.M.E., 1908, for a full description and discussion of various forms of clutches.

89. Permanent Couplings. Where the axes of the two shafts to be connected are parallel and coincident, couplings such as are shown in Figs. 65 (a), 65 (b), 66, and 67 are used. Figure 65 (a) illustrates a type of coupling known as a **split-muff** coupling. The parts *A* and *B* are separated by a small space and can, therefore, be clamped to the shaft by the bolts *C*. For heavy work a key as shown is provided, but in lighter shafting friction alone may suffice to prevent relative rotation.

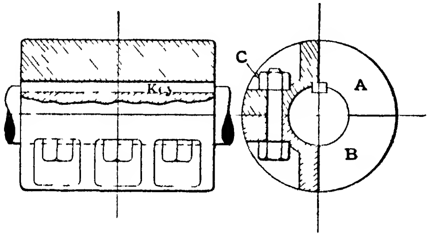

FIG. 65 (a).

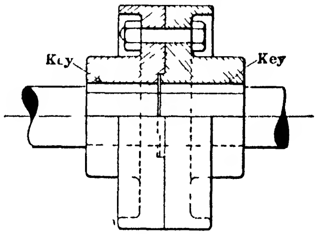

FIG 65 (b).

Figure 66 shows the **Sellers muff coupling.** Here the circular tapered wedges, *B*, *B*, are drawn inward by the bolts, *C*. The wedges are split as shown at *D*, hence the tighter they are drawn inward the more firmly they clasp the shaft. For light work no key is necessary, but for the full capacity of the shaft keys are advisable.

Couplings such as shown in Figs. 65 (a) and 66 are regularly manufactured in standard sizes, and the student is referred to the trade

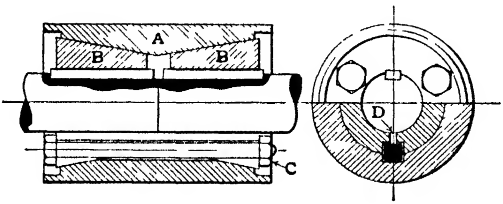

FIG. 66.

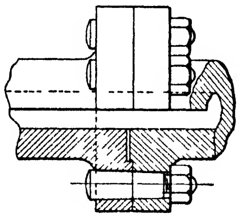

FIG. 67.

catalogues of manufacturers for dimensions and capacities of such couplings.

The **flange coupling,** Fig. 65 (b), is one of the most common and also one of the most effective forms of permanent couplings. The general proportions are usually designed empirically, but the bolts should be designed so that their combined resistance to the turning moment will be at least as great as the torsional strength of the shaft itself; and the bolts should be accurately fitted so as to distribute the load evenly among them.

Let D = diameter of the shaft in inches.
d = diameter of the bolt in inches.
n = the number of bolts.
r = radius of bolt circle in inches.
s_s = allowable shearing stress per square inch, for steel.

Then

$$\frac{\pi D^3}{16} s_s = \frac{\pi d^2}{4} nrs_s$$

whence

$$d = 0.5\sqrt{\frac{D^3}{nr}} \qquad (1)$$

Good practice gives

$$n = 3 + \frac{D}{2}$$

but this number may be modified for convenience in spacing, etc. The bolts should be carefully fitted to insure that each one carries its full share of the load. The projecting outer flange is an important feature as it covers the revolving bolt heads, thus protecting workmen from becoming entangled. For best results the flanges should be pressed on to the shaft and the faces trued up in place, thus insuring greater accuracy of alignment. This is done in all good work.

When great strength and reliability are desired, as in marine work, the flange is sometimes forged solid with the shaft, as in Fig. 67. Here the bolt holes are sometimes bored tapering, and reamed after the flanges are placed together, thus insuring a perfect fit for the bolts, and also facilitating their withdrawal.

When the axes of the two shafts are parallel, but not coincident, or when there is danger of parallel and coincident axes wearing out of coincidence, **Oldham's coupling,** Fig. 68, is often used. It consists of two heavy flanges (A and B), each keyed fast to its own respective shaft, and an intermediate disc, C. The disc has a tongue running diametrically across each face, these tongues being placed at right angles to each other and fitting into grooves cut in the flanges. With this coupling the rate of rotation of the driven shaft is identical with that of the driver, or, in other words, the angular velocity is the same.

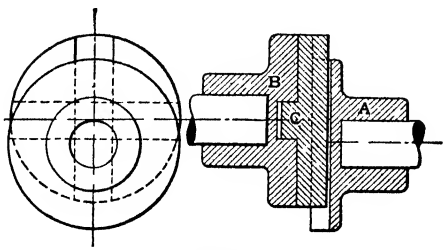

Fig. 68.

If the axes of the two shafts, *A* and *B*, Fig. 69, intersect and make an angle θ with each other they may be coupled together by means of a **Hooke's coupling** or **universal joint**, as it is often called. In this coupling each shaft is fitted with a jaw, *D*, which is pin-connected to an intermediate member, *F*. The holes in this intermediate member for receiving the pins *G* are at right angles to each other. With this arrangement the angular velocity of the driven shaft is not the same at all points of the revolution as that of the driver.* The construction shown in Fig. 69 is sometimes used, but the difference between the angular velocity of the driver and that of the driven shaft is less when the construction is such that the axes of the pins, *G*, intersect. The construction shown in Fig. 69 is that usually adopted where it is desirable to use bolts at *G*, because of the heavy load transmitted. The construction

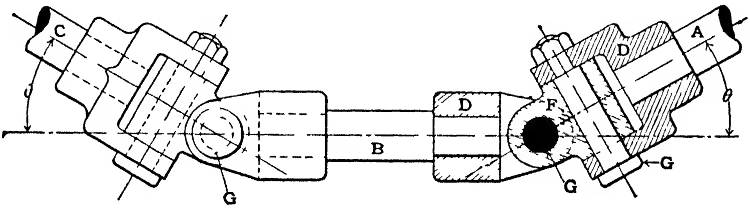

Fig. 69.

required to make the axes of the pins intersect is usually somewhat more complicated than in Fig. 69, and there is a great variety of designs by which this result is attained.

If another shaft, *C*, be coupled to *B* so that *A* and *C* make the same angle θ with *B*; if also the pins, *G*, *G*, in *B* are parallel to each other and all three shafts lie in the same plane; then the angular velocity of *C* will be identical with that of *A* and *vice versa.* Empirical practice makes the diameter of the pin *G* equal to one-half the diameter of the shaft. The universal joint in the form illustrated in Fig. 69 is somewhat cumbersome for some purposes, and many adaptations of the principle have been made in order to secure compactness. The many forms of universal joints in use in automobile propeller shafts are good examples of such adaptations.

90. Positive Clutches. Positive or jaw clutches are much used for starting and stopping such machines as punch presses which must work intermittently. They are made in so many forms that a description of them would be beyond the scope of this work. A very full description of many forms is given in the *Transactions* of the A.S.M.E., vol. 30,

* See "Kinematics of Machinery," J. H. Barr and E. H. Wood, page 214. See also "Kinematics of Machinery," by Albert and Rogers, page 378.

to which reference has already been made. Figure 70 illustrates the most common form of disengaging coupling for heavy work. The part B is made fast to the shaft to be driven; part A, which is compelled to rotate by the feathers, F, can be moved axially along the driving shaft. A ring, R, fitting the groove, G, loosely in a radial direction, is connected by the pins, P, to an operating lever which is not shown. When the part A is moved forward till the jaws, J, engage, A will drive B positively in either direction. In order to facilitate the engaging of the jaws they are often made as in Fig. 71, but in this case the driving can be in one direction only. The total cross-sectional area of the jaws must be such that they will not shear off under the load, and the area of the jaw faces must be sufficient to prevent crushing.

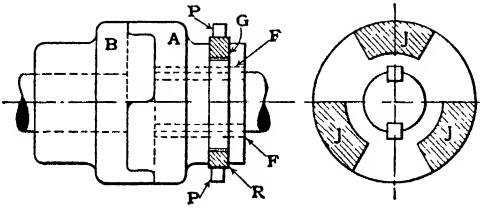

FIG. 70.

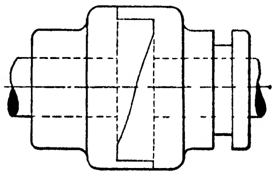

FIG. 71.

Frequently, for light work, only one feather is used, but two feathers are, in general, better, both on account of the driving effort and for ease of operation.

91. Flexible Couplings. Where it is desirable to have a small amount of flexibility in a shaft, a **flexible coupling** is employed. These members are much used for connecting rapidly revolving machines to prime movers, as in the case of a dynamo directly coupled to a steam engine, the object being to prevent undue stress, or bearing pressure, from lack of accurate alignment of the two shafts. Figures 72, 73, and 74 show typical forms of flexible couplings in common use.

In the construction shown in Fig. 72, the shafts, A and B, are fitted with heavy flanges, F, which carry pins, P. Links of leather, L, connect pins on one flange with pins on the other, each set of links having a thickness equal to one-half the length of the pin. The pins on one disc are sometimes placed on a smaller diameter than those on the other, so that in case of failure of the links the pins will not strike and cause breakage. The working stress in the links may be taken at 400 lb per sq in. of cross-section. In small couplings of this type, a single disc of leather is used, this disc being fastened by a ring of bolts alternately to one flange and then to the other.

In the coupling shown in Fig. 73, which is manufactured by the

General Electric Company, the shaft flanges are fitted with steel rims, *R*, one of which is somewhat smaller in diameter than the other. Each rim carries a series of axial slots having well-rounded edges; through these slots an endless leather belt is laced and forms the driving connector. The coupling is readily disconnected by unbolting one of the

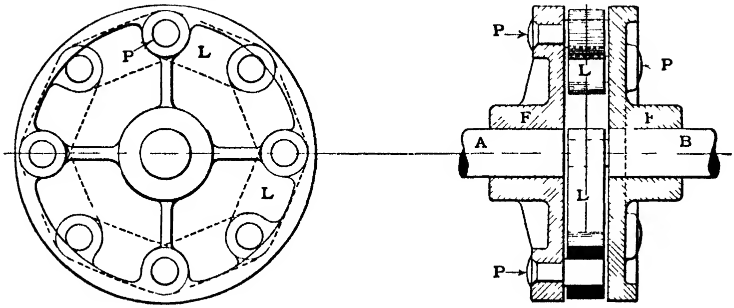

Fig. 72.

steel rims from the main flange. The leather used in these couplings is specially prepared and can be stressed to 400 lb per sq in. of section when in operation.

Couplings with leather connectors and other forms of flexible couplings employing rubber distance pieces between the driving pins serve

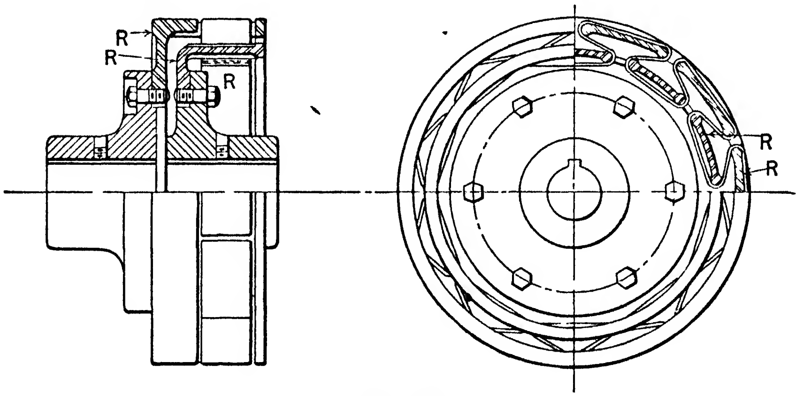

Fig. 73.

also to insulate electric motors or generators from other machines. They should not be used in damp places or where the connectors may be injured by acids, oils or vapors. In the Francke flexible coupling, Fig. 74, the driving connectors are flexible pins made up of thin laminations of steel. These laminated pins are held in place by steel yokes, *A*, the laminations being placed edgewise in a radial position. The steel yokes in turn are held in place by spring retainer rings, *B* (Fig. 74), which snap into a groove in the main member. Retainer pins, *C*, hold the laminations in place in the yokes, *A*. One of the holes made through the bundle of laminations to take the retaining pin is slotted, so that a small amount of endwise motion is permitted between the bundle of laminations and the yoke. This permits the coupling to adjust itself to a small lack of alignment due to the shafts being out of parallel, with each other. The flexible pins permit a small adjustment, if the connected shafts are out of line though parallel with each other. This coupling has proved to be a very satisfactory connection in many lines of work. For very heavy work the manufacturers of this coupling

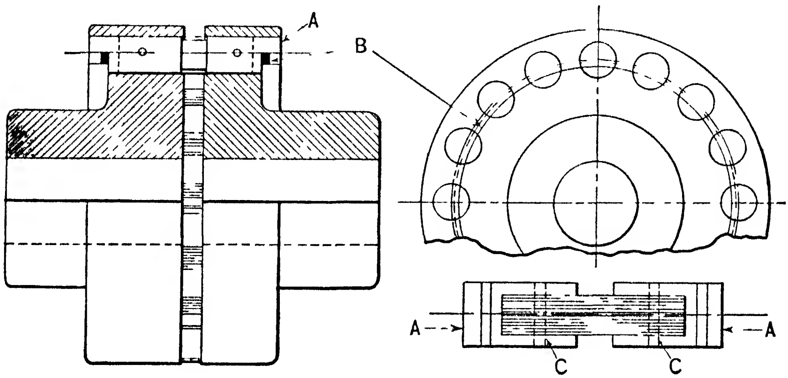

FIG. 74.—Francke Coupling.

have developed a coupling with two sets of flexible pins; a floating ring is placed between the coupling flanges that engage the shafts, and each coupling is engaged with the floating ring by one set of pins. This arrangement gives greater flexibility than can be obtained with the type shown in Fig. 74. In the larger sizes of these couplings, the yokes that retain the flexible pins are fastened in place by bolts instead of snap rings, as shown in Fig. 74. Several other flexible couplings are manufactured commercially, descriptions of which may be found in current advertisements.

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CHAPTER VIII

SPRINGS

92. Distinguishing Characteristic of Springs. Springs are characterized by a considerable distortion under a moderate load. Every machine member is, in a sense, a spring, for no material is absolutely rigid and the application of a load always produces stress and accompanying strain. By proper selection and distribution of material it is possible to control (within wide limits) the degree of distortion under a given load.

An absolutely rigid material would be practically unfit for the construction of any member subject to other than a perfectly quiescent load; for (as shown in Article 26) the stress due to a suddenly applied load would be infinite if the corresponding distortion of the member were zero.

It is usually desirable to restrict the distortions of most machine parts to very small magnitudes, but there are many cases in which considerable distortion under moderate load is desirable or essential. To meet this last requirement the member is often given some one of the forms commonly called springs.

93. The Principal Applications of Springs. Springs are in common use:

I. For weighing forces; as in spring balances, dynamometers, etc.

II. For controlling the motions of members of a mechanism which would otherwise be incompletely constrained; for example, in maintaining contact between a cam and its follower. This constitutes what Reuleaux has called "force closure."

III. For absorbing energy due to the sudden application of a force (shock); as in the springs of railway cars, automobiles, etc.

IV. As a means of storing energy, or as a secondary source of energy; as in clocks, etc.

An important class of mechanisms in which springs are used to weigh forces is a common type of governor for regulating the speed of engines or other motors. In those governors which use springs to oppose the centrifugal, or other inertia actions, the springs automatically weigh forces which are functions of speed, or of change of speed. The links, or other connections, which move relative to the shaft with any variation of the above forces, correspond to the indicating mechanism of ordinary weighing devices.

The first of the above-mentioned applications—the weighing of forces—is usually the most exacting as to the relation between the load and the distortion of the spring throughout the range of action. In the second and third classes of application, it is frequently only required that the maximum load and distortion shall lie within certain limits, which often need not be very precisely defined. The use of springs for storing energy (as the term spring is ordinarily understood) is almost wholly confined to light mechanisms or pieces of apparatus requiring but little power to operate them.

GENERAL SUMMARY OF SPRINGS

Kind of Spring	Load Action	Predominating Stress
Flat or leaf springs	Flexure or bending	Tension and compression
Helical spring under axial load	Torsion of wire	Shear
Helical spring under turning moment tending to wind or unwind it	Flexure	Tension and compression
Spiral spring under turning moment tending to wind or unwind it	Flexure	Tension and compression
Helico-spiral spring under axial load	Flexure combined with torsion	Tension and compression and shear

94. Materials of Springs. Springs are usually of metal; although other solid substances, as wood, are sometimes used. A high grade of steel, designated as spring steel, is the most common material for heavy springs, but brass (or some other alloy) is often used for lighter ones. Of late, springs made of chrome-vanadium, silicon-manganese, or other alloy steels have come into use, but the majority of springs are still made of carbon steel, partly because of the higher cost of these alloys. The design of springs has received much attention by automotive engineers, and reference is made to publications of the S.A.E. for particulars.

A confined quantity of air is used in many important applications to perform the function of a spring. The air-chamber of a pump with its inclosed air is a familiar example of what may be called a fluid spring used to reduce shock ("water hammer"). The characteristic distortion of the solid springs is a change in *form* rather than of volume; the fluid springs are characterized by a change of *volume* with incidental change of form.

Soft-rubber cushions, or buffers, are not infrequently employed as springs, and these are in some respects intermediate in their action between the two classes mentioned above. It is usually not necessary, in these simple buffers, or cushions, to secure a very exact relation between the loads and the distortions under such loads. The discussion of the confined gases (fluid springs) is not within the scope of the present work; hence the following treatment will be limited to solid springs.

95. Flat or Single-leaf Springs. Flat or leaf springs are essentially beams, either cantilevers or beams with more than one support. These

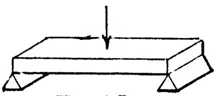

FIG. 75.

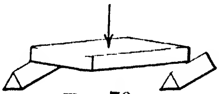

FIG. 76.

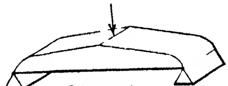

FIG. 77.

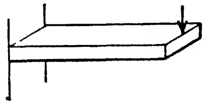

FIG. 78.

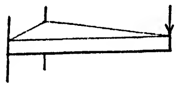

FIG. 79.

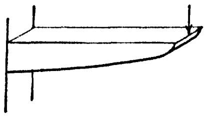

FIG. 80.

springs are subjected to flexure when the load is applied, and the resultant stresses are tension on one side of the neutral axis and compression on the other, with a transverse shear as in all beams. The shear may usually be neglected in computations for strength.

The six forms of rectangular-section beams shown by Figs. 75 to 80, inclusive, are those most commonly used as simple flat springs. These will be designated Types I, II, III, etc., as in the following table.

Let P = load applied to the spring.
l = free length of the spring.
s = intensity of stress in outer fibers.
I = moment of inertia of most-strained section.
h = dimension of this section in plane of flexure.
b = dimension of this section perpendicular to plane of flexure.
E = modulus of elasticity of the material.
δ = deflection of the spring.

The general theory of flexure in beams, gives for strength,

$$M = s\frac{I}{c},$$

which, for beams of rectangular cross-section carrying a single concentrated load, as is most usual with springs, may be expressed,

$$M = Pl = Csbh^2 \qquad (1)$$

where C is a constant dependent upon the beam and the manner of loading. For the deflection of such beams the same theory gives

$$\delta = K' \frac{Pl^3}{EI} = K \frac{Pl^3}{Ebh^3} \qquad (2)$$

where again K is a constant dependent upon the beam and the manner of loading. The following table gives the values of C and K for the cases considered:

TABLE XVII

Type	I	II	III	IV	V	VI
Figure .	75	76	77	78	79	80
C	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
K	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	4	6	8

Formulas (1) and (2) can be used to check the strength and deflection of any existing spring, since both involve the length, breadth, and thickness of the material. In the most usual problem of design, however, the length of the spring, the load, the the allowable deflection are the only known quantities, and the breadth and thickness are to be determined. If either the breadth or the thickness are assumed, the other dimension can be determined from equation (2) and the resultant stress can then be checked by (1). This stress, of course, must be within the elastic limit. If the first estimate gives unsatisfactory results, other assumptions must be made until satisfactory dimensions and stresses are obtained. A more general method will now be deduced by which it is possible to determine the proper dimensions for the requirements given in the foregoing without the necessity of making trial assumptions.

From equation (1),

$$bh^2 = \frac{Pl}{Cs} \quad \therefore \quad bh^3 = \frac{Plh}{Cs} \qquad (3)$$

From equation (2)

$$bh^3 = \frac{KPl^3}{E\delta} \qquad (4)$$

From equations (3) and (4),

$$\frac{Plh}{Cs} = \frac{KPl^3}{E\delta}$$

$$h = \frac{CKsl^2}{E\delta} \qquad (5)$$

From equation (1),

$$b = \frac{Pl}{Csh^2} \qquad (6)$$

The two equations (5) and (6) are in convenient form for designing a flat spring when the span (l), deflection (δ), load (P), and the material are given.

Example. The span of a prismatic flat spring of rectangular section (Type I) is 30 in.; and a load of 1000 lb applied at the middle is to cause a deflection of 1.5 in.

If the modulus of elasticity be 30,000,000 and the safe maximum working stress be taken at 50,000 lb per sq in.,* required the dimensions of the cross-section, h and b.

From equation (5),

$$h = CK\frac{sl^2}{E\delta} = \frac{2}{3} \times \frac{1}{4} \times \frac{50{,}000 \times 900}{30{,}000{,}000 \times 1.5} = \frac{1}{6} \text{ in.}$$

Taking $h = \frac{5}{32}$ in., to use a regular size of stock, s will be somewhat less than 50,000, or since from (5) the stress is proportional to the thickness, h, for a given deflection,

$$s : 50{,}000 :: \tfrac{5}{32} : \tfrac{1}{6}; \quad \therefore\ s = 47{,}000$$

From equation (6),

$$b = \frac{Pl}{Csh^2} = \frac{3}{2} \times \frac{1000 \times 30 \times 1024}{47{,}000 \times 25} = 39.2 \text{ in.}$$

If this width is inadmissible, laminated or plate spring may be used. See next article.

It will be noted that equation (5) does not directly involve either the load P or the breadth of spring b. It is evident that if a beam (flat spring) of given span (l), and thickness (h), is deflected a given amount (δ), the outer fibers will undergo a definite strain which is not dependent

* If the spring is provided with stops to prevent deflection beyond a certain amount, the stress due to such deflection may be nearly equal to the elastic limit of the material. A very small factor of safety is all that is necessary.

upon the width of the beam (b), nor upon the force required to produce this change in relative positions of the molecules. As the unit strain multiplied by the modulus of elasticity equals the unit stress, it follows that this stress may be computed from l, h, and δ (which determine the strain), in connection with E. If the breadth of the beam (b) is increased, the force (P) required to produce the given deflection (δ) will be proportionately increased, but the intensity of stress is not affected by these changes alone.

This same conclusion may be reached from the following relation, in which ρ = the radius of curvature due to load:

$$\rho = \frac{EI}{M} = EI \div \frac{sI}{\frac{1}{2}h} = \frac{Eh}{2s} \tag{7}$$

$$\therefore \; s = \frac{Eh}{2\rho} \tag{8}$$

It appears from equation (8) that the stress is simply proportional to the thickness (h) and the radius of curvature (ρ), for any given value of E. The span l, and the deflection δ, determine ρ, so that equation (7) or (8) may take the place of equation (5).

96. Laminated, or Plate, Springs. It was shown in the preceding article that the maximum thickness of a simple flat spring is fixed when the span, deflection, and modulus of elasticity are known, and the intensity of working stress has been assigned. (See equation (5).) With the value of the thickness (h) thus limited it will frequently happen that a simple spring of rectangular outline will require excessive breadth (b) to sustain the given load, and it is often necessary to use a spring built up of several plates or leaves.

Example. $P = 1000$ lb; $l = 30$ in.; $s = 60{,}000$ lb per sq in.; $\delta = 2$ in., and $E = 30{,}000{,}000$. A simple prismatic spring of rectangular section, with load at the middle of the span (Type I), to meet the above requirements would have:

$$h = CK\frac{sl^2}{E\delta} = \frac{2}{3} \times \frac{1}{4} \times \frac{60{,}000 \times 900}{30{,}000{,}000 \times 2} = 0.15 \text{ in.}$$

$$b = \frac{Pl}{Csh^2} = \frac{3}{2} \times \frac{1000 \times 30}{60{,}000 \times 0.0225} = 33\tfrac{1}{3} \text{ in.}$$

This spring, consisting of a plate 0.15 in. thick and $33\frac{1}{3}$ in. wide, with a span of 30 in., is evidently an impracticable one for any ordinary case. Suppose this plate be split into 6 strips of equal width, each $33.3 \div 6 = 5.5$ in. wide, and that these strips are piled upon each other as in Fig. 81; then, except for friction between the various strips, the spring would be exactly equivalent, as to stiffness and intensity of stress, to the simple spring computed above. Although the form of laminated spring which has just been developed might answer in some cases, another form, based upon the "uniform strength" beam (Type II), is much better for the ordinary conditions.

It may be developed as follows, taking the same data as the preceding example, except that the spring is to be of Type II, Fig. 76.

In the simple spring, Type II,

$$h = CK\frac{sl^2}{E\delta} = \frac{2}{3} \times \frac{3}{8} \times \frac{60{,}000 \times 900}{30{,}000{,}000 \times 2} = 0.225 \text{ in.}$$

$$b = \frac{Pl}{Csh^2} = \frac{3}{2} \times \frac{1000 \times 30}{60{,}000 \times 0.0506} = 14.8 \text{ in.}$$

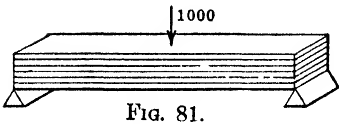

FIG. 81.

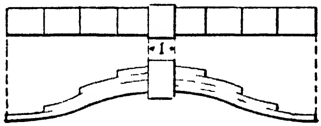

FIG. 83.

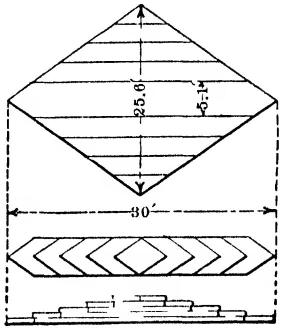

FIG. 82.

A laminated spring for the case under consideration may be derived from this simple spring by imagining the lozenge-shaped plate to be cut into strips which are piled one upon another as indicated in Fig. 82. The thickness of 0.225 in. does not correspond to a regular commercial size of stock, however, and it will usually be better to modify the spring to permit using standard stock. If a thickness of $\frac{1}{4}$ in. be assumed for the leaves or plates, the stress, as found from equation (5) of the preceding article, becomes:

$$s = \frac{hE\delta}{CKl^2} = \frac{4 \times 0.25 \times 30{,}000{,}000 \times 2}{900} = 66{,}700$$

If this stress is considered too great, steel $\frac{3}{16}$ in. thick might be used, when

$$s = \frac{4 \times 3 \times 30{,}000{,}000 \times 2}{16 \times 900} = 50{,}000$$

With $h = \frac{3}{16}$ in. and $s = 50{,}000$,

$$b = \frac{Pl}{Csh^2} = \frac{3}{2} \times \frac{1000 \times 30 \times 256}{50{,}000 \times 9} = 25.6 \text{ in.}$$

If this spring, 30 in. span, $\frac{3}{16}$ in. thick, and 25.6 in. wide at the middle, be replaced by 5 equivalent strips, each $25.6 \div 5 = 5.11$ in. wide (nearly $5\frac{1}{8}$ in.), see Fig. 82, a laminated spring of good form and practical dimensions will result. If the maximum allowable width of spring is fixed, a larger number of plates may be necessary. Thus, in the preceding problem, if the spring width must be kept within $4\frac{1}{4}$ in., it is necessary to use 6 plates, each $25.6 \div 6 = 4.27$ in. wide.

In laminated springs as actually constructed, the full-length, or base, leaf must usually have a square end of some kind, either to support the spring or to fasten it to some other part. It is not unusual for the

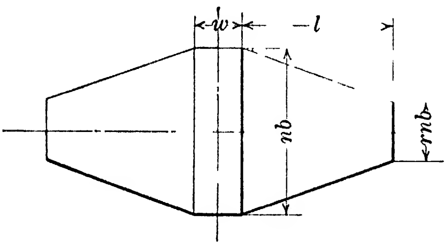

FIG. 84.

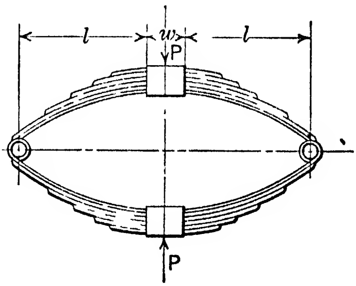

FIG. 85.

ends of the shorter leaves to be cut squarely across, as shown in Fig. 83. While such springs may be designed with safety by equations (5) and (6), the computations will necessarily be less accurate than when the ends are pointed. Some spring manufacturers make the ends of the shorter leaves semicircular * and thin them somewhat, so as to attain the same effect as when the end is pointed. Others compromise by making the ends semi-pointed or trapezoidal in outline. The error introduced by making the end of the base leaf square is probably not great where only one such base leaf is used. If, however, several full-length base leaves are used, the error may be so great as to make equation (5) inapplicable. In laminated springs as actually constructed, furthermore, the laminations must be held together at the center by a band of some kind, as shown in Fig. 85. The free length of the spring, l, used in computations should be as shown in Figs. 84 and 85,

* See "A New Theory of Plate Springs" by Landau and Parr, *Journal Franklin Institute*, December, 1918.

and the plate spring equivalent to a laminated spring with one or more full-length leaves with square ends is trapezoidal in outline, as shown also in Fig. 84.

Professor Peddle has shown that if the usual mathematical methods of finding the deflection of beams is applied to cantilevers of length, l, and of trapezoidal outline as shown in Fig. 84, then for semi-elliptic springs, as shown in Fig. 83,

$$P = \frac{snbh^2}{3l} \tag{1}$$

and

$$\delta = \frac{2l^2sk}{hE} \tag{2}$$

where b = breadth of the leaves.
n = number of leaves.

$$k = \frac{1}{(1-r)^3}\left[\frac{1-r^2}{2} - 2r(1-r) - r^2 \log_e r\right]^* \tag{3}$$

$$r = \frac{\text{number of full-length leaves}}{\text{total number of leaves}}$$

The strength of a full-elliptic spring is, of course, the same as that of a semi-elliptic spring, but the deflection for the same load will be twice as great.

The preceding discussion has been based upon the assumption that a laminated spring is made from a trapezoidal plate, as shown in Fig. 84, and that all the leaves are given the same radius of curvature. If there are several full-length leaves the spring really consists of two fundamental parts, namely, a beam of uniform cross-section and a beam of uniform strength. Now, the deflection of a beam of uniform strength is 50 per cent greater, for the same load, than a similar beam of uniform cross-section. It is not reasonable, therefore, to assume that in such a composite beam each leaf will carry its proportional share of the load, and be stressed to the same amount as the adjacent leaf. Mr. E. R. Morrison has suggested † that if the two fundamental parts of such a composite beam are separated at the middle, before being banded together, by a space equal to the difference in deflection

* Equation (3) is somewhat cumbersome, but Professor Peddle has devised charts which make its solution easy. See *American Machinist*, April 17, 1913, and also Halsey's "Handbook for Designers and Draftsmen," page 201.

† See *Machinery*, January, 1910, page 343. See also *Journal* S.A.E., June, 1919, for a discussion of stresses in springs when a space, or "nip," is allowed between each adjacent pair of leaves.

of the two fundamental parts at full load, the fiber stresses under full load will be the same in each leaf When this construction is followed there must be, of course, an initial stress in the leaves when unloaded, due to the action of the band

The spring shown in Fig 83 is initially curved (when free), which is common practice The best results are obtained by having the plates straight when the spring is under its normal full load (if this is practicable) because the sliding of the plates upon each other, with the vibrations, is then reduced to a minimum It is not uncommon to make the longest plate thicker than the others, if but one plate is given the full length of the spring This cannot be looked upon as desirable practice, however, as all the plates are subjected to the same change in radius of curvature, hence the thicker plate is subjected to the greater stress. See equation (8). In applying the equations that have

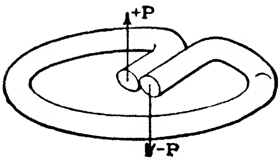

Fig 86

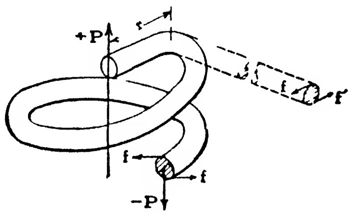

Fig 87.

been developed, it should be remembered that there is always liability of considerable variation in the modulus of elasticity, hence, such computations can only be expected to give approximations to the deflections which will be observed by tests of actual springs These computations will be sufficiently exact for many purposes, but when it is important to determine accurately the scale of the spring (ratio of deflection to load), actual tests must be made

97. Helical Springs. If a rod or wire be wound into a flat ring with the ends bent in to the center, Fig 86, and two equal and opposite forces, $+P$ and $-P$, be applied to these ends (perpendicular to the plane of the ring) as indicated, the rod will be subjected to torsion

If a longer rod be wound into a helix, with the two ends turned in radially to the axis, the typical helical spring is produced If two equal and opposite forces, $+P$ and $-P$, act on these ends, along the axis of the helix, they induce a similar stress (torsion) in the rod, but as the coils do not lie in planes, perpendicular to the line of the forces, there is a component of direct stress along the rod This direct stress increases as the pitch of the coils increases relative to their diameter; but with ordinary proportions of springs, the torsion alone need be considered, when the external forces lie along the axis of the helix.

The following notation will be used in treating of helical springs of circular wire, subjected to an axial load:

P = the force in pounds acting along the axis.
r = the radius of the coils, to center of wire.
d = the diameter of wire in inches.
s_s = the maximum intensity of stress in wire (torsion).
J = the polar moment of inertia of wire.
E_s = the transverse modulus of elasticity.
δ = the " deflection " (elongation or shortening) of spring in inches.
n = the number of effective coils in the spring.
l = the length of wire in inches in the helix = $2\pi rn$ (approximately).

Suppose a helical spring under an axial load to be cut across the wire at any section, and the portion on one side of this section to be considered as a free body, Fig. 87. Neglecting the direct stress, equilibrium demands that the moment (Pr) of the external force shall equal the stress couple, or moment of resistance ($s_s\pi d^3/16$ for circular section). Therefore, the equation for the strength of helical springs of solid circular cross-section is

$$Pr = s_s \frac{\pi d^3}{16} \quad \therefore \quad P = s_s \frac{\pi d^3}{16r} \tag{1}$$

and

$$s_s = \frac{16Pr}{\pi d^3} \tag{2}$$

These equations have been used extensively in the design of springs but have not always proved satisfactory. M. A. M. Wahl,* as the result of analytical and experimental studies, states that the shearing stress is not uniformly distributed over the area but is greatest in the inner surfaces of the coil. He recommends the following equation:

$$s_s = \frac{16Pr}{\pi d^3}\left[\frac{4c-1}{4c-4} + \frac{0.615}{c}\right] = K\frac{16Pr}{\pi d^3} \tag{3}$$

His equation therefore is equation (2) multiplied by a factor which is a function of c which is defined as $2r/d$. Mr. Wahl also noted that if the load is not applied axially but is applied so as also to produce a bending moment upon the spring an additional allowance must be made for the stress due to the moment.

* "Stresses in Heavy Closely-Coiled Helical Springs," *Trans.* A.S.M.E., vols. 51 and 52.

If the free portion of the helix is straightened out, as indicated by the broken lines in Fig. 87, till its direction is perpendicular to the radial end, it will appear that the moment Pr still equals the moment of resistance, $s_s\ \pi d^3/16$. Since the stress and strain are the same in this helix and the straight rod, it appears that the energy expended against the resilience is the same in both cases (the length of wire affected remaining constant). Or, as the force (P) and the arm (r) are the same in both conditions, the distances through which this force acts to produce a given torsional stress (s_s) are equal. If a straight rod of length l is subjected to a torsional moment Pr, the angle of twist in radians being α, then

$$Pr = \frac{\alpha J E_s}{l}$$

The energy expended on the rod is the mean force applied multiplied by the distance through which this force acts. If the load is gradually applied, this energy is $\frac{1}{2}Pr\alpha$. In the corresponding helical spring, the mean force ($\frac{1}{2}P$) acts through a distance equal to the " deflection " of the spring (δ), or the energy expended is $\frac{1}{2}P\delta$. As has been noted, the energy expended in the two cases is the same, or

$$\tfrac{1}{2}Pr\alpha = \tfrac{1}{2}P\delta \quad \therefore\ \alpha = \frac{\delta}{r}$$

$$\therefore\ Pr = \frac{\alpha J E_s}{l} = \frac{\delta}{r} \times \frac{\pi d^4}{32} \times \frac{E_s}{2\pi r n} = \frac{\delta d^4 E_s}{64 r^2 n}$$

$$\therefore\ P = \frac{\delta d^4 E_s}{64 r^3 n} \tag{4}$$

Equation (4) expresses the relation between the external load and the deflection, in terms of the diameter of the wire, the number of turns in the coil, and the mean radius of the coil. Mr. Wahl's experiments indicated that this equation was fairly accurate. This equation for *rigidity* holds good only within the elastic limit of the material, as E_s is simply a *ratio* between stress and strain within this limit. It therefore becomes necessary to check the above indicated computation for strength, and it will often be found, after thus checking, that the stress is either too high for safety, or too low for economy.

Example. The load on a helical spring is 1600 lb, and the corresponding deflection is to be 4 in. Transverse modulus of elasticity of material = 11,000,000, and the maximum intensity of safe torsional stress = 68,000 lb, wire of circular section. To design the spring, assume $d = \frac{5}{8}$ in., and $r = 1\frac{1}{2}$ in., whence $2r/d = c = 4.8$ and $K = 1.33$; from equation (4),

$$n = \frac{4 \times 625 \times 11{,}000{,}000 \times 8}{4096 \times 64 \times 1600 \times 27} = 19.4,$$

Checking for the stress by equation (3),

$$s_s = \frac{16 \times 1600 \times 1.5 \times 512 \times 1.33}{\pi \times 125} = 66{,}766 \text{ lb}$$

This stress is found to be safe, but is somewhat below the limit assigned, and it may be desirable to work up to a rather higher stress. Another computation can be made (with a smaller d or larger r), and by a series of trials, the desired spring can be found. The following order of procedure may be convenient. The load being given, assume a diameter of wire and value of safe stress, then solve in equation (1) for the radius of coil. Give this radius some convenient standard dimension (not exceeding that computed if the assumed stress is considered the maximum safe value). Next substitute these values of d and r (with those given for P, δ, and E_s) in equation (4) to find the number of coils. Then check the stress by equation (3).

The weight of a spring is a matter of some importance, as the material is expensive. The following discussion shows that the weight varies directly as the product of the load and the deflection, inversely as the square of the intensity of stress in the wire, and directly as the transverse modulus of elasticity. Hence, for a given load and deflection, economy calls for a high working stress and a low modulus of elasticity. From equation (1):

$$P = s_s \frac{\pi d^3}{16r}$$

also for a member under torsion,

$$s_s = \frac{d}{2} \times \frac{\alpha E_s}{l}$$

$$\therefore \; s_s = \frac{d}{2} \times \frac{\delta}{r} \times \frac{E_s}{2\pi r n} = \frac{d\delta E_s}{4\pi r^2 n}$$

$$\therefore \; \delta = \frac{4\pi r^2 n s_s}{d E_s}$$

$$\therefore \; P\delta = \frac{\pi^2 d^2 r n s_s^2}{4E_s}$$

But the volume of the spring is

$$v = \frac{1}{4}\pi d^2 l = \frac{1}{2}\pi^2 d^2 rn$$

$$\therefore\ P\delta = \frac{s_s{}^2 v}{2E_s}, \quad \therefore\ v = \frac{2E_s}{s_s{}^2} P\delta \tag{5}$$

The weight is directly proportional to the volume; hence, for given values of E_s and s_s the weight varies simply as the product of the load and the deflection. All possible helical springs (of similar section of wire) have the same weight for a given load and deflection, if of the same material and worked to the same stress. It can be shown that a helical spring of square wire must have 50 per cent greater volume than one of round wire, the stress and modulus of elasticity being the same in both. The round section is generally admitted to be best for helical springs under ordinary conditions.

A small wire of any given steel usually has a higher elastic limit than a larger one, while there is no corresponding change in the modulus

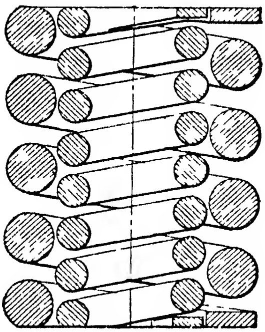

FIG. 88.

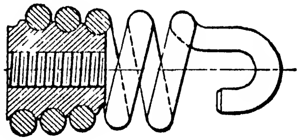

FIG. 89.

of elasticity with change in diameter. This suggests the use of as light a wire as is consistent with other requirements.

Two or more helical springs are often used in a concentric nest (the smaller inside the larger), all being subjected to the same deflection. This is common practice in railway trucks, where the springs are under compression when loaded. If these springs have the same "free" height (when not loaded), and if they are of equal height when closed down, "solid," it can be shown that the length of wire should be the same in each spring of the set for equal intensity of stress. The "solid" height of a spring is $H = dn$, and the length of wire is $l = 2\pi rn$; hence the number of coils of the separate springs of the set are inversely as the diameters of the wire and inversely as the radii of the coils; or the ratio of r to d is the same in each spring of the nest. This conclusion may be somewhat modified when it is remembered that the wire of smaller diameter may usually be subjected to somewhat higher working stress than the larger wire of the outer helices; and also that the wire of these compression springs is commonly flattened at the end to secure a better bearing against the seats. See Fig. 88.

Two common methods of attaching "pull" springs are shown in Fig. 89. One end of the spring shows a plug with a screw thread to fit the wire of the spring. This plug is usually tapered slightly, and the coils of the spring are somewhat enlarged by screwing it in. The other end of the spring shows the wire bent inward to a hook which lies along the axis of the helix. The former method is usually preferable for heavy springs.

Springs of circular cross-section, as has been noted, are usually preferable to those of rectangular cross-section. For helical springs of *square* cross-section,

$$Pr = 0.208b^3s_s \quad \therefore \; P = \frac{0.208b^3s_s}{r} \; * \tag{6}$$

and

$$s_s = \frac{Pr}{0.208b^3} = \frac{4.8Pr}{b^3} \tag{7}$$

Mr. Wahl recommends for axially loaded springs of **rectangular** section with the long side parallel to the load line

$$s_s = K\frac{Pr(3b + 1.8t)}{b^2t^2} \tag{8}$$

where K has the same value as in equation (3). If $b = t$, that is, if the section is square, equation (8) reduces to

$$s_s = K\frac{4.8Pr}{b^3} \tag{9}$$

which is equation (7) multiplied by the factor K. Mr. Wahl recommends for the deflection of rectangular springs

$$\alpha = \frac{19.6Pr^3n}{E_st^3(b - 0.56t)} \tag{10}$$

When $b = t$ the equation reduces to

$$\alpha = \frac{44.6Pr^3n}{E_sb^4} \tag{11}$$

which is the usual form for square sections.

* See "Strength of Materials" by Arthur Morley, pages 302 and 398.

98. Spiral Springs. Spiral springs properly so called are those of the form of the familiar clock spring. These are best adapted for a twist relative to the axis of the spiral, and are usually employed when a very large angle of torsion between the two connections is necessary. In this form of spring, the stress in the material is that due to flexure, or tensile and compressive stress on opposite sides of the neutral axis.

If P = the turning force in pounds applied to the central spindle;
r = the lever arm in inches of force P;
b = breadth of spring in inches;
h = thickness of spring in inches;
δ = distance in inches moved through by force P;
ϕ = angle, in radians, through which spindle turns;
s = stress in outer fibers of spring;
l = total length of spring in inches;

then, for moderate loads,

$$P = \frac{sbh^2}{12r} \qquad (1)$$

and for heavy loads,

$$P = \frac{sbh^2}{6r} \qquad (2)$$

and for moderate loads,

$$\delta = r\phi = \frac{Plr^2}{EI} = \frac{12Plr^2}{Ebh^3} * \qquad (3)$$

99. Helico-spiral Springs. The form of spring represented by the common upholstery spring may be looked upon as a spiral spring which has been elongated, and given a permanent set, in the direction of its axis; or it may be considered as a modified helical spring in which the radii of the successive coils are not equal. It is thus intermediate between the two preceding classes. This last form is not usual in machine construction, though it has the advantage over the common helical spring of considerable lateral resistance, and it may be employed to advantage where it is difficult or undesirable otherwise to constrain the spring against buckling. This spring is used only as a push spring, to resist a compressive action. The springs used on the ordinary disc valves of pumps are often of this form, as they will close up flat between the valve and guard. Car springs are sometimes made of a flat strip or ribbon of steel wound in this general form, with the flat sides of the strip parallel to the axis of the spring.

* See "Strength of Materials" by Arthur Morley, pages 302–398.

100. Allowable Stresses in Springs. Experience shows that thin plates have a higher elastic limit than thick plates of similar grade of material In the practice of a prominent eastern railway company, the values allowed for the maximum intensity of stress in flat steel springs are as follows·

For plates	$\frac{1}{4}$ in thick,	s = 90,000	lb per sq in
"	$\frac{5}{16}$ "	s = 84,000	" "
"	$\frac{3}{8}$ "	s = 80,000	" "
"	$\frac{7}{16}$ "	s = 77,000	" "
"	$\frac{1}{2}$ "	s = 75,000	" "

The above values are satisfied by the equation

$$s = 60{,}000 + \frac{7500}{h} \qquad (1)$$

in which h is the thickness of plate in inches

These values are for the greatest stress to which the material can be subjected, as when the spring is deflected down against the stops

The modulus of elasticity, E, may vary considerably, but its value may be assumed at about 30,000,000 in the absence of more definite data

An extensive set of tests of springs, conducted by Mr E T Adams, in the Sibley College Laboratories, indicates that the steel used in helical governor springs may be subjected to stress varying from about 60,000 lb per sq in with $\frac{3}{4}$-in wire to 80,000 lb per sq in (or more) in wire of $\frac{3}{8}$-in diameter The following expression may be used to find the maximum safe stress in such springs:

$$s_s = 40{,}000 + \frac{15{,}000}{d} \qquad (2)$$

Mr J. W. Cloud presented a most valuable paper on helical springs before the American Society of Mechanical Engineers (*Trans*, vol 5, page 173), in which he shows that for rods used in railway springs ($\frac{3}{4}$-in to $1\frac{5}{16}$-in diam), the stress may be as high as 80,000 lb per sq in, and that the transverse modulus of elasticity is about 12,600,000

For chrome-vanadium steels, s_s may be taken as high as 60,000 to 180,000 lb per sq in, depending upon the treatment; for phosphor bronze wire as used in helical springs, s_s may be taken as 30,000 to 40,000 lb per sq in Springs that may be overstressed should be provided with stops to limit their deflection; and if they are to be in constant service, as the springs on the valves of gas engines, the stresses should be kept low.

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CHAPTER IX

TUBES, PIPES, CYLINDERS, FLUES, AND THIN PLATES

101. Resistance of Thin Cylinders to Internal Pressure. If a hollow circular cylinder, whose walls are very thin compared to its diameter, is subjected to an internal bursting pressure, a tensile stress is induced in the walls. This tensile stress is reduced near the ends by the action of the ends themselves, which tend to hold the walls together. Let Fig. 90 represent one-half of a portion of a thin cylinder, so far removed from the ends that their effect may be neglected.

Let p = the unit internal pressure in pounds.
d = the internal diameter of the cylinder.
r = the radius of the cylinder in inches.
t = the thickness of the cylinder walls in inches.
s = the unit tensile stress in the longitudinal section.
s_t = the unit tensile stress in the transverse section.
l = the length of the part considered in inches.

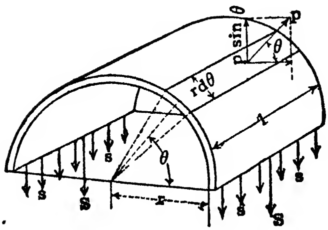


FIG. 90.

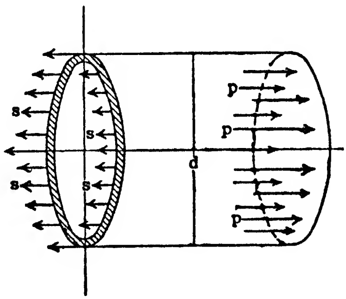


FIG. 91.

Consider the half of the cylinder as a free body, and resolve all forces perpendicular to the cutting plane. The normal pressure on a longitudinal strip of length l and width $rd\theta$ is $plrd\theta$. The component of this force perpendicular to the cutting plane is $plrd\theta \sin \theta$. The total pressure normal to this plane is

$$\int_0^\pi plrd\theta \sin \theta = plr \int_0^\pi \sin \theta \, d\theta = 2plr = pld$$

For equilibrium this normal force must equal the resisting stress in the two sides of the cylinder. Hence,

$$2stl = pld$$

or

$$s = \frac{pd}{2t} \tag{1}$$

$$p = \frac{2st}{d} \tag{2}$$

or

$$t = \frac{pd}{2s} \tag{3}$$

In other words, the unit longitudinal stress in the walls of a thin cylinder is equal to the product of the diameter and the unit internal pressure, divided by twice the thickness of the cylinder walls, and is independent of the length of the cylinder.

If a transverse section of the cylinder (Fig. 91) be considered, it will be seen that the total pressure on the head, which tends to cause rupture along a transverse section, is $\pi d^2 p/4$, and this must be equal to the intensity of the transverse stress produced multiplied by the area of the metal in such a section, or,

$$\frac{\pi d^2 p}{4} = \pi d t s_t$$

$$\therefore \quad s_t = \frac{pd}{4t} \tag{4}$$

$$p = \frac{4s_t t}{d} \tag{5}$$

or

$$t = \frac{pd}{4s_t} \tag{6}$$

A comparison of equations (1) and (4) shows the stress in transverse sections to be only one-half of that in longitudinal sections. For this reason it is very common practice to make the circumferential seams of a boiler shell with a single riveted joint, when the longitudinal seams are double or triple riveted.

Equations (1) and (4) and their several forms, as shown in the foregoing, are commonly accepted and used for the design of thin cylinders, so called, subjected to internal pressure. They assume that the stress is uniform throughout the cross-section of the wall of the cylinder, an assumption that can be made with safety for very thin walls only. Professor R. T. Stewart, as a result of testing * several hundred steel and wrought-iron pipes, of commercial sizes up to 10 in. in diameter, recommends the use of Barlow's formula for the design of cylinders and tubes of this character. If d_2 be the outside diameter of the cylinder, then by Barlow's equation

$$s = \frac{pd_2}{2t} \tag{7}$$

$$p = \frac{2st}{d_2} \tag{8}$$

$$t = \frac{pd_2}{2s} \tag{9}$$

The relation between the circumferential stress and the longitudinal stress may be assumed to be the same as in the foregoing discussion.

102. Thin Spheres. Since all the meridian sections of a sphere are the same as the transverse section of a cylinder of equal diameter, it is evident that the stress in the walls of a sphere is given by (4). If spherical heads, of the same thickness as the shell, are placed on a cylinder which is to withstand internal pressure, they will be subjected to a maximum stress equal to the transverse stress in the shell.

103. Resistance of Non-circular Thin Cylinders to Internal Pressure. Suppose a cylinder to have a cross-section made up of circular arcs, as in Fig. 92. Take the upper half as a free body (section along the major axis). Let the resultants of the components of pressure which are normal to the plane of the section be P_1, P_2, P_3, for the portions marked I, II, III, respectively. Then these resultant forces per unit of length of the cylinder are as follows:

$$P_1 = pr\int_0^{\phi'} \sin\phi \, d\phi = pr(-\cos\phi' + \cos o) = pm_1$$

$$P_2 = pR\int_{\theta''}^{\theta'} \sin\theta \, d\theta = pR(-\cos\theta' + \cos\theta'') = pm_2$$

$$P_3 = pr\int_{\phi''}^{\pi} \sin\phi \, d\phi = pr(-\cos\pi + \cos\phi'') = pm_3$$

Therefore,

$$P_1 + P_2 + P_3 = p(m_1 + m_2 + m_3) = pA$$

* See *Trans.* A.S.M.E., vol. 34, page 297.

In a similar way, if the section is taken along the minor axis, the resultant force normal to this axis is found to be pB. In like manner the resultant force normal to any section is (per unit of length of cylinder) equal to the intensity of pressure multiplied by the axis of that section. As B is less than A, the resultant force pB is less than pA; or the force tending to elongate the minor axis is greater than the force tending to elongate the major axis. If the tube were perfectly flexible, its form of cross-section would become, under pressure, one in which all axes are equal, or circular. A rigid material offers resistance to such change of form, and a flexural stress is produced in addition to the direct tension, but it approaches nearer to the circular form as the pressure increases. The existence of this flexural stress in a non-circular cylinder becomes

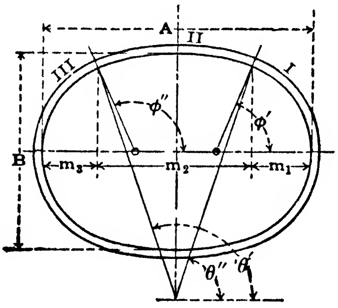


FIG. 92.

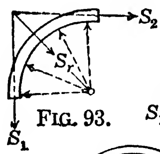


FIG. 93.

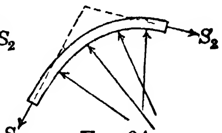


FIG. 94.

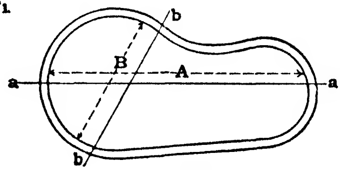


FIG. 95.

apparent from a comparison of Figs. 93 and 94. In Fig. 93 (circular section) the lines of normal pressure all pass through a single point (the center of the circle); the resultant (S_r) of the tensions (S_1 and S_2) also passes through this same point, hence these forces form a concurrent system, and they are in equilibrium. In Fig. 94, however, the pressures do not in themselves form a concurrent, nor parallel system of forces, hence they cannot be balanced by a single force (as the resultant S_r), but there must be a moment, or moments, of stress for equilibrium. A similar course of reasoning could be applied to a cylinder of any non-circular cross-section, for such a section (Fig. 95) could be considered as made up of circular arcs, each of which could be treated (like the special case of Fig. 92) by integrating between proper limits. A direct inspection will also show that in any cylinder of non-circular section, subjected to internal pressure, the pressure tends to reduce the cylinder to a circular cross-section. Suppose the original cylinder (Fig. 95) to be cut along the greatest axis of its cross-section, and that a flat bottom coinciding with the section-plane be secured to it, the lower portion of the cylinder being entirely removed. The total pressure on this bottom evidently balances the components of the pressure on the curved surface which lie normally to this flat bottom; hence, the resultant of these normal components of pressure equals $p(a \ . \ . \ a) = pA$, per unit of length of cylinder. In a similar way, the resultant of components of pressure acting normally to any other section (as $b \ . \ . \ b$, Fig. 95) equals $p(b) \ . \ . \ b = pB < pA$. This direct method might have been used in the preceding cases (Figs. 90 and 92) without recourse to the calculus.

It is apparent, then, that any cylinder under internal pressure tends to assume a circular cross-section. A cylinder of nominal circular section, but departing from the true form to some extent, tends to correct this departure under internal pressure; or if a circular cylinder under internal pressure is deformed by any external force, it tends to resume its circular shape. Thus a circular cylinder under internal pressure is in "stable equilibrium." If the section is other than a true circle there is a flexural stress, as well as tension, when under pressure.

RESISTANCE OF THIN CYLINDERS TO EXTERNAL PRESSURE

104. Theoretical Considerations. If a thin hollow cylinder of circular section is subjected to an external pressure, it is obvious that a course of reasoning similar to that in Article 101 will show that a compressive stress is induced in the walls of the cylinder, the value of which will be given by formula (1), Article 101, or

$$s_c = \frac{pd}{2t}$$

where s_c is a compressive stress.

If the cylinder were perfectly cylindrical, of uniform thickness, and of homogeneous material, there seems to be no reason why failure should occur until the compressive stress reaches the yield point of the material. But tubes are never absolutely circular in form, uniform in thickness, or homogeneous in character; and hence failure occurs long before the compressive yield point is reached. A tube which fails under external pressure is said to collapse, and the forms of collapsed tubes are very characteristic. Figure 96 shows the form of cross-section of collapsed tubes, and Unwin * has shown that the number of lobes depends on the ratio of length to diameter, the smaller this ratio the greater being the number of lobes. This peculiarity is undoubtedly due to the influence of the heads placed in the ends. For values of l/d greater than about 4 to 6, only the forms of collapse shown at c and d, Fig. 96, appear.

* See "Elements of Machine Design," page 101, 1901 edition.

If the non-circular cylinders of either Figs. 92 or 95 be considered as subjected to external pressure, the force tending to increase the major axis will be seen to be greater than that tending to increase the minor axis; hence the external pressure will cause collapse, unless the flexural rigidity of the material is sufficient to prevent this action. In a cylinder of nominal circular section any departure from the ideal section will be increased by the external pressure. Or, if a cylinder of true circular section is deformed in any way while under external pressure, this pressure will tend still further to increase the deformation. In other words, a cylinder under external pressure is in "unstable equilibrium." As perfectly true circular sections and homogeneous materials are not attainable in practice, the danger of collapse must be taken into consid-

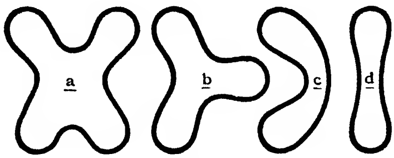


FIG. 96.

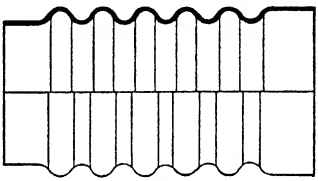


FIG. 97.

eration in designing pipes, tubes, or flues to withstand external fluid pressure.

Since the wall of an ideal thin tube is subjected to a uniform compressive stress, it may be considered as being in the same condition as a long column; and theoretical equations expressing the relation between the external pressure, the stress, and the dimensions of the tube have been developed on this basis. In view of the fact that the theory of long columns is itself most unsatisfactory, it is not surprising that such equations do not accord with actual results, and they may be safely disregarded; but the analogy between long compression members and tubes subjected to external pressure is instructive. Other deductions based upon the theory of elasticity, though throwing some light on the form of rational equations expressing these relations, are not as yet applicable to practical problems.

105. Long Tubes, Pipes, etc. Long tubes are defined as those so long that the influence of the heads is negligible in resisting collapse. The best known experimental results on the collapse of such tubes are those of Professors Carman and Stewart, respectively.

In 1906 Professor A. P. Carman published * the results of a set of experiments made at the Engineering Experiment Station of the University

* See *Bulletin* of the University of Illinois Engineering Experiment Station, vol. III, No. 17, June, 1906.

versity of Illinois, which prove conclusively that Fairbairn's equations (see Article 107) hold only for tubes whose lengths are from four to six times their diameters; and that beyond that ratio the collapsing pressure is independent of the length. He found that the results of his experiments could not be well expressed by a single equation, but devised two equations to cover the range; these equations expressing the relation which exists between the unit pressure p and t d, where t is the thickness and d the outside diameter of the tube. Thus, for values of $\frac{t}{d} > 0.025$, and length greater than four to six times the diameter, he gives

$$p = k\frac{t}{d} - c$$

where k and c are constants to be determined experimentally and depending upon the material.

For brass tubes,

$$p = 93{,}365\frac{t}{d} - 2474 \tag{1}$$

For seamless drawn cold steel,

$$p = 95{,}520\frac{t}{d} - 2090 \tag{2}$$

For lap-welded steel,

$$p = 83{,}270\frac{t}{d} - 1025 \tag{3}$$

Professor R. T. Stewart,* in an elaborate set of experiments on lap-welded steel boiler tubes made for the National Tube Company, found that for values of $\frac{t}{d} > 0.023$, the results of his work could be expressed by the following:

$$p = 86{,}670\frac{t}{d} - 1386 \tag{4}$$

which corresponds closely with (3) of Professor Carman's work, showing the accuracy of the experimental work.

For values of $\frac{t}{d} < 0.925$, Professor Carman found that the results of his work could be expressed by an equation of the form

$$p = k'\left(\frac{t}{d}\right)^3 \tag{5}$$

* See *Trans.* A.S.M.E., vol. 27, 1906.

where k', as before, is an experimentally determined constant, whose value for thin brass tubes is 25,150,000, and for thin cold-drawn seamless steel tubes 50,200,000.

Professor Stewart found that for values of t/d below 0.023, or practically the same limit as above, his results were expressed by

$$p = 1000\left(1 - \sqrt{1 - 1600\frac{t^2}{d_2}}\right) \tag{6}$$

The value of p for $t/d = 0.023$ is about 600 lb, which corresponds closely with the upper limiting value of p obtained from (5).

For values of t/d less than 0.023, the corresponding values of p, as found by either (5) or (6), do not differ materially. Furthermore, tubes in which $\frac{t}{d} < 0.02$ are not much used in engineering work under external pressure, and for convenience, therefore, equation (5) will be adopted.

106. Summary of Equations for Long Tubes. The work of Stewart and Carman deals entirely with tubes which are so long that the supporting effect of the heads is negligible, or in which the length is at least four times the diameter. Their experiments, conducted separately, supplement and corroborate each other. As given above, the equations are not in the most convenient form for use by the designer, since usually l, d, and p are known and t is required. Transposing these equations, therefore, they may be written as follows:

For values of $\frac{t}{d} < 0.025$ and pressures less than 600, equation (5) becomes

$$t = d\sqrt[3]{\frac{p}{k}} \tag{7}$$

where $k = 25{,}150{,}000$ for thin brass tubes, and 50,200,000 for thin cold-drawn seamless tubes or lap-welded steel tubes.

For values of $\frac{t}{d} > 0.025$ and pressures greater than 600 lb, equation (1) becomes

$$t = \frac{d(p + c)}{k} \tag{8}$$

where for brass tubes	$k = 93{,}365$ and	$c = 2474$
" " seamless cold-drawn steel,	$k = 95{,}520$ and	$c = 2090$
" " lap-welded steel, . . .	$k = 83{,}270$ and	$c = 1025$

The following approximate formula, which covers practically the whole range of values of t/d, is suggested by Professor Carman as useful in making rough calculations

$$p = k''\left(\frac{t}{d}\right)^2 \tag{8a}$$

where $k'' = 1{,}000{,}000$ for cold-drawn seamless tubes and 1,250,000 for lap-welded steel tubes. From this, the following formula, which is usually more convenient, can be derived:

$$t = d\sqrt{\frac{p}{k''}} \tag{8b}$$

Example. A lap-welded steel boiler tube,* 4 in. outside diameter and 10 ft long, is subjected to an external pressure of 300 lb per sq in. What must the thickness be in order to have a factor of safety of at least 6?

Here, the assumed collapsing pressure is

$$300 \times 6 = 1800 \text{ lb per sq in.}$$

Applying equation (8)

$$t = \frac{d(p + c)}{k} = \frac{4(1800 + 1025)}{83{,}270} = 0.14 \text{ in.}$$

Here the ratio

$$\frac{t}{d} = \frac{0.14}{4} = 0.035$$

and hence equation (8) applies.

In case this ratio should be less than 0.025, which will seldom occur, a second solution, using equation (7), should be made.

Messrs. Jasper and Sullivan † have made an extended résumé of the work of Stewart and Carman and have added thereto the results of experimental work designed to evaluate the effect of " out-of-roundness " of tubing. Space does not permit a discussion of the results presented. It should be noted, however, that if such a refinement were incorporated in design formulas it would entail corresponding refinement in inspection. Stewart's work was done upon commercial tubes in which as he states the out-of-roundness was complex and confusing so far as analytical

* See also the Boiler Code of the American Society of Mechanical Engineers. The Regulations of the U. S. Board of Supervising Inspectors and the rules of several insurance companies give equations for fixing the minimum thickness of tubes.

† See *Trans.* A.S.M.E., vol. 53, 1931.

conclusions were concerned. They, no doubt, represent average conditions as presented to the designer.

107. Short Cylinders, Flues, etc. When the cylinder or flue is short, that is, $\frac{l}{d} < 4$ to 6, the effect of the heads cannot be neglected and resort must be had to short-tube formulas. The best known experimental work on short tubes of moderate size is that of Sir William Fairbairn, who in 1858 made a series of carefully conducted experiments from which he deduced the following equation:

$$p = 9{,}675{,}600 \frac{t^{2.19}}{ld} \tag{9}$$

where p is the unit external collapsing pressure in pounds per square inch; and t, l, and d are the thickness, length, and outside diameter, respectively, in inches. Fairbairn himself modified this equation, for simplicity, to the form

$$p = 9{,}675{,}600 \frac{t^2}{ld} \tag{10}$$

or, transposing,

$$t = \sqrt{\frac{pld}{9{,}675{,}600}} \tag{11}$$

Many other equations have been deduced from the experiments of Fairbairn, usually of the same form but with different exponents. Thus Professor Unwin gives the following as the result of a careful résumé of Fairbairn's work:

For tubes with a **longitudinal lap-joint**

$$p = 7{,}363{,}000 \frac{t^{2.21}}{l^{0.9} d^{1.16}} \tag{12}$$

For tubes with a **longitudinal butt-joint**

$$p = 9{,}614{,}000 \frac{t^{2.21}}{l^{0.9} d^{1.16}} \tag{13}$$

For tubes with **longitudinal and cross-joints,** like an ordinary boiler flue

$$p = 15{,}547{,}000 \frac{t^{2.35}}{l^{0.9} d^{1.16}} \tag{14}$$

Other writers have deduced similar equations from the same data.

Fairbairn's experiments were conducted with tubes whose lengths were small compared to their diameters. In such tubes the effect of the supporting action of the head is noticeable; hence, his equations make the allowable pressure vary inversely as some function of the length. Now it is reasonable to suppose that if the tube were long enough the head would have no effect, except near the ends, and the collapsing pressure would be independent of the length. In a similar way, if the tube were very short, the walls should theoretically yield by crushing, and the intensity of the compressive stress would be given by formula (1), Article 101, or,

$$s = \frac{pd}{2t} \tag{15}$$

If this equation gives a lower working pressure than equation (10) the flue designed by it will be safe against collapse since (10) takes into consideration the supporting effects of the ends.

Many variations of Fairbairn's equation are in use. Thus the rules of Lloyd's Marine Register give the following:

$$p = \frac{C(t - 1)^2}{(l + 24)d} \tag{16}$$

where t = thickness of plate in thirty-seconds of an inch; l = length of flue between substantial supports in inches; d = external diameter of flue in inches; and C = a constant = 1450 when the longitudinal seams are welded and 1300 when the longitudinal seams are riveted. The Register also gives alternate equations for variations in conditions.

The Boiler Construction Code of the A.S.M.E. gives for furnaces with riveted longitudinal joints not over 18 in. in diameter, and where the length does not exceed 120 times the thickness of the plate,

$$p = \frac{51.5}{d}(18.75t - 1.03l) \tag{17}$$

When the length exceeds 120 times the thickness of the plate

$$p = \frac{4250t^2}{ld}$$

where l and d are in inches and t in sixteenths of an inch. If t is expressed in inches

$$p = \frac{1{,}088{,}000t^2}{ld} \tag{18}$$

which is Fairbairn's equation with a factor of safety of 9.

For furnaces over 18 in. and up to 38 in. in diameter the same formulas apply, but the code specifies the character of the riveted joints and if the furnace is over 6 diameters in length l is to be taken as $6d$.

For welded or seamless circular flues from 5 to 18 in. in diameter and where the thickness of the wall is not greater than 0.023 times the diameter the Code specifies

$$p = \frac{10,000,000t^3}{d^3} \tag{19}$$

Where the thickness of the wall is greater than 0.023 times the diameter

$$p = \frac{17,300t}{d} - 275 \tag{20}$$

where t and d are both in inches.

It will be noted that all these equations are applicable only to comparatively short flues. In practice, long flues are reinforced at short intervals by heavy rings of rolled section known as **collapse rings,** thus making the flue consist of a series of short flues to which these equations may be applied. Various insurance and governmental inspection departments give rules for proportioning flues and furnaces. These rules change from time to time, and if the construction of which the flue is a part is to be insured in any company the specific rules prescribed by it should be consulted. The student is referred particularly to the Boiler Construction Code of the A.S.M.E., which gives rules also for stayed and unstayed flues and pipes.

The foregoing discussion deals with flues and cylinders of moderate size. For larger vessels subjected to external pressure, reference may be made to the Proposed Rules for the Construction of Unfired Vessels Subjected to External Pressure by the A.S.M.E. Boiler Code in *Mechanical Engineering* for April, 1934. See also "Strength of Thin Cylindrical Shells under External Pressure," by Saunders and Windenburg, *Trans.* A.S.M.E., 1931, vol. 53.

108. Corrugated Furnace Flues. Flues corrugated as in Fig. 97 are very much stiffer against collapse than plain cylindrical flues, and with proper dimensions of corrugations may be safely made of any desired length. Their peculiar shape also permits of expansion and contraction under the influence of heat. The A.S.M.E. Boiler Code formula for such furnaces is

$$p = \frac{Ct}{t} \tag{21}$$

where p is in pounds per square inch and t and d are in inches; C is a constant that varies from 10,000 to 17,300, depending upon the type of corrugation.

In the Adamson flue short, flanged sections not less than 18 in. long are separated by reinforcing rings through which adjacent flanges are riveted. The code prescribes equation (17) for their design with the constant 57.6 substituted for 51.5.

The following references contain valuable practical information on this subject:

Lloyd's Register of British and Foreign Shipping.
"Steam Boilers," by Peabody and Miller.
Rules and Regulations of U. S. Board of Supervising Inspectors.
Rules and Regulations of the American Bureau of Shipping.
Rules of the British Board of Trade.
Boiler Construction Code of A.S.M.E.

THICK CYLINDERS

109. When the wall of a cylinder, which is subjected to internal or external fluid pressure, is thick, relatively to the internal diameter, it can no longer be assumed that the stress in the wall is uniformly distributed over the cross-section. In such cylinders, the stress is greater at the inner surface and decreases to a minimum at the outer surface, whether the pressure is internal or external. When the pressure is internal the stress is tensile, and when the pressure is external the stress is compressive.

Many formulas have been deduced to express the relations between pressure, stress, and cylinder thickness. Of these, the formulas deduced by Lamé, Birnie, and Clavarino are the best known. Lamé's equation, which was once much used, neglects the factor of lateral contraction (Poisson's ratio) and has therefore been superseded by others. Birnie's equation assumes that the ends of the cylinder are open and that, as a consequence, there is no longitudinal stress. It applies, therefore, to shrink fits, such as the jackets of large guns. For the most usual case of design, namely, where the ends of the cylinder are closed, Clavarino's modification of Lamé's equation is now much used, and it will be adopted in the work.

Ordinarily, a cylinder of this character is subjected to either an external or an internal pressure, but not to both. But in a gun tube, for example, which has a hoop shrunk upon it and has an internal pressure applied to it by the explosion of the charge, the more general case occurs, in which the cylinder is subjected to both internal and external pressure.

Let p_1 = the internal unit pressure.
p_2 = the external unit pressure.

r_1 = the internal radius of the cylinder.
r_2 = the external radius of the cylinder.
s_1 = the unit stress at the inner surface.
s_2 = the unit stress at the outer surface.

Then, by Clavarino's * equation, the unit stress at any radius r is

$$s = \frac{\left[r_1^2 p_1 - r_2^2 p_2 + \frac{4 r_1^2 r_2^2}{r^2}(p_1 - p_2)\right]^\dagger}{3(r_2^2 - r_1^2)} \qquad (1)$$

If the external pressure p_2 be zero, which is the most usual case, the greatest tensile stress is at the inner surface, and is

$$s_1 = \frac{p_1}{3}\left[\frac{r_1^2 + 4r_2^2}{r_2^2 - r_1^2}\right] \qquad (2)$$

or

$$r_2 = r_1\left[\frac{3s_1 + p_1}{3s_1 - 4p_1}\right]^{1/2} \qquad (3)$$

Example. A cast-iron cylinder, 20 in. in internal diameter, is to withstand an internal pressure of 1000 lb per sq in. How thick must the wall be in order that the stress at the inner surface may not exceed 4000 lb per sq in.?

Here $r = 10$, $p_1 = 1000$, and $s_1 = 4000$. Hence, substituting in (3)

$$r_2 = r_1\left[\frac{3s_1 + p_1}{3s_1 - 4p_1}\right]^{1/2} = 10\left[\frac{3 \times 4000 + 1000}{3 \times 4000 - 4 \times 1000}\right]^{1/2} = 12.8 \text{ in.}$$

or the cylinder walls must be 2.8 in. thick.

From (1) it is found that s_2, the stress in the cylinder walls at the outer fiber is 2620 lb.

PRACTICAL CONSIDERATIONS

110. Cast-iron pipes are widely used for underground water pipes and to some extent also for gas pipes, largely on account of their durability against corrosion. For steam, or for high pressures generally, cast-iron pipes are now seldom used because of their unreliability. For all ordinary purposes pipes made of wrought iron or steel are most used,

* The student is advised to read the discussion of thick cylinders given in Merriman's "Mechanics of Materials," eleventh edition, 1914, page 383. See also "Combined Stresses in Thick Cylinders," by E. B. Norris, *Trans.* A.S.M.E., vol. 51.

† In the derivation of this equation the induced stress s was assumed to be tensile. If the pressure is external, or if the external exceeds the internal pressure, the induced stress is compressive and will be negative in value. If the equation is applied to cylinders that are subjected to a greater external than internal pressure and a limiting stress is substituted for s, it must be written with the negative sign.

although in special applications, such as marine work where corrosion is to be resisted, copper and brass are preferred.

Wrought-iron or steel pipes may be either lap-welded or butt-welded, the latter being commonly used for the smaller diameters, while steel piping may be "drawn" so that there is no seam, when it is known as "seamless drawn tubing."

Standard piping is designated by its nominal internal diameter. Thus, standard 1-in. wrought-iron or steel pipe has a nominal internal diameter of 1.049 in., and an external diameter of 1.315 in. So-called standard wrought-iron piping may be used for pressures up to 100 lb, with safety. For still higher pressures, such as are found in high-class steam plants, thicker pipes, known as **extra strong,** are used. For hydraulic work, where pressures up to several thousand pounds per square inch must be withstood, still thicker piping, known as **double extra strong,** is used. These heavy pipes are made by decreasing the internal diameter of the standard pipe, thus keeping the outside diameter, and hence the screw threads for the flanges, to one standard.* Thus an extra strong 1-in. pipe (nominal size) would have an internal diameter of 0.957 in., and a double extra strong of the same nominal size would have an internal diameter of 0.599 in., the external diameter remaining 1.315 in. in all cases.

For large cylinders, both for steam and hydraulic service, cast iron is still much used and probably will be for some time, on account of the ease with which complicated iron castings can be made and machined. In steam-engine cylinders, the thickness of the walls is fixed by considerations other than those of strength, such as stiffness and the possibility of securing good castings. The proportions of steam cylinders, as fixed by practice, are the best guide. An examination of current practice shows the average thickness of low-speed engines to be given by the following, $t = 0.05d + 0.3$ in., † where t = thickness and d = diameter in inches, when the steam pressure does not exceed 125 lb per sq in.

Kent's "Mechanical Engineers' Pocket Book" gives the following as representing current practice, $t = 0.0004dp + 0.3$, where d = diameter in inches and p = pressure in pounds per square inch. If p be taken as 125 lb, this equation reduces to that given by Barr.

Cast iron is also much used for the cylinders of hydraulic machines,

* The student is referred to Kent's "Mechanical Engineers' Pocket Book," or similar works, for full tables of standard sizes of pipes, flanges, etc. See also current trade catalogues.

† See "Current Practice in Engine Proportions," by J. H. Barr, *Trans.* A.S.M.E., vol. 18.

although steel castings are better in general In such cases equations (1) to (3) of Article 109, in common with all equations based on the theory of elasticity, should be used with caution when cast iron is selected for the cylinder Furthermore, it must be borne in mind that the thicker the cylinder walls, the more likely they are to be porous in the interior, when made of castings It is safer, therefore, as a rule, to carry a high working stress, within safe limits and insure sound castings, than to design thick walls which are open to suspicion, in order to get a theoretically lower stress A 3-in wall, for instance, with a working stress of 5000 lb per sq in , is preferable to a 4-in wall with a working stress of 3000 lb per sq in Care should also be exercised in cast cylinders to avoid excessive thickness of metal at any point, thus insuring sound castings Thick castings of any metal are very likely to give trouble by leaking on account of porosity, if subjected to high pressures, and cast-iron cylinders are often fitted with brass or bronze liners to obviate this difficulty.

111. Pipe Couplings, Flanges, etc. Methods for securing the ends of pipes together have become of greater importance as higher steam pressures have been employed The most usual method has been to thread the ends of the pipes (see Article 135) and secure them together with either a cylindrical pipe **coupling,** a pipe **union,** or a pair of pipe **flanges.** All of these are in very common use For pressures up to 100 lb per sq in. and pipes not over 12 in. in diameter these may be used with success, but for higher pressures and larger diameters they are not so satisfactory. The pipe connection known as a union is used on small pipes only.

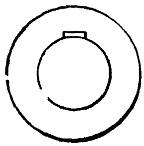
In the ordinary screwed fitting of large size it is difficult to cut the thread accurately, and to screw the fitting on tight enough to prevent leakage at A, Fig 98 (a). This can be remedied to some extent by making the threaded portion of the pipe long enough to project through the flange slightly, and then facing off pipe and flange so as to make a smooth surface, and permitting the packing or gasket (P) to cover up the screwed joint, as shown at B, Fig 98 (a). Even this joint, however, is likely to leak if the workmanship is poor or if the flanges do not align properly. For high-pressure piping, threads of somewhat finer pitch than ordinary are sometimes used so as to reduce the torque due to screwing up (see Article 133).

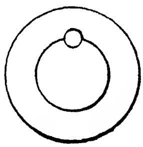
To obviate the difficulties of the screwed joint on pipe of larger diameter, the flanges are sometimes shrunk on as shown in Fig. 98 (b) (see also Article 153). In order to insure tightness, and secure a firmer grip on the flange, the end of the pipe is usually expanded into the flange, as shown in Fig. 98 (b). The gasket usually covers up the joint between

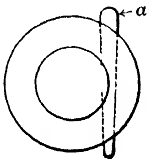
CHAPTER XII

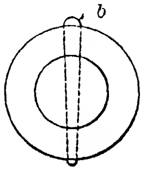
KEYS, SPLINES AND COTTERS

146. Forms of Keys. Keys are wedge-shaped pieces, generally made of steel, which are used primarily to prevent relative rotation between shafts and the pulleys, gears, etc., which they carry. On account of the frictional resistance which they induce between the surface of the shaft and the member which is keyed to it, they also often prevent relative sliding of the parts. Keys are most usually rectangular in cross-section; but occasionally they are made in circular form. A **saddle key** is shown in Fig. 124. This form of key does not require the shaft to be cut; but its holding power is so small that it is used only for









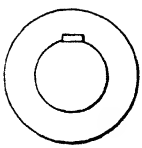


Fig. 124. Fig. 125. Fig. 126 (a). Fig. 126 (b). Fig. 127.

light work. Where the hub is to be fastened to the end of a projecting shaft, a round key, as shown in Fig. 125, is often used. Formerly this form of key was used for light loads only, but the Nordberg Manufacturing Company has used them very successfully for heavy loads. In the practice of this company, such pins are tapered $\frac{1}{16}$ in. for every foot of length, and the holes are carefully reamed, thus securing a good fit at every point. Figures 126 (a) and 126 (b) show two other methods of applying round taper pins as a substitute for keys where light loads are to be resisted. A **flat key** is shown in Fig. 127. This form requires a small portion of the shaft to be cut away, and its holding power is much greater than that of the saddle key. The **sunk key,** Fig. 128, is the most secure form of key fastening, and is more used than any other. It is so called because it is sunk into a keyway or groove cut in the shaft. It thus requires more metal to be cut away from the shaft than the flat key, and this must be taken into account in designing shafting, since the metal is removed from the outer fiber where it is most serviceable for resisting applied loads. The keyway cut in a shaft for a sunk key is made parallel to the axis of the shaft; but the keyway in the hub of the pulley or gear which is to be made fast is cut tapering, as shown in Fig. 128 (b). The sides of the key are parallel, as shown, and should fit well in both shaft and hub. When the key is driven in, the shaft and hub are drawn tightly together on the side of the shaft opposite to the key, and the frictional resistance thus set up helps to prevent relative sliding of the parts lengthwise of the shaft. If the bore of the hub is tapering, or if the key fits more tightly at one end than at the other, the part keyed on may be thrown out of alignment so that its plane is not perpendicular to the axis of the shaft. Where great accuracy is required, as in flanged couplings on shafting, owing to this tendency, the faces of

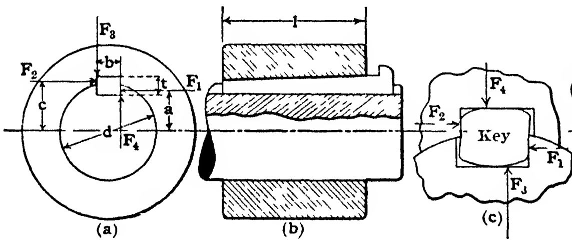


FIG. 128.

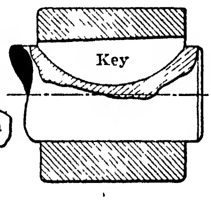


FIG. 129.

the flange or part secured to the shaft should be faced in place after the key is driven. If the part keyed on does not have to be removed often, the hub may be made a tight or press fit on the shaft, thereby preventing largely the tilting action of the key which might otherwise occur. In the **Woodruff system** (Fig. 129), the key is a circular segment and the keyway may be cut with a milling cutter. This allows the key to adjust itself to the taper of the keyway in the hub, hence it will not throw the keyed part out of perpendicular alignment. With this system, the hub must be forced on over the key. These keys are used largely in machine tools.

In general, the part to be secured on the shaft is placed in position and the key driven in. This makes it necessary to extend the keyway along the shaft at least the length of the key (except when the hub is at the end of the shaft), unless the diameter of the shaft is enlarged under the hub, sufficiently to allow the keyway to be cut without cutting into the shaft proper. Where it is desirable to withdraw the key occasionally, it is often provided with a head, as shown in Fig. 128 (b), in which case it is called a **draw key*** or **gib head key.** The head of the draw key should always be carefully guarded, or covered in some manner, in all fast-running machinery, as there is danger that the clothing of a workman may be entangled, with fatal results. Sometimes, however, it is not desirable to extend the keyway beyond the hub, in which case, the keyway in the shaft is made the same length as the key, and the hub is driven over the key into its correct position. Much more force is necessary to drive the hub into place in this manner than to drive the key, on account of the friction between the shaft and the hub When the hub is a sliding or an easy fit on the shaft, and only one key is used, there is a tendency to throw the hub eccentric to the shaft Under these circumstances there is a tendency for the hub to rock and work loose on the shaft, especially if the direction of motion be reversed In such cases two keys set 90° apart make a much more secure fastening, as this gives three lines of contact and prevents rocking If one of these keys is a saddle key, as shown in Fig 124, the fitting is greatly facilitated and the fastening is almost as secure as with two sunk keys.

147. Stresses in Common Sunk Keys. Since keys are designed to prevent relative rotation, it is evident that every key must transmit a certain torsional moment, or torque. This torsional moment may be equal to the total torque transmitted by the shaft, or the key may be required to transmit only a part of it This would indicate that keys of different sizes should be used with any given diameter of shaft, depending on the load which the key must transmit. For practical reasons, however, such as standardization and interchangeability, it is desirable that the dimensions of the shaft and key should bear a fixed relation to each other All practical systems of keys, therefore, give a fixed size of keys for each diameter of shaft, the dimensions of the key, presumably, being such that its strength is equal to the torsional strength of the shaft Shafts are usually designed for torsional stiffness rather than torsional strength, which results in a shaft considerably larger than necessary as far as strength is concerned. If, under these circumstances, the key is designed as indicated above, it will also have excess strength. Where the shaft is short and is designed for strength alone, the key should be more carefully considered.

Keys resisting a torsional moment are subjected to simple crushing, or to crushing and shearing, depending on the manner of their application and manner of fitting. The ordinary sunk key (Fig 128 (a)), is subjected to a force, F_1, due to the pressure from the shaft, and to a resisting force, F_2, due to the reaction from the hub which it secures. The effect

* Where a draw key cannot be used the point of the key is sometimes case-hardened so that it will not upset so readily in being driven out.

of these two forces is to set up a shearing stress along the middle section of the key at the outer surface of the shaft. They also form a couple which tends to rotate the key in the keyway. This tendency to rotate should be, for best results, resisted by the pressure of the hub and shaft against the top and bottom of the key. If the key is not a tight fit on the top and bottom these resisting pressures, F_3 and F_4, will be concentrated near the corners. This concentrated pressure may be sufficient to crush the key at these points, and allow it to roll in the keyway, deforming both the keyway and key and subjecting the key to a severe crushing action rather than simple shear. If the conditions of service require a continual reversal of motion, a state similar to that shown in Fig 128 (c) is induced, where the resisting forces F_3 and F_4, have been moved inward and their moment arm made so short that their magnitude must be very great to hold the key in position. This may bring a severe bursting stress on the hub. It is evident, therefore, that keys which fit sidewise only, cannot be depended on to carry as great a load as those which fit well on the top and bottom. Where great accuracy is required, as in machine tool construction, the hub is often made a force fit on the shaft and the key fitted only on the sides, so that it cannot throw the parts out of relative alignment by radial pressure.

Referring to Fig. 128 (a):

Let l = the length of the key or hub.
t = the thickness of the key.
b = the breadth of the key.
T = the torsional moment applied to the shaft.
P = the force acting at the radius of the shaft so that $P\frac{d}{2} = T$.

Then for shearing stress s_s.

$$P = s_s lb \tag{1}$$

and since the torsional moment applied to the shaft must equal the moment of the crushing load applied to the side of the key,

$$T = P\frac{d}{2} = F_1 a$$

or

$$P\frac{d}{2} = s_c l\frac{t}{2}\left(\frac{d}{2} - \frac{t}{4}\right) \tag{2}$$

If F_1 be considered to act at the radius of the shaft (which can be done without serious error for keys as ordinarily proportioned) equation (2) reduces to

$$P = s_c l \frac{t}{2} \tag{3}$$

Equations (1) and (3) may be used to compute the stresses in any sunk key.

If the shearing resistance of the key is to equal the crushing resistance, then from (1) and (3),

$$s_s lb = s_c l \frac{t}{2}$$

$$\therefore \frac{b}{t} = \frac{s_c}{2s_s} \quad \text{or} \quad t = b\frac{2s_s}{s_c} \tag{4}$$

If $s_c = 2s_s$, $t = b$, and the key is square for equal resistance to shearing and crushing. For machinery steel, *at the elastic limit,*

$$\frac{s_s}{s_c} = 0.6$$

and hence from (4) for equal strength in shearing and compression $t = 1.2b$. If, in addition, the moment of the shearing resistance of the key is to be equal to the torsional resisting moment of the shaft, then,

$$T = s_s lb\frac{d}{2} = s'_s \frac{\pi d^3}{16} \tag{5}$$

where s'_s is the shearing stress in the outer fiber of the shaft. For steel shafts and keys, which are most common, $s_s = s'_s$, whence from (5),

$$lb\frac{d}{2} = \frac{\pi d^3}{16} \tag{6}$$

The minimum length of hub (l), as determined by practice, which is necessary to give a good grip on the shaft, should not be less than $3d/2$. Substituting this value of l in equation (6),

$$\frac{3bd^2}{4} = \frac{\pi d^3}{16}$$

$$\therefore b = \frac{\pi}{12}d = \frac{d}{4}, \text{ nearly} \tag{7}$$

The above would, therefore, give keys of breadth, $b = d/4$, depth or thickness $t = 1.2b = .3d$, and minimum length $\frac{3}{2}d$. Flat keys as used in practice conform closely to these rules as far as length and breadth are concerned; but, to avoid cutting away so much of the shaft, the thickness is usually much less than that given above. An average value of the thickness may be taken at $\frac{5}{8}b$. This gives a key considerably thinner than it is wide and makes it weakest in crushing. The crushing resistance can, however, be increased by lengthening the key or by using a hard grade of steel.

Keys designed as above usually have an excess of strength, since the friction between the shaft and the hub materially decreases the load actually brought upon the key. In addition, as has been noted, shafts are most usually designed for stiffness or angular distortion, and therefore are greater in diameter than would be required for strength alone. If the key is made proportional to the shaft diameter as above, it must, therefore, have excess strength against rupture; and such keys seldom fail unless subjected to severe shock or extraordinary loads.

There are no fixed standards for the dimensions of keys, various machine builders having their own standards.* The dimensions listed in Table XXI are recommended by the American Standards Association. See standard sheet B17e–1927 of A.S.M.E. If the length must be less than $\frac{3}{2}d$, the crushing stress should be computed, as it may be necessary to use two keys.

The **taper of sunk keys** is usually about $\frac{1}{8}$ in. per foot of length, which is the taper recommended by the A.S.A. standard.

Another form of sunk key, known as the Kennedy Key, is shown in Fig. 130. It has been used with great success in very heavy work, such as rolling-mill machinery. For such heavy work the width of the keys is made approximately one-fourth of the shaft diameter and they are located in the hub, so that the diagonals through the corners of the key pass through the center of the shaft. The taper of the key is about $\frac{1}{8}$ in. to the foot of length and is made on the top of the key, the sides being a snug fit.

In applying these keys for very heavy work, the hub is first bored for a pressed fit and then rebored eccentrically to provide a small clearance on the side to which the keys are fitted, so that when the keys are driven the hub will be concentric with the shaft. Such keys act as compression members or as struts. The keyways are more difficult to cut and the shaft is cut deeper than for the ordinary sunk key, but the holding power of such keys is very great. For diameters of shaft up to about 6 in., a single key is usually sufficient, but above that two keys are standard practice. For comparatively light loads the keys are sometimes made somewhat thinner.

* See Kent's "Mechanical Engineers' Handbook," page 1328.

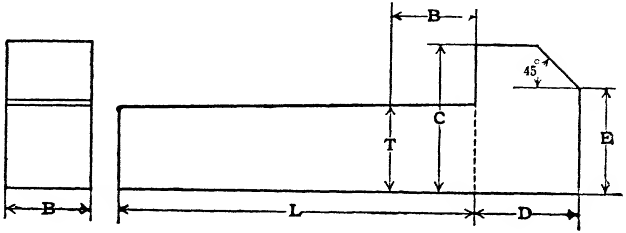


TABLE XXI

A S A STANDARD KEYS

Diameter of Shaft, Inches	Square Keys					Flat Keys					Tolerances	
	Key		Gib Head			Key		Gib Head				
	Width B	Thickness T	Height C	Length D	Height E	Width B	Thickness T	Height C	Length D	Height E	Width (Minus)	Thickness (Plus)
$\frac{1}{2}$ – $\frac{9}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{7}{32}$	$\frac{5}{32}$	$\frac{1}{8}$	$\frac{3}{32}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	0 0020	0 0020
$\frac{5}{8}$ – $\frac{7}{8}$	$\frac{3}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{9}{32}$	$\frac{7}{32}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{5}{32}$	0 0020	0 0020
$\frac{15}{16}$–$1\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{7}{16}$	$\frac{11}{32}$	$\frac{11}{32}$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{1}{4}$	$\frac{3}{16}$	0 0020	0 0020
$1\frac{5}{16}$–$1\frac{3}{4}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{11}{16}$	$\frac{15}{32}$	$\frac{15}{32}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{7}{16}$	$\frac{3}{8}$	$\frac{5}{16}$	0 0020	0 0020
$1\frac{13}{16}$–$2\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{7}{8}$	$\frac{19}{32}$	$\frac{5}{8}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{5}{8}$	$\frac{1}{2}$	$\frac{7}{16}$	0 0025	0 0025
$2\frac{5}{16}$–$2\frac{3}{4}$	$\frac{5}{8}$	$\frac{5}{8}$	$1\frac{1}{16}$	$\frac{23}{32}$	$\frac{3}{4}$	$\frac{5}{8}$	$\frac{7}{16}$	$\frac{3}{4}$	$\frac{5}{8}$	$\frac{1}{2}$	0 0025	0 0025
$2\frac{7}{8}$ –$3\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$1\frac{1}{4}$	$\frac{7}{8}$	$\frac{7}{8}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{7}{8}$	$\frac{3}{4}$	$\frac{5}{8}$	0 0025	0 0025
$3\frac{3}{8}$ –$3\frac{3}{4}$	$\frac{7}{8}$	$\frac{7}{8}$	$1\frac{1}{2}$	1	1	$\frac{7}{8}$	$\frac{5}{8}$	$1\frac{1}{16}$	$\frac{7}{8}$	$\frac{3}{4}$	0 0030	0 0030
$3\frac{7}{8}$ –$4\frac{1}{2}$	1	1	$1\frac{3}{4}$	$1\frac{3}{16}$	$1\frac{3}{16}$	1	$\frac{3}{4}$	$1\frac{1}{4}$	1	$\frac{13}{16}$	0 0030	0 0030
$4\frac{3}{4}$ –$5\frac{1}{2}$	$1\frac{1}{4}$	$1\frac{1}{4}$	2	$1\frac{7}{16}$	$1\frac{7}{16}$	$1\frac{1}{4}$	$\frac{7}{8}$	$1\frac{1}{2}$	$1\frac{1}{4}$	1	0 0030	0 0030
$5\frac{3}{4}$ –6	$1\frac{1}{2}$	$1\frac{1}{2}$	$2\frac{1}{2}$	$1\frac{3}{4}$	$1\frac{3}{4}$	$1\frac{1}{2}$	1	$1\frac{3}{4}$	$1\frac{1}{2}$	$1\frac{1}{4}$	0 0030	0 0030

148. Feathers or Splines. Sometimes it is desirable to have the hub free to slide axially along the shaft, but constrained to rotate with it. In such cases a **feather,** or **spline,** is used. The sides of the spline are parallel, and it may either be fastened rigidly to the shaft or move with the hub. Small splines are frequently dovetailed into the shaft (or hub), as shown in Fig. 131 (a); larger ones are often held in place by means of countersunk screws (Fig. 131 (b)), or rivets.

A common way of securing the feather so that it will move with the hub is shown in Fig. 133. Splines are subjected to a shearing stress across the mid-section at the radius of the shaft, and to a crushing stress on the sides in the same way as sunk keys. Being fitted loosely on the top and bottom, they do not produce any friction between the hub and

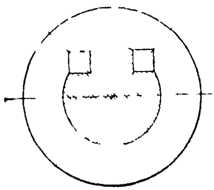


Fig. 130

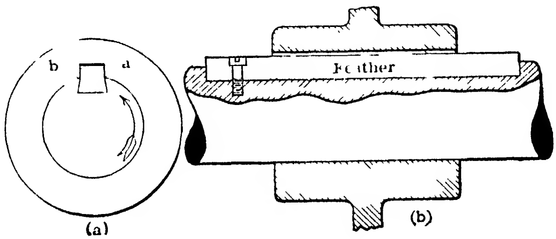


Fig. 131.

the shaft and, therefore, offer much less resistance than sunk keys to the rolling action imposed upon them (see Article 147). This rolling action tends to bring a concentrated crushing force at *a* and *b* (Fig. 131 (a)), if the feather is not rigidly secured to either the hub or the shaft. For this reason, and in order also to provide ample wearing surfaces, feathers are usually given a greater radial depth than sunk keys, and from their general proportions are often distinguished as square keys. It is evident that the holding power of splines is not equal to that of sunk keys.

The following table gives dimensions of feathers which agree with common practice:

TABLE XXII

Dimensions of Feather Keys in Inches

Diameter of shaft, d..	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	5	6	7	8	9	10
Breadth of feather, b	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{3}{8}$	$\frac{7}{16}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1	$1\frac{1}{8}$	$1\frac{3}{8}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{4}$
Thickness of feather, t	$\frac{3}{8}$	$\frac{7}{16}$	$\frac{1}{2}$	$\frac{9}{16}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1	$1\frac{1}{4}$	$1\frac{3}{8}$	$1\frac{5}{8}$	$1\frac{3}{4}$	2	$2\frac{1}{2}$	$2\frac{3}{4}$

The length of feather keys is, in general, greater than that of sunk keys, for the same size of shaft, in order to reduce the bearing pressure and increase the wearing surface on the sides.

Where very great torsional loads are to be transmitted, the ordinary splines described in the foregoing have not been found to be satisfactory. In such cases the shaft is sometimes made square, as illustrated in Fig. 132 (a), or hexagonal, or, of some other special shape that will effectively resist rotation under heavy load. The squared or hexagonal shaft induces a bursting stress in the hub, *H*, and when such shapes are used special attention should be given to the strength of the hub. In recent years, especially in automobile driving shafts, where the load transmitted is very great compared to the size of the shaft, multiple-spline

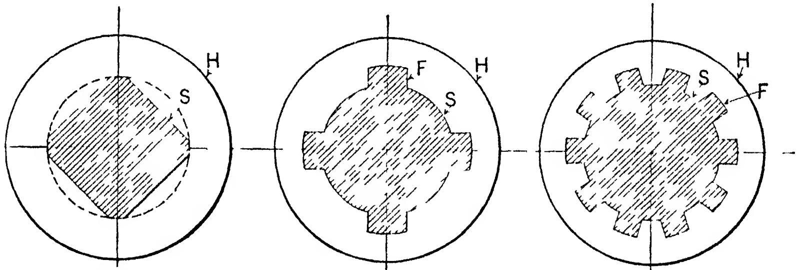


Fig 132.

shafts with the splines made integral with the shaft have come into extensive use. The usual manner of constructing such splined shafting is shown in Fig. 132 (b) and (c). The splines are made by milling or hobbing processes, and the spaces in the hub or collar, *H*, are usually made by the broaching process. If a considerable number of such parts are to be made, they can probably be produced more cheaply than the ordinary spline or key. These forms of splined shafting are also coming into more common use in industries other than automobile construction. The Society of Automotive Engineers has established standard dimensions for squared shafts and for multiple-splined shafts with four, six, ten, and sixteen integral splines, both for sliding and for permanent fit of the collar on the shaft. The student is referred to the S.A.E. standards for such dimensions.

149. Cotters. A cotter is a form of key used to prevent relative sliding between two members. Figure 134 shows a method of securing a piston rod to a piston by means of a cotter. In this case the connection is permanent in character, the cotter being removed only when the piston or piston rod is repaired or renewed. In other forms of cottered joints of this character, the rod is not tapered, but is prevented from sliding into the boss by means of a shoulder or by the cotter alone. The cotter is usually rectangular in section, but sometimes the edges are rounded so as to avoid sharp corners in the opening cut through the rod or to facilitate machining. In light work a taper pin of circular section is often used as a cotter. Figure 135 shows an arrangement of a **gib and cotter** (commonly known as a **gib and key**), such as is used on the ends of the connecting-rod of steam engines. The function of the gib is to prevent spreading of the strap. This arrangement permits a small amount of adjustment between the strap and the connecting-rod for taking up wear on the pin and brasses.

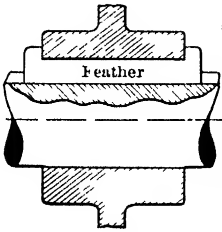


FIG. 133.

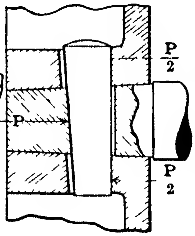


FIG. 134.

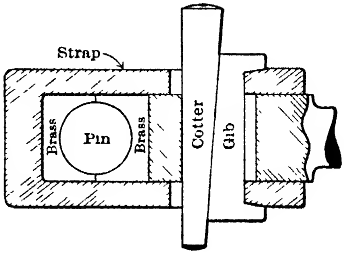


FIG. 135.

150. Stresses in Cotters. A cotter of the form shown in Fig. 134 is a beam supported at the ends. The exact distribution of the loading is indeterminate, as the bending of the cotter tends to concentrate the load near the points of support. It is sufficiently accurate, however, to consider the load as uniformly distributed. The area of the surface of the cotter where it bears on the rod, and also on the hub, should be sufficiently great to prevent crushing of the material. This indicates that the diameter of the hub should, for similar materials, be twice that of the rod, which is the usual proportion. The section of the cotter at the point of support should be great enough to prevent shearing, and in many cases it is sufficient to compute the section for shear alone, neglecting the bending action.

When a cottered joint of this character is made, the cotter must be driven in tight enough to prevent its backing out. This is especially true when the load is a reversed one, as in a steam-engine piston. This induces an initial stress in the cotter and rod, over and above that due to the load P. The conditions, in fact, are somewhat similar to those which exist in screwed fastenings (see Article 138). The initial stress due to the driving of the cotter cannot be accurately computed, though it may be very great. For this reason all calculations of dimensions based on the maximum applied load should be modified to suit the conditions of service and the materials of which the joint is made. Thus if the rod be of brass and the hub or boss of steel, as is common in pump work, the proportions would be different from those employed if all the materials were of steel or of steel and cast iron.

Let d = the diameter of the rod where the cotter passes through.
t = thickness of cotter.
b = breadth of cotter.

Then, in order that the net cross-section of the rod may be as strong in tension as the cotter and rod, where they bear upon each other, are in crushing,

$$s_t\left(\frac{\pi d^2}{4} - td\right) = tds_c$$

$$\therefore\ t = \frac{\pi d}{4\left(1 + \frac{s_c}{s_t}\right)} \qquad (1)$$

For a steel rod and steel cotter where $s_c = s_t$,

$$t = 0.4d \qquad (2)$$

Good practice gives

$$b = 4t = 1.57d \qquad (3)$$

The taper of cotters, as shown in Fig. 134, should be so small that there is no danger of backing out and should not exceed $\frac{1}{2}$ in. per foot of length. An auxiliary locking device is often used in arrangements such as shown in Fig. 135, in which case the taper may be as great as 1 in 8.

In the form of cotter shown in Fig. 135, the stress due to driving the key may be disregarded, and the design based on the maximum applied load. The student is referred to treatises on steam-engine design for relative proportions of this form.

It is often necessary to allow a rather high bearing pressure on the cotter to avoid large and clumsy proportions. An examination of successful practice shows an allowable pressure of 15,000 lb per sq in. as computed from the applied load.

CHAPTER XIII

MACHINE FITS—FORCE AND SHRINKAGE FITS

151. Interchangeable Manufacturing. The degree of closeness with which two machine parts engage each other is known as **the fit** and, as will be seen, is necessarily of considerable variation. In building certain classes of machinery and especially where only a limited number are produced the exact *size* of the mating parts is not, within close limits, a matter of great importance, and considerable fitting together is done during assembly. In mass production, as in producing typewriters, automobiles, etc., it is essential that all fits be made to known dimensions, with great accuracy, for the following reasons:

(1) To obviate all fitting at assembly.

(2) To enable manufacturers of accessory parts and appliances such as bolts, nuts, shafting, taps, dies, etc., to coordinate their work with that of various producers of machinery.

(3) To make it possible to supply repair parts that will fit with accuracy.

The basic requirement for such a system of interchangeable parts is furnished by the Johannsen block gages. These blocks are made of steel about $\frac{3}{8}$ in. wide and 1 in. long, hardened and ground to very exact thickness and parallelism between faces. In the best sets the dimensional errors are not greater than 0.000004 in. A set consists of about 80 blocks varying in thickness from 0.05 in. to 4 in. by means of which a large number of dimensions can be built up. Such accuracy of course assumes a fixed temperature, taken in this case as 65° F. From these master blocks, working gages * of great accuracy are made with which to gage the machine fits.

The most important case of machine fits is where circular parts fit into circular holes. Obviously if such fits are to be of varying degrees of looseness either the hole or the shaft or both must vary in diameter. Space forbids a discussion of the relative merits of hole or shaft diameter

* See "Methods of Gaging and Specifications for Plain Limit Gages" published by A.S.M.E.

as a basis of such variation. The American Standards Association * has adopted the diameter of the hole as a basic size and has established the following definitions:

1. **Basic size:** the exact theoretical size from which all variations are made.
2. **Allowance:** an intentional difference in the dimensions of mating parts to provide for different classes of fits.
3. **Tolerance:** the amount of variation permitted in the size of a part.

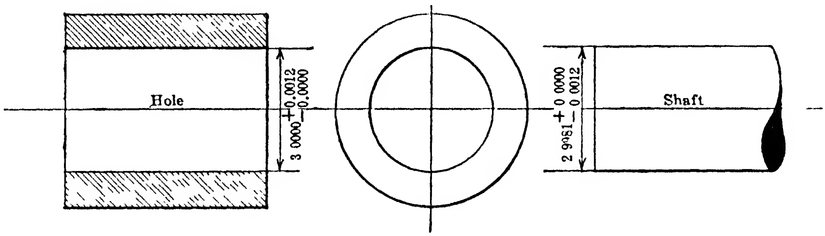


FIG. 136.

Thus, referring to Fig. 136, the allowance is 3.000 − 2.9981 = 0 0019. The tolerance for the hole is plus 0.0012, and the tolerance for the shaft is minus 0.0012. Therefore

	Hole	Shaft	Difference
For tightest fit:	3 0000	2 9981	0 0019 = allowance
For loosest fit:	3 0012	2 9969	0 0043 = allowance + tolerances

The Standard referred to recommends eight classes of fits, namely:

Class 1. **Loose fit,** where accuracy is not essential.

Class 2. **Free fit,** as in ordinary engine and dynamo bearings where the rpm are 600 or over and bearing pressure 600 lb per sq in. or over.

Class 3. **Medium fit,** for running fits where rpm are under 600 and bearing pressures less than 600 lb per sq in.

Class 4. **Snug fit,** to be used where no perceptible shake is permissible and where mating parts are not intended to move freely.

Class 5. **Wringing fit,**† which is practically metal-to-metal contact.

Class 6. **Tight fit,** which requires light pressure to assemble and is intended for more or less permanent assembly.

* See "American Standard Tolerances, Allowances and Gages for Metal Fits," B4*a*–1925, by A.S.M.E.

† The report states that this is sometimes called a "tunking fit." The term is neither elegant nor necessary and should be dropped.

Class 7. **Medium force fit,** in which the allowance is negative and considerable force is necessary to assemble the parts. These fits are the tightest recommended for cast-iron hubs.

Class 8. **Heavy force and shrink fit,** for steel hubs and where very great holding force is required.

The first four of these classes of fits are intended for use in interchangeable work. The last four, where the allowance is zero or a minus quantity, which is defined by the A.S.A. as **interference,** are more permanent in character and **selective** rather than interchangeable.

Table XXIII * lists the formulas recommended by the report for calculating allowances and tolerances, and the report tabulates allow-

TABLE XXIII

ALLOWANCES AND TOLERANCES

Class of Fit	Method of Assembly	Allowance	Selected Average Interference of Metal	Hole Tolerance	Shaft Tolerance
(1) Loose	Strictly interchangeable	$0.0025\sqrt[3]{d^2}$		$0.0025\sqrt[3]{d}$	$0.0025\sqrt[3]{d}$
(2) Free	Strictly interchangeable	$0.0014\sqrt[3]{d^2}$		$0.0013\sqrt[3]{d}$	$0.0013\sqrt[3]{d}$
(3) Medium	Strictly interchangeable	$0.0009\sqrt[3]{d^2}$		$0.0008\sqrt[3]{d}$	$0.0008\sqrt[3]{d}$
(4) Snug	Strictly interchangeable	0.0000		$0.0006\sqrt[3]{d}$	$0.0004\sqrt[3]{d}$
(5) Wringing	Selective assembly		0.0000	$0.0006\sqrt[3]{d}$	$0.0004\sqrt[3]{d}$
(6) Tight	Selective assembly		$0.00025d$	$0.0006\sqrt[3]{d}$	$0.0006\sqrt[3]{d}$
(7) Medium force	Selective assembly		$0.0005d$	$0.0006\sqrt[3]{d}$	$0.0006\sqrt[3]{d}$
(8) Heavy force or shrink	Selective assembly		$0.001d$	$0.0006\sqrt[3]{d}$	$0.0006\sqrt[3]{d}$

* See also "Machinery's Handbook," page 981, for other data from practice for tolerances, allowances, and fits.

ances and tolerances in detail for each class of fit. The report further lists the forcing pressure for force fits and also the stress at the inner surface of the hub. These values are given only for hubs twice the diameter of the shaft. However, for shafting other than the small sizes, hubs are rarely twice the shaft diameter even in heavy shrink fits such as are used in marine crankshafts. It may be well, therefore, to discuss the general theory of force and shrink fits.

152. General Considerations. Crank discs, the hubs of heavy flywheels, impulse water wheels, and in general, work which is to be subjected to shock or vibration, must be fastened to the shaft more securely than is possible with a key, when the hub is a sliding fit on the shaft. In such cases the bore of the hub is made slightly smaller than the diameter of the shaft, and the shaft is forced cold into the hub; or the hub is expanded by heating till the bore is slightly larger than the shaft, then slipped over the shaft and allowed to cool in place. The first method is known as a **force or pressure fit** (class 7), and the second as a **shrinkage fit** (class 8). The degree of tightness or " grip " required between shaft and hub depends largely on the service. Thus, with shafts up to 3 or 4 in. in diameter, a difference between the diameter of the shaft and the bore, such that the parts may be driven together with a hand sledge, is often satisfactory. Such a fit is called a **driving fit** (class 6), and the difference between the shaft diameter and the bore is very small. With such work as armature spiders and flywheel hubs, the allowance for the press fit depends largely on the facilities for erection. If the parts can be forced together in the shop, where adequate means, in the form of a powerful hydraulic press, is to be had, an allowance requiring a pressure of 100 tons or more may be made. But if the parts must be erected in the field, this allowance may have to be reduced on account of the difficulties of erection. It is usually possible in armature spiders, flywheel hubs, etc., to obtain a sufficiently tight grip on the shaft by means of a press fit without inducing undue stress in the parts. Dependence for preventing relative rotation may, in a large measure, be placed upon the key in all such cases.

In such work as crankshafts, when built up from separate parts, it is often necessary to insure as strong a grip upon the shaft as is possible without inducing undue stress. A greater difference between the shaft diameter and the bore of the hub is then allowed than in force fits and the parts are usually put together by shrinking. In such fits the stresses induced are of importance and should be carefully considered.

153. Stresses Due to Force Fits. If x be the elongation or contraction of any radius r, then $2\pi x$ is the corresponding elongation or contraction of the circumference $2\pi r$. The elongation or contraction of the circumference per unit of length is $2\pi x \; 2\pi r$. If s be the stress which would induce this strain, and E be the coefficient of elasticity of the material, then,

$$E = \frac{\text{Stress}}{\text{Strain}} = \frac{s}{\dfrac{2\pi x}{2\pi r}} \quad \text{or} \quad x = \frac{sr}{E} \tag{1}$$

In Fig. 137 let A represent a hollow shaft on which has been forced or shrunk a hub or boss B, the radius of the contact surface being r_2. Before the operation of pressing, the outer radius of the shaft was

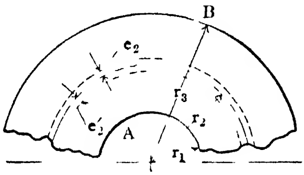


FIG. 137.

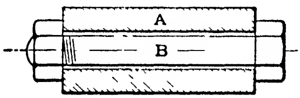


FIG. 138.

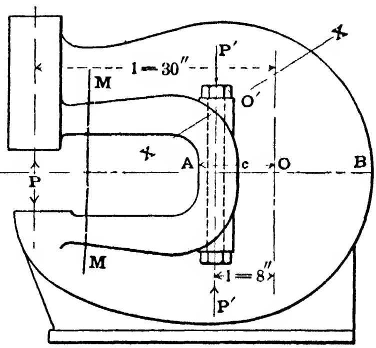


FIG. 139.

$r_2 + e_2$, and the inner radius of the hub was $r_2 - e'_2$. The hub B is, therefore, in the condition of a thick cylinder without ends subjected to an internal pressure, and the shaft A is in the condition of a thick cylinder subjected to an external pressure. The greatest tensile stress will be found at the inside surface of the hub, and the greatest compressive stress at the inside surface of the shaft. If, therefore, e be the difference between the outer radius of the shaft and the inner radius of the hub, before pressing, then $e = e_2 + e'_2$.

Let s_t = the unit tensile stress in the hub at a radius r_2.
s_c = the unit compressive stress in the shaft at a radius r_2.
w_2 = the unit radial pressure between A and B.
r_1 = the internal radius of the shaft.
r_3 = the external radius of the hub.

With negligible error the final common radius can be used for the original inside radius of the hub and the original outside radius of the shaft; hence from (1),

$$e = \frac{s_t}{E} r_2 + \frac{s_c}{E} r_2 \quad \text{or} \quad s_t + s_c = \frac{Ee}{r_2} \tag{2}$$

Birnie's equation for stresses in thick cylinders without ends, and hence without longitudinal stress, is much used for the design of such members and will be adopted in this work. This equation is:

$$*\ s = \frac{2r_1^2w_1 - 2r_2^2w_2 + \dfrac{4r_1^2r_2^2}{r^2}(w_1 - w_2)}{3(r_2^2 - r_1^2)} \tag{3}$$

Where r_1 is the inner radius of the cylinder, r_2 the outer radius, w_1 the internal unit pressure, w_2 the external unit pressure, and s the tensile or compressive stress at any radius r. Applying this equation to the shaft $w_1 = 0$, $r = r_2$, whence the compressive stress at the surface of the shaft is

$$s_c = \frac{w_2(2r_2^2 + 4r_1^2)}{3(r_2^2 - r_1^2)} = \alpha w_2 \tag{4}$$

In a similar way, substituting r_2 for r_1, r_3 for r_2, w_2 for w_1, and w_3 for w_2, the unit tensile stress on the inner surface of the hub is

$$s_t = \frac{w_2(2r_2^2 + 4r_3^2)}{3(r_3^2 - r_2^2)} = \beta w_2 \tag{5}$$

Dividing (4) by (5),

$$\frac{s_c}{s_t} = \frac{\alpha}{\beta} \tag{6}$$

From (2) and (6),

$$s_t = \frac{Ee\beta}{r_2(\alpha + \beta)} \tag{7}$$

and

$$s_c = \frac{Ee\alpha}{r_2(\alpha + \beta)} \tag{8}$$

When the shaft is solid, r_1 in the above equation becomes zero and the equations are much simplified.

Example. A hollow steel shaft 10 in. outside diameter and 2 in. inside diameter is to have a steel crank with a hub 15 in. long shrunk upon its end. The hub of the crank is 18 in. in diameter. What must be the difference between the diameter of the shaft and the bore of the

* The following treatment is from Professor Merriman's "Mechanics of Materials," eleventh edition, page 594. The notation has been changed to agree with that adopted in this work. The student should compare this equation and its application with Clavarino's equation and its application in Article 109.

crank so that the tensile stress at the inner surface of the hub shall not exceed 20,000 lb per sq in.? What will be the corresponding compressive stresses at the outer and inner surfaces of the shaft? Take $E = 30{,}000{,}000$.

Here $r_1 = 1$, $r_2 = 5$, $r_3 = 9$ and $s_t = 20{,}000$. Whence,

$$\alpha = \frac{2r_2^2 + 4r_1^2}{3(r_2^2 - r_1^2)} = \frac{(2 \times 5^2) + (4 \times 1^2)}{3(5^2 - 1^2)} = \frac{3}{4}$$

and

$$\beta = \frac{2r_2^2 + 4r_3^2}{3(r_3^2 - r_2^2)} = \frac{(2 \times 5^2) + (4 \times 9^2)}{3(9^2 - 5^2)} = 2.23$$

Then from (7),

$$e = \frac{s_t r_2(\alpha + \beta)}{E\beta} = \frac{20{,}000 \times 5(\frac{3}{4} + 2.23)}{30{,}000{,}000 \times 2.23} = 0.0044$$

From (8),

$$s_c = \frac{Ee\alpha}{r_2(\alpha + \beta)} = \frac{30{,}000{,}000 \times 0.0044 \times 0.75}{5 \times 2.98} = 6700$$

From (4),

$$w_2 = \frac{s_c}{\alpha} = \frac{6700}{\frac{3}{4}} = 8900 \text{ lb}$$

and, substituting this value in (3), making $r = r_1$ and $w_1 = 0$, it is found that the compressive stress at the inner surface of the shaft is 18,500 lb per sq in.

Equation (3) and the discussion that follows it assume that the materials are elastic and that Hooke's law applies. Professor A. Lewis Jenkins * noted that the most usual combination is a cast-iron hub on a steel shaft and that Hooke's law does not hold for cast iron. He made a careful study of the records of force and shrink fits made by several manufacturing companies and deduced therefrom practical coefficients which he applied to a theoretical equation based upon Lamé's theory. These equations are as follows, the nctation being changed to agree with this discussion. For steel hubs on steel shafts

$$s = \frac{15{,}000{,}000\frac{e}{r_2}\left[\left(\frac{r_3}{r_2}\right)^2 + 1\right]}{\left(\frac{r_3}{r_2}\right)^2} \tag{9}$$

* See *American Machinist*, March 4, 1915, for Professor Jenkins' important discussion of this problem.

and for cast-iron hubs on steel shafts,

$$s = \frac{102{,}900{,}000 \dfrac{e}{r_2}}{7.16 + \dfrac{r_3}{r_2}} \tag{10}$$

Assuming the data of the problem discussed in the preceding section, $e = 0.0044$, $r_3 = 9$ in., $r_2 = 5$ in., and, solving equation (9), the stress at the inner surface of the hub is found to be 17,274 lb per sq in., instead of 20,000 lb as assumed in the original example. For a cast-iron hub, assuming the same data, equation (10) gives 10,106 lb per sq in. as the stress at the inner surface of the hub.

It is evident that if e be assumed, as is usual, the resulting pressure and stresses can be computed. It should be noted that s_t must be well within the elastic limit to prevent the hub's yielding and relieving the pressure. It appears, as pointed out by Professor Merriman, that the allowances made in practice for force fits induce stresses which should be considered if other stresses are to act on the members. Thus, in the example given, the total *allowance* or difference between the *diameter* of the shaft and the bore of the hub would be $2 \times 0.0044 = 0.0088$; and the allowance per inch of diameter would be $0.0088 \div 10 = 0.00088$ in., which is close to average practice for *force* fits, where 0.001 in. per in. of diameter is often allowed. A somewhat greater allowance is generally made for shrinkage fits, as here the difficulty of forcing on the hub does not occur.

The A.S.A. Report formula for stress in steel hubs *twice* the diameter of the shaft is: $s_t = 29{,}000{,}000\ A/d$, where A is the allowance and d the diameter of the shaft. If, in equation (7), $r_1 = 0$ and the hub is taken as twice the diameter of the shaft, $s_t = \frac{3}{4}EA/d$, which gives smaller values than those of the A.S.A. Report.

154. Forcing Pressures and Allowances. The foregoing equations, while giving the probable stresses and radial pressure resulting from a force or shrink fit made with an allowance e, are limited in their application to the practical making of *force* fits. They are important, however, as indicating the great stress that may be induced by a small allowance or difference in diameters. There is, generally speaking, no difficulty in making shrink fits, with any practical allowance, as far as getting the parts together is concerned, although greater skill is required in handling shrink fits than force fits. In making force fits, however, the amount of pressure that can be applied to the parts is often a controlling factor. The probable radial pressure between the shaft and hub (w_2) may be found as above, but little is known of the coefficient of friction in such work, and it is evident that this quantity will vary greatly with the character of the material, the finish of the surface, and the lubricant applied. Experimental data are lacking on this point; hence, it is almost impossible to compute accurately the resistance to slipping offered by force or shrink fits. In general, shrink fits are superior to force fits, since their surfaces are very dry and unlubricated, while those of a force fit are lubricated. Total dependence is, therefore, seldom placed on the forced fit itself, but a key is also used for safety.

Experience shows that the pressure required to make a force fit will vary, for any given diameter,

(a) Directly as the length of the hub.
(b) Directly as the allowance e.
(c) As some function of the radial thickness of hub.
(d) With the character of the materials and the finish of the surfaces.

If the radial pressure for a given value of e is computed and the value of the coefficient of friction can be assumed, then the axial pressure necessary to force the shaft into the hub is

$$P = \pi d l \mu w_2 \tag{11}$$

where d is the diameter of the shaft and l the length of the hub.

Professor Jenkins, as the result of the investigations already referred to, gave the value of P for steel shafts in steel hubs as

$$P = \frac{4120\left[\left(\frac{r_3}{r_2}\right)^2 - 1\right]le}{\left(\frac{r_3}{r_2}\right)^2} \text{ tons} \tag{12}$$

and for steel shafts in cast-iron hubs,

$$P = \frac{200{,}000\left(\frac{r_3}{r_2} + 0.03\right)le}{33\frac{r_3}{r_2} + 209} \text{ tons} \tag{13}$$

Assuming the allowance of 0.0044 computed in the example discussed in the preceding section, and taking $r_3 = 9$ in., $r_2 = 5$ in., and $l = 15$ in., as in that example, the force, P, required to press the shaft into the hub the entire length is found by equation (12) to be 188 tons. In the previous example referred to, the radial pressure w_2 was found to be 8900 lb per sq in. Substituting this value in equation (11) and taking $d = 10$ in., $l = 15$ in., and $\mu = 0.085$, the forcing pressure is found to be 178.2 tons. These results agree quite closely, as might be expected, since Professor Jenkins used a value of $\mu = 0.085$ in his determinations.

It is evident, as stated, that the value of the coefficient of friction in work of this character depends upon the character of the materials, the finish of the surfaces, the lubricant, and the radial pressure. If the pressure is too high the lubricant may be squeezed out and abrasion may occur. As might be expected, therefore, practical data on this value vary considerably; although such data indicate that values ranging from 0.08 to 0.125 are not infrequent in practice, it is very difficult to assign an accurate value of μ for any given set of conditions, and it is always very desirable to check press fit allowances by actual practical experience.

An allowance of 0.001 in. per in. of diameter will represent average practice in this country for such work as crankshafts, crankpins, and, in general, where a tight fit is required. For armature spiders, or flywheels, one-half this allowance is often sufficient. For shrink fits a greater allowance is often made, although the foregoing discussion indicates that this should not be much exceeded considering the stresses induced. The following table, which represents the practice of the General Electric Company, has been used successfully for many years, and may be helpful as a guide:

TABLE XXIV

Nominal Diameter, Inches	Allowance for Fit in Inches					
	Sliding Fit	Commutator and Split Hub	Press Fit for Solid Armature Spider Steel	Press Fit for Solid Armature Spider Cast Iron	Press Fit for Couplings	Shrink Fit
2	−0.0015	+0.0005	+0.00075	+0.0015	+0.00175	+0.0025
4	−0.0020	+0.0005	+0.00150	+0.0025	+0.00300	+0.0040
8	−0.0040	+0.0010	+0.00200	+0.0035	+0.00450	+0.0060
12	−0.0050	+0.0010	+0.00250	+0.0045	+0.00550	+0.0075
16	−0.0055	+0.0010	+0.00300	+0.0050	+0.00600	+0.0090
20	−0.0060	+0.0015	+0.00350	+0.0055	+0.00700	+0.0100
24	−0.0070	+0.0015	+0.00350	+0.0060	+0.00750	+0.0110
28	−0.0075	+0.0015	+0.00400	+0.0065	+0.00850	+0.0120
32	−0.0080	+0.0015	+0.00450	+0.0070	+0.00900	+0.0125
36	−0.0085	+0.0020	+0.00450	+0.0075	+0.00950	+0.0135
40	−0.0090	+0.0020	+0.00500	+0.0080	+0.01000	+0.0140
44	−0.0095	+0.0020	+0.00500	+0.0085	+0.01050	+0.0145
48	−0.0100	+0.0020	+0.00550	+0.0090	+0.01100	+0.0150

It will be obvious that the difficulties of pressing parts together and the uncertainty due to possible abrasion during pressure are lessened if the mating surfaces are tapering in form. The increased cost of machining such surfaces usually makes such a procedure undesirable.

155. Thin Bands or Hoops. If the ring or band which is forced or shrunk on to a member be thin, radially, compared to its diameter, the assumption can be made, without appreciable error, that the stress is uniform throughout the cross-section of the ring. The change of form, due to compression, in the member on which the band is placed, is so small in such cases that it may be neglected, and the stress in the band may be taken as that due to stretching it over an incompressible body. This is practically applicable to any ordinary shape of band, but rigidly true for circular shapes only. Thin bands of this character are usually shrunk into position.

Example. A thin steel band is to be shrunk on to a casting whose external perimeter where the band is to be placed is 48 in. What must be the length of the inside face of the band so that the stress per unit area due to shrinking will be 30,000 lb? What will be the area of the cross-section of the band in order that the total stress in the band may be 60,000 lb?

Let l = the internal perimeter of band before shrinking.

Then

$$48 - l = \text{Total amount of elongation of band}$$

and

$$\frac{48 - l}{l} = \text{Unit elongation of band}$$

Whence, if E, the coefficient of elasticity, be taken as 30,000,000, then,

$$E = 30{,}000{,}000 = \frac{\text{Unit stress}}{\text{Unit strain}} = \frac{30{,}000}{\dfrac{48 - l}{l}}, \quad \text{or} \quad l = 47.95 \text{ in.}$$

The total area of the cross-section of the band will be

$$A = \frac{60{,}000}{30{,}000} = 2 \text{ sq in.}$$

which may be distributed in any convenient proportions.

If the part on which the band is to be shrunk is circular in form, the band is in the condition of a thin cylinder subjected to an internal pressure w per unit area, where w is the radial pressure between the band and the part on which it is shrunk.

Therefore by Article 101,

$$wd = 2S$$

where S is the *total* stress per unit width of the band, or

$$w = \frac{2S}{d}$$

Thus, in the above problem, let the band be shrunk upon a circular hub of diameter $48/\pi$, and let the cross-section of the band be $\frac{1}{2}$ in. by 4 in. Then,

$$S = \frac{60{,}000}{4} = 15{,}000$$

and

$$w = \frac{2S}{d} = \frac{2 \times 15{,}000}{\frac{48}{\pi}} = 1962 \text{ lb per sq in.}$$

The steel tires of locomotive driving wheels are usually shrunk on with an allowance for shrinkage of 0.001 in. per inch of diameter, which gives 0.001 in. elongation per inch of circumference. Taking $E = 30{,}000{,}000$, and considering the tire a thin band, the unit stress in the tire is

$$s = E\Delta = 30{,}000{,}000 \times 0.001 = 30{,}000 \text{ lb}$$

156. Other Forms of Shrink Fits. Many machine parts, such as flywheel rims, are held together by steel links or bands shrunk into place. The theory outlined in the preceding article is clearly applicable to these members, and their dimensions should be carefully calculated so that they will not be overstrained by the shrinking alone. If such members are so designed that they will be stressed up to the elastic limit from shrinkage alone, they are liable to be strained beyond the elastic limit, when an external load greater than the total shrinkage stress is applied to the parts which they hold together, and the link, taking a permanent set, becomes ineffective. In computing the dimensions of such links, allowance must sometimes be made for the compression of the parts held together, but ordinarily this is small and may be neglected.

Occasionally a bolt or link is used to reinforce a cast-iron member against tensile stress. Thus, in open frames (Fig. 139) that are used normally as punch press frames, a pair of removable reinforcing bolts are occasionally applied at some point, M, near the location where the work is done, so that heavier work, such as shearing of metal plates, may be performed without danger of breaking the frame which has not been designed, primarily, for such heavy loads. If it is desired to reinforce the frame, and still keep the throat of the frame clear, a large bolt is sometimes placed on each side of the throat, as shown also in Fig. 139. These bolts are usually put in hot and allowed to cool in place. As ordinarily applied, the benefit derived from them is questionable. If they are designed and fitted so as to put the frame in compression at A, an amount equal to the tension induced by the working load, P, at this same point, without being themselves strained beyond the elastic limit when the load is applied, then no stress can come upon the frame itself from the force P. If, however, the bolts and frame are each to carry part of the load, care should be exercised that the stress induced in the bolts by the *initial* load due to shrinking or screwing up is so low that the additional stress due to the external load does not raise this initial stress beyond the elastic limit, thus giving the bolts a permanent set and destroying their usefulness.

Let A, Fig. 138, represent a cast-iron member of uniform cross-section which is to be reinforced against tensile stress by the bolt B. Suppose, first, that the nut is screwed up till it just bears firmly on the casting. If now an external tensile load is applied to the casting, the bolt and casting will be elongated the same amount Δ. But the coefficient of elasticity of cast iron is only about one-half that of steel. Hence, since $s = E \times \text{strain} = E\,\Delta/l$, the stress per *unit area* in the casting will only be one-half that in the steel. If 2000 lb is the allowable unit stress in the casting, 4000 lb per unit area is all that can be thus obtained in the bolt. This would lead to unnecessarily large bolts.

Suppose, however, that the nut is set up till a total compressive load, W, is applied to the cast iron. The bolt will be elongated * and the casting compressed, the amount of elongation or compression depending on the cross-section of the respective members. The unit stress induced in the bolt and casting will also be proportional to the area of their respective cross-sections. If now an external tensile load, W', is applied to the *bolt*, the tendency is to relieve the compressive stress in the casting and to increase the tensile stress in the bolt. When the load applied is sufficient to elongate the bolt as much as the casting was originally compressed, the casting will be relieved of all stress. If the external load, W', is applied to the bolt through the casting itself, it is evident that practically the same result is obtained; and after the compressive stress in the casting is fully relieved, any further addition to W' induces a tensile stress in the casting and still further increases the tension in the bolt. Usually the cross-sectional area of the casting is very much greater than that of the bolt. Furthermore, the compressive stress

* See Article 137.

induced in the casting by the initial load on the bolt is usually very small compared to the tensile stress induced by the working load. For these reasons the compressive deformation in the casting can usually be neglected without appreciable error; and the bolt may be designed on the basis of the external load alone. (See Article 138, Case a.)

Example. In Fig. 139 let the section AB be stressed by the load P whose arm is l. Let O be the location of the gravity axis of the section AB. It is desired to keep the stress at A not greater than 3000 lb per sq in. The material is to be cast iron.

Let $P = 60{,}000$.
I = moment of inertia of section = 4500.
$c = 10$ in.

Also let the area of the section be 200 sq in. Then from (M),* page 93, the *tensile* stress at A due to P is

$$s = \left(\frac{P}{A} + \frac{Plc}{I}\right) = \left(\frac{60{,}000}{200} + \frac{60{,}000 \times 30 \times 10}{4500}\right) = 4300 \text{ lb}$$

and it is desired to reduce this to 3000 lb by reinforcing bolts. These reinforcing bolts serve the double purpose of increasing the factor of safety by reducing the fiber stress, and also of decreasing the deflection of the frame at the point where the work is done. Let these bolts be located 8 in. from O. Then the *compressive* stress induced at A by P' is

$$s' = \left(\frac{P'}{A} + \frac{P'l'c}{I}\right) = \left(\frac{P'}{200} + \frac{P' \times 8 \times 10}{4500}\right)$$

But $s - s'$ must equal 3000; therefore,

$$4300 - \left(\frac{P'}{200} + \frac{P' \times 8}{450}\right) = 3000$$

whence $P' = 57{,}000$. This is the total tensile load on both bolts, when the full working load, P, is applied. If the maximum stress at the root of the thread be taken at 15,000 lb, then the area of each bolt at the root of the thread is

$$\frac{57{,}000}{2 \times 15{,}000} = 1.9 \text{ sq in.}$$

which corresponds closely to a $1\frac{3}{4}$-in. bolt. The area of the body of a

* The effect of the curvature of the frame is neglected for simplicity.

$1\frac{3}{4}$-in. bolt, where most of the stretching takes place, is 2.4 sq in. Hence the working stress in the body of the bolt is

$$\frac{57{,}000}{2 \times 2.4} = 11{,}800 \text{ lb per sq in.}$$

That portion of the boss which immediately adjoins the throat is subjected to an average tensile stress nearly equal to the fiber stress at the surface of the throat, or 3000 lb per sq in. The upper and lower portions of the boss have little or no tensile stress induced in them, as a consideration of a section such as XX, whose gravity axis is at O', will show. It will be reasonable to estimate that the stress in the boss is equivalent to the full stress of 3000 lb per sq in. through 14 in. of its length, the total length being 19 in. The increase in the length of the bolts due to loading the frame will be the same as the increase in the length of the boss. Neglecting the compressive deformation of the boss due to the initial load from the bolts, the elongation of the boss due to loading the frame will be

$$\Delta = (\text{effective length})\ (\text{strain}) = l\frac{s}{E} = 14\frac{3000}{15{,}000{,}000} = 0.0028$$

The increase in the stress in the bolts will be

$$s = E(\text{strain}) = 30{,}000{,}000 \times \frac{0.0028}{19} = 4420 \text{ lb}$$

whence the initial stress in the bolt will be $11{,}880 - 4420 = 7450$. The allowance for shrinkage necessary to give this initial stress will be

$$\Delta = \frac{ls}{E} = \frac{19 \times 7460}{30{,}000{,}000} = 0.0047 \text{ in.}$$

The number of threads per inch on a $1\frac{3}{4}$-in. bolt is 5. Hence, after the nut has been set up snugly it should be given $0.0047 \div \frac{1}{5} = 0.0235$ of a turn, or should be turned through $360 \times 0.0235 = 8.45$ degrees. This is most easily done in the case of large bolts by first marking the nut with reference to the bolt when set up snug in a cold state, and then heating the body of the bolt, if necessary, and rotating the nut the desired amount, allowing it to cool in position.

It is to be especially noted that a very small shrinkage allowance is needed to induce a great stress in the bolt. If too great an allowance is made, the bolts may be stressed beyond the elastic limit, and take permanent set the first time the external load is applied. When the external load is again applied, a force much smaller than the total load, *P*, will strain the casting to the point where the bolt becomes effective. The total load, *P*, will strain the casting further than it did originally and even if the stresses induced are not sufficient to rupture the casting, the stiffness of the frame is materially decreased.

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CHAPTER XIV

TOOTHED GEARING

157. General Principles. Power may be transmitted from one shaft to another by means of belts, friction wheels, gearing, or chain drives. When it is not essential that rotation of one shaft shall produce definite and positive rotation of another, belts and friction wheels may be used to advantage. When the velocity ratio must be constant, however, some form of gear wheels or chain drive must be employed. It is evident that any pair of surfaces which will roll together with pure rolling motion, so as to give the required velocity ratio, may serve as a basis for the design of a pair of toothed gears; and works on mechanism treat fully of the methods of drawing the sections of such surfaces for various conditions and velocity ratios. Whether the elements of the surface thus outlined shall be parallel or otherwise will depend on the angle which the shafts make with each other, as in the case of friction wheels, and tooth gearing may be classified according to the character of the pitch surfaces, and the relation of the axes, thus:

Kind of Gear	Relation of Axes	Pitch Surfaces	Tooth Elements	Tooth Contact
Plain spur	Parallel	Cylinders	Straight lines	Straight line
Helical, or twisted, spur	Parallel	Cylinders	Helices	Straight line
Plain bevel	Intersecting	Cones	Straight lines	Straight line
Twisted bevel	Intersecting	Cones	Curved lines	Curved line
Helical gears, in general	Any angle not intersecting	Cylinders	Helices	Point
Worm and wheel (special helical gear) hobbed	Axes at 90° not intersecting	Cylinder for worm	Helices for worm	Curved line
Hyperboloidal * gears	Any angle not intersecting	Hyperboloids	Straight lines	Straight line

* Sometimes called skew gears.

The most important of these are spur, bevel, and a few special forms of twisted and helical gears. The motion transmitted by a pair of properly designed toothed gears is identical with that of the base curves or surfaces rolling together. In order that a pair of curves may move upon one another with pure rolling motion, the point of tangency of the curves must always lie upon the straight line joining the centers of rotation of the curves. And, as a consequence, a pair of surfaces whose axes lie in the same plane will roll together with pure rolling motion when the line of tangency lies in the plane passing through the axes of rotation. If r_1 and r_2 be the instantaneous radii of such a pair of rolling surfaces at the point of contact, and ω_1 and ω_2 be their instan-

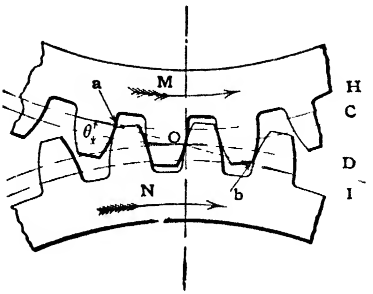

FIG. 140.

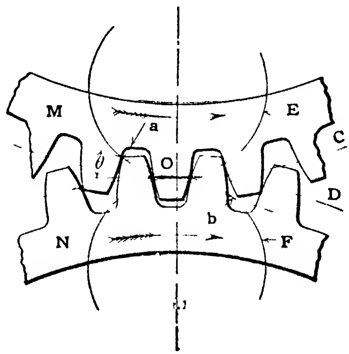

FIG. 141.

taneous angular velocities, then $\omega_1/\omega_2 = r_2/r_1$. In the most common case, the angular velocity of both shafts is constant, and hence r_1 and r_2 are constant, and the rolling surfaces are circular in cross-section. Thus, Fig. 140 shows a portion of two gears whose rolling surfaces are a pair of circular cylinders, represented in cross-section by the circles C and D. If the teeth are properly proportioned the motion transmitted will be identical with that produced by the rolling of C on D. It can be shown that the condition which such tooth outlines must fulfill, in order that the velocity ratio may be constant, is that *the common normal to the tooth outlines at the point of contact must always pass through the point of tangency of the rolling circles.* There are many curves that can be used for tooth outlines, and which would fulfill the condition, but in practice only two are commonly employed, namely, the **involute** and the **cycloid.**

Cycloidal tooth outlines are formed by rolling a circle upon the outside and inside of the rolling circles or **pitch circles** C and D, Fig. 141. Involute tooth outlines are formed by rolling a straight line upon the **base circles** H and I, Fig. 140.

Figure 140 illustrates a portion of two gears with involute teeth. The upper wheel, M, is the driver. Contact between two teeth has just begun at a, and the common normal to the point of contact aOb passes through the pitch point O. As the wheels rotate, the point of contact will move along the line aOb till contact ceases at b. Hence, in the involute system, the normal to the point of contact makes a fixed angle with the common tangent to the pitch circles.

Figure 141 shows a portion of two gears with cycloidal teeth. Contact is just beginning at a, and as the gears rotate the point of contact will move along the curved path aOb, contact ceasing at b. The normal to the first point of contact is drawn, and it is clear that the inclination of the normal to the common tangent of the pitch circles is a maximum at this point, and continually varies in direction though always passing through the point O. It can be shown that in the involute system the angular velocity ratio will remain constant, within the limits of action, whether the pitch circles are tangent or not; but for the transmission of constant velocity ratio with cycloidal gearing the pitch circles must remain tangent. This practical advantage in favor of the involute tooth outline has brought it into extended use, and it has almost superseded the cycloidal tooth. In cycloidal teeth the contact is between convex and concave surfaces, whereas in the involute the contact is between convex surfaces or between convex and plane surfaces. The lubrication of cycloidal teeth is, therefore, somewhat more efficient than in involute teeth, and this property is of value in worm drives that carry heavy loads. A fuller treatment of the theory of gear-tooth outlines, which is beyond the scope of this work, will be found in treatises on mechanism.*

158. Interference in Involute Teeth. In cycloidal teeth the tooth outline is theoretically correct for conjugate action both above and below the pitch line, regardless of the size of the gear. In involute teeth, however, the true tooth outline is entirely outside of the base circle, and in order to obtain a tooth of sufficient depth below the pitch circle it has been customary to extend the outline inside the base circle by drawing radial lines from the end of the involute curve. This part of the tooth outline is not theoretically correct and may interfere with the action of the involute outline of the mating tooth. In gears of fair size this causes no difficulty, but it will be noted that a decrease in the pressure angle or a decrease in the radius of the gear reduces the length of the involute that lies below the pitch circle, thus bringing more of the incorrect radial outline into operation. This action is known as **inter-**

* See "Kinematics of Machinery," by J. H. Barr and E. H. Wood, also "Kinematics of Machinery," by Albert and Rogers.

ference, and in small gears the tooth outline must be corrected either by cutting away a portion of the lower face or **flank** of the tooth or by removing a part of the upper face near the point. This latter method is preferable as undercutting the lower part of the tooth naturally weakens it.

159. Interchangeable Systems of Gearing: Standard Forms. It is desirable in practical work that any gear of a given pitch shall run properly with any other gear of the same pitch. In order that this may be so, certain limitations must be placed upon the form and dimensions of the tooth. In the cycloidal system, interchangeability may be accomplished, so far as the tooth outlines are concerned, by keeping the diameter of the describing circle constant for all gears of the series.

Any involute tooth outline will run properly with any other similar outline; and any gear with involute teeth will run with any other gear having similar teeth, so far as the height and thickness of teeth will allow them to mesh. In order to obtain involute outlines of sufficient length, and a series of gears with fixed nominal pitch circles, the angle θ, Fig. 140, made by the line of action with the common tangent to the pitch circles must have a proper value, and be constant for all gears of the series. In the systems in most common use this angle is $14\frac{1}{2}°$, though there is a tendency in modern work toward a greater angle.

It is found undesirable in practice to make gears with less than 12 teeth; and in some cycloidal systems the radius of a 12-tooth gear of the required pitch is taken as the diameter of the describing circle. For a 12-toothed gear this will result in radial lines for the tooth outlines below the pitch circle, i.e., the tooth will have radial flanks. In the practice of the Brown and Sharpe Manufacturing Company, the diameter of the describing circle is the radius of the 15-toothed gear of the series. This gives spaces between the flanks of the teeth on the 12-toothed, or smallest gear, so nearly parallel that they may be cut with a rotary cutter.

It is evident from Figs. 140 and 141 that the tooth outlines of either system may be extended below the pitch line as far as the involute base circle or cycloidal generating circle will permit and above the base circle until they meet at the point of the tooth. It is also clear that the longer the teeth the earlier will they engage with each other, the greater will be the arc of contact and the greater will be the number of teeth continually in contact. The distribution of the load over a number of pairs of teeth is in itself conducive to smooth running; but on the other hand, extending the arc of contact away from the pitch point increases the sliding between teeth, and also the velocity with which the teeth approach each other. The tooth also becomes weaker as it is lengthened, the thickness remaining the same, and for these reasons a practical limit is placed on the length of teeth. The length of tooth adopted in practice is, therefore, a compromise between conflicting conditions, which experience has shown will give good results.

The distance along the pitch line from any point on a tooth to a corresponding point on the next tooth, is called the **circular pitch,** and will be denoted by p_c. The **thickness of the tooth** along the pitch line will be denoted by t, Fig. 142. For cut gears, where no backlash is allowed between teeth, $t = p_c/2$. In some forms of gears, such as shown in Fig. 154, where a metal pinion engages with a gear having wooden teeth, the pitch may not be equally divided, but the metal tooth may be thinner than the wooden tooth. If N be the number of teeth and D the pitch diameter, then evidently $Np_c = \pi D$. If the number of teeth N be divided by the pitch diameter, the quotient, or *the teeth per inch of diameter,* is called the **diametral pitch** and will be denoted by p_d. Since

$$p_d = \frac{N}{D} \quad \text{and} \quad p_c = \frac{\pi D}{N}$$

$$p_d \times p_c = \pi. \quad \therefore \; p_d = \frac{\pi}{p_c} \quad \text{and} \quad p_c = \frac{\pi}{p_d}$$

The diametral pitch is, ordinarily, the most convenient for use, and in this country practically all interchangeable systems are based upon the diametral pitch. Thus a gear 24 in. in diameter and 3 diametral pitch would have $24 \times 3 = 72$ teeth, and the circular pitch would be $\pi/3 = 1.047$ in. The following notation will be used in this discussion:

D_1 = the outside diameter of the gear.
D = the pitch diameter of the gear.
D_2 = the diameter of a circle through bottom of space.
p_d = the diametral pitch.
p_c = the circular pitch.
a = the addendum = height of tooth above pitch line.
c = the clearance between top of tooth and bottom of space when gears are in mesh.
d = the dedendum, or depth of tooth below the pitch line.
$a + d = h$ = total height of tooth.
b = breadth of tooth.
t = the thickness of tooth on pitch line = width of space on pitch line in cut teeth.
N = the number of teeth in gear.
h = the total height of tooth.

Many attempts to formulate standard gear systems have been made in this country and the 1923 edition of this book contains a comparative description of the most important of these early efforts. The best known and most widely used of these systems is that devised by the Brown and Sharpe Manufacturing Company. Involute gears made to this standard have been found to be very satisfactory for average conditions. For more extreme conditions, however, in order to obtain greater strength other proportions have come into use. Three methods of meeting this difficulty are available. The first is to increase the pressure angle (Fig. 140), thus thickening the flanks by carrying the true outline nearer to the root of the tooth and also thereby reducing the interference. With a pressure angle of $14\frac{1}{2}°$ interference begins with a pinion having about 32 teeth, but if the pressure angle is made $22\frac{1}{2}°$ interference does not begin until the number of teeth is 12, which is about as small a number as is ordinarily used. Most authorities on this subject have hesitated to recommend such a large angle of obliquity because of the added radial pressure which brings additional load upon the bearings, and some have adopted 20° as a compromise.

A second method of obviating undercutting because of interference is to make the tooth outline below the base circle of cycloidal form, thus securing accurate conjugate action throughout and markedly thickening the root of the tooth. This also requires that the face of the tooth near the upper end must be cycloidal for a short distance. Such a system is known as a **composite system** and is now in use in this country.

The third method of strengthening the tooth is to decrease its length, thus lessening the bending moment applied at the root. Experience has shown that teeth considerably shorter than Brown and Sharpe standard teeth have sufficient arc of contact and run smoothly and quietly. In fact, the claim is made that they operate more smoothly than teeth of standard height. Obviously, the height of the involute teeth may be reduced and the angle of obliquity increased simultaneously, and this has been done in some of the new standard systems that have been developed or proposed. The name **stub teeth** has been given to any system of teeth where the length of the tooth is markedly shorter than those of the Brown and Sharpe standard. Mr. C. W. Hunt was one of the first to use such short teeth; he reported to the A.S.M.E. in 1897 (vol. 18) the results of the adoption of his system for heavy loads, and gave full information for designing the tooth outlines. One of the most prominent of these so-called stub-tooth systems is that advocated by the Fellows Gear Shaper Company. In this system an involute tooth with a pressure angle of 20° is used. The addendum is fixed, not by the reciprocal of the diametral pitch as in the Brown and Sharpe system, but by the reciprocal of a diametral pitch somewhat larger. Thus, in this system, a $\frac{7}{9}$ gear is a gear of 7 diametral pitch having an addendum $\frac{1}{9}$ in. in height. These relations are as follows:

Diametral pitch	4	5	6	7	8	9	10	12
Addendum of Fellows gears	$\frac{1}{5}$ in	$\frac{1}{7}$ in	$\frac{1}{8}$ in.	$\frac{1}{9}$ in	$\frac{1}{10}$ in	$\frac{1}{11}$ in	$\frac{1}{12}$ in	$\frac{1}{14}$ in

The principal criticism made of this system is that the depth of tooth is not a direct function of the circular pitch, and advocates of other systems have adopted proportions that depend upon the pitch. The Fellows system is much used, however, and appears to give excellent results. In order to clarify this situation the American Standards Association has approved four systems of interchangeable gearing, full description of which will be found in its publication B6.1–1932. These systems are, respectively:

(1) A **composite** system with an involute pressure angle of $14\frac{1}{2}°$.
(2) A **full-depth** involute system with a pressure angle of $14\frac{1}{2}°$.
(3) A **full-depth** involute system with a pressure angle of 20°.
(4) A **stub-tooth** system with a pressure angle of 20°.

Table XXV gives comparative proportions of these several systems; in this connection it should be noted that in the original Brown and Sharpe System the addendum is $1/p_d$.

160. Other Systems. As noted, there have been other attempts to devise standard systems. Of these the Maag System is interesting in that full involute action is obtained with all combinations of gears. The system is not interchangeable, however, and this is a great defect in these days of standardization. Space does not permit description of this system.

In all this discussion reference is to *cut* or machined gear teeth. In *rough* gear teeth, cast from a wooden pattern, the thickness of the tooth must be less than the width of the space, and the clearance at the bottom of the space must be greater than in cut teeth. If the gears are **machine-molded,** the difference need not be quite so great as in pattern-molded gears. For pattern-molded gears good practice gives $t = 0.45p_c$ for large gears, to $0.47p_c$ for small gears, and the corresponding width of the space would be $0.55p_c$ to $0.53p_c$. For machine-molded gears $t = 0.46p_c$ to $0.48p_c$ and the corresponding space would be $0.54p_c$ to $0.52p_c$. The difference between the thickness of the tooth and the width of the space is commonly called **back-lash.**

TABLE XXV

COMPARISON OF GEAR TOOTH SYSTEMS

	$14\frac{1}{2}°$ Brown & Sharpe System	$14\frac{1}{2}°$ A S A. Composite System	$14\frac{1}{2}°$ A S.A. Full-Depth Involute	20° A.S.A. Full-Depth Involute	20° A S A. Stub Tooth	$22\frac{1}{2}°$ Full-Depth Involute	$22\frac{1}{2}°$ Stub Tooth
Pressure angle	$14\frac{1}{2}°$	$14\frac{1}{2}°$	$14\frac{1}{2}°$	20°	20°	$22\frac{1}{2}°$	$22\frac{1}{2}°$
Addendum	$\frac{1\ 0}{p_d}$	$\frac{1\ 0}{p_d}$	$\frac{1\ 0}{p_d}$	$\frac{1\ 0}{p_d}$	$\frac{0\ 80}{p_d}$	$\frac{0\ 875}{p_d}$	$\frac{0\ 7854}{p_d}$
Working depth	$\frac{2\ 0}{p_d}$	$\frac{2\ 0}{p_d}$	$\frac{2\ 0}{p_d}$	$\frac{2\ 0}{p_d}$	$\frac{1\ 6}{p_d}$	$\frac{1\ 75}{p_d}$	$\frac{1\ 5708}{p_d}$
Whole depth	$\frac{2\ 1571}{p_d}$	$\frac{2\ 1571}{p_d}$	$\frac{2\ 1571}{p_d}$	$\frac{2\ 1571}{p_d}$	$\frac{1\ 8}{p_d}$	$\frac{1\ 875}{p_d}$	$\frac{1\ 7279}{p_d}$
Clearance	$\frac{0\ 1571}{p_d}$	$\frac{0\ 1571}{p_d}$	$\frac{0\ 1571}{p_d}$	$\frac{0\ 1571}{p_d}$	$\frac{0\ 2}{p_d}$	$\frac{0\ 125}{p_d}$	$\frac{0\ 1571}{p_d}$
Minimum number of teeth[1]	32	12	32	18	14	12	11

[1] Number of teeth in smallest pinion to mesh with rack without interference

161. Forces Acting on Spur Gears. In Fig. 142 let the gear A drive the gear B. Let V_a be the velocity of the pitch circle of A; and V_b be the velocity of the pitch circle of B. Also let W_a be the *equivalent driving force* acting at the pitch circle of A, and let W_b be the *equivalent resisting force* acting at the pitch circle of B. If now the tooth outlines are properly constructed, the line of action of the *actual* driving force, W_1, will always pass through the pitch point, and the angular velocity ratio of A to B will be constant. The action of the pitch circles will be as though they rolled upon each other, and their linear velocity will be the same or $V_a = V_b$. The corresponding tangential driving forces at the pitch line, W_a and W_b, must therefore be equal, also, since $W_aV_a = W_bV_b$. The actual tangential driving force on the tooth varies somewhat throughout the period of contact, but this variation may be neglected and it may be assumed that the action is the same as though a pair of teeth were continually in action at the pitch point.

The pressure at the tooth contact is opposed by the supporting bearings. The greater the angle of obliquity the more will the pressure at tooth contact and at the bearings exceed the effective force required to turn the gear. In the involute system the angle of obliquity is constant for any given system and is equal usually to $14\frac{1}{2}°$ or 20°. Neglecting the friction due to sliding between teeth the pressure angle at tooth contact will be equal to the angle of obliquity. The increase in the pressure at the bearing due to an angle of 20° is about (sec 20° − 1) or 6.4 per cent. In cycloidal gearing the obliquity varies from a maximum at the beginning of contact to zero when the contact point lies in the line of centers; and during recess it increases again to a max-

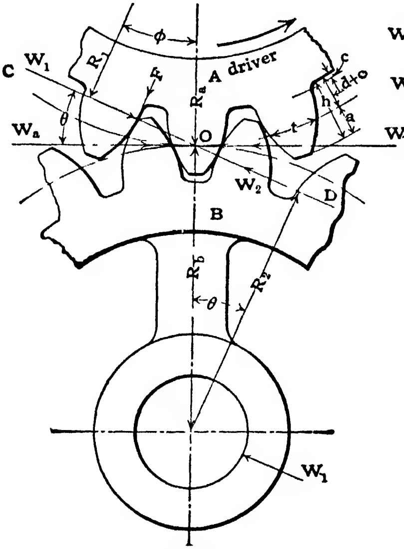

FIG. 142.

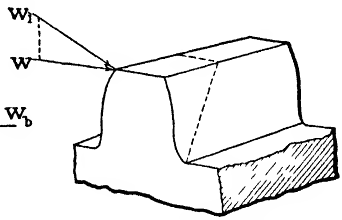

FIG. 143.

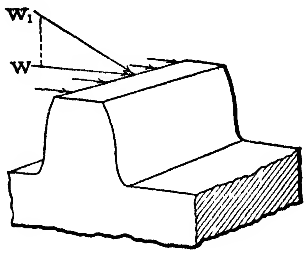

FIG. 144.

imum at the end of contact. For the usual cycloidal systems, the maximum value of the angle of obliquity is about 22° and the increase in pressure at the bearings over and above the required tangential force is about 8 per cent.

In the above discussion the influence of friction has been neglected. During the arc of approach the frictional force F (Fig. 142) deflects the line of action of W_1 in such a way as to increase the effective obliquity. During the arc of recess it acts in the opposite direction and decreases the obliquity. The influence of this frictional force is small and may, usually, be neglected, but its action accounts, to a certain degree, for the well-known fact that gears run more smoothly during recess than during approach.

It is usually intended that more than one pair of teeth shall be in action at all times, but, owing to the unavoidable inaccuracy of form and spacing, it is not safe to depend upon a distribution of the load between two or more teeth of a gear. It is safest to provide sufficient strength for carrying the entire load on a single tooth. In the rougher classes of work, this load may be concentrated at one corner of the tooth, as indicated in Fig. 143, and all such gears should be carefully inspected and corrected, if intended to carry heavy and important loads. With well-supported bearings and machine-molded or cut gears, it is not unreasonable to consider the load as fairly well distributed across the face of the gear, if the face does not exceed in width about three times the circular pitch (see Fig. 144).

The obliquity of the line of pressure gives rise to a crushing action on the teeth (due to the radial component of the normal force), in addition to the flexural stress which results from the tangential component.

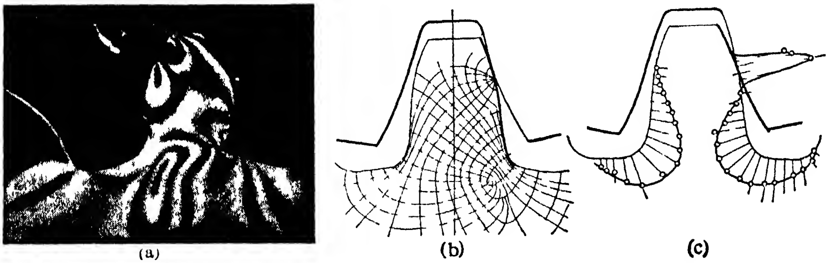

Fig. 145.

This crushing component, with the ordinary proportions of teeth, does not exceed 10 per cent of the normal pressure. Its effect is to reduce the tensile stress due to flexure, and to increase the compressive stress. Since cast iron is far stronger in compression than in tension, this may be neglected in gears made of that metal, and with steel or composition gears, the margin of safety assumed usually makes it unnecessary to consider this component.

162. Stresses in Spur Gear Teeth. The assumption often made in determining the strength of spur gears that the teeth of such gears can be considered as rectangular cantilevers is wholly unsatisfactory especially in gears with a small number of teeth. Furthermore, the stresses in gear teeth are very complex. Figure 145 (a) * shows a photo-elastic picture of a gear tooth under stress and gives an idea of the complexity of the stress distribution. Figure 145 (b) shows a study of a gear

* "Stress Distribution in Electric Railway Pinions," A. L. Kimball, *Trans. A.S.M.E.*, 1922.

tooth reported by E. G. Coker * indicating the lines of the principal stresses as derived from a photo-elastic picture; Fig. 145 (c) shows the relative values of the contour stresses both in tension and compression. These last refer of course to the stresses in the outer surfaces. The compressive stress directly under the point of contact should be particularly noted since it is this stress that causes pitting or flaking of the surfaces. The theoretical discussions that follow should therefore be considered in the light of these complex stresses. Figure 146 shows four gear

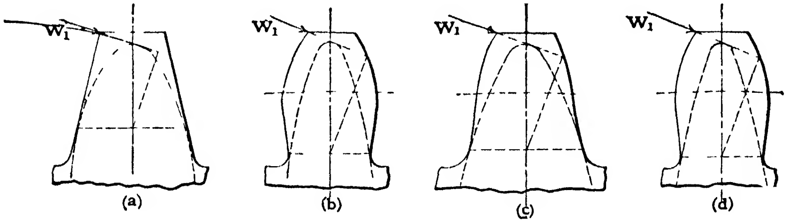

FIG. 146.

teeth which have the same thickness at the pitch line and the same height. The tooth marked (a) is one of an involute rack; (b) is one of an involute pinion having 12 teeth, (c) is one of a cycloidal gear having 30 teeth; (d) is one of a cycloidal pinion of 12 teeth.

Mr. Wilfred Lewis, of Wm. Sellers and Company, seems to have been the first to investigate the strength of spur gear teeth with due regard to the actual forms used in modern gearing. His work was published originally in the *Proceedings* of the Engineers' Club of Philadelphia, in January, 1893, and his method of investigation was as follows: Accurate drawings of gear teeth were made on a large scale, and the line of action of the normal force, when acting on the point of a tooth, was drawn in (see Fig. 147). From the intersection of this line of action with the center line of the tooth, a parabola was drawn tangent to the sides of the tooth, thus locating a beam of uniform strength equal to the effective strength of the tooth (see Article 16). The points of tangency *a*, *a*, locate the weakest section of the tooth, and the bending moment applied to this section is *Wl*. Then from equation (J), page 93:

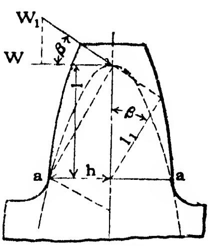

FIG. 147.

* "Engineering Problems Solved by Photo-elastic Methods," *Journal* of the Franklin Institute, 1923. See also "Contact Stresses in Gears" by R. V. Baud *Mechanical Engineering*, 1931, page 666.

$$Wl = s\frac{I}{c} = \frac{sb(2h)^2}{6} = \frac{2}{3}bsh^2 = bsp_c\left(\frac{2}{3}\frac{h^2}{p_c}\right)$$

or

$$W = bsp_c\left(\frac{2}{3}\frac{h^2}{p_c l}\right) = bsp_c(y) \qquad (1)$$

where b = the breadth of the tooth in inches, s = the tensile stress, and p_c = the circular pitch. The factor y is a variable, depending on the shape of the tooth. Mr. Lewis found that its value is practically independent of the pitch (since p_c, h and l are proportional to the pitch), but dependent mainly on the number of teeth in the gear. Tabulated values of this coefficient may be found in Kent's "Mechanical Engineers' Pocketbook." From these tabulated values, Mr. Lewis deduced the following equations in which N = the number of teeth in the gear.

For the Brown and Sharpe $14\frac{1}{2}°$ composite system and the cycloidal system using a generating circle whose diameter equals the radius of the 12-tooth * pinion:

$$W = bsp_c\left(0.124 - \frac{0.684}{N}\right) \qquad (2)$$

For the 20° involute and addendum equal to $\frac{1}{p_d}$:

$$W = bsp_c\left(0.154 - \frac{0.912}{N}\right) \qquad (3)$$

For the Brown and Sharpe cycloidal system with a generating circle equal to the radius of the 15-tooth pinion:

$$W = bsp_c\left(0.106 - \frac{0.678}{N}\right) \qquad (4)$$

It will be obvious that Mr. Lewis' analyses can be applied to stub teeth, but it appears that the value of (y) cannot be expressed accurately for all systems of stub teeth in the comparatively simple manner illustrated in equations (2), (3), and (4). Table XXVI gives the tabulated values of (y) for Fellows stub teeth and for other standard forms.

The Lewis formula is convenient for determining W, b, p_c, or s, where the number of teeth (N) is known; but a very common problem in design is to determine the *pitch* (p_c), when the pitch diameter of the gear is given and the *number of teeth is unknown.* The formula may

* The 12-tooth involute pinion may have its teeth weakened by a correction for interference; but it is usually better to correct the points of the mating gear.

TABLE XXVI

VALUES OF y FOR STANDARD GEAR TOOTH FORMS

No. of Teeth N	$14\frac{1}{2}°$ Composite $a = \frac{1}{p_d}$	20° Full Depth $a = \frac{1}{p_d}$	20° Stub $a = \frac{0\ 8}{p_d}$	Fellows System							
				$\frac{4}{5}$	$\frac{5}{7}$	$\frac{6}{8}$	$\frac{7}{9}$	$\frac{8}{10}$	$\frac{9}{11}$	$\frac{10}{12}$	$\frac{12}{14}$
10	0 056	0 064	0 083								
11	0 061	0 072	0 092								
12	0 067	0 078	0 099	0 096	0 111	0 102	0 100	0 096	0 100	0 093	0 092
13	0 071	0 083	0 103	0 101	0 115	0 107	0 106	0 101	0 104	0 098	0 096
14	0 075	0 088	0 108	0 105	0 119	0 112	0 111	0 106	0 108	0 102	0 100
15	0 078	0 092	0 111	0 108	0 123	0 115	0 115	0 110	0 111	0 105	0 103
16	0 081	0 094	0 115	0 111	0 126	0 119	0 118	0 113	0 114	0 109	0 106
17	0 084	0 096	0 117	0 114	0 129	0 122	0 121	0 116	0 116	0 111	0 109
18	0 086	0 098	0 120	0 117	0 131	0 124	0 124	0 119	0 119	0 114	0 111
19	0 088	0 100	0 123	0 119	0 133	0 127	0 127	0 122	0 121	0 116	0 113
20	0 090	0 102	0 125	0 121	0 135	0 129	0 129	0 124	0 123	0 118	0 115
21	0 092	0 104	0 127	0 123	0 137	0 131	0 131	0 126	0 125	0 120	0 117
23	0 094	0 106	0 130	0 126	0 141	0 134	0 135	0 129	0 128	0 123	0 120
25	0 097	0 108	0 133	0 129	0 143	0 137	0 138	0 133	0 130	0 126	0 123
27	0 099	0 111	0 136	0 132	0 146	0 140	0 140	0 135	0 133	0 129	0 125
30	0 101	0 114	0 139	0 135	0 149	0 143	0 144	0 138	0 136	0 132	0 128
34	0 104	0 118	0 142	0 137	0 150	0 145	0 146	0 140	0 137	0 134	0 130
38	0 106	0 122	0 145	0 140	0 154	0 149	0 149	0 144	0 141	0 138	0 133
43	0 108	0 126	0 147	0 145	0 159	0 154	0 154	0 149	0 145	0 142	0 138
50	0 110	0 130	0 151	0 147	0 161	0 156	0 156	0 151	0 147	0 144	0 140
60	0 113	0 134	0 154	0 150	0 164	0 159	0 159	0 154	0 150	0 148	0 143
75	0 115	0 138	0 158	0 153	0 166	0 161	0 161	0 156	0 152	0 150	0 145
100	0 117	0 142	0 161	0 158	0 171	0 166	0 166	0 160	0 156	0 154	0 150
150	0 119	0 146	0 165	0 162	0 174	0 170	0 169	0 164	0 160	0 158	0 154
300	0 122	0 150	0 170	0 164	0 176	0 172	0 171	0 166	0 162	0 160	0 156
Rack	0 124	0 154	0 175	0 173	0 184	0 179	0 176	0 172	0 170	0.168	0 166

be adapted to this last stated problem as follows. To accord with modern practice, circular pitch will also be transformed to diametral pitch.

Let D = the pitch diameter.

w = the load per inch of face.

p_d = the diametral pitch = π/p_c or $p_c = \pi/p_d$.

Then,

$$N = D \times p_d$$

Therefore

$$W = bp_cs\left(0.124 - \frac{0.684}{N}\right) = b \times \frac{\pi}{p_d} \times s\left(0.124 - \frac{0.684}{Dp_d}\right)$$

or

$$W = bs\left(\frac{0.389}{p_d} - \frac{2.15}{Dp_d^2}\right)$$

or since $w = W \div b$,

$$w = s\left(\frac{0.389}{p_d} - \frac{2.15}{Dp_d^2}\right) \tag{5}$$

and therefore

$$p_d = \frac{s}{w}\left(0.194 + \sqrt{0.038 - \frac{2.15w}{sD}}\right) \tag{6}$$

If $b = kp_c$ equation (1) may be written

$$s = \frac{Wp_d^2}{k\pi^2 y} \tag{6a}$$

This is a convenient form where the power to be transmitted and the diameter of the gears are known.

Since also $W = \frac{2T}{D} = \frac{2Tp_d}{N}$ the equation may be written

$$s = \frac{2Tp_d^3}{k\pi^2 yN} \tag{6(b)}$$

which is a convenient form where the power to be transmitted and the ratio of the gears are known.

The Lewis equation in its original form considers only the *static* or *beam* strength of the tooth. It is obvious that the stress so determined must be modified for the effects of the impinging of the teeth upon each other.*

One of the first attempts to make allowance for this increase in stress was that of Carl Barth, who formulated the equation

$$s = \frac{600s_1}{600 + V} \tag{7}$$

where s = safe working stress for a velocity at the pitch line of V ft per min and s_1 = the allowable stress for a pitch line velocity of zero, assumed by Barth to be one-third the ultimate strength. Table XXVII gives values of s_1 that have been used successfully with equation (7) for moderate speeds.

For well-cut metal gears and pitch line velocities up to 4000 ft per min the following modification of this equation is recommended:

$$s = \frac{1200s_1}{1200 + V} \tag{8}$$

For pitch line velocities of 4000 ft and over the American Gear Manufacturers Association recommends

$$s = \frac{78s_1}{78 + V^{1/2}} \tag{9}$$

* "Effect of Rotation on Stress Distribution in Electric Railway Motor Pinions," *General Electric Review*, February, 1924.

TABLE XXVII

VALUES OF s_1

Materials	s_1 in lbs per sq in
Wood (beech or maple)	3,000
Rawhide	8,000
Fabroil	8,000
Bakelite micarta	8,000
Cast iron	8,000 to 10,000
Semi-steel	10,000
Bronze	12,000 to 15,000
Steel castings	20,000
Mild steel, untreated	25,000
Alloy steels, case-hardened	50,000
Chrome-nickel steel, hardened throughout	100,000
Chrome-vanadium steel, hardened throughout	100,000

TABLE XXVII (*a*)

VALUES OF s_t IN LB PER SQ IN

Material	Brinell Hardness Number	s_t	Material	Brinell Hardness Number	s_t
Gray iron	160	12,000	Steel	240	60,000
Semi-steel	190	18,000	Steel	280	70,000
Phosphor bronze	100	24,000	Steel	320	80,000
			Steel	360	90,000
Steel	150	36,000	Steel	400	100,000
Steel	200	50,000			

Professor Earle Buckingham remarks that it has been found by experience that after gears reach a pitch line velocity of over 5000 ft per min their load-carrying capacity is practically constant for any higher speeds and states that the following equation is in common use in this country for high velocities.

$$W = bkD \tag{10}$$

where W = transmitted load, b = width of gear face, D = pitch diameter of smallest gear, and k = a constant For heat-treated gears

k = 62 5 for single reduction gears steady load continuous service.

k = 100 for single reduction gears steady load but full load reached only occasionally

The Lewis formula as presented in the foregoing has been used with success for moderate speeds However, it is obvious that the effects of variation in velocity, inertia, inaccuracy of tooth outlines, wear, etc., naturally are magnified by increased velocity. The best information on this point has been supplied by an extended series of tests conducted by Professor Earle Buckingham under the suspices of the A.S.M.E., the results of which are published by that society under the title "Dynamic Loads on Gear Teeth." The principal object of these tests was to separate the **increment load** due to inertia and imperfections of tooth outline from the basic load due to the power transmitted; and to evaluate the effect of repeated local stress upon wear. For the increment load Professor Buckingham gives

$$Wi = \frac{0.05V(W + bC)}{0.05V + (W + bC)^{1/2}} \tag{11}$$

where Wi = total increment load, W = load due to the power transmitted, V = pitch line velocity, b = breadth of the tooth, and C = a constant dependent upon the error in the tooth faces. The total load upon the tooth is therefore

$$Wd = Wi + W = \frac{0.05V[W + bC]}{0.05V + [W + bC]^{1/2}} + W \tag{12}$$

The error in the tooth face, of course, will depend upon the degree of refinement used in cutting the teeth. The American Gear Manufacturers Association publications state that this error will vary from 0.002 to 0.0048 in. in well-cut commercial gears. The Association sets up three classes of gears and establishes tolerances for gear manufacture. Class 1 includes ordinary good commercial gears; Class 2, gears cut with great care; and Class 3 is for ground teeth. These values are listed in Table XXVIII, and Table XXIX gives corresponding values of C. The values shown in Table XXVII (b) should keep the noise and the intensity of the dynamic load within reasonable limits.

TABLE XXVII (b)

MAXIMUM ERROR IN ACTION BETWEEN GEARS

V	Error	V	Error	V	Error
250	0 0037	1750	0 0017	3250	0 0008
500	0 0032	2000	0 0015	3500	0 0007
750	0 0028	2250	0 0013	4000	0 0006
1000	0 0024	2500	0 0012	4500	0 0006
1250	0 0021	2750	0 0010	5000	0 0005
1500	0 0019	3000	0 0009	and over	

TABLE XXVIII

Errors in Teeth

Diametral Pitch	Class 1	Class 2	Class 3
1	0 0048	0 0024	0 0012
2	0 0040	0 0020	0 0010
3	0 0032	0 0016	0 0008
4	0 0026	0 0013	0 0007
5	0 0022	0 0011	0 0006
6 and finer	0 0020	0 0010	0 0005

TABLE XXIX

Values of C

Materials of Gear and Pinion	Tooth Form	Error in Gear Teeth					
		0 0005	0 001	0 002	0 003	0 004	0 005
C I and C I	$14\frac{1}{2}°$	400	800	1600	2400	3200	4000
C I and St	$14\frac{1}{2}°$	550	1100	2200	3300	4400	5500
St and St	$14\frac{1}{2}°$	800	1600	3200	4800	6400	8000
C I and C I.	20° full depth	415	830	1660	2490	3320	4150
C I and St.	20° full depth	570	1140	2280	3420	4560	5700
St and St.	20° full depth	830	1660	3320	4980	6640	8300
C I. and C I.	20° stub tooth	430	860	1720	2580	3440	4300
C.I. and St.	20° stub tooth	590	1180	2360	3540	4720	5900
St. and St.	20° stub tooth	860	1720	3440	5160	6880	8600

For safety the static load W_b for safe beam strength should exceed the total dynamic load W_d.

For steady loads. . . . . $W_b = 1.25W_d$
For pulsating loads. . . . . $W_b = 1.35W_d$
For shock loads. . . . . $W_b = 1.50W_d$

The foregoing discussion has reference to gears with cut teeth. Rough-cast gears are still used in agricultural machinery and where the speed is low and precision not important. An empirical rule for rough-cast teeth is

$$W = 200 \times p_c \times b \tag{13}$$

where W, as before, is the *total* load p_c the circular pitch, and b the width of face of tooth. The strength of wooden mortise teeth, made of beech or maple, may be taken as about one-half that of cast iron, under the same circumstances; and the strength of good rawhide gears may be taken as equal to that of similar gears made of cast iron. It is to be noted that a rawhide gear will endure considerably more shock than one made of cast iron.

163. Wear of Gear Teeth. The wear on gear teeth may be divided into two kinds, namely, the wear from abrasion due to sliding between surfaces and that due to fatigue failure from concentrated compressive stress between the surfaces. The first of these may be obviated by proper lubrication provided the load is not excessive. The second is dealt with in some detail in the report by Professor Buckingham to which reference has been made. From this report,

Let D = pitch diameter of pinion.
b = width of face in inches.
n = number of teeth in pinion.
N = number of teeth in gear.
E_1 and E_2 = modulus of elasticity of materials of gears.
θ = pressure angle.
s_c = compressive endurance stress limit.

Then the limiting load on the tooth for wear is

$$Ww = DbK\left[\frac{2N}{n + N}\right] \tag{14}$$

where

$$K = \frac{s_c^2 \sin \theta}{1.40}\left[\frac{1}{E_1} + \frac{1}{E_2}\right]$$

Values of K are given in Table XXX.

TABLE XXX

VALUES OF K

Material in Pinion	Brinell Number	Material in Gear	Brinell Number	s_c Fatigue Limit of Surface of Material	K for $14\frac{1}{2}°$ System	K for 20° System
Steel	150	Steel	150	50,000	30	41
"	200	"	150	60,000	43	58
"	250	"	150	70,000	58	79
"	200	"	200	70,000	58	79
"	250	"	200	80,000	76	103
"	300	"	200	90,000	96	131
"	250	"	250	90,000	96	131
"	300	"	250	100,000	119	162
"	350	"	250	110,000	144	196
"	300	"	300	110,000	144	196
"	350	"	300	120,000	171	233
"	400	"	300	125,000	186	254
"	350	"	350	130,000	201	275
"	400	"	350	140,000	233	318
"	500	"	350	145,000	250	342
"	400	"	400	170,000	344	470
"	500	"	400	175,000	364	497
"	600	"	400	180,000	385	526
"	500	"	500	190,000	430	588
"	600	"	600	230,000	630	861
"	150	Cast iron	....	50,000	44	60
"	200	Cast iron	....	70,000	87	119
"	250	Cast iron	....	90,000	144	196
"	150	Ph. bronze	....	50,000	46	62
"	200	Ph. bronze	....	70,000	91	124
"	250	Ph. bronze	....	85,000	135	204
Cast iron	....	Cast iron	....	90,000	193	284

164. Width of Face of Gears. There is no fixed rule governing the width of gear faces and this factor may vary considerably depending upon conditions. However, if the face is over-long and the supporting framework yielding, concentrated loading due to lack of alignment will occur. On the other hand, the tooth must not be so short that overloading and excessive wear may occur.

Experimental data on the durability of teeth are few, those of Marx and Buckingham being the most instructive. It is evident, however, that the allowable load will depend largely on the character of the service, velocity of rubbing, lubrication, and the material used. Thus, for ordinary cut cast-iron teeth under constant service, the value given above (200 lb) is probably conservative; with teeth of high-grade steel much greater loads may be carried. Cases are on record where loads above 2000 lb per in. of face were successfully carried, with a peripheral velocity of more than 2000 ft per min, the pinion being of forged steel and the gear a steel casting, 4.92-in. circular pitch. Well-made **gears of rawhide** may be loaded up to 150 lb per in. of face, per inch of circular pitch; but the load should never exceed 250 lb per in. of face.*

In machines such as punching-machines, which work intermittently and whose operation extends over a short space of time, the element of wear is not so important in the design of the teeth; but in such gears as those connecting street-railway or automobile motors with the driving axles, where the work is both continuous and severe, wearing qualities may be fully as important as strength; and gears made of steel or other hard materials may have to be used solely on this account.

The actual width of face for ordinary pattern-cast gears, machine molded, and cut gears is usually **three to three and a half times the circular pitch.** So far as computations for strength are concerned, the width of the face should not be assumed greater than two to two and a half times the circular pitch for pattern-cast and machine-molded gears nor greater than three to four times the circular pitch for cut gears as ordinarily aligned.

Examples. The pitch can be found from equation (6) for any values of w, D, and s, when the face of the gear is known or assumed. A common problem is as follows: The distance † between two shafts and their velocity ratio is known; required the pitch of spur gears to connect these shafts for a given load and working stress on the teeth. The center distance of the shafts, and the velocity ratio, fix the diameter of the gears. The face of the gears may be governed by the space available, or it may be assumed by the designer upon other considerations. To illustrate: Suppose $W = 15{,}000$ lb and s_1 the allowable stress at low velocity $= 8000$ lb per sq in. Let the diameter of the smaller gear be 40 in. and the rpm be 30. Assume the width of face as three times the circular pitch. Then the velocity at the pitch line will be

$$V = \pi \frac{40}{12} \times 30 = 314 \text{ ft per min}$$

and from equation (7) of Article 162 the allowable stress s will be

$$s = s_1 \left(\frac{600}{600 + V}\right) = 8000 \left(\frac{600}{600 + 314}\right) = 5250 \text{ lb per sq in.}$$

Hence

$$w = \frac{W}{b} = \frac{W}{3p_c} = \frac{Wp_d}{3\pi} = \frac{15{,}000p_d}{3\pi} = 1590p_d$$

* Private communication from the New Process Raw Hide Co.

† It is better, if possible, to keep the distance between shaft centers somewhat flexible, as it may be difficult to find a suitable pitch that will give an integral number of teeth.

If, now, trial values of p_d are assumed the resulting values of w can be substituted in equation (6) of Article 162. Thus, let $p_d = 1$, whence $w = 1590$; then

$$p_d = \frac{s}{w}\left(0.194 + \sqrt{0.038 - \frac{2.15w}{sD}}\right)$$

$$= \frac{5250}{1590}\left(0.194 + \sqrt{0.038 - \frac{2.15 \times 1590}{5250 \times 40}}\right) = 1.15$$

The result indicates that the assumed diametral pitch is a little on the side of safety; whence $p_d = 1$, $N = 40$, and $b = 3\pi/p_d = 9.42$ or say $9\frac{1}{2}$.

A more common problem is that in which the distance between shaft centers is not fixed but the velocity ratio and the load to be transmitted are known. Thus let it be required to design a pair of spur gears with $14\frac{1}{2}°$ composite teeth to transmit 24 hp with a gear ratio of 1 to 5. Assume pinion to have 24 teeth and gear 120 teeth. The pinion shaft rotates 480 rpm and it is desirable to have quiet running conditions. Assume a steel pinion and a bronze gear. From Table XXVII the allowable stress for bronze is 12,000 to 15,000 lb per sq in. and for steel 25,000 lb per sq in., hence gear will have weaker teeth.

$$\text{The torque on gear} = T = \frac{24 \times 33{,}000 \times 12}{2\pi \times 96} = 15{,}750 \text{ in lbs.}$$

The width of face $= b = 3p_c$ or $k = 3$ and $y = 0.1178$ for $N = 120$. The induced stress (equation 6(b)

$$= s = \frac{2Tp_d^3}{k\pi^2 yN} = \frac{2 \times 15{,}750p_d^3}{3\pi^2 \times 0.1178 \times 120} = 75p_d^3$$

If $p_d = 4$, $s = 4800$ lb; $D = \frac{24}{4} = 6'$ and $V = \pi(\frac{6}{12})480 = 753$ ft min.

$$\text{Hence from equation (7) } s = 12{,}000 \frac{600}{(600 + 753)} = 5300 \text{ lb sq in.}$$

Therefore for the Lewis equation the design is satisfactory and the diameters are 6″ and 30″ and $b = 2.35''$ or say 2.5″.

For beam strength Table XXVII (a),

$$W_b = s_t b p_c y = 24{,}000 \times 2.5 \left(\frac{\pi}{4}\right) 0.1178 = 5550 \text{ lb.}$$

$$\text{For static load } W = \frac{T}{r} = \frac{15{,}750}{15} = 1050 \text{ lb.}$$

For quiet running and dynamic effect within reason tooth error for $V = 753$ may by Table XXVII (b) be 0.0028, hence for $p_d = 4$, Table XXVIII Class I gears would be satisfactory for strength and dynamic effect. However, wear may demand a higher class of gear. Assuming Class II gears the allowable error is 0.0013. Since the elasticity of bronze is about the same as cast iron, for steel pinion and bronze gear from Table XXIX, $C = 1430$.

$$\text{The dynamic load } W_d = \frac{0.05V\,(W + bC)}{0.05V + \sqrt{W + bC}} + W$$

$$= \frac{0.05 \times 753(1050 + 2.5 \times 1430)}{0.05 \times 753 + \sqrt{1050 + 2.5 \times 1430}} + 1050 = 2700 \text{ lb.}$$

The ratio $\frac{W_b}{W_d} = \frac{5550}{2700} = 2.06$, hence gear is safe for shock load.

For wear W_w should be equal to or greater than W_d. Assuming pinion of steel of 250 Brinell hardness and bronze gear, from Table XXX, $K = 135$

Hence for wear (Equation 14),

$$W_w = DbK\left[\frac{2N}{n + N}\right] = 6 \times 2\tfrac{1}{2} \times 135 \times \tfrac{5}{3} = 3375$$

Since W_w exceeds W_d the solution is satisfactory.

It should also be noted that the teeth of the smaller gear of a mating pair are weaker in form than those of the larger. The wear, also, is greater on the teeth of the smaller gear, since they come in contact more frequently. Hence, in general, if the small gear is properly designed the larger gear will have sufficient strength. This does not apply to certain forms of reinforced or "shrouded" teeth discussed later, nor, necessarily, where the thickness of teeth and spaces are unequal, nor where the mating gears are of different material.

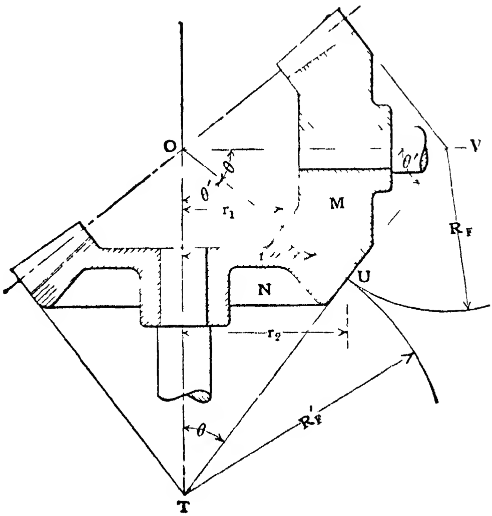

FIG. 148.

165. Strength of Bevel Gear Teeth. If a pair of bevel gear teeth, Fig. 148, have just come into contact as shown at *a*, Fig. 140, then the driving force is applied to the *point* of the driven tooth by the *root* of the driver. The tooth of the driven wheel will be deflected a certain amount, while the deflection of the driving tooth will be negligible. Since the deflection of the driven tooth is caused by the rotative effort of the driving gear, the magnitude of this deflection at any point on the line of contact of the two teeth will be proportional to the movement of the corresponding contact point of the driver or to its distance from the axis of rotation of the driver; hence, from similar triangles, this deflection will also be proportional to the distance from its own axis of rotation. Now, the cross-sectional outlines of the tooth are similar at all points, and it can be shown that, in simple cantilevers of similar form, the load applied is proportional to deflection. It has just been shown, however, that the deflection of the tooth at any point is proportional to the distance of that point from the axis of rotation. Hence the load on the tooth at any point must also be proportional to the distance from the axis, being least at the small end, and greatest at the large end, the *mean* value being at the middle of the tooth. Therefore, a spur gear which has the same width of face, and teeth of the same form and pitch as the *mean* section, will have, theoretically, the same strength as the bevel gear. It can also be shown that, in simple cantilevers of *equal breadth* and *similar outline*, the stresses induced at corresponding points on the cantilevers are equal, if the load applied is proportional to the linear dimensions. Hence, the maximum stresses are the same at all sections of the tooth.

It is evident that the relation thus established between the mean section of a bevel gear and a spur gear with similar teeth may be (and often is) used as a means of designing bevel gears. It is much more convenient, however, to deal with the teeth at the outer or large end. If, also, the pitch radii are used instead of the addendum radii, the error will not be great.

Let r_1 = the pitch radius at the small end of the tooth.
r_2 = the pitch radius at the large end of the tooth.
r = the mean pitch radius of the tooth.
b = the width of the face of the tooth along the pitch elements.
w = the load per inch of face at the radius r.
w_2 = the load per inch of face at the radius r_2.
W = the resultant load on the tooth = wb.
w_e = the equivalent load per inch of face which, if acting at a radius r_2, would produce the same rotative effect as the actual load.

Since the intensity of the load on the tooth varies as the radii, the *total resultant* load will act at a radius

$$R = \frac{2}{3}\frac{(r_2^3 - r_1^3)}{(r_2^2 - r_1^2)}$$

and the torsional moment due to the resultant force, W, will be

$$WR = \frac{2}{3} wb \frac{(r_2^3 - r_1^3)}{(r_2^2 - r_1^2)}$$

Now by definition

$$w_e b r_2 = WR$$

Therefore,

$$w_e = \frac{2}{3} w \frac{(r_2^3 - r_1^3)}{(r_2^2 - r_1^2) r_2} \tag{15}$$

Also, since the load varies with the radius,

$$\frac{w_2}{r_2} = \frac{w}{r} = \frac{w}{\frac{r_2 + r_1}{2}} = \frac{2w}{r_2 + r_1}. \quad \therefore \; w_2 = \frac{2wr_2}{r_2 + r_1} \tag{16}$$

and from (15) and (16),

$$\frac{w_2}{w_e} = \frac{2wr_2}{r_2 + r_1} \div \frac{2}{3} w \frac{(r_2^3 - r_1^3)}{(r_2^2 - r_1^2) r_2} = \frac{3(r_2 - r_1) r_2^2 \,*}{(r_2^3 - r_1^3)} = k \tag{17}$$

The actual load, w_2, will always be greater than w_e in the ratio shown above. A bevel gear, therefore, will be more heavily loaded at the large end than a spur gear of the same diameter, and carrying the same torque, in the ratio shown above. If, however, w_e is known, w_2 can be computed, and used in equations (2) and (6) of Article 162 instead of w. Usually r_1 is made not less than $\frac{2}{3} r_2$. When

$$r_1 = \frac{2}{3} r_2, \quad \frac{w_2}{w_e} = k = 1.4$$

and this value can be used in computing w_2 unless the face of the gear is excessively long. It should be especially noted that in solving problems in bevel gearing, by equations (2) and (6) of Article 162, the diameter, D, which must be substituted therein, is that corresponding to the *formative circle*, whose radius is $R_f = r_2 \sec \theta$, Fig. 148, as the *form* of the tooth is fixed by this radius and *not* by the radius r_2. The computations should be made for the smaller of the two gears, as in spur gears.

Example. Design a pair of bevel gears to transmit 100 hp with a velocity ratio of 3 to 2; the gears to be of cast iron, and the maximum fiber stress to be 4000 lb per sq in. The revolutions per minute of the shafts are to be 300 and 200 respectively.

* See also Mr. Lewis's article, *Proceedings* Engineers' Club of Philadelphia, January, 1893.

Lay off the axes OV and OT, Fig. 148, and draw OU so that corresponding radii of N and M are in the proportion of 3 to 2. Then it is found that $\theta = 34°$ and $\theta' = 56°$. Assume tentatively that r_2 for the large gear = 15 in. and r_2 for the small gear = 10 in., whence

$$r_1 = \tfrac{2}{3} \times 10 = 6.666 \text{ in.}$$

and the width of the face = 6 in.

The velocity at the radius r_2

$$= \frac{2 \times \pi \times 10 \times 300}{12} = 1575 \text{ ft per min}$$

Hence the equivalent *total* load at a radius r_2

$$= W_e = \frac{100 \times 33{,}000}{1575} = 2100, \quad \text{or} \quad w_e = \frac{2100}{6} = 350 \text{ lb}$$

Therefore

$$w_2 = w_e \times k = 350 \times 1.4 = 490 \text{ lb per in. of face.}$$

Since $\sec \theta = \sec 34° = 1.2$, the diameter of the formative circle is $20 \times 1.2 = 24$ in., and from equation (6) it is found that for $D = 24$ in., $s = 4000$ lb, and $w = 490$ lb, the diametral pitch is very nearly 3, which may therefore be selected. This would give 60 teeth for the small gear and 90 for the large gear.

166. More Refined Considerations. The methods outlined in the preceding section are sufficiently accurate for ordinary bevel gear problems where the speed is not high and where wear and noise are not important factors. However, the turbine, the automobile, and other accurately constructed machines have made a demand for more refined methods of production and more accurate methods of computing the characteristics of bevel gears. The Gleason Works have long been leaders in this field, and their system of tooth outlines has been adopted by the American Gear Manufacturers Association as a standard. Space does not permit the insertion of a detailed statement of the dimensional characteristics, and reference is made for these to the publications of the Gleason Works or of the American Gear Manufacturers Association. It should be noted that bevel gears are not interchangeable, and in the Gleason system three pressure angles are employed, namely, $14\frac{1}{2}°$, 17°, and 20°, depending upon the gear ratio. The proportions of the addendum and dedendum are not standardized but are varied to reduce the sliding effect. The Gleason system is the result of a long and varied experience in this field with bevel gears of all kinds and no doubt represents the best information to be had.

For the strength of Gleason gears the A G M A recommends the Gleason equation

$$W_s = \frac{sby[L - b]}{p_d L} \tag{18}$$

where W_s = tangential load; b = length of tooth face, L = cone length = OU (Fig 148); p_d = diametral pitch, s = the allowable stress as calculated by equation (8), and y = the Lewis coefficient taken from Table XXXI It should be remembered that the *actual* number of teeth is used in selecting y in Table XXXI.

TABLE XXXI

Outline Factors Y for Gleason Works System

Number of Teeth in Pinion	Ratios: 1.00 to 1.25	1.25 to 1.50	1.50 to 1.75	1.75 to 2.00	2.00 to 2.25	2.25 to 2.50	2.50 to 2.75	2.75 to 3.00	3.00 to 3.25	3.25 to 3.50	3.50 to 3.75	3.75 to 4.00	4.00 to 4.50	4.50 to 5.00	5.00 to ∞
	Values of Y for Lewis Formula														
10	0.231	0.260	0.280	0.294	0.305	0.315	0.324	0.332	0.340	0.347	0.353	0.358	0.365	0.371	0.377
11	0.268	0.264	0.273	0.286	0.296	0.303	0.309	0.315	0.320	0.324	0.328	0.332	0.336	0.340	0.342
12	0.248	0.265	0.281	0.295	0.308	0.318	0.328	0.335	0.341	0.345	0.348	0.351	0.353	0.355	0.356
13	0.264	0.278	0.291	0.280	0.278	0.286	0.291	0.295	0.298	0.299	0.301	0.303	0.305	0.307	0.310
14	0.242	0.254	0.263	0.272	0.281	0.288	0.294	0.299	0.304	0.307	0.310	0.313	0.316	0.318	0.319
15	0.248	0.258	0.266	0.274	0.283	0.290	0.296	0.301	0.305	0.308	0.312	0.315	0.318	0.319	0.320
16	0.252	0.261	0.269	0.277	0.285	0.292	0.298	0.304	0.308	0.312	0.314	0.317	0.319	0.321	0.323
17 to 18	0.257	0.265	0.273	0.281	0.288	0.295	0.302	0.307	0.311	0.315	0.318	0.320	0.322	0.325	0.326
19 to 21	0.265	0.272	0.279	0.286	0.294	0.300	0.307	0.312	0.317	0.320	0.324	0.326	0.328	0.330	0.332
22 to 25	0.274	0.281	0.288	0.295	0.301	0.307	0.314	0.319	0.321	0.327	0.331	0.332	0.335	0.337	0.338
26 to 30	0.284	0.291	0.297	0.304	0.310	0.317	0.322	0.327	0.332	0.336	0.339	0.342	0.344	0.346	0.347

For the allowable wearing load the same authority gives

$$W_w = 376bC_mC_s\left(\frac{N}{p_d}\right)^{1/2} \tag{19}$$

where C_m = a material factor from Table XXXII and C_s = a service factor from Table XXIII, N = number of teeth in the pinion, and p_d = the diametral pitch.

TABLE XXXII

Material Factors (C_m) Used in Wear Load Formula

Pinion	Gear	C_m
Cast iron or soft steel	Cast iron	0.30
Heat-treated steel	Heat-treated steel	0.35
Case-hardened steel	Cast iron	0.40
Oil-hardened steel	Cast iron	0.40
Case-hardened steel	Unhardened steel	0.45
Oil-hardened steel	Unhardened steel	0.45
Case-hardened steel	Heat-treated steel	0.50
Oil-hardened steel	Heat-treated steel	0.50
Oil-hardened steel	Oil-hardened steel	0.80
Case-hardened steel	Oil-hardened steel	0.85
Case-hardened steel	Case-hardened steel	1.00

The material factors (C_m) are based upon the hardness factors listed below

Condition of Steel	Brinell	Scleroscope	S A E Steels Commonly Used
Unhardened steel	160–190	25–28	1035
Heat-treated steel	200–260	30–36	2335 3140
Oil-hardened steel		70–80	3245
Case-hardened steel		80–90	2315 2512 3312X

TABLE XXXIII

Nature of Load and Service Factors (C_s)

Intermittent service	C_s = 1.3 for non-pulsating load; 1.0 for light shock; 0.65 for heavy shock
Continuous service.	C_s = 1.0 for non-pulsating load; 0.75 for light shock; 0.50 for heavy shock
Starting	C_s = 1.5 for infrequent loads of short duration

Example. A pair of steel bevel gears carefully cut are required to transmit 5 hp with a velocity ratio of 1 to 3. The pinion shaft rotates 500 rpm. The service factor may be taken as unity and the material factor as 0.3 or 0.8 depending on whether the gears are oil hardened or not. The error in the tooth outlines may be assumed as 0.001 in., whence C in equation (11) may be taken as 1600. A preliminary layout assumes the largest pitch diameters as 4 and 12 in., cone length as 6.32 in., and length of tooth as 2 in. Assuming a diametral pitch of 6, there will be 24 teeth in the pinion and 72 teeth in the gear. Hence also the pitch line velocity will be $4 \times \pi \times 500/12 = 524$ ft per min and the transmitted load will be

$$W = \frac{5 \times 33,000}{524} = 317 \text{ lb}$$

From equation (8) and Table XXVII the allowable stress will be

$$s = 25.000\left[\frac{1200}{1200 + 524}\right] = 17,000 \text{ lb}$$

From Table XXXI the value of y for a Gleason system tooth for these conditions is 0.324, whence from equation (18) the tooth strength will be

$$W_s = \frac{17,000 \times 2 \times 0.324(6.32 - 2)}{6 \times 6.32} = 1250 \text{ lb}$$

For $C_m = 0.3$ and $C_s = 1$, from equation (19) the allowable wearing load will be

$$W_w = 376 \times 2 \times 0.3 \times 1 \times (\tfrac{24}{6})^{1/2} = 451 \text{ lb}$$

If the gears are oil hardened and $C_m = 0.8$, then $W_w = 1200$ lb.

From Table XXIX, $C = 1600$, whence the dynamic load will be from equation (11),

$$W_d = \frac{0.05 \times 524(317 + 2 \times 1600)}{0.05 \times 524 + (317 + 2 \times 1600)^{1/2}} = 1082 \text{ lb}$$

and the total load $W_t = 317 + 1082 = 1399$ lb. This is somewhat higher than the beam strength but may be accepted. Since, also, the allowable wearing load for unhardened gears is much less than the dynamic load the gears should be hardened, whence $W_w = 1200$ lb. Since this allowable wearing load is in excess of the transmitted load and not much less than the dynamic load these proportions may be considered satisfactory, though a somewhat larger tooth would be desirable.

HELICAL OR TWISTED SPUR GEARING

167. General Principles. Suppose a spur gear to be cut into n small sections by a series of planes perpendicular to the axis of rotation. If each section be then placed a proper distance ahead or behind the adjacent section, Fig. 149 (a), it is evident that they may be so arranged that some one section is just coming into contact with its mating section when the nth section in advance of it is in contact at the pitch point. With such an arrangement some section will always be in contact near the pitch point, and there will always be approximately n points of contact with the mating gear between the pitch point and the point which marks the beginning of tooth action. Since the action of gear teeth is smoothest when contact is near the pitch point, this arrangement of gearing runs more quietly and smoothly than ordinary spur gearing, and it was at one time used in machine tool and similar work where smooth action is very desirable.

As the number of sections is increased, the total width of the gear remaining the same, the spacing of these sections being kept uniform as before, the form of the stepped tooth approaches that shown in Fig. 149 (c). When the number becomes infinite the teeth become helical in form, and contact is continuous along that portion of the face which is within the arc of contact. It is evident, however, that since the relative position of adjacent laminæ is arbitrary, and may follow any desired law, the outline of the tooth in an axial direction is not necessarily helical, but may have any desired shape; although these teeth are most usually made helical, this form being more practical to cut. This form of gearing is also known as *twisted gearing*, for an obvious reason. The action of such gears is identical with that of common spur gearing, and should not be confused with that of screw gearing, though certain limiting forms of the latter are also twisted gears. A screw gear must have regular or uniform helical teeth; a twisted gear does not necessarily have this limitation. In plain and twisted spur gears the relative sliding between teeth is in the plane of rotation only; in screw gears there is also relative sliding along the tooth elements.

Since the pressure, W, between mating teeth must be normal to the surface, there is a component, Fig. 149 (c), which tends to move the gear in an axial direction causing end thrust on the shaft collars. This can be obviated by placing two sets of helical gears upon the shafts to be connected, one gear on each shaft being made with a right-hand helix and one with a left-hand helix. The gears on the same shaft may or may not be placed close to each other. A very common method is to make the two sets of teeth on the same shaft integral with each other, as shown in Fig. 150. This makes a very strong form of tooth, and this construction has been much used in heavy hoisting and haulage machinery where a high velocity ratio between the shafts is desired. Gears of this type are known as **herring-bone gears.** In recent years the term has been extended to include all similar combinations of right- and left-hand helical teeth whether those on the same shaft are made integral with each other or not.

Gears such as are shown in Fig. 150 cannot be machined by the ordinary rotary cutters or hobs. They can be machined, however, by planing processes, and at least one manufacturer in this country is producing true herring-bone gears of large size with teeth accurately finished in this manner. Herring-bone gears with cast teeth for very heavy work are often shrouded, either the pinion being shrouded to the top of the tooth or both pinion and gear shrouded to the pitch line. Gears of this construction, running at low velocities and with very heavy lubricant, have

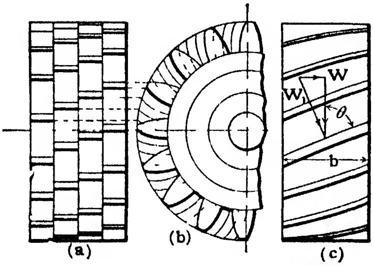

Fig 149.

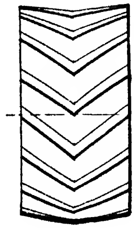

Fig. 150.

been found to operate very well. It is obvious that where a pair of right-hand and left-hand helical gears are used to connect a pair of shafts, it is not necessary that they be placed close to eath other. Economy of space, however, often makes it desirable that they be so placed, and in such a case the problem of machining the teeth becomes a troublesome one. Figure 151 shows the developed surfaces of several types of machined

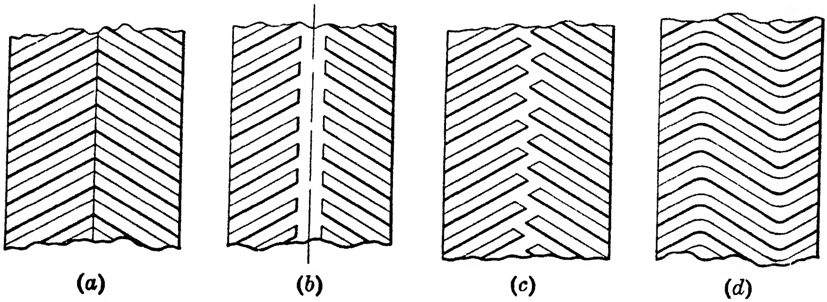

Fig. 151.

herring-bone gears. In Fig. 151 (a) the two gears are machined independently and are keyed to the shaft so that the two sets of helical teeth mate properly at the adjoining surfaces. For light work this arrangement is satisfactory. In Fig. 151 (b) the two sets of teeth are cut integral with the wheel, and a clearance space is left between the two sets of teeth to permit the cutter to run out of the cut. In the Wuest system, shown in Fig. 151 (c), the two sets of teeth are staggered, thus saving much of the space allowed in (b), while still permitting a circular cutter to be used in machining the teeth. Since there is no connection between the two sets of teeth in any of these three constructions, they are obviously not as strong as that shown in Fig. 150. In the Citroen system, illustrated in Fig. 151 (d), the teeth are machined with an end-milling cutter and hence there is almost no limit to the form of twist that can be given to the teeth. There are serious mechanical difficulties in machining teeth in this manner, but the system has been used to a considerable extent in Europe. With properly machined teeth herring-bone gears may be used at high velocities under heavy loads.

Care must be used that the alignment in an axial direction is accurate, or end play must be provided so that each gear carries its own share of the load. Because of the extended continuity of action, due to twisting the teeth, pinions may operate successfully with fewer teeth than are necessary in the case of plain spur gears, provided of course that tooth interference is cared for. For this reason, herring-bone gears are now much used for reducing the rotative speed of steam turbines in connection with the driving of marine propellers and for similar conditions where a high velocity ratio is necessary. In such installations the pitch of the teeth is rather fine and the face of the gears comparatively wide. Installations of this character have been operated at a speed of 6000 ft per min at the pitch line.

168. Strength of Helical Spur Gears. If the effective load which one tooth of a helical gear transmits to its mate be W, Fig. 149 (c), then the total load normal to the face is $W_1 = W \operatorname{cosec} \theta$. If the length of the tooth be denoted by l, and the breadth of the gear by b, then $l = b \operatorname{cosec} \theta$. Hence, the load per inch of face on a helical tooth $= W_1/l = W \operatorname{cosec} \theta / b \operatorname{cosec} \theta = W/b$ or the same as in a spur gear of face b. This would be strictly true if all points in the line of contact were at the same distance from the axis of rotation, as in a spur gear. This is never so in helical gears, the line of contact always extending diagonally across the tooth face. The error due to this, however, is small and on the side of safety, and it may be assumed that the load per inch of face in helical gears is the same as that of a spur gear of equal width and equally loaded. This diagonal distribution of the load across the tooth face decreases the lever arm of the force which tends to break the tooth, the amount of decrease depending on the amount of twist in the tooth. If the twist is so great that when the end in advance is going out of contact the other end is just coming into contact, the line of contact will run diagonally across the tooth from point to flank, and the average arm of the driving force will be about one-half the height of the tooth. If the twist be made equal to the pitch, tooth action is continuous at every point of the arc of action and this or a somewhat greater twist is generally used. The cross-section of the tooth in line with W_1, Fig. 149, is narrower, however, than the true theoretical outline in line with W, this difference becoming greater and greater as the angle of twist is increased. It is clear, therefore, that the assumption often made that helical teeth are twice as strong as spur teeth of the same pitch is not true for teeth of usual proportions, a difference of 25 per cent being, perhaps, as much as can safely be assumed. On account of continuous tooth action and consequent smoother operation in helical gears, the effect of shock is lessened somewhat.

It will be obvious that the Lewis equation could be applied to helical gears if the tooth-form factor y were known. Trautschold * gives an empirical equation for the factor as

$$y = \frac{T^2}{6L} \tag{20}$$

where T = the width of the tooth flank just above the fillets and L = radial distance from the intersection of the line of action and the center line of the tooth to the tooth-flank width.

For the beam strength of helical spur gears the A.G.M.A. recommends the following

$$W_s = \frac{sby}{p_d C_w} \tag{21}$$

where C_s is a service factor = 1.15 for enclosed gearing and $s = 78s'/(78 + \sqrt{V})$. Values of s', the allowable static stress in the material, are given as follows:

Material	s'
High-carbon or alloy steels heat-treated to an elastic limit of approximately 60,000 lb per sq in.	15,000
0.40 to 0.50 carbon steel heat-treated to an elastic limit of approximately 50,000 lb per sq in	12,500
0.40 to 0.50 carbon steel untreated with an elastic limit of approximately 40,000 lb per sq in.	10,000
Cast steel A.S.T.M. Class B, elastic limit approximately 36,000 lb per sq in.	7,500
Cast iron of tensile strength approximately 24,000 lb per sq in	4,000
Bronze 88–10–2 of tensile strength approximately 27,000 lb per sq in	4,000

Reduction gears with herring-bone teeth are now manufactured as a regular commercial product by several manufacturers in this coun-

* See "Standard Gear Book," by R. Trautschold, page 92.

try. Practice as to tooth proportions vary somewhat, but the angle of twist is usually made 23°, the angle of pressure 20°, with stub teeth. A detailed description of Wuest gears as made by the Falk Company and an excellent discussion of their strength will be found in the *Transactions* A.S.M.E., vol. 33. A similar description and discussion of herring-bone gears, as manufactured by the Fawcus Company, will be found in the *American Machinist,* Nov. 18, 1915. The Farrell-Birmingham Company manufactures a machine-cut herring-bone gear of high quality. In general, such gears are a specialized product, and it is well to consult the manufacturers where conditions are at all complex. The A.G.M.A. has done considerable work in standardizing helical gears, and their publications give ratings for various classes of gear reductions using helical gears. These publications should be consulted.

169. Spiral Bevel Gears. It will be obvious that the teeth of bevel gears can be twisted in the same way as those of spur gears. In the Citroen system of gearing, referred to in the foregoing, herring-bone bevel gears have been manufactured commercially. In this country the spiral bevel gear has been much used for driving the rear axle of automobiles. In the well-known Gleason spiral bevel gear the curve of twist is circular, since the teeth are cut with a hollow circular cutter. As yet, this form of gear has not been used to any great extent outside of the automobile industry, but its many advantages would appear to commend it for a wider service.

As these gears are highly specialized reference is made to the publications of the Gleason Works and of the A.G.M.A. for data on proportions and strength. The same remarks apply to the Gleason **hypoid** gear which may be used with non-intersecting axes.

170. Other Forms of Gear Teeth. The foregoing discussion has been confined largely to standard gear tooth systems now in use and to the several methods employed for strengthening them. One method of securing stronger teeth which has already been noted is to make the addendum of the pinion longer and the dedendum shorter than the standard tooth and the addendum of the gear shorter and the dedendum longer than standard, the total length of the tooth remaining the same. Thus, in certain bevel gears made by the Gleason Company the addendum of the pinion is 0.7 of the working depth, while the addendum of the gear is 0.3 of the working depth, the total working depth of the tooth being the same as the Brown and Sharpe standard. It is thus possible with high velocity ratios to have theoretically correct outlines without interference and to secure pinion teeth having a thickness at the root greater than at the pitch line, and the thickness of the teeth of the mating gear will exceed the pitch line thickness by an increasing amount as the root is approached. The Gleason Company also varies the angle of obliquity, thereby still further strengthening the teeth at the roots. Obviously, interchangeability is not a factor in bevel gearing.

A very common way of reinforcing teeth of cast gears is by **shrouding,** which consists in casting an annular ring of metal on one or both ends of the teeth, as shown in Fig. 152. This ring is cast as an integral part of the gear casting, and hence strengthens the gear tooth by practically twice the shearing strength of the cross-section of the tooth, when both ends are shrouded to the top. The teeth of the pinion are, from their outline, always weaker than those of the gear, and the wear on them is also greater. The shrouding should, therefore, be put on the pinion; and if carried to the top of the teeth on both ends it will give them an excess of strength over those of the gear, with usual widths of face. If the gears to be reinforced do not differ greatly in diameter, the teeth of both may be shrouded half way up. Shrouding is used mostly on rough-cast gears, the shroud practically prohibiting the cutting of the teeth by the usual methods.

If the gears are to run in one direction only, and very heavy pressures are to be withstood, a form of tooth shown in Fig. 153, and known as a **buttress** tooth, may be, but seldom is, employed. The driving face, A, is made of correct theoretical outline; the back face, B, may be of any outline that will give the required strength, and clear the teeth of the mating gear. The front face should be of standard cycloidal or involute form and the backs are preferably involute forms, with a much greater angle of obliquity than would be permissible in driving.

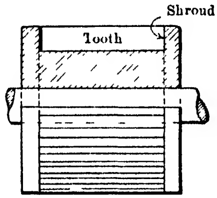

FIG. 152.

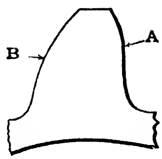

FIG. 153.

171. Strength of Gear Rims and Arms. The rim of the gear wheel must not only be strong enough to resist the forces brought upon it, but also stiff enough to prevent improper action of the teeth due to springing of the rim. A section of rim between two arms may be considered as a beam fixed at the ends and carrying a load at the middle, the value of which is $W_1 \sin \theta$, Fig. 142. Good practice makes the thickness of the rim at least $1.25t$, where t is the thickness of the tooth on the pitch line. For small gears this proportion gives ample stiffness, but for very large gears the thickness should be $1.5t$, and stiffening ribs are also sometimes necessary. In many cases the thickness should be sufficient to allow of dovetailing a tooth into the rim, in case of accidental breakage of one or more teeth. The R. D. Nuttall Company of Pittsburgh expresses the thickness t of the rim of its gears in terms of the diametral pitch p_d, the number of teeth n, and the number of arms a, as,

$$t = \frac{1}{p_d}\sqrt[3]{\frac{n}{2a}} \tag{1}$$

And Professor C. D. Albert has found the following equation very satisfactory:

$$t = \frac{1}{p_d}\sqrt{\frac{n}{4a} + 2.5} \tag{2}$$

The depth of the stiffening rib, or bead, for the rim may be taken about $1.25t$. Gear wheels are seldom run at peripheral velocities which induce dangerous centrifugal stresses. The principles governing the design of such wheels are discussed, however, in Chapter XVIII.

It is usual to assume that each arm carries its share of the torque to be transmitted and that each arm acts as a cantilever free at the outer, or rim, end and fixed at the center of the wheel. Where the rim is comparatively thin, as in gears and pulleys, the distribution of the load between the arms is uncertain. Equal distribution is the most tenable assumption where the rim is sufficiently strong and rigid for its purpose. The action of the arms is intermediate between cantilevers free at the outer ends and cantilevers free but guided at the outer ends. The assumption that they are free at the outer ends is on the safe side.

To allow for possible shrinkage and contraction stresses incident to casting and to obtain sufficient rigidity, it is quite usual in the design of the arms to take the allowable stress from 0.50 to 0.60, the allowable stress used for the teeth of the gear. Another and perhaps superior method is to use the same stress as for the teeth but to base the bending moment for each arm on the maximum permissible load on the teeth instead of the working load. The maximum load corresponds to zero velocity and may be found by substituting the allowable stress for zero velocity in equation (7), Article 162, or it may be found by multiplying the working load by the ratio of the allowable stresses for zero velocity and the velocity corresponding to the working conditions. The dimensions as dictated by the two methods should be determined and contrasted. Computations for strength of their arms or rims must, however, be considered as giving minimum dimensions, stiffness being the prime requirement, and due regard must be paid to proportions of rim, arms, and hub, to minimize shrinkage stresses due to cooling.

172. Efficiency of Spur Gearing. The experimental data on the efficiency of spur gearing are very meager. Probably the best available data are those obtained by Mr. Wilfred Lewis, for details of which see *Trans.* A.S.M.E., vol. 7. His investigation was made with a cut spur pinion of 12 teeth meshing with a gear of 39 teeth. The circular pitch was $1\frac{1}{2}$ in. and the face $3\frac{3}{8}$ in. The load varied from 430 to 2500 lb per tooth, and the peripheral speed ranged from 3 to 200 ft per min. The measurements included the friction at the teeth, and the friction at the bearings. The efficiency, as observed, varied from 90 per cent at a velocity of 3 ft per min to over 98 per cent at 200 ft per min. It appears that the friction at the teeth is a small part of the loss with good cut gears, the greater portion of the loss being at the journals. The efficiency of bevel gears is somewhat less than that of spur gears, on account of the axial thrust, which induces friction between the hub of the gear and the collar at the supporting bearing.

173. Materials and Methods of Manufacture. The principal materials used for gear teeth are cast iron, steel castings, wrought steel, bronze, wood, rawhide, cloth, and fiber. Metallic gear teeth may be cast to the required form or cut from the solid metal. Gear teeth made from other materials, in most cases, must be cut from the solid. Where the gear teeth are cast, it is very important that the pattern itself be very accurately made; for even with the greatest care in molding, it is impossible to obtain true spacing, on account of shrinkage and displacement due to "rapping" the pattern in the sand. For this reason, and on account of the difficulty of obtaining smooth surfaces, greater clearance must be allowed in cast gears than in cut gears, as already noted. Wooden patterns are very unreliable for such work, on account of their tendency to warp and shrink, and permanent patterns should be made of metal. If the pattern for a spur gear is withdrawn from the sand with a movement *parallel* to the length of the tooth, the tooth pattern must have draft, or be slightly tapering to facilitate drawing, and consequently the cast tooth must also be tapering. Care should be taken, in assembling such gears, that the tapers in the two gears are reversed to avoid having the thick ends of both sets of teeth come together, thus concentrating the pressure at one end. Rough-cast gears, of the kind described above, are used only for rough or large work, and not for high speed. The particular defect of spur gears due to draft does not exist in bevel gearing.

In gear-molding machines the pattern consists of a segment of the gear pattern, carrying several teeth. The pattern is mounted on an axis in such a manner that it can be rotated accurately through any portion of a complete revolution, or "indexed." In forming the mold the segmental pattern is placed in position and sand is rammed around it. The pattern is then withdrawn *radially* and rotated to the next succeeding position (the indexing device insuring accurate spacing), the operation being repeated till the whole circumference is molded. The mold for the hub and arms is then completed, in large work this last being often accomplished by means of cores If machine molding is well done the results are far superior to those obtained by pattern molding, and gears may be made that can be run at moderately high speeds Obviously, however, all cast gears are much more inaccurate than cut gears, and the latter are preferable where high speeds and smoothness of action are required

It is not difficult to mold fairly satisfactory gear teeth of cast iron. They have the advantage of retaining the hard strong surface which is lost when the teeth are cut by a machine Steel castings, however, are not quite so satisfactory It is difficult to obtain a smooth tooth surface in a steel casting, and the errors in the finished gear, due to shrinkage and to warping out of shape in cooling, are more marked in steel castings than in those made of cast iron The development of more powerful gear-cutting machinery has greatly extended the use of gears, especially pinions, cut from bar steel or from steel forgings The softer grades of steel have not proved to be very durable, and quite frequently gears made of wrought steel are case-hardened or are made of a quality of steel that can be hardened without the use of carbonizing materials Alloy steels carrying chromium and nickel have been used with success for this work As stated in Article 45, materials of the same texture do not usually rub together satisfactorily It is advisable, therefore, where both pinion and gear are of steel, to make them of somewhat different carbon content A bronze pinion meshing with a cast iron or steel gear is an excellent combination, and if the teeth are accurately cut such combinations may be run at comparatively high speed without trouble from noise In high-grade worm-wheels, the worm is usually made of steel and the worm wheel of bronze.

Metallic gearing, even when accurately cut and aligned, is inclined to be very noisy when run at a peripheral speed of more than 1200 ft per min, especially if any appreciable "back-lash" exists. If the points of the teeth are slightly relieved, the tendency to produce noise is reduced If high speeds are unavoidable the teeth of one of the mating gears is sometimes made of wood. Wheels with wooden teeth are known as **mortise wheels.** They are not as much used as formerly, because modern methods of gear-cutting produce metallic gears of such accurate form that they may be run in places where mortise gears were formerly considered indispensable. In making mortise wheels the wooden teeth are roughed out and the shank is fitted into openings cast in the rim of the wheel, as shown in Figs. 154 and 155. The teeth are held in place by the keys, K, or pins, P, as shown. The teeth proper are dressed to correct form with hand tools or by special machines using a fine circular saw for a cutter.

As a rule, only the large gear is made with wooden or "mortise" teeth, the pinion being made of metal. This is rational, since the pinion, on account of the shape of its teeth, is the weaker of the two, and also because of the teeth of the pinion come into contact more frequently, and hence suffer greater wear. In such combinations, the metal gear frequently has teeth of thickness less than $p/2$ and the wooden gear teeth of thickness greater than $p/2$, to equalize strength. (See Fig. 155.) In recent years, gears made of rawhide have been much

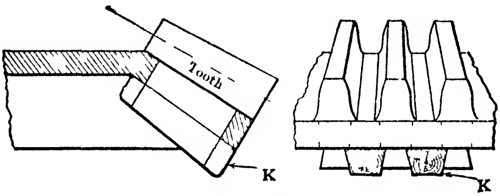

Fig. 154

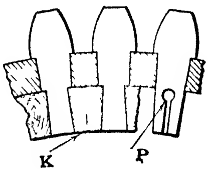

Fig. 155.

used for high speeds. The blanks for rawhide gears are made by cementing specially prepared rawhide discs together under great pressure. Metallic discs, on each side of the blank, held together by rivets passing through the blank, assist the rawhide teeth in retaining their form. The teeth are cut in the blank in the same manner that metallic teeth are cut. When rawhide gearing is used, the pinion is almost always made of rawhide and the larger gear of cast iron or brass. Such a combination may be run at a very high rate of speed, 3000 ft per min being a not unusual velocity. Rawhide gears are almost noiseless in operation, but care must be used that they are not subjected to extreme moisture or run in too dry an atmosphere.

The so-called "Fabroil" gears are manufactured by the General Electric Co. The blanks for these gears are built up by compressing cotton fabric impregnated with oil, under a very heavy pressure. The gears are held together axially by metallic shrouding, in very much the same manner as rawhide gears. These gears are as strong as ordinary cast iron; they are impervious to moisture, will run in oil, are vermin-proof, and can be run up to 3000 ft per min at the pitch line. The blanks are machined and the teeth cut in the same manner as for metallic gears.

The Bakelite Micarta gears, manufactured by the Westinghouse Electric and Manufacturing Company, are made of a fibrous material that has strength and wearing qualities similar to fabroil. This material will also run in oil, is vermin-proof, and is unaffected by atmospheric changes. Several other fibrous materials are used for gears where the reduction of noise is a factor.

Formerly it was cheaper to cast gear teeth, but the development of gear-cutting machinery has changed the situation where a large number of gears with small teeth are to be made. Modern methods of gear-cutting produce teeth of great accuracy, and have also so greatly reduced the cost of production that for high speeds, and where smoothness of action is necessary, cut gears have largely superseded cast gears, even in large work.

There are many methods of cutting gear teeth in practical operation, the most common method of cutting spur gears being by the use of a rotating cutter.* The outlines of gear teeth vary with the number of teeth in the gear, the pitch or thickness of tooth remaining constant, and, theoretically, a different cutter is required for every different diameter of gear in a series of the same pitch. To meet this requirement would lead to an excessive number of cutters for each pitch. It is found in practice, however, that the same cutter can be used, without serious error, for several sizes of gears of a given pitch. In the system adopted by the Brown and Sharpe Manufacturing Company only 24 cutters are used for each pitch in the cycloidal system, and only 8 cutters for each pitch in the involute system.

When gear cutting is carefully done, very accurate work may be accomplished. It is to be noted, however, that the form of the teeth when cut with a set of cutters, as above, is not theoretically correct; and even in the best practice the error in the gear-cutting machine itself, coupled with that due to dullness of cutters and deviation due to different degrees of hardness in the metal, may be considerable.

Another method much used for cutting spur gear teeth is known as the **generating process.** In this method the cutter is made in the form of a gear or rack and the gear-cutting machine is built so as to roll the cutter and the gear blank together at the proper velocity ratio. The cutter is given a reciproacting motion and cuts its own way into the blank. Provision must be made in the machine for feeding the cutter radially into the blank while keeping the relative rotative positions of the cutter and blank the same. It should be noted that in the generating process only one cutter is needed for each pitch, since in such a system every gear of a given pitch must run with every gear of the same

* See "Gear-Cutting Machinery," by Ralph E. Flanders.

pitch. This method of generation is employed in the Fellows gear shaper.

Another generating method now much used to cut spur gears is based upon the use of a **hob.*** A hob for this purpose is a hardened-steel worm on which the cross-section of the thread is the same as that of a rack in the particular system of teeth to be cut. Cutting edges are formed upon the thread by notching it transversely. The hob is set with reference to the blank so that the cutting surfaces in actual operation move parallel to the axis of the gear blank. The gear-cutting machine rotates the hob and gear at the proper velocity ratio and also moves the hob slowly across the face of the blank. The action is equivalent to forming the gear by means of a cutter made in the form of a rack, though obviously the cutting action is much more rapid. One form of gear cutter uses a cutting tool made in the form of a rack, the cutter having a reciprocating motion, as in the Fellows shaper.

Bevel gears, which are discussed in the succeeding section, can, of course, be cast, in the same way as spur gears. The problem of cutting bevel gear teeth is not the same, however, as that of cutting spur gear teeth. By reference to Fig. 148, it can be seen that the linear tooth elements of bevel gears meet in the common center, O, and hence the cross sections of the teeth and of the spaces between teeth decrease in size as the tooth approaches O. It is not possible, therefore, to cut accurate bevel tooth outlines with ordinary rotating cutters. It is common practice, however, to cut such outlines approximately correct with circular cutters. Sometimes the inaccuracies of such methods are corrected by filing the teeth after machining. In all methods that produce accurate bevel gear teeth the teeth are planed out by a reciprocating tool guided so as to move toward the common center, O, and controlled by guiding surfaces or other mechanisms so as to generate the required tooth form as it moves backward and forward. There is a great variety of machines for producing bevel gears, and even a brief description † of this interesting machinery is beyond the scope of this work.

HYPERBOLOIDAL AND WORM GEARING

174. General Principles. When the axes of two shafts are not parallel and do not intersect, it is possible to lay out contact surfaces on which gear teeth may be constructed which will give line contact. Gears of this kind are known as **hyperboloidal** gears. They are difficult to construct, and are very rarely used. If the load can be carried on

* See also Article 174.

† See "Gear-Cutting Machinery," by Ralph Flanders.

point contact, pitch cylinders may be described on the axes, Fig. 156, and on these surfaces helical teeth may be constructed which will transmit the desired motion. Such gears are known as **helical** or **spiral** * gears, the latter name being really a misnomer. Although the teeth of such gears resemble those of helical twisted gears, their theory and action are quite different, for, in addition to the conjugate rolling and sliding action, as in spur gears, there is also a sliding component along the elements between contact surfaces. The action of such gearing is very smooth. The special case where the axes are at right angles, and where a large wheel having many helical teeth meshes with a small one having a very few helical teeth, is an important one on account of the great reduction in velocity ratio that may thus be obtained. This last arrange-

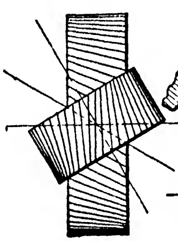

FIG. 156.

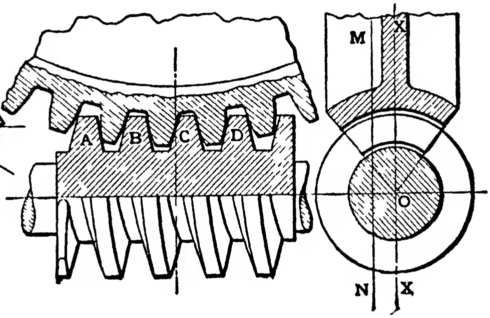

FIG. 157.

ment is commonly known as a **worm and worm wheel.** Figure 158 illustrates such a worm and worm wheel, the teeth on the worm wheel being truly helical in form and cut at angle to suit the worm thread or helix. The contact in these cases is point contact, and on the worm wheel tooth is confined to points in a line cut from the working surface of the tooth by a plane passing through the axis of the worm at right angles to the axis of the worm wheel. In operation the point of contact becomes a limited area. The advantage of this form of worm wheel, like all spur gears, is that the teeth can be cut with a rotary cutter, and patterns for rough-cast teeth are comparatively easy to construct. The same result is sometimes obtained by using a plain spur gear, and setting the axis of the worm at the proper angle with the plane of the gear.†

* For a full discussion of the methods of laying out and producing so-called spiral gears, see a "Practical Treatise on Gearing," by Brown and Sharpe Manufacturing Company, and also "Worm and Spiral Gearing," by F. A. Halsey.

† A highly successful form of this arrangement is the worm-and-rack drives on planing machines first used by Wm. Sellers and Company.

It is possible, however, to construct a worm wheel in such a manner as to secure line contact, as in spur gearing. Referring to Fig. 157, it can be seen that, when the single-threaded worm shown is rotated through 360°, any median section, as A, is moved forward an amount equal to the pitch of the worm wheel to a position B; and that rotation of the worm, in general, is equivalent to a translation of these sections backward or forward. The action is equivalent to translating a rack of similar proportions, and, in fact, if the worm itself is moved axially it will engage with the teeth of the worm wheel in the same manner as a rack does with a gear. In the involute system of gear teeth the rack has straight sides,* and this property is usually taken advantage of in making worm gearing, since a worm thread of such a cross-section is easily machined. The sides of the involute rack face are at right angles to the line of contact, aOb, Fig. 140, and hence the inclination of the sides to each other is 2θ, Fig 140, and in the standard system $2\theta = 29°$. If other planes, such MN, be passed through the worm and worm wheel parallel to the median plane, XX, Fig. 157, it will cut a trapezoid from the worm somewhat different from that cut by the median plane. The rack-like action of these trapezoids would, however, be similar to those on the median plane, and it is clear that the shape of the worm-wheel tooth in the plane MN may be so made as to mesh correctly with this new trapezoidal section. It is evident that if enough such sections be taken, a complete tooth outline may be formed that will give line contact with a worm across its full face. It is evident also that any other form of worm thread may be similarly treated.

The preceding discussion demonstrates the possibility of line contact for the worm and worm wheel, and suggests a method by which the teeth of such gears could be drawn, and hence constructed. There is no practical value in actually making such drawings; but teeth having this property of line contact are automatically produced by what is known as the **hobbing process.** A worm of tool steel is made of the exact form of the desired worm. This worm is made into a cutter by cutting flutes across the face as in Fig. 161. This is known as a **hob**; and when hardened and tempered it is used as a milling cutter. The wheel blank, which has been turned to correspond to the outside of the teeth, is mounted in a gear cutter, or a special hobbing machine, and the hob is also mounted in correct relation to the wheel, but with the axes of the wheels a little greater distance apart than the required final distance. The hob is then rotated and at the same time fed toward the worm wheel till the proper distance between the axes is reached, thus cutting

* See "Kinematics of Machinery," by John H. Barr and E. H. Wood, page 125.

the teeth in the worm wheel in a very accurate manner. Sometimes the wheel is caused to rotate simply by the action of the hob, but much better results are obtained if it is driven positively, with the proper velocity ratio, from the cutter spindle by means of positive gearing. In heavy work the teeth of the wheel are roughed out or "gashed" before hobbing. Figure 159 shows a worm wheel which has been hobbed, and its mating worm. Figure 160 * shows a form of wheel occasionally used where the wheel is sometimes rotated by hand or when the projecting teeth are undesirable. Such wheels may be hobbed, but are

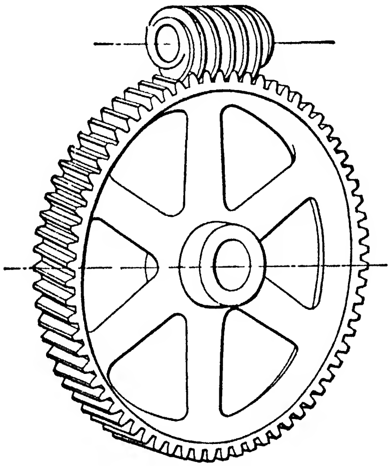

FIG. 158.

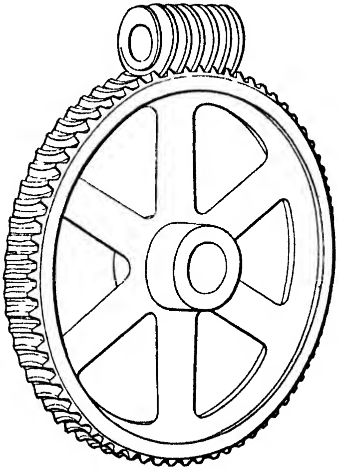

FIG. 159.

usually cut by the approximate method shown in Fig. 162, where a cutter is fed radially inward toward the axis of the worm wheel, producing what is known as a **drop-cut** wheel. In the **Hindley** worm the pitch line of the worm is curved to coincide with the pitch line of the wheel, thus obtaining contact on several teeth at the same time.†

175. Velocity Ratio of Worm Gearing. The axial advance per turn of the worm thread is called the **lead.** Thus in Fig. 157 the lead of the single-threaded worm shown is the distance, parallel to the axis, from any point on the tooth section A, to a corresponding point on the section B, and is equal to the circumferential pitch of the worm wheel. If the worm were double-threaded the lead would be twice this amount, or equal to the distance between corresponding points on A and C, and

* Figures 158, 159, and 160 are reproduced from Brown and Sharpe's "Treatise on Gearing."

† See "Worm and Spiral Gearing," by F. A. Halsey.

would then be twice the pitch of the worm wheel. The lead of the triple-threaded worm would be three times the pitch, and so on. If a single-threaded worm makes one revolution, a tooth of the worm wheel is moved a distance equal to the pitch. In a double-threaded worm the tooth would be moved twice the pitch; and in general, if N be the number of teeth in the worm wheel, and n the number of threads on the worm, then,

$$\frac{\text{Angular velocity of worm}}{\text{Angular velocity of worm wheel}} = \frac{N}{n}$$

Evidently a very great velocity ratio is possible with a comparatively small worm wheel. It is to be especially noted that the angular velocity ratio is independent of the diameter of the worm. The pitch of the worm wheel, which must be decided upon by consideration of the strength of the teeth, fixes the radius of the worm wheel for a given number of teeth; but the radius of the worm may then be varied to suit other conditions.

176. Efficiency of Worm Gearing. The general expressions for the efficiency of screws, deduced in Article 130, Chapter XI, apply also to

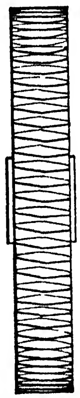

FIG. 160.

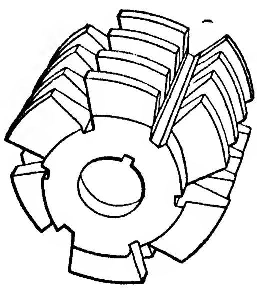

FIG. 161.

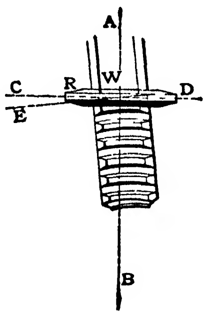

FIG. 162.

worm gearing. Since the worm thread is usually a so-called angular thread, equation 13 (a) of that article would strictly apply. However, the inclination of the face of worm threads is so small that the error introduced in using the simpler equations (9) and (10) of that article, which were deduced from the square thread, is small. These equations show that the efficiency of all screw gears is a function of the angle which the thread makes with a plane perpendicular to the axis, and of the coefficient of friction, assuming that the coefficient of friction at the thrust collar is the same as at the tooth.

One of the most valuable contributions to this subject is the experimental work of Mr. Wilfred Lewis.* The full lines in Fig. 163 have been plotted from the diagram on which he has summarized his results. They show clearly the increase of efficiency with increase of thread or lead angle at all velocities. They also show a remarkable agreement with the theoretical equations of Article 130. The dotted curve is reproduced from curve (2) of Fig. 123, and its close agreement with Mr. Lewis's curves is to be noted. This dotted curve was plotted for a value of $\mu = 0.05$. Mr. Lewis's calculated average value of this coefficient for a

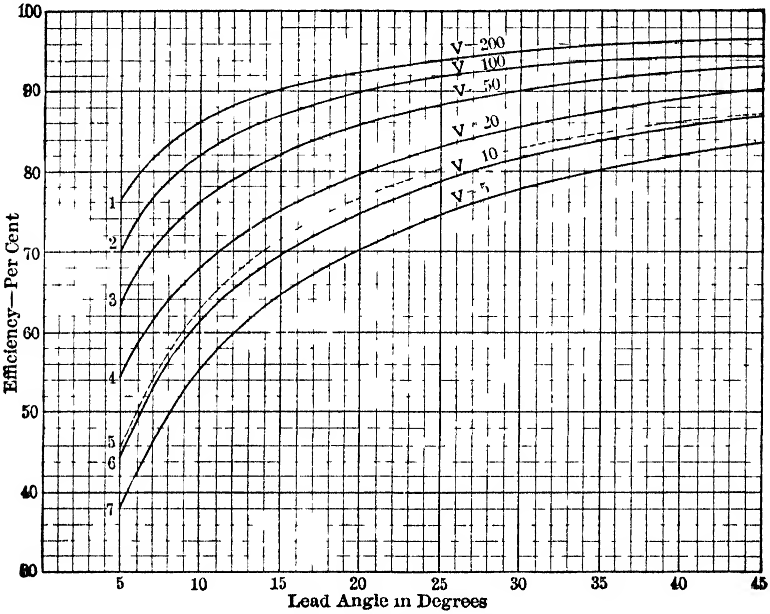

Fig 163

velocity of 20 ft per min is 0.059 and for 10 ft per min 0.074. Curves (4) and (5) in Fig. 123 may, therefore, be taken as supplementary to those in Fig. 163, and may be used, as they were intended, for designing slow-moving and poorly lubricated screws. A theoretical curve plotted from equation (9), Article 130, with a value of $\mu = 0.014$ (which would be obtained only at high speeds), will coincide very closely with Curve 1, Fig. 163. This coincidence is closer than might be expected from the nature of the problem and the assumptions on which equation (9) is based. Mr. Lewis's value of μ for these velocities † (200 ft per min)

* *Trans.* A.S M.E., vol. 7, page 297.

† Velocity here means velocity of rubbing at the point of contact between worm and worm wheel.

ranged from 0.026 to 0.015, his average value being 0.02 Professor Kennerson experimenting for the Brown and Sharpe Manufacturing Company (see *Trans.*, A.S.M.E., vol. 34), developed efficiencies that are in accord with the foregoing discussion. His experiments were conducted with steel worms mating with bronze worm wheels. The helix angles of the worms were 45° and 38° 16′ respectively. Efficiencies as high as 97.9 per cent were reported.

Mr. Halsey * has examined the design of a number of successful and unsuccessful worms used for transmitting power, and has found that every worm among those examined whose lead angle was greater than 12° 30′ was successful, and every worm whose lead angle was less than 9° was unsuccessful. He quotes Mr. James Christie, who has had considerable experience with this form of gearing, as giving 17° 15′ as the lower limit for successful design, which still further corroborates the general theory given. It is to be noted, on the other hand, that there is little to be gained in using a lead angle above 30°, the increase in efficiency being very small, while the side thrust on the wheel is increased. It is not to be understood that it is never proper to design a worm with a lead angle less than 9°, for there are many cases, not primarily for power transmission, and where the velocity is low, in which worms of less pitch are not only effective but necessary. In Mr. Lewis's experiments the worms ran in a bath of oil, and the efficiencies given include journal friction, the thrust being taken at the end of the worm shaft by a loose brass washer running between two hardened and ground steel washers.

The effect of the velocity of rubbing on the coefficient of friction of imperfectly lubricated surfaces was noted in Article 41, and Fig. 18 of that article indicates, in a general way, what may be expected with sliding *surfaces:* all experimental results going to show that the lowest coefficient was obtained at about 200 ft per min. Mr. Lewis, as the result of his work, fixes 200 ft per min as the point of maximum efficiency of worm gearing, which is in perfect accord with the general theory of lubrication. The surfaces of worm gearing, although running in an oil bath, must, from the nature of the contact, be classified as imperfectly lubricated surfaces. An increase of velocity may, up to a certain limit, decrease the coefficient of friction, but it is not possible at any speed, with the small amount of surface contact obtainable in screw gearing, to create a true oil film so that the load would be fluid-borne (Article 42).

177. Limiting Pressures and Velocities in Worm Gearing. It was stated in the last two articles that the best results are obtained from worm gearing when the rubbing velocity is about 200 ft per min and the

* See "Worm and Spiral Gearing," page 38.

lead angle not less than 12° 30′. It is not always possible, however, to keep the design within these limits. Thus, in order to obtain *mechanical advantage* (see Article 142), it may be necessary to use a worm with a very small lead angle, and kinematic requirements may necessitate a much higher velocity than 200 ft at the pitch line.

The allowable axial load that may be applied to a worm under varying velocities has not been very accurately determined, the law undoubtedly being complex (see Article 41). Enough experimental work has been done, however, to show that the pressure varies, approximately, inversely with the velocity; or the law may be roughly expressed as $WV = K$, where W = the axial load on the worm, V = the velocity of rubbing in feet per minute, and K = a constant to be determined by experiment (see also Article 54). In Lewis's experiments, made on cast-iron worms and worm wheels, running in an oil bath, it was found that the *limiting* value of K, i.e., where cutting began, was about 1,500,000. Smith and Marx * quote corresponding pressures and velocities, attributed to Stribeck, obtained with hardened-steel worm and bronze worm wheel running in an oil bath, which give an average allowable value of 690,000 for K. Bach and Roser, experimenting with soft-steel worms and bronze worm wheels, succeeded in carrying a pressure of 800 lb at a velocity of 1700 ft per min, which gives $K = 1{,}360{,}000$. It would appear, therefore, that for average conditions and bath lubrication of the worm it will be safe, for velocities up to 1500 ft per min, to take

$$WV = 750{,}000 \tag{1}$$

This value is probably conservative for accurately machined worms and worm wheels under best conditions of operation. Bach and Roser made a series of experiments with a steel worm mating with a bronze wheel and on the basis of these experiments have developed an equation † giving the relation between the axial load, the rubbing velocity and the temperature. The helix angle of the worm was 17° 34′, which is very close to the allowable limit; the formula is based upon the result of experiments with this worm only and therefore there is reason to doubt its accuracy except for the particular conditions under which it was developed. The above discussion has reference to worms as ordinarily constructed with straight-sided threads. Mr. Robert Bruce ‡ has shown that if the sides of the worm are made *concave* a much greater load may be carried. With improved threads of this form he has succeeded

* "Machine Design," fourth edition, page 415.

† *American Machinist*, July 16 and 23, 1903.

‡ *Proceedings* of Institution of Mechanical Engineers (British), page 57 of the year 1906.

in carrying 25 tons at a velocity of 120 ft per min, corresponding to $K = 6{,}720{,}000$. This great gain is due, without doubt, to the improved lubrication obtained by what practically amounts to surface contact, between the mating convex and concave surfaces of the teeth.

178. Design of Worm Gearing. In general, the strength of the worm exceeds the strength of the teeth in the worm wheel; and where the worm is made of a harder material, as is usual, the wear is greatest on the worm-wheel teeth. It is usually sufficient, therefore, to design the wheel teeth alone, considering them as simple spur gear teeth as in Article 162. With rough-cast or drop-cut teeth, it must be assumed that the entire load is carried by a single tooth; but in hobbed gearing it is safe to assume that the load is distributed between two, or even three, teeth, depending on the number of teeth in the wheel.

Example. Design a worm gear to connect two shafts which are 11 in. apart, and to transmit $7\frac{1}{2}$ hp. The velocity ratio is to be 20 to 1, the worm shaft is to make 320 rpm, the lead angle is to be not less than 15°, and the worm wheel is to be cut with a hob.

The solution of problems in worm gearing must generally be tentative. If the velocity ratio is to be 20 to 1, the worm wheel will have 20, 40, or 60 teeth, depending on whether the worm is single-, double-, or triple-threaded. It is difficult to obtain a high lead angle with a single-threaded worm without making a very large thread; therefore, a trial assumption will be made with a triple-threaded worm, and 60 teeth in the wheel. Twenty inches may be taken as a trial diameter for the wheel, and the trial pitch circumference will therefore be 63 in. approximately. If the circumferential pitch be taken as 1 in., the lead of the worm thread will be 3 in., and can therefore be easily cut in a lathe. The corrected circumference of the wheel will then be 60 in., corresponding to a pitch diameter of 19.11 in. The pitch diameter of the worm, with the given distance between centers, will be 2.9 in.; hence, the tangent of the lead angle $= \dfrac{3}{\pi \times 2.9} = 0.33$, or the lead angle is 18° 15′, which is an efficient angle.

The number of revolutions per minute of the worm wheel will be $320 \div 20 = 16$. Hence the velocity of the *worm wheel* at the pitch line $= 60 \times 16 \div 12 = 80$ ft per min. The total axial thrust on the worm will be $7\frac{1}{2} \times 33{,}000 \div 80 = 3100$ lb. The velocity of rubbing equals the length of one turn of the worm thread multiplied by the number of revolutions per minute, or

$$V = \frac{\pi \times 2.9 \times 320}{(\cos 18' - 15'') \times 12} = \frac{\pi \times 2.9 \times 320}{0.95 \times 12} = 255 \text{ ft per min}$$

The product of velocity and axial pressure on the worm $= 255 \times 3100 = 790{,}000$ which by equation (1) is a safe value, although somewhat high.

The load may be considered as distributed between two teeth, and each tooth will have a face or length at the root somewhat less than the pitch diameter of the worm (see Fig. 157), or say 2.75 in. Hence the load per inch of face of tooth $= \dfrac{3100}{2 \times 2.75} = 560$ lb. From equation (6), Article 162, it is seen that this load corresponds to a fiber stress of about 5000 lb per sq in. with 1 in. circular pitch. From equation (7) of Article 162, however, it is seen that for the velocity, 80 ft, the allowable stress is 7000 lb, hence, the tooth has an excess of strength to provide against the wear, which falls heaviest on the worm wheel.

From curve (1), Fig. 163, it is found that the efficiency is about 90 per cent; hence the horsepower which must be supplied to furnish 7.5 hp at the worm-wheel shaft will be

$$\frac{7.5}{0.90} = 8.4 \text{ hp} = 277{,}200 \text{ ft-lb per min}$$

or 866 ft-lb per revolution of the worm. The torque, T, which must be applied to the worm-wheel shaft, will be

$$T = \frac{866 \times 12}{2\pi} = 1650 \text{ in-lb}$$

The depth below the pitch line of a standard tooth of 1-in. circular pitch is, from Table XXV, 0.3857 in.; therefore, the diameter of the worm at the root of the thread $= 2.9 - (2 \times 0.3857) = 2.13$ in., and from equation (E), page 93, the torsional stress

$$s_s = \frac{16T}{\pi d^3} = \frac{16 \times 1650}{\pi \times (2.13)^3} = 850 \text{ lb per sq in.}$$

which is very low. The design may, therefore, be considered satisfactory if the worm is to be cut integral with the shaft. If, however, it is to be bored out and fitted over the shaft, further calculation as to the strength of the shaft which may be fitted is necessary.

The A.G.M.A. recommends certain standards for the proportions of worm drives and has established ratings for them. Its publications should be consulted for details. In high-speed worm gears of the enclosed type, heating may be a major factor in the design. The A.G.M.A. also gives coefficients and ratings for this factor.

179. Thrust Bearings for Worms. An important frictional loss in worm gearing occurs in the thrust bearing, which therefore deserves special attention. The general discussion in Article 62 applies in this case. The type of bearing shown in Fig. 41 is much used, and of late ball bearings have met with considerable success in such places.

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Machinery's Handbook.

CHAPTER XV

BELT AND ROPE TRANSMISSION

180. General Considerations. When power is to be transmitted from one shaft to another, especially when such shafts are not far apart, in such a manner that the velocity ratio of the two must be constant, some form of toothed gearing is usually employed. When, however, it is not necessary that the velocity ratio remain constant, flexible elastic connectors are much used. When the distance through which power is to be transmitted is comparatively short (50 ft or less), flat belts, or ropes of cotton or manila, are most common; for longer distances steel ropes have certain advantages. For small amounts of power, round belts of leather are much used. Belts with a V cross-section are also used for transmitting moderate amounts of power. Chain drives, which are virtually flexible connectors running on toothed wheels, have lately come into extended use for transmitting power over comparatively short distances. They are very efficient, maintain positive velocity ratio between the two shafts, and can be used when the distance between shafts is too great for convenient use of gears.

Leather belts are most usually made by cementing, sewing, or riveting together strips of leather cut from oak-tanned ox-hides. In recent years, however, chemical processes have come into use for leather belts that must operate under abnormal conditions as to heat, water, gas, etc. Such tanning is known as **mineral** or **chrome tanning.** Combination processes with both oak bark and chemicals are also in use. Where only one thickness is used belts are known as **single leather** belts; where two, three, or four thicknesses are needed to obtain a heavy belt, they are known respectively as **double, triple,** and **quadruple** belts. Cotton belts are made either by weaving in a loom, or are built up of several layers of canvas, sewed together, with a special composition between each fold. They are very little used in this country. Rubber belts are made of several layers of canvas, held together with, and completely covered by, a rubber composition which is vulcanized after the belt is constructed. They are very effective in wet places. Balata belts, so called, are made in the same way as rubber belts, except that balata gum is used instead of rubber. This compound is not vulcanized. Balata is water-proof, like rubber, and is said to resist acids somewhat better than rubber. It is not serviceable in hot places, as the balata, when hot, becomes sticky.

The ends of all belts are joined, to make them continuous, either by lacing or sewing, or by some kind of special fastening of which there are many on the market, or by making a permanent joint by cementing and riveting. The last method is much preferable where it can be applied, as it makes the joint practically as strong as the rest of the belt, and gives a smooth surface which runs better than any joint. Other kinds of joints reduce the strength of the belt from 60 to 75 per cent, but inasmuch as the lacing can be replaced and the belt itself has its life prolonged by reduced load, this initial loss of efficient strength is not as wasteful as it at first appears.

181. Theoretical Consideration of Belts and Ropes. In Fig. 164, let A represent a pulley whose center is at O, and which is connected by

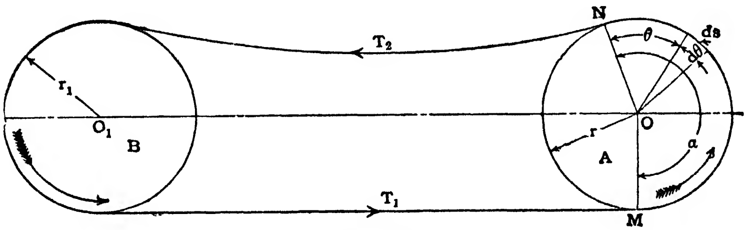

FIG. 164.

a belt, as shown, to the pulley B, whose center is at O_1. When no turning moment is applied to the driving pulley A, the tensions in the two parts of the belt are the same, except possibly for friction of the bearings, and is that due to the initial tension with which the belt is placed upon the pulleys. Let this *total* initial tension on each side of the belt be called T_3.

It is evident that this initial tension will cause the belt to exert a pressure upon the pulley, and this pressure will induce a frictional resistance opposing relative sliding between the belt and the pulley. If now a turning moment is applied to A, and a resisting moment to B, the pull upon the belt due to this frictional resistance will *increase* the tension in the lower part of the belt, and *decrease* the tension in the upper part. Let these new *total* tensions be called T_1 and T_2 respectively. It is evident that the tendency of the belt to slip around the pulley, owing to the difference in tension on the two parts of the belt, is resisted by the frictional resistance between the belt and pulley. The difference in tensions tends to rotate the pulley B, and when the turning moment

$(T_1 - T_2)r_1$ becomes equal to the resisting moment applied to B, rotation will take place.

If the difference between T_1 and T_2 which is necessary to overcome the resisting moment is small compared to the frictional resistance between the pulley and belt, no slipping of the belt on the pulley will occur. To obtain this result in practice would necessitate the use of very large belts, relatively for the power transmitted. It has been found to be better practice to use smaller belts and allow the belt to slip somewhat.

In addition to the slipping action noted above, all belts are subjected to what is known as **creep.** Referring again to Fig. 164, consider a piece of the belt of unit length moving on to the pulley under a tension T_1. As this piece of belt, of unit length, moves around with the pulley from M to N, the tension to which it is subjected decreases from T_1 to T_2 and the piece, owing to its elasticity, shrinks in length accordingly. The pulley A, therefore, continually receives a *greater* length of belt than it delivers, and the velocity of the surface of the pulley is *faster* than that of the belt which moves over it. In a similar way the pulley B receives a *lesser* length of belt than it delivers, and its surface velocity is *slower* than that of the belt which moves over its surface. This creeping of the belt, as it moves over the pulley, results in some loss of power, and is unavoidable. The total loss of speed due to both slip and creep should not exceed 3 per cent; that is, the surface speed of the driving pulley should not exceed that of the driven pulley by more than 3 per cent. Good practice limits this value to about 2 per cent. When the total slip approaches 20 per cent, there is danger of the belt's sliding off the pulley entirely.

Since the pulling power of a belt is proportional to the difference between T_1 and T_2, it is necessary to know the relation which exists between these quantities.

Let t = the tension per square inch of belt *section* at *any* point on the pulley.

t_1 = the tension per square inch of belt *section* on the *tight* side, in pounds.

t_2 = the tension per square inch of belt *section* on the *slack* side, in pounds.

f = effective pull of belt per square inch of cross-section = $(t_1 - t_2)$, in pounds.

v = the velocity of the belt, in feet per second.

w = the weight of 1 cu in. of belt, in pounds.

q = the reaction of pulley against 1 linear inch of belt of the width considered, in pounds.

c = the centrifugal force of 1 cu in. of belt, in pounds, at the given speed.
μ = the coefficient of friction between belt and pulley.
r = the radius of the pulley in inches.
α = the angle of belt contact in degrees.
θ = the angle of belt contact in radians = 0.0175α.

The centrifugal force of 1 cu in. of belt will be

$$c = \frac{12wv^2}{gr}$$

hence the centrifugal force of 1 linear inch of belt having 1 sq in. of cross-section will be $12wv^2/gr$.

Let the cross-sectional area of the belt be 1 sq in., and consider an elemental portion of its length as shown in Fig. 165. It is held in equilibrium, when slipping is impending, by the following forces:

(a) The centrifugal force = cds.

(b) The radial reaction of the pulley against the belt = qds.

(c) The frictional force = μqds.

(d) The tensions t and $t + dt$.

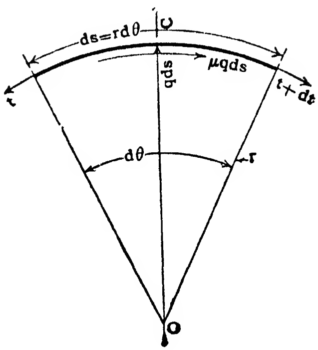

FIG. 165.

Resolving all forces vertically,

$$qds + cds = t \sin \frac{d\theta}{2} + (t + dt) \sin \frac{d\theta}{2} \tag{1}$$

Here $d\theta$ is so small that $\sin d\theta/2$ may be taken as equal to $d\theta/2$ in radians, without appreciable error, and the product of dt and $\sin d\theta/2$ may be neglected.

Hence (1) may be written

$$qds + cds = td\theta \tag{2}$$

but

$$c = \frac{12wv^2}{gr} \quad \text{and} \quad ds = rd\theta$$

$$\therefore \; cds = \frac{12wv^2}{gr} ds = \frac{12wv^2}{g} d\theta = zd\theta, \text{ for convenience}$$

Hence from (2),

$$qds = td\theta - zd\theta = (t - z)d\theta \tag{3}$$

From equality of moments around O,

$$t + dt = t + \mu q ds$$

$$\therefore \quad dt = \mu q ds \tag{4}$$

Substituting in (4) the value of qds obtained from (3),

$$dt = \mu(t - z)d\theta$$

$$\therefore \quad \int_{t_2}^{t_1} \frac{dt}{t - z} = \mu \int_0^\theta d\theta$$

or

$$\log_e \frac{t_1 - z}{t_2 - z} = \mu\theta \tag{5}$$

and

$$\text{common log} \frac{t_1 - z}{t_2 - z} = 0.434\mu\theta$$

$$\therefore \quad \frac{t_1 - z}{t_2 - z} = 10^{0.434\mu\theta} = 10^{0.0076\mu\alpha} = 10^k, \text{ for convenience} \tag{6}$$

If the effect of centrifugal action is neglected (Article 184) z becomes zero, whence equation (6) reduces to

$$\frac{t_1}{t_2} = 10^{0.0076\mu\alpha} = 10^k \tag{6'}$$

Now

$$f = t_1 - t_2. \quad \therefore \ t_2 = t_1 - f$$

and, substituting this value of t_2 in (6) and reducing,

$$t_1 = \frac{[f + z]10^k - z}{10^k - 1} = \frac{f}{C} + z \tag{7}$$

where

$$C = \frac{10^k - 1}{10^k}$$

and

$$f = [t_1 - z]\left[\frac{10^k - 1}{10^k}\right] = [t_1 - z]C \tag{8}$$

If (8) be multiplied through by $v/550$ it will express the horsepower (Hp) which a belt of 1 sq in. cross-sectional area will transmit, or

$$\frac{fv}{550} = \text{Hp} = [t_1 - z]\frac{Cv}{550} \tag{9}$$

If, as before, the effect of centrifugal action is neglected (Article 184) equation (9) reduces to

$$\mathrm{Hp} = t_1 \frac{Cv}{550} \tag{9'}$$

182. Practical Coefficients. In the above equations, the quantities α, μ, and z must be known or assumed before a solution for t_1 or f can be made. The **angle of contact,** α, can be taken from the drawing of the drive in question, and some allowance should be made for the conditions of operation. Thus, if the belt is to run in a horizontal position, with the slack side on top, the full theoretical value of α may be taken. If, however, the slack side must be on the bottom (an arrangement which should be avoided if possible) or if the belt is to be run in a vertical position, some reduction must often be made in the theoretical value of α to allow for sagging of the belt. This also applies to belts running at high speed, where centrifugal force tends to lessen the arc of contact.

The **coefficient of friction,** μ, is an exceedingly variable quantity, changing with the character and the condition of the surfaces of contact, the initial tension of the belt, the speed of the belt, and the rate of slip. It has been found by experiment that, within reasonable limits, the coefficient increases with the slip and that, as before stated, a maximum rate of slip, including creep, not in excess of about 3 per cent is good practice. Rational equations expressing the value of μ in terms of these variables have not been developed, but the following equation, proposed by Mr. Carl Barth and based upon his own experience and that of Professor Bird, has been used with considerable success for leather belts on iron pulleys:

$$\mu = 0.54 - \frac{140}{500 + V} \tag{10}$$

where V is the velocity of the belt in feet per minute. Values of μ as computed by equation (10) are tabulated in Table XXXIV.

Experiments made by Professor Diederichs in the laboratories of Sibley College gave average values of μ shown in the following table:

For pulleys made of pulp....................	$\mu = 0.29$
For pulleys made of wood....................	$\mu = 0.31$
For pulleys made of cast iron................	$\mu = 0.46$

Values considerably above these were found for paper pulleys of special construction. These values may serve as guides in modifying values of μ determined by equation (10) when pulleys made of wood or pulp are to be used.

The quantity z is proportional to the **weight of the belt per cubic inch.** For ordinary leather (which is most commonly used), w may be taken from 0.03 to 0.04, an average value being 0.035 lb.

Table XXXV has been calculated with a value of $w = 0.035$; Table XXXVI is abbreviated from "Transmission of Power by Belting," * by Wilfred Lewis.

TABLE XXXIV

Coefficients of Belt Friction by Barth's Formula

Velocity of Belt, Feet per Minute	Coefficient of Friction	Velocity of Belt, Feet per Minute	Coefficient of Friction	Velocity of Belt, Feet per Minute	Coefficient of Friction
0	0 260	700	0 423	2000	0.484
50	0 285	800	0 432	2500	0 493
100	0 307	900	0 440	3000	0.500
200	0 340	1000	0 446	3500	0 505
300	0 365	1200	0 458	4000	0 509
400	0 384	1400	0 466	4500	0.512
500	0 400	1600	0 473	5000	0.514
600	0 413	1800	0 479	5500	0 517

TABLE XXXV

Values of $z = \frac{12wv^2}{g}$, for v = feet per second, or

V = feet per minute, $w = 0.035$

v	30	40	50	60	70	80	90	100	110	120	130	140
V	1800	2400	3000	3600	4200	4800	5400	6000	6600	7200	7800	8400
z	11 75	20.9	32 5	47.0	64.2	83 4	105.5	130.5	157 6	187.6	220 2	255 5

Example. Design a belt to operate a dynamo of 15-hp capacity, when the belt velocity is 2400 ft per min. Assume $\alpha = 180°$ and $t_1 = 200$ lb/sq in. From Table XXXIV, $\mu = 0.49$.

* *Trans.* A.S.M.E., vol. 7, page 579.

From equation (9) the horsepower transmitted by a belt having a cross-sectional area of 1 sq in. is, for these conditions:

$$\text{Hp} = [t_1 - z]\frac{Cv}{550} = [200 - 20.9]\frac{0\ 783 \times 40}{550} = 10.2$$

$$\therefore \text{ the cross section required } = \frac{15}{10.2} = 1.47 \text{ sq in.}$$

which is equivalent to a belt $\frac{7}{32}$ in. thick and 6.75 in. wide.

The total tension (T_1) in the tight side of the belt will be 1 47 × 200 = 294 lb. The total tension (T_2) in the slack side will be this value minus the required effective pull, P, which is found by dividing the foot-pounds of work to be done by the velocity of the belt, or

$$P = \frac{15 \times 33{,}000}{2400} = 206$$

Hence

$$T_2 = T_1 - P = 294 - 206 = 88 \text{ lb}$$

TABLE XXXVI

Values of $C = \frac{10^k - 1}{10^k}$ (Nagle)

μ	Degrees of Contact = α									
	90	100	110	120	130	140	150	160	170	180
0 15	0 210	0 230	0 250	0 270	0 288	0 307	0 325	0 342	0 359	0 376
0 20	0 270	0 295	0 319	0 342	0 364	0 386	0 408	0 428	0 448	0 467
0 25	0 325	0 354	0 381	0 407	0 432	0 457	0 480	0 503	0 524	0 544
0 30	0 376	0 408	0 438	0 467	0 494	0 520	0 544	0 567	0 590	0 610
0 35	0 423	0 457	0 489	0 520	0 548	0 575	0 600	0 624	0 646	0 667
0 40	0 467	0 502	0 536	0 567	0 597	0 624	0 649	0 673	0 695	0 715
0 45	0 507	0 544	0 579	0 610	0 640	0 667	0 692	0 715	0 737	0 757
0 55	0 578	0 617	0 652	0 684	0 713	0 739	0 763	0 785	0 805	0 822

Equations (7) and (8) involve the relations that exist between T_1 and T_2 for a given set of conditions, but they do not indicate the relation between them and the initial tension T_3. It was formerly supposed that the sum of T_1 and T_2 was constant and equal to $2T_3$; and this relation may still be used for very rough calculations. Mr. Wilfred Lewis * has shown, experimentally, that this is not true. The ratio of stress to strain in leather and rubber *increases* with the strain instead of being *proportional* to it as in ductile metals. When a belt transmits power the tension is *increased* on the tight side and *decreased* on the slack side till the difference in tension is equal to the required driving force. This is accomplished by what virtually amounts to shortening the belt on the tight side, a given amount, by transferring this amount to the slack side. Because, however, of the relation between stress and strain noted above, the increase of tension on the tight side, due to this amount of shortening, is greater than the decrease of tension on the slack side due to an equal amount of lengthening, and, as a consequence, the sum of the two tensions is increased † as the effective pull is increased. Suggestion: Place a rubber band over the fingers of the two hands and stretch it moderately; then twist one of the hands in either direction and the increase of force tending to bring the hands together will be apparent. As the result of an extended study of these relations, Mr. Barth concludes that, under any variation of the effective pull of a belt, the sum of the square roots of the tensions remains constant and equal to twice the square root of the initial tension, or

$$\sqrt{T_1} + \sqrt{T_2} = 2\sqrt{T_0} \tag{11}$$

where T_0 is the initial tension when the belt is at rest. In the best modern practice the stress in belts is carefully measured, when they are fitted to the pulleys, on weighing machines specially devised for this purpose. Equation (11) gives the value of the initial tension when T_1 and T_2 are known.

In a long horizontal belt, the increase in the sum of the tensions is still further augmented in driving, because the tension on the slack side (with a proper initial tension in the belt) is largely due to the sag of the belt from its own weight; and thus the tension on the slack side tends to remain nearly constant, while the tension on the tight side increases with the power transmitted, at a given speed. It is found that the sum of the tensions on the two sides, when driving, may exceed the sum of the initial tensions by about 33 per cent in vertical belts, and

* *Trans.* A.S.M.E., vol. 7, page 566.
† *Trans.* A.S.M.E., vol. 7, page 569.

in horizontal belts the increase may be limited only by the strength of the belt. In addition to the causes discussed, the tensions on both parts of the belt are increased by the centrifugal action due to the mass of that portion of the belt which is rotating round the pulley axis. This latter cause increases the stresses on both the tight and slack sides of the belt, and decreases adhesion between the belt and the pulley, but does not increase the loads on the shafts which produce pressure at the bearings and flexure of the shafts.

Large belts should therefore be put on with care, as to initial tension. Ordinarily, the initial tension is left to trained judgment, but it would seem that the more advanced practice of splicing the belt under a known initial tension will add to the life of large and important belts.

183. Strength of Belting. The ultimate strength of good oak-tanned leather belting will vary from 3500 to 6000 lb per sq in. Chrome-tanned leather belting has a tensile strength varying from 7000 to 12,000 lb per sq in., or about twice that of oak-tanned leather. The ultimate strength of belting seldom enters as a factor in belt design, as the real strength of the belt is in the joint. If the ends of the belt are laced together, a maximum working stress of 200 to 300 lb per sq in. is found to be good practice; and if the belt is cemented together, and thus made "endless," a working stress of 400 lb per sq in. may be used. Leather belting is classified as **single-ply, double-ply,** and **triple-ply.** Each of these classes is again divided into **light, medium,** and **heavy,** according to thickness, by sixty-fourths of an inch. The allowable stress on the tight side of the belt, per inch of width, will depend upon the thickness. The subjoined table gives average tension based upon the allowable stresses given in the foregoing:

Grade	Average Thickness		Allowable Tension per Inch of Width			
	Single-ply	Double-ply	Single-ply Laced	Double-ply Laced	Single-ply Cemented	Double-ply Cemented
Light......	$\frac{1}{8}$ to $\frac{5}{32}$	$\frac{15}{64}$ to $\frac{17}{64}$	30	60	50	90
Medium....	$\frac{5}{32}$ to $\frac{3}{16}$	$\frac{19}{64}$ to $\frac{21}{64}$	40	75	70	120
Heavy.....	$\frac{3}{16}$ to $\frac{7}{32}$	$\frac{23}{64}$ to $\frac{25}{64}$	60	100	90	150

Lower tension than these are often advocated, and undoubtedly lower tension increase the life of the belt.

Professor Benjamin * gives the strength of cotton belting as about

* See "Machine Design," by Benjamin, page 186.

the same as that of good leather He also found that four-ply rubber belting had a tensile strength of 840 to 930 lb per in of width Professor Leutwiler quotes the following values from a catalogue of the Diamond Rubber Company as the allowable net tensions, or effective driving force, for rubber belts

For a three-ply belt	40 lb per in of width
For a four- or five-ply belt	50 lb per in of width
For a six-ply belt	60 lb per in of width
For a seven-ply belt	70 lb per in of width
For an eight-ply belt	80 lb per in of width
For a ten-ply belt	120 lb per in of width

184. Velocity of Belting. In equation (8), when $z = t_1$, $f = 0$ and the belt will exert no turning force, the centrifugal force relieving all frictional resistance between the belt and pulley

If t_1 be taken as high as 400 lb, and $w = 0\ 035$, this will occur when $z = 400$ or when $12wv^2/g = 400$, whence $v = 175$ ft per sec or 10,500 ft per min

If equation (8) be multiplied through by v, the velocity of the belt, it will express the rate at which energy is being delivered, or

$$fv = v[t_1 - z]C = v\left[t_1 - \frac{12wv^2}{g}\right]C$$

If now $\mu = 0\ 3$, $w = 0\ 035$, $\alpha = 180$, which are average conditions, the equation becomes

$$fv = v[t_1 - 0\ 013v^2] \times 0\ 6 = 0\ 6t_1v - 0\ 0078v^3$$

Differentiating the right-hand side with respect to v and equating to zero,

$$0\ 6t_1 - 0\ 0234v^2 = 0 \quad \text{or} \quad v = 5\ 1\sqrt{t_1} \qquad (12)$$

which gives the relation between v and t_1 for maximum power When $t_1 = 400$, $v = 102$ ft per sec or 6120 ft per min, and when $t_1 = 275$ lb, $v = 85$ ft per sec or 5100 ft per min It is often necessary to run belts at much lower speeds than these, but it is not economical to exceed these limits A speed of a mile per minute may be taken as about the economical maximum limit, and it so happens that this is also about the limit of safety for ordinary cast-iron pulley rims For durability combined with efficiency, a speed of 3000 to 4000 ft per min may be taken as a fair value, though practical limitations, such as speed for shafting and diameter of pulleys, often fix belt velocities at much lower values The effect of centrifugal action for velocities up to 2000 ft per min is usually neglected in applying equations (6) to (9) of Article 181.

185. Efficiency of Belting. The losses of power in belt transmission consist of the loss due to slip and creep, that due to bending the belt over the pulley, and the frictional losses at the shaft bearings, due to belt pull. The first two, slip and creep, should not exceed 3 per cent, and 2 per cent is better. The loss due to bending the belt is usually negligible, although the effect on the life of thick belting running on small pulleys is important. The losses at the bearings may be considerable if the belt must be laced on under great initial tension in order to carry the load, and this condition should be avoided except where it is absolutely necessary to use a short belt. A well-designed belt transmission should have an efficiency at least as high as 95 per cent, and it may be as high as 97 per cent, including bearing losses.

186. Other Equations, Common Rules. If in equation (9), w be taken as 0.032 and t_1 as 305 lb, the equation reduces to

$$\text{hp} = [0.55 - 0.0000216v^2]vC \tag{13}$$

where $C = (10^k - 1)/10^k$ as before and hp = horsepower per square inch of belt area. If the equation be multiplied by A, the area of the belt cross-section, it will express the total horsepower transmitted, or

$$\text{Hp} = [0.55 - 0.0000216v^2]vCA \tag{14}$$

Professor Diederichs has pointed out that equation (14) is identical with that reported by Mr. Nagle to the A.S M.E.* and commonly known by his name. Values of C have already been given in Table XXXVI.

In the *Transactions* A S.M.E., January, 1909, Mr. Carl Barth presents a more extended mathematical treatment of the driving capacity of belts. He also presents scientific methods for measuring the tension in belting. Many other formulas of a strictly empirical character are given by different authorities, and some of them are very convenient. In general these last formulas neglect centrifugal action and are hence applicable only to belt speeds below 2500 ft per min. Thus, a common rule is that a single leather belt 1 in. wide traveling 1000 ft per min will transmit 1 hp. Kent's "Mechanical Engineers' Pocket Book," gives a number of these so-called practical rules.

187. Practical Considerations. One of the most valuable contributions to the literature of the subject is " Notes on Belting " by Mr. F. W. Taylor, in vol. 15 of the *Transactions* A.S.M.E. Mr. Taylor kept an accurate record of measurements and observations on belts in use at the Midvale Steel Company's works, for nine years, and gives many valuable

* Vol. 2, page 91.

facts and practical suggestions in his paper. A satisfactory abstract of it is not possible here. Mr. Taylor advocates thick narrow belts rather than thin wide belts.* He sums up his investigation in thirty-six "Conclusions," among which are:

"A double leather belt having an arc of 180° will give an effective pull on the face of the pulley per inch of width of belt of 35 lb for oak-tanned and fulled leather, or 30 lb for other types of leather belts and 6- to 7-ply rubber belts.

"The number of lineal feet of double belting, 1 in. wide, passing around a pulley per minute, required to transmit one horsepower is 950 ft for oak-tanned and fulled leather belt, and 1100 ft for other types of leather belts, and 6- to 7-ply rubber belts.

"The most economical average total load for double belting is 65 to 73 lb per in. of width, i.e., 200 to 225 lb per sq in. of section. This corresponds to an effective pulling power of 30 lb per in. of width.

"The speed at which belting runs has comparatively little effect on its life, till it passes 2500 or 3000 ft per min.

"The belt speed for maximum economy should be from 4000 to 4500 ft per min."

It should be especially noted that Mr. Taylor advocates a maximum belt tension about one-half that ordinarily used. This would, of course, increase the first cost of the installation materially. His values, however, are not based on the minimum size of belt required to simply transmit a given horsepower, but on the size of belt which will transmit that horsepower for a given time with minimum wear and loss of time due to breakage or taking up to restore tension. Whether his practice is followed or not, it indicates the true aspect of the problem, and is a step in advance.

In laying out belt drives, care should be taken to keep the diameters of pulleys reasonably large. The constant bending action to which the belt is subjected as it runs around the pulley is a great source of wear, and if the pulley is very small, compared to the thickness of the belt, this may be excessive. For this reason also, it is probably better to run the hair side of the belt next to the face of the pulley, as this side is more easily cracked by bending than the flesh side, which is softer and more pliable. Mr. Taylor says it is safe to run double leather belts on pulleys 12 in. in diameter.

The total length of the belt, or distance between shaft centers, also deserves attention. A belt, being elastic, acts like a spring when tension

* Although in general this conclusion is justifiable, care should be taken that it is not carried to the extreme where the life of the belt may be shortened by excessive bending.

is applied to it. The longer the belt the greater will be the total stretch for a given load. Suddenly applied loads, therefore, produce less stress in long belts than in short ones (see Article 29). If, however, the distance between centers is too great, compared to the size of the belt, the belt is liable to flap and run unevenly on the pulleys. For small, narrow belts a maximum distance of 15 ft is good practice; for heavier belts 25 ft is found satisfactory. The minimum distance between shafts is sometimes given as 3.5 times the largest pulley. The Rockwood Manufacturing Company has devised an ingenious drive for short-center motor drives in which the belt tension is applied by the weight of the motor. When the tension is correctly adjusted it remains so unless excessive stretching of the belt occurs. Tightener pulleys should be avoided if possible.

A number of important investigations of belt transmission have been reported to the A.S.M.E. See the following papers in the transactions of the Society by: Mr. A. F. Nagle, vol. 2, page 91; Professor G. Lanza, vol. 7, page 347; Mr. Wilfred Lewis, vol. 7, page 549; Mr. F. W. Taylor, vol. 15, page 204; Professor W. S. Aldrich, vol. 20, page 136. Abstracts of these, as well as other valuable data, are given in Kent's "Mechanical Engineers' Pocket Book."

188. Steel Belting.* Steel belting originated in Germany and has been used in that country and in England, to some extent, with considerable success. So far, steel belts have not been used in America. As used abroad these belts are made of tempered-steel bands varying in thickness from about 0.0078 in. to about 0.035 in. and in width from $\frac{1}{2}$ in. to 8 in. The tensile strength of the steel used is reported to be about 200,000 lb per sq in. To form the joint a steel plate of special design is brazed to each end of the band. These plates lock together and are held in contact by removable screws.

The principal advantages claimed for steel belts are as follows: They do not stretch while in service and they are not affected by ordinary changes in temperature. They are lighter than leather belts and can be operated at velocities as high as 19,000 ft per min. Hence, for a given amount of power transmitted, they are narrower than leather belts, and consequently they, and the pulleys over which they run, occupy less space. The slip is said to be very small and the efficiency consequently high. They are more sensitive, however, than leather belts, and the pulleys over which they run must be kept in alignment. These pulleys, also, for best results, must be covered with leather, canvas, or some similar material.

* For more extended discussions of steel belting, see Halsey's "Handbook for Designers," and also O. A. Leutwiler's "Machine Design," Chapter VII.

FIBROUS ROPE DRIVES

189. General Considerations. If the amount of power to be transmitted is large, the width of belt required may be excessive, even when the belt is made very thick. To run wide belts successfully, the shafting must be kept in perfect parallel alignment, and the distance between shaft centers must not be too great. For these reasons rope drives have been found very satisfactory where the amount of power to be transmitted is large, and the distance of transmission relatively great. They are also particularly serviceable for connecting shafts that are not parallel, as in the case of " quarter-turn " drives, especially where a belt would have to be of considerable width and would, as a consequence, run badly.

In all fibrous rope drives the surfaces of the pulleys, or " sheaves," are provided with wedge-shaped grooves to receive the rope and thereby give the rope a better grip on the sheave. For drives of moderate length, 40 to 150 ft., fibrous ropes of cotton, hemp or manila fiber are chiefly employed. For transmitting power comparatively great distances, wire rope is more common, although fibrous ropes are also used for comparatively long transmissions. In all long-distance transmission the rope must be supported at intervals by idler pulleys.

Figure 166 * shows a typical rope drive, where the line shafting of each floor of a mill is driven by its own rope drive from the main shaft of the engine.

190. Materials for Fibrous Ropes. Round ropes of **leather,** or **rawhide,** are used to a limited extent, when the amount of power to be transmitted is small. Rawhide is especially useful in damp places, but since it costs about six times as much as vegetable fiber rope, its application is very limited. Leather belts or ropes of square † or wedge-shaped section have also been used to a limited extent. In certain localities in Great Britain, **hemp,** which is a local product, is used quite extensively; but **cotton** and **manila** fiber are by far the most common for transmissions of any considerable size. In this country manila fiber is used almost exclusively; in England and on the Continent cotton rope is also much employed.

It is obvious that, as a twisted rope of any fibrous material bends while passing over the sheave, there must be a certain amount of internal friction. The result of this action is very noticeable in any old manila rope which has been used without lubrication. When such a rope is

* Reproduced by permission from "The Blue Book of Rope Transmission," by American Manufacturing Company.

† For a fuller discussion of such ropes see "Machine Design," by H. J. Spooner.

broken open it is found to be filled with powdered fiber, the result of the internal chafing. For this reason manila fiber, which is naturally rough, is usually lubricated, while being twisted into rope, with tallow, paraffin, soapstone, graphite, or some such lubricant.

Cotton fibers, on the other hand, are smoother and hence give rise to less internal friction. They are, therefore, usually laid up dry into rope, a dressing or lubricant being applied to the exterior to prevent small fibers from rising on the outside, thus starting the rope to fraying. This dressing also excludes moisture and retains the natural oils in the interior fibers. Cotton rope is not as strong as manila.

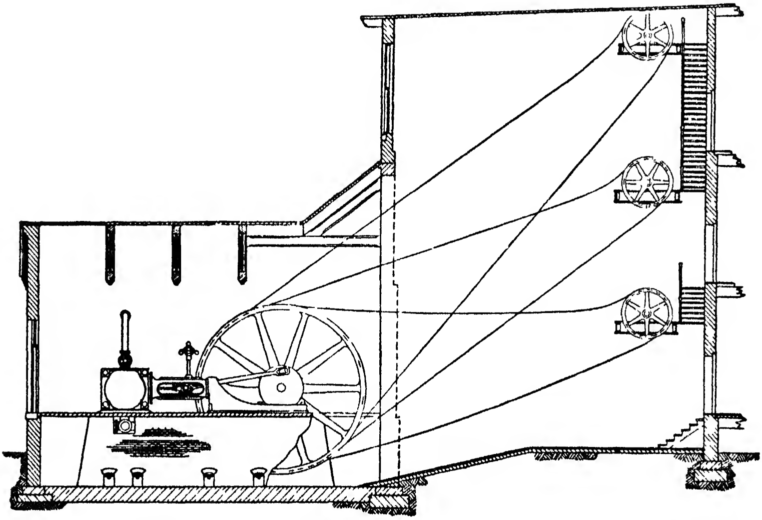

FIG. 166.

Professor Flather * makes the following comparison between cotton and manila rope: " As compared with manila, then, the advantages of cotton ropes of the same diameter are: Greater flexibility, greater elasticity, less internal wear and loss of power due to bending of the fibers, and the use of smaller pulleys for a given diameter of rope. Its disadvantages are: Greater first cost, lesser strength, and possibly a greater loss of power due to pulling the ungreased rope out of the groove—in any case this is usually small with speeds over 2000 ft per min."

Manila ropes, as used in this country for transmitting power, are made specially for that purpose. For ropes less than $\frac{7}{8}$ in. in diameter, the rope is usually made with three strands, larger ropes having four or

* "Rope Driving," by J. J. Flather, page 81.

six strands with a central core. Extra long manila fiber is used, and the inner core and inner strands are treated during manufacture with a lubricant of fish oil and graphite. Tallow, which was formerly used for this purpose, has been found to be unsatisfactory, partly because it usually contains acid. The outer yarns of each strand are laid so as to form fairly smooth protective covering for inner yarns that compose the body of the strand. In designing a rope transmission it is advisable to consult manufacturers who furnish such apparatus.

191. Theoretical Considerations. The general equations (7), (8) and (9), of Article 181, which were deduced for flat belts, hold also for round ropes if the proper notation be substituted. In these equations the unit mass of belt was taken as 1 cu in. With ropes it is more convenient to take a piece of rope 1 in. in length and 1 in. in diameter. With the following exceptions, therefore, the notation used here will be the same as that used in Article 181.

Let w' = the weight of a piece of rope 1 in. in diameter and 1 in. long.
z' = $12w'v^2/g$, where w' has the value above.
t'_1 = the tension in a rope of 1 in. diameter on the tight side.
C' = a new coefficient = C modified on account of wedging effect of groove.

Then equations (8) and (9) become

$$f = [t'_1 - z']C' \tag{15}$$

and

$$\text{Hp} = [t'_1 - z']\frac{C'v}{550} \tag{16}$$

In equations (8) and (9) the frictional force between the pulley and the belt for a flat belt is taken as μq, where q is the radial pressure between the pulley and the belt. In a grooved pulley the pressure between the pulley and the rope is greater than the radial pressure in the ratio of cosec $\theta/2$ to unity, where θ is the angle between the sides of the groove. The frictional resistance between the rope and sheave is therefore μq cosec $\theta/2$. If μ cosec $\theta/2$ be substituted for μ in the quantity C (equations 8 and 9) the result C' may be used as indicated in equations (15) and (16) for rope drives. The value of μ for rope sheaves has not been determined with any degree of accuracy. Professor Flather * after reviewing what experimental data there are on the subject, concludes that 0.12 is a fair value and computes the following values of $\phi = \mu \operatorname{cosec} \theta/2 = 0.12 \operatorname{cosec} \theta/2$.

* "Rope Driving," page 112.

TABLE XXXVII

ϕ = coefficient of friction = 0.12 cosec $\frac{\theta}{2}$

Angle of groove	30°	35°	40°	45°	50°	55°	60°
ϕ	0 46	0 40	0 35	0 31	0 28	0 26	0 24

It is obvious that if ϕ be used instead of μ in Table XXXVI, the corresponding values of C in Table XXXVI will be the new constant C'. Thus if $\theta = 45°$, $\phi = 0.31$. If also $a = 180°$, C' from Table XXXVI = 0.61 about. The angle 45° has been found to be the most satisfactory and is most commonly used.* If the angle θ be less than 45°, the wedging action, and hence the pulling capacity, is increased, but the power loss and wear of rope due to drawing it out of the grooves is greater. For such sheaves, with $\theta = 45°$ and $\alpha = 180°$.

$$\text{Hp} = 0.61[t'_1 - z'] \frac{v}{550} \tag{17}$$

As before stated, reliable data on the coefficient of friction for ropes are scarce, and designing engineers have approached the problem of rope drives without regard to this coefficient. One of the most important contributions to the subject is that of Mr. C. W. Hunt (see *Transactions* A.S.M.E., vol. 12). The notation of Mr. Hunt's article has been changed somewhat to correspond with that used in this text.

Let d diameter of the rope in inches.

δ = sag of rope in inches.

L = distance between pulleys in feet.

w' = weight of 1 in. of rope *of 1-in. diameter.*

W = weight of 1 ft of rope of diameter d.

T_1 = total tension in rope on *tight* side.

T_2 = total tension in rope on *slack* side.

T_0 = tension necessary to give the rope adhesion.

K = the total tension applied to each side of the rope due to centrifugal force.

P = effective turning force = $T_1 - T_2$.

Then

$$T_1 = T_0 + K + P$$

and

$$T_2 = T_0 + K$$

* The Dodge Manufacturing Company, which is well known in this line of work, prefers an angle of 60°.

Mr. Hunt says that "when a rope runs in a groove whose sides are inclined toward each other at an angle of 45° there is sufficient adhesion when $T_1 \div T_2 = 2$.' However, he assumes a somewhat different ratio in the development of his equation, for which he assumes "that the tension on the slack side necessary for giving adhesion is equal to one-half the force doing useful work on the driving side of the rope." Or,

$$T_0 = \frac{P}{2} \quad \text{and} \quad T_1 = T_0 + K + P = \frac{P}{2} + K + P = \frac{3}{2}P + K$$

and

$$T_2 = T_0 + K = \frac{P}{2} + K \text{ by assumption}$$

$$\therefore\ P = \tfrac{2}{3}[T_1 - K] \tag{18}$$

If equation (18) be multiplied through by $v/550$ it will express the total horsepower transmitted, or

$$\text{Hp} = \tfrac{2}{3}[T_1 - K]\frac{v}{550} \tag{19}$$

The tension K on each side of the rope for an arc of contact of 180° and a rope of *one-inch diameter* is $12w'v^2/g$, which is identical with the constant z' in equation (16). Mr. Hunt's formula therefore may be written

$$\text{Hp} = \tfrac{2}{3}[t'_1 - z']\frac{v}{550} = \frac{v}{825}[t'_1 - z'] \tag{20}$$

where Hp is the horsepower transmitted by a rope 1 in. in diameter. This is identical in form with the theoretical equation (17) and differs from it only by a negligible amount in the value of the coefficient.

It would seem therefore that Mr. Hunt's assumptions give results very close to those obtained by using the value 0.12 for μ, as recommended by Professor Flather.

It is to be noted that the values of z given in Table XXXV may be used in computing values of z'. The quantities are the same except for the weight w'. In Table XXXV, w = the weight of 1 *cu in. of leather* = 0.035. In equation (20), w' = the *weight of* 1 *in. of rope of* 1 *in. diameter* = 0.028 for manila rope and 0.022 for cotton rope. If, therefore, the values given in Table XXXV are multiplied by $\frac{4}{5}$ they are applicable to manila ropes, and if multiplied by $\frac{3}{5}$ they may be used for cotton ropes.

Example. What diameter of manila rope is necessary to transmit 25 hp, when running 4000 ft per min, in grooves having an angle of 45°. Take t'_1 = 200 lb, and w' = 0.028. From Table XXXV, z, for the given velocity = 64, nearly.

$$\therefore\ z' = 64 \times \tfrac{4}{5} = 51$$

From equation (20) the horsepower which a rope 1 in. in diameter will deliver under these conditions is

$$\text{Hp} = [t'_1 - z']\,\frac{v}{825} = [200 - 51]\,\frac{66\frac{2}{3}}{825} = 12.1 \text{ hp}$$

$\therefore$ the cross-section required $= 25 \div 12.1$ or about twice the area of a 1-in. rope which corresponds to a rope $1\frac{3}{8}$ in. in diameter.

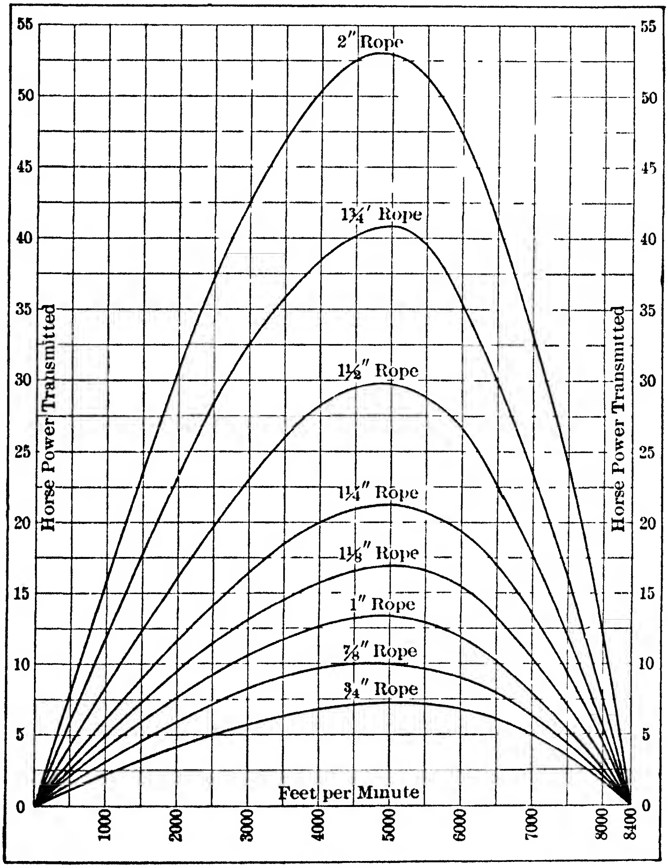

FIG. 167.

Figure 167 * shows curves based on equation (19), giving the *total* horsepower transmitted by ropes of various sizes for $T_1 = 200d^2$, and will be found convenient for making calculations.

* From "The Blue Book of Rope Transmission," by the American Manufacturing Company.

192. Strength of Fibrous Ropes. The ultimate strength of manila transmission ropes may be taken as about $7000d^2$ and for cotton rope as about $4600d^2$, where d = diameter of rope in inches. The working stress must be taken very much less than these values, or otherwise the life of the rope is much shortened. For manila rope Mr. Hunt recommends that the working tension (T_1) be not over $200d^2$. The same factor of safety would give $130d^2$ as the allowable working tension for cotton ropes; but since cotton ropes are somewhat less affected by internal chafing the working tension may, perhaps, be safely taken at a rather higher value.

193. Velocity of Fibrous Ropes. The centrifugal force produces a tension in a rope of 1-in. diameter of $z' = 12w'v^2/g$, or in a rope of diameter d the centrifugal force $= 12w'd^2v^2/g$. The allowable stress in the rope is $200d^2$. The centrifugal force will equal the allowable tensile stress when $12w'd^2v^2/g = 200d^2$, or when $v = 140$ ft per sec, at which speed the effective pull becomes zero for this allowable working stress.

If equation (20) be differentiated and the differential be equated to zero as in Article 184, the resultant equation will give the value of the velocity where the work done is a maximum, for a rope 1 in. in diameter. This is found to be about 4900 ft per min. Since the centrifugal force and the total working stress both vary as the area of the rope, this limiting velocity applies to all sizes of ropes, a conclusion which is borne out by the curves of Fig. 167.

It has been found, in practice, that the most economical speed for ropes is from 4000 to 5000 ft per min. If speeds greater than this are used, the wear on the rope is excessive. For a fixed value of $T_1 = 200d^2$ the first cost of a rope is a minimum at about 4900 ft as above, and this first cost is greater by 10 per cent if the velocity is increased to 6000 or decreased to 3700 ft per min. The first cost is increased 50 per cent when the velocity is reduced to 2400 ft per min with $T_1 = 200d^2$, but the reduction in speed increases the life of the rope.

194. Systems of Rope-driving. There are two methods of placing fibrous ropes on the sheaves. In the **multiple** or **English** system, several separate ropes run side by side, each rope forming a closed circuit in exactly the same manner as a flat belt, and running constantly in its own particular groove on each pulley. In the **continuous** or **American** system, one rope only is used, being carried continuously from one pulley to the other till all the grooves are filled, and it is then spliced; so that the rope, as it leaves the last groove of the driven sheave, is returned to the first groove of the driver, or driving pulley, by means of an idler, or guiding sheave. This idler is usually arranged so that through it a suitable tension may be put upon the rope (see Fig. 168).

Regarding the merits of the two systems it may be said that the multiple system is the simpler, and that it also provides considerable security against the loss of time due to breakdowns, as it is not likely that more than one rope will break at a time. When failure of a rope does occur, the broken rope may be removed and repaired at a more convenient opportunity, allowing the other ropes to carry the load temporarily. Occasionally, however, the breaking of a rope in a multiple system may cause great delay, on account of the broken rope becoming entangled in one of the rope sheaves and winding up upon it before the machinery can be stopped. In this system the individual ropes must be respliced occasionally to take up the sag in the rope due to stretching. The velocity ratio transmitted by a new rope will be different from that

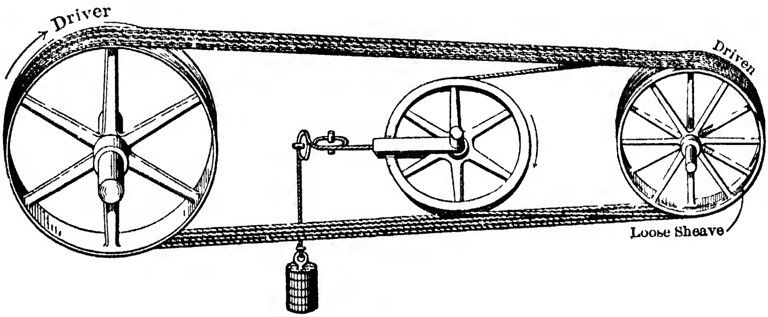

FIG. 168.

transmitted by an old one which has worn smaller, and hence fits down farther into the grooves, thereby changing its effective radius. The velocity ratio of the two sheaves can, however, have but one value, and, therefore, the tendency will be for either the old or the new ropes to carry the whole load. When the driving sheave is the larger, this will result in a tendency to throw more load on the old ropes; when the driving sheave is the smaller, the tendency is to throw more load on the new and larger ropes. The unequal speed of the ropes, of course, leads to unequal stress; and slipping and consequent wear are sure to occur.

The continuous system is more flexible in its application than the multiple system; for, owing to the limited sag in the ropes due to the action of the weighted idler, the rope may be run safely at any angle. This form of drive is, therefore, much used for vertical and quarter-turn drives, and, generally, where the transmission is of a complicated nature. The principal objections to the system are the danger of loss of time due to a breakdown, and the unequal straining of the various spans of the rope, particularly with a varying load or inequality of grooves. When a load is suddenly applied to the continuous system all the spans on the slack side become slacker, except that which runs over the idler and which is kept at a fixed tension. A much greater load is hence brought on the driving span of rope next to the idler, and some time must elapse before this load can be equalized over all the spans. Mr. Spencer Miller * has pointed out that the general tendency to unequal straining may be somewhat obviated, where the sheaves are of different diameters, by making the angle of the groove in the small sheave somewhat sharper than that in the larger, so that the product of the arc of

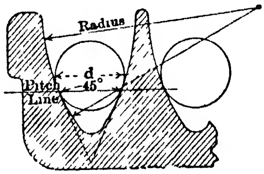

FIG. 169 (a).

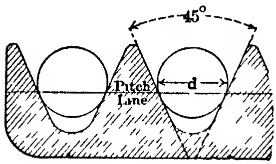

FIG. 169 (b).

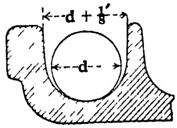

FIG. 169 (c).

contact and the cosecant of half the groove angle are equal; thus making the tendency to slip equal.

The above are the principal points of difference between the two systems. The particular conditions of the installation must be considered in making a choice between them.

195. Sheaves for Fibrous Ropes. The sheaves over which ropes are to run deserve special attention. Care should be taken that the form of the grooves and the effective diameters are the same for all grooves of the same sheave, and the surfaces should be accurately finished and well polished, as any roughness or unevenness seriously affects the life of the rope. As the result of much experimentation, two forms of grooves as shown in Fig. 169 (a) and 169 (b) have become most common. In Fig. 169 (b) the sides of the groove are straight, while in 169 (a) the sides are curved. This curving of the sides makes the angle of the groove somewhat flatter at the bottom, and hence when the rope has been reduced in diameter from wear it lies lower in the groove and will slip a little more readily than when it is new and occupies a higher position. This is of importance in relieving the old rope of a tendency to pull harder as indicated in the preceding article. The curved outline is also said to assist the rope to roll in the groove, a very desirable feature since it distributes the wear on the rope. The curved groove is therefore much used in the multiple system. In the continuous system the rope necessarily rotates as it passes round the idler to the first groove.

* *Trans.* A.S.M.E., vol. 12, page 243.

The angle of the groove, as before stated, is usually 45°. The grooves of idler pulleys for simply supporting the rope when the stretch is great are not made V-shaped but as shown in Fig. 169 (c).

The wear of fibrous ropes is both internal and external, the internal wear being due largely to chafing of the fibers on each other in bending the rope over the sheaves. For this reason sheaves should be as large as possible, and, in general, should not have a diameter less than forty diameters of the rope.

196. Deflection or Sag. If the span between the pulleys is considerable, the amount of deflection is sometimes of importance. Since the deflection varies with the distance between pulleys, the size and speed of the rope, and the difference in elevation of the pulleys, it is impossible to express the relation existing between them in a single formula. For the simple case of the horizontal drive the approximate deflection on the *driving side* may be determined both for the continuous and multiple systems and also the deflection of the *slack* side of the continuous system, where uniform tension is maintained by a tension weight. In the multiple system, however, ample allowance must be made on the slack side, as new ropes stretch very rapidly, and the deflection may become excessive before resplicing can be performed. Mr. Hunt gives the following equation (transformed) for computing the deflection in horizontal drives:

$$\Delta = \frac{T}{2W} - \sqrt{\frac{T^2}{4W^2} - \frac{L^2}{8}} \qquad (21)$$

where T is the total tension on either the slack or tight side, depending on the side for which it is desired to compute the deflection; W the weight of rope per foot; L the span in feet; and Δ the deflection in feet. Where the tension on the driving side is assumed to be equal to $200d^2$, regardless of speed, the deflection on the driving side will be constant for a given span. As the tension in the rope due to centrifugal action increases as the square of the velocity, there is an increasing total tension T_2 on the slack side for a fixed value of T_1; and hence the deflection on the slack side decreases with the velocity, the span remaining constant. The value of T_2 may be computed and substituted in equation (21) to find the deflection.

Mr. Frederick Green * gives the following approximate formula for computing the deflection:

$$\Delta = \frac{W \times L^2}{8T} \qquad (22)$$

* See "The Blue Book of Rope Transmission," by American Manufacturing Company.

where the symbols are the same as in equation (21), and from which he has calculated the following table on the assumption that $T_1 = 200d^2$:

TABLE XXXVIII

Distance between Pulleys, Feet	Sag on Driving Side, All Speeds, Feet	Sag on Slack Side in Feet				
		Velocity, Feet per Minute				
		3000	4000	4500	5000	5500
30	0 19	0 45	0 39	0 36	0 33	0 30
40	0 34	0 80	0 69	0 64	0 59	0 53
50	0 53	1 2	1 1	1 0	0 92	0 84
60	0 76	1 8	1 7	1 4	1 3	1 2
70	1 0	2 4	2 1	1 9	1 7	1 6
80	1 4	3 2	2 9	2 5	2 3	2 1
90	1 7	4 0	3 5	3 2	3 0	2 7
100	2 1	5 0	4 3	4 0	3 7	3 3
120	3 0	7 2	6 2	5 7	5 3	4.8
140	4 1	9 9	8 5	7 8	7 2	6 6
160	5 4	12 9	11 1	10 2	9 5	8 6

197. Efficiency of Manila Rope Transmission. Data on the efficiency of manila rope transmission are somewhat rare. The best available data are those collected by the Dodge Manufacturing Company and reported by Mr. E. H. Ahara in *Trans.* A.S.M.E., vol. 35. These results indicate that a good American rope drive of the general design shown in Fig. 168 developed an efficiency ranging from 95 per cent at 2500 ft per min to 90 per cent at about 5000 ft per min. Corresponding drives of the English or multiple type gave efficiencies about 5 per cent lower throughout the same range. Other drives of both kinds, but with more complicated means of returning the slack rope to the driver, gave efficiencies considerably lower. The reasons for an increase in efficiency with decrease in rope speed are not clear. The experiment showed that rope drives are efficient at light loads, the efficiency at half load being almost equal to that at full load. It should be noted that the sheave grooves in the testing apparatus were made with 60° angle for the American system and 45° angle for the English system, which is in accord with the practice of the Dodge Company. The cost of rope transmission compares favorably with that of belt driving.

198. Fibrous Ropes for Hoisting. In power transmission it is usually possible to install sheaves large enough to prevent the bending action from seriously affecting the life of the rope; but in hoisting work this is not always possible, on account of the size and clumsiness of the resulting tackle Thus, a manila rope of 1-in diameter, if used for power transmission, should run over a sheave at least 40 in in diameter, but if used for hoisting it might be required to run over a block sheave 12 in. or even 8 in in diameter The internal friction and external chafing are, in such cases, very great, and the life of the rope, even when working at a lower stress, is greatly shortened, but in hoisting tackle, the frequency with which any portion of the rope passes over the sheaves is much less than is ordinarily the case in power transmission, on account of lower speed.

Theoretical considerations are of little or no help in hoisting installations, and recourse must be had to successful practice on which, fortunately, there are considerable data The following table, from a paper presented by Mr C W Hunt, before the A S M E, gives the results of a long series of observations and indicates the most economical size of rope for a given load It has been found, by experience, that ropes larger or smaller than those recommended in the table are shorter-lived under the load indicated The speeds indicated in the table are defined as follows:

"Slow"—Derrick, crane, and quarry work, 50 to 100 ft per min.
"Medium"—Wharf and cargo work, 150 to 300 ft per min
"Rapid"—400 to 800 per min

TABLE XXXIX

WORKING LOAD FOR MANILA ROPE

A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Diameter of Rope,	Ultimate Strength,	Working Load in Pounds			Minimum Diameter of Sheaves in Inches		
Inches	Pounds	Rapid	Medium	Slow	Rapid	Medium	Slow
1	7,100	200	400	1000	40	12	8
$1\frac{1}{8}$	9,000	250	500	1250	45	13	9
$1\frac{1}{4}$	11,000	300	600	1500	50	14	10
$1\frac{3}{8}$	13,400	380	750	1900	55	15	11
$1\frac{1}{2}$	15,800	450	900	2200	60	16	12
$1\frac{5}{8}$	18,800	530	1100	2600	65	17	13
$1\frac{3}{4}$	21,800	620	1250	3000	70	18	14

199. Other Forms of Belts. Belts with trapezoidal cross-section erroneously known as V-belts have long been in use for special purposes. Such belts run in grooves with inclined sides like those used for rope drives, and the general theory as concerns their action is the same. A modern application is that for driving the fans on automobiles. A multiple V-belt transmission for short center drives has also been found useful. One of the best-known applications of this principle is the Reeves variable-speed transmission. The belts used in V-drives, so called, may be made of either rubber or leather. When made of leather the required thickness is obtained by lamination and riveting. Belts of both round and square section are also in use for special conditions. For a fuller description of such belting as used in England see "Machine Design Construction and Drawing" by H. J. Spooner. Since the application of these several forms of belting is usually of special character the publications of manufacturers should be consulted as to strength and driving power.

WIRE ROPE TRANSMISSION

200. General. Ropes made of iron or steel wire have been used to a considerable extent for transmitting power over comparatively great distances. The introduction of electrical transmission, however, has greatly curtailed the field as far as power transmission is concerned, although wire ropes are still much used for conveying materials such as coal, rock, etc., by means of buckets attached at intervals along the rope. The rope in such installations moves at very low velocities and constitutes a different problem from that of power transmission. Wire ropes are also much used for hoisting and haulage work, such as elevator and mine work and for carrying static loads, as in supporting smokestacks, masts, and suspension bridges. The problem of conveying materials by means of wire rope usually involves special apparatus beyond the scope of this book. Haulage is a special form of hoisting; hence this discussion will be confined to the problems of hoisting vertical loads and of power transmission by wire ropes.

201. Materials for Wire Ropes. Wire ropes are usually made of wrought iron or crucible steel. The American Steel and Wire Company gives the following values for the tensile strength of the several kinds of wire rope which it produces, the values referring to the strength of the individual wires composing the rope.

Wrought iron .	85,000 lb per sq in.
Crucible cast steel	170,000 to 200,000 " " "
Extra strong crucible cast steel	190,000 to 220,000 " " "
Plow steel	210,000 to 260,000 " " "
Monitor steel	230,000 to 280,000 " " "

Its publications also state that it is difficult to obtain from a sample of rope, in a testing machine, more than 95 per cent of the aggregate strength of all the wires, and that with some ropes this percentage may be as low as 80. This is due to the difficulty of getting a perfect grip on the rope so that all the wires will carry their full share of the load; and also because the inner wires of a strand are shorter than the outer wires and are therefore more quickly overloaded. The wires, on account of the twisted construction, also tend to cut into each other, thus rendering them more liable to fracture under heavy loads. On account of this latter action, ropes made with a short twist break at a lower percentage of their full strength than those of a longer twist. For these reasons and because of the great variety of ways in which wire ropes are made, it is always advisable to consult manufacturers' publications in selecting such ropes.

202. Wire Hoisting Ropes. Wire ropes for hoisting and haulage are made with a soft core of hemp, to increase their flexibility and to reduce the bending stresses incident to going over sheaves. They are made in a variety of forms, so that it is essential, as previously noted, to consult manufacturers' publications in selecting such rope. For rough service, deep mine work, or wherever great strength is necessary, the better grades of crucible steel, plow steel, and monitor steel are used. Wire ropes used for hoisting may be subjected to the following principal stresses:

(*a*) Static stress due to the weight of the load.
(*b*) Stress due to accelerating the load.
(*c*) Stress due to bending the rope over sheave or drum.
(*d*) Stresses due to shock from sudden loading.

The static stress can be determined if the load and the true cross-sectional area of the rope are known; but since these ropes vary widely it is advisable to take the carrying strength of wire ropes from manufacturers' publications. Carrying strengths, as quoted by reliable manufacturers, are based upon actual breaking tests.

203. Stresses Due to Accelerating the Load. If the load is lifted very slowly, only the static stress need be considered; but in nearly all hoisting problems the load must be accelerated at least for a short period after starting, and with some forms of hoisting drums the acceleration may vary throughout the entire distance. If the velocity curve at which hoisting is performed is known, the acceleration diagram can be deduced from it. (See Article 5.) If the acceleration is known, the accelerating force can be calculated from the well-known relation: Force = Mass $\times$ Acceleration.

204. Stresses in Wire Rope Due to Bending. The stress due to bending a rope over a sheave or drum may be great and should never be neglected in design. If a straight circular bar of diameter d is bent around a cylinder of diameter D, then it can be shown that

$$s = E\frac{d}{D}$$

where s is the greatest tensile stress in the bar and E is the coefficient of elasticity. (See also Article 9.) Obviously, the equation cannot be expected to give accurately the stresses induced in the wires of a rope due to bending the rope around a sheave, and it has been demonstrated experimentally that it does not do so. This equation is therefore usually written

$$s = CE\frac{d}{D} \tag{23}$$

for wire ropes, where C is a constant varying, according to different writers, from 0.35 to .05. The American Steel and Wire Company, as the result of an extended series of tests on full-sized ropes, recommend a value of $C \times E = 12{,}000{,}000$. The carrying strength of the more common kinds of hoisting cables listed in their publications, namely, 6×7 (six strands of seven wires each) 6×19, 6×37, and 8×19, are based upon the use of this value in equation (23). In this equation d is the diameter of the wire of which the rope is constructed, and it may be noted for convenience in making calculations that $d = \frac{1}{9}$ the outside diameter of the rope in 6×7 ropes and $\frac{1}{15}$ the outside diameter in 6×19 ropes. It will appear from the foregoing that the stress due to bending is inversely proportional to the diameter of the sheave and hence this diameter should be as large as possible. Prominent American manufacturers of wire rope recommend a minimum diameter of sheave of forty-eight times the diameter of the rope.

205. Stresses in Wire Rope Due to Sudden Loading. Suddenly applied loads are most likely to occur in hoisting ropes, because of the reeling up of slack in the rope just previous to lifting the load. Consider first a rope so short that the amount of stretching under normal loading is very small. If the slack of such a rope is reeled in slowly until the rope is taut and if, then, the power is applied suddenly, the stress in the rope will approximate that due to applying the load suddenly but without initial velocity. By Article 29 this stress will be twice that due to the load if applied slowly. If, however, the slack of any rope is reeled in so rapidly that the rope is applied with a jerk, the stress in the rope will depend upon the energy stored in the rope and the reeling machinery, the elasticity of the rope, and the mass of the load. If these quantities are known the stress in the rope can be calculated.

In actual design, however, it is seldom necessary to make such calculations. It should be noted that suddenly applied loads in hoisting ropes are caused almost always by the inertia of the reels and other moving parts, rather than by the application of hoisting torque. In very long hoisting ropes, the elasticity of the rope makes it difficult to apply a sudden load even when the reels and other moving parts of the hoisting machinery are very heavy. In short hoisting ropes, the inertia of the moving parts is small and, except for sheer carelessness, jerking effects can be avoided, though the load must often be started suddenly without initial velocity of the rope. These general conclusions are borne out by the factors of safety assigned in practice to such ropes. Thus, for mining hoists, overhead traveling cranes, and similar service, where human life is involved, a factor of safety of 5 has been found to be sufficient. In elevators and other applications where extreme safety is desired, a factor of safety of 10 or even 12 is often used.

Example. The ram of a pile driver weighs 2000 lb, and the hoisting machinery is capable of accelerating it at the rate of 8 ft per sec/sec. Select a 6 × 19 steel cable for lifting it, assuming that the sheave and drum are forty-eight times the diameter of the rope, the factor of safety as 5, the maximum tensile strength of the wire as 250,000 lb per sq in., and the effective area of the rope as 95 per cent of the total area. Assume also that the load may be applied suddenly but without initial velocity.

Since the factor of safety is 5, the maximum allowable working stress will be 50,000 lb per sq in.; and the sum of the stresses due to bending, lifting the load suddenly, and accelerating it must not exceed this amount. A tentative computation indicates that a rope $\frac{5}{8}$ in. in diameter will serve. The diameter of the wire in the rope will be $\frac{5}{8} \times \frac{1}{15} = \frac{1}{24}$ in. The total cross-sectional area of the rope will be 0.155. The *effective* area will be 0.155 × 0.95 = 0.147. The bending *stress* will be, from equation (23),

$$s = 12{,}000{,}000 \times \frac{1}{24} \times \frac{1}{30} = 16{,}666 \text{ lb per sq in.}$$

The force necessary to accelerate the ram = mass × acceleration $= \dfrac{2000 \times 8}{32.2} = 497$ lb. Hence,

$$\text{Total } \textit{load} \text{ due to ram} = (2 \times 2000) + 497 = 4497 \text{ lb}$$

$$\text{The } \textit{stress} \text{ due to this load} = \frac{4497}{0.147} = 30{,}600 \text{ lb}$$

Therefore,

$$\text{Total stress} = 16{,}666 + 30{,}600 = 47{,}266 \text{ lb per sq in.}$$

and the selection is satisfactory.

206. Power Transmission by Wire Rope. Wire ropes for power transmission are usually made of iron or soft steel and are laid up with a soft core of hemp in order to give greater flexibility. They cannot be run on metallic surfaces, and the sheaves must be lined at the bottom with soft rubber or similar yielding material. Great care must be taken that the rope does not chafe and, unlike the sheaves for fibrous ropes, the grooves in sheaves used for wire rope are so formed that the sides of the groove do not compress the ropes. In wire-ropes sheaves, the radius at the bottom of the groove is always greater than that of the rope itself, so that wire-ropes drive, like flat belts, simply through the friction on the bottom of the groove, due to the tension of the rope. The lining of the bottom of the groove (leather, wood, or some other comparatively soft material) gives increased friction as well as less wear of the rope. The sheaves should be as large as possible to minimize the bending effect on the rope, one hundred rope diameters being often taken as the minimum diameter of the sheave.

The general theory and equations developed for fibrous rope hold also for wire rope, proper constants being substituted. It is evident from this discussion that wire ropes can safely transmit a greater amount of power than fibrous ropes of the same diameter, because of the much higher allowable tensile stress. Refined calculations, however, are seldom necessary in determining the transmitting capacity of wire ropes, since dependence for such capacity is placed upon the tension due to the weight of the rope between sheaves rather than upon elastic tension such as is employed in leather belts. The John A. Roebling Sons Company recommends that transmission ropes be permitted to deflect or sag one thirty-sixth of the span between sheaves, and states that this amount of deflection will provide ample friction on the driving sheaves. If less than this amount of sag is permitted, the resulting tension is likely to cause undue stress in the rope and undue wear in the sheaves. If more than this amount is permitted, the rope may sway and jerk. It will be clear that under such a rule a rope of a given size will transmit more power when the span is long than when it is short, and with short spans it is sometimes necessary to use a heavier rope in order to secure the necessary amount of tractive force. Spans less than 70 ft are not very satisfactory, and single spans of 400 ft are not uncommon.

The Roebling Company also states that with this amount of sag the difference in tension between the tight side and the slack side of the rope may be taken as three times the weight of a single span of the rope for the deflection stated in the foregoing. This difference in tensions, multiplied by the speed of the rope, will give the foot-pounds that the rope will safely transmit. The velocity should not exceed 5000 ft per min.

The American Steel and Wire Co. manufactures **marlin-clad** transmission ropes. Each strand of these ropes is wound with marlin or tarred hemp cord. This covering protects the wire from wear and makes a hard wearing surface that has exceptionally good tractive qualities. The lining of the bottoms of the grooves in the sheaves should be maintained in good repair. If it becomes irregular, through wear, the rope may be bent at a sharp angle in passing over the high spots of the lining, with a resultant increase in the stress of the wires. This last action, however, is not equivalent, so far as the life of the rope is concerned, to running over a correspondingly smaller sheave, for every portion of each wire is bent around each sheave once during every circuit of the rope; and it is not likely that the same portion of the rope will frequently come in contact with any single irregularity in the lining.

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Treatise on Leather Belting, Haven and Swett.

CHAPTER XVI

CHAINS AND CHAIN TRANSMISSION

207. Classification. Chains may be conveniently divided into three classes:

(a) Chains for raising and supporting loads.

(b) Chains for conveying purposes.

(c) Chains for power-transmission purposes.

208. Chains for Hoists. In the first class are such chains as are used on cranes and hoisting appliances. Chains of this character are made with elliptical-shaped links and should be manufactured of the best **wrought iron** to insure perfect welding where the link is joined. The links themselves should be as small as possible, to minimize the collapsing action or bending due to the pull of the adjacent links, and also that due to winding the chain upon a circular drum. Such chains are sometimes called **short-link, close,** or **crane chains.** In so-called **stud-link** chain, a transverse bar, or **stud,** prevents the sides of the link from straightening under load.

The strength of a chain link in tension is less than twice that of a bar of the iron from which the chain is made, on account of the curvature and the bending action due to the manner in which the load is applied, and also on account of the weld. The following empirical equation, in which W = the breaking load in pounds and d = the diameter in inches of the bar from which the link is made, has been much used for iron crane chains.

$$W = 54{,}000d^2 \tag{1}$$

The working load (W') should not be more than one-third this value or

$$W' = 18{,}000d^2 \tag{2}$$

Professors Goodenough and Moore, as a result of experimental work,* conclude that the stresses allowed by equation (1) are too high and recommend the following:

* See *Bulletin* 18 of Engineering Experiment Station, Univ. of Illinois.

$$W = 0.4sd^2 \text{ for open links} \tag{3}$$

and

$$W = 0.5sd^2 \text{ for stud links} \tag{4}$$

where s is the allowable tensile stress in the link.

These writers also deduced the following conclusions from their experiments: " In the formulas for the safe design of chains given by the leading authorities on machine design, the maximum stresses to which the link is subjected seem to be underestimated, and the constants are such as to give maximum stresses of from 30,000 to 40,000 lb per sq in. for full load.

" The introduction of a stud in the link equalizes the stresses throughout the link, reduces the maximum tensile stresses about 20 per cent and reduces the excessive compressive stress at the end of the link about 50 per cent.

" The stud-link chain of equal dimensions will, within the elastic limit, bear from 20 to 25 per cent more load than the open link. The ultimate strength of stud-link chain is, however, probably less than that of the open link."

In many cases, no doubt, a lower stress than that indicated by (2) should be adopted. Whenever the load is not a direct pull, but severe bending stresses are also induced, as in chain "slings" for handling heavy iron castings, the chain should have great excess of strength. Chains should be carefully inspected and tested or " proved " before using. The " proof " usually applied is one-half the ultimate load. Where chains are used for hoisting work, they are likely to become badly strained. Annealing by heating allows a readjustment of the structure of the iron, and this should be done periodically with all such chains, particularly chains used for slings. This also affords an opportunity to thoroughly inspect chains which are greased in operation. The uncertainty regarding the exact condition of a chain in service, and the fact that it gives no warning of weakness, but may break at a load below the normal working load, have caused them to be largely replaced, on such appliances as overhead cranes, by steel rope. The state of the strength of the latter is more easily determined by inspection.

Weldless * steel chain rolled from a bar of special shape has lately come into use to some extent. The chain is made in lengths of 60 to 90 ft, and the lengths are joined by a link made of special welding steel. They are said to be much stronger than iron chains.

209. Chain Drums and Sheaves. Drums on which crane chains are to wind should be carefully grooved so that alternate links lie flat

* See "Machine Design," by H. J. Spooner, page 452.

on the surface of the drum; and they should have sufficient capacity to receive the chain in one layer, as overwinding brings severe stresses on the parts wound upon the drum. The diameter of the drum should in no case be less than twenty times the diameter of the chain used, and thirty times this diameter is better. If it is not possible to have the chain wind upon a drum, **pocket chain wheels** are often used. These wheels are made with pockets around the periphery into which the links fit. The links are prevented from coming out by a guide over a portion of the wheel; and hence cannot slip on the sheave. Anchor chains, and the chains of certain forms of chain blocks for raising weights, run over such sheaves.

210. Hoisting-hooks. The hooks used for raising heavy weights deserve special attention. They are usually made of steel or iron forg-

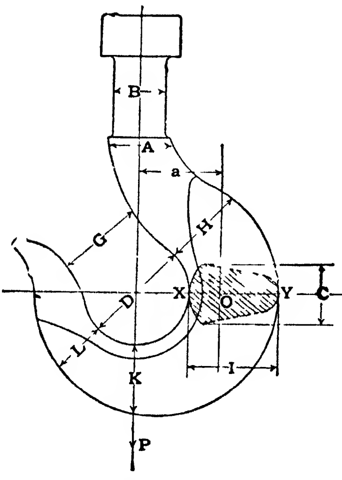

Fig. 170 (a).

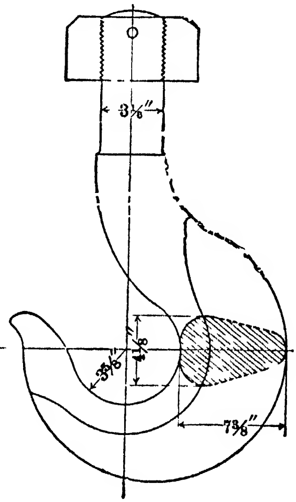

Fig. 170 (b).

ings, although steel castings are employed to some extent. If the stress in the hook can be kept low, the use of steel castings may be justified; but where the load is great and the fiber stress in the hook necessarily high, to avoid clumsy proportions, the hook should be forged from ductile material.

Let the hook in Fig. 170 (a) be subjected to a vertical load P; then XY, the most dangerous section, is apparently acted upon by a direct stress $s' = P/A$ (where A is the area of the section) and by a flexural stress s'' due to the moment Pa, the stress s'' being tensile at X

and compressive at Y. Apparently, therefore, the theory of Article 21 applies, and equation (M_1) (Table VI) may be used to design the section, or

$$s_t = \frac{P}{A}\left[1 + \frac{a}{R - c_t}\left(\frac{c_t}{R} \times \frac{A'}{A' - A} - 1\right)\right]$$

The resulting tensile stress at X is equal to the sum of the direct stress and the maximum tensile stress due to bending; the resulting compressive stress at Y is equal to the difference between the maximum compressive stress due to bending and the direct stress. The resulting tensile stress at X will, therefore, always exceed the resulting compressive stress at Y if the gravity axis is midway between X and Y or nearer Y than X. For this reason section XY is made unsymmetrical, with the gravity axis nearer X than Y for materials equally strong in tension and compression or stronger in compression than in tension.

Hooks for small cranes and hoists are much more likely to be loaded frequently to their full capacity than hooks for raising large loads; thus, a hook on a 5-ton crane may be loaded to its full capacity several times every day, while the hook of a 20-ton crane would be thus loaded only at rare intervals. The stresses in small hooks must therefore be kept low, and fortunately this can be done without making the hook clumsy. As the size of the hook increases, however, the stresses must necessarily be increased to avoid clumsiness; but the larger the hook the less frequently will it be fully loaded, and a working stress as high as 15,000 lb per sq in., or more, is as safe in a 50-ton hook as 10,000 lb per sq in. would be in a 10-ton hook. (See Article 27.)

The most valuable data on crane hooks are those given by Mr. Henry R. Towne in his " Treatise on Cranes," as a result of both mathematical and experimental work. Figure 170 (a) and the following formulas give the most important dimensions of a hook according to this work, and these proportions have been much used with uniform success. The basis for each size is a commercial size of round iron or dimension A. In the following formula Δ is the nominal capacity of the hook in tons of 2000 lb. The dimension D is assumed arbitrarily but so as to provide ample room for the slings. The following measurements are then expressed in inches:

$D = 0.5\Delta + 1.25$ $\quad H = 1.08A$ $\quad K = 1.13A$

$G = 0.75D$ $\quad I = 1.33A$ $\quad L = 1.05A$

The following gives the capacity of the hooks made from various sizes of bar stock:

TABLE XL

Capacity of hook in tons	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	3	4	5	6	8	10
Size of bar A in inches.	$\frac{5}{8}$	$\frac{11}{16}$	$\frac{3}{4}$	$1\frac{1}{16}$	$1\frac{1}{4}$	$1\frac{3}{8}$	$1\frac{3}{4}$	2	$2\frac{1}{4}$	$2\frac{1}{2}$	$2\frac{7}{8}$	$3\frac{1}{4}$

It is to be noticed that the stresses allowed by Mr. Towne's proportions are very low. Thus in a 10-ton hook the dimension A is $3\frac{1}{4}$ in. or, allowing for finishing, the dimension B may be taken as 3 in., which would give a tensile stress in the shank of only 3000 lb per sq in. It should be borne in mind, however, that hooks are subjected to much

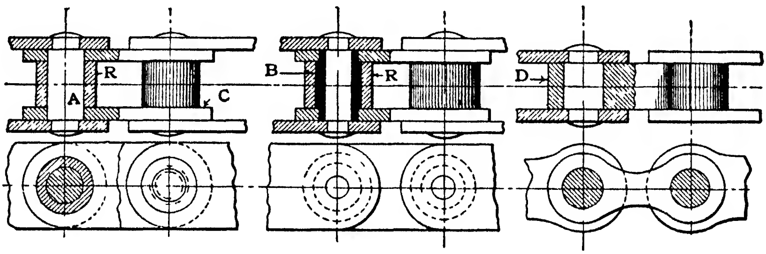

FIG. 171 (a). FIG. 171 (b). FIG. 171 (c).

abuse, and the designer has no assurance that they will always be loaded with a true axial load, for improper arrangement of the sling often throws the load more toward the point of the hook, and the member is called upon to carry a bending moment greatly in excess of that for which it is intended.

When, however, hooks larger than those covered by Mr. Towne's work are to be designed, his proportions lead to clumsy dimensions. Thus a 20-ton hook would require a shank $4\frac{1}{2}$ in. in diameter and a 50-ton shank would be $6\frac{1}{2}$ in. in diameter. Figure 170 (b) shows a 20-ton hook of Norway iron which has been successfully used in practice. The threaded shank, being $3\frac{1}{8}$ in. in diameter, is therefore stressed to about 6000 lb per sq in., but yet is only as large as the shank of a 10-ton hook as given by Mr. Towne's dimensions. Examination of current practice * and measurements taken from a number of large hooks in successful service indicate an allowable tensile stress at X, ranging from 10,000 lb per sq in. in 10-ton hooks, to 15,000 lb per sq in. in 50-ton hooks.

* For tabulated dimensions of hooks see "Machinery's Handbook," page 814.

211. Conveyor Chains. Chains for conveying and elevating materials, such as grain, coal, ashes, etc., are usually made of malleable iron, the links *hooking* together in some manner. This style of chain is known as link belt. On account of the diverse purposes to which they are applied, they are made in many forms, and the particular form for a given problem is usually selected in conference with the manufacturer or taken from trade catalogues giving the desired information. This form of chain is also extensively used in rough machinery, such as agricultural implements, for the transmission of power. Such chains must be run at low speeds, as they become noisy and unreliable even at moderate velocities.

212. Chains for Power Transmission. The chains heretofore discussed move, necessarily, at low velocities, but a demand arose for chains which may be run at high speeds for the purpose of transmitting power. Such chains are used when a positive velocity ratio must be maintained between the connected shafts, and where the distance between shafts is so great as to make tooth gearing inconvenient. Of this class there are at present three principal types, namely, **roller chains, block chains,** and so-called **silent chains.** Figure 171 (a) illustrates the simplest form of roller chain, in which the pin, A, is riveted fast in the outer links, and rotates in the inner links. The roller, R, lessens the friction against the tooth. In this form of chain the wear between the pin and the inner link is excessive, and for this reason it is now little used for power transmission. It is sometimes made without the roller and with several inside links and is then known as **stud chain.** In this form it is used for very low velocities only. The form shown in Fig. 171 (b) is most common. Here the bushing, B, is pressed into the inner links, and the pin, which is riveted fast to the outer links, bears over the whole length of the bushing. The roller, R, rotates on the bushing. In the block chain, Fig. 171 (c), the pin also bears over the whole thickness of the block, D, but since the roller is necessarily omitted, there is more friction against the tooth. Roller chains may be used for velocities up to about 800 ft per min, and block chains up to about 500 ft per min.

The defect in the operation of the roller or block chain may be seen by referring to Figs. 172 and 173. When the chain is new, and has the same pitch as the wheel, it fits down on the wheel as shown in Fig. 172, but in a very short time the chain stretches slightly, owing to wear of the joints, thus increasing the pitch of the links. The wheel, on the other hand, may *wear,* but this does not change the *pitch.* The operation of the chain is then as shown in Fig. 173, the increased pitch causing the rollers to ride higher and higher on the back of the tooth as they move round the sprocket. The roller A is shown fully seated while B is just coming down to its seat. Before B can become fully seated A must rise, and this action takes place when A and B are carrying full load. As a consequence the chain does not run quietly and smoothly and the wear

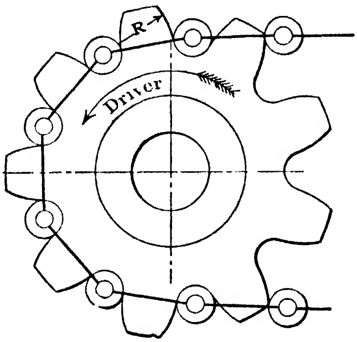

FIG. 172.

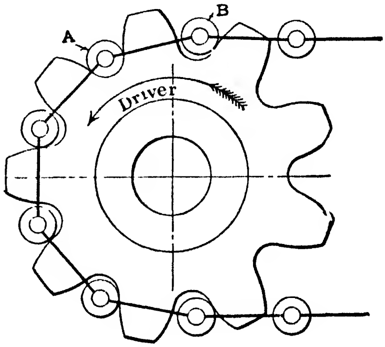

FIG. 173.

is excessive, thus limiting the speed at which the chain may be run. This difficulty is sometimes met by the arrangement shown in Fig. 174. Here the pitch of the chain when new is made a little *less* than the pitch of the driving sprocket, and clearance is allowed between the roller and the tooth, so that the driving is done by the last tooth, L, the pitch of

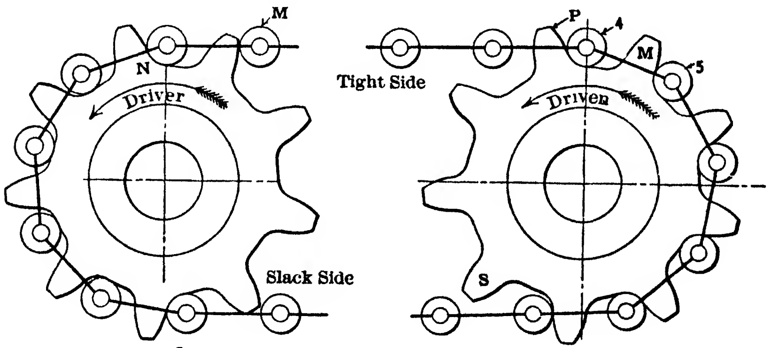

FIG. 174.

the chain being such that the incoming roller, M, just clears the *back* of the first tooth and seats itself close to it at the root as at N. As the chain stretches, the rollers move backward toward the *faces* of the teeth, till a condition like that in Fig. 172 is reached, and riding commences. The pitch of the driven sprocket wheel is made equal to that of the chain, and the condition when new is that shown in Fig. 174. As the chain stretches, the rollers move gradually backward *away* from the driving faces of the tooth, the driving being done on the last tooth, P. It is evident that this construction extends the time preceding the condition shown in Fig. 173. When this construction is used, the form of the tooth must be slightly modified. Referring to Fig. 175, it is obvious that if the outline of the tooth, M, be an arc of a circle struck from the center of the roller (2), this roller will swing from its position 1′ rolling on the face of the tooth, and this is the usual outline. But before roller (3) can take the load, which (2) is about to give up, it must be fully rooted against the next tooth; whereas (from Fig. 175) a small distance now separates the two. Therefore, as (2) rolls up the curve of the tooth it should allow (3) to slowly settle back in place. The tooth outline is therefore struck (as shown on M), from a point a little inside the pitch polygon so as to give a curve tangent to the first and last positions of the roller. This outline is also necessary for the back of the tooth in order to allow the incoming roller to swing in without striking. The velocity of the chain is, therefore, a little *less* than the theoretical velocity on account of this continual slipping backward. Brief reflection will show that the tooth outlines of the driven sprocket may be struck from the center of the roller when rooted in place; and that when the chain is stretched a little it will *creep* as it is wound upon the driven sprocket.

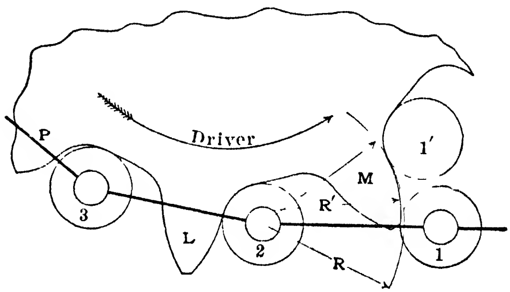

FIG. 175.

When the roller (4), Fig. 174, is about to roll up the face of P, roller (5) is not in contact with M (wear having begun); hence the chain will move ahead till (5) is in full contact with M.

The greatest defect in this construction is the fact that the load is carried entirely on one tooth and hence the wear is excessive. This may be so great that the chain creeps forward on the driven wheel, so as to cause the incoming roller to strike the tooth S, Fig. 174.

The above difficulties are overcome in the so-called *silent chains.* In these chains the inevitable stretching of the links is compensated for in a peculiar manner. The true theory of the action of these chains is very complex; but the general action is as follows: as the chain stretches, the links continually tend to take up a position farther and farther away from the center of the sprocket, thus increasing the length of the sides of the pitch polygon to suit the elongation of the link. Each link, therefore, remains in constant contact with its own tooth, from the time of engagement till release takes place. The links seat themselves without sliding action and the operation is nearly noiseless.

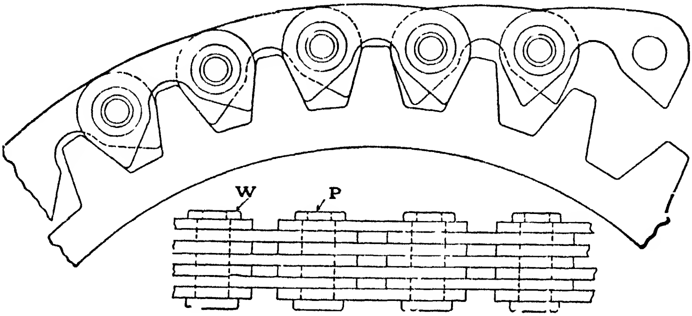

FIG. 176.

In the Renold chain of this type, Fig. 176, the links move relative to each other on a round pin, *P*, the shouldered ends of which are riveted into a washer, *W*, thus holding the chain together. In a later form half bushings of case-hardened steel are so fitted to the links that the case-hardened pin has a bearing over its full length; but the relative motion of the pin to the bush is still a *sliding* motion. In the Morse chain this sliding is eliminated by an ingenious form of rocker joint shown in Fig. 177. The hardened-steel parts, *A* and *B*, are fitted respectively to the sets of links, *D* and *C*. While keeping contact along a fixed line they *rock* on each other as the links *C* and *D* move relatively to each other, and sliding is thus eliminated. When transmitting simple tension between the sprockets, the parts *A* and *B* are in contact on flat surfaces as shown at *E*. This construction has the advantage of requiring little or no lubrication, hence the chain may be run at higher speeds than others requiring lubrication, the speeds of

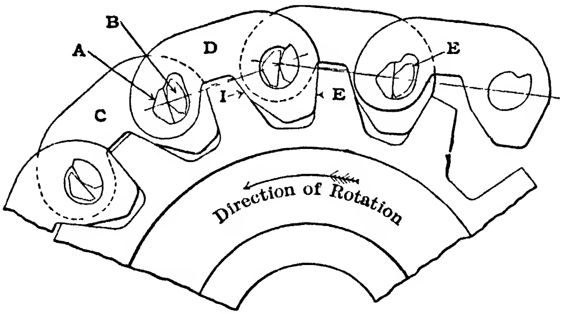

FIG. 177.

which are limited by the velocity at which centrifugal action throws off the lubricant. The Morse chains also work well in dusty places.

The efficiency of both of these chains is very high, the makers of the Morse chain claiming an efficiency of nearly 99 per cent. Such chains are particularly useful for connecting shafts which are too far apart for gearing, and not far enough for a belt, and in places where positive connection is desirable, as in motors driving heavy machine tools. It is to be especially noted that this form of transmission requires no definite tension on the slack side of the chain to produce a certain driving force on the tight side; and hence the pressure on the bearings is much reduced, for a given effective pull on the wheel rim.

213. Standard Transmission Chains. Transmission chains are for the most part a standard product and are *selected* by the user rather than *designed.* Manufacturers' catalogues give much information concerning size and capacity of chains, but in case of doubt the manufacturer should be consulted. The American Gear Manufacturers Association has developed recommended standards for roller and block chains. The Association recommends the following equations for allowable load and speed subject to modification to suit conditions:

$$T = \frac{2,600,000A}{V + 600} - \frac{WV^2}{115,900}$$

where T = allowable working load in pounds, A = projected pin-bearing area, V = velocity of chain in feet per minute, and W = the weight in pounds of 1 ft of chain. If the speed of the chain is less than 800 ft per min the centrifugal effect represented by the second fraction may be neglected. Under best conditions of installation, lubrication, etc., this value may be increased 25 per cent. If the conditions of service are bad and where the load may be applied suddenly or where long life is desired this value should be reduced one-half.

The Association further recommends that on account of impact between the chain rollers and the sprocket teeth the allowable revolutions per minute should not exceed

$$\text{RPM} = \frac{1920}{P}\sqrt{\frac{A}{WP}}$$

where P = pitch of chain.

The velocity of silent chains, so called, may be as high as 1200 to 1600 ft per min.

REFERENCES

Publications of American Gear Manufacturers Association.

A New Bases for the Rating of Roller Chain Drives, G. M. Bartlett: *Trans.* A.S.M.E., April, 1935.

Machinery's Handbook.

CHAPTER XVII

APPLICATIONS OF FRICTION

FRICTION WHEELS FOR POWER TRANSMISSION

214. General Considerations. When it is required to drive a rotating member intermittently, and the rate of driving is not necessarily positive, friction wheels have been found very useful. They are particularly applicable where the amount of power is comparatively small, as in feed mechanisms, but they may also be used for heavy work when properly constructed. For continuous driving, the transverse sections of friction wheels must be circular in cross-section, and this form, only, is used in practice.

Figures 178 and 179 show common forms of friction wheels. In Fig. 178 let A be the driving wheel which rotates continuously and let B

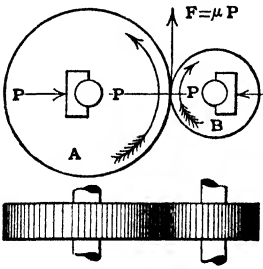

FIG. 178.

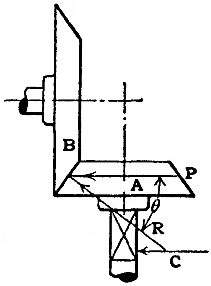

FIG. 179.

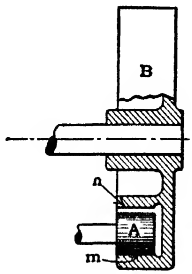

FIG. 180.

be the driven wheel which is required to be driven intermittently. The shaft of A is so mounted that, by means of a lever attached to the bearing, A may be pressed up against B with a force P, or it can be moved slightly away from B until no contact exists. If now the force P is applied to the bearing (which should be close to A), an equal and opposite force is set up in the bearing of B, and the wheels are pressed together at the line of contact. The resistance to slipping at the line of contact will be μP, where μ is the coefficient of friction of the materials of which the wheels are made; and if μP is equal to, or greater than, the resisting force at the surface of B, A will cause B to rotate. Theoretically, A and B

will roll together with pure rolling motion, but practically this cannot be attained, as even with very hard materials the wheels flatten slightly at the line of contact. (See Article 67.)

Figure 179 illustrates the application of friction wheels to shafts which are not parallel to each other, the wheels here having the form of rolling cones. Obviously, the principle is of wide application and many combinations of friction wheels are used. Figure 180 illustrates a friction wheel arranged so that the driver, A, can rotate the driven wheel, B, in either direction, depending on whether it is pressed against the surface m or the surface n.

Figure 181 shows a form of friction mechanism much used for imparting variable speed to the driven shaft. The driver, A, may be moved along the shaft, C, at will. When at A' the angular velocity of B is a

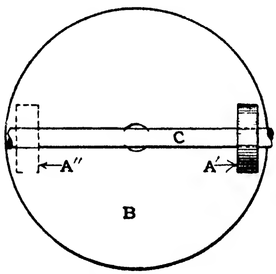

FIG. 181.

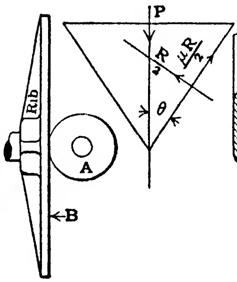

FIG. 182 (b).

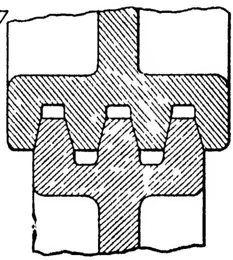

FIG 182 (a).

minimum. As A is moved inward, the rotative velocity of B increases. When A is moved across the center of B to the other side, the direction of rotation of B is reversed. If A were infinitely thin, it would, theoretically, roll upon B with pure rolling motion. Since, however, it must have an appreciable width of face, and since the velocity of B varies with the radius, it is evident that there must be some sliding at the line of contact. For this reason the thickness of A must, for best results, be kept small compared to the radius of B.

215. Materials for Friction Wheels. The driven wheels of friction devices should always be made of a harder material than the driver, for the reason that the driven wheel is likely at any time to be held stationary by the load, while the driving wheel revolves against it under pressure. This action, though severe on the driver, does not tend to wear it out locally, but it does rapidly wear flat spots on the driven wheel. Driven wheels are, therefore, almost universally made of iron, and driving wheels of wood, leather, paper, rubber, or of some composition of these, the most common being leather and various forms of paper.

216. Practical Coefficients. The tangential force, F, exerted by A upon B, Fig. 178, is dependent on the pressure P and the coefficient of friction μ. It is, therefore, necessary to know the allowable pressure per unit of length along the contact elements and also the value of μ for the particular materials used. The most comprehensive investigation of these relations is that made by Professor Goss,* whose experiments cover a variety of materials. He recommends the following pressures, which are about one-fifth of the ultimate crushing strength of the respective materials. Professor Goss found that the coefficient of

SAFE WORKING PRESSURES PER INCH OF CONTACT

Material	*Pressure*
Straw fiber	150
Leather fiber	240
Tarred fiber	240
Leather	150
Wood †	100 to 150

friction for all the wheels tested approached a maximum value when the slip between the two wheels was about 2 per cent, and, within narrow limits, was practically independent of the pressure of contact. He found these values to range for different combinations from low values up to 0.515. In these experiments the friction due to the bearings was neglected. The bearings, however, were of the roller type and, probably, absorbed less power than the ordinary bearing. Making due allowance for the difference between laboratory conditions and those found in practice, Professor Goss recommends the following approximate values of μ‡ for the various combinations. In this connection it is to be noted that allowance must be made for a decrease in the value of this coefficient when the linear velocity of the driver is great, in the case where the driver is starting the driven wheel under load (see Article 37):

WORKING VALUES OF COEFFICIENT OF FRICTION

Materials	*Coefficient of Friction*
Straw fiber and cast iron	0.26
Straw fiber and aluminum	0.27
Leather fiber and cast iron	0.31
Leather fiber and aluminum	0.30
Tarred fiber and cast iron	0.15
Tarred fiber and aluminum	0.18
Leather and cast iron	0.14
Leather and aluminum	0.22
Leather and typemetal	0.25
Wood and metal	0.25

* See *Trans.* A.S.M.E., vol. XXIX.

† The value for wood is not from Professor Goss's paper.

‡ The coefficient for wood is not from Professor Goss's paper.

217. Power Transmitted by Friction Wheels. If V be the velocity at the surface of the friction wheels in feet per minute, P the total normal pressure in pounds, F the resulting tangential force, and μ the coefficient of friction, then since $F = \mu P$, the rate at which power is transmitted in foot-pounds per minute is μPV, and the horsepower is

$$\text{Hp} = \frac{\mu PV}{33{,}000} \tag{1}$$

or, if d be the diameter of the driver in inches, l the width of face in inches, w the allowable load per inch of face, and N the number of revolutions per minute, the horsepower is

$$\text{Hp} = \frac{\mu wl \times \pi dN}{12 \times 33{,}000} = 0.000008\mu wldN \tag{2}$$

Example. How many horsepower can be transmitted by a straw-fiber friction pulley of 8-in. diameter and 6-in. face, when running at 500 rpm, the driven wheel to be of cast iron?

Here $d = 8$ in., $l = 6$ in., $N = 500$, $\mu = 0.26$, $w = 150$.

$$\therefore \text{Hp} = 0.000008 \times 0.26 \times 150 \times 6 \times 8 \times 500 = 7.5$$

It may be noted that the horsepower per inch of width of face is a little more than unity, for a surface speed of 1000 ft, as in the above example. This corresponds closely to the empirical rule given for belts in Article 186, and corroborates the empirical rule often used that the same width of face is necessary for a friction wheel as for a belt, to transmit a given horsepower at the given speed.

In the case of bevel wheels (see Fig. 179) the component R of the applied force, P, presses the wheels together, and $R = \dfrac{P}{\cos\theta}$. The velocity of the mean circumference of the driver may be taken as the velocity of transmission.

In face friction driving as in Fig. 181, the width of the driving wheels should be kept as narrow as possible for best results. If the velocity of the outer edge of the driving wheel is not more than 4 per cent greater than that of the inner edge, the coefficients listed may be used. Where the driver must, at times, drive at a short distance from the center, lower values of the coefficient of friction must be taken.

218. Wedge-faced Friction Wheels. The faces of a pair of metal friction wheels are sometimes formed as shown in Fig. 182 (a), and are then known as **wedge-faced friction wheels.** The object of this construction is to secure a greater resistance to slipping, with a given radial pressure. It is to be noted that the number of wedges does not affect this ratio, but decreases the wear by distributing it over several surfaces, and thus reducing the difference in the velocities of the mating bottoms of the grooves and tops of the wedges. This last item is important, as it is easily seen that the contact surfaces of the driver and the driven wheel can have the same velocity at one point only, and that at all other points slipping or a grinding action occurs and wear must result.* The teeth, therefore, should not be very long.

In Fig. 182 (b), let P be the radial force applied to the wedged surface, F the tangential force transmitted, $R/2$ the reaction on each face and 2θ the angle of the wedge; then the wedge is held in equilibrium by the force P, the reactions $R/2$, and the frictional resistances $\mu R/2$ due to the wedging action. Equating vertical forces,

$$P = 2\left(\frac{R}{2}\sin\theta + \frac{\mu R}{2}\cos\theta\right)$$

or since

$$F = 2\left(\frac{\mu R}{2}\right) \quad \text{or} \quad R = \frac{F}{\mu}$$

$$P = \frac{F\sin\theta}{\mu} + F\cos\theta \qquad (1)$$

or

$$F = \frac{\mu P}{\sin\theta + \mu\cos\theta} \qquad (2)$$

To avoid sticking, the angle 2θ should not be less than 30°.

Equation (2), as will be seen, applies also to other wedge-shaped frictional surfaces, such as conical friction clutches and wedge-faced brakes. It appears, however, that when such frictional surfaces are brought into contact while under motion, the frictional component due to sliding at right angles to the direction of motion ($\mu R/2$, Fig. 182 (b)), is greatly decreased if not eliminated. Professor Leutwiler † quotes experiments by Professor Bonte which would indicate that such is the case. If this be assumed, equation (2) becomes

$$F = \frac{\mu P}{\sin\theta} \qquad (3)$$

* See "Kinematics of Machinery," by John H. Barr and E. H. Wood, page 101.
† "Machine Design," page 418.

FRICTION BRAKES

219. Friction brakes are used for controlling and stopping machinery by absorbing energy through frictional resistance from some moving part, and dissipating it as heat. Brakes used in heavy work, and as dynamometers for measuring energy, must often be fitted with water circulation to carry away the heat. The student is referred to treatises on power measurement for a discussion of dynamometers.

220. Block Brakes. The simplest form of brake is the block brake, as shown in Fig. 183. Here the force P, acting on the lever A, presses the block C against the wheel B. Let the reaction between the wheel and the block be R. Then if B be rotating, a tangential frictional resistance $\mu R = F$ will oppose its motion. With the arrangement shown

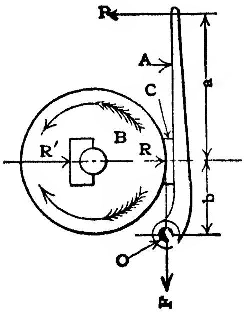

FIG. 183.

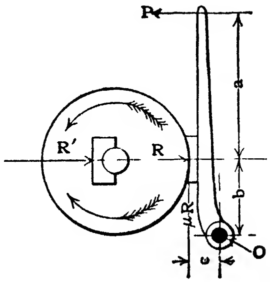

FIG. 184.

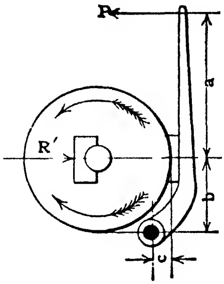

FIG. 185.

in Fig. 183, the line of action of F passes through O, the center of the fulcrum of A. Considering A as a free body and taking moments around O, then for rotation in either direction

$$P(a + b) = Rb$$

or since

$$R = \frac{F}{\mu}$$

$$P = \frac{Fb}{\mu(a + b)} \tag{1}$$

or

$$F = \frac{\mu P(a + b)}{b} \tag{2}$$

In Fig. 184 the line of action of F does not pass through O, and therefore in writing the equation for the equilibrium of A its effect must be considered, whence

$$P = \frac{Fb}{a + b}\left[\frac{1}{\mu} \pm \frac{c}{b}\right] \tag{3}$$

or

$$F = \frac{P(a + b)}{b\left[\frac{1}{\mu} \pm \frac{c}{b}\right]} \tag{4}$$

The minus sign is to be used for rotation in a clockwise direction, *for the arrangement shown,* and the plus sign for rotation in the opposite direction. It is to be especially noted that for clockwise rotation when $1/\mu = c/b$, or when $b = \mu c$, $P = 0$; that is, the brake is *self-acting* and if put in contact the moment of the frictional force will apply it with ever-increasing pressure. Obviously such proportions should be avoided.

In a similar manner for Fig. 185,

$$P = \frac{Fb}{a + b}\left[\frac{1}{\mu} \pm \frac{c}{b}\right] \tag{5}$$

or

$$F = \frac{P(a + b)}{b\left[\frac{1}{\mu} \pm \frac{c}{b}\right]} \tag{6}$$

the plus sign referring to clockwise rotation, *for the arrangement shown,* and the minus sign to rotation in the opposite direction.

In this class of brakes the pressure of the brake R against the wheel is opposed by an equal force R' at the bearing near the wheel. In the calculations above, the braking effect due to friction of the journal is neglected, as its lever arm is, usually, small. It cannot be neglected in designing the bearing, and for this reason this form of brake is not well adapted to heavy work. Most usually, two block brakes placed diametrically opposite each other are used. The methods of applying and relieving the brakes vary widely; where safety against accidents is a factor the brakes are usually held in action by a spring or a weight until relieved by the operator, going into action automatically when left to themselves.

Figure 186 shows a brake such as is often used in hoisting engines of moderate size. Such brakes are usually operated by hand levers or foot pedals in the manner indicated. The frictional resistance, F, passes through the fulcrum, O, as in Fig. 183. There is no tendency toward automatic action, and the brake pressures balance, thus, relieving the bearings of all braking load. This type of brake is often constructed with an arrangement of levers as shown in Figs. 184 and 185, the line of action of the frictional force passing either inside or outside of the fulcrum. In such constructions the total braking force is the sum of the two values given by equation (4) or (6), as the case may be, and is equal to twice the value of the frictional force given by equation (2). It should be noted, however, that the braking load in these cases is unequally distributed between the two blocks, and this unequal load must be resisted by the bearings. Usually this is not difficult to provide for.

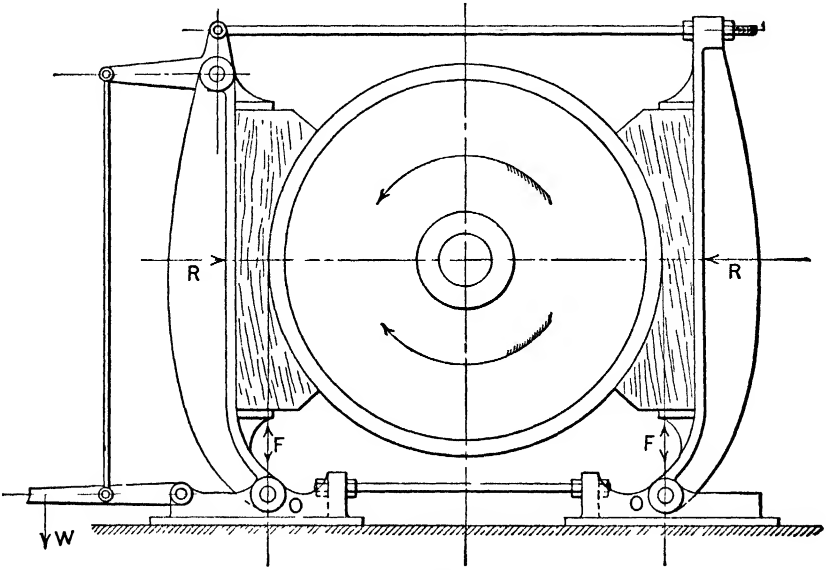

FIG. 186.

It may be noted that if the brake on one side of the brake wheel is arranged as in Fig. 184 and the brake on the other side arranged as in Fig. 185 the radial pressure will be the same on both sides of the wheel no matter what the direction of rotation may be. This arrangement of brake beams is unusual.

Figure 187 shows an arrangement of brakes much used on large mining engines. The brake beams, B, are supported by the links, LL', through the pins, N, these pins being placed as near the frictional surface as possible, so as to reduce the tendency of the frictional force to rotate the beams. The rods, A, are adjustable in length, and by means of these rods and the adjustable link, M, the brake beams can be adjusted to fit the brake wheel correctly. The links, M and L', being approximately the same length, constitute a parallel motion, compelling the beams to open and close without tipping. The brakes are applied by the load, W, and relieved by the action of the small cylinder, E, which may be actuated by steam or air. If $T/2$ be the tension in each of the rods, A and A, the frictional force exerted on the wheel is $F = 2\mu T$. If the pin, O, is so located that when the load, W, is applied it moves to O', and the center line of the rod, A, passes through the center of the pin, P, a toggle effect is obtained and the tension in the rods, A and A, may be made any desirable value; in fact, with such an arrangement care must be exercised in adjusting the brake that such pressures are not brought on the pins as will cause

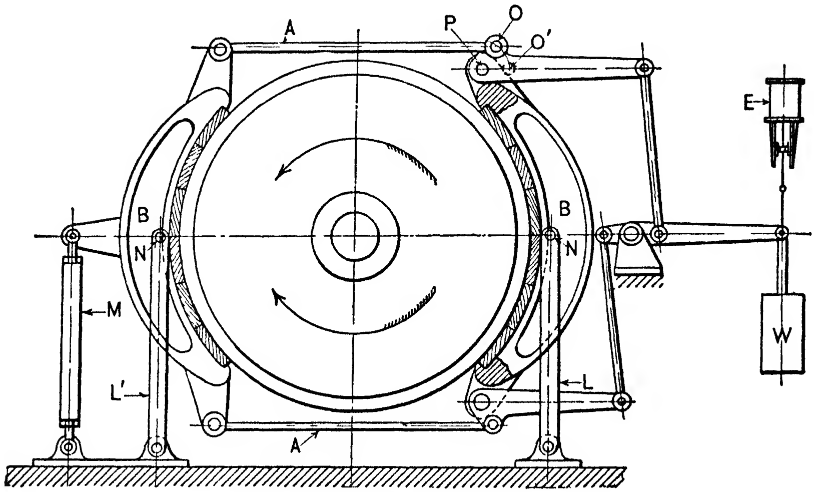

FIG. 187.

failure by shearing. When O moves down to O' the brake is "locked" in position and the operating force may be removed. This last feature is often a valuable quality in a brake. Brakes of this character are generally lined with wooden blocks as shown. If the face of the brake wheel is wedge-faced the value of μ in the foregoing equation should be replaced by $\frac{\mu}{\sin\theta + \mu\cos\theta}$ from equation (2), or $\frac{\mu}{\sin\theta}$ from equation (3), Article 218, as the circumstances may dictate.

221. Strap Brakes. If the effect of centrifugal force is neglected (see Article 181), and the total tensions in the band (T_1 and T_2) be taken instead of the tensions per inch of width, equation (6) of that article reduces to

$$\frac{t_1}{t_2} = \frac{T_1}{T_2} = 10^k \qquad (1)$$

where $k = 0.0076\mu\alpha$, α being the arc of contact in *degrees*. If, also, F is the total frictional force exerted by the band upon the wheel,

$$F = T_1 - T_2 \tag{2}$$

It is obvious that these equations are applicable to the discussion of band brakes. Figures 188, 189, and 190 show the most usual arrangement of band brakes. In Fig. 188 the end of the strap which is subjected to the greatest tension, T_1, is anchored, for convenience, at the

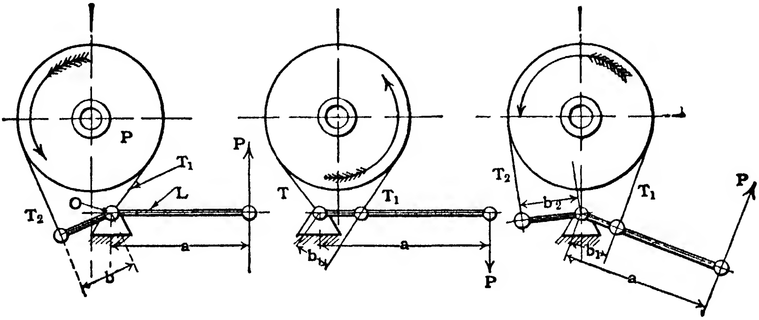

FIG. 188 FIG. 189 FIG. 190

pin which serves as a fulcrum for the operating lever, L, it could be anchored to any other convenient part of the frame.

From (1) and (2),

$$T_2 = \frac{F}{10^k - 1}$$

Taking moments around O,

$$Pa = T_2 b = \frac{Fb}{10^k - 1}$$

or

$$P = \frac{Fb}{a(10^k - 1)} \tag{3}$$

which expresses the relation between the applied force, P, and the frictional resistance applied to the wheel.

In Fig. 189 the end under greatest tension is attached to the lever and the end of least tension is anchored, hence for this case

$$P = \frac{Fb}{a}\left[\frac{10^k}{10^k - 1}\right] \tag{4}$$

In Fig. 190 the end under greatest tension is anchored to the lever at a *shorter* radius than the end of least tension, so that the force which it exerts *assists* the operating force P. This is known as a **differential brake.** For this case in a similar manner as above

$$P = \frac{F}{a}\left[\frac{b_2 - 10^k b_1}{10^k - 1}\right] \tag{5}$$

It is to be especially noted that if $10^k b_1 = b_2$, $P = 0$, and the band will brake automatically; that is, if any force is applied to the lever, the brake will continue to set itself up with ever-increasing force till motion ceases or rupture occurs. This form of brake is exceedingly dangerous on account of its tendency to "grab," especially if μ is materially increased through a change in the character of the friction surfaces.

Strap brakes are usually made of wrought iron or steel. In light work they may engage with a cast-iron surface or may be lined with leather; but in very heavy work they should be lined with wood.

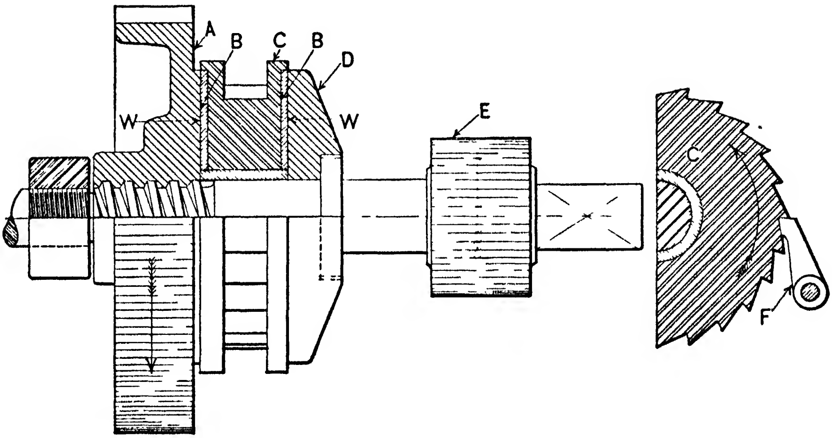

FIG. 191.

Figure 191 shows a form of brake much used on overhead traveling cranes. The driving gear, A, meshes with the motor gear, and the pinion, E, actuates the train of gearing that operates the hoisting drum. In hoisting, the gear, A, is rotated in the direction of the arrow, advancing upon the thread on the main shaft, thus clamping the friction faces of the ratchet wheel, C, between itself and the collar, D. This collar is fast to the shaft and hence the latter is rotated against the resistance of the load applied through the pinion, E. The pawl, F, of the ratchet wheel, C, permits the latter to rotate freely during hoisting, but prevents its rotation in a reverse direction, and the load is sustained so long as the pressure between A and D is continued. In lowering, C is stationary.

The motor reverses the direction of rotation of A, thus unscrewing A and relieving the frictional resistance between A and D. But the load immediately rotates the shaft, screwing the shaft into the gear, A, and tending to restore the frictional resistance. In actual operation these two actions tend to offset each other and the load is lowered only as fast as the rotation of the motor will permit, and hence is always under perfect control. There are many variations of this general principle; in some constructions multiple-friction discs, such as are described in Article 226, are used.

This mechanism is essentially a screw bolt and nut with a loose friction ring placed between the nut, A, and the bolt head or collar, D. Therefore, equations (6) and (6′) of Article 130 apply, as does also the discussion in that article. The relation between the moment of the load applied through E and the normal force, W, is given by equation (6) of Article 130, while the reversing moment that must be applied by the motor for lowering is given by (6′) of the same article. In computing the moment of the load, due account should be taken of the acceleration that it is desired to give the load to be raised. If the speed of hoisting is low this may be very small and negligible, but in cranes of moderate size the acceleration applied to the load is sometimes considerable. The accelerating force, it will be remembered, is Pa/g, where a is in foot-seconds and P is the load in pounds. This accelerating force should be added to the actual load to be hoisted in computing the moment upon E. The radius r_c of the equivalent frictional force, $\mu_1 W$, is $\frac{2}{3}\left[\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right]$,* where r_1 and r_2 are the outer and inner radii, respectively, of the friction surfaces. The value of W can be determined from equation (6), Article 130, and applied in equation (6′) of the same article to determine the turning moment that must be exerted in lowering.

222. Absorption of Energy. Brakes may be designed for one of two purposes, namely (a) to absorb the energy of a moving mass, or (b) to resist the torque of a load as in cranes and hoisting engines. Let F = the braking force at the surface of the brake wheel, D the diameter of the brake wheel, N the number of revolutions during which the brake acts, K_1 the energy of the body when the brake is applied, K_2 the energy that may be added during the application of the brake, and K_3, the energy of the body at the end of the application of the brake. Then in general†

$$K_1 + K_2 - K_3 = F\pi DN \tag{6}$$

* See Article 165. † See Article 2.

If no energy is added during the operation $K_2 = 0$. If the mass is brought to a full stop $K_3 = 0$. Ordinarily $K_2 = 0$, but there are cases such as the automobile where the brakes are applied before the propelling power is cut off. A falling body such as a mine cage is constantly acquiring energy until it is stopped. For any given distance of drop it is possible to compute the force F which, acting simultaneously, will stop the cage. Or if it has acquired a kinetic energy K_1 before the brake is applied, the force F may be computed which will arrest the cage when it has fallen an additional distance in which it would acquire an additional amount of energy K_2. In all such machinery as cranes, and hoisting machinery in general, the torque of the brake must exceed the torque of the load to be sustained. In such machinery the braking force is arranged so as to keep the brake on unless removed to allow of operation.

Brakes operate by absorbing energy in the form of heat and dissipating it by radiation and conduction. For a given coefficient of friction the heat generated will be proportional to the normal force P and the velocity of rubbing V. Obviously if heat is generated faster than it can be dissipated excessive heating and possibly combustion will occur. Marks (page 747) gives the following limiting values of PV for wooden blocks:

For intermittent service and poor radiation $PV = 1000$.
For fairly continuous service and poor radiation $PV = 500$.
For continuous service and artificial cooling $PV = 1400$.

FRICTION CLUTCHES

223. Friction clutches, though made in a great variety of forms, can, in a large measure, be classified under four principal types, namely, **conical, radially expanding, disc, and band.** A well-designed clutch should start its load quickly but without shock, and should disengage quickly. It should be "self-sustained," that is, when the clutch is driving, no *external* force should be necessary to hold the contact surfaces together. In addition, it is often necessary that the clutch should "lock" in place, after the manner of the brake in Fig. 187.

224. Conical Clutches. Figure 192 shows the elements of a conical clutch which is self-sustained. The cone, K, is fast to the shaft, S, and rotates with it. The pulley, H, rotates upon K and carries with it the levers, E. When the thimble, B, is forced under the rollers, C, the levers, E, force the cone surfaces into contact. Heavy springs at G (not shown) throw the surfaces apart when the thimble is withdrawn. The relation between the transmitted frictional force, F, and the force P, applied to the cone, in a direction parallel to the axis, is the same as that of the wedge gearing in Article 218, or

$$F = \frac{\mu P}{\sin \theta + \mu \cos \theta}. \tag{7}$$

The angle θ should not be less than 10°, unless some mechanism is provided for separating the cone surfaces positively, when desired. For clutches that do not operate frequently, metal surfaces are often used; but where the operation of clutching is frequent, one surface is usually lined with wood, cork, or leather.

As noted in Article 218, there is a question as to how much the power that may be transmitted is affected by the apparent frictional com-

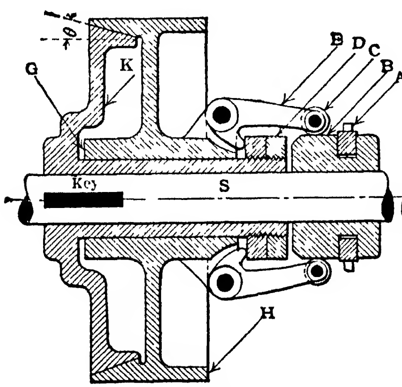

FIG. 192.

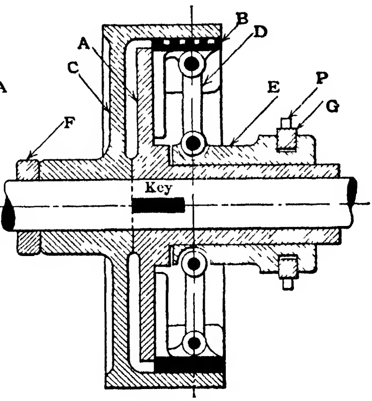

FIG. 193.

ponents along the contact elements. From that discussion it would appear that this frictional action should be neglected in making computations for metallic surfaces, especially after the load has been fully picked up.

225. Radially Expanding Clutches. Figure 193 shows the elements of a radially expanding, self-sustained clutch. The clutch body, A, is keyed to the shaft; the pulley, C, rotates loosely upon the shaft. The circular segment, B, which fits the inner surface of C, can be moved radially upon A. The loose ring, G, is operated axially by a forked lever fitting on the pins, P. When the sleeve, E, is forced inward by the ring, G, the links, D, force the segments, B, outward against C. In the arrangement shown the links have a toggle effect and can exert enormous pressure against B; hence adjustment must be carefully performed. This is usually accomplished by making the length of the link, D, adjustable, by means of turn-buckles or similar devices, which also provide a means of compensating for wear. Usually the sleeve has motion enough to carry the inner end of the link slightly past the center position shown, thus locking the clutch in place.

226. Disc Clutches. Figure 194 shows the elements of a **multiple-disc clutch** as sometimes used in automobile work for connecting the engine to the transmission shaft, A being fast to the engine shaft and B to the transmission shaft. The part A carries a number of discs, C, which fit loosely in an axial direction but are prevented from rotating

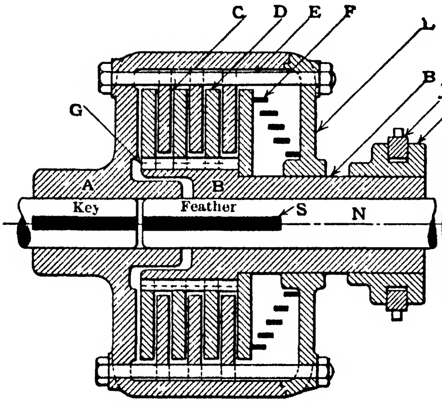

FIG. 194.

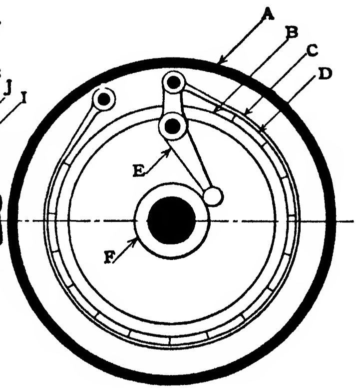

FIG. 195.

relatively to A by bolts, E, which also hold L, the cover of the case, in place. A second set of discs, D, placed alternately between the discs C, are carried on the part B and compelled to rotate with it by the keys, G. A heavy helical spring, F (sometimes made of rectangular section as shown), presses the two sets of discs together with a known load, P, when the clutch is in and the shafts connected. The sleeve, B, while compelled by the feather, S, to rotate with the transmission shaft, N, can be moved axially by means of the grooved collar, I, and the ring, J, I being made fast to B but built separately from it for constructive purposes only. When B is moved to the right the spring is compressed and the pressure on the discs relieved. The discs often run in an oil bath to prevent " grabbing." It is readily seen that while the force, P, which presses each pair of contact surfaces together is the same, the total frictional force transmitted is proportional to the number of pairs of contact surfaces n, or

$$F = \mu n P \tag{8}$$

Assuming P to be uniformly distributed and that μ does not vary in value the frictional radius will be $R = \frac{2}{3}\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)$ (see Article 64) and the moment that may be transmitted will be $FR = \mu nPR$. In Fig. 194, $n = 7$. The above form of clutch is known as the Weston clutch. Obviously, any number of pairs of discs may be used. For large work the discs are sometimes made of iron and wood (or wood-faced). For small work, alternate discs of steel and brass are employed. Many pairs of contact surfaces are then used and the discs run in oil to prevent "grabbing." The width of the wearing faces of the discs should be made small to prevent undue wear toward the outer edges of the discs, D, as in a thrust block (Article 63). It is better to use a number of smaller discs than a few large ones.

227. Band Clutches. Figure 195 illustrates the elements of a **band clutch.** The clutch wheel, A (which may be fast to one shaft), carries the wood-lined band, C. When the thimble, F (which slides on the shaft), is forced under the lever, E, the iron band, C, is tightened and clutches the rim of the driven wheel, B. Obviously, the principles involved are identical with those of the strap brake, Fig. 188 of Article 221. For light work the band may be lined with leather, but in heavy work, such as mine hoisting, blocks of basswood, or other soft wood, are used. The wood lining is usually made fast to the strap, though occasionally on very large diameters they are attached to the wheel so that they may be turned true in place. These clutches are made self-locking by arranging for a toggle effect in some one of the operating levers.

Occasionally the band is made to *expand inside* of the rim of the wheel to be driven. It is to be noted that this case is not the same as the one just discussed, but is a special case of a radially expanding clutch. The outward force exerted by the band may be computed by the theory of Article 101, considering the band as a thin cylinder under compression, the compressive stress at any section being that due to the pressure applied by the operating lever.

228. Magnetic Clutches. A number of clutches have recently appeared which are operated magnetically. These are most generally of the disc type. Evidently the general principles above, regarding transmissive power, apply also to these clutches. In magnetic brakes, the load is usually applied by a spring, or weight, and released by magnetic action, thus insuring safety against accident should the electric service fail.

229. Practical Coefficients for Brakes and Clutches. The most usual combinations of friction surfaces for brakes and clutches are wood, leather, or cork with iron; and iron with iron. In the multiple-disc type, brass or bronze on iron or steel are sometimes used. Mr. C. W.

Hunt gives the following values of μ, as the result of considerable experience in designing clutches, namely: cork on iron, 0.35; leather on iron, 0.3; and wood on iron, 0.2. For iron on iron μ may be taken as 0.25 to 0.3. It should be remembered that if the friction surfaces are to be engaged under load and at high velocity, lower values must be assumed than for lower speeds (see Article 37).

The pressure per unit area of surface is also an important feature in the design of friction machinery, for if this is taken too high, excessive wear will result. Thus, in disc clutches the pressure is usually taken at not more than 25 to 30 lb per sq in., and lower values are desirable. Wooden surfaces should not be loaded beyond 20 to 25 lb per sq in. If the clutch or brake is to operate frequently, ample surface must be provided properly to radiate the heat generated.

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they refer to specific conditions. It is to be noted that the weight of the flywheel is directly proportional to the energy to be absorbed and inversely proportional to $(v_1^2 - v_2^2)$. The latter is usually a small quantity and, therefore, if E is large the weight of the flywheel may be excessive, which is undesirable because of the cost, and also because heavy wheels bring great loads on the bearings, causing frictional losses. For this reason it is always desirable so to arrange the sequence of events in the energy supply and work to be done as to minimize the excess energy to be absorbed. This is illustrated in Article 6, Fig. 5, where the area K may be greatly decreased (or increased) by changing the relative positions of the crankpins. This procedure is of great importance to avoid wheels of great weight in large steam engines when variation in velocity must be closely restricted.

The allowable variation in velocity is fixed with reference to the character of the work to be done. It is evident that some classes of work require much more constant velocity than others, and experience has shown what the limits in variation of velocity may be for successful operation. The following limiting values of the proportionate variation $(v_1 - v_2)/v$ represent average practice. The particular case of direct driving of alternating generators in parallel must, in general, be treated with reference to the allowable variation per pole, and when, therefore, the number of poles is great the total allowable variation is correspondingly small.

TABLE XLI

Values of $\dfrac{v_1 - v_2}{v}$

For punching machines and similar machines	0.10 to 0.15
For engines driving stamps, crushers, etc.	0.20
For engines driving pumps, sawmills	0.03 to 0.05
For engines driving machine tools, weaving and paper mills	0.025 to 0.03
For engines driving spinning mills for coarse thread	0.016 to 0.025
For engines driving spinning mills for fine thread	0.01 to 0.02
For engines driving single dynamos	0.007
For engines driving alternators in parallel	0.003 to 0.0003

232. Stresses in Flywheels. The velocity of the rims of all flywheels is, from the nature of their requirements, very high. If the wheel is to act as a band wheel, the desirability of obtaining high belt speed (Article 184) brings the peripheral velocity up to 4000 or 5000 ft per min. It has been shown that the *capacity* of a given wheel is proportional to the square of its velocity and, therefore, when the wheel is to act as a flywheel alone, economy in the use of material, or the limiting of the external dimensions, makes high speed very desirable. Great care should be used in the design of such wheels, for a flywheel which breaks at normal speed is exceedingly dangerous to life and limb, and when such wheels "explode" or break from overspeeding, the results are usually very disastrous.

Unfortunately, mathematical analysis of the stresses in flywheels and pulleys is not satisfactory or conclusive. In small wheels cast in one piece, unknown shrinkage stresses of great magnitude may exist, which render useless any refined calculations. In large wheels built up of sections, the presence of joints vitiates any calculations based on the elastic theory of the strength of materials; and when the parts are of cast materials and of large sectional area, there is no assurance that the character of the material is uniform throughout. It is important, however, to understand fully the general *character* of the stresses, even though no accurate computations can be made as to their magnitude.

Consider that the rim of the wheel in Fig. 197 is free to expand radially, the arms exerting no restraining force in a radial direction. If the wheel be rotated on its axis, the action of centrifugal force is such as to cause an outward pressure on every part of the rim, in exactly the same manner as in a boiler shell acted on by an internal pressure (see Article 101), the rim expanding until the tensile stress induced in any section, AA, Fig. 197, balances the tendency of the wheel to separate along that section. If, on the other hand, the arms are rigidly attached to both hub and rim, and are so inelastic that their stretch, under the action of the centrifugal pull due to their own mass and that of the rim, is negligible, it is clear that they may be placed so close together that the rim cannot expand, and practically no stress will exist in the rim, the centrifugal action being balanced by the stress in the arms.

Flywheels approximating both of these conditions are sometimes built, but in the most usual case the arms stretch a certain amount and are not placed close together, so that a condition results similar to that shown, in an exaggerated manner, in Fig. 198. Here, the arms, though stretching somewhat, do not stretch enough to allow the rim to expand freely, and, therefore, the hoop tension is somewhat less than that in the free ring. The section of rim between each pair of arms is so long that it becomes a beam fixed at the ends and loaded uniformly by the unbalanced centrifugal action, the greatest bending moment being at the arm, and a bending moment of half the maximum occurring at the center of the span. The maximum tensile stress will be the sum of the hoop tension and the tensile stress due to the bending action. The relative values of the hoop tension and bending stress will, evidently, depend upon the amount which the arms stretch. If they should stretch enough, owing to their own centrifugal force, so that the rim expands freely, no bending action will occur; if they are so inelastic as completely to restrain the rim, no hoop tension will be induced, but the full centrifugal force will be applied to bend the rim. With any intermediate amount of stretch of the arms, the rim will be held in equilibrium, partly by the hoop tension and partly by the restraining action of the arms, the latter being a measure of the unbalanced centrifugal force of the rim, and of the bending stress caused thereby. Since the expansion of the rim is directly proportional to the stretch of the arms,

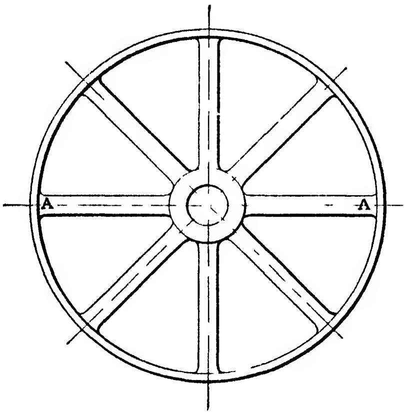

FIG. 197.

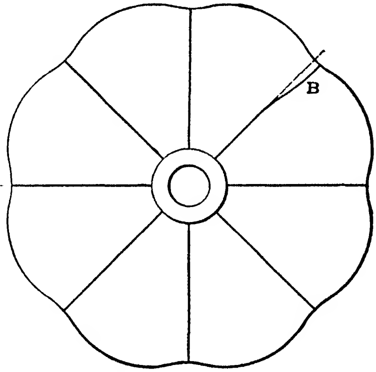

FIG. 198.

it is clear that the hoop tension is also directly proportional to the stretch. If, for instance, the arms stretch one-quarter the amount necessary for free expansion, the hoop tension will be one-quarter that due to free expansion, and the bending stress will be proportional to three-quarters of the centrifugal force of the rim. The mathematical relation which exists between these stresses is complex, and will, of course, vary with the relative size and shape of the rim and the arms. If the rim is of a wide thin section, and the arms are few, the bending stress may be very serious. Professor Lanza* has shown that, with the proportions ordinarily used, the arm, theoretically, stretches about **three-quarters** the amount necessary for free expansion. It is also to be noted that if the wheel is to act as a band wheel, and has a wide thin rim, the bending action on the arms as at B, Fig. 198, still further distorts the rim and increases the bending on the forward side.

* *Trans.* A.S.M.E., vol. 16, page 208.

Let D = the mean diameter of the rim in feet.
R = the mean radius of the rim in feet.
t = the thickness of the rim in inches.
v = the velocity of the rim in feet per second.
w = the weight of the material per cubic inch.
l = the length of the rim between arms in inches.

Consider a section of the rim 1 in. wide on the face. The centrifugal force per unit of length (1 in.), circumferentially, of this section is $f = wtv^2/Rg$ and, therefore, by Article 101, the total load which tends to separate such a ring along a diameter is $wtv^2/Rg \times 12D$, and the unit stress in the section, if no bending exists, is therefore,

$$s_1 = \frac{12wtv^2D}{2tRg} = \frac{12wv^2}{g} = \frac{v^2}{10}\text{, nearly, for iron wheels} \qquad (5)$$

The maximum bending moment in the rim occurs at the arms, and its value is $M = fl^2/12$,* considering the rim as a straight beam. The stress due to the bending moment, when no hoop tension exists, is, therefore,

$$s_2 = \frac{Mc}{I} = \frac{fl^2c}{12I} \qquad (6)$$

where c is the distance to the outer fiber, and I† the moment of inertia of the *cross-section of the elementary ring.*

If now the stretch of the arms be taken as three-quarters that necessary for free expansion of the rim, the total unit tension in the rim will be

$$s = \tfrac{3}{4}s_1 + \tfrac{1}{4}s_2 = \left(\frac{3v^2}{40} + \frac{fl^2c}{48I}\right) \qquad (7)$$

if n be the number of arms in the wheel, $l = 12\pi D/n$, and if the cross-section of the rim be rectangular $c/I = 6/t^2$, whence equation (7) reduces to

$$s = \left(\frac{3v^2}{40} + \frac{3Dv^2}{tn^2}\right) = 3v^2\left(\frac{D}{tn^2} + 0.025\right) \qquad (8)$$

For $t = \frac{9}{16}$, $v = 88$, $D = 4$ ft, and $n = 6$, Professor Lanza found the stress due to hoop tension = 575 and the stress due to bending = 5060,

* See Table I, case 17.

† It should be noted that this I is for a unit (1 in.) of width of rim and not for the entire cross-section.

or s, the total stress, $=5635$. For the same data, equation (8) gives $\frac{3}{4}s_1 = 581$ and $\frac{1}{4}s_2 = 4600$, or a total stress $s = 5181$, which agrees quite closely.

The above equation may be used for roughly checking the allowable stress in flywheel rims, but implicit faith must not be placed upon it for the reasons given in the first paragraph of this article, and all results obtained from this or similar formulas should be checked by successful practice wherever a doubt arises. The equation does, however, show clearly that in wheels having thin rims, or few arms, the bending stress is much greater than that due to hoop tension, and care should be exercised accordingly when such wheels must run at high speed. Equation (5) is often taken as a basis for the design of flywheels, a large factor of safety being used therewith, to cover uncertainties. If s_1 in equation (5) be taken as 1000 (a factor of safety = 20), then $v = 6000$ ft per min, and this is found to be a safe peripheral speed for ordinary cast-iron wheels. It is to be noted, however, that this speed is safe only because experience has shown it to be so, and not, as will be seen, because the stress is necessarily as low as 1000 lb.

Example. Compute the stress in the rim of the cast-iron flywheel discussed in Example 2 of Article 230, assuming that the arms stretch three-quarters the amount necessary for free expansion of the rim. Here

$$n = 6, \quad t = 2.33, \quad D = 8, \quad \text{and} \quad v = \frac{4000}{60} = 66.6$$

$\therefore$ from (8),

$$s = 3v^2\left(\frac{D}{tn^2}+0.025\right)=3\times 66.6^2\left(\frac{8}{2.33 \times 6^2}+0.025\right)=1590 \text{ lb per sq in.}$$

The stress, if based on equation (5), would be 444 lb per sq in.

When a flywheel is being accelerated from rest, or when the energy supply is suddenly cut off, as it may be in a steam engine, the arms may be called upon to carry the full torque load. Each arm of a wheel with a very stiff rim approximates a cantilever beam fixed at one end, free but guided at the other, and carrying a concentrated load at the free or rim end (see Table I, case 7). If the rim is thin and flexible, the arms approximate a simple cantilever loaded at the free end. In addition, the arm is subjected to a tensile stress due to the centrifugal action of its own weight, and that part of the rim which it supports, so that apparently equation M (Table VI) applies. The direct stress is difficult to compute, however, and since the bending stress in the simple cantilever is twice that of a cantilever with the free end guided, it is considered sufficiently accurate to compute the arm as a simple cantilever and neglect the direct stress.

Let P = the greatest force due to the net belt pull at the rim.
a = the length of the arm.
n = the number of arms.

Then from J, Table VI:

$$s = \frac{Pac}{nI} \tag{9}$$

from which the stress, s, or the moment of inertia, I, may be determined. The stress allowed should not exceed 2000 lb per sq in., for cast iron, on account of the uncertainties of the material, and a lower value is sometimes desirable. The statement sometimes made that the arms should be as strong against bending stress as the shaft is against torsional stress is misleading as, in general, shafts are designed for *stiffness*, and not for torsional strength. The shaft of a steam engine may have to be very large to avoid excessive deflection and, as a consequence, may have great excess of torsional strength.

233. Construction of Wheels. Flywheels and band wheels, for velocities below 5000 ft per min, are usually made of cast iron on account of low cost. For higher velocities steel castings are used, and in extreme cases wheels made of steel plates, or wire-wound wheels, have been constructed. Equation (5) may be written

$$v = 1.64\sqrt{\frac{s_1}{w}}$$

The allowable unit tensile strength divided by the weight per cubic unit is, therefore, a measure of the value of the material for this purpose. For this reason some woods are superior to cast iron for wheel rims, and cast-iron wheels which have burst have been successfully replaced with wheels having rims made of wood.* As has been noted, however, wheels made of cast iron are cheaper for ordinary speeds, and for very high speeds steel is preferable for reasons of strength.

Difficulties in transportation limit the diameter of wheels cast in one piece to about 10 ft, and the diameter of wheels cast in two parts to about 20 ft. Wheels from about 16 ft in diameter upward are usually made in several sections. Small flywheels and band wheels are usually cast in one piece, or made in two parts for convenience in erecting. In either of the latter cases, unknown shrinkage stresses will most probably exist. These shrinkage stresses are sometimes relieved by casting the

* *Trans.* A.S.M.E., vol. 13, page 618.

hub in several pieces, each piece being cast integral with one or more arms. The openings between the parts are afterward filled with lead, and rings are shrunk upon the hub to hold the parts in place. Experience shows that solid cast-iron wheels, when properly proportioned, are safe up to 6000 ft per min, which, fortunately, is also about the limit of efficient belt speed. If, however, the wheel has a very wide, thin rim it cannot be considered safe at this speed, particularly if balance weights are attached to the rim between the arms, thus increasing the centrifugal bending force. If joints exist in the rim, their relative strength must be considered. Band wheels of wrought-steel construction can now be obtained up to about 4 ft in diameter; they are light and strong, and are rapidly coming into favor.

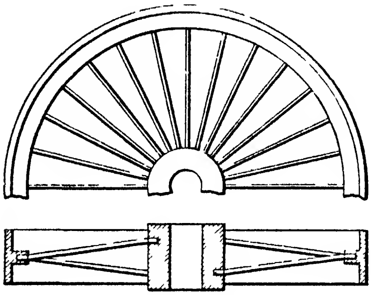

FIG. 199.

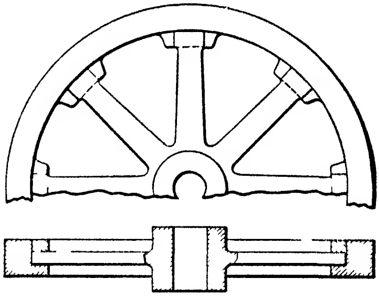

FIG. 200.

Where speeds above 6000 ft per min are necessary, wheels such as shown in Fig. 199 are sometimes built. Here the rim and hub are of cast iron, each a separate casting, and the spokes are of steel. The spokes are placed in the mold, and the metal poured around them, so that on cooling they are gripped very firmly. The spokes are placed close together so that there is practically no bending of the rim, and the rim is also prevented from expanding freely. Wheels of this construction are used for large band saws at velocities above 10,000 ft per min under heavy service, with perfect success.

In Fig. 200 the rim is cast separately in one or more pieces. The arms do not constrain the rim radially, but leave it free to expand. The stresses in the rim, when cast in several pieces so that shrinkage is not a factor, are those due to centrifugal force only, and the arms are simple cantilevers. Wheels of this character have been used with success in rolling-mill work.* Figures 199 and 200 illustrate wheels which correspond closely to the limiting types discussed in Article 232. The con-

* *Trans.* A.S.M.E., vol. 20, page 944.

struction of most wheels lies between these types. Figure 201 illustrates a band wheel with the arms and hub cast in one piece and the rim in sections. The joints in the rim are simple flange joints, placed midway between the arms. This is the most dangerous location possible, on account of the added bending effect due to the centrifugal force of the flanges which add to the mass without contributing to the strength. The best location is at the arm, and many wheels are built thus, the arm being bolted to each segment, and the segments themselves bolted together as well. Where the joint is placed between the arms, it should be about one-quarter the length of the span away from the arm, as at A, Fig. 201, where, by the theory of elasticity, the bending moment is zero. Figure 202 shows a heavy flywheel with an arm and a segment of the rim cast together. The arms are secured in the hub by means of fitted bolts. The hub may be solid, or the flange on one side may be

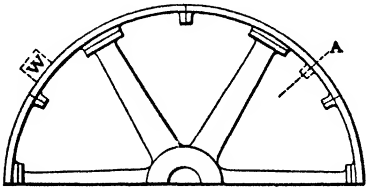

Fig. 201.

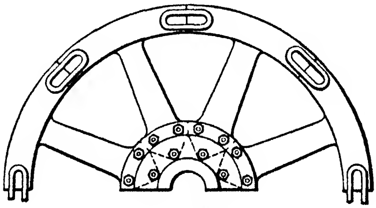

Fig. 202.

movable axially so as to clamp the arms more firmly. The segments are held together at the rim by means of links of rectangular cross-section shrunk in place. This construction is very common. Occasionally, links are also shrunk into recesses on the outer face of the wheel. In Fig. 203 the segments are held together by T-headed links, sometimes called "prisoners," shrunk in place. The segments are joined at the arms, which are fastened to them by through bolts. This construction is simple and the machining is easier than with flanged connections. The construction of the hub is similar to that in Fig. 202.

It is evident that the manner of joining segments in built-up wheels is most important. Wheels seldom fail at the hub. Wheels with thin, wide sections are almost always joined by flanges as shown in Fig. 205. When such joints are used they should be well ribbed for stiffness, as indicated, and the bolts should be placed as near the rim as possible, so that the lever arm, a, shall be as great as possible compared to the arm b (see Article 141). A much better arrangement is shown in Fig. 206, where an arm is placed on each side of the joint. This is particularly applicable to wheels cast in two parts. It may be noted that thin rims are often stiffened by light *circumferential* ribs at the outer edges.

Mr. A. K. Mansfield has pointed out (*Trans.* A.S.M.E., vol. 20) that these ribs *may* be a source of weakness. The greatest bending moment is near the arm, where these ribs are on the *tension* side of the beam. A rim having such ribs is not necessarily as strong against bending in this direction, as one of rectangular cross-section having the same area; and when ribs are used the section modulus should be calculated.

The prisoner link shown in Fig. 203 has certain advantages over the link shown in Fig. 202. It is evident that the depth of the recesses in

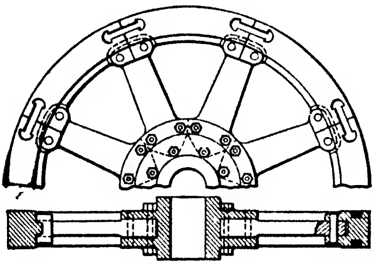

Fig. 203.

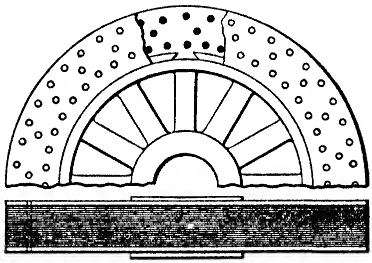

Fig. 204.

Fig. 202 is limited, while in Fig. 203 the slot can extend entirely across the section and the link can be made as wide as the rim itself. Furthermore, it is possible to machine both wheel and link in Fig. 203 accurately, which is difficult to do with the construction in Fig. 202. This permits of greater accuracy in computing the initial stress induced in the link by shrinking it in place, the importance of which has been noted in

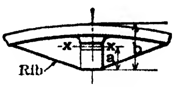

Fig. 205.

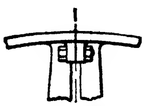

Fig. 206.

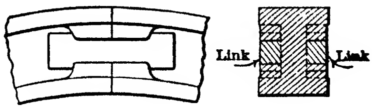

Fig. 207.

Article 156. If the rim be made I-shaped,* as in Fig. 207, the links can be so proportioned that the joint will be as strong as the rim proper or even stronger.

While, evidently, the relative strength of the joint compared to the solid rim will vary with the exact proportions selected, average practice gives the following apparent values:

* See *Trans.* A.S.M.E., vol. 20, page 944.

Flanged joint, bolted, midway between arms	0.25
Flanged joint, bolted, at end of arms	0 50
Linked joint as in Fig. 202	0 60
Linked joint as in Fig. 203	0 65
Linked joint as in Fig. 207	1 00
Solid rim	1.00

It must not be inferred from the above that a solid rim is necessarily the best; obviously, a wide thin rim with unknown shrinkage stress may not be as safe as a narrow deep rim of the same sectional area if held together by a good joint.

For extreme velocities, wheels built up of steel plates, or wheels with rims made of plates fastened to a central spider made of steel castings, are now used. Figure 204 shows a flywheel of the latter type used in rolling-mill work (see *Power*, Feb. 4, 1908). The rim is made of laminations held to the spider by dovetails, as shown, the laminations being assembled with overlapping joints. Heavy outside plates clamp the whole structure together by means of through bolts. In the particular case noted above, the velocity of the wheel rim is 250 ft per sec. Descriptions of a number of examples of such wheels are to be found in the *Transactions* A.S.M.E., the magazine *Power*, and other periodicals. Wheels for great speed have also been constructed by winding the rim with many turns of steel wire.

The rotors of some forms of electric generators, steam turbine rotors, and similar revolving members are often loaded as shown at W, Fig. 201. Such loads add to the centrifugal force acting on the rim, but do not add to the strength of the rim. Due allowance should be made in such cases, particularly if the load or loads are placed near a joint as shown in Fig. 201. The teeth of gear wheels constitute such a load, and if the wheel is large, and the peripheral speed high, this should be considered. Balance weights, placed between the arms, should be carefully considered, especially when the rim is thin and the velocity high.

234. Experiments on Flywheels. The best experimental data upon the strength of flywheels are from tests conducted by Professor Benjamin and reported to the A.S.M.E.* While these experiments were made to determine the *bursting speed* of small cast-iron wheels only, and throw no light on the increase of stress with an increase of speed, they are very valuable as indicating the manner in which various types of wheels fail. As they were conducted on small wheels, due allowance must be made for the difference in quality between the metal of small and large castings in estimating probable bursting stresses. These experiments go to show that solid cast wheels will burst at a peripheral

* See *Trans.*, vols. 20 and 23. See also "Machine Design," by C. H. Benjamin.

velocity somewhere near 400 ft per sec. A factor of safety of 3 on the bursting speed corresponds to a factor of safety of about 9 on the stress. Such a factor of safety would permit a peripheral velocity of 135 ft per sec. It is common practice to limit the rim speed of such wheels to 100 ft per sec. The bursting speed was about 300 ft per sec for two-piece cast-iron wheels with thick heavy rims and about 225 ft per sec for two-piece wheels with wide thin rims divided along a diameter through diametrically opposite arms. With the same factor of safety as in the foregoing the corresponding safe peripheral velocities would be 100 and 75 ft per sec, respectively. Rim joints midway between the arms, particularly the common flange joints, were found to reduce the strength materially. The strength of various joints was found to be about as tabulated in Article 233. Extra loads, such as balance weights located between the arms, were found to be very dangerous, on account of the added bending effect.

235. Rotating Discs. If the radial depth of a wheel rim be great compared to its diameter, the equations deduced in the preceding articles do not apply, the difference being analogous to that existing between thick and thin cylinders. Mathematical analysis of the stresses in a rotating disc, in common with those existing in thick cylinders under internal pressure, are complicated and not altogether satisfactory. Experimental data, corroborating the theories, are also lacking. A full mathematical treatment of these stresses is beyond the scope of this treatise, and only enough will be inserted to show the general character of the problem.

When a disc of uniform thickness is rapidly rotated on its axis, the principal stresses induced are a tangential tension, and a radial stress, at every point in the disc.

Let r_2 = the outer radius of the disc in inches.
r_1 = the inner radius of the disc in inches.
r = the radius at any point.
λ = Poisson's ratio = 0.3 for steel and 0.25 for cast iron.
N = revolutions per minute.
w = weight of 1 cu in. of the material.
s = the tangential stress at any radius r.
s' = the radial stress at any radius r.

Then it can be shown* that, for a flat disc of uniform thickness, having a hole at the center of radius r_1,

* See "Theory of the Steam Turbine," by A. Jude, pages 237 and 249. The notation and units have been changed to correspond to those used in this text.

$$s = 0.00000355wN^2\left[(3 + \lambda)\left(r_2^2 + r_1^2 + \frac{r_2^2r_1^2}{r^2}\right) - (1 + 3\lambda)r^2\right] \quad (11)$$

and

$$s' = 0.00000355wN^2\left[(3 + \lambda)\left(r_2^2 + r_1^2 - \frac{r_2^2r_1^2}{r^2} - r^2\right)\right] \quad (12)$$

For a solid disc,

$$s = 0.00000355wN^2[(3 + \lambda)r_2^2 - (1 + 3\lambda)r^2 \quad (13)$$

and

$$s' = 0.00000355wN^2[(3 + \lambda)(r_2^2 - r^2)] \quad (14)$$

It is to be noted that the radial stress is less at any point than the corresponding tangential stress; and an examination of equation (11) shows that this tangential stress is a maximum at the surface of the bore and a minimum at the outer periphery. At the surface of the bore, or where $r = r_1$, the stress

$$s = 0.00000355wN^2[(3 + \lambda)(2r_2^2 + r_1^2) - (1 + 3\lambda)r_1^2]$$

If now r_1 be taken so small that r_1^2 is negligible, it appears that the tangential stress is

$$s = 2 \times 0.00000355wN^2[(3 + \lambda)r_2^2]$$

which is just twice that obtained by making $r = o$ in equation (13). The effect of even a very small hole at the center of a rotating disc is, therefore, greatly to increase the stresses.

Example. A circular steel saw $\frac{1}{4}$ in. in thickness and 80 in. in diameter has a hole 4 in. in diameter in the center and runs at the rate of 500 rpm. Determine the tangential stress at rim and also at the hole.

Here $N = 500$, $w = 0.28$, $\lambda = \frac{1}{3}$, $r_2 = 40$ and $r_1 = 2$. Whence in (11), making $r = r_2 = 40$, the tangential stress at the rim is

$$s = 0.00000355 \times 0.28 \times 500^2[(3 + \tfrac{1}{3})(40^2 + 2^2 + 2^2) - (1 + 1)40^2] = 535 \text{ lb per sq in.}$$

and at the hole, making $r = r_1 = 2$,

$$s = 0.00000355 \times 0.28 \times 500^2[(3 + \tfrac{1}{3})(40^2 + 2^2 + 40^2) - (1 + 1)2^2] = 2643 \text{ lb per sq in.}$$

It is sometimes desirable to design a revolving disc in which the stresses are uniform at every point. From the foregoing it will be clear that such a disc must be of varying thickness and, in general, thicker at the hub than at the rim.

Let t_1 = thickness of disc at hub.
t_2 = thickness of disc at rim.

t = thickness of disc at any point.
e = Napierian base.
ω = angular velocity in radians.

Then the following equation is given by several authors,*

$$t = t_1 e^{\frac{-w\omega^2 r^2}{2gs}} \qquad (15)$$

Obviously, if t_2 is fixed by practical conditions, the corresponding value of t_1 can be computed from equation (15), by making $r = r_2$ and $t = t_2$, and the outline of the disc can then be determined.

Mr. Jude, commenting on equation (15), says that its accuracy is open to grave doubt because of the neglect of certain variables in deriving the expression. It should be remembered also that all these equations, (11) to (15), are deduced upon the hypothesis that the material is perfectly elastic and homogeneous. It is clear that they cannot be applied intelligently to built-up wheels of the disc type, and must be applied with caution to brittle materials. They are of value, however, in showing the general character of these stresses and in indicating the general outline of discs with approximately uniform stress. For complete mathematical analysis of discs of different shapes and discs carrying peripheral loads, reference may be made to the various works on the steam turbine. It is evident that great care should be used in selecting and working the material for high-speed discs. Rolled sheets are not good for very high speeds on account of their seamy structure, which is conducive to incipient cracks, and cast materials of brittle structure must be of first-class quality. Discs forged down from much thicker ingots give the safest construction.

Discs rotating at high speeds and particularly turbine discs and others that are loaded intermittently are subjected to severe vibratory stresses. A discussion of this phenomenon is beyond the scope of this treatise. Reference is made in particular to the work of Mr. A. L. Kimball in this field.

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* See "Theory of the Steam Turbine," by A. Jude, page 242; and also "Strength of Materials," by Ewart S. Andrews, page 562.

CHAPTER XIX

MACHINE FRAMES AND ATTACHMENTS

236. Stresses in Machine Frames. Since machine frames must, in general, receive the reactions from the forces applied to the various moving members by the energy transmitted, it is obvious that most of the stresses induced in frame members are very complex and beyond mathematical analysis. If it is essential that the moving members be held in accurate alignment, as in machine tools, the predominating requirement for the frame is *stiffness* and not strength. For these reasons the design of machine frames, in general, must be governed largely by judgment and experience, the cases where complete mathematical analysis is possible being rare. However, even where judgment must be the guide, it is not only helpful, but sometimes necessary, to check, as closely as possible, the stresses in certain important sections, by applying those fundamental formulas of Table VI, page 93, which apparently fit the circumstances. In all cases, what may be termed a "qualitative analysis" of the frame is very desirable as a guide in properly distributing the material and in determining the *forms* of the various sections.

If the character, value, and line of action of every force acting upon a given section are known, the stresses in the section can be determined by applying the fundamental requirements for static equilibrium of the section, namely:

(*a*) The algebraic sum of all horizontal component forces must $= 0$.
(*b*) The algebraic sum of all vertical component forces must $= 0$.
(*c*) The algebraic sum of all the moments must $= 0$.

The stress, in any direction, at any point, will be the algebraic sum of all the stresses acting in that direction, at that point, as found by applying (*a*), (*b*), and (*c*). It is impossible to make a classification of machine frames that would be of any service, but the principles will be illustrated by applying them to typical cases. It is to be noted that it is seldom possible to find the required dimensions of a section directly, by solving the particular equations from Table VI which apply; but, in general, the section must be assumed from the conditions given, and then checked for strength or stiffness.

Figure 208 illustrates a type of frame which is quite common and known as an **open frame.** It is one of the few types where a mathematical analysis can be made with some degree of completeness. In the punching-machine frame illustrated in Fig. 208, great stiffness is not essential and the design may be based on the strength required. Suppose the frame to be outlined as shown, so that the dimensions of the cross-section at any place may be assigned. Evidently, if the stresses are checked at the sections *BC*, *DE*, and *FG* the strength of the frame will be fully determined.

In the section *BC*, whose gravity axis is at O_1, consider the portion of the frame above *BC* as a **free body.** It is in equlibrium under

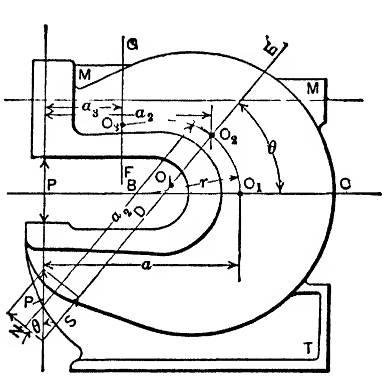

Fig. 208.

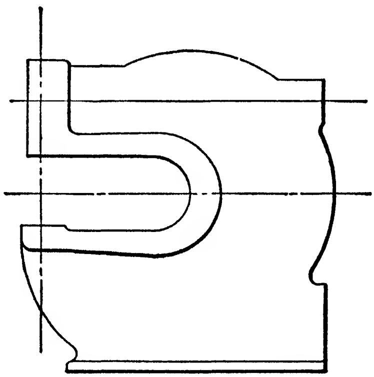

Fig. 209.

the action of the *exterior* force *P*, due to punching, and the *internal* forces exerted upon it by the lower half of the frame. There are no horizontal forces. The vertical force *P* must be balanced by an equal and opposite force at the section *BC*, which induces a tensile stress uniformly distributed over the section, the intensity of which is

$$s_1 = \frac{P}{A} \text{ lb per sq in.}$$

where *A* is the area of the section. The only moment acting on the part is *Pa*, due to the action of *P*, which tends to rotate the upper part of the frame around O_1, the gravity axis of the section, causing a resisting tension, s_2, at *B*, and a resisting compression at *C*. Frames of this kind, as usually designed, approximate curved beams in their outline, and, therefore, equation (M_1), of Article 21, applies; hence,

$$s = s_1 + s_2 = \frac{P}{A}\left[1 + \frac{a}{R - c_t}\left(\frac{c_t}{R} \times \frac{A'}{A' - A} - 1\right)\right]$$

using the same notation as in that article.

If the locus of the gravity axes of cross-sections of the frame, normal to the inner surface, is plotted, it forms a curved surface whose radius of curvature constantly changes in length. The curve O_1, O_2, O_3 in Fig. 208 indicates such a locus. Consider the section DE, which is approximately normal to the inner surface; it passes through the gravity axis O_2, and includes the instantaneous radius of curvature r_2, whose center is at O_4. The difference between this section and one truly normal to the inner surface through O_2 is not very great. Consider, again, that the portion of the frame to the left of the section DE is a free body. There are no horizontal forces. The external load P is held in equilibrium by the induced stresses distributed over the section DE. The action of P is equivalent to the action of the two component forces $N = P \cos \theta$ and $S = P \sin \theta$. The component N is held in equilibrium by a uniformly distributed stress

$$s_1 = \frac{N}{A_2} = \frac{P \cos \theta}{A_2}$$

and by a stress couple balancing the moment Na'_2 or

$$Na'_2 = P \cos \theta \times \frac{a_2}{\cos \theta} = Pa_2$$

The component $S = P \sin \theta$ is balanced by a shearing stress that is not distributed uniformly over the section. At the outer fibers this stress is a minimum. At the juncture of the side webs with the flanges of the section there is a rapid increase in the intensity of the stress and then a rather slow increase to a maximum intensity at the neutral axis which, it should be noted, in a curved beam does not coincide with the gravity axis. The maximum intensity of the shear stress at the neutral axis may be approximated by dividing $S = P \sin \theta$ by the area of the section of the side webs. This value may also be taken as representing, roughly, the value of the shear stress at the junction of the side webs with the flanges. As the intensity of the shear stress is small at the outer fibers the maximum induced tensile and compressive stresses are not affected by it. With the ordinary proportions of web and flanges, this shearing stress may be neglected. But since the value of the shearing stress varies with the width* of the cross-section, care should be used that the web is not made unduly thin near the gravity axis, or failure may result from the shearing stress, particularly if the sides of the frame are used as a support for attachments.

By equation (M) the maximum induced tensile stress equal to the sum of the direct stress s_1 and the maximum bending stress s_2 due to Pa_2 is

$$s = s_1 + s_2 = \frac{P}{A_2}\left[\cos\theta + \frac{a_2}{R - c_t}\left(\frac{c_t}{R} \times \frac{A'_2}{A'_2 - A_2} - 1\right)\right]$$

where c_t, A_2, A'_2 refer to the section DE, and θ is the angle between the section DE and the horizontal.

Consider, lastly, the section FG. As before, there are no horizontal forces and the vertical force P is held in equilibrium by a shearing stress, which is not distributed uniformly over the section, and a stress couple due to the moment Pa_3. The intensity of the shearing stress may be approximated as indicated in the foregoing. If the section FG is considerably smaller than section DE, it may be well to so check it. As section FG is well beyond the curved portion of the frame the maximum tensile and compressive stresses may be computed by equation (M) of Table VI.

In unimportant cases where only approximate results are desired, or where the radius of curvature is great, or where a large factor of safety can be applied, equation (M) of Table VI is often used, because of its comparative simplicity, to check selections such as BC and DE. It should not be relied upon, however, where accurate results are desired and the curvature of the beam is appreciable.

Figure 210 illustrates an open frame as applied to a power riveter. The rivet which is to be "driven" is placed between the dies, D and D_1, and pressure is applied to the movable die, D, by means of the power cylinder, R. The pressure which is applied may be very great (150 tons or more), and unless the jaws are properly designed they may spring so much that the dies will fail to align properly, and faulty work will result (see Article 126). Stiffness and not strength is, therefore, the essential factor in the design, for if the parts are stiff enough they will, in general, be strong enough. The yielding which most affects the alignment is that due to the bending of the frame, B, and the stake, C, and that which may result from the elongation of the bolts which hold these members together. When the riveting

* See "Strength of Materials," by Arthur Morley, page 136; and also "Materials of Construction," by G. B. Upton, page 67.

pressure, P, is applied, the beams, B and C, tend to rotate around the point O, this tendency being resisted by the tension in the bolts. The load which may be applied to the bolts by the force P will be $P_1 = P(a + b)/b$. If the nuts on the bolts are set up so that a combined total initial tension somewhat greater than P_1 is induced in the bolts, the stretching of the bolts, and the consequent opening up between the frame and the stake, will be negligible. (See Article 138 and Fig. 114 and also Article 156.) The *intensity* of stress in the bolts should not exceed 6000 lb per sq in. The upper part of the frame, B, approximates a cantilever of *uniform strength* of length a. (See Article 16 and

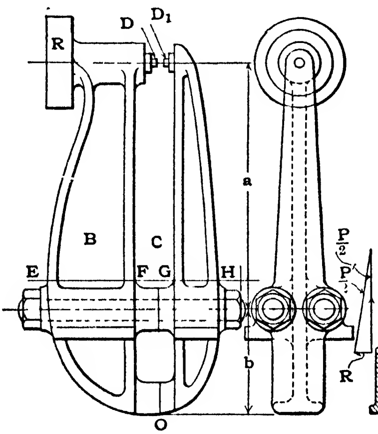

Fig. 210.

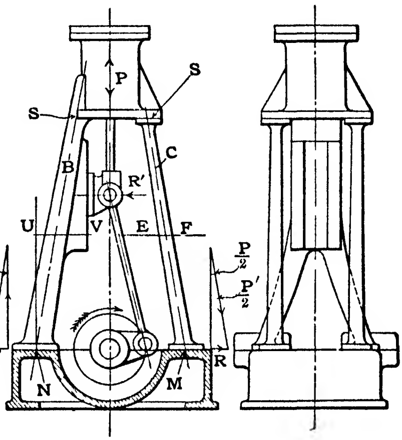

Fig. 211.

Case 1 of Table II.) The maximum deflection which occurs at D may, therefore, be computed, and the maximum stress which occurs at EF may be checked by equation J of Table VI. The stake, C, approximates a cantilever of *uniform cross-section,* and may therefore be treated in a similar manner. (See Case 1, Table I, and equation J, Table VI.)

Figure 211 illustrates a **closed frame** as applied to a vertical steam engine. The back column, B, which carries the crosshead guide is of cast iron, while the front columns, C, are of steel. It is required to check the stresses in these columns when the piston is ascending and also when it is descending, the rotation of the engine to be taken in a clockwise direction as indicated.

When the piston is ascending, the steam pressure tends to draw the cylinder and bed closer together. This tendency is resisted by P', the combined thrust on all the columns, the vertical component of which must equal P, the total steam pressure on the piston. It may be reasonably assumed that the back column carries one-half of the total thrust, and that each of the front columns carries one-quarter. The thrust of the back column, $P'/2$, may be resolved into components perpendicular and parallel to the face of the foot. The vertical component will equal $P/2$. The horizontal component, R, tends to spread the foot of the column outward and induce a bending stress in it. The column should, therefore, be secured to the bed by fitted bolts, or, if the bolts are loose in the holes, the foot should be well doweled to the bed; or, better still, the foot should fit against a ledge cast on the bed plate. R will then be balanced by an equal and opposite reaction at the feet of the front columns, thus setting up a negligible tension in the bed and leaving a compressive force only on the column. By similar reasoning, each front column is subjected to a compressive load $P'/4$, and the total horizontal component, R, is balanced by that of the back column through the bed.

The tension or compression in the piston-rod and connecting-rod either ascending or descending, has a resultant, R', normal to the guide, which may have a large value where the connecting-rod is short compared to the crank. This resultant tends to bend B, and hence C also, in a left-hand direction, the bending being resisted by the fastenings at the feet. The columns and cylinder, however, constitute a very stiff structure, and, except where the frame is made up of light construction, this effect may be neglected. This reaction, R', however, also bends the column B locally, that is, as a beam encastré at S and N, the effect of R being greatest when the crosshead is near half stroke. (See Case 18, Table I.) If then it be desired to check the central section UV of the column, the long column stress due to $P'/2$ must be added to the flexural stress due to R'. The sum of these stresses should not, of course, be greater than the allowable stress for the material used. The columns, C, need be checked only as long columns (see equation (N_2), Table VI).

When the piston is descending, the steam pressure tends to separate the bed and the cylinder. The reactions at M and N are reversed in direction and the columns are put in tension, the horizontal components inducing negligible compression in the bed. The most dangerous section in this case will be under R', and the stress will be that due to $P'/2$ plus the tensile stress due to the bending effect of R'. The fastenings of the columns to the cylinder and to the bed plate must, of course, be sufficiently strong in tension to resist the force tending to separate the cylinder and bed.

In the foregoing examples, the lines of action of all forces, acting on the section considered, lay in a *plane* of *symmetry* of the section, and the section tended to rotate around a gravity axis at right angles to this plane. This is the most usual case, but occasionally the force or forces acting are not in a plane of symmetry. Thus, Fig. 212 may represent the cross-section of the column of a radial drilling machine, in which it is required to check the stresses when the force P, due to drilling, is in the position shown. If C be the center of gravity of the section, the tendency to rotate will be around the axis $X'X'$ at right angles to PC, the arm of the force P, and the resistance of the section against such rotation will be measured by the moment of inertia of the section with reference to this axis. The maximum tensile and compressive stresses will occur at the fiber farthest removed from $X'X'$ or at M and N, the stress at M being tensile when the direction of P is upward to the

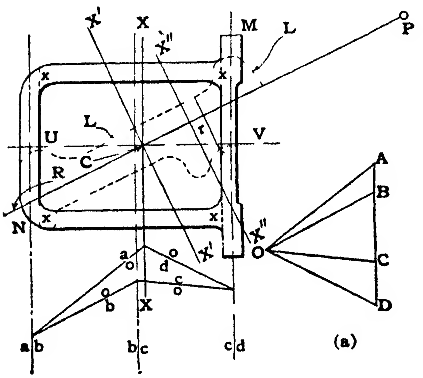

FIG. 212.

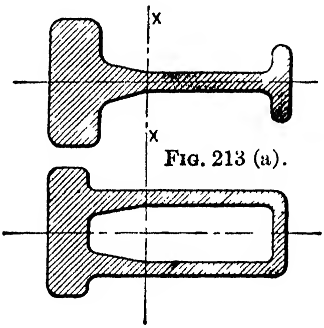

FIG. 213 (a).

FIG. 213 (b).

plane of the paper, and compressive when its direction is downward. The center of gravity, C, may be located readily, by finding the intersection of any pair of gravity axes. If the section has an axis of symmetry, as UV, Fig. 212, it is necessary only to find the axis at right angles to UV. This is most readily done graphically as follows: Divide the section into small areas, as shown by dotted lines at xx in Fig. 212. From the center of gravity of each area draw parallel lines ab, bc, cd, preferably at right angles to the known axis UV. In Fig. 212 (a), lay off AB, BC, etc., proportional to the respective areas whose gravity axes are ab, bc, etc. Take any pole O and draw AO, BO, etc. From any point on ab, draw ao indefinitely, parallel to AO. From the same point draw ob parallel to OB. From the intersection of ob and bc draw co parallel to CO, and from its intersection with cd, draw od parallel to OD. The intersection of ao and od locates the gravity axis XX (see also Article 80). It is evident that this method may be applied when both axes are unknown.

The moment of inertia of the section around $X'X'$ may be most readily found by transforming the area of the figure into an *equivalent figure* with RP as a base, as follows: Draw lines, as $X''X''$, parallel to $X'X'$, and plot the intercepts made by it on the given section, on each side of CB as ordinates of an equivalent section, shown in Fig. 212 by the dotted line L. The accuracy of the work may be checked with a planimeter, as it is evident that the area of the transformed section will be equal to that of the original. Divide this equivalent figure into approximate rectangles, by lines drawn parallel to $X'X'$, as shown at r. Then the moment of inertia of r around the axis $X'X'$ will be its moment of inertia round its own gravity axis parallel to $X'X'$, plus its area into the square of the distance between these axes. The sum of the moments of inertia of all such small areas will be the required moment of inertia of the section.

237. Distribution of Metal in Frames. Machine frames are usually made of castings, on account of their complicated shapes, cast iron being the material most used, though steel castings are rapidly coming into use for severe work. In addition to the stresses induced in the frame by the energy transmitted by the machine, it may also be subjected to severe *accidental* stresses due to such causes as shrinkage, or the settling of a part of the foundation. Both these classes of stresses are, in general, very complex and generally beyond mathematical analysis, and the problem must frequently be left to the judgment of the designer, especially if stiffness is a large factor. Economy in the use of metal, however, demands that its distribution throughout the frame shall be in accord with the best analysis possible, and, therefore, the general principles governing the forms of sections must be kept in mind.

The most trying stresses to which a frame may be subjected are torsion, flexure, or a combination of these. It has been noted in Article 13 that the hollow section (Fig. 7) is most effective for resisting torsion, and, if this be the predominating stress, sections such as are shown in Fig. 7, or modified sections as shown in Fig. 212, are correct. It was also noted in Article 20 (Fig. 10) that with cast iron, or other metal whose tensile strength is much less than its compressive strength, a great saving of material is effected by massing the metal on the tension side as shown in Fig. 213 (a), thus making the tensile and compressive stresses more in proportion to the strength of the material. If then the predominating action on a frame is simple flexure (in a given plane), a section like that shown in Fig. 213 (a) is allowable, but if, in addition, torsional action must be withstood, or if the plane of flexure may change, a section similar to that shown in Fig. 213 (b) is better design, since it combines the merits of both Figs. 212 and 213 (a). Sometimes it is better to make the section so large that the flexural stress can be safely withstood by a wall of uniform thickness, as in Fig. 212, as the construction of the pattern is simpler and the shrinkage stresses less serious than in such sections as shown in Fig. 213. The metal in the walls will be much sounder, also, as the thick sections of Fig. 213 are very likely to have a porous interior, due to shrinkage. Cast-iron parts more than 4 or 5 in. thick are almost sure to be defective in this manner. Where a considerable change in the thickness of metal in a section is found desirable, as in Fig. 213, the transition from the thin to the thicker portions should be gradual, as shown, and not abrupt, so as to secure sound castings and to minimize the effects of contraction and shrinkage. For the same reasons inner corners should be filleted and outer corners well rounded. Thin, wide flanges or webs should not be cast integral with thick, heavy parts, as unequal shrinkage and porosity are sure to result. This is especially true of thin ribs cast on the tension side of large sections, as the edge of the rib may crack through shrinkage, thus starting rupture across the entire section. Small brackets or other attachments of thin sections should never be cast on a large frame, as they seldom cast well. A section of moderate thickness is often stronger than a thicker one, since the greatest strength of cast iron is in the outer skin. It should also be remembered that, even when a frame is both strong and stiff enough to do the required work at low speeds, it may not have mass enough to absorb the vibrations set up when running more rapidly. This may call for more metal in the frame than is dictated by other requirements. Openings for supporting or removing cores should be placed near the gravity axis so as to reduce the strength as little as possible (see Fig. 213).

238. Attachments and Supports. The general appearance of a machine is affected more by the outline of the main frame than by that of any other member. This outline should, therefore, be clearly shown, and not obliterated at places by the various attachments which restrain the moving parts or support the frame. In Fig. 208 is shown the outline of a frame in which the various sections have been proportioned in accordance with the loads brought upon them, and the various bosses, *M*, and the support, *T*, appear as attachments to the main member. Figure 209 illustrates the same machine with the attachments merged into the main member, thereby destroying the character of the design, and also making it more difficult to judge of the relative strength of various sections of the frame.

The form of an attachment will, of course, be governed by the service it is required to render and the manner in which it is loaded and supported. If the outline of the attachment is based on theoretical considerations, care should be exercised that all the modifying influences are duly considered. Thus, if parabolic outlines are given to an attachment, such as the housings, H, for supporting the tool in Fig. 217, the upper end of the housing must be modified from the theoretical parabolic outline indicated by the bending effect of the force P, so as to provide for the shearing effect at the upper end, which is frequently neglected. (See also Article 16.)

If the frame rests directly on the floor, its outlines should be carried down to the floor in such a manner as will give an appearance of stability. Figure 214 shows such a machine frame on which the vertical outline of the back of the frame is undercut; Fig. 215 shows the same machine with the outline carried straight to the floor, and the improvement in

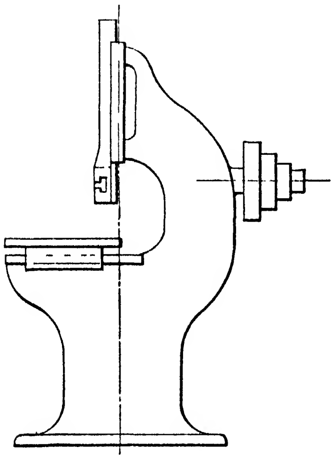

Fig. 214.

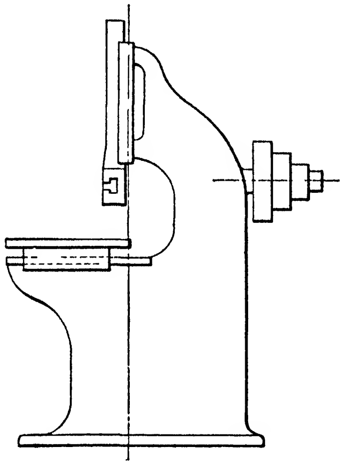

Fig. 215.

appearance, so far as stability is concerned, is obvious. Figure 216 shows the outline of a planing machine in which the upright, U, is carried to the floor at V, in the form of a leg. This construction is not correct, as U is an attachment to the bed, designed to resist the force of the cut and transfer it to the bed, which should itself be stiff enough to withstand all such stress thus brought upon it. Any settling of the foundation might affect the alignment of U, and hence the arrangement shown in Fig. 217 is more nearly correct.

In large machines the frame usually rests directly upon the foundation and should have sufficient stiffness to resist distortion due to the settling of the foundation, which is very difficult to avoid. In smaller machines the frame is carried on supports, which may be of two general types, (a) cabinet or box pillar supports (Fig. 217), and (b) legs as shown in Fig. 218. The choice of support will, of course, depend on the type and size of the machines. In any case the number of points of support should be as few as possible. If the machine can be supported on three points it is evident that the frame cannot be affected by settling of the foundation. It is difficult, in general, to obtain three-point support, but it is seldom necessary to place supports as close together as in Fig. 216 (which is taken from an actual design), where the frame is carried on eight points. Figure 217 shows the same frame properly carried on box supports, the supports themselves being so stiff as materially to assist the frame and practically reducing the support to so-called two-point support. Small machines can often be supported on a single box-pillar, the overhanging parts of the frame having a parabolic out-

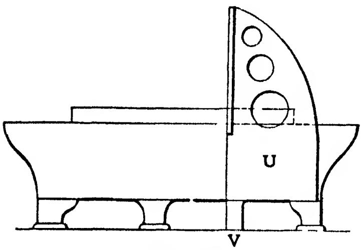

Fig. 216.

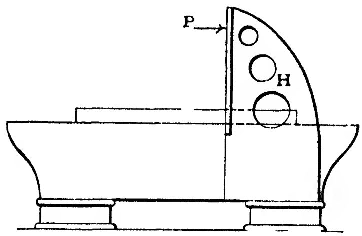

Fig. 217.

line, as suggested in Fig. 215. If the box pillar is of considerable height, the sides should taper slightly toward the top, for if made parallel the pillar will appear wider at the top than at the bottom. It is preferable to use one form of support throughout, i.e., all box pillars or all legs, and not one or more of each.

When the frame must be supported on legs, as in Fig. 220, these should not curve outward as in Fig. 218, unless it is absolutely essential in order to obtain stability. Spreading the legs, as in Fig. 218 lengthens the distance between the reactions, R, R, and, therefore, increases the bending effect on the bed and legs as a whole. The leg shown in profile in Fig. 219 is better and much easier to make. The legs should be so placed that the outline, L, forms a continuation of the principal vertical outline, L', of the frame, as shown in Fig. 219. The same remarks apply to the end view of the legs as shown in Figs. 220 and 221. The complex curves and ornate features of Fig. 220 are not only useless but expensive. It is not always possible or desirable to make machine frames and supports with simple straight line outlines; but where curves are necessary they should be as simple as possible; and in general the best results can be obtained by using arcs of circles or parabolas. Ornamentation of a fanciful nature is not permissible anywhere, as it really detracts from the appearance of the machine and adds to the cost of production. *Harmony* of design can be attained by making the various members of

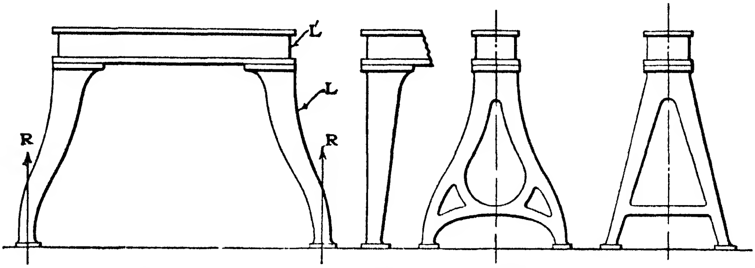

Fig. 218. Fig. 219. Fig. 220. Fig. 221.

correct proportions to withstand the loads brought upon them, and by using the simplest and most direct design with smooth transition curves between straight lines which intersect. It is a proverb in design that "what is right looks right."

CHAPTER XX

BALANCING OF MACHINE PARTS

239. General. Machine members such as reciprocating parts and unbalanced rotating masses tend to set up vibrations in the machines to which they belong. In many cases the inertia forces due to variations of velocity or to changes in the direction of motion are not serious, as they are so small that they are practically absorbed by the framework, and foundation of the machine. Whether vibrating inertia forces may be neglected depends upon the masses involved, the rate of change of velocity, the rate of change in the direction of motion, and the moments of such forces. The effects of such forces are always deserving of consideration and especially so in the case of high-speed machinery. The general subject of vibrations in machines is an extensive one, and only the elementary theory of the balancing of vibratory forces will be considered here. Rotation and reciprocation are by far the most common motions in machines. The more common problems of balancing vibratory forces may, therefore, be treated under two heads—the balancing of rotating parts and the balancing of reciprocating parts.

240. Coplanar Masses: Discs. When a mass that is mounted on a frictionless axle does not, in any position, show a tendency to rotate, it is said to be in **static balance.** A rotating mass that shows no tendency to set up vibrations is said to be in **dynamic** or **running balance.** Consider first an unbalanced mass, W_1, in Fig. 222. Such a body is clearly not in static balance with reference to its axis, OO'. If rotated around the axis, OO', it will develop a centrifugal force, $W_1r_1\omega^2/g$, which will tend to move the axis in a direction radial to the mass, and hence to set up vibrations in the system. Suppose, however, that a mass, W_2, is placed radially opposite to W_1 and that its radius, r_2, is so chosen that $W_1r_1 = W_2r_2$. Since the moments of these two masses balance, the system will be in static equilibrium. Furthermore, if these masses be rotated, their centrifugal forces will balance, since

$$\frac{W_1r_1\omega^2}{g} = \frac{W_2r_2\omega^2}{g}$$

If, therefore, such a set of coplanar forces are balanced statically they are also in balance dynamically, and if balanced dynamically at one speed they are balanced dynamically at any other speed. By similar reasoning it is clear that any number of coplanar forces, W_1, W_2, W_3, W_4,

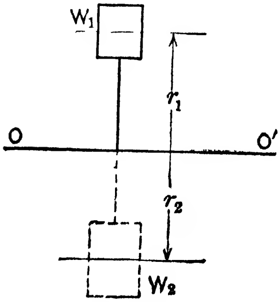

FIG. 222.

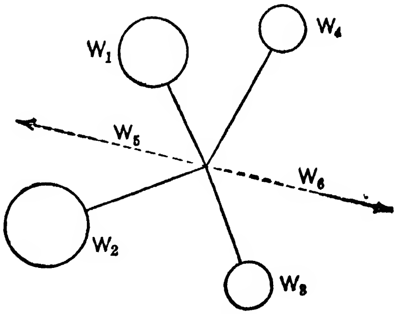

FIG. 223.

Fig. 223, can be balanced statically and dynamically by finding the resultant centrifugal force, W_5, and balancing this force with an equal and opposite centrifugal force, W_6. It follows, also, that thin discs are in dynamic balance when they are in static balance, and that a long cylinder made up of such discs will be in both static and dynamic balance. This property is made use of in the construction and balancing of such structures as the runners of steam turbines.

241. Balancing of Rotating Masses Not in the Same Plane. Quite frequently it is impossible to balance a single mass or a number of coplanar masses by the simple expedient described in the foregoing paragraph; in such cases, the balance weight must be in some other plane, and, in general, two balance weights must be used. Thus, in Fig. 224, let it be required to balance the mass W. So far as centrifugal action is concerned, this may be balanced by a mass W_2 at a radius r_2.

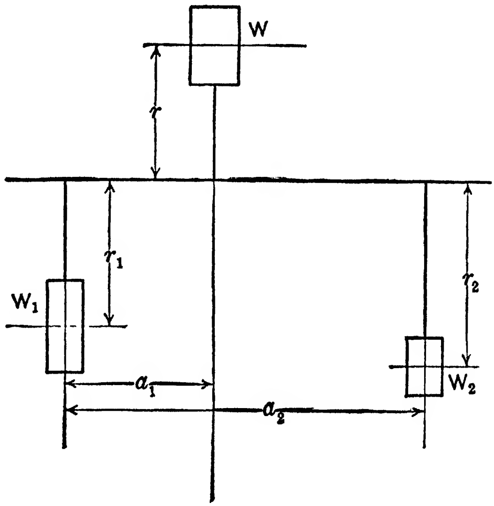

FIG. 224.

But W and W_2 would constitute an unbalanced couple that would vibrate the shaft in the plane of the couple. For dynamic balance the sum of all the forces must equal zero and the sum of all the moments must equal zero. Suppose now another mass, W_1, to be located at a distance a_1 from the plane of W, and let W, W_1, and W_2 be in the same longitudinal plane. Then, if all forces are to balance,

$$(W_1r_1 + W_2r_2 - Wr)\frac{\omega^2}{g} = 0 \quad \text{or} \quad W_1r_1 + W_2r_2 - Wr = 0 \qquad (1)$$

Taking moments around the plane containing W_1,

$$[Wra_1 - W_2r_2a_2]\frac{\omega^2}{g} = 0 \quad \text{or} \quad Wra_1 - W_2r_2a_2 = 0 \qquad (2)$$

Whence, from equations (1) and (2)

$$W_1r_1 = W_2\left(1 - \frac{a_1}{a_2}\right) \quad \text{and} \quad W_2r_2 = W_2\frac{a_1}{a_2}$$

and if r_1, r_2, a_1, and a_2 are assigned, W_1 and W_2 are known. It should be noted that the locations of W_1 and W_2 are arbitrary, the effect of a couple being the same without reference to its location on the shaft. Obviously, however, if the balanced couples are any appreciable distance apart the rigidity of the shaft becomes a factor in design.

The principles discussed in the foregoing are applicable to the balancing of any number of masses not in the same longitudinal plane, and in such cases graphic analysis is most convenient. Usually it is necessary to introduce two balance weights, the case in which one weight is sufficient being a special one. In balancing the moments it is usually necessary to take the origin of moments in one of the normal planes that contains the path of one of the balance weights, thus eliminating one variable from the equations.

In Fig. 225 four revolving masses, W_1, W_2, W_3, and W_4, are shown rotating at the respective radii, r_1, r_2, r_3, and r_4. It is desired to balance the system by means of two additional unknown masses, W and W_5, rotating in the planes indicated and at unknown radii, r and r_5. As indicated in the preceding paragraph, the quantity ω^2/g may be omitted from the calculations, and the centrifugal forces of the several known weights may be taken as proportional to the vectors, W_1r_1, W_2r_2, W_3r_3 and W_4r_4, shown in Fig. 225 (d). The direction of each of these forces is, of course, radial.

Let the origin of moments be taken in the plane which contains

W, whence the several masses are at distances a_1, a_2, a_3, a_4, and a_5 from this plane. Since for equilibrium all moments must balance,

$$W_1r_1a_1 + W_2r_2a_2 + W_3r_3a_3 + W_4r_4a_4 + W_5r_5a_5 = 0$$

Here the only unknown is $W_5r_5a_5$. If the known moments be taken as proportional to the respective vectors in Fig. 225 (c), the vector polygon can be drawn, since these moments act in the longitudinal planes containing the several forces and hence their direction of action is known. The unknown moment, $W_5r_5a_5$, is thus determined in magnitude and

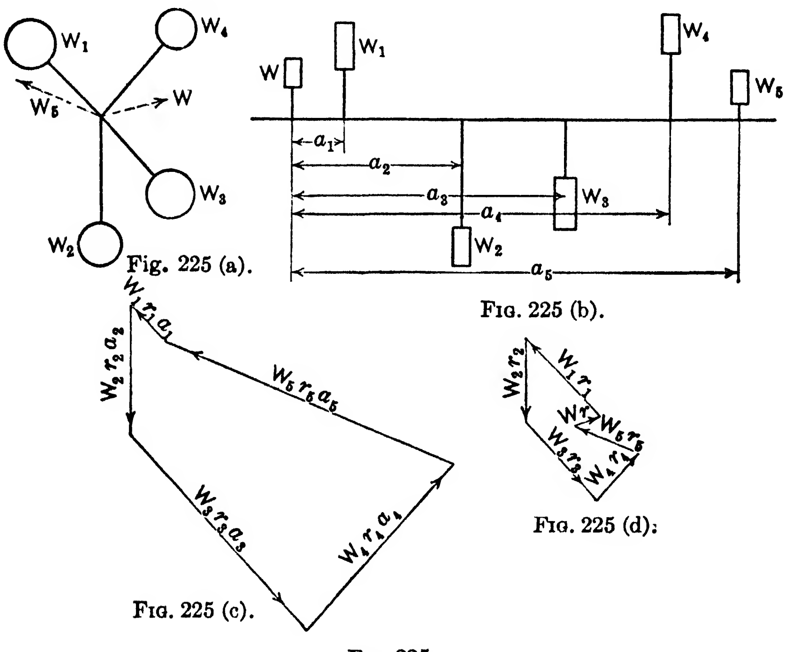

Fig. 225.

direction as the closing link of the polygon, and since a_5 is a known quantity W_5r_5 can be computed. For equilibrium also

$$Wr + W_1r_1 + W_2r_2 + W_3r_3 + W_4r_4 + W_5r_5 = 0$$

and since W_5r_5 is now known, Wr can be determined by the force polygon Fig. 225 (d) in which the known vectors are taken as proportional to Wr, W_1r_1, etc. The closing link determines Wr in magnitude and direction and, as before, if the weight of W and W_5 are selected, the radii can be computed.

242. Balancing Unknown Moments: Balancing Machines. In the foregoing discussion it has been assumed that the masses to be balanced are known in magnitude and position, but in many structures this is not the case. Thus, in large cast forms of any appreciable length, unbalanced moments usually exist, due to lack of homogeneity of the material. Such unbalanced moments are found to be present in carefully machined automobile shafts which normally should run in balance. Built-up structures, particularly those that are to carry loads other than their own weight, usually have such unbalanced moments initially, if they are of appreciable length compared to their diameter. Thus, the long rotors used in the electric generators of fast-running turbo-generators may have unbalanced moments initially, either because of lack of homogeneity in the metal of the rotor itself, or because of irregularities in the heavy wire coils that they carry. As previously noted, such structures, if made up of discs, can be balanced by balancing each disc statically; but in all other cases the balancing of unknown moments in revolving structures of appreciable length can usually be accomplished only by experimental methods. It should be noted that balancing a long rotating member statically may make the running balance worse. Static balance of such a member may be attained by placing the proper balancing weight along the axis; but from the discussion in Article 241 it will be clear that such a weight is more than likely to make the unbalanced moment worse instead of better.

Static balancing of moderately heavy members can be fairly well accomplished by mounting the member on a mandrel which rests upon level, hardened, knife-edged parallels. This method is not satisfactory, however, for very heavy bodies. In the case of members that may be considered as discs, special balancing machines are now much used. In some of these machines* the part to be balanced is supported in a horizontal plane by a flexible support that permits the heavy side of the member to tilt, thus indicating the location of the unbalanced mass.

Running balance, where the unbalanced moments are unknown, is accomplished by rotating the member and marking the points where it runs out of true. These marked points, properly interpreted, are an indication as to where weight should be added or taken away to secure balance. Such balancing requires skill and experience. A number of balancing machines are now upon the market for simplifying this process. In the Norton balancing machine the member is rotated under slight restraint and special scribers mark the " high " spots on the shaft. In using the Akimoff† balancing machine the part to be balanced

* See "Handbook for Machine Designers," by F. A. Halsey, page 301.

† See *Trans.* A.S.M.E., 1916.

is first put into static balance. It is then mounted in the balancing machine and driven in unison with a balanced squirrel-cage rotor This squirrel-cage carries a number of bars on its periphery that can be moved laterally, thus introducing a moment into the system By experiment a moment is thus found that balances the unknown moment in the machine member to be balanced, and vibration in the system ceases The location of the bars laterally determines the value and plane of the moment to be balanced

243. Relation to the Critical Speed. It was stated in Article 240 that if a body were in dynamic balance at one speed it would be in balance at any other speed This statement does not hold, of course, if the mass has been balanced at the critical speed as, obviously, a member might run in perfect balance at the critical speed, but be out of balance at any other speed (see Article 86) Most machinery, however, runs well below the critical speed It should be noted that up to the critical speed the heavy side runs " out," but at and above that speed the light side runs out.

244. Balancing Reciprocating Masses. The balancing of reciprocating and oscillating masses presents an entirely different problem from that of rotating masses and, in general, a much more difficult one. In the majority of such cases, the mass to be balanced is known in magnitude and direction of motion and the balancing forces necessary can usually be calculated It is difficult, however, to arrange balance weights that are effective in all positions and, generally, the most that can be done is to secure partial balance and to minimize the disturbing forces

These difficulties may be made more apparent by considering the horizontal steam engine mechanism shown in Fig 226 The static load due to the steam pressure between the piston and the cylinder head is balanced in both directions, but the force necessary to accelerate the piston is an unbalanced load and tends to set up horizontal vibrations in the machine The most obvious method of balancing this disturbing force is to arrange a heavy reciprocating weight moving in a horizontal plane and in opposite phase to the moving parts This is done occasionally, but the device is a clumsy one that occupies space and involves machine parts that must be cared for; hence it is little used In fact, this method of balancing reciprocating parts usually involves undesirable construction. Occasionally two machines exactly alike are opposed in such a way as to secure balance; thus, in Fig. 226, if another engine, exactly like that shown, were connected to a pin at P_1, the two machines would mutually balance each other.

A more usual plan is to secure partial balance by the use of revolving balance weights. There is an essential difference between revolving masses and reciprocating masses, in that the force exerted by a revolving mass is usually constant in magnitude but varying in direction, whereas the force exerted by a reciprocating mass is variable in magnitude and operates only in the fixed plane of action of the mass. In general, therefore, a revolving mass cannot balance a reciprocating mass, and *vice versa*.

Let W be the weight of the reciprocating parts in Fig. 226; then, neglecting the angularity of the connecting-rod, for simplicity, the force necessary to accelerate the reciprocating parts is $\frac{W}{g}(\omega^2 r \cos\theta)$, where r is the radius of the crank and θ the angle made by the crank radius with the line of action of the piston (see Article 5). If the angularity of the connecting-rod is neglected, as assumed, this force acts

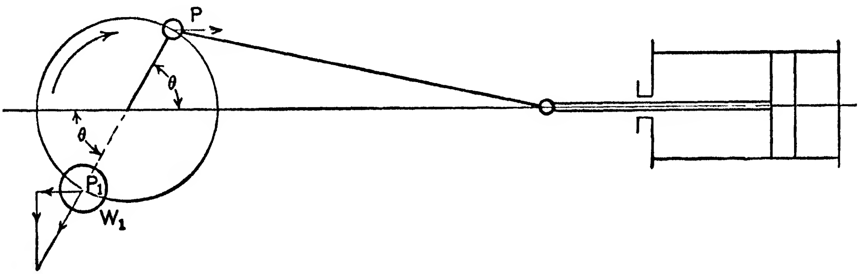

Fig. 226.

upon the crankpin in a direction parallel to the path of the piston. Suppose a rotating balance weight, W_1, equal to W, to be placed diametrically opposite to the crankpin and at the same distance from the center. The centrifugal force of this mass is $\frac{W_1}{g}\omega^2 r$. The horizontal component of this force is $\frac{W_1}{g}\omega^2 r \cos\theta$, and hence it just balances the inertia force at the crankpin. The vertical component of this force, $\frac{W}{g}\omega^2 r \sin\theta$, still remains unbalanced, so that, virtually, horizontal balance has been secured at the expense of vertical equilibrium. These vertical components are opposed, however, by the inertia of the foundation, which is usually of sufficient mass, in land engines, to absorb them, and this method of balance is much used in such machines.

It will be clear that if a single-cylinder vertical engine is balanced in the manner described in the foregoing, an unbalanced horizontal force will be introduced; and, since this will not be opposed by any stationary mass, undesirable vibrations may be set up. For this reason, such engines are often entirely unbalanced, the foundations being so constructed as to absorb vertical vibrations. Where such massive foundations cannot be secured, as in vertical marine engines, good balancing can be attained by the use of several cylinders. For a more complete discussion of this problem, reference should be made to treatises on the balancing of engines.

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